On the space of morphisms between étale groupoids

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ON THE SPACE OF MORPHISMS BETWEEN ÉTALE GROUPOIDS

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Abstract. Given two étale groupoids $\mathcal{G}$ and $\mathcal{G}'$, we consider the set of pointed morphisms from $\mathcal{G}$ to $\mathcal{G}'$. Under suitable hypothesis we introduce on this set a structure of Banach manifold which can be considered as the space of objects of an étale groupoid whose space of orbits is the space of morphisms from $\mathcal{G}$ to $\mathcal{G}'$.

This is a drawing up of a talk given at the IHP Paris in february 2007, in the framework of the trimester "Groupoids and Stacks". In this revised version, we correct the paragraph IV.4 and give more details on the proofs.

Let $\mathcal{G}$ and $\mathcal{G}'$ be two étale groupoids. The aim of this talk is to show that, under suitable conditions, the set of continuous morphisms from $\mathcal{G}$ to $\mathcal{G}'$ can be considered as the space of orbits of an étale groupoid. This was done and used in the paper *Closed geodesics on orbifolds* [4] by K. Guruprasad and A. Haefliger in the particular case where $\mathcal{G}$ is the circle (considered as a trivial groupoid) and $\mathcal{G}'$ an orbifold. The same problem has been also considered independentlly by Weimin Chen in [2]. See also the paper [8] by E. Lupercio and B. Uribe. Our basic references are [1] and [5].

Plan of the talk.

I. Étale groupoids (homomorphisms, equivalence, localization, developability)
II. Morphisms and cocycles
III. Pointed morphisms
IV. Topology on the set of pointed morphisms
V. Selfequivalences and extensions.

I. ÉTALE GROUPOIDS

1. Definition. An étale groupoid $\mathcal{G}$ with space of objects T is a topological groupoid such that the source and target projections $s, t : \mathcal{G} \to T$ are étale maps (local homeomorphisms). For $x \in T$, we note $1_x \in \mathcal{G}$ the corresponding unit element. An arrow $g \in \mathcal{G}$ with source $s(g) = x$ and target $t(g) = y$ is pictured as an arrow $y \leftarrow x$. Accordingly, the space of composable arrows, namely the subspace of $\mathcal{G} \times \mathcal{G}$ formed by the pairs (g, g') with $s(g) = t(g')$, is noted $\mathcal{G} \times_T \mathcal{G}$. We shall assume throughout that the space of objects T is arcwise locally connected and locally simply connected. The groupoid $\mathcal{G}$ is connected iff its space of orbits $\mathcal{G} \backslash T$ is connected.

2. Homomorphisms and equivalences. If $\mathcal{G}'$ is an étale groupoid with space of objects T', a homomorphism $\phi : \mathcal{G} \to \mathcal{G}'$ is a continuous functor. It induces a continuous map $\phi_0 : T \to T'$, and passing to the quotient, a continuous map $\mathcal{G}\backslash T \to \mathcal{G}'\backslash T'$ of spaces of orbits. Two homomorphisms ϕ and ϕ' from $\mathcal{G}$ to $\mathcal{G}'$ are equivalent if they are related by a natural transformation, i.e a continuous map $h : T \to \mathcal{G}'$ such that the following is defined and satisfied for each $g \in \mathcal{G}$:

$$\phi'(g) = h(t(g))\phi(g)h^{-1}(s(g)).$$

We say that ϕ is an equivalence if the induced map on the spaces of orbits is bijective and if ϕ is locally an isomorphism. This means that each point has an open neighbourhood U such that the restriction of ϕ_0 to U is a homeomorphism onto an open set U' of T' and that ϕ restricted to U is an isomorphism from the restriction of $\mathcal{G}$ to U to the restriction of $\mathcal{G}'$ to U'. This generates an equivalence relation among étale groupoids. The natural philosophy coming out from foliation theory is to consider étale groupoids only up to equivalence.

For instance if T_0 is an open subset of T meeting every orbit of $\mathcal{G}$, then the inclusion in $\mathcal{G}$ of the restriction of $\mathcal{G}$ to T_o is an equivalence. If $\mathcal{U} = \{U_i\}_{i \in I}$ is an open cover of T, the localization of $\mathcal{G}$ over $\mathcal{U}$ is the étale groupoid $\mathcal{G}_{\mathcal{U}}$ whose space of objects is the disjoint union $T_{\mathcal{U}}$ of the U_i, i.e the space of pairs (i, x) with $x \in U_i$. The morphisms are the triple (j, g, i) such that $s(g) \in U_i$ and $t(g) \in U_j$. The source and target of (j, g, i) are $(i, s(g))$ and $(j, t(g))$ and the composition $(k, g', j)(j, g, i)$ whenever defined is equal to $(k, g'g, i)$. The natural projection $(j, g, i) \to g$ is an equivalence from $\mathcal{G}_{\mathcal{U}}$ to $\mathcal{G}$. Note that two étale groupoids $\mathcal{G}$ and $\mathcal{G}'$ are equivalent iff there are open cover $\mathcal{U}$ of T and $\mathcal{U}'$ of T' such that $\mathcal{G}_{\mathcal{U}}$ is isomorphic to $\mathcal{G}'_{\mathcal{U}'}$.

3. Developability. A connected étale groupoid $\mathcal{G}$ is developable if it is equivalent to a groupoid $\Gamma \ltimes X$ given by a discrete group Γ acting by homeomorphisms on the space X. If this is so, we can always assume that X is simply connected. In that case the groupoid $\Gamma \ltimes X$ is unique up to isomorphism: Γ is isomorphic to the fundamental group of $\mathcal{G}$ and X is equivalent to its universal covering. This is specific to étale groupoids.

4. Groupoid of germs. Another feature specific to étale groupoids is that we can associate to an étale groupoid $\mathcal{G}$ with space of objects T its étale groupoid of germs $\mathcal{H}$ constructed as follows. Given $g \in \mathcal{G}$ with source x, let $\tilde{g} : U \to \mathcal{G}$ be a local (continuous) section of the source projection s defined on an open neighbourhood U of $s(g)$ such that $s(\tilde{g}(x)) = g$. The germs of $t \circ \tilde{g} : U \to T$ at the various points of U form an open set in $\mathcal{H}$. The map $g \mapsto$ germ at x of $t \circ \tilde{g}$ is an étale surjective homomorphism $\mathcal{G} \to \mathcal{H}$. A geometric structure on T, as for instance a differentiable or analytic or Riemannian structure, invariant by $\mathcal{H}$ induces the corresponding geometric structure on $\mathcal{G}$.

II. Morphisms and cocycles

1. $(\mathcal{G}', \mathcal{G})$**-bundles.** Let $\mathcal{G}$ and $\mathcal{G}'$ be étale groupoids with space of objects T and T' respectively. We recall the notion of $(\mathcal{G}', \mathcal{G})$-bundle sketched to me by Georges Skandalis after my Toulouse talk in 1982 (see [5] for a general definition in case of topological groupoids and [7] for differentiable groupoids). A $(\mathcal{G}', \mathcal{G})$-bundle is a topological space E together with a right action of $\mathcal{G}$ with respect to a continuous projection $s : E \to T$ and a left action of $\mathcal{G}$ with respect to a continuous map $t : E \to T'$. The following conditions must be satisfied.

a) E is a left principal $\mathcal{G}'$-bundle with base space T. This means that $s : E \to T$ is surjective and that each point of T has an open neighbourhood U with a continuous section $\sigma : U \to E$ (i.e. $s \circ \sigma$ is the identity of U) such that the map $\mathcal{G}' \times_{T'} U \to s^{-1}(U)$ sending (g', x) to $g'.\sigma(x)$ is an isomorphism. Here $\mathcal{G}' \times_{T'} U$ is the subspace of $\mathcal{G}' \times U$ formed by the pairs (g', x) with $s(g') = t(\sigma(x))$.

b) The right action of $\mathcal{G}$ commutes with the left action of $\mathcal{G}'$.

