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Karemera, Mucyo

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Université de Genève
Section de Mathématiques

Faculté des Sciences
Professeur Rinat KASHAEV

Quantum Invariants of 3-Manifolds from a Quantum Group related to $U_q(\mathfrak{sl}_3)$

THÈSE

présentée à la Faculté des Sciences de l'Université de Genève
pour obtenir le grade de Docteur ès Sciences, mention Mathématiques

par

Mucyo KAREMERA

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"Quantum Invariants of 3-Manifolds from a Quantum Group Related to $U_q(\mathfrak{sl}_3)$"

La Faculté des sciences, sur le préavis de Monsieur R. KASHAEV, professeur associé et directeur de thèse (Section de mathématiques), Monsieur M. MARINO BEIRAS, professeur ordinaire (Section de mathématiques et Département de physique théorique), Monsieur B. PATUREAUD-MIRAND, docteur (Laboratoire de mathématiques de Bretagne Atlantique, Université de Bretagne-Sud, Vannes, France) et Monsieur A. VIRELIZIER, professeur (Laboratoire Paul Painlevé, Université Lille 1, Villeneuve d'Ascq, France), autorise l'impression de la présente thèse, sans exprimer d'opinion sur les propositions qui y sont énoncées.

Genève, le 8 décembre 2016

Thèse - 5051 -

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N.B. - La thèse doit porter la déclaration précédente et remplir les conditions énumérées dans les "Informations relatives aux thèses de doctorat à l'Université de Genève".

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Résumé

La théorie que nous développons dans cette thèse trouve ses origines dans la littérature de physique quantique basée sur la théorie du moment cinétique et ses développements ultérieurs dans la littérature mathématiques, notamment par Turaev et Viro. Ces derniers ont été les premiers à observer que la catégorie des représentations du groupe quantique $U_q(\mathfrak{sl}_2)$ permet de construire des invariants topologique de 3-variétés. Ces invariants sont obtenus comme somme d'état sur 3-variétés triangulées. Les ingrédients principaux des sommes d'état sont les $6j$-symboles. Les $6j$-symboles sont naturellement associés aux tétraèdres combinatoires. Dans le cas de la théorie de représentation de dimension finie du moment cinétique, ainsi que de sa q-déformation, la valeur numérique d'un $6j$-symbole est déterminée par six représentations irréductibles associées aux arêtes d'un tétraèdre. En particulier, les $6j$-symboles réalisent explicitement les symétries tétraédriques. Dans la théorie de Turaev-Viro, une spécification du paramètre de déformation aux racines de l'unité permet de considérer un ensemble fini de représentations irréductibles dans la catégorie de représentation. L'invariant de 3-variétés est alors obtenu en sommant sur tous les coloriages des arêtes d'une triangulation par des représentations de cet ensemble.

Une construction analogue aux invariants de Turaev-Viro est due à Kashaev. Ses invariants furent définis comme des sommes d'état sur les triangulations de paire (M, L), où M est une 3-variété et $L \subset M$ est un entrelacs, en utilisant des versions chargées des $6j$-symboles associées à certains modules de la sous-algèbre de Borel de $U_q(\mathfrak{sl}_2)$. Ses invariants le conduisirent à la fameuse conjecture du volume. La particularité de ses $6j$-symboles, construits à l'aide du dilogarithme quantique défini par Faddeev et Kashaev, est qu'ils ne dépendent que d'une variable continue et que les symétries du tétraèdres sont implicitement réalisées par des matrices de transformation non-triviales.

Ces invariants ont été généralisés de deux façons. Une généralisation topologique permit à Baseilhac et Benedetti de définir des invariants quantiques hyperbolique de 3-variétés en utilisant les mêmes $6j$-symboles chargés. Ils interprétèrent les variables continues complexes déterminant les $6j$-symboles comme les paramètres tétraédriques de tétraèdres idéaux dans les équations de recollement de Thurston. Cela leur permit de construire des invariants quantiques qui sont des fonctions sur les variétés de deformation de 3-variétés à cusp.

Ce carde topologique a été utilisé, dans une certaine mesure, dans l'autre généralisation due à Geer, Kashaev et Turaev. Cette construction est une généralisation au niveau de la théorie des catégories de la construction de Kashaev-Baseilhac-Benedetti. La notion de $\hat{\Psi}$-système dans une catégorie monoïdale abélienne y est introduite. Elle fournit un carde général pour la construction de $6j$-symboles chargés. Cependant, dans ce contexte, la dépendance des $6j$-symboles en leurs variables continues ne se réduit pas nécessairement à une seule variable. Ainsi, leur interprétation en termes de paramètres tétraédriques de tétraèdres idéaux hyperboliques n'est pas évidente.

Le résultat principal de cette thèse est la construction d'un $\hat{\Psi}$-système dans la catégorie de modules d'un groupe quantique relié à $U_q(\mathfrak{sl}_3)$ donnant lieu à une famille d'invariants de 3-variétés. Bien que les $6j$-symboles de cette construction proviennent d'un groupe quantique différent de $U_q(\mathfrak{sl}_2)$, ils sont similaires à ceux de la théorie de Kashaev-Baseilhac-Benedetti du fait d'être construits à l'aide du dilogarithme quantique. En effet, ils dépendent aussi d'une seule variable permettant une interprétation en termes de paramètres tétraédriques de tétraèdres idéaux hyperboliques.

Contents

Notations

In this thesis we will use the following notations:

(1) $\mathbb{R}_{\neq 0} = \mathbb{R} \setminus \{0\}$
(2) $\mathbb{R}_{\neq 0,1} = \mathbb{R} \setminus \{0, 1\}$
(3) $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$
(4) $\mathcal{V} = \mathbb{C}^N \otimes \mathbb{C}^N$
(5) For any linear basis $\{v_\alpha\}_{\alpha \in \mathbb{Z}_N}$ of $\mathbb{C}^N$, we will denote by $\left\{v_{(\alpha,\beta)}\right\}_{\alpha,\beta \in \mathbb{Z}_N}$ the associated basis $\{v_\alpha \otimes v_\beta\}_{\alpha,\beta \in \mathbb{Z}_N}$ of $\mathcal{V}$
(6) $\mathcal{A} = \mathrm{End}(\mathcal{V})$
(7) If $x, y \in \mathbb{C}$ are such that $x^N = y^N$, we will write $x \equiv y$

Introduction

The theory we develop in this dissertation can be traced back to original constructions in quantum physics literature based on the theory of quantum angular momentum [**11**] and subsequent developments in mathematics literature, notably by Turaev and Viro [**13**] who first observed that the category of representations of the quantum group $U_q(\mathfrak{sl}_2)$ gives rise to topological invariants of 3-manifolds. The invariants are obtained as state sums on triangulated 3-manifolds. The key ingredients of the state sums are the 6j-symbols. The $6j$-symbols are naturally associated with combinatorial tetrahedra. In the case of finite dimensional representation theory of the angular momentum, as well as of its q-deformation, the numerical values of 6j-symbols are specified by six irreducible representations associated with the edges of the tetrahedron. In particular, they realize explicitly the tetrahedral symmetries. In the Turaev-Viro theory, a specification of the deformation parameter to roots of unity allows to separate a sector in the representation category with a finite set of irreducible representations, and the 3-manifold invariant is obtained by summing over all labelings of the edges of a triangulation by representations from this finite set.

A related construction to the Turaev-Viro invariants was done by Kashaev in [**6**]. His invariants were defined as state sums on triangulations of the pair (M, L), where M is a 3-manifold and $L \subset M$ is a link, using charged versions of $6j$-symbols associated to certain modules of the Borel subalgebra of $U_q(\mathfrak{sl}_2)$. These invariants led him to the famous volume conjecture [**7**]. The particularity of his $6j$-symbols, constructed using the quantum dilogarithm function defined by Faddeev and Kashaev in [**4**], is that they depend on a continuous variable and the tetrahedral symmetries are realized implicitly through non-trivial transformation matrices.

These invariants have been generalized in two ways. A topological generalization has been done by Baseilhac and Benedetti in [**1**] where they define quantum hyperbolic invariants of 3-manifolds using the same charged $6j$-symbols. They interpreted the complexified continuous parameters, entering the 6j-symbols, as shape variables of ideal tetrahedra in Thurston's gluing equations. This enabled them to construct quantum invariants which are functions on deformation varieties of cusped 3-manifolds.

