Testing for Equality Between Two Copulas

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ABSTRACT. We develop a test of equality between two dependence structures estimated through empirical copulas. We provide inference for independent or paired samples. The multiplier central limit theorem is used for calculating p-values of the Cramér-von Mises test statistic. Finite sample properties are assessed with Monte Carlo experiments. We apply the testing procedure on empirical examples in finance, psychology, insurance and medicine.

1. Introduction

Copulas are omnipresent in statistics and other fields like actuarial science, finance, reliability and hydrology to name a few. This presence is explained by the copula being a summary of the full dependence structure between random variables. From a methodological point of view, most papers concentrate on parameter estimation, using ranks as in Genest et al. (1995), and Shih and Louis (1995), or using estimated parametric margins, as in Joe (2005).

However functional nonparametric estimation of the copula is also examined. It was first studied by Deheuvels in a series of papers (Deheuvels 1979, 1980, 1981a,c,b), for the independent copula, and studied in full generality in Gänler and Stute (1987). Recent work on copula processes include Fermanian et al. (2004) and Ghoudi and Rémillard (2004). Copula processes help to develop tests for goodness-of-fit in semi-parametric models, e.g. Fermanian (2005), Genest et al. (2006), Genest and Rémillard (2008), and Scaillet (2007).

Another statistical issue related to copula modelling is the problem of testing for equality between two copulas. This yet unsolved issue aims at checking the validity of the hypothesis of two dependence structures being identical. For example, we could argue in credit risk that the copula of the joint default times of firms is the same as the copula of their respective asset values. See Dupuis et al. (2006) for an illustration.

Our method to gauge the similarity between dependence structures has several advantages. First it is applicable to any dimension. It is not restricted to the two dimensional case only. Second it is not affected by strict monotonic transformations of the variables like log or exp transforms. Copulas enjoy an invariance property with respect to such mappings. This is a clear benefit over using a standard correlation to measure dependence. Third it is model free. We rely on empirical estimation

of copulas following a nonparametric approach. Fourth finite sample properties are expected to be well behaved since we rely on a simulation strategy induced by a multiplier method. Our Monte Carlo results confirm this conjecture. The testing procedure performs well in samples as small as $n_1 = n_2 = 50$ and $d = 2$. Fifth the test statistic takes a closed form. This improves the numerical speed of the simulation based testing procedure.

In this paper we illustrate the testing procedure on several empirical examples. We investigate questions arising in finance, psychology, insurance and medicine. The first application concerns the dependence structure between expense ratio and turnover level within two categories of US mutual funds. The second application examines the links between emotional experience and life satisfaction in the Chinese culture vis-à-vis the American culture. The third one is dedicated to the analysis of losses and allocated loss adjustment expenses (ALAEs, in short). In the last application, we investigate the dependence structure over time between two methods of assessment of depression. Other potential applications include investigating dependence between product sales in different retail stores (marketing), between income and consumption in different countries (economics), between reported items on corporate balance sheets in different countries (accounting), etc.

To describe the problem at hand, suppose we face two independent samples of $\mathbb{R}^d$-valued vectors. The first sample, $X_1, \ldots, X_{n_1}$ is taken from a distribution function F with continuous margins F_1, ..., F_d, and the second sample $Y_1, \ldots, Y_{n_2}$ is taken from a distribution function G with continuous margins G_1, ..., G_d. The vectors X_i, $i = 1, .., n_1$, and Y_i, $i = 1, .., n_2$, have size d, and entries denoted by X_{il} and Y_{il}, $l =, 1..., d$. Then the unique copulas C and D associated with the first and second

sample are determined, for any $x = (x_1, \ldots, x_d)$, by

$$F(x) = C\{F_1(x_1), \ldots, F_d(x_d)\}, \qquad G(x) = D\{G_1(x_1), \ldots, G_d(x_d)\}.$$

The aim of the paper is to show how we can test the hypotheses

$$H_0: \ C = D \quad \text{vs} \quad H_1: \ C \neq D.$$

Obviously this is not equivalent to testing for $F = G$. We focus here on the equality between the dependence structure as posited by $C = D$, leaving the behavior of the margins out of interest. By construction our method is invariant with respect to strict monotonic transformations of the data.

To obtain consistent tests, we rely on a statistic based on the integrated square difference between the empirical copulas C_{n_1} and D_{n_2} defined for any $u = (u_1, \ldots, u_d) \in [0,1]^d$ by

$$C_{n_1}(u) = \frac{1}{n_1} \sum_{i=1}^{n_1} \mathbb{I}(U_{i,n_1} \leq u) = \frac{1}{n_1} \sum_{i=1}^{n_1} \prod_{l=1}^{d} \mathbb{I}(U_{il,n_1} \leq u_l),$$

and

$$D_{n_2}(u) = \frac{1}{n_2} \sum_{i=1}^{n_2} \mathbb{I}(V_{i,n_2} \leq u) = \frac{1}{n_2} \sum_{i=1}^{n_2} \prod_{l=1}^{d} \mathbb{I}(V_{il,n_2} \leq u_l),$$

where $U_{i,n_1} = (U_{i1,n_1}, \ldots, U_{id,n_1})$, $V_{i,n_2} = (V_{i1,n_2}, \ldots, U_{id,n_2})$, and for any $l \in \{1, \ldots, d\}$,

$$\begin{aligned}
U_{il,n_1} &= \tfrac{n_1}{n_1+1} F_{l,n_1}(X_{il}) &&= \mathrm{rank}(X_{il})/(n_1+1), && 1 \leq i \leq n_1, \\
V_{il,n_2} &= \tfrac{n_2}{n_2+1} G_{l,n_2}(Y_{il}) &&= \mathrm{rank}(Y_{il})/(n_2+1), && 1 \leq i \leq n_2,
\end{aligned}$$

with

$$F_{l,n_1}(x_l) = \frac{1}{n_1} \sum_{i=1}^{n_1} \mathbb{I}(X_{il} \leq x_l) \text{ and } G_{l,n_2}(x_l) = \frac{1}{n_2} \sum_{i=1}^{n_2} \mathbb{I}(Y_{il} \leq x_l),$$

being the empirical distribution functions of $(X_{il})_{i=1}^{n_1}$ and $(Y_{il})_{i=1}^{n_2}$, respectively, defined for any $x_l \in \mathbb{R}$.