Let $\mathcal{G}''$ be an étale groupoid with space of objects T'' and let E' be a $(\mathcal{G}'', \mathcal{G}')$-bundle. Then the composition $E' \circ E$ is the $(\mathcal{G}'', \mathcal{G})$-bundle $E' \times_{\mathcal{G}'} \mathcal{G}$-bundle which is the quotient of $E' \times_{T'} E$ by the equivalence relation which identifies $(e'.g', e)$ to $(e', g'.e)$ for $g' \in \mathcal{G}'$ with $s(e') = t(g')$ and $s(g') = t(e)$. The equivalence class of (e', e) is noted $[e', e]$. The projections source and target map $[e', e]$ to $s(e)$ and $t(e')$. The actions of $g'' \in \mathcal{G}''$ and $g \in \mathcal{G}$ are given by $g''.[e', e].g = [g''.e', e.g]$.

2. Example. Let $\phi : \mathcal{G} \to \mathcal{G}'$ be a homomorphism inducing the map $\phi_0 : T \to T'$ on the spaces of objects. The associated $(\mathcal{G}', \mathcal{G})$-bundle E_ϕ is the space $E_\phi = \mathcal{G}' \times_{T'} T$ made up of pairs $(g', x) \in \mathcal{G}' \times T$ such that $s(g') = \phi_0(x)$. The source (resp. target) map sends (g', x) to x (respectively $t(g')$). The action of $g'_1 \in \mathcal{G}'$ with $s(g'_1) = t(g')$ on (g', x) is given by $g'_1.(g', x) = (g'_1 g', x)$. The action of $g \in \mathcal{G}$ with $t(g) = x$ and $s(g) = y$ on (g', x) is given by $(g', x).g = (g'\phi(g), y)$. The projection $s : E_\phi \to T$ has a global section $\sigma : T \to E_\phi$ given by $\sigma(x) = (\phi(1_x), x)$. If $\phi' : \mathcal{G}' \to \mathcal{G}''$ is a homomorphism, then $E_{\phi'} \circ E_\phi$ is naturally isomorphic to $E_{\phi' \circ \phi}$.

Conversely if a $(\mathcal{G}', \mathcal{G})$-bundle E has a global section $\sigma : X \to E$, then it is canonically isomorphic to E_ϕ, where $\phi : \mathcal{G} \to \mathcal{G}'$ is the homomorphism defined by

$$\sigma(t(g)).g = \phi(g).\sigma(s(g)).$$

Another section $\sigma' : T \to E$ gives a homomorphism related to ϕ by the natural transformation h defined by $\sigma'(x) = h(x)\sigma(x)$.

3. Morphisms and equivalences. An isomorphism between two $(\mathcal{G}', \mathcal{G})$-bundles E and E' is a homeomorphism from E to E' commuting with s and t and which is $\mathcal{G}'$ and $\mathcal{G}$-equivariant. The isomorphism class of a $(\mathcal{G}', \mathcal{G})$-bundle E is noted $[E]$ and is called a morphism from $\mathcal{G}$ to $\mathcal{G}'$. The set of morphisms from $\mathcal{G}$ to $\mathcal{G}'$ is noted $\mathcal{M}or(\mathcal{G}', \mathcal{G})$.

A $(\mathcal{G}', \mathcal{G}')$-bundle is invertible if it is also a right principal $\mathcal{G}$-bundle over T' with respect to the projection t. It is also called a $(\mathcal{G}', \mathcal{G})$-bibundle. The inverse E^{-1} of E is the $(\mathcal{G}, \mathcal{G}')$-bundle obtained from E by exchanging the role of s and t and the actions by their opposite. Note that $E \times_{\mathcal{G}} E^{-1}$ is isomorphic to $\mathcal{G}'$ and $E^{-1} \times_{\mathcal{G}'} E$ is isomorphic to $\mathcal{G}$. The isomorphism class $[E]$ of an invertible bundle is called an equivalence from $\mathcal{G}$ to $\mathcal{G}'$. The bundle associated to a homomorphism ϕ represents an equivalence iff ϕ is an equivalence.

The morphisms from étale groupoids to étale groupoids form a category. Note that if T and T' have invariant geometric structures, one can also consider morphisms and equivalences compatible with those structures.

4. Description in terms of cocycles. Given a $(\mathcal{G}', \mathcal{G})$-bundle E, let $\mathcal{U} = \{U_i\}_{i \in I}$

each $i \in I$. To this we can associate the homomorphism $\phi : \mathcal{G}_{\mathcal{U}} \to \mathcal{G}'$ mapping (j, g, i), with $s(g) = x$ and $t(g) = y$, to the unique element $\phi(j, g, i)$ such that

$$\sigma_j(y)g = \phi(j, g, i)\sigma_i(x).$$

Conversely, a homomorphism $\phi : \mathcal{G}_{\mathcal{U}} \to \mathcal{G}'$ gives a $(\mathcal{G}', \mathcal{G})$-bundle $E_{\phi,\mathcal{U}}$ as follows. Let $\mathcal{G}' \times_{T'} U_i$ be the subspace of $\mathcal{G}' \times U_i$ consisting of pairs (g', x) such that $\phi_0(x, i) = s(g')$. In the disjoint union $\coprod_{i \in I} \mathcal{G}' \times_{T'} U_i$ consider the equivalence relation identifying, for $x \in U_i \cap U_j$, the element $(j, g', x) \in \{j\} \times \mathcal{G}' \times_{T'} U_j$ to the element $(i, g'\phi(j, 1_x, i), x) \in \{i\} \times \mathcal{G}' \times_{T'} U_i$. The equivalence class of (i, g', x) is noted $[i, g', x]$ and the quotient of $\coprod \mathcal{G}' \times_{T'} U_i$ by this equivalence relation is noted $E_{\phi,\mathcal{U}}$. The projections s and t map $[i, g', x]$ to x and $t(g')$ respectively. For $g'_1 \in \mathcal{G}'$ with $s(g'_1) = t(g')$ we define $g'_1.[i, g', x] = [i, (g'_1 g').x]$. For $g \in \mathcal{G}$ wiht source $y \in U_j$ and target $x \in U_i$, we define $[i, g', x].g = [j, g'\phi(i, g, j), y]$. This defines on $E_{\phi,\mathcal{U}}$ the structure of a $(\mathcal{G}', \mathcal{G})$-bundle, called the $(\mathcal{G}', \mathcal{G})$-bundle constructed from ϕ.