This topological framework has been used, to some extend, in the other generalization done by Geer, Kashaev and Turaev in [**5**]. This construction is a categorical generalization of the Kashaev-Baseilhac-Benedetti construction. It introduces the notion of a $\hat{\Psi}$-system in a monoidal abelian category which provides a general framework for charged $6j$-symbols. However, in this general context, the dependence of 6j-symbols on continuous variables does not necessarily reduce only to one variable, and thus the interpretation in terms of the shapes of hyperbolic ideal tetrahedra is not evident.

The main result of this thesis is the construction of a $\hat{\Psi}$-system in the category of modules of a quantum group related to $U_q(\mathfrak{sl}_3)$ leading to a family of 3-manifolds invariants. Although the $6j$-symbols involved in this construction come from another quantum group than $U_q(\mathfrak{sl}_2)$, they are similar to the one used in the Kashaev-Baseilhac-Benedetti theory in the sense that they are also constructed using the quantum dilogarithm. They also only depend on one parameter allowing interpretation in terms of shapes variables of ideal hyperbolic tetrahedra.

This dissertation splits into two Chapters. Chapter 1 is devoted to the construction of the $\hat{\Psi}$-system. Chapter 2 is devoted to technical computations. In the rest of this introduction, we explain the definition of our invariants.

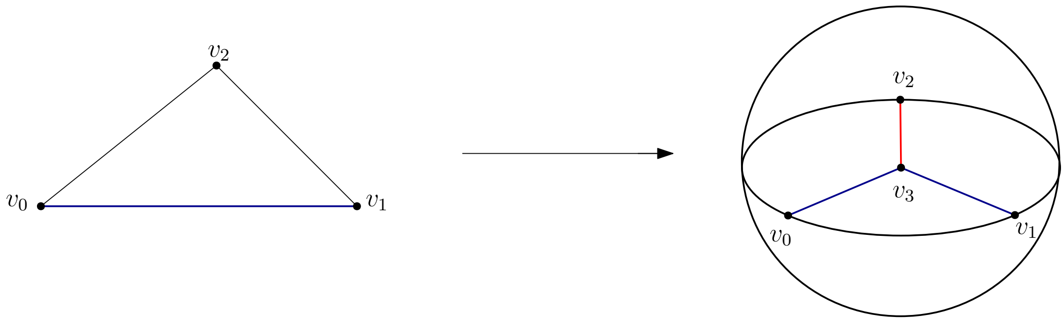
1. H-triangulation of (M, L)

Let M be a closed connected oriented 3-manifold. A *quasi-regular triangulation* $\mathcal{T}$ *of* M is a decomposition of M as a union of embedded tetrahedra (3-simplices) such that the intersection of any two tetrahedra is a union (possibly, empty) of several of their vertices (0-simplices), edges (1-simplices) and faces (2-simplices). Quasi-regular triangulations differ from the usual triangulations in that they may have tetrahedra meeting along several vertices, edges, and faces. Note that each edge of a quasi-regular triangulation has two distinct endpoints. In the sequel, we denote $\Lambda_i(\mathcal{T})$ the set of i-simplices of $\mathcal{T}$ for $i \in \{0, 1, 2, 3\}$.

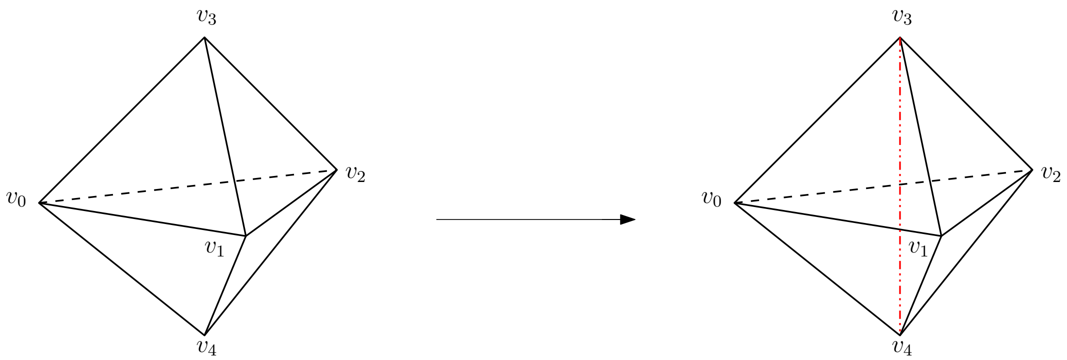
Consider a non-empty link $L \subset M$. An *H-triangulation of* (M, L) is a pair $(\mathcal{T}, \mathcal{L})$ where $\mathcal{T}$ is a quasi-regular triangulation of M and $\mathcal{L} \subset \Lambda_1(\mathcal{T})$ is such that each vertex of $\mathcal{T}$ belongs to exactly two edges of $\mathcal{L}$ and L is the union of the elements of $\mathcal{L}$.

Proposition 1.1 ([**1**], Proposition 4.20). *For any non-empty link L in M, the pair (M, L) admits an H-triangulation.*

H-triangulations of (M, L) can be related by elementary moves of two types, the *H-bubble moves* and the *H-Pachner* 2-3 *moves*. The *positive* H-bubble move on an H-triangulation $(\mathcal{T}, \mathcal{L})$ starts with a choice of a face $F = v_0v_1v_2 \in \Lambda_2(\mathcal{T})$, where $v_0, v_1, v_2 \in \Lambda_0(\mathcal{T})$, such that at least one of its edges, say v_0v_1, is in $\mathcal{L}$. Consider two tetrahedra of T meeting along F. We unglue these tetrahedra along F and insert a 3-ball between the resulting two copies of F. We triangulate this 3-ball by adding a vertex v_3 at its center and three edges connecting v_3 to v_0, v_1, and v_2. The edge v_0v_1 is removed from $\mathcal{L}$ and replaced by the edges v_0v_3 and v_1v_3. This move can be visualized as the transformation



where the blue edges belong to $\mathcal{L}$. The inverse move is the *negative H-bubble move*. The *positive H-Pachner* 2-3 *move* can be visualized as the transformation



This transformation preserve the set $\mathcal{L}$. The inverse move is the *negative H-Pachner move*; it is allowed only when the edge common to the three tetrahedra on the right is not in $\mathcal{L}$.

PROPOSITION 1.2 ([**1**], Proposition 4.23). *Let L be a non-empty link in M. Any two H-triangulations of (M, L) can be related by a finite sequence of H-bubble moves and H-Pachner moves in the class of H-triangulations of (M,L).*

2. Charge on $(\mathcal{T},\mathcal{L})$ and $\mathbb{R}$-coloring of $\mathcal{T}$

A *charge on* $T \in \Lambda_3(\mathcal{T})$ is a map $c : \Lambda_1(T) \to \frac{1}{2}\mathbb{Z}$ such that

(1) $c(e) = c(e')$ if e, e' are opposite edges,
(2) $c(e_1) + c(e_2) + c(e_3) = \frac{1}{2}$ if e_1, e_2, e_3 are edges of a face of T.

We denote $\Lambda_3^1(\mathcal{T}) = \{(T, e) | T \in \Lambda_3(\mathcal{T}), e \in \Lambda_1(T)\}$ and consider the obvious projection $\epsilon_{\mathcal{T}} : \Lambda_3^1(\mathcal{T}) \to \Lambda_1(\mathcal{T})$. For any edge e of $\mathcal{T}$, the set $\epsilon_{\mathcal{T}}^{-1}(e)$ has n elements, where n is the number of tetrahedron adjacent to e.

A *charge on* $(\mathcal{T},\mathcal{L})$ is a map $c : \Lambda_3^1(\mathcal{T}) \to \frac{1}{2}\mathbb{Z}$ such that

(1) the restriction of c to any tetrahedron T of $\mathcal{T}$ is a charge on T,
(2) for each edge e of $\mathcal{T}$ not belonging to $\mathcal{L}$ we have $\sum_{e' \in \epsilon_{\mathcal{T}}^{-1}(e)} c(e') = 1$,
(3) for each edge e of $\mathcal{T}$ belonging to $\mathcal{L}$ we have $\sum_{e' \in \epsilon_{\mathcal{T}}^{-1}(e)} c(e') = 0$.

Each charge c on $(\mathcal{T},\mathcal{L})$ determines a cohomology class $[c] \in H^1(M; \mathbb{Z}/2\mathbb{Z})$.

