Geometry of tropical curves and application to real algebraic geometry

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Introduction

Dans cette thèse, on s'intéresse aux questions d'approximabilité des courbes tropicales dans les espaces affines. Originellement, ces questions ont mené à l'un des résultats les plus marquants de la récente théorie qu'est la géométrie tropicale: dans [Mik05], Mikhalkin montre que l'on peut résoudre des problèmes de géométrie énumérative dans le plan en adoptant un point de vue tropical. La version tropical de tels problèmes s'avère être bien plus abordable et tend à se réduire à des questions de combinatoire plus simples. La profondeur d'une telle approche réside dans l'existence d'une correspondance entre les objets classiques et les objets tropicaux que l'on considère. Classiquement, on s'intéresse à compter les courbes planaires de degré fixé et soumises à des contraintes bien choisies; on entend par là que l'on devrait compter un nombre fini d'objets. La même question possède une contrepartie tropicale qui admet une réponse combinatoire. Il faut alors se demander en quoi résoudre le problème tropicalement nous aide à résoudre notre problème de départ. Le théorème de correspondance de Mikhalkin nous dit que les solutions du problème tropical correspondent à un certain nombre de solution du problème d'origine, et donc qu'un compte tropical avec les multiplicités appropriées nous mène au résultat escompté. Pour autant, nous n'avons pas expliqué ce qu'était cette correspondance : il s'agit de la possiblité d'approximer les courbes tropical par des familles de courbes algébriques classiques, modulo une dégénérescence de la structure complexe ambiante.
Suivant les idées développées par Krichever dans [Kri], on abordera la question d'approximabilité avec une perspective plus large : au lieu de nous restreindre aux applications algébriques, nous considérerons plutôt des applications harmoniques, nous permettant ainsi une plus grande flexibilité. À étendre notre point de vue, on peut espérer pouvoir approximer une plus grande variété d'objets : nous les appellerons courbes tropicales harmoniques, ainsi que les morphismes harmoniques associés. Dans la première partie de ce travail, nous montrerons que tout morphisme tropical harmonique est approximable par une famille d'applications harmoniques sur des surfaces de Riemann. Il nous faudra passer par un théorème de convergence sur les familles de différentielles imaginaires normalisées sur les surfaces de Riemann, dans le style de [GK10]. À l'aide de cette machinerie, on donnera une démonstration alternative du théorème d'approximation des courbes tropicales complexe dans le plan, originellement du à Mikhalkin.

Un autre de nos intérêts pour les questions d'approximabilité se rattache à la topologie des courbes algébriques réelles, comme suggéré par le seizième problème de Hilbert. Cela nous ramène à la fin des années soixante-dix, lorsque Viro inventa les techniques dites aujourd'hui de patchworking, *cf* [Vir08] par exemple. On ne pourrait donner ici une liste exhaustive des applications de ces techniques comme [Vir84], [IV96], [Mik00] ou encore [ABD14]. Le but de telles méthodes est de construire des courbes algébriques réelles planaires, ou plus généralement dans n'importe quelle surface torique, tout en controllant leur topologie. Originellement, ces techniques servaient à produire des courbes lisses, mais on put produire après quelques améliorations certaines courbes singulières, *cf* [MMS12] et [Shu12]. Concernant la description des courbes singulières, il y à deux manières de s'y prendre : soit en en donnant une équation, soit en en donnant une paramétrisation. À noter que la seconde a l'avantage d'être stable : en effet, la déformation d'une paramétrisation ne fait pas disparaître les singularités, contrairement à la déformation générique d'une équation. C'est en partie pour cela que le théorème d'approximation de Mikhalkin est très pratique pour construire des courbes singulières avec une topologie métrisée. Dans la seconde partie de cette thèse, nous utiliserons ces techniques pour construire et classifier une classe très particulière de courbes. Les courbes de Harnack simples ont été introduites par Mikhalkin dans [Mik00]. Dès lors, on leur a trouvé une multitude de définitions equivalentes et très intéressantes (voir [MR01], [MO07] et [PR04]) et on leur a trouvé quelques étonnantes interprétations physiques (voir [KOS06] et [CD13]). Ici, nous donnerons une généralisation naturelle de ces courbes. En particulier, nous permettons à ces courbes d'acquérir des singularités de tous types, contrairement aux courbes originelles, *cf* [MR01]. Nous montrerons que ces courbes admettent une contrepartie tropicale et que leur approximation donne lieu à toute une faune de nouvelles courbes de Harnack. Enfin, nous donnerons la classifiaction topologique des courbes de Harnack possédant un unique noeud hyperbolique. Il s'agit du cas le plus simple n'ayant pas été traité jusqu'à là. Nous décrirons complètement leur classification topologique et établirons une relation forte entre ces courbes et leur avatars tropicaux, dans le style de [KO06]. Nous pensons que ces courbes ont encore beaucoup de secrets à nous livrer. Nous appuyerons nos spéculations dans la toute dernière section de ce travail.

Cette thèse se divise en deux parties. Elles peuvent être lues indépendamment, bien qu'elles soient très connectées dans leur contenu.

Introduction

In this work, we investigate a generalization of the question of approximability of tropical curves in affine spaces. Originally, this question led to one of the most striking result in the early days of tropical geometry: in [Mik05], Mikhalkin showed that one can give tropical solutions to classical problems in enumerative geometry in the plane. The tropical version of such enumerative problems turns out to be more tractable as the whole question reduces to simple combinatorics. The deepness of such an approach lies in the existence of a correspondence between the classical and tropical objects involved in the picture. Classically, one is interested in counting algebraic curves in the plane with a fixed degree and submitted to well chosen point constraints, meaning that the number of solution should be finite. The very same question can be considered tropically and solved by combinatorial means. Now comes the question of how much does it help to solve the original problem. Mikhalkin's correspondence theorem states any tropical solution corresponds to a certain number of classical solutions and that counting the number of tropical solutions with multiplicities leads to the classical count. Still, we did not explain how this correpondence arises: it comes from the ability to approximate tropical curves by a family of classical ones while degenerating the ambient complex structure.
Following the ideas introduced by Krichever in [Kri], we approach the question of approximability from a wider point of view by dealing not only with algebraic maps but also with harmonic ones. This extension allows for a lot more flexibility. As a consequence, one approximates a wider class of objects that will be called harmonic tropical curves and associated harmonic morphisms. In the first part of this work, we prove that basically any tropical harmonic morphism is approximable by families of harmonic maps on Riemann surfaces. This goes by the way of a convergence theorem on sequence of imaginary normalized differentials on Riemann surfaces, in the spirit of [GK10]. Using this framework, we give an alternative proof of Mikhalkin's approximation theorem for simple complex tropical curves in the plane.

Another motivation of the question of approximability that will be of interest to us is related to the topology of real algebraic curves, as suggested by Hilbert's sixteenth problem. It brings us back to the late seventies when Viro invented the now called patch-working techniques, see [Vir08] for instance.

One could give but a partial list of the applications of such techniques, as [Vir84], [IV96], [Mik00] or [ABD14]. The aim of such techniques is to produce real algebraic curves in the plane, or in any toric surface, with prescribed topology. Originally designed to produce smooth curves, some enhancement of Viro's patch-working allows to construct singular ones, see [MMS12] and [Shu12]. Regarding construction of singular curves, one can proceed in 2 ways : either by using equations or by using parametrizations. The latter has the advantage of stability, namely that singularities of a curve do not disappear by perturbation of a parametrization, unlike equations. That is why Mikhalkin's approximation theorem is very usefull and tractable for the construction of singular curves with prescribed topology. In the second part of this work, we use these techniques to construct and classify a class of very particular planar curves. Simple Harnack curves have been introduced by Mikhalkin in [Mik00]. Since then, it has been shown they possess many equivalent and interesting definitions (see [MR01], [MO07] and [PR04]) and that they have surprising physical interpretations (see [KOS06] and [CD13]). Here, we provide a natural extension of this class of curves. In particular, we allow such curves to possess singularities of any kind, unlike the classical definition, see [MR01]. We show that these curves have tropical counterparts and that their approximation produces a whole zoo of new Harnack curves. Finally we undertake the topological classification of Harnack curves with a single hyperbolic node. This is the simplest instance that has yet to be considered. We give a complete description of them and show a strong connection with their tropical counterparts, in the fashion of [KO06]. We believe these curves have yet to reveal all their secrets. We motivate our speculation in the very last section of this work.

The content of this thesis is two-folded. Both part can be read quite independently, however they are strongly connected in spirit.

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Contents

Part I

On convergence of imaginary normalized differentials, harmonic tropical curves and their approximation

Introduction In his paper [Kri], I.Krichever introduced a generalization of the notion of amoebas due to Gelfand, Kapranov and Zelevinsky [GKZ08]. Imaginary normalized differential are meromorphic differentials on Riemann surface having simple poles and purely imaginary periods. It is a classical fact due to Riemann that imaginary normalized differentials are determined by their collection of residues at the poles. Integrating the real part of such differentials provides real valued harmonic functions. Any n-tuple of those functions gives a map very similar to the classical amoeba map for algebraic varieties immersed in any complex torus. In this work, we suggest the terminology of *harmonic amoebas* for the image of such maps. Krichever shows that harmonic amoebas are very similar to classical ones: in the planar case, they have thin tentacles going off to infinity and define convex connected components in their complement. They also possess a logarithmic gauss map, a Ronkin function, gradient of which takes values in a convex polygon, as in [PR04]. They have finite area bounded in term of the area of their "Newton" polygon. The bound provided is sharp and characterizes a generalization of simple Harnack curves as studied in [Mik00] and [MR01].

One distinction from the algebraic picture introduced by [GKZ08] is that there is no restriction on the Riemann surfaces under consideration. For example, admitting an embedding in a toric surface imposes a constraint on the moduli spaces of curves, see [CV09] for example. A second distinction is non integrality: the Legendre transform of the Ronkin function defines a piecewise linear function. In turn, the corner locus of this function defines the *harmonic spine* of the harmonic amoebas. The facets of the Legendre trans-

form, and thus the spine of a harmonic amoeba are no longer constrained to have integral slopes. Consideration of such object is our motivation to introduce here the notion of *harmonic tropical curves*.
To be more precise, we introduce the notion of harmonic morphism from abstract tropical curves to affine spaces. The definition is a mimic of the one given in [Mik05], except that one allows irrational slopes. Let us mention here that we won't give a general definition of what an harmonic tropical curve is. In the usual tropical setting, an immersed (or parametrized) tropical curve is more than its underlying point set. It is generally equipped with weights that can be defined in a very natural way, thanks to integrality. Such data gives extra information about the possible parametrization of such objects. In the harmonic case, things are more complicated and the data one should add to the underlying point set is more cumbersome. Therefore, we give the precise definition only in the case of *simple harmonic tropical curves*, yet still focus on morphisms in great generality. .
In this part, we show that tropical harmonic morphism can be approximated by harmonic amoeba map from Riemann surfaces to affine spaces. This goes by the study of convergence of imaginary normalized differentials (i.n.d. for short) along well chosen families of Riemann surfaces. Such approximability implies that harmonic tropical curves of any type can be obtained as limit of families of harmonic amoebas. With a bit of extra work, it provides an alternative proof of a theorem due to Mikhalkin: the approximability of simple complex tropical curves in the plane.
Let us now describe more precisely the content of the first part of this thesis, section by section.

1. We introduce the necessary material. We recall the main facts of the paper [Kri] which is the starting point of our study. Then, we introduce the necessary settings in order to define convergence of differentials. We also reviews some classical facts about hyperbolic geometry in the plane with a view toward uniformization of Riemann surfaces, their decomposition into pairs of pants and the introduction of Fenchel- Nielsen coordinates. We end up with the usual definitions surrounding tropical curves.

2. We define convergence of a sequence of Riemann surfaces toward an abstract tropical curve in terms of Fenchel-Nielsen coordinates. In spirit, this definition first exhibit the limiting curve, the maximally degenerate stable curve dual to the abstract tropical curve, when simply considered as a graph. Secondly, it specify how fast vanishing cycles are contracted in time. In a way, it describes an abstract tropical curve as a linear subspace of the tangent space of the corresponding stable curve. Now fixing a collection of residues specifies both a sequence of i.n.d. defined respectively on the sequence of Riemann surfaces, and and exact (tropical) 1-form on the limiting tropical curve. We prove

> **Theorem 1.** *" The sequence of i.n.d. converges to an i.n.d. on the limiting stable curve. The differential at the limit is totally determined by the exact 1-form on the limiting tropical curve."*

We then refine the definition of convergence to abstract complex tropical curves: one demands convergence of the twists in the Fenchel-Nielsen coordinates. In a way, it describes abstract complex tropical curves as tangent direction in the tangent space of the corresponding stable curve. We obtain

> **Theorem 2.** *" The sequence of period matrices associated to the sequence of i.n.d. converges. The limiting period matrix is totally determined by the limiting complex tropical curve and the exact 1-form on it."*

3. Fix a sequence of Riemann surface converging to an abstract tropical curve. As before, choosing a m-tuples of collection of residues defines a sequence of m i.n.d.'s on the sequence of Riemann surface, and m exact 1-forms on the tropical curve. Integration of these forms corresponds to an harmonic morphism on the tropical curve, giving rise to an immersed harmonic tropical curve. Integration of the i.n.d.'s gives a sequence of harmonic amoebas. Using theorem 1, one has

> **Theorem 3.** *" The sequence of harmonic amoebas converges in Hausdorff distance to the harmonic tropical curve."*

4. In the last section, we undertake the question of integrality in the planar case. It was a remark in [Kri] that exponentials of harmonic maps given by i.n.d.'s admits a well define harmonic conjugate if and only if the periods of the differentials are integer multiples of $2\pi i$. In such case it gives rise to an holomorphic function. Then, any m-tuples of such differentials defines a classical amoeba in the affine m-space. Theorem 2 implies that the limiting period matrix is integer when the sequence of Riemann surfaces converges to the normalization of a simple complex tropical curve in the plane. The fact that the period matrix is integral at the limit does not guarantee intregality close to the limit. Then, in order to prove Mikhalkin's approximation theorem, see theorem 4.9, one look for sequences of Riemann surfaces included in some leaf of the moduli space. Such leaves are those on which the specified i.n.d.'s have the desired integer periods. We show that these leaves have the required transversality property at the limit and give an alternative proof of Mikhalkin's theorem.

1 Prerequisites

1.1 Imaginary normalized differentials and harmonic amoebas

In this section, one briefly overviews the facts of [Kri] of main interest for us. Unless specified otherwise, the proofs of the statements to follow can be found there.

Definition 1.1. *Let n and g be natural numbers such that $2g-2+n>0$ with $n\geq 2$, and $S\in\mathcal{M}_{g,n}$ a Riemann surface. An imaginary normalized differential ω on S is an holomorphic differential on S having simple poles at the n punctures of S and such that*

$$Re\left(\int_\gamma \omega\right)=0$$

for any $\gamma\in H_1(S,\mathbb{Z})$.

Theorem 1.2. *Let n and g be natural numbers such that $2g-2+n>0$ with $n\geq 2$, and $S\in\mathcal{M}_{g,n}$ a Riemann surface. Denote by $p_1,...,p_n$ the n punctures of S. For any collection of real number $r_1,...,r_n$ such that $\sum_j r_j=0$, there exists a unique imaginary normalized differential ω on S such that*

$$Res_{p_j}\omega=r_j$$

for any $1\leq j\leq n$.

Proof See [Lan82].

□

Definition 1.3. *A collection of residues R is the data $R := \left\{(r_1^{(j)}, ..., r_n^{(j)})\right\}_{1\leq j\leq m}$ for some natural numbers n and m and real numbers $r_k^{(j)}$ such that for any $1 \leq j \leq m$*

$$\sum_{1\leq k\leq n} r_k^{(j)} = 0.$$

The number m is called the dimension of R. According to the previous theorem, a collection of residues R defines a collection of imaginary normalized differentials $\omega_{R,S} := (\omega_{R,S}^{(1)}, ..., \omega_{R,S}^{(m)})$ on any curve $S \in \mathcal{M}_{g,n}$ defined by

$$Res_{p_k}\omega_{R,S}^{(j)} = r_k^{(j)}$$

for any $1 \leq k \leq n$ and $1 \leq j \leq m$, where p_k is the k-th puncture of S. Given a collection of residues R, a curve $S \in \mathcal{M}_{g,n}$ and an initial point $z_0 \in S$, one defines the map

$$\begin{array}{rcl} \mathcal{A}_R \,:\, S & \to & \mathbb{R}^m \\ z & \mapsto & \left(Re(\int_{z_0}^{z}\omega_{R,S}^{(1)}), ..., Re(\int_{z_0}^{z}\omega_{R,S}^{(m)})\right) \end{array}.$$

One introduces here the following terminology

Definition 1.4. *Let $S \in \mathcal{M}_{g,n}$ and R be a collection of residues of dimension m. The set $\mathcal{A}_R(S) \subset \mathbb{R}^m$ is the harmonic amoeba of S with respect to R.*

Those objects have been introduced by Krichever in [Kri]. Recall that classical amoebas as introduced in [GKZ08] are defined as images of algebraic subvarieties $V \subset (\mathbb{C}^*)^m$ by the map

$$\begin{array}{rcl} \mathcal{A} \,:\, (\mathbb{C}^*)^m & \to & \mathbb{R}^m \\ (z_1, ..., z_k) & \mapsto & (\log|z_1|, ..., \log|z_k|) \end{array}.$$

Suppose for simplicity that V is smooth. The embedding of V in $(\mathbb{C}^*)^m$ is given by the m coordinates functions z_j's. The map $\mathcal{A}_{|V}$ is given coordinate wise by integrating the real part of the imaginary normalized differentials $d\log z_j$. Note that any period of such differential is an integer multiple of $2i\pi$, it applies in particular for its residues at the punctures.
Here, one restricts to the case of amoebas of Riemann surface S. The major difference with harmonic amoebas is that S can be taken arbitrarily. In

particular, S need not to be immersed in the torus $(\mathbb{C}^*)^m$. Moreover, the collection of residues R can take arbitrary values at a puncture.
The terminology is motivated by the fact that any coordinate function of the map $\mathcal{A}_R$ is harmonic on the punctured Riemann surface, and that the definition of harmonic amoebas generalizes the one of classical amoebas in the case of Riemann surfaces.
The first analogy with classical amoebas is given by the following

Proposition 1.5. *Let $S \in \mathcal{M}_{g,n}$ and R be a collection of residues of dimension 2. Then, all the connected components of the complement of $\mathcal{A}_R(S) \subset \mathbb{R}^2$ are convex, and the unbounded components are separated by tentacle-like asymptotes of $\mathcal{A}_R(S)$.*

The similarity goes far beyond. To such harmonic amoebas, once can associate a logarithmic Gauss map, a Ronkin function and extend many classical properties of those objects. One again, one refers to [Kri]. Even though it hasn't been defined there, the latter reference provide the material to generalize the notion of spine over verbatim, as it was introduced in [PR04]. Such consideration leads to introducing a more general class of immersed tropical curve with non rational slopes. They will be introduced further in this text as harmonic tropical curves.

Proposition 1.6. *Let $S \in \mathcal{M}_{g,n}$ and R be a collection of residues of dimension 2. There exists a constant M_R independent of S, such that*

$$\mathcal{A}rea\left(\mathcal{A}_R(S)\right) \leq M_R$$

where $\mathcal{A}rea$ is the euclidean area in $\mathbb{R}^2$.

Let $S \in \mathcal{M}_{g,n}$ and R be a collection of residues of dimension m. Fixing a basis $\gamma_1, ..., \gamma_{2g+n-1}$ of $H_1(S, \mathbb{Z})$, one can define the period matrix $\mathcal{P}_{R,S} \in M_{(2g+n-1)\times m}(\mathbb{R})$ by

$$\left(\mathcal{P}_{R,S}\right)_{l,k} := \frac{1}{2i\pi} \int_{\gamma_l} \omega_{R,S}^{(k)}.$$

A change of basis of $H_1(S, \mathbb{Z})$ is given by a matrix $A \in Sl_{2g+n-1}(\mathbb{Z})$. The period matrix relative to the new basis is given by $A \cdot \mathcal{P}_{R,S}$.

Definition 1.7. *For $S \in \mathcal{M}_{g,n}$ and R be a collection of residues of dimension m, the period matrix of S with respect to R is the equivalence class of $\mathcal{P}_{R,S} \in$*

$M_{(2g+n-1)\times m}(\mathbb{R})$ by the left action of $Sl_{2g+n-1}(\mathbb{Z})$. One still denote this class by $\mathcal{P}_{R,S}$.
$\mathcal{P}_{R,S}$ is an integer period matrix if one of its representative (and then all of them) has all its entries in $\mathbb{Z}$. In such case, one can define the holomorphic map

$$\begin{array}{rcll} \iota_R : S & \to & (\mathbb{C}^*)^m \\ z & \mapsto & \left(e^{\int_{z_0}^z \omega_{R,S}^{(1)}}, ..., e^{\int_{z_0}^z \omega_{R,S}^{(m)}}\right) \end{array}.$$

1.2 Relative differentials and the Hodge bundle

One of he main ingredient of the results to follow is the study of sequences of imaginary normalized differentials on Riemann surfaces. In the present case, we want to study the convergence of a family $\{S_t, \omega_t\}_{t\in\mathbb{N}}$, where ω_t is an imaginary normalized differential on a Riemann surface S_t, such that S_t converges to maximally degenerated stable curve of the compact moduli space $\overline{\mathcal{M}}_{g,n}$. One has to precise what we mean by convergence, and in which space.
Recall that the classical Hodge bundle $\Lambda_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is the rank g vector bundle whose fiber at a point $(S; p_1, ..., p_n)$ is the space of holomorphic sections of the dualizing sheaf over S, see [ELSV01]. Geometrically, the fiber of $\Lambda_{g,n}$ over a smooth curve $(S; p_1, ..., p_n) \in \mathcal{M}_{g,n}$ is the vector space of holomorphic differentials. Now for a singular curve $(S; p_1, ..., p_n) \in \overline{\mathcal{M}}_{g,n}$ with simple node q_1, ..., q_k, the fiber of $\Lambda_{g,n}$ over S is the vector space of meromorphic differentials such that their pullback on the normalization $\tilde{S}$ have at most simple poles at the preimages of the nodes of S, and such that their residues at the 2 preimages of any of the q_j's are opposite to each other. Note that this vector space is also g-dimensional. The fact that $\Lambda_{g,n}$ is a vector bundle on $\overline{\mathcal{M}}_{g,n}$ can be found in [ACG11], and references therein. Now, consider the following twisted version of the latter bundle.

Definition 1.8. *One defines $\Lambda_{g,n}^m \to \overline{\mathcal{M}}_{g,n}$ to be the vector bundle of rank $g+n-1$ obtained by tensoring $\Lambda_{g,n}$ by $\mathcal{O}_S(p_1 + ... + p_n)$. An point of $\Lambda_{g,n}^m$ lying over a curve S will be called a generalized meromorphic differential on S.*

The fibers of $\Lambda_{g,n}^m$ are easily described in terms of the one of the Hodge bundle: we simply allowed extra simple poles at the punctures. Then, a sequence $\{(S_t, \omega_t)\}_{t\in\mathbb{N}}$ is just a sequence of points in the total space of $\Lambda_{g,n}^m$. We naturally define

Definition 1.9. *A sequence* $\left\{\left(S_t, \omega_t\right)\right\}_{t\in\mathbb{N}}$ *convergence if it converges point-wise in the total space of the bundle* $\Lambda^m_{g,n}$

Of course, it would be very practical to have a set of coordinates, at least locally, to determine such point wise convergence. Convergence on the base space will be described in term of Fenchel-Nielsen coordinates, see next subsection. We should only care about a coherent coordinate system on the fibers over the sequence $\{S_t\}_{t\in\mathbb{N}}$. Assume for example that there is a simply connected open subset $\mathscr{U} \subset \mathcal{M}_{g,n}$ such that $\{S_t\}_{t>t_0} \subset \mathscr{U}$, for some large t_0. In such case, $H_1(S, \mathbb{Z})$ can be trivialized on $\mathscr{U}$. In the sequel, we will consider sequences converging to maximally degenerated stable curves. Irreducible components of such curves are Riemann spheres with 3 marked points standing for either nodes or marked points of the curve. There are exactly $3g-3$ nodes. Hence, the sequence $\{S_t\}_{t>t_0}$ specifies $3g-3$ vanishing cycles in $H_1(S, \mathbb{Z})$, cycles against which one can integrate the differentials ω_t, for any time $t > t_0$. Now, choose g of these cycles and $(n-1)$ small loops around all the punctures except one such that they are linearly independant in $H_1(S, \mathbb{Z})$. Denote them by $\gamma_1, ..., \gamma_{g+n-1}$, then computing the periods along those cycles gives the map

$$\begin{array}{rcl} \Lambda^m_{g,n}\big(\mathscr{U}\big) & \to & \mathscr{U} \times \mathbb{C}^{g+n-1} \\ (S, \omega) & \mapsto & \Big(S, \big(\int_{\gamma_1} \omega, ..., \int_{\gamma_{g+n-1}} \omega\big)\Big) \end{array} .$$

This provides a system of coordinates on the fibers of the twisted Hodge bundle $\Lambda^m_{g,n}$ above $\mathscr{U}$. In particular, we have the following lemma

Lemma 1.10. *A sequence* $\left\{\left(S_t, \omega_t\right)\right\}_{t>t_0}$ *converges if and only if the sequence* $\{S_t\}_{t>t_0}$ *converges in* $\overline{\mathcal{M}}_{g,n}$ *and if the period vectors* $\big(\int_{\gamma_1} \omega_t, ..., \int_{\gamma_{g+n-1}} \omega_t\big)$ *converges in* $\mathbb{C}^{g+n-1}$.

We conclude this section by recalling the notion of relative differential in a particular case of interest for us. In order to determine global convergence of differentials, one need to study their local behaviour on the Riemann surface around the nodes to be. Generically, simple nodes appear by shrinking cycles in cylinders. A canonical model can be chosen as

$$\begin{array}{rcl} \mathcal{F} \,:\, D(1,1) & \to & D(1) \\ (x, y) & \mapsto & xy \end{array}$$

where $D(1,1) := \big\{(x,y) \in \mathbb{C}^2 \,\big|\, |x| < 1,\ |y| < 1\big\}$ and $D(1) := \big\{\lambda \in \mathbb{C} \,\big|\, |\lambda| < 1\big\}$. The fibers of $\mathcal{F}$ over a non zero λ are holomorphic cylinders and the fiber over 0 is the union of the 2 unit discs in $\mathbb{C}^2$. We will denote them by

$$C_\lambda := \mathcal{F}^{-1}(\lambda).$$

Consider the relative dualizing sheaf $\omega_\mathcal{F}$ of the morphism $\mathcal{F}$, see [ACG11]. As before, sections of such sheaf are just holomorphic differentials on C_λ for $\lambda \neq 0$. The fiber over C_0 is the space of meromorphic differentials having simple poles with opposite residues at the origin of both irreducible component of C_0. Roughly speaking, global sections are given by differential on the total space $D(1,1)$ mod out by differentials coming from functions on the base. The latter are exactly those that are identically zero on every fiber C_λ. For example

$$d\lambda = ydx + xdy$$

implies that $\frac{dx}{x}$ is equivalent to $-\frac{dy}{y}$ as a section of $\omega_\mathcal{F}$. They correspond to the same differential while restricted to any fiber C_λ. In particular, the induced differential on C_0 is given by $\frac{dx}{x}$ on the x-unit disc and $-\frac{dy}{y}$ on the y-unit disc.

Let us now define what we will mean by the convergence of a sequence of holomorphic differentials in this local model. It is a classical fact that the moduli space of holomorphic cylinders is described by the family $\mathcal{C}_\lambda$ for $0 < \lambda < 1$. In the sequel, we will always restrict to this case.

Definition 1.11. *Let $\{\lambda_t\}_{t\in\mathbb{N}} \subset\ ]0,1[$ be a sequence converging to zero and $\{\omega_t\}_{t\in\mathbb{N}}$ be a sequence of holomorphic differentials on $\mathcal{C}_{\lambda_t}$. The sequence $\{\omega_t\}_{t\in\mathbb{N}}$ converges if it converges point wise in in the total space $H_0\big(D(1), \omega_\mathcal{F}\big)$*

Denote by γ_λ the oriented loop in C_λ given by

$$\gamma_\lambda(\theta) = \big(\sqrt{\lambda}e^{i\theta}, \sqrt{\lambda}e^{-i\theta}\big) \text{ for } 0 \leq \theta \leq 2\pi.$$

The latter definition implies that the sequence of periods $\Big\{\int_{\lambda_t} \omega_t\Big\}_{t\in\mathbb{N}}$ on the family of holomorphic cylinders does converges. Unlike before, convergence of the sequence of periods does not imply point wise convergence. Indeed, adding a global section that is holomorphic everywhere does no change the periods.

1.3 A bit of hyperbolic geometry

In this subsection, one recalls some basic facts about geometry of hyperbolic surfaces, uniformization of punctured Riemann surfaces and Fenchel-Nielsen coordinates on moduli spaces of curves. One refers to [Bus10] for the proofs of the statements given here. Another reference is [ACG11].

Definition 1.12. *A pair-of-pants Y ("pop" for short) is a Riemannian surface such that*

- *Y is homeomorphic to $\mathbb{CP}^1 \setminus \big(E_1 \cup E_2 \cup E_3\big)$ where each of the E_i's is either a point or an open disc, the E_i's being pairwise disjoint,*
- *the boundary components of Y are geodesics,*
- *Y is a complete metric space of constant curvature -1.*

Recall that the upper half-space $\mathbb{H} := \big\{z \in \mathbb{C} \,\big|\, Im(z) > 0\big\}$ equipped with the Riemannian metric

$$g := \frac{(dx^2 + dy^2)}{y^2}$$

where $z = x + iy$, is a complete metric space with constant curvature -1.

Theorem 1.13. *For any $\alpha, \beta, \gamma > 0$, there exists an hyperbolic right-angled hexagon $H \subset \mathbb{H}$ with consecutive side lengths $a, \alpha, b, \beta, c, \gamma$. The lengths a, b, c are determined by α, β, γ, and H is unique up to isometry.*

Definition 1.14. *A generalized hyperbolic right-angled hexagon $H \subset \mathbb{H}$ is the limit for the Hausdorff distance on compact sets of a sequence of hyperbolic right-angled hexagons $\{H_t\}_{t>>1}$ of respective lengths $\alpha_t, \beta_t, \gamma_t$ converging in $\mathbb{R}_{\geq 0}$.*

From now on, a boundary components of a "pop" Y will be meant to be a boundary geodesic as well as a puncture of Y.

Theorem 1.15. *For any "pop" Y, there exists a unique orientation reversing isometry σ fixing the boundary components. The quotient Y/σ is isometric to a generalized hyperbolic right-angled hexagon $H \subset \mathbb{H}$.*

A cyclical order on the 3 boundary components of a "pop" Y is equivalent to an orientation on Y/σ. Hence, it specifies an initial vertex on the sides of Y/σ corresponding to boundary geodesics of Y. In other words, it specifies one of the 2 fixed points of σ on every boundary geodesic. Moreover, every boundary geodesic is naturally oriented. Therefore, such cyclical order provides a natural isomorphism from any boundary geodesic of Y to

$$S^1 := \big\{ z \in \mathbb{C} \,\big|\, |z| = 1 \big\}$$

equipped with the canonical complex orientation. Notice that the fixed locus of σ is made of 3 geodesics. Each such geodesic joins 2 of the boundary components, and meets them orthogonally. Note that orthogonality make sense even in the case of a puncture if one consider the real oriented blow up of the "pop" at the puncture.

Definition 1.16. *For any "pop" Y, we define $\mathbb{R}Y$ to be the fixed locus of the isometry σ of the previous theorem.*

Definition 1.17. *Let Y be a "pop". For any boundary geodesic γ of Y, the half-collar associated to γ is the tubular neighbourhood*

$$K_\gamma^{1/2} := \big\{ z \in Y \,\big|\, d(z,\gamma) \leq w\big(l(\gamma)\big) \big\}$$

where $w\big(l(\gamma)\big) := arcsinh\left(1/\sinh\left(\frac{1}{2}l(\gamma)\right)\right)$.
For any puncture p of Y, the cusp K_p associated to p is the neighbourhood of p in Y isometric to the space $]-\infty, \log(2)] \times \mathbb{R}/\mathbb{Z}$ of coordinates (ρ, t) with the metric $d\rho^2 + e^{2\rho}dt^2$.

Theorem 1.18. *Let Y be a "pop". The half-collars and cusps of Y are pairwise disjoint.*

Definition 1.19. *Let Y be a "pop". Define Y^{bd} to be the closure in Y of the complement of all the cusps and half-collars of Y.*

Now, let Y_1 and Y_2 be 2 "pop", with cyclical order of their boundary components, and let $\gamma_i \subset Y_i$ ($i = 1, 2$) be boundary geodesics of the same length. Gluing Y_1 to Y_2 by an orientation reversing isometry

$$\begin{array}{ccc} \gamma_1 \simeq S^1 & \to & \gamma_2 \simeq S^1 \\ z & \mapsto & -\overline{e^{i\theta} z} \end{array}$$

one obtains a complete hyperbolic surface with geodesic boundaries.

This is a building block for the following construction.

Definition 1.20. *A cubic graph G is a graph with only 3-valent and 1-valent vertices. The edges adjacent to a 1-valent vertex are called leaves. A ribbon structure $\mathscr{R}$ on G is a the data for every vertex of G of a cyclical ordering of its adjacent leaves-edges.*

Consider a cubic graph G equipped with a ribbon structure. Denote by g the genus of G and by n its number of leaves. A quick Euler characteristic computation shows that G has $(3g-3+n)$ edges and $(2g-2+n)$ 3-valent vertices. Assume the latter quantity is strictly positive. Now consider 2 functions

$$\begin{array}{rcl} l : E(G) & \to & \mathbb{R}_{>0} \\ \Theta : E(G) & \to & S^1 \end{array}.$$

From this data, one can construct a Riemann surface of genus g with n punctures as follows: for each $v \in V(G)$, consider a "pop" Y_v together with a bijection between the set of boundary components of Y_v and the set of leaf-edge $e \in LE(G)$ adjacent to v such that

* punctures of Y_v are in bijection with leaves adjacent to v,
* boundary geodesics are in bijection with edges adjacent to v and their length is given by the map l.

The ribbon structure on G is equivalent to a cyclical order on the boundary components of every Y_v, via these bijections. According to this cyclical ordering, let us fix the following framing on each boundary geodesic of Y_v: first orient each geodesic such that the normal vector field points inward Y_v. Define the origin of each boundary geodesic γ to be its intersection point with the unique connected component of $\mathbb{R}Y_v$ connecting γ to the previous boundary component of Y_v. Equipped with this framing, each boundary geodesic is a group, and there is a unique orientation preserving isomorphism between each geodesic and S^1.
For 2 nearby vertices $v_1, v_2 \in V(G)$ connected by an edge $e \in E(G)$, glue the corresponding geodesics in Y_{v_1} and Y_{v_1} by the isometry

$$\begin{array}{rcl} S^1 & \to & S^1 \\ z & \mapsto & -\overline{\Theta(e)z}. \end{array}$$

The element $\Theta(e)$ is classically referred as the twist parameter along the edge e. The result is a complete Riemannian surface $S(G, l, \Theta)$ of genus g with n punctures and constant curvature -1. This surface is locally modelled on $\mathbb{H}$ with holomorphic transition functions, hence it is a Riemann surface.