Let $\mathcal{U}' = \{U'_{i'}\}_{i' \in I'}$ be another open cover of T and $\phi' : \mathcal{G}_{\mathcal{U}'} \to \mathcal{G}'$ be a homomorphism. We assume that the sets of indices I and I' are disjoint. Let $\mathcal{U} \coprod \mathcal{V}$ be the union of these open covers. We can identify $\mathcal{G}_{\mathcal{U}}$ and $\mathcal{G}_{\mathcal{U}'}$ to disjoint subgroupoids of $\mathcal{G}_{\mathcal{U} \coprod \mathcal{U}'}$. The $(\mathcal{G}', \mathcal{G})$-bundles $E_{\phi,\mathcal{U}}$ and $E_{\phi',\mathcal{U}'}$ associated to ϕ and ϕ' are isomorphic iff there is a homomorphism $\psi : \mathcal{G}_{\mathcal{U} \coprod \mathcal{U}'} \to \mathcal{G}'$ extending ϕ and ϕ'. Such an extension gives a well defined isomorphism between these bundles.

III. Pointed morphisms

1. Definition. A pointed $(\mathcal{G}', \mathcal{G})$-bundle over a point $* \in T$ (see [10]) is a pair (E, e_0), where E is a $(\mathcal{G}', \mathcal{G})$-bundle and $e_0 \in E$ a point such that $s(e_0) = *$. A pointed morphism from $\mathcal{G}$ to $\mathcal{G}'$ over $*$ is the isomorphism class $[E, e_0]$ of a pointed bundle (E, e_0) over $*$. The set of pointed morphisms over $*$ is noted $\mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$. In the case where $\mathcal{G}$ is the circle (considered as a trivial groupoid), a pointed morphism from $\mathcal{G}$ to $\mathcal{G}'$ was called in [4] a based $\mathcal{G}'$-loop.

Note that the bundle $E_\phi = \mathcal{G}' \times_{T'} T$ associated to a homomorphism $\phi : \mathcal{G} \to \mathcal{G}'$ inherits naturally a base point over $* \in T$, namely the point $(\phi(1_*), *)$. Also the bundle $E_{\phi,\mathcal{U}}$ associated to a homomorphism $\phi : \mathcal{G}_{\mathcal{U}} \to \mathcal{G}'$ inherits a base point once we have chosen $U \in \mathcal{U}$ containing $*$, namely the class of the point $(\phi(1_*), *) \in \mathcal{G}' \times_{T'} U$.

$\mathcal{G}'$ acts on the left on $\mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$ with respect to the map $t : \mathcal{M}or(\mathcal{G}', \mathcal{G}, *) \to T'$ sending $[E, e_0]$ to $t(e_0)$. The action of $g' \in \mathcal{G}'$ on $[E, e_0]$ is defined by $g'.[E, e_0] = [E, g'.e_0]$. The associated groupoid with set of objects $\mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$ will be noted $\mathcal{G}' \times_{T'} \mathcal{M}(\mathcal{G}', \mathcal{G}, *)$. We shall introduce below, under suitable conditions, a topology which makes it an étale groupoid. For that the following lemma will be crucial. Note that the set of orbits $\mathcal{G}' \backslash \mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$ is isomorphic to $\mathcal{M}or(\mathcal{G}', \mathcal{G})$.

2. Lemma. *Suppose that $\mathcal{G}$ is connected and that $\mathcal{G}'$ is Hausdorff. Then any automorphism h of E preserving a base point $e_o \in E$ is the identity.*

Proof. Let $F \subset E$ be the set of points $e \in E$ such that $h(e) = e$. The set F is open because it preserves the projection s which is étale. The set F is closed because $\mathcal{G}'$ is Hausdorff. It is non empty and invariant under $\mathcal{G}$ and $\mathcal{G}'$. Hence it is equal to E because $(\mathcal{G}' \times \mathcal{G})\backslash E = \mathcal{G}\backslash T$ is assumed to be connected. □

In terms of cocycles, with the notations at the end of II.4, the lemma is formulated as follows. Assume that we have chosen $i \in I$ and $i' \in I'$ such that $* \in U_i \cap U'_{i'}$. Let $(E_{\phi,\mathcal{U}}, e_0)$ and $(E_{\phi',\mathcal{U}'}, e'_0)$ be the pointed bundles associated as above to ϕ and ϕ'. Then these two pointed bundles are isomorphic iff there is a homomorphism $\psi : \mathcal{G}_{\mathcal{U} \coprod \mathcal{U}'} \to \mathcal{G}'$ extending ϕ and ϕ' such that $\psi(i, 1_*, i') = \phi(i, 1_*, i) = \phi'(i', 1_*, i')$. Under the hypothesis of the lemme, the assertion is that such a ψ is unique.

3. Proposition. *For $\mathcal{G}$ connected , the equivalence class of $\mathcal{G} \times_{T'} \mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$ does not change if we replace $\mathcal{G}$ and $\mathcal{G}'$ by equivalent étale groupoids, or if we change the base point $* \in T$.*

Proof. For instance, assume that $\mathcal{E}'$ is a bibundle giving an equivalence from $\mathcal{G}'$ to an étale groupoid $\overline{\mathcal{G}}'$ with space of objects $\overline{T}'$. Then an equivalence from $\mathcal{G}' \times_{T'} \mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$ to $\overline{\mathcal{G}}' \times_{T'} \mathcal{M}or(\overline{\mathcal{G}}', \mathcal{G}, *)$ will be represented by the bibundle $\mathcal{E}' \times_{T'} \mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$. The projection s maps $(e', [E, e_0])$ to $[E, e_0]$ and the projection t maps it to $(\mathcal{E}' \times_{\mathcal{G}'} E, [e', e_0])$. An element of $\mathcal{G}' \times_{T'} \mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$ with target $[E.e_0]$ is of the form $(g', [E, {g'}^{-1} e_0])$, where $g' \in \mathcal{G}'$ with $t(g') = t(e_0)$. Its right action on $(e', [E, e_0])$ is equal to $(e'g', [E, {g'}^{-1} e_0])$. An element of $\overline{\mathcal{G}}' \times_{T'} \mathcal{M}or(\overline{\mathcal{G}}', \mathcal{G}, *)$ with source $[\mathcal{E}' \times_{\mathcal{G}'} E, [e', e_0])$ is of the form $(\overline{g}', [\mathcal{E}' \times_{\mathcal{G}'} E, [e', e_0])$, where $s(\overline{g}') = t(e')$. Its left action on $(e', [E, e_0])$ is equal to $(\overline{g}' e', [E, e_0])$. With those definitions, it is easy to check that $\mathcal{E}' \times_{T'} \mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$ is a bibundle giving an equivalence from $\mathcal{G}' \times_{T'} \mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$ to $\overline{\mathcal{G}'} \times_{\overline{T}'} \mathcal{M}or(\overline{\mathcal{G}}', \mathcal{G}, *)$.

The other claims of the proposition are similarly easy to prove. For instance, if $*' \in T$ is another base point, the elements of the bibundle giving the equivalence from $\mathcal{G}' \times_{T'} \mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$ to $\mathcal{G}' \times_{T'} \mathcal{M}or(\mathcal{G}', \mathcal{G}, *')$ are the isomorphism classes of bipointed $(\mathcal{G}', \mathcal{G})$-bundles over $*$ and $*'$. □

As an example, we consider below the developable case.

