

Archive ouverte UNIGE

https://archive-ouverte.unige.ch

Thèse 2015

Open Access

This version of the publication is provided by the author(s) and made available in accordance with the copyright holder(s).

Spanning trees in discrete tori, hypercubic lattices and circulant graphs

Louis, Justine

How to cite

LOUIS, Justine. Spanning trees in discrete tori, hypercubic lattices and circulant graphs. Doctoral Thesis, 2015. doi: 10.13097/archive-ouverte/unige:81877

Publication DOI:

This publication URL: <u>https://archive-ouverte.unige.ch/unige:81877</u> 10.13097/archive-ouverte/unige:81877

© This document is protected by copyright. Please refer to copyright holder(s) for terms of use.

Spanning Trees in Discrete Tori, Hypercubic Lattices and Circulant Graphs

THESE

présentée à la Faculté des Sciences de l'Université de Genève pour obtenir le grade de Docteur ès Sciences, mention Mathématiques

 par

Justine Louis

 de

Bruxelles (Belgique)

Thèse $N^{\circ}4867$

GENEVE Atelier d'impression ReproMail 2015



Doctorat ès sciences Mention mathématiques

Thèse de Madame Justine LOUIS

intitulée :

"Spanning Trees in Discrete Tori, Hypercubic Lattices and Circulant Graphs "

La Faculté des sciences, sur le préavis de Monsieur A. KARLSSON, professeur associé et directeur de thèse (Section de mathématiques), Madame T. SMIRNOVA-NAGNIBEDA, professeure associée (Section de mathématiques), et Monsieur D. LENZ, professeur (Friedrich-Schiller-Universität Jena, Fakultät für Mathematik und Informatik, Jena, Germany), autorise l'impression de la présente thèse, sans exprimer d'opinion sur les propositions qui y sont énoncées.

Genève, le 1^{er} décembre 2015

Thèse - 4867 -

Le Doyen

N.B.- La thèse doit porter la déclaration précédente et remplir les conditions énumérées dans les "Informations relatives aux thèses de doctorat à l'Université de Genève".

A ma Maman

 $\mathbf{2}$

The more I learn, the more I realise how much I don't know. – Albert Einstein

Acknowledgements

J'aimerais en premier lieu remercier très chaleureusement Anders Karlsson pour son enthousiasme, sa disponibilité, ses encouragements tout au long de ma thèse. Je suis très reconnaissante de la confiance qu'il m'a accordée ainsi que pour sa très grande gentillesse. Je suis très heureuse d'avoir eu l'opportunité de faire ma thèse sous sa direction, c'était un grand plaisir. Je remercie également Tatiana Smirnova-Nagnibeda et Daniel Lenz pour avoir accepté d'être membres du jury de ma thèse. Je remercie le Fonds National Suisse 200021 132528/1 de m'avoir soutenue durant ma thèse ainsi que le Subside Tremplin pour m'avoir déchargée des charges d'enseignement lors de mon dernier semestre.

Je remercie la section de Mathématiques de l'Université de Genève où il est très agréable de travailler. Merci à Rinat Kashaev et Sylvain Sardy, je suis très heureuse d'avoir été votre assistante. Merci pour votre soutien en 2012. Je remercie les bibliothécaires Anne-Sophie et Valérie, ainsi que les secrétaires Najat, Joselle, Mervat et Annick, pour leur gentillesse, bonne humeur et efficacité. I am very grateful to Teresa, Hui and Yingxiang with whom I shared the office during most of my thesis. Thank you so much for your kindness and support, especially in 2012. Thank you for all these good moments that we shared and for cooking Chinese food for me. I could not dream to have better office mates. Je remercie également mes collègues et amis qui font la joie de ce département. Merci à ceux présents lors du début de ma thèse, Yacine pour sa gentillesse et son humour, Loren pour son sourire et sa bonne humeur, Renaud, Stefan, Paola, Zsolt, Yves, Cyrille et Mucyo. Un très grand merci à Máté pour sa gentillesse et sa sensibilité. Un très grand merci à Caterina pour tout ce qu'elle est, sa gentillesse, son sourire, sa bonne humeur, elle est le rayon de soleil de la section. Thank you so much to Soheil for being my best friend, for all the good moments that we had and his constant support, kindness and jokes. Many thanks to Aran, for his great humour and kindness. Merci à Hester, Faycal, Parisa, Jhih-Huang, Anthony et Jérémy. Merci à Fabien pour nos quelques discussions mathématiques. Je remercie de tout cœur en particulier Caterina, Soheil, Máté, Loren, Hester, Paola, Jean-René, Marie-Charlotte et Rebecca pour tout le soutien que vous m'avez apporté en 2012.

Un très grand merci au Ring Star Vernier et en particulier à Larbi, Salah, Jean-Pierre, Stefano et Massak. Merci pour tout ce que vous m'avez appris et surtout pour votre grand soutien vital en 2012. Pour terminer, je remercie du fond du cœur ma Maman pour tout ce qu'elle fait pour moi, d'être toujours là pour moi et pour son soutien inconditionnel. Je ne serais jamais arrivée jusque-là sans ses encouragements.

Finalement, merci à Dieu, pour tout.

Contents

1	Introduction							
	1.1 Graphs	9						
	1.2 Matrix tree theorem	11						
	1.3 Theta functions, heat kernels and theta inversion formulas	15						
	1.4 Spectral zeta function	18						
	1.5 Results	18						
2	Asymptotics for the number of spanning trees in circulant graphs and degenerating d-dimensional discrete tori							
3	A formula for the number of spanning trees in circulant graphs with non-fixed generators and discrete tori	. <u>.</u> 50						
4	Spanning trees in directed circulant graphs and cycle power graphs	59						
5	Asymptotics for the determinant of the combinatorial Laplacian on hypercubic lattice	s 70						
6	Low temperature ratchet current	90						
7	A formula for the energy of circulant graphs with two generators	95						
8	Summary	101						
9	Résumé	103						

1 Introduction

The number of spanning trees of a finite graph is an interesting invariant which arises in many various fields. A spanning tree is an efficient way to connect a set of nodes such as computers, telephone lines, railroads, fiber-optic cable, cities, without redundancy in the network and in an economical way. While studying electrical resistor networks, Kirchhoff established in 1847 the matrix tree theorem relating the number of spanning trees in a graph to the determinant of the associated combinatorial Laplacian [18]. A graph can represent an electrical network, that is, the edges represent the wires and the vertices the junctions at which the wires are connected with one another. The number of spanning trees of the corresponding graph measures the network reliability and is related to the effective resistances of the network, see for instance [5, 7, 27]. Spanning trees are also of interest in theoretical chemistry, as a graph can represent the connectivity of atoms that constitute the conjugation network of an unsaturated molecule. The number of spanning trees of such labelled molecular graphs gives the complexity of a molecular structure and enables to classify them, as the most complex structure possesses the highest number of spanning trees, see [8, 17, 23]. Spanning trees also appear in graph theoretical problems, such as counting Hamiltonian walks [11], can sometimes enumerate the dimer coverings of a graph [16] and as a special value of the Tutte polynomial [1, 29]. The number of spanning trees is as well interesting in statistical physics. It corresponds to the limit $q \rightarrow 0$ of the partition function of the q-state Potts model [14, 30]. Spanning trees have applications in quantum field theory as well since they are related to the Feynman graph polynomials which define the integrand of a Feynman integral [6].

Considering an increasing sequence of graphs converging in some sense to some infinite graph, one can study the asymptotic behaviour of the number of spanning trees. The rescaled logarithm of the lead term of it is referred as the tree entropy by Lyons [20]. This quantity is also of interest in physics in the calculation of the thermodynamic limit in different lattices, such as square, honeycomb and triangular lattices [26]. As we study the next terms in the asymptotic development of the number of spanning trees, interesting terms appear such as the regularized determinant associated to the limiting object, see [24].

One type of graphs, circulant graphs, are of particular interest as they appear in various contexts, see for instance [19, 22]. They are also known as multi-loop networks. In this thesis we mainly consider circulant graphs which have first generator equal to 1, that is, they begin with a cycle. It turns out that this is not too restrictive since for a given circulant graph, if one of his generators is relatively prime to the number of vertices, then it is isomorphic to a circulant graph with first generator equal to 1.

Below we define a graph and its related notions, then state the matrix tree theorem, its proof and an example to illustrate it. In the following sections, theta functions, heat kernels, inversion formulas and spectral zeta functions are discussed. In the last section, we briefly describe the results obtained in the thesis.

1.1 Graphs

Following Serre's definition [25], a graph G consists of a set of vertices V(G), a set of edges E(G) and two maps

$$\begin{array}{ll} \mathsf{E}(\mathsf{G}) \longrightarrow \mathsf{V}(\mathsf{G}) \times \mathsf{V}(\mathsf{G}) & \qquad \mathsf{E}(\mathsf{G}) \longrightarrow \mathsf{E}(\mathsf{G}) \\ e \longmapsto (\mathsf{o}(e), \mathsf{t}(e)) & \qquad e \longmapsto \bar{e} \end{array}$$

such that for all edge $e \in E(G)$, $\overline{e} = e$, $\overline{e} \neq e$ and $o(e) = t(\overline{e})$. The edge \overline{e} is called the inverse edge of e, the vertices o(e) and t(e) are respectively the origin and the terminus of \underline{e} . An orientation of the graph G is a subset $E(G)_+$ of the edge set E(G) such that $E(G) = E(G)_+ \sqcup \overline{E(G)}_+$, where the union is disjoint. The couple (V(G), E(G)) with an orientation $E(G)_+$ and a map $E(G)_+ \longrightarrow V(G) \times V(G)$ defines an oriented graph. Two vertices $v_1, v_2 \in V(G)$ are said to be adjacent if there exists an edge $e \in E(G)$ such that $v_1 = o(e)$ and $v_2 = t(e)$. Let $n \in \mathbb{N}_{\geq 3}$ and $\{i_k\}_{k=1}^n \subset \{1, \ldots, Card(E(G))\}$. The graph G contains an n-cycle if there exists a sequence of n edges $e_{i_1} \ldots e_{i_n}$ such that $t(e_{i_k}) = o(e_{i_{k+1}})$ for all $k \in \{1, \ldots, n-1\}$, and $t(e_{i_n}) = o(e_{i_1})$. An n-path is defined similarly without the condition $t(e_{i_n}) = o(e_{i_1})$. A graph is connected if any two vertices are the extremities of at least one path. A tree is a connected graph without cycles. A spanning subgraph of G is a couple (V(G), F(G)) with the same vertex set as G and such that $F(G) \subset E(G)$. A spanning tree is thus a spanning subgraph of G without cycles. The combinatorial Laplacian on G defined as an operator acting on the space of functions $f: V(G) \to \mathbb{R}$ is given by

$$\Delta_{G} f(v) = \sum_{\substack{e \in E(G):\\ o(e) = v}} (f(v) - f(t(e))).$$

The combinatorial Laplacian has a matrix representation given by the difference of the degree matrix and the adjacency matrix, namely $\Delta_G = D_G - A_G$. Let n denote the cardinal of the vertex set V(G). These matrices are $n \times n$ matrices with rows and columns indexed by the vertices v_1, \ldots, v_n of G. The degree of a vertex $v \in V(G)$ is defined by

$$\deg(v) = \operatorname{Card} \{ e \in E(G) \text{ such that } o(e) = v \}.$$

The degree matrix $D_G = (D_{ij})$ is defined by $D_{ii} = deg(v_i)$ and $D_{ij} = 0$ for $i \neq j$. The adjacency matrix $A_G = (A_{ij})$ is defined by

$$A_{ij} = -Card\{e \in E(G) \text{ such that } o(e) = v_i \text{ and } t(e) = v_j\}.$$

In this thesis, we mainly consider three types of graphs that we describe below, namely hypercubic lattices, discrete tori and circulant graphs.

• Let n_1, \ldots, n_d be non-zero positive integers. The d-dimensional hypercubic lattice $L(n_1, \ldots, n_d)$ is defined by the d-fold cartesian product of the n_i -path graphs for $i = 1, \ldots, d$.

Figure 1: The 2-dimensional lattice L(11, 7).

 A d-dimensional discrete torus is defined by the quotient Z^d/ΛZ^d where Λ is a d×d invertible integer matrix with nearest neighbours connected.



Figure 2: The discrete torus $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/|\log n|\mathbb{Z}$ with n = 43.

• Let $1 \leq \gamma_1 \leq \cdots \leq \gamma_d \leq \lfloor n/2 \rfloor$ be integers. The circulant graph C_n^{Γ} generated by the set $\Gamma = \{\gamma_1, \ldots, \gamma_d\}$ on n vertices labelled by the elements of $\mathbb{Z}/n\mathbb{Z}$ is the 2d-regular graph such that each vertex $\nu \in \mathbb{Z}/n\mathbb{Z}$ is connected to $\nu - \gamma_i \mod n$ and to $\nu + \gamma_i \mod n$, for all $i \in \{1, \ldots, d\}$.



Figure 3: Examples of circulant graphs.

1.2 Matrix tree theorem

In 1847, Kirchhoff established the matrix tree theorem which relates the number of spanning trees to the determinant of the combinatorial Laplacian. We follow the proof given in [13].

Theorem 1.1. Let G be a connected labelled graph. Then all cofactors of the Laplacian matrix Δ_G are equal and their common value is the number of spanning trees in G, $\tau(G)$. More precisely, we have that

$$\tau(\mathsf{G}) = \frac{\det^* \Delta_\mathsf{G}}{|\mathsf{V}(\mathsf{G})|}$$

where $det^* \Delta_G$ is the product of the non-zero eigenvalues of the Laplacian.

To show the matrix tree theorem we need the two following results:

Lemma 1.2 (Cauchy-Binet theorem). Let A be a $k \times n$ matrix and B a $n \times k$ matrix with $k \leq n$. Then

$$\det(AB) = \sum_{J} \det(A(J)) \det(B(J))$$
(1)

where $J = (j_1, ..., j_k)$, $1 \leq j_1 < \cdots < j_k \leq n$ runs through all such multi-indices, A(J) denotes the matrix formed from A using columns J and B(J) denotes the matrix formed from B using rows J in that order.

Lemma 1.3. Let A be an $n \times n$ matrix such that the row sums and column sums of A are all zero, then all cofactors of A are equal.

This result can be shown using Cramer's rule. Considering the equation AX = 0 we have that X = (1, ..., 1) is a solution by hypothesis. Let B be the submatrix consisting of the first n-1 columns and rows of A and consider the system $(BY)_i = A_{in}$ where A_{in} , i = 1, ..., n-1, are the elements of the last column of A, then Y = (-1, ..., -1) is a solution. Using this solution in Cramer's rule gives the equality between all cofactors of the last row of the matrix A. This procedure can be done for any row of A which gives the equality between all the cofactors of A.

Proof of the matrix tree theorem. Assume that G has no multiple edges and no loops. Let |V(G)| = p and |E(G)| = q. Since G is connected, for every $v_i \in V(G)$, i = 1, ..., p, there exists a $y_j \in E(G)$, j = 1, ..., q, such that $v_i = o(y_i)$ or $v_i = t(y_i)$. Let N be the $p \times q$ incidence matrix of G defined by

$$N_{ij} := \begin{cases} 1 & \text{if } \nu_i = o(y_j) \text{ or } \nu_i = t(y_j) \\ 0 & \text{otherwise} \end{cases}$$
(2)

Let E be the $p \times q$ matrix obtained from N by replacing one of the two ones by -1 in each column of N. Then we have that $\Delta_G = EE^T$. Indeed, $(EE^T)_{ii} = \sum_{k=1}^{q} E_{ik}^2 = deg(v_i)$ and for $i \neq j$, $(EE^T)_{ij} = -1$ if v_i and v_j are adjacent, and 0 otherwise. Let H be a spanning subgraph of G with p vertices and p-1 edges. From the submatrix of E corresponding to H, we remove an arbitrary row, that we call the k-th, which corresponds to the vertex v_k . This forms a square matrix, F, of order p-1.

Suppose H is not a tree, thus it is disconnected since he has p vertices and p-1 edges. So there exists a component of H which does not contain v_k . Each column of F represents an edge of H and has exactly one 1 one -1 and 0 elsewhere. So the sum of the rows corresponding to the vertices of this component is the 0-row because they must all have a 1 and a -1 in their columns since they are connected to each other. These rows are thus linearly dependent and so $\det(F) = 0$.

Suppose now that H is a tree. We relabel the edges and vertices in the following way. Let $u_1 \neq v_k$ be an endpoint of H and let $y_1 \in E(G)$ be such that $o(y_1) = u_1$. Let $u_2 \neq v_k$ be an endpoint of $H - u_1$ and $y_2 \in E(G)$ such that $o(y_2) = u_2$. We continue relabelling in this way with u_i being an endpoint of $H - \{u_1, \ldots, u_{i-1}\}$ and $y_i \in E(G)$ being such that $o(y_i) = u_i$ until the last vertex left is v_k . So we obtain a new matrix F' which is lower triangular since u_i cannot be connected to y_j with j > i by construction and with 1 or -1 on the diagonal. Thus, $|\det(F')| = |\det(F)| = 1$ because F' is a permutation of rows and columns of F.

Let E_1 be the $(p-1) \times q$ matrix obtained from E without his first row. Let $\Delta_G(i|j)$ denote the Laplacian matrix where row i and column j are removed. To evaluate the first cofactor of the Laplacian, L_{11} , given by

$$L_{11} := (-1)^{1+1} \det(\Delta_{G}(1|1)) = \det(E_{1}E_{1}^{1}),$$

we apply Cauchy-Binet theorem, that is

$$L_{11}=\sum_J \left(\det(\mathsf{E}_1(J))\right)^2$$

with $J = (j_1, \ldots, j_{p-1})$ such that $1 \leq j_1 < \cdots < j_{p-1} \leq q$. The $(p-1) \times (p-1)$ matrix $E_1(J)$ corresponds to the matrix F previously defined with determinant equals to ± 1 or 0 depending on whether the corresponding graph is a spanning tree or not. As a consequence, the sum over all the possibilities gives the number of spanning trees of G. The equality between all cofactors comes from Lemma 1.3 since the sum of all the rows and columns of L is 0. Thus,

$$\tau(G) = L_{ij}$$

where L_{ij} , i, j = 1, ..., p, denote the cofactors of Δ_G . The cofactors can be expressed in terms of the eigenvalues of the Laplacian. Let λ_i , i = 0, ..., p - 1, denote the eigenvalues of Δ_G . Since G is connected, the Laplacian has exactly one zero eigenvalue, $\lambda_0 = 0$. The characteristic polynomial of the Laplacian, $\chi(t)$, is given by

$$\chi(t) = t(t - \lambda_1) \cdots (t - \lambda_{p-1}) = t^p + \ldots + t(-1)^{p-1} \prod_{i=1}^{p-1} \lambda_i.$$
(3)

Denote by $\Delta_{G,i}$, i = 1, ..., p, the columns of Δ_G and by Γ_i , i = 1, ..., p, the canonical vectors of \mathbb{R}^p , such that $(\Gamma_i)_j = \delta_{ij}$. Let f_{t^k} , $k \ge 1$, denote a polynomial in t which can be factorised by t^k . By developing the characteristic polynomial with respect to the columns $\Delta_{G,i}$, i = 1, ..., p, we get

$$\begin{split} \chi(t) &= \det(tI - \Delta_G) \\ &= \det[t\Gamma_1 - \Delta_{G,1}, t\Gamma_2 - \Delta_{G,2}, \dots, t\Gamma_p - \Delta_{G,p}] \\ &= t\det[\Gamma_1, t\Gamma_2 - \Delta_{G,2}, \dots, t\Gamma_p - \Delta_{G,p}] - \det[\Delta_{G,1}, t\Gamma_2 - \Delta_{G,2}, \dots, t\Gamma_p - \Delta_{G,p}] \\ &= t(-\det[\Gamma_1, \Delta_{G,2}, \dots, t\Gamma_p - \Delta_{G,p}] + f_t) \\ &- (t\det[\Delta_{G,1}, \Gamma_2, \dots, t\Gamma_p - \Delta_{G,p}] - \det[\Delta_{G,1}, \Delta_{G,2}, \dots, t\Gamma_p - \Delta_{G,p}]) \\ &= (-1)^{p-1} \sum_{i=1}^p \det[\Delta_{G,1}, \dots, \Delta_{G,i-1}, \Gamma_i, \Delta_{G,i+1}, \dots, \Delta_{G,p}] t + f_{t^2} \\ &= (-1)^{p-1} \sum_{i=1}^p (-1)^{i+i} \det(\Delta_G(i|i)) t + f_{t^2}. \end{split}$$

Since all cofactors are equal, it comes

$$\chi(t) = (-1)^{p-1} p L_{ij} t + f_{t^2}.$$
(4)

By identifying the coefficient in t in equations (3) and (4), it follows that

$$\tau(G) = L_{ij} = \frac{1}{p} \prod_{i=1}^{p-1} \lambda_i.$$

Example. To illustrate the matrix tree theorem, we consider the following graph on 5 vertices.



Figure 4: The butterfly graph.

The Laplacian matrix associated to the butterfly graph is

	(2	-1	-1	0	0	
	-1	2	-1	0	0	
$\Delta =$	-1	-1	4	-1	-1	
	0	0	-1	2	-1	
	0	0	-1	-1	2)

with eigenvalues given by 0, 1, 3, 3, 5. As a consequence of the matrix tree theorem, there are 9 spanning trees that we enumerate in the figure below.



Figure 5: The 9 spanning trees of the butterfly graph.

In this thesis, we study the number of spanning trees in the previously mentioned graphs in section 1.1. The matrix tree theorem expresses it as a product of n terms, n being the number of vertices of the graph. Therefore as n grows, it becomes difficult to estimate it. In the following section we define the theta function of a graph which contains the spectral information and establish theta inversion formulas relating it to the modified I-Bessel function, see [9, 10, 15].

1.3 Theta functions, heat kernels and theta inversion formulas

Let λ_k , k = 0, 1, ..., n - 1, be the eigenvalues of the combinatorial Laplacian on a graph G with n vertices. The theta function of G is defined by

$$\theta_{\rm G}(t) = \sum_{k=0}^{n-1} e^{-\lambda_k t}.$$

By considering the heat equation below we can express the theta function in terms of a series of modified I-Bessel functions,

$$(\Delta_{\rm G} + \partial_{\rm t}) f({\rm t}, {\rm x}) = 0.$$

The heat kernel $K_G(t,x)$ is the unique bounded solution to the heat equation such that $K_G(0,x) = \delta_0(x)$ for all x where δ_0 is the delta function, that is $\delta_0(0) = 1$ and $\delta_0(x) = 0$ for $x \neq 0$. Hence it can be written as

$$\mathsf{K}_{\mathsf{G}}(\mathsf{t},\mathsf{x}) = e^{-\mathsf{t}\Delta_{\mathsf{G}}}\delta_0(\mathsf{x}).$$

Denote by ϕ_k , k = 0, 1, ..., n - 1, the orthonormal eigenvectors of the combinatorial Laplacian. The delta function is expressed in the eigenbasis as

$$\delta_0 = \sum_{k=0}^{n-1} \overline{\varphi_k(0)} \varphi_k,$$

implying that the heat kernel is given by

$$\mathsf{K}_{\mathsf{G}}(\mathsf{t},\mathsf{x}) = \sum_{\mathsf{k}=0}^{n-1} e^{-\lambda_{\mathsf{k}} \mathsf{t}} \overline{\varphi_{\mathsf{k}}(0)} \varphi_{\mathsf{k}}(\mathsf{x}).$$

Following [15], we solve the heat equation on the discrete line \mathbb{Z} and on the discrete n-cycle $\mathbb{Z}/n\mathbb{Z}$.

Heat kernel on $\ensuremath{\mathbb{Z}}$

On \mathbb{Z} , the heat equation is

$$(\Delta_{\mathbb{Z}} + \partial_{t}) \mathsf{K}_{\mathbb{Z}}(t, \mathbf{x}) = 0$$

where the combinatorial Laplacian is given by

$$\Delta_{\mathbb{Z}} f(x) = 2f(x) - (f(x-1) + f(x+1))$$

We solve the heat equation by Fourier transform. Let $g(t, \omega) = \sum_{x \in \mathbb{Z}} K_{\mathbb{Z}}(t, x) e^{ix\omega}$. We have

$$\Delta_{\mathbb{Z}}g(t,\omega) = \sum_{x\in\mathbb{Z}} (2\mathsf{K}_{\mathbb{Z}}(t,x) - (\mathsf{K}_{\mathbb{Z}}(t,x-1) + \mathsf{K}_{\mathbb{Z}}(t,x+1)))e^{ix\omega} = (2-2\cos\omega)g(t,\omega).$$

The Fourier transform of the heat equation is thus

$$(2-2\cos\omega)g(t,\omega) + \partial_t g(t,\omega) = 0, \quad g(0,\omega) = 1,$$

which is solved by $g(t, \omega) = e^{-(2-2\cos\omega)t}$. As a consequence, the heat kernel on \mathbb{Z} is given by

$$\mathsf{K}_{\mathbb{Z}}(\mathsf{t},\mathsf{x}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathsf{g}(\mathsf{t},\omega) e^{-\mathsf{i}\mathsf{x}\omega} \, \mathsf{d}\omega = \frac{1}{2\pi} e^{-2\mathsf{t}} \int_{-\pi}^{\pi} e^{2\mathsf{t}\cos\omega} e^{-\mathsf{i}\mathsf{x}\omega} \, \mathsf{d}\omega = e^{-2\mathsf{t}} \mathsf{I}_{\mathsf{x}}(2\mathsf{t})$$

where I_x is the modified I-Bessel function of order x given by

$$I_{x}(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{t \cos \omega} \cos(\omega x) d\omega.$$

Heat kernel on $\mathbb{Z}/n\mathbb{Z}$

The heat kernel on the n-cycle is n-periodic in x, that is

$$\mathsf{K}_{\mathbb{Z}/n\mathbb{Z}}(\mathsf{t},\mathsf{x}+\mathsf{n})=\mathsf{K}_{\mathbb{Z}/n\mathbb{Z}}(\mathsf{t},\mathsf{x}).$$

Thus it is given by

$$\mathsf{K}_{\mathbb{Z}/n\mathbb{Z}}(t,x) = \sum_{k\in\mathbb{Z}}\mathsf{K}_{\mathbb{Z}/n\mathbb{Z}}(t,x+kn) = e^{-2t}\sum_{k\in\mathbb{Z}}\mathsf{I}_{x+kn}(2t).$$

On the other hand, since the n-cycle is the Cayley graph of the group $\mathbb{Z}/n\mathbb{Z}$ with generators -1 and 1, the eigenvectors of the Laplacian are given by the characters

$$\chi_k(x) = e^{2\pi i k x/n}, \quad k = 0, 1, \dots, n-1,$$

with corresponding eigenvalues $\lambda_k = 2 - 2\cos(2\pi k/n)$, $k = 0, 1, \dots, n-1$. Thus the heat kernel can be expressed as

$$K_{\mathbb{Z}/n\mathbb{Z}}(t,x) = \frac{1}{n} \sum_{k=0}^{n-1} e^{-(2-2\cos(2\pi k/n))t} e^{2\pi i k x/n}.$$

As a consequence of the two derivations of the heat kernel on $\mathbb{Z}/n\mathbb{Z}$, we deduce

$$\frac{1}{n}\sum_{k=0}^{n-1}e^{-(2-2\cos(2\pi k/n))t}e^{2\pi i kx/n}=e^{-2t}\sum_{k\in\mathbb{Z}}I_{x+kn}(2t).$$

By letting x = 0 gives the beautiful theta inversion formula on $\mathbb{Z}/n\mathbb{Z}$,

$$\frac{1}{n}\sum_{k=0}^{n-1}e^{-(2-2\cos(2\pi k/n))t} = e^{-2t}\sum_{k\in\mathbb{Z}}I_{kn}(2t).$$

Theta inversion formula on $\mathbb{Z}/n_1\mathbb{Z}\times \cdots \times \mathbb{Z}/n_d\mathbb{Z}$

Let $\Lambda = \text{diag}(n_1, \ldots, n_d)$ be a diagonal matrix with positive integer coefficients. Let $k = (k_1, \ldots, k_d)$, $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d / \Lambda \mathbb{Z}^d$ and $k_{\Lambda} = \Lambda^{-1}k$. The eigenvectors of the Laplacian on the discrete torus $\mathbb{Z}^d / \Lambda \mathbb{Z}^d$ are given by

$$g_{k_{\Lambda}}(x) = e^{2\pi i \langle k_{\Lambda}, x \rangle}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product. Denote by e_i , i = 1, ..., d, the canonical basis of \mathbb{Z}^d . Since each vertex $x \in \mathbb{Z}^d / \Lambda \mathbb{Z}^d$ is connected to his nearest neighbours, that is x is adjacent to $x - e_i$ and to $x + e_i$, for all i = 1, ..., d, the eigenvalues on $\mathbb{Z}^d / \Lambda \mathbb{Z}^d$ are obtained by applying the Laplacian on the eigenvectors $g_{k_\Lambda}(x)$:

$$\lambda_k = 2d - 2\sum_{i=1}^d \cos(2\pi k_i/n_i) \text{ where } k \in \mathbb{Z}^d/\Lambda\mathbb{Z}^d.$$

On the other hand, the heat kernel on the cartesian product of graphs is the product of the heat kernels on each graph. Thus, the theta inversion formula on $\mathbb{Z}^d/\Lambda\mathbb{Z}^d$ is

$$\theta_{\mathbb{Z}^d/\Lambda\mathbb{Z}^d}(t) = \sum_{k \in \mathbb{Z}^d/\Lambda\mathbb{Z}^d} e^{-(2d-2\sum_{i=1}^d \cos(2\pi k_i/n_i))t} = e^{-2dt} \sum_{k \in \mathbb{Z}^d} \prod_{i=1}^d n_i I_{k_i n_i}(2t).$$

More generally, on a d-dimensional discrete torus $\mathbb{Z}^d/\Lambda\mathbb{Z}^d$ where Λ is a $d \times d$ invertible integer matrix non-necessarily diagonal, the theta inversion is stated in [10, Proposition 5] as

$$\theta_{\mathbb{Z}^d/\Lambda\mathbb{Z}^d}(t) = \sum_{\nu \in \Lambda^*\mathbb{Z}^d/\mathbb{Z}^d} e^{-(2d-2\sum_{i=1}^d \cos(2\pi\nu_i))t} = |\det \Lambda| e^{-2dt} \sum_{y \in \Lambda\mathbb{Z}^d} \prod_{i=1}^d I_{y_i}(2t)$$
(5)

where $\Lambda^*\mathbb{Z}^d$ is the dual lattice of $\Lambda\mathbb{Z}^d$ defined by

$$\Lambda^*\mathbb{Z}^d = \{ y \in \mathbb{R}^d | \langle x, y \rangle \in \mathbb{Z}^d, \ \forall x \in \Lambda\mathbb{Z}^d \}.$$

Theta inversion formula on the circulant graph $C_n^{\gamma_1,\ldots,\gamma_d}$

The circulant graph $C_n^{\gamma_1,...,\gamma_d}$ is the Cayley graph of the group $\mathbb{Z}/n\mathbb{Z}$ with generators $-\gamma_i$ and γ_i , i = 1,...,d, thus the eigenvectors of the Laplacian are the characters

$$\chi_k(x) = e^{2\pi i k x/n}, \quad k = 0, 1, \dots, n-1.$$

Consequently, the eigenvalues are given by

$$\lambda_k = 2d-2\sum_{i=1}^d\cos(2\pi k\gamma_i/n), \quad k=0,1,\ldots,n-1.$$

In the special case of circulant graphs with first generator equal to 1, one can show that C_n^{Γ} , $\Gamma = \{1, \gamma_1, \dots, \gamma_{d-1}\}$ is isomorphic to the discrete torus $\mathbb{Z}^d / \Lambda_{\Gamma} \mathbb{Z}^d$ where Λ_{Γ} is the following matrix

$$\Lambda_{\Gamma} = \begin{pmatrix} \begin{array}{c|c} n & -\gamma_{1} & \cdots & -\gamma_{d-1} \\ \hline 0 & & I_{d-1} \\ \end{array} \end{pmatrix},$$

 I_{d-1} being the identity matrix of order d-1. From the theta inversion formula on the discrete torus (5), we deduce the theta inversion formula for C_n^{Γ}

$$\begin{split} \theta_{C_n^{\Gamma}}(t) &= \sum_{k=0}^{n-1} e^{-(2d-2\cos(2\pi k/n)-2\sum_{i=1}^{d-1}\cos(2\pi\gamma_i k/n))t} \\ &= ne^{-2dt} \sum_{(k_1,\dots,k_d)\in\mathbb{Z}^d} I_{nk_1-\sum_{i=1}^{d-1}\gamma_i k_{i+1}}(2t) \prod_{i=2}^d I_{k_i}(2t) \end{split}$$

1.4 Spectral zeta function

Let λ_k , $k \ge 0$, denote the eigenvalues of the Laplace-Beltrami operator $\Delta_{\mathcal{M}} = -\sum_{i=1}^d \partial^2 / \partial x_i^2$ on a d-dimensional manifold \mathcal{M} . The theta function, also called the partition function in [28], is defined for $\Re(t) > 0$ by

$$\Theta_{\mathcal{M}}(t) = \sum_{k=0}^{\infty} e^{-\lambda_k t}.$$

The spectral zeta function on \mathcal{M} is defined as the Mellin transform of the theta function and converges for $\Re(s) > d/2$

$$\zeta_{\mathcal{M}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty (\Theta_{\mathcal{M}}(t) - 1) t^s \frac{dt}{t} = \sum_{\lambda_k \neq 0} \frac{1}{\lambda_k^s}.$$

It can be shown that the spectral zeta function admits a meromorphic continuation to the whole complex plane and is analytic at 0, see [28]. Consequently, one can define the regularized determinant of Δ_M as

$$\log \det^* \Delta_{\mathcal{M}} = -\zeta_{\mathcal{M}}'(0).$$

1.5 Results

Below, we summarise the results obtained in this thesis.

- In Asymptotics for the number of spanning trees in circulant graphs and degenerating d-dimensional discrete tori, page 22, we extend the method from [9] to two other types of graphs. First we derive asymptotics for the number of spanning trees in a sequence of circulant graphs $C_n^{1,\gamma_1,\ldots,\gamma_{d-1}}$ which improves the results in [3, 12]. The second part of the paper concerns a sequence of d-dimensional discrete tori $\mathbb{Z}^d/\Lambda\mathbb{Z}^d$, with Λ a positive integer matrix, when $|\det \Lambda|$ tends to infinity but with dimensions not all growing at the same rate. In [9], the sequence of discrete tori considered $\mathbb{Z}^d/\Lambda_n\mathbb{Z}^d$, Λ_n being a $d \times d$ integer matrix, is such that $\det \Lambda_n \to \infty$ and $\Lambda_n/(\det \Lambda_n)^{1/d} \to A \in SL_d(\mathbb{R})$ as $n \to \infty$. In the present work, this condition does not hold anymore.
- In A formula for the number of spanning trees in circulant graphs with non-fixed generators and discrete tori, page 50, we consider circulant graphs of the form C^{1,γ1n,...,γd-1n}, where 1 ≤ γ₁ ≤ ... ≤ γ_{d-1} ≤ [β/2] are integers. We derive a closed formula for the number of spanning trees as a product of β values of a function of the eigenvalues on the subgraph C^{γ1...,γd-1}_β. This improves a result from [32]. The technique used here is also applied to d-dimensional discrete tori Z^d/AZ^d, where Λ is the diagonal integer matrix Λ = diag(α₁,...,α_{d-1}, n), which leads to a formula of a product of det(A) terms of a function of the eigenvalues on the subgraph Z^{d-1}/AZ^{d-1}, where A = diag(α₁,...,α_{d-1}). These formulas are interesting when n is larger than the other parameters of the graphs. As a consequence of these results, the tree entropy of these sequences of graphs is derived.
- In Spanning trees in directed circulant graphs and cycle power graphs, page 59, we find closed formulas for the number of spanning trees in directed circulant graphs with generators depending linearly on the number of vertices, that is $\overrightarrow{C}_{\beta n}^{p,\gamma_1 n+p,\ldots,\gamma_{d-1} n+p}$. This improves the results from [31], and partially answers an open question posed in [2]. In the second part of the paper, we derive a formula for the number of spanning trees in the n-th and (n-1)-th power graphs of the βn -cycle as a product of $|\beta/2|$ terms.

- In Asymptotics for the determinant of the combinatorial Laplacian on hypercubic lattices, page 70, we compute asymptotics for the determinant of the Laplacian on a sequence of d-dimensional orthotope square lattices as the number of vertices tends to infinity. This is done by expressing the theta function of these graphs in terms of the theta function of d-dimensional discrete tori and then using the asymptotics results from [9]. We also compute asymptotics for the number of spanning trees in the quartered Aztec diamond.
- In *Low temperature ratchet current*, page 90, we give an explicit expression for the low temperature ratchet current in a multilevel system and its limit as the number of states goes to infinity. The calculation is reduced to evaluating the number of spanning trees in a directed graph, which is given by the Tutte matrix tree theorem. This problem is a continuation of [21] where the authors found numerical values for the ratchet current while in this work we derive a formula which is consistent with numerics.
- In A formula for the energy of circulant graphs with two generators, page 95, we give a formula for the energy in circulant graphs $C_n^{1,\gamma}$. This problem has interesting applications in theoretical chemistry, see for example [4].

References

- [1] Martin Aigner. A course in enumeration, volume 238 of Graduate Texts in Mathematics. Springer, Berlin, 2007.
- [2] Talip Atajan, Naohisa Otsuka, and Xuerong Yong. Counting the number of spanning trees in a class of double fixed-step loop networks. *Appl. Math. Lett.*, 23(3):291–298, 2010.
- [3] Talip Atajan, Xuerong Yong, and Hiroshi Inaba. Further analysis of the number of spanning trees in circulant graphs. *Discrete Math.*, 306(22):2817–2827, 2006.
- [4] R. Balakrishnan and K. Ranganathan. A textbook of graph theory. Universitext. Springer, New York, second edition, 2012.
- [5] Norman Biggs. Algebraic potential theory on graphs. Bull. London Math. Soc., 29(6):641–682, 1997.
- [6] Christian Bogner and Stefan Weinzierl. Feynman graph polynomials. Internat. J. Modern Phys. A, 25(13):2585–2618, 2010.
- [7] Béla Bollobás. Modern graph theory, volume 184 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1998.
- [8] T. J. N. Brown, R. B. Mallion, P. Pollak, and A. Roth. Some methods for counting the spanning trees in labelled molecular graphs, examined in relation to certain fullerenes. *Discrete Appl. Math.*, 67(1-3):51–66, 1996.
- [9] Gautam Chinta, Jay Jorgenson, and Anders Karlsson. Zeta functions, heat kernels, and spectral asymptotics on degenerating families of discrete tori. *Nagoya Math. J.*, 198:121–172, 2010.

- [10] Gautam Chinta, Jay Jorgenson, and Anders Karlsson. Complexity and heights of tori. In Dynamical systems and group actions, volume 567 of Contemp. Math., pages 89–98. Amer. Math. Soc., Providence, RI, 2012.
- [11] Bertrand Duplantier and François David. Exact partition functions and correlation functions of multiple Hamiltonian walks on the Manhattan lattice. J. Statist. Phys., 51(3-4):327–434, 1988.
- [12] Mordecai J. Golin, Xuerong Yong, and Yuanping Zhang. The asymptotic number of spanning trees in circulant graphs. *Discrete Math.*, 310(4):792–803, 2010.
- [13] Frank Harary. Graph theory. Addison-Wesley Publishing Co., Reading, Mass.-Menlo Park, Calif.-London, 1969.
- [14] Jesper Lykke Jacobsen, Jesús Salas, and Alan D. Sokal. Spanning forests and the q-state Potts model in the limit $q \rightarrow 0$. J. Stat. Phys., 119(5-6):1153–1281, 2005.
- [15] Anders Karlsson and Markus Neuhauser. Heat kernels, theta identities, and zeta functions on cyclic groups. In *Topological and asymptotic aspects of group theory*, volume 394 of *Contemp. Math.*, pages 177–189. Amer. Math. Soc., Providence, RI, 2006.
- [16] Richard W. Kenyon, James G. Propp, and David B. Wilson. Trees and matchings. *Electron. J. Combin.*, 7:Research Paper 25, 34 pp. (electronic), 2000.
- [17] Edward C Kirby, Douglas J Klein, Roger B Mallion, Paul Pollak, and Horst Sachs. A theorem for counting spanning trees in general chemical graphs and its particular application to toroidal fullerenes. Croatica chemica acta, 77(1-2):263–278, 2004.
- [18] G. Kirchhoff. Ueber die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Vertheilung galvanischer Ströme geführt wird. Annalen der Physik, 148(12):497–508, 1847.
- [19] Qiaoliang Li and Qiao Li. Reliability analysis of circulant graphs. Networks, 31(2):61-65, 1998.
- [20] Russell Lyons. Asymptotic enumeration of spanning trees. Combin. Probab. Comput., 14(4):491– 522, 2005.
- [21] Christian Maes, Karel Netočný, and Winny O'Kelly de Galway. Low temperature behavior of nonequilibrium multilevel systems. *Journal of Physics A: Mathematical and Theoretical*, 47(3):035002, 2014.
- [22] Jens Marklof and Andreas Strömbergsson. Diameters of random circulant graphs. Combinatorica, 33(4):429–466, 2013.
- [23] Sonja Nikolić, Nenad Trinajstić, Albin Jurić, Zlatko Mihalić, and Goran Krilov. Complexity of some interesting (chemical) graphs. Croat. Chem. Acta, 69:883–897, 1996.
- [24] Peter Sarnak. Determinants of Laplacians; heights and finiteness. In Analysis, et cetera, pages 601–622. Academic Press, Boston, MA, 1990.
- [25] Jean-Pierre Serre. Trees. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation.