Definition 1.21. *Define* $\mathcal{FN}_G : (\mathbb{C}^*)^{3g-3+n} \to \mathcal{M}_{g,n}$ *to be the map that associates to any couple of functions* (l, Θ) *the Riemann surface* $S(G, l, \theta)$.

Theorem 1.22. *Let* G *be a cubic graph of genus* g*, with* n *leaves and equipped with a fixed ribbon structure. Assume* G *has at least one vertex, guaranteeing* $2g - 2 + n > 0$.
The map $\mathcal{FN}_G : (\mathbb{C}^*)^{3g-3+n} \to \mathcal{M}_{g,n}$ *is surjective.*

The 2 maps l and Θ are known as Fenchel-Nielsen coordinates for the curve S, relatively to G.

Remark. The latter map turns out to be an intermediate covering to the universal covering of $\mathcal{M}_{g,n}$ by the Teichmüller space $\mathcal{T}_{g,n} \simeq \mathbb{R}^{6g-6+2n}$. As one knows that the topology of $\mathcal{M}_{g,n}$ is not trivial, the latter covering is far from being injective.

Remark. Ribbon structures are used here as a technical tool and are not of primary interest for us. For this reason, our notation will never refer explicitly to the choice of such structure. Nevertheless, the reader should be aware that this underlying structure has to be fixed.
As an example, a Riemann surface with twists in $\{0, \pi\}$ can always be presented as a curve with only 0-twists by an appropriate change of the ribbon structure.

Definition 1.23. *For any Riemann surface* $S \in \mathcal{M}_{g,n}$*, and any geodesic* $\gamma \subset S$*, the collar associated to* γ *is the tubular neighbourhood*

$$K_\gamma := \left\{ z \in S \;\middle|\; d(z, \gamma) \leq w\big(l(\gamma)\big) \right\}.$$

1.4 Tropical curves

In this subsection, one introduces abstract (complex) tropical curves and tropical 1-form on them. As abstract objects, the definition of (complex) tropical curves follows the classical framework. There are several ways to broach these objects, and those known to the author can be found in [Mik05] and [Vir11]. However, the point of view adopted here on complex tropical curves is a bit different. The definitions are designed such that complex tropical curves arise naturally as limiting object of families of Riemann surfaces described in the Fenchel-Nielsen fashion. The reader could be surprised to

see that Riemann surfaces and complex tropical curves are encoded here by the very same combinatorial data. Nevertheless, they should be considered as different objects in spirit, as suggested by definitions 2.1 and 2.6.
Another point is that one restrict our attention to 3-valent tropical curves. They stands for smooth curves. There are several reason for that : first, it is a necessary restriction in order to fit perfectly to the hyperbolic framework given previously. Secondly, considering general tropical curves would obscure vainly both statements and proofs. Finally, all the results in the present paper can be generalized to every tropical curve by simple density arguments, as 3-valent tropical curves are dense in their respective moduli space, see [Mik05].

Definition 1.24. *An abstract tropical curve $\tilde{C}$ is a topological space homeomorphic to a cubic graph with all 1-valent vertices removed, and equipped with a complete inner metric.*

- *We denote by $V(\tilde{C})$ the set of vertices of $\tilde{C}$.*
- *The connected components of $\tilde{C} \setminus V(\tilde{C})$ isometric to $\big(]0;a[\,,\, d_{eucl}\big)$ for some $a > 0$ are called the edge of $\tilde{C}$ and form the set $E(\tilde{C})$, where d_{eucl} stands for the euclidean distance on $\mathbb{R}$.*
- *The connected components of $\tilde{C} \setminus V(\tilde{C})$ isometric to $\big(]-\infty;0[\,,\, d_{eucl}\big)$ are called the leaves of $\tilde{C}$ and form the set $L(\tilde{C})$.*
- *The union of $L(\tilde{C})$ and $E(\tilde{C})$ is denoted $LE(\tilde{C})$.*
- *The length of an edge $e \in E(\tilde{C})$ is denoted $l(e)$.*

In the sequel, we will replace the word "tropical" by the symbol $\mathbb{T}$ whenever it is not confusing. Similarly, we will replace the word "complex" (*resp.* "real") by the symbol $\mathbb{C}$ (*resp.* $\mathbb{R}$).

We are about to define exact 1-forms on abstract $\mathbb{T}$-curves. In order to do so, let us introduce some necessary definitions. Such 1-forms will be modelled on the following local construction:
let $\tilde{C}$ be an abstract $\mathbb{T}$-curve and $e \in LE(\tilde{C})$. A 1-form ω on e is a classical constant 1-form $a \cdot dx$ where $a \in \mathbb{R}$ and $x \,:\, e \to \mathbb{R}$ is an isometric coordinate. For any other isometric coordinate $y \,:\, e \to \mathbb{R}$, one has $\omega = \pm a\, dy$ depending

whether $x \circ y^{-1}$ preserves orientation or not.
Hence, a 1-form ω on e is equivalent to the data of an orientation $\vec{e}$ on e and a real number $\omega_{\vec{e}}$. If $-\vec{e}$ denotes the opposite orientation on e, then $\omega_{-\vec{e}} = -\omega_{\vec{e}}$.

Definition 1.25. *Let $\tilde{C}$ be an abstract $\mathbb{T}$-curve. A 1-form ω on $\tilde{C}$ is the data of a 1-form on every $e \in LE(\tilde{C})$ such that for any $v \in V(\tilde{C})$, and $\vec{e_1}, \vec{e_2}, \vec{e_3}$ its 3 adjacent elements in $LE(\tilde{C})$ oriented toward v, one has*

$$w_{\vec{e}_1} + w_{\vec{e}_2} + w_{\vec{e}_3} = 0.$$

The set of 1-forms on $\tilde{C}$ is denoted by $\Omega(\tilde{C})$. For any $e \in L(\tilde{C})$ and its orientation $\vec{e}$ toward its adjacent vertex, the number $w_{\vec{e}}$ is called the residue of ω at e. An element $\omega \in \Omega(\tilde{C})$ is holomorphic if all its residues are zero. The set of holomorphic 1-form on $\tilde{C}$ is denoted by $\mathcal{H}\Omega(\tilde{C})$.

Definition 1.26. *Let $\tilde{C}$ be an abstract $\mathbb{T}$-curve. A path c in $\tilde{C}$ is an injective map $c : [\![1, m]\!] \to LE(\tilde{C})$, for some $m \in \mathbb{N}$, such that each $c(j)$ is oriented and such that the terminal vertex of $c(j)$ is the initial vertex of $c(j+1)$ for all $1 \leq j < m$.*
A loop λ in $\tilde{C}$ is an injective map $\lambda : \mathbb{Z}/m\mathbb{Z} \to E(\tilde{C})$, for some $m \in \mathbb{N}$, such that each $c(j)$ is oriented and such that the terminal vertex of $c(j)$ is the initial vertex of $c(j+1)$ for all $j \in \mathbb{Z}/m\mathbb{Z}$.
The leaves-edges of a path, or a loop ρ in $\tilde{C}$ are oriented by definition. Hence, one can integrate any 1-form ω on $\tilde{C}$ along ρ, and one has the formula

$$\int_\rho \omega = \sum_{e \in \rho} l(e)\omega_{\vec{e}} \, .$$

For a path from a leaf to another, or a loop ρ, one defines the 1-form ω^ρ dual to ρ by

$$\omega^\rho_{\vec{e}} = \begin{cases} 1 & if \ e \in Im(\rho) \\ 0 & otherwise \end{cases} .$$

Definition 1.27. *Let $\tilde{C}$ be an abstract $\mathbb{T}$-curve. An exact 1-form ω on $\tilde{C}$ is an element of $\Omega(\tilde{C})$ such that*

$$\int_\rho \omega = 0$$

for any loop ρ in $\tilde{C}$. The set of exact 1-forms on $\tilde{C}$ is denoted by $\Omega_{exact}(\tilde{C})$.

Remark. The terminology is justified by the fact that exact 1-forms are exactly those obtained as gradient of tropical functions. This is a part of the tropical Abel-Jacobi theorem, see [MZ08].

Before ending this subsection, we introduce some useful results on exact 1-forms.

Proposition 1.28. *Let $\tilde{C}$ be an abstract $\mathbb{T}$-curve of genus g with n leaves such that $n \geq 1$ and $2g-2+n>0$. Then, one has the following*

1) *For any 1-form ω, the sum of all its residues is zero.*

2) *$\Omega(\tilde{C})$ is a real vector space of dimension $g+n-1$.*

3) *$\mathcal{H}\Omega(\tilde{C})$ is a real vector space of dimension g.*

4) $\Omega(\tilde{C}) = \Omega_{exact}(\tilde{C}) \oplus \mathcal{H}\Omega(\tilde{C})$.

5) *$\Omega_{exact}(\tilde{C})$ is a real vector space of dimension $n-1$ and each element of $\Omega_{exact}(\tilde{C})$ is determined by its residues.*

Proof Under our assumptions on g and n, $\tilde{C}$ can be considered as the dual graph of an hyperbolic pair-of-pants decomposition of a Riemann surface of genus g with n marked points. As we have already said, $\tilde{C}$ has exactly $2g-2+n$ 3-valent vertices. Cut $\tilde{C}$ at g points in such a way that the result is a metric tree $\tilde{C}_{cut}$. Hence $\tilde{C}_{cut}$ has still $2g-2+n$ 3-valent vertices, n leaves of infinite length and $2g$ leaves of finite length. Any 1-form on $\tilde{C}_{cut}$ can be constructed as follows : pick one of the n infinite leaves, and prescribe a residue on it, and travel along the tree $\tilde{C}_{cut}$. Each time a 3-valent vertex is encountered, one has $\mathbb{R}$-many ways to locally define a 1-form, hence the space of 1-forms on $\tilde{C}_{cut}$ is naturally isomorphic to $\mathbb{R}^{(2g-2+n)+1}$. In order to get a 1-form on $\tilde{C}$, one has to pick a 1-form on $\tilde{C}_{cut}$ that has opposite residues at every of the g pairs of finite leaves. These are g linearly independent conditions on the space of 1-forms on $\tilde{C}_{cut}$, as long as $n>0$. Indeed, consider a labelling $l_1, ...l_{2g+n}$ of the leaves of $\tilde{C}_{cut}$ such that $l_1, ...l_n$ correspond to the infinite leaves and l_{n+2i-1}, l_{n+2i} correspond to the 2 finite leaves glued together in $\tilde{C}$ (for $1 \leq i \leq g$). Let $\omega_2, ..., \omega_{g+n}$ be the 1-forms dual to the paths from l_1 to l_i for $2 \leq i \leq n$, and from l_{2i-n-1} to l_{2i-n} for $n+1 \leq i \leq g+n$, see definition 1.26. It forms a generating family for the space of 1-forms on $\tilde{C}_{cut}$ inducing a 1-form on $\tilde{C}$. By dimension reasoning, it is then a basis and 2) is

proven.
The sum of the residues for any element of the basis $\omega_2, ..., \omega_{g+n}$ is zero, hence it holds for any 1-form on $\tilde{C}$ and 1) is proven.
It follows that $\mathcal{H}\Omega(\tilde{C})$ is a g-dimensional vector space generated by $\omega_{n+1}, ..., \omega_{g+n}$, which is 3).
The map $H_1(\tilde{C}, \mathbb{R}) \times \Omega(\tilde{C}) \to \mathbb{R}$ defined by $(\gamma, \omega) \mapsto \int_\gamma \omega$ is a perfect pairing when restricted to $\mathcal{H}\Omega(\tilde{C})$. It implies both that a holomorphic 1-form with all periods equal to zero is necessarily zero and that for any $\omega \in \Omega(\tilde{C})$, there is a unique $\omega_0 \in \mathcal{H}\Omega(\tilde{C})$ having exactly the same periods. In other words, $\Omega(\tilde{C}) = \Omega_{exact}(\tilde{C}) \oplus \mathcal{H}\Omega(\tilde{C})$.
Finally, the difference between two elements of $\Omega_{exact}(\tilde{C})$ having the same residues is an element of $\Omega_{exact}(\tilde{C}) \cap \mathcal{H}\Omega(\tilde{C})$, hence it is zero, and 5) is proven.

□

Definition 1.29. *According to the point* 5) *of proposition 1.28, a collection of residues R of dimension m defines a collection of exact 1-forms $\omega_{R,\tilde{C}} := (\omega^{(1)}_{R,\tilde{C}}, ..., \omega^{(m)}_{R,\tilde{C}})$ on any abstract $\mathbb{T}$-curve $\tilde{C}$ defined by*

$$Res_{l_k} \omega^{(j)}_{R,\tilde{C}} = r^{(j)}_k$$

for any $1 \leq k \leq n$ and $1 \leq j \leq m$, where l_k is the l-th leaf of $\tilde{C}$.
Given a collection of residues R, an abstract $\mathbb{T}$-curve $\tilde{C}$ and an initial point $p_0 \in \tilde{C}$, one defines the map

$$\begin{array}{rcl} \pi_R \, : \, \tilde{C} & \to & \mathbb{R}^m \\ p & \mapsto & \left(Re(\int_{p_0}^p \omega^{(1)}_{R,\tilde{C}}), ..., Re(\int_{p_0}^p \omega^{(m)}_{R,\tilde{C}}) \right) \end{array} .$$

Definition 1.30. *An abstract complex tropical curve $\tilde{V}$ is the data of an abstract $\mathbb{T}$-curve $\tilde{C}$ equipped with a ribbon structure and a collection of twist parameters $\Theta \, : \, E(\tilde{C}) \to S^1$. An abstract $\mathbb{CT}$-curve will simply be denoted by the pair $\tilde{V} = (\tilde{C}, \Theta)$, without referring explicitly to the underlying ribbon structure.*

Remark. Note that an abstract $\mathbb{T}$-curve $\tilde{C}$ can be seen as a cubic graph together with a length function

$$l \, : \, E(\tilde{C}) \to \mathbb{R}_{\geq 0}.$$

Then, an abstract $\mathbb{CT}$-curve can be described by Fenchel-Nielsen coordinates in the same fashion as theorem 1.22.

2 Convergence of imaginary normalized differentials

The purpose of this section is to study the limit of imaginary normalized differentials on punctured Riemann surfaces while moving to maximally degenerate stable curves in the Deligne-Mumford moduli space $\overline{\mathcal{M}_{g,n}}$.
In the sequel, we will be looking at the following class of sequences of curves in $\mathcal{M}_{g,n}$.

Definition 2.1. *Let $\tilde{C}$ be an abstract $\mathbb{T}$-curve of genus g with n leaves such that $2g-2+n>0$ and $n \geq 2$. One says that a sequence $\{S_t\}_{t\in\mathbb{N}} \subset \mathcal{M}_{g,n}$ converges to $\tilde{C}$ if for a fixed ribbon structure on $\tilde{C}$*

- *there exists a length function l_t and a twist function Θ_t such that $S_t \simeq S(\tilde{C}, l_t, \Theta_t)$ for any t,*
- $l_t(e) \underset{t\to\infty}{\sim} \dfrac{4\pi}{l(e)\log(t)}$ *for any $e \in E(\tilde{C})$.*

Remark. If one forgets the metric, an abstract $\mathbb{T}$-curve $\tilde{C}$ can be thought of as the dual graph of a maximally degenerate stable curve where any element $v \in V(\tilde{C})$ is dual to a sphere with 3 special points that are either nodes (dual to elements $e \in E(\tilde{C})$) or marked points (dual to elements $e \in L(\tilde{C})$). In the latter definition, every geodesic dual to an edge $e \in E(\tilde{C})$ in S_t gets contracted to a node in the limit. Set theoretically, $\{S_t\}_{t\in\mathbb{N}}$ is a sequence of points in $\mathcal{M}_{g,n}$ converging to the maximally degenerate stable curve $S_{\tilde{C}}$ in $\overline{\mathcal{M}}_{g,n}$ corresponding to $\tilde{C}$. The asymptotic of the lengths of these geodesics is a partial information on the tangent direction with which $\{S_t\}_{t\in\mathbb{N}}$ approaches $S_{\tilde{C}}$.

Considering a collection of residues $R := (r_1, ..., r_n)$ and a sequence $\{S_t\}_{t\in\mathbb{N}} \subset \mathcal{M}_{g,n}$ give rise to the sequence $\{(S_t, \omega_{R,S_t})\}_{t\in\mathbb{N}} \subset \mathcal{M}_{g,n}$ in the total space of the twisted Hodge bundle $\Lambda^m_{g,n}$, see definition 1.3.
As we already explain, an abstract $\mathbb{T}$-curve $\tilde{C}$ gives a point $S_{\tilde{C}} \in \overline{\mathcal{M}}_{g,n}$ simply by considering the stable curve dual to the underlying cubic graph of $\tilde{C}$.
Now for $\omega \in \Omega_{exact}(\tilde{C})$, let us associate a generalized imaginary normalized differential on $S_{\tilde{C}}$ as follows:
each irreducible component of the normalization $\tilde{S}_{\tilde{C}}$ of $S_{\tilde{C}}$ corresponds to a

vertex $v \in V(\tilde{C})$, denote it $\mathcal{C}_v$. It is a sphere with 3 punctures corresponding to the 3 elements $e_1, e_2, e_3 \in LE(\tilde{C})$ adjacent to v. If one orients them toward v and consider the unique imaginary normalized differential on $\mathcal{C}_v$ having residue $\omega_{\vec{e}_j}$ at the j-th puncture, it defines a generalized imaginary normalized differential on $S_{\tilde{C}}$.
According to this construction, a pair $(\tilde{C}, \omega)$ of an abstract $\mathbb{T}$-curve and an element $\omega \in \Omega_{exact}(\tilde{C})$ gives a point in the total space $\Lambda_{g,n}^m$ over the curve $S_{\tilde{C}}$.

Definition 2.2. *For an abstract $\mathbb{T}$-curve $\tilde{C}$ and a 1-form $\omega \in \Omega(\tilde{C})$, one denote by $\left[\tilde{C}, \omega\right] \in \Lambda_{g,n}^m$ the image of the pair $(\tilde{C}, \omega)$ by the above construction.*

The main result of this subsection is the following

Theorem 1. *Let $\{S_t\}_{t\in\mathbb{N}} \subset \mathcal{M}_{g,n}$ be a sequence converging to an abstract $\mathbb{T}$-curve $\tilde{C}$ and $R := (r_1, ..., r_n)$ be a collection of residues. Let $\{\omega_{R,S_t}\}_{t\in\mathbb{N}}$ be the associated sequence of imaginary normalized differentials, see definition 1.3, and $\omega_{R,\tilde{C}}$ the associated 1-form on $\tilde{C}$, see definition 1.29.*
Then, the sequence $\{(S_t, \omega_{R,S_t})\}_{t\in\mathbb{N}} \subset \Lambda_{g,n}^m$ converges to $\left[\tilde{C}, \omega_{R,\tilde{C}}\right]$.

Remark. The latter theorem suggest that one should think about exact 1-forms on tropical curves as the tropical avatar of imaginary normalized differentials on Riemann surfaces. Forgetting the phases is a common process in tropical geometry. In the case of an imaginary normalized differential ω, it seems at first glance that the whole information is encapsulated in its imaginary part, as every period of the real part is zero. This is equivalent to the exactness of $Re(\omega)$, that is $Re(\omega) = df$ for some harmonic function f. Considered in family, it turns out that one controls the rate of growth of the f_t's near the nodes to be. Those rates can be used to determine the imaginary normalized differential in the limit. So then, exact 1-forms on tropical curves should be considered more precisely as tropical avatars of real parts of imaginary normalized differentials. Exactness appears then naturally.

The rest of this section is devoted to the proof of the latter theorem. Let us recall that

$$C_\lambda := \left\{(x, y) \in \mathbb{C}^2 \,|\, xy = \lambda,\ |x| < 1,\ |y| < 1\right\}$$

for $0 \leq \lambda < 1$. It is a classical fact that for $\lambda > 0$, any biholomorphism of C_λ that preserves the boundary components is a rotation, see for example [Rud87]. Hence, one can define an argument function $Arg : C_\lambda \to \mathbb{R}/2\pi\mathbb{Z}$ that is unique up to an additive constant.
For $0 < \lambda < 1$, a transversal path $\rho_\lambda : [a_\lambda, b_\lambda] \to C_\lambda$ is a smooth injective map such that $\rho_\lambda(a_\lambda)$ belongs to the connected component of ∂C_λ on which $|x| = \lambda$ and $\rho_\lambda(b_\lambda)$ belongs to the connected component of ∂C_λ on which $|x| = 1$. An admissible family of transversal paths is a family $\{\rho_\lambda\}_{0<\lambda<1}$ such that there exists a uniform bound $M > 0$ such that for $0 < \lambda < 1$ one has

$$\left| \int_{a_\lambda}^{b_\lambda} \frac{d}{ds} Arg\big(\rho_\lambda(s)\big) ds \right| < M.$$

It just prevents ρ_λ from wrapping infinitely many time around C_λ as $\lambda \to 0$.
For $0 < \lambda < 1$, recall also that γ_λ is the oriented loop on C_λ defined by

$$\gamma_\lambda(\theta) = \big(\sqrt{\lambda}e^{i\theta}, \sqrt{\lambda}e^{-i\theta}\big) \text{ for } 0 \leq \theta \leq 2\pi.$$

Note that the signed intersection number $\gamma_\lambda \cap \rho_\lambda$ is always 1.

Lemma 2.3. *Let $\{\lambda_t\}_{t\in\mathbb{N}} \subset]0,1[$ be a sequence converging to zero and $\{\omega_t\}_{t\in\mathbb{N}}$ be a converging sequence of holomorphic differentials on $\mathcal{C}_{\lambda_t}$, see definition 1.11. Define*

$$\Lambda := \lim_{t\to\infty} \frac{1}{2\pi i} \int_{\gamma_{\lambda_t}} \omega_t.$$

For any admissible family of transversal paths $\{\rho_{\lambda_t}\}_{t\in\mathbb{N}}$, one has

$$\lim_{t\to\infty} \frac{\int_{\rho_{\lambda_t}} \omega_t}{\log(\lambda_t)} = -\Lambda.$$

Proof First, let us prove the lemma on a particular case : consider $\omega_t^\bullet$ to be the restriction of the holomorphic differential $x^{-1}dx$ on $D(1,1)$ to C_{λ_t}. Consider the following family of transversal paths

$$\begin{array}{rcl} \rho^\bullet_{\lambda_t} : [\lambda, 1] & \to & C_{\lambda_t} \\ s & \mapsto & \big(\sqrt{s}, \sqrt{s^{-1}\lambda_t}\big) \end{array}$$

It is obviously an admissible family. Remark that this family converges when t goes to ∞. The limit path is a union of 2 straight segments $\rho^\bullet_{0,1}$ and $\rho^\bullet_{0,2}$

sitting in the x- and y-unit discs respectively. On one hand, we have

$$\Lambda^\bullet := \lim_{t\to\infty} \frac{1}{2\pi i}\int_{\gamma_{\lambda_t}} \omega_t^\bullet = \lim_{t\to\infty} \frac{1}{2\pi i}\oint_{|z|=\sqrt{\lambda_t}} \frac{dz}{z} = 1,$$

and on the other

$$\int_{\rho^\bullet_{\lambda_t}} \omega_t^\bullet = \int_{\lambda_t}^1 \frac{ds}{s} = -\log(\lambda_t).$$

This proves the lemma for our particular case.
Now let us prove the lemma in the general case. Remark that any transversal path ρ_{λ_t} is homotopic to $\gamma^2_{\lambda_t}\circ\gamma^\bullet_{\lambda_t}\circ\gamma^1_{\lambda_t}$, with $\gamma^1_{\lambda_t}$ and $\gamma^2_{\lambda_t}$ being 2 locally injective paths in the appropriate connected components of ∂C_{λ_t}. Hence

$$\int_{\rho_{\lambda_t}} \omega_t = \int_{\rho^1_{\lambda_t}} \omega_t + \int_{\rho^\bullet_{\lambda_t}} \omega_t + \int_{\rho^2_{\lambda_t}} \omega_t.$$

But the integrals along the 2 paths on the boundary are uniformaly bounded in λ, by the admissibility condition and the convergence of ω_t. It follows that

$$\lim_{t\to\infty} \frac{\int_{\rho_{\lambda_t}} \omega_t}{\log(\lambda_t)} = \lim_{t\to\infty} \frac{\int_{\rho^\bullet_{\lambda_t}} \omega_t}{\log(\lambda_t)}$$

Now, define $\tilde{\omega}_t := \omega_t - \Lambda\,\omega_t^\bullet$ so that $\tilde{\omega}_t$ converges to an holomorphic differential on C_0. Then, there exist 2 holomorphic functions $h_1(x)$ and $h_2(y)$ defined on the respective unit discs such that, by the above remark, one has

$$\lim_{t\to\infty}\int_{\rho_{\lambda_t}} \tilde{\omega}_t = \int_{\rho^\bullet_{0,1}} h_1(x)\,dx + \int_{\rho^\bullet_{0,2}} h_2(y)\,dy \;+\; O(1).$$

Each of the integrals of the right-hand side is a well defined and bounded quantity. As a consequence

$$\lim_{t\to\infty} \frac{\int_{\rho_{\lambda_t}} \tilde{\omega}_t}{\log(\lambda_t)} = 0,$$

and

$$\lim_{t\to\infty} \frac{\int_{\rho_{\lambda_t}} \omega_t}{\log(\lambda_t)} = \Lambda \lim_{t\to\infty} \frac{\int_{\rho_{\lambda_t}} \omega_t^\bullet}{\log(\lambda_t)}.$$

Using the above remark one again, one has that

$$\begin{aligned}\lim_{t\to\infty}\frac{\int_{\rho_{\lambda_t}}\omega_t^\bullet}{\log(\lambda_t)} &= \lim_{t\to\infty}\frac{\int_{\rho^\bullet_{\lambda_t}}\omega_t^\bullet}{\log(\lambda_t)}\\ &= -1,\end{aligned}$$

and the lemma is proved.

□

Lemma 2.4. *Let $a\in\mathbb{R}$, $a>0$ and $\{a_t\}_{t\in\mathbb{N}}$ be a sequence of positive numbers such that*

$$a_t \underset{t\to\infty}{\sim} \frac{4\pi}{a\log(t)}.$$

Let λ_t be the unique real number such that C_{λ_t} is biholomorphic to the hyperbolic collar around a geodesic of length a_t, see definition 1.23.
Consider a converging sequence $\{\omega_t\}_{t\in\mathbb{N}}$ of holomorphic differentials on C_{λ_t} and define

$$\Lambda := \lim_{t\to\infty}\frac{1}{2\pi i}\int_{\gamma_{\lambda_t}}\omega_t.$$

For any admissible family of transversal paths $\{\rho_{\lambda_t}\}_{t\in\mathbb{N}}$, one has

$$\lim_{t\to\infty}\frac{\int_{\rho_{\lambda_t}}\omega_t}{\log(t)} = a\Lambda.$$

Proof The proof is about finding the asymptotic of λ_t in terms of t. On one hand, the map $z\mapsto(\sqrt{\lambda}z,\sqrt{\lambda}z^{-1})$ defines a biholomorphism between the annulus $\left\{z\in\mathbb{C}|\sqrt{\lambda}\le|z|\le 1/\sqrt{\lambda}\right\}$ and C_λ. For an annulus of type $r<|z|<R$ in the complex plane, one can choose R/r as a conformal invariant, see [Ahl10]. This invariant determines the annulus up to biholomorphism. According to this, C_λ has conformal invariant λ^{-1}.
On the other hand, following [Bus10], an hyperbolic collar around a geodesic of length a_t can be isometrically presented as the space

$$\big([-w(a_t),w(a_t)]\times\mathbb{R}/\mathbb{Z},\ d\rho^2+a_t^2\cosh(\rho)^2ds^2\big)$$

where $w(a_t) = arcsinh\big(1/\sinh(a_t/2)\big)$. Applying the change of variable $\rho(u) = arcosh\big(a_t/\cos(a_t u)\big)$, one gets

$$\left(\left[-\rho^{-1}\big(w(a_t)\big), \rho^{-1}\big(w(a_t)\big)\right] \times \mathbb{R}/\mathbb{Z}, \left(\frac{a_t}{\cos(a_t u)}\right)^2 (du^2 + ds^2)\right).$$

Applying now $\big(u(z), s(z)\big) = (2\pi)^{-1}\big(Re(\log(z)), Im(\log(z))\big)$, one gets

$$\left(\left\{z \in \mathbb{C} | \eta(a_t)^{-1} \leq |z| \leq \eta(a_t)\right\} , \left(\frac{a_t}{2\pi|z|\cos(a_t \log(z))}\right)^2 |dz|^2\right),$$

where

$$\eta(a_t) = \exp\left(\frac{2\pi}{a_t} \arccos\left(\frac{1}{\cosh\big(w(a_t)\big)}\right)\right).$$

Comparing conformal invariants, one deduces that $\lambda_t = \eta(a_t)^{-2}$. Let us compute the asymptotic of λ_t : first notice that $w(a_t)$ goes to $+\infty$ as t goes to $+\infty$, hence

$$\lim_{t\to\infty} \arccos\left(\frac{1}{\cosh\big(w(a_t)\big)}\right) = 1.$$

It follows that

$$\lambda_t = \eta(a_t)^{-2} \underset{t\to\infty}{\sim} \exp\left(\frac{-4\pi}{a_t}\right) \underset{t\to\infty}{\sim} e^{-a\log(t)}.$$

Applying the previous lemma, one concludes that

$$\begin{aligned} a\Lambda &= \lim_{t\to\infty} -a\,\frac{\int_{\rho_{\lambda_t}} \omega_t}{\log(\lambda_t)} \\ &= \lim_{t\to\infty} -a\,\frac{\int_{\rho_{\lambda_t}} \omega_t}{\log(e^{-a\log(t)})} \\ &= \lim_{t\to\infty} \frac{\int_{\rho_{\lambda_t}} \omega_t}{\log(t)}. \end{aligned}$$

□

Before proving theorem 1, let us introduce a technical definition : to a loop $\rho_{\mathbb{T}}$ in an abstract $\mathbb{T}$-curve $\tilde{C}$, we will associate a piecewise geodesic loop in any curve $S \simeq S(\tilde{C}, l, \Theta)$. Recall that a loop is a map $\rho_{\mathbb{T}} : \mathbb{Z}/m\mathbb{Z} \to E(\tilde{C})$. It can be presented as well as a map $V : \mathbb{Z}/m\mathbb{Z} \to V(\tilde{C})$ where $V(j)$ is the vertex between $\rho_{\mathbb{T}}(j)$ and $\rho_{\mathbb{T}}(j+1)$. Denote by $Y_{V(j)}$ the "pop" in the decomposition of S corresponding to the vertex $V(j)$, by γ_j the geodesic in the decomposition of S corresponding to the edge $\rho_{\mathbb{T}}(j)$ and orient it such that the associated normal vector field points toward $Y_{V(j)}$.
Now construct the piecewise geodesic loop $\breve{\rho}$ in S as follows : $\breve{\rho} \cap Y_{V(j)}$ is the oriented connected component of $\mathbb{R}Y_{V(j)}$ going from $\gamma(j)$ to $\gamma(j+1)$, and $\breve{\rho} \cap \gamma_j$ is the positive arc of γ_j connecting $\breve{\rho} \cap Y_{V(j-1)}$ to $\breve{\rho} \cap Y_{V(j)}$.

Definition 2.5. *For any loop $\rho_{\mathbb{T}}$ in an abstract $\mathbb{T}$-curve $\tilde{C}$ and any curve $S \simeq S(\tilde{C}, l, \Theta)$, define by $\breve{\rho} \subset S$ the loop in S associated to $\rho_{\mathbb{T}}$, as constructed above.*

Proof of theorem 1 Recall that from definition 2.1, S_t is presented as $S(\tilde{C}, l_t, \Theta_t)$.
Suppose first that for any geodesic γ_t of the "pop" decomposition of S_t, the sequence $\left\{\int_{\gamma_t} \omega_{R,S_t}\right\}_{t\in\mathbb{N}}$ converges. It is clearly a necessary and sufficient condition for the sequence $\{(S_t, \omega_{R,S_t})\}_{t\in\mathbb{N}}$ to converge in the total space of the bundle $\Lambda^m_{g,n}$, see lemma 1.10. Let us show that the limit is $\left[\tilde{C}, \omega_{R,\tilde{C}}\right]$.
Pick g loops $\rho^1_{\mathbb{T}}, ..., \rho^g_{\mathbb{T}}$ in $\tilde{C}$ forming a basis of $H_1(\tilde{C}, \mathbb{Z})$. For any $1 \leq j \leq g$ and any t, construct a loop $\rho^j_t \subset S_t$ as follows: as an intermediate step, consider the piecewise geodesic loop $\breve{\rho}^j_t \subset S_t$ associated to $\rho^j_{\mathbb{T}}$ (see definition 2.5). Now define ρ^j_t to be the unique geodesic in the free homotopy class of $\breve{\rho}^j_t$, see theorem 1.6.6 in [Bus10]. Now for $1 \leq j \leq g$, index coherently the edges $\vec{e}^{jk}$ as they are encountered in the loop $\gamma^j_{\mathbb{T}}$ for $1 \leq k \leq m_j$, where the orientation on the $\vec{e}^{jk}$'s is induced by $\gamma^j_{\mathbb{T}}$. Denote also by γ^{jk}_t the geodesic of the "pop" decomposition of S_t associated to $\vec{e}^{jk}$ oriented coherently to the orientation of $\vec{e}^{jk}$, and K^{jk}_t the collar around γ^{jk}_t. Define $\psi^{jk}_t : K^{jk}_t \to C_{\lambda^{jk}_t}$ the biholomorphism such that :

* $\psi^{jk}_t\left(\rho^j_t \cap K^{jk}_t\right)$ is a transversal path in $C_{\lambda^{jk}_t}$,
* $\psi^{jk}_t\left(\rho^j_t \cap K^{jk}_t\right) \cap \{(x, y) \in \mathbb{C}^2 | \, |x| = 1\} = (1, \lambda^{jk}_t)$.