4. Proposition. *Assume that $\mathcal{G} = \Gamma \ltimes T$ and $\mathcal{G}' = \Gamma' \ltimes T'$, with T simply connected.*

Then $\mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$ is the set of pairs (f, ψ), where $\psi : \Gamma \to \Gamma'$ is a group homomorphism and $f : T \to T'$ is a ψ-equivariant continuous map.

The groupoid $\mathcal{G}' \times_{T'} \mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$ is isomorphic to $\Gamma' \ltimes \mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$, where the action of γ' on the pair (f, ψ) is given by

$$\gamma'.(f, \psi) = (t_{\gamma'} \circ f, Ad(\gamma') \circ \psi),$$

and $t_{\gamma'}$ is the translation of T' by γ'.

Proof. Let (E, e_o) be a pointed bundle representing an element of $\mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$. With respect to the projection $s : E \to T$, the bundle E is a Γ'-principal covering. As T is simply connected, there is a unique lifting $\sigma : T \to E$ of the projection s such that $\sigma(*) = e_0$. Then to σ is associated as in II.2 a continuous homomorphism $\phi : \Gamma \ltimes T \to \Gamma' \ltimes T'$ such that E is canonically isomorphic to E_ϕ. Let $f : T \to T'$ be the map $t \circ \sigma$ induced by ϕ on the space of objects. As T is connected and Γ' is discrete, $\phi(\gamma, x)$ is of the form $(\psi(\gamma), f(x))$. Composing with the target projection, we get $f(\gamma.x) = \psi(\gamma).f(x)$. Conversely such a pair (f, ψ) defines a homomorphism $(\gamma, x) \mapsto (\psi(\gamma), f(x))$ from $\mathcal{G}$ to $\mathcal{G}'$.

The choice of another base point over $*$ leads to a homomorphism ϕ' related to ϕ by a natural transformation $h : T \to \mathcal{G}'$ of the form $h(x) = (\gamma', f(x))$. So $\phi'(\gamma, x) = (\gamma' \psi(\gamma) {\gamma'}^{-1}, \gamma'.f(x))$. □

5. The exponential morphism. We want to define a morphism from the groupoid product of $\mathcal{G}' \times_{T'} \mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$ with $\mathcal{G}$ to the groupoid $\mathcal{G}'$, which is the usual exponential map ${T'}^T \times T \to T'$ when the groupoids $\mathcal{G}$ and $\mathcal{G}'$ are the trivial groupoids T and T'. We first describe the principle of the construction on the set theoretical level, under the hypothesis of the lemma. In IV we shall consider topologies.

For each $z \in Mor(\mathcal{G}', \mathcal{G}, *)$, choose a pointed $(\mathcal{G}', \mathcal{G})$-bundle (E^z, e_0^z) whose isomorphism class $[E^z, e_0^z]$ is z. Let EXP be the disjoint union $\coprod_{z \in Mor(\mathcal{G}', \mathcal{G}, *)} E^z$. We have projections $s : EXP \to Mor(\mathcal{G}', \mathcal{G}, *) \times T$ and $t : EXP \to T'$ mapping $(z, e \in E^z)$ respectively to $(z, s(e))$ and $t(e)$. For $(g', z') \in \mathcal{G}' \times_{T'} Mor(\mathcal{G}', \mathcal{G}, *)$ with target z we have an isomorphism $m_{(g'.z')}$ from the pointed bundle $(E^{z'}, g'.e_0^{z'})$ to the pointed bundle (E^z, e_0^z). The lemma implies that this isomorphism is unique. Commuting left action of $\mathcal{G}'$ and right action of $\mathcal{G}' \times_{T'} Mor(\mathcal{G}', \mathcal{G}, *) \times \mathcal{G}$ on EXP are defined as follows. For $(z, e) \in (z, E^z) \subset EXP$, $g' \in \mathcal{G}'$ with source $t(e)$,$(g'', z') \in \mathcal{G}' \times_{T'} Mor(\mathcal{G}', \mathcal{G}, *)$ with target z and $g \in \mathcal{G}$ with target $s(e)$ we define

$$g'.(z, e).((g'', z'), g) = (z', m^{-1}_{(g'', z')}(g'.e.g)) \in (z', E^{z'}) \subset EXP.$$

Note that $m^{-1}_{(g'', z')}(g'.e.g) = g'.m^{-1}_{(g'', z')}(e).g$. These actions define on EXP the structure of a $(\mathcal{G}', \ (\mathcal{G}' \times_{T'} Mor(\mathcal{G}', \mathcal{G}, *)) \times \mathcal{G})$-bundle defining the desired exponential morphism.

Example. Assume that $\mathcal{G} = \Gamma \ltimes T$ and $\mathcal{G}' = \Gamma' \ltimes T'$ with T simply connected as in III.4. Then EXP is the disjoint union of the $(\mathcal{G}', \mathcal{G})$-bundles $E^{(f,\psi)}$ indexed by the set of pairs $(f, \psi) \in \mathcal{M}(\mathcal{G}', \mathcal{G}.*)$. The bundle $E^{(f,\psi)}$ is the product $\Gamma' \times T$ with projections s and t mapping (γ', x) to x and $\gamma'.f(x)$ respectively. The right action of $(\gamma, \gamma^{-1}.x) \in \Gamma \ltimes T$ on (γ', x) is equal to $(\gamma'\psi(\gamma), \gamma^{-1}.x)$ and the left action of $(\gamma'', \gamma'.f(x)) \in \Gamma' \ltimes T'$ on (γ', x) is equal to $(\gamma''\gamma', x)$. In that case the bundle EXP has a canonical section and therefore the exponential morphism can be described directly by the homomorphism from $(\mathcal{G}' \times_{T'} \mathcal{M}or(\mathcal{G}', \mathcal{G}, *)) \times \mathcal{G} = (\Gamma' \ltimes \mathcal{M}or(\mathcal{G}', \mathcal{G}, *)) \times (\Gamma \ltimes T)$ to $\mathcal{G} = \Gamma' \ltimes T'$ mapping $(\gamma'', (f, \psi), \gamma, x)$ to $(\gamma''\psi(\gamma), f(x))$.

IV. A topology on $\mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$

To get some feeling we begin with the particular case considered in III.4 .

1. The developable case. We assume that

1) $\mathcal{G}' = \Gamma' \ltimes T'$, where T' is differentiable manifold with a Γ'-invariant Riemannian metric.

2) $\mathcal{G} = \Gamma \ltimes T$, where T is 1-connected and there is a compact subset of T meeting every orbit.

We want to define an open neighbourhood of an element $(f, \psi) \in \mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$ (see III.4). Let $\tau T'$ be the tangent bundle of T' and let $f^*(\tau T') = \{(x, v), x \in T, v \in \tau_{f(x)} T'\}$ be its pull back by f. The group Γ acts naturally on it, namely $\gamma.(x, v) = (\gamma.x, \psi(\gamma).v)$, where $\psi(\gamma).v$ denote the image of v by the differntial of the isometry defined by $\psi(\gamma)$. Let $V_{(f,\psi)}$ be the Banach space of Γ-invariant continuous sections of $f^*(\tau T')$ with the sup-norm. Let ϵ be a small enough positive number such that the exponential map exp ... $T' \to T'$ restricted to the open ball of radius ϵ is a diffeomorphism on a convex geodesic ball of T', and this for all $x \in T$ (this is possible by condition 2)). Let $V^{\epsilon}_{(f,\psi)}$ be the open ball of radius ϵ in $V_{(f,\psi)}$.