- [26] Robert Shrock and F. Y. Wu. Spanning trees on graphs and lattices in d dimensions. J. Phys. A, 33(21):3881–3902, 2000.
- [27] Elmar Teufl and Stephan Wagner. Spanning forests, electrical networks, and a determinant identity. In 21st International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2009), Discrete Math. Theor. Comput. Sci. Proc., AK, pages 847–858. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2009.
- [28] A. Voros. Spectral functions, special functions and the Selberg zeta function. Comm. Math. Phys., 110(3):439–465, 1987.
- [29] Dominic Welsh. The Tutte polynomial. Random Structures Algorithms, 15(3-4):210–228, 1999. Statistical physics methods in discrete probability, combinatorics, and theoretical computer science (Princeton, NJ, 1997).
- [30] F. Y. Wu. The Potts model. Rev. Modern Phys., 54(1):235-268, 1982.
- [31] Xuerong Yong, Yuanping Zhang, and Mordecai J. Golin. The number of spanning trees in a class of double fixed-step loop networks. *Networks*, 52(2):69–77, 2008.
- [32] Yuanping Zhang, Xuerong Yong, and Mordecai J. Golin. Chebyshev polynomials and spanning tree formulas for circulant and related graphs. *Discrete Math.*, 298(1-3):334–364, 2005.

Asymptotics for the number of spanning trees in circulant graphs and degenerating d-dimensional discrete tori*

Justine Louis

1 December 2014

Abstract

In this paper we obtain precise asymptotics for certain families of graphs, namely circulant graphs and degenerating discrete tori. The asymptotics contain interesting constants from number theory among which some can be interpreted as corresponding values for continuous limiting objects. We answer one question formulated in a paper from Atajan, Yong and Inaba in [1] and formulate a conjecture in relation to the paper from Zhang, Yong and Golin [23]. A crucial ingredient in the proof is to use the matrix tree theorem and express the combinatorial Laplacian determinant in terms of Bessel functions. A non-standard Poisson summation formula and limiting properties of theta functions are then used to evaluate the asymptotics.

1 Introduction

The number of spanning trees of a finite graph is an interesting invariant which has many applications in different fields such as network reliability (for example see [9]), statistical physics [18], designing electrical circuits; for more applications see [10]. In 1847 Kirchhoff established the matrix tree theorem [15] which relates the number of spanning trees $\tau(G)$ in a graph G with |V(G)| vertices to the determinant of the combinatorial Laplacian on G by the following relation

$$\tau(G) = \frac{1}{|V(G)|} \det^* \Delta$$

where $det^* \Delta$ is the product of the non-zero eigenvalues of the Laplacian on G.

One type of graphs, so-called circulant graphs, also known as loop networks, has been much studied. Let $1 \leq \gamma_1 \leq \cdots \leq \gamma_d \leq \lfloor n/2 \rfloor$ be positive integers. A circulant graph $C_n^{\gamma_1,\dots,\gamma_d}$ is the 2d-regular graph with n vertices labelled $0, 1, \dots, n-1$ such that each vertex $v \in \mathbb{Z}/n\mathbb{Z}$ is connected to $v \pm \gamma_i$ mod n for all $i \in \{1, \dots, d\}$. Figure 1 illustrates two examples. The problem of computing the number of spanning trees in these graphs can be approached in several ways. One of the first results, proved by Kleitman and Golden [16], see also [4] and [20], states that $\tau(C_n^{1,2}) = nF_n^2$, where F_n are the Fibonacci numbers. Boesch and Prodinger [5] computed the number of spanning trees for different classes of graphs with algebraic techniques using Chebyshev polynomials. Zhang, Yong and Golin [21, 23] used this technique for circulant graphs. The same authors showed in [22] that the number of spanning trees in circulant graphs with fixed generators

^{*}published in Annals of Combinatorics, 19(3):513-543, 2015. The author was supported in part by the Swiss NSF grant 200021 132528/1.

satisfies a recurrence relation, that is $\tau(C_n^{\gamma_1,\dots,\gamma_d}) = na_n^2$ where a_n satisfies a recurrence relation of order 2^{γ_d-1} . This was also proved combinatorially later by Golin and Leung in [11]. They extended their method to circulant graphs with non-fixed generators in [12]. In [1], Atajan, Yong and Inaba improved the order of the recurrence relation for a_n and found the asymptotic behaviour of a_n , *i.e.* $a_n \sim c\varphi^n$, where c and φ are constants which are obtained from the smallest modulus root of the generating function of a_n . They again improved this in [2] by finding an efficient way of solving the recurrence relation of a_n .



Figure 1: The circulant graphs $C_7^{1,2}$ and $C_{13}^{1,3}$

In this work we are interested in studying the asymptotic behaviour of the number of spanning trees in circulant graphs with fixed generators and in d-dimensional discrete tori. This will be done by extending the work of Chinta, Jorgenson and Karlsson in [6] and [7] to these cases. In their papers, the authors developed a technique to compute the asymptotic behaviour of spectral determinants of sequences of discrete tori $\mathbb{Z}^d/\Lambda_n\mathbb{Z}^d$ where Λ_n is a $d \times d$ integer matrix such that det $\Lambda_n \to \infty$ and $\Lambda_n/(\det \Lambda_n)^{1/d} \to A \in SL_d(\mathbb{R})$ as $n \to \infty$. The two families of graphs which will be considered here do not satisfy this condition. An important ingredient is the theta inversion formula (see Proposition 2.1 below) which relates the eigenvalues of the combinatorial Laplacian to the modified I-Bessel functions. The method then consists in studying the asymptotics of integrals involving these Bessel functions. In the first part of this work we apply it to the case of circulant graphs with fixed generators. We will prove the following theorem:

Theorem 1.1. Let C_n^{Γ} be a circulant graph with n vertices and d generators independent of n given by $\Gamma := \{1, \gamma_1, \dots, \gamma_{d-1}\}$, such that $1 \leq \gamma_1 \leq \dots \leq \gamma_{d-1} \leq \lfloor \frac{n}{2} \rfloor$, and let $\det^* \Delta_{C_n^{\Gamma}}$ be the product of the non-zero eigenvalues of the Laplacian on C_n^{Γ} . Then as $n \to \infty$

$$\log \det^* \Delta_{C_n^{\Gamma}} = n \int_0^\infty (e^{-t} - e^{-2dt} I_0^{\Gamma}(2t, \dots, 2t)) \frac{dt}{t} + 2\log n - \log c_{\Gamma} + o(1)$$

where $c_{\Gamma} = 1 + \sum_{i=1}^{d-1} \gamma_i^2$ and

$$I_0^{\Gamma}(2t,\ldots,2t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{2t(\cos w + \sum_{i=1}^{d-1} \cos(\gamma_i w))} dw$$

is the d-dimensional modified I-Bessel function of order zero.

The function I_0^{Γ} appearing in the lead term is a generalization of the 2-dimensional J-Bessel function in [17] and will be defined in section 2.4.

Theorem 1.1 can be compared to Lemma 2 of Golin, Yong and Zhang in [13] where they find the lead term of the asymptotic number of spanning trees. With our method we derived also the second term of the asymptotic. These are consistent with numerics given in [22, 1] by these authors. In particular, this answers one of their open problems stated in the conclusion of [1] that asks whether we can find out the exact value of the asymptotic constants. Indeed we show that

$$c^2 = \frac{1}{c_{\Gamma}}.$$

Let Λ_n be a $d \times d$ invertible diagonal integer matrix. In the second part of this work we extend the method used in [6] to study, in two different cases, the asymptotic behaviour of spectral determinants of a sequence of d-dimensional degenerating discrete tori, that is, the Cayley graphs of the groups $\mathbb{Z}^d/\Lambda_n\mathbb{Z}^d$ with respect to the generators corresponding to the standard basis vectors of \mathbb{Z}^d . In the first case, it is degenerating in the sense that d-p sides of the torus are tending to infinity at the same rate while p sides tend to infinity sublinearly with respect to the d-p sides. More precisely, let α_i , $i = 1, \ldots, p$, and β_i , $i = 1, \ldots, d-p$, be positive non-zero integers and a_n be a sequence of positive integers, the matrix Λ_n considered is then given by $\Lambda_n = \text{diag}(\alpha_1 a_n, \ldots, \alpha_p a_n, \beta_1 n, \ldots, \beta_{d-p} n)$. In the first case, a_n goes to infinity sublinearly with respect to n, that is

$$\frac{a_n}{n} \to 0 \text{ as } n \to \infty.$$

In the second case, the size of the p sides of the torus stay constant $(a_n = 1 \text{ for all } n)$ while the d - p other sides go to infinity at the same rate. The matrix considered, denoted by Λ_n^0 , is therefore given by $\Lambda_n^0 = \text{diag}(\alpha_1, \dots, \alpha_p, \beta_1 n, \dots, \beta_{d-p} n)$. Figure 2 illustrates an example.



Figure 2: The discrete torus $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/|\log n|\mathbb{Z}$ with n = 43.

Let M be an $r \times r$ invertible matrix. We define the spectral or Epstein zeta function associated to the real torus $\mathbb{R}^r/M\mathbb{Z}^r$, for $\Re(s) > r/2$ by

$$\zeta_{\mathbb{R}^r/\mathcal{M}\mathbb{Z}^r}(s) = rac{1}{(2\pi)^{2s}}\sum_{\mathbf{k}\in\mathbb{Z}^r\setminus\{0\}} (\mathbf{k}^\mathsf{T} \mathcal{M}^{-1} \mathbf{k})^{-s}.$$

It has an analytic continuation to the whole complex plane except for a simple pole at s = r/2. The regularized determinant of the Laplacian on the real torus $\mathbb{R}^r/M\mathbb{Z}^r$ is then defined through the spectral zeta function evaluated at s = 0 by

$$\log \det^* \Delta_{\mathbb{R}^r / \mathcal{M}\mathbb{Z}^r} = -\zeta'_{\mathbb{R}^r / \mathcal{M}\mathbb{Z}^r}(0).$$

From now on, the matrices A, B and Λ will denote:

$$A = \operatorname{diag}(\alpha_1, \ldots, \alpha_p), \quad B = \operatorname{diag}(\beta_1, \ldots, \beta_{d-p}), \quad \Lambda = \operatorname{diag}(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_{d-p}).$$

We will prove the two following theorems.

Theorem 1.2. Let det^{*} $\Delta_{\mathbb{Z}^d/\Lambda_n\mathbb{Z}^d}$ be the product of the non-zero eigenvalues of the Laplacian on the discrete torus $\mathbb{Z}^d/\Lambda_n\mathbb{Z}^d$. Then as $n \to \infty$

$$\log \det^* \Delta_{\mathbb{Z}^d / \Lambda_n \mathbb{Z}^d} = n^{d-p} a_n^p \det(\Lambda) c_d - \left(\frac{n}{a_n}\right)^{d-p} \left(\det(\Lambda) (4\pi)^{d/2} \Gamma(d/2) \zeta_{\mathbb{R}^p / A^{-1} \mathbb{Z}^p}(d/2) + o(1)\right)^{d/2} C_{\mathbb{R}^p / A^{-1} \mathbb{Z}^p}(d/2) + o(1)^{d/2} C_{\mathbb{$$

where c_d is the following integral

$$c_d = \int_0^\infty \left(e^{-t} - e^{-2dt}I_0(2t)^d\right) \frac{dt}{t}.$$

We recall the special values for the gamma function for odd d, $\Gamma(d/2) = (d-2)!!\sqrt{\pi}/2^{(d-1)/2}$, and for even d, $\Gamma(d/2) = (d/2 - 1)!$.

Theorem 1.3. Let det^{*} $\Delta_{\mathbb{Z}^d/\Lambda_n^0\mathbb{Z}^d}$ be the product of the non-zero eigenvalues of the Laplacian on the discrete torus $\mathbb{Z}^d/\Lambda_n^0\mathbb{Z}^d$. Then as $n \to \infty$

$$\begin{split} \log \det^* \Delta_{\mathbb{Z}^d/\Lambda_n^0 \mathbb{Z}^d} &= n^{d-p} \det(B) \sum_{j=0}^{\det(A)-1} \int_0^\infty \left(e^{-t} - I_0(2t)^{d-p} e^{-(2(d-p)+\lambda_j)t} \right) \frac{dt}{t} \\ &+ 2\log n - \zeta'_{\mathbb{R}^{d-p}/B\mathbb{Z}^{d-p}}(0) + o(1) \end{split}$$

where λ_i , $j = 0, 1, \dots, \det(A) - 1$, are the eigenvalues of the Laplacian on $\mathbb{Z}^p / A \mathbb{Z}^p$ given by

$$\{\lambda_j\}_j=\{2p-2\sum_{i=1}^p\cos(2\pi j_i/\alpha_i):j_i=0,1,\ldots,\alpha_i-1, \text{ for } i=1,\ldots,p\}.$$

The second term in Theorem 1.2 is new in the asymptotic development which comes from the degeneration. In Theorem 1.3 the terms are similar to the usual ones appearing in the asymptotic behaviour of spectral determinants (see [6] and [7]). As mentioned above the last term is the logarithm of the spectral determinant of the Laplacian on the real torus $\mathbb{R}^{d-p}/B\mathbb{Z}^{d-p}$ where p dimensions are lost because of the degeneration of the sequence of tori. Indeed one can rescale the discrete torus by dividing the number of vertices per dimension by n. Therefore the d-dimensional sequence of discrete tori converges in some sense to the (d-p)-dimensional real torus $\mathbb{R}^{d-p}/B\mathbb{Z}^{d-p}$.

Example. To illustrate Theorem 1.2 we consider the graphs $\mathbb{Z}^3/\Lambda_n\mathbb{Z}^3$ where

$$\Lambda_{n} = \left(\begin{array}{ccc} \lfloor \log n \rfloor & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & n \end{array} \right)$$

Then as $n \to \infty$

$$\log \det^* \Delta_{\mathbb{Z}^3/\Lambda_n \mathbb{Z}^3} = c_3 n^2 \lfloor \log n \rfloor - \left(\frac{n}{\lfloor \log n \rfloor} \right)^2 \left(\frac{1}{\pi} \zeta(3) + o(1) \right).$$

This work is structured as follows. In subsection 2.1 we define the combinatorial Laplacian, and then the spectral zeta function and the theta function in subsection 2.2. In subsection 2.3 we

recall some results on modified I-Bessel functions and in the next subsection we define the ddimensional modified I-Bessel function which will be used in the computation of the asymptotics for the circulant graphs. In the two next subsections we recall some upper bounds on modified I-Bessel functions and briefly describe the method used in [6]. In section 3 we show Theorem 1.1 and compare the results with other papers. In section 4 we treat the case of the degenerating sequence of tori, show Theorems 1.2 and 1.3 and give some examples. In the last section we formulate a conjecture on the number of spanning trees in $C_{5n}^{1,n}$, for $n \ge 2$.

Acknowledgements: The author gratefully thanks Anders Karlsson for valuable discussions, comments and a careful reading of the manuscript. The author also thanks Fabien Friedli for useful discussions. The author is grateful to the referees for useful comments.

2 Preliminary results

2.1 Laplacians

We define a d-dimensional discrete torus to be the quotient $\mathbb{Z}^d/M\mathbb{Z}^d$ where M is an invertible $d \times d$ matrix with coefficients in \mathbb{Z} and a d-dimensional real torus by the quotient $\mathbb{R}^d/C\mathbb{Z}^d$ where $C \in GL_d(\mathbb{R})$. Let C^{*} be the matrix generating the dual lattice of $C\mathbb{Z}^d$ defined by

$$C^*\mathbb{Z}^d = \{y \in \mathbb{R}^d | \langle x, y \rangle \in \mathbb{Z}, \ \forall x \in C\mathbb{Z}^d\}$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product, which satisfies the two following conditions:

- $span(C) = span(C^*)$
- $C^{T}C^{*} = 1.$

The eigenfunctions of the Laplace-Beltrami operator $-\sum_{j=1}^{d} \partial^2 / \partial x_j^2$ on the real torus are given by $\phi(x) = \exp(2\pi i \langle \mu, x \rangle)$, for some $\mu \in \mathbb{R}^d$, with the condition that the opposite sides of the parallelogram generated by \mathbb{CZ}^d are identified. So for all $x \in \mathbb{R}^d$ we have $\phi(x + \mathbb{CZ}^d) = \phi(x)$. Hence $\exp(2\pi i \langle \mu, \mathbb{CZ}^d \rangle) = 1$ and therefore $\langle \mu, \mathbb{CZ}^d \rangle \in \mathbb{Z}$ if and only if $\mu = \mathbb{C}^* \mathfrak{m}$ for $\mathfrak{m} \in \mathbb{Z}^d$. It follows that the eigenvalues are given by

$$\lambda_{\mathfrak{m}} = (2\pi)^{2} \mu^{\mathsf{T}} \mu = (2\pi)^{2} \| \mathbb{C}^{*} \mathfrak{m} \|^{2} \text{ with } \mathfrak{m} \in \mathbb{Z}^{d}.$$
(1)

Let $V(\mathbb{Z}^d/M\mathbb{Z}^d)$ be the set of vertices of the torus $\mathbb{Z}^d/M\mathbb{Z}^d$ and $f: V(\mathbb{Z}^d/M\mathbb{Z}^d) \to \mathbb{C}$. The combinatorial Laplacian on $\mathbb{Z}^d/M\mathbb{Z}^d$ is defined by

$$\Delta_{\mathbb{Z}^d/\mathcal{M}\mathbb{Z}^d} f(x) = \sum_{y \sim x} (f(x) - f(y))$$

where the sum is over the vertices adjacent to x. Recall Proposition 5 of [7]:

Proposition 2.1. Let λ_{ν} , with $\nu \in M^* \mathbb{Z}^d / \mathbb{Z}^d$, be the eigenvalues of $\Delta_{\mathbb{Z}^d / M \mathbb{Z}^d}$. The following formula holds for $t \in \mathbb{R}_{\geq 0}$

$$\left|\det(\mathsf{M})\right|\sum_{y\in\mathsf{M}\mathbb{Z}^d}e^{-2dt}I_{y_1}(2t)\ldots I_{y_d}(2t)=\sum_{\nu\in\mathsf{M}^*\mathbb{Z}^d/\mathbb{Z}^d}e^{-t\lambda_\nu}$$

where I_{u_i} is the modified I-Bessel function of order y_i .

2.2 Spectral zeta function and theta function

In this section we define the spectral zeta function and the theta function and give the relations that will enable us to compute the asymptotics in sections 3 and 4.

Let $\{\lambda_j\}_{j\geq 0}$ be the eigenvalues of the combinatorial Laplacian, respectively the Laplace-Beltrami operator, on a discrete torus, respectively a real torus, T, with $\lambda_0 = 0$. The associated theta function on T is defined by

$$\sum_{j} e^{-\lambda_{j} t}.$$
(2)

It will be denoted by $\theta_T(t)$ when T denotes a discrete torus and by $\Theta_T(t)$ when T denotes a real torus. The relation in Proposition 2.1 is then called the theta inversion formula on $\mathbb{Z}^d/M\mathbb{Z}^d$. The associated spectral zeta function on a real torus T is defined for $\Re(s) > d/2$ by

$$\zeta_{\mathsf{T}}(s) = \sum_{j \neq 0} \frac{1}{\lambda_j^s}.$$

It is related to the theta function through the Mellin transform:

$$\zeta_{\mathsf{T}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty (\Theta_{\mathsf{T}}(t) - 1) t^s \frac{dt}{t}$$

where the -1 in the integral comes from the fact that the zero eigenvalue is kept in the definition of the theta function, and where $\Gamma(s) = \int_0^\infty e^{-t} t^s dt/t$ is the gamma function. Let $M \in GL_d(\mathbb{R})$ be a matrix. By splitting the above integral one can show that the zeta function

Let $M \in GL_d(\mathbb{R})$ be a matrix. By splitting the above integral one can show that the zeta function admits a meromorphic continuation to $s \in \mathbb{C}$ (see section 2.6 in [6]). By differentiating $\zeta_{\mathbb{R}^d/M\mathbb{Z}^d}$ and evaluating at s = 0, one has

$$\begin{aligned} \zeta'_{\mathbb{R}^{d}/M\mathbb{Z}^{d}}(0) &= \int_{0}^{1} (\Theta_{\mathbb{R}^{d}/M\mathbb{Z}^{d}}(t) - |\det(M)| (4\pi t)^{-d/2}) \frac{dt}{t} + \Gamma'(1) \\ &- \frac{2}{d} |\det(M)| (4\pi)^{-d/2} + \int_{1}^{\infty} (\Theta_{\mathbb{R}^{d}/M\mathbb{Z}^{d}}(t) - 1) \frac{dt}{t}. \end{aligned}$$
(3)

In section 3 a limiting torus will be the circle $S^1 = \mathbb{R}/\mathbb{Z}$. In this case it is convenient to split the integral at c_{Γ} . The spectral zeta function is defined for $\Re(s) > 1/2$:

$$\begin{split} \zeta_{S^1}(s) &= \frac{1}{\Gamma(s)} \int_0^\infty (\Theta_{S^1}(t) - 1) t^s \frac{dt}{t} \\ &= \frac{1}{\Gamma(s)} \int_0^{c_{\Gamma}} \left(\Theta_{S^1}(t) - \frac{1}{\sqrt{4\pi t}} \right) t^s \frac{dt}{t} + \frac{1}{\Gamma(s)} \int_0^{c_{\Gamma}} \left(\frac{1}{\sqrt{4\pi t}} - 1 \right) t^s \frac{dt}{t} \\ &\quad + \frac{1}{\Gamma(s)} \int_{c_{\Gamma}}^\infty (\Theta_{S^1}(t) - 1) t^s \frac{dt}{t} \\ &= \frac{1}{\Gamma(s)} \int_0^{c_{\Gamma}} \left(\Theta_{S^1}(t) - \frac{1}{\sqrt{4\pi t}} \right) t^s \frac{dt}{t} + \frac{1}{\Gamma(s)} \left(\frac{c_{\Gamma}^{s-1/2}}{\sqrt{4\pi}(s-1/2)} - \frac{c_{\Gamma}^s}{s} \right) \\ &\quad + \frac{1}{\Gamma(s)} \int_{c_{\Gamma}}^\infty (\Theta_{S^1}(t) - 1) t^s \frac{dt}{t}. \end{split}$$

This defines a meromorphic continuation of ζ_{S^1} to the whole complex plane, hence the limit of $\zeta_{S^1}(s)$ at s = 0 exists. Near s = 0 the gamma function behaves as $1/\Gamma(s) = s + O(s^2)$. Therefore

$$\zeta_{S^{1}}'(0) = \int_{0}^{c_{\Gamma}} \left(\Theta_{S^{1}}(t) - \frac{1}{\sqrt{4\pi t}}\right) \frac{dt}{t} - \frac{1}{\sqrt{\pi c_{\Gamma}}} - \log c_{\Gamma} + \Gamma'(1) + \int_{c_{\Gamma}}^{\infty} (\Theta_{S^{1}}(t) - 1) \frac{dt}{t}.$$
 (4)

As mentioned in the introduction, we notice that for a real torus T the regularized determinant of the Laplacian, $det^* \Delta_T$, is defined by the following identity (for more details see [19]):

$$\log \det^* \Delta_{\mathsf{T}} = -\zeta_{\mathsf{T}}'(0).$$

Let $s \in \mathbb{C}$ with $\Re(s) > d/2$, and $M = diag(m_1, \dots, m_d)$ be a positive diagonal matrix. Using (1), the zeta function can be rewritten as

$$\zeta_{\mathbb{R}^{d}/M\mathbb{Z}^{d}}(s) = \frac{1}{(4\pi^{2})^{s}} \sum_{(k_{1},...,k_{d})\in\mathbb{Z}^{d}\setminus\{0\}} \frac{1}{\left(\sum_{i=1}^{d} k_{i}^{2}/m_{i}^{2}\right)^{s}}.$$
(5)

Let ζ be the Riemann zeta function. In the case of the circle $\mathbb{R}/\beta\mathbb{Z}$ the eigenvalues of the Laplacian are given by $\lambda_j = (2\pi)^2 (j/\beta)^2$ for $j \in \mathbb{Z}$, so the spectral zeta function is related to the Riemann zeta function by

$$\zeta_{\mathbb{R}/\beta\mathbb{Z}}(s) = 2(\beta/2\pi)^{2s}\zeta(2s).$$

Using the special values of the Riemann zeta function $\zeta(0) = -1/2$ and $\zeta'(0) = -(1/2)\log(2\pi)$, the derivative evaluated at zero is given by

$$\zeta'_{\mathbb{R}/\beta\mathbb{Z}}(0) = 4\log(\beta/2\pi)\zeta(0) + 4\zeta'(0) = -2\log\beta.$$
(6)

In particular for the unit circle $S^1 = \mathbb{R}/\mathbb{Z}$, one has

$$\zeta_{S^1}'(0) = 0. (7)$$

2.3 Modified I-Bessel functions

Let I_x be the modified I-Bessel function of the first kind of index x. For positive integer values of x, $I_x(t)$ has the following series representation

$$I_{x}(t) = \sum_{n=0}^{\infty} \frac{(t/2)^{2n+x}}{n!\Gamma(n+1+x)}$$
(8)

and the integral representation

$$I_{x}(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{t \cos \theta} \cos(\theta x) d\theta.$$

For negative values of x we have that $I_x(t) = I_{-x}(t)$ for all t. From Theorem 9 in [14] which is a special case of Proposition 2.1, we have the theta inversion formula on $\mathbb{Z}/m\mathbb{Z}$, that is, for every integer m > 0 and all t,

$$e^{-t} \sum_{k \in \mathbb{Z}} I_{km}(t) = \frac{1}{m} \sum_{j=0}^{m-1} e^{-(1 - \cos(2\pi j/m))t}.$$
(9)

The two following propositions give some results on the asymptotics of the I-Bessel function. The first result has been proved in [6].

Proposition 2.2. Let b(n) be a sequence of positive integers parametrised by $n \in \mathbb{N}$ such that $b(n)/n \to \beta > 0$ as $n \to \infty$. Then for any t > 0 and non-negative integer $k \ge 0$, we have

$$\lim_{n \to \infty} b(n) e^{-2n^2 t} I_{b(n)k}(2n^2 t) = \frac{\beta}{\sqrt{4\pi t}} e^{-(\beta k)^2/(4t)}.$$

Proposition 2.3. Let a_n be a sequence of positive integers tending to infinity sublinearly with respect to n. Then we have that

$$\lim_{n\to\infty} a_n e^{-2n^2t} \sum_{k\in\mathbb{Z}} I_{a_nk}(2n^2t) = 1.$$

Proof. From the theta inversion formula on \mathbb{Z} ,

$$a_n e^{-2n^2t} \sum_{k \in \mathbb{Z}} I_{a_n k}(2n^2t) = 1 + \sum_{j=1}^{a_n - 1} e^{-4\sin^2(\pi j/a_n)n^2t}.$$

If a_n is even,

$$\sum_{j=1}^{a_n-1} e^{-4\sin^2(\pi j/a_n)n^2 t} = e^{-4n^2 t} + 2\sum_{j=1}^{a_n/2-1} e^{-4\sin^2(\pi j/a_n)n^2 t}.$$

If a_n is odd,

$$\sum_{j=1}^{a_n-1} e^{-4\sin^2(\pi j/a_n)n^2t} = 2\sum_{j=1}^{(a_n-1)/2} e^{-4\sin^2(\pi j/a_n)n^2t}.$$

Since $e^{-4n^2t} \to 0$ as $n \to \infty$ both cases behave the same, so we only treat the case where a_n is odd. Using the fact that $\sin x \ge x/2$ for all $x \in [0, \pi/2]$, we have

$$\sum_{j=1}^{(a_n-1)/2} e^{-4\sin^2(\pi j/a_n)n^2 t} \leqslant \sum_{j=1}^{(a_n-1)/2} e^{-\pi^2 j^2 t n^2/a_n^2}$$
$$\leqslant \sum_{j=1}^{\infty} e^{-\pi^2 j t n^2/a_n^2} = \frac{1}{e^{\pi^2 t n^2/a_n^2} - 1} \to 0$$

since $n/a_n \to \infty$ as $n \to \infty$.

Proposition 2.4. For all $x \ge 2$,

$$\int_0^\infty \left(e^{-t}-e^{-x\,t}I_0(2t)\right)\frac{dt}{t}=\mathrm{argcosh}(x/2).$$

Proof. Setting x = 0 in (8), we have

$$I_0(2t)=\sum_{n\geqslant 0}\frac{t^{2n}}{(n!)^2}.$$

It follows

$$\begin{split} \int_0^\infty e^{-xt} (I_0(2t) - 1) \frac{dt}{t} &= \int_0^\infty e^{-xt} \sum_{n \ge 1} \frac{t^{2n}}{(n!)^2} \frac{dt}{t} \\ &= \sum_{n \ge 1} \frac{(2n - 1)!}{(n!)^2} \frac{1}{x^{2n}}. \end{split}$$

Let $y = 1/x^2$ with $y \leq 1/4$, so the above is equivalent to the following sum $\sum_{n \geq 1} y^n (2n-1)!/(n!)^2$. Let $C_n = C_{2n}^n/(n+1) = (2n)!/(n+1)!n!$ be the Catalan numbers, $n \geq 0$, where $C_m^n = m!/n!(m-n)!$ is the binomial coefficient. The generating function of the Catalan numbers is given by

$$\sum_{n \ge 0} C_n y^n = \frac{2}{1 + \sqrt{1 - 4y}}.$$
(10)

The integration over y of the above leads to

$$\sum_{n \geqslant 0} \frac{C_n}{n+1} y^{n+1} = \log(1 + \sqrt{1-4y}) - \sqrt{1-4y} + \text{constant.}$$

Taking the limit $y \to 0$ on both sides gives the constant $= 1 - \log 2.$ Hence,

$$\begin{split} \sum_{n \ge 0} \frac{C_n}{n+1} y^{n+1} &= y + \sum_{n \ge 2} \frac{(2n-2)!}{(n!)^2} y^n \\ &= \log(1 + \sqrt{1-4y}) - \sqrt{1-4y} + 1 - \log 2. \end{split}$$

Let $\alpha_n = C_{n-1}/n = (2n-2)!/(n!)^2$, $n \ge 2$, and $\alpha_1 = 1$, and let $g(y) = \log(1 + \sqrt{1-4y}) - \sqrt{1-4y} + 1 - \log 2$. So the previous equation can be written as

$$\sum_{n\geqslant 1}\alpha_ny^n=g(y).$$

So (10) is equivalent to

$$\sum_{n \ge 1} n \alpha_n y^{n-1} = g'(y).$$

Finally,

$$\begin{split} \sum_{n \geqslant 1} \frac{(2n-1)!}{(n!)^2} y^n &= \sum_{n \geqslant 1} (2n-1) \alpha_n y^n \\ &= 2y \sum_{n \geqslant 1} n \alpha_n y^{n-1} - \sum_{n \geqslant 1} \alpha_n y^n \\ &= 2y g'(y) - g(y) \\ &= \log \left(\frac{2}{1 + \sqrt{1 - 4y}} \right). \end{split}$$

Writing the above in terms of x gives for all $x \ge 2$,

$$\int_0^\infty e^{-xt} (I_0(2t) - 1) \frac{dt}{t} = \log \frac{x}{2} + \log(x - \sqrt{x^2 - 4}).$$

Notice that the above is the generating function of the Catalan numbers, and therefore is equal to
$$\begin{split} \log(\sum_{n\geqslant 0}C_nx^{-2n}).\\ \text{Using the following integral identity for all } x\in\mathbb{C} \text{ with } \Re(x)>0 \end{split}$$

$$\int_0^\infty \left(e^{-t} - e^{-xt}\right) \frac{dt}{t} = \log x$$

one has

$$\int_0^\infty \left(e^{-t}-e^{-xt}I_0(2t)\right)\frac{dt}{t} = \log\left(\frac{x+\sqrt{x^2-4}}{2}\right) = \operatorname{argcosh}(x/2).$$

2.4 d-dimensional modified I-Bessel function

Let m, p_1, \ldots, p_d be positive integers. By analogy with the two-dimensional J-Bessel function defined in [17] we define the d-dimensional modified I-Bessel function of order $m, I_m^{p_1,\ldots,p_d}(u_1,\ldots, u_d)$, by the generating function $e^{\sum_{i=1}^d u_i \cos(p_i t)}$, that is

$$e^{\sum_{i=1}^{d} u_i \cos(p_i t)} = \sum_{m=-\infty}^{\infty} I_m^{p_1,\dots,p_d}(u_1,\dots,u_d) e^{imt}$$

In our computation we will only need $u_1 = \ldots = u_d = 2n^2t$ so we set $u_1 = \ldots = u_d = u$. We have

$$I_{m}^{p_{1},...,p_{d}}(u,...,u) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{(\mu_{1},...,\mu_{d})\in\mathbb{Z}^{d}} \prod_{i=1}^{d} I_{\mu_{i}}(u) e^{i\left(\sum_{i=1}^{d} \mu_{i} p_{i} - m\right)t} dt.$$

The integral is non-zero only for $\sum_{i=1}^{d} \mu_i p_i = m$. Let $(\mu_1, \dots, \mu_d) = (M_1, \dots, M_d)$ be a particular solution, then the set of solutions is given by

$$\mu_1=M_1-\sum_{i=2}^d p_ik_i,\quad \mu_i=M_i+p_1k_i,\quad i=2,\ldots,d,\quad k_2,\ldots,k_d\in\mathbb{Z}.$$

So we have

$$I_{\mathfrak{m}}^{p_{1},\ldots,p_{d}}(\mathfrak{u},\ldots,\mathfrak{u})=\sum_{(k_{2},\ldots,k_{d})\in\mathbb{Z}^{d-1}}I_{M_{1}-\sum_{\mathfrak{i}=2}^{d}p_{\mathfrak{i}}k_{\mathfrak{i}}}(\mathfrak{u})\prod_{\mathfrak{i}=2}^{d}I_{M_{\mathfrak{i}}+p_{1}k_{\mathfrak{i}}}(\mathfrak{u}).$$

Let $\Gamma := \{1, \gamma_1, \dots, \gamma_{d-1}\}$ be a set of integral parameters, and $k_1 \in \mathbb{N}$. We set $M_1 = nk_1$, $M_2 = \dots = M_d = 0$, $p_1 = 1$, $p_i = \gamma_{i-1}$, $i = 2, \dots, d$, then the d-dimensional modified I-Bessel function of order nk_1 and parameters set Γ is given by

$$I_{nk_{1}}^{\Gamma}(\mathfrak{u},\ldots,\mathfrak{u}) := I_{nk_{1}}^{1,\gamma_{1},\ldots,\gamma_{d-1}}(\mathfrak{u},\ldots,\mathfrak{u}) = \sum_{(k_{2},\ldots,k_{d})\in\mathbb{Z}^{d-1}} I_{nk_{1}-\sum_{\mathfrak{i}=1}^{d-1}\gamma_{\mathfrak{i}}k_{\mathfrak{i}+1}}(\mathfrak{u}) \prod_{\mathfrak{i}=2}^{d} I_{k_{\mathfrak{i}}}(\mathfrak{u})$$

which has the integral representation

$$I_{nk_{1}}^{\Gamma}(u,...,u) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{u\left(\cos w + \sum_{i=1}^{d-1} \cos(\gamma_{i}w)\right)} e^{-ink_{1}w} dw.$$
 (11)

Since $I_{-n}(u) = I_n(u)$, notice that

$$I^{\Gamma}_{-\mathbf{n}\mathbf{k}_1}(\mathfrak{u},\ldots,\mathfrak{u}) = I^{\Gamma}_{\mathbf{n}\mathbf{k}_1}(\mathfrak{u},\ldots,\mathfrak{u}).$$
⁽¹²⁾

2.5 Upper bounds for I-Bessel functions

Recall Remark 4.2 in [6]: For all t > 0 the following upper bound holds:

$$0 \leqslant n e^{-n^2 t} I_0(n^2 t) \leqslant C t^{-1/2} \tag{13}$$

for some positive constant C. Recall Lemma 4.6 in [6]: **Lemma 2.5.** Fix $t \ge 0$ and non-negative integers x and n_0 . Then for all $n \ge n_0$, we have the uniform bound

$$0\leqslant \sqrt{n^2t}e^{-n^2t}I_{nx}(n^2t)\leqslant \left(\frac{n_0t}{x+n_0t}\right)^{n_0x/2}=\left(1+\frac{x}{n_0t}\right)^{-n_0x/2}\leqslant 1.$$

2.6 Method

The method developed in [6] consists in studying the asymptotic behaviour of the Gauss transform of the theta function evaluated at zero in order to obtain the product of the Laplacian eigenvalues. This leads to the two following theorems which are adapted from Theorem 3.6 in [6]. They express the logarithm of the determinant of the combinatorial Laplacian on the corresponding discrete torus in terms of integrals of theta and I-Bessel functions. The study of the asymptotics of these integrals will therefore lead to the asymptotic behaviour of the number of spanning trees. In the case of the circulant graph we have:

Theorem 2.6. We have the identity

$$\log \Big(\prod_{\lambda_j \neq 0} \lambda_j \Big) = n \mathcal{I}_d^{\Gamma} + \mathcal{H}_{C_n^{\Gamma}}$$

where

$$\mathcal{I}^{\Gamma}_d = \int_0^\infty \left(e^{-t} - e^{-2dt} I^{\Gamma}_0(2t,\ldots,2t) \right) \frac{dt}{t}$$

and

$$\mathcal{H}_{C_{\mathfrak{n}}^{\Gamma}} = -\int_{0}^{\infty} \left(\theta_{C_{\mathfrak{n}}^{\Gamma}}(t) - \mathfrak{n} e^{-2dt} I_{0}^{\Gamma}(2t,\ldots,2t) - 1 + e^{-t} \right) \frac{dt}{t}.$$

And in the case of the diagonal discrete torus we have:

Theorem 2.7. We have the identity

$$\log\Big(\prod_{\lambda_j\neq 0}\lambda_j\Big)=\det(\Lambda_n)\mathcal{I}_d^{\{\alpha_i\}_{i=1}^p}+\mathcal{H}_{\Lambda_n}$$

where

$$\mathcal{I}_{d}^{\{\alpha_{i}\}_{i=1}^{p}} = \int_{0}^{\infty} \left(e^{-t} - e^{-2dt} I_{0}(2t)^{d-p} \sum_{(k_{1},...,k_{p})\in\mathbb{Z}^{p}} \prod_{i=1}^{p} I_{k_{i}\alpha_{i}\alpha_{n}}(2t) \right) \frac{dt}{t}$$

and

$$\mathcal{H}_{\Lambda_{n}} = -\int_{0}^{\infty} \left(\theta_{\Lambda_{n}}(t) - e^{-2dt} I_{0}(2t)^{d-p} \sum_{(k_{1},...,k_{p}) \in \mathbb{Z}^{p}} \prod_{i=1}^{p} I_{k_{i}\alpha_{i}\alpha_{n}}(2t) - 1 + e^{-t} \right) \frac{dt}{t}.$$
 (14)

3 Asymptotic behaviour of spectral determinant on circulant graphs

3.1 Computation of the asymptotics

Let $1 \leqslant \gamma_1 \leqslant \cdots \leqslant \gamma_{d-1} \leqslant \lfloor n/2 \rfloor$ be positive integers and C_n^{Γ} denote the circulant graph where $\Gamma := \{1, \gamma_1, \dots, \gamma_{d-1}\}$ is the set of generators. In this work we only consider circulant graphs with



first generator equals to 1. In this case one can verify that C_n^{Γ} is isomorphic to the d-dimensional discrete torus $\mathbb{Z}^d / \Lambda_{\Gamma} \mathbb{Z}^d$ where Λ_{Γ} is the following matrix



where I_{d-1} is the identity matrix of order d-1. Indeed, all the points on the lattice $\Lambda_{\Gamma}\mathbb{Z}^{d}$ are identified according to the numbers, where the nearest neighbours are connected to each other. Denote by e_{i} , i = 1, ..., d, the canonical basis of \mathbb{Z}^{d} . Then $0 \in \mathbb{Z}^{d}$ is connected to e_{i} and $-e_{i}$ for i = 1, ..., d. For $v \in \mathbb{Z}/n\mathbb{Z}$, all the points $ve_{1} + \Lambda_{\Gamma}\mathbb{Z}^{d}$ are identified to v. Hence 0 is connected to 1. Since $-e_{1} = (n-1)e_{1} - \Lambda_{\Gamma}e_{1}$, 0 is connected to n-1. Using that $e_{i+1} = \gamma_{i}e_{1} + \Lambda_{\Gamma}e_{i+1}$, i = 1, ..., d-1, 0 is connected to γ_{i} for all i = 1, ..., d-1. Finally, $-e_{i+1} = -\gamma_{i}e_{1} - \Lambda_{\Gamma}e_{i+1}$, i = 1, ..., d-1, so that 0 is connected to $-\gamma_{i}$ mod n, for all i = 1, ..., d-1. Similarly, all $v \in \mathbb{Z}/n\mathbb{Z}$ are connected to $v \pm \gamma_{i}$ mod n for all i = 1, ..., d. Therefore the quotient $\mathbb{Z}^{d}/\Lambda_{\Gamma}\mathbb{Z}^{d}$ with nearest neighbours connected to each other is isomorphic to the circulant graph C_{n}^{Γ} . Figure 3 illustrates the lattice corresponding to the circulant graph $C_{1}^{\Gamma^{2}}$ represented in Figure 1. The fact that the matrix is almost diagonal simplifies the expression of the theta function. Indeed from Proposition 2.1 the theta function on C_{n}^{Γ} is given by

$$\theta_{C_n^{\Gamma}}(n^2 t) = n e^{-2dn^2 t} \sum_{(k_1, \dots, k_d) \in \mathbb{Z}^d} I_{nk_1 - \sum_{i=1}^{d-1} \gamma_i k_{i+1}}(2n^2 t) \prod_{i=2}^d I_{k_i}(2n^2 t).$$