Note that the push-forward of ω_{R,S_t} on $C_{\lambda^{jk}_t}$ by ψ^{jk}_t gives rise to a convergent sequence, according to definition 1.11. Indeed, the limit is given on each

connected component of the normalization of $S_{\tilde{C}}$ by a unique meromorphic differential. This differential admits a unique representation as $f(z)dz$ once a coordinate z is chosen.
It implies first that ω_{R,S_t} converges toward an holomorphic differential on the complement of the collars, so the integral on this complement converges to a finite quantity. Hence one has

$$\int_{\rho_t^j} \omega_{R,S_t} = \sum_{k=1}^{m_j} \int_{\rho_t^j \cap K_t^{jk}} \omega_{R,S_t} \quad + \; O(1).$$

Now, applying lemma 2.4 for each of the collars involved in the latter formula, one gets

$$\lim_{t\to\infty} \frac{\int_{\rho_t^j} \omega_{R,S_t}}{\log(t)} = \sum_{k=1}^{m_j} a_{jk}\Lambda_{jk} \tag{1}$$

where a_{jk} is the length of the edge e_{jk} of $\tilde{C}$ and $\Lambda_{jk} = \lim_{t\to\infty} \frac{1}{2\pi i} \int_{\gamma_t^{jk}} \omega_{R,S_t}$ which exist by assumption.
As ω_{R,S_t} is an imaginary normalized differential for any t, it implies that $\Lambda_{jk} \in \mathbb{R}$ and that

$$Re\Big(\int_{\rho_t^j} \omega_{R,S_t} \Big) = 0, \; \forall t, \; \forall j.$$

Considering the real part on both sides in (1) gives in turn that

$$\sum_{k=1}^{m_j} a_{jk}\Lambda_{jk} = 0, \; \forall j. \tag{2}$$

By definition of the Λ_{jk}, one can construct a 1-form on $\tilde{C}$ taking the value Λ_{jk} along $\vec{e}_{jk}$ and having residue r_l at the l-th leaf of $\tilde{C}$. The equation (2) states exactly that this 1-form belongs to $\Omega_{exact}(\tilde{C})$. According to proposition 1.28, this 1-form is exactly $\omega_{R,\tilde{C}}$ and the theorem is proved in this case.
Suppose now that there is a bound for all the periods $\int_{\gamma_t} \omega_{R,S_t}$ that is uniform in t. For any subsequence $\{t_k\}_{t\in I}$ such that all these periods converge, one can reduce to the previous case and deduce that $\omega_{R,S_{t_k}}$ converges to $\omega_{R,\tilde{C}}$. As any converging subsequence converges to the same limit, the original sequence does converge.
Suppose finally that $\lim_{t\to\infty} M_t = +\infty$ where M_t is defined as $\max \left| \int_{\gamma_t} \omega_{R,S_t} \right|$.

Consider the sequence of imaginary normalized differential $\tilde{\omega}_t := \frac{1}{M_t}\omega_{R,S_t}$. Considering a subsequence if necessary, assume that the periods $\int_{\gamma_t}\tilde{\omega}_t$ converge. Applying the same argument as in the first case, one construct a limit element $\tilde{\omega}^{\mathbb{T}} \in \Omega_{exact}(\tilde{C})$. The way we rescaled ω_{R,S_t} to get $\tilde{\omega}_t$ implies that

* there is an $\vec{e}$ in $\tilde{C}$ such that $|\tilde{\omega}^{\mathbb{T}}_{\vec{e}}| = 1$,
* $\tilde{\omega}^{\mathbb{T}}$ has no residues.

In other word, $\tilde{\omega}^{\mathbb{T}}$ is a non zero exact 1-form which is holomorphic. This is in contradiction with 1.28 and the theorem is proved.

□

In the rest of this section, one shows that if one refines the definition of convergence of 2.1, one controls every period of the sequence of imaginary normalized differentials at the limit.

Definition 2.6. *Let $(\tilde{C}, \Theta)$ be an abstract $\mathbb{CT}$-curve of genus g and n leaves with $2g - 2 + n \geq 1$ and $n \geq 1$. One says that a sequence $\{S_t\}_{t\in\mathbb{N}} \subset \mathcal{M}_{g,n}$ converges to $(\tilde{C}, \Theta)$ as t goes to infinity if :*

- *$S_t \simeq S(\tilde{C}, l_t, \Theta_t)$ converges to $\tilde{C}$, see definition 2.1,*
- *the sequence of functions $\{\Theta_t\}_{t\in\mathbb{N}}$ converges to Θ.*

Theorem 2. *Let $\{S_t\}_{t\in\mathbb{N}} \subset \mathcal{M}_{g,n}$ be a sequence converging to an abstract $\mathbb{CT}$-curve $(\tilde{C}, \Theta)$ and $R := (r_1, ..., r_n)$ be a collection of residues. Let $\{\omega_{R,S_t}\}_{t\in\mathbb{N}}$ be the associated sequence of imaginary normalized differentials, see definition 1.3, and $\omega_{R,\tilde{C}}$ the associated 1-form on $\tilde{C}$, see definition 1.29.*
For any loop $\rho_{\mathbb{T}} \subset \tilde{C}$, and $\breve{\rho}_t$ the associated loop in S_t, see definition 2.5, one has

$$\lim_{t\to\infty} \int_{\breve{\rho}_t} \omega_{R,S_t} = \sum_{e\in\rho_{\mathbb{T}}} \log\big(\Theta(e)\big)\big(\omega_{R,\tilde{C}}\big)_{\vec{e}}$$

where the branch of log *is chosen such that* $\log : S^1 \to [0, 2i\pi[\subset \mathcal{C}$.

In order to prove this statement, let us briefly study imaginary normalized differentials ω on $S \in \mathcal{M}_{0,3}$. More precisely, we are interested in paths ρ in S for which $Im(\omega)_{|_\rho} \equiv 0$. With the help of the uniformizing metric g on S, one can consider the vector field $Im(\omega)^\vee$ dual to $Im(\omega)$ defined by

$Im(\omega) = g\big(\,\cdot\,, Im(\omega)^\vee\big)$. Similarly, denote by $Re(\omega)^\vee$ the vector field dual to $Re(\omega)$ with respect to g. The vector field $Im(\omega)^\vee$ is obtained by rotating $Re(\omega)^\vee$ by $\pi/2$. As a consequence, a path ρ in S is such that $Im(\omega)_{|\rho} \equiv 0$ if and only if the tangent vector field of ρ is parallel to $Re(\omega)^\vee$ at any point, *i.e* ρ is a flow line of $Re(\omega)^\vee$.
Let us choose the model $S \simeq \mathbb{CP}^1 \setminus \{-1, 1, \infty\}$. Up to multiplication of ω by a constant and an automorphism of S (exchanging the punctures), one can assume that

$$\omega = \left(\frac{\lambda_1}{z-1} + \frac{\lambda_{-1}}{z+1}\right) dz \tag{3}$$

with $\lambda_{-1}, \lambda_1 > 0$. Consider the real oriented blow-up S_{Bl} of S at -1, 1 and ∞, and denote by γ_{-1}, γ_1 and γ_∞ the 3 corresponding boundary components of S_{Bl}. The vector field $Re(\omega)^\vee$ does not extend to the boundary of S_{Bl} as its modulus tends to infinity, nevertheless its asymptotic direction is well defined.

Lemma 2.7. *Let ω be an imaginary normalized differential on S as in* (3)*. Then, one has*

* *the 3 connected components of $\mathbb{R}S$ are parallel to $Re(\omega)^\vee$,*
* *$Re(\omega)^\vee$ is asymptotically orthogonal to γ_{-1}, γ_1 and γ_∞. It is oriented inward S_{Bl} at γ_{-1} and γ_1 and outward S_{Bl} at γ_∞.*
* *For any point p in γ_{-1} or γ_1 out of $\mathbb{R}S_{Bl}$, the flow line of $Re(\omega)^\vee$ starting at p ends in γ_∞.*
* $[-1, 1]$ *is the unique flow line in S_{Bl} connecting γ_{-1} to γ_1.*

Proof The first point is an easy consequence of the fact that ω is defined over $\mathbb{R}$.
For the second point, consider a complex coordinate z centred at one of the 3 punctures of S and consider the circular integral

$$\begin{aligned}
\oint_{|z|=r} \omega &= \oint_{|z|=r} \left(\frac{\lambda}{z} + h(z)\right) dz \\
&= \int_0^{2\pi} \left(i\lambda + h(re^{i\theta}) ire^{i\theta}\right) d\theta \\
&= \int_0^{2\pi} \left(i\lambda + rO(1)\right) d\theta
\end{aligned}$$

where h is an holomorphic function near the origin. The integrand converges uniformly to a purely imaginary function as r goes to 0. It is equivalent to say that the vector field $Re(\omega)^\vee$ becomes everywhere orthogonal to the tangent vector field of the circle of radius r as r goes to 0.
As $Im(\omega)^\vee$ is obtained by rotating $Re(\omega)^\vee$ by $\pi/2$, it gets asymptotically parallel to the tangent vector field of the circle of radius r as r goes to 0. If $\lambda > 0$, the latter integral is a positive multiple of i. It means then that these vector fields point in the same direction. If $\lambda < 0$, they point in opposite direction.
For the third point, notice that $Re(\omega)^\vee$ has a unique zero at the point $\zeta := \frac{\lambda_1 - \lambda_{-1}}{\lambda_1 + \lambda_{-1}} \in]-1,1[$. As this vector field is the gradient field of the harmonic function $\int Re(\omega)$, by classical Morse lemma, it is locally isotopic to the gradient field of $x^2 - y^2$. There are exactly 2 flow lines passing through ζ. One is $]-1,1[$ and the flow is oriented towards η on each connected component of $]-1,1[\setminus \zeta$. The other flow line intersects$]-1,1[$ transversally at ζ, and the flow is locally oriented outward η. Hence, any flow line starting from p never reaches the only singular point ζ of the vector field $Re(\omega)^\vee$. Hence, it can be extended until it reaches the boundary of S_{Bl}. According to the different orientation of the field at the boundary components, it has to end up in γ_∞. The last point is a consequence of the previous points.

□

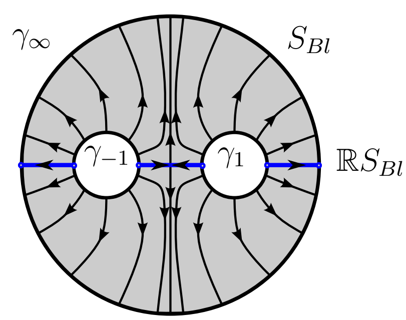


Figure 1: Phase portrait of $Re(\omega)^\vee$ on S_{Bl}.

Lemma 2.8. *Let $\{Y_t\}_{t\in\mathbb{N}}$ be a sequence of "pop" converging to $S \in \mathcal{M}_{0,3}$ and let $\{\omega_t\}_{t\in\mathbb{N}}$ be a sequence of imaginary normalized differential on Y_t converging to an imaginary normalized differential on S. For any connected* *component* $\rho_t \subset Y_t$ *of* $\mathbb{R}Y_t$*, one has*

$$\lim_{t\to\infty} \int_{\rho_t} Im(\omega_t) = 0.$$

Proof Assume that the 3 residues of ω_t are non zero at the limit, the general case can be deduced by continuity. According to the proof of the previous lemma, one can choose 3 sequences of positive numbers $r_{t,-1}$, $r_{t,1}$ and $r_{t,\infty}$ such that up to multiplication of ω_t by a constant independent of t, one has

* $Y_t \simeq \overline{\mathcal{B}_0(r_{t,\infty})} \setminus (\mathcal{B}_{-1}(r_{t,-1}) \cup \mathcal{B}_1(r_{t,1}))$,
* if one denotes by $\gamma_{t,-1}$ (*resp.* $\gamma_{t,1}$, *resp.* $\gamma_{t,\infty}$) the boundary of $\overline{\mathcal{B}_0(r_{t,\infty})}$ (*resp.* $\overline{\mathcal{B}_{-1}(r_{t,-1})}$, *resp.* $\overline{\mathcal{B}_1(r_{t,1})}$), there exists $N_1 > 0$ such that for $t > N$, one has $\int_{\gamma_{t,-1}} Im(\omega_t)$, $\int_{\gamma_{t,1}} Im(\omega_t) > 0$.
* There exists $N_2 > N_1$ such that for $t > N_2$, $Re(\omega_t)^\vee$ is everywhere transversal to the tangent vector fields of $\gamma_{t,-1}$, $\gamma_{t,1}$ and $\gamma_{t,\infty}$.
* There exists $N_3 > N_2$ such that for $t > N_3$, ω_t has a single zero on Y_t.

The same treatment as in the proof of the previous lemma implies that

* for $t > N_3$, there is a unique flow line of $Re(\omega_t)$ in Y_t joining $\gamma_{t,-1}$ and $\gamma_{t,1}$, converging in Hausdorff distance to $]-1+r_{t,-1}, 1-r_{t,1}[\subset Y_t$,
* for $t > N_3$, one can construct 2 sequences of flow line of $Re(\omega_t)$ in Y_t going from $\gamma_{t,-1}$ (*resp.* $\gamma_{t,1}$) to $\gamma_{t,\infty}$ converging in Hausdorff distance to $]-r_{t,\infty}, -1-r_{t,-1}[$ (*resp.* $]1+r_{t,1}, r_{t,\infty}[$).

If for example $\rho_t :=]-1+r_{t,-1}, 1-r_{t,1}[$ and $\breve{\rho}_t$ is the sequence of flow lines converging to ρ_t, then ρ_t is homotopic to $\rho_{t,1} \circ \breve{\rho}_t \circ \rho_{t,-1}$ where $\rho_{t,-1}$ is an arc in $\gamma_{t,-1}$ and $\rho_{t,1}$ is an arc in $\gamma_{t,1}$. As $\breve{\rho}_t$ converges to ρ_t, the integrals of $Im(\omega_t)$ over $\rho_{t,-1}$ and $\rho_{t,1}$ tend to 0 as t goes to ∞. Hence

$$\lim_{t\to\infty} \int_{\rho_t} Im(\omega_t) = \lim_{t\to\infty} \int_{\breve{\rho}_t} Im(\omega_t) = 0$$

as $\breve{\rho}_t$ is a flow line of $Re(\omega_t)$ for any t. The same argument apply for any choice of the sequence ρ_t and the lemma is proved.

$\square$

Proof of theorem 2 For any loop $\rho_{\mathbb{T}} \subset \tilde{C}$, the associated loop $\breve{\rho}_t \subset S_t$ is piecewise geodesic, see 2.5. According to the definition, $\breve{\rho}_t$ is made out of connected components of $\mathbb{R}Y_t$ for some "pop" Y_t of the decomposition of S_t and arcs in some of the geodesics γ_t of the same decomposition. By the previous lemma, parts of $\breve{\rho}_t$ contained in the different $\mathbb{R}Y_t$'s do not contribute in the limit of the integral of $Im(\omega_t)$ along $\breve{\rho}_t$. Let $\vec{e}$ be an edge of $\rho_{\mathbb{T}}$, and γ_t the corresponding geodesic in S_t. By definition, $\breve{\rho}_t \cap \gamma_t$ is an arc of length $\frac{1}{2\pi} \cdot \log\big(\Theta_t(e)\big) \cdot l_t(e)$ of γ_t. If one shows that

$$\lim_{t\to\infty} \int_{\breve{\rho}_t \cap \gamma_t} Im(\omega_t) = \log\big(\Theta(e)\big) \cdot \big(\omega_{R,\tilde{C}}\big)_{\vec{e}} \, ,$$

then the proposition is proved.

For t big enough, and $\varepsilon > 0$ small enough, consider the tubular neighbourhood $\mathcal{U}_\varepsilon$ of width ε around γ_t. Let $c_{t,1}$ and $c_{t,2}$ be the flow lines of $Re(\omega_t)$ starting at the end points of $\breve{\rho}_t \cap \gamma_t$ and ending on $\partial\mathcal{U}_\varepsilon$. Denote by $\breve{\rho}_{t,\varepsilon}$ the arc on $\partial\mathcal{U}_\varepsilon$ between the end points of $c_{t,1}$ and $c_{t,2}$ such that $\breve{\rho}_t \cap \gamma_t$ is homotopic to $c_{t,2}^{-1} \circ \breve{\rho}_{t,\varepsilon} \circ c_{t,1}$. For any t, one has

$$\int_{\breve{\rho}_t \cap \gamma_t} Im(\omega_t) = \int_{c_{t,2}^{-1} \circ \breve{\rho}_{t,\varepsilon} \circ c_{t,1}} Im(\omega_t) = \int_{\breve{\rho}_{t,\varepsilon}} Im(\omega_t)$$

as $c_{t,1}$ and $c_{t,2}$ are flow lines of $Re(\omega_t)$. Now, one clearly has

$$\begin{aligned} \lim_{\varepsilon\to 0} \lim_{t\to\infty} \int_{\breve{\rho}_{t,\varepsilon}} Im(\omega_{R,\tilde{C}}) &= \lim_{\varepsilon\to 0} \int_{\alpha_{\varepsilon,\Theta(e)}} \frac{\big(\omega_{R,\tilde{C}}\big)_{\vec{e}}}{z} dz \\ &= \log\big(\Theta(e)\big) \cdot \big(\omega_{R,\tilde{C}}\big)_{\vec{e}} \end{aligned}$$

where $\alpha_{\varepsilon,\Theta(e)} := \big\{ z \in \mathbb{C} \, \big| |z| = \varepsilon, \; \theta \leq Arg(z) \leq \theta + \log\big(\Theta(e)\big) \big\}$ for some argument θ.

$\square$

3 Approximation of harmonic tropical curves

3.1 Harmonic tropical curves

Definition 3.1. *Let $\tilde{C}$ be an abstract tropical curve. An harmonic morphism $\pi : \tilde{C} \to \mathbb{R}^m$ is a continuous map such that*

* *for any $e \in LE(\tilde{C})$, the restriction map $\pi_{|e}$ is affine linear.*

* *For any $v \in V(\tilde{C})$, denote by $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ the 3 unitary tangent vectors of the 3 leaves-edges adjacent to v. Then*

$$\pi(\vec{e}_1) + \pi(\vec{e}_2) + \pi(\vec{e}_3) = 0.$$

The latter condition is referred as the balancing condition. The harmonic morphism π is called simple if moreover

* *for any $e \in LE(\tilde{C})$, the restriction map $\pi_{|e}$ is an embedding.*

* *π is at most 2-to-1 and injective out of a finite set of points disjoint from $V(\tilde{C})$.*

In the case of a simple harmonic morphism $\pi : \tilde{C} \to \mathbb{R}^m$, its image is naturally a metric space. Indeed, the injectivity assumptions ensure that one can push forward the metric of $\tilde{C}$ onto its image.

Definition 3.2. *A simple harmonic tropical curve $C \in \mathbb{R}^m$ is the metric space obtained as the image of a simple harmonic morphism $\pi : \tilde{C} \to \mathbb{R}^m$.*

Remark. A simple harmonic tropical curve is fully determined by its point set together with a unitary tangent vector at one of its vertices v. Indeed, this tangent vector is supported by one of the elements e of $LE(\tilde{C})$ adjacent to v. As harmonic morphisms are locally affine linear, the data of this tangent vector determined the metric on e. Now, the balancing condition in 3.1 allows to recovers the unitary tangent vectors for the 2 other elements of $LE(\tilde{C})$ adjacent to v. Hence, the metric can be recovered inductively on the whole curve C.

Proposition 3.3. *For any simple harmonic $\mathbb{T}$-curve $C \in \mathbb{R}^m$, there exists a unique abstract $\mathbb{T}$-curve $\tilde{C}$ and a simple harmonic morphism $\pi : \tilde{C} \to \mathbb{R}^m$ such that $\pi(\tilde{C}) = C$ as metric space. The $\mathbb{T}$-morphism π is unique up to isometry of the source.*
The abstract $\mathbb{T}$-curve $\tilde{C}$ is called the normalisation of C.

Proof Set theoretically, a simple harmonic $\mathbb{T}$-curve $C \in \mathbb{R}^m$ is just a graph with straight edges, and straight leaves. It has only 3-valent and 4 valent vertices. By definition, there exists such simple harmonic morphism $\pi : \tilde{C} \to \mathbb{R}^m$ such that $\pi(\tilde{C}) = C$ as metric space. One simply reconstruct $\tilde{C}$ from C by splitting up the 4-valent vertices of C into 2 edges-leaves in $\tilde{C}$ and pulling back the metric. The details are left to the reader.

□

Remark. For the sake of simplicity, one does not introduce a definition for general harmonic tropical curve in $\mathbb{R}^m$. It is already clear that the only consideration of the set theoretical image of an harmonic morphism is not satisfactory, as one would lake too much structure. In fair generality, some leaves-edges can be contracted by an harmonic morphism. It implies in particular that the combinatorial type of an abstract $\mathbb{T}$-curve cannot be recovered by its image. One should reasonably equip every leaf-edge of the image with positive multiplicities μ together with a collection of μ unitary tangent vectors such that the all setting satisfies a balancing condition. We postpone the general treatment of harmonic tropical curves to a further paper.

Proposition 3.4. *Let $\tilde{C}$ be an abstract $\mathbb{T}$-curve and R a collection of residues of dimension m. Then the map $\pi_R : \tilde{C} \to \mathbb{R}^m$ is an harmonic morphism. Moreover, for any vertex $v \in V(\tilde{C})$ and any of its adjacent leaf-edge $\vec{e}$ oriented outward, the corresponding unitary tangent vector is sent to the vector $\left(\omega_{R,\tilde{C}}\right)_{\vec{e}}$.*
Reciprocally, for any harmonic morphism $\pi : \tilde{C} \to \mathbb{R}^m$, there exists a unique collection of residues R of dimension m such that $\pi = \pi_R$.

Proof As every coordinates of $\omega_{R,\tilde{C}}$ is a constant 1-form on any leaf-edge of $\tilde{C}$, it is clear that π_R is affine linear on every leaf-edge. By definition, an oriented segment of length l included in $\vec{e} \in LE(\tilde{C})$ is sent to the vector $l \cdot (\omega_{R,\tilde{C}})_{\vec{e}}$. Hence the image of a unitary tangent vector is as predicted. The balancing condition of 3.1 follows from the definition 1.25. The first part of the statement is proven.
Reciprocally, it is now clear that the map π is given by integration of an m-tuple of local 1-forms on any $e \in LE(\tilde{C})$. The balancing condition of 3.1 is clearly equivalent to the fact that these m-tuples of local 1-forms give rise to m-tuple of 1-forms on $\tilde{C}$, see 1.25. Exactness follows from the fact that this m-tuple is given by integrating the m-tuple of the coordinate functions of π. Uniqueness is clear.

□

Definition 3.5. *An element $\omega \in \Omega_{exact}(\tilde{C})$ for a $\mathbb{T}$-curve $\tilde{C}$ is integer if $\omega_{\vec{e}} \in \mathbb{Z}$ for any element $e \in LE(\tilde{C})$. A collection of residues R of dimension m is integer on $\tilde{C}$ if the m coordinates of $\omega_{R,\tilde{\mathcal{C}}}$ are integer elements of $\Omega_{exact}(\tilde{C})$.*

3.2 Degeneration of harmonic amoebas

Theorem 3. *Let* $\{S_t\}_{t\in\mathbb{N}} \subset \mathcal{M}_{g,n}$ *be a sequence converging to an abstract* $\mathbb{T}$*-curve* $\tilde{C}$ *and let* $R := \left\{(r_1^{(j)}, ..., r_n^{(j)})\right\}_{1\leq j\leq m}$ *be a collection of residues. Then, for an appropriate choice of a sequence* $z_{0,t} \in S_t$ *and* $p_0 \in \tilde{C}$ *of initial points (see 1.3 and 1.29), one has*

$$\lim_{t\to\infty} \frac{1}{\log(t)} \mathcal{A}_R(S_t) = \pi_R(\tilde{C})$$

in Hausdorff distance.

Remark. In addition to the remark following theorem 1, the latter theorem justifies the terminology of harmonic tropical curves as limits of harmonic amoebas.

Proof For $m = 1$, both $\mathcal{A}_R$ and π_R are surjective on $\mathbb{R}$, hence there is nothing to prove. Assume $m \geq 2$. One can reduce to the case $m = 2$. Indeed, if for any $1 \leq i < j \leq m$ and any projection

$$\begin{array}{rcl} \rho_{ij} \ : \ \mathbb{R}^m & \to & \mathbb{R}^2 \\ \underline{x} := (x_1, ..., x_m) & \mapsto & (x_i, x_j) \end{array}$$

$\frac{1}{\log(t)}\rho_{ij}\big(\mathcal{A}_R(S)\big)$ converges to $\rho_{ij}\big(\pi_R(\tilde{C})\big)$, then $\frac{1}{\log(t)}\mathcal{A}_R(S)$ converges to $\pi_R(\tilde{C})$. Assume $m = 2$. According to proposition 1.6, there exists a constant M_R independent of t such that

$$\mathcal{A}rea\left(\mathcal{A}_R(S_t)\right) \leq M_R.$$

It implies that

$$\lim_{t\to\infty} \mathcal{A}rea\left(\frac{1}{\log(t)}\mathcal{A}_R(S_t)\right) = 0. \tag{4}$$

For each $v \in V(\tilde{C})$, denote by $Y_{v,t}$ the "pop" corresponding to v in the decomposition of S_t, and $Y_{v,t}^{bd}$ the subset of $Y_{v,t}$ defined in 1.19. For each $e \in E(\tilde{C})$, denote by $K_{e,t}$ the collar associated to the geodesic corresponding e in the decomposition of S_t. For each $e \in L(\tilde{C})$, denote by $K_{e,t}$ the cusp associated to the puncture of S_t corresponding to e. As $\mathcal{A}_R$ is defined by integration and depends on the choice of an initial point, let us move

successively this point on each of the pieces $Y^{bd}_{v,t}$ and $K_{e,t}$ and study the convergence of their respective images. The convergence of ω_{R,S_t} to $\omega_{R,\tilde{C}}$ given by theorem 1 ensures that $\mathcal{A}_R$ is bounded on $Y^{bd}_{v,t}$, uniformly in v and t. It implies that $\frac{1}{\log(t)}\mathcal{A}_R(Y^{bd}_{v,t})$ converges to a point when t goes to ∞. Now, according to lemma 2.4 and theorem 1, the projections on the x- and y-axis of $\frac{1}{\log(t)}\mathcal{A}_R(K_{e,t})$ for any $e \in E(\tilde{C})$ are segments of respective asymptotic length $\left|l(e)\cdot\left(\omega^{(1)}_{R,\tilde{C}}\right)_{\vec{e}}\right|$ and $\left|l(e)\cdot\left(\omega^{(2)}_{R,\tilde{C}}\right)_{\vec{e}}\right|$. Using (4) and the convexity of the connected component of the complement of $\mathcal{A}_R(S_t)$ (see 1.5), one deduces that $\frac{1}{\log(t)}\mathcal{A}_R(K_{e,t})$ converges to a segment of slope $\left(\left(\omega^{(1)}_{R,\tilde{C}}\right)_{\vec{e}},\left(\omega^{(2)}_{R,\tilde{C}}\right)_{\vec{e}}\right)$ and length $l(e)\cdot\sqrt{\left(\omega^{(1)}_{R,\tilde{C}}\right)^2_{\vec{e}}+\left(\omega^{(2)}_{R,\tilde{C}}\right)^2_{\vec{e}}}$, for any $e \in E(\tilde{C})$. For $e \in L(\tilde{C})$, $\frac{1}{\log(t)}\mathcal{A}_R(K_{e,t})$ converges to a half-line of slope $\left(\left(\omega^{(1)}_{R,\tilde{C}}\right)_{\vec{e}},\left(\omega^{(2)}_{R,\tilde{C}}\right)_{\vec{e}}\right)$. Note that this is precisely the image of e by π_R for any $e \in LE(\tilde{C})$.
Now, choose an initial vertex $v_0 \in V(\tilde{C})$, and any sequence of initial points $z_{0,t} \in Y^{bd}_{v_0,t}$ for S_t, and consider the maps $\mathcal{A}_R$ and π_R with respect to these choices of initial points. Putting the pieces $Y^{bd}_{v,t}$ and $K_{e,t}$ together, one deduces that $\frac{1}{\log(t)}\ \mathcal{A}_R(S_t)$ converges to $\pi_R(\tilde{C})$.

□

4 Approximation of complex tropical curves in the plane

4.1 Immersed complex tropical curves

Recall that by proposition 3.4, every simple harmonic morphism on an abstract $\mathbb{T}$-curve is given by a unique collection of residues.

Definition 4.1. *Let $\tilde{C}$ be an abstract $\mathbb{T}$-curve. A simple harmonic morphism $\pi_R : \tilde{C} \to \mathbb{R}^m$ is a simple tropical morphism if R is integer, see definition 3.5.*
A simple tropical curve is the image of an abstract tropical curve by a simple $\mathbb{T}$-morphism, see definition 3.1.

Remark. The latter definition of simple $\mathbb{T}$-morphism and $\mathbb{T}$-curve gives the same objects as in classical theory, see [Mik05] for instance. In the present text, we introduced the more general notions of simple hamonic morphisms and simple harmonic tropical curves and the fact that we added the specific term "harmonic" to descibe more general object can be a bit misleading. One makes such choice here in order to fit with classical terminology and point out once again that simple tropical curves are in fact special instances of simple harmonic tropical curves.

Proposition 4.2. *For any simple $\mathbb{T}$-curve $C \in \mathbb{R}^2$, there exists a unique abstract $\mathbb{T}$-curve $\tilde{C}$ and a $\mathbb{T}$-morphism $\pi : \tilde{C} \to \mathbb{R}^2$ such that $\pi(\tilde{C}) = C$. The $\mathbb{T}$-morphism π is unique up to isometry of the source.*
The abstract $\mathbb{T}$-curve $\tilde{C}$ is called the normalisation of C.

Proof This is a specific instance of proposition 3.3.

□

Definition 4.3. *From now on, we define the vertices (resp. edges, resp. leaves) of a simple $\mathbb{T}$-curve $C \in \mathbb{R}^2$ to be the image of the vertices (resp. edges, resp. leaves) of its normalization $\tilde{C}$, and carry the previous notations $V(C)$, $E(C)$, $L(C)$ and $LE(C)$.*
The points of C having 2 preimages in $\tilde{C}$ are called the nodes of C and form the set $N(C)$.

One comes now to the definition of simple complex tropical curves in the plane. Recall that for a polynomial $f \in \mathbb{C}[x, y]$, the Newton polygon $New(f)$ is the convex hull in $\mathbb{R}^2$ of the set of monomials of appearing in f, seen as points in $\mathbb{Z}^2 \subset \mathbb{R}^2$.

Definition 4.4. *A binomial cylinder $\varsigma \subset (\mathbb{C}^*)^2$ is a set defined by an equation $z^a w^b = c$ with $c \in \mathbb{C}^*$ and $(a, b) \in \mathbb{Z}^2$ a primitive integer vector.*
A general line $\mathcal{L} \in (\mathbb{C}^*)^2$ is a set defined by a polynomial equation $f(z, w) = 0$ where $New(f)$ has euclidean area $1/2$.

Recall that the coamoeba of a curve $\mathcal{C} \subset (\mathbb{C}^*)^2$ is the set denoted $\mathcal{C}o\mathcal{A}(\mathcal{C})$ obtained as the image of $\mathcal{C}$ by the argument map $Arg : (\mathbb{C}^*)^2 \to T$ where $T := \left(\mathbb{R}/2\pi\mathbb{Z}\right)^2$ is the argument torus.
Consider the general line $\mathcal{L}_0 := \left\{(z, w) \in (\mathbb{C}^*)^2 \mid z + w + 1 = 0\right\}$. Its coamoeba $\mathcal{C}o\mathcal{A}(\mathcal{L}_0)$ is the union of the 2 open triangles delimited by the 3 geodesics $\gamma_1 := \{Arg(z) = \pi\}$, $\gamma_2 := \{Arg(w) = \pi\}$ and $\gamma_3 := \{Arg(w) = Arg(z) + \pi\}$ plus their 3 common vertices $(\pi, 0)$, $(0, \pi)$ and (π, π), see figure 2.
As a convention, we choose a framing on each of the γ_i's as follows

* γ_1 is oriented downward, with origin (π, π),
* γ_2 is oriented leftward, with origin $(0, \pi)$,
* γ_3 is oriented up-rightward, with origin $(\pi, 0)$.

These framings are coherent with the up-left triangle of $\mathcal{C}o\mathcal{A}(\mathcal{L}_0)$ (see figure 2). The argument map is orientation preserving on this triangle and orientation reversing on the other. For this reason, the framings are kept globally unchanged by any of the 6 toric automorphisms of $\mathcal{L}_0$.
With this framing, any boundary geodesic is canonically isomorphic as an abelian group to

$$S^1 := \left\{ z \in \mathbb{C} \;\middle|\; |z| = 1 \right\}$$

equipped with the counter-clockwise orientation.

Proposition 4.5. *For any general line $\mathcal{L}$, there exists a toric transformation of $(\mathbb{C}^*)^2$, that is an element $\sigma \in Sl_2(\mathbb{Z}) \rtimes (\mathbb{C}^*)^2$ such that $\sigma(\mathcal{L}) = \mathcal{L}_0$.*
The induced transformation $\sigma_T \in Sl_2(\mathbb{Z}) \rtimes T$ on the argument torus T sends $\mathcal{C}o\mathcal{A}(\mathcal{L})$ onto $\mathcal{C}o\mathcal{A}(\mathcal{L}_0)$.
There is a unique framing of the boundary geodesics of $\mathcal{C}o\mathcal{A}(\mathcal{L})$ that maps to the canonical framings on $\mathcal{C}o\mathcal{A}(\mathcal{L}_0)$ by any $\sigma \in Sl_2(\mathbb{Z}) \rtimes (\mathbb{C}^*)^2$ such that $\sigma(\mathcal{L}) = \mathcal{L}_0$.
For any general line $\mathcal{L}$, we equip $\mathcal{C}o\mathcal{A}(\mathcal{L})$ with these framings.

Proof By Pick's formula, the Newton polygon of any polynomial equation defining $\mathcal{L}$ contains 3 integer points spanning the lattice $\mathbb{Z}^2$. Then, there exists an element of $Sl_2(\mathbb{Z})$ sending this polygon onto the standard simplex of size 1. Translating the coordinates by an appropriate element of $(\mathbb{C}^*)^2$ gives the desired σ. The details are left to the reader.

$\square$

Definition 4.6. *A simple complex tropical curve $V \subset (\mathbb{C}^*)^2$ is a topological Riemann surface such that :*

* *its amoeba $\mathcal{A}(V)$ is a simple $\mathbb{T}$-curve $C \subset \mathbb{R}^2$,*
* *for any $e \in LE(C)$, there exists a binomial cylinder ς such that $\left(\mathcal{A}_{|V}\right)^{-1}(e) = \mathcal{A}^{-1}(e) \cap \varsigma$,*

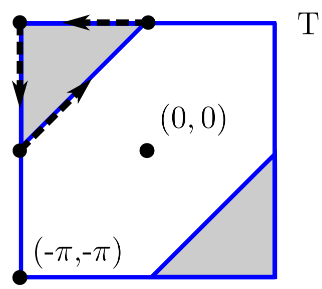


Figure 2: The coamoeba of $\mathcal{L}_0$ (grey), and the framings (black) of its 3 boundary geodesics (blue).

$*$ *for any* $v \in V(C)$, *there exists a general line* $\mathcal{L}$ *such that* $\left(\mathcal{A}_{|V}\right)^{-1}(v) = \overline{\mathcal{C}o\mathcal{A}(\mathcal{L})}$.