We define a structure of Banach manifold on $\mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$. A chart at (f, ψ) is the map $V^{\epsilon}_{(f,\psi)} \to \mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$ sending a section $\nu : x \mapsto (x, v(x))$ to (f^{ν}, ψ), where $f^{\nu}(x) = exp_{f(x)} v(x)$. One verifies as usual (see Eells [3]) that the changes of charts are differentiable.

With this topology, the exponential homomorphism as described at the end of III is continuous.

2. A more general case. We make the following hypothesis which is satisfied for $\mathcal{G}'$ the transverse holonomy or monodromy groupoid of a Riemannian foliation on a complete Riemannian manifold (see [9] and the appendix by E. Salem), in particular for Riemannian orbifolds.

1) $\mathcal{G}'$ is a Hausdorff and complete Riemannian étale groupoid. This means that T' has a $\mathcal{G}$-invariant Riemannian metric, that $\mathcal{G}$ is Hausdorff and that the following condition is satisfied: for each $g' \in \mathcal{G}'$ with source x' and target y' which are centers of convex geodesic balls $B(x', \epsilon)$ and $B(y', \epsilon)$, there is a unique continuous map $\tilde{g'} : B(x', \epsilon) \to \mathcal{G}'$ such that, for each $z \in B(x', \epsilon)$, the source of $\tilde{g'}(z)$ is z and $g' = \tilde{g'}(x')$. In particular $t \circ \tilde{g'} \circ exp_{x'} = exp_{y'} \circ Dg'$, where Dg' is the differential of $t \circ \tilde{g'}$ at x' (it depends only on g').

It implies the following. If $g'' \in \mathcal{G}'$ with source y' and target the center of a convex geodesic ball of radius ϵ, then

$$(*) \qquad \widetilde{g''g'}(z) = \tilde{g''}(t(\tilde{g'}(z)))\tilde{g'}(z) \quad \forall z \in B(x', \epsilon).$$

2) $\mathcal{G}$ is connected and there is a relatively compact open subset $T_0 \subseteq T$ meeting every orbit of $\mathcal{G}$.

3. Theorem. *Under the above hypothesis, there is a natural structure of Banach manifold on $\mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$ so that $\mathcal{G}' \times_{T'} \mathcal{M}(\mathcal{G}', \mathcal{G}, *)$ becomes a differentiable Banach étale groupoid.*

Proof. The proof will be in three steps.

i) Given a pointed $(\mathcal{G}', \mathcal{G})$-bundle (E, e_0) over $* \in T$, we describe a chart at $[E, e_0]$. Let $\mathcal{G}_0$ be the restriction of $\mathcal{G}$ to T_0. The restriction E_0 of E above T_0 is a pointed $(\mathcal{G}', \mathcal{G}_0)$-bundle and (E, e_0) is naturally isomorphic to $E_0 \times_{\mathcal{G}_0} \mathcal{E}$, where $\mathcal{E} = \{g \in \mathcal{G} : t(g) \in T_0\}$, with the base point $[e_0, 1_*]$.

Choose a finite open cover $\mathcal{U}$ of T_0 consisting of $U_0, U_1 \ldots, U_k$ such that $* \in U_0$ and such that there are continuous sections $\sigma_i : U_i \to E$ with $\sigma_0(*) = e_0$. Let $\mathcal{G}_{\mathcal{U}}$ be the localization of $\mathcal{G}_0$ over $\mathcal{U}$. As in II.4, we get a homomorphism $\phi : \mathcal{G}_{\mathcal{U}} \to \mathcal{G}'$ whose restriction to the space of objects is a continuous map $\phi_0 : T_{\mathcal{U}} \to T'$. The groupoid $\mathcal{G}_{\mathcal{U}}$ acts on $\phi_0^*(\tau T')$: the action of $g \in \mathcal{G}_{\mathcal{U}}$ with source x and target y on $(x, v(x))$, where $v(x) \in \tau_{\phi_0(x)} T'$, is defined by $g.(x, v(x)) = (y, D\phi(g).v(x))$. Let V_{ϕ} be the Banach space of continuous $\mathcal{G}_{\mathcal{U}}$-invariant sections of $\phi_0^*(T')$. Thanks to hypothesis 2), we can find a positive number ϵ such that, for all $x \in T_{\mathcal{U}}$, the exponential map $exp_{\phi_0(x)}$ gives a diffeomorphism from the ϵ-ball in $\tau_{\phi_0(x)} T'$ to a convex geodesic ball in T'. Let V^{ϵ}_{ϕ} be the open subset of V_{ϕ} formed by the $\mathcal{G}_{\mathcal{U}}$-invariant sections of $\phi_0^*(T')$ of norm $< \epsilon$.

Given $g \in \mathcal{G}_{\mathcal{U}}$ with source x, let $\widetilde{\phi(g)}$ be the extension of $\phi(g)$ to the geodesic ball of radius ϵ centered at the source $\phi_0(x)$ of $\phi(g)$ as defined in 1). Given a section

$\nu : x \mapsto (x, v(x))$ in V^ϵ_ϕ, let $\phi^\nu : \mathcal{G}_\mathcal{U} \to \mathcal{G}'$ be defined, for g with source x, by

$$\phi^\nu(g) = \widetilde{\phi(g)}(exp_{\phi_0(x)} v(x)).$$

From ϕ^ν, we reconstruct as in II.4 a $(\mathcal{G}', \mathcal{G}_0)$-bundle $E^\nu_0 = E_{\phi^\nu, \mathcal{U}}$ with base point e^ν_0 (see III.1) and then, as above, a $(\mathcal{G}', \mathcal{G})$-bundle E^ν with a base point still noted e^ν_0.

The map $\nu \mapsto [E^\nu, e^\nu_0]$ from V^ϵ_ϕ to $\mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$ will be a chart h_ϕ for the structure of Banach manifold. This map is injective because for $\nu, \nu' \in V^\epsilon_\phi$, the set of points x in the space of objects of $\mathcal{G}_\mathcal{U}$ such that $v(x) = v'(x)$ is a closed $\mathcal{G}_\mathcal{U}$-invariant subset which is also open. If $h_\phi(\nu) = h_\phi(\nu')$ it is non empty because it contains the base point, hence it is the whole of the space of objects because $\mathcal{G}_\mathcal{U}$ is connected.

ii) Let $\phi' : \mathcal{G}_{\mathcal{U}'} \to \mathcal{G}'$ be another pointed homomorphism, where $\mathcal{U}'$ is another open cover $\{U'_0, \ldots, U'_{k'}\}$ of T_0 such that $* \in U'_0$, and let $h_{\phi'} : V^{\epsilon'}_{\phi'} \to \mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$ be a corresponding chart. One has to check that the change of charts $h^{-1}_{\phi'} {h_\phi}^{-1}$ is differentiable. We first consider some particular cases.

a) Assume that $\phi : \mathcal{G}_\mathcal{U} \to \mathcal{G}'$ and $\phi' : \mathcal{G}_{\mathcal{U}'} \to \mathcal{G}'$ are pointed equivalent (see III,2). Then there is a naturel linear isomorphism from V_ϕ to $V_{\phi'}$ and the change of charts $h^{-1}_{\phi'} h_\phi$ is the restriction of this isomorphism to the balls of radius the minimum of ϵ and ϵ'

b) If $\phi' = \phi^\nu : \mathcal{G}_\mathcal{U} \to \mathcal{G}'$, where $\nu \in V^\epsilon_\phi$, the usual argument shows that the change of charts $h^{-1}_{\phi'} h_\phi$ is differentiable at ν.