Rewriting it in terms of the d-dimensional modified I-Bessel function defined in section 2.4 we get

$$\theta_{C_n^{\Gamma}}(n^2t) = ne^{-2dn^2t}\sum_{k_i\in\mathbb{Z}}I_{nk_i}^{\Gamma}(2n^2t,\ldots,2n^2t).$$

A circulant graph is the Cayley graph of a finite abelian group, so the eigenvectors of the Laplacian on C_n^{Γ} are the characters

$$\chi_j(x) = e^{2\pi i j x/n}, \quad j = 0, 1, \dots, n-1.$$

By applying the Laplacian on the characters, we obtain the eigenvalues

$$\lambda_j=2d-2\cos(2\pi j/n)-2\sum_{i=1}^{d-1}\cos(2\pi\gamma_i j/n),\quad j=0,1,\ldots,n-1.$$
Therefore, by definition of the theta function (2) it can also be written as

$$\theta_{C_{n}^{r}}(n^{2}t) = \sum_{j=0}^{n-1} e^{-(2d-2\cos(2\pi j/n)-2\sum_{i=1}^{d-1}\cos(2\pi\gamma_{i}j/n))n^{2}t}$$
$$= \sum_{j=0}^{n-1} e^{-4(\sin^{2}(\pi j/n)+\sum_{i=1}^{d-1}\sin^{2}(\pi\gamma_{i}j/n))n^{2}t}.$$
(15)

Proposition 3.1. With the above notation we have for all $t \ge 0$,

$$\lim_{n\to\infty}\theta_{C_n^{\Gamma}}(n^2t)=\Theta_{S^1}(c_{\Gamma}t)$$

where Θ_{S^1} is the theta function on the circle $S^1 = \mathbb{R}/\mathbb{Z}$ given by

$$\Theta_{S^1}(t) = \frac{1}{\sqrt{4\pi t}} \sum_{k=-\infty}^{\infty} e^{-k^2/(4t)}.$$

Proof. From the theta inversion formula on $\mathbb{Z}/m\mathbb{Z}$ (Theorem 10 in [14]) we have for any $z \in \mathbb{C}$, and integers x and m > 0,

$$\sum_{k=-\infty}^{\infty} I_{x+km}(z) = \frac{1}{m} \sum_{j=0}^{m-1} e^{\cos(2\pi j/m)z + 2\pi i j x/m}.$$
 (16)

Using the expression of the theta function in terms of I-Bessel functions, it follows that for all $n \ge 1$ and t > 0,

$$\begin{split} |\theta_{C_n^{\Gamma}}(n^2 t)| &= |ne^{-2dn^2 t} \sum_{(k_2,\dots,k_d) \in \mathbb{Z}^{d-1}} \frac{1}{n} \sum_{j=0}^{n-1} e^{2n^2 t \cos(2\pi j/n) - 2\pi i j \sum_{i=1}^{d-1} \gamma_i k_{i+1}/n} \prod_{i=2}^d I_{k_i}(2n^2 t)| \\ &\leqslant \prod_{i=2}^d \sum_{k_i \in \mathbb{Z}} e^{-2n^2 t} I_{k_i}(2n^2 t) \sum_{j=0}^{n-1} e^{-2n^2 t (1 - \cos(2\pi j/n))} \\ &\leqslant \sum_{j=0}^{n-1} e^{-8\pi^2 c t j^2} \leqslant \sum_{j=0}^{n-1} e^{-c' t j} \leqslant \frac{1}{1 - e^{-c' t}} \end{split}$$

where c'>0. In the second inequality we used the fact that for all $\nu\in[0,\pi],~(1-\cos\nu)/\nu^2\geqslant c,$ with $c=1/2-\pi^2/24>0,$ and $e^{-t}\sum_{x\in\mathbb{Z}}I_x(t)=1.$ It follows that

$$\lim_{n\to\infty}\theta_{C_n^{\Gamma}}(n^2t) = \sum_{k_l\in\mathbb{Z}}\lim_{n\to\infty}ne^{-2dn^2t}I_{nk_l}^{\Gamma}(2n^2t,\ldots,2n^2t). \tag{17}$$

Let $k_1 > 0$. From the integral representation of the d-dimensional I-Bessel function we have

$$ne^{-2dn^{2}t}I_{nk_{1}}^{\Gamma}(2n^{2}t,\ldots,2n^{2}t) = \frac{1}{2\pi k_{1}}\int_{-\pi nk_{1}}^{\pi nk_{1}} e^{iw}e^{-2n^{2}t\left(d-\cos(w/(nk_{1}))-\sum_{i=1}^{d-1}\cos(\gamma_{i}w/(nk_{1}))\right)}dw_{i}dw$$

Since $(1-\cos\nu)/\nu^2 \geqslant c > 0$ for all $\nu \in [0,\pi]$, we have that

$$n^2(d-\cos(w/(nk_1))-\sum_{i=1}^{d-1}\cos(\gamma_iw/(nk_1))) \geqslant c\left(w/k_1\right)^2$$

for all $w \in [0, \pi nk_1]$. Hence for all $n \ge 1$,

$$|\mathbf{n}e^{-2dn^{2}t}\mathbf{I}_{nk_{1}}^{\Gamma}(2n^{2}t,\ldots,2n^{2}t)| \leqslant \frac{1}{2\pi k_{1}} \int_{-\pi nk_{1}}^{\pi nk_{1}} e^{-2tcw^{2}/k_{1}^{2}} dw \leqslant \frac{1}{2\pi k_{1}} \int_{-\infty}^{\infty} e^{-2tcw^{2}/k_{1}^{2}} dw = \sqrt{\frac{2}{\pi ct}} \int_{-\infty}^{\infty} e^{-2tcw^{2}/k_{1}^{2}} dw = \sqrt{\frac{2}{\pi ct}} \int_{-\infty}^{\infty} e^{-2tcw^{2}/k_{1}^{2}} dw$$

We also have that

$$\lim_{n\to\infty} n^2(d-\cos(w/(nk_1)) - \sum_{i=1}^{d-1} \cos(\gamma_i w/(nk_1))) = \frac{c_{\Gamma}}{2}(w/k_1)^2$$

So by the Lebesgue dominated convergence Theorem, we have for all $k_{1} > 0 \label{eq:kappa}$

$$\lim_{n \to \infty} n e^{-2dn^2 t} I_{nk_1}^{\Gamma}(2n^2 t, \dots, 2n^2 t) = \frac{1}{2\pi k_1} \int_{-\infty}^{\infty} e^{-c_{\Gamma} t w^2/k_1^2} e^{iw} dw$$
$$= \frac{1}{\sqrt{4\pi c_{\Gamma} t}} e^{-k_1^2/(4c_{\Gamma} t)}.$$
(18)

Let $k_1 = 0$. From the integral representation of the d-dimensional I-Bessel function we have

$$ne^{-2dn^{2}t}I_{0}^{\Gamma}(2n^{2}t,\ldots,2n^{2}t) = \frac{1}{2\pi}\int_{-\pi n}^{\pi n} e^{-2n^{2}t\left(d-\cos(w/n)-\sum_{i=1}^{d-1}\cos(\gamma_{i}w/n)\right)}dw.$$

With the same argument as in the case $k_1 > 0$ we can apply the Lebesgue dominated convergence Theorem and we get

$$\lim_{n \to \infty} n e^{-2dn^2 t} I_0^{\Gamma}(2n^2 t, \dots, 2n^2 t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-c_{\Gamma} t w^2} dw$$
$$= \frac{1}{\sqrt{4\pi c_{\Gamma} t}}.$$
(19)

Putting (18) and (19) in (17) and using (12), the result follows.

Proposition 3.2. With the above notation we have

$$\begin{split} &\lim_{n\to\infty}\int_0^1(\theta_{C_n^{\Gamma}}(n^2t)-ne^{-2dn^2t}I_0^{\Gamma}(2n^2t,\ldots,2n^2t))\frac{dt}{t}\\ &=\int_0^1\Big(\Theta_{S^1}(c_{\Gamma}t)-\frac{1}{\sqrt{4\pi c_{\Gamma}t}}\Big)\frac{dt}{t}. \end{split}$$

Proof. For a given positive integer $k_1 \geqslant 1,$ let D_{k_1} denote the following set

$$\mathsf{D}_{k_1} = \{(k_2, \dots, k_d) \in \mathbb{Z}^{d-1} : |nk_1 - \sum_{i=1}^{d-1} \gamma_i k_{i+1}| \leqslant nk_1/2\}$$

and let $D_{k_1}^c=\mathbb{Z}^{d-1}\setminus D_{k_1}$ denote the complement of $D_{k_1}.$ From the theta inversion formula we have

$$\begin{split} \theta_{C_{n}^{\Gamma}}(n^{2}t) &- ne^{-2dn^{2}t}I_{0}^{\Gamma}(2n^{2}t,\ldots,2n^{2}t) \\ &= 2ne^{-2dn^{2}t}\sum_{k_{1}=1}^{\infty}\sum_{(k_{2},\ldots,k_{d})\in\mathbb{Z}^{d-1}}I_{nk_{1}-\sum_{i=1}^{d-1}\gamma_{i}k_{i+i}}(2n^{2}t)\prod_{i=2}^{d}I_{k_{i}}(2n^{2}t) \\ &= 2ne^{-2dn^{2}t}\sum_{k_{1}=1}^{\infty}\left[\sum_{(k_{2},\ldots,k_{d})\in\mathbb{D}_{k_{1}}^{c}}+\sum_{(k_{2},\ldots,k_{d})\in\mathbb{D}_{k_{1}}}\right]I_{nk_{1}-\sum_{i=1}^{d-1}\gamma_{i}k_{i+i}}(2n^{2}t)\prod_{i=2}^{d}I_{k_{i}}(2n^{2}t) \end{split}$$

Since the modified Bessel function I_k is decreasing in the index k [8], for $(k_2,\ldots,k_d)\in D_{k_l}^c,$

$$I_{nk_{l}-\sum_{\mathfrak{i}=1}^{d-1}\gamma_{\mathfrak{i}}k_{\mathfrak{i}+l}}(2n^{2}t)=I_{|nk_{l}-\sum_{\mathfrak{i}=1}^{d-1}\gamma_{\mathfrak{i}}k_{\mathfrak{i}+l}|}(2n^{2}t)\leqslant I_{nk_{l}/2}(2n^{2}t).$$

Using that $e^{-2n^2t}\sum_{k\in\mathbb{Z}}I_k(2n^2t)=1$, it follows that

$$2ne^{-2dn^{2}t}\sum_{k_{1}=1}^{\infty}\sum_{(k_{2},...,k_{d})\in D_{k_{1}}^{c}}I_{nk_{1}-\sum_{i=1}^{d-1}\gamma_{i}k_{i+1}}(2n^{2}t)\prod_{i=2}^{d}I_{k_{i}}(2n^{2}t) \leqslant 2ne^{-2dn^{2}t}\sum_{k_{1}=1}^{\infty}I_{nk_{1}/2}(2n^{2}t).$$

Using Lemma 2.5, for all $n \ge n_0$ the above is less or equal than

$$\sqrt{\frac{2}{t}} \sum_{k=1}^{\infty} \left(1 + \frac{k}{4n_0 t} \right)^{-n_0 k/4} \leqslant \sqrt{\frac{2}{t}} \frac{1}{(1 + 1/(4n_0 t))^{n_0/4} - 1} \leqslant \sqrt{2} (4n_0)^{n_0/4} t^{n_0/4 - 1/2}.$$
(20)

For $(k_2, \ldots, k_d) \in D_{k_1}$, we have

$$\|nk_1-\sum_{i=1}^{d-1}\gamma_i|k_{i+1}\|\leqslant \frac{nk_1}{2}.$$

Since $1 \leqslant \gamma_1 \leqslant \cdots \leqslant \gamma_{d-1}$, it follows that

$$\frac{nk_1}{2} \leqslant \sum_{i=1}^{d-1} \gamma_i |k_{i+1}| \leqslant \gamma_{d-1} (d-1) \max_{i \in \{2,\ldots,d\}} \lvert k_i \rvert$$

so that

$$\max_{i \in \{2,...,d\}} |k_i| \ge \frac{nk_1}{2(d-1)\gamma_{d-1}}.$$
(21)

Let S_{d-1} denote the set of permutations of $\{2, \ldots, d\}$. By ordering the k_i 's in the second summation we obtain

$$\begin{split} |2ne^{-2dn^{2}t} \sum_{k_{1}=1}^{\infty} \sum_{(k_{2},...,k_{d})\in D_{k_{1}}} I_{nk_{1}-\sum_{i=1}^{d-1}\gamma_{i}k_{i+1}} (2n^{2}t) \prod_{i=2}^{d} I_{k_{i}} (2n^{2}t)| \\ &= |2ne^{-2dn^{2}t} \sum_{k_{1}=1}^{\infty} \sum_{\sigma\in S_{d-1}} \left[\sum_{\substack{(k_{2},...,k_{d})\in D_{k_{1}}\\|k_{\sigma(2)}|<\cdots<|k_{\sigma(d)}|}} + \sum_{\substack{(k_{2},...,k_{d})\in D_{k_{1}}\\|k_{\sigma(2)}|=\cdots=|k_{\sigma(d)}|}} \right] I_{nk_{1}-\sum_{i=2}^{d}\gamma_{\sigma(i)-1}k_{\sigma(i)}} (2n^{2}t) \prod_{i=2}^{d} I_{k_{\sigma}(i)} (2n^{2}t)| \end{split}$$

Using inequality (21) and the fact that I_k is decreasing in $\boldsymbol{k},$ we have

$$I_{k_{\sigma(d)}}(2n^{2}t) = I_{|k_{\sigma(d)}|}(2n^{2}t) \leqslant I_{nk_{l}/(2(d-1)\gamma_{d-1})}(2n^{2}t)$$

hence the above is less or equal than

$$2ne^{-2n^{2}t}\sum_{k_{1}=1}^{\infty}I_{nk_{1}/(2(d-1)\gamma_{d-1})}(2n^{2}t)$$

$$\times\sum_{\sigma\in S_{d-1}}\sum_{(k_{\sigma(2)},...,k_{\sigma(d-1)})\in\mathbb{Z}^{d-2}}\left(|e^{-2n^{2}t}\sum_{k_{\sigma(d)}\in\mathbb{Z}}I_{nk_{1}-\sum_{i=2}^{d}\gamma_{\sigma(i)-i}k_{\sigma(i)}}(2n^{2}t)|\right)$$

$$\times\prod_{i=2}^{d-1}e^{-2n^{2}t}I_{k_{\sigma(i)}}(2n^{2}t)\right).$$
(22)

Using the theta inversion formula (16) with $m = \gamma_{\sigma(d)-1}$ and $x = nk_1 - \sum_{i=2}^{d-1} \gamma_{\sigma(i)-1} k_{\sigma(i)}$, we have that

$$\begin{split} |e^{-2n^2t} \sum_{\substack{k_{\sigma(d)} \in \mathbb{Z} \\ i=2}} I_{nk_1 - \sum_{i=2}^{d} \gamma_{\sigma(i)-i}k_{\sigma(i)}} (2n^2t)| \\ &= \frac{1}{\gamma_{\sigma(d)-1}} |\sum_{k=0}^{\gamma_{\sigma(d)-1}-1} e^{-2n^2t(1 - \cos(2\pi k/\gamma_{\sigma(d)-1})) + 2\pi i k(nk_1 - \sum_{i=2}^{d-1} \gamma_{\sigma(i)-1}k_{\sigma(i)})/\gamma_{\sigma(d)-1}}| \leqslant 1. \end{split}$$

Putting the above in (22) and using that $e^{-2n^2t}\sum_{k\in\mathbb{Z}}I_k(2n^2t)=1$, the second summation is less or equal than

$$2ne^{-2n^{2}t} \sum_{k_{1}=1}^{\infty} I_{nk_{1}/(2(d-1)\gamma_{d-1})}(2n^{2}t)(d-1)!$$

$$\leq \sqrt{2}(d-1)!(4(d-1)\gamma_{d-1}n_{0})^{n_{0}/(4(d-1)\gamma_{d-1})}t^{n_{0}/(4(d-1)\gamma_{d-1})-1/2}$$
(23)

for all $n \ge n_0$, where we used Lemma 2.5 in the second inequality. Inequalities (20) and (23) together lead to

$$\begin{split} |\theta_{C_n^{\Gamma}}(n^2t) - ne^{-2dn^2t} I_0^{\Gamma}(2n^2t, \dots, 2n^2t)| \\ &\leqslant \sqrt{2}(4n_0)^{n_0/4} t^{n_0/4-1/2} + \sqrt{2}(d-1)! (4(d-1)\gamma_{d-1}n_0)^{n_0/(4(d-1)\gamma_{d-1})} t^{n_0/(4(d-1)\gamma_{d-1})-1/2} \end{split}$$

which is integrable on (0,1) with respect to the measure dt/t for all $n \ge n_0 = 2(d-1)\gamma_{d-1} + 1$. The proposition then follows from the Lebesgue dominated convergence Theorem and from the pointwise convergence.

Recall the following lemma from [6]:

Lemma 3.3. For $n \in \mathbb{R}$, we have the asymptotic formula

$$\int_0^1 (e^{-n^2t}-1)\frac{dt}{t}=\Gamma'(1)-2\log n+o(1)\quad \text{as $n\to\infty$}.$$

Proposition 3.4. With the above notation we have that

$$\lim_{n\to\infty}\int_1^\infty \left(\theta_{C_n^{\Gamma}}(n^2t)-1\right)\frac{dt}{t}=\int_1^\infty \left(\Theta_{S^1}(c_{\Gamma}t)-1\right)\frac{dt}{t}.$$

Proof. From Proposition 3.1 we have for all t > 0, the pointwise limit

$$\lim_{n\to\infty}\theta_{C_n^{\Gamma}}(n^2t)-1=\Theta_{S^1}(c_{\Gamma}t)-1$$

From (15) we have

$$\theta_{C_n^{\Gamma}}(n^2 t) = 1 + \sum_{j=1}^{n-1} e^{-4\sin^2(\pi j/n)n^2 t} \prod_{i=1}^{d-1} e^{-4\sin^2(\pi \gamma_i j/n)n^2 t}$$

Since the product on i is smaller than 1, we have

$$\theta_{C_n^{\Gamma}}(n^2 t) \leqslant 1 + \sum_{j=1}^{n-1} e^{-4\sin^2(\pi j/n)n^2 t} = 1 + 2\sum_{j=1}^{\lfloor n/2 \rfloor} e^{-4\sin^2(\pi j/n)n^2 t}.$$

Using the elementary bound

$$\sin(\pi x) \geqslant \pi x \left(1 - \pi^2 x^2 / 6\right) \geqslant c \pi x$$

for all $x \in [0, 1/2]$, where $c = 1 - \pi^2/24 > 0$, we have

$$\theta_{C_n^{\Gamma}}(n^2 t) - 1 \leqslant 2 \sum_{j=1}^{\lfloor n/2 \rfloor} e^{-4c^2 \pi^2 j^2 t} \leqslant 2 \sum_{j=1}^{\infty} e^{-djt} = \frac{2}{e^{dt} - 1} \leqslant \frac{2}{1 - e^{-d}} e^{-dt},$$

for all $t \ge 1$, where $d = 4c^2\pi^2 > 0$. Since it is integrable on $(1, \infty)$ with respect to the measure dt/t, the proposition follows from the Lebesgue dominated convergence Theorem.

Proposition 3.5. With the above notation we have

$$\lim_{n\to\infty}\int_1^\infty ne^{-2dn^2t}I_0^{\Gamma}(2n^2t,\ldots,2n^2t)\frac{dt}{t}=\frac{1}{\sqrt{\pi c_{\Gamma}}}.$$

Proof. By definition, we have

$$I_0^{\Gamma}(2n^2t,\ldots,2n^2t) = \sum_{(k_2,\ldots,k_d)\in\mathbb{Z}^{d-1}} I_{-\sum_{i=1}^{d-1}\gamma_i k_{i+1}}(2n^2t) \prod_{i=2}^d I_{k_i}(2n^2t).$$

From Lemma 2.5 we have the uniform upper bound

$$\mathfrak{n} e^{-2\mathfrak{n}^2 \mathfrak{t}} \mathbb{I}_{-\sum_{i=1}^{d-1} \gamma_i k_{i+1}}(2\mathfrak{n}^2 \mathfrak{t}) \leqslant \frac{1}{\sqrt{2\mathfrak{t}}}.$$

Hence

$$ne^{-2dn^2t}I_0^{\Gamma}(2n^2t,\ldots,2n^2t)\leqslant \frac{1}{\sqrt{2t}}(e^{-2n^2t}\sum_{k\in\mathbb{Z}}I_k(2n^2t))^{d-1}=\frac{1}{\sqrt{2t}}$$

which is integrable on $(1,\infty)$ with respect to the measure dt/t. By the Lebesgue dominated convergence Theorem it follows that

$$\lim_{n\to\infty}\int_1^{\infty}ne^{-2dn^2t}I_0^{\Gamma}(2n^2t,\ldots,2n^2t)\frac{dt}{t}=\int_1^{\infty}\frac{1}{\sqrt{4\pi c_{\Gamma}t}}\frac{dt}{t}=\frac{1}{\sqrt{\pi c_{\Gamma}}}.$$

Since $\int_1^\infty e^{-n^2t} dt/t$ converges to zero as $n \to \infty$, putting Lemma 3.3 and Propositions 3.2, 3.4 and 3.5 together in Theorem 2.6 leads to the asymptotic of the $\mathcal{H}_{C_n^{\Gamma}}$ term as $n \to \infty$:

$$\mathcal{H}_{C_{n}^{\Gamma}} = 2\log n - \int_{0}^{c_{\Gamma}} (\Theta_{S^{1}}(t) - \frac{1}{\sqrt{4\pi t}}) \frac{dt}{t} - \Gamma'(1) - \int_{c_{\Gamma}}^{\infty} (\Theta_{S^{1}}(t) - 1) \frac{dt}{t} + \frac{1}{\sqrt{\pi c_{\Gamma}}} + o(1).$$

Using equation (4) we can then rewrite:

$$\mathcal{H}_{C_n^{\,\Gamma}} = 2\log n - \zeta_{S^1}'(0) - \log c_{\Gamma} + o(1) \quad \text{as } n \to \infty.$$

Since $\zeta'_{S^1}(0) = 0$ (7) we get

$$\mathcal{H}_{C_n^{\Gamma}} = 2\log n - \log c_{\Gamma} + o(1)$$
 as $n \to \infty$

and so

$$\log \det^* \Delta_{C_n^{\Gamma}} = n \int_0^\infty (e^{-t} - e^{-2dt} I_0^{\Gamma}(2t, \dots, 2t)) \frac{dt}{t} + 2\log n - \log c_{\Gamma} + o(1) \quad \text{as } n \to \infty$$

which proves Theorem 1.1.

3.2 Asymptotic number of spanning trees and comparison of the results

Notice that in the trivial case d = 1, the cycle has n spanning trees so $\log \det^* \Delta_{C_n} = \log n^2$. On the other hand, from Proposition 2.4

$$\int_0^\infty (e^{-t} - e^{-2t} I_0(2t)) \frac{dt}{t} = 0$$

and so the right hand side of the asymptotic development is $2\log n$. Therefore the theorem is verified in this particular case.

From Kirchhoff's matrix tree theorem and Theorem 1.1, the number of spanning trees in the circulant graph C_n^{Γ} with $\Gamma = \{1, \gamma_1, \dots, \gamma_{d-1}\}$ is asymptotically given by

$$\tau(C_n^{\Gamma}) = \frac{n}{c_{\Gamma}} e^{n \mathcal{I}_d^{\Gamma} + o(1)} \text{ as } n \to \infty.$$
(24)

The lead term can be rewritten as

$$\mathcal{I}_{d}^{\Gamma} = \int_{0}^{\infty} (e^{-t} - e^{-2dt} I_{0}^{\Gamma}(2t, \dots, 2t)) \frac{dt}{t} = \log(2d) + \int_{0}^{\infty} e^{-2dt} (1 - I_{0}^{\Gamma}(2t, \dots, 2t)) \frac{dt}{t}.$$

From the integral representation of I_0^{Γ} (11) and writing the exponential as a series one has

$$\begin{split} \int_{0}^{\infty} e^{-2dt} (1 - I_{0}^{\Gamma}(2t, \dots, 2t)) \frac{dt}{t} &= -\frac{1}{2\pi} \int_{0}^{\infty} e^{-2dt} \sum_{n=1}^{\infty} \frac{2^{n}}{n!} \int_{-\pi}^{\pi} (\cos w + \sum_{i=1}^{d-1} \cos(\gamma_{i}w))^{n} dw t^{n} \frac{dt}{t} \\ &= -\frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{d^{n}} \frac{1}{n} \int_{-\pi}^{\pi} (\cos w + \sum_{i=1}^{d-1} \cos(\gamma_{i}w))^{n} dw \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left(1 - \frac{\cos w + \sum_{i=1}^{d-1} \cos(\gamma_{i}w)}{d} \right) dw \\ &= \int_{0}^{1} \log(\sin^{2}(\pi w) + \sum_{i=1}^{d-1} \sin^{2}(\pi \gamma_{i}w)) dw + \log \frac{2}{d}. \end{split}$$

Hence the lead term is given by

$$\mathcal{I}_d^{\Gamma} = \log 4 + \int_0^1 \log(\sin^2(\pi w) + \sum_{i=1}^{d-1} \sin^2(\pi \gamma_i w)) dw$$

which corresponds to Lemma 2 of [13].

As mentioned in the introduction, the authors show in [22] that the number of spanning trees in a circulant graph is given by

$$\tau(C_n^{\gamma_1,\ldots,\gamma_d}) = n \mathfrak{a}_n^2$$

where a_n satisfies a recurrence relation which behaves asymptotically as $c\phi^n$ for some constants c and ϕ which can be determined numerically. Comparing with (24) it follows that

$$c^2 = \frac{1}{c_{\Gamma}}$$

which is numerically verified with the values in Table 1 in [22]. This answers to one of the questions asked in the conclusion of [1].

4 Asymptotic behaviour of spectral determinant on degenerating tori

We consider the sequence of d-dimensional discrete tori described in the introduction. For simplicity, we denote by θ_{Λ_n} the theta function associated to $\mathbb{Z}^d/\Lambda_n\mathbb{Z}^d$. It is given by

$$\theta_{\Lambda_n}(t) = \sum_{\lambda_j} e^{-\lambda_j \, t}$$

where

$$\begin{split} \{\lambda_j\}_{j=0,1,\dots,\det(\Lambda_n)-1} = & \{2d-2\sum_{i=1}^p\cos(2\pi m_i/(\alpha_i\mathfrak{a}_n)) - 2\sum_{i=1}^{d-p}\cos(2\pi m_i'/(\beta_in)):\\ & m_i = 0,1,\dots,\alpha_i\mathfrak{a}_n - 1, i = 1,\dots,p \text{ and } m_i' = 0,1,\dots,\beta_in - 1, i = 1,\dots,d-p \end{split}$$

are the eigenvalues of the combinatorial Laplacian on $\mathbb{Z}^d/\Lambda_n\mathbb{Z}^d$. From the theta inversion formula on $\mathbb{Z}^d/\Lambda_n\mathbb{Z}^d$ (Proposition 2.1) we have for all $t \ge 0$

$$\theta_{\Lambda_{n}}(t) = \left(\prod_{i=1}^{p} \alpha_{i} a_{n} e^{-2t} \sum_{k \in \mathbb{Z}} I_{k\alpha_{i}a_{n}}(2t)\right) \left(\prod_{i=1}^{d-p} \beta_{i} n e^{-2t} \sum_{k \in \mathbb{Z}} I_{k\beta_{i}n}(2t)\right).$$
(25)

4.1 Computation of the lead term when a_n grows sublinearly with respect to n

Let c_d be the integral below. A numerical estimation of it is discussed in section 7.2 of [6].

$$c_d = \int_0^\infty \left(e^{-t} - e^{-2dt}I_0(2t)^d\right) \frac{dt}{t}.$$

The lead term of $\log \det^* \Delta_{\mathbb{Z}^d / \Lambda_n \mathbb{Z}^d}$ in Theorem 2.7 is given by

$$\begin{split} \det(\Lambda_n)\mathcal{I}_d^{\{\alpha_i\}_{i=1}^p} &= \det(\Lambda_n) \int_0^\infty \left(e^{-t} - e^{-2dt} I_0(2t)^{d-p} \sum_{(k_1,\dots,k_p) \in \mathbb{Z}^p} \prod_{i=1}^p I_{k_i \alpha_i \alpha_n}(2t) \right) \frac{dt}{t} \\ &= \det(\Lambda_n) c_d - \det(\Lambda_n) \int_0^\infty e^{-2dt} I_0(2t)^{d-p} \sum_{(k_1,\dots,k_p) \in \mathbb{Z}^p \setminus \{0\}} \left(\prod_{i=1}^p I_{k_i \alpha_i \alpha_n}(2t) \right) \frac{dt}{t} \\ &= n^{d-p} \alpha_n^p \det(\Lambda) c_d - \left(\frac{n}{\alpha_n} \right)^{d-p} \det(B) \int_0^\infty J(\alpha_n, t) dt \end{split}$$

where in the last equality the integration variable t is changed into $a_n^2 t$ and $J(a_n, t)$ is given by

$$J(\mathfrak{a}_n,t) \coloneqq \frac{1}{t} \left(\mathfrak{a}_n e^{-2\mathfrak{a}_n^2 t} I_0(2\mathfrak{a}_n^2 t) \right)^{d-p} \sum_{(k_1,\ldots,k_p) \in \mathbb{Z}^p \setminus \{0\}} \prod_{i=1}^p \left(\alpha_i \mathfrak{a}_n e^{-2\mathfrak{a}_n^2 t} I_{k_i \alpha_i \mathfrak{a}_n}(2\mathfrak{a}_n^2 t) \right).$$

From Proposition 2.2 we have that

$$\lim_{n\to\infty} \mathfrak{a}_n e^{-2\mathfrak{a}_n^2 t} I_0(2\mathfrak{a}_n^2 t) = \frac{1}{\sqrt{4\pi t}}$$

and

$$\lim_{n\to\infty}\alpha_i\mathfrak{a}_ne^{-2\mathfrak{a}_n^2t}I_{k_i\alpha_i\mathfrak{a}_n}(2\mathfrak{a}_n^2t)=\frac{\alpha_i}{\sqrt{4\pi t}}e^{-\alpha_i^2k_i^2/(4t)}$$

From the definition of the zeta function (5), we have that

$$\begin{split} \int_{0}^{\infty} \frac{1}{(4\pi t)^{d/2}} \sum_{(k_{1},...,k_{p})\in\mathbb{Z}^{p}\setminus\{0\}} e^{-\sum_{i=1}^{p}\alpha_{i}^{2}k_{i}^{2}/(4t)} \frac{dt}{t} &= \frac{1}{\pi^{d/2}}\Gamma(d/2) \sum_{(k_{1},...,k_{p})\in\mathbb{Z}^{p}\setminus\{0\}} \frac{1}{(\sum_{i=1}^{p}\alpha_{i}^{2}k_{i}^{2})^{d/2}} \\ &= (4\pi)^{d/2}\Gamma(d/2)\zeta_{\mathbb{R}^{p}/A^{-1}\mathbb{Z}^{p}}(d/2). \end{split}$$

So as $n \to \infty$,

$$\int_0^\infty J(\mathfrak{a}_n,t)dt = \det(A)(4\pi)^{d/2}\Gamma(d/2)\zeta_{\mathbb{R}^p/A^{-1}\mathbb{Z}^p}(d/2) + o(1).$$

The exchange of the limit as n goes to infinity with the integral over t can be justified using the same argument as in the proof of Proposition 4.2 below on (0,1) and Lemma 4.3 with inequality (13) on $(1,\infty)$. Hence as $n \to \infty$ the lead term behaves as

$$\det(\Lambda_n)\mathcal{I}_d^{\{\alpha_i\}_{i=1}^p} = n^{d-p} a_n^p \det(\Lambda) c_d - \left(\frac{n}{a_n}\right)^{d-p} \left(\det(\Lambda)(4\pi)^{d/2} \Gamma(d/2) \zeta_{\mathbb{R}^p/A^{-1}\mathbb{Z}^p}(d/2) + o(1)\right).$$

Remark. To find one more term in the asymptotic development we would need to show that one can exchange the limit as $n \to \infty$ with the integration over t of

$$\mathfrak{a}_n^2(\mathfrak{l}(\mathfrak{a}_n,\mathfrak{t})-(4\pi)^{d/2}\Gamma(d/2)\zeta_{\mathbb{R}^p/A^{-1}\mathbb{Z}^p}(d/2)).$$

We were not able to do that. One way of proving this is to find an upper bound of the above for all n which is integrable over t on $(0,\infty)$ and apply the Lebesgue dominated convergence Theorem. This means that we need a sharp integrable upper bound of $e^{-t}I_v(t)$. To the best of our

knowledge, the best upper bound of $e^{-t}I_{\nu}(t)$ is given in [3] and is not sharp enough. Assuming that one can exchange the limit with integration, the asymptotic development would be as $n \to \infty$:

$$\begin{split} \log \det^* &\Delta_{\mathbb{Z}^d / \Lambda_n \mathbb{Z}^d} = \mathfrak{n}^{d-p} \mathfrak{a}_n^p \det(\Lambda) \mathfrak{c}_d \\ &- \left(\frac{\mathfrak{n}}{\mathfrak{a}_n}\right)^{d-p} \det(\Lambda) (4\pi)^{d/2} \Biggl[\Gamma(d/2) \zeta_{\mathbb{R}^p / A^{-1} \mathbb{Z}^p} (d/2) \\ &\quad + \frac{1}{\mathfrak{a}_n^2} \Biggl(- (d+4) \Gamma(d/2+1) \zeta_{\mathbb{R}^p / A^{-1} \mathbb{Z}^p} (d/2+1) \\ &\quad + \frac{4}{3} \Gamma(d/2+3) \sum_{i=1}^p x_i^2 \frac{\partial^2}{\partial x_i^2} \zeta_{\mathbb{R}^p / A(x)^{-1/2} \mathbb{Z}^p} (d/2+1) \Biggr|_{\substack{x_i = \alpha_i^2 \\ i = 1, \dots, p}} \Biggr) + o\left(\frac{1}{\mathfrak{a}_n^2}\right) \Biggr] \end{split}$$

where A(x) is the diagonal matrix $A(x) := diag(x_1, \dots, x_p)$.

4.2 Computation of the lead term when a_n is constant

When $a_n = 1$, the lead term is given by

$$\det(\Lambda_n)\mathcal{I}_d^{\{\alpha_i\}_{i=1}^p} = \det(\Lambda_n) \int_0^\infty \Big(e^{-t} - e^{-2dt} I_0(2t)^{d-p} \sum_{(k_1,\ldots,k_p)\in\mathbb{Z}^p} \prod_{i=1}^p I_{k_i\alpha_i}(2t) \Big) \frac{dt}{t}.$$

From the theta inversion formula (9) we have for $i = 1, \ldots, p$

$$\alpha_i e^{-2t} \sum_{k_i \in \mathbb{Z}} I_{k_i \alpha_i}(2t) = \sum_{j_i=0}^{\alpha_i-1} e^{-(2-2\cos(2\pi j_i/\alpha_i))t}.$$

Hence

$$\det(\Lambda_n)\mathcal{I}_d^{\{\alpha_i\}_{i=1}^p} = n^{d-p}\det(B)\sum_{j=0}^{\det(A)-1} \int_0^\infty \left(e^{-t} - I_0(2t)^{d-p}e^{-(2(d-p)+\lambda_j)t}\right) \frac{dt}{t}$$

where

$$\{\lambda_{j}\}_{j} = \{2p - 2\sum_{i=1}^{p} \cos(2\pi j_{i}/\alpha_{i}) : j_{i} = 0, 1, \dots, \alpha_{i} - 1, \text{ for } i = 1, \dots, p\},\$$

 $j = 0, 1, \dots, \det(A) - 1$, are the eigenvalues of the Laplacian on $\mathbb{Z}^p / A \mathbb{Z}^p$.

4.3 Asymptotic behaviour of the second term

In this section we compute the asymptotics of the \mathcal{H}_{Λ_n} term when a_n indifferently goes to infinity sublinearly with respect to n or is constant. To do this we change the integration variable t into $n^2 t$ in (14)

$$\mathcal{H}_{\Lambda_n} = -\int_0^\infty \Bigl(\theta_{\Lambda_n}(n^2 t) - \det(\Lambda_n) e^{-2dn^2 t} I_0(2n^2 t)^{d-p} \sum_{\substack{(k_1, \dots, k_p) \\ \in \mathbb{Z}^p}} \prod_{i=1}^p I_{k_i \alpha_i \alpha_n}(2n^2 t) - 1 + e^{-n^2 t} \Bigr) \frac{dt}{t}$$

Proposition 4.1. With the above notation, we have for all $t \ge 0$,

$$\lim_{n\to\infty}\theta_{\Lambda_n}(n^2t)=\Theta_{\mathbb{R}^{d-p}/B\mathbb{Z}^{d-p}}(t).$$

Proof. The theta function (25) with the change of variable is given by

n

$$\theta_{\Lambda_n}(n^2 t) = \left(\prod_{i=1}^p \alpha_i a_n e^{-2n^2 t} \sum_{k \in \mathbb{Z}} I_{k\alpha_i a_n}(2n^2 t)\right) \left(\prod_{i=1}^{d-p} \beta_i n e^{-2n^2 t} \sum_{k \in \mathbb{Z}} I_{k\beta_i n}(2n^2 t)\right).$$

From Proposition 2.3 we have that

$$\lim_{n\to\infty}\prod_{i=1}^{P}\alpha_{i}\mathfrak{a}_{n}e^{-2n^{2}t}\sum_{k\in\mathbb{Z}}I_{k\alpha_{i}\mathfrak{a}_{n}}(2n^{2}t)=1$$

and from Proposition 2.2 we have

$$\lim_{n\to\infty}\prod_{i=1}^{d-p}\beta_i n e^{-2n^2t}I_{k\beta_i n}(2n^2t) = \prod_{i=1}^{d-p}\frac{\beta_i}{\sqrt{4\pi t}}e^{-(\beta_i k)^2/(4t)}.$$

The proposition follows if we can exchange the limit with the sum. This can be justified in the same way as the proof of Proposition 5.2 in [6]. $\hfill \Box$

Proposition 4.2. With the above notation, we have that

$$\begin{split} &\lim_{n\to\infty}\int_0^1 \Big(\theta_{\Lambda_n}(n^2t) - \det(\Lambda_n)e^{-2dn^2t}I_0(2n^2t)^{d-p}\sum_{(k_1,\dots,k_p)\in\mathbb{Z}^p}\prod_{i=1}^p I_{k_i\alpha_i\alpha_n}(2n^2t)\Big)\frac{dt}{t}\\ &=\int_0^1 \left(\Theta_{\mathbb{R}^{d-p}/B\mathbb{Z}^{d-p}}(t) - \frac{\det(B)}{(4\pi t)^{(d-p)/2}}\right)\frac{dt}{t}. \end{split}$$

Proof. From Propositions 4.1, 2.2 and 2.3 we have the pointwise convergence:

$$\begin{split} &\lim_{n\to\infty}\theta_{\Lambda_n}(n^2t)-\det(\Lambda_n)e^{-2dn^2t}I_0(2n^2t)^{d-p}\,\sum_{(k_1,\ldots,k_p)\in\mathbb{Z}^p}\,\prod_{i=1}^p I_{k_i\alpha_i\alpha_n}(2n^2t)\\ &=\Theta_{\mathbb{R}^{d-p}/B\mathbb{Z}^{d-p}}(t)-\frac{\det(B)}{(4\pi t)^{(d-p)/2}}. \end{split}$$

We have

$$\begin{split} \theta_{\Lambda_n}(n^2 t) &- \det(\Lambda_n) e^{-2dn^2 t} I_0(2n^2 t)^{d-p} \sum_{(k_1, \dots, k_p) \in \mathbb{Z}^p} \prod_{i=1}^p I_{k_i \alpha_i \alpha_n}(2n^2 t) \\ &= \Bigg(\sum_{(k_1, \dots, k_p) \in \mathbb{Z}^p} \prod_{i=1}^p \alpha_i \alpha_n e^{-2n^2 t} I_{k_i \alpha_i \alpha_n}(2n^2 t) \Bigg) \Bigg(\sum_{\substack{(k_1, \dots, k_d - p) \\ \in \mathbb{Z}^{d-p} \setminus \{0\}}} \prod_{i=1}^{d-p} \beta_i n e^{-2n^2 t} I_{k_i \beta_i n}(2n^2 t) \Bigg). \end{split}$$

The first product of the above can be bounded using Proposition 2.3. Indeed we have that for all i = 1, ..., p there exists an $n_{i,0}$ such that for all $n \ge n_{i,0}$

$$\alpha_{\mathfrak{i}}\mathfrak{a}_{\mathfrak{n}}\mathfrak{e}^{-2\mathfrak{n}^{2}t}\sum_{k_{\mathfrak{i}}\in\mathbb{Z}}I_{k_{\mathfrak{i}}\alpha_{\mathfrak{i}}\mathfrak{a}_{\mathfrak{n}}}(2\mathfrak{n}^{2}t)<\frac{3}{2}.$$

The second product can be rewritten in d-p sums with exactly r of the k_i which are non-zero and d-p-r which are zero. Since the $(k_1,\ldots,k_{d-p})=0$ is taken off the sum, we have $1\leqslant r\leqslant d-p$. Let $n_0=\max_{1\leqslant i\leqslant p}n_{i,0}.$ From inequality (13) and Lemma 2.5 we have that for t>0 and all $n\geqslant n_0$ the above is less equal than

$$\begin{split} & 2^{d-p} \det(B) \sum_{r=1}^{d-p} C^{d-p-r} t^{-(d-p)/2} \prod_{i=1}^r \sum_{k_i=1}^\infty \left(1 + \frac{\beta_i k_i}{2n_0 t}\right)^{-n_0 \beta_i k_i/2} \\ & \leqslant 2^{d-p} \det(B) \sum_{r=1}^{d-p} C^{d-p-r} t^{-(d-p)/2} \prod_{i=1}^r \frac{1}{(1+\beta_i/(2n_0t))^{n_0 \beta_i/2} - 1} \\ & \leqslant 2^{d-p} \det(B) \sum_{r=1}^{d-p} C^{d-p-r} t^{-(d-p)/2} \left(\prod_{i=1}^r (\beta_i/(2n_0))^{n_0 \beta_i/2}\right) t^{rn_0 \min_{1 \leqslant i \leqslant d-p} \beta_i/2} \end{split}$$

Hence if we choose $n_0 = 2(d-p)/\min_{1 \le i \le d-p} \beta_i + 1$ the above is integrable on (0,1) with respect to the measure dt/t. The proposition then follows from the Lebesgue dominated convergence Theorem.