Remark. The fact that such simple $\mathbb{CT}$-curve does exist could be formulated as an easy proposition. In order $V \subset (\mathbb{C}^*)^2$ to be a topological Riemann surface, the binomial cylinder ς sitting above an edge $e \in LE(C)$ has to be defined by an equation $z^a w^b = c$ where $(a, b) \in \mathbb{Z}^2$ is a primitive integer vector supporting e. The fact that such cylinders can be glued together by the coamoeba of general line above a vertex $v \in V(C)$ holds on an appropriate choice of the c's and the second point of definition 3.1.

Let us consider a simple $\mathbb{CT}$-curve V, with $C := \mathcal{A}(V)$. For any $e \in E(C)$ and $v_1, v_2 \in V(C)$ its 2 adjacent vertices, the topological cylinder $\left(\mathcal{A}_{|V}\right)^{-1}(e)$ is glued to 2 boundary geodesics γ_1 and γ_2 in $\left(\mathcal{A}_{|V}\right)^{-1}(v_1)$ and $\left(\mathcal{A}_{|V}\right)^{-1}(v_2)$ respectively. Let τ_1, τ_2 be the canonical group isomorphisms $\tau_i : S^1 \to \gamma_i$. Identifying the 2 argument torus $\mathcal{A}^{-1}(v_1)$ and $\mathcal{A}^{-1}(v_2)$, one has that γ_1 and γ_2 are set theoretically identical. This is due to the fact that the coamoeba of a binomial cylinder is single geodesic in T. Under this identification, one has that the map $\tau_2^{-1} \circ \tau_1$ is an orientation reversing isometry of the form

$$\begin{array}{rcl} \tau_2^{-1} \circ \tau_1 \, : \, S^1 & \to & S^1 \\ z & \mapsto & -\overline{e^{i\theta} z}. \end{array}$$

As this isometry is self inverse, the datum $e^{i\theta}$ does not depend on the order we have chosen for γ_1 and γ_2.

Definition 4.7. *Let V be a simple $\mathbb{CT}$-curve, with $C := \mathcal{A}(V)$. For any $e \in E(C)$, the element $e^{i\theta} \in S^1$ constructed above is called the twist parameter of the edge e.*

Remark. A simple $\mathbb{T}$-curve $C \subset \mathbb{R}^2$ is naturally equipped with a ribbon structure: for any $v \in V(C)$, its adjacent edges are ordered cyclically with respect to the counter-clockwise orientation of $\mathbb{R}^2$.

Proposition 4.8. *For a simple $\mathbb{CT}$-curve $V \subset (\mathbb{C}^*)^2$, there exists a unique abstract $\mathbb{CT}$-curve $\tilde{V} := (\tilde{C}, \Theta)$ such that*

* *there exists a $\mathbb{T}$-morphism $\pi : \tilde{C} \to \mathbb{R}^2$ such that $\pi(\tilde{C}) = C$ with $C := \mathcal{A}(C)$,*
* *the ribbon structure on $\tilde{C}$ is the pull-back by π of the natural ribbon structure of C,*
* *Θ is the pull-back by π of the collection of twist parameters of V $\big($recall that π induces a bijection between $E(\tilde{C})$ and $E(C)\big)$.*

The abstract $\mathbb{CT}$-curve $\tilde{V}$ is called the normalization of V.

Proof It falls from proposition 6.27.

□

Remark. A simple $\mathbb{CT}$-curve $V \subset (\mathbb{C}^*)^2$ is a topological oriented Riemann surface obtained by patching triangles and cylinders equipped with a conformal structure. Its normalization $\tilde{V}$ is the necessary and sufficient data to reconstruct abstractly V.

The study of simple $\mathbb{CT}$-curves is motivated by the fact that they arise as limit of classical algebraic curves in the plane in the following way :
as in [Mik05], consider the diffeomorphism

$$\begin{array}{rccl} H_t : & (\mathbb{C}^*)^2 & \to & (\mathbb{C}^*)^2 \\ & (z,w) & \mapsto & \left(|z|^{\frac{1}{\log(t)}}\frac{z}{|z|}, |w|^{\frac{1}{\log(t)}}\frac{w}{|w|}\right) \end{array}.$$

This corresponds to a change of the holomorphic structure of $(\mathbb{C}^*)^2$, see section 6 in [Mik05]. Note that for any holomorphic curve $S \subset (\mathbb{C}^*)^2$,

$$\mathcal{A}\big(H_t(S)\big) = \frac{1}{\log(t)}\mathcal{A}(S).$$

One has the following

Proposition 4.9. *Let $V \subset (\mathbb{C}^*)^2$ be a simple $\mathbb{CT}$-curve, $C := \mathcal{A}(V)$, $\tilde{V} = (\tilde{C}, \Theta)$ its normalization, and R the collection of residue giving the normalization $\mathbb{T}$-morphism $\pi_R : \tilde{C} \to C$. Let $\{S_t\}_{t\in\mathbb{N}, t>>1}$ be a sequence of Riemann surface converging to $\tilde{V}$. Assume moreover that $\mathcal{P}_{R,S_t}$ is a constant family of integer period matrix, then*

$$\lim_{t\to\infty} H_t\big(\iota_R(S_t)\big) = V$$

in Hausdorff distance, see definition 1.7.

Proof To see that $H_t\big(\iota_R(S_t)\big)$ converges to a simple $\mathbb{CT}$-curve, one adopt the same startegy as in theorem 3, cutting S_t into "pop"'s. Let $Y_{v,t}$ be the "pop" of the decomposition of S_t associated to $v \in V(C)$. Up to the action of $Sl_2(\mathbb{Z})$ on the coordinates of ω_{R,S_t}, one can suppose that $(\omega_{R,S_t})_{\vec{e}} = (1,0)$, $(0,1)$ and $(-1,-1)$ for the 3 inward edges adjacent to v. We have already seen in the proof of 3 that choosing the initial point z_0 in $Y_{v,t}^{bd}$, $\mathcal{A}\big(H_t(\iota_R(S_t))\big)$ converges to a tripod T centred at the origin with 3 rays of finite or infinite length pointing in the directions $(-1,0)$, $(0,-1)$ and $(1,1)$. The pair $(Y_{v,t}, \omega_{R,S_t})$ converges to $\big(\mathbb{CP}^1 \setminus \{-1,1,\infty\}, (\frac{dz}{z-1}, \frac{dz}{z+1})\big)$. As this pair of imaginary normalized differentials gives the embedding of $\mathbb{CP}^1 \setminus \{-1,1,\infty\}$ as $\mathcal{L}_0$ in $(\mathbb{C}^*)^2$, it is clear that $Arg\big(H_t(\iota_R(S_t))\big)$ converges in Hausdorff distance to $\overline{\mathcal{C}o\mathcal{A}(\mathcal{L}_0)}$ in the argument torus T. It is an easy exercise to check that $H_t\big(\iota_R(S_t)\big)$ converges in Hausdorff distance to a surface with boundary $V_{v,\infty} \subset (\mathbb{C}^*)^2$ such that $Arg(V_{v,\infty}) = \overline{\mathcal{C}o\mathcal{A}(\mathcal{L}_0)}$, $\mathcal{A}(V_{v,\infty}) = \mathsf{T}$ and such that the fibration of $V_{v,\infty}$ over T is the one of a simple $\mathbb{CT}$-curve over its amoeba, that is $Arg\big(\mathcal{A}^{-1}_{|V_{v,\infty}}(0,0)\big) = \overline{\mathcal{C}o\mathcal{A}(\mathcal{L}_0)}$ and that the fibration over $\mathsf{T} \setminus (0,0)$ is a locally trivial fibration of geodesics in T.
Note that $V_{v,\infty}$ has a unique anti-holomorphic involution, which is the limit of the unique such involution on $Y_{v,t}$. Its fixed locus is also the limit of $\mathbb{R}Y_{v,t}$. It consists of the fiber in $V_{v,\infty}$ by the map Arg over the 3 special points of $\mathcal{C}o\mathcal{A}(\mathcal{L}_0)$.
Gluing back the $Y_{v,t}$ together, one deduces first that $V_\infty := \lim_{t\to\infty} H_t\big(\iota_R(S_t)\big)$ is a simple $\mathbb{CT}$-curve. Secondly, the twists of V_∞ are exactly the twists of V. As we already know from theorem 3 that $\mathcal{A}(V_\infty) = \mathcal{A}(V)$, the proposition is proved.

□

Let us conclude this section with a useful technical lemma

Lemma 4.10. *Let $V \subset (\mathbb{C}^*)^2$ be a simple $\mathbb{CT}$-curve, and $\pi_R : \tilde{V} \to V$ be its normalization. Denote $\tilde{V} := \big(\tilde{C}, \Theta\big)$, and $\omega_{R,\tilde{C}} := \big(\omega^{(1)}_{R,\tilde{C}}, \omega^{(2)}_{R,\tilde{C}}\big)$. For any loop $\rho_{\mathbb{T}} \subset \tilde{C}$ and $1 \leq j \leq 2$, one has*

$$\sum_{e \in \rho_{\mathbb{T}}} \log\big(\Theta(e)\big)\big(\omega^{(j)}_{R,\tilde{C}}\big)_{\vec{e}} \in 2i\pi\mathbb{Z}.$$

Proof Let us denote $C := \mathcal{A}(V)$. Recall that edges, leaves and vertices of C and $\tilde{C}$ are in bijection, and we will use it implicitly here. For any vertex $v \in \rho_{\mathbb{T}}$, there is a distinguished point among the 3 special points of $\big(\mathcal{A}_{|V}\big)^{-1}(v)$, namely the intersection point of the 2 geodesics corresponding to the 2 edges in $\rho_{\mathbb{T}}$ adjacent to v. Let us look at the position of this distinguished point in the argument torus T while going around $\rho_{\mathbb{T}}$. Going from a vertex v to the next one via an edge $\vec{e}$, the distinguished point is translated in the argument torus $T = \big(\mathbb{R}/2\pi\mathbb{Z}\big)^2$ by $\frac{1}{i}\Theta(e)\big(\omega_{R,\tilde{C}}\big)^{\perp}_{\vec{e}}$ where $\big(\omega_{R,\tilde{C}}\big)^{\perp}_{\vec{e}}$ is the rotation of $\big(\omega_{R,\tilde{C}}\big)_{\vec{e}}$ by $\pi/2$. After a full cycle along $\rho_{\mathbb{T}}$, the distinguished point has to end up at its initial place in T. Summing these displacements in the universal cover $\mathbb{R}^2$ of T, it is equivalent to say that

$$\sum_{e \in \rho_{\mathbb{T}}} \frac{1}{i} \log\big(\Theta(e)\big)\big(\omega_{R,\tilde{C}}\big)^{\perp}_{\vec{e}} \in 2\pi\mathbb{Z}^2,$$

which is equivalent to the statment.

□

4.2 A Mikhalkin's approximation theorem

Theorem 4.11 (Mikhalkin). *Let $V \subset (\mathbb{C}^*)^2$ be a simple $\mathbb{CT}$-curve, with normalization $\tilde{V} := (\tilde{C}, \Theta)$, such that $\tilde{C}$ has genus g with n leaves. There exists a family $\{S_t\}_{t \in \mathbb{N}, t >> 1} \subset \mathcal{M}_{g,n}$ together with immersions $\iota_t : S_t \to (\mathbb{C}^*)^2$ such that*

$$\lim_{t\to\infty} H_t\big(\ \iota_R(S_t)\big) = V$$

in Hausdorff distance. Moreover, the sequence of twist functions $\{\Theta_t\}_{t \in \mathbb{N}, t>>1}$ of S_t can be chosen to be constant and equal to Θ.

In the rest of this section, $V \subset (\mathbb{C}^*)^2$ will be a simple $\mathbb{CT}$-curve, the $\mathbb{T}$-curve $\mathcal{A}(V) \subset \mathbb{R}^2$ will be denoted C, and the normalization of V will be denoted $\tilde{V} := (\tilde{C}, \Theta)$. The genus and number of leaves of $\tilde{C}$ will be denoted g and n respectively. At the level of $\mathbb{T}$-curves, the collection of residue giving the normalization morphism will be denoted by R, that is $\pi_R : \tilde{C} \to C$.

According to theorem 3, one is able to approximate any $\mathbb{T}$-curve in the plane by a family of harmonic amoebas $\{\mathcal{A}_R(S_t)\}_{t\in\mathbb{N}}$, for a suitable family $\{S_t\}_{t\in\mathbb{N}} \subset \mathcal{M}_{g,n}$.
We have seen in the prerequisites that for $S \in \mathcal{M}_{g,n}$, and the pair of imaginary normalized differentials $\omega_{R,S} := (\omega_{R,S}^{(1)}, \omega_{R,S}^{(2)})$, if $\int_\gamma \omega_R^{(j)} \in 2i\pi\mathbb{Z}$ for every loop $\gamma \subset S$ and $1 \leq j \leq 2$, then

$$\begin{array}{rcll} \iota_R : & S & \to & (\mathbb{C}^*)^2 \\ & z & \mapsto & \left(e^{\int_{z_0}^z \omega_{R,S}^{(1)}}, e^{\int_{z_0}^z \omega_{R,S}^{(2)}}\right) \end{array}$$

is a well defined holomorphic function such that

$$\mathcal{A}_R(S) = \mathcal{A}\big(\iota_R(S)\big).$$

In the conditions of theorem 4.11, the collection of residues R is determined by the $\mathbb{C}\mathbb{T}$-curve $V \subset (\mathbb{C}^*)^2$. One then need to find a family $\{S_t\}_{t\in\mathbb{N}} \subset \mathcal{M}_{g,n}$ converging to $\tilde{V}$ and such that ω_{R,S_t} has period vectors in $2i\pi\mathbb{Z}^2$ for every loop in S_t, for t large enough.
A first step in this direction is the following

Proposition 4.12. *Let $V \subset (\mathbb{C}^*)^2$ be a simple $\mathbb{C}\mathbb{T}$-curve, $C := \mathcal{A}(V)$, $\tilde{V} := (\tilde{C}, \Theta)$ its normalization and R the collection of residues giving the normalization $\mathbb{T}$-morphism $\pi_R : \tilde{C} \to C$. For any family $\{S_t\}_{t\in\mathbb{N}} \subset \mathcal{M}_{g,n}$ converging to $\tilde{V}$, the family of period matrices $\{\mathcal{P}_{R,S_t}\}_{t\in\mathbb{N}}$ of S_t with respect to R converges to an integer period matrix $\mathcal{P}_{R,\tilde{C}}$.*

Proof According to theorem 1, the period vectors $\frac{1}{2i\pi}\big(\int_{\gamma_t} \omega_{R,S_t}^{(1)}, \int_{\gamma_t} \omega_{R,S_t}^{(2)}\big)$ tends to the vector $\big(\omega_{R,\tilde{C}}\big)_{\vec{e}}$, where γ_t is the geodesic of the decomposition of S_t corresponding to $e \in E(\tilde{C})$. As V is a simple $\mathbb{C}\mathbb{T}$-curve, the collection of residues R is integer on $\tilde{C}$. Hence the latter period vectors tend to be integers. Now let us consider a basis $\rho_{\mathbb{T}}^{(1)}, ..., \rho_{\mathbb{T}}^{(g)} \in H_1(\tilde{C}, \mathbb{Z})$ and consider the associated family of piecewise geodesic loops $\breve{\rho}_t^{(1)}, ..., \breve{\rho}_t^{(g)} \subset S_t$ (see definition 2.5). By theorem 2 and lemma 4.10, one has that

$$\lim_{t\to\infty} \int_{\breve{\rho}_t^{(k)}} \omega_{R,t}^{(j)} \in 2i\pi\mathbb{Z}$$

for $1 \leq k \leq g$ and $1 \leq j \leq 2$. As the γ_t and the $\breve{\rho}_t^{(k)}$ generate $H_1(S_t, \mathbb{Z})$ for any t, the proposition is proved.

$\square$

As we have seen in the prerequisites, the consideration of $\tilde{C}$ with the ribbon structure induced from the one of C, allows to describe $\mathcal{M}_{g,n}$ locally in terms of Fenchel-Nielsen coordinates. Recall that $\tilde{C}$ has exactly $3g-3+n$ edges, and that together with a ribbon structure on $\tilde{C}$, the data of length and twists on every edges of $\tilde{C}$ allows to construct every element of $\mathcal{M}_{g,n}$ via the map

$$\begin{array}{rcl} \mathcal{FN}_{\tilde{C}} \,:\, \left(\mathbb{C}^*\right)^{3g-3+n} & \to & \mathcal{M}_{g,n} \\ (l,\Theta) & \mapsto & S(\tilde{C},l,\Theta) \end{array},$$

see theorem 1.22. Now consider the following partial compactification of $(\mathbb{C}^*)^{3g-3+n}$ at the origin : consider the length factor $(\mathbb{R}_{>0})^{3g-3+n}$ of its polar coordinate system, and embed it in the real oriented blowup of $\mathbb{R}^{3g-3+n}$ at the origin. The latter ambiant space is diffeomorphic to $\left\{\underline{x} \in \mathbb{R}^{3g-3+n} \,\middle|\, |\underline{x}| \geq 1\right\}$ and $(\mathbb{R}_{>0})^{3g-3+n}$ is presented there as

$$\left\{\underline{x} \in (\mathbb{R}_{>0})^{3g-3+n} \,\middle|\, |\underline{x}| > 1\right\}.$$

Consider its partial compactification

$$\left\{\underline{x} \in (\mathbb{R}_{>0})^{3g-3+n} \,\middle|\, |\underline{x}| \geq 1\right\}$$

and denote by $(\mathbb{C}^*)_0^{3g-3+n}$ the product space of $(S^1)^{3g-3+n}$ with the latter partial compactification, and define also the following subset

$$F_0 := (S^1)^{3g-3+n} \times \left\{\underline{x} \in (\mathbb{R}_{>0})^{3g-3+n} \,\middle|\, |\underline{x}| = 1\right\}.$$

The points of F_0 can be naturally considered as equivalence classes of sequences $\{\underline{z}_n\}_{n\in\mathbb{N}} \subset (\mathbb{C}^*)^{3g-3+n}$ converging to the origin and such that $\lim\limits_{n\to\infty} Arg(\underline{z}_n)$ exists coordinate wise, up to the equivalence relation

$$\begin{gathered} \{\underline{z}_n\} \sim \{\underline{\mathrm{z}}_n\} \text{ if } \lim_{n\to\infty} Arg(\underline{z}_n) = \lim_{n\to\infty} Arg(\underline{\mathrm{z}}_n) \\ \text{and } \exists \lambda > 0 \text{ such that } \lim_{n\to\infty} \frac{|\underline{z}_n|}{|\underline{\mathrm{z}}_n|} = (\lambda, ..., \lambda). \end{gathered}$$

Let $\tilde{\mathscr{C}}$ be an abstract $\mathbb{T}$-curve combinatorially equivalent to $\tilde{C}$, that is supported on the same cubic graph. According to the definition 2.6, any sequence $\{S_t\}_{t\in\mathbb{N}}$ converging to a $\mathbb{CT}$-curve $\tilde{\mathscr{V}} = (\tilde{\mathscr{C}}, \mathscr{O})$ can be lifted by $H_{\tilde{C}}$ to a sequence $\{\underline{z}_n\}_{n\in\mathbb{N}} \subset (\mathbb{C}^*)^{3g-3+n}$ converging in $(\mathbb{C}^*)_0^{3g-3+n}$. Hence, F_0 can be naturally considered as equivalence classes of $\mathbb{CT}$-curves $\tilde{\mathscr{V}} = (\tilde{\mathscr{C}}, \mathscr{O})$ such that $\tilde{\mathscr{C}}$ is combinatorially equivalent to $\tilde{C}$, up to the equivalence relation

$$\begin{array}{c} \tilde{\mathscr{V}_1} = (\tilde{\mathscr{C}_1}, \mathscr{O}_1) \sim \tilde{\mathscr{V}_2} = (\tilde{\mathscr{C}_2}, \mathscr{O}_2) \text{ if } \mathscr{O}_1 = \mathscr{O}_2 \text{ and} \\ \exists \lambda > 0 \text{ such that } \tilde{\mathscr{C}_1} = \lambda \tilde{\mathscr{C}_2} \text{ as metric spaces.} \end{array}$$

Clearly, the map $\mathcal{FN}_{\tilde{C}}$ extends to $(\mathbb{C}^*)_0^{3g-3+n} \to \overline{\mathcal{M}}_{g,n}$, and F_0 is mapped to $S_{\tilde{C}}$, the stable curve of dual graph $\tilde{C}$. Now, consider the map

$$\begin{array}{rcl} \Pi_R \, : \, (\mathbb{C}^*)^{3g-3+n} & \to & M_{(2g+n-1)\times 2}(\mathbb{R})/Sl_{(2g+n-1)}(\mathbb{Z}) \\ (l, \mathscr{O}) & \mapsto & \mathcal{P}_{R,S(\tilde{C},l,\mathscr{O})} \end{array}$$

which associates to $(l, \mathscr{O})$ the period matrix of $S(\tilde{C}, l, \mathscr{O})$ with respect to R, see definition 1.7.

Proposition 4.13. Π_R *extends to* $(\mathbb{C}^*)_0^{3g-3+n}$. *Let* $[\tilde{\mathscr{V}}]$ *be the equivalence class of* $\tilde{\mathscr{V}}$ *in* F_0, *then the level set* $\Pi_R^{-1}\big(\Pi([\tilde{\mathscr{V}}])\big) \subset (\mathbb{C}^*)_0^{3g-3+n}$ *is an analytic subset of real codimension* $4g$ *intersecting* F_0 *at the locus of classes* $[\tilde{\mathscr{V}}]$ *with* $\tilde{\mathscr{V}} = (\tilde{\mathscr{C}}, \mathscr{O})$ *satisfying*

$$\sum_{e\in\rho} l(e)(\omega_{R,\tilde{\mathscr{C}}})_{\vec{e}} = (0,0) \in \mathbb{R}^2 \tag{5}$$

$$\sum_{e\in\rho} \log\big(\Theta(e)\big)(\omega_{R,\tilde{\mathscr{C}}})_{\vec{e}} = (0,0) \in (S^1)^2 \tag{6}$$

for any loop $\rho \subset \tilde{\mathscr{C}}$.

Proof From the proof of lemma 4.10, one remark that the limiting period matrix of a family S_t converging to a $\mathbb{CT}$-curve $\tilde{\mathscr{V}} = (\tilde{\mathscr{C}}, \mathscr{O})$ depends only on the pair of exact 1-forms $\omega_{R,\tilde{\mathscr{C}}}$ on $\tilde{\mathscr{C}}$ and $\mathscr{O}$. These objects are kept fixed within the class $[\tilde{\mathscr{V}}]$. It follows that Π_R extends to $(\mathbb{C}^*)_0^{3g-3+n}$.
For any simply connected domain $\mathcal{U} \subset (\mathbb{C}^*)^{3g-3+n}$, one can trivialize $H_1(S,\mathbb{Z}) \simeq H$ for any $S \in \mathcal{FN}_{\tilde{C}}(\mathcal{U})$. Now fix a basis $\rho_1, ..., \rho_{2g+n-1}$ of H such that $\rho_1, ..., \rho_{n-1}$ are given by small loops around $n-1$ of n punctures, and consider the period matrix $\mathcal{P}_{R,S} \in M_{(2g+n-1)\times 2}(\mathbb{R})$ of any curve $S \in \mathcal{U}$

with respect to the latter basis of H. This gives a map to $\mathbb{R}^{2(2g+n-1)}$, but the $2(n-1)$ coordinates corresponding to $\rho_1, ..., \rho_{n-1}$ are constant as they just compute the residues at the punctures. Hence, one has at most $4g$ non trivial coordinate functions. This functions are in fact analytic and define smooth level sets of codimension exactly $4g$. One refers to theorem 2.6 and lemma 2.4 of [GK10], for the proof.

Now, let $\tilde{\mathscr{V}} = (\tilde{\mathscr{C}}, \mathscr{O})$ be a $\mathbb{CT}$-curve such that $[\tilde{\mathscr{V}}] \in \Pi_R^{-1}\big(\Pi([\tilde{V}])\big)$. Recall that $\tilde{\mathscr{C}}$ has to have the same combinatorial type as $\tilde{C}$. As a consequence, their edges are in natural bijection. The bunch of equation (5) is equivalent to saying that the pairs of exact 1-forms $\omega_{R,\tilde{C}}$ on $\tilde{C}$ and $\omega_{R,\tilde{\mathscr{C}}}$ on $\tilde{\mathscr{C}}$ are equal in the following sense : for any $\vec{e} \in E(\tilde{\mathscr{C}}) = E(\tilde{C})$, $(\omega_{R,\tilde{C}})_{\vec{e}} = (\omega_{R,\tilde{\mathscr{C}}})_{\vec{e}}$. These are necessary conditions, as these values are indeed limits of corresponding periods by theorem 1. For the bunch of equations (6), the left-hand side can be seen as a limit period vector in $\mathbb{R}^2$ for a well chosen basis $\rho_1, ..., \rho_{2g+n-1}$ of H, see theorem 2. The equations (6) are equivalent to the fact that the associated periods are in the lattice $2i\pi\mathbb{Z}^2$, which is necessary by lemma 4.10. In order to see that those conditions are also sufficient, one can extract out of them a subset of $4g$ linearly independent conditions. Clearly the conditions (5) are independent of the conditions (6), and both have the same rank. The fact that the conditions of (5) are $2g$ dimensional is shown in [Mik05]. Proposition 2.23 of the latter reference shows that (5) defines a polyhedral domain on the l-factor, interior of which is of the expected codimension.

□

Proof of theorem 4.11 Let $p_1 : (\mathbb{C}^*)_0^{3g-3+n} \to (S^1)^{3g-3+n}$ be the projection on the factor of twists. Note that, as $\tilde{V}$ is the normalization of the simple $\mathbb{CT}$-curve V, the class $[\tilde{V}]$ is in the interior of the polyhedral domain $F_0 \cap \Pi_R^{-1}\big(\Pi_R([\tilde{V}])\big)$. By the previous lemma, one has the following transversal intersection

$$\Big(\Pi_R^{-1}\big(\Pi_R([\tilde{V}])\big) \cap F_0\Big) \pitchfork \big(p_1^{-1}(\Theta) \cap F_0\big)\,.$$

Hence, the intersection of $\Pi_R^{-1}\big(\Pi_R([\tilde{V}])\big)$ and $p_1^{-1}(\Theta)$ stays transversal for a small enough neighbourhood $\mathcal{U} \subset (\mathbb{C}^*)_0^{3g-3+n}$ of $[\tilde{V}]$ and $\mathcal{U} \cap p_1^{-1}(\Theta) \cap \Pi_R^{-1}\big(\Pi_R([\tilde{V}])\big)$ is a smooth subvariety of dimension $g-3+n$. Hence, for t large enough, it is possible to construct a sequence of points $\{\underline{z}_t\}_{t\in\mathbb{N},t>>1} \subset \mathcal{U} \cap p_1^{-1}(\Theta) \cap \Pi_R^{-1}\big(\Pi_R([\tilde{V}])\big)$ converging to $[\tilde{V}]$ and such that the sequence

$$S_t := \mathcal{FN}_{\tilde{C}}(\underline{z}_t)$$

converges to $\tilde{V}$. The question of finding such a sequence is just about choosing $|z_t|$ coherently to definition 2.1. Now, S_t has been constructed such that the period matrix $\mathcal{P}_{R,S_t}$ is constant and integer, thanks to lemma 4.10. Hence, one can conclude by applying proposition 4.9.

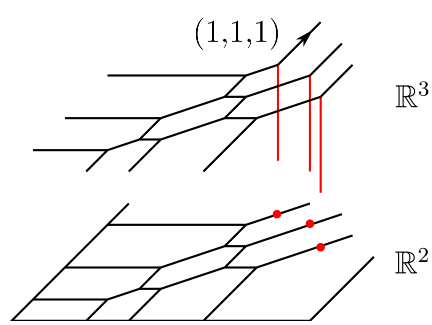
□

5 Discussions

5.1 Superabundancy and transversality of the phases

A tropical immersion $\pi : \tilde{\mathcal{C}} \to \mathbb{R}^m$ is called superabundant if its deformation space is bigger than expected, see [Mik05]. A simple instance of superabundancy is given when a cycle of the tropical curve $\tilde{\mathcal{C}}$ is mapped to a strict linear subspace of $\mathbb{R}^m$.

Superabundant immersions are possible candidate for not being approximable by algebraic immersions. Let us give an example : consider the tautological embedding of a regular projective tropical cubic in $\mathbb{R}^2$. Now embed this plane as $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. By the above remark, the resulting embedding in $\mathbb{R}^3$ is superabundant. So far, this tropical embedding is still approximable by algebraic ones as the embedding $\pi : \tilde{\mathcal{C}} \to \mathbb{R}^2$ is. Now choose 3 points on the planar cubic, and consider the modification of the cubic at this points. Assume for simplicity that these points are chosen on the leaves of direction $(1, 1)$. It gives a tautological embedding of an abstract tropical elliptic curve in $\mathbb{R}^3$, as shown below.



Let us denote by $\pi_3 : \tilde{\mathcal{C}} \to \mathbb{R}^3$ this tropical embedding. By classical theory, one knows that every cubic in the 3-space is contained in a plane. Hence, in order our tropical elliptic curve to be approximable by algebraic ones, it has

to sit in a tropical plane. This condition is equivalent for the 3 chosen points on the planar cubic to sit on a tropical line.
By theorem 3, the tropical morphism π_3 is approximable by a sequence of harmonic maps, and $\pi_3(\tilde{\mathcal{C}})$ arise as the Hausdorff limit of the corresponding sequence of harmonic amoebas. In order to approximate π_3 by algebraic map, one also have to deal with phases, as we did to prove theorem 4.11. More precisely one need to construct a sequence of imaginary normalized differentials having integer periods. Obviously, it can be done for the 2 differentials giving the first 2 coordinate functions, as the projection of $\pi_3(\tilde{\mathcal{C}})$ to the (x, y)-plane is approximable by algebraic curves. For the z-coordinate, we are facing a non transversality problem. For all the possible deformation of $\tilde{\mathcal{C}}$ within the same combinatorial type, the exact 1-forms having residues 1 at the 3 vertical leaves and -1 at the leaves of slope $(1, 1, 1)$ are supported exactly on this 6 leaves and are zero everywhere else.
Coming back to the formalism of the last subsection, denote by R the collection of residues just given above on these 6 leaves. The observation we made means that the level set of the period map Π_R we are interested in does not intersect transversally the exceptional divisor F_0 of the partial compactification $(\mathcal{C}^*)_0^{12}$. This level set is indeed reducible, one of its components being F_0. There is another irreducible component not contained in F_0 and having a transversal intersection with it. It corresponds to the deformations of $\tilde{\mathcal{C}}$ for which the 3 points of the modification lie on a tropical line. Equivalently, these are the curves for which the harmonic function giving the z-coordinate comes from a holomorphic function. Hence, one cannot leave F_0 while staying in the specified level set if one does not already lie in the latter irreducible component.

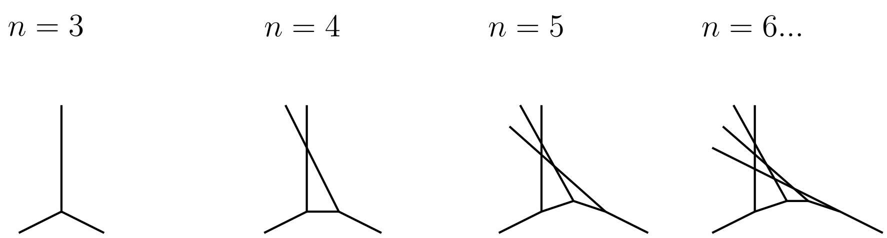
5.2 The complement of harmonic amoebas

In [Kri], Krichever asked whether one can determine an upper bound $\nu(g, n)$ for the number of connected components of the harmonic amoeba $\mathcal{A}_R(S) \subset \mathbb{R}^2$ independent of the collection of residues R and the Riemann surface $S \in \mathcal{M}_{g,n}$. This question boils down to the study of the complement of their underlying spines. It is not clear a priori how one can equip such spine with extra structure in order to make it an harmonic tropical curve in a natural way. Nevertheless, one can already ask the latter question while restricting to simple harmonic tropical curves of genus g and with n leaves.
Note first that one can assume that all the leaves have pairwise distinct

directions. Indeed, if 2 leaves have the same direction, slanting these leaves one toward the other would produce an extra component of the complement. Let us denote by $\nu_{\mathbb{T}}(g,n)$ the number of connected components of the the complement of simple harmonic tropical of genus g with n leaves. One has

$$\nu_{\mathbb{T}}(g,n) \leq \nu(g,n).$$

Determining the $\nu_{\mathbb{T}}(g,n)$ seems to be already a chalenging combinatorial question. For $g=0$ at least, one can provide a simple recursive formula : in order to maximalize the number of connected components of the complement, one has to maximize the number of intersection points between the leaves. There are at least 2 leaves that cannot intersect. Starting from $n=3$, one construct a sequence of simple harmonic tropical curves we claim to be maximal regarding to our problem. The construction is pictured below. Note



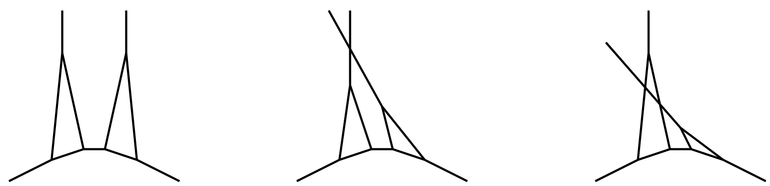
that as long as the $(n-2)$ leaves pointing upward does intersect pairwise, the combinatorial type of their arrangement doesn't matter. It gives the recursive formula $\nu_{\mathbb{T}}(0,n+1)=\nu_{\mathbb{T}}(0,n)+(n-1)$ for $n\geq 3$, and $\nu_{\mathbb{T}}(0,3)=3$, hence

$$\nu_{\mathbb{T}}(0,n)=\frac{(n-2)(n-1)}{2}+2$$

for $n\geq 3$. In the case $g=1$, no leaf can intersect the only cycle of the curve. Similarily to the rational case, one gets

$$\nu_{\mathbb{T}}(1,n)=\frac{(n-2)(n-1)}{2}+3.$$

Starting from $g=2$, some leaves can intersect cycles of the curve, and the number of connected component depends on it, as shown below.