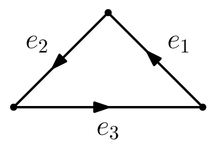
Proposition 2.1 ([**1**], Lemma 4.10). *Let $(\mathcal{T},\mathcal{L})$ and $(\mathcal{T}',\mathcal{L}')$ be H-triangulations of (M,L) such that $(\mathcal{T}',\mathcal{L}')$ is obtained from $(\mathcal{T},\mathcal{L})$ by an H-Pachner move or an H-bubble move. Let c be a charge on $(\mathcal{T},\mathcal{L})$. Then there exists a charge c' on $(\mathcal{T}',\mathcal{L}')$ such that c' equals c on all pairs $(T,e)\in\Lambda_3^1(\mathcal{T})$ such that T is not involved in the move and for any common edge e of $\mathcal{T}$ and $\mathcal{T}'$,*

$$\sum_{a\in\epsilon_{\mathcal{T}}^{-1}(e)} c(a) = \sum_{a'\in\epsilon_{\mathcal{T}'}^{-1}(e)} c'(a').$$

Moreover, $[c]=[c']$.

A $\mathbb{R}$-*coloring of* $\mathcal{T}$ is a map Φ from the oriented edges of $\mathcal{T}$ to $\mathbb{R}^*$ such that

(1) $\Phi(-e)=-\Phi(e)$ for any oriented edge e of $\mathcal{T}$, where $-e$ is e with the opposite orientation,
(2) $\Phi(e_1)+\Phi(e_2)+\Phi(e_3)=0$ if e_1,e_2,e_3 are edges of a face of $\mathcal{T}$ with the following orientations:



A $\mathbb{R}$-coloring Φ represents a cohomology class $[\Phi]\in H^1(M,\mathbb{R})$.

3. State sum

The main result of this thesis is the construction of a $\hat{\Psi}$-*system* in a particular monoidal abelian category. This allows us to construct a state sum invariant of any tuple $(M,L,[\Phi],[c])$. The key ingredients in this state sum are the *charged 6j-symbols* $S(z|a,c)^{\pm1}\in\mathcal{A}^{\otimes 2}$ and more precisley their matrix elements. We give explicit expression of the matrix elements, using a suitable basis $\{u_\alpha\}_{\alpha\in\mathbb{Z}_N^2}$ of $\mathcal{V}$ and its dual basis $\{\bar{u}_\alpha\}_{\alpha\in\mathbb{Z}_N^2}$(defined in Lemma 2.9). Fix an integer t, a natural number $N\notin 3\mathbb{N}$ which divides t^2+t+1 and a primitive N-th root of unity ω. Remark that N is odd. Therefore, for any $a\in\frac{1}{2}\mathbb{Z}$, we will write ω^a instead of $\omega^{a(N+1)}$. For any $z\in\mathbb{R}_{\neq 0,1}$, any $a,c\in\frac{1}{2}\mathbb{Z}$ and any $\alpha,\beta,\mu,\nu\in\mathbb{Z}_N^2$ we have

$$\Big\langle \bar{u}_\mu\otimes\bar{u}_\nu\Big|S(z|a,c)\Big|u_\alpha\otimes u_\beta\Big\rangle = \Big\langle \bar{u}_\mu\otimes\bar{u}_\nu\Big|S(z)\Big|u_\alpha\otimes u_{(\beta_1-2ta,\beta_2)}\Big\rangle\times$$
$$\left(z^a(1-z)^c\right)^{\frac{4(1-N)}{N}}\omega^{a(2c-2t\beta_1-\beta_2)+2(t+1)\{a(a+\mu_1)+c(\beta_1-\nu_1)\}+(2t+1)\{a(\mu_2+1)+c(\beta_2-\nu_2)\}}$$

and

$$\Big\langle \bar{u}_\mu\otimes\bar{u}_\nu\Big|S(z|a,c)^{-1}\Big|u_\alpha\otimes u_\beta\Big\rangle = \Big\langle \bar{u}_\mu\otimes\bar{u}_{(\nu_1+2ta,\nu_2)}\Big|S(z)^{-1}\Big|u_\alpha\otimes u_\beta\Big\rangle\times$$
$$\left(z^a(1-z)^c\right)^{\frac{4(1-N)}{N}}\omega^{a(2c-2t\nu_1-\nu_2)+2(t+1)\{a(a-\alpha_1)+c(\beta_1-\nu_1)\}+(2t+1)\{a(\alpha_2+1)+c(\nu_2-\beta_2)\}}$$

where $\langle \bar{u}_\mu \otimes \bar{u}_\nu | S(z) | u_\alpha \otimes u_\beta \rangle$ and $\langle \bar{u}_\mu \otimes \bar{u}_\nu | S(z)^{-1} | u_\alpha \otimes u_\beta \rangle$ are given in Proposition 2.11 and where the Nth root is chosen to be the unique real root.

Let $(\mathcal{T}, \mathcal{L})$ be an H-triangulation of (M, L). Fix a total order on $\Lambda_0(\mathcal{T})$ and consider a $\mathbb{R}$-coloring Φ of $\mathcal{T}$, a charge c on $(\mathcal{T}, \mathcal{L})$ and a map $\alpha : \Lambda_2(\mathcal{T}) \to \mathbb{Z}_N^2$. From this data, we define the state sum as follows. Let T be a tetrahedron of $\mathcal{T}$ with order vertices v_0, v_1, v_2, v_3. We say that T is *right oriented* if the vertices v_0, v_1 and v_2 go round in the counter-clockwise direction when we look at them from v_3 in the increasing order. Otherwise, T is *left oriented.* Set

$$p = \Phi(\overrightarrow{v_0 v_1}), \quad q = \Phi(\overrightarrow{v_1 v_2}), \quad r = \Phi(\overrightarrow{v_2 v_3})$$

where $\overrightarrow{v_i v_j}$ is the oriented edge of T going from v_i to v_j. Then set

$$z = \frac{pr}{(p+q)(q+r)}$$

and denote $c_{ij} = c(v_i v_j)$ and $\alpha_i = \alpha(v_j v_k v_l)$ for $\{i, j, k, l\} = \{0, 1, 2, 3\}$. We associate the following matrix element to the tetrahedron T

$$T(\Phi, c, \alpha) = \begin{cases} \left\langle \bar{u}_{\alpha_2} \otimes \bar{u}_{\alpha_0} \middle| S(z | c_{12}, c_{23}) \middle| u_{\alpha_3} \otimes u_{\alpha_1} \right\rangle & \text{if } T \text{ is right oriented,} \\ \left\langle \bar{u}_{\alpha_3} \otimes \bar{u}_{\alpha_1} \middle| S(z | c_{12}, c_{23})^{-1} \middle| u_{\alpha_2} \otimes u_{\alpha_0} \right\rangle & \text{if } T \text{ is left oriented.} \end{cases} \tag{3.1}$$

and we define the state sum as follows

$$\mathsf{K}_N(\mathcal{T}, \mathcal{L}, \Phi, c) = N^{-2|\Lambda_0(\mathcal{T})|} \sum_{\alpha} \prod_{T \in \mathcal{T}} T(\Phi, c, \alpha) \in \mathbb{C}. \tag{3.2}$$

THEOREM 3.1. *Up to multiplication by integer powers of ω, the state sum $\mathsf{K}_N(\mathcal{T}, \mathcal{L}, \Phi, c)$ depends only on the isotopy class of L in M and the cohomology classes $[\Phi] \in H^1(M, \mathbb{R})$ and $[c] \in H^1(M, \mathbb{Z}/2\mathbb{Z})$ (and does not depend on the choice of Φ and c in their cohomology classes, the H-triangulation $(\mathcal{T}, \mathcal{L})$ of (M, L), and the ordering of the vertices of $\mathcal{T}$).*

The statement of our Theorem is a direct adaptation of Theorem 29 of [**5**]. The reason why $\mathsf{K}_N(\mathcal{T}, \mathcal{L}, \Phi, c)$ is invariant only up to multiplication by integer powers of ω is explained in Section 5 of Chapter 1.