We now study the convergence of the integral over $(1, \infty)$. The theta function can be written as the product of two theta functions, that is

$$\theta_{\Lambda_{\mathfrak{n}}}(\mathfrak{n}^{2}t) = \theta_{\text{diag}(\beta_{1}\mathfrak{n},...,\beta_{d-p}\mathfrak{n})}(\mathfrak{n}^{2}t)\theta_{\text{diag}(\alpha_{1}\alpha_{n},...,\alpha_{p}\alpha_{n})}(\mathfrak{n}^{2}t)$$

The first theta function can be bounded using Lemma 5.3 in [6] that we recall below.

Lemma 4.3. Let

$$\theta_{abs}(t) = 2\sum_{j=1}^{\infty} e^{-cj^2 t}$$

with $c = 4\pi^2(1 - \pi^2/24)^2$. Let n_0 be a positive integer. Then for any t > 0 and $n \ge n_0$ we have the bound

$$\theta_{\text{diag}(\beta_1 n, \ldots, \beta_{d-p} n)}(n^2 t) \leqslant \prod_{i=1}^{d-p} \left(1 + e^{-4n_0^2 t} + \theta_{\text{abs}}(t/(4\beta_i^2))\right).$$

It is easy to verify that similarly the second theta function can be bounded by the following

$$\theta_{\operatorname{diag}(\alpha_{1}a_{n},\ldots,\alpha_{p}a_{n})}(n^{2}t) \leqslant \left(1+e^{-4t}+\theta_{\operatorname{abs}}(t)\right)^{p}.$$
(26)

Therefore it follows that $\theta_{\Lambda_n}(n^2t) - 1$ is dt/t-integrable on $(1, \infty)$. So by the Lebesgue dominated convergence Theorem we can exchange the limit and integral. Hence the following proposition is proved:

Proposition 4.4. With the above notation we have that

$$\lim_{n\to\infty}\int_1^\infty \left(\theta_{\Lambda_n}(n^2t)-1\right)\frac{dt}{t}=\int_1^\infty \left(\Theta_{\mathbb{R}^{d-p}/B\mathbb{Z}^{d-p}}(t)-1\right)\frac{dt}{t}.$$

Proposition 4.5. With the above notation we have that

$$\lim_{n \to \infty} \int_1^\infty \det(\Lambda_n) e^{-2dn^2t} I_0(2n^2t)^{d-p} \sum_{(k_1, \dots, k_p) \in \mathbb{Z}^p} \prod_{i=1}^p I_{k_i \alpha_i \alpha_n}(2n^2t) \frac{dt}{t} = \frac{2}{d-p} \frac{\det(B)}{(4\pi)^{(d-p)/2}}.$$

Proof. Combining (13) with (26) we have

$$\det(\Lambda_n)e^{-2dn^2t}I_0(2n^2t)^{d-p}\sum_{(k_1,\ldots,k_p)\in\mathbb{Z}^p}\prod_{i=1}^pI_{k_i\alpha_i\,\alpha_n}(2n^2t)\leqslant Ct^{-(d-p)/2}(1+e^{-4t}+\theta_{abs}(t))^p$$

for some constant C > 0, which is dt/t-integrable on $(1, \infty)$. The result follows from the pointwise convergence and from the Lebesgue dominated convergence Theorem.

Since $\int_{1}^{\infty} e^{-n^2 t} dt/t \to 0$ as $n \to \infty$, the asymptotic of the \mathcal{H}_{Λ_n} term then follows from Lemma 3.3, Propositions 4.2, 4.4 and 4.5:

$$\begin{split} \mathcal{H}_{\Lambda_n} &= 2\log n - \int_0^1 \left(\Theta_{\mathbb{R}^{d-p}/B\mathbb{Z}^{d-p}}(t) - \frac{\det(B)}{(4\pi t)^{(d-p)/2}} \right) \frac{dt}{t} - \Gamma'(1) \\ &\quad - \int_1^\infty \left(\Theta_{\mathbb{R}^{d-p}/B\mathbb{Z}^{d-p}}(t) - 1 \right) \frac{dt}{t} + \frac{2}{d-p} \frac{\det(B)}{(4\pi)^{(d-p)/2}} + o(1) \quad \text{as } n \to \infty. \end{split}$$

Rewriting it in terms of the spectral zeta function with the help of equation (3) yields

$$\mathcal{H}_{\Lambda_{\mathfrak{n}}} = 2\log \mathfrak{n} - \zeta'_{\mathbb{R}^{d-p}/\mathbb{BZ}^{d-p}}(0) + \mathfrak{o}(1) \text{ as } \mathfrak{n} \to \infty.$$
(27)

The calculation of the lead term in section 4.2 together with equation (27) gives Theorem 1.3. For the case where a_n grows sublinearly with respect to n, the error in the lead term, $(n/a_n)^{d-p}o(1)$, is bigger than the \mathcal{H} term (27), therefore the asymptotic is given by

$$\log \det^* \Delta_{\mathbb{Z}^d/\Lambda_n \mathbb{Z}^d} = \mathfrak{n}^{d-p} \mathfrak{a}_n^p \det(\Lambda) \mathfrak{c}_d - \left(\frac{\mathfrak{n}}{\mathfrak{a}_n}\right)^{d-p} \left(\det(\Lambda)(4\pi)^{d/2} \Gamma(d/2) \zeta_{\mathbb{R}^p/A^{-1} \mathbb{Z}^p}(d/2) + \mathfrak{o}(1)\right)$$

as $\mathfrak{n} \to \infty$

as $n \to \infty$.

4.4 Examples

The following examples are here to illustrate the general formula and to highlight the interesting constants appearing in some particular cases. In the examples below, α_i and β_i denote non-zero positive integers.

4.4.1 Example with p = 1 and d = 2

Let $\Lambda_n = \text{diag}(\alpha a_n, \beta n)$ be a sequence of diagonal matrices where a_n grows sublinearly with respect to n. In [6] the authors showed that $c_2 = 4G/\pi$ where G is the Catalan constant. Then as $n \to \infty$

$$\log \det^* \Delta_{\mathbb{Z}^2/\Lambda_n \mathbb{Z}^2} = \mathfrak{n} \mathfrak{a}_n \alpha \beta \frac{4\mathsf{G}}{\pi} - \frac{\mathfrak{n}}{\mathfrak{a}_n} \left(\frac{\beta}{\alpha} \frac{\pi}{3} + \mathfrak{o}(1) \right).$$

4.4.2 Example with p = 1 and any d

Let $\Lambda_n = \text{diag}(\alpha a_n, \beta_1 n, \dots, \beta_{d-1} n)$ be a sequence of diagonal matrices where a_n grows sublinearly with respect to n. Then as $n \to \infty$

$$\log \det^* \Delta_{\mathbb{Z}^d / \Lambda_n \mathbb{Z}^d} = \mathfrak{n}^{d-1} \mathfrak{a}_n \det(\Lambda) \mathfrak{c}_d - \left(\frac{\mathfrak{n}}{\mathfrak{a}_n}\right)^{d-1} \left(\frac{\beta_1 \cdots \beta_{d-1}}{\alpha^{d-1}} \frac{2}{\pi^{d/2}} \Gamma(d/2) \zeta(d) + \mathfrak{o}(1)\right)$$

where ζ is the Riemann zeta function.

4.4.3 Example with a_n constant and p = d - 1

Let $\Lambda^0_n=\text{diag}(\alpha_1,\ldots,\alpha_{d-1},\beta n)$ be a sequence of diagonal matrices. From (6), $-\zeta'_{\mathbb{R}/\beta\mathbb{Z}}(0)=2\log\beta.$ Using Proposition 2.4 one has as $n\to\infty$

$$\log \det^* \Delta_{\mathbb{Z}^d / \Lambda^0_n \mathbb{Z}^d} = n\beta \sum_{j=0}^{\det(A)-1} \mathrm{argcosh}\left(1 + \frac{\lambda_j}{2}\right) + 2\log n + 2\log \beta + o(1)$$

where

$$\{\lambda_j\}_j = \{2(d-1) - 2\sum_{i=1}^{d-1} \cos(2\pi j_i/\alpha_i) : j_i = 0, 1, \dots, \alpha_i - 1, \text{ for } i = 1, \dots, d-1\},$$

 $j=0,1,\ldots,\det(A)-1\!\!$, are the eigenvalues of the Laplacian on $\mathbb{Z}^{d-1}\!/A\mathbb{Z}^{d-1}\!.$

4.4.4 Example with a_n constant, p = 1 and d = 3

Let $\Lambda^0_n=diag(\alpha,\beta_1n,\beta_2n)$ be a sequence of diagonal matrices. From section 6.3 in [6], we have that

$$-\zeta'_{\mathbb{R}^2/\text{diag}(\beta_1,\beta_2)\mathbb{Z}^2}(0) = 2\log(\beta_2\eta(i\beta_2/\beta_1)^2)$$

where η is the Dedekind eta function defined for $z \in \mathbb{C}$ with $\Im(z) > 0$ by

$$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}).$$

Hence as $n\to\infty$

$$\begin{split} \log \det^* & \Delta_{\mathbb{Z}^3/\Lambda_n^0 \mathbb{Z}^3} = n^2 \beta_1 \beta_2 \sum_{j=0}^{\alpha-1} \int_0^\infty \left(e^{-t} - I_0 (2t)^2 e^{-(6-2\cos(2\pi j/\alpha))t} \right) \frac{dt}{t} \\ &+ 2\log n + 2\log(\beta_2 \eta (i\beta_2/\beta_1)^2) + o(1). \end{split}$$

Using the special value of η at z = i, $\eta(i) = \Gamma(1/4)/(2\pi^{3/4})$, one has for the special case $\beta_1 = \beta_2 =: \beta$ the asymptotic behaviour as $n \to \infty$

$$\begin{split} \log \det^* & \Delta_{\mathbb{Z}^3/\Lambda_n^0 \mathbb{Z}^3} = n^2 \beta^2 \sum_{j=0}^{\alpha-1} \int_0^\infty \left(e^{-t} - I_0 (2t)^2 e^{-(6-2\cos(2\pi j/\alpha))t} \right) \frac{dt}{t} \\ &+ 2\log n + \log(\beta^2 \Gamma(1/4)^4/(16\pi^3)) + o(1). \end{split}$$

5 A comment on circulant graphs with non-fixed generators^{*}

In [13, 23] the authors considered circulant graphs with non-fixed generators. In [13] they computed the lead term of the asymptotic number of spanning trees. It is conceivable that the techniques used here could be extended to improve their result and compute the second term. In [23] they computed the exact number of spanning trees in $C_{\beta n}^{1,n}$ for $\beta \in \{2,3,4,6,12\}$ via Chebyshev polynomials, but

^{*}At the time of reviewing this paper, this conjecture has been proved and will appear in a forthcoming paper.

were not able to generalize to other values of β . We propose a conjecture for the case $\beta = 5$: For all $n \ge 2$,

$$\begin{aligned} \tau(C_{5n}^{1,n}) &= \frac{n}{5} \left(\left(\frac{9 - \sqrt{5} + \sqrt{70 - 18\sqrt{5}}}{4} \right)^n + \left(\frac{9 - \sqrt{5} + \sqrt{70 - 18\sqrt{5}}}{4} \right)^{-n} + \frac{1 - \sqrt{5}}{2} \right)^2 \\ & \times \left(\left(\frac{9 + \sqrt{5} + \sqrt{70 + 18\sqrt{5}}}{4} \right)^n + \left(\frac{9 + \sqrt{5} + \sqrt{70 + 18\sqrt{5}}}{4} \right)^{-n} + \frac{1 + \sqrt{5}}{2} \right)^2. \end{aligned}$$

Notice that the coefficients in the formula can be expressed in terms of integrals involving modified I-Bessel function. Indeed, let

$$J_k^\beta = \int_0^\infty \left(e^{-t} - e^{-2t(2-\cos(2\pi k/\beta))} I_0(2t) \right) \frac{dt}{t}, \quad k = 1, \dots, \beta - 1.$$

Then from Proposition 2.4, the above can be rewritten as

$$\begin{split} \tau(C_{5n}^{1,n}) &= \frac{n}{5} \left(e^{nJ_1^5} + e^{-nJ_1^5} + \frac{1}{2}(1-\sqrt{5}) \right) \left(e^{nJ_2^5} + e^{-nJ_2^5} + \frac{1}{2}(1+\sqrt{5}) \right) \\ & \times \left(e^{nJ_3^5} + e^{-nJ_3^5} + \frac{1}{2}(1+\sqrt{5}) \right) \left(e^{nJ_4^5} + e^{-nJ_4^5} + \frac{1}{2}(1-\sqrt{5}) \right). \end{split}$$

Therefore for other values of β the general formula might have the form

$$\tau(C^{1,n}_{\beta n}) = \frac{n}{\beta} \prod_{k=1}^{\beta-1} \left(e^{nJ_k^\beta} + e^{-nJ_k^\beta} + \alpha_k^\beta \right), \text{ for all } n \geqslant 1,$$

where α_k^{β} are coefficients which are not known for $\beta \ge 7$.

References

- [1] Talip Atajan, Xuerong Yong, and Hiroshi Inaba. Further analysis of the number of spanning trees in circulant graphs. *Discrete Math.*, 306(22):2817–2827, 2006.
- [2] Talip Atajan, Xuerong Yong, and Hiroshi Inaba. An efficient approach for counting the number of spanning trees in circulant and related graphs. *Discrete Math.*, 310(6-7):1210–1221, 2010.
- [3] Prakash Balachandran, Weston Viles, and Eric D Kolaczyk. Exponential-type inequalities involving ratios of the modified bessel function of the first kind and their applications. *arXiv* preprint arXiv:1311.1450, 2013.
- [4] G. Baron, H. Prodinger, R. F. Tichy, F. T. Boesch, and J. F. Wang. The number of spanning trees in the square of a cycle. *Fibonacci Quart.*, 23(3):258–264, 1985.
- [5] F. T. Boesch and H. Prodinger. Spanning tree formulas and Chebyshev polynomials. *Graphs Combin.*, 2(3):191–200, 1986.
- [6] Gautam Chinta, Jay Jorgenson, and Anders Karlsson. Zeta functions, heat kernels, and spectral asymptotics on degenerating families of discrete tori. *Nagoya Math. J.*, 198:121–172, 2010.

- [7] Gautam Chinta, Jay Jorgenson, and Anders Karlsson. Complexity and heights of tori. In Dynamical systems and group actions, volume 567 of Contemp. Math., pages 89–98. Amer. Math. Soc., Providence, RI, 2012.
- [8] James Alan Cochran. The monotonicity of modified Bessel functions with respect to their order. J. Math. and Phys., 46:220–222, 1967.
- [9] Charles J. Colbourn. The combinatorics of network reliability. International Series of Monographs on Computer Science. The Clarendon Press Oxford University Press, New York, 1987.
- [10] Dragoš M. Cvetković, Michael Doob, and Horst Sachs. Spectra of graphs, volume 87 of Pure and Applied Mathematics. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1980. Theory and application.
- [11] Mordecai J. Golin and Yiu Cho Leung. Unhooking circulant graphs: a combinatorial method for counting spanning trees and other parameters. In *Graph-theoretic concepts in computer science*, volume 3353 of *Lecture Notes in Comput. Sci.*, pages 296–307. Springer, Berlin, 2004.
- [12] Mordecai J. Golin, Yiu Cho Leung, and Yajun Wang. Counting spanning trees and other structures in non-constant-jump circulant graphs. In *Algorithms and computation*, volume 3341 of *Lecture Notes in Comput. Sci.*, pages 508–521. Springer, Berlin, 2004.
- [13] Mordecai J. Golin, Xuerong Yong, and Yuanping Zhang. The asymptotic number of spanning trees in circulant graphs. *Discrete Math.*, 310(4):792–803, 2010.
- [14] Anders Karlsson and Markus Neuhauser. Heat kernels, theta identities, and zeta functions on cyclic groups. In *Topological and asymptotic aspects of group theory*, volume 394 of *Contemp. Math.*, pages 177–189. Amer. Math. Soc., Providence, RI, 2006.
- [15] G. Kirchhoff. Ueber die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Vertheilung galvanischer Ströme geführt wird. Annalen der Physik, 148(12):497–508, 1847.
- [16] D. J. Kleitman and B. Golden. Counting trees in a certain class of graphs. Amer. Math. Monthly, 82:40–44, 1975.
- [17] H. J. Korsch, A. Klumpp, and D. Witthaut. On two-dimensional Bessel functions. J. Phys. A, 39(48):14947–14964, 2006.
- [18] Leslie M. McDonald and Iain Moffatt. On the Potts model partition function in an external field. J. Stat. Phys., 146(6):1288–1302, 2012.
- [19] A. Voros. Spectral functions, special functions and the Selberg zeta function. Comm. Math. Phys., 110(3):439–465, 1987.
- [20] Xue-rong Yong, Talip, and Acenjian. The numbers of spanning trees of the cubic cycle C_N^3 and the quadruple cycle C_N^4 . *Discrete Math.*, 169(1-3):293–298, 1997.
- [21] Yuanping Zhang and Mordecai J. Golin. Further applications of Chebyshev polynomials in the derivation of spanning tree formulas for circulant graphs. In *Mathematics and computer science*, *II (Versailles, 2002)*, Trends Math., pages 541–553. Birkhäuser, Basel, 2002.
- [22] Yuanping Zhang, Xuerong Yong, and Mordecai J. Golin. The number of spanning trees in circulant graphs. *Discrete Math.*, 223(1-3):337–350, 2000.

[23] Yuanping Zhang, Xuerong Yong, and Mordecai J. Golin. Chebyshev polynomials and spanning tree formulas for circulant and related graphs. *Discrete Math.*, 298(1-3):334–364, 2005.

A formula for the number of spanning trees in circulant graphs with non-fixed generators and discrete tori^{*}

Justine Louis

15 December 2014

Abstract

We consider the number of spanning trees in circulant graphs of βn vertices with generators depending linearly on n. The matrix tree theorem gives a closed formula of βn factors, while we derive a formula of $\beta - 1$ factors. Using the same trick, we also derive a formula for the number of spanning trees in discrete tori. Moreover, the spanning tree entropy of circulant graphs with fixed and non-fixed generators is compared.

1 Introduction

A spanning tree of a connected graph G is a connected subgraph of G without cycles with the same vertex set as G. The number of spanning trees in a graph G, $\tau(G)$, is an important graph invariant and is widely studied. It can be computed from the well-known matrix tree theorem due to Kirchhoff (e.g. see [1]). Let G be a graph on n vertices labelled by v_1, \ldots, v_n . The adjacency matrix A of G is defined by the $n \times n$ matrix in which $(A)_{ij} = 1$ if v_i and v_j are adjacent and $(A)_{ij} = 0$ otherwise. The degree matrix D is defined by the $n \times n$ diagonal matrix in which the diagonal element $(D)_{ii}$ is the degree of the corresponding vertex v_i . In this paper we only consider 2d-regular graphs, so that $D = 2dI_n$, where I_n is the $n \times n$ identity matrix. The combinatorial Laplacian matrix Δ_G of a 2d-regular graph on n vertices G is defined by

$$\Delta_{\rm G} = 2 d I_{\rm n} - A$$

The matrix tree theorem states that

$$\tau(G) = \frac{1}{n} \det^* \Delta_G \tag{1}$$

where det^{*} Δ_G denotes the product of the non-zero eigenvalues of the Laplacian on G. In this paper we prove closed formulas for $\tau(G)$ for two types of graphs in terms of eigenvalues of the Laplacian on a subgraph of G. The formulas are particularly interesting when the number of vertices is larger than the other parameters of the graph.

Let $1 \leq \gamma_1 \leq \cdots \leq \gamma_d \leq \lfloor n/2 \rfloor$ be positive integers. A circulant graph $C_n^{\gamma_1,\dots,\gamma_d}$ is the 2dregular graph with n vertices labelled $0, 1, \dots, n-1$ such that each vertex $v \in \mathbb{Z}/n\mathbb{Z}$ is connected to $v \pm \gamma_i$ mod n for all $i \in \{1, \dots, d\}$. The first type of graphs studied is the circulant graph with the first generator equal to one and the d-1 others linearly depending on the number of vertices, that is $C_{\beta n}^{l,\gamma_1 n,\dots,\gamma_{d-1} n}$, where $1 \leq \gamma_1 \leq \cdots \leq \gamma_{d-1} \leq \lfloor \beta/2 \rfloor$ and β are integers. Two examples

^{*}published in Bulletin of the Australian Mathematical Society, 92(3):365–373, 2015. The author was supported in part by the Swiss NSF grant 200021 132528/1.

are illustrated in Figure 1 below. It is known that the number of spanning trees in circulant graphs with n vertices satisfies a linear recurrence relation with constant coefficients in n, this has been shown by Golin, Leung and Wang in [4]. For $\beta \in \{2, 3, 4, 6, 12\}$, closed formulas have been obtained by Zhang, Yong and Golin in [8] where the authors used techniques inspired from Boesch and Prodinger [2] using Chebyshev polynomials. As noted in [8] this method does not work for other values of β . In section 2, we derive Theorem 2.1 in a simple way which gives a closed formula for all integer values of β . This gives an answer to an open question in [3] and [8] and proves the conjecture stated in [6]. The second type of graphs studied is the d-dimensional discrete torus defined by the quotient $\mathbb{Z}^d/\Lambda\mathbb{Z}^d$, where Λ is a diagonal integer matrix, with nearest neighbours connected. In the last section, we deduce the tree entropy for a sequence of non-fixed generated circulant graphs and compare it to the one with fixed generators.



Figure 1: Examples of circulant graphs.

Acknowledgements: The author thanks Anders Karlsson for encouraging and helpful discussions and support. The author also thanks the anonymous referee for useful comments which improved the quality of the paper.

2 Spanning trees in circulant graphs with non-fixed generators

Let V(G) be the set of vertices of a graph G and $f: V(G) \to \mathbb{R}$ a function. To derive the eigenvalues of the Laplacian it is more convenient to use the variant definition of the combinatorial Laplacian defined as an operator acting on the space of functions, that is

$$\Delta_{G} f(x) = \sum_{y \sim x} (f(x) - f(y))$$

where the sum is over all vertices adjacent to x. Since the circulant graph $C_{\beta n}^{l,\gamma_1 n,...,\gamma_{d-1}n}$ is the Cayley graph of the group $\mathbb{Z}/\beta n\mathbb{Z}$, the eigenvectors of the Laplacian are given by the characters

$$\chi_{\mathbf{k}}(\mathbf{x}) = e^{2\pi \mathbf{i} \mathbf{k} \mathbf{x}/(\beta \mathbf{n})}, \quad \mathbf{k} = 0, 1, \dots, \beta \mathbf{n} - 1,$$

where $x \in \mathbb{Z}/\beta n\mathbb{Z}$. Therefore the eigenvalues are given by (see also [1, Proposition 3.5])

$$\lambda_{k} = 2d - 2\cos(2\pi k/(\beta n)) - 2\sum_{m=1}^{d-1}\cos(2\pi k\gamma_{m}/\beta), \quad k = 0, 1, \dots, \beta n - 1.$$
(2)

Theorem 2.1. Let $1 \leq \gamma_1 \leq \cdots \leq \gamma_{d-1} \leq \lfloor \beta/2 \rfloor$ be positive integers and $\mu_k = 2(d-1) - 2\sum_{m=1}^{d-1} \cos(2\pi k \gamma_m/\beta)$, $k = 1, \ldots, \beta - 1$, be the non-zero eigenvalues of the Laplacian on the circulant graph $C_{\beta}^{\gamma_1, \ldots, \gamma_{d-1}}$. For all $n \in \mathbb{N}_{\geq 1}$, the number of spanning trees in the circulant graph $C_{\beta n}^{\gamma_1, \ldots, \gamma_{d-1}}$ is given by

$$\tau(C_{\beta n}^{1,\gamma_{1}n,...,\gamma_{d-1}n}) = \frac{n}{\beta} \prod_{k=1}^{\beta-1} \left((\mu_{k}/2 + 1 + \sqrt{\mu_{k}^{2}/4 + \mu_{k}})^{n} + (\mu_{k}/2 + 1 - \sqrt{\mu_{k}^{2}/4 + \mu_{k}})^{n} - 2\cos(2\pi k/\beta) \right).$$

Remark. It would be interesting to see if this pattern appears in other types of graphs, that is, the number of spanning trees could be expressed in terms of the eigenvalues of the Laplacian on a subgraph of the original graph.

Proof. Applying the matrix tree theorem (1) to the graph $C_{\beta n}^{1,\gamma_1 n,\ldots,\gamma_{d-1}n}$, with eigenvalues given by (2), gives

$$\tau(C_{\beta\mathfrak{n}}^{1,\gamma_{1}\mathfrak{n},\ldots,\gamma_{d-1}\mathfrak{n}}) = \frac{1}{\beta\mathfrak{n}} \prod_{k=1}^{\beta\mathfrak{n}-1} \bigg(2d - 2\cos(2\pi k/(\beta\mathfrak{n})) - 2\sum_{\mathfrak{m}=1}^{d-1}\cos(2\pi k\gamma_{\mathfrak{m}}/\beta) \bigg).$$

Since there are n spanning trees in the cycle C_n^1 , we have

$$n = \tau(C_n^1) = \frac{1}{n} \prod_{k=1}^{n-1} (2 - 2\cos(2\pi k/n)).$$
(3)

The product over $k = 1, ..., \beta n - 1$ can be split as a product over multiples of β , that is $k = \beta k'$ with k' = 1, ..., n - 1, and over non-multiples of β , that is $k = k' + l\beta$ with $k' = 1, ..., \beta - 1$ and l = 0, 1, ..., n - 1. The product over the multiples of β reduces to equation (3), so it follows that

$$\begin{aligned} \tau(C_{\beta n}^{1,\gamma_{1}n,...,\gamma_{d-1}n}) &= \frac{n}{\beta} \prod_{\substack{k=1\\\beta \nmid k}}^{\beta n-1} \left(2d - 2\cos(2\pi k/(\beta n)) - 2\sum_{m=1}^{d-1}\cos(2\pi k\gamma_{m}/\beta) \right) \\ &= \frac{n}{\beta} \prod_{k=1}^{\beta-1} \prod_{l=0}^{n-1} \left(2d - 2\cos(2\pi (k+l\beta)/(\beta n)) - 2\sum_{m=1}^{d-1}\cos(2\pi (k+l\beta)\gamma_{m}/\beta) \right) \\ &= \frac{n}{\beta} \prod_{k=1}^{\beta-1} \prod_{l=0}^{n-1} \left(2\cosh(\operatorname{Argcosh}(d-\sum_{m=1}^{d-1}\cos(2\pi k\gamma_{m}/\beta))) - 2\cos(2\pi k/(\beta n) + 2\pi l/n) \right). \end{aligned}$$
(4)

We now evaluate the product over l by the following calculation

$$\prod_{l=0}^{n-1} (2\cosh\theta - 2\cos((\omega + 2\pi l)/n)) = e^{-n\theta} \prod_{l=0}^{n-1} (e^{2\theta} - 2\cos((\omega + 2\pi l)/n)e^{\theta} + 1)$$
$$= e^{-n\theta} \prod_{l=0}^{n-1} (e^{\theta} - e^{i(\omega + 2\pi l)/n})(e^{\theta} - e^{-i(\omega + 2\pi l)/n}).$$
(5)

The complex numbers $e^{i(\omega+2\pi l)/n}$ and $e^{-i(\omega+2\pi l)/n}$ for l = 0, 1, ..., n-1 are the 2n roots of the following polynomial in e^{θ}

$$e^{2n\theta} - 2e^{n\theta}\cos\omega + 1 = 0$$

Therefore the product (5) is equal to

$$e^{-n\theta}(e^{2n\theta}-2e^{n\theta}\cos\omega+1)=2\cosh(n\theta)-2\cos\omega.$$

Using this relation in (4) with $\theta = \operatorname{Argcosh}(d - \sum_{m=1}^{d-1} \cos(2\pi k \gamma_m / \beta))$ and $\omega = 2\pi k / \beta$, we have

$$\tau(C_{\beta n}^{1,\gamma_1 n,\dots,\gamma_{d-1} n}) = \frac{n}{\beta} \prod_{k=1}^{\beta-1} \left(2\cosh(n\operatorname{Argcosh}(d - \sum_{m=1}^{d-1}\cos(2\pi k\gamma_m/\beta))) - 2\cos(2\pi k/\beta) \right).$$
(6)

The theorem then follows by expressing the formula in terms of the eigenvalues on $C_{\beta}^{\gamma_1,\ldots,\gamma_{d-1}}$ and from the relation $\operatorname{Argcosh} x = \log(x + \sqrt{x^2 - 1})$ for $x \ge 1$. Indeed, writing $\mu_k = 2(d - 1) - 2\sum_{m=1}^{d-1} \cos(2\pi k \gamma_m / \beta)$, we have that

$$\begin{split} \tau(C_{\beta\pi}^{1,\gamma_{1}n,...,\gamma_{d-1}n}) &= \frac{n}{\beta} \prod_{k=1}^{\beta-1} \left(2\cosh(n\operatorname{Argcosh}(1+\mu_{k}/2)) - 2\cos(2\pi k/\beta) \right) \\ &= \frac{n}{\beta} \prod_{k=1}^{\beta-1} \left(2\cosh(n\log(1+\mu_{k}/2+\sqrt{\mu_{k}^{2}/4+\mu_{k}})) - 2\cos(2\pi k/\beta) \right) \\ &= \frac{n}{\beta} \prod_{k=1}^{\beta-1} \left((\mu_{k}/2+1+\sqrt{\mu_{k}^{2}/4+\mu_{k}})^{n} + (\mu_{k}/2+1-\sqrt{\mu_{k}^{2}/4+\mu_{k}})^{n} - 2\cos(2\pi k/\beta) \right). \end{split}$$

Remark. The techniques used here to derive Theorem 2.1 might not be generalisable to circulant graphs with two or more fixed generators. As an example, to compute the number of spanning trees in the graph $C_{\beta n}^{1,2,\gamma n}$ we would need to find a closed formula for the product

$$\prod_{l=0}^{n-1} (2\cosh\theta - 2\cos((\omega + 2\pi l)/n) - 2\cos(2(\omega + 2\pi l)/n))$$

where $\theta = \operatorname{Argcosh}(3 - \cos(2\pi k\gamma/\beta))$ and $\omega = 2\pi k/\beta$. We were not able to do that.

Examples. This formula reproves Theorems 4, 5, 6, 8 and corrects a typographical error in Theorem 7 in [8]. For example, [8, Theorem 5] states that

$$\tau(C_{3n}^{1,n}) = \frac{n}{3} \left[(\sqrt{7/4} + \sqrt{3/4})^{2n} + (\sqrt{7/4} - \sqrt{3/4})^{2n} + 1 \right]^2$$

which is a particular case of the formula with d = 2, $\gamma_1 = 1$, $\beta = 3$ and $\mu_k = 2 - 2\cos(2\pi k/3)$, k = 1, 2, being the non-zero eigenvalues on the cycle C_3^1 . As another example, [8, Theorem 8] states that

$$\begin{split} \tau(\mathbf{C}_{6n}^{1,2n,3n}) &= \frac{n}{6} \left[(\sqrt{11/4} + \sqrt{7/4})^{2n} + (\sqrt{11/4} - \sqrt{7/4})^{2n} - 1 \right]^2 \left[(\sqrt{2} + 1)^n + (\sqrt{2} - 1)^n \right]^2 \\ & \times \left[(\sqrt{7/4} + \sqrt{3/4})^{2n} + (\sqrt{7/4} - \sqrt{3/4})^{2n} + 1 \right]^2 \end{split}$$

which is a particular case with d = 3, $\gamma_1 = 2$, $\gamma_2 = 3$, $\beta = 6$ and $\mu_k = 4-2\cos(2\pi k/3)-2\cos(\pi k)$, $k = 1, \ldots, 5$, being the non-zero eigenvalues on the circulant graph $C_6^{2,3}$.

Remark. We emphasize that the circulant graph $C_{\beta n}^{1,\gamma_1 n,\dots,\gamma_{d-1}n}$ consists of n copies of $C_{\beta}^{\gamma_1,\dots,\gamma_{d-1}}$ which are embedded in the cycle $C_{\beta n}^1$. This explains the eigenvalues on $C_{\beta}^{\gamma_1,\dots,\gamma_{d-1}}$ appearing in the formula.

3 Spanning trees in discrete tori

In this section we establish a formula for the number of spanning trees in the discrete torus $\mathbb{Z}^d/\Lambda\mathbb{Z}^d$, where $\Lambda = diag(\alpha_1, \ldots, \alpha_{d-1}, n)$ is a diagonal matrix with positive integer coefficients, with nearest neighbours connected. Let $k = (k_1, \ldots, k_d)$, $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d/\Lambda\mathbb{Z}^d$ and $k_\Lambda = \Lambda^{-1}k$. The eigenvectors of the Laplacian are given by

$$g_{k_{\Lambda}}(x) = e^{2\pi i \langle k_{\Lambda}, x \rangle}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product. Denote by e_i , i = 1, ..., d, the canonical basis of \mathbb{Z}^d . Since each vertex $x \in \mathbb{Z}^d / \Lambda \mathbb{Z}^d$ is connected to his nearest neighbours, that is x is adjacent to $x - e_i$ and to $x + e_i$, for all i = 1, ..., d, we obtain the eigenvalues on $\mathbb{Z}^d / \Lambda \mathbb{Z}^d$ by applying the Laplacian on the eigenvectors $g_{k_\Lambda}(x)$:

$$\lambda_k = 2d - 2\sum_{i=1}^{d-1} \cos(2\pi k_i/\alpha_i) - 2\cos(2\pi k_d/n) \text{ where } k \in \mathbb{Z}^d/\Lambda\mathbb{Z}^d.$$

The formula given in the following theorem is interesting when n is larger than det(A). It improves the asymptotic result given in [6, Example 4.4.3].

Theorem 3.1. Let $A = \text{diag}(\alpha_1, \ldots, \alpha_{d-1})$ and $\{\mu_\ell\}_\ell = \{2(d-1) - 2\sum_{i=1}^{d-1}\cos(2\pi k_i/\alpha_i) : k_i = 0, 1, \ldots, \alpha_i - 1, i = 1, \ldots, d-1, (k_1, \ldots, k_{d-1}) \neq 0\}, \ell = 1, \ldots, \det(A) - 1$, be the non-zero eigenvalues of the Laplacian on $\mathbb{Z}^{d-1}/A\mathbb{Z}^{d-1}$. For all $n \in \mathbb{N}_{\geq 1}$, the number of spanning trees in the discrete torus $\mathbb{Z}^d/A\mathbb{Z}^d$ is given by

$$\tau(\mathbb{Z}^d/\Lambda\mathbb{Z}^d) = \frac{n}{\det(A)} \prod_{\ell=1}^{\det(A)-1} \left(\left(\mu_\ell/2 + 1 + \sqrt{\mu_\ell^2/4 + \mu_\ell} \right)^n + \left(\mu_\ell/2 + 1 - \sqrt{\mu_\ell^2/4 + \mu_\ell} \right)^n - 2 \right) \cdot \frac{1}{\det(A)} \left(\frac{1}{2} \left(\frac{1}{2} - \frac{1}{2} \right)^n + \frac{1}{2} \left(\frac{1}{2} - \frac$$

Proof. From the matrix tree theorem, we have

$$\begin{split} \tau(\mathbb{Z}^d/\Lambda\mathbb{Z}^d) &= \frac{1}{\det(A)n} \prod_{\substack{i=1 \\ (k_1,\dots,k_d) \neq 0}}^{d-1} \prod_{\substack{k_i=0 \\ (k_1,\dots,k_d) \neq 0}}^{n-1} \prod_{\substack{i=1 \\ (k_1,\dots,k_d) \neq 0}}^{n-1} \left(2d - 2\sum_{i=1}^{d-1} \cos(2\pi k_i/\alpha_i) - 2\cos(2\pi k_d/n) \right) \\ &= \frac{n}{\det(A)} \prod_{\substack{i=1 \\ (k_1,\dots,k_d-1) \neq 0}}^{d-1} \prod_{\substack{k_i=0 \\ (k_1,\dots,k_d-1) \neq 0}}^{n-1} \prod_{\substack{k_i=0 \\ (k_1,\dots,k_d-1) \neq 0}}^{n-1} \left(2\cosh(\operatorname{Argcosh}(d - \sum_{i=1}^{d-1}\cos(2\pi k_i/\alpha_i))) - 2\cos(2\pi k_d/n) \right) \\ &= \frac{n}{\det(A)} \prod_{\substack{i=1 \\ (k_1,\dots,k_{d-1}) \neq 0}}^{d-1} \prod_{\substack{k_i=0 \\ (k_1,\dots,k_{d-1}) \neq 0}}^{n-1} \left(2\cosh(n\operatorname{Argcosh}(d - \sum_{i=1}^{d-1}\cos(2\pi k_i/\alpha_i))) - 2 \right) \end{split}$$

where the second equality comes from equation (3) and the third equality comes from the same trick as in the proof of Theorem 2.1,

$$\prod_{k=0}^{n-1} \left(2\cosh\theta - 2\cos(2\pi k/n) \right) = 2\cosh(n\theta) - 2.$$

4 Spanning tree entropy of circulant graphs

For a sequence of regular graphs G_n with vertex set $V(G_n)$, one can consider the number of spanning trees as a function of n. Assuming that the following limit exists

$$z = \lim_{n \to \infty} \frac{\log \tau(\mathsf{G}_n)}{|\mathsf{V}(\mathsf{G}_n)|},$$

it is sometimes called the associated tree entropy [7]. From Theorem 2.1, the tree entropy for the non-fixed generated circulant graph $C_{\beta n}^{l,\gamma_1 n,\ldots,\gamma_{d-1} n}$ as $n \to \infty$, denoted by $z_{NF}(\beta;\gamma_1,\ldots,\gamma_{d-1})$, is given in the following corollary.

Corollary 4.1. Let $1 \leq \gamma_1 \leq \cdots \leq \gamma_{d-1} \leq \lfloor \beta/2 \rfloor$ and β be positive integers. The tree entropy of the circulant graph $C^{1,\gamma_1n,\ldots,\gamma_{d-1}n}_{\beta n}$ as $n \to \infty$ is given by

$$\begin{split} z_{\mathsf{NF}}(\beta;\gamma_1,\ldots,\gamma_{d-1}) &= \frac{1}{\beta}\sum_{k=1}^{\beta-1}\mathrm{Argcosh}(d-\sum_{m=1}^{d-1}\cos(2\pi k\gamma_m/\beta)) \\ &= \int_0^\infty (e^{-t}-\frac{1}{\beta}\sum_{k=0}^{\beta-1}e^{-\mu_k\,t}e^{-2t}I_0(2t))\frac{dt}{t} \end{split}$$

where $\mu_k=2(d-1)-2\sum_{m=1}^{d-1}\cos(2\pi k\gamma_m/\beta),\ k=0,1,\ldots,\beta-1,$ are the eigenvalues of the Laplacian on the circulant graph $C_\beta^{\gamma_1,\ldots,\gamma_{d-1}}$, and I_0 is the modified I-Bessel function of order zero.