Test statistics for the equality between two copulas rely on functionals of the empirical process

$$\mathbb{E}_{n_1,n_2} = (C_{n_1} - D_{n_2}) / \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}.$$

The asymptotic behavior of $\mathbb{E}_{n_1,n_2}$ is given in Section 2, together with a simulation based method for computing p-values. Some numerical results are given in Section 3 to illustrate the finite sample properties of the testing procedure. Section 4 is dedicated to empirical applications. The proof of the theoretical results are relegated to Appendix A while explicit expressions for calculating the simulated Cramér-von Mises test statistics are available in Appendix B of Rémillard and Scaillet (2006).

2. Test statistic and main results

If the mappings $u \mapsto \partial_{u_l} C(u)$ are continuous on $[0,1]^d$, then it is known, see, e.g., Gänler and Stute (1987), Tsukahara (2005), that $\mathbb{C}_{n_1} = \sqrt{n_1}(C_{n_1} - C)$ converges weakly in $\mathcal{D}([0,1]^d)$ to a continuous centered Gaussian process $\mathbb{C}$, denoted by $\mathbb{C}_{n_1} \rightsquigarrow \mathbb{C}$, where $\mathbb{C}$ has the representation

$$(1) \qquad \mathbb{C}(u) = \alpha(u) - \sum_{l=1}^{d} \beta_l(u_l) \partial_{u_l} C(u),$$

with

$$\alpha_{n_1}(u) = \frac{1}{\sqrt{n_1}} \sum_{i=1}^{n_1} \{\mathbb{I}(U_i \leq u) - C(u)\} \rightsquigarrow \alpha(u),$$

$\beta_l(u_l) = \alpha(1, \ldots, 1, u_l, 1, \ldots, 1)$, $1 \leq l \leq d$, and $U_i = (F_1(X_{i1}), ..., F_d(X_{id}))$. Note that the extra term $\sum_{l=1}^{d} \beta_l(u_l) \partial_{u_l} C(u)$ comes from the marginal distributions $F_1, \ldots, F_d$ being unknown.

Similarly, $\mathbb{D}_{n_2} = \sqrt{n_2}(D_{n_2} - D) \rightsquigarrow \mathbb{D}$ in $\mathcal{D}([0,1]^d)$ where $\mathbb{D}$ is a continuous centered Gaussian process represented by

$$(2) \qquad \mathbb{D}(u) = \gamma(u) - \sum_{l=1}^{d} \delta_l(u_l) \partial_{u_l} D(u),$$

with

$$\gamma_{n_2}(u) = \frac{1}{\sqrt{n_2}} \sum_{i=1}^{n_2} \{\mathbb{I}(V_i \leq u) - D(u)\} \rightsquigarrow \gamma(u),$$

$\delta_l(u_l) = \gamma(1, \ldots, 1, u_l, 1, \ldots, 1)$, $1 \le l \le d$, and $V_i = (G_1(Y_{i1}), \ldots, G_d(Y_{id}))$.

If $\min(n_1, n_2) \to \infty$, in such a way that $n_1/(n_1 + n_2) \to \lambda \in [0, 1]$, then (see the proofs of the theorems below)

$$\mathcal{E}_{n_1,n_2} = \sqrt{\frac{n_2}{n_1 + n_2}} \mathbb{C}_{n_1} - \sqrt{\frac{n_1}{n_1 + n_2}} \mathbb{D}_{n_2} \rightsquigarrow \mathcal{E} = \sqrt{1 - \lambda}\, \mathbb{C} - \sqrt{\lambda}\, \mathbb{D}.$$

Under the null hypothesis $H_0 : C = D$, we have $\mathbb{E}_{n_1,n_2} = \mathcal{E}_{n_1,n_2}$, and thus $\mathbb{E}_{n_1,n_2} \rightsquigarrow \mathcal{E}$.

To test the null hypothesis $H_0 : C = D$, we propose to use the Cramér-von Mises principle, and build

$$
\begin{aligned}
S_{n_1,n_2} &= \int_{[0,1]^d} \mathbb{E}^2_{n_1,n_2}(u) du \\
&= \left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1} \times \left\{ \frac{1}{n_1^2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_1} \prod_{s=1}^{d} (1 - U_{is,n_1} \vee U_{js,n_1}) \right. \\
&\quad - \frac{2}{n_1 n_2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \prod_{s=1}^{d} (1 - U_{is,n_1} \vee V_{js,n_2}) \\
&\quad \left. + \frac{1}{n_2^2} \sum_{i=1}^{n_2} \sum_{j=1}^{n_2} \prod_{s=1}^{d} (1 - V_{is,n_2} \vee V_{js,n_2}) \right\},
\end{aligned}
$$

where $a \vee b$ stands for $\max(a, b)$. When $C = D$, then

$$S_{n_1,n_2} \rightsquigarrow S = \int_{[0,1]^d} \mathcal{E}^2(u) du,$$

while if $C \ne D$, then $S_{n_1,n_2} \xrightarrow{Pr} \infty$. This yields consistency of the testing procedure.