CHAPTER 1

Construction of a $\hat{\Psi}$-system

1. The Hopf algebra $\mathcal{A}_{\omega,t}$ and its reduced cyclic modules

The two-paramerter quantum groups $U_{r,s}(\mathfrak{sl}_n)$ have been introduced by Takeuchi in [**12**]. In [**2**], r and s are non zero elements in a filed $\mathbb{K}$ such that $r \neq s$ and $U_{r,s}(\mathfrak{sl}_n)$ is defined as the unital associative algebra over $\mathbb{K}$ generated by elements $k_i^{\pm 1}, (k_i')^{\pm 1}, e_i, f_i$ $(1 \leq i < n)$ which satisfy the following relations

(R1) The $k_i^{\pm 1}, (k_j')^{\pm 1}$ all commutes with one another and $k_i k_i^{-1} = k_j'(k_j')^{-1} = 1$,

(R2) $k_i e_j = r^{\delta_{i,j}-\delta_{i,j+1}} s^{\delta_{i+1,j}-\delta_{i,j}} e_j k_i$ and $k_i f_j = r^{\delta_{i,j+1}-\delta_{i,j}} s^{\delta_{i,j}-\delta_{i+1,j}} f_j k_i$,

(R3) $k_i' e_j = r^{\delta_{i+1,j}-\delta_{i,j}} s^{\delta_{i,j}-\delta_{i,j+1}} e_j k_i'$ and $k_i' f_j = r^{\delta_{i,j}-\delta_{i+1,j}} s^{\delta_{i,j+1}-\delta_{i,j}} f_j k_i'$,

(R4) $[e_i, f_j] = \dfrac{\delta_{i,j}}{r-s}(k_i - k_i')$,

(R5) $[e_i, e_j] = [f_i, f_j] = 0$ if $|i-j| > 1$,

(R6) $e_i^2 e_{i+1} = (r+s) e_i e_{i+1} e_i + rs e_{i+1} e_i^2$,
$e_i e_{i+1}^2 = (r+s) e_{i+1} e_i e_{i+1} + rs e_{i+1}^2 e_i$,

(R7) $f_i^2 f_{i+1} = (r^{-1}+s^{-1}) f_i f_{i+1} f_i + (rs)^{-1} f_{i+1} f_i^2$,
$f_i f_{i+1}^2 = (r^{-1}+s^{-1}) f_{i+1} f_i f_{i+1} + (rs)^{-1} f_{i+1}^2 f_i$.

As a Hopf algebra, the coproduct $\Delta : U_{r,s}(\mathfrak{sl}_n) \to U_{r,s}(\mathfrak{sl}_n) \otimes U_{r,s}(\mathfrak{sl}_n)$ is such that $k_i^{\pm 1}, (k_i')^{\pm 1}$ are group-like elements, and the remaining coproducts are determined by

$$\Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = 1 \otimes f_i + f_i \otimes k_i'.$$

Therefore, the counit $\varepsilon : U_{r,s}(\mathfrak{sl}_n) \to \mathbb{K}$ and the antipode $\gamma : U_{r,s}(\mathfrak{sl}_n) \to U_{r,s}(\mathfrak{sl}_n)$ are given by

$$\begin{gathered}\varepsilon(k_i) = \varepsilon(k_i') = 1, \quad \gamma(k_i) = k_i^{-1}, \quad \gamma(k_i') = (k_i')^{-1}, \\ \varepsilon(e_i) = \varepsilon(f_i) = 0, \quad \gamma(e_i) = -k_i^{-1} e_i, \quad \gamma(f_i) = -f_i (k_i')^{-1}.\end{gathered}$$

When $r = q$ and $s = q^{-1}$, $U_{r,s}(\mathfrak{sl}_n)$ modulo the (Hopf) ideal generated by the elements $k_i' - k_i^{-1}, 1 \leq i < n$, is $U_q(\mathfrak{sl}_n)$.

The algebra $\mathcal{A}_{\omega,t}$ that we are about to introduce, is obtained as a certain quotient of the Borel subalgebra $BU_{r,s}(\mathfrak{sl}_3)$ of $U_{r,s}(\mathfrak{sl}_3)$ generated by $k_i^{\pm 1}, e_i$, for a certain choice of $r, s \in \mathbb{C}$.

1.1. The Hopf algebra $\mathcal{A}_{\omega,t}$. We fix an integer t, a natural number $N \notin 3\mathbb{N}$ which divides t^2+t+1 and a primitive N-th root of unity ω. Remark that N is odd and N and t are relatively prime.

The Hopf algebra $\mathcal{A}_{\omega,t}$ is defined as the quotient of $BU_{\omega^t,\omega^{t+1}}(\mathfrak{sl}_3)$ by the Hopf ideal generated by $k_1^N - 1$ and $k_2 - k_1^t$. Therefore, the generators k_1, e_1 and e_2 satisfy the following relations

$$\begin{aligned} k_1^N = 1, \quad k_1 e_1 = \omega^{-1} e_1 k_1, \quad k_1 e_2 = \omega^{t+1} e_2 k_1, \\ e_1^2 e_2 = (\omega^t + \omega^{t+1}) e_1 e_2 e_1 + \omega^{2t+1} e_2 e_1^2, \\ e_1 e_2^2 = (\omega^t + \omega^{t+1}) e_2 e_1 e_2 + \omega^{2t+1} e_2^2 e_1. \end{aligned} \tag{1.1}$$

The coproduct $\Delta : \mathcal{A}_{\omega,t} \to \mathcal{A}_{\omega,t} \otimes \mathcal{A}_{\omega,t}$ is given by

$$\Delta(k_1) = k_1 \otimes k_1, \quad \Delta(e_1) = e_1 \otimes 1 + k_1 \otimes e_1, \quad \Delta(e_2) = e_2 \otimes 1 + k_1^t \otimes e_2, \tag{1.2}$$

the counit $\varepsilon : \mathcal{A}_{\omega,t} \to \mathbb{C}$ by

$$\varepsilon(k_1) = 1, \quad \varepsilon(e_1) = 0, \quad \varepsilon(e_2) = 0, \tag{1.3}$$

and the antipode $\gamma : \mathcal{A}_{\omega,t} \to \mathcal{A}_{\omega,t}$ by

$$\gamma(k_1) = k_1^{-1}, \quad \gamma(e_1) = -k_1^{-1} e_1, \quad \gamma(e_2) = -k_1^{-t} e_2. \tag{1.4}$$

1.2. Reduced cyclic $\mathcal{A}_{\omega,t}$-modules. In what follows, all the vector spaces are finite dimensional $\mathbb{C}$-vector spaces.

Let $\{v_\alpha\}_{\alpha \in \mathbb{Z}_N}$ be the canonical basis of $\mathbb{C}^N$, where the index α taking its values in the set $\{0, 1, \ldots, N-1\}$ is interpreted as an element of $\mathbb{Z}_N$. Let $\mathsf{X}, \mathsf{Y} \in \mathrm{End}(\mathbb{C}^N)$ be the invertible operators defined by

$$\mathsf{X} v_\alpha = \omega^{-\alpha} v_\alpha, \quad \mathsf{Y} v_\alpha = v_{\alpha+1}. \tag{1.5}$$

LEMMA 1.1. *The algebra* $\mathrm{End}(\mathbb{C}^N)$ *is generated by* $\{\mathsf{X}, \mathsf{Y}\}$. *In particular,* $\mathcal{A}$ *is generated by* $\mathsf{X}_1, \mathsf{X}_2, \mathsf{Y}_1, \mathsf{Y}_2$ *where*

$$\mathsf{X}_1 = \mathsf{X} \otimes \mathrm{Id}_{\mathbb{C}^N}, \quad \mathsf{Y}_1 = \mathsf{Y} \otimes \mathrm{Id}_{\mathbb{C}^N}, \quad \mathsf{X}_2 = \mathrm{Id}_{\mathbb{C}^N} \otimes \mathsf{X}, \quad \mathsf{Y}_2 = \mathrm{Id}_{\mathbb{C}^N} \otimes \mathsf{Y}.$$

PROOF. We consider the canonical basis $\{v_\alpha\}_{\alpha \in \mathbb{Z}_N}$ of $\mathbb{C}^N$ and its dual basis $\{\bar{v}_\alpha\}_{\alpha \in \mathbb{Z}_N}$. The set $\{E_{i,j}\}_{i,j \in \mathbb{Z}_N} \subset \mathrm{End}(\mathbb{C}^N)$ defined by

$$\langle \bar{v}_\beta | E_{i,j} | v_\alpha \rangle = \delta_{\alpha,j} \delta_{\beta,i}, \quad \forall i, j, \alpha, \beta \in \mathbb{Z}_N,$$

where δ is Kronecker's delta, is a linear basis of $\mathrm{End}(\mathbb{C}^N)$. It is easy to check that we have the following relation

$$E_{i,j} E_{k,l} = \delta_{j,k} E_{i,l}, \quad \forall i, j, k, l \in \mathbb{Z}_N. \tag{1.6}$$

First, we compute, for $i, \alpha, \beta \in \mathbb{Z}_N$,

$$\frac{1}{N}\sum_{j\in\mathbb{Z}_N} \left\langle \bar{v}_\beta | \omega^{ij}\mathsf{X}^j | v_\alpha \right\rangle = \frac{1}{N}\sum_{j\in\mathbb{Z}_N} \omega^{ij}\omega^{-\alpha j}\delta_{\alpha,\beta} = \frac{1}{N}\sum_{j\in\mathbb{Z}_N} \omega^{j(i-\alpha)}\delta_{\alpha,\beta}$$
$$= \frac{1}{N} N\delta_{i,\alpha}\delta_{\alpha,\beta} = \langle \bar{v}_\beta | E_{i,i} | v_\alpha \rangle .$$

Hence, for all $i \in \mathbb{Z}_N$ we have $E_{i,i} = \frac{1}{N}\sum_{j\in\mathbb{Z}_N}\omega^{ij}\mathsf{X}^j$.