Proof. Let $f_k := \operatorname{Argcosh}(1 + \mu_k/2) > 0$, $k = 1, \dots, \beta - 1$. From equation (6), the number of spanning trees in $C_{\beta n}^{1,\gamma_1 n, \dots, \gamma_{d-1} n}$ is given by

$$\begin{aligned} \tau(C_{\beta n}^{1,\gamma_1 n, \dots, \gamma_{d-1} n}) &= \frac{n}{\beta} \prod_{k=1}^{\beta - 1} (e^{nf_k} + e^{-nf_k} - 2\cos(2\pi k/\beta)) \\ &= \frac{n}{\beta} e^{n\sum_{k=1}^{\beta - 1} f_k} \prod_{k=1}^{\beta - 1} (1 + e^{-2nf_k} - 2\cos(2\pi k/\beta)e^{-nf_k}) \end{aligned}$$

We have that

$$\lim_{n\to\infty}\log(1+e^{-2\pi f_k}-2\cos(2\pi k/\beta)e^{-\pi f_k})=0, \ \text{for} \ k=1,\ldots,\beta-1.$$

Hence

$$\prod_{k=1}^{\beta-1} (1 + e^{-2\pi f_k} - 2\cos(2\pi k/\beta)e^{-\pi f_k}) = e^{o(1)} \text{ as } n \to \infty.$$

Therefore, the asymptotic number of spanning trees in $C_{\beta n}^{1,\gamma_1 n,...,\gamma_{d-1}n}$ is given by

$$\tau(C^{1,\gamma_1n,\ldots,\gamma_{d-1}n}_{\beta n}) = \frac{n}{\beta} e^{n \sum_{k=1}^{\beta-1} \operatorname{Argcosh}(d - \sum_{m=1}^{d-1} \cos(2\pi k \gamma_m/\beta)) + o(1)} \quad \text{as } n \to \infty.$$

This shows the first equality. The second equality comes from [6, Proposition 2.4] which expresses the Argcosh in terms of an integral of modified I-Bessel function: for all $x \ge 2$,

$$\int_0^\infty (e^{-t} - e^{-xt} I_0(2t)) \frac{dt}{t} = \operatorname{Argcosh}(x/2).$$

As mentioned in section 2 the circulant graph $C_{\beta n}^{1,\gamma_1 n,...,\gamma_{d-1}n}$ consists of n copies of $C_{\beta}^{\gamma_1,...,\gamma_{d-1}}$ which are embedded in the cycle $C_{\beta n}^1$. This structure is reflected by the appearance of the term $\theta_{C_{\beta}^{\gamma_1,...,\gamma_{d-1}}}(t)e^{-2t}I_0(2t)$ in the asymptotic formula, where $\theta_{C_{\beta}^{\gamma_1,...,\gamma_{d-1}}}(t) = \sum_{k=0}^{\beta-1} e^{-\mu_k t}$ is the theta function on $C_{\beta}^{\gamma_1,...,\gamma_{d-1}}$ and $e^{-2t}I_0(2t)$ is the typical term appearing in the asymptotics of the number of spanning trees in the cycle. Indeed, the tree entropy on the cycle is (see section 3.2 in [6])

$$z_{\texttt{cycle}} = \int_0^\infty (e^{-t} - e^{-2t} \mathrm{I}_0(2t)) \frac{\mathrm{d}t}{t} = 0.$$

Consider the sequence of circulant graphs $C_{\beta n}^{1,n,\gamma_1 n,...,\gamma_{d-1}n}$ when $n \to \infty$ with $z_{NF}(\beta; 1, \gamma_1, ..., \gamma_{d-1})$ denoting the corresponding tree entropy. In the following proposition we show that it is greater than the one of fixed generated circulant graphs.

Proposition 4.2. For all positive integers $\gamma_1, \ldots, \gamma_d$, there exists an integer $B \ge 2$ such that for all $\beta \ge B$,

$$z_{\mathsf{NF}}(\beta;1,\gamma_1,\ldots,\gamma_{d-1}) > z_{\mathsf{F}}(1,\gamma_1,\ldots,\gamma_d)$$

where $z_F(1, \gamma_1, \ldots, \gamma_d)$ is the tree entropy of the fixed generated circulant graph $C_n^{1,\gamma_1,\ldots,\gamma_d}$.

Proof. By letting $\beta \to \infty$ in the corollary, the sum over the Laplacian eigenvalues converges to a Riemann integral, so that

$$\lim_{\beta \to \infty} z_{\mathsf{NF}}(\beta; 1, \gamma_1, \dots, \gamma_{d-1}) = \int_0^\infty (e^{-t} - e^{-2(d+1)t} I_0(2t) I_0^{1, \gamma_1, \dots, \gamma_{d-1}}(2t, \dots, 2t)) \frac{dt}{t}$$

where $I_0^{1,\gamma_1,\ldots,\gamma_{d-1}}$ is the d-dimensional modified I-Bessel function of order zero defined by (see section 2.4 in [6])

$$I_0^{1,\gamma_1,...,\gamma_{d-1}}(2t,...,2t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{2t(\cos w + \sum_{m=1}^{d-1} \cos(\gamma_m w))} dw.$$

It can be expressed in terms of a series of modified I-Bessel functions

$$I_0^{l,\gamma_1,\ldots,\gamma_{d-l}}(2t,\ldots,2t) = \sum_{(k_1,\ldots,k_{d-l})\in\mathbb{Z}^{d-l}} I_{\sum_{i=1}^{d-l}\gamma_i k_i}(2t) \prod_{i=1}^{d-l} I_{k_i}(2t).$$

On the other hand, from [6, Theorem 1.1], the tree entropy of the fixed generated circulant graph $C_n^{1,\gamma_1,\ldots,\gamma_d}$ as $n \to \infty$ is given by

$$z_{\mathsf{F}}(1,\gamma_{1},\ldots,\gamma_{d}) = \int_{0}^{\infty} (e^{-t} - e^{-2(d+1)t} I_{0}^{1,\gamma_{1},\ldots,\gamma_{d}}(2t,\ldots,2t)) \frac{dt}{t}$$

where

$$\begin{split} I_0^{l,\gamma_1,\dots,\gamma_d}(2t,\dots,2t) &= \sum_{(k_1,\dots,k_d)\in\mathbb{Z}^d} I_{\sum_{i=1}^d\gamma_i k_i}(2t) \prod_{i=1}^d I_{k_i}(2t) \\ &> I_0(2t) \sum_{(k_1,\dots,k_{d-1})\in\mathbb{Z}^{d-1}} I_{\sum_{i=1}^{d-1}\gamma_i k_i}(2t) \prod_{i=1}^{d-1} I_{k_i}(2t) \\ &= I_0(2t) I_0^{l,\gamma_1,\dots,\gamma_{d-1}}(2t,\dots,2t), \quad \forall t > 0. \end{split}$$

Therefore

$$\lim_{\beta \to \infty} z_{\mathsf{NF}}(\beta; 1, \gamma_1, \dots, \gamma_{d-1}) > z_{\mathsf{F}}(1, \gamma_1, \dots, \gamma_d).$$

Related to this comparison between fixed and non-fixed generated circulant graphs one might wonder, for example in the simplest case of $C^{1,n}_{\beta n}$, how taking limits first in β then in n would compare to taking limits first in n then in β . From [5, Lemma 5] and by letting $\beta \to \infty$ in [5, Theorem 4], one easily sees that for all positive integers $\gamma_1, \ldots, \gamma_{d-1}$,

$$\lim_{\gamma_d \to \infty} \lim_{n \to \infty} \frac{\log \tau(C_n^{\gamma_1, \dots, \gamma_d})}{n} = \lim_{\beta \to \infty} \lim_{n \to \infty} \frac{\log \tau(C_{\beta n}^{1, \gamma_1 n, \dots, \gamma_{d-1} n})}{\beta n}$$

which by definition is

$$\lim_{\gamma_d\to\infty} z_{\mathsf{F}}(\gamma_1,\ldots,\gamma_d) = \lim_{\beta\to\infty} z_{\mathsf{N}\mathsf{F}}(\beta;\gamma_1,\ldots,\gamma_{d-1}).$$

In the particular case of d = 2 it shows that the limits over n and β commute, that is,

$$\lim_{\beta \to \infty} \lim_{n \to \infty} \frac{\log \tau(C_{\beta n}^{l,n})}{\beta n} = \lim_{n \to \infty} \lim_{\beta \to \infty} \frac{\log \tau(C_{\beta n}^{l,n})}{\beta n}$$

which does not seem obvious a priori.

References

- [1] Norman Biggs. Algebraic graph theory. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 1993.
- [2] F. T. Boesch and H. Prodinger. Spanning tree formulas and Chebyshev polynomials. Graphs Combin., 2(3):191–200, 1986.
- [3] Mordecai J. Golin and Yiu Cho Leung. Unhooking circulant graphs: a combinatorial method for counting spanning trees and other parameters. In *Graph-theoretic concepts in computer science*, volume 3353 of *Lecture Notes in Comput. Sci.*, pages 296–307. Springer, Berlin, 2004.
- [4] Mordecai J. Golin, Yiu Cho Leung, and Yajun Wang. Counting spanning trees and other structures in non-constant-jump circulant graphs. In *Algorithms and computation*, volume 3341 of *Lecture Notes in Comput. Sci.*, pages 508–521. Springer, Berlin, 2004.
- [5] Mordecai J. Golin, Xuerong Yong, and Yuanping Zhang. The asymptotic number of spanning trees in circulant graphs. *Discrete Math.*, 310(4):792–803, 2010.

- [6] Justine Louis. Asymptotics for the Number of Spanning Trees in Circulant Graphs and Degenerating d-Dimensional Discrete Tori. *Ann. Comb.*, 19(3):513–543, 2015.
- [7] Russell Lyons. Asymptotic enumeration of spanning trees. Combin. Probab. Comput., 14(4):491– 522, 2005.
- [8] Yuanping Zhang, Xuerong Yong, and Mordecai J. Golin. Chebyshev polynomials and spanning tree formulas for circulant and related graphs. *Discrete Math.*, 298(1-3):334–364, 2005.

Spanning trees in directed circulant graphs and cycle power graphs*

Justine Louis

10th July 2015

Abstract

The number of spanning trees in a class of directed circulant graphs with generators depending linearly on the number of vertices βn , and in the n-th and (n-1)-th power graphs of the βn -cycle are evaluated as a product of $\lceil \beta/2 \rceil - 1$ terms.

1 Introduction

In this paper we study the number of spanning trees in a class of directed and undirected circulant graphs. Let $1 \leq \gamma_1 \leq \cdots \leq \gamma_d \leq \lfloor n/2 \rfloor$ be positive integers. A circulant directed graph, or circulant digraph, on n vertices generated by $\gamma_1, \ldots, \gamma_d$ is the directed graph on n vertices labelled $0, 1, \ldots, n-1$ such that for each vertex $v \in \mathbb{Z}/n\mathbb{Z}$ there is an oriented edge connecting v to $v + \gamma_m$ mod n for all $m \in \{1, \ldots, d\}$. We will denote such graphs by $\overrightarrow{C}_n^{\gamma_1, \ldots, \gamma_d}$. Similarly, a circulant graph on n vertices generated by $\gamma_1, \ldots, \gamma_d$, denoted by $C_n^{\gamma_1, \ldots, \gamma_d}$. Similarly, a circulant graph on n vertices generated by $\gamma_1, \ldots, \gamma_d$, denoted by $C_n^{\gamma_1, \ldots, \gamma_d}$, is the undirected graph on n vertices labelled $0, 1, \ldots, n-1$ such that each vertex $v \in \mathbb{Z}/n\mathbb{Z}$ is connected to $v \pm \gamma_m$ mod n for all $m \in \{1, \ldots, d\}$. Circulant graphs and digraphs are used as models in network theory. In this context, they are called multi-loop networks, or double-loop networks when they are 2-generated, see for example [7, 8]. The number of spanning tree measures the reliability of a network.

The evaluation of the number of spanning trees in circulant graphs and digraphs has been widely studied, were both exact and asymptotic results have been obtained as the number of vertices grows, see [2, 6, 11, 12, 13] and references therein. In [3, 5], the authors showed that the number of spanning trees in such graphs satisfy linear recurrence relations. Yong, Zhang and Golin developed a technique in [13] to evaluate the number of spanning trees in a particular class of double-loop networks $\vec{C}_{\beta n}^{p,\gamma n+p}$. In the first section of this work, we derive a closed formula for these graphs, and more generally for d-generated circulant digraphs with generators depending linearly on the number of vertices, that is $\vec{C}_{\beta n}^{p,\gamma_1 n+p}$, where $p, \gamma_1, \ldots, \gamma_{d-1}, \beta, n$ are positive integers. This partially answers an open question posed in [2] by simplifying the formula given in [2, Corollary 1].

In the second section we calculate the number of spanning trees in the n-th and (n-1)-th power graphs of the β n-cycle which are the circulant graphs generated by the n, respectively n-1, first consecutive integers, denoted by $C_{\beta n}^n$ and $C_{\beta n}^{n-1}$ respectively, where $\beta \in \mathbb{N}_{\geq 2}$. As a consequence, the asymptotic behaviour of it is derived. Cycle power graphs appear, for example, in graph colouring problems, see [9, 10].

The results obtained here are derived from the matrix tree theorem (see [1, 4]) which provides a

^{*}arXiv:1507.02990. The author was supported in part by the Swiss NSF grant 200021 132528/1.

closed formula of a product of $\beta n - 1$ terms for a graph on βn vertices. Our formulas are a product of $\lceil \beta/2 \rceil - 1$ terms and are therefore interesting when n is large. In both cases, the symmetry of the graphs is reflected in the formulas which are expressed in terms of eigenvalues of subgraphs of the original graph. This fact was already observed in [12].

Acknowledgements: The author thanks Anders Karlsson for reading the manuscript and useful discussions.

2 Spanning trees in directed circulant graphs

Let G be a directed graph and V(G) its vertex set. A spanning arborescence converging to $v \in V(G)$ is an oriented subgraph of G such that the out-degree of all vertices except v equals one, and the out-degree of v is zero. We define the combinatorial Laplacian of a directed graph G as an operator acting on the space of functions defined on V(G), by

$$\Delta_{G}^{-}f(x) = \sum_{y: x \to y} (f(x) - f(y))$$
(1)

where the sum is over all vertices y such that there is an oriented edge from x to y. Equivalently, the combinatorial Laplacian can be defined as a matrix by $\Delta_G^- = D^- - A$, where D^- is the outdegree matrix and A is the adjacency matrix such that $(A)_{ij}$ is the number of directed edges from i to j. Let $\tau^-(G, \nu)$ denote the number of arborescences converging to ν . The Tutte matrix tree theorem (see [1]) states that for all $\nu \in V(G)$,

$$\tau^{-}(G, v) = \det \Delta_{G, v}^{-}$$

where det $\Delta_{G,\nu}^-$ is the v-th cofactor of the Laplacian Δ_G^- obtained by deleting the row and column of Δ_G^- corresponding to the vertex v. For a regular directed graph G, we define the number of spanning trees in G, $\tau(G)$, by the sum over all vertices $\nu \in V(G)$ of the number of arborescences converging to v, that is

$$\tau(G) = \sum_{\nu \in V(G)} \tau^{-}(G, \nu).$$

Notice that we could have defined the number of spanning trees by the sum over all vertices $v \in V(G)$ of the number of spanning arborescences diverging from v.

By symmetry, all cofactors of the Laplacian of a directed circulant graph are equal and are equal to the product of the non-zero eigenvalues of the Laplacian divided by the number of vertices. Therefore we have that

$$\tau(G) = \prod_{k=1}^{|V(G)|} \lambda_k$$

where λ_k , k = 1, ..., |V(G)|, denote the non-zero eigenvalues of the Laplacian of G. The non-zero eigenvalues of the Laplacian of the directed circulant graph $\overrightarrow{C}_n^{\gamma_1,...,\gamma_d}$ are given by (see [4, Proposition 3.5])

$$\lambda_k = d - \sum_{m=1}^d e^{2\pi i \gamma_m k/n}, \quad k = 1, \dots, n-1.$$

This can also be derived by noticing that the eigenvectors are given by the characters $\chi_k(x) = e^{2\pi i k x/n}$, $k = 0, 1, \dots, n-1$, and then applying the Laplacian (1) on it.

In this section, we establish a formula for the number of spanning trees in directed circulant graphs

 $\overrightarrow{C}_{\beta n}^{\Gamma}$ generated by $\Gamma = \{p, \gamma_1 n + p, \dots, \gamma_{d-1} n + p\}$ and in the particular case of two generators $\overrightarrow{C}_{\beta n}^{p,\gamma n+p}$. Figure 1 illustrates a 2 and a 3 generated directed circulant graphs. We denote by $\mu_k = d - 1 - \sum_{m=1}^{d-1} e^{2\pi i \gamma_m k/\beta}$, $k = 1, \dots, \beta - 1$, the non-zero eigenvalues of the Laplacian on the directed circulant graph $\overrightarrow{C}_{\beta}^{\gamma_1,\dots,\gamma_{d-1}}$ and by $\eta_k = 2(d-1) - 2\sum_{m=1}^{d-1} \cos(2\pi \gamma_m k/\beta)$, $k = 1,\dots,\beta-1$, the non-zero eigenvalues of the Laplacian on the circulant graph $C_{\beta}^{\gamma_1,\dots,\gamma_{d-1}}$. Let A be a statement and δ_A be defined by

$$\delta_A = \begin{cases} 1 & \text{if } A \text{ is satisfied} \\ 0 & \text{otherwise} \end{cases}$$



Figure 1: Examples of directed graphs

Theorem 2.1. Let $1 \leq \gamma_1 \leq \cdots \leq \gamma_{d-1} \leq \beta$ and p, n be positive integers. For all even $n \in \mathbb{N}_{\geq 2}$ such that (p, n) = 1, the number of spanning trees in the directed circulant graph $\overrightarrow{C}_{\beta n}^{\Gamma}$, where $\Gamma = \{p, \gamma_1 n + p, \dots, \gamma_{d-1} n + p\}$, is given by

$$\begin{split} \tau(\overrightarrow{C}_{\beta n}^{\Gamma}) &= n d^{\beta n - 1} \Big(1 - \delta_{\beta \text{ even}} \frac{(-1)^{p}}{d^{n}} \Big(1 + \sum_{m=1}^{d-1} (-1)^{\gamma_{m}} \Big)^{n} \Big) \\ &\times \prod_{k=1}^{\lceil \beta/2 \rceil - 1} \left(1 - 2 \Big| 1 - \frac{\mu_{k}}{d} \Big|^{n} \cos\left(\frac{2\pi p k}{\beta} + n \operatorname{Arctg}\left(\frac{\sum_{m=1}^{d-1} \sin(2\pi \gamma_{m} k/\beta)}{d - \eta_{k}/2}\right) \right) \\ &+ \left| 1 - \frac{\mu_{k}}{d} \right|^{2n} \Big) \end{split}$$

and for odd $n \in \mathbb{N}_{\geq 1}$,

$$\begin{split} \tau(\overrightarrow{C}_{\beta n}^{\Gamma}) &= n d^{\beta n - 1} \Big(1 - \delta_{\beta \text{ even}} \frac{(-1)^p}{d^n} \Big(1 + \sum_{m=1}^{d-1} (-1)^{\gamma_m} \Big)^n \Big) \\ \times \prod_{k=1}^{\lceil \beta/2 \rceil - 1} \left(1 - 2 \text{sgn}(d - \eta_k/2) \Big| 1 - \frac{\mu_k}{d} \Big|^n \cos\left(\frac{2\pi pk}{\beta} + n \operatorname{Arctg}\left(\frac{\sum_{m=1}^{d-1} \sin(2\pi \gamma_m k/\beta)}{d - \eta_k/2}\right)\right) \right) \\ &+ \left| 1 - \frac{\mu_k}{d} \right|^{2n} \Big) \end{split}$$

where $\lceil x \rceil$ is the smallest integer greater or equal to x, |.| denotes the modulus and we set gn(0) = 1. The number of spanning trees in $\overrightarrow{C}_{\beta n}^{\Gamma}$ is zero if either (p, n) = 1 and β , p, γ_m , $m = 1, \ldots, d - 1$, are all even or either $(p, n) \neq 1$.

Proof. From the Tutte matrix tree theorem, the number of spanning trees in $\overrightarrow{C}_{\beta n}^{\Gamma}$ is given by

$$\tau(\overrightarrow{C}_{\beta\mathfrak{n}}^{\Gamma}) = \prod_{k=1}^{\beta\mathfrak{n}-1} (d - e^{2\pi i\mathfrak{p}k/(\beta\mathfrak{n})} - \sum_{m=1}^{d-1} e^{2\pi i(\gamma_m\mathfrak{n}+\mathfrak{p})k/(\beta\mathfrak{n})}).$$

By splitting the product over $k = 1, ..., \beta n - 1$ into two products, when k is a multiple of β , that is $k = l\beta$ with l = 1, ..., n - 1, and over non-multiples of β , that is, $k = k' + l'\beta$ with $k' = 1, ..., \beta - 1$ and l' = 0, 1, ..., n - 1, we have

$$\tau(\vec{C}_{\beta n}^{\Gamma}) = \prod_{l=1}^{n-1} (d - de^{2\pi i p l/n}) \prod_{k=1}^{\beta-1} \prod_{l'=0}^{n-1} (d - (1 + \sum_{m=1}^{d-1} e^{2\pi i \gamma_m k/\beta}) e^{2\pi i p k/(\beta n)} e^{2\pi i p l'/n}).$$
(2)

We have that

$$\prod_{l=1}^{n-1} (d - de^{2\pi i p l/n}) = d^{n-1} \prod_{l=1}^{n-1} (1 - e^{2\pi i p l/n}) = n d^{n-1} \delta_{(p,n)=1}.$$

This equality comes from the fact that $\prod_{l=1}^{n-1}(1-e^{2\pi ipl/n})$ is the number of spanning trees of the directed graph \overrightarrow{C}_n^p , which is isomorphic to the directed cycle on n vertices if (p,n) = 1, and is not connected if $(p,n) \neq 1$. Therefore the product is equal to $n\delta_{(p,n)=1}$. Hence, if $(p,n) \neq 1$, we have

$$\tau(\overrightarrow{C}_{\beta n}^{\Gamma}) = 0.$$

Let p be relatively prime to n. Using that the complex numbers $e^{2\pi i l/n}$, l = 0, 1, ..., n-1, are the n non-trivial roots of unity, we have for all x,

$$\prod_{l=0}^{n-1} (x - e^{2\pi i l p/n}) = x^n - 1.$$
(3)

since (p, n) = 1. Equivalently we have,

$$\prod_{l=0}^{n-1} (1 - x e^{2\pi i l p/n}) = 1 - x^n.$$

Using this identity in (2) enables to evaluate the product over l', it comes

$$\tau(\overrightarrow{C}_{\beta n}^{\Gamma}) = n d^{\beta n-1} \prod_{k=1}^{\beta-1} (1 - \frac{1}{d^n} (1 + \sum_{m=1}^{d-1} e^{2\pi i \gamma_m k/\beta})^n e^{2\pi i p k/\beta}).$$
(4)

For odd β we write the product over k, $k = 1, ..., \beta - 1$, as a product from 1 to $(\beta - 1)/2$, and for even β we write it as a product from 1 to $\beta/2 - 1$ and add the $k = \beta/2$ factor which is given by $1 - (-1)^p (1 + \sum_{m=1}^{d-1} (-1)^{\gamma_m})^n / d^n$. Writing the above expression in terms of

 $\mu_k = d - 1 - \sum_{m=1}^{d-1} e^{2\pi i \gamma_m k/\beta},$ it comes

$$\begin{aligned} \tau(\overrightarrow{C}_{\beta n}^{\Gamma}) &= n d^{\beta n - 1} \Big(1 - \delta_{\beta \ \text{even}} \frac{(-1)^{p}}{d^{n}} (1 + \sum_{m=1}^{d-1} (-1)^{\gamma_{m}})^{n} \Big) \\ &\times \prod_{k=1}^{\lceil \beta/2 \rceil - 1} (1 - (1 - \mu_{k}/d)^{n} e^{2\pi i p k/\beta}) (1 - (1 - \mu_{k}^{*}/d)^{n} e^{-2\pi i p k/\beta}) \\ &= n d^{\beta n - 1} \Big(1 - \delta_{\beta \ \text{even}} \frac{(-1)^{p}}{d^{n}} (1 + \sum_{m=1}^{d-1} (-1)^{\gamma_{m}})^{n} \Big) \\ &\times \prod_{k=1}^{\lceil \beta/2 \rceil - 1} (1 - 2|1 - \mu_{k}/d|^{n} \cos(2\pi p k/\beta + n \varphi_{k}) + |1 - \mu_{k}/d|^{2n}) \end{aligned}$$
(5)

where ϕ_k is the phase of the complex number $1 - \mu_k/d$ such that $1 - \mu_k/d = |1 - \mu_k/d| e^{i \phi_k}$. We have

$$|1 - \mu_k/d| = \frac{1}{d} \Big((d - \eta_k/2)^2 + \Big(\sum_{m=1}^{d-1} \sin(2\pi\gamma_m k/\beta)\Big)^2 \Big)^{1/2}$$

and

$$\cos\varphi_k = \frac{d - \eta_k/2}{|d - \mu_k|}, \quad \sin\varphi_k = \frac{\sum_{m=1}^{d-1} \sin(2\pi\gamma_m k/\beta)}{|d - \mu_k|}$$

Therefore for k such that $d-\eta_k/2\neq 0,$ the phase is given by

$$\phi_{k} = \operatorname{Arctg}\left(\frac{\sum_{m=1}^{d-1} \sin(2\pi\gamma_{m}k/\beta)}{d - \eta_{k}/2}\right) + \epsilon\pi$$
(6)

where $\varepsilon=0$ if $\text{sgn}(d-\eta_k/2)=1$ and $\varepsilon\in\{-1,1\}$ if $\text{sgn}(d-\eta_k/2)=-1$. For k such that $d-\eta_k/2=0$, we take the limit as $d-\eta_k/2\to 0$ in (6), with $\varepsilon=0$. The theorem follows by putting equation (6) into equation (5).

When β , p and γ_m , m = 1, ..., d-1 are all even, the directed circulant graph $\overrightarrow{C}_{\beta n}^{\Gamma}$ is not connected and therefore the number of spanning trees is zero, this is reflected in the formula. \Box

In the following theorem we state the particular case on two-generated directed circulant graphs.

Theorem 2.2. Let $1 \leq \gamma \leq \beta$ and p, n be positive integers. For odd β and all $n \in \mathbb{N}_{\geq 1}$ such that (p, n) = 1, the number of spanning trees in the directed circulant graph $\overrightarrow{C}_{\beta n}^{p,\gamma n+p}$ is given by

$$\tau(\overrightarrow{C}_{\beta n}^{p,\gamma n+p}) = n2^{\beta n-1} \prod_{k=1}^{(\beta-1)/2} \left(1 - 2\cos(2\pi(p+\gamma n/2)k/\beta)\cos^n(\pi\gamma k/\beta) + \cos^{2n}(\pi\gamma k/\beta)\right)$$

and for even β , if γ or p is odd, then

$$\tau(\overrightarrow{C}_{\beta n}^{p,\gamma n+p}) = n2^{\beta n-1+\delta_{\gamma \text{ even }}} \prod_{k=1}^{\beta/2-1} \Big(1 - 2\cos(2\pi(p+\gamma n/2)k/\beta)\cos^{n}(\pi\gamma k/\beta) + \cos^{2n}(\pi\gamma k/\beta)\Big).$$

The number of spanning trees in $\overrightarrow{C}_{\beta n}^{p,\gamma n+p}$ is zero if either (p,n) = 1 and β , p and γ are all even or either $(p,n) \neq 1$.

Proof. From equation (4) it follows

$$\tau(\overrightarrow{C}_{\beta\mathfrak{n}}^{p,\gamma\mathfrak{n}+p}) = \mathfrak{n}2^{\beta\mathfrak{n}-1}\prod_{k=1}^{\beta-1}(1-e^{2\pi\mathfrak{i}(p+\gamma\mathfrak{n}/2)k/\beta}\cos^{\mathfrak{n}}(\pi\gamma k/\beta)).$$

For odd β , we have

$$\begin{split} \tau(\overrightarrow{C}_{\beta n}^{p,\gamma n+p}) &= n 2^{\beta n-1} \prod_{k=1}^{(\beta-1)/2} (1 - e^{2\pi i (p+\gamma n/2)k/\beta} \cos^n(\pi \gamma k/\beta)) \\ &\qquad \times (1 - e^{-2\pi i (p+\gamma n/2)k/\beta} \cos^n(\pi \gamma k/\beta)) \\ &= n 2^{\beta n-1} \prod_{k=1}^{(\beta-1)/2} (1 - 2\cos(2\pi (p+\gamma n/2)k/\beta) \cos^n(\pi \gamma k/\beta) + \cos^{2n}(\pi \gamma k/\beta)). \end{split}$$

For even β , the factor $k = \beta/2$ is added:

$$1 - e^{\pi i(p + \gamma n/2)} \cos^n(\pi \gamma/2) = \begin{cases} 0 & \text{if } p \text{ and } \gamma \text{ are even} \\ 1 & \text{if } \gamma \text{ is odd} \\ 2 & \text{otherwise} \end{cases}$$

•

For even β , p and γ , the graph $\overrightarrow{C}_{\beta n}^{p,\gamma n+p}$ is not connected and therefore the number of spanning trees is zero. Therefore if p or γ is odd, we have

$$\begin{aligned} \tau(\overrightarrow{C}_{\beta n}^{p,\gamma n+p}) &= n2^{\beta n-1+\delta_{\gamma \text{ even }}} \prod_{k=1}^{\beta/2-1} (1-e^{2\pi i(p+\gamma n/2)k/\beta}\cos^{n}(\pi\gamma k/\beta)) \\ &\qquad \times (1-e^{-2\pi i(p+\gamma n/2)k/\beta}\cos^{n}(\pi\gamma k/\beta)) \\ &= n2^{\beta n-1+\delta_{\gamma \text{ even }}} \prod_{k=1}^{\beta/2-1} (1-2\cos(2\pi (p+\gamma n/2)k/\beta)\cos^{n}(\pi\gamma k/\beta) + \cos^{2n}(\pi\gamma k/\beta)). \end{aligned}$$

Examples. Consider the case when $p = \beta = 3$ and $\gamma = 2$. It follows from Theorem 2.2 that $\tau(\overrightarrow{C}_{3n}^{3,2n+3}) = 0$ if n is a multiple of 3, otherwise,

$$\tau(\overrightarrow{C}_{3n}^{3,2n+3}) = n2^{3n-1}(1 - 2\cos(2\pi n/3)\cos^{n}(2\pi/3) + \cos^{2n}(2\pi/3))$$
$$= n(2^{3n-1} - 2^{2n}\cos(\pi n/3) + 2^{n-1})$$

as stated in [13, Example 4.(iii)]. As another example, consider the case when p = 2, $\gamma = 5$ and $\beta = 6$. From Theorem 2.2, for even n, $\tau(\overrightarrow{C}_{6n}^{2,5n+2}) = 0$, and for odd n,

$$\begin{split} \tau(\overrightarrow{C}_{6n}^{2,5n+2}) &= n2^{6n-1}(1-2\cos(2\pi(2+5n/2)/6)\cos^n(5\pi/6) + \cos^{2n}(5\pi/6)) \\ &\times (1-2\cos(4\pi(2+5n/2)/6)\cos^n(10\pi/6) + \cos^{2n}(10\pi/6)) \\ &= \frac{n}{2}(2^{3n}+2^{2n}3^{n/2}\cos(\pi n/6) - 2^{2n}3^{(n+1)/2}\sin(\pi n/6) + 6^n) \\ &\times (2^{3n}-2^{2n-1}3^{n/2}\cos(\pi n/3) + 2^{n-1}3^{(n+1)/2}\sin(\pi n/3) + 2^n). \end{split}$$

3 Spanning trees in cycle power graphs

The k-th power graph of the n-cycle, denoted by \mathbf{C}_n^k , is the graph with the same vertex set as the n-cycle where two vertices are connected if their distance on the n-cycle is at most k. It is therefore the circulant graph on n vertices generated by the first k consecutive integers. In this section, we derive a formula for the number of spanning trees in the n-th and (n-1)-th power graphs of the β n-cycle, where $\beta \in \mathbb{N}_{\geq 2}$. As a consequence we derive the asymptotic behaviour of it as n goes to infinity.

The combinatorial Laplacian of an undirected graph G with vertex set V(G) defined as an operator acting on the space of functions is

$$\Delta_G f(x) = \sum_{y \sim x} (f(x) - f(y))$$

where the sum is over all vertices adjacent to x. The matrix tree theorem [4] states that the number of spanning trees in G, $\tau(G)$, is given by

$$\tau(G) = \frac{\prod_{k=1}^{|V(G)|-1} \lambda_k}{|V(G)|}$$

where λ_k , $k = 1, \ldots, |V(G)| - 1$, are the non-zero eigenvalues of Δ_G . The eigenvectors of the Laplacian on the circulant graph $C^{1,\ldots,n}_{\beta n}$ are given by the characters $\chi_k(x) = e^{2\pi i k x/(\beta n)}$, $k = 0, 1, \ldots, \beta n - 1$. Therefore the non-zero eigenvalues are given by

$$\lambda_k = 2n - 2\sum_{m=1}^n \cos(2\pi km/(\beta n)), \quad k = 1, \dots, \beta n - 1.$$

Similarly, the non-zero eigenvalues on $C^{1,\ldots,n-1}_{\beta\,n}$ are given by

$$\lambda_k = 2(n-1) - 2\sum_{m=1}^{n-1} \cos(2\pi km/(\beta n)), \quad k = 1, \dots, \beta n - 1.$$

Figure 2 below illustrates two power graphs of the 24-cycle.



Figure 2: 8-th and 7-th power graphs of the 24-cycle

Theorem 3.1. Let $\beta \ge 2$ be an integer and $\mu_k = 2 - 2\cos(2\pi k/\beta)$, $k = 1, \dots, \beta - 1$, be the non-zero eigenvalues of the Laplacian on the β -cycle. The number of spanning trees in the n-th power graph of the β n-cycle $\mathbf{C}_{\beta n}^n$ for $\beta \ge 3$, is given by

$$\begin{split} \tau(\mathbf{C}_{\beta\,n}^n) &= \frac{2^{\beta\,(n+1)}}{(2\beta)^2} n^{\beta\,n-2} \left(1 + \frac{1}{2n}\right)^{\beta\,n} (1 - (2n+1)^{-\beta})^n \\ &\times \prod_{k=1}^{\lceil\beta/2\rceil - 1} \sin^2\left(\frac{\pi(n+1)k}{\beta} - n\operatorname{Arcsin}\left(\frac{n+1}{\sqrt{4n^2/\mu_k + 2n + 1}}\right)\right) \end{split}$$

where $\lceil x \rceil$ denotes the smallest integer greater or equal to x. For $\beta = 2$, it is given by

$$\tau(\mathbf{C}_{2n}^n) = (2n)^{2n-2}(1+1/n)^n.$$

The number of spanning trees in the (n-1)-th power graph of the β n-cycle $C_{\beta n}^{n-1}$, for $\beta \ge 3$, is given by

$$\begin{aligned} \tau(\mathbf{C}_{\beta n}^{n-1}) &= \frac{2^{\beta (n+1)}}{(2\beta)^2} n^{\beta n-2} \left(1 - \frac{1}{2n} \right)^{\beta n} |(-1)^{\beta} - (2n-1)^{-\beta}|^n \\ &\times \prod_{k=1}^{\lceil \beta/2 \rceil - 1} \sin^2 \left(\frac{\pi (n-1)k}{\beta} - n \operatorname{Arcsin} \left(\frac{n-1}{\sqrt{4n^2/\mu_k - (2n-1)}} \right) \right). \end{aligned}$$

For $\beta = 2$, it is given by

$$\tau(\mathbf{C}_{2n}^{n-1}) = (2n)^{2n-2}(1-1/n)^n.$$

Remark. We emphasise that in the cycle power graphs $C_{\beta n}^{n-1}$ and $C_{\beta n}^{n}$ there are β copies of n-cliques as subgraphs of the original graph. This fact appears in the formula by the factor $n^{\beta n-2} = (n^{n-2})^{\beta} n^{2(\beta-1)}$ since the number of spanning trees in the complete graph on n vertices is n^{n-2} .

Proof. We prove the theorem only for the first type of graphs $C_{\beta n}^n$. The proof of the second type $C_{\beta n}^{n-1}$ is very similar to the first one. The matrix tree theorem states that

$$\tau(\mathbf{C}_{\beta n}^{n}) = \frac{1}{\beta n} \prod_{k=1}^{\beta n-1} (2n - 2\sum_{m=1}^{n} \cos(2\pi km/(\beta n))).$$

Lagrange's trigonometric identity expresses the sum of cosines appearing in the above formula in terms of a quotient of sines:

$$2\sum_{m=1}^{n}\cos(2\pi km/(\beta n)) = \frac{\sin((n+1/2)2\pi k/(\beta n))}{\sin(\pi k/(\beta n))} - 1.$$

Hence,

$$\tau(\mathbf{C}_{\beta n}^{n}) = \frac{1}{\beta n} \prod_{k=1}^{\beta n-1} (\sin(\pi k/(\beta n)))^{-1} ((2n+1)\sin(\pi k/(\beta n)) - \sin(\pi k/(\beta n) + 2\pi k/\beta))$$

Using that there are βn spanning trees in the βn -cycle, that is $\frac{1}{\beta n} \prod_{k=1}^{\beta n-1} (2-2\cos(2\pi k/(\beta n))) = \beta n$, it follows that

$$\prod_{k=1}^{\beta n-1} \sin(\pi k/(\beta n)) = \frac{\beta n}{2^{\beta n-1}}.$$
(7)

For the second factor, as in the proof of Theorem 2.1, we split the product over $k = 1, ..., \beta n - 1$ into two products, first when k is a multiple of β , that is $k = l\beta$ with l = 1, ..., n - 1, and second when k is not a multiple of β , that is, $k = k' + l'\beta$ with $k' = 1, ..., \beta - 1$ and l' = 0, 1, ..., n - 1. The product over the multiples of β reduces to

$$\prod_{l=1}^{n-1} 2n\sin(\pi l/n) = n^n.$$

We have

$$\tau(\mathbf{C}_{\beta n}^{n}) = \frac{2^{\beta n - 1} n^{n}}{(\beta n)^{2}} \prod_{k=1}^{\beta - 1} \prod_{l=0}^{n-1} ((2n+1)\sin(\pi k/(\beta n) + \pi l/n) - \sin(\pi k/(\beta n) + \pi l/n + 2\pi k/\beta).$$
(8)

The difference of sines in the above product can be written as

 $(2n+1)\sin(\pi k/(\beta n) + \pi l/n) - \sin(\pi k/(\beta n) + \pi l/n + 2\pi k/\beta) = |z_k|\sin(\pi(n+1)k/(\beta n) + \theta_k + \pi l/n)$ (9) where

$$z_{k} = 2n\cos(\pi k/\beta) - i(2n+2)\sin(\pi k/\beta) \Longrightarrow |z_{k}|e^{i\theta_{k}}.$$

Let $\omega_k = \pi(n+1)k/(\beta n) + \theta_k$, we have

$$\begin{split} \prod_{l=0}^{n-1} \sin(\omega_{k} + \pi l/n) &= \frac{1}{(2i)^{n}} \prod_{l=0}^{n-1} (e^{i(\omega_{k} + \pi l/n)} - e^{-i(\omega_{k} + \pi l/n)}) \\ &= \frac{1}{(2i)^{n}} e^{-i\omega_{k}n} e^{\pi i(n-1)/2} \prod_{l=0}^{n-1} (e^{2i\omega_{k}} - e^{-2\pi i l/n}) \\ &= \frac{\sin(\omega_{k}n)}{2^{n-1}} \end{split}$$
(10)

where in the last equality we used equation (3). Putting equations (8), (9) and (10) together yields

$$\tau(\mathbf{C}_{\beta n}^{n}) = \frac{2^{\beta n-1}n^{n}}{(\beta n)^{2}} \prod_{k=1}^{\beta-1} \frac{|z_{k}|^{n}}{2^{n-1}} \sin(\pi(n+1)k/\beta + n\theta_{k}).$$

Notice that for even β , the phase of $z_{\beta/2}$ is $\theta_{\beta/2} = -\pi/2$, so that $\sin(\pi(n+1)/2 + n\theta_{\beta/2}) = 1$. For $\beta = 2$, $z_1 = -2(n+1)i$, hence

$$\tau(\mathbf{C}_{2n}^n) = (2n)^{2n-2}(1+1/n)^n.$$

For $\beta \ge 3$, we have

$$\tau(\mathbf{C}_{\beta n}^{n}) = \frac{2^{n+\beta-2}n^{n}}{(\beta n)^{2}} \left(\prod_{k=1}^{\beta-1} |z_{k}|^{n}\right) \prod_{k=1}^{\lceil \beta/2\rceil-1} \sin(\pi(n+1)k/\beta + n\theta_{k}) \sin(\pi(n+1)(\beta-k)/\beta + n\theta_{\beta-k}).$$

For $1 \leq k \leq \lceil \beta/2 \rceil - 1$, the phase of z_k is $\theta_k = -\operatorname{Arcsin}((2n+2)\sin(\pi k/\beta)/|z_k|)$. The phase of $z_{\beta-k}$ satisfies

 $\cos\theta_{\beta-k}=-\cos\theta_k,\quad \sin\theta_{\beta-k}=\sin\theta_k$

so that, $\theta_{\beta-k} = \pi - \theta_k$. The modulus of z_k is given by

$$|z_k| = ((2n+1)^2 + 1 - 2(2n+1)\cos(2\pi k/\beta))^{1/2} = (4n^2 + (2n+1)\mu_k)^{1/2}$$

where $\mu_k = 2 - 2\cos(2\pi k/\beta)$, $k = 1, \dots, \beta - 1$, are the non-zero eigenvalues of the Laplacian on the β -cycle. We have $\sin(\pi k/\beta) = \mu_k^{1/2}/2$. Hence for $1 \le k \le \lceil \beta/2 \rceil - 1$, the phase is given by $\theta_k = -\operatorname{Arcsin}((n+1)/\sqrt{4n^2/\mu_k + 2n + 1})$. Therefore

$$\tau(\mathbf{C}_{\beta n}^{n}) = \frac{2^{n+\beta-2}n^{n}}{(\beta n)^{2}} \Big(\prod_{k=1}^{\beta-1} |z_{k}|^{n}\Big) \prod_{k=1}^{\lceil \beta/2 \rceil - 1} \sin^{2}\left(\frac{\pi(n+1)k}{\beta} - n\operatorname{Arcsin}\left(\frac{(n+1)}{\sqrt{4n^{2}/\mu_{k} + 2n + 1}}\right)\right).$$
(11)

The product of the modulus of z_k is given by

$$\begin{split} \prod_{k=1}^{\beta-1} &|z_k| = \frac{(2n+1)^{\beta/2}}{2n} \prod_{k=0}^{\beta-1} (2n+1+1/(2n+1)-2\cos(2\pi k/\beta))^{1/2} \\ &= \frac{(2n+1)^{\beta/2}}{2n} (2\cosh(\beta\operatorname{Argcosh}(n+1/2+1/(4n+2)))-2)^{1/2} \\ &= \frac{(2n+1)^{\beta}}{2n} (1-(2n+1)^{-\beta}) \end{split}$$
(12)

where the second equality comes from the identity (see [12, section 2])

$$\prod_{k=0}^{\beta-1}(2\cosh\theta-2\cos(2\pi k/n))=2\cosh(\beta\theta)-2.$$

Putting equality (12) into (11) gives the theorem.

