Limits of graphs and number theory: a case for spectral zeta functions

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This publication URL: https://archive-ouverte.unige.ch/unige:92363

Publication DOI: 10.13097/archive-ouverte/unige:92363

Université de Genève
Section de Mathématiques

Faculté des Sciences
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Limits of graphs and number theory: a case for spectral zeta functions

THESE

présentée à la Faculté des Sciences de l'Université de Genève
pour obtenir le grade de Docteur ès Sciences, mention Mathématiques

par

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de

Genève (Suisse)

Thèse N°5018

GENEVE
Atelier d'impression ReproMail
2017

UNIVERSITÉ DE GENÈVE

FACULTÉ DES SCIENCES

Doctorat ès Sciences
Mention mathématiques

Thèse de *Monsieur Fabien FRIEDLI*

intitulée :

"Limits of Graphs and Number Theory: a Case for Spectral Zeta Functions"

La Faculté des sciences, sur le préavis de Monsieur A. KARLSSON, professeur associé et directeur de thèse (Section de mathématiques), Monsieur A. DIETMAR, professeur (Mathematisch-Naturwissenschaftliche Fakultät, Universität Tübingen, Deutschland), Monsieur J. JORGENSON, professeur (The City College of New York, U.S.A.) et Madame L. SMAJLOVIC, professeure (Département de mathématiques, Université de Sarajevo, Bosnie Herzégovine), autorise l'impression de la présente thèse, sans exprimer d'opinion sur les propositions qui y sont énoncées.

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N.B. - La thèse doit porter la déclaration précédente et remplir les conditions énumérées dans les "Informations relatives aux thèses de doctorat à l'Université de Genève".

Résumé

Dans cette thèse, on s'intéresse principalement aux fonctions zetas spectrales de graphes. Ce sont des fonctions d'une variable complexe associées à des graphes qui font intervenir le spectre du Laplacien discret. On peut également les définir via le noyau de la chaleur (c'est-à-dire la solution d'une équation différentielle appelée l'équation de la chaleur faisant intervenir le Laplacien), comme une transformée de Mellin de ce dernier, ce qui permet d'étendre la définition à des graphes infinis.

Dans le premier chapitre, le résultat principal est une formule asymptotique pour les fonctions zetas spectrales de graphes correspondant à des tores discrets dont le nombre de sommets tend vers l'infini. Dans cette formule apparaissent les fonctions zetas du graphe de Cayley des entiers $\mathbb{Z}^d$ ainsi que du tore continu. En dimension 1, on montre que la fonction zeta des entiers, $\zeta_{\mathbb{Z}}$, satisfait une équation fonctionnelle du type $s \longleftrightarrow 1-s$ très similaire à celle satisfaite par la fonction zeta de Riemann. De plus, en utilisant la formule asymptotique et des résultats récents de Matiyasevich-Saidak-Zwengrowski, on montre qu'une certaine équation fonctionelle asymptotique pour les fonctions zetas des tores discrets est équivalente à l'hypothèse de Riemann. Ainsi, ces fonctions zetas sont intéressantes tant d'un point de vue combinatoire (via le spectre du Laplacien) que d'un point de vue de théorie analytique des nombres. Le cas particulier de la valeur de la dérivée en zéro, qui correspond au nombre d'arbres couvrants du graphe dans le cas d'un graphe fini, est également lié à des questions de mathématique physique. Nous montrons également comment retrouver certaines valeurs spéciales connues de la fonction zeta de Riemann à l'aide de notre formule asymptotique et discutons le cas des fonctions zetas des arbres réguliers.

Dans le deuxième chapitre, on montre que cette équivalence avec l'hypothèse de Riemann est vraie dans un contexte plus large, celui des fonctions L de Dirichlet. En effet, on peut définir des fonctions L spectrales sur des tores discrets en ajoutant un caractère de Dirichlet et faire la même étude asymptotique. L'équivalence avec l'hypothèse de Riemann généralisée subsiste dans ce cas. On note également que le problème très important de l'existence de zéros réels de certaines fonctions L de Dirichlet admet une reformulation en termes de positivité des fonctions L de tores discrets.

Dans le troisième chapitre, on garde la même suite de graphes, à savoir des tores discrets, mais l'on ajoute des poids sur les arêtes et l'on étudie une modification du Laplacien agissant sur ces graphes. Cela revient à considérer des fibrés vectoriels attachés au graphes, comme expliqué dans un article de Kenyon datant de 2011. Le déterminant de cet opérateur a une interprétation combinatoire en termes de "cycle-rooted spanning forests", résultat similaire au théorème de Kirchhoff sur les arbres couvrants. On établit une formule asymptotique pour ce déterminant lorsque les tores discrets approchent le tore continu, qui fait intervenir la fonction zeta d'Epstein-Hurwitz, en particulier la dérivée en zéro, qui peut être étudiée par une identité ressemblant à la formule limite de Kronecker. On montre aussi une formule asymptotique pour les fonctions zetas correspondantes.

Enfin, dans le dernier chapitre, on initie une discussion sur la fonction $\zeta_{\mathbb{Z}^2}$ en montrant qu'elle satisfait une sorte d'équation fonctionnelle sur les entiers.

Acknowledgements

Je tiens tout d'abord à remercier mon directeur de thèse Anders Karlsson pour toute son aide, ses encouragements, ses idées, nos discussions mathématiques et également pour sa gentillesse. Cela fut un plaisir de pouvoir travailler avec lui, sans pression et dans une ambiance très agréable.

Je remercie également Anton Deitmar, Jay Jorgenson et Lejla Smajlovic d'avoir consenti à être jurés pour la soutenance et pour leur lecture de ma thèse et leurs commentaires.

Durant mon travail de recherche, j'ai été supporté par le FNS 200021 132528/1 et j'en suis très reconnaissant.

J'ai eu la chance de pouvoir bénéficier de conditions de travail très confortables au sein de la Section de mathématiques de l'Université de Genève. J'en remercie tous les membres qui m'ont entouré tout au long de mes années d'études et de doctorat.

Enfin, mille mercis à Cécile, à ma famille et à ma belle-famille.

Contents

Introduction

The magic of zeta functions

Riemann, Dedekind, Hurwitz, Epstein, Hasse, Weil, Selberg, Ruelle, Ihara: what do these great mathematicians have in common? They all have a zeta function named after them! This is an impressive list of people, which is an indication of the importance of this mathematical object.
The use of generating functions in order to study sequences of numbers goes back at least to the early 18th century and de Moivre, according to [15]. Euler used them a lot, for example in the theory of partitions. Since then it has become a very important tool which allows for the translation of a discrete problem into an analytic one. Often the generating function exhibits some analytic structure or symmetries which yield informations about the sequence we started with. In number theory this philosophy has proven to be particularly profound and successful, where the generating functions considered are often Dirichlet series and called zeta or L functions. There is not really a precise definition of a zeta function, but "we know one when we see one", in the words of M. Huxley. Usually, a generating function is called a zeta function if it has a meromorphic continuation, a functional equation, an Euler product and interesting special values. In many cases, a zeta function is both a fruitful construction and at the same time a mysterious entity for which some of the deepest conjectures in mathematics are avalaible.

In this thesis, motivated by this way of thinking and inspired by the definition of spectral zeta functions on manifolds, we study spectral zeta functions associated to the spectrum of the Laplacian on graphs.

The Riemann zeta function

It all started with the work of Euler first and his very important Euler product formula and then, many years later, with Riemann's famous 1859 paper "Über die Anzahl der Primzahlen unter einer gegebenen Grösse". In this masterpiece Riemann proved, among other things, the functional equation for the Riemann zeta function, established a link between the prime numbers counting function and the zeros of the zeta function and of course stated his Hypothesis. The key idea is that information about the *analytic continuation* of zeta, in particular the location of its zeros, can be very useful although the function is initially not defined everywhere. In order to put some perspective to the work in the present thesis, we recall in the sequel the main results (and some proofs) involving the Riemann zeta function, which is historically (and arguably also in terms of importance) the first instance of a zeta function.

Definition 0.1 *For complex numbers s such that $Re(s) > 1$ we define the* Riemann zeta function *by*

$$\zeta(s) := \sum_{n\geq 1} \frac{1}{n^s}.$$

In 1737, Euler observed that this sum admits a representation as a product over the prime numbers. Let $\mathcal{P} = \{2, 3, 5, \dots\}$ denote the set of prime numbers.

Theorem 0.2 (Euler, 1737) *For $Re(s) > 1$ we have*

$$\zeta(s) = \prod_{p\in\mathcal{P}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

It is an easy consequence of the geometric series and the fundamental theorem of arithmetic.
This formula is fundamental and exhibits an important link between $\zeta(s)$ and the prime numbers. In particular, we can deduce from it the fact that the series

$$\sum_{p\in\mathcal{P}} \frac{1}{p}$$

is divergent, which is a first quantitative result about the distribution of primes. Furthermore we can easily deduce from this formula that $\zeta(s) \neq 0$ if $\mathrm{Re}(s) > 1$.
Euler also computed the special values at even integers: for $n \geq 0$, we have

$$\zeta(2n) = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!},$$

where B_{2n} is a Bernoulli number. By contrast, the numbers $\zeta(2n+1)$ are still very mysterious, in particular we don't know if they are irrational or not (except for the so-called Apéry constant $\zeta(3)$).
Finally, long before Riemann, Euler guessed the functional equation.

Then came Riemann, who introduced the idea of using the analytic continuation of $\zeta(s)$ to get information about prime numbers. One way to see that the Riemann zeta function can be analytically continued to the whole complex plane, except for a simple pole at $s = 1$, is to write it as a Mellin transform of a theta function, which is an important idea that we use in this thesis. Define the theta function by

$$\theta(t) := \sum_{n\in\mathbb{Z}} e^{-n^2 t}.$$

It has a beautiful symmetry, known as the Poisson-Jacobi theta inversion formula, which is a consequence of the Poisson summation formula: for all $t > 0$ we have

$$\theta(t) = \sqrt{\frac{\pi}{t}}\,\theta(\pi^2/t). \tag{0.1}$$

Using the definition of the Gamma function it is easy to see that, for $\mathrm{Re}(s) > \frac{1}{2}$, we have

$$2\zeta(2s) = \frac{1}{\Gamma(s)} \int_0^\infty (\theta(t) - 1) t^{s-1} dt. \tag{0.2}$$

The integral is indeed convergent for $\mathrm{Re}(s) > \frac{1}{2}$ in view of the two following asymptotics. First,

$$\theta(t) = 1 + O(e^{-c_1 t}) \quad (t \longrightarrow \infty)$$

for a constant $c_1 > 0$, which follows from the definition of $\theta(t)$. Second,

$$\theta(t) = \sqrt{\frac{\pi}{t}} + O(e^{-c_2/t}) \quad (t \longrightarrow 0^+)$$

for a constant $c_2 > 0$, which follows from (0.1).
If we change variables in (0.2) and write $\frac{s}{2}$ instead of s we get

$$
\begin{aligned}
2\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) &= \int_0^\infty (\theta(\pi t) - 1)t^{\frac{s}{2}-1}dt \\
&= \int_1^\infty (\theta(\pi t) - 1)t^{\frac{s}{2}-1}dt + \int_0^1 \Big(\theta(\pi t) - \frac{1}{\sqrt{t}}\Big)t^{\frac{s}{2}-1}dt + \frac{2}{s(s-1)} \\
&= \int_0^1 \Big(\theta(\frac{\pi}{t}) - 1\Big)t^{-1-\frac{s}{2}}dt + \int_0^1 \Big(\theta(\pi t) - \frac{1}{\sqrt{t}}\Big)t^{\frac{s}{2}-1}dt + \frac{2}{s(s-1)} \\
&= \int_0^1 \Big(\theta(\pi t) - \frac{1}{\sqrt{t}}\Big)\Big(t^{\frac{s}{2}-1} + t^{-\frac{1}{2}-\frac{s}{2}}\Big)dt + \frac{2}{s(s-1)},
\end{aligned}
$$

where we used (0.1) in the last equality.
Due to the asymptotics we just mentioned, the right-hand side of this equality defines a meromorphic function on the complex plane with simple poles at $s = 0$ and $s = 1$. Since $\Gamma(s)$ also has a simple pole at $s = 0$, this shows that the Riemann zeta function can be analytically continued to $\mathbb{C}\backslash\{1\}$. Moreover it is obvious from the last expression that the function $\xi(s) := \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ satisfies the so-called *functional equation*

$$\xi(s) = \xi(1-s),$$

for any $s \in \mathbb{C}\backslash\{0,1\}$. Finally, since $\Gamma(s)$ has simple poles at $s = -2n$ ($n \in \mathbb{N}^*$), we deduce that the Riemann zeta function has zeros at negative even integers, We call them *trivial zeros*.
At the end of the 19th century it was shown by Hadamard and de la Vallée Poussin that the Riemann zeta function does not vanish on the line $\text{Re}(s) = 1$, which was a great achievement since it implies the Prime Number Theorem, which asserts that

$$\pi(x) \sim \frac{x}{\log x}$$

when $x \longrightarrow \infty$, with $\pi(x) := |\{p \leq x : p \text{ is prime}\}|$ the prime numbers counting function.

To summarize, we know by Euler's formula, the result of Hadamard and de la Vallée Poussin and the functional equation that $\zeta(s) \neq 0$ for $\text{Re}(s) \geq 1$ and $\text{Re}(s) \leq 0$, if $s \neq -2n$. So what about the strip $0 < \text{Re}(s) < 1$? It is called the *critical strip* and $\text{Re}(s) = \frac{1}{2}$ is the *critical line*. The location of the zeros of the Riemann zeta function inside the critical strip is one of the greatest mystery in mathematics.

Conjecture 1 (Riemann Hypothesis) *If $\zeta(s) = 0$ and $0 < Re(s) < 1$, then $Re(s) = \frac{1}{2}$. In words, all non trivial zeros lie on the critical line.*

This would greatly improve on the Prime Number Theorem, giving the best possible error term and implying that the prime numbers seem to behave as if they were "randomly" distributed.

Many other zeta functions have been considered since then and one example is particularly relevant for the content of this thesis, the Ihara zeta function. Indeed, this is a zeta function associated to graphs and modeled on the Euler product of the Riemann zeta function, whereas spectral zeta functions of graphs we consider here are modeled on the Dirichlet series representation of the Riemann zeta function.

The Ihara zeta function

Let G be a finite graph. Roughly speaking, a prime in G is an equivalence class of (finite) closed paths in G (with two paths being equivalent if one is a cyclic permutation of the other one) which have no backtracking (the same edge cannot be used twice consecutively in the path) and no tail (the first edge must be different from the last one) and which are not powers of other closed paths without backtracking or tail. The length of a prime is the number of edges in the path. See [22],[1] or [3] for a precise definition.
For u a complex variable in a disk with sufficiently small radius, the *Ihara zeta function* of G is defined by

$$\zeta_G(u) = \prod_P \left(1 - u^{\nu(P)}\right)^{-1},$$

where the product is over all primes P in G and $\nu(P)$ is the length of P. This product can be written as a formal power series in u with positive coefficients and so, by a theorem of Landau, the product is convergent in some disk in the complex plane, see [22].
We have the following spectral interpretation of the Ihara zeta function ([22], [1], [3]).

Theorem 0.3 *Let $G = (V, E)$ be a finite graph, A its adjacency matrix and D the degree matrix. Then*

$$\zeta_G(u)^{-1} = (1 - u^2)^{|E|-|V|} \det(I - Au + (D - I)u^2).$$

This gives an analytic continuation for the Ihara zeta function, which is in fact the inverse of a polynomial.
Ihara zeta functions of regular graphs satisfy functional equations. There is also an analog of the Riemann Hypothesis, which is satisfied by the Ihara zeta function if and only if the graph is Ramanujan, see [22].
This function has attracted a lot of attention and usually, when one speaks about the zeta function of a graph, one means the Ihara zeta function. The main purpose of this thesis is to initiate the study of a different zeta function that we can associate to a graph, the *spectral zeta function.*

Discrete Laplacian, heat kernel and spanning trees

The basic object in this thesis (in addition to the graphs we will consider) is the discrete (or combinatorial) Laplacian. This is an operator which acts on functions defined on the vertex set of a given graph.

Definition 0.4 *Let $G = (V, E)$ be a countable graph with bounded degree, that is the degree of every vertex is less or equal than some fixed constant B. The* discrete Laplacian $\Delta : L^2(V(G)) \longrightarrow L^2(V(G))$ *is defined by*

$$\Delta f(x) = \sum_{y \sim x} (f(x) - f(y)).$$

If the graph is finite, the Laplacian can be written as a matrix and is closely related to the adjacency matrix of the graph:

$$\Delta = D - A,$$

where the D is the diagonal matrix with the degree of the i-th vertex at the i-th entry on the diagonal and A is the adjacency matrix.
In that case the Laplacian is a real symmetric diagonally-dominant matrix and thus all its eigenvalues are real and non-negative. It is easy to see that 0 is always an eigenvalue (any constant vector is sent to 0 by Δ). In fact the multiplicity of 0 as an eigenvalue is equal to the number of connected components of the graph. The spectrum of the Laplacian is a very interesting invariant of graphs and contains a great quantity of combinatorial and geometric informations about it. An important example is the matrix-tree theorem of Kirchhoff. Before stating the theorem we recall that a spanning tree in a graph G is a subgraph of G which is a tree and which contains every vertex of G. The number of spanning trees in a graph G is called the *complexity* of G and is written $\tau(G)$.

Theorem 0.5 (Kirchhoff) *Let G be a connected finite graph with n vertices. Then we have*

$$\tau(G) = \frac{1}{n}\det{}^*\Delta = \frac{1}{n}\lambda_1\lambda_2\ldots\lambda_{n-1},$$

where $\lambda_1,\ldots,\lambda_{n-1}$ are the non-zero eigenvalues of Δ.

This theorem relates spectral information to combinatorial information on the graph. This leads to the use of analytical techniques in order to estimate $\tau(G)$ for some particular graphs. A very important example of this philosophy is the paper [4] by Chinta, Jorgenson and Karlsson, in which they compute precise asymptotics for the number of spanning trees in sequences of discrete tori with the number of vertices tending to infinity (see also [5]). The techniques used in [4] are important ingredients for the papers we present in this thesis.
The main idea in [4] is to compute $\tau(G)$ via some integral transform of the discrete heat kernel and a nice formula which allows to write this heat kernel in terms of modified Bessel functions.
For a countable graph G with bounded degree and a fixed vertex $x_0 \in V(G)$, the (discrete) *heat kernel* at x_0 is the unique bounded solution $K_{x_0}(t,x) : \mathbb{R}_+ \times G \longrightarrow \mathbb{C}$ of the heat equation

$$\Big(\Delta + \frac{\partial}{\partial t}\Big)K_{x_0}(t,x) = 0$$

which satisfies the initial condition $K_{x_0}(0,x) = 1$ if $x = x_0$ and $K_{x_0}(0,x) = 0$ otherwise (see [7] and [6] for the existence and unicity).
If the graph is finite with n vertices and with eigenvalues of the Laplacian given by $\lambda_0,\ldots,\lambda_{n-1}$, then it is easy to see that

$$\sum_{j=0}^{n-1} e^{-\lambda_j t} = \sum_{x\in V(G)} K_x(t,x). \tag{0.3}$$

Indeed ([13]) in this case the solution to the differential equation above defining the heat kernel is given by

$$K_{x_0}(t,x) = e^{-t\Delta}\delta_{x_0}(x),$$

where δ_{x_0} is the Kronecker delta. Since the Laplacian Δ is symmetric, it has an orthonormal basis of eigenvectors $\phi_0, \dots, \phi_{n-1}$ and we have $\delta_{x_0} = \sum_{j=0}^{n-1} \overline{\phi_j(x_0)}\phi_j$. Thus we can write

$$K_{x_0}(t,x) = \sum_{j=0}^{n-1} e^{-\lambda_j t}\phi_j(x)\overline{\phi_j(x_0)},$$

and we can deduce (0.3).
The technique developed in [4] consists of starting from the heat equation to deduce a formula for the theta function $\sum_{j=0}^{n-1} e^{-\lambda_j t}$ using (0.3) and then applying some integral transform to get the (star)-determinant of the Laplacian and so the number of spanning trees. Later, this procedure has also been successfully used to compute asymptotics of the number of spanning trees in other graphs, such as circulant graphs, see [17] and [16].
In this thesis we will make use of another integral transform (namely the Mellin tansform) of the theta function to arrive at the definition of the spectral zeta function of a graph.

Spectral zeta functions of graphs

Given a compact Riemannian manifold we can define a spectral zeta function associated to the spectrum of the Laplace-Beltrami operator on this manifold:

$$\zeta(s) = \sum_{\lambda \neq 0} \frac{1}{\lambda^s},$$

where λ takes values in the spectrum of the Laplacian.
Such a function was considered first by Carleman for bounded domains in $\mathbb{R}^2$ ([2]) and then by Pleijel and Minakshisundaram for more general manifolds ([19], [18]). It has proven to be a very useful object, for example for the definition of the analytic torsion or in different areas in theoretical physics, see for example [8], [12] and [20]. For instance, if we consider perhaps the most simple example, the circle, we obtain the Riemann zeta function, as the eigenvalues of the Laplacian acting on the circle are given by the squares of the integers.

Some graphs can naturally be viewed as discrete analogs to manifolds, hence it seems interesting to consider spectral zeta functions for graphs modeled on spectral zeta functions for manifolds. In particular, if a sequence of graphs approximates in some sense a manifold in the limit, it is expected that the asymptotics of the number of spanning trees reflects this convergence and contains geometric information on both the graphs and the manifold. See the discussion in [21] for a more precise formulation of this philosophy and for motivation from a physical point of view (especially from quantum field theory).

Another reason to be interested in such zeta functions for graphs (instead of the Ihara zeta function) is simply the fact that it encodes the spectrum of the (discrete) Laplacian. In particular, many symmetric functions of the eigenvalues have combinatorial interpretations and can be recovered from special values of the spectral zeta function. For example, for a finite graph, we will have

$$e^{-\zeta_G'(0)} = \lambda_1\lambda_2\dots\lambda_{n-1},$$

so that the number of spanning trees in G is a particular value of the spectral zeta function of G.

We will give a precise definition of each spectral zeta function of graph we will consider, but the general informal definition is the following. If G is a finite graph, the *sepctral zeta function* of G is the Mellin transform of the trace of the heat kernel, that is

$$\begin{aligned}\zeta_G(s) &= \frac{1}{\Gamma(s)}\int_0^\infty \Big(\sum_{x\in V(G)} K_x(t,x) - 1\Big)t^{s-1}dt\\ &= \sum_{j=0}^{n-1}\frac{1}{\lambda_j^s},\end{aligned}$$

using (0.3) and the definition of the Euler Gamma function.
If the graph is infinite and transitive, the heat kernel $K_x(t,x)$ evaluated at the base point is independent of the base point and we define

$$\zeta_G(s) = \frac{1}{\Gamma(s)}\int_0^\infty K_x(t,x)t^{s-1}dt$$

for some $x \in V(G)$.

Structure of the thesis and results

The present thesis is divided into four sections, which are ordered in chronological order of writing. The first three ones correspond to three different papers.

In the first one [11], in collaboration with Karlsson, we define and initiate the study of spectral zeta functions of discrete tori, $\mathbb{Z}^d$ and regular trees. We consider a sequence of discrete tori whose number of vertices goes to infinity and establish an asymptotic formula which relates the spectral zeta functions of the discrete tori, the spectral zeta function of $\mathbb{Z}^d$ and the Epstein zeta function, which is in fact the spectral zeta function of a continuous torus.