We claim at least that

$$\nu_{\mathbb{T}}(g,n) \geq g(n-2) + \frac{(n-2)(n-1)}{2} + 2.$$

Part II
On generalized Harnack curves

Intoduction The main subject of this second part is the study of a generalization of simple Harnack curves as introduced by Mikhalkin in [Mik00]. In the latter, he considered the question of existence of real projective plane curve of degree d in maximal position with respect to a given set of lines. Maximal position means that there exists one oval of its real locus intersecting each line d times, one after the other. He also demanded the curve to be an M-curve. This mean that its real locus has the maximal number of connected components, which is $\frac{(d-1)(d-2)}{2} + 1$ by Harnack's inequality. He showed that for 3 lines, that can be taken to be the coordinate axes, there exists such curves and that the topological type of the arrangement is unique. For 4 lines and more, no such curve exists. He noticed that the question can also be asked for any toric surface, where lines are replaced by the toric divisors at infinity. In this situation one obtains similar existence and uniqueness results.
In the projective case, these curves were constructed by Harnack, before Hilbert announced his sixteenth problem. For general toric surfaces, the existence is guaranteed by using Viro's combinatorial patch-working. The uniqueness of the topological type of arrangement proved in [Mik00] follows from a careful study of amoebas of plane curves. Since then, simple Harnack curves have been shown to enjoy many different equivalent definitions, of different nature. In [MR01], the authors show that simple Harnack curves are the only one for which the inequality provided by [PR04] on area of amoebas is sharp. Simple Harnack curves are also characterized by the fact that their amoeba map is at most 2-to-1. They also have maximal logarithmic curvature, see [PR11]. Finally, one can show that after appropriate compactification, the argument map realizes a covering from simple Harnack curves to the argument torus, blown up at the origin, see [MO07].
Simple Harnack curves surprisingly appeared as spectral curves in physical model. One example is given in [CD13] for the Ising model on the torus. Going backwards, the authors of [KOS06] presented simple Harnack curves as spectral curves of dimer configurations on the torus, in relation with random surfaces and crystallography. A bit later, some of the previous authors gave a surprising parametrization of the space of projective simple Harnack curves,

see [KO06]. As we have seen before, amoebas have thin tentacles going off to infinity, and the connected components of their complement are convex. The intercept of the tentacles together with the area of the compact components of the complement give global coordinates on the space of simple Harnack curves. They also show that consideration of their spine as defined in [PR04] gives a local diffeomorphism between such curves and planar tropical curve of the same degree.
We believe that simple Harnack curves have yet to reveal all of their properties. This is one reason why we suggest a simple generalization. Let us now describe more precisely the content of the second part of this thesis, section by section.

5.First we recall more precisely the general theory of Harnack curves, and tropical curves. Some of the prerequisites might be repeated from the first part of this work.

6. We give the definition of generalized Harnack curves. The characterization given in [PR11] can be reformulated by saying that the logarithmic Gauss map of a simple Harnack curve is totally real, this means that the pull-back of the real part of the target is exactly the real part of the source. We simply keep this characterization and relax any smoothness assumption. Note that Mikhalkin's Harnack curve were originally assumed to be smooth in [Mik00]. Later in [MR01], the authors showed that in the closure, simple Harnack curves can only get real isolated double points as singularities. Going forth, we generalize in a straightforward way the characterization given in [MO07].

> **Theorem 4** *"Up to an appropriate compactification, Harnack curves are the only planar curves for which the argument map realizes a covering on the argument torus, blown up at the origin."*

This characterization is not the most tractable, but it has a very useful application. It "determines" the area of the amoeba and the coamoeba of Harnack curves.

7. We define tropical Harnack curves in the most practical way. We show that one can equipped these curves with phases in order to get very particular complex tropical curves in the plane. Using Mikhalkin's approximation theorem, one shows

Theorem 5 *"Approximation of tropical Harnack curve produces Harnack curves with prescribed topology."*

As an illustration, we show how to construct Harnack curves with hyperbolic nodes, cusps, or complex conjugated double points. We end up this section by classifying all topological type that can be obtained from tropical Harnack curves with a single hyperbolic node.

8. In the last section, we undertake the study of Harnack curves with a single hyperbolic node. The main result is

Theorem 6 *"All the topological type for Harnack curves with a single hyperbolic node can be constructed by tropical methods."*

On the way we show

Theorem 7 *"The spine of a generic simple Harnack curve with a single hyperbolic node is a tropical Harnack curve with a single hyperbolic node, and that these two spaces of curves are locally diffeomorphic."*

and

Theorem 8 *"The argument map lift to the universal covering of the argument torus while restricted to half of the curve.It realizes a diffeomorphism onto the complementary in the Newton Polygon of the curve, of a unit square located in one of the corners."*

6 Prerequisites

6.1 Subtropical geometry of planar curves

In this text, $\mathcal{C}^\circ \subset (\mathbb{C}^*)^2$ will denote an algebraic curve in the complex 2-torus. Such curve can be defined as the zero set of a Laurent polynomial

$$f \in \mathbb{C}\left[z^{\pm 1}, w^{\pm 1}\right].$$

Any monomial $z^\alpha w^\beta$ is a point of the space of characters over $(\mathbb{C}^*)^2$, this space being naturally isomorphic to $\mathbb{Z}^2$. The Newton polygon $New(f)$ is defined as the convex hull in $\mathbb{R}^2 = \mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{R}$ of the monomials $z^\alpha w^\beta$ appearing in f. Two Laurent polynomials g and f define the same zero set in $(\mathbb{C}^*)^2$ if and only if there exists $a \in \mathbb{C}^*$ and $(\alpha, \beta) \in \mathbb{Z}^2$ such that

$$f(z, w) = a z^\alpha w^\beta g(z, w).$$

Then, $New(f)$ is the translation of $New(g)$ by (α, β). As a convention, one will always consider polynomials $f \in \mathbb{C}[z, w]$ such that $New(f)$ touches both the z- and w- axes. According to this convention, any curve $\mathcal{C}^\circ \subset (\mathbb{C}^*)^2$ is defined by a polynomial f, unique up to a multiplicative constant $a \in \mathbb{C}^*$. The Newton polygon $New(f)$ is uniquely determined by $\mathcal{C}^\circ$ and will be denoted Δ.

The Newton polygon Δ of $\mathcal{C}^\circ$ induces a toric compactification $(\mathbb{C}^*)^2 \subset \mathcal{T}_\Delta$, whenever the interior of Δ is 2-dimensional. It can be constructed as the closure of the image of the following map

$$\begin{array}{ccc} (\mathbb{C}^*)^2 & \to & \mathbb{CP}^m \\ (z, w) & \mapsto & \left[z^{\alpha_0} w^{\beta_0} : ... : z^{\alpha_m} w^{\beta_m} \right] \end{array}$$

where $\Delta \cap \mathbb{Z}^2 =: \{(\alpha_0, \beta_0), ..., (\alpha_m, \beta_m)\}$. The action of $(\mathbb{C}^*)^2$ onto itself extends to $\mathcal{T}_\Delta$. $(\mathbb{C}^*)^2$ is an open dense orbit in $\mathcal{T}_\Delta$. In addition, there is a $\mathbb{C}^*$-orbit for each side of Δ, and an orbit reduced to a single point for each vertex of Δ. Denote

$$S^1 := \left\{ z \in \mathbb{C} \;\middle|\; |z| = 1 \right\}.$$

We will also describe S^1 as $\mathbb{R}/2\pi\mathbb{Z}$ without any difference. As a subgroup of $(\mathbb{C}^*)^2$, $\left(S^1\right)^2$ acts on $\mathcal{T}_\Delta$. The moment map $\mu : \mathcal{T}_\Delta \to \Delta$ is the quotient map of $\mathcal{T}_\Delta$ by the latter action. On the torus, it is given by

$$\mu(z, w) = \frac{\displaystyle\sum_{1 \le j \le m} |z^{\alpha_j} w^{\beta_j}| (\alpha_j, \beta_j)}{\displaystyle\sum_{1 \le j \le m} |z^{\alpha_j} w^{\beta_j}|}.$$

For any side s of Δ, the toric divisor at infinity associated to s is defined by $\mathcal{D}_s := \mu^{-1}(s)$. It is isomorphic to $\mathbb{CP}^1$ and compactify one of the $\mathbb{C}^*$-orbit mentioned above. For any vertex v of Δ, $\mu^{-1}(v)$ is one of the orbits reduced to a single point. We will refer to it as the vertices of $\mathcal{T}_\Delta$.

We will denote by $\mathcal{C} \subset \mathcal{T}_\Delta$ the closure of $\mathcal{C}^\circ$. The curve $\mathcal{C}$ intersects every divisor $\mathcal{D}_s$ with multiplicity $|s \cap \mathbb{Z}^2| - 1$. A curve $\mathcal{C} \subset \mathcal{T}_\Delta$ constructed as the closure of a curve $\mathcal{C}^\circ \subset (\mathbb{C}^*)^2$ of Newton polygon Δ never contains any vertex of $\mathcal{T}_\Delta$. We will always restrict to this case while considering curve $\mathcal{C} \subset \mathcal{T}_\Delta$. We define the points at infinity of $\mathcal{C}$ as $\mathcal{C}_\infty := \mathcal{C} \setminus \mathcal{C}^\circ$.We will also denote

$$b := |\partial\Delta \cap \mathbb{Z}^2| \quad \text{and} \quad g := |Int\,\Delta \cap \mathbb{Z}^2|.$$

By [Kho78], g is the arithmetic genus of $\mathcal{C}$ and b is the intersection multiplicity of $\mathcal{C}$ with the union of the divisor at infinity. In particular, if $\mathcal{C}$ intersects each of them transversally, then $b = |\mathcal{C}_\infty|$.

Up to a well chosen diffeomorphism $\mathbb{R}^2 \to Int\,\Delta$, the moment map μ extends the map

$$\begin{array}{rccc} \mathcal{A} : & (\mathbb{C}^*)^2 & \to & \mathbb{R}^2 \\ & (z,w) & \mapsto & \big(\log|z|, \log|w|\big) \end{array}.$$

Definition 6.1. *Let $\mathcal{C}^\circ \subset (\mathbb{C}^*)^2$ be an algebraic curve. The amoeba of $\mathcal{C}^\circ$ is the subset $\mathcal{A}(\mathcal{C}^\circ) \subset \mathbb{R}^2$. With a slight abuse, we will as well refer to the latter subset as the amoeba of $\mathcal{C}$ and denote it $\mathcal{A}(\mathcal{C})$.*

Proposition 6.2 (see [FPT00])**.** *For an algebraic curve $\mathcal{C}^\circ \subset (\mathbb{C}^*)^2$, its amoeba $\mathcal{A}(\mathcal{C}^\circ)$ is a closed subset of $\mathbb{R}^2$. Moreover, every connected component of $\mathbb{R}^2 \setminus \mathcal{A}(\mathcal{C}^\circ)$ is convex.*

Denote by $arg(z) \in S^1$ the argument of the complex number $z \in \mathbb{C}^*$, the argument torus $T := \big(S^1\big)^2$ and

$$\begin{array}{rccc} Arg : & (\mathbb{C}^*)^2 & \to & T \\ & (z,w) & \mapsto & \big(arg(z), arg(w)\big) \end{array}.$$

Definition 6.3. *Let $\mathcal{C}^\circ \subset (\mathbb{C}^*)^2$ be an algebraic curve. The coamoeba $\mathcal{C}o\mathcal{A}(\mathcal{C}^\circ)$ of $\mathcal{C}^\circ$ is the subset $Arg(\mathcal{C}^\circ) \subset \mathbb{R}^2$. With a slight abuse, we will as well refer to the latter subset as the coamoeba of $\mathcal{C}$ and denote it $\mathcal{C}o\mathcal{A}(\mathcal{C})$.*

Amoebas and coamoebas are related by the fact that they are respectively real and imaginary part of algebraic curves in logarithmic coordinates. Interplays between them can be often described in term of the logarithmic

Gauss map. For a smooth curve $\mathcal{C} \subset \mathcal{T}_\Delta$, the logarithmic Gauss $\gamma : \mathcal{C} \to \mathbb{CP}^1$ is given on $\mathcal{C}^\circ$ by

$$\gamma(z, w) = [z \cdot \partial_z f(z, w) : w \cdot \partial_w f(z, w)]$$

where f is a polynomial equation for $\mathcal{C}^\circ$. Geometrically, this map is locally the composition of the coordinate wise complex logarithm with the classical Gauss map which associates to every point of a smooth hypersurface its tangent hyperplane. Even though the complex logarithm is multivalued, this map is well defined. In the case of singular curves $\mathcal{C}$, γ might not be define globally. It is always well defined on the smooth part of $\mathcal{C}$. If $\pi : \tilde{\mathcal{C}} \to \mathcal{C}$ is the normalization of $\mathcal{C}$, γ defines then a rational map on $\tilde{\mathcal{C}}$. By removable singularity theorem, it extends to an algebraic map $\tilde{\gamma} : \tilde{\mathcal{C}} \to \mathbb{CP}^1$.

Definition 6.4. *Let $\mathcal{C} \subset \mathcal{T}_\Delta$ be a curve, possibly singular, and $\pi : \tilde{\mathcal{C}} \to \mathcal{C}$ its normalization. The logarithmic Gauss map of $\tilde{\mathcal{C}}$ is the map $\tilde{\gamma} : \tilde{\mathcal{C}} \to \mathbb{CP}^1$ defined above.*

For a singular curve $\mathcal{C} \subset \mathcal{T}_\Delta$, we will always denote its normalization by $\pi : \tilde{\mathcal{C}} \to \mathcal{C}$. We will also denote $\tilde{\mathcal{C}}^\circ := \pi^{-1}(\mathcal{C}^\circ)$ and $\tilde{\mathcal{C}}_\infty := \tilde{\mathcal{C}} \setminus \tilde{\mathcal{C}}^\circ$. Remark that our assumptions guarantees that $\mathcal{C}$ can only have singularities inside $(\mathbb{C}^*)^2$. In particular, $\mathcal{C}_\infty = \tilde{\mathcal{C}}_\infty$.

An alternative description of γ (and then $\tilde{\gamma}$) can be given. Looking at z and w as 2 meromorphic functions on $\tilde{\mathcal{C}}$, consider the 2 meromorphic differentials $d\log(z)$ and $d\log(w)$ on $\mathcal{C}$. The quotient of 2 such differentials defines a meromorphic function on $\tilde{\mathcal{C}}$. One has

Proposition 6.5. *The Logarithmic Gauss map $\tilde{\gamma} : \tilde{\mathcal{C}} \to \mathbb{CP}^1$ is given by*

$$p \mapsto \left[-d\log\big(w(p)\big) : d\log\big(z(p)\big)\right].$$

Moreover, the degree of $\tilde{\gamma}$ is $-\chi\big(\tilde{\mathcal{C}}^\circ\big)$.

Proof If p is a local coordinate on $\tilde{\mathcal{C}}^\circ$, then

$$\left[-d\log\big(w(p)\big) : d\log\big(z(p)\big)\right] = \left[-\frac{d}{dp}\log\big(w(p)\big) : \frac{d}{dp}\log\big(z(p)\big)\right].$$

Obviously, as $\tilde{\mathcal{C}}^\circ$ is immersed in $(\mathbb{C}^*)^2$, $\big(\frac{d}{dp}\log\big(z(p)\big), \frac{d}{dp}\log\big(w(p)\big)\big)$ is a non zero tangent vector for the coordinatewise logarithm of $\mathcal{C}^\circ$ at the point corresponding to p. Hence, this tangent plane is given by the equation

$$-d\log\big(w(p)\big)\cdot u + d\log\big(z(p)\big)\cdot v = 0,$$

which proves the first part of the statement.
The degree of $\tilde{\gamma}$ can be computed as the number of zeroes of $d\log(w)\cdot u + d\log(z)\cdot v$ for a generic non zero vector (u,v). By genericity, $\tilde{\mathcal{C}}_\infty$ is exactly the set of poles of this differential and all of them are simple. By Riemann-Roch, such differential has degree $2\,b_1(\tilde{\mathcal{C}})-2$. Hence, it has $2\,b_1(\tilde{\mathcal{C}})+|\tilde{\mathcal{C}}_\infty|-2$ zeroes.

□

Lemma 6.6. *Let s be a side of Δ and $(a,b)\in\mathbb{Z}^2$ a primitive integer vector supporting s. For any curve $\mathcal{C}\subset\mathcal{T}_\Delta$, and any point $p\in\mathcal{D}_s\cap\mathcal{C}$, one has*

$$\gamma(p) = [a:b]\,.$$

Proof Suppose first that neither a nor b is zero. By implicit function theorem, for any local coordinate t of $\mathcal{C}$ centred at p, there exists 2 holomorphic functions $h_z(t)$ and $h_w(t)$ having a simple zero at the origin and a positive integer m such that

$$z(t) = \big(h_z(t)\big)^{-bm} \text{ and } w(t) = \big(h_w(t)\big)^{am}.$$

The number m is exactly the intersection multiplicity $\mathcal{D}_e\cap\mathcal{C}$ at p. By 6.5,

$$\begin{aligned}\gamma(p) &= \lim_{t\to 0}\left[-d\log\big(w(t)\big) : d\log\big(z(t)\big)\right]\\ &= \lim_{t\to 0}\left[-am\cdot\frac{h'_w(t)}{h_w(t)} : -bm\cdot\frac{h'_z(t)}{h_z(t)}\right]\\ &= [a:b]\,.\end{aligned}$$

If a (*resp.* b) is zero, h_w (*resp.* h_z) is a non vanishing holomorphic function. The same computation leads to the result.

□

The map $\mathcal{A} : \tilde{\mathcal{C}}^\circ \to \mathbb{R}^2$ is a map between smooth surfaces. Following [Mik00], denote by $\tilde{F}^\circ\subset\tilde{\mathcal{C}}^\circ$ the critical locus of $\mathcal{A}$, that is the set of points where $\mathcal{A}$ is not submersive. Denote by $\tilde{F}$ its closure in $\tilde{\mathcal{C}}$.

Lemma 6.7.
$$\tilde{F} = \tilde{\gamma}^{-1}\big(\mathbb{RP}^1\big).$$
Moreover, $\tilde{F}$ is also the closure of the critical locus of the map $Arg : \tilde{\mathcal{C}}^\circ \to T$.

Proof Let $\tilde{p} \in \tilde{\mathcal{C}}^\circ$ and denote $\pi(\tilde{p}) =: (p_1, p_2) =: p$. The point $\tilde{p}$ is in $\tilde{F}^\circ$ if $T_p\mathcal{C}^\circ$ contains a vector v tangent to the torus $|z| = |p_1|$, $|w| = |p_2|$ (if p is a singular point of $\mathcal{C}^\circ$, consider the tangent line in $T_p\mathcal{C}^\circ$ corresponding to $\tilde{p}$). Equivalently, z has purely imaginary logarithmic coordinates. This hold if and only if $\tilde{\gamma}(\tilde{p}) \in \mathbb{RP}^1$.
Similarly, $\tilde{p}$ is a critical point for the map Arg if $T_p\mathcal{C}^\circ$ contains a vector v tangent to $arg(z) = arg(p_1)$, $arg(w) = arg(p_2)$, *i.e.* z has real logarithmic coordinates. Once again, it holds if and only if $\tilde{\gamma}(\tilde{p}) \in \mathbb{RP}^1$.

□

Remark. Any point of $\tilde{\mathcal{C}}$ mapped to the boundary of $\mathcal{A}(\mathcal{C})$ belongs to $\tilde{F}$. By the above lemma, the real part $\mathbb{R}\tilde{\mathcal{C}}$ of any curve $\tilde{\mathcal{C}}$ defined over $\mathbb{R}$ is always a subset of $\tilde{F}$.

Corollary 6.8. *One has the following*

* *$\tilde{\mathcal{C}}_\infty \subset \tilde{F}$.*
* *$\tilde{F} \subset \tilde{\mathcal{C}}$ is smooth if and only if $\tilde{\gamma}$ has no branching point on $\mathbb{RP}^1$. In this case, $\tilde{F}$ is a disjoint union of smoothly embedded circle in $\tilde{\mathcal{C}}$.*
* *If $\tilde{F}$ is smooth, both $\mathcal{A}_{|_{\tilde{F}^\circ}}$ and $Arg_{|_{\tilde{F}^\circ}}$ have a well defined Gauss map given by $\tilde{\gamma}$. In particular, $\mathcal{A}_{|_{\tilde{F}^\circ}}$ and $Arg_{|_{\tilde{F}^\circ}}$ have no inflection points.*

Proof The first point is a direct consequence of lemmas 6.6 and 6.7. For the second point, if $\tilde{\gamma}$ has no branching point on $\mathbb{RP}^1$, $\tilde{\gamma}_{|_{\tilde{F}}}$ is a local diffeomorphism, and $\tilde{F}$ is a topological covering of $\mathbb{RP}^1$. It implies that $\tilde{F}$ is a disjoint union of smoothly embedded circle in $\tilde{\mathcal{C}}$. If $\tilde{\gamma}$ has a branching point $q \in \mathbb{RP}^1$, then there exists $p \in \tilde{\gamma}^{-1}(q)$ such that $\tilde{F}$ near p is diffeomorphic to the preimage of $\mathbb{R} \subset \mathbb{C}$ by $z \mapsto z^n$ for some $n \geq 2$. Hence, $\tilde{F}$ is not smooth at p. The last point falls from the geometric interpretation of the logarithmic Gauss map. Consider the tangent bundle of $\mathbb{C}^2$ restricted to the coordinate wise complex logarithm $Log\big(\tilde{\mathcal{C}}^\circ\big)$. Considering real and imaginray parts gives a splitting $\mathbb{R}^2 \oplus i\mathbb{R}^2$ of the latter bundle. We have seen in the previous lemma that the tangent bundle of $Log\big(\tilde{\mathcal{C}}^\circ\big)$ splits in a direct sum of 2 line bundles while restricted to $Log\big(\tilde{F}^\circ\big)$. One of them is contained in the $\mathbb{R}^2$ factor of the previous splitting, and the other one is contained in the $i\mathbb{R}^2$ factor. Denote them by $\mathcal{L}_{Re}$ and $\mathcal{L}_{Im}$ respectively. Note that the maps $\mathcal{A}$ and Arg are just linear projections on $\mathbb{R}^2$ and $i\mathbb{R}^2$ in these logarithmic coordinates. Therefore, $\mathcal{A}$ (*resp.* Arg) maps $\mathcal{L}_{Re}$ (*resp.* $\mathcal{L}_{Im}$) to the tangent line bundle of $\mathcal{A}(\tilde{F}^\circ)$ (*resp.* $Arg(\tilde{F}^\circ)$). It follows that the Gauss maps of $\mathcal{A}(\tilde{\mathcal{C}}^\circ)$ and $Arg(\tilde{\mathcal{C}}^\circ)$ are both given by $\tilde{\gamma}$. By assumption, $\tilde{\gamma}$ has no critical point, that is $\mathcal{A}(\tilde{\mathcal{C}}^\circ)$ and $Arg(\tilde{\mathcal{C}}^\circ)$ have no inflection point.

□

Now let us recall that to any holomorphic function f on $\big(\mathbb{C}^*\big)^2$, one can associate its Ronkin function

$$N_f(x,y) := \frac{1}{(2i\pi)^2}\int_{\mathcal{A}^{-1}(x,y)} \frac{\log|f(z,w)|}{zw} dz\wedge dw$$

defined on $\mathbb{R}^2$. The function N_f allows to describe the geometry of the amoeba $\mathcal{A}\big(\{f=0\}\big)$. It is a convex function , that is affine linear on any connected component of the complement of $\mathcal{A}\big(\{f=0\}\big)$, see [PR04]. The gradient $grad\, N_f$ is then a constant function on such components. One defines the order of such component to be the value $grad\, N_f$ on it.
In the case where f is a polynomial, $grad\, N_f$ takes values inside of $New(f)$. Moreover, its image is dense there.

Proposition 6.9. *Let $\mathcal{C}\subset\mathcal{T}_\Delta$ be an algebraic curve. The order map defines an injection from the set of connected components of $\mathbb{R}^2\setminus\mathcal{A}(\mathcal{C})$ to $\Delta\cap\mathbb{Z}^2$. Compact components are sent in the interior of Δ and non-compact components are sent on its boundary. Moreover, the order map is surjective on the vertices of Δ.*

Consideration of the Hessian of N_f gives rise to a so-called Monge-Ampère measure supported on $\mathcal{A}(\mathcal{C})$. Comparison of this measure with the standard Lebesgue measure gives the following interesting result.

Proposition 6.10. *[PR04] Let $\mathcal{C}\subset\mathcal{T}_\Delta$ be an algebraic curve. Then*

$$\mathcal{A}rea\big(\mathcal{A}(\mathcal{C})\big)\le\pi^2\mathcal{A}rea\big(\Delta\big)$$

where the area is computed with respect to the standard Lebesgue measure.

There are other "areas" one can compute about amoebas and coamoebas. Let x_1,x_2 be the coordinates of $\mathbb{R}^2$ and y_1, y_2 be the coordinates on T. Define first

$$\mathcal{A}rea_s\big(\mathcal{A}(\mathcal{C})\big) = \int_{\tilde{\mathcal{C}}^\circ \setminus \tilde{F}^\circ} \mathcal{A}^*\big(dx_1 \wedge dx_2\big)$$

and

$$\mathcal{A}rea_s\big(\mathcal{C}o\mathcal{A}(\mathcal{C})\big) = \int_{\tilde{\mathcal{C}}^\circ \setminus \tilde{F}^\circ} Arg^*\big(dy_1 \wedge dy_2\big).$$

It consists of computing areas of amoebas and coamoebas with multiplicities. For both maps, the source and the target spaces are canonically oriented, and the maps are local diffeomorphisms. Hence one can associate to each point in the image its signed number of preimages, counting in how many sheets upstairs the map is orientation preserving minus the number of sheets where it is orientation reversing.
Define as well

$$\mathcal{A}rea_m\big(\mathcal{A}(\mathcal{C})\big) = \int_{\tilde{\mathcal{C}}^\circ \setminus \tilde{F}^\circ} \big|\mathcal{A}^*\big(dx_1 \wedge dx_2\big)\big|$$

and

$$\mathcal{A}rea_m\big(\mathcal{C}o\mathcal{A}(\mathcal{C})\big) = \int_{\tilde{\mathcal{C}}^\circ \setminus \tilde{F}^\circ} \big|Arg^*\big(dy_1 \wedge dy_2\big)\big|.$$

Here, one compute areas with multiplicities just by counting the number of preimages over any point of the respective target spaces. The following observation is due to Mikhalkin :

Lemma 6.11. *For any algebraic curve $\mathcal{C} \subset \mathcal{T}_\Delta$, the 2-forms $\mathcal{A}^*\big(dx_1 \wedge dx_2\big)$ and $Arg^*\big(dy_1 \wedge dy_2\big)$ are equal on $\tilde{\mathcal{C}}^\circ \setminus \tilde{F}^\circ$. In implies that*

$$\mathcal{A}rea_s\big(\mathcal{A}(\mathcal{C})\big) = \mathcal{A}rea_s\big(\mathcal{C}o\mathcal{A}(\mathcal{C})\big) = 0$$

and

$$\mathcal{A}rea_m\big(\mathcal{A}(\mathcal{C})\big) = \mathcal{A}rea_m\big(\mathcal{C}o\mathcal{A}(\mathcal{C})\big).$$

Proof Consider locally the coordinate wise complex logarithm $Log(\tilde{\mathcal{C}})$. It is a holomorphic curve in $\mathbb{C}^2$. It implies that the the restriction of the complex 2-form $dz_1 \wedge dz_2$ on $\mathbb{C}^2$ to $Log(\tilde{\mathcal{C}})$ is zero. Write $z_j = x_j + iy_j$ for $j = 1, 2$, then $Re\big(dz_1 \wedge dz_2\big) = dx_1 \wedge dx_2 - dy_1 \wedge dy_2$ is also zero on $Log(\tilde{\mathcal{C}})$, meaning that the 2-forms $dx_1 \wedge dx_2$ and $dy_1 \wedge dy_2$ are equal on $Log(\tilde{\mathcal{C}})$. As

we already said, the projection on the x-plane (*resp.* y-plane) is nothing but $\mathcal{A}$ (*resp.* Arg). It implies the first part of the statement.
Equalities of $\mathcal{A}rea_s$ and $\mathcal{A}rea_m$ are direct consequences. In the case of $\mathcal{A}rea_s$, the moment map μ extends $\mathcal{A}$ and has a compact source space. Then it has a well defined degree d which is zero as μ is not surjective. This degree is precisely the number of preimages of $\mathcal{A}$ counted with signs. Hence $\mathcal{A}rea_s\big(\mathcal{A}(\mathcal{C})\big) = 0$.

□

Denote by A_f the set of connected components of the complement of $\mathcal{A}\big(\{f = 0\}\big)$. For an element $\alpha \in A_f$ denote by N_f^α the affine linear function on $\mathbb{R}^2$ extending $(N_f)_{|\alpha}$. Then the spine $\mathscr{S}_f$ is defined as the corner locus of the piecewise affine linear and convex function

$$S_f := \max_{\alpha \in A_f} N_f^\alpha.$$

S_f is a piecewise linear graph in the plane. Equipped with some natural collection of weight, the spine turns out to be a tropical curve, see next subsection.

Theorem 6.12 ([PR04]). *Let $\mathcal{C} \subset \mathcal{T}_\Delta$ be an algebraic curve defined by a polynomial f. Then, $\mathcal{A}(\mathcal{C})$ deformation retracts on $\mathscr{S}_f$.*

we end up this section with a short description of the maps $\mathcal{A}$ and Arg near the points of $\mathcal{C}_\infty$. The map $\mathcal{A}$ has to go to infinity. By convexity and finiteness of the area of $\mathcal{A}(\mathcal{C})$, one deduces that a neighbourhood of any point of $\mathcal{C}_\infty$ is mapped onto a thin tentacle going off to infinity along a certain asymptotic direction. If this point belongs to $\mathcal{D}_s$, lemma 6.6 implicitly states that this direction is orthogonal to the corresponding side s of Δ.
In the case of Arg, define $\tilde{\mathcal{C}}_{Arg}$ to be the real oriented blow-up of $\tilde{\mathcal{C}}$ at every point of $\tilde{\mathcal{C}}_\infty$. Denote by $S_p \subset \tilde{\mathcal{C}}_{Arg}$ the fiber of the blow-up over $p \in \tilde{\mathcal{C}}_\infty$.

Lemma 6.13. *The map Arg extends to $\tilde{\mathcal{C}}_{Arg}$. Moreover, if s is a side of Δ, (a, b) a primitive integer vector supporting s, and p belongs to $\mathcal{D}_s$, then $Arg : S_p \to T$ is an m-covering over a geodesic of slope $(-b, a)$, where m is the intersection multiplicity of $\tilde{\mathcal{C}}_\infty \cap \mathcal{D}_s$ at p.*

Proof As in the proof of lemma 6.6, use a local coordinate t and consider the expressions

$$z(t) = \big(h_z(t)\big)^{-bm} \text{ and } w(t) = \big(h_w(t)\big)^{am}.$$

For $t = re^{i\theta}$, one has

$$z(t) = r^{-bm}\big(z_0 e^{-ibm\theta} + o(1)\big) \text{ and } w(t) = r^{am}\big(w_0 e^{iam\theta} + o(1)\big).$$

It follows that for any $e^{i\theta} \in S^1$

$$\lim_{r\to 0} arg\big(h_z(re^{i\theta})\big) = arg(z_0) - bm\cdot\theta$$

and

$$\lim_{r\to 0} arg\big(h_w(re^{i\theta})\big) = arg(w_0) + am\cdot\theta.$$

This gives the result when a and b are non zero. Otherwise, replace $z(t)$ or $w(t)$ by a non vanishing function and repeat the same computation, see 6.6.

□

6.2 Tropical curves

Let us recall briefly some classical notions about tropical curves in the plane. All definitions, statements and their proofs can be found in [Mik05] and [IMS09].
Consider the set $\mathbb{T} := \mathbb{R} \cup \{-\infty\}$ and the 2 tropical operations

$$\text{“}x + y\text{”} = \max\{x, y\} \text{ and “}xy\text{”} = x + y.$$

Tropical operations will always be distinguished from usual ones by quotation marks. Equipped with this 2 operations, $\mathbb{T}$ is a semifield : the tropical semifield.
A tropical Laurent polynomial in 2 variables x and y is a function

$$f(x, y) = \text{“}\sum_{(\alpha,\beta)\in A} c_{(\alpha,\beta)} x^\alpha y^\beta\text{”}$$

where $A \subset \mathbb{Z}^2$ is a finite subset. Such function is piecewise affine linear and convex. As for classical polynomials, the Newton polygon $New(f)$ of f is the convexhull of A in $\mathbb{R}^2 = \mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{R}$. The tropical zero set $V(f)$ of a tropical Laurent polynomial f is defined as the subset of $\mathbb{R}^2$ where f is not smooth. Equivalently,

$$V(f) = \Big\{(x, y) \in \mathbb{R}^2 \,\Big|\, \exists(\alpha,\beta) \neq (a, b) \in A\, s.t.\ \text{“}c_{(\alpha,\beta)} x^\alpha y^\beta\text{”} = \text{“}c_{(a,b)} x^a y^b\text{”}\Big\}.$$

As a first observation, a tropical zero set is a graph embedded in $\mathbb{R}^2$ with straight edges and leaves (unbounded edges) with rational slopes. If g is another tropical Laurent polynomial given by

$$g(x,y) := \text{“}c_{(\alpha,\beta)}x^\alpha y^\beta \cdot f(x,y)\text{”},$$

then obviously

$$V(f) = V(g),$$

but the converse fails to be true. Without loss of generality, we can and we do restrict once again to Newton polygon Δ contained in the positive quadrant and touching the 2 coordinate axes.
For a tropical polynomial f of Newton polygon Δ, consider its extended Newton polygon

$$\tilde{\Delta} := ConvexHull\left\{\big((\alpha,\beta),t\big) \in \mathbb{R}^3 \,\middle|\, (\alpha,\beta) \in A,\ t \geq c_{(\alpha,\beta)}\right\}.$$

Projecting on the first 2 coordinates maps down all closed bounded faces of $\tilde{\Delta}$ homeomorphically on Δ. It induces a subdivision $Subdiv_f$ of Δ. It can be seen using Legendre transform that there is the following duality

Proposition 6.14. *Let f be a tropical polynomial in 2 variables. The subdivision of $\mathbb{R}^2$ by $V(f)$ is dual to the subdivision $Subdiv_f$ of Δ in the following way*

* *2-cells of $\mathbb{R}^2 \setminus V(f)$ are in bijection with vertices of $Subdiv_f$, and 2-cells of $Subdiv_f$ are in bijection with vertices of $V(f)$,*
* *leaves-edges of $V(f)$ are in bijection with edges of $Subdiv_f$, and their directions are orthogonal to each other,*
* *incidence relations are reversed.*

Moreover, unbounded 2-cells of $\mathbb{R}^2 \setminus V(f)$ are dual to boundary points of Δ and leaves of $V(f)$ are dual to edges on the boundary of Δ.

Remark. Note that this duality rests only on the knowledge of the tropical zero set $V(f)$. A defining tropical equation f need not to be chosen.

This duality implies that the geometry of Δ is fixed by $V(f)$, and that up to our convention on Newton polygons, a tropical zero set has a unique Newton polygon, not depending on its defining equation.

Definition 6.15. *Let f be a tropical polynomial in 2 variables. For any leaf-edge ε of $V(f)$, the weight $w(\varepsilon)$ of ε is the integer length of its dual edge $\varepsilon^\vee$ in $Subdiv_f$, that is $|\varepsilon^\vee \cap \mathbb{Z}^2| - 1$.*

Definition 6.16. *A tropical curve $C \subset \mathbb{R}^2$ is a tropical zero set equipped with the weights defined in 6.15. If Δ is its Newton polygon, denote by $Subdiv_C$ the subdivision of Δ dual to C.*

By the above remark, the notion of weight does not depend on the polynomial f but only on its tropical zero set. Hence, the latter definition makes sense.