Remark. We point out that the proof above could not be easily applied to other powers of the β n-cycle, like $C_{\beta n}^{n-p}$, where $p \ge 2$ or $p \le -1$, because in this case z_k defined in equation (9) would also depend on l and the phase θ_k of z_k cannot be easily determined. As a consequence, the product over l cannot be evaluated in the same way as it is done in the proof. It would be interesting to find a formula for this class of more general circulant graphs.

From Theorem 3.1, we derive the asymptotic behaviour of the number of spanning trees in the n-th, respectively (n-1)-th, power graph of the β n-cycle as $n \to \infty$.

Corollary 3.2. Let $\beta \in \mathbb{N}_{\geq 2}$. The asymptotic number of spanning trees in the n-th and (n-1)-th power graphs of the β n-cycle $C_{\beta n}^n$ and $C_{\beta n}^{n-1}$ as $n \to \infty$ is respectively given by

$$\tau(C_{\beta n}^{n}) = \frac{2^{\beta n}}{2\beta} n^{\beta n - 2} (e^{\beta/2} + o(1))$$

and

$$\tau(\mathbf{C}_{\beta n}^{n-1}) = \frac{2^{\beta n}}{2\beta} n^{\beta n-2} (e^{-\beta/2} + o(1)).$$

Proof. By observing that for all $k \in \{1, \ldots, \lceil \beta/2 \rceil - 1\}$,

$$\lim_{n\to\infty}\frac{n+1}{\sqrt{4n^2/\mu_k+2n+1}}=\sin(\pi k/\beta)\quad\text{and}\quad\lim_{n\to\infty}\frac{n-1}{\sqrt{4n^2/\mu_k-(2n-1)}}=\sin(\pi k/\beta)$$

where $\mu_k=2-2\cos(2\pi k/\beta)$ and using relation (7) the corollary is a direct consequence of Theorem 3.1. $\hfill\square$

References

- [1] Martin Aigner. A course in enumeration, volume 238 of Graduate Texts in Mathematics. Springer, Berlin, 2007.
- [2] Talip Atajan, Naohisa Otsuka, and Xuerong Yong. Counting the number of spanning trees in a class of double fixed-step loop networks. *Appl. Math. Lett.*, 23(3):291–298, 2010.
- [3] Talip Atajan, Xuerong Yong, and Hiroshi Inaba. Further analysis of the number of spanning trees in circulant graphs. *Discrete Math.*, 306(22):2817–2827, 2006.
- [4] Norman Biggs. Algebraic graph theory. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 1993.
- [5] Xiebin Chen. The number of spanning trees in directed circulant graphs with non-fixed jumps. Discrete Math., 307(15):1873–1880, 2007.
- [6] Mordecai J. Golin, Xuerong Yong, and Yuanping Zhang. The asymptotic number of spanning trees in circulant graphs. *Discrete Math.*, 310(4):792–803, 2010.
- [7] F. K. Hwang. A complementary survey on double-loop networks. *Theoret. Comput. Sci.*, 263(1-2):211–229, 2001. Combinatorics and computer science (Palaiseau, 1997).
- [8] F. K. Hwang. A survey on multi-loop networks. Theoret. Comput. Sci., 299(1-3):107-121, 2003.
- [9] Deming Li, Mingju Liu, and Yumei Peng. Hajós' conjecture and cycle power graphs. *European J. Combin.*, 31(3):759–764, 2010.
- [10] Ko-Wei Lih, Daphne Der-Fen Liu, and Xuding Zhu. Star extremal circulant graphs. SIAM J. Discrete Math., 12(4):491–499, 1999.
- [11] Justine Louis. Asymptotics for the Number of Spanning Trees in Circulant Graphs and Degenerating d-Dimensional Discrete Tori. Ann. Comb., 19(3):513–543, 2015.
- [12] Justine Louis. A formula for the number of spanning trees in circulant graphs with nonfixed generators and discrete tori. *Bulletin of the Australian Mathematical Society*, 92(3):365–373, 2015.
- [13] Xuerong Yong, Yuanping Zhang, and Mordecai J. Golin. The number of spanning trees in a class of double fixed-step loop networks. *Networks*, 52(2):69–77, 2008.
Asymptotics for the determinant of the combinatorial Laplacian on hypercubic lattices*

Justine Louis

30th July 2015

Abstract

In this paper, we compute asymptotics for the determinant of the combinatorial Laplacian on a sequence of d-dimensional orthotope square lattices as the number of vertices in each dimension grows at the same rate. It is related to the number of spanning trees by the well-known matrix tree theorem. Asymptotics for 2 and 3 component rooted spanning forests in these graphs are also derived. Moreover, we express the number of spanning trees in a 2-dimensional square lattice in terms of the one in a 2-dimensional discrete torus and also in the quartered Aztec diamond. As a consequence, we find an asymptotic expansion of the number of spanning trees in a subgraph of \mathbb{Z}^2 with a triangular boundary.

1 Introduction

In this paper we study the asymptotic behaviour of the number of spanning trees in a discrete d-dimensional orthotope square lattice and in the quartered Aztec diamond. Let $L(n_1, ..., n_d)$ denote the d-dimensional orthotope square lattice defined by the cartesian product of the d path graphs P_{n_i} , i = 1, ..., d, where n_i , i = 1, ..., d, are positive non-zero integers. We set $n_i = \alpha_i n$, i = 1, ..., d, and write indifferently n_i or $\alpha_i n$ throughout the paper. By rescaling the distance between two vertices on the lattice $L(n_1, ..., n_d)$ with a factor of 1/n, the limiting object as n goes to infinity is a d-dimensional orthotope of size $\alpha_1 \times \cdots \times \alpha_d$, that we denote by K_d :

$$\mathsf{K}_{\mathsf{d}} \coloneqq [0, \alpha_1[\times \cdots \times [0, \alpha_{\mathsf{d}}[.$$

The volume of K_d is

$$V_d^d := \prod_{i=1}^d \alpha_i.$$

Let $m \in \{1, ..., d-1\}$ and let S_d denote the symmetric group on $\{1, ..., d\}$. An m-dimensional face of \overline{K}_d is defined by

$$\begin{split} \{(x_1,\ldots,x_d)\in \overline{K}_d|\ \exists \{i_q\}_{q=1}^d\in S_d \text{ such that } x_{i_q}\in \left[0,\alpha_{i_q}\right],\ q=1,\ldots,m\\ \text{ and } x_{i_q}\in \{0,\alpha_{i_q}\},\ q=m+1,\ldots,d\}. \end{split}$$

The volume of the sum of all the m-dimensional faces of \overline{K}_d is given by

$$V_m^d := 2^{d-\mathfrak{m}} \sum_{1\leqslant \mathfrak{i}_1 < \cdots < \mathfrak{i}_m \leqslant d} \prod_{q=1}^m \alpha_{\mathfrak{i}_q}.$$

^{*}arXiv:1507.08652. The author was supported in part by the Swiss NSF grant 200021 132528/1.

For example, V_1^d is the perimeter and V_2^d the area of K_d .

Asymptotics for the determinant of the combinatorial Laplacian on graphs have been widely studied, see for example [2, 3, 5, 9, 11]. It is related to the number of spanning trees of a graph G, denoted by $\tau(G)$, through the matrix tree theorem due to Kirchhoff (see [1])

$$\tau(G) = \frac{1}{|V(G)|} \det^*\!\Delta_G$$

where det^{*} Δ_G is the product of the non-zero eigenvalues of the Laplacian on G and |V(G)| the number of vertices in G. In [2], the authors developed a technique to compute the asymptotic behaviour of spectral determinants of the combinatorial Laplacian associated to a sequence of discrete tori. The technique consists in studying the asymptotic behaviour of the associated theta function which contains the spectral information of the graph. Consider a graph G with vertex set V(G). For a function f defined on V(G), the combinatorial Laplacian is defined by

$$\Delta_G f(x) = \sum_{y \sim x} (f(x) - f(y))$$

where the sum is over all vertices adjacent to x. Let $\{\lambda_k\}_k$ denote the spectrum of the Laplacian on G. The associated theta function is defined by

$$\theta_{G}(t) = \sum_{k \in V(G)} e^{-\lambda_{k} t}.$$

To compute the asymptotic behaviour of spectral determinants on a sequence of d-orthotope square lattices, we express the associated theta function in terms of the theta function associated to the discrete torus with twice vertices at each side of the torus. This can be done because of the similarity of their spectrum. We then use the asymptotic results from [2]. The formula obtained relates the determinant of the Laplacian on the discrete lattice $L(n_1, \ldots, n_d)$ to the regularized determinant of the Laplacian on the rescaled limiting object, which is the real d-dimensional orthotope K_d , and to the ones on the m-dimensional boundary faces of K_d , $m = 1, \ldots, d - 1$. Moreover, we compute asymptotic results for the number of rooted spanning forests with 2 and 3 components.

We will prove the following theorem.

Theorem 1.1. Given positive integers α_i , i = 1, ..., d, let $\det^* \Delta_{L(\alpha_1 n, ..., \alpha_d n)}$ be the product of the non-zero eigenvalues of the Laplacian on the d-dimensional orthotope square lattice $L(\alpha_1 n, ..., \alpha_d n)$. Then as $n \to \infty$

$$\begin{split} \log \det^* \Delta_{L(\alpha_1 n, \dots, \alpha_d n)} &= c_d V_d^d n^d - \sum_{m=1}^{d-1} \frac{1}{4^{d-m}} \left(\int_0^\infty (1 - e^{-4t})^{d-m} e^{-2mt} I_0(2t)^m \frac{dt}{t} \right) V_m^d n^m \\ &+ (2 - 2^{1-d}) \log n + \sum_{m=1}^d \sum_{I \leqslant i_1 < \dots < i_m \leqslant d} \log \det^* \Delta_{\alpha_{i_1} \times \dots \times \alpha_{i_m}} \\ &+ \frac{1}{2^d} \sum_{m=1}^d C_d^m (-1)^m \log(4m) + o(1) \end{split}$$

where $\det^* \Delta_{\alpha_{i_1} \times \cdots \times \alpha_{i_m}}$ is the regularized determinant of the Laplacian on the m-orthotope $\alpha_{i_1} \times \cdots \times \alpha_{i_m}$ with Dirichlet boundary conditions. The constant c_d is

$$c_{d} = \int_{0}^{\infty} (e^{-t} - e^{-2dt} I_{0}(2t)^{d}) \frac{dt}{t}$$

where I_0 is the modified I-Bessel function of order zero.

Notice that the limiting object K_d can be decomposed into a disjoint union of m orthotopes, $m \in \{0, 1, \ldots, d\}$. More precisely, let $\alpha_{i_1} \times \cdots \times \alpha_{i_m}$, $\{i_q\}_{q=1}^m \subset \{1, \ldots, d\}$, denote the m-dimensional orthotope of side lengths $\alpha_{i_1}, \ldots, \alpha_{i_m}$ which is open in \mathbb{R}^m . Then

$$\mathsf{K}_{d} = \{0\} \sqcup \bigsqcup_{\mathfrak{m} = 1}^{d} \bigsqcup_{1 \leqslant \mathfrak{i}_{1} < \cdots < \mathfrak{i}_{\mathfrak{m}} \leqslant d} \alpha_{\mathfrak{i}_{1}} \times \cdots \times \alpha_{\mathfrak{i}_{\mathfrak{m}}}.$$

This decomposition is reflected in the theorem by the appearance of the sum over this decomposition of the logarithm of the regularized determinant of the Laplacian on the m-dimensional faces, m = 1, ..., d.

By expressing the eigenvalues of the Laplacian on the square lattice $L(n_1, n_2)$ in terms of the one on the two-dimensional discrete torus $\mathbb{Z}^2/\text{diag}(2n_1, 2n_2)\mathbb{Z}^2$, we derive a relation, which is stated below, between the number of spanning trees on these two lattices.

Theorem 1.2. Given positive integers n_1, n_2 , let $\tau(L(n_1, n_2))$ denote the number of spanning trees on the rectangular square lattice $L(n_1, n_2)$ and $\tau(T(2n_1, 2n_2))$ the number of spanning trees on the discrete torus $\mathbb{Z}^2/\text{diag}(2n_1, 2n_2)\mathbb{Z}^2$. We have

$$\tau(L(\mathfrak{n}_1,\mathfrak{n}_2)) = \frac{2^{5/4}\tau(T(2\mathfrak{n}_1,2\mathfrak{n}_2))^{1/4}}{(\mathfrak{n}_1\mathfrak{n}_2)^{1/4}((3+2\sqrt{2})^{\mathfrak{n}_1}-(3-2\sqrt{2})^{\mathfrak{n}_1})^{1/2}((3+2\sqrt{2})^{\mathfrak{n}_2}-(3-2\sqrt{2})^{\mathfrak{n}_2})^{1/2}}.$$

In [9], Kenyon computed asymptotics for spectral determinants on a simply-connected rectilinear region in \mathbb{R}^2 . Here we compute it in the particular case of a triangular region. More precisely, we consider the quartered Aztec diamond of order n, denoted by QAD_n , which is the subgraph of \mathbb{Z}^2 with nearest neighbours connected induced by the vertices (k_1, k_2) such that $k_1 + k_2 \leq n$ and $k_1, k_2 \geq 0$. Figure 1 illustrates QAD_7 . In [4], Ciucu derived a relation between the characteristic polynomials of the rectangular square lattice and of the quartered Aztec diamond using combinatorial arguments. From this one can deduce a relation for the number of spanning trees. In the second part of this work, we present an alternative approach for it. Consequently, we derive the asymptotic behaviour of it, stated in the following theorem, which shows that it is related to the regularized determinant of the Laplacian on the triangle with Dirichlet boundary conditions.



Figure 1: Quartered Aztec diamond of order 7.

Theorem 1.3. Let $\tau(QAD_n)$ denote the number of spanning trees in the quartered Aztec diamond of order n. Then as $n \to \infty$

$$\log(\tau(\text{QAD}_n)) = \frac{2G}{\pi}n^2 - \log(2 + \sqrt{2})n - \frac{3}{4}\log n + \log \det^* \Delta_\Delta + \frac{23}{8}\log 2 + o(1)$$

where G is the Catalan constant and det^{*} Δ_{Δ} is the regularized determinant of the Laplacian on the right-angled isosceles unit triangle with Dirichlet boundary conditions.

1.1 Regularized determinant

Let M be a Riemannian manifold with or without boundary and let Δ_M be the Laplace-Beltrami operator associated to M. If M has a boundary we associate Dirichlet boundary conditions to Δ_M . Denote by $\{\lambda_k\}_k$ the eigenvalues of Δ_M . The spectral zeta function associated to M is defined for $\Re(s) > \dim M/2$ by

$$\zeta_{\mathsf{M}}(s) = \sum_{\lambda_k \neq 0} \frac{1}{\lambda_k^s}.$$

It admits a meromorphic continuation to the whole complex plane (see for example [13] for M with a boundary and [2] for the case where M is a torus). The regularized determinant of Δ_M can then be defined by

$$\log \det^* \Delta_{\mathcal{M}} = -\zeta_{\mathcal{M}}'(0).$$

1.2 Preliminary result

We prove the following lemma which will be useful in the next section to invert relations between theta functions. Throughout this paper, we set an empty summation to be one by convention.

Lemma 1.4. Let $\{i_q\}_{q \ge 1}$ be an increasing sequence of positive integers. Let $\{n_q\}_{q \ge 1}$ be a sequence of positive integers and $\{n_{i_q}\}_{q \ge 1}$ a subsequence of it. Let $f, g : \mathbb{N}^{<\mathbb{N}} \to \mathbb{R}$ be two sequences of variadic functions such that for all $l \in \mathbb{N}_{\ge 1}$,

$$f(\mathfrak{n}_{\mathfrak{i}_{1}},\ldots,\mathfrak{n}_{\mathfrak{i}_{1}}) = \sum_{k=0}^{l} \sum_{\substack{\mathfrak{a}_{1} < \cdots < \mathfrak{a}_{k} \\ \{\mathfrak{a}_{q}\}_{\mathfrak{q}=1}^{k} \subset \{\mathfrak{i}_{q}\}_{\mathfrak{q}=1}^{l}} g(\mathfrak{n}_{\mathfrak{a}_{1}},\ldots,\mathfrak{n}_{\mathfrak{a}_{k}}).$$
(1)

Then the following inversion formula holds: for all $l \in \mathbb{N}_{\geq 1}$,

$$g(\mathfrak{n}_{\mathfrak{i}_1},\ldots,\mathfrak{n}_{\mathfrak{i}_1})=\sum_{k=0}^l(-1)^{l-k}\sum_{\substack{\mathfrak{a}_1<\cdots<\mathfrak{a}_k\\\{\mathfrak{a}_q\}_{\mathfrak{q}=1}^k\subset\{\mathfrak{i}_q\}_{\mathfrak{q}=1}^l}}f(\mathfrak{n}_{\mathfrak{a}_1},\ldots,\mathfrak{n}_{\mathfrak{a}_k}).$$

Proof. Let $l \in \mathbb{N}_{\geq 1}$. From relation (1) between f and g, we have

$$\sum_{k=0}^{L} (-1)^{l-k} \sum_{\substack{j_{1} < \dots < j_{k} \\ \{j_{q}\}_{q=1}^{k} \subset \{i_{q}\}_{q=1}^{l}}} f(n_{j_{1}}, \dots, n_{j_{k}})$$

$$= \sum_{k=0}^{l} (-1)^{l-k} \sum_{\substack{j_{1} < \dots < j_{k} \\ \{j_{q}\}_{q=1}^{k} \subset \{i_{q}\}_{q=1}^{l}}} \left(\sum_{\substack{m=0 \\ \{a_{q}\}_{q=1}^{m} \subset \{j_{q}\}_{q=1}^{k}}} \sum_{\substack{q_{1}, \dots, q_{m} \\ \{a_{q}\}_{q=1}^{m} \subset \{j_{q}\}_{q=1}^{k}}} g(n_{a_{1}}, \dots, n_{a_{m}}) \right)$$

$$= \sum_{k=1}^{l} (-1)^{l-k} \sum_{\substack{m=1 \\ \{j_{q}\}_{q=1}^{k} \subset \{i_{q}\}_{q=1}^{l} \in \{a_{q}\}_{q=1}^{m} \subset \{j_{q}\}_{q=1}^{k}}} \sum_{\substack{q_{1}, \dots, q_{m} \\ \{a_{q}\}_{q=1}^{m} \subset \{j_{q}\}_{q=1}^{k}}} g(n_{a_{1}}, \dots, n_{a_{m}})$$
(2)

where in the second equality we used that

$$\sum_{k=0}^{\iota} (-1)^{l-k} \sum_{\substack{j_1 < \dots < j_k \\ \{j_q\}_{q=1}^k \subset \{i_q\}_{q=1}^l}} 1 = \sum_{k=0}^{\iota} (-1)^{l-k} C_l^k = 0$$

from the binomial theorem, where $C_l^k = l!/(k!(l-k)!)$. For $l \ge k \ge m$, the double summation can be rewritten in one summation as

$$\sum_{\substack{j_1 < \dots < j_k \\ \{j_q\}_{q=1}^k \subset \{i_q\}_{q=1}^l \{\alpha_q\}_{q=1}^m \subset \{j_q\}_{q=1}^k}} \sum_{\substack{\alpha_1 < \dots < \alpha_m \\ \alpha_q \in \alpha_q \\ \alpha_q \in \alpha_q}} \sum_{\substack{j_1 < \dots < j_m \\ \{j_q\}_{q=1}^m \subset \{i_q\}_{q=1}^l}} (3)$$

Thus (2) is equal to

$$\begin{split} \sum_{k=1}^{l} (-1)^{l-k} \sum_{m=1}^{k} C_{l-m}^{k-m} \sum_{\substack{j_1 < \dots < j_m \\ \{j_q\}_{q=1}^m \subset \{i_q\}_{q=1}^l}} g(n_{j_1}, \dots, n_{j_m}) \\ &= \sum_{m=1}^{l} \sum_{k=0}^{l-m} (-1)^{l-m-k} C_{l-m}^k \sum_{\substack{j_1 < \dots < j_m \\ \{j_q\}_{q=1}^m \subset \{i_q\}_{q=1}^l}} g(n_{j_1}, \dots, n_{j_m}) = g(n_{i_1}, \dots, n_{i_l}) \end{split}$$

where the second equality comes from the fact that the only non-zero term in the summation over m is when m = l from the binomial theorem.

Remark. If we set $f(n_{i_1}, \ldots, n_{i_k}) = f_1$ and $g(n_{i_1}, \ldots, n_{i_k}) = g_k$ for all $l, k \in \mathbb{N}_{\geq 1}$, we recover the standard binomial inversion:

$$f_l = \sum_{k=0}^l C_l^k g_k \text{ if and only if } g_l = \sum_{k=0}^l (-1)^{l-k} C_l^k f_k, \text{ for all } l \in \mathbb{N}_{\geqslant l},$$

with $f_0 = g_0 = 1$.

Acknowledgements: The author thanks Anders Karlsson for suggesting this problem to her and for valuable discussions. She also thanks Larry Glasser for pointing reference [6] to her.

2 Asymptotic number of spanning trees in the d-orthotope lattice

2.1 Theta function

The eigenvalues of the Laplacian on the square lattice $L(n_1, \ldots, n_d)$ are given by (see [5])

$$\{\lambda_k^L\}_{k=0,1,\ldots,N-1} = \{2d-2\sum_{i=1}^d \cos(\pi k_i/n_i), \ k_i = 0, 1, \ldots, n_i - 1, \ \text{for} \ i = 1, \ldots, d\}$$

where $N = \prod_{i=1}^{d} n_i$. The d-dimensional discrete torus of size $2n_1 \times \cdots \times 2n_d$ is defined by the quotient $\mathbb{Z}^d/\text{diag}(2n_1,\ldots,2n_d)\mathbb{Z}^d$ with nearest neighbours connected. We denote it by $T(2n_1,\ldots,2n_d)$. The eigenvalues of the Laplacian on $T(2n_1,\ldots,2n_d)$ are given by (see [12])

$$\{\lambda_k^{\mathsf{T}}\}_{k=0,1,\ldots,2^d\,\mathsf{N}-1} = \{2d-2\sum_{i=1}^d\cos(\pi k_i/n_i),\ k_i=0,1,\ldots,2n_i-1,\ \text{for}\ i=1,\ldots,d\}.$$

Notice that $\{\lambda_k^L\}_k \subset \{\lambda_k^T\}_k$.

The theta function on the d-orthotope lattice $L(n_1, \ldots, n_d)$ is given by

$$\theta_{L(n_1,...,n_d)}(t) = \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_d=0}^{n_d-1} e^{-(2d-2\sum_{i=1}^d \cos(\pi k_i/n_i))t}$$

and on the discrete torus $T(2n_1, \ldots, 2n_d)$ by

$$\theta_{\mathsf{T}(2n_1,\ldots,2n_d)}(t) = \sum_{k_1=0}^{2n_1-1} \cdots \sum_{k_d=0}^{2n_d-1} e^{-(2d-2\sum_{i=1}^d \cos(\pi k_i/n_i))t}.$$

Therefore by expressing the theta function on the d-orthotope square lattice $L(n_1, \ldots, n_d)$ in terms of the one on the torus $T(2n_1, \ldots, 2n_d)$, one can deduce the asymptotic behaviour from the results obtained in [2].

Let $[\![a,b]\!]$ denote the set of successive integers $\{a,a+1,\ldots,b\}$. In the theta function on $L(n_1,\ldots,n_d)$, the summation is over the discrete d-orthotope $[\![0,n_1-1]\!] \times \cdots \times [\![0,n_d-1]\!]$ that we denote by J_d , while for the torus $T(2n_1,\ldots,2n_d)$ it is over the discrete d-orthotope $[\![0,2n_1-1]\!] \times \cdots \times [\![0,2n_d-1]\!]$, denoted by \widetilde{J}_d . We decompose J_d as a disjoint union of l-dimensional faces, $l = 0, 1, \ldots, d$. The 0 dimension is the point $0 \in \mathbb{Z}^d$, we call it the root of J_d . For $l \in \{1,\ldots,d-1\}$, the l-dimensional faces are defined by a subset of \mathbb{Z}^d , $(k_1,\ldots,k_d) \subset J_d$, such that $\exists \{i_q\}_{q=1}^d \in S_d$ such that $k_{i_q} \in [\![1,n_{i_q}-1]\!]$, $q = 1,\ldots,l$ and $k_{i_q} = 0$, $q = l+1,\ldots,d$. We call the d-dimensional face the interior of J_d where no coordinate is zero, that is $[\![1,n_1-1]\!] \times \cdots \times [\![1,n_d-1]\!]$. For example, in the 2 dimensional case, J_2 decomposes as:

$$J_2 = \{0\} \sqcup (\llbracket 1, n_1 - 1 \rrbracket \times \{0\}) \sqcup (\{0\} \times \llbracket 1, n_2 - 1 \rrbracket) \sqcup (\llbracket 1, n_1 - 1 \rrbracket \times \llbracket 1, n_2 - 1 \rrbracket).$$

In the theta function of the torus $T(2n_1, \ldots, 2n_d)$, the summation is over 2^d copies of J_d , namely:

$$\widetilde{J}_{d} = \bigsqcup_{\substack{\varepsilon_{i} \in \{0, n_{i}\}\\i=1, \dots, d}} (J_{d} + (\varepsilon_{1}, \dots, \varepsilon_{d}))$$

where the unions are disjoint. Figure 2 illustrates the decompositions of J_d and \tilde{J}_d in the case d = 2.



Figure 2: J_2 and J_2 with $n_1 = 4$ and $n_2 = 3$. The big black dot is the root of J_2 , the small black dots are the two 1-dimensional faces and the white dots are the interior of J_2 .

Let define the theta-star function on the lattice $L(n_{i_1}, \ldots, n_{i_1})$, with $1 \leq i_1 < \cdots < i_l \leq d$, by the following expression

$$\theta^*_{L(n_{i_1},\ldots,n_{i_1})}(t) = \sum_{k_{i_1}=1}^{n_{i_1}-1} \cdots \sum_{k_{i_1}=1}^{n_{i_1}-1} e^{-(2l-2\sum_{i=i_1}^{i_1}\cos(\pi k_i/n_i))t}$$

where the summation is over the interior of the l-orthotope of size $n_{i_1} \times \cdots \times n_{i_l}$, that is, the k_i 's start at 1 instead of 0 for all $i \in \{i_1, \ldots, i_l\}$. Therefore the theta function is related to the theta-star function by the relation

$$\theta_{\mathsf{L}(\mathfrak{n}_{1},\ldots,\mathfrak{n}_{d})}(t) = \sum_{l=0}^{d} \sum_{1 \leqslant \mathfrak{i}_{1} < \cdots < \mathfrak{i}_{l} \leqslant d} \theta^{*}_{\mathsf{L}(\mathfrak{n}_{\mathfrak{i}_{1}},\ldots,\mathfrak{n}_{\mathfrak{i}_{l}})}(t).$$
(4)

To evaluate the theta function on $T(2n_1, \ldots, 2n_d)$ we sum over the roots of the 2^d discrete orthotopes J_d , the l-dimensional boundary faces of J_d and the interior of J_d . Summing over the roots of the J_d 's means that for all $i = 1, \ldots, d$, k_i is either 0 or n_i . There are $C_d^j = d!/(j!(d-j)!)$ number of ways to take j of the k_i 's to be zero. In this case the corresponding exponential term of the theta function is $e^{-4(d-j)t}$. The term l = 0 is then the sum of all the possibilities:

$$\sum_{j=0}^{d} C_{d}^{j} e^{-4(d-j)t} = (1+e^{-4t})^{d}.$$

Then for each l-dimensional boundary face of J_d , where $l \in \{1, \ldots, d-1\}$, there are d-l of the k_i 's which are either 0 or n_i , this positions the l-dimensional face in \widetilde{J}_d . For this l, there are C_{d-l}^j number of ways where j of the k_i 's are zero and the exponential term is then $e^{-4(d-l-j)t}$. And we sum over the interior of the l-dimensional face $L(n_{i_1}, \ldots, n_{i_l})$, where $l \leq i_l < \cdots < i_l \leq d$, which is by definition the theta-star function $\theta^*_{L(n_{i_1}, \ldots, n_{i_l})}(t)$ with a factor of 2^l since this configuration appears 2^l times. So for $l \in \{1, \ldots, d-1\}$, we have

$$\sum_{j=0}^{d-l} C_{d-l}^{j} e^{-4(d-l-j)t} \sum_{1 \leqslant i_1 < \cdots < i_l \leqslant d} 2^l \theta_{L(n_{i_1}, \dots, n_{i_l})}^*(t).$$

Finally we sum over the interior of J_d , that is, when all the k_i 's are greater or equal to one, which appears 2^d times. This gives the l = d term

$$2^{\mathfrak{a}}\theta^*_{L(\mathfrak{n}_1,\ldots,\mathfrak{n}_d)}(t).$$

The theta function on $T(2n_1, \ldots, 2n_d)$ is then the sum over the l-dimensional faces of the 2^d orthotopes J_d :

$$\theta_{\mathsf{T}(2\mathfrak{n}_1,\ldots,2\mathfrak{n}_d)}(t) = 2^d \sum_{l=0}^d \left(\frac{1+e^{-4t}}{2}\right)^{d-l} \sum_{1\leqslant \mathfrak{i}_1<\cdots<\mathfrak{i}_l\leqslant d} \theta^*_{\mathsf{L}(\mathfrak{n}_{\mathfrak{i}_1},\ldots,\mathfrak{n}_{\mathfrak{i}_l})}(t)$$

which is equivalent to

$$(1+e^{-4t})^{-d}\theta_{\mathsf{T}(2\mathfrak{n}_1,\ldots,2\mathfrak{n}_d)}(t) = \sum_{l=0}^d 2^l (1+e^{-4t})^{-l} \sum_{1\leqslant \mathfrak{i}_1<\cdots<\mathfrak{i}_l\leqslant d} \theta^*_{\mathsf{L}(\mathfrak{n}_{\mathfrak{i}_1},\ldots,\mathfrak{n}_{\mathfrak{i}_l})}(t).$$

By setting

$$f(n_1,\ldots,n_d) = (1+e^{-4t})^{-d}\theta_{\mathsf{T}(2n_1,\ldots,2n_d)}(t) \text{ and } g(n_{i_1},\ldots,n_{i_l}) = 2^l(1+e^{-4t})^{-l}\theta^*_{\mathsf{L}(n_{i_1},\ldots,n_{i_l})}(t)$$

in Lemma 1.4, it comes

$$\theta^*_{L(\mathfrak{n}_{i_1},\ldots,\mathfrak{n}_{i_1})}(t) = 2^{-l} \sum_{\mathfrak{m}=0}^{l} (-1 - e^{-4t})^{l-\mathfrak{m}} \sum_{\substack{j_1 < \cdots < j_\mathfrak{m} \\ \{j_q\}_{q=1}^{\mathfrak{m}} \subset \{i_q\}_{q=1}^{l}}} \theta_{\mathsf{T}(2\mathfrak{n}_{j_1},\ldots,2\mathfrak{n}_{j_\mathfrak{m}})}(t).$$

From the above relation and relation (4), the theta function on the d-orthotope square lattice is expressed in terms of the theta function on the d-dimensional torus:

$$\theta_{L(n_1,\ldots,n_d)}(t) = \sum_{l=0}^d \sum_{1 \leqslant i_1 < \cdots < i_l \leqslant d} \sum_{m=0}^l 2^{-l} (-1 - e^{-4t})^{l-m} \sum_{\substack{j_1 < \cdots < j_m \\ \{j_q\}_{q=1}^m \subset \{i_q\}_{q=1}^l}} \theta_{T(2n_{j_1},\ldots,2n_{j_m})}(t).$$

Rewriting the double multi-index summation in one summation using (3), we have

$$\theta_{L(n_1,...,n_d)}(t) = \sum_{l=0}^{d} \sum_{m=0}^{l} C_{d-m}^{l-m} 2^{-l} (-1 - e^{-4t})^{l-m} \sum_{1 \leqslant i_1 < \cdots < i_m \leqslant d} \theta_{T(2n_{i_1},...,2n_{i_m})}(t)$$

Therefore the theta functions are related by

$$\theta_{L(n_1,...,n_d)}(t) = \frac{1}{2^d} \sum_{m=0}^d (1 - e^{-4t})^{d-m} \sum_{1 \le i_1 < \dots < i_m \le d} \theta_{T(2n_{i_1},\dots,2n_{i_m})}(t).$$
(5)

2.2 Preliminary calculation

Let $\{\lambda_j^L\}_{j=0,\dots,N-1}$ be the eigenvalues of the combinatorial Laplacian on the square lattice $L(n_1,\dots,n_d)$. For small t>0, the theta function on the torus $\theta_{T(2n_{i_1},\dots,2n_{i_m})}$ behaves as

$$\theta_{\mathsf{T}(2\mathfrak{n}_{\mathfrak{i}_{1}},\ldots,2\mathfrak{n}_{\mathfrak{i}_{m}})}(t) - 2^{\mathfrak{m}} \Big(\prod_{q=1}^{\mathfrak{m}} \mathfrak{n}_{\mathfrak{i}_{q}} \Big) e^{-2\mathfrak{m} t} I_{0}(2t)^{\mathfrak{m}} = O(t), \quad t \to 0. \tag{6}$$

We follow the method derived in [2]. From relation (5) and the behaviour of the theta function at small t > 0 (6), we start by writing the theta function on the lattice $L(n_1, \ldots, n_d)$ as

$$\begin{split} \sum_{j \neq 0} e^{-\lambda_j^L t} &= \sum_{m=1}^d \left(\frac{1-e^{-4t}}{4}\right)^{d-m} e^{-2mt} I_0(2t)^m V_m^d n^m \\ &+ \left[\theta_{L(n_1,\dots,n_d)}(t) - \sum_{m=1}^d \left(\frac{1-e^{-4t}}{4}\right)^{d-m} e^{-2mt} I_0(2t)^m V_m^d n^m - 1\right], \end{split}$$

to ensure the convergence of the integral of the Gauss transform that will appear below. By taking the Gauss transform of the above, that is, multiplying by $2se^{-s^2t}$ and then integrating with respect

to t from zero to infinity, we have

$$\begin{split} \sum_{j\neq 0} \frac{2s}{s^2 + \lambda_j^{\rm L}} &= \sum_{m=1}^d V_m^d n^m 2s \int_0^\infty e^{-s^2 t} \left(\frac{1 - e^{-4t}}{4}\right)^{d-m} e^{-2mt} I_0(2t)^m dt \\ &+ 2s \left[\sum_{m=1}^d \int_0^\infty e^{-s^2 t} (1 - e^{-4t})^{d-m} \right. \\ &\quad \times \left(\frac{1}{2^d} \sum_{1 \leqslant i_1 < \dots < i_m \leqslant d} \theta_{\mathsf{T}(2n_{i_1}, \dots, 2n_{i_m})}(t) - \frac{1}{4^{d-m}} V_m^d n^m e^{-2mt} I_0(2t)^m - \frac{1}{2^d} C_d^m \right) dt \\ &+ \int_0^\infty e^{-s^2 t} ((1 - e^{-4t})^d - 2^d + \sum_{m=1}^d C_d^m (1 - e^{-4t})^{d-m}) dt \right]. \end{split}$$
(7)

For $\mathfrak{m} \in \{1, \ldots, d\},$ define the functions \mathcal{I}_m^d and $\mathcal{H}_{\mathfrak{m}, \mathfrak{n}}^d$ such that

$$\partial_{s} \mathcal{I}_{m}^{d}(s) = 2s \int_{0}^{\infty} e^{-s^{2}t} \left(\frac{1-e^{-4t}}{4}\right)^{d-m} e^{-2mt} I_{0}(2t)^{m} dt$$
(8)

and

$$\begin{split} \partial_{s}\mathcal{H}^{d}_{m,n}(s) &= 2s \int_{0}^{\infty} e^{-s^{2}t} (1-e^{-4t})^{d-m} \\ &\times \bigg(\frac{1}{2^{d}} \sum_{1 \leqslant i_{1} < \dots < i_{m} \leqslant d} \theta_{\mathsf{T}(2n_{i_{1}},\dots,2n_{i_{m}})}(t) - \frac{1}{4^{d-m}} V^{d}_{m} \mathfrak{n}^{m} e^{-2mt} I_{0}(2t)^{m} - \frac{1}{2^{d}} C^{m}_{d} \bigg) dt. \end{split}$$

$$(9)$$

Equation (7) can then be written as

$$\sum_{j\neq 0} \frac{2s}{s^2 + \lambda_j^{\mathsf{L}}} = \sum_{m=1}^d V_m^d \mathfrak{n}^m \vartheta_s \mathcal{I}_m^d(s) + \sum_{m=1}^d \vartheta_s \mathcal{H}_{m,n}^d(s) + \frac{1}{2^d} 2s \int_0^\infty e^{-s^2 t} ((2 - e^{-4t})^d - 2^d) dt.$$
(10)

By integrating over s equations (8) and (9) we get

$$\begin{split} \mathcal{I}_{d}^{d}(s) &= \int_{0}^{\infty} (e^{-t} - e^{-(s^{2} + 2d)t} I_{0}(2t)^{d}) \frac{dt}{t}, \\ \mathcal{H}_{d,n}^{d}(s) &= -\frac{1}{2^{d}} \int_{0}^{\infty} (e^{-s^{2}t} (\theta_{\mathsf{T}(2n_{1},...,2n_{d})}(t) - \mathsf{V}_{d}^{d}(2n)^{d} e^{-2dt} I_{0}(2t)^{d} - 1) + e^{-t}) \frac{dt}{t} \end{split}$$

and for $m \neq d$,

$$\begin{split} \mathcal{I}_{m}^{d}(s) &= -\frac{1}{4^{d-m}} \int_{0}^{\infty} (1-e^{-4t})^{d-m} e^{-(s^{2}+2m)t} I_{0}(2t)^{m} \frac{dt}{t}, \\ \mathcal{H}_{m,n}^{d}(s) &= -\int_{0}^{\infty} e^{-s^{2}t} (1-e^{-4t})^{d-m} \\ & \qquad \times \left(\frac{1}{2^{d}} \sum_{1 \leqslant i_{1} < \cdots < i_{m} \leqslant d} \theta_{\mathsf{T}(2n_{i_{1}}, \ldots, 2n_{i_{m}})}(t) - \frac{1}{4^{d-m}} V_{m}^{d} n^{m} e^{-2mt} I_{0}(2t)^{m} - \frac{1}{2^{d}} C_{d}^{m} \right) \frac{dt}{t}. \end{split}$$

By integrating equation (10) above we get

$$\sum_{j \neq 0} \log(s^2 + \lambda_j^L) = \sum_{m=1}^d V_m^d n^m \mathcal{I}_m^d(s) + \sum_{m=1}^d \mathcal{H}_{m,n}^d(s) + \sum_{k=1}^d C_d^k(-2)^{-k} \log(s^2 + 4k) + \text{constant.}$$
(11)

$$\mathcal{I}^d_d(s) = 2\log s + o(1) \text{ and } \mathcal{I}^d_m(s) = o(1) \text{ for } m \neq d \text{ as } s \rightarrow \infty$$

and

$$\mathcal{H}^d_{d,n}(s) = -\frac{1}{2^{d-1}}\log s + o(1) \text{ and } \mathcal{H}^d_{m,n}(s) = o(1) \text{ for } m \neq d \text{ as } s \rightarrow \infty$$

and

$$\sum_{\neq 0} \log(s^2 + \lambda_j^L) = 2(N-1)\log s + o(1) \text{ as } s \to \infty.$$