Because C and D are unknown, computing p-values appears difficult at first sight. However, due to a powerful multiplier technique, we can estimate the p-value via simulations. In a single copula context the idea is already suggested in Scaillet (2005), and further developed in Rémillard (2006). The trick is to use a multiplier central limit theorem (van der Vaart and Wellner 1996) to approximate each random term appearing in (1) and (2). Note that a bootstrap approach would be inappropriate here

since it fails to deliver consistency when applied to Cramér-von Mises test statistics (see Example 7 of Bickel et al. (1997), Bickel and Freedman (1981), and Bretagnolle (1983)).

To see how it works, suppose that for any $k \in \{1, \ldots, N\}$, $\xi_1^{(k)}, \ldots, \xi_{n_1}^{(k)}, \zeta_1^{(k)}, \ldots, \zeta_{n_2}^{(k)}$ are independent and identically distributed variables with mean zero and variance one.

Set

$$
\begin{aligned}
\hat{\alpha}_{n_1}^{(k)}(u) &= \frac{1}{\sqrt{n_1}} \sum_{i=1}^{n_1} \xi_i^{(k)} \{\mathbb{I}(U_{i,n_1} \leq u) - C_{n_1}(u)\} \\
&= \frac{1}{\sqrt{n_1}} \sum_{i=1}^{n_1} \left(\xi_i^{(k)} - \bar{\xi}^{(k)}\right) \mathbb{I}(U_{i,n_1} \leq u), \\
\hat{\gamma}_{n_2}^{(k)}(u) &= \frac{1}{\sqrt{n_2}} \sum_{i=1}^{n_2} \left(\zeta_i^{(k)} - \bar{\zeta}^{(k)}\right) \mathbb{I}(V_{i,n_2} \leq u),
\end{aligned}
$$

where $\bar{\xi}^{(k)} = \frac{1}{n_1} \sum_{i=1}^{n_1} \xi_i^{(k)}$, $\bar{\zeta}^{(k)} = \frac{1}{n_2} \sum_{i=1}^{n_2} \zeta_i^{(k)}$, and for any $l \in \{1, \ldots, d\}$,

$$
\begin{aligned}
\hat{\beta}_{l,n_1}^{(k)}(u_l) &= \hat{\alpha}_{n_1}^{(k)}(1, \ldots, 1, u_l, 1, \ldots, 1) \\
&= \frac{1}{\sqrt{n_1}} \sum_{i=1}^{n_1} \left(\xi_i^{(k)} - \bar{\xi}^{(k)}\right) \mathbb{I}(U_{il,n_1} \leq u_k), \\
\hat{\delta}_{l,n_2}^{(k)}(u_l) &= \hat{\gamma}_{n_2}^{(k)}(1, \ldots, 1, u_l, 1, \ldots, 1) \\
&= \frac{1}{\sqrt{n_2}} \sum_{i=1}^{n_2} \left(\zeta_i^{(k)} - \bar{\zeta}^{(k)}\right) \mathbb{I}(V_{il,n_2} \leq u_k).
\end{aligned}
$$

To approximate the partial derivatives ∇C and ∇D, we proceed as in Ghoudi and Rémillard (2004). For any $l \in \{1, \ldots, d\}$, set

$$
\widehat{\partial_{u_l} C_{n_1,h_1}}(u) = \frac{C_{n_1}(u + h_1 e_l) - C_{n_1}(u - h_1 e_l)}{2h_1}
$$

and

$$
\widehat{\partial_{u_l} D_{n_2,h_2}}(u) = \frac{D_{n_2}(u + h_2 e_l) - D_{n_2}(u - h_2 e_l)}{2h_2},
$$

where e_l is the l-th column of the $d \times d$ identity matrix. We could also rely on a kernel based estimate of the derivative (Fermanian and Scaillet 2003), but this would

impede the writing of explicit expressions for the simulated test statistic, and slow down the procedure. These expressions are available on request from the authors, and also written down in Appendix B of Rémillard and Scaillet (2006).

Finally, for all $u \in [0,1]^d$, and for all $k \in \{1 \ldots, N\}$, let

$$\hat{\mathbb{C}}^{(k)}_{n_1,h_1}(u) = \hat{\alpha}^{(k)}_{n_1}(u) - \sum_{l=1}^{d} \hat{\beta}^{(k)}_{l,n_1}(u_l)\partial_{u_l}\widehat{C_{n_1,h_1}}(u),$$

$$\hat{\mathbb{D}}^{(k)}_{n_2,h_2}(u) = \hat{\gamma}^{(k)}_{n_2}(u) - \sum_{l=1}^{d} \hat{\delta}^{(k)}_{l,n_2}(u_l)\partial_{u_l}\widehat{D_{n_2,h_2}}(u),$$

and

$$\hat{\mathcal{E}}^{(k)}_{n_1,n_2} = \sqrt{\frac{n_2}{n_1+n_2}}\hat{\mathbb{C}}^{(k)}_{n_1,h_1} - \sqrt{\frac{n_1}{n_1+n_2}}\hat{\mathbb{D}}^{(k)}_{n_2,h_2}.$$

Further set

$$S^{(0)}_{n_1,n_2} = \int_{[0,1]^d} \mathcal{E}^2_{n_1,n_2}(u)du$$

and

$$\hat{S}^{(k)}_{n_1,n_2} = \int_{[0,1]^d} \left\{\hat{\mathcal{E}}^{(k)}_{n_1,n_2}\right\}^2 (u)du, \qquad k \in \{1,\ldots,N\}.$$