Theorem 0.6 *As $n \longrightarrow \infty$ we have (for $s \neq \frac{d}{2}$)*

$$\zeta_{\mathbb{Z}^d/A_n\mathbb{Z}^d}(s) = \zeta_{\mathbb{Z}^d}(s)\det A\, n^d + \zeta_{\mathbb{R}^d/A\mathbb{Z}^d}(s)n^{2s} + o(n^{2s}).$$

This "three zetas" formula can be used to recover some special values of the d-dimensional Epstein zeta function, which reduces to Riemann zeta if $d = 1$. For instance, we show how to deduce the trivial zeros from this result or how to give a combinatorial proof of the Basel problem $\zeta(2) = \frac{\pi^2}{6}$.
We then focus on dimension $d = 1$ and prove that $\zeta_{\mathbb{Z}}$ admits a functional equation of the type $s \longleftrightarrow 1-s$ very similar to the functional equation satisfied by the Riemann zeta function.

Theorem 0.7 *Let the completed zeta function for $\mathbb{Z}$ be defined as*

$$\xi_{\mathbb{Z}}(s) = 2^s\cos(\pi s/2)\zeta_{\mathbb{Z}}(s/2).$$

Then this is an entire function that satisfies for all $s \in \mathbb{C}$

$$\xi_{\mathbb{Z}}(s) = \xi_{\mathbb{Z}}(1-s).$$

Wondering if such a relation could hold for the spectral zeta function of the cycle $\zeta_{\mathbb{Z}/n\mathbb{Z}}$, at least asymptotically, we arrive at a surprising equivalence with the Riemann Hypothesis. Indeed, for $0 < \mathrm{Re}(s) < 1$ let

$$h_n(s) = (4\pi)^{s/2}\Gamma(s/2)n^{-s}\Big(\zeta_{\mathbb{Z}/n\mathbb{Z}}(s/2) - n\zeta_{\mathbb{Z}}(s/2)\Big),$$

which is essentially the difference between the two spectral zeta functions of the cycle and of the integers, normalized and completed by some usual fudge factors.

Theorem 0.8 *The asymptotic functional equation (as $n \longrightarrow \infty$)*

$$|h_n(s)| \sim |h_n(1-s)|$$

is satisfied if and only if the Riemann Hypothesis is true.

In the second paper [10], we consider the same graphs in dimension one but treat the case of L-functions instead of zeta functions, that is the same functions twisted with a Dirichlet character. More precisely, given an even Dirichlet character χ modulo k and an integer n we define

$$L_n(s,\chi) = \sum_{j=1}^{kn-1} \frac{\chi(j)}{\sin(\frac{\pi j}{kn})^s}.$$

This is the L-function of the cyclic graph with kn vertices (recall that the Laplace eigenvalues of this graph are given by the set $\{4\sin^2(\frac{\pi j}{kn}), 0 \leq j \leq kn-1\}$).
We show that similar asymptotics are avalaible in this context, with the notable difference that there is no term corresponding to the $\zeta_{\mathbb{Z}}$ term in Theorem 0.6.

Theorem 0.9 *For an even primitive Dirichlet character modulo $k \geq 3$ we have*

$$L_n(s,\chi) = 2\left(\frac{kn}{\pi}\right)^s \left(L(s,\chi) + \frac{s}{6}\left(\frac{kn}{\pi}\right)^{-2} L(s-2,\chi) + O\Big(\frac{1}{n^4}\Big)\right)$$

as $n \longrightarrow \infty$.

This formula leads to an analog of Theorem 0.8. Due to the absence of a first term corresponding to $\zeta_{\mathbb{Z}}$, the equivalence with the Riemann Hypothesis below is even more suggestive. As usual we complete (and normalize) L_n by writing

$$\xi_n(s,\chi) = n^{-s}(\pi/k)^{s/2}\Gamma(s/2)L_n(s,\chi).$$

Then we have the following equivalence.

Theorem 0.10 *The completed L-function ξ_n satisfies the asymptotic functional equation*

$$|\xi_n(s,\chi)| \sim |\xi_n(1-s,\overline{\chi})|$$

for $0 < Re(s) < 1$ and $Im(s) \geq 8$ if and only if the Dirichlet L-function $L(s,\chi)$ satisfies the Generalized Riemann Hypothesis in the same region.

This should be compared with the standard functional equation satified by Dirichlet L-functions of even characters:

$$|\xi(s,\chi)| = |\xi(1-s,\overline{\chi})|,$$

where $\xi(s,\chi) = (\pi/k)^{-s/2}\Gamma(s/2)L(s,\chi)$.
We also remark that Theorem 0.9 gives a link between the famous and important problem of the existence of real zeros for Dirichlet L-functions and our graph L-functions.

Corollary 0.11 *If χ is primitive, even and real, then we have*

$$L(s,\chi) > 0$$

if and only if

$$L_n(s,\chi) \geq 0$$

for infinitely many n.

We end that paper by stating two open questions about the positivity of some character sums related to $L_n(s,\chi)$.

In the third paper [9], we are concerned with the line bundle Laplacian on discrete tori, see [14]. This more or less amounts to consider weighted dicrete tori. The determinant of the bundle Laplacian also has a combinatorial interpretation in terms of cycle-rooted spanning forests. We establish the following asymptotics for the determinant of the bundle Laplacian $\det\Delta$ on the graph G_n defined in [9].

Theorem 0.12 *As $n \longrightarrow \infty$ we have*

$$\log\det\Delta = \Big(\prod_{i=1}^{d} a_i(n)\Big)c_d - \zeta'_{EH}(0;\alpha_1,\ldots,\alpha_d;\lambda_1,\ldots,\lambda_d) + o(1),$$

where

$$c_d = -\int_0^\infty \Big(e^{-2dt}I_0(2t)^d - e^{-t}\Big)\frac{dt}{t}$$

and

$$\zeta_{EH}(s;\alpha_1,\ldots,\alpha_d;\lambda_1,\ldots,\lambda_d) = (2\pi)^{-2s}\sum_{K\in\mathbb{Z}^d}\Big(\Big(\frac{k_1+\lambda_1}{\alpha_1}\Big)^2+\ldots+\Big(\frac{k_d+\lambda_d}{\alpha_d}\Big)^2\Big)^{-s}.$$

The Epstein-Hurwitz zeta function ζ_{EH} which appears here is an important quantity in physics and especially its derivative at $s=0$, see [8]. When the dimension is equal to 2 we give an expression for this quantity, reminiscent of the Kronecker limit formula.

Theorem 0.13 *We have*

$$\zeta'_{EH}(0;\alpha_1,\alpha_2;\lambda_1,\lambda_2) = 2\pi\frac{\alpha_1}{\alpha_2}B_2(\lambda_2) - 2\log\prod_{n\in\mathbb{Z}}\Big|1-e^{2\pi i\lambda_1}e^{-2\pi\frac{\alpha_1}{\alpha_2}|n+\lambda_2|}\Big|,$$

where $B_2(x) = x^2 - x + \frac{1}{6}$ stands for the second Bernoulli polynomial.

This leads to nice looking identities, such as the following one.

Corollary 0.14

$$\frac{\prod_{n\geq 1}(1+e^{-2n\pi})}{\prod_{n\geq 0}(1-e^{-(2n+1)\pi})} = \frac{e^{\pi/8}}{\sqrt{2}}.$$

Then we prove the zeta-version of the asymptotics in Theorem 0.12, similar to Theorems 0.6 and 0.9.

Theorem 0.15 *Let $s \in \mathbb{C}$ such that $s \neq m + \frac{d}{2}$ for $m \in \mathbb{N}$. Then as $n \longrightarrow \infty$ we have*

$$\zeta_{G_n}(s) = \Big(\prod_{i=1}^{d} a_i(n)\Big)\zeta_{\mathbb{Z}^d}(s) + \zeta_{EH}(s; \alpha_1, \ldots, \alpha_d; \lambda_1, \ldots, \lambda_d)n^{2s} + o(n^{2s}).$$

The fourth section is devoted to a small discussion about the function $\zeta_{\mathbb{Z}^2}(s)$ and whether it might satisfy a functional equation of the same type as the others ($s \longleftrightarrow 1-s$) or not. We show that a surprising relation holds at the integers.

Proposition 0.16 *Let $n \geq 0$ be an integer. Then*

$$Res(f, n+1) = -\frac{\sqrt{2}}{\pi} f(-n),$$

where

$$f(s) := 2^{5s/2}\zeta_{\mathbb{Z}^2}(s).$$

To guess the functional equation of the Riemann zeta function, Euler proceeded in the same way: he noticed a relation between values of Riemann zeta at integers and then conjectured the general formula. We were hopeful to try the same approach using Proposition 0.16, but we have not succeed until now. This leads to an interesting problem: is there another graph, different from $\mathbb{Z}$, whose spectral zeta function exhibits a functional equation of the type $s \longleftrightarrow 1-s$ as $\zeta_{\mathbb{Z}}$ does ?

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Spectral zeta functions of graphs and the Riemann zeta function in the critical strip

Fabien Friedli and Anders Karlsson*

Abstract

We initiate the study of spectral zeta functions ζ_X for finite and infinite graphs X, instead of the Ihara zeta function, with a perspective towards zeta functions from number theory and connections to hypergeometric functions. The Riemann hypothesis is shown to be equivalent to an approximate functional equation of graph zeta functions. The latter holds at all points where Riemann's zeta function $\zeta(s)$ is non-zero. This connection arises via a detailed study of the asymptotics of the spectral zeta functions of finite torus graphs in the critcal strip and estimates on the real part of the logarithmic derivative of $\zeta(s)$. We relate $\zeta_{\mathbb{Z}}$ to Euler's beta integral and show how to complete it giving the functional equation $\xi_{\mathbb{Z}}(1-s) = \xi_{\mathbb{Z}}(s)$. This function appears in the theory of Eisenstein series although presumably with this spectral intepretation unrecognized. In higher dimensions d we provide a meromorphic continuation of $\zeta_{\mathbb{Z}^d}(s)$ to the whole plane and identify the poles. From our aymptotics several known special values of $\zeta(s)$ are derived as well as its non-vanishing on the line $Re(s) = 1$. We determine the spectral zeta functions of regular trees and show it to be equal to a specialization of Appell's hypergeometric function F_1 via an Euler-type integral formula due to Picard.

Introduction

In order to study the Laplace eigenvalues λ_n of bounded domains D in the plane, Carleman employed the function

$$\zeta_D(s) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s}$$

taking advantage of techniques from the theory of Dirichlet series including Ikehara's Tauberian theorem [Ca34]. This was followed-up in [P39], and developed further in [MP49] for the case of compact Riemannian manifolds. These zeta functions have since played a role in the definitions of determinants of Laplacians and analytic

2010 *Mathematics Subject Classification.* Primary 11M; Secondary 10H05, 05C50, 33C65.

Key words and phrases. Zeta functions, combinatorial Laplacian, functional equations, hypergeometric functions, Riemann Hypothesis.

*The authors were supported in part by the Swiss NSF grant 200021 132528/1.

torsion, and they are important in theoretical physics [Ha77, El12, RV15]. For graphs it has been popular and fruitful to study the Ihara zeta function, which is an analog of the Selberg zeta function in turn modeled on the Euler product of Riemann's zeta function. Serre noted that Ihara's definition made sense for any finite graph and this suggestion was taken up and developed by Sunada, Hashimoto, Hori, Bass and others, see [Su86, Te10].

The present paper has a three-fold objective. First, we advance the study of spectral zeta functions of graphs, instead of the Ihara zeta function. We do this even for infinite graphs where the spectrum might be continuous. For the most fundamental infinite graphs, this study leads into the theory of hypergeometric function in several variables, such as those of Appell, and gives rise to several questions.

Second, we study the asymptotics of spectral zeta functions for finite torus graphs as they grow to infinity, in a way similar to what is often considered in statistical physics (see for example [DD88]). The study of limiting sequences of graphs is also a subject of significant current mathematical interest, see [Lo12, Ly10, LPS14]. Terms appearing in our asymptotic expansions are zeta functions of lattice graphs and of continuous torus which are Epstein zeta function from number theory. This relies to an important extent on the work of Chinta, Jorgenson, and the second-named author [CJK10], in particular we quote and use without proof several results established in this reference.

Third, we provide a new perspective on some parts of analytic number theory, in two ways. In one way, this comes via replacing partial sums of Dirichlet series by zeta functions of finite graphs. Although the latter looks somewhat more complicated, they have more structure, being a spectral zeta function, and are decidedly easier in some respects. We show the equivalence of the Riemann hypothesis with a conjectural functional equation for graph spectral zeta functions, and this seems substantially different from other known reformulations of this important problem [RH08]. In a second way, the spectral zeta function of the graph $\mathbb{Z}$ enjoys properties analogous to the Riemann zeta function, notably the relation $\xi_{\mathbb{Z}}(1-s) = \xi_{\mathbb{Z}}(s)$, and it appears *incognito* as fudge factor in a few instances in the classical theory, such as in the Fourier development of Eisenstein series.

For us, a spectral zeta function ζ_X of a space X is the Mellin transform of the heat kernel of X at the origin, removing the trivial eigenvalue if applicable, and divided by a gamma factor (*cf.* [JL12]). Alternatively one can define this function by an integration against the spectral measure.

Consider a sequence of discrete tori $\mathbb{Z}^d/A_n\mathbb{Z}^d$ indexed by n and where the matrices A_n are diagonal with entries $a_i n$, and integers $a_i > 0$. The matrix A is the diagonal matrix with entries a_i. We show the following for any dimension $d \geq 1$:

Theorem 0.1. *The following asymptotic expansion as $n \to \infty$ is valid for $Re(s) < d/2 + 1$, and $s \neq d/2$,*

$$\zeta_{\mathbb{Z}^d/A_n\mathbb{Z}^d}(s) = \zeta_{\mathbb{Z}^d}(s) \det A\, n^d + \zeta_{\mathbb{R}^d/A\mathbb{Z}^d}(s) n^{2s} + o(n^{2s}).$$

The formula reflects that as n goes to infinity the finite torus graph can be viewed as converging to $\mathbb{Z}^d$ on the one hand, and rescaled to the continuous torus

$\mathbb{R}^d/\mathbb{Z}^d$ on the other hand. For $Re(s) > d/2$ one has

$$\lim_{n\to\infty} \frac{1}{n^{2s}} \zeta_{\mathbb{Z}^d/A_n\mathbb{Z}^d}(s) = \zeta_{\mathbb{R}^d/A\mathbb{Z}^d}(s), \tag{1}$$

as already shown in [CJK10], see also Section 4 below. One can verify that it is legitimate to differentiate in the asymptotics in Theorem 0.1 and if we then set $s = 0$, we recover as expected the main asymptotic formula in [CJK10] in the case considered. The asymptotics of the determinant of graph Laplacians is a topic of significant interest, see [RV15, Conclusion] for a recent discussion from the point of view of quantum field theory, and see [L02] for related determinants in the context of L^2-invariants.

We now specialize to the case $d = 1$. In particular, the spectral zeta function of the finite cyclic graph $\mathbb{Z}/n\mathbb{Z}$ (see e.g. [CJK10] for details and section 1) is

$$\zeta_{\mathbb{Z}/n\mathbb{Z}}(s) = \frac{1}{4^s} \sum_{k=1}^{n-1} \frac{1}{\sin^{2s}(\pi k/n)}.$$

The spectral zeta function of the graph $\mathbb{Z}$ is

$$\zeta_{\mathbb{Z}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-2t} I_0(2t) t^s \frac{dt}{t},$$

where it converges, which it does for $0 < Re(s) < 1/2$. From this definition it is not immediate that its meromorphic continuation admits a functional equation much analogous to classical zeta functions:

Theorem 0.2. *Let the completed zeta function for $\mathbb{Z}$ be defined as*

$$\xi_{\mathbb{Z}}(s) = 2^s \cos(\pi s/2) \zeta_{\mathbb{Z}}(s/2).$$

Then this is an entire function that satisfies for all $s \in \mathbb{C}$

$$\xi_{\mathbb{Z}}(s) = \xi_{\mathbb{Z}}(1 - s).$$

This raises the question: Are there other spectral zeta functions of graphs with similar properties?

The function $\zeta_{\mathbb{Z}}$ actually appears implicitly in classical analytic number theory. Let us exemplify this point. To begin with

$$\zeta_{\mathbb{Z}}(s) = \frac{1}{4^s\sqrt{\pi}} \frac{\Gamma(1/2 - s)}{\Gamma(1 - s)},$$

which is the crucial fact behind the result above. Now, in the main formula of Chowla-Selberg in [SC67] the following term appears:

$$\frac{2^{2s} a^{s-1} \sqrt{\pi}}{\Gamma(s) \Delta^{s-1/2}} \zeta(2s - 1) \Gamma(s - 1/2).$$

Here lurks $\zeta_{\mathbb{Z}}(1 - s)$, not only by correctly combining the two gamma factors, but also incorporating the factor 2^{2s} and explaining the appearance of $\sqrt{\pi}$. In other words, the term above equals

$$\frac{4\pi a^{s-1}}{\Delta^{s-1/2}} \zeta(2s - 1) \zeta_{\mathbb{Z}}(1 - s).$$

Upon dividing by the Riemann zeta function $\zeta(s)$, this term is called scattering matrix (function) in the topic of Fourier expansions of Eisenstein series and is complicated or unknown for discrete groups more general than $SL(2,\mathbb{Z})$, see [IK04, section 15.4] and [M08]. We believe that the interpretation of such fudge factors as spectral zeta functions is new and may provide some insight into how such factors arise more generally.

The Riemann zeta function is essentially the same as the spectral zeta function of the circle $\mathbb{R}/\mathbb{Z}$, more precisely one has

$$\zeta_{\mathbb{R}/\mathbb{Z}}(s) = 2(2\pi)^{-2s}\zeta(2s). \tag{2}$$

Here is a specialization of Theorem 0.1 to $d = 1$ with explicit functions and some more precision:

Theorem 0.3. *For $s \neq 1$ with $Re(s) < 3$ it holds that*

$$\sum_{k=1}^{n-1} \frac{1}{\sin^s(\pi k/n)} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(1/2 - s/2)}{\Gamma(1 - s/2)} n + 2\pi^{-s}\zeta(s)n^s + o(n^s)$$

as $n \to \infty$. In the critical strip, $0 < Re(s) < 1$, more precise asymptotics can be found, such as

$$\sum_{k=1}^{n-1} \frac{1}{\sin^s(\pi k/n)} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(1/2 - s/2)}{\Gamma(1 - s/2)} n + 2\pi^{-s}\zeta(s)n^s + \frac{s}{3}\pi^{2-s}\zeta(s-2)n^{s-2} + o(n^{s-2})$$

as $n \to \infty$.

For example, with $s = 0$ the sum on the left equals $n - 1$, and the asymptotic formula hence confirms the well-known values $\Gamma(1/2) = \sqrt{\pi}$ and $\zeta(0) = -1/2$. On the line $Re(s) = 1$, the asymptotics is critical in the sense that the two first terms on the right balance each other in size as a power of n. As a consequence, for all $t \neq 0$ we have that $\zeta(1 + it) \neq 0$ if and only if

$$\frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{\sin^{1+it}(\pi k/n)}$$

diverges as $n \to \infty$. The latter sum does indeed diverge. We do not have a direct proof of this at the moment, but it does follow from a theorem of Wintner [W47] since the improper integral $\int \sin^{-1-it}(x)dx$ diverges at $x = 0$. So we have that the Riemann zeta function has no zeros on the line $Re(s) = 1$, which is the crucial input in the standard proof of the prime number theorem. It should however be said that Wintner's theorem is known to already be intimately related to the prime number theorem via works of Hardy-Littlewood.

As suggested to us by Jay Jorgenson, one may differentiate the formula in Theorem 0.1 for $d = 1$, as can be verified via the formulas in section 4, and get a criterion for multiple zeros:

Corollary 0.4. *Let*

$$c(s) = \frac{1}{2\sqrt{\pi}} \frac{\Gamma(1/2 - s/2)}{\Gamma(1 - s/2)} \left(\frac{\Gamma'(1/2 - s/2)}{\Gamma(1/2 - s/2)} - \frac{\Gamma'(1 - s/2)}{\Gamma(1 - s/2)} \right)$$

and

$$S(s,n) = c(s)n - \sum_{k=1}^{n-1} \frac{\log(\sin(\pi k/n))}{\sin^s(\pi k/n)}.$$

Then ζ has a multiple zero at s, $0 < Re(s) < 1$ if and only if $S(s,n) \to 0$ as $n \to \infty$, and otherwise $S(s,n) \to \infty$ as $n \to \infty$.

It is believed that all Riemann zeta zeros are simple.

Similarily to the above discussion about the prime number theorem, the Riemann hypothesis has a formulation in terms of the behaviour of the sum of sines (here we can refer to [So98] for comparison). It turns out that with some further investigation there is, what we think, a more intriguing formulation of the Riemann hypothesis. This is in terms of functional equations and provides perhaps some further heuristic evidence for its validity. Let

$$h_n(s) = (4\pi)^{s/2}\Gamma(s/2)n^{-s}\left(\zeta_{\mathbb{Z}/n\mathbb{Z}}(s/2) - n\zeta_{\mathbb{Z}}(s/2)\right).$$

Conjecture. *Let $s \in \mathbb{C}$ with $0 < Re(s) < 1$. Then*

$$\lim_{n\to\infty} \left| \frac{h_n(1-s)}{h_n(s)} \right| = 1.$$

This is an asymptotic or approximative functional equation, and it is true almost everywhere as follows from the asymptotics above, see Corollary 0.5 below. Although we came to this via graph zeta functions, it is important to emphasize, as a referee pointed out, that the asymptotics in one dimension hold with the same proofs for more general sums, instead of the inverse sine sums coming from cyclic graphs. More precisely, let f be an analytic function being real and positive on the open interval $(0,1)$, satisfying $f(z) = f(1-z)$ for any $z \in \mathbb{C}$, with $f(0) = 0$, $f'(0) > 0$, $f''(0) = 0$ and $f^{(3)}(0) \neq 0$.

Now let for $0 < Re(s) < 1$

$$h_n[f](s) = f'(0)^s \pi^{-s/2}\Gamma(s/2)n^{-s}\left[\sum_{j=1}^{n-1} \frac{1}{f(j/n)^s} - n\int_0^1 \frac{dx}{f(x)^s}\right].$$

As in section 5 applying [Si04] one gets

$$h_n[f](s) = 2\xi(s) - \frac{f^{(3)}(0)}{f'(0)\pi^2}\alpha(s)n^{-2} + o(n^{-2})$$

as $n \to \infty$ and where α is the function appearing in section 8. So one may formulate the same conjecture above, and Corollary 0.5 and Theorem 0.6 below hold for $h_n[f]$. This being said, we still feel that there might be something special with the sum of reciprocal sines in this context since, as discussed above, $\zeta_{\mathbb{Z}}(s/2)$ has a functional equation of the desired type, $s \longleftrightarrow 1-s$, and also $\zeta_{\mathbb{Z}/n\mathbb{Z}}(s/2)$ in an asymptotic sense, see section 7. These functional relations do not depend on the Riemann zeta function and may not hold for the corresponding sum and integral defined by f as above. Independently of number theory, we are interested in functional equations for graph zeta functions, which ultimately may also reflect similar relations for spectral zeta functions of manifolds. See also our concluding remarks below.

Corollary 0.5. *The conjecture holds in the crticial strip wherever* $\zeta(s) \neq 0$.

So the question is whether it also holds at the Riemann zeros. Here is the relation to the Riemann hypothesis:

Theorem 0.6. *The conjecture is equivalent to the Riemann hypothesis.*

Section 8 is devoted to the proof of this statement. This relies in particular on properties of the logarithmic derivative of ζ, in the proof of Lemma 8.4, and the Riemann functional equation.

Some concluding remarks.