Therefore, equation (11) as $s \to \infty$ yields

$$2(\mathsf{N}-1)\log s + \mathsf{o}(1) = 2\mathsf{N}\log s - \frac{1}{2^{d-1}}\log s + \frac{1}{2^{d-1}}(1-2^d)\log s + \text{constant} + \mathsf{o}(1)$$

implying that the constant is zero. Evaluating equation (11) at s = 0 gives the logarithm of the product of the non-zero eigenvalues of the Laplacian on the d-dimensional square lattice $L(n_1, \ldots, n_d)$:

$$\log\left(\prod_{j\neq 0}\lambda_{j}^{L}\right) = c_{d}V_{d}^{d}n^{d} + \sum_{m=1}^{d-1}V_{m}^{d}\mathcal{I}_{m}^{d}(0)n^{m} + \sum_{m=1}^{d}\mathcal{H}_{m,n}^{d}(0) + \sum_{k=1}^{d}C_{d}^{k}(-2)^{-k}\log(4k)$$
(12)

where

$$c_d = \int_0^\infty (e^{-t} - e^{-2dt} I_0(2t)^d) \frac{dt}{t}$$

2.3 Asymptotic expansion

By expanding $(1 - e^{-4t})^{d-m} = \sum_{k=0}^{d-m} C_{d-m}^k (-1)^k e^{-4kt}$ in $\mathcal{H}_{m,n}^d(0)$, it can be rewritten as

$$\begin{split} \mathcal{H}^{d}_{m,n}(0) &= -\frac{1}{2^{d}} \sum_{k=0}^{d-m} C^{k}_{d-m}(-1)^{k} \\ &\times \sum_{1 \leqslant i_{1} < \cdots < i_{m} \leqslant d} \int_{0}^{\infty} \big(e^{-4kt} (\theta_{\mathsf{T}(2n_{i_{1}}, \dots, 2n_{i_{m}})}(t) - \prod_{q=1}^{m} (2\alpha_{i_{q}} n e^{-2t} I_{0}(2t)) - 1) + e^{-t} \big) \frac{dt}{t}, \end{split}$$

for all m = 1, ..., d, where the e^{-t} term is added to make the integral converge. It can be added since $\sum_{k=0}^{d-m} C_{d-m}^k (-1)^k = 0$ for m = 1, ..., d-1. By splitting the sum over k we have

$$\begin{split} &\sum_{m=1}^{d} \mathcal{H}_{m,n}^{d}(0) = \\ &- \frac{1}{2^{d}} \sum_{m=1}^{d} \sum_{I \leqslant i_{1} < \dots < i_{m} \leqslant d} \int_{0}^{\infty} \left(\theta_{\mathsf{T}(2n_{i_{1}},\dots,2n_{i_{m}})}(t) - \prod_{q=1}^{m} (2\alpha_{i_{q}} n e^{-2t} I_{0}(2t)) - 1 + e^{-t} \right) \frac{dt}{t} \\ &- \frac{1}{2^{d}} \sum_{m=1}^{d} \sum_{k=1}^{d-m} \sum_{I \leqslant i_{1} < \dots < i_{m} \leqslant d} \int_{0}^{\infty} \left(e^{-4kt} (\theta_{\mathsf{T}(2n_{i_{1}},\dots,2n_{i_{m}})}(t) - \prod_{q=1}^{m} (2\alpha_{i_{q}} n e^{-2t} I_{0}(2t)) - 1) + e^{-t} \right) \frac{dt}{t}. \end{split}$$

From [2, Theorem 5.8], the asymptotic behaviour as $n \to \infty$ of the k = 0 term is given by

$$\begin{split} &-\frac{1}{2^d}\sum_{m=1}^d\sum_{\substack{i_1<\dots< i_m\leqslant d}}\int_0^\infty(\theta_{\mathsf{T}(2\mathfrak{n}_{i_1},\dots,2\mathfrak{n}_{i_m})}(t)-\prod_{q=1}^m(2\alpha_{i_q}\mathfrak{n} e^{-2t}I_0(2t))-1+e^{-t})\frac{dt}{t}\\ &=\frac{1}{2^d}\sum_{m=1}^d\sum_{\substack{i_1<\dots< i_m\leqslant d}}(2\log\mathfrak{n}-\zeta'_{\mathbb{R}^m/diag(2\alpha_{i_1},\dots,2\alpha_{i_m})\mathbb{Z}^m}(0))+\mathfrak{o}(1). \end{split}$$

After a change of variable $t \rightarrow n^2 t$, the sum over the non-zero k's can be split as

$$\begin{split} &-\frac{1}{2^{d}}\sum_{k=1}^{d-i}\sum_{m=1}^{d-k}\sum_{1\leqslant i_{1}<\dots< i_{m}\leqslant d}C_{d-m}^{k}(-1)^{k}\\ &\times\left[\int_{0}^{1}e^{-4kn^{2}t}(\theta_{\mathsf{T}(2n_{i_{1}},\dots,2n_{i_{m}})}(n^{2}t)-\prod_{q=1}^{m}(2\alpha_{i_{q}}ne^{-2n^{2}t}I_{0}(2n^{2}t)))\frac{dt}{t}+\int_{0}^{1}(e^{-n^{2}t}-e^{-4kn^{2}t})\frac{dt}{t}\right.\\ &+\int_{1}^{\infty}e^{-4kn^{2}t}(\theta_{\mathsf{T}(2n_{i_{1}},\dots,2n_{i_{m}})}(n^{2}t)-1)\frac{dt}{t}\\ &+\int_{1}^{\infty}(e^{-n^{2}t}-e^{-4kn^{2}t}\prod_{q=1}^{m}(2\alpha_{i_{q}}ne^{-2n^{2}t}I_{0}(2n^{2}t)))\frac{dt}{t}\right]. \end{split}$$
(13)

From the propositions in [2, section 5], the first, third and fourth integrals tend to zero as $n \to \infty$. The second integral tends to

$$\int_0^\infty (e^{-t} - e^{-4kt}) \frac{dt}{t} = \log(4k).$$

The limit as $n \to \infty$ of (13) is then

. . . .

$$-\frac{1}{2^{d}}\sum_{k=1}^{d-1}\sum_{m=1}^{d-k}C_{d}^{m}C_{d-m}^{k}(-1)^{k}\log(4k) = -\frac{1}{2^{d}}\sum_{k=1}^{d-1}C_{d}^{k}(2^{d-k}-1)(-1)^{k}\log(4k) = -\frac{1}{2^{d}}\sum_{k=1}^{d-1}C_{d}^{k}(2^{d-k}-1)(-1)^{k}(2^{d-k}-1)(-1)^{k}(2^{d-k}-1)(-1)^{k}(2^{d-k}-1)(-1)^{k}(2^{d-k}-1)(-1)^{k}(2^{d-k}-1)$$

Therefore, the $\mathcal{H}_{m,n}^d(0)$ term together with the constant term of equation (12) behave as

$$\left(2 - \frac{1}{2^{d-1}}\right)\log n - \frac{1}{2^d} \sum_{m=1}^d \sum_{1 \leqslant i_1 < \dots < i_m \leqslant d} \zeta'_{\mathbb{R}^m/\text{diag}(2\alpha_{i_1}, \dots, 2\alpha_{i_m})\mathbb{Z}^m}(0) + \frac{1}{2^d} \sum_{k=1}^d C_d^k(-1)^k \log(4k)$$

 $\text{ as }n\rightarrow\infty.$

We will now express the derivative of the spectral zeta function on the m-dimensional real torus $\mathbb{R}^m/\text{diag}(2\alpha_{i_1},\ldots,2\alpha_{i_m})\mathbb{Z}^m$, that is $\zeta'_{\mathbb{R}^m/\text{diag}(2\alpha_{i_1},\ldots,2\alpha_{i_m})\mathbb{Z}^m}$, in terms of the derivative of the spectral zeta function on the m-dimensional orthotope of size $\alpha_{i_1} \times \cdots \times \alpha_{i_m}$. The eigenvalues of the Laplace-Beltrami operator with Dirichlet boundary conditions on the m-dimensional orthotope of size $\alpha_{i_1} \times \cdots \times \alpha_{i_m}$ are given by

$$\lambda_k = \pi^2 \sum_{q=1}^m \left(\frac{k_{\mathfrak{i}_q}}{\alpha_{\mathfrak{i}_q}}\right)^2 \text{ with } k = (k_{\mathfrak{i}_1}, \dots, k_{\mathfrak{i}_m}) \in (\mathbb{N}^*)^m.$$

So that the spectral zeta function on the m-dimensional orthotope of size $\alpha_{i_1} \times \cdots \times \alpha_{i_m}$ with Dirichlet boundary conditions, denoted by $\zeta_{\alpha_{i_1} \times \cdots \times \alpha_{i_m}}$, is given by

$$\zeta_{\alpha_{\mathfrak{i}_{1}}\times\cdots\times\alpha_{\mathfrak{i}_{m}}}(\mathfrak{s})=\frac{1}{\pi^{2\mathfrak{s}}}\sum_{k_{\mathfrak{i}_{1}},\ldots,k_{\mathfrak{i}_{m}}\geqslant 1}\left(\sum_{q=1}^{\mathfrak{m}}\left(\frac{k_{\mathfrak{i}_{q}}}{\alpha_{\mathfrak{i}_{q}}}\right)^{2}\right)^{-\mathfrak{s}}.$$

80

The spectral zeta function on the real torus $\mathbb{R}^m/\text{diag}(2\alpha_{i_1},\ldots,2\alpha_{i_m})\mathbb{Z}^m$ is given by

$$\zeta_{\mathbb{R}^m/\text{diag}(2\alpha_{\mathfrak{i}_1},\ldots,2\alpha_{\mathfrak{i}_m})\mathbb{Z}^m}(s) = \frac{1}{(2\pi)^{2s}} \sum_{(k_{\mathfrak{i}_1},\ldots,k_{\mathfrak{i}_m})\in\mathbb{Z}^m\setminus\{0\}} \left(\sum_{q=1}^m \left(\frac{k_{\mathfrak{i}_q}}{2\alpha_{\mathfrak{i}_q}}\right)^2\right)^{-s}.$$

The spectral zeta functions are then related by

$$\zeta_{\mathbb{R}^m/\text{diag}(2\alpha_{i_1},\ldots,2\alpha_{i_m})\mathbb{Z}^m}(s) = \sum_{l=1}^m 2^l \sum_{\substack{j_1 < \cdots < j_l \\ \{j_q\}_{q=1}^l \subset \{i_q\}_{q=1}^m}} \zeta_{\alpha_{j_1} \times \cdots \times \alpha_{j_l}}(s).$$

Summing the above over all i_q , q = 1, ..., m and over all m, m = 1, ..., d, gives

$$\begin{split} &\frac{1}{2^{d}}\sum_{m=1}^{d}\sum_{\substack{i_{1}<\dots< i_{m}\leqslant d}}\zeta_{\mathbb{R}^{m}/diag(2\alpha_{i_{1}},\dots,2\alpha_{i_{m}})\mathbb{Z}^{m}}(s) \\ &=\frac{1}{2^{d}}\sum_{m=1}^{d}\sum_{\substack{i_{1}<\dots< i_{m}\leqslant d}}\sum_{\substack{l=1}}^{m}2^{l}\sum_{\substack{j_{1}<\dots< j_{l}\\\{j_{q}\}_{q=1}^{l}\subset\{i_{q}\}_{q=1}^{m}}\zeta_{\alpha_{j_{1}}\times\dots\times\alpha_{j_{1}}}(s) \\ &=\frac{1}{2^{d}}\sum_{m=1}^{d}\sum_{\substack{l=1}}^{m}C_{d-l}^{m-l}2^{l}\sum_{\substack{1\leqslant i_{1}<\dots< i_{1}\leqslant d}}\zeta_{\alpha_{i_{1}}\times\dots\times\alpha_{i_{1}}}(s) \\ &=\sum_{m=1}^{d}\sum_{\substack{1\leqslant i_{1}<\dots< i_{m}\leqslant d}}\zeta_{\alpha_{i_{1}}\times\dots\times\alpha_{i_{m}}}(s) \end{split}$$

where in the last equality we exchanged the sums over m and l and used the fact that $\sum_{m=1}^{d} C_{d-1}^{m-l} = 2^{d-l}$. By expressing the derivative of the spectral zeta function evaluated at zero in terms of the regularized determinant of the Laplace-Beltrami operator on m-dimensional orthotopes, $m = 1, \ldots, d$, with Dirichlet boundary conditions, we have

$$\frac{1}{2^d}\sum_{m=1}^d\sum_{1\leqslant i_1<\dots< i_m\leqslant d}\zeta'_{\mathbb{R}^m/\text{diag}(2\alpha_{i_1},\dots,2\alpha_{i_m})\mathbb{Z}^m}(0)=-\sum_{m=1}^d\sum_{1\leqslant i_1<\dots< i_m\leqslant d}\log\det^*\Delta_{\alpha_{i_1}\times\dots\times\alpha_{i_m}}.$$

Putting everything together gives the asymptotic behaviour of the determinant of the Laplacian on the d-dimensional square lattice $L(n_1, \ldots, n_d)$

$$\begin{split} \log \det^* &\Delta_{L(\alpha_1 n, \dots, \alpha_d n)} = c_d V_d^d n^d - \sum_{m=1}^{d-1} \frac{1}{4^{d-m}} \left(\int_0^\infty (1 - e^{-4t})^{d-m} e^{-2mt} I_0(2t)^m \frac{dt}{t} \right) V_m^d n^m \\ &+ (2 - 2^{1-d}) \log n + \sum_{m=1}^d \sum_{1 \leqslant i_1 < \dots < i_m \leqslant d} \log \det^* \Delta_{\alpha_{i_1} \times \dots \times \alpha_{i_m}} \\ &+ \frac{1}{2^d} \sum_{m=1}^d C_d^m (-1)^m \log(4m) + o(1) \end{split}$$

as $n \to \infty$. Notice that in the bulk limit the lead term is the same as in the case of the torus but lower order terms are deducted; for each m-dimensional face, m = 1, ..., d - 1, a term

proportional to $V_m^d n^m$ is deducted. This can be explained by observing that in the torus case we have a periodic lattice while for the d-dimensional hypercubic lattice the periodicity is substituted by free boundary conditions. Spectral determinants of the limiting d-dimensional orthotope and each of its m-dimensional faces, m = 1, ..., d-1, also appear. Moreover the last constant term is new in the development. The terms in $V_m^d n^m$ appearing from the boundary effect can be written in the following way:

$$-\int_{0}^{\infty} (1-e^{-4t})^{d-m} e^{-2mt} I_{0}(2t)^{m} \frac{dt}{t} = \sum_{k=0}^{d-m} C_{d-m}^{k} (-1)^{k} \int_{0}^{\infty} (e^{-t} - e^{-(2k+m)t} I_{0}(t)^{m}) \frac{dt}{t}$$

These integrals are denoted by $J_m(2k + m)$ in [6] by Glasser and are related to the Mahler measure of the hypercubic polynomial $P(x_1, \ldots, x_m) = 4k + 2m + \sum_{j=1}^{m} (x_j + x_j^{-1})$ by the relation $m(P) = \log 2 + J_m(2k + m)$. In [6], Glasser calculates these integrals in terms of hypergeometric functions for m = 2 and 3. For m = 1, one explicitly has

$$\frac{1}{4^{d-1}}\int_0^\infty (1-e^{-4t})^{d-1}e^{-2t}I_0(2t)\frac{dt}{t} = \frac{1}{4^{d-1}}\sum_{k=1}^{d-1}C_{d-1}^k(-1)^k\log(2k+1+2\sqrt{k^2+k})$$

(see [11, Proposition 2.4]).

Example. Consider the two dimensional rectangular grid $\alpha_1 n \times \alpha_2 n$. The volume of the limiting rectangle of size $\alpha_1 \times \alpha_2$ is $V_2^2 = \alpha_1 \alpha_2$ with perimeter $V_1^2 = 2\alpha_1 + 2\alpha_2$. From Theorem 1.1 it comes

$$\begin{split} \log \det^*\!\Delta_{L(\alpha_1 n, \alpha_2 n)} &= \frac{4G}{\pi} V_2^2 n^2 - \frac{1}{2} \log(1 + \sqrt{2}) V_1^2 n + \frac{3}{2} \log n + \log \det^*\!\Delta_{\alpha_1 \times \alpha_2} \\ &+ \log \det^*\!\Delta_{\alpha_1} + \log \det^*\!\Delta_{\alpha_2} - \frac{1}{4} \log 2 + o(1) \end{split}$$

as $n \to \infty$, which is equivalent to the formula derived in [5, section 4.2].

2.4 Asymptotic number of rooted 2- and 3-spanning forests

In this section, we derive asymptotics for 2 and 3 component rooted spanning forests in d-orthotope square lattices. Let N_k^d denote the number of rooted k-spanning forests on $L(n_1, \ldots, n_d)$ which is given by the k-th power in s^2 of the characteristic polynomial:

$$\prod_{j=0}^{\prod_{i=1}^{d} n_i - 1} \left(\lambda_j^L + \frac{s^2}{n^2} \right).$$

Following [5], by expanding the above in powers of $(s/n)^2$, one finds that N_2^d and N_3^d are related to N_1^d by

$$\frac{N_2^d}{N_1^d} = \sum_{j \neq 0} \frac{1}{\lambda_j^L} \quad \text{and} \quad \frac{N_3^d}{N_1^d} = \frac{1}{2} \left(\left(\frac{N_2^d}{N_1^d} \right)^2 - \sum_{j \neq 0} \frac{1}{(\lambda_j^L)^2} \right).$$

The number of rooted spanning trees N_1^d is related to the number of unrooted spanning trees $\tau(L(n_1,\ldots,n_d))$ by the relation

$$\mathsf{N}_1^d = \Big(\prod_{i=1}^d \mathfrak{n}_i\Big) \tau(\mathsf{L}(\mathfrak{n}_1,\ldots,\mathfrak{n}_d)).$$

Recall that from equation (11), we have that

$$\sum_{j \neq 0} \log((s/n)^2 + \lambda_j^L) = \sum_{m=1}^d V_m^d n^m \mathcal{I}_m^d(s/n) + \sum_{m=1}^d \mathcal{H}_{m,n}^d(s/n) + \sum_{k=1}^d C_d^k (-2)^{-k} \log((s/n)^2 + 4k)$$

where

$$\mathcal{I}_{d}^{d}(s/n) = \int_{0}^{\infty} (e^{-t} - e^{-((s/n)^{2} + 2d)t} I_{0}(2t)^{d}) \frac{dt}{t}.$$

We have

$$\lim_{n\to\infty}\mathcal{I}^d_d(s/n)=c_d.$$

For $d \ge 3$,

$$\lim_{n \to \infty} n^2 (\mathcal{I}_d^d(s/n) - c_d) = \lim_{n \to \infty} n^2 \int_0^\infty (1 - e^{-(s/n)^2 t}) e^{-2dt} I_0(2t)^d \frac{dt}{t} = \frac{s^2}{2} W_d$$

where W_d is the so-called Watson integral for the d-dimensional hypercubic lattice

$$W_d = \int_0^\infty e^{-dt} I_0(t)^d dt.$$

In [8], Joyce and Zucker introduced the generalised lattice Green function defined by

$$G_{d}(\mathbf{n};k,w) = \frac{1}{\Gamma(k)} \int_{0}^{\infty} t^{k-1} e^{-wt} \prod_{i=1}^{d} I_{n_{i}}(t) dt$$

where $\mathbf{n} = \{n_1, \ldots, n_d\}$ is a set of non-negative integers, $w \ge d$, k > 0 and Γ is the gamma function. Here the lattice Green function will only appear with $\mathbf{n} = 0$, hence we denote it shortly by $G_d(k, w)$. In [8], numerical evaluations of the integrals c_d and W_d are computed and also in [2] for c_d and in [7] for W_d and $G_d(1, w)$. For $d \ge 5$,

$$\begin{split} \lim_{n \to \infty} (n^4 (\mathcal{I}_d^d(s/n) - c_d) - n^2 s^2 W_d) &= \lim_{n \to \infty} n^4 \int_0^\infty \left(1 - e^{-(s/n)^2 t} - \frac{s^2}{n^2} t \right) e^{-2dt} I_0(2t)^d \frac{dt}{t} \\ &= -\frac{s^4}{8} G_d(2, d). \end{split}$$

Continuing in this way, we arrive at the following expansion for $\mathcal{I}_d^d(s/n)$ as $n\to\infty$

$$n^{d} \mathcal{I}_{d}^{d}(s/n) = c_{d} n^{d} + \sum_{k=1}^{\lfloor (d-1)/2 \rfloor} (-1)^{k+1} n^{d-2k} \frac{s^{2k}}{k2^{k}} G_{d}(k,d) + o(n).$$
(14)

Recall that for $m \in \{1, \ldots, d-1\}$,

$$\mathcal{I}_{\mathfrak{m}}^{\mathfrak{d}}(s/\mathfrak{n}) = -\frac{1}{4^{\mathfrak{d}-\mathfrak{m}}} \int_{0}^{\infty} (1-e^{-4t})^{\mathfrak{d}-\mathfrak{m}} e^{-((s/\mathfrak{n})^{2}+2\mathfrak{m})t} I_{0}(2t)^{\mathfrak{m}} \frac{dt}{t}.$$

We have

$$\lim_{n\to\infty}\mathcal{I}_{\mathfrak{m}}^{\mathfrak{d}}(\mathfrak{s}/\mathfrak{n})=\mathcal{I}_{\mathfrak{m}}^{\mathfrak{d}}(0).$$

Similarly as for m=d, we obtain as $n\to\infty$

$$n^{m} \mathcal{I}_{m}^{d}(s/n) = \mathcal{I}_{m}^{d}(0) n^{m} + \frac{1}{4^{d-m}} \sum_{k=1}^{\lfloor (m-1)/2 \rfloor} (-1)^{k} n^{m-2k} \frac{s^{2k}}{k! 2^{k}} \int_{0}^{\infty} (1 - e^{-2t})^{d-m} e^{-mt} I_{0}(t)^{m} t^{k-1} dt + o(n).$$
(15)

The above integral can be expressed in terms of the generalised lattice Green function:

$$\frac{1}{(k-1)!}\int_0^\infty (1-e^{-2t})^{d-m}e^{-mt}I_0(t)^mt^{k-1}dt = \sum_{l=0}^{d-m}C_{d-m}^l(-1)^lG_m(k,m+2l).$$

Putting equations (14) and (15) together gives the expansion for $d \geqslant 3$

$$\begin{split} \sum_{j \neq 0} \log((s/n)^2 + \lambda_j^L) &= V_d^d c_d n^d + \sum_{m=1}^{d-1} V_m^d \mathcal{I}_m^d(0) n^m + \sum_{k=1}^{\lfloor (d-1)/2 \rfloor} \frac{s^{2k}}{k2^k} \bigg((-1)^{k+1} V_d^d G_d(k, d) n^{d-2k} \\ &+ \delta_{d \geqslant 4} \frac{(-1)^k}{(k-1)!} \sum_{m=2k+1}^{d-1} V_m^d n^{m-2k} \frac{1}{4^{d-m}} \int_0^\infty (1 - e^{-2t})^{d-m} e^{-mt} I_0(t)^m t^{k-1} dt \bigg) + o(n) \end{split}$$

where $\delta_{d\geqslant d_0}=1 \text{ if } d\geqslant d_0 \text{ and } 0$ otherwise. For d=3,

$$\sum_{j \neq 0} \log((s/n)^2 + \lambda_j^L) = V_3^3 c_3 n^d + V_1^3 \mathcal{I}_1^3(0) n + V_2^3 \mathcal{I}_2^3(0) n^2 + \frac{s^2}{2} V_3^3 W_3 n + o(n)$$

with the special values (W_3 is given in [8])

$$\mathcal{I}_{1}^{3}(0) = \frac{1}{16} \log((17 + 2\sqrt{2})(5 - 2\sqrt{6})) \text{ and } W_{3} = \frac{1}{96\pi^{3}}(\sqrt{3} - 1)\left(\Gamma(1/24)\Gamma(11/24)\right)^{2}.$$

For $d \ge 4$,

$$\begin{split} &\sum_{j \neq 0} \log((s/n)^2 + \lambda_j^L) = V_d^d c_d n^d + \sum_{m=1}^{d-1} V_m^d \mathcal{I}_m^d(0) n^m \\ &+ \frac{s^2}{2} \left(V_d^d W_d n^{d-2} - \sum_{m=3}^{d-1} V_m^d n^{m-2} \frac{1}{4^{d-m}} \int_0^\infty (1 - e^{-2t})^{d-m} e^{-mt} I_0(t)^m dt \right) \\ &+ \frac{s^4}{8} \left(-\delta_{d \geqslant 5} V_d^d G_d(2, d) n^{d-4} + \delta_{d \geqslant 6} \sum_{m=5}^{d-1} V_m^d n^{m-4} \frac{1}{4^{d-m}} \int_0^\infty t (1 - e^{-2t})^{d-m} e^{-mt} I_0(t)^m dt \right) \\ &+ (\text{terms in } s^k \text{ with } k \geqslant 3) + o(n) \end{split}$$

 $\text{ as }n\to\infty.$

On the other hand, the formal expansion of $\sum_{j\neq 0}\log((s/n)^2+\lambda_j^L)$ gives

$$\sum_{j\neq 0} \log((s/n)^2 + \lambda_j^L) = \log\left(\prod_{j\neq 0} \lambda_j^L\right) + \sum_{p\geqslant 1} \frac{(-1)^{p-1}}{p} \left(\frac{s}{n}\right)^{2p} \sum_{j\neq 0} \frac{1}{(\lambda_j^L)^p}.$$

84

By identification of the terms in $s^2,$ we find the asymptotic number of rooted 2-spanning forests for d=3, as $n\to\infty$

$$N_2^3 = \left(\frac{V_3^3}{2}W_3n^3 + o(n^3)\right)N_1^3$$

and for $d \ge 4$, we have

$$\sum_{j \neq 0} \frac{1}{\lambda_j^L} = \frac{V_d^d}{2} W_d n^d - \sum_{m=3}^{d-1} \frac{V_m^d}{2} n^m \frac{1}{4^{d-m}} \int_0^\infty (1 - e^{-2t})^{d-m} e^{-mt} I_0(t)^m dt + o(n^3)$$

so that

$$N_{2}^{d} = \left(\frac{V_{d}^{d}}{2}W_{d}n^{d} - \sum_{m=3}^{d-1}\frac{V_{m}^{d}}{2}n^{m}\frac{1}{4^{d-m}}\int_{0}^{\infty}(1-e^{-2t})^{d-m}e^{-mt}I_{0}(t)^{m}dt + o(n^{3})\right)N_{1}^{d}$$

where N_1^d is asymptotically given by Theorem 1.1. By identification of the terms in $s^4,$ we find that as $n\to\infty$

$$\sum_{j \neq 0} \frac{1}{(\lambda_j^L)^2} = O(n^d)$$

so that the asymptotic number of rooted 3-spanning forests for d = 4 is given by

$$N_3^4 = \left(\frac{(V_4^4 W_4)^2}{8}n^8 - \frac{V_3^4 V_4^4}{8}W_4 n^7 \int_0^\infty (1 - e^{-2t})e^{-3t}I_0(t)^3 dt + o(n^7)\right)N_1^4, \text{ as } n \to \infty.$$

Remark. It would be interesting to find the next terms in the development. For the 2-dimensional case, we would need to find the asymptotic development as $n \to \infty$ of the following integral

$$\int_0^\infty n^2 (1 - e^{-(s/n)^2 t}) e^{-4t} I_0(2t)^2 \frac{dt}{t}.$$

In [5], the authors computed the asymptotic development of $\sum_{j \neq 0} \log((s/n)^2 + \lambda_j)$ in the case of the torus with other techniques. To generalise their result to higher dimensions with our techniques, for example in the 3-dimensional case, we would need to find the asymptotic development of

$$\int_{0}^{\infty} (n^{3}(1 - e^{-(s/n)^{2}t}) - ns^{2}t)e^{-6t}I_{0}(2t)^{3}\frac{dt}{t}$$

as $n \to \infty$. This would enable us to derive asymptotics for the number of rooted k-spanning forests with $k \ge 4$.

2.5 Spanning trees in two-dimensional square lattices

In the two-dimensional case, one can derive an exact relation between the number of spanning trees on the rectangular square lattice $n_1 \times n_2$ and the one on the torus of size $2n_1 \times 2n_2$. The product of the non-zero eigenvalues on $T(2n_1, 2n_2)$ is given by

$$\det^*\Delta_{\mathsf{T}(2n_1,2n_2)} = \prod_{\substack{k_1=0\\(k_1,k_2)\neq 0}}^{2n_1-1} \prod_{\substack{k_2=0\\(k_1,k_2)\neq 0}}^{2n_2-1} (4-2\cos(\pi k_1/n_1)-2\cos(\pi k_2/n_2)).$$

The product over k_1 , k_2 is a disjoint union of products over four squares of size $n_1 \times n_2$. We split this product as a product over the 0, 1 and 2 dimensional faces of the squares. It comes

$$det^{*}\Delta_{T(2n_{1},2n_{2})} = 4^{2}8 \prod_{k_{1}=1}^{n_{1}-1} (2 - 2\cos(\pi k_{1}/n_{1}))^{2} \prod_{k_{2}=1}^{n_{2}-1} (2 - 2\cos(\pi k_{2}/n_{2}))^{2} \\ \times \prod_{k_{1}=1}^{n_{1}-1} (6 - 2\cos(\pi k_{1}/n_{1}))^{2} \prod_{k_{2}=1}^{n_{2}-1} (6 - 2\cos(\pi k_{2}/n_{2}))^{2} \\ \times \prod_{k_{1}=1}^{n_{1}-1} \prod_{k_{2}=1}^{n_{2}-1} (4 - 2\cos(\pi k_{1}/n_{1}) - 2\cos(\pi k_{2}/n_{2}))^{4}.$$
(16)

On the other hand, the product of the non-zero eigenvalues on the square lattice $L(n_1, n_2)$ is given by

$$\det^* \Delta_{L(n_1,n_2)} = \prod_{\substack{k_1=0 \\ (k_1,k_2) \neq 0}}^{n_1-1} \prod_{\substack{k_2=0 \\ (k_1,k_2) \neq 0}}^{n_2-1} (4 - 2\cos(\pi k_1/n_1) - 2\cos(\pi k_2/n_2)).$$

By splitting the above product as a product when $k_1 = 0$, then $k_2 = 0$, then $1 \le k_1 \le n_1 - 1$, $1 \le k_2 \le n_2 - 1$, we get

$$det^* \Delta_{L(n_1,n_2)} = \prod_{k_1=1}^{n_1-1} (2 - 2\cos(\pi k_1/n_1)) \prod_{k_2=1}^{n_2-1} (2 - 2\cos(\pi k_2/n_2)) \\ \times \prod_{k_1=1}^{n_1-1} \prod_{k_2=1}^{n_2-1} (4 - 2\cos(\pi k_1/n_1) - 2\cos(\pi k_2/n_2)).$$
(17)

Using the matrix tree theorem and putting the following identities coming from relations for Chebyshev polynomials of the second kind

$$\prod_{k=1}^{n-1} (2 - 2\cos(\pi k/n)) = n \text{ and } \prod_{k=1}^{n-1} (6 - 2\cos(\pi k/n)) = \frac{(3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n}{4\sqrt{2}},$$

in equations (16) and (17), it follows that

$$\tau(L(\mathfrak{n}_1,\mathfrak{n}_2)) = \frac{2^{5/4}\tau(T(2\mathfrak{n}_1,2\mathfrak{n}_2))^{1/4}}{(\mathfrak{n}_1\mathfrak{n}_2)^{1/4}((3+2\sqrt{2})^{\mathfrak{n}_1} - (3-2\sqrt{2})^{\mathfrak{n}_1})^{1/2}((3+2\sqrt{2})^{\mathfrak{n}_2} - (3-2\sqrt{2})^{\mathfrak{n}_2})^{1/2}}$$

Remark. It would be interesting to see if one could generalise the above relation to higher dimensions. It could not be done in the same way as it is done above. More precisely, when splitting the product in 3 dimensions as a product over 0, 1, 2 and 3 dimensional faces, one would need for example to evaluate the following product

$$\prod_{k_1=1}^{n_1-1} \prod_{k_2=1}^{n_2-1} (8 - 2\cos(\pi k_1/n_1) - 2\cos(\pi k_2/n_2))$$

appearing for the 2-dimensional face defined by $k_3 = n_3$ and $k_1 = 1, \ldots, n_1 - 1, k_2 = 1, \ldots, n_2 - 1$.

3 Asymptotic number of spanning trees in the quartered Aztec diamond

3.1 A relation between the number of spanning trees on the quartered Aztec diamond and on the square lattice

In [4, 10], the authors showed that the number of spanning trees in the quartered Aztec diamond of side length n is given by

$$\tau(\text{QAD}_n) = \prod_{0 < k_1 < k_2 < n} (4 - 2\cos(\pi k_1/n) - 2\cos(\pi k_2/n)).$$

The product of the non-zero eigenvalues on the square grid of side n is given by

$$\det^*\Delta_{L(n,n)} = \prod_{\substack{k_1=0\\(k_1,k_2)\neq 0}}^{n-1} \prod_{\substack{k_2=0\\(k_1,k_2)\neq 0}}^{n-1} (4-2\cos(\pi k_1/n)-2\cos(\pi k_2/n)).$$

By splitting this product as a product when $k_1 = 0$, then $k_2 = 0$, then $k_1 = k_2$, $k_1 = 1, \ldots, n-1$, then $k_1 < k_2$ and $k_2 < k_1$, we have

$$\begin{split} \det^* \Delta_{L(n,n)} &= \prod_{k=1}^{n-1} (2 - 2\cos(\pi k/n))^2 \prod_{k=1}^{n-1} (4 - 4\cos(\pi k/n)) \\ &\times \prod_{1 \leqslant k_1 < k_2 \leqslant n-1} (4 - 2\cos(\pi k_1/n) - 2\cos(\pi k_2/n))^2. \end{split}$$

From the matrix tree theorem, it follows that

$$\tau(\text{QAD}_{n}) = \frac{\tau(L(n,n))^{1/2}}{\sqrt{n2^{(n-1)/2}}}.$$
(18)

3.2 Asymptotic expansion

From (18), we have

$$\log(\tau(QAD_n)) = \frac{1}{2}\log \det^* \Delta_{L(n,n)} - \frac{n}{2}\log 2 - \frac{3}{2}\log n + \frac{1}{2}\log 2$$
(19)

where the asymptotic behaviour of $\log \det^* \Delta_{L(n,n)}$ is given by

$$\log \det^* \Delta_{\mathsf{L}(\mathfrak{n},\mathfrak{n})} = \frac{4\mathsf{G}}{\pi} \mathfrak{n}^2 - 2\mathfrak{n}\log(1+\sqrt{2}) + \frac{3}{2}\log\mathfrak{n} - \zeta_{1\times 1}'(0) - 2\zeta_1'(0) - \frac{1}{4}\log 2 + \mathfrak{o}(1)$$
(20)

as $n \to \infty$.

Consider the right-angled isosceles triangle with the sides of same length equal to 1. The eigenvalues of the Laplace-Beltrami operator with Dirichlet boundary conditions are given by

$$\lambda_k = \pi^2 (k_1^2 + k_2^2), \quad \text{with } k = (k_1, k_2) \in (\mathbb{N}^*)^2 \ \text{and} \ k_1 > k_2.$$

The associated spectral zeta function with Dirichlet boundary conditions, denoted by ζ_{Δ} , is then given by

$$\zeta_{\Delta}(s) = \frac{1}{\pi^{2s}} \sum_{1 \leqslant k_2 < k_1} \frac{1}{(k_1^2 + k_2^2)^s}.$$

The spectral zeta function on the unit square with Dirichlet boundary conditions is given by

$$\zeta_{1\times 1}(s) = \frac{1}{\pi^{2s}} \sum_{k_1, k_2 \geqslant 1} \frac{1}{(k_1^2 + k_2^2)^s} = 2\zeta_{\Delta}(s) + 2^{-s}\zeta_1(s).$$

The spectral zeta function on the unit interval with Dirichlet boundary conditions is related to the Riemann zeta function by $\zeta_1(s) = (2/\pi^{2s})\zeta(2s)$ with special values in 0, $\zeta(0) = -1/2$ and $\zeta'(0) = -(1/2)\log(2\pi)$. Thus we have that $\zeta'_1(0) = -2\log 2$. By differentiating the above and evaluating in s = 0, we get

$$\zeta_{1\times 1}'(0) = 2\zeta_{\Delta}'(0) - \log 2.$$
⁽²¹⁾

Putting (19), (20), (21) together and writing the derivative of the spectral zeta function in 0 in terms of the regularized determinant of the Laplace-Beltrami operator on the right-angled isosceles unit triangle with Dirichlet boundary conditions, that is

$$\zeta_{\Delta}'(0) = -\log \det^* \Delta_{\Delta},$$

gives the asymptotic behaviour of the number of spanning trees on the quartered Aztec diamond, namely

$$\log(\tau(\text{QAD}_n)) = \frac{2\mathsf{G}}{\pi}n^2 - n\log(2+\sqrt{2}) - \frac{3}{4}\log n + \log \det^*\Delta_\Delta + \frac{23}{8}\log 2 + o(1)$$

as $n \to \infty$.

References

- Norman Biggs. Algebraic graph theory. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 1993.
- [2] Gautam Chinta, Jay Jorgenson, and Anders Karlsson. Zeta functions, heat kernels, and spectral asymptotics on degenerating families of discrete tori. *Nagoya Math. J.*, 198:121–172, 2010.
- [3] Gautam Chinta, Jay Jorgenson, and Anders Karlsson. Complexity and heights of tori. In Dynamical systems and group actions, volume 567 of Contemp. Math., pages 89–98. Amer. Math. Soc., Providence, RI, 2012.
- [4] Mihai Ciucu. Symmetry classes of spanning trees of Aztec diamonds and perfect matchings of odd squares with a unit hole. J. Algebraic Combin., 27(4):493–538, 2008.
- [5] Bertrand Duplantier and François David. Exact partition functions and correlation functions of multiple Hamiltonian walks on the Manhattan lattice. J. Statist. Phys., 51(3-4):327–434, 1988.
- [6] M. L. Glasser. A note on a hyper-cubic Mahler measure and associated Bessel integral. J. Phys. A, 45(49):494002, 4, 2012.
- [7] G. S. Joyce. Singular behaviour of the lattice Green function for the d-dimensional hypercubic lattice. J. Phys. A, 36(4):911–921, 2003.
- [8] G. S. Joyce and I. J. Zucker. Evaluation of the Watson integral and associated logarithmic integral for the d-dimensional hypercubic lattice. J. Phys. A, 34(36):7349–7354, 2001.

- [9] Richard Kenyon. The asymptotic determinant of the discrete Laplacian. Acta Math., 185(2):239–286, 2000.
- [10] Richard W. Kenyon, James G. Propp, and David B. Wilson. Trees and matchings. *Electron. J. Combin.*, 7:Research Paper 25, 34 pp. (electronic), 2000.
- [11] Justine Louis. Asymptotics for the Number of Spanning Trees in Circulant Graphs and Degenerating d-Dimensional Discrete Tori. *Ann. Comb.*, 19(3):513–543, 2015.
- [12] Justine Louis. A formula for the number of spanning trees in circulant graphs with nonfixed generators and discrete tori. Bulletin of the Australian Mathematical Society, 92(3):365–373, 2015.
- [13] Nicolai Reshetikhin and Boris Vertman. Combinatorial quantum field theory and gluing formula for determinants. *Lett. Math. Phys.*, 105(3):309–340, 2015.

Low temperature ratchet current*

Justine Louis

5th August 2015

Abstract

In [3], the low temperature ratchet current in a multilevel system is considered. In this note, we give an explicit expression for it and find its numerical value as the number of states goes to infinity.

1 Introduction

In this note, we compute the stationary ratchet current in the large system size limit. In [3], the authors derive a formula for the occupation of a general multilevel system at low temperature. As an application, they consider a continuous time version of Parrondo's game at low temperature (see [4]) and give an expression for the ratchet current. We consider a multilevel system determined by a finite number of states. The set of all states is denoted by K. The ratchet is modelised by two rings of N states. In the present section, we recall the definitions and results from [3, section 3] and in the next section we give an explicit expression for the ratchet current using the Tutte matrix tree theorem and find its limit as the number of states goes to infinity. The states on the outer ring are denoted by (0, i) and on the inner ring by (1, i), where i = 1, ..., N. The energies are denoted by E_i , i = 1, ..., N and are such that $E_1 < \cdots < E_N$. The transition rates on the outer ring are given by

$$\lambda((i,0),(i+1,0)) = e^{\beta(E_i - E_{i+1})/2}, \quad \lambda((i+1,0),(i,0)) = e^{\beta(E_{i+1} - E_i)/2}$$

where β is the inverse temperature. On the inner ring, the transition rates are constant and equal to one, that is,

$$\lambda((i, 1), (i + 1, 1)) = \lambda((i + 1, 1), (i, 1)) = 1.$$

The two rings are connected with transition rates constant equal to one,

$$\lambda((i, n), (i, 1 - n)) = 1$$
, where $n = 0, 1$.

The zero-temperature logarithmic limit denoted by $\phi(x, y)$ is given by

$$\phi(\mathbf{x},\mathbf{y}) := \lim_{\beta \to \infty} \frac{1}{\beta} \log \lambda(\mathbf{x},\mathbf{y}).$$

The zero-temperature logarithmic limit of the escape rates of state x is denoted by $\Gamma(x)$ and given by

$$\Gamma(x) := -\lim_{\beta \to \infty} \frac{1}{\beta} \log \big(\sum_y \lambda(x,y) \big) = -\max_y \varphi(x,y).$$

^{*}arXiv:1508.01189. The author was supported in part by the Swiss NSF grant 200021 132528/1.

The logarithmic-asymptotic transition probability is given by $e^{-\beta U(x,y)}$ where

$$U(x, y) := -\Gamma(x) - \phi(x, y).$$

We have $U(x,y) \ge 0$ for all $x, y \in K$. The smaller U(x,y) is, the larger is the probability of transition from state x to state y. Hence, the set of preferred successors of x is defined by

$$\{y \in K \mid U(x,y) = 0\}.$$

When U(x, y) = 0, the probability of transition from x to y is high. Thus we consider the directed graph K^D defined by the vertex set K and edge set $\{(x, y) | U(x, y) = 0\}$ where (x, y) indicates an oriented edge from x to y. The digraph K^D is represented in Figure 1 below. The low temperature asymptotic of the stationary occupation is given in the following theorem from [3]:



Figure 1: The directed graph K^{D} .