Theorem 2.1 (Independent samples). *Suppose that ∇C and ∇D are continuous on $[0,1]^d$. If $h_i = n_i^{-1/2}$, $i = 1,2$ and if $\min(n_1,n_2) \to \infty$ in such a way that $n_1\big/(n_1+n_2) \to \lambda \in (0,1)$, then*

$$\left(\mathcal{E}_{n_1,n_2}, \hat{\mathcal{E}}^{(1)}_{n_1,n_2},\ldots\hat{\mathcal{E}}^{(N)}_{n_1,n_2}\right) \rightsquigarrow \left(\mathcal{E}, \tilde{\mathcal{E}}^{(1)},\ldots\tilde{\mathcal{E}}^{(N)}\right) \ \text{in} \ \mathcal{D}\left([0,1]^d\right)^{\otimes(N+1)},$$

where $\tilde{\mathcal{E}}^{(1)},\ldots\tilde{\mathcal{E}}^{(N)}$ are independent copies of $\mathcal{E}$. In particular,

$$\left(S^{(0)}_{n_1,n_2}, \hat{S}^{(1)}_{n_1,n_2},\ldots\hat{S}^{(N)}_{n_1,n_2}\right) \rightsquigarrow \left(S, \tilde{S}^{(1)},\ldots\tilde{S}^{(N)}\right) \ \text{in} \ [0,\infty)^{\otimes(N+1)},$$

where $\tilde{S}^{(1)},\ldots\tilde{S}^{(N)}$ are independent copies of $S = \int_{[0,1]^d} \mathcal{E}^2(u)du$. An approximate p-value for S_{n_1,n_2} is then given by

$$\frac{1}{N}\sum_{k=1}^{N} \mathbb{I}\left(\hat{S}^{(k)}_{n_1,n_2} > S_{n_1,n_2}\right).$$

The proof is given in Appendix A.1.

The previous theorem holds true for two independent populations. What about paired observations, i.e., X_i is not independent of Y_i, but $n_2 = n_1 = n$? It is easy to check that the previous methodology applies, provided we draw $\xi_i^{(k)}$ and set $\zeta_i^{(k)} = \xi_i^{(k)}$, for all $i = 1, \ldots, n$, and all $k = 1, \ldots, N$. In the next theorem we shorten the subscript n,n as n.

Theorem 2.2 (Paired samples). *Suppose that ∇C and ∇D are continuous on $[0,1]^d$. If $h_i = h = n^{-1/2}$, $i = 1, 2$ and if $n \to \infty$, then*

$$\left(\mathcal{E}_n, \hat{\mathcal{E}}_n^{(1)}, \ldots \hat{\mathcal{E}}_n^{(N)} \right) \rightsquigarrow \left(\mathcal{E}, \tilde{\mathcal{E}}^{(1)}, \ldots \tilde{\mathcal{E}}^{(N)} \right) \text{ in } \mathcal{D}\left([0,1]^d\right)^{\otimes(N+1)},$$

where $\tilde{\mathcal{E}}^{(1)}, \ldots \tilde{\mathcal{E}}^{(N)}$ are independent copies of $\mathcal{E}$. In particular,

$$\left(S_n^{(0)}, \hat{S}_n^{(1)}, \ldots \hat{S}_n^{(N)} \right) \rightsquigarrow \left(S, \tilde{S}^{(1)}, \ldots \tilde{S}^{(N)} \right) \text{ in } [0, \infty)^{\otimes(N+1)},$$

where $\tilde{S}^{(1)}, \ldots \tilde{S}^{(N)}$ are independent copies of $S = \displaystyle\int_{[0,1]^d} \mathcal{E}^2(u)du$. An approximate p-value for S_n is then given by

$$\frac{1}{N} \sum_{k=1}^{N} \mathbb{I}\left(\hat{S}_n^{(k)} > S_n \right).$$

The proof is given in Appendix A.3.

3. NUMERICAL EXPERIMENTS

From Theorem 2.1 we know that the level of the test should be correct when $n_1, n_2 \to \infty$. Here we check the finite sample properties of the testing procedure in terms of size and power. To this end, we have chosen three bivariate copula families (Clayton, Frank and Gumbel), all indexed by the Kendall tau $\tau(\theta)$ depending on the copula parameter θ. Recall that the Clayton copula is defined by all $u, v \in (0, 1)$ and parameter $\theta > 0$ by

$$C_\theta(u, v) = \left(u^{-\theta} + v^{-\theta} - 1 \right)^{-1/\theta}.$$

Here $\tau(\theta) = \theta/(\theta + 2)$.

The Frank copula is defined for all $u, v \in (0, 1)$ and $\theta > 0$ by

$$C_\theta(u, v) = \log\left(\frac{\theta + \theta^{u+v} - \theta^u - \theta^v}{\theta - 1}\right) \Big/ \log(\theta).$$

Then $\tau(\theta) = \dfrac{\log(\theta)^2 + 4\log(\theta) + 4\mathrm{dilog}(\theta)}{\log(\theta)^2}$, where $\mathrm{dilog}(x) = \displaystyle\int_1^x \frac{\log t}{1-t} dt$.

Finally, the Gumbel copula is defined for all $u, v \in (0, 1)$ and $0 < \theta < 1$ by

$$C_\theta(u, v) = \exp\left[-\left\{(-\log u)^{1/\theta} + (-\log v)^{1/\theta}\right\}^\theta\right],$$

which gives $\tau(\theta) = 1 - \theta$.

As we can see from Table 1 for Clayton copulas, even for sample sizes as small as $n_1 = n_2 = 50$, the empirical level of the test (4.9%) is close to the theoretical one (5%). Moreover, the power of the test increases as expected, when D goes away from C, i.e., when τ_D increases, τ_C being fixed. It is close to 100% when τ_D is above .7 and τ_C is kept equal to .2. These results are confirmed by Table 2 for the Frank copula and by Table 3 for the Gumbel copula. Similar results also hold true for the other pairs of sample sizes $(n_1, n_2) = (50, 100), (100, 50), (100, 100)$.