Why do we think that the study of sums like

$$\sum_{k=1}^{n-1} \frac{1}{\sin^s(\pi k/n)}$$

could in some ways be better than the standard Dirichlet series $\sum_1^n k^{-s}$, or some other sum of similar type for that matter? For example, it has been pointed out to us that we could also derive version of Theorems 0.3 and 0.6 for more general functions, as described above, for example replacing sine with $x - 2x^3 + x^4 = x(1-x)(1+x-x^2)$. In this case the function corresponding to our $\zeta_{\mathbb{Z}}(s)$, say in the definition of h_n, would be

$$\int_0^1 \frac{1}{x^{2s}(1-x)^{2s}(1+x-x^2)^{2s}} dx,$$

which is a less standard function.

Let us now address this legitimate question with several answers that reinforce each other:

1. The graph zeta functions are defined in a parallel way to the definition of Riemann's zeta. Functions arising in this way may have greater chance to have more symmetries and structure, for example, keep in mind the remarkable relation
$$\xi_{\mathbb{Z}}(1-s) = \xi_{\mathbb{Z}}(s),$$
which is far from being just an abstract generality. Furthermore, it appears in the theory of Eisenstein series as observed above in a way that is difficult to deny, and in our opinon, unwise to dismiss. On the other hand, it is not clear whether the integral above satisfies a functional equation. In Section 7, we obtain an asymptotic functional relation of the desired type for the completed finite $1/\sin$ sums:
$$\lim_{n\to\infty} \frac{1}{n} \left(\xi_{\mathbb{Z}/n\mathbb{Z}}(1-s) - \xi_{\mathbb{Z}/n\mathbb{Z}}(s) \right) = 0$$
in the critical strip. We do not see a similar relation for, say
$$\sum_{k=1}^{n-1} \frac{1}{((k/n)(1-k/n)(1+k/n-k^2/n^2)^{2s}}.$$
Relations when $s \longleftrightarrow 1-s$ is at the heart of the matter for our reformulation of the Riemann Hypothesis.

2. Symmetric functions of graph eigenvalues often have combinatorial interpretations as counting something (starting with Kirchhoff's matrix tree theorem), see our Section 6.2 for a small illustration. This also motivates further study of spectral zeta functions for graphs. In particular, the analogous functions for manifolds play a role in various branches of mathematical physis. In this connection, Theorem 0.1 is of definite interest, see the comments after this theorem.

3. It is also noteworthy to recall that for $s = 2m$, the even positive integers, our finite sums admit a closed form expression as a polynomial in n, for example (which can be shown combinatorially in line with the previous point),

$$\sum_{k=1}^{n-1} \frac{1}{\sin^2(\pi k/n)} = \frac{1}{3}n^2 - \frac{1}{3},$$

while $\sum_1^n k^{-2m}$ does not admit such a formula. The sine series evaluation implies, in view of (1) and (2) above, Euler's formulas for $\zeta(2m)$, for example $\zeta(2) = \pi^2/6$. See section 6 for more about how our asymptotical relations imply known special values, and also references to contexts where the finite $1/\sin$ sums are studied.

Higher dimensions. For $d > 1$ the torus zeta functions are Epstein zeta functions also appearing in number theory. Some of these are known not to satisfy the Riemann hypothesis, the statement that all non-trivial zeros lie on one vertical line (see [RH08] and [PT34]). It seems interesting to understand this difference between $d = 1$ and certain higher dimensional cases from our perspective. Theorem 0.1 gives precise asymptotics in higher dimensions, but to get even further terms in the expansion, as in Theorem 0.3, there are some complications, especially when trying to assemble a nice expression, like $\zeta(s-2)$ as in Theorem 0.3. Therefore this is left for future study.

Generalized Riemann Hypothesis (GRH). In a forthcoming sequel about Dirichlet L-functions [F15], by the first-named author, it similarily emerges that the GRH is essentially equivalent to an expected asymptotic functional equation of the corresponding graph L-function. More precisely, spectral L-functions for graphs (different from those considered in [H92] and [STe00]) are introduced, and in the case of $\mathbb{Z}/n\mathbb{Z}$, the L-functions completed with suitable fudge factors, and denoted $\Lambda_n(s,\chi)$, satisfy

$$\lim_{n\to\infty} \left| \frac{\Lambda_n(s,\chi)}{\Lambda_n(1-s,\overline{\chi})} \right| = 1,$$

for $0 < Re(s) < 1$ and $Im(s) \geq 8$, if and only if the GRH holds (for s in the same range) for Dirichlet's L-function $L(s,\chi)$.

Zeta functions of graphs. As recalled in the beginning, the more standard zeta function of a graph is the one going back to a paper by Ihara. Ihara zeta functions for infinite graphs appear in a few places, three recent papers are [D14, CJK15, LPS14], which contain further generalizations and where references to papers by Grigorchuk-Zuk and Guido, Isola, and Lapidus on this topic can be found. A two variable extension of the Ihara zeta function was introduced by Bartholdi [B99] developed out of a formula in [G78]. Zeta functions more closely related to the ones considered in the present paper, are the spectral zeta functions of fractals in works by Teplyaev, Lapidus and van Frankenhuijsen.

Acknowledgement. The second-named author thanks Jay Jorgenson, Pär Kurlberg and Andreas Strömbergsson for valuable discussions related to this paper. We thank Franz Lehner for suggesting the use of the spectral measure in the calculation of the spectral zeta function of regular trees. We thank the referee for insightful comments.

1 Spectral zeta functions

At least since Carleman [Ca34] one forms a spectral zeta function

$$\sum_j \frac{1}{\lambda_j^s}$$

over the set of non-zero Laplace eigenvalues, convergent for s in some right half-plane. For a finite graph the elementary symmetric functions in the eigenvalues admit a combinatorial interpretation starting with Kirchhoff, see *e.g.* [CL96] for a more recent discussion. For infinite graphs or manifolds one does at least not a priori have such symmetric functions (since the spectrum may be continuous or the eigenvalues are infinite in number). This is one reason for defining spectral zeta functions, since these are symmetric, and via transforms one can get the analytic continued intepretations of the elementary symmetric functions, such as the (restricted) determinant. As has been recognized at least for the determinant, the combinatorial intepretation persists in a certain sense, see [Ly10].

As often is the case, since Riemann, in order to define its meromorphic continuation one writes the zeta function as the Mellin transform of the associated theta series, or trace of the heat kernel. For this reason and in view of that some spaces have no eigenvalues but continuous spectrum, a case important to us in this paper, we suggest (as advocated by Jorgenson-Lang, see for example [JL12]) to start from the heat kernel to define spectral zeta functions. Recall that the Mellin transform of a function $f(t)$ is

$$\mathbf{M}f(s) = \int_0^\infty f(t)t^s \frac{dt}{t}.$$

For example when $f(t) = e^{-t}$, the transform is $\Gamma(s)$.

More precisely, for a finite or compact space X we can sum over x_0 of the unique bounded fundamental solution $K_X(t, x_0, x_0)$ of the heat equation (see for example [JL12, CJK10] for more background on this), which gives the heat trace $Tr(K_X)$, typically on the form $\sum e^{-\lambda t}$, and define

$$\zeta_X(s) = \frac{1}{\Gamma(s)} \int_0^\infty (Tr(K_X) - 1)t^s \frac{dt}{t}.$$

When the spectrum is discrete this formula gives back Carleman's definition above. For a non-compact space with a heat kernel independent of the point x_0, for example a Cayley graph of an infinite, finitely generated group, it makes sense to take Mellin transform of $K_X(t, x_0, x_0)$ without the trace. Moreover since zero is no longer an eigenvalue for the Laplacian acting on $L^2(X)$ we should no longer subtract 1, so the definition in this case is

$$\zeta_X(s) = \frac{1}{\Gamma(s)} \int_0^\infty K_X(t, x_0, x_0) t^s \frac{dt}{t}.$$

Let us also note that in the graph setting as shown in [CJK15], it holds that if we start with the heat kernel one may via instead a Laplace transform obtain the Ihara zeta function and the fundamental determinant formula.

An alternative, equivalent, definition is given by the spectral measure $d\mu = d\mu_{x_0,x_0}$, see [MW89],

$$\zeta_X(s) = \int \lambda^{-s} d\mu(\lambda).$$

Here and in the next two sections we provide some examples:

Example 1.1. *For a finite torus graph defined as in the introduction we have by calculating the eigenvalues (see for example [CJK10])*

$$\zeta_{\mathbb{Z}^d/A\mathbb{Z}^d}(s) = \frac{1}{2^{2s}} \sum_k \frac{1}{\left(\sin^2(\pi k_1/a_1) + ... + \sin^2(\pi k_d/a_d)\right)^s},$$

where the sum runs over all $0 \leq k_i \leq a_i - 1$ except for all k_is being zero.

Example 1.2. *For real tori we have again by calculating the eigenvalues (see [CJK12]) as is well known*

$$\zeta_{\mathbb{R}^d/A\mathbb{Z}^d}(s) = \frac{1}{(2\pi)^{2s}} \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{\|A^* k\|^{2s}},$$

where $A^* = \left(A^{-1}\right)^t$.

In the following sections we will discuss the zeta function of some infinite graphs, namely the standard lattice graphs $\mathbb{Z}^d$. Before that let us mention yet another example, that we again do not think one finds in the literature.

Example 1.3. *The $(q+1)$-regular tree T_{q+1} with $q \geq 2$ is a fundamental infinite graph ($q = 1$ corresponds to $\mathbb{Z}$ treated in the next section). Also here the spectral measure is well-known, our reference is [MW89]. Thus*

$$\zeta_{T_{q+1}}(s) = \int_{-2\sqrt{q}}^{2\sqrt{q}} \frac{1}{(q+1-\lambda)^s} \frac{(q+1)}{2\pi} \frac{\sqrt{4q-\lambda^2}}{((q+1)^2 - \lambda^2)} d\lambda$$

$$= \frac{q+1}{2\pi} \int_{-2\sqrt{q}}^{2\sqrt{q}} \frac{1}{(q+1-\lambda)^{s+1}} \frac{\sqrt{4q-\lambda^2}}{(q+1+\lambda)} d\lambda.$$

We change variable $u = 2\sqrt{q} - \lambda$. So

$$\zeta_{T_{q+1}}(s) = \frac{q+1}{2\pi} \int_0^{4\sqrt{q}} \frac{1}{(q+1-2\sqrt{q}+u)^{s+1}} \frac{\sqrt{4\sqrt{q}u - u^2}}{(q+1+2\sqrt{q}-u)} du$$

$$= \frac{q+1}{2\pi}\int_0^{4\sqrt{q}} \frac{u^{1/2}}{(q+1-2\sqrt{q}+u)^{s+1}} \frac{\sqrt{4\sqrt{q}-u}}{(q+1+2\sqrt{q}-u)}du.$$

We change again: $u = 4\sqrt{q}t$, so

$$\zeta_{T_{q+1}}(s) = \frac{q+1}{2\pi}\int_0^1 \frac{(4\sqrt{q})^{1/2}t^{1/2}}{(q+1-2\sqrt{q}+4\sqrt{q}t)^{s+1}} \frac{\sqrt{4\sqrt{q}-4\sqrt{q}t}}{(q+1+2\sqrt{q}-4\sqrt{q}t)}4\sqrt{q}dt$$

$$= \frac{d}{2\pi}\frac{16q}{(q+1-2\sqrt{q})^{s+1}(q+1+2\sqrt{q})}\int_0^1 \frac{t^{1/2}\sqrt{1-t}}{(1-ut)^{s+1}(1-vt)}dt,$$

where $u = -4\sqrt{q}/(q+1-2\sqrt{q})$ and $v = 4\sqrt{q}/(q+1+2\sqrt{q})$. This is an Euler-type integral that Picard considered in [Pi1881] and which lead him to Appell's hypergeometric function F_1,

$$\zeta_{T_{q+1}}(s) = \frac{q+1}{2\pi}\frac{16q}{(q+1-2\sqrt{q})^{s+1}(q+1+2\sqrt{q})}\frac{\Gamma(3/2)\Gamma(3/2)}{\Gamma(3)}F_1(3/2, s{+}1, 1, 3; u, v).$$

Simplifing this somewhat we have proved:

Theorem 1.4. *For $q > 1$, the spectral zeta function of the $(q+1)$-regular tree is*

$$\zeta_{T_{q+1}}(s) = \frac{q(q+1)}{(q-1)^2(\sqrt{q}-1)^{2s}}F_1(3/2, s+1, 1, 3; u, v),$$

with $u = -4\sqrt{q}/(\sqrt{q}-1)^2$ and $v = 4\sqrt{q}/(\sqrt{q}+1)^2$, and where F_1 is one of Appell's hypergeometric functions.

The topic of functional relations between hypergeometric functions is a very classical one. In spite of the many known formulas, we were not able to derive a functional equation for $\zeta_{T_{q+1}}$ with $s \longleftrightarrow 1-s$.

2 The spectral zeta function of the graph $\mathbb{Z}$

The heat kernel of $\mathbb{Z}$ is $e^{-2t}I_x(2t)$ where I_x is a Bessel function (see [CJK10] and its references). Therefore

$$\zeta_{\mathbb{Z}}(s) = \frac{1}{\Gamma(s)}\int_0^\infty e^{-2t}I_0(2t)t^s\frac{dt}{t},$$

which converges for $0 < Re(s) < 1/2$. It is not so clear why this function should have a meromorphic continuation and functional equation very similar to Riemann's zeta.

Proposition 2.1. *For $0 < Re(s) < 1/2$ it holds that*

$$\zeta_{\mathbb{Z}}(s) = \frac{1}{4^s\sqrt{\pi}}\frac{\Gamma(1/2-s)}{\Gamma(1-s)} = \frac{1}{4^s\pi}B(1/2, 1/2-s),$$

where B denotes Euler's beta function. This formula provides the meromorphic continuation of $\zeta_{\mathbb{Z}}(s)$.

Proof. By formula 11.4.13 in [AS64], we have

$$\mathbf{M}(e^{-t}I_x(t))(s) = \frac{\Gamma(s+x)\Gamma(1/2-s)}{2^s\pi^{1/2}\Gamma(1+x-s)},$$

valid for $Re(s) < 1/2$ and $Re(s+x) > 0$. This implies the first formula. Finally, using that $\Gamma(1/2) = \sqrt{\pi}$ and the definition of the beta function the proposition is established. □

We proceed to determine a functional equation for this zeta function. Recall that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

Therefore

$$2^s\sqrt{\pi}\zeta_{\mathbb{Z}}(s/2) = \frac{\Gamma(1-(1/2+s/2))}{\Gamma(1-s/2)} = \frac{\sin(\pi s/2)\Gamma(s/2)}{\pi}\frac{\pi}{\sin(\pi(s+1)/2)\Gamma(1/2+s/2)}$$

$$= \tan(\pi s/2)\frac{\Gamma(1/2-(1-s)/2)}{\Gamma(1-(1-s)/2)} = 2^{1-s}\sqrt{\pi}\tan(\pi s/2)\zeta_{\mathbb{Z}}((1-s)/2).$$

Hence in analogy with Riemann's case we have

$$\zeta_{\mathbb{Z}}(s/2) = 2^{1-2s}\tan(\pi s/2)\zeta_{\mathbb{Z}}((1-s)/2).$$

(The passage from s to $s/2$ is also the same.) If we define the completed zeta to be

$$\xi_{\mathbb{Z}}(s) = 2^s\cos(\pi s/2)\zeta_{\mathbb{Z}}(s/2),$$

then one verifies that the above functional equation can be written in the familiar more symmetric form

$$\xi_{\mathbb{Z}}(s) = \xi_{\mathbb{Z}}(1-s)$$

for all $s \in \mathbb{C}$. Moreover, note that this is an entire function since the simple poles coming from Γ are cancelled by the cosine zeros and it takes real values on the critical line. We call $\xi_{\mathbb{Z}}$ the *entire completion* of $\zeta_{\mathbb{Z}}$.

Let us determine some special values. In view of that for integers $n \geq 0$,

$$\Gamma(1/2+n) = \frac{(2n)!}{4^n n!}\sqrt{\pi}$$

and $\Gamma(1+n) = n!$, we have for $s = -n$,

$$\zeta_{\mathbb{Z}}(-n) = \frac{1}{4^{-n}\sqrt{\pi}}\frac{\Gamma(1/2+n)}{\Gamma(1+n)} = \frac{(2n)!}{n!n!} = \binom{2n}{n}.$$

This number equals the number of paths of length $2n$ from the origin to itself in $\mathbb{Z}$.

Furthermore, in a similar way for $n \geq 1$,

$$\zeta_{\mathbb{Z}}(-n+1/2) = \frac{1}{4^{-n}\sqrt{\pi}}\frac{\Gamma(n)}{\Gamma(1/2+n)} = \frac{4^{2n}}{2\pi n}\frac{n!n!}{(2n)!} = \frac{4^{2n}}{2\pi n\binom{2n}{n}}.$$

It is well-known that the gamma function is a meromorphic function in the whole complex plane with simple poles at the negative integers and no zeros. Note that if we pass from s to $s/2$ we have that $\zeta_{\mathbb{Z}}(s/2)$ has simple poles at the positive odd integers, and the special values determined above appear at the even negative numbers.

We may thus summarize:

Theorem 2.2. *The spectral zeta function $\zeta_{\mathbb{Z}}(s)$ can be extended to a meromorphic function on $\mathbb{C}$ satisfying*

$$\zeta_{\mathbb{Z}}(s) = \frac{1}{4^s\sqrt{\pi}}\frac{\Gamma(1/2-s)}{\Gamma(1-s)}.$$

It has zeros for $s = n$, $n = 1, 2, 3, \ldots$, and simple poles for $s = 1/2+n$, $n = 0, 1, 2, \ldots$ Moreover, its completion $\xi_{\mathbb{Z}}$, which is entire, admits the functional equation

$$\xi_{\mathbb{Z}}(s) = \xi_{\mathbb{Z}}(1-s).$$

Finally we have the special values

$$\zeta_{\mathbb{Z}}(-n) = \binom{2n}{n} \text{ and } \zeta_{\mathbb{Z}}(-n+1/2) = \frac{4^{2n}}{2\pi n\binom{2n}{n}},$$

where $n \geq 0$ is an integer ($n \geq 1$ for the second equality).

3 The spectral zeta function of the lattice graphs $\mathbb{Z}^d$

The heat kernel on $\mathbb{Z}^d$ is the product of heat kernels on $\mathbb{Z}$ and this gives that

$$\zeta_{\mathbb{Z}^d}(s) = \frac{1}{\Gamma(s)}\int_0^\infty e^{-2dt} I_0(2\mathrm{t})^d t^s \tfrac{dt}{t},$$

which converges for $0 < Re(s) < d/2$. For $d = 2$ taking instead the equivalent definition with the spectral measure, the spectral zeta function is a variant of the Selberg integral with two variables.

The integrals like

$$\int_0^\infty e^{-zt} I_0(2t)^d t^s \frac{dt}{t},$$

and more general ones, have been studied by Saxena in [Sa66], see also the discussion in [SK85, sect. 9.4]. For $Re(z) > 2d$ and $Re(s) > 0$ one has

$$\int_0^\infty e^{-zt} I_0(2t)^d t^s \frac{dt}{t} = \frac{2^{s-1}}{\sqrt{\pi}} z^{-s+1/2}\Gamma(\frac{s+1}{2})F_C^{(d)}(s/2, (s+1)/2; 1, 1, \ldots, 1; 4/z^2, 4/z^2, \ldots, 4/z^2),$$

where $F_C^{(d)}$ is one of the Lauricella hypergeometric functions in d variables [Ex76]. The condition $Re(z) > 2d$ can presumably be relaxed by the principle of analytic continuation giving up the multiple series definition of $F_C^{(d)}$. This point is discussed in [SE79]. Formally we would then have that

$$\zeta_{\mathbb{Z}^d}(s) = \frac{d^{-s+1/2}}{\sqrt{2\pi}}\frac{\Gamma((s+1)/2)}{\Gamma(s)}F_C^{(d)}(s/2, (s+1)/2; 1, 1, \ldots, 1; 1/d^2, 1/d^2, \ldots, 1/d^2),$$

which is rather suggestive as far as functional relations go. It is however not clear at present time that for $d > 1$ there is a relation as nice as the functional equation in the case $d = 1$. Related to this, it is remarked in [Ex76, p. 49] that no integral representation of Euler type has been found for F_C. We note that if one instead of the heat kernel start with the spectral measure in defining $\zeta_{\mathbb{Z}^d}(s)$, we do get such an integral representation, at least for special parameters. This aspect is left for future investigation.

We will now provide an independent and direct meromorphic continuation of these functions. To do this, we take advantage of the heat kernel definition of the zeta function. Fix a dimension $d \geq 1$. Recall that on the one hand there are explicit positive non-zero coefficients a_n such that

$$e^{-2dt} I_0(2t)^d = \sum_{n \geq 0} a_n t^n$$

which converges for every positive t, and on the other hand we similarly have an expansion at infinity,

$$e^{-2dt} I_0(2t)^d = \sum_{n=0}^{N-1} b_n t^{-n-d/2} + O(t^{-N-d/2})$$

as $t \to \infty$ for any integer $N > 0$.

Therefore we write

$$\int_0^\infty e^{-2dt} I_0(2t)^d t^{s-1} dt = \int_0^1 \sum_{n=0}^{N-1} a_n t^n t^{s-1} dt + \int_0^1 \sum_{n \geq N} a_n t^n t^{s-1} dt +$$

$$+ \int_1^\infty \left(e^{-2dt} I_0(2t)^d - \sum_{n=0}^{N-1} b_n t^{-n-d/2} \right) t^{s-1} dt + \int_1^\infty \sum_{n=0}^{N-1} b_n t^{-n-d/2} t^{s-1} dt$$

$$= \sum_{n=0}^{N-1} \frac{a_n}{s+n} - \sum_{n=0}^{N-1} \frac{b_n}{s - (n + d/2)} + \int_0^1 O(t^N) t^{s-1} dt + \int_1^\infty O(t^{-N-d/2}) t^{s-1} dt.$$

This last expression defines a meromorphic function in the region $-N < Re(s) < N + d/2$, with simple poles at $s = -n$ and $s = n + d/2$.

The spectral zeta function $\zeta_{\mathbb{Z}^d}(s)$ is the above integral divided by $\Gamma(s)$. In view of that the entire function $1/\Gamma(s)$ has zeros at the non-positive integers, this will cancel the simple poles at $s = -n$. Since we can take N as large as we want we obtain in this way the meromorphic continuation of $\zeta_{\mathbb{Z}^d}(s)$. Moreover, thanks to that the coefficients b_n are non-zero we have established:

Proposition 3.1. *The function $\zeta_{\mathbb{Z}^d}(s)$ admits a meromorphic continuation to the whole complex plane with simple poles at the points $s = n + d/2$ with $n \geq 0$.*

It is natural to wonder whether this function also for $d > 1$ can be completed like in the case $d = 1$ giving an entire function with functional relation $\xi_{\mathbb{Z}^d}(1 - s) =$

$\xi_{\mathbb{Z}^d}(s)$. Indeed, more generally we find the question interesting for which graph, finite or infinite, the zeta functions have a functional relation in some way analogous to the classical type of functional equations.

Finally we point out a non-trivial special value that we derive in a later section:

$$\zeta_{\mathbb{Z}^d}(0) = 1.$$

4 Asymptotics of the zeta functions of torus graphs

We consider a sequence of torus graphs $\mathbb{Z}^d/A_n\mathbb{Z}^d$ indexed by n and where the matrices A_n are diagonal with entries $a_i n$, with integers $a_i > 0$. (A more general setting could be considered (*cf.* [CJK12]) but it will not be important to us in the present context.) We denote by ζ_n the corresponding zeta function defined as in the previous section. We let the matrix A be the diagonal matrix with entries a_i. In this section we take advantage of the theory developed in [CJK10] without recalling the proofs which would take numerous pages.