In the general case, if $h_\phi(\nu) = h_{\phi'}(\nu')$, we express the change of charts $h^{-1}_{\phi'} h_\phi$ around ν as the composition $(h^{-1}_{\phi'} h_{\phi'^{\nu'}})(h^{-1}_{\phi'^{\nu'}} h_{\phi^\nu})(h^{-1}_{\phi^\nu} h_\phi)$ and we apply a) and b).

The map $\mathcal{M}(\mathcal{G}', \mathcal{G}, *) \to T'$ sending $[E, e_0]$ to $t(e_0)$ is continuous because, for each chart h_ϕ, the map $\nu \mapsto t(e^\nu_0) = exp_{\phi_0(*)}(v(*))$ is continuous.

iii) Let us check now that the action of $\mathcal{G}'$ on $\mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$ is differentiable. Let (E, e_0) be a pointed bundle and let $g' \in \mathcal{G}'$ be such that $s(g') = t(e_0)$. Choose an open neighbourhood U_0 of $*$ in T_0 such that there are continuous sections $\sigma_0, \sigma'_0 : U_0 \to E$ such that $\sigma_0(*) = e_0$ and $\sigma'_0(*) = g'.*$. We can find a finite open cover $\mathcal{U} = \{U_0, \ldots, U_k\}$ of T_0 such that there are continuous sections $\sigma_i : U_i \to E$ for $i \neq 0$. Let $\phi, \phi' : \mathcal{G}_\mathcal{U} \to \mathcal{G}'$ be the homomorphisms associated to the choice of the sections σ_i and σ'_i, where $\sigma'_i = \sigma_i$ for $i \neq 0$. The associated pointed bundles are naturally isomorphic to (E, e_0) and $(E, g'.e_0)$. For $\epsilon > 0$ small enough, the charts $h_\phi : V^\epsilon_\phi \to \mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$ and $h_{\phi'} : V^\epsilon_{\phi'} : \mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$ are defined. For each $x \in U_0$, let $g'(x)$ be the unique element of $\mathcal{G}'$ such that $\sigma'(x) = g'(x).\sigma(x)$. The extension $\widetilde{g'(x)}$ of $g'(x)$ is defined on the geodesic ball of radius ϵ and center $s(g'(x)$. Let $f : V_\phi \to V_{\phi'}$ be the linear isometry mapping the $\mathcal{G}_\mathcal{U}$-invariant section $\nu : (i, x) \in T_\mathcal{U} \mapsto ((i, x), v(i, x))$ in V_ϕ to the $\mathcal{G}_{\mathcal{U}'}$- invariant section $f(\nu) \in V_{\phi'} : (i, x) \mapsto ((i, x), f(v)(i, x))$, where $f(v)(i, x)$ is equal to $v(i, x)$ for $i \neq 0$ and $f(v)(0, x) = Dg'(x).v(0, x)$. Let σ be the section of $s : \mathcal{G}' \times_{T'} \mathcal{M}or(\mathcal{G}', \mathcal{G}.*) \to \mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$ above $h_\phi(V^\epsilon_\phi)$ given by $h_\phi(\nu) \mapsto (\widetilde{g'}(t(e^\nu_0), h_\phi(\nu))$. The differentiable structure on $\mathcal{M}(\mathcal{G}', \mathcal{G}, *)$ is invariant by $\mathcal{G}'$ because $h^{-1}_{\phi'} \circ t \circ \sigma \circ h_\phi = f$ is differentiable. $\square$

4. The exponential morphism. The canonical morphism EXP from $(\mathcal{G}' \times_{T'} \mathcal{M}or(\mathcal{G}', \mathcal{G}, *)) \times \mathcal{G}$ to $\mathcal{G}'$ is represented in the above framework by a tautological bundle over $\mathcal{M}or(\mathcal{G}', \mathcal{G}, *) \times T$ described as follows. As $\mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$ is canonically isomorphic to $\mathcal{M}or(\mathcal{G}', \mathcal{G}_0, *)$, we can replace $\mathcal{G}$ by it s rectriction $\mathcal{G}_0$ to T_0 (see the beginning of the proog of 3). To each chart h_ϕ we associate the family of pointed $(\mathcal{G}', \mathcal{G})$-bundles $(E_{\phi^\nu, \mathcal{U}}, e_0^\nu)$ parametrized by the elements $\nu \in V_\phi^\epsilon$. We note EXP_ϕ the disjoint union $\coprod_{\nu \in V_\phi^\epsilon} E_{\phi^\nu, \mathcal{U}}$ topologized as the quotient of the subspace of $V_\phi^\epsilon \times \mathcal{G}' \times T_\mathcal{U}$ made up of the triples $(\nu, g', (i, x))$ such that $s(g') = \phi_0^\nu(i, x)$. We have a continuous projection $s_\phi : EXP_\phi \to V_\phi^\epsilon \times T_0$ sending (ν, e) to $(\nu, s(e))$.

Let $h_{\phi'} : V_{\phi'}^{\epsilon'} \to \mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$ be another chart. The change of charts $h_{\phi'}^{-1} h_\phi$: $W \subset V_\phi^\epsilon \to W' \subset V_{\phi'}^{\epsilon'}$ lifts functorially to a family of pointed isomorphisms $f_{\phi', \phi}$: $s_\phi^{-1}(W \times T_0) \to s_{\phi'}^{-1}(W' \times T_0)$. The bundle over $\mathcal{M}or(\mathcal{G}', \mathcal{G}, *) \times T_0$ representing EXP is the quotient of the disjoint union of the EXP_ϕ under the equivalence relation which identifies points corresponding to each other by the isomorphisms $f_{\phi'\phi}$. This bundle is also denoted EXP and its restriction above the range of the chart ϕ is canonically isomorphic to EXP_ϕ. The map $f_\phi : E_\phi \to EXP$ associating to a point its equivalence class is injective and can be considered as a chart. The change of charts $f_{\phi'}^{-1} \circ f_\phi$ is the map $f_{\phi'\phi}$.

The left action of $\mathcal{G}'$ and the commuting right action of $(\mathcal{G}' \times \mathcal{M}or(\mathcal{G}', \mathcal{G}_0, *)) \times \mathcal{G}_0$ on EXP is described in the charts like in III.4.

Let $\mathcal{H}$ be an étale groupoid with space of objects a topological space S. Let Ψ be a morphism from $\mathcal{H}$ to $\mathcal{G}' \times_{T'} \mathcal{M}(\mathcal{G}', \mathcal{G}, *)$. Let $\Psi \times id_\mathcal{G}$ be the morphism from $\mathcal{H} \times \mathcal{G}$ to $(\mathcal{G}' \times_{T'} \mathcal{M}(\mathcal{G}', \mathcal{G}, *)) \times \mathcal{G}$ which is the cartesian product of Ψ with the identity of $\mathcal{G}$.