TABLE 1. Size and power of the Cramér-von Mises test based on a multiplier technique with $N = 1,000$, when $n_1 = 50, 100$, $n_2 = 50, 100$, $d = 2$, and Clayton copulas parameterized such that the Kendall tau is $\tau_C = 0.2$ for C, and $\tau_D = 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$ for D. The significance level is 5%, and empirical levels are computed with 1000 replicates.

(n_1, n_2)	Kendall tau τ_D	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$(50, 50)$	Power (%)	4.9	9.4	28.9	58	87.4	97.6	99.9
$(50, 100)$	Power (%)	4.6	12.7	37.2	73.6	95.4	99.6	100
$(100, 50)$	Power (%)	5.4	14.3	40.3	74.4	95.7	99.9	100
$(100, 100)$	Power (%)	4.5	13.5	53.1	88.5	99.2	100	100

TABLE 2. Size and power of the Cramér-von Mises test based on a multiplier technique with $N = 1,000$, when $n_1 = 50, 100$, $n_2 = 50, 100$, $d = 2$, and Frank copulas parameterized such that the Kendall tau is $\tau_C = 0.2$ for C, and $\tau_D = 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$ for D. The significance level is 5%, and empirical levels are computed with 1000 replicates.

(n_1, n_2)	Kendall tau τ_D	0.2	0.3	0.4	0.5	0.6	0.7	0.8
(50,50)	Power (%)	4.7	10	32.9	55.9	89.9	99.1	99.9
$(50, 100)$	Power (%)	5.7	12.9	36.9	72.1	96.6	99.8	100
$(100, 50)$	Power (%)	4.8	15	45.1	74.9	98	100	100
$(100, 100)$	Power (%)	4.4	16.3	59.2	89.4	99.8	100	100

TABLE 3. Size and power of the Cramér-von Mises test based on a multiplier technique with $N = 1,000$, when $n_1 = 50, 100$, $n_2 = 50, 100$, $d = 2$, and Gumbel copulas parameterized such that the Kendall tau is $\tau_C = 0.2$ for C, and $\tau_D = 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$ for D. The significance level is 5%, and empirical levels are computed with 1000 replicates.

(n_1, n_2)	Kendall tau τ_D	0.2	0.3	0.4	0.5	0.6	0.7	0.8
(50,50)	Power (%)	4.2	8.4	26.5	57.1	85.6	98.5	99.9
$(50, 100)$	Power (%)	4.9	10.3	36.6	70.4	94.1	99.9	100
$(100, 50)$	Power (%)	4.6	14.7	39.4	73.1	96.4	99.9	100
$(100, 100)$	Power (%)	4.7	16.4	53.1	87.8	99.8	100	100

4. EMPIRICAL APPLICATIONS

In this section we illustrate the testing procedures on empirical examples in finance, psychology, insurance and medicine. A generic MATLAB code and its C add-in are available upon request from the authors for applied work. We have used $N = 1,000$.

4.1. **Expense ratio and turnover level.** The data set is made of expense ratio and turnover level reported by 222 "Growth and Income" funds and 333 "Aggressive Growth" funds at the end of year 1994 (see, e.g., Wermers (2000) for a detailed description of the data). A higher turnover induces higher transaction costs, and funds charge expenses partly to cover these costs. In 1994 growth-oriented funds maintain roughly 90% of their portfolios in equities, while income-oriented funds maintain a

lower proportion around 80%. We want to study whether funds having different investment objectives share the same link between turnover level and expense ratio. The p-value is 0.425, and we conclude that the null hypothesis of equal dependence structure is not rejected at a 5% level. This means that the two categories of funds act in a similar way when adjusting the expenses they charge to recover their transaction costs.

4.2. **Emotional experience and life satisfaction.** The data set consists of positive affect scores (positive emotional mood) and life satisfaction scores (subjective well-being) recorded in China (559 university students) and the United States (443 university students) in the early 90's. We refer to the paper of Suh et al. (1998) for data description and background on the psychological concepts. The question is whether the dependence structure for a collectivist culture, i.e., where a significant part of one's identity is made of collective elements, versus an individualistic culture, i.e., where one's internal attributes are emphasized over the evaluations and expectations of others, can be considered as equal or not. The p-value is 0, and we conclude that the null hypothesis of equal dependence structure is rejected at a 5% level. Hence the underlying culture has a significant impact.

4.3. **Losses and ALAEs.** Often actuaries have to price insurance contracts involving pairs of dependent variables. A classical example consists of computing the premium of a reinsurance treaty on a policy with unlimited liability, some retention level of the losses and a prorata sharing of ALAEs. Here ALAEs are type of insurance company expenses that are specifically attributable to the settlement of individual claims such as lawyers fees and claims investigation expenses. The data are extracted from a database about medical insurance claims available from the Society of Actuaries. A thorough description of the data can be found in the monograph Grazier

and G'Sell (1997). We analyze the dependence structure between losses (hospital charges) and ALAEs (other charges) for dependent females (967 observations) versus employee females (1116 observations) aged 30 to 39 in 1991 and insured by a Preferred Provider Organization (PPO) plan. The p-value is 0.065, and the null hypothesis of equal dependence structure is not rejected at a 5% level. We conclude that the status of the policy holder is here irrelevant (at a 5% level), and that premiums charged to both types of individuals should be the same if margins are roughly identical.