Following [CJK10] we have

$$\theta_n(t) := \sum_m e^{-\lambda_m t} = \det(A_n) \sum_{k\in\mathbb{Z}^d} \prod_{1\le j\le d} e^{-2t} I_{a_j n k_j}(2t),$$

where λ_m denotes the Laplace eigenvalues. From the left hand side it is clear that this function is entire. Let

$$\theta_A(t) = \sum_\lambda e^{-\lambda t},$$

where the sum is over the eigenvalues of the torus $\mathbb{R}^d/A\mathbb{Z}^d$. The meromorphic continuation of the corresponding spectral zeta function is, as is well-known (see *e.g.* [CJK10]),

$$\zeta_{\mathbb{R}^d/A\mathbb{Z}^d}(s) = \frac{1}{\Gamma(s)} \int_1^\infty (\theta_A(t) - 1)\, t^s \frac{dt}{t} + \frac{1}{\Gamma(s)} \int_0^1 \left(\theta_A(t) - \det A (4\pi t)^{-d/2}\right) t^s \frac{dt}{t}$$

$$+ \frac{(4\pi)^{-d/2} \det A}{\Gamma(s)(s-d/2)} - \frac{1}{s\Gamma(s)}.$$

Recall the asymptotics for the I-Bessel functions:

$$I_n(x) = \frac{e^x}{\sqrt{2\pi x}} \left(1 - \frac{4n^2-1}{8x} + O(x^{-2})\right)$$

as $x \to \infty$.

For $0 < Re(s) < d/2$ we may write

$$\Gamma(s)\zeta_n(s) = \int_0^\infty (\theta_n(t) - 1)\, t^s \frac{dt}{t} = n^{2s} \int_0^\infty \left(\theta_n(n^2 t) - 1\right) t^s \frac{dt}{t}.$$

We decompose the integral on the right and let $n \to \infty$, the first piece being

$$S_1(n) := \int_1^\infty \left(\theta_n(n^2 t) - 1\right) t^s \frac{dt}{t} \to \int_1^\infty (\theta_A(t) - 1)\, t^s \frac{dt}{t}$$

for every $s \in \mathbb{C}$ as $n \to \infty$. The convergence is proved in [CJK10]. The second piece is for $Re(s) > -n$,

$$S_2(n) := \int_0^1 \left(\theta_n(n^2 t) - \det A_n e^{-2dn^2 t} I_0(2n^2 t)^d\right) t^s \frac{dt}{t} \to \int_0^1 \left(\theta_A(t) - \det A (4\pi t)^{-d/2}\right) t^s \frac{dt}{t},$$

as $n \to \infty$ which is proved in [CJK10].

What remains is now the third piece

$$S_3(n) := \int_0^1 \left(\det A_n e^{-2dn^2 t} I_0(2n^2 t)^d - 1\right) t^s \frac{dt}{t} = n^{-2s} \int_0^{n^2} \left(\det A_n e^{-2dt} I_0(2t)^d - 1\right) t^s \frac{dt}{t}.$$

This we write as follows

$$S_3(n) = \left(\det A_n \int_0^\infty e^{-2dt} I_0(2t)^d t^s \frac{dt}{t} - \det A_n \int_{n^2}^\infty e^{-2dt} I_0(2t)^d t^s \frac{dt}{t} - \int_0^{n^2} t^s \frac{dt}{t}\right) n^{-2s}.$$

The first integral is the spectral zeta of $\mathbb{Z}^d$ times $\Gamma(s)$ and the last integral is

$$\int_0^{n^2} t^s \frac{dt}{t} = \frac{n^{2s}}{s}.$$

We continue with the middle integral here:

$$\int_{n^2}^\infty e^{-2dt} I_0(2t)^d t^s \frac{dt}{t} = \int_{n^2}^\infty \left(e^{-2dt} I_0(2t)^d - (4\pi t)^{-d/2}\right) t^s \frac{dt}{t} + \int_{n^2}^\infty (4\pi t)^{-d/2} t^s \frac{dt}{t},$$

hence

$$\int_{n^2}^\infty e^{-2dt} I_0(2t)^d t^s \frac{dt}{t} = \int_{n^2}^\infty \left(e^{-2dt} I_0(2t)^d - (4\pi t)^{-d/2}\right) t^s \frac{dt}{t} - (4\pi)^{-d/2} \frac{n^{2s-d}}{s - d/2}.$$

We denote

$$S_{rest}(n) = \int_{n^2}^\infty \left(e^{-2dt} I_0(2t)^d - (4\pi t)^{-d/2}\right) t^s \frac{dt}{t},$$

which is a convergent integral for $Re(s) < d/2 + 1$ in view of the asymptotics for $I_0(t)$. Notice also that for fixed s with $Re(s) < d/2 + 1$ the integral is of order n^{2s-2-d} as $n \to \infty$.

Taken all together we have

$$n^{-2s} \zeta_n(s) = \frac{1}{\Gamma(s)} S_1(n) + \frac{1}{\Gamma(s)} S_2(n) - \frac{1}{s\Gamma(s)} + (4\pi)^{-d/2} \frac{\det A}{\Gamma(s)(s - d/2)}$$

$$+ n^{d-2s} \det A \, \zeta_{\mathbb{Z}^d}(s) - n^{d-2s} \frac{\det A}{\Gamma(s)} S_{rest}(n).$$

This is valid for all s in the intersection of where $\zeta_{\mathbb{Z}^d}(s)$ is defined, $-n < Re(s) < d/2 + 1$, and $s \neq d/2$. As remarked above coming from [CJK10] as $n \to \infty$ the first four terms combines to give $\zeta_{\mathbb{R}^d/A\mathbb{Z}^d}(s)$. This means that we have in particular proved Theorem 0.1.

5 The one dimensional case

We now specialize to $d = 1$ and $A_n = n$. In this case recall that

$$\zeta_n(s) = \zeta_{\mathbb{Z}/n\mathbb{Z}}(s) = \frac{1}{4^s}\sum_{k=1}^{n-1}\frac{1}{\sin(\pi k/n)^{2s}}$$

and

$$\zeta_{\mathbb{R}/\mathbb{Z}}(s) = 2(2\pi)^{-2s}\zeta(2s),$$

where ζ is the Riemann zeta function. Moreover,

$$\zeta_{\mathbb{Z}}(s) = \frac{1}{4^s\sqrt{\pi}}\frac{\Gamma(1/2-s)}{\Gamma(1-s)}.$$

In view of the previous section the first part of Theorem 0.3 is established. Let us remark that this can also be viewed as a special case of Gauss-Chebyshev quadrature but with a more precise error term.

With more work one can also find the next term in the asymptotic expansion in the critical strip. This can be achieved with some more detailed analysis, in particular of Proposition 4.7 in [CJK10] and an application of Poisson summation. For the purpose of the present discussion we only need to look at the more precise asymptotics in the critical strip and here for $d = 1$ there is an alternative approach available by using a non-standard version of the Euler-Maclaurin formula established in [Si04]. See also [Si12] and [BHS09], where these sums of sines are studied and called *Riesz s-energy* of a collection of points on the unit circle. The asymptotics is:

$$\sum_{k=1}^{n-1}\frac{1}{\sin(\pi k/n)^s} = \frac{1}{\sqrt{\pi}}\frac{\Gamma(1/2-s/2)}{\Gamma(1-s/2)}n + 2\pi^{-s}\zeta(s)n^s + \frac{s}{3}\pi^{2-s}\zeta(s-2)n^{s-2} + o(n^{s-2})$$

where $0 < Re(s) < 1$ as $n \to \infty$. This is the second statement in Theorem 0.3.

Example 5.1. *Although we did not verify this asymptotics outside of the critical strip, it may nevertheless be convincing to specialize to $s = 2$, we then would have*

$$\frac{1}{3}n^2 - \frac{1}{3} = \frac{1}{\sqrt{\pi}}0\cdot n + 2\pi^{-2}\zeta(2)n^2 + \frac{2}{3}\zeta(0) + o(1),$$

which confirms the values $\zeta(0) = -1/2$ and $\zeta(2) = \pi^2/6$. As remarked in the introduction, from [CJK10], the value of $\zeta(2)$ can also be derived via

$$\frac{2}{\pi^2}\zeta(2) = \lim_{n\to\infty}\frac{1}{n^2}\left(\frac{1}{3}n^2 - \frac{1}{3}\right).$$

6 Special values

6.1 The case of $s = 0$

Setting $s = 0$ in Theorem 0.1 we clearly have

$$\det A\, n^d - 1 = \zeta_{\mathbb{Z}^d}(0)\det A\, n^d + \zeta_{\mathbb{R}^d/A\mathbb{Z}^d}(0) + o(1),$$

which implies that $\zeta_{\mathbb{Z}^d}(0) = 1$, and that $\zeta_{\mathbb{R}^d/A\mathbb{Z}^d}(0) = -1$, which is a known special value of Epstein zeta functions.

6.2 The case of s being negative integers

Let us now recall some known results about the sums:

$$\sum_{k=1}^{n-1} \frac{1}{\sin(\pi k/n)^s}$$

for special s. We begin with a simple calculation (see for example [BH10, Lemma 3.5]) namely that for integers $0 < m < n$

$$\sum_{k=1}^{n-1} \sin^{2m}(\pi k/n) = \frac{n}{4^m} \binom{2m}{m}.$$

In view of the asymptotics in Theorem 0.3 this immediately imply that $\zeta(-2m) = 0$, the so-called trivial zeros of Riemann's zeta function. It also verifies with the special values of $\zeta_{\mathbb{Z}}$ stated in section 2. There is a probabilistic interpretation for this: when the number of steps m is smaller than n, the random walker cannot tell the difference between the graphs $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$.

Conversely, for s being an odd negative integer our asymptotic formula gives information about the sine sum which is somewhat more complicated in this case, as the fact that ζ does not vanish implies. For low exponent m one can find formulas in [GR07], the simplest one being

$$\sum_{k=1}^{n-1} \sin(k\pi/n) = \cot(\pi/2n).$$

6.3 The case of s being even positive integers

In view of the elementary equality

$$\frac{1}{\sin^2 x} = 1 + \cot^2 x,$$

one sees that for positive integers a,

$$\sum_{k=1}^{n-1} \frac{1}{\sin^{2a}(\pi k/n)}$$

can be expresed in terms of higher Dedekind sums considered by Zagier [Z73]. There is also a literature more specialized on this type of finite sums which can be evaluated with a closed form expression already mentioned in the introduction (see [CM99, BY02]):

$$\sum_{k=1}^{n-1} \frac{1}{\sin^{2a}(\pi k/n)} = -\frac{1}{2} \sum_{m=0}^{2a} \frac{(-4)^a}{n^m} \binom{2a+1}{m+1}$$

$$\times \sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} \frac{m+1-2k}{m+1} \binom{a+kn+(m-1)/2}{2a+m}.$$

These sums apparently arose in physics in Dowker's work and in mathematical work of Verlinde (see [CS12]). The first order asymptotics is known to be

$$\sum_{k=1}^{n-1} \sin^{-2m}(\pi k/n) \sim (-1)^{m+1}(2n)^{2m}\frac{B_{2m}}{(2m)!},$$

where m is a positive integer, see for example [BY02, CS12] and their references. As explained in the introduction these evaluations together with the asymptotics formulated in the introduction re-proves Euler's celebrated calculations of $\zeta(2m)$.

At $s=1$, the point where our asymptotic expansion does not apply because of the pole of ζ, one has (see [He77, p. 460] attributed to J. Waldvogel)

$$\zeta_{\mathbb{Z}/n\mathbb{Z}}(1) = \frac{2n}{\pi}\left(\log(2n/\pi) - \gamma\right) + O(1),$$

where as usual γ is Euler's constant.

6.4 Further special values

Recall the values $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma'(1) = -\gamma$ and $\Gamma'(1/2) = -\gamma\sqrt{\pi} - \log 4$, or in the logarithmic derivative, the psi-function, $\psi(1) = -\gamma$ and $\psi(1/2) = -\gamma - 2\log 2$. We differentiate $\zeta_{\mathbb{Z}}(s)$ which gives

$$\zeta'_{\mathbb{Z}}(s) = \zeta_{\mathbb{Z}}(s)\left(-2\log 2 - \psi(1/2-s) + \psi(1-s)\right).$$

Setting $s=0$ and inserting the special values mentioned we see that

$$\zeta'_{\mathbb{Z}}(0) = 0.$$

This value has the interpretation of being the tree entropy of $\mathbb{Z}$, which is the exponential growth rate of spanning trees of subgraphs converging to $\mathbb{Z}$, see e.g. [DD88, Ly10, CJK10], studied via the Fuglede-Kadison determinant of the Laplacian. This has also a role in the theory of operator algebras, but in any case it is not evaluated in this way in the literature. Of course one could in our way compute other special values of $\zeta'_{\mathbb{Z}}$. For example, at positive integers and half-integers this function has zeros and poles, respectively, and at negative integers we have for integers $n>0$ the following:

Proposition 6.1. *It holds that*

$$\zeta'_{\mathbb{Z}}(-n) = \binom{2n}{n}\left(1+\frac{1}{2}+\cdots+\frac{1}{n} - 2\left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right)\right)$$

and

$$\zeta'_{\mathbb{Z}}(-n+1/2) = \frac{4^{2n}}{2\pi n\binom{2n}{n}}\left(-4\log 4 - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n-1} + 2\left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right)\right).$$

We remark that this section concerned mostly $d=1$, we have not investigated the case of higher dimensions.

6.5 No real zero in the crticial strip

The non-vanishing of number theoretic zeta functions on the real line in the critical strip is of importance, see e.g. [SC67]. We outline one possible strategy for this problem in general from our asymptotics. We treat here only Rieman's zeta function for illustration, in this case there are however other more elementary arguments available.

Already the beginning of this section indicates that certain Epstein zeta functions have a tendency to be negative on the real line in the critical strip. It is as if the number of terms in the finite graph zeta is not enough to account for the limit graph zeta function, leaving the relevant Epstein zeta function negative.

The function $\sin(\pi x)^{-s}$ for $0 < s < 1$ is positive, convex and symmetric around $x = 1/2$. The graph zeta funciton in question, $\zeta_{\mathbb{Z}}(s)$ is via a change of variables

$$\int_0^1 \sin^{-s}(\pi x)dx.$$

If we compare this with the sum, using the symmetry, we have for odd n

$$2\left(\frac{1}{n}\sum_{k=1}^{(n-1)/2}\sin^{-s}(\pi k/n) - \int_0^{1/2}\sin^{-s}(\pi x)dx\right) = \frac{2\zeta(s)}{\pi^s}n^{s-1} + o(n^{s-1}).$$

If we interpret the sum as the Riemann sum of the integral (with not enough terms) the integral can be thought of as always lying above the rectangles. Ignoring all but one rectangle then gives

$$\begin{aligned}&\frac{1}{n}\sum_{k=1}^{(n-1)/2}\sin^{-s}(\pi k/n) - \int_0^{1/2}\sin^{-s}(\pi x)dx\\ &< \frac{1}{n}\sin^{-s}(\pi/n) - \int_0^{1/n}\sin^{-s}(\pi x)dx = \frac{1}{n}\frac{n^s}{\pi^2} - \frac{1}{\pi^s}\int_0^{1/n}x^{-s}dx + o(n^{s-1})\\ &= \frac{n^{s-1}}{\pi^s}\left(1 - \frac{1}{1-s}\right) + o(n^{s-1}).\end{aligned}$$

This shows by letting n go to infinity that

$$\zeta(s) \leq -\frac{s}{1-s} < 0,$$

which is consistent with numerics, for example, $\zeta(1/2) = -1.460\ldots < -1$.

As with several other aspects of this paper, we leave higher dimensions to future study.

7 Approximative functional equations

It is natural to wonder about to what extent $\zeta_{\mathbb{Z}/n\mathbb{Z}}$ has a functional equation. In view of our asymptotics and the, in this context crucial, relation $\xi_{\mathbb{Z}}(s) = \xi_{\mathbb{Z}}(1-s)$, one could expect at least an asymptotic version. Indeed, we start by completing the finite torus zeta functions as $\xi_{\mathbb{Z}/n\mathbb{Z}}(s) := 2^s\cos(\pi s/2)\zeta_{\mathbb{Z}/n\mathbb{Z}}(s/2)$, and multiply the asymptotics at s in the critical strip with the corresponding fudge factors, and do the similar thing for the corresponding formula at $1-s$. After that, we subtract the two expressions, the one at s with the one at $1-s$, and obtain after further calculations, notably using $\xi_{\mathbb{Z}}(s)=\xi_{\mathbb{Z}}(1-s)$:

$$\xi_{\mathbb{Z}/n\mathbb{Z}}(s)-\xi_{\mathbb{Z}/n\mathbb{Z}}(1-s)=X(s)n^s-X(1-s)n^{1-s}$$

$$-\frac{s}{6}X(s-2)n^{s-2}+\frac{1-s}{6}X((1-s)-2)n^{(1-s)-2}+o(n^a),$$

where $a=\max\{Re(s)-2,-1-Re(s)\}$ and $X(s)=2\pi^{-s}\cos(\pi s/2)\zeta(s)$. Thus:

Corollary 7.1. *The Riemann zeta function has a zero at s in the critical strip iff*

$$\lim_{n\to\infty}\left(\xi_{\mathbb{Z}/n\mathbb{Z}}(1-s)-\xi_{\mathbb{Z}/n\mathbb{Z}}(s)\right)=0$$

as $n\to\infty$, unless $s=1/2$. In any case, for all s in the critical strip

$$\lim_{n\to\infty}\frac{1}{n}\left(\xi_{\mathbb{Z}/n\mathbb{Z}}(1-s)-\xi_{\mathbb{Z}/n\mathbb{Z}}(s)\right)=0.$$

As is well known there is a very useful approximative functional equation for $\zeta(s)$, sometimes called the Riemann-Siegel formula, which states that

$$\zeta(s)=\sum_{k=1}^{n}\frac{1}{k^s}+\pi^{s-1/2}\frac{\Gamma((1-s)/2)}{\Gamma(s/2)}\sum_{k=1}^{m}\frac{1}{k^{1-s}}+R_{n,m}(s),$$

where $R_{m,n}$ is the error term. Notice that the two partial Dirichlet series here have the same sign, which is a different feature from the formulas above. A question here is what functional equations prevail in higher dimension d.

8 The Riemann hypothesis

From the asymptotics given in the theorems above there is a straightforward reformulation of the Riemann hypothesis in terms of the asymptotical behaviour of

$$\sum_{k=1}^{n-1}\frac{1}{\sin(\pi k/n)^s}$$

as $n\to\infty$ as a function of s. It turns out however, that there is a more unexpected, nontrivial, and, what we think, more interesting equivalence with the Riemann hypothesis.

To show this we begin from the second asymptotical formula in Theorem 0.3:

$$\sum_{k=1}^{n-1}\frac{1}{\sin(\pi k/n)^s}=\frac{1}{\sqrt{\pi}}\frac{\Gamma(1/2-s/2)}{\Gamma(1-s/2)}n+2\pi^{-s}\zeta(s)n^s+\frac{s}{3}\pi^{2-s}\zeta(s-2)n^{s-2}+o(n^{s-2})$$

for $0<Re(s)<1$ as $n\to\infty$.

Let

$$h_n(s)=(4\pi)^{s/2}\Gamma(s/2)n^{-s}\left(\zeta_{\mathbb{Z}/n\mathbb{Z}}(s/2)-n\zeta_{\mathbb{Z}}(s/2)\right)$$

$$= \pi^{s/2}\Gamma(s/2)n^{-s}\left(\sum_{k=1}^{n-1}\frac{1}{\sin(\pi k/n)^s} - \frac{1}{\sqrt{\pi}}\frac{\Gamma(1/2-s/2)}{\Gamma(1-s/2)}n\right).$$

Using the completed Riemann zeta function $\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$ the above asymptotics can be restated as

$$h_n(s) = 2\xi(s) + \alpha(s)n^{-2} + o(n^{-2}),$$

where $\alpha(s) := \frac{s}{3}\pi^{2-s/2}\Gamma(s/2)\zeta(s-2)$.

From this asymptotics and in view of $\xi(1-s) = \xi(s)$ we conclude immediately:

Proposition 8.1. *Let $s \in \mathbb{C}$ with $0 < Re(s) < 1$ and $\zeta(s) \neq 0$. Then $h_n(1-s) \sim h_n(s)$ in the sense that*

$$\lim_{n\to\infty}\frac{h_n(1-s)}{h_n(s)} = 1.$$

We now conjecture that a weakened version of this asymptotic functional relation is valid even at zeta zeros:

Conjecture. *Let $s \in \mathbb{C}$ with $0 < Re(s) < 1$. Then*

$$\lim_{n\to\infty}\left|\frac{h_n(1-s)}{h_n(s)}\right| = 1.$$

From now on we will prove that this is equivalent to the Riemann hypothesis:

Theorem. *The conjecture above is equivalent to the statement that all non-trivial zeros of ζ have real part $1/2$.*

We begin the proof with a simple observation:

Lemma 8.2. *Suppose $\zeta(s) = 0$. Then the asymptotic relation*

$$\lim_{n\to\infty}\left|\frac{h_n(1-s)}{h_n(s)}\right| = 1$$

is equivalent to $|\alpha(1-s)| = |\alpha(s)|$.

Next we have:

Lemma 8.3. *The equation $|\alpha(1-s)| = |\alpha(s)|$ holds for all s on the critical line $Re(s) = 1/2$.*

Proof. Recall that

$$\alpha(s) = \frac{s}{3}\pi^{2-s/2}\Gamma(s/2)\zeta(s-2).$$

Since $\zeta(\overline{s}) = \overline{\zeta(s)}$ and $\Gamma(\overline{s}) = \overline{\Gamma(s)}$, we have that $\alpha(\overline{s}) = \overline{\alpha(s)}$. Therefore if $s = 1/2 + it$, then

$$\alpha(1-s) = \alpha(1-1/2-it) = \alpha(\overline{1/2+it}) = \overline{\alpha(s)},$$

which implies the lemma. □

Note that using $\xi((1-s)-2) = \xi(s+2)$ and Euler's reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ we have

$$\left|\frac{\alpha(1-s)}{\alpha(s)}\right| = \left|\frac{\frac{(s-1)(s+1)}{6}\pi\pi^{-(s+2)/2}\Gamma((s+2)/2)\zeta(s+2)}{\frac{s(s-2)}{6}\pi\pi^{-(s-2)/2}\Gamma((s-2)/2)\zeta(s-2)}\right| = \left|\frac{\zeta(s+2)(s-1)(s+1)}{\zeta(s-2)4\pi^2}\right|.$$

As a consequence $|\alpha(1-s)| = |\alpha(s)|$ is equivalent to

$$\left|\frac{\zeta(s+2)}{\zeta(s-2)}\right| = \frac{4\pi^2}{|s^2-1|}.$$

We will study the right and left hand sides as functions of σ, in the interval $0 < \sigma < 1$, with $s = \sigma + it$ and $t > 0$ fixed. In view of that

$$\frac{1}{|s^2-1|^2} = \frac{1}{\sigma^4 + 2\sigma^2 + (t^2-1)^2},$$

we see that the right hand side is strictly decreasing in σ. On the other hand we have the following:

Lemma 8.4. *Let $s = \sigma + it$, with t fixed such that $|t| > 26$. Then the function*

$$\left|\frac{\zeta(s+2)}{\zeta(s-2)}\right|$$

is strictly increasing in $0 < \sigma < 1$.

Proof. As remarked in [MSZ14], for a homolorphic function f, a simple calculation, using the Cauchy-Riemann equation, leads to

$$Re(f'(s)/f(s)) = \frac{1}{|f(s)|}\frac{\partial\,|f(s)|}{\partial\sigma},$$

in any domain where $f(z) \neq 0$. This implies that for $|f|$ to be increasing in σ we should show that the real part of its logarithmic derivative is positive.

