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Topological Strings on Local Curves

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Marino Beiras, Marcos

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Marcos Mariño

Département de Physique Théorique and Section de Mathématiques
Université de Genève, CH 1211 Genève 4, Switzerland

`marcos.marino@unige.ch`

Abstract

We review some perturbative and nonperturbative aspects of topological string theory on the Calabi–Yau manifolds $X_p = \mathcal{O}(-p) \oplus \mathcal{O}(p-2) \to \mathbb{P}^1$. These are exactly solvable models of topological string theory which exhibit a nontrivial yet simple phase structure, and have a phase transition in the universality class of pure two–dimensional gravity. They don't have conventional mirror description, but a mirror B model can be formulated in terms of recursion relations on a spectral curve typical of matrix model theory. This makes it possible to calculate nonperturbative, spacetime instanton effects in a reliable way, and in particular to characterize the large order behavior of string perturbation theory.

Contents

1 Introduction

Topological string theory on non-compact Calabi–Yau manifolds (also known as local Calabi–Yau's) has taught us many interesting things about topological strings and about string theory in general. Although non-compact backgrounds are quite special, their relative simplicity makes them also a fascinating laboratory. From the mathematical point of view, the theory of topological strings on local backgrounds has made a myriad of connections to other fields of mathematics and mathematical physics, including matrix models, integrable systems, and combinatorics.

Among these local backgrounds, perhaps the simplest and the most peculiar are what I will call *local curves*. This is the family of non-compact Calabi–Yau manifolds given by the total space of a bundle over a sphere. More concretely, they have the form

$$X_p = \mathcal{O}(p-2) \oplus \mathcal{O}(-p) \to \mathbb{P}^1, \quad p \in \mathbb{Z}. \tag{1.1}$$

This family includes two of the most studied local backgrounds: for $p = 1$ one recovers the resolved conifold, while X_2 is the $\mathbb{C} \times A_1$ singularity. Therefore, the family (1.1) provides the simplest generalization of these well studied examples.

The study of topological string theory on X_p has led to many insights. For example, the conjecture relating toric backgrounds to matrix models stated in [24] and further refined in [5] was motivated to a large extent by the genus zero solution on X_p presented in [8]. These backgrounds have been an important testing ground for recent techniques and ideas, but one has to keep in mind that they are rather unconventional in many respects.

In this note we review some properties of topological string theory on X_p discovered in [8, 24, 25]. We summarize them briefly in this introduction by emphasizing both their connections to general aspects of topological string theory, as well as their idiosyncrasies.

- *A-model and topological vertex.* The Gromov–Witten theory for these spaces was developed by Bryan and Pandharipande [6] and can be reproduced, in the equivariant case, by the topological vertex of [1]. The total partition sum in the A model is a sum over partitions. This is reviewed in section 2.1.

- *Sum over partitions and mirror symmetry.* The backgrounds X_p do not have standard mirror manifolds (see however [16] for some progress along this direction). However, a mirror geometry can be extracted by studying the saddle-point of the sum over partitions [8] and it is encoded in a complex curve which we will call the spectral curve of the model. This is similar to the way in which the Seiberg–Witten curve emerges from the sum over partitions in Nekrasov's computation [26, 27]. The mirror geometry is reviewed in section 2.3

- *Matrix models.* The generating functionals of Gromov–Witten invariants at genus g, F_g, can be obtained by applying the matrix model formalism of [15] to the spectral, mirror curve. This was conjectured and tested to lower genus in [24]. Therefore, topological strings on X_p can be described by a matrix model formalism. The matrix model/topological string correspondence was first found by [12] in some special affine backgrounds and later generalized to toric manifolds [24, 5]. In the case of X_p the existence of a matrix model description was proved by Eynard in [14]. This development is briefly mentioned in section 2.4.

- *Phase transitions.* As for other topological string models, the free energies F_g of X_p exhibit singular behavior for $p > 2$ at a particular point $t_c \neq 0$ in the Kähler moduli space. For most topological string models, this point is the conifold point and the singular behavior is described by the $c = 1$ string at self-dual radius [17]. In the case of topological string theory on local curves, however, the singular behavior is described by the $c = 0$ string, i.e. by two-dimensional gravity [8]. Phase transitions on X_p are described in section 3.

- *Instantons and large order behavior.* The matrix model formalism makes possible to compute spacetime instanton corrections to the partition function [25]. Using the connection between instantons and large order behavior, one obtains conjectural, precise descriptions of the asymptotic behavior of the couplings F_g at large g [24, 25] which can be tested numerically. This aspect is reviewed in section 4.

- *Hurwitz theory.* In the limit $p \to \infty$, topological strings on local curves encode the simple Hurwitz numbers of $\mathbb{P}^1$. This is established in section 2.2. Most of the properties above can be seen to be inherited by this Hurwitz model (like in particular

the critical behavior and the instanton effects describing large order), and we refer for this to the original papers [8, 25].