4.4. **St John's wort vs sertaline.** In van Gurp et al. (2002) the authors want to compare the change in severity of depressive symptoms and occurrence of side effects in primary care patients treated with St John's wort and sertaline using a double-blind randomized 12-week trial. For each of the two treatment groups, depression was measured every two weeks with two different instruments: Hamilton raring scale for Depression (Ham-D) and Beck Depression Inventory (BDI). The authors conclude that there is no significant difference between the two treatments. By looking at the two groups, we now ask whether there is no change on the dependence structure of the two measures of depression over time. To this end, we use the methodology developed for paired samples. All ten pairs of measures corresponding to weeks 2,4,6,8,10 are compared, and we find that the largest estimated p-value is 0.001. Thus we have that the null hypothesis of equal dependence structure is rejected at a 5% level. This rejection might impact the conclusion on the no difference between the two treatments since the relationship between the two measurement instruments is not the same in the two groups.

Appendix A. Proofs of the results

Let $\xi_1, \ldots, \xi_n$ be independent and identically distributed random variables with mean zero and variance one. Suppose also that $X_1, \ldots, X_n$ are independent random

vectors with continuous marginals $F_1, \ldots, F_d$ and copula C. Set $U_{ij} = F_j(Z_{ij})$, $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, d\}$.

Then, for any $u = (u_1, \ldots, u_d) \in (0,1)^d$, α_n and C_n can be expressed as

$$\alpha_n(u) = \sqrt{n}\left\{\tilde{C}_n(u) - C(u)\right\},$$

with

$$\tilde{C}_n(u) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{I}\left(U_{i1} \leq u_1, \ldots, U_{id} \leq u_d\right),$$

and

$$
\begin{aligned}
C_n(u) &= \frac{1}{n}\sum_{i=1}^{n}\mathbb{I}\left(F_{n1}(Z_{i1}) \leq u_1, \ldots, F_{nd}(Z_{id}) \leq u_d\right) \\
&= \frac{1}{n}\sum_{i=1}^{n}\mathbb{I}\left(U_{i1} \leq E_{n1}^{-1}(u_1), \ldots, U_{id} \leq E_{nd}^{-1}(u_d)\right) \\
&= \tilde{C}_n\left(E_{n1}^{-1}(u_1), \ldots, E_{nd}^{-1}(u_d)\right),
\end{aligned}
$$

where for any $j \in \{1, \ldots, d\}$,

$$E_{nj}(u_j) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{I}(U_{ij} \leq u_j), \quad u_j \in [0,1].$$

Furthermore, for any $u = (u_1, \ldots, u_d) \in [0,1]^d$, set

$$\tilde{\alpha}_n(u) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi_i\left\{\mathbb{I}\left(U_{i1} \leq u_1, \ldots, U_{id} \leq u_d\right) - \tilde{C}_n(u)\right\}.$$

Then, for any $u = (u_1, \ldots, u_d) \in [0,1]^d$,

$$
\begin{aligned}
\hat{\alpha}_n(u) &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi_i\left[\mathbb{I}\left(F_{n1}(Z_{i1}) \leq u_1, \ldots, F_{nd}(Z_{id}) \leq u_d\right) - C_n(u)\right] \\
&= \tilde{\alpha}_n\left(E_{n1}^{-1}(u_1), \ldots, E_{nd}^{-1}(u_d)\right).
\end{aligned}
$$

It follows from the classical multiplier central limit theorem (van der Vaart and Wellner 1996) that $(\alpha_n, \tilde{\alpha}_n) \rightsquigarrow (\alpha, \tilde{\alpha})$ in $\mathcal{D}([0,1]^d) \times \mathcal{D}([0,1]^d)$, where $\tilde{\alpha}$ is an independent copy of α, and α is a C-Brownian bridge.

Next, since for any $j \in \{1, \ldots, d\}$, $\sup_{u_j \in [0,1]}\left|E_{nj}^{-1} - u_j\right| = \sup_{u_j \in [0,1]}\left|E_{nj} - u_j\right| \to 0$ as $n \to \infty$, e.g., Shorack and Wellner (1986), the following result holds.

Lemma A.1. $(\alpha_n, \hat{\alpha}_n) \rightsquigarrow (\alpha, \tilde{\alpha})$ *in* $\mathcal{D}([0,1]^d) \times \mathcal{D}([0,1]^d)$, *where* $\tilde{\alpha}$ *is an independent copy of* α, *and* α *is a* C-*Brownian bridge.*

A.1. **Proof of Theorem 2.1.**

Proof. The proof is closely related to the one in Scaillet (2005). Here we can simply use Lemma A.1 to conclude that, as $n_1 \to \infty$,

$$\left(\alpha_{n_1}, \hat{\alpha}_{n_1}^{(1)}, \ldots, \hat{\alpha}_{n_1}^{(N)}\right) \rightsquigarrow \left(\alpha, \tilde{\alpha}^{(1)}, \ldots, \tilde{\alpha}^{(N)}\right)$$

in $\mathcal{D}([0,1]^d)^{\otimes(N+1)}$, where $\tilde{\alpha}^{(1)}, \ldots, \tilde{\alpha}^{(N)}$ are independent copies of α, and α is a C-Brownian bridge.

Also, as $n_2 \to \infty$,

$$\left(\gamma_{n_2}, \hat{\gamma}_{n_2}^{(1)}, \ldots, \hat{\gamma}_{n_2}^{(N)}\right) \rightsquigarrow \left(\gamma, \tilde{\gamma}^{(1)}, \ldots, \tilde{\gamma}^{(N)}\right)$$

in $\mathcal{D}([0,1]^d)^{\otimes(N+1)}$, where $\tilde{\gamma}^{(1)}, \ldots, \tilde{\gamma}^{(N)}$ are independent copies of γ, and γ is a D-Brownian bridge.