Now, using relation (1.6) and since $\mathsf{Y} = \sum_{j\in\mathbb{Z}_N} E_{j+1,j}$, we can compute, for all $j, k \in \mathbb{Z}_N$

$$E_{k,k}\mathsf{Y}E_{i,i} = E_{k,k}\sum_{j\in\mathbb{Z}_N} E_{j+1,j}E_{i,i} = E_{k,k}\sum_{i\in\mathbb{Z}_N}\delta_{i,j}E_{j+1,i} = E_{k,k}E_{i+1,i} = \delta_{k,i+1}E_{i+1,i}.$$

Hence, for all $i \in \mathbb{Z}_N$, $E_{i+1,i} = E_{i+1,i+1}\mathsf{Y}E_{i,i}$.

Finally, for any $i, j \in \mathbb{Z}_N$ we have

$$E_{i,i-1}E_{i-1,i-2}\cdots E_{i-(i-j)+1,i-(i-j)} = E_{i,i-1}E_{i-1,i-2}\cdots E_{j+1,j} = E_{i,j}.$$

Therefore, the operators X and Y generate $\mathrm{End}(\mathbb{C}^N)$. □

We now consider the following operators in $\mathcal{A}$

$$X = \mathsf{X}_1,\quad Y = \mathsf{Y}_1,\quad U = \mathsf{Y}_1^{-t}\mathsf{X}_1^{t}\mathsf{X}_2^{-t},\quad V = \mathsf{Y}_1^{-t}\mathsf{X}_1^{t+1}\mathsf{Y}_2. \tag{1.7}$$

We can easily see that we have $X^N = Y^N = U^N = V^N = \mathrm{Id}_{\mathcal{V}}$. Moreover, by Lemma 1.1, $\mathcal{A}$ is generated by $\{X, Y, U, V\}$ since

$$\mathsf{X}_1 = X,\quad \mathsf{Y}_1 = Y,\quad \mathsf{X}_2 = (X^{-t}Y^tU)^{t+1},\quad \mathsf{Y}_2 = X^{-t-1}Y^tV.$$

Proposition-Definition 1.2. For any $p \in \mathbb{R}_{\neq 0}$, let V_p be the space $\mathcal{V}$ provided with a $\mathcal{A}_{\omega,t}$-module structure $\pi_p : \mathcal{A}_{\omega,t} \to \mathrm{End}(V_p)$, defined by

$$\begin{aligned} &\pi_p(k_1) = X,\quad \pi_p(e_1) = p^{\frac{1}{N}}Y,\\ &\pi_p(e_2) = \left(\tfrac{1}{2}p\right)^{\frac{1}{N}}(U+V)\,Y^{-1}, \end{aligned} \tag{1.8}$$

where the Nth root is chosen to be the unique real root. Furthermore, V_p is cyclic, i.e. the operators $\pi_p(k_1), \pi_p(e_1)$ and $\pi_p(e_2)$ are invertible. We call V_p a *reduced cyclic $\mathcal{A}_{\omega,t}$-module of parameter p.*

Remark 1.3. The reason why we have chosen to use the word "reduced" to name the cyclic $\mathcal{A}_{\omega,t}$-modules V_p is because the set $\{V_p\}_{p\in\mathbb{R}_{\neq 0}}$ is included in a much larger set of cyclic $\mathcal{A}_{\omega,t}$-modules. Indeed, for any $\mathfrak{p} = (p_1, p_2, p_3) \in \mathbb{C}^3$ such that $p_1 \in \mathbb{C}^*$ and $p_2 \notin \{-\omega^i p_3 | i \in \mathbb{Z}_N\}$, the space $\mathcal{V}$ can be provided with a cyclic $\mathcal{A}_{\omega,t}$-module structure $\pi_{\mathfrak{p}} : \mathcal{A}_{\omega,t} \to \mathrm{End}(\mathcal{V})$, defined by

$$\begin{aligned} &\pi_{\mathfrak{p}}(k_1) = X,\quad \pi_{\mathfrak{p}}(e_1) = p_1 Y\\ &\pi_{\mathfrak{p}}(e_2) = p_1^{-t-1}(p_2U + p_3V)Y^{-1}. \end{aligned}$$

The set of reduced cyclic $\mathcal{A}_{\omega,t}$-module is determined by the triples of the form $\mathfrak{p} = \left(p^{\frac{1}{N}}, \left(\frac{1}{2}p^2\right)^{\frac{1}{N}}, \left(\frac{1}{2}p^2\right)^{\frac{1}{N}}\right) \in \mathbb{R}^3$ where $p \in \mathbb{R}_{\neq 0}$.

Before the proof, we need to make a quick recall on the *q-integers*, the *q-factorials*, the *q-binomial coefficients* and the *q-binomial formula*. Fix a non-zero complex number q.

(1) For any integer $n > 0$, we define the *q-integer* $(n)_q \in \mathbb{C}$ to be

$$(n)_q = 1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q}.$$

Note that if q is a nth root of the unit, then $(n)_q = 0$.

(2) The *q-factorial* $(n)!_q$ of an integer $n \geq 0$ is defined by $(0)!_q = 1$ and when $n > 0$ by

$$(n)!_q = (1)_q(2)_q \cdots (n)_q$$

(3) The *q-binomial coefficient* of integers $0 \leq k \leq n$ is defined by

$$\binom{n}{k}_q = \frac{(n)!_q}{(k)!_q(n-k)!_q}.$$

Clearly we have

$$\binom{n}{k}_q = \binom{n}{n-k}_q$$

(4) Finally, for any variables x and y satisfying the relation $yx = qxy$, the *q-binomial formula* (shown in [**10**, Proposition IV.2.2]) is given by

$$(x+y)^n = \sum_{0 \leq k \leq n} \binom{n}{k}_q x^k y^{n-k}. \tag{1.9}$$

Note that if q is a nth root of the unit, then we have,

$$(x+y)^n = x^n + y^n. \tag{1.10}$$

This last formula will be used in the sequel.

Proof of Proposition-Definition 1.2. Using equalities (1.5), one can check that

$$\mathsf{XY} = \omega^{-1}\mathsf{YX}.$$

Then, using equalities (1.7), one easily sees that π_p is an algebra morphism.

The operators $\pi_p(k_1)$ and $\pi_p(e_1)$ are clearly invertible since $p \in \mathbb{R}_{\neq 0}$ and $\mathsf{X}, \mathsf{Y} \in \mathrm{End}(\mathbb{C}^N)$ are invertible operators. For the invertibility of $\pi_p(e_2)$ we compute $\pi_q(e_2)^N$. Since

$$UY^{-1} \cdot VY^{-1} = \omega^{-1} VY^{-1} \cdot UY^{-1}$$

we can compute $\pi_q(e_2)^N$ using the q-binomial formula. Indeed, we have

$$\begin{aligned}\pi_p(e_2)^N = \tfrac{1}{2}p\left[UY^{-1}+VY^{-1}\right]^N &= \tfrac{1}{2}p\left[\left(UY^{-1}\right)^N+\left(VY^{-1}\right)^N\right]\\ &= \tfrac{1}{2}p\left[\mathrm{Id}_{\mathcal{V}}+\mathrm{Id}_{\mathcal{V}}\right]=p\,\mathrm{Id}_{\mathcal{V}}\,.\end{aligned}\tag{1.11}$$

We conclude that the operator $\pi_p(e_2)$ is invertible since $p\in\mathbb{R}_{\neq 0}$. □

In order to study the reduced cyclic $\mathcal{A}_{\omega,t}$-modules, we need to introduce particular elements of $\mathcal{A}_{\omega,t}$. First, we define $a_1, a_2 \in \mathcal{A}_{\omega,t}$ by

$$\begin{aligned}\omega^t(1-\omega)a_1 &= e_1e_2-\omega^{t+1}e_2e_1,\\ (1-\omega)a_2 &= e_2e_1-\omega^{-t}e_1e_2.\end{aligned}\tag{1.12}$$