We begin with one of the two terms in the logarithmic derivative of $\zeta(s+2)/\zeta(s-2)$:

$$Re(\zeta'(s+2)/\zeta(s+2)) = -Re(\sum_{n\geq 1}\Lambda(n)n^{-s-2}) = -\sum_{n\geq 1}\Lambda(n)n^{-\sigma-2}\cos(t\log n),$$

where $\Lambda(n)$is the von Mangoldt function. So

$$\left|Re(\zeta'(s+2)/\zeta(s+2))\right| \leq \sum_{n\geq 1}\Lambda(n)n^{-2} = -\frac{\zeta'(2)}{\zeta(2)} = \gamma + \log(2\pi) - 12\log A < 0.57,$$

by known numerics. We are therefore left to show that the other term

$$Re(-\zeta'(s-2)/\zeta(s-2)) \geq 0.57.$$

On the one hand, following the literature, see [L99, SD10, MSZ14], from the Mittag-Leffler expansion we have

$$\frac{\tilde{\xi}'(s)}{\tilde{\xi}(s)} = \sum_{\rho}\frac{1}{s-\rho}$$

where the sum is taken over the zeros which all lie in the critical strip. (The function $\tilde{\xi}(s)$ is defined by $\tilde{\xi}(s) = (s-1)\Gamma(1+s/2)\pi^{-s/2}\zeta(s)$.) This implies by a simple termwise calculation ([MSZ14]) that since $s-2$ is to the left of the critical strip, we have $Re(\tilde{\xi}'(s-2)/\tilde{\xi}(s-2)) < 0$ in the interval $0 < \sigma < 1$. On the other hand

$$0 > Re(\tilde{\xi}'(s-2)/\tilde{\xi}(s-2)) = Re(1/(s-3)) + \frac{1}{2}Re(\psi(s/2)) - \frac{1}{2}\log\pi + Re(\zeta'(s-2)/\zeta(s-2)),$$

where ψ is the logarithmic derivative of the gamma function. We estimate

$$Re(1/(s-3)) = \frac{\sigma-3}{(\sigma-3)^2+t^2} > \frac{-3}{4+t^2} > -\frac{3}{4+144} > -0.03$$

and $-\log\pi > -1.2$. Hence

$$Re(-\zeta'(s-2)/\zeta(s-2)) > -0.7 + Re(\psi(s/2))/2.$$

The last thing to do is to estimate the psi-function. Following [MSZ14], we have using Stirling's formula for ψ,

$$Re(\psi(s)) = \log|s| - \frac{\sigma}{2|s|^2} + Re(R(s)),$$

where $|R(s)| \leq \sqrt{2}/(6|s|^2)$. This is valid for any $s = \sigma + it$ in the critical strip. We observe that

$$-\frac{\sigma}{2|s|^2} \geq -\frac{1}{2t^2}$$

so

$$Re(\psi(s/2)) \geq \log\frac{|t|}{2} - \frac{2}{t^2} - \frac{2\sqrt{2}}{3t^2} \geq 2.56$$

if $|t| \geq 26$. This completes the proof. $\square$

Note that by numerics one can see that the lemma does not hold for small t. The lemma implies that the left and right hand sides can be equal only once for a fixed t, and this occurs at $Re(s) = 1/2$ as shown above. We summarize this in the following statement which concerns just the Riemann zeta function:

Proposition 8.5. *For $s \in \mathbb{C}$ with $0 < Re(s) < 1$, with $|Im(s)| > 26$, the equality $|\alpha(1-s)| = |\alpha(s)|$ holds if and only if $Re(s) = 1/2$.*

Therefore, since it is known that the Riemann zeta zeros in the critical strip having imaginary part less than 26 in absolute value all lie on the critcal line and in view of Lemma 8.2, the equivalence between the graph zeta functional equation and the Riemann hypothesis is established.

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A functional relation for L-functions of graphs equivalent to the Riemann Hypothesis for Dirichlet L-functions

Fabien Friedli*

Abstract

In this note we define L-functions of finite graphs and study the particular case of finite cycles in the spirit of a previous paper that studied spectral zeta functions of graphs. The main result is a suggestive equivalence between an asymptotic functional equation for these L-functions and the corresponding case of the Generalized Riemann Hypothesis. We also establish a relation between the positivity of such functions and the existence of real zeros in the critical strip of the classical Dirichlet L-functions with the same character.

Keywords: L-functions of graphs, Dirichlet L-functions, functional equations, Generalized Riemann Hypothesis, cyclic graphs

1 Introduction

In a previous paper with Karlsson ([6]), we initiated the study of spectral zeta functions of graphs. In particular, we studied the behaviour of the zeta function of the cyclic graph with n vertices as the parameter n goes to infinity. It turned out that the classical Riemann zeta function (or Epstein zeta functions in more than one dimension) appeared in the asymptotics, together with the spectral zeta function of the graph $\mathbb{Z}$. This lead to a surprising reformulation of the Riemann Hypothesis in terms of an asymptotic functional equation $s \longleftrightarrow 1-s$ for spectral zeta functions. It is natural to wonder whether this could be generalized to L-functions.

In this work, we propose a definition of L-functions for finite graphs different from those in Stark-Terras [10]. Given a finite graph G with $m \geq 3$ vertices and a primitive character modulo $k \geq 3$ one can define a *(spectral) L-function* of G by

$$L_G(s,\chi) := \sum_{j=1}^{m-1} \frac{\chi(j)}{\lambda_j^s},$$

where the sum runs over all non-zero ordered eigenvalues of the combinatorial Laplacian on G and s is any complex number, in analogy with the definition of spectral zeta functions in [6]. For our purposes here we normalize things a bit differently in the case of the Cayley graph of $\mathbb{Z}/kn\mathbb{Z}$ and a character modulo k.

*The author was supported in part by the Swiss NSF grant 200021 132528/1.

Definition 1.1 Let $k \geq 3$ and let χ be a primitive and even ($\chi(-1) = 1$) Dirichlet character modulo k. For $n \geq 1$, the function $L_n : \mathbb{C} \longrightarrow \mathbb{C}$ defined by

$$L_n(s,\chi) := \sum_{j=1}^{kn-1} \frac{\chi(j)}{\sin(\frac{\pi j}{kn})^s}$$

is the *(normalized) L-function* of the cyclic graph with kn vertices $\mathbb{Z}/kn\mathbb{Z}$ associated to the character χ.

In contrast to [6], it is not clear to us how to define appropriate L-functions for infinite graphs.
Denote by $L(s,\chi)$ the classical Dirichlet L-function associated to the character χ, which is defined by

$$L(s,\chi) = \sum_{j\geq 1} \frac{\chi(j)}{j^s}$$

when $\mathrm{Re}(s) > 0$ and by analytic continuation elsewhere. We will prove in section 2 the following asymptotics:

Proposition 1.2 *For any $s \in \mathbb{C}$ we have, as $n \longrightarrow \infty$,*

$$L_n(s,\chi) = 2\left(\frac{kn}{\pi}\right)^s \left(L(s,\chi) + \frac{s}{6}\left(\frac{kn}{\pi}\right)^{-2} L(s-2,\chi) + O(\frac{1}{n^4})\right).$$

This leads us into an interesting reformulation of the Generalized Riemann Hypothesis (GRH) (reminiscent to Riemann's case in [6], although there one also has a $\zeta_{\mathbb{Z}}$ term).

Theorem 1.3 *Let χ be a primitive and even Dirichlet character modulo $k \geq 3$. For $0 < Re(s) < 1$, let*

$$\xi_n(s,\chi) = n^{-s}(\frac{\pi}{k})^{\frac{s}{2}}\Gamma(\frac{s}{2})L_n(s,\chi)$$

be the (normalized) completed L-function of $\mathbb{Z}/kn\mathbb{Z}$. The following are equivalent:

(i) *For all s such that $0 < Re(s) < 1$ and $Im(s) \geq 8$ we have*

$$\lim_{n\to\infty} \frac{|\xi_n(s,\chi)|}{|\xi_n(1-s,\overline{\chi})|} = 1;$$

(ii) *In the region $Im(s) \geq 8$, $L(s,\chi)$ satisfies the Generalized Riemann Hypothesis (GRH), that is all zeros in this region have real part $\frac{1}{2}$.*

We will see in the proof that the relation in (i) holds in any case for all s such that $L(s,\chi) \neq 0$ and for obvious reasons on the critical line $\mathrm{Re}(s) = \frac{1}{2}$ for each n.

For comparison, recall that if we let $\xi(s,\chi) = (\frac{\pi}{k})^{\frac{-s}{2}}\Gamma(\frac{s}{2})L(s,\chi)$ then the functional equation implies that $|\xi(s,\chi)| = |\xi(1-s,\overline{\chi})|$.

In view of Proposition 1.2 there is a more obvious way to relate properties of our L-functions of cyclic graphs to GRH. Indeed, since the holomorphic functions $\frac{1}{2}\left(\frac{\pi}{kn}\right)^s L_n(s,\chi)$ converge to the corresponding Dirichlet L-function, it would be possible to study zeros of these finite sums to get information about zeros of L (for example using Hurwitz theorem). Also we see that the rate of convergence of the sequence L_n changes according to whether $L(s,\chi) = 0$ or not. Nervertheless we believe the result given in Theorem 1.3 is more unexpected and structural, in that it unveils a relationship between zeros of L and an asymptotic functional equation for L_n of the usual type $s \longleftrightarrow 1-s$. A weaker (additive) version of the latter holds unconditionally by Proposition 1.2, but the remarkable thing is that a strengthening of this property is ruled by GRH.

Acknowledgement. The author gratefully thanks Anders Karlsson for useful discussions, comments and corrections to this paper. This work was also partly inspired by a question raised by François Ledrappier following a lecture about [6]. The author thanks the referee for his comments.

2 Asymptotics of $L_n(s,\chi)$

In this section we derive an asymptotic formula for $L_n(s,\chi)$ when the parameter $n \longrightarrow \infty$.

Proof of Proposition 1.2 The proof is based on an Euler-MacLaurin formula established in [9]. We recall the result we need here. Let $f \in \mathcal{C}^\infty(a,b)$ such that, for every $K \in \mathbb{N}^*$,

$$f(x) = \sum_{k=0}^{K-1} c_k (x-a)^{\gamma_k} + O((x-a)^{\gamma_K})$$

when $x \longrightarrow a^+$ and

$$f(x) = \sum_{k=0}^{K-1} d_k (b-x)^{\delta_k} + O((b-x)^{\delta_K})$$

when $x \longrightarrow b^-$. Suppose that $-\gamma_k \notin \mathbb{N}^*$, $\mathrm{Re}(\gamma_k)$ is increasing, $\lim_{k\to\infty} \mathrm{Re}(\gamma_k) = \infty$ and similarly for δ_k. Then as $n \longrightarrow \infty$ we have, for every $0 < \theta < 1$,

$$\begin{aligned}\frac{b-a}{n}\sum_{i=0}^{n-1} f\Big(a + (i+\theta)\frac{(b-a)}{n}\Big) = \int_a^b f(x)dx &+ \sum_{k=0}^{K-1} c_k \zeta(-\gamma_k,\theta)\Big(\frac{b-a}{n}\Big)^{\gamma_k+1} \\ &+ \sum_{k=0}^{K-1} d_k \zeta(-\delta_k, 1-\theta)\Big(\frac{b-a}{n}\Big)^{\delta_k+1} \\ &+ O\Big(\Big(\frac{b-a}{n}\Big)^{\min\{\gamma_K+1,\delta_K+1\}}\Big),\end{aligned}$$

where $\zeta(s,\theta)$ is the Hurwitz zeta function. This is the content of the first part of Theorem 2.1 in [9]. More general formulas can be found in that paper.
We apply this to the function

$$f(x) = \frac{1}{(\sin(\pi x))^s}$$

on the interval $(0,1)$, with $0 < \mathrm{Re}(s) < 1$. Since $f(x) = x^{-s}(\pi^{-s} + \frac{s}{6}\pi^{2-s}x^2 + O(x^4))$ near $x = 0$ and in view of the symmetry of the sine, we have

$$\frac{1}{n}\sum_{j=0}^{n-1}\frac{1}{(\sin(\frac{\pi}{n}(j+\theta)))^s} = \int_0^1 \frac{1}{(\sin(\pi x))^s}dx + \pi^{-s}\left(\zeta(s,\theta)+\zeta(s,1-\theta)\right)\frac{1}{n^{1-s}}$$
$$+\pi^{2-s}\frac{s}{6}\left(\zeta(s-2,\theta)+\zeta(s-2,1-\theta)\right)\frac{1}{n^{3-s}}$$
$$+O\left(\frac{1}{n^{5-s}}\right).$$

Now we notice that

$$L_n(s,\chi) = \sum_{r=1}^{k-1}\chi(r)\sum_{j=0}^{n-1}\frac{1}{\sin(\frac{\pi}{n}(j+\frac{r}{k}))^s}$$

using periodicity of characters. The following identity is well-known:

$$L(s,\chi) = \frac{1}{k^s}\sum_{m=1}^{k}\chi(m)\zeta(s,m/k), \tag{2.1}$$

with χ as before and $s \neq 1$. Putting everything together, we obtain

$$\begin{aligned} L_n(s,\chi) &= \left(\frac{n}{\pi}\right)^s\sum_{r=1}^{k-1}\chi(r)\left(\zeta(s,r/k)+\zeta(s,1-r/k)\right) \\ &+\left(\frac{n}{\pi}\right)^{s-2}\frac{s}{6}\sum_{r=1}^{k-1}\chi(r)\left(\zeta(s-2,r/k)+\zeta(s-2,1-r/k)\right)+O\left(n^{s-4}\right) \\ &= 2\left(\frac{n}{\pi}\right)^s\sum_{r=1}^{k-1}\chi(r)\zeta(s,r/k)+2\left(\frac{n}{\pi}\right)^{s-2}\frac{s}{6}\sum_{r=1}^{k-1}\chi(r)\zeta(s-2,r/k)+O\left(n^{s-4}\right), \end{aligned}$$

where we used the fact that

$$\sum_{r=1}^{k-1}\chi(r)\zeta(s,1-\frac{r}{k}) = \sum_{r=1}^{k-1}\chi(k-r)\zeta(s,\frac{k-r}{k}) = \sum_{r=1}^{k-1}\chi(r)\zeta(s,\frac{r}{k}).$$

Now we can apply (2.1) and conclude by analytic continuation. □

Remark 2.1 We can obtain as many terms as we want in the proposition by looking further in the asymptotics of f near 0.

3 Relation with GRH

A natural question to ask is whether there is a kind of functional equation for L_n. Indeed, Proposition 1.2 trivially implies $\lim_{n\to\infty}\left(\xi_n(s,\chi)-\xi_n(1-s,\overline{\chi})\right) = 0$. But [6] suggests looking for a stronger version and an equivalence to GRH. This is the content of Theorem 1.3 which we now begin to prove.

In what follows, we write $G(\chi) = \sum_{l=0}^{k-1} \chi(l) e^{\frac{2\pi i l}{k}}$ for the Gauss sum of χ. Recall that $L(s,\chi)$ satisfies a functional equation: write $\xi(s,\chi) = (\frac{\pi}{k})^{\frac{-s}{2}} \Gamma(\frac{s}{2}) L(s,\chi)$ for the completed L-function. Then (see [3]),

$$\xi(s,\chi) = \frac{G(\chi)}{\sqrt{k}} \xi(1-s,\overline{\chi}).$$

We begin the proof with the following lemma which concerns only Dirichlet L-functions.

Lemma 3.1 *Let $t \in \mathbb{R}$ be fixed with $|t| \geq 8$ and write $s = \sigma + it$. Then the function*

$$\left| \frac{L(s+2,\chi)}{L(s-2,\chi)} \right|$$

is strictly increasing in $0 < \sigma < 1$.

Proof The proof relies on a recent paper by Matiyasevich, Saidak and Zvengrowski ([7]). By Lemma 2.1 in that paper, it is enough to prove that the real part of the logarithmic derivative of $\frac{L(s+2,\chi)}{L(s-2,\chi)}$ is positive for all s as in the statement of the lemma. Thus we are done if we show that

$$\mathrm{Re}\left(-\frac{L'(s-2,\chi)}{L(s-2,\chi)} \right) > \left| \mathrm{Re}\left(\frac{L'(s+2,\chi)}{L(s+2,\chi)} \right) \right|.$$

The right-hand side is easy to estimate. Indeed, since $\mathrm{Re}(s+2) > 1$ we have

$$\left| \mathrm{Re}\left(\frac{L'(s+2,\chi)}{L(s+2,\chi)} \right) \right| = \left| -\mathrm{Re}\left(\sum_{n\geq 1} \frac{\Lambda(n)\chi(n)}{n^{s+2}} \right) \right| \leq \left| \sum_{n \geq 1} \frac{\Lambda(n)\chi(n)}{n^{s+2}} \right|$$
$$\leq \sum_{n\geq 1} \frac{\Lambda(n)}{n^{\sigma+2}} \leq \sum_{n \geq 1} \frac{\Lambda(n)}{n^2} = -\frac{\zeta'(2)}{\zeta(2)} < 0.57.$$

For the left-hand side we first use the definition of ξ to see that

$$\mathrm{Re}\left(\frac{\xi'(s-2,\chi)}{\xi(s-2,\chi)} \right) = \frac{1}{2} \log\left(\frac{k}{\pi} \right) + \frac{1}{2} \mathrm{Re}\left(\psi\left(\frac{s-2}{2} \right) \right) + \mathrm{Re}\left(\frac{L'(s-2,\chi)}{L(s-2,\chi)} \right),$$

where $\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}$ is the logarithmic derivative of Γ.
Since $0 < \sigma < 1$ we have $\mathrm{Re}\left(\frac{\xi'(s-2,\chi)}{\xi(s-2,\chi)} \right) < 0$, as is shown in [7] using the Hadamard product. Thus,

$$\mathrm{Re}\left(-\frac{L'(s-2,\chi)}{L(s-2,\chi)} \right) > \frac{1}{2} \log\left(\frac{k}{\pi} \right) + \frac{1}{2} \mathrm{Re}\left(\psi\left(\frac{s-2}{2} \right) \right)$$
$$> \frac{1}{2} \mathrm{Re}\left(\psi\left(\frac{s-2}{2} \right) \right) - 0.03$$
$$> 0.57,$$

where we used $k \geq 3$ in the second inequality and Lemma 3.2 (v) of [7] (with our hypothesis that $|t| \geq 8$) in the last one. □

We now proceed to prove the equivalence stated in Theorem 1.3.

Proof of Theorem 1.3 By Proposition 1.2 we have

$$\begin{aligned}\xi_n(s,\chi) &= 2\left(\frac{\pi}{k}\right)^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\left(L(s,\chi)+\frac{s}{6}\left(\frac{kn}{\pi}\right)^{-2}L(s-2,\chi)+O\left(\frac{1}{n^4}\right)\right)\\ &= 2\xi(s,\chi)+\alpha(s,\chi)\frac{1}{n^2}+O\left(\frac{1}{n^4}\right),\end{aligned} \tag{3.1}$$

where we denote $\alpha(s,\chi) \coloneqq \frac{s}{3}(\frac{\pi}{k})^{2-\frac{s}{2}}\Gamma(\frac{s}{2})L(s-2,\chi)$.
Since $G(\chi)$ has modulus $\sqrt{k}$, the functional equation for ξ together with (3.1) implies that part (i) of the theorem is true for any s which is not a zero of L. We want to know when it is true also for other values of s, namely those corresponding to zeros of L. For this we observe that

$$\begin{aligned}\left|\alpha(1-s,\overline{\chi})\right| &= \left|\frac{1-s}{3}\left(\frac{\pi}{k}\right)^{2-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)L(-s-1,\overline{\chi})\right|\\ &= \left|\frac{1-s}{3}\left(\frac{\pi}{k}\right)^{\frac{3}{2}+\frac{s}{2}}\Gamma\left(\frac{1-s}{2}\right)\xi(-s-1,\overline{\chi})\left(\frac{\pi}{k}\right)^{\frac{-s-1}{2}}\frac{1}{\Gamma(\frac{-s-1}{2})}\right|\\ &= \left|\frac{(s-1)(s+1)}{6}\frac{\pi}{k}\Gamma\left(\frac{s+2}{2}\right)L(s+2,\chi)\left(\frac{\pi}{k}\right)^{-\frac{s+2}{2}}\right|\\ &= \left|\frac{s(s-1)(s+1)}{12}\left(\frac{\pi}{k}\right)^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)L(s+2,\chi)\right|\end{aligned}$$

and so the identity

$$|\alpha(s,\chi)| = |\alpha(1-s,\overline{\chi})| \tag{3.2}$$

is equivalent to the following:

$$\left|\frac{L(s+2,\chi)}{L(s-2,\chi)}\right| = \frac{4\pi^2}{k^2|s^2-1|}. \tag{3.3}$$

We note that identities (3.2) and (3.3) are true when $\mathrm{Re}(s) = \frac{1}{2}$, because

$$\alpha(1-(1/2+it),\overline{\chi}) = \alpha(\overline{1/2+it},\overline{\chi}) = \overline{\alpha(1/2+it,\chi)}.$$

Clearly, the right-hand side of (3.3) is strictly decreasing in $0<\sigma<1$. By Lemma 3.1 the left-hand side is strictly increasing in $0<\sigma<1$ provided $\mathrm{Im}(s)\geq 8$. Thus, (3.3) can be true for only one value of σ and this is $\sigma = \frac{1}{2}$.

This finishes the proof of Theorem 1.3, since when s is a zero of L, the assertion

$$\lim_{n\to\infty}\frac{|\xi_n(s,\chi)|}{|\xi_n(1-s,\overline{\chi})|} = 1$$

is equivalent to (3.2). □

4 On real zeros of $L(s,\chi)$

It is an important and difficult problem to show that Dirichlet L-functions have no real zeros. Even partial answers to this question have many consequences, in particular in the theory of primes in arithmetic progressions and in the theory of quadratic forms and class numbers initiated by Gauss. Particular cases such as the non-vanishing of $L(\frac{1}{2},\chi)$ or the existence of a Siegel zero are much studied and still open in general.

Consider χ a primitive, even and *real* character modulo $k \geq 3$ (if k is odd this is the Jacobi symbol modulo k). It is interesting to observe that if we suppose that $0 < s < 1$ is a real zero of the associated Dirichlet L-function then Proposition 1.2 tells us that $L_n(s,\chi) < 0$ for all n sufficiently large, since $L(s,\chi)$ is negative for $-2 < s < 0$. Thus the possible existence of real zeros of $L(s,\chi)$ is encoded in the L-function of a very simple graph.

Corollary 4.1 *Let χ be a primitive, even and real character modulo $k \geq 3$ and let $0 < s < 1$. The following are equivalent:*

(i) $L(s,\chi) > 0$;

(ii) For infinitely many $n \geq 1$ we have $L_n(s,\chi) \geq 0$, that is $\sum_{j=1}^{kn-1} \frac{\chi(j)}{\sin(\frac{\pi j}{kn})^s} \geq 0$.

A similar result is avalaible if we consider the standard partial sums $\sum_{j=1}^{kn-1} \frac{\chi(j)}{j^s}$ instead, see [4]. We refer to [6] for a heuristic explanation as to why these partial sums with sines, although being seemingly more involved, may be an interesting alternative to the usual partial sums in the study of Dirichlet L-functions.

Now we would like to mention an interesting problem related to the study of the sign of L_n and say a few words about its analog in the classical case. The problem comes from the following simple observation: if we write

$$\frac{1}{(1-x)^s} = \sum_{m\geq 0} a_m(s) x^m$$

for $0 < s < 1$, we see that

$$\sum_{j=1}^{kn-1} \frac{\chi(j)}{\sin(\frac{\pi j}{kn})^s} = \sum_{m\geq 1} a_m(s/2) \sum_{j=1}^{kn-1} \chi(j) \cos^{2m}\left(\frac{\pi j}{kn}\right).$$

It is then natural to ask:

Question 1 *Let χ be a primitive, even and real character modulo $k \geq 3$. Does there exist a sequence (a_n) of positive integers such that*

$$\sum_{j=1}^{ka_n-1} \chi(j) \cos^{2m}\left(\frac{\pi j}{ka_n}\right) \geq 0 \tag{4.1}$$

for every $m \geq 1$?