We finally mention that the study of these backgrounds was originally motivated by the results of [2] and the connection to the OSV conjecture [28].

2 Topological strings on local curves

First, we notice that since X_p is invariant under $-p \leftrightarrow p - 2$ we can restrict ourselves to the case $p > 0$.

2.1 A model

The first step in understanding topological string theory on X_p is to determine the genus g free energies of closed strings in the A model. Already this is nontrivial, since as explained in [6], the A model has to be defined equivariantly with respect to an action of $\mathbb{C}^* \times \mathbb{C}^*$ on the bundles. Therefore, the most general topological string theory on X_p will depend on two equivariant parameters s_1, s_2. The most natural choice (also called the equivariant Calabi–Yau case) corresponds to the antidiagonal action, in which $s_1 = -s_2$. It can be shown that in this case the dependence on the equivariant parameters drops out, and one obtains topological closed string amplitudes $F_g(t)$ which only depend on the Kähler parameter t (corresponding to the $\mathbb{P}^1$ in the base). We recall that, when expanded around $t = \infty$, the $F_g(t)$ are generating functionals of Gromov–Witten invariants at genus g,

$$F_g(t) = \sum_{k=1}^{\infty} N_{g,k} \mathrm{e}^{-kt}, \tag{2.1}$$

where k corresponds to the degree of the map. As usual in topological string theory, the $F_g(t)$ are put together into a single total free energy,

$$F_{X_p}(g_s, t) = \sum_{g=0}^{\infty} g_s^{2g-2} F_g^{X_p}(t) \tag{2.2}$$

and its exponential is the closed partition function

$$Z_{X_p} = \exp F_{X_p}(g_s, t). \tag{2.3}$$

This partition function was computed in [6] and we can equivalently calculate it by using the theory of the topological vertex [1]. We collect here some formulae from this theory will be useful in the following. First of all, we define the q–number $[n]$ as

$$[n] = q^{n/2} - q^{-n/2}, \qquad q = \mathrm{e}^{g_s}. \tag{2.4}$$

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A representation R of $U(\infty)$ is encoded by a Young tableau, labeled by the lengths of its rows $\{l_i\}$. The quantity

$$\ell(R) = \sum_i l_i \tag{2.5}$$

is the total number of boxes of the tableau. Another important quantity associated to a tableau is

$$\kappa_R = \sum_i l_i(l_i - 2i + 1). \tag{2.6}$$

We also introduce the quantity

$$W_R = q^{-\kappa_R/4} \prod_{\square \in R} \frac{1}{[\mathrm{hook}(\square)]}. \tag{2.7}$$

We can now write the topological string partition function on X_p. It is given in terms of the W_R by

$$Z_{X_p} = \sum_R W_R W_{R^t} q^{(p-1)\kappa_R/2} Q^{\ell(R)}, \quad Q = (-1)^p \mathrm{e}^{-t}. \tag{2.8}$$

Although (2.8) gives an all–genus expression, it is effectively an expansion in powers of Q. One can easily compute the first few terms in the expansion:

$$
\begin{aligned}
F_0^{X_p}(t) &= (-1)^p \mathrm{e}^{-t} + \frac{1}{8}(2\,p^2 - 4\,p + 1)\mathrm{e}^{-2t} + \frac{(-1)^p}{54}(1 - 6\,p + 3\,p^2)(2 - 6\,p + 3\,p^2)\mathrm{e}^{-3t} \\
&\quad + \mathcal{O}(\mathrm{e}^{-4t}), \\
F_1^{X_p}(t) &= -\frac{(-1)^p}{12}\mathrm{e}^{-t} + \frac{1}{48}(p^4 - 4\,p^3 + p^2 + 6\,p - 2)\mathrm{e}^{-2t} \\
&\quad + \frac{(-1)^p}{72}(-2 + 14\,p - 19\,p^2 - 20\,p^3 + 45\,p^4 - 24\,p^5 + 4\,p^6)\mathrm{e}^{-3t} + \mathcal{O}(\mathrm{e}^{-4t}),
\end{aligned}
\tag{2.9}
$$

and so on.