Using (1.1) and (1.12) one can check that

$$\begin{array}{ll} k_1a_1=\omega^ta_1k_1, & k_1a_2=\omega^ta_2k_1,\\ e_1a_1=\omega^ta_1e_1, & e_1a_2=\omega^{t+1}a_2e_1,\\ e_2a_1=\omega^{-t}a_1e_2, & e_2a_2=\omega^{-(t+1)}a_2e_2,\\ a_1a_2=a_2a_1, & e_2e_1=a_1+a_2.\end{array}\tag{1.13}$$

and with (1.2) we have

$$\begin{aligned}\Delta(a_1)&=a_1\otimes 1+k_1^{t+1}\otimes a_1+k_1^te_1\otimes e_2,\\ \Delta(a_2)&=a_2\otimes 1+k_1^{t+1}\otimes a_2+e_2k_1\otimes e_1.\end{aligned}\tag{1.14}$$

Next, we consider elements $c_1, c_2\in\mathcal{A}_{\omega,t}$ defined by

$$c_1=a_1k_1^{-t}e_1^t,\quad c_2=a_2k_1^{-(t+1)}e_1^t.\tag{1.15}$$

By using (1.1) and (1.13), we deduce the following relations

$$\begin{array}{lll} k_1c_1=c_1k_1, & k_1c_2=c_2k_1, & \\ e_1c_1=c_1e_1, & e_1c_2=c_2e_1, & c_1c_2=\omega^tc_2c_1\end{array}\tag{1.16}$$

Finally, for a reduced cyclic $\mathcal{A}_{\omega,t}$-module V_p, using (1.8), (1.12) and (1.15), we have

$$\begin{array}{ll}\pi_p(a_1)=\left(\tfrac{1}{2}p^2\right)^{\frac{1}{N}}U, & \pi_p(a_2)=\left(\tfrac{1}{2}p^2\right)^{\frac{1}{N}}V,\\[2ex] \pi_p(c_1)=\left(\tfrac{1}{2}p^{t+2}\right)^{\frac{1}{N}}\mathsf{X}_2^{-t}, & \pi_p(c_2)=\left(\tfrac{1}{2}p^{t+2}\right)^{\frac{1}{N}}\mathsf{Y}_2.\end{array}\tag{1.17}$$

We note that $\{v_\alpha\}_{\alpha\in\mathbb{Z}_N^2}$ is a basis of common eigenvectors of $\pi_p(k_1)$ and $\pi_p(c_1)$.

Lemma 1.4. *A reduced cyclic $\mathcal{A}_{\omega,t}$-module V_p is simple and V_q is equivalent to V_p only if $p=q$.*

Proof. The simplicity is clear. Indeed, since

$$\begin{array}{ll}\pi_p(k_1)=\mathsf{X}_1, & \pi_p(c_1^{t+1})=\left(\tfrac{1}{2}p^{t+2}\right)^{\frac{t+1}{N}}\mathsf{X}_2,\\[2ex] \pi_p(e_1)=p^{\frac{1}{N}}\mathsf{Y}_1, & \pi_p(c_2)=\left(\tfrac{1}{2}p^{t+2}\right)^{\frac{1}{N}}\mathsf{Y}_2,\end{array}$$

by Lemma 1.1, the only invariant subspaces of V_p are $\{0\}$ and V_p.

If $V_p \cong V_q$ then, by definition, there is an isomorphism $S : V_p \to V_q$ such that for all $a \in \mathcal{A}_{\omega,t}$, $S\pi_p(a) = \pi_q(a)S$. In particular, we have

$$S(p\,\mathrm{Id}_{\mathcal{V}}) = S\pi_p(e_1^N) = \pi_q(e_1^N)S = (q\,\mathrm{Id}_{\mathcal{V}})S \Rightarrow p = q.$$

□

Let $\mathrm{Hom}_{\mathcal{A}_{\omega,t}}(V_p, V_q)$ be the set of morphisms of $\mathcal{A}_{\omega,t}$-modules between V_p and V_q i.e., the set of linear maps $f : V_p \to V_q$ such that for all $a \in A_{\omega,t}$, we have

$$\pi_q(a)f = f\pi_p(a).$$

Schur's Lemma implies that

(1) if $p \neq q$ then $\mathrm{Hom}_{\mathcal{A}_{\omega,t}}(V_p, V_q) = 0$,
(2) if $p = q$ then $\mathrm{Hom}_{\mathcal{A}_{\omega,t}}(V_p, V_p) = \mathrm{End}_{\mathcal{A}_{\omega,t}}(V_p) = \mathbb{C}\,\mathrm{Id}_{\mathcal{V}}$.

Hence, for any $f \in \mathrm{End}_{\mathcal{A}_{\omega,t}}(V_p)$, there is a unique $c \in \mathbb{C}$ such that $f = c\,\mathrm{Id}_{\mathcal{V}}$.

NOTATION 1.5. This c is denoted $\langle f \rangle$.

1.3. Tensor product of reduced cyclic $\mathcal{A}_{\omega,t}$-modules. The tensor product $V_p \otimes V_q$ is provided with a $\mathcal{A}_{\omega,t}$-module structure through

$$(\pi_p \otimes \pi_q)\Delta : \mathcal{A}_{\omega,t} \to \mathrm{End}(V_p \otimes V_q).$$

DEFINITION 1.6. An *admissible pair* (V_p, V_q) is a pair of reduced cyclic $\mathcal{A}_{\omega,t}$-modules such that $p \neq -q$. In such case, we will also say that the pair (p, q) is admissible.

The reason we are interested in admissible pairs is that their tensor product decomposes as a direct sum of reduced cyclic $\mathcal{A}_{\omega,t}$-modules.

LEMMA 1.7. *We have*

$$(\pi_p \otimes \pi_q)\Delta(e_1)^N = (p+q)\,\mathrm{Id}_{\mathcal{V}} \otimes \mathrm{Id}_{\mathcal{V}},$$
$$(\pi_p \otimes \pi_q)\Delta(c_1)^N = (\pi_p \otimes \pi_q)\Delta(c_2)^N = \tfrac{1}{2}(p+q)^{t+2}\,\mathrm{Id}_{\mathcal{V}} \otimes \mathrm{Id}_{\mathcal{V}}\,.$$

PROOF. Let (V_p, V_q) be a pair of reduced cyclic $\mathcal{A}_{\omega,t}$-modules. For $(\pi_p \otimes \pi_q)\Delta(e_1)$ we compute

$$(\pi_p \otimes \pi_q)\Delta(e_1)^N = (\pi_p \otimes \pi_q)\left(\Delta(e_1)^N\right) = (\pi_p \otimes \pi_q)\left((e_1 \otimes 1 + k_1 \otimes e_1)^N\right)$$
$$= (\pi_p \otimes \pi_q)\left(e_1^N \otimes 1 + k_1^N \otimes e_1^N\right) = pY^N \otimes \mathrm{Id}_{\mathcal{V}} + qX^N \otimes Y^N = (p+q)\,\mathrm{Id}_{\mathcal{V}} \otimes \mathrm{Id}_{\mathcal{V}}$$

where we used the q-binomial formula for the third equality.

For $(\pi_p \otimes \pi_q)\Delta(c_1)$ we have

$$\begin{aligned}(\pi_p \otimes \pi_q)\Delta(c_1)^N &= (\pi_p \otimes \pi_q)\Delta\left((a_1 k_1^{-t} e_1^t)^N\right) = (\pi_p \otimes \pi_q)\Delta\left(a_1^N k_1^{-Nt} e_1^{Nt}\right)\\ &= (\pi_p \otimes \pi_q)\Delta(a_1^N)(\pi_p \otimes \pi_q)\Delta(k_1^N)^{-t}(\pi_p \otimes \pi_q)\Delta(e_1^N)^t\\ &= (\pi_p \otimes \pi_q)\Delta(a_1)^N(\mathrm{Id}_{\mathcal{V}} \otimes \mathrm{Id}_{\mathcal{V}})^{-t}(p+q)^t \,\mathrm{Id}_{\mathcal{V}} \otimes \mathrm{Id}_{\mathcal{V}}\\ &= (\pi_p \otimes \pi_q)\Delta(a_1)^N(p+q)^t \,\mathrm{Id}_{\mathcal{V}} \otimes \mathrm{Id}_{\mathcal{V}}\end{aligned}$$

where we used (1.1) and (1.13) for the second equality. Using (1.13) again, one can use the q-binomial formula to compute $(\pi_p \otimes \pi_q)\Delta(a_1)^N$ since t and $t+1$ are invertible in $\mathbb{Z}_N$. Using (1.14), we have