Note that this claim is independent of s. Since $a_m(s/2) > 0$ for all $0 < s < 1$ and $m \geq 1$, a positive answer to Question 1 would imply that $L(s,\chi)$ has no real zero. Obviously it is a much stronger statement but it is not clear to us whether it is true or not and we think it is an interesting problem to investigate.

It may be a good idea to draw a comparison with the classical case, that is when we consider the sign of the partial sums $\sum_{j=1}^{kn-1} \frac{\chi(j)}{j^s}$. We already mentioned the paper [4] where it is shown that it is enough to establish that $\sum_{j=1}^{k-1} \frac{\chi(j)}{j^s}$ is positive in order to prove that there is no real zero. We can make the same computation as above:

$$\sum_{j=1}^{k-1} \frac{\chi(j)}{(\frac{j}{k})^s} = \sum_{m\geq 0} a_m(s) \sum_{j=1}^{k-1} \chi(j) \left(1 - \frac{j}{k}\right)^m = \sum_{m\geq 0} a_m(s) \sum_{j=1}^{k-1} \chi(j) \left(\frac{j}{k}\right)^m,$$

where we used the fact that χ is even in the last equality. Thus if

$$S(m,\chi) := \sum_{j=1}^{k-1} \chi(j) j^m$$

was non-negative for all $m \geq 0$ (it is well-known that $S(0,\chi) = S(1,\chi) = 0$) it would imply that $L(s,\chi)$ has no real zero. This was already observed by Rosser in [8]. The analog to Question 1 is then:

Question 2 *Let χ be a primitive, even and real character modulo $k \geq 3$. Is it true that*

$$S(m,\chi) \geq 0$$

for every $m \geq 1$?

It is possible to show that $S(m,\chi) > 0$ if $m \geq k-2$, simply by showing that the term with $j = k-1$ dominates the rest of the sum. Thus if we are given a character only a finite number of sums need to be checked in order to answer Question 2. On the other hand, there is a nice formula available for $S(m,\chi)$, probably well-known, which we prove now.

Proposition 4.2 *Let χ be a primitive, even and real character modulo $k \geq 3$. For any $n \geq 1$ and $m \geq 2$ we have*

$$\frac{1}{(kn)^m} \sum_{j=1}^{kn-1} \chi(j) j^m = 2n\sqrt{k} \sum_{j=1}^{[\frac{m}{2}]} \frac{(-1)^{j+1} m(m-1)\dots(m-2j+2)}{(4\pi^2 n^2)^j} L(2j,\chi). \tag{4.2}$$

This can be thought of as a kind of Faulhaber formula twisted by a Dirichlet character.

Proof We use once again Euler-MacLaurin formula (see for example [11]):

$$\begin{aligned}\sum_{j=1}^{kn-1}\chi(j)\left(\frac{j}{kn}\right)^m &= \sum_{j=0}^{n-1}\sum_{r=0}^{k-1}\chi(jk+r)\Big(\frac{jk+r}{kn}\Big)^m \\ &= \sum_{r=1}^{k-1}\chi(r)\sum_{j=0}^{n-1}\left(\frac{1}{n}\Big(j+\frac{r}{k}\Big)\right)^m \\ &= \sum_{r=1}^{k-1}\chi(r)\left(\frac{n}{m+1}+\sum_{j=1}^{m}\frac{(-1)^{j+1}\zeta(1-j,\frac{r}{k})}{(j-1)!n^{j-1}}m(m-1)\dots(m-j+2)\right) \\ &= -\sum_{j=1}^{[\frac{m}{2}]}\frac{m(m-1)\dots(m-2j+2)}{(2j-1)!(kn)^{2j-1}}L(1-2j,\chi).\end{aligned}$$

We used the fact that χ is k-periodic in the second equality and, in the last equality, (2.1) and the fact that $L(-2m,\chi)=0$ for $m\geq 0$ when χ is even. We obtain (4.2) by applying the functional equation. □

Corollary 4.3 *Let χ be a primitive, even and real character modulo $k\geq 3$. For $m=2,3,4,5,6,7$ we have*

$$S(m,\chi)>0.$$

Proof Apply (4.2) with $n=1$ and $m=2,3,4,5,6$ or 7 and use the bounds

$$\frac{\zeta(2s)}{\zeta(s)}\leq L(s,\chi)\leq\zeta(s),$$

valid for all $s>1$ and all real characters. □

With a more delicate analysis it is in principle possible to extend Corollary 4.3 to greater values of m but it becomes more and more difficult as m grows. To our knowledge, no example of a character χ as in the corollary and an integer m such that $S(m,\chi)<0$ has been found.

The same problem for an *odd* character has received much more attention, starting with a paper by Chowla, Ayoub and Walum ([1]) where they show that the sign of $S(3,\chi)$ changes infinitely often when χ varies. See [5] for a stronger result and [12] for subsequent work on this problem. These kind of sums also appear in [2]. Of course, it makes no sense in our situation to consider odd characters since $L_n(s,\chi)\equiv 0$ in that case. We believe that the study of the sign of both sums $S(m,\chi)$ for even χ and (4.1) are worthy problems and we hope this note will stimulate work on this topic.

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The bundle Laplacian on discrete tori

Fabien Friedli*

Abstract

We prove an asymptotic formula for the determinant of the bundle Laplacian on discrete d-dimensional tori as the number of vertices tends to infinity. This determinant has a combinatorial interpretation in terms of cycle-rooted spanning forests. We also establish a relation (in the limit) between the spectral zeta function of a line bundle over a discrete torus, the spectral zeta function of the infinite graph $\mathbb{Z}^d$ and the Epstein-Hurwitz zeta function. The latter can be viewed as the spectral zeta function of the twisted continuous torus which is the limit of the sequence of discrete tori.

Keywords: bundle Laplacian, heat kernel, cycle-rooted spanning forest, spectral zeta function, Kronecker limit formula

1 Introduction

The number of spanning trees in a graph is an important quantity in combinatorics, probability, statistical physics and other fields, and has been studied extensively. The main tool used to count spanning trees is the matrix-tree theorem by Kirchhoff, which relates their number to the determinant of the combinatorial Laplacian. Thus this combinatorial problem can be translated into a spectral one. In [11] Kenyon develops the theory of the *vector bundle Laplacian*, first studied by Forman [7], in order, among other things, to obtain results on the loop-erased random walk on lattices (see also his paper [13]). There is an analog of the matrix-tree theorem in this setting, relating the determinant of the bundle Laplacian to cycle-rooted spanning forests, see Section 2.

In statistical physics in particular, it is often interesting to look at sequences of graphs whose number of vertices go to infinity and to relate the combinatorics of such sequences to continuous objects in the limit. If the graphs are discrete tori and we are interested in the number of spanning trees, this was carried out, in all dimensions, by Chinta, Jorgenson and Karlsson in [2]. They show in particular that the constant term in the asymptotics is the regularized determinant of the continuous torus. We refer to [15] for an explanation of what is expected to hold in general (for other graphs) and relations with quantum field theory.

In the present paper we use the ideas of [2] to establish an asymptotic formula for the determinant of the bundle Laplacian on discrete tori when the number of vertices goes to infinity.

Let G_n be the line bundle over the discrete torus defined in Section 2 and write Δ

*The author was supported in part by the Swiss NSF grant 200021 132528/1.

for the bundle Laplacian on G_n. Suppose that $\lambda_i \in [0,1]$ for each $i \in \{1,\dots,d\}$ and that $\lambda_i \notin \{0,1\}$ for some index i. Our main result is

Theorem 1.1 *For an integer $d \geq 1$ write*

$$c_d = -\int_0^\infty \left(e^{-2dt} I_0(2t)^d - e^{-t}\right) \frac{dt}{t}$$

and, for $Re(s) > \frac{d}{2}$,

$$\zeta_{EH}(s;\alpha_1,\dots,\alpha_d;\lambda_1,\dots,\lambda_d) = (2\pi)^{-2s} \sum_{K\in\mathbb{Z}^d} \left(\left(\frac{k_1+\lambda_1}{\alpha_1}\right)^2 + \dots + \left(\frac{k_d+\lambda_d}{\alpha_d}\right)^2\right)^{-s}.$$

Then, as $n \longrightarrow \infty$,

$$\log\det\Delta = \Big(\prod_{i=1}^d a_i(n)\Big) c_d - \zeta'_{EH}(0;\alpha_1,\dots,\alpha_d;\lambda_1,\dots,\lambda_d) + o(1).$$

The constant c_d is the same as for the spanning trees and it is known that $c_1 = 0$ and $c_2 = \frac{4G}{\pi}$, where G is Catalan's constant (see [2] or [17]). The difference lies in the second term (if we forget about the $\log(u^2)$ term in [2]).
In dimension $d = 2$ there is a nice expression for $\zeta'_{EH}(0;\alpha_1,\alpha_2;\lambda_1,\lambda_2)$ in the spirit of the famous Kroncecker limit formula, which adds some interest to this asymptotics independently of the combinatorial setting.

Theorem 1.2 *If $d = 2$ we have*

$$\zeta'_{EH}(0;\alpha_1,\alpha_2;\lambda_1,\lambda_2) = 2\pi\frac{\alpha_1}{\alpha_2} B_2(\lambda_2) - 2\log\prod_{n\in\mathbb{Z}} \left|1 - e^{2\pi i\lambda_1} e^{-2\pi\frac{\alpha_1}{\alpha_2}|n+\lambda_2|}\right|,$$

where $B_2(x) = x^2 - x + \frac{1}{6}$ stands for the second Bernoulli polynomial.

After giving some definitions in Section 2, we compute the heat kernel on G_n and determine asymptotics for the associated theta functions in Section 3. The proof of Theorem 1.1 is carried out in Section 4.
Finally, in Section 5 we consider the spectral zeta function of G_n as defined in [9] and show the following.

Theorem 1.3 *Let $s \in \mathbb{C}$ such that $s \neq m + \frac{d}{2}$ for $m \in \mathbb{N}$. Then, as $n \longrightarrow \infty$, we have*

$$\zeta_{G_n}(s) = \Big(\prod_{i=1}^d a_i(n)\Big)\zeta_{\mathbb{Z}^d}(s) + \zeta_{EH}(s;\alpha_1,\dots,\alpha_d;\lambda_1,\dots,\lambda_d) n^{2s} + o(n^{2s}).$$

This should be compared with the results obtained in [9] and [8] where similar formulas were obtained in the case of the standard Laplacian and spectral L-functions, respectively.

In this work we only consider diagonal discrete tori, but with a bit of work, more or less the same results should hold for arbitrary discrete tori, see [3].

Acknowledgement. The author is grateful to Anders Karlsson for suggesting this problem to him and for useful discussions and comments on this project. The author also thanks Justine Louis and Pham Anh Minh for interesting discussions related to this work.

2 Definitions

As explained by Kenyon in [11], given a finite graph G, we can construct a vector bundle over it. To each vertex of G we associate a vector space isomorphic to a given vector space. For each oriented edge we can then choose an isomorphism between the two vector spaces attached to the end-points of that edge, with the condition that the isomorphism corresponding to an oriented edge must be the inverse of the isomorphism corresponding to the same edge oriented in the opposite direction. The set of these isomorphisms is called the *connection* of G. We have a natural generalization of the Laplacian operator on such graphs, the *bundle Laplacian*. It acts on the functions $f : VG \longrightarrow \mathbb{C}$ (where VG denotes the set of vertices of G) and it is defined by

$$\Delta f(v) = \sum_{w \sim v} (f(v) - \phi_{w,v} f(w)),$$

where the sum is over all adjacent vertices and $\phi_{w,v}$ denotes the isomorphism for the oriented edge wv.

Remark 2.1 If the bundle is trivial (all the isomorphisms are identity), this is the usual Laplacian.

For the standard Laplacian we know by the matrix-tree theorem of Kirchhoff that its determinant (in fact the product of the non-zero eigenvalues) counts spanning trees in the graph. We have a similar combinatorial interpretation here. For this we only consider *line bundles*, that is bundles of dimension one. In this case we associate a copy of $\mathbb{C}$ to each vertex and the isomorphisms are just multiplication by a non-zero complex number. We can see this process as a choice of a complex weight on each oriented edge (with the inverse weight for the same edge with opposite orientation), but the bundle Laplacian should not be confused with what is usually called the weighted Laplacian (see for example [14]).
Given an oriented cycle, the product of the weights on the oriented edges along the cycle is called the *monodromy* of the cycle.
A subset of the set of the edges of a given graph which spans all the vertices of the graph and such that each connected component has exactly one cycle is called a *cycle-rooted spanning forest* and abreviated $CRSF$. The analog of Kirchhoff theorem is then ([11])

Theorem 2.2 *For a line bundle on a connected finite graph,*

$$\det \Delta = \sum_{CRSF's} \prod_{cycles} (2 - w - \frac{1}{w}),$$

where the sum is over all unoriented CRSF's C, the product is over cycles of C and $w, \frac{1}{w}$ are the monodromies of the two orientations of the cycle.

Remark 2.3 If the weights on the edges are chosen to be of modulus one, the bundle is called *unitary* and the bundle Laplacian becomes Hermitian and positive semidefinite.

We will evaluate $\det \Delta$ for a sequence of line bundles over discrete tori, when the number of vertices goes to infinity. More precisely, let $d \geq 1$ be an integer (the "dimension"). For each $i \in \{1, \ldots, d\}$, let $a_i(n)$ be a sequence of integers such that

$$\lim_{n \to \infty} \frac{a_i(n)}{n} = \alpha_i > 0.$$

For every n and $i \in \{1, \ldots, d\}$ we associate a complex number $w_{i,j}(n)$ of modulus one to the oriented edge between j and $j+1$ (with $0 \leq j \leq a_i(n) - 1$) in the Cayley graph of $\mathbb{Z}/a_i(n)\mathbb{Z}$ with generators $\{\pm 1\}$. We consider the *discrete torus* defined by the Cayley graph of

$$\prod_{i=1}^{d} \mathbb{Z}/a_i(n)\mathbb{Z}$$

(with generators given by $(0, \ldots, 0, \pm 1, 0, \ldots, 0)$) and the natural line bundle which comes with it (that is, the weight of an oriented edge in this graph is given by the weight associated to the corresponding edge in some $Cay(\mathbb{Z}/a_i(n)\mathbb{Z})$). We denote this graph (with the line bundle) by G_n. Note that this graph depends on several parameters, namely d, n, $a_i(n)$ and $w_{i,j}(n)$, but in order to simplify the notation we only write the dependence in n.

3 Heat kernel and theta functions

We adapt the method used in [2] and [9] to compute asymptotics for $\det \Delta$ and for the spectral zeta function associated to G_n for the sequence of discrete tori described above. The first step is to compute the heat kernel of the graph G_n, that is the unique bounded solution

$$K : \mathbb{R}_+ \times G_n \longrightarrow \mathbb{R}$$

of the equation

$$(\Delta + \frac{\partial}{\partial t}) K(t, x) = 0$$

with initial condition $K(0, x) = \delta_0(x)$, where δ is the Kronecker delta and 0 means the vertex corresponding to $(0, \ldots, 0)$ in G_n.
The existence and uniqueness of such a function is established for a general class of graphs and for the standard Laplacian in [5] and [4]. Here we do not need a general theory, because it is possible and quite easy to check the uniqueness of the solution found in Proposition 3.1 by taking the Fourier transform and solving the corresponding differential equation.

Proposition 3.1 *The heat kernel for the graph G_n defined above is given by*

$$K(t, x) = e^{-2dt} \sum_{K \in \mathbb{Z}^d} \prod_{i=1}^{d} I_{x_i + k_i a_i(n)}(2t) \prod_{j=0}^{a_i(n)-1} w_{i,j}(n)^{[\frac{j - x_i}{a_i(n)}] - k_i},$$

where we write $K = (k_1, \ldots, k_d)$, $x = (x_1, \ldots, x_d) \in \prod_{i=1}^{d} \mathbb{Z}/a_i(n)\mathbb{Z}$ and I_x for the modified Bessel function of the first kind of order x.

Proof First note that $K(t,x)$ is well-defined in the sense that we have $K(t,x) = K(t,y)$ if $x \equiv y$ in $\prod_{i=1}^d \mathbb{Z}/a_i(n)\mathbb{Z}$. Also the infinite sum is convergent and is bounded in t, as can be seen using the series representation for the modified Bessel function I_x, see [10].
For $i \in \{1,\dots,d\}$ and $x_i \in \mathbb{Z}/a_i(n)\mathbb{Z}$ define

$$K_i(t,x_i) := e^{-2t} \sum_{k\in\mathbb{Z}} I_{x_i+ka_i(n)}(2t) \prod_{j=0}^{a_i(n)-1} w_{i,j}(n)^{\left[\frac{j-(x_i+ka_i(n))}{a_i(n)}\right]}.$$

With this notation we have $K(t,x) = \prod_{i=1}^d K_i(t,x_i)$, where $x = (x_1,\dots,x_d)$.

Let $x_i \in \mathbb{Z}/a_i(n)\mathbb{Z}$ and write $x_i = m + ra_i(n)$ with $0 \le m \le a_i(n)-1$ (for convenience we do not write the dependence in x_i for m and for r). Since $\left[\frac{j-(m+1)}{a_i(n)}\right] = \left[\frac{j-m}{a_i(n)}\right]$ if $j \ne m$ and $\left[\frac{j-(m+1)}{a_i(n)}\right] = \left[\frac{j-m}{a_i(n)}\right] - 1$ if $j = m$ we observe that

$$\begin{aligned} K_i(t,x_i+1) &= e^{-2t} \sum_{k\in\mathbb{Z}} I_{x_i+1+ka_i(n)}(2t) \prod_{j=0}^{a_i(n)-1} w_{i,j}^{k+r+\left[\frac{j-(m+1)}{a_i(n)}\right]} \\ &= w_{i,m} e^{-2t} \sum_{k\in\mathbb{Z}} I_{x_i+1+ka_i(n)}(2t) \prod_{j=0}^{a_i(n)-1} w_{i,j}^{\left[\frac{j-(x_i+ka_i(n))}{a_i(n)}\right]}. \end{aligned}$$

Similarly we have

$$K_i(t,x_i-1) = w_{i,m-1}^{-1} e^{-2t} \sum_{k\in\mathbb{Z}} I_{x_i-1+ka_i(n)}(2t) \prod_{j=0}^{a_i(n)-1} w_{i,j}^{\left[\frac{j-(x_i+ka_i(n))}{a_i(n)}\right]},$$

where $m-1$ must be understood modulo $a_i(n)$, that is if $m=0$ the weigth above is in fact $w_{i,a_i(n)-1}^{-1}$. We keep this convention for the rest of the proof. Therefore, using the relation $I'_x(2t) = I_{x-1}(2t) + I_{x+1}(2t)$, we have

$$-\frac{\partial}{\partial t} K_i(t,x_i) = 2K_i(t,x_i) - w_{i,m}^{-1} K_i(t,x_i+1) - w_{i,m-1} K_i(t,x_i-1).$$

In other words, K_i is a solution to the heat equation on the i-th copy of our Cayley graph $\prod_{i=1}^d \mathbb{Z}/a_i(n)\mathbb{Z}$. Now we can compute

$$\begin{aligned} -\frac{\partial}{\partial t} K(t,x) &= \sum_{i=1}^d \left(-\frac{\partial}{\partial t} K_i(t,x_i)\right) \prod_{l\ne i} K_l(t,x_l) \\ &= 2dK(t,x) - \sum_{i=1}^d \left(w_{i,m}^{-1} K_i(t,x_i+1) + w_{i,m-1} K_i(t,x_i-1)\right) \prod_{l\ne i} K_l(t,x_l) \\ &= \Delta K(t,x). \end{aligned}$$

Using the fact that $I_0(0) = 1$ and $I_m(0) = 0$ for all $m \in \mathbb{Z}^*$ it is easy to check that $K(t,x) = \delta_0(x)$, which completes the proof. □

Remark 3.2 Notice that in the proof we show in fact that the heat kernel of the product graph is equal to the product of the heat kernels on each cyclic copy. To guess the formula on one copy we computed the heat kernel on $\mathbb{Z}$ with periodic weights and then took the quotient by making the function we obtained periodic. See [10] for more details about this procedure in the standard case without the weights.

From now on, we write $K_i(t)$ for $K_i(t,0)$ and $K(t)$ for $K(t,0)$. As explained in [9] the spectral zeta function of a finite graph is the Mellin transform of the trace of the heat kernel or, equivalently, of the theta function associated to the Laplacian acting on this graph. Here the theta function associated to G_n is given by

$$\theta^{G_n}(t) := \prod_{i=1}^{d} a_i(n) K_i(t),$$

where K_i is as in the proof above, that is

$$K_i(t) = e^{-2t} \sum_{k\in\mathbb{Z}} I_{ka_i(n)}(2t) \prod_{j=0}^{a_i(n)-1} w_{i,j}(n)^{-k}.$$

For convenience we write $\theta_i^{G_n}(t) = a_i(n)K_i(t)$. From this last expression, it is easy to see that $\theta_i^{G_n}(t) \sim a_i(n)e^{-2t}I_0(2t)$ when $t \longrightarrow 0$ (see Lemma 3.7). We will also need to know the behavior of $\theta_i^{G_n}(t)$ when $t \longrightarrow \infty$.

Proposition 3.3 *Let $\lambda_i(n) \in [0,1)$ such that $\prod_{j=0}^{a_i(n)-1} w_{i,j} = e^{2\pi i \lambda_i(n)}$. For any $t \in \mathbb{R}$ we have*

$$\theta_i^{G_n}(t) = \sum_{j=0}^{a_i(n)-1} e^{-4t\sin^2\left(\frac{\pi(j+\lambda_i(n))}{a_i(n)}\right)}.$$

For the proof we need a formula about Bessel functions that we could not find explicitly in the literature.

Lemma 3.4 *For any $t \in \mathbb{C}^*$, $z \in \mathbb{C}$ and $n \geq 1$ we have*

$$\sum_{k\in\mathbb{Z}} t^{kn} I_{kn}(z) = \frac{1}{n}\sum_{j=0}^{n-1} \exp\left(\frac{z}{2}(t^{-1}e^{\frac{2\pi i j}{n}} + te^{\frac{-2\pi i j}{n}})\right).$$

Proof Consider the function of the real variable x defined by $e^{\frac{z}{2}(\frac{1}{t}e^{ix}+te^{-ix})}$. It is 2π-periodic in x and derivable so by Fourier analysis we can write

$$e^{\frac{z}{2}(\frac{1}{t}e^{ix}+te^{-ix})} = \sum_{k\in\mathbb{Z}} L_k(z,t)e^{ikx},$$

where $L_k(z,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\frac{z}{2}(\frac{1}{t}e^{i\tau}+te^{-i\tau})}e^{-ik\tau}d\tau$. If we substitute $x = \frac{2\pi j}{n}$ and sum over $j = 0,\ldots,n-1$ we obtain

$$\sum_{j=0}^{n-1} e^{\frac{z}{2}(\frac{1}{t}e^{\frac{2\pi i j}{n}}+te^{\frac{-2\pi i j}{n}})} = n\sum_{k\in\mathbb{Z}} L_{kn}(z,t).$$

Hence it only remains to show that $\sum_{k\in\mathbb{Z}} L_{kn}(z,t) = \sum_{k\in\mathbb{Z}} t^{kn} I_{kn}(z)$. In order to do this write

$$F_m(z,t) = \sum_{\substack{k\in\mathbb{Z} \\ k\equiv m(n)}} L_k(z,t)$$

and

$$G_m(z,t) = \sum_{\substack{k\in\mathbb{Z} \\ k\equiv m(n)}} t^k I_k(z).$$

We want to prove that $F_0(z,t) = G_0(z,t)$. From the definition of L_k we observe that

$$\frac{d}{dz} L_k(z,t) = \frac{1}{2}\Big(\frac{1}{t} L_{k-1}(z,t) + tL_{k+1}(z,t)\Big)$$

and so we have

$$\frac{d}{dz} F_m(z,t) = \frac{1}{2}\Big(\frac{1}{t} F_{m-1}(z,t) + tF_{m+1}(z,t)\Big),$$

for all $m \in \{0,\dots,n-1\}$. We are left with a simple system of linear differential equations with matrix $A = Circ(0, \frac{t}{2}, 0, \dots, 0, \frac{1}{2t})$, where $Circ(v)$ means a circulant matrix with vector v, that is

$$\frac{d}{dz}\begin{pmatrix} F_0(z,t) \\ \vdots \\ F_{n-1}(z,t) \end{pmatrix} = A \begin{pmatrix} F_0(z,t) \\ \vdots \\ F_{n-1}(z,t) \end{pmatrix}.$$

The solution is given by the vector

$$e^{zA}\begin{pmatrix} F_0(0,t) \\ \vdots \\ F_{n-1}(0,t) \end{pmatrix}.$$

It is obvious from the definition that $L_0(0,t) = 1$ and $L_k(0,t) = 0$ if $k \neq 0$. Therefore we have $F_0(z,t) = c_{1,1}(e^{zA})$, where $c_{1,1}$ stands for the upper left entry of the matrix.