2.2 Relation to Hurwitz theory

The partition function Z_{X_p} can be regarded as a "quantum deformation" of a simpler theory, namely the counting of simple Hurwitz covers of $\mathbb{P}^1$. To see this, we first note that the quantity W_R is a q–deformation of the dimension d_R of the representation R of $S_{\ell(R)}$, the permutation group of $\ell(R)$ elements: as $g_s \to 0$, one has that

$$W_R \to g_s^{-\ell(R)} \frac{d_R}{|\ell(R)|!}. \tag{2.10}$$

This suggests taking the following limit,

$$g_s \to 0, \quad t \to \infty, \quad p \to \infty, \tag{2.11}$$

in such a way that

$$pg_s = \tau_2/N, \quad (-1)^p e^{-t} = (g_s N)^2 e^{-\tau_1},$$

(2.12)

and τ_1, τ_2 and N are new parameters that are kept fixed. In the limit (2.11)–(2.12) the partition function becomes

$$Z_{X_p} \to Z_{\text{Hurwitz}} = \sum_R \left(\frac{d_R}{|\ell(R)|!} \right)^2 N^{2\ell(R)} e^{-\tau_2 \kappa_R/2N} e^{-\tau_1 \ell(R)}.$$

(2.13)

This is the generating functional of simple Hurwitz numbers of $\mathbb{P}^1$ at all genus and degrees. Recall that Hurwitz theory studies branched covers of Riemann surfaces, and Hurwitz numbers enumerate these coverings for fixed genus and degree. When all branch points are simple, the Hurwitz number is called a *simple Hurwitz number*, and for $\mathbb{P}^1$ it is given at genus g and degree d by

$$H_{g,d}^{\mathbb{P}^1}(1^d) = \sum_{\ell(R)=d} \left(\frac{d_R}{\ell(R)!} \right)^2 (\kappa_R/2)^{2g-2+2d},$$

(2.14)

where the sum is over representations R with fixed number of boxes equal to the degree d. Using this formula we can rewrite (2.13) as

$$
\begin{aligned}
Z_{\text{Hurwitz}} &= \sum_{d,m} N^{2d-m} e^{-\tau_1 d} \sum_{\ell(R)=d} \left(\frac{d_R}{\ell(R)!} \right)^2 \frac{(-\tau_2)^m}{m!} (\kappa_R/2)^m \\
&= \sum_{g \geq 0} N^{2-2g} \sum_{d \geq 0} e^{-\tau_1 d} H_{g,d}^{\mathbb{P}^1}(1^d) \frac{\tau_2^{2g-2+2d}}{(2g-2+2d)!},
\end{aligned}
$$

(2.15)

where in the second line we have traded the sum over m by a sum over g. Notice that, since $\kappa_{R^t} = -\kappa_R$, only even powers of τ_2 appear. The model described by (2.13) has been studied in detail due to its connection to Hurwitz theory. From the physical point of view, it was analyzed in [21, 9], and in the mathematical literature it has been studied for example in [18].

The free energy of Z_{Hurwitz} describes *connected*, simple Hurwitz numbers $H_{g,d}^{\mathbb{P}^1}(1^d)^\bullet$:

$$F_{\text{Hurwitz}} = \log Z_{\text{Hurwitz}} = \sum_{g \geq 0} N^{2-2g} \sum_{d \geq 0} e^{-\tau_1 d} H_{g,d}^{\mathbb{P}^1}(1^d)^\bullet \frac{\tau_2^{2g-2+2d}}{(2g-2+2d)!}$$

(2.16)

If we compare this to the total free energy F_{X_p} written in (2.2) in terms of Gromov–Witten invariants, and take the limit (2.11)–(2.12), we find

$$\lim_{p \to \infty} p^{2-2g-2d} (-1)^p N_{g,d}(p) = \frac{H_{g,d}^{\mathbb{P}^1}(1^d)^\bullet}{(2g-2+2d)!}.$$

(2.17)

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The l.h.s. is precisely the coefficient of the highest power in p of $N_{g,d}(p)$. We can therefore interpret the Gromov–Witten invariants of this model as q–deformed connected, simple Hurwitz numbers, since they promote $H_{g,d}^{\mathbb{P}^1}(1^d)^\bullet$ to polynomials of degree $2g - 2 + 2d$ (which is equal to the number of simple branch points).

2.3 Mirror symmetry from large partitions

Sums over partitions of the form (2.8) are sometimes dominated by a single Young tableau which can be regarded as the "saddle point" of the sum over partitions. This approach has been very useful in understanding for example two-dimensional Yang–Mills theory [13] or Nekrasov's instanton sums [27]. Moreover, one expects that, if the total partition function Z_{X_p} can be described by mirror symmetry, the mirror geometry will be encoded in the saddle point of the sum over partitions (this is for example the case in instanton sums, whose mirror description is the special geometry of the Seiberg–Witten curves). Since we don't have an easy way to construct the mirror geometry, we will *deduce it* as a saddle partition of the sum (2.8). We summarize now the results of this deduction, first performed in [8].