Remark. The convex piecewise affine linear function S_f defining the spine of the curve $\{f = 0\} \subset (\mathbb{C}^*)^2$ is a tropical polynomial. Equipped with the corresponding collection of weights, the spine of an algebraic curve in $(\mathbb{C}^*)^2$ is a tropical curve.

It is a classical fact that tropical curves satisfy the so-called balancing given by the following

Proposition 6.17. *Let $C \subset \mathbb{R}^2$ be a tropical curve. For any vertex v of C, let v_1, ..., v_k be the collection of outgoing primitive vectors supporting the k leaves-edges adjacent to v and w_1, ..., w_k their respective weights, then*

$$\sum_{j=1}^{k} w_j \cdot v_j = 0.$$

We end up this subsection by recalling what is the stable intersection multiplicity of 2 tropical curves in the plane C_1 and C_2. Assume first that we are in a generic case : C_1 and C_2 intersect transversally, that is in a finite number of points that are neither vertices of C_1 nor vertices of C_2. Let p be an intersection point, v_1 (*resp.* v_2) a primitive integer vector supporting the leaf-edge of C_1 (*resp.* C_2) containing p. Denote also w_1 (*resp.* w_2) the corresponding weight. Define the local intersection multiplicity of C_1 and C_2 at p by

$$m_p(C_1, C_2) := w_1 w_2 \cdot |det(v_1, v_2)|,$$

and the intersection multiplicity by

$$m(C_1, C_2) := \sum_{p \in C_1 \cap C_2} m_p(C_1, C_2).$$

Now assume that the intersection of C_1 and C_2 is arbitrary. There is a dense open subset $\mathscr{O} \subset \mathbb{R}^2$, such that for any $\vec{v} \in \mathscr{O}$, the intersection of C_1 and $\mathcal{C}_2 + \vec{v}$ is transversal. It is an easy consequence of the balancing condition that $m(C_1, C_2 + \vec{v})$ does not depend on $\vec{v}$.

Definition 6.18. *The stable intersection multiplicity of C_1 and C_2 is given by $m(C_1, C_2 + \vec{v})$ for any $\vec{v} \in \mathscr{O}$.*

The stable intersection considered as a divisor on tropical curve can be defined by using small perturbations. For more details, see [RST05]

6.3 Simple complex tropical curves

Definition 6.19. *A simple tropical curve $C \subset \mathbb{R}^2$ is a tropical curve with all weights 1 and such that its vertices are dual in $Subdiv_C$ to a triangle of area $1/2$ or to any parallelogram.*

A vertex dual to a triangle of area 1/2 is classically referred as a smooth vertex.

Definition 6.20. *For any simple tropical curve $C \in \mathbb{R}^2$, The normalization $\tilde{C}$ of C is the proper transform of $\tilde{C}$ obtained by the real blow-up of $\mathbb{R}^2$ at the 4-valent vertices of C. Denote the blow-up by $\pi : \tilde{C} \to C$.*

Remark. The normalization of a simple tropical curve is a 3-valent graph. It could be made into an abstract tropical curve by adding some extra structure, and the morphism π could be made into a tropical morphism, but this considerations are not needed here, see [Mik05] for example.

Definition 6.21. *A simple tropical curve $C \subset \mathbb{R}^2$ is irreducible if its normalization $\tilde{C}$ is connected.*

Definition 6.22. *From now on, we define the vertices (resp. edges, resp. leaves) of a simple tropical curve $C \in \mathbb{R}^2$ to be the image of the vertices (resp. edges, resp. leaves) of its normalization $\tilde{C}$ by the map π. They form the set $V(C)$ (resp. $E(C)$,resp. $L(C)$) and we denote $LE(C) := L(C) \cup E(C)$.*
The points of C having 2 preimages in $\tilde{C}$ are called the nodes of C and form the set $N(C)$.

Definition 6.23. *Let $C \in \mathbb{R}^2$ be a simple tropical curve and $n \in N(C)$. The multiplicity of the node n is the positive integer number*

$$m(n) := 2 \cdot Area(n^\vee)$$

where $n^\vee$ is the 2-cell dual to n in $Subdiv_C$. An node n is hyperbolic if $m(n) = 2$.

Definition 6.24. *A binomial cylinder $\varsigma \subset (\mathbb{C}^*)^2$ is a set defined by an equation $z^a w^b = c$ with $c \in \mathbb{C}^*$ and $(a, b) \in \mathbb{Z}^2$ a primitive integer vector.*
A general line $\mathcal{L} \in (\mathbb{C}^*)^2$ is a set defined by a polynomial equation $f(z, w) = 0$ where $New(f)$ has euclidean area $1/2$.

Consider the general line $\mathcal{L}_0 := \{(z, w) \in (\mathbb{C}^*)^2 | z + w + 1 = 0\}$. Its coameba $\mathcal{C}o\mathcal{A}(\mathcal{L}_0)$ is the union of the 2 open triangles delimited by the 3 geodesics $\gamma_1 := \{arg(z) = \pi\}$, $\gamma_2 := \{arg(w) = \pi\}$ and $\gamma_3 := \{arg(w) = arg(z) + \pi\}$ plus their 3 common vertices $(\pi, 0)$, $(0, \pi)$ and (π, π), see figure 6.3.
As a convention, we choose a framing on each of the γ_i's as follows

* γ_1 is oriented downward, with origin (π, π),
* γ_2 is oriented leftward, with origin $(0, \pi)$,
* γ_3 is oriented up-rightward, with origin $(\pi, 0)$.

These framings are coherent with the up-left triangle of $\mathcal{C}o\mathcal{A}(\mathcal{L}_0)$ (see figure 6.3). The argument map is orientation preserving on this triangle and orientation reversing on the other. For this reason, the framings are kept globally unchanged by any of the 6 toric automorphisms of $\mathcal{L}_0$.
With this framing, any boundary geodesic is canonically isomorphic as an abelian group to S^1 equipped with the counter-clockwise orientation.

Proposition 6.25. *For any general line $\mathcal{L}$, there exists a toric transformation of $(\mathbb{C}^*)^2$, that is an element $\sigma \in Sl_2(\mathbb{Z}) \rtimes (\mathbb{C}^*)^2$ such that $\sigma(\mathcal{L}) = \mathcal{L}_0$.*
The induced transformation $\sigma_T \in Sl_2(\mathbb{Z}) \rtimes T$ on the argument torus T sends $\mathcal{C}o\mathcal{A}(\mathcal{L})$ onto $\mathcal{C}o\mathcal{A}(\mathcal{L}_0)$.
There is a unique framing of the boundary geodesics of $\mathcal{C}o\mathcal{A}(\mathcal{L})$ that maps to the canonical framings on $\mathcal{C}o\mathcal{A}(\mathcal{L}_0)$ by any $\sigma \in Sl_2(\mathbb{Z}) \rtimes (\mathbb{C}^*)^2$ such that $\sigma(\mathcal{L}) = \mathcal{L}_0$.
For any general line $\mathcal{L}$, we equip $\mathcal{C}o\mathcal{A}(\mathcal{L})$ with these framings.

Proof By Pick's formula, the Newton polygon of any polynomial equation defining $\mathcal{L}$ contains 3 integer points spanning the lattice $\mathbb{Z}^2$. Then, there exists an element of $Sl_2(\mathbb{Z})$ sending this polygon onto the standard simplex of size 1. Translating the coordinates by an appropriate element of $(\mathbb{C}^*)^2$ gives the desired σ. The details are left to the reader.

□

Definition 6.26. *A simple complex tropical curve $V \subset (\mathbb{C}^*)^2$ is a topological Riemann surface such that :*

- *its amoeba $\mathcal{A}(V)$ is a simple tropical curve $C \subset \mathbb{R}^2$,*
- *for any $e \in LE(C)$, there exists a binomial cylinder ς such that $\left(\mathcal{A}_{|V}\right)^{-1}(e) = \mathcal{A}^{-1}(e) \cap \varsigma$,*
- *for any $v \in V(C)$, there exists a general line $\mathcal{L}$ such that $\left(\mathcal{A}_{|V}\right)^{-1}(v) = \overline{\mathcal{C}o\mathcal{A}(\mathcal{L})}$.*

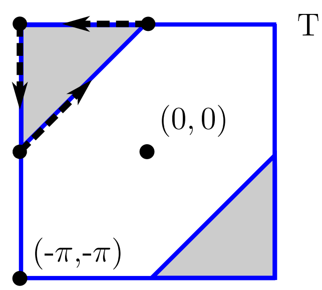


Figure 3: The coamoeba of $\mathcal{L}_0$, and the framings of its 3 boundary geodesics.

Remark. The fact that such complex tropical curve do exists could be formulated as an easy proposition. In order $V \subset (\mathbb{C}^*)^2$ to be a topological Riemann surface, the binomial cylinder ς sitting above an element $e \in LE(C)$ has to be defined by an equation $z^a w^b = c$ where $(a, b) \in \mathbb{Z}^2$ is a primitive integer vector supporting e. The fact that such cylinders can be glued together by the coamoeba of general line above a vertex $v \in V(C)$ holds on an appropriate choice of the c's and the fact that C has only smooth vertices.

Definition 6.27. *For a simple complex tropical curve $V \subset (\mathbb{C}^*)^2$, denote by $\tilde{V}$ the smooth topological Riemann surface obtained by pulling back the fibration $\mathcal{A}_{|V} : V \to C$ to the normalization $\tilde{C}$ of C by the map $\pi : \tilde{C} \to C$.*

Remark. Similarily to definition 6.20, we could add extra structure in the latter definition in order to fit in complex tropical settings, but this is not needed here, either.

Let us consider a simple complex tropical curve V, with $C := \mathcal{A}(V)$. For any $e \in E(C)$ and $v_1, v_2 \in V(C)$ its 2 adjacent vertices, the topological cylinder $\left(\mathcal{A}_{|V}\right)^{-1}(e)$ is glued to 2 boundary geodesics γ_1 and γ_2 in $\left(\mathcal{A}_{|V}\right)^{-1}(v_1)$ and $\left(\mathcal{A}_{|V}\right)^{-1}(v_2)$ respectively. Let τ_1, τ_2 be the canonical group isomorphisms $\tau_i : S^1 \to \gamma_i$. Identifying the 2 argument torus $\mathcal{A}^{-1}(v_1)$ and $\mathcal{A}^{-1}(v_2)$, one has that γ_1 and γ_2 are set theoretically identical. This is due to the fact that the coamoeba of a binomial cylinder is single geodesic in T. Under this identification, one has that the map $\tau_2^{-1} \circ \tau_1$ is an orientation reversing isometry of the form

$$\begin{array}{rcl} \tau_2^{-1} \circ \tau_1 \,:\, S^1 & \to & S^1 \\ z & \mapsto & -e^{i\theta}\overline{z}. \end{array}$$

As this isometry is self inverse, the datum $e^{i\theta}$ does not depend on the order we have chosen for γ_1 and γ_2.

Definition 6.28. *Let V be a simple complex tropical curve, with $C := \mathcal{A}(V)$. For any $e \in E(C)$, the element $e^{i\theta} \in S^1$ constructed above is called the twist parameter of the edge e.*

Definition 6.29. *A simple real tropical curve $V \subset (\mathbb{C}^*)^2$ is a simple complex tropical curve that is invariant under complex conjugation. We denote by $\mathbb{R}V \subset (\mathbb{R}^*)^2$ its real point set and by $\mathbb{T}V$ the image of $\mathbb{R}V$ under the diffeomorphism*

$$\begin{array}{rcl} \mathcal{A}_s \,:\, (\mathbb{R}^*)^2 & \to & \mathbb{R}^2 \times (\mathbb{Z}_2)^2 \\ (x, y) & \mapsto & \left((\frac{x}{|x|} \ln|x|, \frac{y}{|y|} \ln|y|), (\frac{x}{|x|}, \frac{y}{|y|}) \right). \end{array}$$

Proposition 6.30. *Let $V \subset (\mathbb{C}^*)^2$ be a simple complex tropical curve with $C := \mathcal{A}(V)$. Then V a is real if and only if one of the following equivalent conditions holds :*

- *i) For every $e \in LE(C)$, the binomial cylinder $\left\{z^a w^b = c\right\}$ supporting $\left(\mathcal{A}_{|V}\right)^{-1}(e)$ is defined over $\mathbb{R}$, that is $c \in \mathbb{R}^*$.*

ii) For every $v \in V(C)$, the general line $\mathcal{L}$ for which $\left(\mathcal{A}_{|_V}\right)^{-1}(v) = \overline{\mathcal{C}o\mathcal{A}(\mathcal{L})}$ is defined over $\mathbb{R}$.

iii) For every $v \in V(C)$, the 3 special points of the closed coamoeba $\left(\mathcal{A}_{|_V}\right)^{-1}(v)$ are in $\{(0,0), (\pi,0), (0,\pi), (\pi,\pi)\}$.

The proof is straightforward.

Remark. Note that by composing the map $\mathcal{A}_s$ with

$$\begin{array}{rcl} Abs\,:\, \mathbb{R}^2 \times (\mathbb{Z}_2)^2 & \to & \mathbb{R}^2 \\ \big((x,y),(\varepsilon,\delta)\big) & \mapsto & (\varepsilon x, \delta y), \end{array}$$

one recovers the map $\mathcal{A}$. Note moreover that the real locus of any binomial cylinder defined over $\mathbb{R}$ has exactly 2 connected components.

As corollaries of the previous observations, we have

Proposition 6.31. *Let $V \subset (\mathbb{C}^*)^2$ be a simple real tropical curve and denote by $C := \mathcal{A}(V)$ its amoeba. Then $\mathbb{T}V$ is a piecewise linear curve and $Abs : \mathbb{T}V \to C$ is 2-to-1.*

Proposition 6.32. *The twist parameters of a real tropical curve $V \subset (\mathbb{C}^*)^2$ are always contained in $\{-1, 1\} \subset S^1$.*

7 Definition and first properties

Definition 7.1. *A real algebraic curve $\mathcal{C} \subset \mathcal{T}_\Delta$ is a simple Harnack curve if it is irreducible and*

$$\tilde{F} = \mathbb{R}\tilde{\mathcal{C}}.$$

This generalizes the notion of simple Harnack curve given in the previous section. Here, we relax any constraints on the type of singularities of simple Harnack curves. We will see later that it allows many other cases to appear. From now on, we will always refer to the latter definition while speaking about simple Harnack curves.

Recall that given two compact Riemann surfaces equipped with a real structure (X, σ_X) and (Y, σ_Y) together with a holomorphic map $f : X \to Y$, then f is said to be totally real if $f^{-1}(\mathbb{R}Y) = \mathbb{R}X$. By the lemma 6.7, one has the following reformulation

Proposition 7.2. *An irreducible real algebraic curve $\mathcal{C}$ is a simple Harnack curve if and only if its logarithmic Gauss map $\tilde{\gamma}$ is totally real.*

Remark. The latter equivalent definition of simple Harnack curve implies that $\mathbb{R}\tilde{\mathcal{C}} \subset \tilde{\mathcal{C}}$ is of type 1, that is $\mathbb{R}\tilde{\mathcal{C}} \setminus \tilde{\mathcal{C}}$ has exactly 2 connected components.

In order to get the first general properties of such curves, let us slightly generalize a construction due to [MO07] : for a curve $\mathcal{C} \subset \mathcal{T}_\Delta$ consider the map

$$\begin{array}{rccc} Alga : & \mathcal{C}^\circ & \to & T \\ & (z,w) & \mapsto & (2\,arg\,z, 2\,arg\,w). \end{array}$$

Define $\tilde{\mathcal{C}}_{Bl}$ to be the blow-up of $\tilde{\mathcal{C}}$ at every point of $\tilde{\mathcal{C}}_\infty$. For any point $p \in \tilde{\mathcal{C}}_\infty$ denote by $\mathbb{P}_p$ the fiber of $\tilde{\mathcal{C}}_{Bl} \to \tilde{\mathcal{C}}$ over p. By construction, one has the factorisation $\tilde{\mathcal{C}}_{Arg} \to \tilde{\mathcal{C}}_{Bl} \to \tilde{\mathcal{C}}$ inducing a double covering $S_p \to \mathbb{P}_p$ for any $p \in \tilde{\mathcal{C}}_\infty$.

Lemma 7.3. *The map Alga naturally extends to*

$$Alga : \tilde{\mathcal{C}}_{Bl} \to T.$$

Proof By lemma 6.13, $Alga$ extend to $\tilde{\mathcal{C}}_{Arg}$. For any p and any point in $\mathbb{P}_p$, its 2 preimages in S_p are mapped to the same value by $Alga$. Hence, $Alga : \tilde{\mathcal{C}}_{Arg} \to T$ factorizes through $\tilde{\mathcal{C}}_{Bl}$.

□

Define the subset $\tilde{\mathcal{C}}_0 \subset \tilde{\mathcal{C}}_{Bl}$ to be $Alga^{-1}_{|_{\tilde{\mathcal{C}}_{Bl}}}\left(\{0_T\}\right)$. Note that

$$Alga^{-1}\left(\{0_T\}\right) = \left(\mathbb{R}^*\right)^2.$$

Thus, it implies that $\tilde{\mathcal{C}}_0$ is the union of $\mathbb{R}\tilde{\mathcal{C}}_{Bl}$ plus some isolated points, whenever $\mathcal{C}$ is defined over $\mathbb{R}$. In this case, the isolated points of $\tilde{\mathcal{C}}_0$ come either as the trace of real isolated singular points of $\mathbb{R}\mathcal{C}$ or from non transverse intersection with a toric divisor at infinity. Indeed, by lemma 6.13, if $\mathcal{C}$ intersects a divisor $\mathcal{D}_s$ with multiplicity m at a point $p \in \mathcal{C}_\infty$, there are exactly m points in the exceptional divisor $\mathbb{P}_p$ belonging to $\tilde{\mathcal{C}}_0$, and exactly one of these belongs to $\mathbb{R}\tilde{\mathcal{C}}_{Bl}$.

Lemma 7.4. *A real algebraic curve $\mathcal{C} \subset \mathcal{T}_\Delta$ is a simple Harnack if and only if*

$$Alga : \tilde{\mathcal{C}}_{Bl} \setminus \tilde{\mathcal{C}}_0 \to T \setminus \{0_T\}$$

is an unbranched covering.

Proof By lemma 6.7 and the remark above, the latter statement is an equivalent reformulation of definition 7.1.

□

Define at last $\hat{T}$ to be the real blow-up of T at 0_T, and $\hat{\mathcal{C}}$ to be the real blow-up of $\tilde{\mathcal{C}}_{Bl}$ at $\tilde{\mathcal{C}}_0$. As blowing-up at a smooth submanifold of codimension 1 doesn't change the surface, blowing-up is effective only at isolated points of $\tilde{\mathcal{C}}_0$.

Theorem 4. *A real algebraic curve $\mathcal{C} \subset \mathcal{T}_\Delta$ is a simple Harnack if and only if the map $Alga : \tilde{\mathcal{C}}_{Bl} \setminus \tilde{\mathcal{C}}_0 \to T \setminus \{0_T\}$ extends to a covering*

$$Alga : \hat{\mathcal{C}} \to \hat{T}.$$

Proof Let $\mathcal{C}$ be a simple Harnack curve. By definition, the map $Alga : \tilde{\mathcal{C}}_{Bl} \to T$ is regular at the isolated points of $\tilde{\mathcal{C}}_0$. Hence, the map $Alga : \tilde{\mathcal{C}}_{Bl} \setminus \tilde{\mathcal{C}}_0 \to T \setminus \{0_T\}$ extends to $\hat{T}$ in a tautological way at these isolated points. At a point of $\mathbb{R}\tilde{\mathcal{C}}_{Bl}$, blowing-up consists of considering the normal direction to $\mathbb{R}\tilde{\mathcal{C}}_{Bl}$ in the tangent space. This actually specifies the line bundle $\mathcal{L}_{Im}$ introduced in the proof of lemma 6.8. The projectivized tangent map of $Alga$ realizes a covering of the exceptional divisor of $\hat{T}$ by $\mathcal{L}_{Im}$, giving the extension $Alga : \hat{\mathcal{C}} \to \hat{T}$.
Conversely, if $\mathcal{C}$ is such that $Alga : \hat{\mathcal{C}} \to \hat{T}$, lemma 7.4 implies that $\mathcal{C}$ is a simple Harnack curve.

□

Corollary 7.5. *If $\mathcal{C} \subset \mathcal{T}_\Delta$ is a simple Harnack curve, then*

$$Area_m\big(\mathcal{C}o\mathcal{A}(\mathcal{C})\big) = \pi^2\big(-\chi(\hat{\mathcal{C}})\big).$$

If moreover $\mathcal{C}$ has no real isolated singularity and if it intersecting transversally every toric divisors at infinity, then

$$Area_m\big(\mathcal{C}o\mathcal{A}(\mathcal{C})\big) = \pi^2\big(-\chi(\tilde{\mathcal{C}}^\circ)\big).$$

Proof It is clear by definition that $Area_m\big(\mathcal{C}o\mathcal{A}(\mathcal{C})\big) = 4\,Area_m\big(Alga(\mathcal{C})\big)$. By theorem 4, $Area_m\big(Alga(\mathcal{C})\big) = Area(T)\cdot deg\,Alga = 4\pi^2\cdot\big(-\chi(\hat{\mathcal{C}})\big)$. The first part of the statement follows.
For the second one, the assumptions are such that $\tilde{\mathcal{C}}_0$ has no isolated point. It implies that $\hat{\mathcal{C}} = \tilde{\mathcal{C}}_{Bl}$, but $\chi(\tilde{\mathcal{C}}_{Bl}) = \chi(\tilde{\mathcal{C}}^\circ)$.

□

8 Tropical construction of Harnack curves

8.1 Tropical Harnack curves

Definition 8.1. *Let $C \subset \mathbb{R}^2$ be a simple tropical curve with normalization $\tilde{C}$. For every oriented loop $\tilde{\lambda} \subset \tilde{C}$, and λ the corresponding oriented loop in C, denote by $\Gamma_\lambda \subset E(C) \cap \lambda$ the subset of oriented edges that forms, together with its previous and following edges in λ, a non convex piecewise linear curve.*

Definition 8.2. *An irreducible simple tropical curve $C \subset \mathbb{R}^2$ with normalization $\tilde{C}$ is a tropical Harnack curve if for every oriented loop $\tilde{\lambda} \subset \tilde{C}$, one has*

$$\sum_{\vec{e}\in\Gamma_\lambda} v_{\vec{e}} = 0 \quad mod\ 2 \tag{7}$$

where $v_{\vec{e}}$ is the primitive integer vector supporting the oriented edge $\vec{e}$.

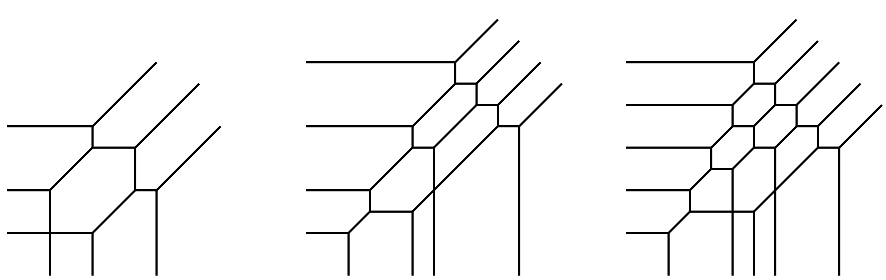


Figure 4: Projective tropical cubic, quadric and quintic. Both the cubic and the quintic are tropical Harnack curves. The quartic in the middle does not satisfy the condition of definition 8.2.

Definition 8.3. *An edge (resp. a leaf) of* $\mathbb{T}V$ *is defined to be one of the 2 connected components mapping onto an edge (resp. a leaf) of* C*. They form a set denoted* $E(\mathbb{T}V)$ *(resp.* $L(\mathbb{T}V)$*). Their union is denoted* $LE(\mathbb{T}V)$*.*

Definition 8.4. *Let* $V \subset (\mathbb{C}^*)^2$ *be* $\mathbb{RT}$ *curve. An inflection pattern of* $\mathbb{T}V$ *is a collection of 3 consecutive elements of* $LE(\mathbb{T}V)$ *that is not convex.*

Proposition 8.5. *An irreducible simple tropical curve* $C \subset \mathbb{R}^2$ *is a Harnack tropical curve if and only if there exists a simple real tropical curve* $V \subset (\mathbb{C}^*)^2$ *such that*

- $\mathcal{A}(V) = C$,
- $\mathbb{T}V$ *has no inflection pattern.*

In such case, V *is unique up to sign change* $(z, w) \mapsto (\pm z, \pm w)$ *of the coordinates.*

Definition 8.6. *Any of the 4 simple complex tropical curve* V *of the previous proposition is defined as a complex tropical Harnack curve sitting above* C*.*

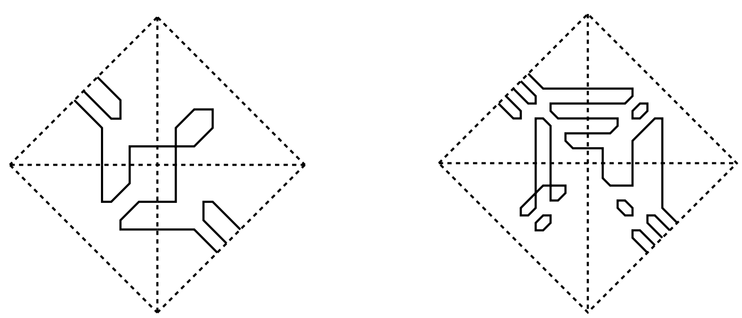


Figure 5: $\mathbb{T}V$ for the the cubic and the quintic of figure 4.

The definition of tropical Harnack curves given in 8.2 is a practical definition, in the sense that it gives a effective method to determine whether a simple tropical curve is Harnack or not. In spirit, the equivalent definition given by the above proposition is the one of interest for us, especially because of the following

Proposition 8.7. *Let* $C \subset \mathbb{R}^2$ *be a tropical Harnack curve, and* V *be one of the 4 complex tropical Harnack curves sitting above* C*. Then the total logarithmic curvature of* $\mathbb{R}V$ *is equal to* $-\chi(\tilde{V})$*.*

The notion of total logarithmic curvature needs to be clarified. Before doing so, let us make some enlightening comments.
The "curve" $\mathbb{T}V$ lives in the logarithmic plane. The interest of such object for us is that one can construct families of real curves $\{\mathcal{C}_t^\circ\}_{t>>1}$ such that $\mathcal{A}_s(\mathbb{R}\mathcal{C}_t^\circ)$ converges to $\mathbb{T}V$, in Hausdorff distance. Reasonably, one should define the total logarithmic curvature of $\mathbb{R}V$ such that it corresponds to the limit of $deg\ \tilde{\gamma}_{|_{\mathbb{R}\tilde{\mathcal{C}}_t^\circ}}$, that is the total logarithmic curvature of $\mathbb{R}\tilde{\mathcal{C}}_t^\circ$. Relating the latter proposition with the propositions 6.5 and 7.2, one forsee that the family $\{\mathcal{C}_t^\circ\}_{t>>1}$ will be a family of simple Harnack curve, topology of which is prescribed by the topology of $\mathbb{T}V$. The question of approximation of tropical Harnack curve will be undertaken in the next subsection.

Now we come to the definition of the total logarithmic curvature on $\mathbb{R}V$. Consider the compactification of $\mathbb{R}V$ in $\mathcal{T}_\Delta$, and normalize it. The result is a disjoint union of topological circle. Up to the change of coordinates $\mathcal{A}_s$, it induces a compactification-normalization of $\mathbb{T}V$. Choose $\varepsilon > 0$ sufficiently small such that the ε-neighbourhood of the vertices of $\mathbb{T}V$ are pairwise disjoint, and smooth the immersion of the topological circles such that

* $\mathbb{T}V$ gets deformed only inside the chosen ε-neighbourhoods,

* the deformation has no inflection inside the chosen ε-neighbourhoods.

Then, each of the smoothed immersions have a well defined logarithmic Gauss map. In particular, it has a well defined degree, up to the choice of an orientation. For each of these circles $\mathcal{O}$, define its total curvature $\kappa_\mathcal{O}$ to be the absolute value of the degree of its Gauss map. Each of these numbers is clearly independent of the deformation and the choice of orientation.

Definition 8.8. *The total curvature of* $\mathbb{R}V$ *is the defined by*

$$\kappa := \sum \kappa_\mathcal{O},$$

where the sum runs over all the topological circles $\mathcal{O}$ *of the compactification-normalization of* $\mathbb{R}V$ *defined above.*

Proof of proposition 8.7 On one hand, $-\chi(\tilde{V})$ is equal to the number of vertices of $\tilde{\mathcal{C}}$. To see this, cut $\tilde{\mathcal{C}}$ at the middle of any of its edges. As $\tilde{\mathcal{C}}$ has only 3-valent vertices, it splits into a collection of tripods, one for each vertex

of $\tilde{\mathcal{C}}$. The part of $\tilde{V}$ sitting above any of these tripods is a topological pair-of-pants, having Euler characteristic -1. Additivity of Euler characteristic implies the above claim.
On the other hand, one can compute the total logarithmic curvature of $\mathbb{R}V$ by computing its local contributions. By the very definition, these contributions are concentrated at the vertices of $\mathbb{T}V$. It corresponds to $1/\pi$ times the measure of the solid angle between the 2 normals at the vertices of $\mathbb{T}V$ (see figure 4 in [BLR13]), counted with signs depending on how $\mathbb{T}V$ changes inflection between to consecutive vertices. As $\mathbb{T}V$ has no inflection pattern, this local contributions should all be counted positively. Now, for each vertex in C, there are 3 vertices in $\mathbb{T}V$. One easily sees that the local contributions at these 3 vertices add up to 1, see figure 5 in [BLR13]. Hence the total logarithmic curvature of $\mathbb{R}V$ is also equal to the number of vertices of C, that is the number of vertices of $\tilde{\mathcal{C}}$ by convention. The result follows.

□

The rest of this subsection is devoted to the proof of proposition 8.5.

Definition 8.9. *The edge of a simple real tropical curve is twisted (or has a twist) if its twist parameter is -1. It is not twisted (or has no twist) otherwise.*

Note that we made a slight abuse of language by speaking about edge of a simple complex tropical curve rather than edge of its underlying simple tropical curve.

Lemma 8.10. *Let $V \subset (\mathbb{C}^*)^2$ be simple real tropical curve. Then the inflection patterns of $\mathbb{T}V$ are in 2-to-1 correspondence with the twisted edges of V.*

Proof Let $C := \mathcal{A}(V)$. The middle element of an inflection pattern has to be an edge. Consider $e \in E(C)$ and v_1, $v_2 \in V(C)$ its 2 adjacent vertices. Let γ_1 be the geodesic corresponding to e in $\left(\mathcal{A}_{|V}\right)^{-1}(v_1)$ and γ_2 the one corresponding to e in $\left(\mathcal{A}_{|V}\right)^{-1}(v_2)$. Denote by $e_1, e_2 \in LE(C)$ the 2 others elements adjacent to v_1 and $\varepsilon_1, \varepsilon_2 \in LE(C)$ the 2 others elements adjacent to v_2, such that e_1, e_2, e and $\varepsilon_1, \varepsilon_2, e$ are cyclically ordered. The origin of γ_1 connects an edge-leaf of $\mathbb{T}V$ above e_2 to an edge of $\mathbb{T}V$above e and the origin of γ_2 connects an edge-leaf of $\mathbb{T}V$ above ε_2 to an edge of $\mathbb{T}V$above e. Similarly, the -1 point of γ_1 connects an edge-leaf of $\mathbb{T}V$ above e_1 to an edge of $\mathbb{T}V$above e and the -1 point of γ_2 connects an edge-leaf of $\mathbb{T}V$ above ε_1

to an edge of $\mathbb{T}V$ above e.

Recall that by definition of the twist parameter, the origin of γ_1 is connected to the origin of γ_2 if and only if there is a twist on e. It follows that: either there is a twist on e and one has 2 inflection patterns in $\mathbb{T}V$, one mapping down to e_2, e, ε_2 and one mapping down to e_1, e, ε_1; or there is no twist and one has 2 patterns in $\mathbb{T}V$, one mapping down to e_2, e, ε_1 and one mapping down to e_1, e, ε_2, and they are not inflected.

□

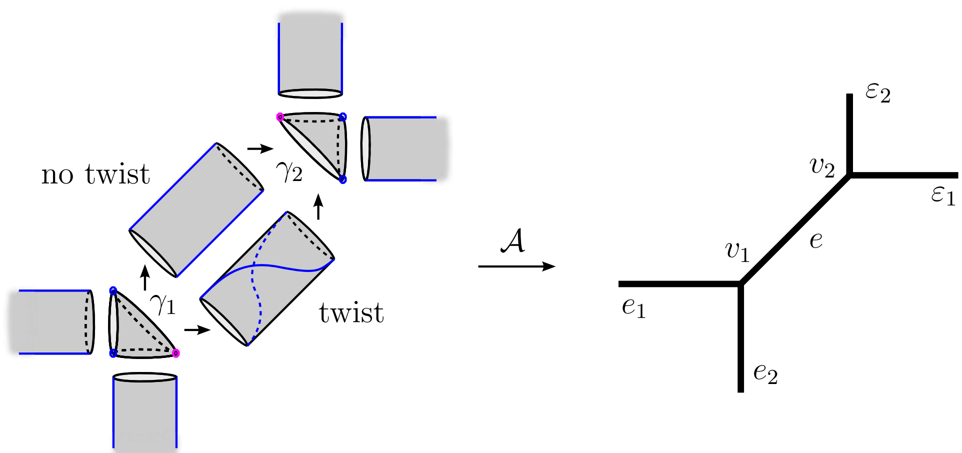


Figure 6: The proof of the latter lemma. On the left, $\mathbb{R}V$ (in blue) sitting inside of V. The origins of γ_1 and γ_2 are drawn in purple.

We also refers to section 3 in [Bru13], for a description of twisted edges in combinatorial patch-working.

Proof of proposition 8.5 Suppose there is a simple complex tropical curve $V \subset (\mathbb{C}^*)^2$ such that $\mathcal{A}(V) = C$ and $\mathbb{T}V$ has no inflection pattern. By the previous lemma, it is equivalent to the fact that V has no twisted edges. For any vertex $v \in \lambda$, there is a distinguished point among the 3 special points of the coamoeba $\left(\mathcal{A}_{|V}\right)^{-1}(v)$, namely the intersection point of the 2 geodesics corresponding to the 2 edges in λ adjacent to v. Let us look at the position of this distinguished point in the argument torus T while going around λ. Going from a vertex v to the next one via an edge e, the point is moved according to the following rule : if e is not in Γ_λ, then this point is fixed; if e is in Γ_λ, this point is moved by $\pi \cdot r_{-\pi/2}(\vec{e})$ in T, where $r_{-\pi/2}$ is

the rotation of angle $-\pi/2$. After a full cycle, the distinguished point has to end up its initial place. This is clearly equivalent to the condition stated in definition 8.2 on the loop λ. Hence C is a tropical Harnack curve.
Conversely, if C satisfies the condition of definition 8.2 for any cycle, pick an initial vertex v_0 on C. There is exactly one possible general line $\mathcal{L}$ such that $\left(\mathcal{A}_{|V}\right)^{-1}(v_0) = \overline{\mathcal{C}o\mathcal{A}(\mathcal{L})}$, up to the 4 changes of signs of the coordinates. The adjacent binomial cylinders are determined by $\mathcal{L}$, and the general lines above the nearby vertices are determined as well, because there are no twist. The first part of the proof shows that the condition of definition 8.2 is necessary and sufficient for this construction to close along every cycle λ. The proposition is proved.