As a consequence of independence between

$$\left(\alpha_{n_1}, \hat{\alpha}_{n_1}^{(1)}, \ldots, \hat{\alpha}_{n_1}^{(N)}\right) \quad \text{and} \quad \left(\gamma_{n_2}, \hat{\gamma}_{n_2}^{(1)}, \ldots, \hat{\gamma}_{n_2}^{(N)}\right),$$

we may conclude that as $\min(n_1, n_2) \to \infty$,

$$\left(\alpha_{n_1}, \gamma_{n_2}, \hat{\alpha}_{n_1}^{(1)}, \hat{\gamma}_{n_2}^{(1)}, \ldots, \hat{\alpha}_{n_1}^{(N)}, \hat{\gamma}_{n_2}^{(N)}\right) \rightsquigarrow \left(\alpha, \gamma, \tilde{\alpha}^{(1)}, \tilde{\gamma}^{(1)}, \ldots, \tilde{\alpha}^{(N)}, \tilde{\gamma}^{(N)}\right)$$

in $\mathcal{D}([0,1]^d)^{\otimes 2(N+1)}$, where $\left(\tilde{\alpha}^{(1)}, \tilde{\gamma}^{(1)}\right), \ldots, \left(\tilde{\alpha}^{(N)}, \tilde{\gamma}^{(N)}\right)$ are independent copies of (α, γ), α is independent of γ.

Next, since the conditions of Proposition A.2 of the next section are met, we obtain that for any $l \in \{1, \ldots, d\}$, $\widehat{\partial_{u_l} C_{n_1, h_1}}$ and $\widehat{\partial_{u_l} D_{n_2, h_2}}$ converge uniformly in probability to $\partial_{u_l} C$ and $\partial_{u_l} D$.

Hence $\left(\mathcal{E}_{n_1, n_2}, \hat{\mathcal{E}}_{n_1, n_2}^{(1)}, \ldots, \hat{\mathcal{E}}_{n_1, n_2}^{(N)}\right) \rightsquigarrow (\mathcal{E}, \tilde{\mathcal{E}}^{(1)}, \ldots, \tilde{\mathcal{E}}^{(N)})$ in $\mathcal{D}\left([0,1]^d\right)^{\otimes(N+1)}$, where $\tilde{\mathcal{E}}^{(1)}, \ldots, \tilde{\mathcal{E}}^{(N)}$ are independent copies of $\mathcal{E}$. Since the mapping $g \mapsto \displaystyle\int_{[0,1]^d} g^2(u) du$

is continuous, whenever g is continuous on $[0,1]^d$, it follows that

$$\left(S^{(0)}_{n_1,n_2}, \hat{S}^{(1)}_{n_1,n_2}, \dots \hat{S}^{(N)}_{n_1,n_2}\right) \rightsquigarrow \left(S, \tilde{S}^{(1)}, \dots \tilde{S}^{(N)}\right) \text{ in } [0,\infty)^{\otimes(N+1)},$$

where $\tilde{S}^{(1)}, \dots \tilde{S}^{(N)}$ are independent copies of $S = \int_{[0,1]^d} \mathcal{E}^2(u)du$. An approximate

p-value for S_{n_1,n_2} is then given by $\dfrac{1}{N}\sum_{k=1}^{N}\mathbb{I}\left(\hat{S}^{(k)}_{n_1,n_2} > S_{n_1,n_2}\right)$. $\square$

A.2. Uniform convergence of partial derivatives estimates.

Proposition A.2. *Suppose that ∇C and ∇D are continuous on $[0,1]^d$. Take $h_i = n_i^{-1/2}$, $i = 1,2$. Then, as $\min(n_1,n_2) \to \infty$,*

$$\max_{1\le l\le d}\sup_{u\in[0,1]^d}\left|\widehat{\partial_{u_l}C_{n_1,h_1}}(u) - \partial_{u_l}C(u)\right| \xrightarrow{Pr} 0$$

and

$$\max_{1\le l\le d}\sup_{u\in[0,1]^d}\left|\widehat{\partial_{u_l}D_{n_2,h_2}}(u) - \partial_{u_l}D(u)\right| \xrightarrow{Pr} 0.$$

Proof. Let $l \in \{1, \dots, d\}$ be fixed. Then,

$$\begin{aligned}
\widehat{\partial_{u_l}C_{n_1,h_1}}(u) &= \frac{C_{n_1}(u+h_1e_l) - C_{n_1}(u-h_1e_l)}{2h_1}\\
&= \frac{C(u+h_1e_l) - C(u-h_1e_l)}{2h_1}\\
&\quad + \frac{\mathbb{C}_{n_1}(u+h_1e_l) - \mathbb{C}_{n_1}(u-h_1e_l)}{2h_1\sqrt{n_1}}.
\end{aligned}$$

Therefore we get by choosing $h_1 = n_1^{-1/2}$:

$$\begin{aligned}
\sup_{u\in[0,1]^d}&\left|\widehat{\partial_{u_l}C_{n_1,h_1}}(u) - \partial_{u_l}C(u)\right|\\
&= \sup_{u\in[0,1]^d}\left|\frac{C_{n_1}(u+h_1e_l) - C_{n_1}(u-h_1e_l)}{2h_1} - \partial_{u_l}C(u)\right|\\
&\le \sup_{u\in[0,1]^d}\left|\frac{C(u+h_1e_l) - C(u-h_1e_l)}{2h_1} - \partial_{u_l}C(u)\right|\\
&\quad + \frac{1}{2}\sup_{u\in[0,1]^d}\left|\mathbb{C}_{n_1}(u+h_1e_l) - \mathbb{C}_{n_1}(u-h_1e_l)\right|,
\end{aligned}$$

which tends to 0 as $n_1 \to \infty$, since $\partial_{u_l} C(u)$ is assumed to be continuous on $[0,1]^d$, and $\mathbb{C}_{n_1}$ converges in law to a continuous centered gaussian process $\mathbb{C}$. The proof for $\widehat{\partial_{u_l} D_{n_2, h_2}}$ is similar. $\qquad \square$

A.3. **Proof of Theorem 2.2.**

Proof. The proof is similar to the proof of Theorem 2.1. First, consider the independent vectors $Z_1 = (X_1, Y_1), \ldots, Z_n = (X_n, Y_n)$, having copula $\mathcal{C}$ on $[0,1]^{2d}$, with the property that for any $u, v \in [0,1]^d$, $\mathcal{C}(u, 1, \ldots, 1) = C(u)$ and $\mathcal{C}(1, \ldots, 1, v) = D(v)$.