The first step is to notice that (2.8) admits a representation in terms of a q-deformed group theoretical quantity of $U(N)$, similarly to what was done in [21] in a similar context. Let $\{l_i\}$ be the lengths of rows in a Young tableau introduced before, and let $h_i = l_i + N - i$. We can write

$$Z_{X_p} = \sum_R \left(\frac{\dim_q R}{{}_q\Omega_R} \right)^2 q^{(p-1)\kappa_R/2} \mathrm{e}^{-t\ell(R)} \tag{2.18}$$

where

$$_q\Omega_R = \prod_{i=1}^N \frac{[h_i]!}{[N-i]!}, \tag{2.19}$$

and

$$\dim_q R = \prod_{1 \leq i < j \leq N} \frac{[l_i - l_j + j - i]}{[j - i]} \tag{2.20}$$

is the quantum dimension of an irreducible representation R of $U(N)$. If we introduce the auxiliary 't Hooft parameter

$$T = g_s N \tag{2.21}$$

and continuous variables in the standard way:

$$\frac{h_i}{N} = \frac{l_i}{N} - \frac{i}{N} + 1 \rightarrow \ell(x) - x + 1 = h(x), \tag{2.22}$$

we find that at large N the sum over partitions is controlled by an effective action for

continuous variables

$$S = -\int_0^1 \int_0^1 \mathrm{d}x\mathrm{d}y \log\left|2\sinh\frac{T}{2}(h(x)-h(y))\right| + \frac{2}{T}\int_0^1 \mathrm{d}x\mathrm{Li}_2(\mathrm{e}^{-Th})$$
$$+ \int_0^1 \mathrm{d}x h(x)(t-(p-1)T) + \frac{pT}{2}\int_0^1 \mathrm{d}x h^2(x) + (p-1)\frac{T}{3} - \frac{\pi^2}{3T} - \frac{1}{2}t. \tag{2.23}$$

The planar limit is governed by a tableau density

$$\rho(h) = -\frac{\mathrm{d}x(h)}{\mathrm{d}h}, \tag{2.24}$$

which one can find as the saddle-point of the effective action. The result is the following. It is useful to introduce the variable

$$\lambda = \exp\big(1 - hT\big). \tag{2.25}$$

In terms of this variable, the density of tableaux $\rho(\lambda)$ has its support on the interval

$$(x_2, x_1) \cup (x_1, \mathrm{e}), \tag{2.26}$$

where $x_{1,2}$ are nontrivial functions of the Kähler parameter t. To specify these, one introduces the *mirror map*

$$Q = (-1)^p \mathrm{e}^{-t} = (1-\zeta)^{-p(p-2)}\zeta. \tag{2.27}$$

The endpoints of the cut are given in terms of ζ by

$$x_1 = (1-\zeta)^{-p}(1+\zeta^{\frac{1}{2}})^2, \quad x_2 = (1-\zeta)^{-p}(1-\zeta^{\frac{1}{2}})^2. \tag{2.28}$$

The information on $\rho(\lambda)$ is equivalently encoded in the resolvent

$$\omega_0(\lambda) = \int_{x_2}^{\mathrm{e}} \frac{\mathrm{d}v}{v}\frac{\rho(v)}{\lambda-v} - \frac{1}{\lambda}\log\frac{\lambda}{\lambda-\mathrm{e}}, \tag{2.29}$$

which according to [8] is given by

$$\omega_0(\lambda) = \frac{1}{\lambda} + \frac{p}{2\lambda}\log\left[2\frac{\sqrt{(\lambda-x_1)(\lambda-x_2)} - \lambda - \sqrt{x_1 x_2}}{(\sqrt{x_1}+\sqrt{x_2})^2}\right]^2 + \frac{1}{\lambda}\log\left[\frac{(\sqrt{\lambda-x_1}+\sqrt{\lambda-x_2})^2}{4\lambda}\right]. \tag{2.30}$$

This function has a branch cut along $[x_1, x_2]$, and its discontinuity is given by

$$y(\lambda) = \frac{2}{\lambda}\left(\tanh^{-1}\left[\frac{\sqrt{(\lambda-x_1)(\lambda-x_2)}}{\lambda-\frac{x_1+x_2}{2}}\right] - p\tanh^{-1}\left[\frac{\sqrt{(\lambda-x_1)(\lambda-x_2)}}{\lambda+\sqrt{x_1 x_2}}\right]\right). \tag{2.31}$$

It is well known that the mirrors to toric Calabi–Yau threefolds can be reduced to algebraic curves. In [24] it was proposed that $y = y(\lambda)$ is the appropriate mirror curve for the X_p geometry. This can be tested in various ways, and we will review some of them below. We then have a mirror geometry obtained from a saddle-point analysis of a sum over tableaux.