Theorem ([3, Theorem 2.1]). *There is* $\varepsilon > 0$ *so that as* $\beta \to \infty$,

$$\rho(\mathbf{x}) = \frac{1}{\mathcal{Z}} A(\mathbf{x}) e^{\beta(\Gamma(\mathbf{x}) - \Theta(\mathbf{x}))} (1 + O(e^{-\beta \varepsilon}))$$

with

$$\begin{split} \Theta(\mathbf{x}) &\coloneqq \min_{\mathcal{T}} U(\mathcal{T}_{\mathbf{x}}) \quad \text{for} \quad U(\mathcal{T}_{\mathbf{x}}) \coloneqq \sum_{(\mathbf{y}, \mathbf{y}') \in \mathcal{T}_{\S}} U(\mathbf{y}, \mathbf{y}') \quad \text{and} \\ A(\mathbf{x}) &\coloneqq \sum_{\mathcal{T} \in \mathcal{M}(\mathbf{x})} \prod_{(\mathbf{y}, \mathbf{y}') \in \mathcal{T}_{\mathbf{x}}} \alpha(\mathbf{y}, \mathbf{y}') = e^{\mathbf{o}(\beta)} \end{split}$$

where the last sum runs over all spanning trees minimizing $U(\mathcal{T}_x)$ (i.e. $\mathcal{T} \in M(x)$ if $\Theta(x) = U(\mathcal{T}_X)$), and $\mathfrak{a}(x, y)$ are the reactivities, which are the sub-exponential part of the transition rates $\lambda(x, y)$. Here all the reactivities are constant equal to one, a(x, y) = 1 for all $x, y \in K$. In the present case, for all $x \in K$, there exists an in-spanning tree \mathcal{T}_x in K^D , so that $U(\mathcal{T}_x) = 0$, and therefore $\Theta(x) = 0$. Let \mathcal{D} be the set of states for which $\Gamma(x) = 0$, it is given by $\mathcal{D} = \{(1,0), (i,1), i = 1, \ldots, N\}$. We denote $f \simeq g$ if $f = g + O(e^{-\beta \varepsilon})$ as $\beta \to \infty$. For $x \in \mathcal{D}$, we have $\rho(x) \simeq |M(x)|/\mathcal{Z}$, where |M(x)| is the number of in-spanning trees in K^D . For $x \notin \mathcal{D}$, the stationary distribution is exponentially small since from the theorem it is given by $\rho(x) \simeq |M(x)|e^{\beta\Gamma(x)}/\mathcal{Z}$, with $\Gamma(x) < 0$. The stationary ratchet current in the clockwise direction is given by

$$J_R = j((i+1,0),(i,0)) + j((i+1,1),(i,1)), \quad \text{for} \quad i=1,\ldots,N,$$

where $j(x, y) = \lambda(x, y)\rho(x) - \lambda(y, x)\rho(y)$. For i = 1,

$$J_{R} = j((2,0), (1,0)) + j((2,1), (1,1))$$

On the outer ring, we have $j((2,0),(1,0))=\lambda((2,0),(1,0))\rho(2,0)-\lambda((1,0),(2,0))\rho(1,0)$ with

$$\begin{split} \lambda((1,0),(2,0)) &\simeq 0, \quad \lambda((2,0),(1,0)) = e^{(\mathsf{E}_2 - \mathsf{E}_1)\beta/2} \\ \rho(2,0) &\simeq \frac{|\mathsf{M}(2,0)|}{\mathcal{Z}} e^{\beta\Gamma(2,0)} = \frac{|\mathsf{M}(2,0)|}{\mathcal{Z}} e^{-(\mathsf{E}_2 - \mathsf{E}_1)\beta/2}, \end{split}$$

so that $j((2,0), (1,0)) \simeq |M(2,0)|/\mathcal{Z}$.

On the inner ring, we have $j((2,1),(1,1)) = \lambda((2,1),(1,1))\rho(2,1) - \lambda((1,1),(2,1))\rho(1,1)$ with

$$egin{aligned} \lambda((2,1),(1,1)) &= \lambda((1,1),(2,1)) = 1, \ &
ho(2,1) \simeq rac{|\mathsf{M}(2,1)|}{\mathcal{Z}}, \quad &
ho(1,1) \simeq rac{|\mathsf{M}(1,1)|}{\mathcal{Z}} \end{aligned}$$

,

so that $j((2,1),(1,1)) \simeq (|M(2,1)| - |M(1,1)|)/\mathcal{Z}$. The ratchet current is thus given by

$$J_R \simeq \frac{1}{\mathcal{Z}}(|M(2,0)| + |M(2,1)| - |M(1,1)|).$$

Considering converging arborescences, the Laplacian matrix of a directed graph is defined by L = D - A where D is the diagonal out-degree matrix and $A = (A_{ij})$ is the adjacency matrix such that A_{ij} is the number of directed edges from i to j. The rows and columns of L are indexed by the vertices of the graph. Here, we index it first by the states on the outer ring then the ones on the inner ring, that is $(1,0), (2,0), \ldots, (N,0), (1,1), (2,1), \ldots, (N,1)$. The Tutte matrix tree theorem (see [1]) relates the number of spanning arborescences converging to x in K^D to the cofactors of the Laplacian det $L_{x,y}$. Let $x \in K$. Then for all $y \in K$,

$$|M(x)| = (-1)^{x+y} \det L_{x,y}.$$

In particular, for y = x, we have $|M(x)| = \det L_x$. Therefore we have

$$J_R \simeq \frac{1}{\mathcal{Z}} (\det L_{(2,1)} + \det L_{(2,0)} - \det L_{(1,1)}).$$

The Laplacian matrix is given by

$$\mathbf{L} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{Id} & \mathbf{C} \end{pmatrix}$$

where A is the $N \times N$ lower triangular matrix given by

$$A = \begin{pmatrix} 1 & & & \\ -1 & 1 & & & \\ & \ddots & \ddots & & \\ & & -1 & 1 & \\ -1 & & & 0 & 1 \end{pmatrix},$$

B is the N × N matrix such that all coefficients are zero except $B_{(1,0),(1,1)} = -1$, the matrix Id is the N × N identity matrix and C is the following circulant matrix

$$C = \begin{pmatrix} 3 & -1 & & -1 \\ -1 & 3 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ -1 & & -1 & 3 \end{pmatrix}.$$

2 Calculation of the ratchet current

From [3], the numerator of J_R is given by

$$\det L_{(2,1)} + \det L_{(2,0)} - \det L_{(1,1)} = \det B_{N-1} - 2 \det B_{N-2} - 2$$

where B_N is the $N \times N$ tridiagonal matrix with 3 on the diagonal and -1 on the two off-diagonals which satisfies the recurrence relation det $B_N = 3 \det B_{N-1} - \det B_{N-2}$ with det $B_1 = 3$ and det $B_2 = 8$. By solving the associated characteristic equation, it comes

$$\det B_{N} = \frac{5 - 3\sqrt{5}}{10} \left(\frac{3 - \sqrt{5}}{2}\right)^{N} + \frac{5 + 3\sqrt{5}}{10} \left(\frac{3 + \sqrt{5}}{2}\right)^{N}.$$

The normalisation factor is given by

$$\mathcal{Z} = \sum_{x \in K} \sum_{\mathcal{T}_x} \prod_{(y,z) \in \mathcal{T}_x} \lambda(y,z) \simeq \sum_{x \in \mathcal{D}} \lvert M(x) \rvert = \sum_{x \in \mathcal{D}} \det L_x.$$

The sum is over the states in \mathcal{D} since the contribution of the states which are not in \mathcal{D} is exponentially damped. Therefore we have

$$\mathcal{Z} \simeq \det L_{(1,0)} + \sum_{\mathfrak{i}=1}^{N} \det L_{(\mathfrak{i},1)}. \tag{1}$$

We have

$$\det L_{(1,0)} = \det C.$$

The circulant matrix C has eigenvalues given by $\mu_j = 3 - 2\cos(2\pi j/N)$, $j = 0, 1, \dots, N-1$ (see [2]). Hence

$$\det L_{(1,0)} = \prod_{j=0}^{N-1} (3 - 2\cos(2\pi j/N)) = U_{N-1}^2(\sqrt{5}/2)$$

where U_N is the Chebyshev polynomial of the second kind. Thus

$$\det L_{(1,0)} = \left(\frac{3+\sqrt{5}}{2}\right)^{N} + \left(\frac{3-\sqrt{5}}{2}\right)^{N} - 2.$$
(2)

From the Tutte matrix tree theorem, the cofactor $(-1)^{N+i} \det L_{(i,1)}$ is equal to the number of converging arborescences to (i,1) and is equal to the cofactor of the Laplacian where row (i,1)

and any column is removed. Since the only non-zero element of B is in column indexed by (1, 1), we choose to remove that one, so that

$$|\mathsf{M}(\mathfrak{i},1)| = (-1)^{(\mathsf{N}+\mathfrak{i})+(\mathsf{N}+1)} \det \mathsf{L}_{(\mathfrak{i},1),(1,1)} = (-1)^{\mathfrak{i}+1} \det \mathsf{C}_{(\mathfrak{i},1),(1,1)}$$
(3)

since A is lower triangular. On the other hand, by adding to the first column of C all the other ones, we have

$$\det \mathbf{C} = \begin{vmatrix} 1 & -1 & & -1 \\ 1 & 3 & \ddots & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ 1 & & & -1 & 3 \end{vmatrix} = \sum_{i=1}^{N} (-1)^{i+1} \det \mathbf{C}_{(i,1),(1,1)}.$$
(4)

Putting equations (1), (2), (3) and (4) together, we have

$$\mathcal{Z} \simeq 2 \det C = 2 \left(\frac{3 + \sqrt{5}}{2} \right)^N + 2 \left(\frac{3 - \sqrt{5}}{2} \right)^N - 4.$$

Up to exponentially small corrections $e^{-\beta \epsilon}$, the ratchet current is given for all N by

$$\begin{split} J_{\mathsf{R}} \simeq & \left(\frac{5+3\sqrt{5}}{10} \left(\frac{3+\sqrt{5}}{2}\right)^{\mathsf{N}-1} + \frac{5-3\sqrt{5}}{10} \left(\frac{3-\sqrt{5}}{2}\right)^{\mathsf{N}-1} - \frac{5+3\sqrt{5}}{5} \left(\frac{3+\sqrt{5}}{2}\right)^{\mathsf{N}-2} \\ & -\frac{5-3\sqrt{5}}{5} \left(\frac{3-\sqrt{5}}{2}\right)^{\mathsf{N}-2} - 2 \right) \left/ \left(2((3+\sqrt{5})/2)^{\mathsf{N}} + 2((3-\sqrt{5})/2)^{\mathsf{N}} - 4\right). \end{split}$$

As a consequence, in the large system size limit the current saturates and has the following limit

$$\lim_{N\to\infty}J_R\simeq \frac{1}{2}-\frac{1}{\sqrt{5}}.$$

Acknowledgements: The author thanks Anders Karlsson for suggesting this problem to her.

References

- [1] Martin Aigner. A course in enumeration, volume 238 of Graduate Texts in Mathematics. Springer, Berlin, 2007.
- [2] Norman Biggs. Algebraic graph theory. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 1993.
- [3] Christian Maes, Karel Netočný, and Winny O'Kelly de Galway. Low temperature behavior of nonequilibrium multilevel systems. *Journal of Physics A: Mathematical and Theoretical*, 47(3):035002, 2014.
- [4] Juan MR Parrondo. Reversible ratchets as brownian particles in an adiabatically changing periodic potential. *Physical Review E*, 57(6):7297, 1998.

A formula for the energy of circulant graphs with two generators^{*}

Justine Louis

10th August 2015

Abstract

In this note, we derive closed formulas for the energy of circulant graphs generated by 1 and γ , where $\gamma \ge 2$ is an integer. We also find a formula for the energy of the complete graph without a Hamilton cycle.

Let $1 \leq \gamma_1 \leq \cdots \leq \gamma_d$ be integers. The circulant graph $C_n^{\gamma_1,\dots,\gamma_d}$ generated by γ_1,\dots,γ_d on n vertices labelled $0, 1, \dots, n-1$, is the 2d-regular graph such that for all $\nu \in \mathbb{Z}/n\mathbb{Z}$, ν is connected to $\nu + \gamma_i \mod n$ and to $\nu - \gamma_i \mod n$, for all $i = 1, \dots, d$. The adjacency matrix $A = (A_{ij})$ of a graph on n vertices is the $n \times n$ matrix with rows and columns indexed by the vertices such that A_{ij} is the number of edges connecting vertices i and j. Let λ_k , $k = 1, \dots, n$, denote the eigenvalues of the adjacency matrix. The energy of a graph G on n vertices is defined by the sum of the absolute value of the eigenvalues of A, that is

$$\mathsf{E}(\mathsf{G}) = \sum_{k=1}^{n} |\lambda_k|.$$

The energy of circulant graphs and integral circulant graphs is widely studied, see for example [4, 5, 6, 7]. It has interesting applications in theoretical chemistry, namely, it is related to the π -electron energy of a conjugated carbon molecule, see [2]. In the following theorem, we give a formula for the energy of circulant graphs with two generators, 1 and γ , $\gamma \ge 2$. The formula is interesting as n is larger than γ .

Theorem. Let $D_n(x)$ denote the Dirichlet kernel. The energy of the circulant graph $C_n^{1,2}$ is given by

$$\mathsf{E}(\mathsf{C}_{n}^{1,2}) = 4 \left(\mathsf{D}_{|n/6|}(2\pi/n) + \mathsf{D}_{|n/6|}(4\pi/n) \right).$$

For $\gamma \ge 3$, the energy of the circulant graph $C_n^{1,\gamma}$ is given by

$$\mathsf{E}(C_{n}^{1,\gamma}) = 4\sum_{\mathfrak{m}\in\{1,\gamma\}} \left(\sum_{l=0}^{\lceil \gamma/2 \rceil - 1} \mathsf{D}_{\lfloor(2l+1)\mathfrak{n}/(2(\gamma+1))\rfloor} \left(\frac{2\pi\mathfrak{m}}{\mathfrak{n}}\right) - \sum_{l=0}^{\lceil \gamma/2 \rceil - 2} \mathsf{D}_{\lfloor(2l+1)\mathfrak{n}/(2(\gamma-1))\rfloor} \left(\frac{2\pi\mathfrak{m}}{\mathfrak{n}}\right) \right)$$

where $\lfloor x \rfloor$ denotes the greatest integer smaller or equal to x and $\lceil x \rceil$ denotes the smallest integer greater or equal to x.

^{*}arXiv:1508.02348. The author was supported in part by the Swiss NSF grant 200021 132528/1.

Proof. The adjacency matrix of a circulant graph is circulant, it follows that the eigenvalues of $C_n^{1,\gamma}$ are given by $\lambda_k = 2\cos(2\pi k/n) + 2\cos(2\pi\gamma k/n)$, $k = 0, \ldots, n-1$, (see [3]). The energy of $C_n^{1,\gamma}$ is then given by

$$\mathsf{E}(C_n^{l,\gamma}) = 2\sum_{k=0}^{n-1} \lvert \cos(2\pi k/n) + \cos(2\pi \gamma k/n) \rvert.$$

Let $\gamma = 2$. The two roots of the equation $\cos x + \cos(2x) = 0$ for $x \in [0, \pi]$ are $\pi/3$ and π . We write the energy as

$$\begin{split} \mathsf{E}(\mathsf{C}_{\mathsf{n}}^{1,2}) &= 4 + 4\sum_{k=1}^{\lceil n/2 \rceil - 1} \lvert \cos(2\pi k/n) + \cos(4\pi k/n) \rvert \\ &= 4 + 4\sum_{k=1}^{\lfloor n/6 \rfloor} (\cos(2\pi k/n) + \cos(4\pi k/n)) - 4\sum_{k=\lfloor n/6 \rfloor + 1}^{\lceil n/2 \rceil - 1} (\cos(2\pi k/n) + \cos(4\pi k/n)). \end{split}$$

The sum of $\cos(kx)$ over consecutive k's can be expressed in terms of the Dirichlet kernel, namely

$$D_n(x) = 1 + 2\sum_{k=1}^n \cos(kx) = \frac{\sin((n+1/2)x)}{\sin(x/2)}.$$

As a consequence,

$$2\sum_{k=n+1}^{m}\cos(kx) = D_m(x) - D_n(x)$$

The energy of $C_n^{1,2}$ is thus given by

$$\mathsf{E}(\mathsf{C}_{\mathfrak{n}}^{1,2}) = 4\mathsf{D}_{\lfloor \mathfrak{n}/6 \rfloor}(2\pi/\mathfrak{n}) + 4\mathsf{D}_{\lfloor \mathfrak{n}/6 \rfloor}(4\pi/\mathfrak{n}) - 2\mathsf{D}_{\lceil \mathfrak{n}/2 \rceil - 1}(2\pi/\mathfrak{n}) - 2\mathsf{D}_{\lceil \mathfrak{n}/2 \rceil - 1}(4\pi/\mathfrak{n}).$$

The formula then follows from the fact that for odd n, $D_{(n-1)/2}(2\pi m/n) = 0$ for m = 1, 2, and for even n, $D_{n/2-1}(2\pi/n) = 1$ and $D_{n/2-1}(4\pi/n) = -1$.

Let $\gamma \ge 3$. For odd γ , the γ solutions of the equation $\cos x + \cos \gamma x = 0$ for $x \in [0, \pi]$ are given in the increasing order by $\pi/(\gamma+1), \pi/(\gamma-1), 3\pi/(\gamma+1), 3\pi/(\gamma-1), \dots, (\gamma-2)\pi/(\gamma-1), \gamma\pi/(\gamma+1)$. For even γ , they are given by $\pi/(\gamma+1), \pi/(\gamma-1), 3\pi/(\gamma+1), 3\pi/(\gamma-1), \dots, (\gamma-3)\pi/(\gamma-1), (\gamma-1)\pi/(\gamma+1), \pi$. Let n be odd. We split the sum over k of cosines to group the positive terms together and the negative terms together. The energy is given by

$$\begin{split} \mathsf{E}(\mathsf{C}_{n}^{1,\gamma}) &= 4 + 4 \sum_{k=1}^{(n-1)/2} |\cos(2\pi k/n) + \cos(2\pi\gamma k/n)| \\ &= 4 + 4 \sum_{k=1}^{\lfloor n/(2(\gamma+1)) \rfloor} (\cos(2\pi k/n) + \cos(2\pi\gamma k/n)) \\ &+ 4 \sum_{l=0}^{\lceil \gamma/2 \rceil - 2} \sum_{k=\lfloor (2l+1)n/(2(\gamma+1)) \rfloor + 1}^{\lfloor (2l+1)n/(2(\gamma+1)) \rfloor + 1} (\cos(2\pi k/n) + \cos(2\pi\gamma k/n)) \\ &- 4 \sum_{l=0}^{\lceil \gamma/2 \rceil - 1} \sum_{k=\lfloor (2l+1)n/(2(\gamma+1)) \rfloor + 1}^{\lfloor (2l+1)n/(2(\gamma+1)) \rfloor + 1} (\cos(2\pi k/n) + \cos(2\pi\gamma k/n)). \end{split}$$
(1)

96

Writing the above relation in terms of Dirichlet kernels, it comes

$$\begin{split} \mathsf{E}(\mathsf{C}_{n}^{l,\gamma}) &= 2 \sum_{\mathsf{m} \in \{l,\gamma\}} \left(\mathsf{D}_{\lfloor n/(2(\gamma+1)) \rfloor}(2\pi\mathsf{m}/\mathsf{n}) \\ &+ \sum_{l=0}^{\lceil \gamma/2 \rceil - 2} \left(\mathsf{D}_{\lfloor (2l+3)n/(2(\gamma+1)) \rfloor}(2\pi\mathsf{m}/\mathsf{n}) - \mathsf{D}_{\lfloor (2l+1)n/(2(\gamma-1)) \rfloor}(2\pi\mathsf{m}/\mathsf{n}) \right) \\ &- \sum_{l=0}^{\lceil \gamma/2 \rceil - 1} \left(\mathsf{D}_{\lfloor (2l+1)n/(2(\gamma-1)) \rfloor}(2\pi\mathsf{m}/\mathsf{n}) - \mathsf{D}_{\lfloor (2l+1)n/(2(\gamma+1)) \rfloor}(2\pi\mathsf{m}/\mathsf{n}) \right) \right). \end{split}$$
(2)

Hence

$$\begin{split} \mathsf{E}(C_{n}^{1,\gamma}) &= \sum_{m \in \{1,\gamma\}} \bigg(4 \sum_{l=0}^{\lceil \gamma/2 \rceil - 1} \mathsf{D}_{\lfloor (2l+1)n/(2(\gamma+1)) \rfloor} (2\pi m/n) \\ &- 4 \sum_{l=0}^{\lceil \gamma/2 \rceil - 2} \mathsf{D}_{\lfloor (2l+1)n/(2(\gamma-1)) \rfloor} (2\pi m/n) - 2\mathsf{D}_{\lfloor n/2 \rfloor} (2\pi m/n) \bigg). \end{split}$$
(3)

The formula follows from the fact that $D_{\lfloor n/2 \rfloor}(2\pi m/n) = 0$ for $m = 1, \gamma$. Let n be even. As for the case when n is odd, we write the energy as follow

$$\mathsf{E}(C_n^{1,\gamma}) = 4(1 + \delta_{\gamma \text{ odd}}) + 4\sum_{k=1}^{n/2-1} \lvert \cos(2\pi k/n) + \cos(2\pi \gamma k/n) \rvert$$

where $\delta_{\gamma \text{ odd}} = 1$ if γ is odd and 0 otherwise. For even γ , relations (1), (2) and (3) hold. The theorem then follows from the fact that $D_{n/2}(2\pi/n) = -1$ and $D_{n/2}(2\pi\gamma/n) = 1$. For odd γ , we have

$$\begin{split} \mathsf{E}(C_n^{l,\gamma}) = & 8 + 4 \sum_{k=1}^{\lfloor n/(2(\gamma+1)) \rfloor} (\cos(2\pi k/n) + \cos(2\pi \gamma k/n)) \\ & + 4 \sum_{l=0}^{\lceil \gamma/2 \rceil - 2} \sum_{k=\lfloor (2l+3)n/(2(\gamma+1)) \rfloor + 1}^{\lfloor (2l+3)n/(2(\gamma+1)) \rfloor + 1} (\cos(2\pi k/n) + \cos(2\pi \gamma k/n)) \\ & - 4 \sum_{l=0}^{\lceil \gamma/2 \rceil - 2} \sum_{k=\lfloor (2l+1)n/(2(\gamma+1)) \rfloor + 1}^{\lfloor (2l+1)n/(2(\gamma+1)) \rfloor + 1} (\cos(2\pi k/n) + \cos(2\pi \gamma k/n)) \\ & - 4 \sum_{k=\lfloor (2\lceil \gamma/2 \rceil - 1)n/(2(\gamma+1)) \rfloor + 1}^{n/2 - 1} (\cos(2\pi k/n) + \cos(2\pi \gamma k/n)). \end{split}$$

Expressing it in terms of Dirichlet kernels, it comes

$$\begin{split} \mathsf{E}(\mathsf{C}_{n}^{l,\gamma}) &= 4 + 2\sum_{\mathfrak{m}\in\{l,\gamma\}} \left(\mathsf{D}_{\lfloor \mathfrak{n}/(2(\gamma+1)) \rfloor}(2\pi\mathfrak{m}/\mathfrak{n}) \\ &+ \sum_{l=0}^{\lceil \gamma/2 \rceil - 2} \left(\mathsf{D}_{\lfloor (2l+3)\mathfrak{n}/(2(\gamma+1)) \rfloor}(2\pi\mathfrak{m}/\mathfrak{n}) - \mathsf{D}_{\lfloor (2l+1)\mathfrak{n}/(2(\gamma-1)) \rfloor}(2\pi\mathfrak{m}/\mathfrak{n}) \right) \\ &- \sum_{l=0}^{\lceil \gamma/2 \rceil - 2} \left(\mathsf{D}_{\lfloor (2l+1)\mathfrak{n}/(2(\gamma-1)) \rfloor}(2\pi\mathfrak{m}/\mathfrak{n}) - \mathsf{D}_{\lfloor (2l+1)\mathfrak{n}/(2(\gamma+1)) \rfloor}(2\pi\mathfrak{m}/\mathfrak{n}) \right) \\ &- \mathsf{D}_{\mathfrak{n}/2 - l}(2\pi\mathfrak{m}/\mathfrak{n}) + \mathsf{D}_{\lfloor (2\lceil \gamma/2 \rceil - 1)\mathfrak{n}/(2(\gamma+1)) \rfloor}(2\pi\mathfrak{m}/\mathfrak{n}) \right). \end{split}$$

The theorem follows from the fact that $D_{n/2-1}(2\pi m/n) = 1$ for $m = 1, \gamma$.

A graph is called hyperenergetic if his energy is greater than the one of the complete graph K_n . The eigenvalues of K_n are given by n-1 and -1 with multiplicity n-1, so that his energy is given by $E(K_n) = 2(n-1)$.

The figure on the left below shows how the energy of $C_n^{l,\gamma}$ grows with respect to n for $\gamma = 8$. We see that it is not hyperenergetic and that the energy grows more or less linearly with respect to n. The figure on the right shows the energy of $C_n^{l,\gamma}$ with fixed n as γ varies. We observe that the energy stays more or less constant independently of γ .



Figure 1: Energy of circulant graphs.

As a consequence of the theorem, we can carry out the sum of the Dirichlet kernels when the number of vertices is proportional to $2(\gamma - 1)(\gamma + 1)$.

Corollary. Given integers $\gamma \ge 3$ and $\alpha \ge 1$, the energy of the circulant graph $C_{2\alpha(\gamma-1)(\gamma+1)}^{1,\gamma}$ is given by

$$\begin{split} \mathsf{E}\Big(\mathsf{C}_{2\alpha(\gamma-1)(\gamma+1)}^{1,\gamma}\Big) &= 4\sum_{\mathfrak{m}\in\{1,\gamma\}}\bigg(\frac{\sin(\pi\mathfrak{m}(\lceil\gamma/2\rceil+1/(2\alpha(\gamma-1)))/(\gamma+1))\sin(\lceil\gamma/2\rceil\pi\mathfrak{m}/(\gamma+1))}{\sin(\pi\mathfrak{m}/(2\alpha(\gamma-1)(\gamma+1)))\sin(\pi\mathfrak{m}/(\gamma+1))} \\ &- \frac{\sin(\pi\mathfrak{m}(\lceil\gamma/2\rceil-1+1/(2\alpha(\gamma+1)))/(\gamma-1))\sin((\lceil\gamma/2\rceil-1)\pi\mathfrak{m}/(\gamma-1))}{\sin(\pi\mathfrak{m}/(2\alpha(\gamma-1)(\gamma+1)))\sin(\pi\mathfrak{m}/(\gamma-1))}\Big). \end{split}$$

Proof. Let $a \ge 1$ and $K \ge 0$ be integers. The sum over k of Dirichlet kernels of index (2k+1)a is given by

$$\sum_{k=0}^{K} D_{(2k+1)\mathfrak{a}}(x) = \sum_{k=0}^{K} \frac{\sin((2k+1)\mathfrak{a}+1/2)x}{\sin(x/2)}.$$

By multiplying the summation by $\sin(\alpha x)/\sin(\alpha x)$ and using the trigonometric identity $2\sin\theta\sin\phi = \cos(\theta - \phi) - \cos(\theta + \phi)$, it comes

$$\sum_{k=0}^{K} \mathsf{D}_{(2k+1)\mathfrak{a}}(x) = \frac{\cos(x/2) - \cos(((2K+2)\mathfrak{a} + 1/2)x)}{2\sin(x/2)\sin(\mathfrak{a}x)} = \frac{\sin(((2K+2)\mathfrak{a} + 1)x/2)\sin((K+1)\mathfrak{a}x)}{\sin(x/2)\sin(\mathfrak{a}x)}.$$

The corollary then follows by applying the above relation first with $a = \alpha(\gamma - 1)$, $K = \lceil \gamma/2 \rceil - 1$, second with $a = \alpha(\gamma + 1)$, $K = \lceil \gamma/2 \rceil - 2$, and $x = 2\pi m/n$, $m \in \{1, \gamma\}$.

In [1], the author considered the graphs $K_n - H$ where K_n is the complete graph on n vertices and H is a Hamilton cycle of K_n and asked whether these graphs are hyperenergetic. In [7], the author showed that the energy of $K_n - H$ is given by

$$E(K_n-H) = n - 3 + \sum_{k=1}^{n-1} |1 + 2\cos(2\pi k/n)|$$

and that as n goes to infinity, it is hyperenergetic. In the following proposition we give a formula for it for all $n \ge 3$.

Proposition. For all $n \ge 3$, the energy of $K_n - H$ is given by

$$\mathsf{E}(\mathsf{K}_{\mathsf{n}}-\mathsf{H}) = 2(\mathsf{n}-3-(\lfloor 2\mathsf{n}/3 \rfloor - \lfloor \mathsf{n}/3 \rfloor)) + 2\frac{\sin((\lfloor \mathsf{n}/3 \rfloor + 1/2)2\pi/\mathsf{n}) - \sin((\lfloor 2\mathsf{n}/3 \rfloor + 1/2)2\pi/\mathsf{n})}{\sin(\pi/\mathsf{n})}$$

Proof. We have

$$\begin{split} \sum_{k=1}^{n-1} &|1+2\cos(2\pi k/n)| = \sum_{k=1}^{\lfloor n/3 \rfloor} (1+2\cos(2\pi k/n)) - \sum_{k=\lfloor n/3 \rfloor+1}^{\lfloor 2n/3 \rfloor} (1+2\cos(2\pi k/n)) \\ &+ \sum_{k=\lfloor 2n/3 \rfloor+1}^{n-1} (1+2\cos(2\pi k/n)) \\ &= n-3-2(\lfloor 2n/3 \rfloor - \lfloor n/3 \rfloor) + 2D_{\lfloor n/3 \rfloor}(2\pi/n) - 2D_{\lfloor 2n/3 \rfloor}(2\pi/n) \\ &+ D_{n-1}(2\pi/n). \end{split}$$

Since $D_{n-1}(2\pi/n) = -1$, the proposition follows.

By elementary analysis, one can show that $E(K_n - H) - 2(n - 1)$ is increasing in n. As a consequence, we find that $K_n - H$ are hyperenergetics for all $n \ge 10$. This has been previously found in [7].

References

[1] R. Balakrishnan. The energy of a graph. Linear Algebra Appl., 387:287-295, 2004.

- [2] R. Balakrishnan and K. Ranganathan. A textbook of graph theory. Universitext. Springer, New York, second edition, 2012.
- [3] Norman Biggs. *Algebraic graph theory*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 1993.
- [4] Simon R. Blackburn and Igor E. Shparlinski. On the average energy of circulant graphs. *Linear Algebra Appl.*, 428(8-9):1956–1963, 2008.
- [5] Aleksandar Ilić and Milan Bašić. New results on the energy of integral circulant graphs. Appl. Math. Comput., 218(7):3470–3482, 2011.
- [6] Igor Shparlinski. On the energy of some circulant graphs. *Linear Algebra Appl.*, 414(1):378–382, 2006.
- [7] Dragan Stevanović and Ivan Stanković. Remarks on hyperenergetic circulant graphs. *Linear algebra and its applications*, 400:345–348, 2005.

8 Summary

In this thesis we study the number of spanning trees in some classes of graphs. This is made possible by the famous matrix tree theorem established by Kirchhoff in 1847 which states that the number of spanning trees in a finite graph is given by the product of the non-zero eigenvalues of the combinatorial Laplacian of the graph divided by the number of vertices. We study circulant graphs with fixed generators, that is, Cayley graphs of $\mathbb{Z}/n\mathbb{Z}$, for which we obtain precise asymptotics as the number of vertices grow. More precisely, a circulant graph generated by $\gamma_1, \ldots, \gamma_d$ on n vertices, which are labelled by $0, 1, \ldots, n-1$, is the 2d-regular graph such that each vertex $v \in \mathbb{Z}/n\mathbb{Z}$ is connected to $v + \gamma_i$ mod n and to $\nu - \gamma_i \mod n$ for all $i \in \{1, \ldots, d\}$. The lead term of the log-asymptotic behaviour of the number of spanning trees in these graphs has been found in 2010 by Golin, Yong and Zhang while in the present work we derive the constant term. In 2010, Chinta, Jorgenson and Karlsson derived the asymptotic behaviour of the number of spanning trees in d-dimensional discrete tori, which can be seen as quotients $\mathbb{Z}^d / \Lambda_n \mathbb{Z}^d$ with nearest neightbours connected, where Λ_n is a $d \times d$ integral matrix such that $\Lambda_n / (\det \Lambda_n)^{1/d}$ converges in $SL_d(\mathbb{R})$ as n goes to infinity. Considering the theta function associated to the discrete torus, which contains the spectral information of the graph, they establish a theta inversion formula expressing it in terms of modified I-Bessel functions. The study of its asymptotics provides the result. In our work, we adapt these techniques to two different cases. First, one can show that circulant graphs with first generator equals to 1 are isomorphic to d-dimensional discrete tori $\mathbb{Z}^d / \Lambda_{\Gamma} \mathbb{Z}^d$ for some matrix Λ_{Γ} . But in this case the condition on the convergence of the lattice does not hold anymore, so that previous results cannot be applied. The other case considered is a sequence of d-dimensional degenerating discrete tori. They are degenerating in the sense that d - p sides of the tori are tending to infinity at the same rate while the p other sides tend to infinity sublinearly with respect to the d - p sides. Again, the condition on the convergence of the lattice is not satisfied. As a result, the lead term in the asymptotics is the same as in the case of non-degenerating tori, but lower order terms are deducted. In the particular situation where the p sides stay constant instead of converging sublinearly to infinity, the regularized determinant of the Laplacian on a limiting (d - p)-dimensional real torus appears in the constant term of the asymptotic development. Furthermore, the results on d-dimensional discrete tori enable to derive asymptotics for the number of spanning trees on d-dimensional orthotope square lattices. Indeed, the spectrum of the combinatorial Laplacian on this lattice is a subset of the one on the corresponding d-dimensional discrete torus with twice vertices at each side of the torus. As a consequence, the theta function on the orthotope square lattice can be expressed in terms of the one on the torus. Asymptotically, the lead term appearing is the same as in the case of the torus, but lower order terms corresponding to free boundary conditions are deducted. More precisely, for each m-dimensional face, $m = 1, \ldots, d - 1$, a term proportional to the m-volume of the orthotope is deducted. As in the case of the torus, regularized determinants of the Laplacian on limiting objects appear, namely real m-orthotopes with Dirichlet boundary conditions, for $m \in \{1, ..., d\}$.

Other results obtained in this thesis concern closed formulas for the number of spanning trees in directed and non-directed circulant graphs where the generators vary, that is, they linearly depend on the number of vertices. In the case of non-directed circulant graphs on β n vertices, the set of generators is given by $\{1, \gamma_1 n, \ldots, \gamma_{d-1} n\}$, where γ_i , $i = 1, \ldots, d-1$, and β are integers. In the case of directed circulant graphs on β n vertices, the generators are $p, \gamma_1 n + p, \ldots, \gamma_{d-1} n + p$. We also derived formulas for the n-th and (n-1)-th power graphs of the β n-cycle. As a result, the number of spanning trees in these graphs can be expressed in terms of the eigenvalues of the Laplacian on a subgraph of the original graph. The formulas obtained are products of $\lceil \beta/2 \rceil - 1$ terms and therefore

are interesting when n is large.

The problem of counting the number of spanning trees also arises in statistical mechanics in evaluating low temperature ratchet current. Maes *et al* found in 2014 a numerical estimation for the current while we found an exact expression for it.

Finally, we considered the energy of circulant graphs with two generators. The energy of a graph is defined as the sum of the absolute value of the eigenvalues of the adjacency matrix of the graph. Closed formulas are obtained for circulant graphs with generators equal to 1 and γ , for $\gamma \in \mathbb{N}_{\geq 2}$.

9 Résumé

Dans cette thèse, nous étudions le nombre d'arbres couvrants dans différentes classes de graphes. Ceci est rendu possible grâce au célèbre théorème établi par Kirchhoff en 1847 (matrix tree theorem) énonçant que le nombre d'arbres couvrants d'un graphe fini est égal au produit des valeurs propres non-nulles du Laplacien combinatoire du graphe divisé par le nombre de sommets. Dans un premier temps, nous étudions les graphes circulants à générateurs fixes, à savoir, des graphes de Cayley du groupe $\mathbb{Z}/n\mathbb{Z}$, pour lesquels nous obtenons un résultat précis sur le comportement asymptotique à mesure que le nombre de sommets croît. Plus précisément, un graphe circulant à n sommets, dénotés par $0,1,\ldots,n-1$, engendré par γ_1,\ldots,γ_d est le graphe 2d-régulier où chaque sommet $\nu \in \mathbb{Z}/n\mathbb{Z}$ est connecté à $\nu + \gamma_i \mod n$ et à $\nu - \gamma_i \mod n$, pour tout $i \in \{1, \ldots, d\}$. En 2010, Golin, Yong et Zhang ont calculé le terme principal du comportement log-asymptotique du nombre d'arbres couvrants dans ce type de graphes tandis que dans notre travail nous dérivons le terme constant. En 2010, Chinta, Jorgenson et Karlsson ont développé une méthode pour calculer le comportement asymptotique du nombre d'arbres couvrants dans une suite de tores discrets en d dimensions, ou, de manière équivalente, de quotients de $\mathbb{Z}^d/\Lambda_n\mathbb{Z}^d$ avec plus proches voisins connectés, où Λ_n est une matrice d × d à coefficients entiers telle que $\Lambda_n/(\det \Lambda_n)^{1/d}$ converge dans $SL_d(\mathbb{R})$ lorsque n tend vers l'infini. En considérant la fonction theta associée au tore discret, qui contient l'information spectrale du graphe, ils établissent une formule d'inversion theta exprimant celle-ci en termes de fonctions de Bessel modifiées dont l'étude du comportement asymptotique permet d'établir le résultat. Dans notre travail, nous adaptons ces techniques à deux différents cas. Premièrement, il est possible de démontrer que les graphes circulants dont le premier générateur est égal à un sont isomorphes à un tore discret d-dimensionnel $\mathbb{Z}^d/\Lambda_{\Gamma}\mathbb{Z}^d$, pour une certaine matrice Λ_{Γ} . Cependant dans ce cas, la condition de convergence du réseau n'est plus satisfaite, impliquant que les résultats précédents ne peuvent être utilisés. L'autre cas que nous considérons est une suite de tores discrets d-dimensionnels dégénérés. Ils sont dégénérés dans le sens que d - p côtés de ces tores tendent vers l'infini à la même vitesse, alors que les p autres côtés tendent vers l'infini de manière sous-linéaire par rapport aux d - p côtés. A nouveau, la condition de convergence du réseau n'est pas satisfaite. En résultat nous nous apercevons que le terme principal qui apparaît est le même que dans le cas des tores discrets non-dégénérés, mais des termes d'ordres inférieurs sont déduits. Dans la situation particulière où les p côtés des tores restent constants au lieu de converger sous-linéairement vers l'infini, nous avons pu déterminer des termes supplémentaires dans le développement asymptotique, laissant apparaître comme terme constant le déterminant régularisé du Laplacien sur un tore réel (ou continu) (d-p)-dimensionnel limitant. De plus, les résultats sur les tores discrets d-dimensionnels permettent de dériver le comportement asymptotique du nombre d'arbres couvrants sur des orthotopes d-dimensionnels à réseau carré. En effet, le spectre du Laplacien combinatoire sur ce réseau est un sous-ensemble de celui sur le tore discret d-dimensionnel correspondant ayant deux fois plus de sommets à chaque côté du tore. En conséquence, la fonction theta sur le d-orthotope à réseau carré peut être exprimée en termes de celle sur le tore discret. Asymptotiquement, le terme principal apparaissant est le même que pour les tores discrets mais des termes d'ordres inférieurs sont soustraits, ceci étant dû aux conditions de bord libre. Plus précisément, pour chaque face m-dimensionnelle, $m = 1, \ldots, d - 1$, un terme proportionnel au m-volume de l'orthotope est déduit. Comme dans le cas du tore discret, des déterminants régularisés du Laplacien sur des objets limitant apparaissent, à savoir, des m-orthotopes réels avec des conditions de bord de Dirichlet, pour $m \in \{1, \ldots, d\}$.

Dans cette thèse, nous avons également obtenu des formules exactes concernant le nombre d'arbres couvrants de graphes circulants orientés et non-orientés où les générateurs varient, c'est-à-dire dépen-

dent linéairement du nombre de sommets. Dans le cas des graphes circulants non-orientés à βn sommets, l'ensemble des générateurs est donné par $\{1, \gamma_1 n, \ldots, \gamma_{d-1} n\}$, où γ_i , $i = 1, \ldots, d-1$, et β sont des entiers. Dans le cas des graphes circulants orientés à βn sommets, les générateurs sont p, $\gamma_1 n + p, \ldots, \gamma_{d-1} n + p$. Nous avons également obtenu des formules pour la n-ième et la (n - 1)-ième puissance du cycle à βn sommets. En résultat, il s'avère que le nombre d'arbres couvrants dans ce type de graphes s'exprime en termes des valeurs propres du Laplacien combinatoire sur un sous-graphe du graphe d'origine, ceci reflétant la symétrie du graphe. Les formules obtenues sont des produits de $\lceil \beta/2 \rceil - 1$ termes et sont donc intéressantes pour n grand.

L'évaluation du nombre d'arbres couvrants apparaît également en mécanique statistique dans l'évaluation du courant de cliquet (*ratchet current*) à basse température. En 2014, Maes *et al.* ont donné une estimation numérique pour le courant, alors que nous en avons dérivé une expression exacte.

Finalement, nous avons étudié l'énergie des graphes circulants à deux générateurs. L'énergie d'un graphe est définie comme la somme des valeurs absolues des valeurs propres de la matrice d'adjacence du graphe. Nous avons obtenu des formules pour les graphes circulants ayant pour générateurs 1 et γ , où $\gamma \in \mathbb{N}_{\geq 2}$.