□

8.2 Construction by tropical approximation

Theorem 8.11. *(Mikhalkin) Let $V \subset (\mathbb{C}^*)^2$ be a simple real tropical curve of Newton polygon Δ, and let g and n be natural number such that the normalization $\tilde{V}$ is a smooth topological Riemann surface of genus g with n punctures. Then, there exists a family of real Riemann surfaces $\{S_t\}_{t>>1} \subset \mathcal{M}_{g,n}$ together with a family of immersions $\iota_t : S_t \to (\mathbb{C}^*)^2$ such that*

- *$\iota\big(S_t\big)$ is a real algebraic curve of newton polygon Δ,*
- *$\iota\big(S_t\big)$ converges in Hausdorff distance to V.*

The latter theorem allows us to construct real algebraic curve with prescribed topology. In particular, it allows to construct a plenty a simple Harnack curve as we just show below.

Definition 8.12. *Let C be a tropical Harnack curve of Newton polygon Δ. Define $Top(C)$ to be the topological triad*

$$\left(\mathbb{R}\mathcal{T}_\Delta,\ \mathbb{R}V,\ \bigcup_s \mathbb{R}\mathcal{D}_s\right)$$

up to homeomorphism, where s runs over all the sides of Δ and V is one of the 4 simple complex tropical curves of proposition 8.5 sitting above C. As they are obtained one from each others by toric transformations, $Top(C)$ is well defined.

Theorem 5. *Let C be a tropical Harnack curve of Newton polygon Δ, then there exists a simple Harnack curve $\mathcal{C} \subset \mathcal{T}_\Delta$ such that*

$$\left(\mathbb{R}\mathcal{T}_\Delta,\ \mathbb{R}\mathcal{C},\ \bigcup_s \mathbb{R}\mathcal{D}_s\right) = Top(C).$$

Proof Taking t large enough in theorem 8.11 guarantees the existence of $\mathcal{C}$ such that the associated topological triad equals $Top(C)$. It remains to show that, necessarily, $\mathcal{C}$ is a simple Harnack curve. By the very definition of 8.8, one has that the degree of $\tilde{\gamma}_{|\mathbb{R}\mathcal{C}}$ is equal to the total logarithmic curvature of $\mathbb{R}V$, for any complex tropical Harnack curve V sitting above C. This is equal to $-\chi(\tilde{V})$ by proposition 8.7. But $-\chi(\tilde{V}) = -\chi(\tilde{\mathcal{C}}^\circ)$, by construction of $\mathcal{C}$, see theorem 8.11. By proposition 6.5, this is exactly the degree of $\tilde{\gamma}$. Hence, the logarithmic Gauss map $\tilde{\gamma}$ is totally real. By proposition 7.2, it implies that $\mathcal{C}$ is a simple Harnack curve.

□

It was shown in [MR01], that smooth simple Harnack curves as introduced by Mikhalkin cannot degenerate to curves with wild singularities. Indeed, the closure of the space of such curves contains only curves with real isolated double points. At such points, the curve is described by

$$x^2 + y^2 = 0.$$

These are also called elliptic nodes, in contrast with hyperbolic node, locally described by

$$x^2 - y^2 = 0.$$

By approximating the simple tropical curves drawn in figure 4, one can construct simple Harnack curve with such hyperbolic nodes, as shown in figure 5. In particular, such curves has not been considered before. One can ask for the possible singularities of simple Harnack curves. Rather than attacking the question in its generality, let us give examples for the simplest singularities.

Let us first construct a cuspidal Harnack cubic in the projective plane. The recipe is now very simple: consider a family $\{C_t\}_{t>>1}$ of tropical Harnack cubic as pictured in figure 4 such that the unique compact connected component of the complement shrinks as $t \to \infty$. Up to the choice of a sign, there is a unique family of complex tropical Harnack curves $\{V_t\}_{t>>1}$ above the previous family. Even if the tropical curve at the limit C_∞ is very degenerated, one can still recover $\mathbb{T}V_\infty$ by passing to the limit. Approximating each C_t close enough and passing to the limit gives a cuspidal Harnack cubic as pictured in figure 7.

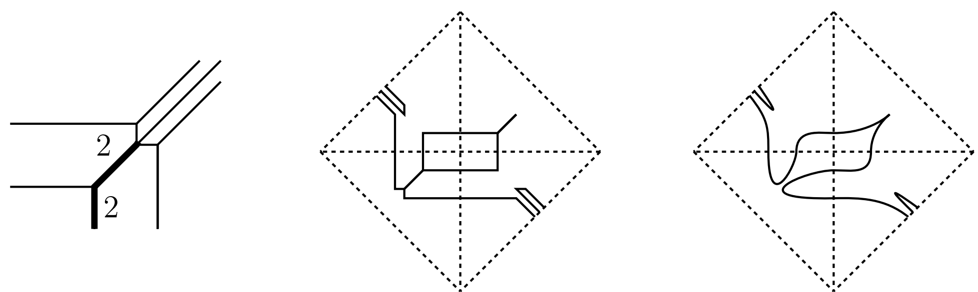


Figure 7: C_∞, $\mathbb{T}V_\infty$ and a cuspidal Harnack cubic.

One can also produces simple Harnack curves with complex conjugated double points. In the figure 8, we illustrate the construction of a curve of bi-degree $(4,2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Such curve has arithmetic genus 3. The curve pictured here has one hyperbolic node on its real part and the 2 edges coloured in blue are responsible for 2 complex conjugated double points.

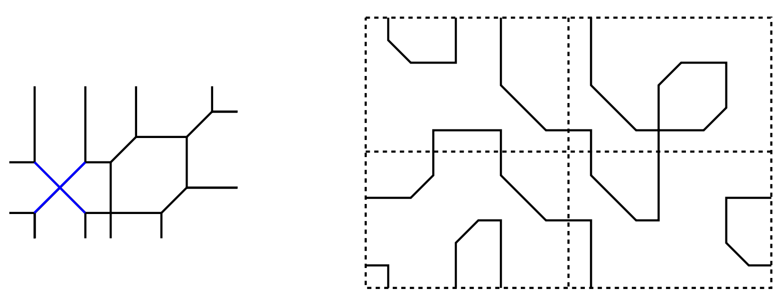


Figure 8: A tropical Harnack curve with a node of multiplicity 2, and its corresponding topological type.

8.3 Tropical Harnack curves with a single hyperbolic node

Proposition 8.13. *Let $C \subset \mathbb{R}^2$ be a tropical Harnack curve of Newton polygon Δ with a single hyperbolic node n. Then the parallelogram $n^\vee$ dual to n in $Subdiv_C$ has exactly 3 of its vertices on the boundary of Δ. This 3 vertices are distributed on 2 sides of Δ that are adjacent to a smooth vertex ν of Δ.*

Proof Suppose $n^\vee$ has at least 2 vertices v_1 and v_2 in the interior of Δ. Consider the polygonal domain P of Δ obtained by taking the union of $n^\vee$ together with all the minimal triangle of $Subdiv_C$ having v_1 or v_2 as a vertex. Consider the subset of C dual to P. It has a single loop $\tilde{\lambda}$ in the normalization $\tilde{\mathcal{C}}$. There are 2 cases : either v_1 and v_2 are consecutive or opposite in $n^\vee$. In the first case, Γ_λ is a singleton. In the second, Γ_λ is exactly composed of the 2 leaves-edges forming the node n. In both case, the condition of definition 8.2 is not fulfilled. Hence, we get a contradiction.
If $n^\vee$ has its 4 vertices on the boundary Δ, then C is reducible. This is a contradiction.
Then, $n^\vee$ exactly 3 of its vertices on the boundary of Δ. Either the middle vertex is a vertex of Δ, and this is then a smooth one, or one of the 2 sides of $n^\vee$ specified by the 3 vertices is on a side of Δ and a minimal triangle is attached to the other side. The vertex of this triangle not contained in $n^\vee$ is the smooth vertex of Δ we are looking for. The statement is proven.

□

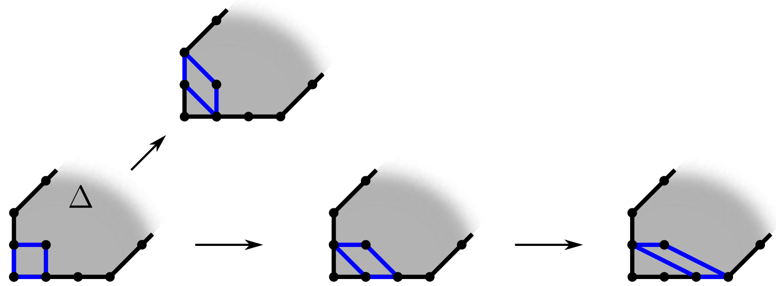
Definition 8.14. *Let $C \subset \mathbb{R}^2$ be a tropical Harnack curve with a single hyperbolic node n, and let ν be the smooth vertex of its Newton polygon given by the previous proposition. One will say that the node of C is next to ν. Denote by $\mathbb{T}\mathcal{H}_{\Delta,\nu}$ the space of tropical Harnack curve with a single hyperbolic node next to ν.*

Proposition 8.15. *Let Δ be a Newton polygon and ν a smooth vertex of Δ. Then, the topological type $Top(C)$ for any $C \in \mathbb{T}\mathcal{H}_{\Delta,\nu}$ is unique and depends only on the pair (Δ, ν).*

Definition 8.16. *The topological type $Top(C)$ of any $C \in \mathbb{T}\mathcal{H}_{\Delta,\nu}$ will be denoted*

$$Top(\Delta, \nu).$$

Proof Up to toric transformation of $(\mathbb{C}^*)^2$ one can assume that $\nu = (0,0)$ and that its 2 adjacent sides are supported on the x- and the y-axes. Consider the subdivision $Subdiv_C$ dual to Δ. By proposition 8.13, the unique parallelogram of $Subdiv_C$ can only be of the following type In particular, $(1,1)$



is always a vertex of this parallelogram. If $g := |\Delta \cap \mathbb{Z}^2|$, each of the $(g-1)$ remaining points is dual to an oval of $\mathbb{R}V$. These ovals are unnested and up to an appropriate choice of sign, the oval corresponding to $(i,j) \in \Delta \cap \mathbb{Z}^2$ sits in the quadrant of $(\mathbb{R}^*)^2$ given by the pair of signs (i,j) *mod* 2. Now, the union of all the 2-cells of $Subdiv_C$ touching either $\partial\Delta$ or the point $(1,1)$ is dual to the unique component of $\mathbb{R}V$ intersecting the toric divisors at infinity. It intersects these divisors in a cyclical order as in the case of smooth Harnack curves, see [Mik00], except that 2 intersection points with the vertical and the horizontal divisors near the origin are exchanged (see the left picture in figure 5 for example), this no matter where the parallelogram sits in Δ. Nesting of the compact ovals with this component is fixed by the geometry of Δ. Details are left to the reader.

□

9 Simple Harnack curves with a single hyperbolic node

9.1 Statements of the main theorems

In this section we undertake the topological classification of the triads

$$\left(\mathbb{R}\mathcal{T}_\Delta,\ \mathbb{R}\mathcal{C},\ \bigcup_s \mathbb{R}\mathcal{D}_s\right)$$

for simple Harnack curves $\mathcal{C}$ with a single hyperbolic node in any toric surface $\mathcal{T}_\Delta$. We go slightly beyond by showing a strong connection between these curves and their tropical avatars, in the fashion of [KO06]. One has the following theorems

Theorem 6. *Let $\mathcal{C} \subset \mathcal{T}_\Delta$ be a simple Harnack curve with a single hyperbolic node. Assume moreover that $\mathcal{C}$ intersects transversally every toric divisor at infinity. Then, there is a smooth vertex ν of Δ such that*

$$\Big(\mathbb{R}\mathcal{T}_\Delta,\ \mathbb{R}\mathcal{C},\ \bigcup_s \mathbb{R}\mathcal{D}_s\Big) = Top(\Delta, \nu).$$

In the latter theorem, one had to specify the intersection profil at infinity, as the topological classification depends on it. Transversality is a genericity assumption, and the general case can be deduced easily from the generic one.

Definition 9.1. *Let $\mathcal{C} \subset \mathcal{T}_\Delta$ be a simple Harnack curve with a single hyperbolic node, and let ν be the smooth vertex of its Newton polygon given by the previous theorem. One will say that the node of $\mathcal{C}$ is next to ν.*
Denote by $\mathcal{H}_{\Delta,\nu}$ the space of simple Harnack curve with a single hyperbolic node next to ν.

Theorem 7. *The spine of a curve in $\overline{\mathcal{H}_{\Delta,\nu}}$ is a tropical curve in $\overline{\mathbb{T}\mathcal{H}_{\Delta,\nu}}$. It defines a map*

$$\mathscr{S}\ :\ \overline{\mathcal{H}_{\Delta,\nu}} \to \overline{\mathbb{T}\mathcal{H}_{\Delta,\nu}},$$

which is a local diffeomorphism.

Remark. In [KO06], The authors proved that consideration of the spine gives a local diffeomorphism from the space of Mikhalkin's Harnack curve to the space of tropical curves. They proved a stronger result : the area of the holes of their amoebas and the coordinates of the points of intersection with the toric divisors gives a global set of coordinates on the space of such Harnack curves. We do believe that it can be reformulated in terms of their spine and obtain that the map $\mathscr{S}$ is a global diffeomorphism. We postpone this study for a further paper.

On the way of proving these main theorems, one goes through the following results :
First one needs to distinguish the different connected components of the critical locus $\tilde{F}$ inside of a cuve $\tilde{\mathcal{C}}$.

Definition 9.2. *Let $\mathcal{C} \subset \mathcal{T}_\Delta$ be a simple Harnack curve. A u-oval is a connected component of $\mathbb{R}\tilde{\mathcal{C}}$ intersecting $\tilde{\mathcal{C}}_\infty$. A b-oval is a connected component of $\mathbb{R}\tilde{\mathcal{C}}$ not intersecting $\tilde{\mathcal{C}}_\infty$.*

This distinction is made because the amoeba map $\mathcal{A}$ has disctinct behaviour on u- and b-ovals : indeed, it is unbounded on u-ovals and bounded on b-ovals.

Proposition 9.3. *Let $\mathcal{C} \subset \mathcal{T}_\Delta$ be a simple Harnack curve with a single hyperbolic node. Then, the normalization $\mathbb{R}\tilde{\mathcal{C}} \subset \tilde{\mathcal{C}}$ is an M-curve. Moreover, $\mathbb{R}\tilde{\mathcal{C}}$ is the union of $(g-1)$ b-ovals and a single u-oval, where $(g-1)$ is the genus of $\tilde{\mathcal{C}}$.*

The latter proposition implies that the normalization of such simple Harnack curve admits the following decomposition

$$\tilde{\mathcal{C}} \ := \ \tilde{\mathcal{C}}^- \ \cup \ \mathbb{R}\tilde{\mathcal{C}} \ \cup \ \tilde{\mathcal{C}}^+$$

where $\tilde{\mathcal{C}}^-$ and $\tilde{\mathcal{C}}^+$ are exchanged by the complex conjugation σ on $\tilde{\mathcal{C}}$. As $Arg \circ \sigma = -id \circ Arg$, the map Arg is orientation preserving on one "half" of $\tilde{\mathcal{C}}$ and orientation reversing on the other. By convention, fix $\tilde{\mathcal{C}}^+$ to be the one where Arg is orientation preserving.
Let us also introduce the following notation : if ν is a smooth vertex of a Newton polygon Δ, denote by $\Delta_\nu \subset \Delta$ the polygonal domain obtained by removing the parallelogram of area 1 in the corresponding corner of Δ. To be more precise, this parallelogram is spanned by the 2 primitive integer vectors supporting the 2 sides of Δ adjacent to ν.

Theorem 8. *Let $\mathcal{C} \subset \mathcal{T}_\Delta$ be a simple Harnack curve with a single hyperbolic node next to ν. Then, the restriction to $\tilde{\mathcal{C}}^+$ of the argument map Arg lifts to the universal covering $\mathbb{R}^2$ of T. Moreover, its lift Arg_0 is a diffeomorphism and*

$$\overline{Arg_0\big(\tilde{\mathcal{C}}^+\big)} = \tau\big(\Delta_\nu\big)$$

where τ is the composition of a rotation by $-\pi/2$ and a homothety by π.

9.2 Some more conventions

In the rest of this section, $\mathcal{C} \subset \mathcal{T}_\Delta$ will be a simple Harnack curve with a

single hyperbolic node p. We denote by φ the connected component of $\mathbb{R}\mathcal{C}^\circ$ containing p, and by $\tilde{\varphi}$ its normalization in $\mathbb{R}\tilde{\mathcal{C}}^\circ$.
Up to toric transformation of $(\mathbb{C}^*)^2$, one can and do assume that Δ has an horizontal side supported on the x-axis. We will denote

$$b := \left|\partial\Delta \cap \mathbb{Z}^2\right| \quad \text{and} \quad g := \left|Int(\Delta) \cap \mathbb{Z}^2\right|$$

and refer to m as the number of sides of Δ.
We will always give $\partial\Delta$ the counter-clockwise orientation. It induces a cyclical order on the set of sides of Δ. We formalize it by a bijection

$$\begin{array}{rcl} \mathbb{Z}/m\mathbb{Z} & \rightarrow & \text{the set of sides of } \Delta \\ j & \mapsto & s_j \end{array}$$

such that s_1 is the horizontal side of Δ supported on the x-axis. For each side s_j, $j \in \mathbb{Z}/m\mathbb{Z}$, there is a unique primitive integer vector v_j, $j \in \mathbb{Z}/m\mathbb{Z}$, supporting s_j and coherent with the orientation of $\partial\Delta$.
By corollary 6.8, we can and we do orient each connected component ϑ of $\mathbb{R}\tilde{\mathcal{C}}$ such that $\mathcal{A}(\mathbb{R}\tilde{\mathcal{C}}^\circ)$ is a locally concave parametrized curve.
For any u-oval ϑ, this orientation induces a cyclical order on the set (with possible repetitions) of the toric divisors $\mathcal{D}_s$ as they are encountered by ϑ. We formalize it by a map

$$\begin{array}{rcccl} \mathbb{Z}/m_\vartheta\mathbb{Z} & \rightarrow & \mathbb{Z}/m\mathbb{Z} & \rightarrow & \text{the set of sides of } \Delta \\ j & \mapsto & \vartheta(j) & \mapsto & s_{\vartheta(j)} \end{array}$$

where m_ϑ is the number of points of $\vartheta_\infty := \tilde{\mathcal{C}}_\infty \cap \vartheta$. Denote also $\vartheta^\circ := \vartheta \setminus \vartheta_\infty$. Note that the map on the left is not necessarily injective, and is defined up to translation. For our purpose, we do not need to specify this map any further.
For 2 vectors u and v in the plane, we denote by $\measuredangle(u,v)$ the measure of the oriented angle from u to v with values in $[0; 2\pi[$.

Definition 9.4. *The index of a u-oval ϑ of $\mathbb{R}\tilde{\mathcal{C}}$ is defined by*

$$ind(\vartheta) = \frac{1}{2\pi} \sum_{j \in \mathbb{Z}/m_\vartheta\mathbb{Z}} \measuredangle\big(v_{\vartheta(j)}, v_{\vartheta(j+1)}\big)$$

9.3 Topological maximality

This subsection is mainly devoted to the proof of proposition 9.3.

Lemma 9.5. *The only possible cases are the following*

(α) *φ is a self-intersecting arc in $\left(\mathbb{R}^*\right)^2$ joining the toric divisor $\mathcal{D}_{s_j}$ to the toric divisor $\mathcal{D}_{s_k}$, with $\pi < \measuredangle(v_j, v_k) < 2\pi$,*

(β) *φ is a self-intersecting arc in $\left(\mathbb{R}^*\right)^2$ joining the toric divisor $\mathcal{D}_{s_j}$ to the toric divisor $\mathcal{D}_{s_k}$, with $0 \leq \measuredangle(v_j, v_k) < \pi$,*

(γ) *φ is an immersed closed curve of rotational index 2 self-intersection at p,*

(δ) *φ is the union of 2 arcs intersecting transversally at p.*

Remark. The distinction we made between case (α) and (β) is not of topological nature *a priori*, but will be motivated later.

Proof Suppose the normalization $\tilde{\varphi}$ of φ is connected. Then it is either an open segment or a topological circle. The first possibility is split between (α) and (β). For the second possibility, φ is an immersed circle self-intersecting at p. One of the 2 possible smoothings of φ gives 2 disjoint circles. If these circle are nested in the plane, then it corresponds to (γ). If they are not, then φ is isotopic to the figure "∞" and has inflection. This contradicts corollary 6.8. Suppose now that the normalization $\tilde{\varphi}$ of φ is not connected. Then it is the union of 2 open segments, but this is case (δ).

$\square$

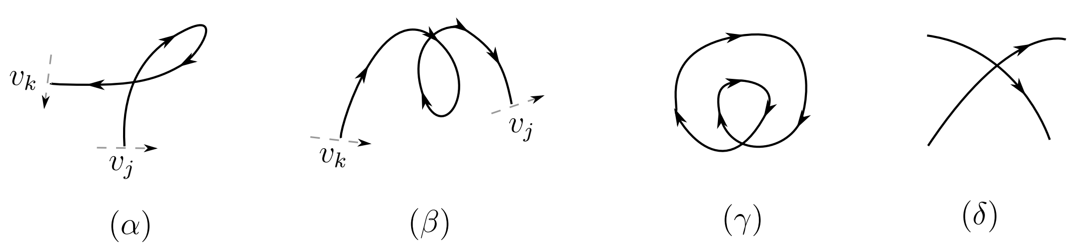


Figure 9: $\mathcal{A}(\varphi)$ in the 4 cases of lemma 9.5

The classification of lemma 9.5 is relevant while computing the contribution to $\tilde{\gamma}$ of the different ovals of $\mathbb{R}\tilde{\mathcal{C}}$. The goal here is to shorten this list using the fact that the sum of all this contributions is constrained by the total reality of $\tilde{\gamma}$.

Remark. Note that all the cases of the list in lemma 9.5 can appear for simple Harnack curve. The case (α) was already illustrated by the cubic of figure 4. We will see in the sequel that the 3 other cases cannot appear solely. Their manifestation forces the curve to have some other singularities, see figure 10.

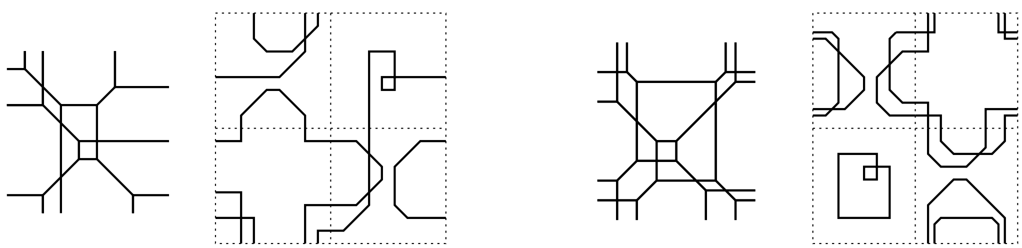


Figure 10: The case (β) on the left, and (γ) on the right. The case (δ) appears in both.

Then we have

Lemma 9.6. *For an u-oval $\vartheta \subset \mathbb{R}\tilde{\mathcal{C}}$ such that $\mathcal{A}$ is an embedding on each connected component of ϑ°, one has*

$$deg\, \gamma_{|\vartheta} = |\vartheta_\infty| - 2 \cdot ind(\vartheta).$$

For a b-oval $\vartheta \subset \mathbb{R}\tilde{\mathcal{C}}$ for which $\mathcal{A}$ is an embedding, one has

$$deg\, \gamma_{|\vartheta} = 2.$$

Proof To prove the first formula, one has to compute the contribution on every connected component of ϑ°. As $\mathcal{A}$ is an embedding, lemma 6.6 implies first that $0 \leq \measuredangle(v_{\vartheta(j)}, v_{\vartheta(j+1)}) \leq \pi$ for each $j \in \mathbb{Z}/m_\vartheta\mathbb{Z}$ and that according to our orientation conventions, the contribution between the j-th and $(j+1)$-th point of ϑ_∞ is given by

$$\frac{1}{\pi}\measuredangle\big(-v_{\vartheta(j+1)}, v_{\vartheta(j)}\big) = \frac{1}{\pi}\Big(\pi - \measuredangle\big(v_{\vartheta(j)}, v_{\vartheta(j+1)}\big)\Big) \geq 0.$$

Summing over all j's gives the desired formula. The second formula is the projective reformulation of the fact that a simple closed curve in the plane has rotational index 1.

□

Lemma 9.7. *In the cases* (α), *and* (β) *of proposition 9.5, let* ϑ *be the unique u-oval in* $\mathbb{R}\tilde{\mathcal{C}}$ *containing* $\tilde{\varphi}$. *Then, one has*

$$deg\,\gamma_{|\vartheta} = |\vartheta_\infty| + 2 - 2\cdot ind(\vartheta).$$

In the case (γ) *of proposition 9.5, let* ϑ *be the unique b-oval in* $\mathbb{R}\tilde{\mathcal{C}}$ *containing the node* p. *Then, one has*

$$deg\,\gamma_{|\vartheta} = 4.$$

Proof In the cases (α), and (β) of proposition 9.5, the proof goes as the one of the first formula of the previous lemma, except that the contribution of the arc $\tilde{\varphi}$ is given for some j by

$$\frac{1}{\pi}\measuredangle\big(-v_{\vartheta(j+1)}, v_{\vartheta(j)} + 2\big).$$

For the case (γ), $\mathcal{A}(\vartheta)$ has rotational index 2 in the plane, that is 4 projectively.

□

Proof of proposition 9.3 By lemma 6.5, one has that

$$deg\,\tilde{\gamma} = 2g + b - 4.$$

Consider first the cases (α) and (β) of proposition 9.5, and denote by ϑ the u-oval of $\mathbb{R}\tilde{\mathcal{C}}$ containing $\tilde{\varphi}$. Let us compute

$$\begin{aligned}
deg\,\tilde{\gamma} &= \sum_{\substack{\mathcal{O}\text{ oval}\\ \text{of }\mathbb{R}\tilde{\mathcal{C}}}} deg\,\tilde{\gamma}_{|\mathcal{O}} = \sum_{\mathcal{O}\text{ u-oval}} deg\,\tilde{\gamma}_{|\mathcal{O}} + \sum_{\mathcal{O}\text{ b-oval}} deg\,\tilde{\gamma}_{|\mathcal{O}} \\
&= \Big(deg\,\tilde{\gamma}_{|\vartheta} + \sum_{\substack{\mathcal{O}\text{ u-oval}\\ \mathcal{O}\neq\vartheta}} deg\,\tilde{\gamma}_{|\mathcal{O}}\Big) + \sum_{\mathcal{O}\text{ b-oval}} deg\,\tilde{\gamma}_{|\mathcal{O}} \\
&= \Big(|\vartheta_\infty| + 2 - 2\cdot ind(\vartheta) + \sum_{\substack{\mathcal{O}\text{ u-oval}\\ \mathcal{O}\neq\vartheta}} (|\mathcal{O}_\infty| - 2\cdot ind(\mathcal{O}))\Big) \\
&\quad +2\,\#\{\mathcal{O}\text{ b-oval}\}
\end{aligned}$$

by lemmas 9.6 and 9.7

$$= \Big(b + 2 - 2 \cdot \sum_{\mathscr{O} \text{ u-oval}} ind(\mathscr{O})\Big) + 2\Big(b_0(\mathbb{R}\tilde{\mathcal{C}}) - \#\{\mathscr{O} \text{ u-oval}\}\Big)$$

$$= 2\, b_0(\mathbb{R}\tilde{\mathcal{C}}) + b + 2 - 2 \cdot \sum_{\mathscr{O} \text{ u-oval}} \big(ind(\mathscr{O}) + 1\big).$$

It follows that

$$2\, b_0(\mathbb{R}\tilde{\mathcal{C}}) - 2 \cdot \sum_{\mathscr{O} \text{ u-oval}} \big(ind(\mathscr{O}) + 1\big) = 2g - 6. \tag{8}$$

Moreover, $\mathbb{R}\tilde{\mathcal{C}}$ is of type 1 inside of $\tilde{\mathcal{C}}$ which is of genus $(g-1)$, hence $b_0(\mathbb{R}\tilde{\mathcal{C}})$ is constrained by

$$b_0(\mathbb{R}\tilde{\mathcal{C}}) \leq g \text{ and } b_0(\mathbb{R}\tilde{\mathcal{C}}) \equiv g \; mod\, 2.$$

If $b_0(\mathbb{R}\tilde{\mathcal{C}}) = g - 2l$, then

$$\begin{aligned} 2g - 6 &= 2\, b_0(\mathbb{R}\tilde{\mathcal{C}}) - 2 \cdot \sum_{\mathscr{O} \text{ u-oval}} \big(ind(\mathscr{O}) + 1\big) \\ &\leq 2g - 4l - 4\,\#\{\mathscr{O} \text{ u-oval}\} \\ &\leq 2g - 4\big(l + 1\big). \end{aligned}$$

It implies that $l = 0$, $b_0(\mathbb{R}\tilde{\mathcal{C}}) = g$, and

$$\sum_{\mathscr{O} \text{ u-oval}} \big(ind(\mathscr{O}) + 1\big) = 3.$$

Hence ϑ is the unique u-oval and $ind(\vartheta) = 2$. We proved the result for cases (α) and (β).
Consider now the case (γ) of proposition 9.5, and denote once again by ϑ the u-oval of $\mathbb{R}\tilde{\mathcal{C}}$ containing $\tilde{\varphi}$. We repeat the same computation

$$\begin{aligned} deg\, \tilde{\gamma} &= \sum_{\mathscr{O} \text{ u-oval}} deg\, \tilde{\gamma}_{|\mathscr{O}} + \sum_{\mathscr{O} \text{ b-oval}} deg\, \tilde{\gamma}_{|\mathscr{O}} \\ &= \sum_{\mathscr{O} \text{ u-oval}} (|\mathscr{O}_\infty| - 2 \cdot ind(\mathscr{O})) + \Big(\sum_{\substack{\mathscr{O} \text{ u-oval} \\ \mathscr{O} \neq \vartheta}} 2 \; + \; 4\Big) \end{aligned}$$

as ϑ contributes to 4 according to 9.7

$$= \Big(b - 2 \cdot \sum_{\mathcal{O} \text{ u-oval}} ind(\mathcal{O})\Big) + \Big(2\,\#\{\mathcal{O} \text{ b-oval}\} + 2\Big)$$

$$= 2\,b_0(\mathbb{R}\tilde{\mathcal{C}}) + b + 2 - 2 \cdot \sum_{\mathcal{O} \text{ u-oval}} \big(ind(\mathcal{O}) + 1\big).$$

Once again, we end up with equation (8). The same arguments as above imply that $b_0(\mathbb{R}\tilde{\mathcal{C}}) = g$, and that there is a unique u-oval ϑ with $ind(\vartheta) = 2$. We proved the result for case (γ).
Consider finally the case (δ) of proposition 9.5. Note that every oval of $\mathbb{R}\tilde{\mathcal{C}}$ satisfies the assumptions of 9.6. We compute as before

$$\begin{aligned} deg\,\tilde{\gamma} &= \sum_{\mathcal{O} \text{ u-oval}} deg\,\tilde{\gamma}_{|\mathcal{O}} + \sum_{\mathcal{O} \text{ b-oval}} deg\,\tilde{\gamma}_{|\mathcal{O}} \\ &= \Big(b - 2 \cdot \sum_{\mathcal{O} \text{ u-oval}} ind(\mathcal{O})\Big) + \Big(2\,\#\{\mathcal{O} \text{ u-oval}\}\Big) \\ &= 2\,b_0(\mathbb{R}\tilde{\mathcal{C}}) + b - 2 \cdot \sum_{\mathcal{O} \text{ u-oval}} \big(ind(\mathcal{O}) + 1\big). \end{aligned}$$

It follows that

$$2\,b_0(\mathbb{R}\tilde{\mathcal{C}}) - 2 \cdot \sum_{\substack{\mathcal{O} \text{ u-oval} \\ \mathcal{O} \neq \vartheta}} \big(ind(\mathcal{O}) + 1\big) = 2g - 4,$$

implying in turn that $b_0(\mathbb{R}\tilde{\mathcal{C}}) = g$, that there is a unique u-oval ϑ and that $ind(\vartheta) = 1$. Equivalently, the cyclical order on the b boundary points of $\mathcal{C}$ induced by ϑ and the one induced by the boundary $\partial\Delta$ of the moment polygon Δ are the same. In such case, the image by $\mathcal{A}$ of any two connected components of ϑ° intersect at 0 or 2 points, but $\mathcal{C}$ has exactly one singular point. This is a contradiction.

□

Note that, along the latter proof, we obtained the following

Lemma 9.8. *For a simple Harnack curve with only one hyperbolic node, the case* (δ) *of lemma 9.5 cannot occur.*

9.4 Lifted coamoebas

This subsection is devoted to the proof of theorem 8.

The idea of lifting the argument map to the universal cover of the argument torus appeared in [Passare,Nilsson]. In this paper, the authors studied A-discriminantal curves : by Horn parametrization theorem, these curves happen to be rational, *i.e* parametrized by a sphere, a simply connected space. As a consequence, the argument map on (half of) such curves can be lifted to $\mathbb{R}^2$, and this has very nice combinatorial repercussions on their coamoebas.
In the present case, our parametrizing curve $\mathbb{R}\tilde{\mathcal{C}}$ is not rational in general. One can consider only half of this space and show that the argument map there factorizes through a disc, implying in particular that the induced map on the fundamental groups is trivial.

Lemma 9.9. *The restriction to $\tilde{\mathcal{C}}^+$ of the argument map Arg lifts to the universal covering $\mathbb{R}^2$ of T. Moreover, its lift Arg_0 is a local diffeomorphism.*

Proof This lemma is a corollary of proposition 9.3. Indeed, the latter implier that $\tilde{\mathcal{C}}^+$ is homeomorphic to an open disc with exactly $(g-1)$ holes. Compactifiying $\tilde{\mathcal{C}}^+$ by attaching back $\mathbb{R}\tilde{\mathcal{C}}$, one see that the fundamental group of $\tilde{\mathcal{C}}^+$ is generated by the $(g-1)$ b-ovals of $\mathbb{R}\tilde{\mathcal{C}}$. The argument map contracts each of these ovals to one of the 4 points $\{(0,0),\ (0,\pi),\ (\pi,0),\ (\pi,\pi)\}$ in T. In other words, the map

$$Arg\ :\ \pi_1\big(\tilde{\mathcal{C}}^+\big) \to \pi_1\big(T\big)$$

is trivial. This is the necessary and sufficient condition for Arg to lift to a map

$$Arg_0\ :\ \tilde{\mathcal{C}}^+ \to \mathbb{R}^2.$$

By the definition of simple Harnack curve and lemma 6.7, Arg is a local diffeomorphism. So is Arg_0.