This analysis makes possible to compute the genus zero free energy in closed form. As detailed in [8], one finds that

$$\frac{\mathrm{d}^2 F_0}{\mathrm{d}t^2} = -\log(1 - \zeta). \tag{2.32}$$

We can also provide a closed expansion for the prepotential F_0 as a series in e^{-t}. This is better done by working out the expansion of $\log(1 - \zeta)$ through Lagrange inversion and integrating (2.32) twice. In this way we obtain

$$F_0^{X_p}(t) = \sum_{d=1}^{\infty} \frac{1}{d!} \frac{1}{d^2} \frac{((p-1)^2 d - 1)!}{(((p-1)^2 - 1)d)!} (-1)^{dt} \mathrm{e}^{-dt}. \tag{2.33}$$

2.4 Higher genus and matrix models

It was conjectured in [24] that the higher F_g of this model can be obtained by using the matrix model formalism of [15] as applied to the mirror curve (2.30). This conjecture was later proved by Eynard [14] by providing an explicit matrix integral representation of Z_{X_p}. For example, using this formalism one finds that

$$F_1 = -\frac{1}{24} \log \left[\frac{(p-1)^2 \zeta (\zeta_c - \zeta)}{(1 - \zeta)^3} \right]. \tag{2.34}$$

This genus one amplitude can be written as

$$F_1 = F_1^{\text{inst}} + \frac{1}{24} \log Q, \tag{2.35}$$

where F_1^{inst} is the instanton part of F_1 which follows from (2.8).

3 Phase transitions, critical behavior and double-scaling limit

3.1 Review of phase transitions in topological string theory

For simplicity, we will assume in this general discussion that the Calabi–Yau X has a single Kähler parameter t, i.e. $h^{1,1}(X) = 1$ (this is in fact the case for the case we are studying, X_p). When t is large (in the so–called large radius regime) the geometry probed

by string theory can be regarded as a classical geometry together with stringy corrections. This is well reflected in the structure of the prepotential $F_0(t)$ or genus zero topological string amplitude, which in the large radius regime is of the form

$$F_0(t) = \frac{C}{6}t^3 + \sum_{k=1}^{\infty} N_{0,k}e^{-kt}. \tag{3.1}$$

In this equation, C is the classical intersection number for the two-cycle whose size is measured by t. The infinite sum in the r.h.s. is given by worldsheet instanton corrections, which are obtained by "counting" (in an appropriate sense) holomorphic maps from $\mathbb{P}^1$ to X. The instanton counting numbers $N_{0,k}$ are genus zero Gromov–Witten invariants, and we have chosen units in which $\ell_s = \sqrt{2\pi}$.

The series of worldsheet instanton corrections, regarded as a power series in e^{-t}, has in general a finite radius of convergence t_c which can be obtained by looking at the asymptotic growth with k of the numbers $N_{0,k}$. We will characterize this asymptotic growth by t_c and by a critical exponent γ:

$$N_{0,k} \sim k^{\gamma-3}e^{kt_c}, \quad k \to \infty. \tag{3.2}$$

When this holds, the prepotential behaves near t_c as

$$F_0(t) \sim (e^{-t_c} - e^{-t})^{2-\gamma}. \tag{3.3}$$

It turns out that typical Gromov–Witten invariants of Calabi–Yau manifolds behave asymptotically as

$$N_{0,k} \sim \frac{e^{kt_c}}{k^3 \log^2 k}, \quad k \to \infty. \tag{3.4}$$

This is of the form (3.2), with critical exponent

$$\gamma = 0 \tag{3.5}$$

and subleading log corrections. This behavior was first established in [7] in the example of the quintic, and since then it has been verified in other examples, like for example in local $\mathbb{P}^2$, where the critical radius is given by [3, 20]

$$t_c = \frac{1}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)} \operatorname{Re} G\left(\frac{1}{3}, \frac{2}{3}, 1; 1\right) \sim 2.90759 \tag{3.6}$$

and G is the Meijer function.

The subleading log in (3.4) leads to log corrections near the critical point (also referred to as scaling violations) of the form

$$F_0(t) \sim (e^{-t_c} - e^{-t})^2 \log(e^{-t_c} - e^{-t}). \tag{3.7}$$

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This is the genus zero free energy of the $c = 1$ string at the self–dual radius, once the scaling variable $e^{-t_c} - e^{-t}$ is identified with the cosmological constant [4, 17].

The behavior of the prepotential gives a precise quantitative meaning to the distinction between classical and quantum geometry. We will refer to the divergence of the large radius expansion at $t = t_c$ as a *phase transition* with a critical exponent γ defined in (3.4). The phase with

$$t > t_c \tag{3.8}$$

where the expansion (3.1) is convergent, is called the large radius or Calabi–Yau phase, where classical geometry makes sense (albeit it is corrected by worldsheet instantons). When $t \leq t_c$, the nonlinear sigma model approach is not well defined, and classical geometric intuition is misleading.