5. Theorem. *The map associating to Ψ the morphism $\overline{\Psi} = EXP \circ (\Psi \times id_\mathcal{G})$ induces a bijection between the set of morphisms from $\mathcal{H}$ to $\mathcal{G}' \times_{T'} \mathcal{M}or(\mathcal{G}', \mathcal{G}, .*)$ and the set of morphisms from $\mathcal{H} \times \mathcal{G}$ to $\mathcal{G}'$.*

Proof. We just indicate how starting from a bundle $\overline{P}$ representing a morphism $\overline{\Psi}$ from $\mathcal{H} \times \mathcal{G}$ to $\mathcal{G}'$ we can construct a morphism Ψ from $\mathcal{H}$ to $\mathcal{G}' \times_{T'} \mathcal{M}(\mathcal{G}', \mathcal{G}, *)$. For each $v \in S$, the pull back of $\overline{P}$ by the inclusion $T \to \{v\} \times T$ is a $(\mathcal{G}', \mathcal{G})$-bundle over T noted $\overline{P}_v$. An element $h \in \mathcal{H}$ with source v and target w gives an isomorphism $\overline{P}_h : \overline{P}_w \to \overline{P}_v$ induced by the right action of $(h, 1_T)$ on $\overline{P}$.

Choose an open cover $\mathcal{V} = \{V_i\}_{i \in I}$ of S such that there exist continuous sections $\sigma_i : V_i \times \{*\} \to \overline{P}$ of the projection $\overline{P} \to S \times T$. For $v \in V_i$, this induces a base point above $*$ on $\overline{P}_v$ noted $\sigma_i(v)$. We note $f(v, i) \in \mathcal{M}(\mathcal{G}', \mathcal{G}, *)$ the isomorphism class of the pointed bundle $(P_v, \sigma_i(v))$.

Let h be an element of $\mathcal{H}$ with source $v \in V_i$ and target $w \in V_j$. Let $f(j, h, i)$ be the element of $\mathcal{G}'$ such that

$$\sigma_j(w, *).(h, 1_*) = f(j, h, i).\sigma_i(v, *).$$

Then the map $\psi : \mathcal{H}_\mathcal{V} \to \mathcal{G}' \times_{T'} \mathcal{M}(\mathcal{G}'\mathcal{G}, *)$ sending $(j, h, i) \in \mathcal{G}_\mathcal{V}$ to $(f(j, h, i), f(v, i))$ is a homomorphism representing the morphism Ψ. $\square$

Remark. A pointed morphism from $\mathcal{H}$ to $\mathcal{G}' \times_{T'} \mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$ above a point $v_0 \in S$ is determined by the choice of a pointed $(\mathcal{G}', \mathcal{G})$-bundle (E, e_0) over $*$. It corresponds to a morphism from $\mathcal{H} \times \mathcal{G}$ to $\mathcal{G}'$ represented by a $(\mathcal{G}', \mathcal{H} \times \mathcal{G})$-bundle $\overline{P}$ over $S \times T$ together with an isomorphism from (E, e_0) to $\overline{P}_{v_0}$. (See [4], 2.2.3 for a general notion of relative morphisms.)

6. Remarks. If $\mathcal{G}$ is replaced by an equivalent groupoid $\overline{\mathcal{G}}$ and $\mathcal{G}'$ by an equivalent groupoid $\overline{\mathcal{G}'}$ with an invariant Riemannian metric, then (see III.2) the groupoids $\mathcal{G}' \times_{T'} \mathcal{M}or(\mathcal{G}', \mathcal{G}, *)$ and $\overline{\mathcal{G}'} \times_{T'} \mathcal{M}or(\overline{\mathcal{G}'}, \overline{\mathcal{G}}, \overline{*})$ are differentiably equivalent.

According to the needs, one can replace in the construction of section 2 the set of continuous pointed morphisms from $\mathcal{G}$ to $\mathcal{G}'$ by a subset with a suitable topology, as it is usual when dealing with functions spaces (see for instance [4]).

V. Selfequivalences and extensions

1. The case of developable groupoids. Let $\mathcal{G} = \Gamma \ltimes T$, where T is simply connected and let $*$ be a point of T. According to III.4. the set $(\mathcal{S}elf(\mathcal{G}), *)$ of pointed selfequivalences of $\mathcal{G}$ is isomorphic to the set $\mathcal{S}$ of pairs (f, ψ), where ψ is an automorphism of Γ and f is a homeomorphism of T which is ψ-equivariant. This set is a group, the composition being defined by $(f, \psi)(f', \psi') = (f \circ f', \psi \circ \psi')$. The groupoid of pointed selfequivalences of $\mathcal{G}$ is isomorphic to the action groupoid $\Gamma \ltimes \mathcal{S}$, where $\gamma \in \Gamma$ acts on a pair by the rule

$$\gamma.(f, \psi) = (t_\gamma \circ f, Ad(\gamma) \circ \psi).$$

In fact we have a **crossed mdule**:

$$\mu : \Gamma \to \mathcal{S},$$

where $\mu(\gamma) = (t_\gamma, Ad(\gamma))$ and the pair $(f, \psi) \in \mathcal{S}$ acts on Γ through the automorphism ψ.

If T has a Γ-invariant Riemannian metric and if there is a compact subset meeting every orbit, then $\mathcal{S}$ has a topology so that μ is a **topological crossed module**.

To illustrate the connection with extensions, we consider two particular cases.

2. The case of a discrete group Γ. Here Γ is considered as a discrete groupoid with one object $*$. The group of pointed selfequivalences is the group $Aut(\Gamma)$ of automorphisms of Γ and the crossed module is

$$\mu : \Gamma \to Aut(\Gamma).$$

where μ sends γ to $Ad(\gamma)$.

As it is well known since the works of J. H. C. Whitehead and S. Maclane in the forties, this crossed module plays the central role in the problem of extensions of groups, and more generally of extensions of groupoids by the discrete group Γ.

Given an étale groupoid $\mathcal{G}$ with space of objects T, an extension of $\mathcal{G}$ by Γ is given by an open cover $\mathcal{U}$ of T, an etale groupoid $\widetilde{\mathcal{G}}$ with space of objects $T_\mathcal{U}$ (notation of I.2) and a homomorphism $\phi : \widetilde{\mathcal{G}} \to \mathcal{G}_\mathcal{U}$ such that ϕ_0 is the identity on the space of objects $T_\mathcal{U}$ and the kernel of ϕ is isomorphic to $\mathcal{T}_\mathcal{U} \times \Gamma$. We say that the extension ϕ is topologically split if there is a continuous map $\sigma : \mathcal{G}_\mathcal{U} \to \tilde{\mathcal{G}}$ such that $\phi(\sigma(g)) = g$ and $\sigma(1_x) = 1_x$ for every $x \in T_\mathcal{U}$.

Such an extension is completely determined by a homomorphism from $\mathcal{G}$ to the 2-group associated to the crossed module μ. The equivalence classes of such extensions is in bijection with the homotopy classes of maps from the classifying space $B\mathcal{G}$ of $\mathcal{G}$ to the classifying space $B(\mu)$ of μ, at least if $\mathcal{G}$ satisfies the condition 1) in IV.2. See the Erratum below.