□

Now, define the topological disc D as follows : consider first the closure of $\tilde{\mathcal{C}}^+$ in $\tilde{\mathcal{C}}_{Arg}$, see lemma 6.13. It is a closed disc with $(g-1)$ open holes bounded by the b-ovals of $\mathbb{R}\tilde{\mathcal{C}}$. Now contract to a point every connected component of $\mathbb{R}\tilde{\mathcal{C}}_{Arg}$ that is in the closure of $\tilde{\mathcal{C}}^+$. The result is clearly a topological disc that we denote D

Lemma 9.10. *Arg_0 extends to a differentiable map $Arg_0 : D \to \mathbb{R}^2$. Denote by ϑ the unique u-oval of $\tilde{\mathcal{C}}$, then Arg_0 maps the boundary of D to the piecewise linear curve with vertices in $\pi\mathbb{Z}^2$ obtained by the concatenation of the vectors $\tau\big(v_{\vartheta(j)}\big)$ according to the cyclical ordering $j \in \mathbb{Z}/m_\vartheta\mathbb{Z}$, τ being defined in 8.*

Remark. The points of $Arg_0(D) \cap \pi\mathbb{Z}^2$ are exactly the points of D coming from the contracted connected components of $\mathbb{R}\tilde{\mathcal{C}}_{Arg}$.

Proof The fact that Arg_0 extends has been proven in lemma 6.13. Differentiability at the boundary is implicitly given in the proof of lemma 6.13. To see differentiability at the points obtained by contraction of the b-ovals, use the same arguments as in the proof of 4.
The second part of the statement falls from lemma 6.13 and the way we defined $\tilde{\mathcal{C}}^+$.

□

Now, we are ready to prove theorem 8. The area enclosed by the piecewise linear curve $Arg_0\big(\partial D\big)$ is given by theorem 4. We somehow have to solve a combinatorial isoperimetrical inequality problem. Here, the constraint is such that the boundary curve is as predicted by theorem 8.

Proof of theorem 8 By theorem 4, one has that

$$\begin{aligned} Area\big(Arg_0(D)\big) &= Area\big(Arg_0(\tilde{\mathcal{C}}^+) = \tfrac{1}{2}Area\big(Arg(\mathcal{C}^\circ)\big) \\ &= -\tfrac{\pi^2}{2}\chi(\tilde{\mathcal{C}}^\circ). \end{aligned}$$

In the present case, Khovanskii's formula [Kho78] gives

$$-\chi(\tilde{\mathcal{C}}^\circ) = (2g + b - 2) - 2.$$

Using Pick's formula gives in turn

$$Area\big(Arg_0(D)\big) = \pi^2\big((g + b/2 - 1) - 1\big) = \pi^2\big(Area(\Delta) - 1\big) \tag{9}$$

Consider the case (α) of lemma 9.5. By the previous lemma, $Arg_0(\partial D)$ is a piecewise linear curve with vertices in $\pi\mathbb{Z}^2$. As we have seen in the proof of lemma 9.6, it is locally convex everywhere except at the vertex coming from $\tilde{\varphi} \subset \tilde{\mathcal{C}}_{Arg}$, where by assumption the angle interior to $Arg_0(D)$ is strictly between π and 2π. Let $j \in \mathbb{Z}/m_\vartheta\mathbb{Z}$ be such that this non convex angle is formed by $\tau\big(v_{\vartheta(j)}\big)$ and $\tau\big(v_{\vartheta(j+1)}\big)$. If one permutes these 2 vectors, the area of the domain of $\mathbb{R}^2$ enclosed by the piecewise linear curve increases at least by π^2, and exactly by π^2 if and only if $v_{\vartheta(j)}$ and $v_{\vartheta(j+1)}$ span a parallelogram of area 1. Suppose the new polygonal domain is still not convex. Then, one can repeat the previous construction, and increase the area until we end up with a convex domain. This convex domain can be nothing but Δ (up to translation). This contradicts (9). Hence, the result of the permutation were already convex and $v_{\vartheta(j)}$ and $v_{\vartheta(j+1)}$ have to span a parallelogram of area 1, by (9). In other words, there exists a smooth vertex ν of Δ such that $Arg_0\big(D\big) = \tau\big(\Delta_\nu\big)$.
In cases (β) and (γ) of lemma 9.5, one has that $Arg_0(\partial\mathcal{D})$ is convex. In such case, its rotational index is exactly computed by $ind(\vartheta)$, where ϑ is the unique u-oval of $\tilde{\mathcal{C}}$. It has been shown in the proof of proposition 9.3 that this rotational index is 2. Then, there exists 2 distinct points p_1 and p_2 on ∂D mapped to the same point in $\mathbb{R}^2$. They cut ∂D into 2 arcs γ_1 and γ_2. Denote by Δ_1 and Δ_2 the polygonal domains enclosed by γ_1 and γ_2 respectively. Then

$$Area\big(Arg_0(D)\big) = Area(\Delta_1) + Area(\Delta_2). \tag{10}$$

2 cases can occurs : either p_1 and p_2 are mapped to a point of $\pi\mathbb{Z}^2$, or not.
In the first case, $\tau^{-1}\big(\Delta_1\big)$ and $\tau^{-1}\big(\Delta_2\big)$ are 2 polygonal domains in $\mathbb{R}^2$ with integer vertices. Reorder their edges in convex position in order to get 2 convex polygon $\Box_1$ and $\Box_2$. Then

$$\Delta = \Box_1 + \Box_2 \tag{11}$$

where the plus sign is a Minkowski sum. By (9), (10), one has

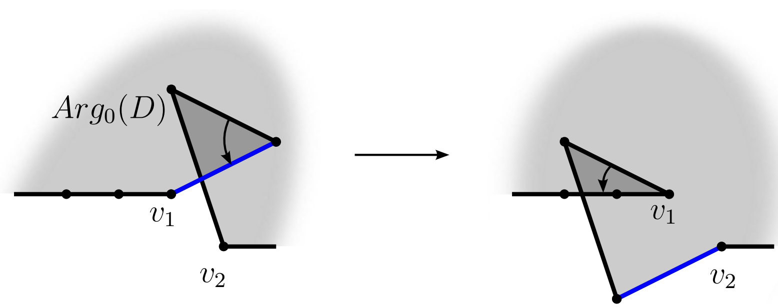
$$Area(\Delta) - 1 \leq Area(\Box_1) + Area(\Box_1),$$

and (11) provide the opposite inequality. It means that the mixed volume $Vol(\Box_1, \Box_2) = 1$. By [Kus76], This is the intersection number of 2 generic curves of respective Newton polygon $\Box_1$ and $\Box_2$. By (11), the union of 2 such curves has Newton polygon Δ. Such reducible curves form a component of

the space of nodal curves in $\mathcal{T}_\Delta$. By Horn parametrization, this space is irreducible. Hence, every nodal curve in $\mathcal{T}_\Delta$ is reducible, in particular, $\mathcal{C}$ is. This is a contradiction.
In the second case, p_1 and p_2 are not mapped on a point of $\pi\mathbb{Z}^2$. Denote by v_1 the closest vertex of $Arg_0(\partial D)$ before the self-intersection point $Arg_0(p_1)$, and v_2 the closest vertex of $Arg_0(\partial D)$ after it. Now cut the first edge of $Arg_0(\partial D)$ after v_1 and past it before v_2 as shown below. This construction has the following properties :

(∗) the area enclosed by the resulting curve is strictly greater than $Area\big(Arg_0(D)\big)$,

(∗) the angle at the vertex next the self-intersection point strictly decreases, if it is still convex.



The latter property implies that, repeating this process, one has to end up either in the case where the self-intersection point is moved to a point of $\pi\mathbb{Z}^2$, then the previous treatment leads to a contradiction; or in the case of a piecewise linear curve of rotational index 1 that is convex except at one vertex. This amounts to the treatment of case (α). The first property implies that

$$Area\big(Arg_0(D)\big) < \pi^2\big(Area(\Delta) - 1\big).$$

This is in contraction with (9). By lemma 9.8, the case (δ) cannot occur. The theorem is proved.

□

Note that during the proof, we obatined the following

Corollary 9.11. *Only the case (α) of lemma 9.5 can occur.*

Remark. Note that in [Mik00], the approach was to determine the topology of smooth simple Harnack curve by using the 2-to-1 property of the map $\mathcal{A}$, or equivalently that this map is 1-to-1 on half of the curve. Obviously, this is no longer true while considering simple Harnack curves with a hyperbolic node. What we have shown here is that on an open half of the curve, the map Arg_0 is 1-to-1 and that we are able to determine precisely its image by using constraints on its area. Note that the same approach works also in the smooth case. Nevertheless, this is not possible to apply such consideration for general simple Harnack curve. There are example where one cannot lift the argument map to $\mathbb{R}^2$, see for instance the quintic of figure 4.

9.5 Spines

This section is devoted to the proofs of theorems 7 and 6

Contrary to the case of smooth Harnack curve, the amoeba map is not at most 2-to-1. It is indeed 4-to-1 over some domain of $\mathbb{R}^2$ as the 2 branches near the singular point are mapped one over the other in a 2-to-1 fashion. It could happen that a hole of one of the branches would be hidden by the other branch in the amoeba-plane. The theorem 7 asserts that it is never the case and that all the holes are "visible". The other striking point is that the spine of $\mathcal{C}$ describes the topology of $\mathcal{C}$. In particular, the node of $\mathcal{C}$ manifests tropically as a node on its spine. It is not clear in general if a singular curve has to have a singular spine.

Up to a toric transformation, one can assume that Δ contains the 3 points $(0,0)$, $(0,1)$ and $(1,0)$ and that $\nu = (0,0)$. It induces the local compactification of $(\mathbb{C}^*)^2$ by $\mathbb{C}^2$ in $\mathcal{T}_\Delta$, where the 2 toric divisors adjacent to ν are the 2 coordinate axis. Hence, the 2 asymptotes of $\mathcal{A}(\varphi)$ are horizontal leftward and verical downward. As before, let $p = (p_1, p_2) \in (\mathbb{C}^*)^2$ be the node of $\mathcal{C}$, and choose ε_1, $\varepsilon_2 > 0$ such that the point $(\log|p_1|+\varepsilon_1, \log|p_2|+\varepsilon_2)$ belongs to the compact connected component of the complement of $\mathcal{A}(\mathbb{R}\mathcal{C})$ delimited $\mathcal{A}(\varphi)$. The next lemma implies that this component is indeed in the complement of $\mathcal{A}(\mathcal{C})$. Define the following sets

$$\begin{aligned} R &:= \left\{(x,y)\in\mathbb{R}^2 \middle|\ x \leq \log|p_1| + \varepsilon_1,\ y \leq \log|p_2| + \varepsilon_2\right\}, \\ H &:= \left\{(x,y)\in\mathbb{R}^2 \middle|\ x \leq \log|p_1| + \varepsilon_1,\ y = \log|p_2| + \varepsilon_2\right\}, \\ V &:= \left\{(x,y)\in\mathbb{R}^2 \middle|\ x = \log|p_1| + \varepsilon_1,\ y \leq \log|p_2| + \varepsilon_2\right\}. \end{aligned}$$

Lemma 9.12. $\mathcal{A}_{|_{\mathcal{C}\setminus\mathcal{A}^{-1}(R)}}$ *is at most 2-to-1. For any connected component* C *of* $\mathcal{A}^{-1}(R)$ *in the normalization* $\tilde{\mathcal{C}}$, $\mathcal{A}_{|_C}$ *is at most 2-to-1.*

Proof The elements of this proof are contained in the proof of lemma 8 in [Mik00]. For self-contentedness, one reproduces almost word by word, the needed arguments. Let $q_1 \in \mathbb{R}^2 \setminus R$ be a point such that $\mathcal{A}^{-1}(q_1)$ consists of more than 2 points. Let L be a line passing through q_1 with a rational slope which is not orthogonal to the slope of a side of Δ. Let $q_2 \in L \cap \left(\mathbb{R}^2 \setminus R\right)$ be a point close to infinity in L so that $\mathcal{A}^{-1}(q_2) = \emptyset$. As each component of $\mathcal{A}(\mathbb{R}\mathcal{C})$ cuts $\mathbb{R}^2$ into a convex and a non convex half, we call the convex half the interior of the component (even if this component is non compact). If q_1 belongs to the interior of a components and q_2 belongs to the interior of b components then the number of points in $\mathcal{A}^{-1}(q_1)$ is $2(b-a)$. But $\mathcal{A}$ is an embedding on the unique u-oval of $\mathcal{C}$ out of R, hence there is only one arc which joins the sides of Δ adjacent to q_2 and only the interior of this arc may contain q_2. Therefore, $b = 1$ and $2(b-a) \leq 2$. The first part of the statement is proven.
Now consider a line segment L in $\mathcal{A}(\mathcal{C}) \setminus R$ having one extremal point on the loop of $\mathcal{A}(\varphi)$ and the other on the boundary of a non compact component of the complement. Such L is easily seen to exist. $\mathcal{A}^{-1}(L)$ is an embedded circle in $\mathcal{C}$. Consider $\mathcal{C}_1$ and $\mathcal{C}_2$ the 2 surfaces with boundary obtained by cutting the normalization $\tilde{\mathcal{C}}$ along this circle. The outer boundaries of $\mathcal{A}(\mathcal{C}_1)$ and $\mathcal{A}(\mathcal{C}_2)$ are given be the line segment L and the image of their respective intersection with the unique u-oval of $\mathcal{C}$. In such case, one can easily adapt the latter arguments to show that $\mathcal{A}$ is at most 2-to-1 on both $\mathcal{C}_1$ and $\mathcal{C}_2$. Each connected component C of the statement is included in $\mathcal{C}_1$ or $\mathcal{C}_2$, hence the second part of the lemma is proved.

□

Lemma 9.13. $\mathcal{C} \cap \mathcal{A}^{-1}(R)$ *is a reducible holomorphic curve such that each of its irreducible components intersects either* $\mathcal{A}^{-1}(H)$ *or* $\mathcal{A}^{-1}(V)$.

Proof This follows from the proof of the latter lemma. Indeed, any such irreducible component is included either in $\mathcal{C}_1$ or $\mathcal{C}_2$. Up to a change of the indices, $\mathcal{A}(\mathcal{C}_1)$ intersects only H and $\mathcal{A}(\mathcal{C}_2)$ intersects only V.

□

Definition 9.14. *Let us denote by $\mathcal{C}_H$ the union of the irreducible components of $\mathcal{C} \cap \mathcal{A}^{-1}(R)$ intersecting $\mathcal{A}^{-1}(H)$ and $\mathcal{C}_V$ the union of the irreducible components of $\mathcal{C} \cap \mathcal{A}^{-1}(R)$ intersecting $\mathcal{A}^{-1}(V)$.*

Lemma 9.15. *There exists 2 functions g and h holomorphic on $\mathcal{A}^{-1}(R)$ such that C_V (resp. C_H) is the zero set of g (resp. h). Moreover they can be chosen such that $g \cdot h = f$ where f is a polynomial defining $\mathcal{C}$.*

Proof The closure of $\mathcal{A}^{-1}(R)$ in $\mathcal{T}$ is the polydisc D centred at the origin, with bi-radius $(|p_1|, |p_2|)$ in the local compactification $\mathbb{C}^2$ of $(\mathbb{C}^*)^2$. Let us consider an irreducible component C of C_V or C_H. If one can show that there exists a holomorphic function on D which zero set is exactly C, the result easily follows. By the classical implicit function theorem, there exists an open covering $D \subset \bigcup_j \mathcal{U}_j$ and a collection of function g_j holomorphic on $\mathcal{U}_j$ such that the zero set of g_j is exactly $\mathcal{U}_j \cap C$. Hence the quotient g_j/g_k is a nowhere vanishing holomorphic function on the overlap $\mathcal{U}j \cap \mathcal{U}_k$. We are looking for a global function g holomorphic on D such that its zero set is exactly C or equivalently such that g/g_j is a nowhere vanishing holomorphic function on $\mathcal{U}_j$. This amounts to solve the second Cousin problem, in the holomorphic case. As D is a Stein manifold, such that $H^2(D, \mathbb{Z}) = 0$, the Cousin problem is always solvable on D, see [H90].

□

The latter local factorization of f induces a splitting of its associated Ronkin function on $\mathcal{A}^{-1}(R)$, that is

$$\begin{aligned} N_f(x,y) &:= \tfrac{1}{(2i\pi)^2} \int_{\mathcal{A}^{-1}(x,y)} \frac{\log|f(z,w)|}{zw} dz \wedge dw \\ &= \tfrac{1}{(2i\pi)^2} \int_{\mathcal{A}^{-1}(x,y)} \frac{\log|g(z,w)| + \log|h(z,w)|}{zw} dz \wedge dw \\ &=: N_g(x,y) + N_h(x,y). \end{aligned}$$

To each of the latter Ronkin functions, one can associate their respective spines $\mathscr{S}_g$ and $\mathscr{S}_h$, as in [PR04]. These are 2 tropical curves in the domain R such that the amoeba $\mathcal{A}(\mathcal{C}_V)$ (*resp.* $\mathcal{A}(\mathcal{C}_H)$) deformation retracts on $\mathscr{S}_g$ (*resp.* $\mathscr{S}_h$), see theorem 6.12.

Recall that A_f is the set of connected components of the complement of $\mathcal{A}(\mathcal{C})$. Similarly, denote A_h (*resp.* A_g) the set of connected components of the complement of $\mathcal{A}(\mathcal{C}_H)$ (*resp.* $\mathcal{A}(\mathcal{C}_V)$ in R. The same properties hold for N_h and N_g and $\mathscr{S}_g$ and $\mathscr{S}_h$ are defined the exact same way as $\mathscr{S}_f$. If one denotes by A_f^R the elements of A_f intersecting R, convexity implies that

$$\begin{aligned}
(S_f)_{|R} &= \max_{\alpha \in A_f^R} N_f^\alpha \\
&= \max_{\alpha \in A_f^R} N_h^\alpha + \max_{\alpha \in A_f^R} N_g^\alpha \\
&\leq \max_{\alpha \in A_h} N_h^\alpha + \max_{\alpha \in A_g} N_g^\alpha \\
&= S_h + S_g,
\end{aligned}$$

where the elements of A_f^R are seen as subsets of the elements of A_h and A_g in the second equality. There is equality if and only the maps $A_f^R \to A_h$ and $A_f^R \to A_g$ given by the inclusion are both surjective, or equivalently no connected component of the complement of $\mathcal{A}(\mathcal{C}_H)$ is included in $\mathcal{A}(\mathcal{C}_V)$ and vice versa. Note that in such case

$$(\mathscr{S}_f)_{|R} = \mathscr{S}_h + \mathscr{S}_g.$$

In order to prove theorem 7, one will need few more lemmas.

Lemma 9.16. *The stable intersection of $\mathscr{S}_g$ and $\mathscr{S}_h$ in R is 1.*

Proof The idea is to deform smoothly $\mathcal{C}_H$ and $\mathcal{C}_V$ "close" to their respective spine in such a way that the intersection number of the deformations is kept constant equal to 1, and such that this intersection number corresponds to the stable intersection of $\mathscr{S}_g$ and $\mathscr{S}_h$.
Let us first deform $\mathcal{C}_H$. Consider a smooth foliation of $\mathcal{A}(\mathcal{C}_H)$ modelled on the foliation $\mathcal{F}$, as pictured in figure 6 of [Mik04]. The example given here is depicted in the neighbourhood of a 3-valent vertex of the spine but can easily be carried out for any higher valency. Now, let us deform $\mathcal{A}(\mathcal{C}_H)$ in time t by applying on each leaf an homothety of ratio $1/t$ centered at the the spine. The result is a smooth deformation retraction of $\mathcal{A}(\mathcal{C}_H)$ to *resp.* $\mathscr{S}_h$, for $t >> 1$. Now, one can deform smoothly $\mathcal{C}_H$ in a smooth surface $\mathcal{C}_{H,t}$ lying above the deformation at time t of $\mathcal{A}(\mathcal{C}_H)$, by keeping constant the argument

of the points in a moving fiber. Now, perform a similar deformation $\mathcal{C}_{V,t}$ of $\mathcal{C}_V$. As $\mathcal{A}(\mathcal{C}_{V,t}) \cap H = \mathcal{A}(\mathcal{C}_{H,t}) \cap V = \emptyset$ and $\partial\mathcal{A}(\mathcal{C}_{V,t}) \subset V$, $\partial\mathcal{A}(\mathcal{C}_{H,t}) \subset H$ for any $t >> 1$, the homological intersection of $\mathcal{C}_{H,t}$ and $\mathcal{C}_{V,t}$ is constant in t, that is equal to 1. At the level of spines, one can use a translation as small as desired in order to assume that the set-theoretical intersection of $\mathscr{S}_g$ and $\mathscr{S}_h$ is a finite collection of points, none of which is a vertex of any of the 2 spines, and that no intersection point went out of R. Then, translating $\mathcal{C}_{H,t}$ and $\mathcal{C}_{V,t}$ accordingly, one clearly has that both intersection numbers under consideration are not affected by these translations.
Hence, for t large enough, $\mathcal{A}(\mathcal{C}_{H,t})$ does not contain any vertex of $\mathscr{S}_g$ and $\mathcal{A}(\mathcal{C}_{V,t})$ does not contain any vertex of $\mathscr{S}_h$. It implies that for any intersection point of $\mathscr{S}_g$ and $\mathscr{S}_h$, there exists a small neighbourhood $\mathcal{U}$ such that the quadruple $(\mathcal{A}(\mathcal{C}_{V,t}) \cap \mathcal{U}, \mathscr{S}_g \cap \mathcal{U}, \mathcal{A}(\mathcal{C}_{H,t}) \cap \mathcal{U}, \mathscr{S}_h \cap \mathcal{U})$ is diffeomorphic to

$$(]-2,2[\times[-1,1]\,,\,]-2,2[\times\{0\}\,,\,[-1,1]\times]-2,2[\,,\,\{0\}\times]-2,2[)$$

in $]-2,2[^2$. By the 2-to-1 property, see lemma 9.12, the preimages of the 2 stripes in $\mathcal{U}$ are 2 cylinders in $\mathcal{C}_{V,t}$ and $\mathcal{C}_{H,t}$. Locally, each of these cylinders separates 2 connected component of the complement. By the proof of lemma 11 in [Mik00], the homolgy class of each cylinder in $H_1(T,\mathbb{Z})$ is given by the difference of the orders of its 2 adjacent components of the complement. By the definition of the spine, the difference of these orders corresponds to the primitive integer vector supporting the corresponding edge of the spine, times its multiplicity. On one side, the homological intersection of the 2 cylinders is given by the intersection of their homology class in $H_1(T,\mathbb{Z})$, that is the corresponding lattice index. By the definition of stable intersection and the latter observations, this is also the local stable intersection of $\mathscr{S}_g$ and $\mathscr{S}_h$ in $\mathcal{U}$. The result follows.

□

Lemma 9.17. *$\mathscr{S}_g$ and $\mathscr{S}_h$ are trees. Their Newton polygons are either segments of integer length 1 or triangles without inner integer points.*

Proof By assumption, $\mathscr{S}_h$ has at least a vertical leaf going downward, and $\mathscr{S}_g$ has at least an horizontal leaf going leftward. Neither $\mathscr{S}_h$ or $\mathscr{S}_g$ could have more than one leaf of the respective kinds, otherwise it would obviously contradict the previous lemma. If Δ_h and Δ_h are the respective Newton polygons, the latter implies that Δ_h is bounded from below by an horizontal side s_H of length 1 and Δ_g is bounded from the left by a vertical side s_V of length 1. For Δ_g, the side attached at the top of s_V, if there is, is strictly slanted toward the right, otherwise the corresponding leaf of $\mathscr{S}_g$ would intersect H. The side attached at the bottom of s_V, if there is, is horizontal or strictly slanted toward the left, otherwise the corresponding leaf of $\mathscr{S}_g$ would intersect the vertical leaf of $\mathscr{S}_h$, up to translation. It contradicts the previous lemma. The only possibility is that Δ_g is either a binomial or a right angled triangle with integer height 1. The same arguments apply for Δ_h.

□

Proof of theorems 6 and 7 As the curves of $\mathcal{H}_{\Delta,\nu}$ having transversal intersection at infinity are dense in the latter space, it is sufficient to prove the result only for this case. By the previous lemma, every connected component of the complement of $\mathcal{A}(\mathcal{C}_H)$ and $\mathcal{A}(\mathcal{C}_V)$ are unbounded in R. It implies that no connected components of the complement of $\mathcal{A}(\mathcal{C}_H)$ is hidden by $\mathcal{A}(\mathcal{C}_V)$ and vice versa. By the previous remarks, it implies that

$$(\mathscr{S}_f)_{|R} = \mathscr{S}_h + \mathscr{S}_g.$$

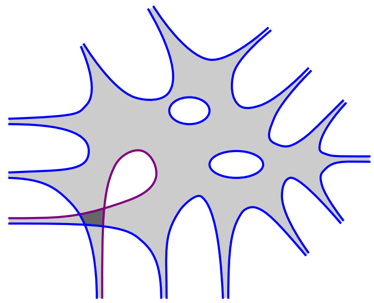
It implies also that $\mathcal{A}(C)$ has exactly g visible holes, which is the maximal possible. Indeed, the previsous lemma implies that none of the $(g-1)$ b-ovals of $\mathbb{R}\tilde{\mathcal{C}}$ intersect $\mathcal{A}^{-1}(R)$, and $\mathcal{A}$ is at most 2-to-1 on this space, by lemma 9.12. Hence, their image by $\mathcal{A}$ bounds a compact component of the complement. The same holds for the singular loop of $\mathcal{A}(\varphi)$. Hence, the complement of $\mathscr{S}_f$ has also g compact connected components.
It implies, together with the transversality assumption, that $\mathscr{S}_f$ has only leaves-edges of weight 1. The only possibility preventing $\mathscr{S}_f$ to be in $\mathbb{T}\mathcal{H}_{\Delta,\nu}$ is the occurrence of vertices of valency higher than 3. Those vertices cannot hide any genus, in the sense that their dual polygon has no interior point. One can perturb them in order to get a 3-valent tree, and this perturbation can be chosen as small as desired. Then the map $\mathscr{S}$ does take values in $\overline{\mathbb{T}\mathcal{H}_{\Delta,\nu}}$. We postpone the fact that $\mathscr{S}$ is a local diffeomorphism to the next section. Up to this, theorem 7 is proven.
Now, theorem 6 falls as a corollary. Thanks to the 2-to-1 property of lemma 9.12, one can recover in which quadrant sits any connected component of $\mathbb{R}\mathcal{C}^\circ$ from its image by $\mathcal{A}$ in the very same fashion as in lemma 11 in [Mik00]. But this is exactly how one recovers $\mathbb{T}V$ from the only datum of the underlying

tropical Harnack curve. Hence $\mathbb{R}\mathcal{C}^\circ$ deforms isotopically to the real locus of a simple tropical Harnack curve, obtained by a small perturbation of its spine.

$\square$

9.6 Spines continued

In order to end the proof of theorem 7, one needs to show that one can reproduce the arguments given in [KO06], in section 2.2.4 and 4.2. We are not any more in the situation studied there. Namely the amoeba map is not 2-to-1 any more, but the situation is not so dramatic. Basically, all of what we have proven yet can be described in the following picture



This represent the amoeba of $\mathcal{C}$, where $\mathcal{A}(\mathbb{R}\mathcal{C})$ is drawn in blue, $\mathcal{A}(\varphi)$ is drawn in purple. Over the light gray region, $\mathcal{A}$ is 2-to-1. It is 4-to-1 over the dark gray region.
For self-contentedness, let us repeat the arguments of [KO06] : take α_1, ..., α_{g-1} to be the $(g-1)$ smooth compact ovals of $\mathbb{R}\mathcal{C}$. For each of these ovals, pick a point (x_j, y_j) in the complement of $\mathcal{A}(\mathcal{C})$ enclosed by $\mathcal{A}(\alpha_j)$, and define

$$\beta_j := \left\{ (z,w) \in (\mathbb{C}^*)^2 \,\middle|\, |z| = e^{x_j}, \ |w| \leq e^{y_j} \right\}.$$

The α_j's and β_j's can be taken as a- and b-cycles for $\tilde{\mathcal{C}}$.
Recall that $\mathcal{C}$ is given by a polynomial f of Newton polygon Δ that contains the origin and has a vertical and an horizontal side adjacent to it. The origin has been chosen to be the vertex next to which $\mathcal{C}$ has an hyperbolic node as shown in the above picture, see definition 9.1.
Then the tangent space to $\mathcal{C}$ with given points at infinity and node is given by the polynomials vanishing at the points at infinity and the node of $\mathcal{C}$, modulo the polynomial f itself. Such polynomials have the form $zw \cdot q(z,w)$ where the Newton polygon of q is obtained from Δ by removing all its boundary points. Hence, the Newton polygon of q has g integer points. As q has to vanish at the node of $\mathcal{C}$, the space of such q's is $(g-1)$ dimensional.
On the other hand, one can check that the holomorphic differentials on $\tilde{\mathcal{C}}$ have the form

$$\omega = \frac{q(z,w)}{\frac{\partial}{\partial w} f(z,w)} dz.$$

The space of tropical curve of $\mathbb{T}\mathcal{H}_{\Delta,\nu}$ with given points at infinity is $(g-1)$ dimensional. It can be described by the intercept of the tropical monomial dominating inside its j-th hole, for $1 \leq j \leq g-1$. If the tropical curve under consideration is the spine of the curve $\mathcal{C}$, this intercept can be computed by

$$N_f(x,y) - (ax + by)$$

where (x, y) lies inside the j-th hole, and (a, b) is the order of this hole. Repeating the computation of [KO06], one has

$$\begin{aligned}
\tfrac{d}{dt}N_{f+tzw\cdot q}(x,y)_{|t=0} &= \frac{1}{(2\pi i)^2}\int_{\substack{|z|=e^x\\|w|=e^y}} \frac{q(z,w)}{f(z,w)}dz\,dw \\
&= \frac{1}{2\pi i}\int_{|z|=e^x} \sum_{\substack{f(z,w_r)=0\\|w_r|\le e^y}} \frac{q(z,w_r)}{\frac{\partial}{\partial w}f(z,w_r)}dz \\
&= \frac{1}{2\pi i}\int_{\beta_j} \frac{q(z,w)}{\frac{\partial}{\partial w}f(z,w)}dz.
\end{aligned}$$

One deduces that the Jacobian of the map that associates to a curve of $\mathcal{H}_{\Delta,\nu}$ the intercepts of its Ronkin function is precisely the period matrix of the curve, in particular it is invertible. One concludes that the map $\mathscr{S}$ is a local diffeomorphism. This ends the proof of theorem 7.

10 Discussions

10.1 Harnack curves and tropical curves

In [KO06], The authors showed that the map $\mathscr{S}$ that associates its spine to a smooth Harnack curve is a local diffeomorphism. They also showed that the area of the holes of their amoebas gives global coordinates on the space of smooth Harnack curves, when the points at infinity are fixed. In the meantime, they show that the map from the intercept of the linear components of the Ronkin function to the area of the holes is a coordinate change. Hence

Theorem 10.1. *[KO06] The map $\mathscr{S}$ from the closure of the space of smooth Harnack curves of degree d to the space of tropical curves of degree d is a diffeomorphism.*

Following the exact same reasoning, we claim that one can strengthen theorem 7 by

Theorem 9. *The map*

$$\mathscr{S} \,:\, \overline{\mathcal{H}_{\Delta,\nu}} \to \overline{\mathbb{T}\mathcal{H}_{\Delta,\nu}}$$

is a global diffeomorphism.

In section 8, we constructed Harnack curves in $\mathcal{H}_{\Delta,\nu}$ from tropical curves. The latter theorem tells in some sense that every Harnack curve can be constructed this way. One can naturally ask the following question

Qu1. Can we construct any Harnack curve by approximating tropical Harnack curves?

It is very tempting to answer positively. Nevertheless, the techniques used here can't be developed further. First, notice that if *Qu1* admits a positive answer, then every Harnack curve has to be an M-curve. This is part of proposition 9.3. Already, the methods used in the proof cannot give the expected result while considering Harnack curves with 2 hyperbolic nodes. Moreover one cannot expect to be able to lift coamoebas to the universal covering of the torus as we did in theorem 8. Indeed, consider the projective Harnack quintic given in figure 5. From a more general point of view, it wouldn't be reasonable trying to answer *Qu1* by treating all the cases one by one.
The approach we would suggest is a more extensive study of the connection between Harnack curves and their tropical avatars. For example

Qu2. Does there exist for any Harnack curve a canonical degeneration towards its spine?

By the local 2-to-1 property of the amoeba map, the pull-back of the spine by $\mathcal{A}$ gives the critical graph of a Strebel foliation in the normalization $\tilde{\mathcal{C}}$, see [HM79]. Such foliation is given by a quadratic differential which in turns correspond to a tangent direction at $\tilde{\mathcal{C}}$ in the appropriate Teichmüller space, see [Nip10]. It would be interesting to study possible tropical degenerations by following the geodesic with respect to the Teichmüller metric given by the specified tangent direction.

10.2 Harnack curves as spectral curves

In [KOS06] and [CD13], Harnack curves arise as spectral curves for different physical models. By Horn-Kapranov theorem, reduced A-discriminantal curves are exactly those curves for which their logarithmic Gauss map are birational isomorphisms, see [Kap91]. For this reason, they are clearly Harnack curves. Such curves are given as spectral curve of the associated discriminantal complex, see [GKZ08]. Naturally one can ask whether every Harnack curve can be presented as the spectral curve of a geometric/physical problem, and if it can be done in a uniform way. For example, can we reasonably relate A-discriminant to dimer configurations on the torus?

10.3 Complement components of coamoebas

Due to their nice properties, amoebas had been more attractive and more extensively studied than their imaginary counterpart, namely coamoebas. They are closed subsets, with convex connected component in their complement, and they are somehow intermediate objects between classical and tropical geometry. They carry nice combinatorial informations. On the other side, coamoebas are apparently not as nice, apparently. We have seen here how useful they can be. One problem coming from the amoeba side is to give a description of the connected components of coamoebas. One knows since [FPT00] that the set of connected components of the complement of amoebas maps injectively in the set of integer point of the Newton polygon. For coamoebas, a similar map has been constructed in [FJ12] by introducing an intermediate object, the lopsided coamoeba.
Lopsided coamoebas are of very combinatorial nature, and for this reason they are easier to study. In fact, one can show that lopsided coamoebas and coamoebas are the same for Harnack curves. More than that, every curve for which its coamoeba and its lopsided coamoeba are the same turns out to be a ramified covering of a Harnack curve. These curves are called multi-Harnack curves. They will be studied by the author in a further paper. Regarding questions on complement components of coamoebas, multi-Harnack curves are very interesting and their classification rely on the classification of Harnack curves. This is one more reason why the latter should be studied extensively.

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