In order to describe the phase structure of the model we have relied on the behavior of the prepotential, i.e. the planar free energy. It is natural to ask what happens when higher genus topological string amplitudes are taken into account. It turns out that the higher genus Gromov–Witten invariants have the asymptotic behavior [4]

$$N_{g,k} \sim k^{(\gamma-2)(1-g)-1} e^{kt_c}, \quad k \to \infty, \tag{3.9}$$

where t_c is the critical radius obtained at genus zero and it is common to all g, and γ is the critical exponent that appears in (3.2). This is equivalent to the following behavior near the critical point

$$
\begin{aligned}
F_1(t) &\sim c_1 \log\left(e^{-t_c} - e^{-t}\right), \\
F_g(t) &\sim c_g (e^{-t_c} - e^{-t})^{(1-g)(2-\gamma)}, \quad g \geq 2.
\end{aligned}
\tag{3.10}
$$

In conventional topological string theory, as we have mentioned, $\gamma = 0$, but the more general form we have written above will be useful later.

We then see that the phase transition at $t = t_c$ is common for all $F_g(t)$, and the critical exponent changes with the genus in the way prescribed by (3.9). This sort of coherent behavior in the genus expansion is not obvious, but seems to characterize a wide variety of systems that admit a genus expansion (like for example matrix models, see [11] for a review). When this is the case, one can define a *double–scaling limit* as follows. Let us consider the total free energy F as a perturbative expansion in powers of the string coupling constant g_s:

$$F(g_s, t) = \sum_{g=0}^{\infty} F_g(t) g_s^{2g-2}. \tag{3.11}$$

We define the double–scaled string coupling as

$$\kappa = a g_s (e^{-t_c} - e^{-t})^{\gamma/2-1}, \tag{3.12}$$

where a is an appropriate constant. We can then consider the limit

$$t \to t_c, \qquad g_s \to 0, \qquad \kappa \text{ fixed.} \tag{3.13}$$

In this limit, only the most singular part of $F_g(t)$ survives at each genus, and the total free energy becomes the *double-scaled free energy*

$$F_{\mathrm{ds}}(\kappa) = f_0 \kappa^{-2} + f_1 \log \kappa + \sum_{g \geq 2} f_g \kappa^{2g-2}, \qquad (3.14)$$

where $f_g = a^{2-2g} c_g$. It is also customary to express the double–scaled free energy in terms of the scaling variable $z = \kappa^{2/(\gamma-2)}$.

It turns out that, in some cases, one can determine the coefficients f_g in closed form. In the double–scaling limit of matrix models, they are governed by a differential equation of the Painlevé type [11]. In the case of topological string theory on Calabi–Yau manifolds, it was conjectured in [17] that, in terms of a natural coordinate

$$\mu \sim \mathrm{e}^{-t_c} - \mathrm{e}^{-t} \qquad (3.15)$$

which in the mirror model measures the distance to the conifold point $\mu = 0$, the double-scaled free energy is universal and reads

$$F_{\mathrm{ds}}(\mu) = \frac{1}{2}\mu^2 \log \mu - \frac{1}{12} \log \mu + \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)} \mu^{2-2g}. \qquad (3.16)$$

This is exactly the all genus free energy of the $c = 1$ string at the self–dual radius. This behavior has been checked in many examples (see, for example, [19] for a recent calculation on the quintic Calabi–Yau).

3.2 Phase transitions for local curves

Surprisingly, the theory of local curves displays a phase transition, but in a different universality class than the usual topological strings on Calabi–Yau threefolds. As shown in [8], the phase transition of local curves belong to the universality class of 2d gravity.

The easiest way to see this is to analyze the asymptotic growth of genus zero Gromov–Witten invariants. We found in (2.33),

$$N_{0,k} = \frac{1}{k!k^2} \frac{((p-1)^2 k - 1)!}{(((p-1)^2 - 1)k)!}, \qquad (3.17)$$

up to a sign $(-1)^{pk}$. By using Stirling's formula, we obtain

$$N_{0,k} \sim \mathrm{e}^{kt_c} k^{-7/2}, \qquad k \to \infty, \qquad (3.18)$$

where

$$t_c = \log\left((p(p-2))^{p(2-p)}(p-1)^{2(p-1)^2}\right). \qquad (3.19)$$

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This corresponds to

$$\zeta_c = \frac{1}{(p-1)^2},$$ (3.20)

By comparing to (3.2) we also deduce that

$$\gamma = -\frac{1}{2}.$$ (3.21)

The above results are valid for $p > 2$. For $p = 1, 2$ the series is convergent for all $t > 0$. The above result for the critical exponent γ is *not* the standard one for Calabi–Yau threefolds, and indicates that we are in a different universality class.