From classical properties of modified Bessel functions, we deduce that a similar system is satisfied by the functions $G_m(z,t)$, namely

$$\frac{d}{dz} G_m(z,t) = \frac{1}{2}\Big(tG_{m-1}(z,t) + \frac{1}{t} G_{m+1}(z,t)\Big)$$

for all $m \in \{0,\dots,n-1\}$. We note that the associated matrix here is A^T. Thus we have $G_0(z,t) = c_{1,1}(e^{zA^T}) = c_{1,1}((e^{zA})^T) = c_{1,1}(e^{zA}) = F_0(z,t)$, since the initial conditions are the same. This completes the proof. □

Now we can easily prove Proposition 3.3.

Proof of Proposition 3.3 In view of Lemma 3.4 we have

$$\begin{aligned}
\theta_i^{G_n}(t) &= a_i(n)e^{-2t}\sum_{k\in\mathbb{Z}} I_{ka_i(n)}(2t)\prod_{j=0}^{a_i(n)-1} w_{i,j}(n)^{-k} \\
&= a_i(n)e^{-2t}\sum_{k\in\mathbb{Z}} I_{ka_i(n)}(2t)\Big(e^{-\frac{2\pi i\lambda_i(n)}{a_i(n)}}\Big)^{ka_i(n)} \\
&= e^{-2t}\sum_{j=0}^{a_i(n)-1}\exp\Big(t\Big(e^{\frac{2\pi i\lambda_i(n)}{a_i(n)}}e^{\frac{2\pi ij}{a_i(n)}} + e^{-\frac{2\pi i\lambda_i(n)}{a_i(n)}}e^{-\frac{2\pi ij}{a_i(n)}}\Big)\Big) \\
&= \sum_{j=0}^{a_i(n)-1} e^{-4t\sin^2\left(\frac{\pi(j+\lambda_i(n))}{a_i(n)}\right)}.
\end{aligned}$$

□

Remark 3.5 As a consequence of this formula we have that the Laplace eigenvalues of G_n are given by the set

$$\{4\sin^2\Big(\frac{\pi(j_1+\lambda_1(n))}{a_1(n)}\Big) + \ldots + 4\sin^2\Big(\frac{\pi(j_d+\lambda_d(n))}{a_d(n)}\Big), 0 \le j_m \le a_i(n)-1\}.$$

Indeed, our conditions on the weights ensure that the bundle Laplacian we consider here is Hermitian and positive definite, so the same reasoning as in [10] on p.180 is valid. This shows that the heat kernel at 0 is in fact the trace of $e^{-t\Delta}$.

For each $1 \le i \le d$ fixed, we will assume without loss of generality that the sequence $\lambda_i(n)$ converges, taking a subsequence if necessary, by compactness. We write

$$\lambda_i := \lim_{n\to\infty}\lambda_i(n) \in [0,1].$$

For $t > 0$, we define

$$\theta_i^\infty(t) = \sum_{k\in\mathbb{Z}} e^{-4(\frac{\pi}{\alpha_i})^2 t(k+\lambda_i)^2}$$

and

$$\theta^\infty(t) := \prod_{i=1}^d \theta_i^\infty(t) = \sum_{K\in\mathbb{Z}^d} e^{-4\pi^2 t\sum_{i=1}^d\left(\frac{k_i+\lambda_i}{\alpha_i}\right)^2}.$$

Lemma 3.6 *For all $t > 0$ we have*

$$\theta_i^\infty(t) = \frac{\alpha_i}{\sqrt{4\pi t}}\sum_{k\in\mathbb{Z}} e^{-\frac{(\alpha_i k)^2}{4t} - 2\pi i\lambda_i k}.$$

Proof The sum in the right-hand side can be written as

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} e^{-\frac{(\alpha_i k)^2}{4t} - 2\pi i\lambda_i k} &= e^{-4(\frac{\pi\lambda_i}{\alpha_i})^2 t}\sum_{k\in\mathbb{Z}} e^{-(\frac{\alpha_i k}{\sqrt{4t}} + i\frac{\sqrt{4t}\pi\lambda_i}{\alpha_i})^2} \\
&= e^{-4(\frac{\pi\lambda_i}{\alpha_i})^2 t}\sum_{k\in\mathbb{Z}} f(k + i\frac{4t\pi\lambda_i}{\alpha_i^2}),
\end{aligned}$$

where $f(y) := e^{-\frac{(\alpha_i y)^2}{4t}}$. This function has a simple Fourier transform, namely $\hat{f}(\nu) = \frac{\sqrt{4\pi t}}{\alpha_i} e^{-4(\frac{\pi\nu}{\alpha_i})^2 t}$. By Poisson summation formula we conclude that

$$e^{-4(\frac{\pi\lambda_i}{\alpha_i})^2 t} \sum_{k\in\mathbb{Z}} f(k + i\frac{4t\pi\lambda_i}{\alpha_i^2}) = e^{-4(\frac{\pi\lambda_i}{\alpha_i})^2 t} \sum_{k\in\mathbb{Z}} \frac{\sqrt{4\pi t}}{\alpha_i} e^{-4(\frac{\pi k}{\alpha_i})^2 t} e^{2\pi i (i\frac{4t\pi\lambda_i}{\alpha_i^2})k}$$
$$= \frac{\sqrt{4\pi t}}{\alpha_i} \sum_{k\in\mathbb{Z}} e^{-4(\frac{\pi}{\alpha_i})^2 t(k+\lambda_i)^2}.$$

□

We consider the case where the bundle does not become trivial asymptotically, that is we suppose that there exists $i \in \{1, \ldots, d\}$ such that $\lambda_i \notin \{0, 1\}$. Taking n big enough, we can always assume that, for this index i, we have $\lambda_i(n) \neq 0$ for every n.

Lemma 3.7 *The following asymptotics hold:*

(a) *When $t \to \infty$ we have, for any n, $\theta^{G_n}(t) = O(e^{-c_1 t})$ for some $c_1 > 0$. We also have $\theta^{\infty}(t) = O(e^{-c_2 t})$ for some $c_2 > 0$.*

(b) *When $t \to 0^+$ we have $\theta^{G_n}(t) = \left(\prod_{i=1}^{d} a_i(n)\right) e^{-2dt} I_0(2t)^d + O(t^{\min a_i(n)})$. We also have $\theta^{\infty}(t) = \frac{\prod_{i=1}^{d}\alpha_i}{(4\pi t)^{\frac{d}{2}}} + O(e^{-c_3/t})$ for some $c_3 > 0$.*

Proof The assertions in point (a) follow from the definition of θ^{∞}, Proposition 3.3 and our hypotheses on $\lambda_i(n)$ and λ_i. The first assertion in point (b) follow from the definition of θ^{G_n}, together with the following estimate:

$$\left|\theta_i^{G_n}(t) - a_i(n) e^{-2t} I_0(2t)\right| = \left| a_i(n) e^{-2t} \sum_{k\neq 0} I_{ka_i(n)}(2t) \prod_{j=0}^{a_i(n)-1} w_{i,j}(n)^{-k} \right|$$
$$\leq 2a_i(n) e^{-2t} \sum_{k\geq 1} I_{ka_i(n)}$$
$$\leq 2a_i(n) e^{-2t} I_0(2t) \frac{t^{a_i(n)}}{1 - t^{a_i(n)}},$$

where we used the bound $I_{ka_i(n)}(2t) = t^{ka_i(n)} \sum_{j\geq 0} \frac{t^{2j}}{j!(j+ka_i(n))!} \leq I_0(2t) t^{ka_i(n)}$. The second assertion is a corollary of Lemma 3.6. □

4 Asymptotics of $\det\Delta$

In this section we establish an asymptotic formula for $\log\det\Delta$ when $n \longrightarrow \infty$, where Δ is the bundle Laplacian on G_n. We follow the steps of [2]. First we notice that, in our setting here, and in view of Remark 3.5, zero is not an eigenvalue. We begin with the following exact result.

Theorem 4.1 *Let*

$$c_d := -\int_0^{\infty} \left(e^{-2dt} I_0(2t)^d - e^{-t}\right) \frac{dt}{t}$$

and

$$\mathcal{H}_{d,n} := -\int_0^\infty \Big(\theta^{G_n}(t) - \Big(\prod_{i=1}^d a_i(n)\Big)e^{-2dt}I_0(2t)^d\Big)\frac{dt}{t}.$$

Then

$$\log(\det\Delta) = \Big(\prod_{i=1}^d a_i(n)\Big)c_d + \mathcal{H}_{d,n}.$$

Proof Thanks to the asymptotics derived in Lemma 3.7, we can proceed exactly in the same fashion as in section 3 of [2]. The only difference is that here 0 is not an eigenvalue, so we do not need to substract 1 in $\mathcal{H}_{d,n}$, whence a slightly different expression for $\mathcal{H}_{d,n}$. □

Now we need to understand the behavior of $\mathcal{H}_{d,n}$ when $n \longrightarrow \infty$. First we observe that the discrete theta function θ^{G_n} converges to the continuous one θ^∞ when suitably normalized.

Proposition 4.2 *For all* $t > 0$

$$\lim_{n\to\infty} \theta^{G_n}(n^2t) = \theta^\infty(t).$$

Proof Our hypotheses on the weights imply that

$$\lim_{n\to\infty} \prod_{j=0}^{a_i(n)-1} w_{i,j}(n)^{-k} = e^{-2\pi ik\lambda_i}$$

for every $i \in \{1,\dots,d\}$. This, together with Proposition 4.7 in [2], leads to

$$\lim_{n\to\infty} a_i(n)e^{-2n^2t}I_{ka_i(n)}(2n^2t)\prod_{j=0}^{a_i(n)-1} w_{i,j}(n)^{-k} = \frac{\alpha_i}{\sqrt{4\pi t}}e^{-\frac{(\alpha_i k)^2}{4t}-2\pi ik\lambda_i}.$$

Since our weights have modulus one the bounds used in the proof of Proposition 5.2 in [2] are valid and allow us to exchange the limit and the infinite sum to deduce that

$$\lim_{n\to\infty} \theta_i^{G_n}(n^2t) = \theta^\infty(t),$$

by Lemma 3.6. We conclude the proof by taking the d-fold product. □

Lemma 4.3 *There exists a constant* $c > 0$ *and an integer* n_0 *such that*

$$\theta^{G_n}(n^2t) \le e^{-ct}$$

for any $t > 0$ *and* $n \ge n_0$.

Proof This is an adaptation of the proof of Lemma 5.3 in [2].
Let $i \in \{1,\dots,d\}$. If $\lambda_i \neq 1$ let ϵ_i be a real number such that $\epsilon_i > 1$ and $\epsilon_i\lambda_i < 1$. If $\lambda_i = 1$ define $\epsilon_i = \frac{3}{2}$. Finally choose n_0 such that $\frac{a_i(n)}{n} \leq 2\alpha_i$ and $\frac{\lambda_i}{2} \leq \lambda_i(n) \leq \epsilon_i\lambda_i$ for every $n \geq n_0$.
For $0 \leq j \leq a_i(n)-1$ such that $\frac{j+\lambda_i(n)}{a_i(n)} \leq \frac{1}{2}$, the elementary estimate $\sin(x) \geq x - \frac{x^3}{6}$ (valid for $0 \leq x \leq \frac{\pi}{2}$) yields

$$\begin{aligned} a_i(n)\sin\Big(\frac{\pi(j+\lambda_i(n))}{a_i(n)}\Big) &\geq \pi(j+\lambda_i(n)) - \frac{\pi^3}{6}(j+\lambda_i(n))^3\frac{1}{a_i(n)^2} \\ &\geq \pi(j+\lambda_i(n)) - \frac{\pi^3}{24}(j+\lambda_i(n)) \\ &\geq \pi(j+\frac{\lambda_i}{2})(1-\frac{\pi^2}{24}) \geq 0. \end{aligned}$$

Hence

$$4n^2\sin^2\Big(\frac{\pi(j+\lambda_i(n))}{a_i(n)}\Big) \geq \frac{1}{\alpha_i^2}\Big(\pi(j+\frac{\lambda_i}{2})(1-\frac{\pi^2}{24})\Big)^2$$

and we have

$$\sum_{\substack{j=0 \\ 2(j+\lambda_i(n))\leq a_i(n)}}^{a_i(n)-1} e^{-4tn^2\sin^2\Big(\frac{\pi(j+\lambda_i(n))}{a_i(n)}\Big)} \leq \sum_{j=0}^{\infty} e^{-c_1(j+\frac{\lambda_i}{2})^2 t},$$

with $c_1 = \Big(\frac{\pi}{\alpha_i}(1-\frac{\pi^2}{24})\Big)^2 > 0$. For the other half of the sum defining $\theta_i^{G_n}(n^2t)$ we use the symmetry of the sine to write

$$\begin{aligned} 4n^2\sin^2\Big(\frac{\pi(j+\lambda_i(n))}{a_i(n)}\Big) &= 4n^2\sin^2\Big(\frac{\pi(a_i(n)-j-\lambda_i(n))}{a_i(n)}\Big) \\ &\geq \frac{1}{\alpha_i^2}\Big(\pi(a_i(n)-j-\lambda_i(n))(1-\frac{\pi^2}{24})\Big)^2 \end{aligned}$$

which leads to

$$\begin{aligned} \sum_{\substack{j=0 \\ 2(j+\lambda_i(n))>a_i(n)}}^{a_i(n)-1} e^{-4tn^2\sin^2\Big(\frac{\pi(j+\lambda_i(n))}{a_i(n)}\Big)} &\leq \sum_{\substack{j=0 \\ 2(j+\lambda_i(n))>a_i(n)}}^{a_i(n)-1} e^{-c_1(a_i(n)-j-\lambda_i(n))^2 t} \\ &= \sum_{\substack{j=1 \\ 2(j-\lambda_i(n))<a_i(n)}}^{a_i(n)} e^{-c_1(j-\lambda_i(n))^2 t} \\ &\leq e^{-c_1(1-\lambda_i(n))^2 t} + \sum_{j\geq 2} e^{-c_1(j-\epsilon_i\lambda_i)^2 t}. \end{aligned}$$

The last expression is less than $1+\sum_{j\geq 2} e^{-c_1(j-\epsilon_i\lambda_i)^2 t}$ if $\lambda_i = 1$. But if $\lambda_i \neq 1$ it is less than $e^{-c_1(1-\epsilon_i\lambda_i)^2 t} + \sum_{j\geq 2} e^{-c_1(j-\epsilon_i\lambda_i)^2 t}$.
Since there is at least one $i \in \{1,\dots,d\}$ such that $\lambda_i \notin \{0,1\}$, at least one of the $\theta_i^{G_n}(n^2t)$ will have have exponential decay thanks to the bounds we just derived. The desired result then follows by taking the d-fold product. □

Definition 4.4 *For $Re(s) > \frac{d}{2}$, we define the* Epstein-Hurwitz zeta function *as*

$$\zeta_{EH}(s;\alpha_1,\ldots,\alpha_d;\lambda_1,\ldots,\lambda_d) = \frac{1}{\Gamma(s)}\int_0^\infty \theta^\infty(t)t^{s-1}dt.$$

The integral is convergent by Lemma 3.7.

We can define a more general version of the Epstein-Hurwitz zeta function, see [6] or [1]. This a particular case where the quadratic form is diagonal. This function is a generalization of both the Epstein zeta function and the Hurwitz zeta function, whence its name. It can be seen as the spectral zeta function of the continuous "twisted" torus $\mathbb{R}^d/A\mathbb{Z}^d$ (where A is the diagonal matrix with α_i on the diagonal), in the sense that functions on this torus are functions u on $\mathbb{R}^d$ which are almost periodic, that is for all $x \in \mathbb{R}^d$:

$$e^{-2\pi i\lambda_i}u(x + (0,\ldots,0,\alpha_i,0,\ldots,0)) = u(x),$$

for every $1 \le i \le d$.
We can also write it in the following, more familiar way (using the definition of θ^∞):

$$\zeta_{EH}(s;\alpha_1,\ldots,\alpha_d;\lambda_1,\ldots,\lambda_d) = (2\pi)^{-2s}\sum_{K\in\mathbb{Z}^d}\left(\left(\frac{k_1+\lambda_1}{\alpha_1}\right)^2 + \ldots + \left(\frac{k_d+\lambda_d}{\alpha_d}\right)^2\right)^{-s}.$$

Thanks to the asymptotic behavior of θ^∞ (see Lemma 3.7) we can compute the analytic continuation of ζ_{EH} by writing

$$\int_0^\infty \theta^\infty(t)t^{s-1}dt = \int_1^\infty \theta^\infty(t)t^{s-1}dt + \int_0^1\left(\theta^\infty(t) - \frac{\prod_{i=1}^d \alpha_i}{(4\pi t)^{\frac{d}{2}}}\right)t^{s-1}dt + \frac{\prod_{i=1}^d\alpha_i}{(4\pi)^{\frac{d}{2}}(s-\frac{d}{2})},$$

where both integrals on the right-hand side define entire functions of s. This expression then provides a meromorphic continuation for ζ_{EH} to $\mathbb{C}$ with a simple pole at $s = \frac{d}{2}$. Note that it also implies that $\zeta_{EH}(-n;\alpha_1,\ldots,\alpha_d;\lambda_1,\ldots,\lambda_d) = 0$ for all integers $n \ge 0$. It is then possible to find an expression for the derivative at $s = 0$, using the fact that $\frac{1}{\Gamma(s)} = s + O(s^2)$ when $s \longrightarrow 0^+$:

$$\begin{aligned}\zeta'_{EH}(0;\alpha_1,\ldots,\alpha_d;\lambda_1,\ldots,\lambda_d) = \int_1^\infty \theta^\infty(t)\frac{dt}{t} + \int_0^1\left(\theta^\infty(t) - \left(\prod_{i=1}^d\alpha_i\right)(4\pi t)^{-\frac{d}{2}}\right)\frac{dt}{t}\\ - \frac{2}{d}\left(\prod_{i=1}^d\alpha_i\right)(4\pi)^{-\frac{d}{2}}. \qquad (4.1)\end{aligned}$$

Proposition 4.5 *We have*

$$\lim_{n\to\infty}\mathcal{H}_{d,n} = -\zeta'_{EH}(0;\alpha_1,\ldots,\alpha_d;\lambda_1,\ldots,\lambda_d).$$

Proof We split the integral in the definition of $\mathcal{H}_{d,n}$ (after changing variables) in the following way:

$$\begin{aligned}\mathcal{H}_{d,n} &= \int_0^\infty \Big(\theta^{G_n}(n^2 t) - \Big(\prod_{i=1}^d a_i(n)\Big)e^{-2dn^2 t} I_0(2n^2 t)^d\Big)\frac{dt}{t} \\ &= \int_1^\infty \theta^{G_n}(n^2 t)\frac{dt}{t} - \Big(\prod_{i=1}^d a_i(n)\Big)\int_1^\infty e^{-2dn^2 t} I_0(2n^2 t)^d \frac{dt}{t} \\ &+ \int_0^1 \Big(\theta^{G_n}(n^2 t) - \Big(\prod_{i=1}^d a_i(n)\Big)e^{-2dn^2 t} I_0(2n^2 t)^d\Big)\frac{dt}{t}.\end{aligned}$$

Thanks to the bound obtained in Lemma 4.3 we can change the limit with the integration sign in the first integral, which then converges to

$$\int_1^\infty \theta^\infty(t)\frac{dt}{t},$$

by Proposition 4.2.
The second term converges to

$$\frac{2}{d}\Big(\prod_{i=1}^d \alpha_i\Big)(4\pi)^{-\frac{d}{2}},$$

as proved in [2].
The third integral converges to

$$\int_0^1 \Big(\theta^\infty(t) - \Big(\prod_{i=1}^d \alpha_i\Big)(4\pi t)^{-\frac{d}{2}}\Big)\frac{dt}{t},$$

using again the same result of [2] (the bounds used in their Proposition 5.5 can be used here thanks to the fact that our weights have modulus one). We conclude by using (4.1). □

Thus we have proved that

$$\log\det\Delta = \Big(\prod_{i=1}^d a_i(n)\Big)\mathcal{I}_d - \zeta'_{EH}(0;\alpha_1,\ldots,\alpha_d;\lambda_1,\ldots,\lambda_d) + o(1),$$

which is Theorem 1.1.

This should be compared with the main theorem in [2]. In particular we see that the bundle has an influence on the second term only, the leading term being independent of the weights. In our opinion this formula has several interesting aspects. First it has a combinatorial interpretation in that, as explained previously, the determinant of the bundle Laplacian counts (with weights) the number of cycle-rooted spanning forests. Second it contains geometric information by relating the determinant of the bundle Laplacian on a line bundle over discrete weighted tori on the one hand and over a continuous torus on the other. Third it may have some number theoretic value, due to the Kronecker-type formula in Theorem 1.2. Finally it seems that physicists are also interested in quantities like $\zeta'_{EH}(0;\alpha_1,\ldots,\alpha_d;\lambda_1,\ldots,\lambda_d)$, see [6].

Example 4.6 If $d = 1$ the graph is a cycle and there is exactly one cycle-rooted spanning forest. It is then elementary to compute $\det\Delta$ using Theorem 2.2. We obtain

$$\det\Delta = 4\sin^2(\pi\lambda(n)) = 4\sin^2(\pi\lambda) + o(1),$$

when $n \longrightarrow \infty$.
On the other hand, we have

$$\zeta_{EH}(s;\alpha;\lambda) = \sum_{k\in\mathbb{Z}} \frac{1}{\left(\frac{k+\lambda}{\alpha}\right)^{2s}} = \alpha^{2s}(\zeta(2s,\lambda) + \zeta(2s,1-\lambda)),$$

where we write $\zeta(s,\lambda)$ for the standard Hurwitz zeta function. Using the formulas $\zeta(0,a) = \frac{1}{2} - a$, $\zeta'(0,a) = \log\Gamma(a) - \frac{1}{2}\log(2\pi)$ and $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$, we see that

$$\zeta'_{EH}(0;\alpha;\lambda) = -2(\log\sin(\pi\lambda) + \log(2)).$$

Since $c_1 = 0$ (see [2]) this small computation confirms Theorem 1.1 in dimension one.
Note that going in the opposite direction, this computation together with Theorem 1.1 consitutes a proof of the reflection formula for the gamma function

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

When the dimension is $d = 2$ there is a nice formula for the derivative of the Epstein-Hurwitz zeta function at $s = 0$, stated in Theorem 1.2. It is very similar to the Kronecker limit formula, which has important applications in number theory, see for example the paper by Chowla and Selberg [16]. The classical formula (which corresponds to the case with no bundle) involves the Dedekind eta-function, which is ubiquitous in the theory of modular form. The infinite product in Theorem 1.2 can be considered as a generalization of the latter. As far as we know, the Epstein-Hurwitz zeta function has received little attention in the literature, with the exception of the papers [6] and [1]. The expression in Theorem 1.2 does not appear explicitly in [1] and the formula proposed in [6] does not make apparent the analogy with the classical Kronecker limit formula.