$$\begin{aligned}(\pi_p \otimes \pi_q)\Delta(a_1)^N &= (\pi_p \otimes \pi_q)\left(a_1 \otimes 1 + k_1^{t+1} \otimes a_1 + k_1^t e_1 \otimes e_2\right)^N\\ &= (\pi_p \otimes \pi_q)\left(a_1^N \otimes 1 + k_1^{N(t+1)} \otimes a_1^N + k_1^{Nt} e_1^N \otimes e_2^N\right)\\ &= \tfrac{1}{2}p^2 U^N \otimes \mathrm{Id}_{\mathcal{V}} + \tfrac{1}{2}q^2 \,\mathrm{Id}_{\mathcal{V}} \otimes U^N + pq \,\mathrm{Id}_{\mathcal{V}} \otimes \mathrm{Id}_{\mathcal{V}}\\ &= \tfrac{1}{2}(p+q)^2 \,\mathrm{Id}_{\mathcal{V}} \otimes \mathrm{Id}_{\mathcal{V}}\end{aligned}$$

where we used (1.1), (1.13) and the q-binomial formula for the second equality, (1.11) and (1.17) for the third one and (1.1) and (1.8) for the last one. Hence we have

$$(\pi_p \otimes \pi_q)\Delta(c_1)^N = \tfrac{1}{2}(p+q)^{t+2} \,\mathrm{Id}_{\mathcal{V}} \otimes \mathrm{Id}_{\mathcal{V}}\,.$$

Finally the computation of $(\pi_p \otimes \pi_q)\Delta(c_2)^N$ is similar to the previous one. We have

$$\begin{aligned}(\pi_p \otimes \pi_q)\Delta(c_2)^N &= (\pi_p \otimes \pi_q)\Delta\left((a_2 k_1^{-(t+1)} e_1^t)^N\right)\\ &= (\pi_p \otimes \pi_q)\Delta\left(a_2^N k_1^{-N(t+1)} e_1^{Nt}\right) = (\pi_p \otimes \pi_q)\Delta(a_2)^N(p+q)^t \,\mathrm{Id}_{\mathcal{V}} \otimes \mathrm{Id}_{\mathcal{V}}\\ &= (\pi_p \otimes \pi_q)(a_2 \otimes 1 + k_1^{t+1} \otimes a_2 + e_2 k_1 \otimes e_1)^N(p+q)^t \,\mathrm{Id}_{\mathcal{V}} \otimes \mathrm{Id}_{\mathcal{V}}\\ &= (\pi_p \otimes \pi_q)(a_2^N \otimes 1 + k_1^{N(t+1)} \otimes a_2^N + e_2^N k_1^N \otimes e_1^N)(p+q)^t \,\mathrm{Id}_{\mathcal{V}} \otimes \mathrm{Id}_{\mathcal{V}}\\ &= \tfrac{1}{2}(p+q)^{(t+2)} \,\mathrm{Id}_{\mathcal{V}} \otimes \mathrm{Id}_{\mathcal{V}}\,.\end{aligned}$$

□

PROPOSITION 1.8. *Let (V_p, V_q) be an admissible pair. Then the $\mathcal{A}_{\omega,t}$-module $V_p \otimes V_q$ is equivalent to the direct sum of N^2 copies of the reduced cyclic $\mathcal{A}_{\omega,t}$-module V_{p+q}.*

PROOF. In order to find the submodules of $V_p \otimes V_q$, we only consider the action of k_1, c_1, e_1 and c_2 on $V_p \otimes V_q$.

First, $(\pi_p \otimes \pi_q)\Delta(k_1) = X \otimes X$ is clearly diagonalizable and its spectrum is the set of all N-th roots of unity $\{\omega^\alpha | \alpha \in \mathbb{Z}_N\}$. We write

$$V_p \otimes V_q = \bigoplus_{\alpha \in \mathbb{Z}_N} W_\alpha$$

where $W_\alpha = \operatorname{Ker}((\pi_p \otimes \pi_q)\Delta(k_1) - \omega^\alpha \operatorname{Id}_\mathcal{V} \otimes \operatorname{Id}_\mathcal{V})$. We have $\dim W_\alpha = N^3$ for all $\alpha \in \mathbb{Z}_N$.

Now we show that for each $\alpha \in \mathbb{Z}_N$, we can decompose W_α in the following way

$$W_\alpha = \bigoplus_{\beta \in \mathbb{Z}_N} W_{(\alpha,\beta)}$$

where

$$W_{(\alpha,\beta)} = \operatorname{Ker}\left((\pi_p \otimes \pi_q)\Delta(c_1)|_{W_\alpha} - s\omega^{t\beta} \operatorname{Id}_{W_\alpha}\right) \text{ and } s = \left(\tfrac{1}{2}(p+q)^{t+2}\right)^{\frac{1}{N}} \in \mathbb{R}^*.$$

By Lemma 1.7, we have

$$((\pi_p \otimes \pi_q)\Delta(c_1))^N = (\pi_p \otimes \pi_q)\Delta(c_1)^N = s^N \operatorname{Id}_\mathcal{V} \otimes \operatorname{Id}_\mathcal{V}.$$

This means that the minimal polynomial of $(\pi_p \otimes \pi_q)\Delta(c_1)$ divides

$$x^N - s^N = \prod_{\beta \in \mathbb{Z}_N} (x - s\omega^\beta)$$

This implies that the minimal polynomial of $(\pi_p \otimes \pi_q)\Delta(c_1)$ has only simple zeros, which means that $(\pi_p \otimes \pi_q)\Delta(c_1)$ is diagonalizable. Moreover, it also implies that the spectrum of $(\pi_p \otimes \pi_q)\Delta(c_1)$ is contained in $\left\{s\omega^\beta | \beta \in \mathbb{Z}_N\right\}$. Actually, its spectrum is exactly $\left\{s\omega^\beta | \beta \in \mathbb{Z}_N\right\}$, since $(\pi_p \otimes \pi_q)\Delta(c_2)$ is invertible, $c_1c_2 = \omega^t c_2 c_1$ and t is invertible in $\mathbb{Z}_N$. Futhermore, since $k_1c_1 = c_1k_1$, $k_1c_2 = c_2k_1$ and that $(\pi_p \otimes \pi_q)\Delta(c_1)$ and $(\pi_p \otimes \pi_q)\Delta(c_2)$ are invertible, we have

$$(\pi_p \otimes \pi_q)\Delta(k_2)(W_\alpha) = (\pi_p \otimes \pi_q)\Delta(c_2)(W_\alpha) = W_\alpha, \quad \forall \alpha \in \mathbb{Z}_N,$$

which implies that the spectrum of $(\pi_p \otimes \pi_q)\Delta(c_1)|_{W_\alpha}$ is $\left\{s\omega^\beta | \beta \in \mathbb{Z}_N\right\}$. This allows us to define the eigenspaces $W_{(\alpha,\beta)}$ for all $\alpha, \beta \in \mathbb{Z}_N$ as announced.

Now, one can see that for all $\alpha, \beta \in \mathbb{Z}_N$, we have $\dim W_{(\alpha,\beta)} = N^2$. Indeed, this comes from the fact that $\dim W_\alpha = N^3$, $(\pi_p \otimes \pi_q)\Delta(c_2)|_{W_\alpha}$ is invertible, $c_1c_2 = \omega^t c_2 c_1$ and t is invertible in $\mathbb{Z}_N$.

Let $\{u_i\}_{i \in \mathbb{Z}_N^2}$ be a basis of $W_{0,0}$ and consider for all $\alpha, \beta \in \mathbb{Z}_N$ and all $i \in \mathbb{Z}_N^2$

$$\xi_{(\alpha,\beta),i} = \frac{1}{r^\alpha s^\beta}(\pi_p \otimes \pi_q)\Delta(e_1^\alpha c_2^\beta)u_i \in W_{(\alpha,\beta)},$$

where $r = (p+q)^{\frac{1}{N}} \in \mathbb{R}_{\neq 0}$. By construction, we have for all $\alpha, \beta \in \mathbb{Z}_N$ and all $i \in \mathbb{Z}_N^2$

$$(\pi_p \otimes \pi_q)\Delta(k_1)\xi_{(\alpha,\beta),i} = \omega^{-\alpha}\xi_{(\alpha,\beta),i}, \quad (\pi_p \otimes \pi_q)\Delta(c_1)\xi_{(\alpha,\beta),i} = s\omega^{t\beta}\xi_{(\alpha,\beta),i},$$

$$(\pi_p \otimes \pi_q)\Delta(e_1)\xi_{(\alpha,\beta),i} = r\xi_{(\alpha+1,\beta),i}, \quad (\pi_p \otimes \pi_q)\Delta(c_2)\xi_{(\alpha,\beta),i} = s\xi_{(\alpha,\beta+1),i}.$$

This clearly shows that for each $i \in \mathbb{Z}_N^2$, the subspace $\Xi_i \subset V_p \otimes V_q$ generated by $\left\{\xi_{(\alpha,\beta),i}\right\}_{\alpha,\beta\in\mathbb{Z}_N}$ is an irreducible submodule. Furthermore, Ξ_i is a reduced cyclic $\mathcal{A}_{\omega,t}$-module of parameter $p+q$. By Lemma 1.4, the reduced cyclic $\mathcal{A}_{\omega,t}$-modules Ξ_i are all equivalent.