What is this universality class? (3.21) is the exponent typical of 2d gravity (see for example [11] for a review and references), and in fact, if one takes the double–scaling limit

$$\zeta \to \zeta_c, \quad g_s \to 0, \quad z \text{ fixed,}$$ (3.22)

where

$$z^{5/2} = g_s^{-2} \frac{(p-1)^8}{4(1-\zeta_c)^3}(\zeta_c - \zeta)^5,$$ (3.23)

then the total free energy (2.2) becomes the free energy of 2d gravity,

$$F_{(2,3)}(z) = -\frac{4}{15}z^{5/2} - \frac{1}{48}\log z + \sum_{g \geq 2} a_g z^{-5(g-1)/2},$$ (3.24)

where the coefficients a_g can be obtained by solving the Painlevé I equation

$$u^2 - \frac{1}{6}u'' = z$$ (3.25)

satisfied by the specific heat

$$u(z) = -F''_{(2,3)}(z).$$ (3.26)

Evidence for this result was given in [8]. One can test it at lower genus for all p, and for all genera in the limit $p \to \infty$ (i.e. Hurwitz theory) by using for example [18]. In fact, this result follows from the description of this theory in terms of a matrix model conjectured in [24] and proved in [14]. It follows from [15] that the computation of symplectic invariants F_g of a given spectral curve commutes with the double-scaling limit. Therefore, it is enough to show that the curve (2.31) becomes the spectral curve characterizing 2d gravity. Let us verify this.

We first notice that near the critical point the endpoints of the curve behave as

$$x_1 = x_1^{(c)} + \theta(\zeta - \zeta_c) + \mathcal{O}(\zeta - \zeta_c)^2, \quad x_2 = x_2^{(c)} + \mathcal{O}(\zeta - \zeta_c)^2$$ (3.27)

where

$$x_1^{(c)} = (1 - \zeta_c)^{-p} \frac{p^2}{(p-1)^2},$$

$$x_2^{(c)} = (1 - \zeta_c)^{-p} \frac{(p-2)^2}{(p-1)^2}, \tag{3.28}$$

$$\theta = 2(1 - \zeta_c)^{-p} \frac{p(p-1)}{p-2}.$$

The coordinate λ in (2.31) must scale in this limit like

$$\lambda = x_1^{(c)} - \theta(\zeta - \zeta_c)s + \mathcal{O}(\zeta - \zeta_c)^2. \tag{3.29}$$

This defines the "renormalized" coordinate s. We now reexpress (2.31) in terms of the scaling variables s and z, which is given in (3.23). After some cancellations, we find that

$$\frac{1}{g_s} y(\lambda) \mathrm{d}\lambda \to y(s) \mathrm{d}s = -\frac{4\sqrt{2}}{3} z^{\frac{5}{4}} (2s-1)\sqrt{1+s} \, \mathrm{d}s, \tag{3.30}$$

for all $p > 2$. This is the spectral curve of the $(2,3)$ model, therefore we have proved our claim. We can also interpret the r.h.s. as the Laplace transform of the macroscopic loop operator of 2d gravity, which corresponds to the disk amplitude of the FZZT brane of Liouville theory (see [23] and references therein).

4 Non-perturbative effects and large order behavior

It was pointed out in [24] that the matrix model description of this topological string models can be used to describe spacetime instantons. In matrix models, instanton effects are associated to eigenvalue tunneling [29, 10], and in fact it is possible to write down explicit formulae for the instanton amplitudes up to two loops by using only information from the spectral curve [25].

Let us describe the instanton amplitudes for a matrix model described by a curve of the form

$$y(z) = M(z)\sqrt{(z-x_1)(z-x_2)}, \tag{4.1}$$

The saddle points where the eigenvalues tunnel are located at

$$M(x_0) = 0. \tag{4.2}$$

The instanton action for an instanton tunneling to x_0 is simply given by

$$A = \int_{x_1}^{x_0} y(p)\mathrm{d}p. \tag{4.3}$$

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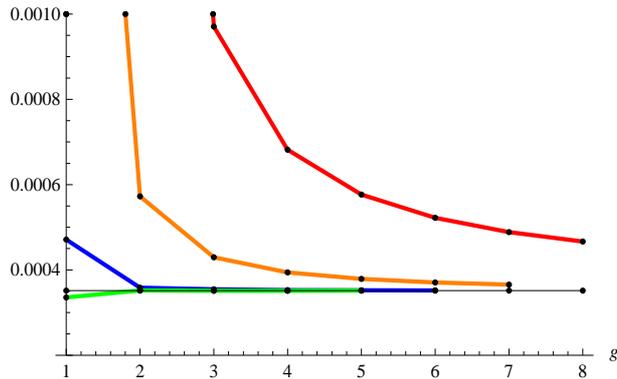Figure 1: The sequence S_g and the first three Richardson transforms for the local curve X_3, at the fixed value $\zeta = 0.24$. The leading asymptotics are predicted to be given by the instanton action $A(\zeta)$, shown as a straight line. The error for the available degree $g = 8$ is 0.014%

There might be many possible saddles x_0, and as usual the leading contribution comes from the instanton with larger action (in absolute value). The one-loop fluctuation around the instanton is given by

$$\mu = -\mathrm{i} \frac{x_1 - x_2}{4} \sqrt{\frac{1}{2\pi M'(x_0)\Big[(x_0 - x_2)(x_0 - x_1)\Big]^{\frac{5}{2}}}}. \tag{4.4}$$