Proof of Theorem 1.2 In this proof we write $\zeta_{EH}(s)$ for $\zeta_{EH}(s;\alpha_1,\alpha_2;\lambda_1,\lambda_2)$ to simplify the notations.
First we note that the infinite product on the right-hand side is always positive, since we assumed that $\lambda_i \notin \{0,1\}$ for some i, so that the expression on the right-hand side is well-defined.
We can use Theorem 2 in the paper [1] by Berndt to write the Epstein-Hurwitz zeta function as an infinite sum of modified Bessel functions. There are three different cases that we have to treat separately: $\lambda_1 \notin \{0,1\}$ and $\lambda_2 \notin \{0,1\}$, $\lambda_1 \in \{0,1\}$ and $\lambda_2 \notin \{0,1\}$, $\lambda_1 \notin \{0,1\}$ and $\lambda_2 \in \{0,1\}$. We will explain the computations for the first case, the other ones being similar. So suppose $\lambda_1, \lambda_2 \notin \{0,1\}$. Then, by Theorem 2

in [1] we have

$$(4\pi^3\alpha_1\alpha_2)^{-s}\Gamma(s)\zeta_{EH}(s) = \left(\frac{\alpha_1}{\alpha_2}\right)^{1-s}\pi^{\frac{1}{2}-s}\Gamma(s-1/2)\Big(\zeta(2s-1,\lambda_2)+\zeta(2s-1,1-\lambda_2)\Big)$$
$$+2\sqrt{\frac{\alpha_1}{\alpha_2}}\sum_{\substack{m,n\\ m\neq 0}} e^{-2\pi i m\lambda_1}\left|\frac{m}{n+\lambda_2}\right|^{s-\frac{1}{2}}K_{s-\frac{1}{2}}\Big(2\pi\frac{\alpha_1}{\alpha_2}|m||n+\lambda_2|\Big),$$

where $K_{s-\frac{1}{2}}$ is a modified Bessel function.
Alternatively, we can also start with the paper by Terras [18] and adapt the computations to our function ζ_{EH} to obtain the same representation in terms of Bessel functions.
Then we develop around $s=0$ using the fact that $\frac{1}{\Gamma(s)} = s + O(s^2)$ and the identity $K_{-\frac{1}{2}}(z) = \frac{1}{\sqrt{2}}\sqrt{\frac{\pi}{z}}e^{-z}$. Since the coefficient of the linear term in s is the derivative at $s=0$, this leads to

$$\zeta'_{EH}(0) = \frac{\alpha_1}{\alpha_2}\sqrt{\pi}\Gamma(-1/2)\Big(\zeta(-1,\lambda_2)+\zeta(-1,1-\lambda_2)\Big) + \sum_{\substack{m,n\\ m\neq 0}}\frac{e^{-2\pi i m\lambda_1}}{|m|}e^{-2\pi\frac{\alpha_1}{\alpha_2}|m||n+\lambda_2|}$$
$$= 2\pi\frac{\alpha_1}{\alpha_2}B_2(\lambda_2) + \sum_{n\in\mathbb{Z}}\sum_{m\geq 1}\left(\frac{e^{2\pi i m\lambda_1}}{m}e^{-2\pi\frac{\alpha_1}{\alpha_2}m|n+\lambda_2|} + \overline{\frac{e^{2\pi i m\lambda_1}}{m}e^{-2\pi\frac{\alpha_1}{\alpha_2}m|n+\lambda_2|}}\right)$$
$$= 2\pi\frac{\alpha_1}{\alpha_2}B_2(\lambda_2) - \sum_{n\in\mathbb{Z}}\left(\log\left(1-e^{2\pi i\lambda_1-2\pi\frac{\alpha_1}{\alpha_2}|n+\lambda_2|}\right)+\log\left(\overline{1-e^{2\pi i\lambda_1-2\pi\frac{\alpha_1}{\alpha_2}|n+\lambda_2|}}\right)\right)$$
$$= 2\pi\frac{\alpha_1}{\alpha_2}B_2(\lambda_2) - 2\log\prod_{n\in\mathbb{Z}}\left|1-e^{2\pi i\lambda_1}e^{-2\pi\frac{\alpha_1}{\alpha_2}|n+\lambda_2|}\right|,$$

where we used the special value $\zeta(-1,\lambda) = -\frac{B_2(\lambda)}{2}$, with $B_2(\lambda) = \lambda^2-\lambda+\frac{1}{6}$ the second Bernoulli polynomial. □

With the same kind of computation we could in fact write a more general Kronecker-type formula for Epstein-Hurwitz zeta functions having non-diagonal quadratic form, see [18] and [1].
An amusing consequence of Theorem 1.2 is the following.

Corollary 4.7 *The following identity is true:*

$$\frac{\prod_{n\geq 1}(1+e^{-2n\pi})}{\prod_{n\geq 0}(1-e^{-(2n+1)\pi})} = \frac{e^{\pi/8}}{\sqrt{2}}.$$

Proof If $\alpha_1=\alpha_2$ the function ζ_{EH} is symmetric in λ_1 and λ_2 by definition. By Theorem 1.2 this implies that

$$2\pi B_2(\lambda_1) - 2\log\prod_{n\in\mathbb{Z}}\left|1-e^{2\pi i\lambda_2}e^{-2\pi|n+\lambda_1|}\right| = 2\pi B_2(\lambda_2) - 2\log\prod_{n\in\mathbb{Z}}\left|1-e^{2\pi i\lambda_1}e^{-2\pi|n+\lambda_2|}\right|.$$

Taking $\lambda_1 = 0$ and $\lambda_2 = \frac{1}{2}$ yields the result. □

Obviously we could write a whole family of similar identities, using other values for λ_1 and λ_2. It is likely that these formulas, or at least some of them, can be derived from the theory of Jacobi theta functions.

An interesting (and simple) choice of bundle is the following. We choose d positive integers $m_1, \dots, m_d$ and define, for each $i \in \{1, \dots, d\}$, $w_{i,j} = 1$ for all $0 \le j \le m_i - 2$ and $w_{i,m_i-1} = e^{2\pi i \lambda_i} =: z_i$. In words, we consider the d-fold cartesian product of the cyclic graphs $\mathbb{Z}/m_i\mathbb{Z}$ (as explained in Section 2), where, in each cycle, all the edges have trivial weight 1 except for one edge which have weight $z_i = e^{2\pi i \lambda_i}$. One can think of this graph as a discrete d-dimensional torus constructed as follows: start with a d-dimensional cubic grid of size $m_1 \times \dots \times m_d$ with all edges having weight 1 and add edges linking opposite boundaries, according to toric boundary conditions. For each pair of opposite boundaries, the corresponding edges all have weight $e^{2\pi i \lambda_i}$. For this example only, we allow all the weights to be trivial (we will come back to our earlier convention in Section 5).
Write

$$F_{(m_1,\dots,m_d)}(z_1, \dots, z_d) := \det \Delta$$

for the determinant of the bundle Laplacian on the graph we just defined, if there exists i such that $z_i \neq 1$. If $z_i = 1$ for all i write

$$F_{(m_1,\dots,m_d)}(z_1, \dots, z_d) := \det{}^* \Delta$$

for the product of the non-zero eigenvalues of the standard Laplacian. We record the following easy result.

Proposition 4.8 *Let $m_1, \dots, m_d$ and n be positive integers.*
For any choice of complex numbers $z_1, \dots, z_d$ of modulus one, we have

$$F_{(m_1 n,\dots,m_d n)}(z_1, \dots, z_d) = \prod_{u_1^{m_1}=z_1} \cdots \prod_{u_d^{m_d}=z_d} F_{(n,\dots,n)}(u_1, \dots, u_d).$$

Proof Suppose first that not all z_i are equal to 1. Since the m_i-th roots of z_i are given by $e^{2\pi i \frac{k+\lambda_i}{m_i}}$ for $0 \le k \le m_i - 1$ and in view of Remark 3.5, the logarithm of the right-hand side is equal to

$$\sum_{k_1=0}^{m_1-1} \dots \sum_{k_d=0}^{m_d-1} \sum_{j_1=0}^{n-1} \dots \sum_{j_d=0}^{n-1} \log\Big(4\sin^2\Big(\frac{\pi}{n}(j_1 + \frac{k_1+\lambda_1}{m_1}\Big) + \dots + 4\sin^2\Big(\frac{\pi}{n}(j_d + \frac{k_d+\lambda_d}{m_d})\Big)\Big),$$

which can be rewritten as

$$\sum_{j_1=0}^{m_1 n-1} \dots \sum_{j_d=0}^{m_d n-1} \log\Big(4\sin^2\Big(\frac{\pi}{m_1 n}(j_1 + \lambda_1)\Big) + \dots + 4\sin^2\Big(\frac{\pi}{m_d n}(j_d + \lambda_d)\Big)\Big),$$

which is $\log(LHS)$.
If $z_i = 1$ for all i almost the same computation works, taking into account the slightly different meaning of F in that case. □

For instance, if we take $d = 2$, $m_1 = m_2 = 2$ and $z_1 = z_2 = 1$ we get

$$F_{(2n,2n)}(1,1) = F_{(n,n)}(1,1) F_{(n,n)}(1,-1) F_{(n,n)}(-1,1) F_{(n,n)}(-1,-1).$$

Since in that situation all cycles have monodromy 1 or −1, Theorem 2.2 tells us that $F_{(n,n)}(1,-1)$, $F_{(n,n)}(-1,1)$ and $F_{(n,n)}(-1,-1)$ are all integers multiple of 4. We deduce that the number of spanning trees in the $2n \times 2n$ discrete torus is an integer multiple of the number of spanning trees in the $n \times n$ discrete torus (and the multiplicative constant is itself a multiple of 4 determined by the cycle-rooted spanning forests in the $n \times n$ torus).
In dimension 2, a very similar formula holds for the characteristic polynomial of the dimer model of any toroidal graph, see for example [12]. It would be interesting to investigate the case of other graphs. For which graphs does such a product formula for the bundle Laplacian hold?

5 Asymptotics of zeta functions

Now we give an asymptotic result about the spectral zeta function of G_n, in the same spirit as in [9].

Definition 5.1 *For $Re(s) > 0$, the* spectral zeta function *associated to the graph G_n with bundle defined above is defined by*

$$\zeta_{G_n}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \theta^{G_n}(t) t^{s-1} dt.$$

In view of the asymptotics of the integrand obtained in Lemma 3.7, this integral is convergent in the domain specified in the definition. Note that, in order to simplify the notation, we do not write the dependence on the various parameters introduced earlier. More precisely, θ^{G_n} depends on the dimension d, the integers $a_i(n)$ and the weights $w_{i,j}$ and so does ζ_{G_n}.

In fact ζ_{G_n} is entire since Proposition 3.3 implies that

$$\zeta_{G_n}(s) = \frac{1}{4^s} \sum_K \frac{1}{\left(\sin^2\left(\frac{\pi(k_1+\lambda_1)}{a_1(n)}\right) + \ldots + \sin^2\left(\frac{\pi(k_d+\lambda_d)}{a_d(n)}\right)\right)^s},$$

where the index K runs over all vectors $(k_1, \ldots, k_d)$, with $0 \le k_j \le a_j(n) - 1$.

The spectral zeta function of $\mathbb{Z}^d$ is given by

$$\zeta_{\mathbb{Z}^d}(s) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-2dt} I_0(2t)^d t^{s-1} dt$$

for $0 < \mathrm{Re}(s) < \frac{d}{2}$ and admits a meromorphic continuation with simple poles at $s = m + \frac{d}{2}$ $(m \ge 0)$, see [9].

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3 The proof is practically the same as in [9], using results from [2]. Recall that

$$\theta^{G_n}(n^2 t) \longrightarrow \theta^\infty(t)$$

when $n \longrightarrow \infty$ for all $t > 0$, by Proposition 4.2. Next we write, for $0 < Re(s) < \frac{d}{2}$,

$$\zeta_{G_n}(s) = \frac{n^{2s}}{\Gamma(s)} \int_0^\infty \theta^{G_n}(n^2 t) t^{s-1} dt = \frac{n^{2s}}{\Gamma(s)} \Big(S_1(n) + S_2(n) + S_3(n) + S_4(n) + S_5(n) \Big), \tag{5.1}$$

where

$$S_1(n) := \int_1^\infty \theta^{G_n}(n^2 t) t^{s-1} dt,$$

$$S_2(n) := \int_0^1 \Big(\theta^{G_n}(n^2 t) - \Big(\prod_{i=1}^d a_i(n) \Big) e^{-2dn^2 t} I_0(2n^2 t)^d \Big) t^{s-1} dt,$$

$$S_3(n) := n^{-2s} \Big(\prod_{i=1}^d a_i(n) \Big) \Gamma(s) \zeta_{\mathbb{Z}^d}(s),$$

$$S_4(n) := -n^{-2s} \Big(\prod_{i=1}^d a_i(n) \Big) \int_{n^2}^\infty \Big(e^{-2dt} I_0(2t)^d - (4\pi t)^{-\frac{d}{2}} \Big) t^{s-1} dt$$

and

$$S_5(n) = -n^{-2s} \Big(\prod_{i=1}^d a_i(n) \Big) \int_{n^2}^\infty (4\pi t)^{-\frac{d}{2}} t^{s-1} dt = \Big(\prod_{i=1}^d a_i(n) \Big) \frac{n^{-d}}{(4\pi)^{\frac{d}{2}} (s - \frac{d}{2})}.$$

Note that the equality (5.1) is in fact valid for $-\min a_i(n) < \mathrm{Re}(s) < \frac{d}{2} + 1$, due to the analytic continuation of $\zeta_{\mathbb{Z}^d}$, Lemma 3.7 and the asymptotic behavior of the modified Bessel function. Letting n go to infinity, $S_1(n)$, $S_2(n)$ and $S_5(n)$ combine to give the second term in the asymptotics, for all $s \neq \frac{d}{2}$ (and a term $o(n^{2s})$). The first term is $S_3(n)$ and $S_4(n)$ contributes to the error term, as can be seen from the asymptotic behavior of I_0, if $\mathrm{Re}(s) < \frac{d}{2} + 1$ *a priori*. But we can use more terms from the asymptotics of I_0 at infinity to split the integral in $S_4(n)$ further, so that the validity of the equality (5.1) and, consequently, of the final asymptotics, extends to any s such that $s \neq m + \frac{d}{2}$. □

This is in principle a more general result, because the function ζ_{G_n} contains a lot of information about the Laplace spectrum of G_n. In particular, if only the error term would be a bit smaller, say for example $o(\frac{n^{2s}}{\log n})$, we would obtain Theorem 1.1 as a simple consequence by taking the derivative at $s = 0$ and using $\zeta_{EH}(0; \alpha_1, \ldots, \alpha_d; \lambda_1, \ldots, \lambda_d) = 0$ as observed above. Indeed, the derivative at $s = 0$ of ζ_{G_n} is equal to $-\log\det\Delta$, because $\zeta_{G_n}(s) = \sum \frac{1}{\lambda_j^s}$ where the λ_j run over all the eigenvalues of Δ. Moreover, if we take the derivative at $s = 0$ in the similar asymptotic formula obtained in [9] and compare with the results in [2], we see that $-\zeta'_{\mathbb{Z}^d}(0) = c_d$.

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The zeta function of $\mathbb{Z}^2$ at integers

In our opinion it is an interesting problem to try to find a graph, different from $\mathbb{Z}$, whose spectral zeta function satisfies a nice functional equation of the form $s \longleftrightarrow 1-s$. In [1] we observed that $\zeta_{\mathbb{Z}}$ possesses such a symmetry. So what about $\zeta_{\mathbb{Z}^2}$? It is natural to investigate this zeta function as it plays an important role in the various formulas in [1].
Here we try our luck with Euler's method, which consists of guessing the shape of a possible functional equation from what we observe at the integers, where exact computations are avalaible. And indeed there is a nice $s \longleftrightarrow 1-s$ symmetry for $\zeta_{\mathbb{Z}^2}$ at integers: for $s \neq n+1$ $(n \geq 0)$ define

$$f(s) := 2^{5s/2}\zeta_{\mathbb{Z}^2}(s).$$

As was shown in [1] it is a meromorphic function with simple poles at $s = n+1$ (see Proposition 3.1).

Proposition 0.1 *Let $n \geq 0$ be an integer. Then*

$$Res(f, n+1) = -\frac{\sqrt{2}}{\pi} f(-n),$$

where $Res(f, s_0)$ denotes the residue of f at the point s_0.

Proof Let $n \geq 0$ be an integer. We start with the following asymptotic expansion of the modified Bessel function I_0 when $t \longrightarrow \infty$ ([4]):

$$\begin{aligned}\sqrt{4\pi t}e^{-2t}I_0(2t) &\sim \sum_{k\geq 0} \frac{1}{(2t)^k}\frac{(2k-1)!!^2}{k!8^k}\\ &= \sum_{k\geq 0}\binom{2k}{k}\frac{(2k)!}{k!}\frac{1}{(64t)^k},\end{aligned}$$

where the sign $\sim$ means that if we stop the sum at $k = K-1$ then the error is of order $O(\frac{1}{t^K})$.
Taking the square we get

$$e^{-4t}I_0(2t)^2 \sim \frac{1}{4\pi t}\sum_{k\geq 0}\frac{1}{k!(64t)^k}\sum_{j=0}^{k}\binom{k}{j}\binom{2j}{j}\binom{2(k-j)}{k-j}(2j)!(2(k-j))!. \tag{0.1}$$

Choose K big enough so that $K > n$ and write

$$e^{-4t}I_0(2t)^2 = \sum_{k=0}^{K-1} b_k t^{-k-1} + O(t^{-K-1})$$

when $t \longrightarrow \infty$ as in section 3 of [1]. The computation there shows that

$$\int_0^\infty e^{-4t}I_0(2t)^2 t^{s-1}dt$$

has a simple pole at $s = n+1$ with residue $-b_n$. Since $\zeta_{\mathbb{Z}^2}$ is equal to that integral divided by $\Gamma(s)$ we deduce that

$$Res(\zeta_{\mathbb{Z}^2}, n+1) = -\frac{b_n}{n!} = -\frac{1}{4\pi 64^n n!^2}\sum_{j=0}^{n}\binom{n}{j}\binom{2j}{j}\binom{2(n-j)}{n-j}(2j)!(2(n-j))!,$$

using $\Gamma(n+1) = n!$ and (0.1).

On the other hand the same computation in [1] shows that

$$\int_0^\infty e^{-4t} I_0(2t)^2 t^{s-1} dt$$

has a simple pole at $s = -n$ with residue a_n, where a_n is defined by the series

$$e^{-4t} I_0(2t)^2 = \sum_{k\geq 0} a_k t^k.$$

Since $\Gamma(s)$ has a simple pole at $s = -n$ with residue $\frac{(-1)^n}{n!}$ we have

$$\zeta_{\mathbb{Z}^2}(-n) = (-1)^n n! a_n. \tag{0.2}$$

Now we could compute the values of a_n by writing the Cauchy product of the exponential and the modified Bessel function. We prefer to use our knowledge of the values at integers of $\zeta_{\mathbb{Z}}$. Indeed we know that (see Theorem 2.2 in [1])

$$\zeta_{\mathbb{Z}}(-n) = \binom{2n}{n}.$$

Thus if we write

$$e^{-2t} I_0(2t) = \sum_{k\geq 0} c_k t^k,$$

then we have

$$c_k = \frac{(-1)^k}{k!}\binom{2k}{k},$$

by the same reasoning as for (0.2).
Since obviously $a_n = \sum_{j=0}^n c_j c_{n-j}$ we get

$$\zeta_{\mathbb{Z}^2}(-n) = \sum_{j=0}^n \binom{n}{j}\binom{2j}{j}\binom{2(n-j)}{n-j}.$$

The numbers $2^n \zeta_{\mathbb{Z}^2}(-n)$ are known as the *Catalan-French-Larcombe* numbers. They were in fact studied by Catalan and initially called Catalan numbers. Catalan himself used the term "Segner numbers" for what we call now Catalan numbers! Larcombe and French proved the following identity involving these numbers, see [2] and [3] (this was pointed to us by Jeremy Dubout):

$$2^n \zeta_{\mathbb{Z}^2}(-n) = \frac{1}{n!^2}\sum_{j=0}^n \binom{2j}{j}\binom{n}{j}\binom{2(n-j)}{n-j}(2j)!(2(n-j))!. \tag{0.3}$$

This readily implies the Proposition. □

Remark 0.2 From the proof we see that we can compute the values at the negative integers recursively for $\zeta_{\mathbb{Z}^d}$, using the values of $\zeta_{\mathbb{Z}^{d-1}}$. See the forthcoming work by Jeremy Dubout for an easier proof of (0.3) than in [2] and [3].

Now the question is: can we complete $f(s)$ with some fudge factor to cancel the poles at the positive integers while preserving this nice functional relation at all the integers? And is it possible to generalize this identity to the whole complex plane in some way?
Suppose there is a "good completion", that is a function $g(s)$ such that $\Lambda(s) := 2^{5s/2}g(s)\zeta_{\mathbb{Z}^2}(s) = f(s)g(s)$ is entire and satisfies the functional equation

$$\Lambda(s) = \Lambda(1-s). \tag{0.4}$$

Then $g(s)$ must have zeros at the points $s = n+1$ and we can write

$$g(s) = d_{n+1}(s-(n+1)) + O((s-(n+1)))$$

when $s \longrightarrow n+1$. From Proposition 0.1 and (0.4) we see that $g(s)$ must satisfy the following relation:

$$g(-n) = -\frac{\sqrt{2}}{\pi}d_{n+1}.$$

For example if we take $g(s) = \frac{1}{\Gamma(1-s)}$, which has zeros where we need and so that $\Lambda(s) = \frac{2^{5s/2}\zeta_{\mathbb{Z}^2}(s)}{\Gamma(1-s)}$, then Proposition 0.1 leads to

$$\Lambda(n+1) = \frac{\sqrt{2}}{\pi}(-1)^n n!^2\Lambda(-n).$$

It is not clear how to generalize this to other complex numbers s. Indeed, the natural way would be to write $\Gamma(s+1)^2$ instead of $n!^2$, but this would add new poles.
Another interesting candidate for $g(s)$ is the function $\cos^2(\frac{\pi s}{2})\zeta_{\mathbb{Z}}(\frac{s}{2})$ near the odd positive integers and even negative integers. Indeed in that case we have (using the explicit formula for $\zeta_{\mathbb{Z}}$ derived in [1])

$$d_{2n+1} = -\frac{\pi}{4}g(-2n),$$

even explaining the factor π! This is almost exactly what we need. Unfortunately this only works for half of the integers. But there is also a nice candidate for the other half of the integers, which is very similar to the previous one, since it also involves $\zeta_{\mathbb{Z}}$. That is, if $g(s)$ is defined to be equal to $4^{s/2}\zeta_{\mathbb{Z}}(\frac{s}{2})$ near the even positive integers and odd negative integers, then

$$d_{2n} = \frac{\pi}{2}g(1-2n).$$

It is not clear how to "glue" these two functions together and it is mysterious to us if Proposition 0.1 is just a coincidence or if there is some good fudge factor g to be found.
As remarked in [1] another graph which could be a good candidate for satisfying the desired functional equation is the regular tree, due to the hypergeometric form of the associated spectral zeta function.

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