Next, for all $u, v \in [0,1]^d$, define

$$\mathcal{C}_n(u, v) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(U_{i,n} \leq u, V_{i,n} \leq v),$$

$$\upsilon_n(u, v) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(U_i \leq u, V_i \leq v),$$

and

$$\hat{\upsilon}_n^{(k)}(u, v) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i^{(k)} \left\{ \mathbb{I}(U_{i,n} \leq u, V_{i,n} \leq v) - \mathcal{C}_n(u, v) \right\}.$$

It follows from Lemma A.1 that, as $n \to \infty$,

$$\left(\upsilon_n, \hat{\upsilon}_n^{(1)}, \ldots, \hat{\upsilon}_n^{(N)} \right) \rightsquigarrow \left(\upsilon, \tilde{\upsilon}^{(1)}, \ldots, \tilde{\upsilon}^{(N)} \right)$$

in $\mathcal{D}([0,1]^d)^{\otimes(N+1)}$, where $\tilde{\upsilon}^{(1)}, \ldots, \tilde{\upsilon}^{(N)}$ are independent copies of υ, and υ is a $\mathcal{C}$-Brownian bridge.

Since for any $u, v \in [0,1]$, we have

$$C_n(u) = \mathcal{C}_n(u, 1, \ldots, 1), \quad \tilde{\alpha}_n(u) = \upsilon_n(u, 1, \ldots, 1), \quad \hat{\alpha}_n^{(k)}(u) = \hat{\upsilon}_n^{(k)}(u, 1, \ldots, 1)$$

and

$$D_n(u) = \mathcal{C}(1, \ldots, 1, v), \quad \tilde{\gamma}_n(v) = \upsilon_n(1, \ldots, 1, v), \quad \hat{\gamma}_n^{(k)}(u) = \hat{\upsilon}_n^{(k)}(1, \ldots, 1, v),$$

we may conclude that as $n \to \infty$,

$$\left(\alpha_n, \gamma_n, \hat{\alpha}_n^{(1)}, \hat{\gamma}_n^{(1)}, \ldots, \hat{\alpha}_n^{(N)}, \hat{\gamma}_n^{(N)} \right) \rightsquigarrow \left(\alpha, \gamma, \tilde{\alpha}^{(1)}, \tilde{\gamma}^{(1)}, \ldots, \tilde{\alpha}^{(N)}, \tilde{\gamma}^{(N)} \right)$$

in $\mathcal{D}([0,1]^d)^{\otimes 2(N+1)}$, where $\left(\tilde{\alpha}^{(1)}, \tilde{\gamma}^{(1)}\right), \ldots, \left(\tilde{\alpha}^{(N)}, \tilde{\gamma}^{(N)}\right)$ are independent copies of (α, γ), where for any $u, v \in [0,1]^d$, $\alpha(u) = v(u, 1, \ldots, 1)$ is a C-Brownian bridge and $\gamma(v) = v(1, \ldots, 1, v)$ is a D-Brownian bridge.

Next, since the conditions of Proposition A.2 of the previous section are met, we obtain that for any $l \in \{1, \ldots, d\}$, $\widehat{\partial_{u_l} C_{n,h}}$ and $\widehat{\partial_{u_l} D_{n,h}}$ converge uniformly in probability to $\partial_{u_l} C$ and $\partial_{u_l} D$.

Hence, defining $\mathcal{E}_n = \mathbb{C}_n - \mathbb{D}_n$ and $\hat{\mathcal{E}}_n^{(k)} = \hat{\mathbb{C}}_{n,h}^{(k)} - \hat{\mathbb{D}}_{n,h}^{(k)}$, it follows that

$$\left(\mathcal{E}_n, \hat{\mathcal{E}}_n^{(1)}, \ldots, \hat{\mathcal{E}}_n^{(N)}\right) \rightsquigarrow (\mathcal{E}, \tilde{\mathcal{E}}^{(1)}, \ldots, \tilde{\mathcal{E}}^{(N)}) \text{ in } \mathcal{D}\left([0,1]^d\right)^{\otimes(N+1)},$$

where $\tilde{\mathcal{E}}^{(1)}, \ldots, \tilde{\mathcal{E}}^{(N)}$ are independent copies of $\mathcal{E}$. Since the mapping $g \mapsto \int_{[0,1]^d} g^2(u) du$ is continuous, whenever g is continuous on $[0,1]^d$, it follows that

$$\left(S_n^{(0)}, \hat{S}_n^{(1)}, \ldots \hat{S}_n^{(N)}\right) \rightsquigarrow \left(S, \tilde{S}^{(1)}, \ldots \tilde{S}^{(N)}\right) \text{ in } [0, \infty)^{\otimes(N+1)},$$

where $\tilde{S}^{(1)}, \ldots \tilde{S}^{(N)}$ are independent copies of $S = \int_{[0,1]^d} \mathcal{E}^2(u) du$. An approximate p-value for S_n is then given by $\frac{1}{N} \sum_{k=1}^{N} \mathbb{I}\left(\hat{S}_n^{(k)} > S_n\right)$. $\square$

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