3. The case where Γ is a dense subgroup of a Lie group G. Let G be a simply connected Lie group G and let Γ be a dense subgoup of G endoved with the discrete topology. Let $\mathcal{G}$ be the action groupoid $\Gamma \ltimes G$, where Γ acts on G by left translations. Let $Aut(G, \Gamma)$ be the group of automorphism of G preserving the subgroup Γ. The group of pointed selfequivalences of $\mathcal{G}$ is isomorphic to the Lie group $G \rtimes Aut(G, \Gamma)$, where $Aut(G, \Gamma)$ is considered as a discrete group acting in the obvious way on the Lie group G. To $(g, \psi) \in G \rtimes Aut(G, \Gamma)$ corresponds the pointed self-equivalence of $\Gamma \ltimes G$ given by the ψ-equivariant homeomorphism $G \to G$ sending h to $\psi(h)g^{-1}$.

The group $Aut(G, \Gamma)$ depends of the arithmetic properties of Γ as a subgroup of G. For instance, if $G = \mathbb{R}$ and if Γ is a subgroup generated by two elements whose ratio is an irrational number α, then $Aut(G, \Gamma)$ is the group acting on $\mathbb{R}$ by the multiplication by ∓ 1 when α is transcendental, otherwise by the units of the ring of integers of the number field $\mathbb{Q}(\alpha)$.

The crossed module associated to the group of selfequivalence of $\Gamma \ltimes G$ is

$$\mu : \Gamma \to G \rtimes Aut(G, \Gamma),$$

defined by $\mu(\gamma) = (\gamma^{-1}, Ad(\gamma))$.

The corresponding extension problem has been studied in the thesis of Ana Maria da Silva ([11]). Let W be a paracompact differentiable manifold considered as a trivial étale groupoid. An extension of W by $\Gamma \ltimes G$ is given by an open cover $\mathcal{U}$ of W and a homomorphism ϕ from an etale groupoid $\tilde{\mathcal{G}}$ to $W_{\mathcal{U}}$. The space of units of $\tilde{\mathcal{G}}$ is assumed to be isomorphic to $W_{\mathcal{U}} \times G$ and the kernel of ϕ is the action groupoid given by Γ acting on $T_{\mathcal{U}} \times G$ by left translations on the factor G and trivially on the first factor.

This problem is motivated by Molino's structure theorem of Riemannian foliations on a complete Riemannian manifold M which are transversally complete (see [9]). In that case Molino showed that the closure of the leaves are fibers of a fibration of M with base space a smooth manifold W. The transverse holonomy groupoid of the foliation restricted to a fiber is equivalent to an action groupoid $\Gamma \ltimes G$. The transverse holonomy groupoid of the foliation is equivalent to an extension of W by $\Gamma \ltimes G$ in the above sense.

Ana Maria da Silva showed in her thesis that the set of equivalence classes of extensions of W by $\Gamma \ltimes G$ are in bijection with the set of homotopy classes of maps from W to a topological space $B_{(G,\Gamma)}$. This space should be the classifying space of the crossed module μ (or equivalently the geometric realization of the nerve of the 2-group associated to the topological crossed module μ).

4. Erratum to [6]. The problem of the classification of extensions of an étale groupoid by a discrete group was briefly discussed at the end of my expository paper [6]. We take this opportunity to list a few corrections of the last pages of this paper.

p.97 line -3 read $\sigma : \mathcal{G}_{\mathcal{U}} \to \tilde{\mathcal{G}}_{\mathcal{U}}$ such that $\phi \circ \sigma$ is the identity of $\mathcal{G}_{\mathcal{U}}$ and $\sigma(1_x) = 1_x$.
p.98 lines 3-8 should be replaced by

The kernel $T_{\mathcal{U}} \times N$ of ϕ is a $\tilde{\mathcal{G}}_{\mathcal{U}}$-sheaf of groups. For $\tilde{g} \in \tilde{\mathcal{G}}_{\mathcal{U}}$, with source x and target y, its action is given by the relation

$$\tilde{g}(x, n) = (y, \tilde{\psi}(\tilde{g})(n))\tilde{g}$$

where $\tilde{\psi}$ is a continuous homomorphism from $\tilde{\mathcal{G}}_{\mathcal{U}}$ to $Aut(N)$. Passing to the quotient we get a homomorphism $\psi : \mathcal{G}_{\mathcal{U}} \to Out(N)$ which determines a $Out(N)$ -principal bundle over T with a compatible $\mathcal{G}$-action, in other words a (generalized) morphism $\Psi : \mathcal{G} \to Out(N)$. Let $\mathcal{C}$ be the associated locally constant $\mathcal{G}$-sheaf over T with stalk C (the center of N) associated to the action of $Out(N)$ on C.

Theorem 8.2 should be stated as follows: Let $\Psi : \mathcal{G} \to Out(N)$ be a morphism and let $\mathcal{C}$ be the associated $\mathcal{G}$- sheaf of abelian groups with stalk the center C of N. There is a topologically split extension of $\mathcal{G}$ by N with associated morphism Ψ iff an obstruction in $\check{H}^3(\mathcal{G}, \mathcal{C})$ vamishes. If this is the case, the set of equivalence classes of locally topologically split extensions of $\mathcal{G}$ by N with associated morphism Ψ is in bijection with the set $\check{H}^2(\mathcal{G}, \mathcal{C})$.

p.99 last line before the references

" if and only if this homomorphism has a discrete image in G."

References

1. M. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der matematischen Wissenshaften **319** (1999), Springer Verlag.
2. Weimin Chen, *On a notion of maps between orbifolds.I. Functions spaces. II. Homotopy and CW-complexes*, Commun. Contemp. Math. **8** (2006), 569-620, 763-821.
3. James Eells, *On the geometry of function spaces*, 1958 A symposium internacional de topologia algebraica, Mexico, Universtad Nacional Autonoma de Mexico and UNESCO, 303-308.
4. K. Guruprasad and A. Haefliger, *Closed geodesics on orbifolds*, Topology **45** (2007), 611-641.
5. A. Haefliger, *Groupoides d'holonomie et classifiants. In: Structures transverses des feuilletages (Toulouse 1982)*, Asterisque **116** (1984), 70-97.
6. A. Haefliger, *Groupoids and foliations*, Contemp. Mathematics **282** (2001), 83-100.
7. M. Hilsum et G. Skandalis, *Morphisms K-orientés d'espaces de feuilles et fonctorialité en théorie de Kasparov*, Ann. Scient. Ec. Norm Sup **20** (1987), 325-390.
8. E. Lupercio and B. Uribe, *Loop groupoids, Gerbes, and Twisted Sectors on Orbifolds. In: Orbifolds in Mathematics and Physics (Madison, WI, 2001)*, Contemporary Mathematics **310** (2002), Amer. Math. Soc., Providence, RI, 163-184.
9. P. Molino, *Riemannian Foliations. Progress in Math.*, vol. 73, Birkhäuser, Boston, 1988.
10. Janez Mrcun, *Stability and Invariants of Hilsum-Skandalis Maps. PHD Thesis, Universiteit Utrecht,1996*, math.AT/0506484.
12. Ana Maria da Silva, *Classification des pseudogroupes localement* (G, Γ)*-principaux*, Thèse, Université de Genève (1989).