The instanton action and one-loop fluctuation are in general complex,

$$A = |A|\mathrm{e}^{\mathrm{i}\theta_A}, \qquad \mu = |\mu|\mathrm{e}^{\mathrm{i}\theta_\mu}. \tag{4.5}$$

Using the standard connection between instantons and large order behaviour (see for example [22]) one finds that the F_g behave at large g as

$$F_g \sim \frac{|A|^{-2g-b}}{\pi} \Gamma(2g + b) |\mu| \cos\big((2g + b)\theta_A + \theta_\mu\big). \tag{4.6}$$

where in the case of one-cut Hermitian matrix models and topological strings on local curves [25]

$$b = -\frac{5}{2}. \tag{4.7}$$

As we explained in section 2, the F_g amplitudes of topological string theory on local curves can be computed by the matrix model formalism of [15] applied to the curve (2.31), as discussed in [24]. Therefore, we can apply the general expressions above to compute instanton amplitudes in terms of spectral curve data, and we can indeed verify that the

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large order of the F_g is governed by (4.6). In order to apply formulae (4.3), (4.4) we have to find the location of the saddle x_0 in (4.2). For the cases $p = 3$ and $p = 4$ the relevant solutions have been determined in [24]; they are given by

$$x_0 = \frac{4x_1 x_2}{(\sqrt{x_1} - \sqrt{x_2})^2}, \qquad p = 3, \tag{4.8}$$

and

$$x_0 = \frac{2\sqrt{x_1} x_2}{\sqrt{x_1} - \sqrt{x_2}}, \qquad p = 4. \tag{4.9}$$

An explicit expression for the instanton action was also computed in [24]; it is given by the expression

$$A(Q) = F(x_0) - F(x_1), \tag{4.10}$$

where

$$
\begin{aligned}
F(x) = & -\log\left(f_1(x)\right)\left(\log\left(f_1(x)\right) - 2\log\left(1 + \frac{2f_1(x)}{(\sqrt{x_1} - \sqrt{x_2})^2}\right) + \log\left(1 + \frac{2f_1(x)}{(\sqrt{x_1} + \sqrt{x_2})^2}\right)\right) \\
& - 2\mathrm{Li}_2\left(-\frac{2f_1(x)}{(\sqrt{x_1} - \sqrt{x_2})^2}\right) - 2\mathrm{Li}_2\left(-\frac{2f_1(x)}{(\sqrt{x_1} + \sqrt{x_2})^2}\right) - \log\frac{(x_1 - x_2)^2}{4}\log x \\
& - p\log\left(f_2(x)\right)\left(\log\left(f_2(x)\right) + 2\log\left(1 - \frac{f_2(x)}{2\sqrt{x_1 x_2}}\right) - \log\left(1 - \frac{2f_2(x)}{(\sqrt{x_1} + \sqrt{x_2})^2}\right)\right) \\
& - 2p\mathrm{Li}_2\left(-\frac{f_2(x)}{2\sqrt{x_1 x_2}}\right) + 2p\mathrm{Li}_2\left(\frac{2f_2(x)}{(\sqrt{x_1} + \sqrt{x_2})^2}\right) + \frac{p}{2}(\log x)^2 + p\log(\sqrt{x_1} + \sqrt{x_2})^2\log x,
\end{aligned}
\tag{4.11}
$$

and

$$
\begin{aligned}
f_1(x) &= \sqrt{(x - x_1)(x - x_2)} + x - \frac{x_1 + x_2}{2}, \\
f_2(x) &= \sqrt{(x - x_1)(x - x_2)} + x + \sqrt{x_1 x_2}.
\end{aligned}
\tag{4.12}
$$

We can now numerically compare the behavior of the sequence F_g with the instanton prediction (4.6). Since we have only computed ten terms in the sequence F_g, we need standard acceleration methods to extract the asymptotics with some precision. For example, assuming (4.6) holds, we can extract numerically the value of A and compare to the instanton prediction. In order to extract A, we consider the sequence

$$S_g = 2g\sqrt{\frac{F_g}{F_{g+1}}} = A + \mathcal{O}(1/g) \tag{4.13}$$

for a fixed value of ζ, and its Richardson transforms $S_g^{(N)}$ for $N = 1, 2, \cdots$, which help to eliminate the subleading tail $\mathcal{O}(1/g)$. The resulting sequences $S_g^{(N)}(\zeta)$ should converge to $A(\zeta)$, therefore they define numerical approximations to $A(\zeta)$. In Fig. 1 we show, for $p = 3$, the sequence S_g at $\zeta = 0.24$ and its Richardson transforms for $N = 1, 2, 3$. The instanton prediction for A is the straight line. As we can see, the agreement between the numerical extrapolation and the instanton prediction is remarkable.

Acknowledgments

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