□

REMARK 1.9. The dual space V_p^* of a reduced cyclic $\mathcal{A}_{\omega,t}$-module V_p can be provided with a $\mathcal{A}_{\omega,t}$-module structure $\pi_p^* : \mathcal{A}_{\omega,t} \to \mathrm{End}(V_p^*)$ defined, for all $a \in \mathcal{A}_{\omega,t}$ and all $\alpha, \beta \in \mathbb{Z}_N^2$, by

$$\left(\pi_p^*(a)v_\alpha^*\right) v_\beta = v_\alpha^* \left(\pi_p(\gamma(a))v_\beta\right).$$

This $\mathcal{A}_{\omega,t}$-module is actually equivalent to the reduced cyclic $\mathcal{A}_{\omega,t}$-module V_{-p}.

2. A Ψ-system in the category of $\mathcal{A}_{\omega,t}$-modules

Since $\mathcal{A}_{\omega,t}$ is a Hopf algebra over $\mathbb{C}$, the category of $\mathcal{A}_{\omega,t}$-modules, as a subcategory of the category of $\mathbb{C}$-vector spaces, is a monoidal abelian category. This is the framework required to construct a Ψ-system in a category.

In order to make this construction, we need to introduce certain sets of morphisms called the multiplicity spaces.

2.1. The multiplicity spaces. In the previous section we have shown that

(1) $V_p \cong V_q$ if and only if $p = q$,
(2) if $p = q$ then $\mathrm{End}_{\mathcal{A}_{\omega,t}}(V_p) = \mathbb{C}\,\mathrm{Id}_{\mathcal{V}}$,
(3) if $p \neq q$ then $\mathrm{Hom}_{\mathcal{A}_{\omega,t}}(V_p, V_q) = 0$,
(4) If (V_p, V_q) is an admissible pair then $V_p \otimes V_q \cong (V_{p+q})^{\oplus N^2}$.

DEFINITION 2.1. Let (V_p, V_q) be an admissible pair, the following vector spaces of linear maps are called *multiplicity spaces*

$$\mathcal{H}_{p,q} = \{f : V_{p+q} \to V_p \otimes V_q \mid f\pi_{p+q}(a) = (\pi_p \otimes \pi_q)\Delta(a)f, \;\; \forall a \in A_{\omega,t}\}$$
$$\bar{\mathcal{H}}_{p,q} = \{f : V_p \otimes V_q \to V_{p+q} \mid \pi_{p+q}(a)f = f(\pi_p \otimes \pi_q)\Delta(a), \;\; \forall a \in A_{\omega,t}\}.$$

In other words, the elements of the multiplicity spaces are morphisms in the category of $A_{\omega,t}$-modules,

$$\mathcal{H}_{p,q} = \mathrm{Hom}_{A_{\omega,t}}(V_{p+q}, V_p \otimes V_q) \;\; \text{and} \;\; \bar{\mathcal{H}}_{p,q} = \mathrm{Hom}_{A_{\omega,t}}(V_p \otimes V_q, V_{p+q}).$$

By Proposition 1.8 and Schur's lemma, we clearly have $\dim \mathcal{H}_{p,q} = \dim \bar{\mathcal{H}}_{p,q} = N^2$.

2.2. Two operator valued functions. In order to construct bases for the multiplicity spaces, we start by introducing two operator valued functions.

Let $\mathsf{U} \in \mathcal{A}^{\otimes 2}$ be an operator such that $\mathsf{U}^N = -\mathrm{Id}_{\mathcal{V}^{\otimes 2}}$ and $x \in \mathbb{R}_{\neq 0,1}$. The first function is the *quantum dilogarithm* $\Psi_x(\mathsf{U})$ introduced by Faddeev and Kashaev

[**4**]. It is defined as a solution of the functional equation

$$\frac{\Psi_x(\omega^{-1}\mathsf{U})}{\Psi_x(\mathsf{U})} = (1-x)^{\frac{1}{N}} - x^{\frac{1}{N}}\mathsf{U}. \tag{2.1}$$

Following [**5**], we can write

$$\Psi_x(\mathsf{U}) = \sum_{\alpha\in\mathbb{Z}_N} \psi_{x,\alpha}(-\mathsf{U})^\alpha$$

where $\psi_{x,\alpha} \in \mathbb{C}$. For all $\alpha \in \mathbb{Z}_N$, (2.1) implies that

$$\psi_{x,\alpha} = \frac{x^{\frac{1}{N}}}{\omega^{-\alpha} - (1-x)^{\frac{1}{N}}}\psi_{x,\alpha-1}.$$

Then, for all $\alpha \in \mathbb{Z}_N$, we write

$$\psi_{x,\alpha} = \psi_{x,0}\prod_{j=1}^{\alpha} \frac{x^{\frac{1}{N}}}{\omega^{-j} - (1-x)^{\frac{1}{N}}}, \tag{2.2}$$

where $\psi_{x,0} \in \mathbb{C}^*$ is chosen so that $\det(\Psi_x(\mathsf{U})) = 1$. Using the notation of [**9**], we have

$$\psi_{x,\alpha} = \psi_{x,0}\omega^{\frac{1}{2}\alpha(\alpha+1)}x^{\frac{\alpha}{N}} w\left((1-x)^{\frac{1}{N}}\middle|\alpha\right) \tag{2.3}$$

where

$$w(x|\alpha) = \prod_{j=1}^{\alpha}\frac{1}{1-x\omega^j},$$

is defined for all $x \in \mathbb{C}$ such that $x^N \neq 1$ and all $\alpha \in \{0, \cdots, N-1\} \subset \mathbb{Z}$.

The computation of the determinant of $\Psi_x(\mathsf{U})^N$ is made possible due to the following formula that will be shown in Chapter 2.

$$\Psi_x(\mathsf{U})^N = \psi_{x,0}^N(1-x)^{1-N}\frac{D(1)}{D\left((1-x)^{-\frac{1}{N}}\right)D\left(\mathsf{U}\omega\left(\frac{x}{1-x}\right)^{\frac{1}{N}}\right)} \tag{2.4}$$

where

$$D(x) = \prod_{j=1}^{N-1}(1-x\omega^j)^j. \tag{2.5}$$

LEMMA 2.2. *For any $x \in \mathbb{R}_{\neq 0,1}$ and any operator $\mathsf{U} \in \mathcal{A}^{\otimes 2}$ such that there exists an invertible opertator $\mathsf{M} \in \mathcal{A}^{\otimes 2}$ such that $\mathsf{U} = -\mathsf{M}(X \otimes \mathrm{Id}_{\mathcal{V}})\mathsf{M}^{-1}$, we have*

$$\psi_{x,0}^N = (1-x)^{\frac{N-1}{2}}D\left((1-x)^{-\frac{1}{N}}\right)D(1)^{-1} \quad\Rightarrow\quad \det\left(\Psi_x(\mathsf{U})^N\right) = 1. \tag{2.6}$$

PROOF. First we compute $\det\left(D\left(\mathsf{X}\omega\left(\frac{x}{x-1}\right)^{\frac{1}{N}}\right)\right)$ using the matrix form of X in the canonical basis $\{v_\alpha\}_{\alpha\in\mathbb{Z}_N^2}$ of $\mathbb{C}^N$.

$$D\left(\mathsf{X}\omega\left(\tfrac{x}{x-1}\right)^{\frac{1}{N}}\right) = \prod_{j=1}^{N-1}\left(\mathrm{Id}_{\mathbb{C}^N} - \left(\tfrac{x}{x-1}\right)^{\frac{1}{N}}\omega^{j+1}\mathsf{X}\right)^j$$

$$= \prod_{j=1}^{N-1}\begin{pmatrix} 1-\left(\tfrac{x}{x-1}\right)^{\frac{1}{N}}\omega^{j+1} & & & \\ & 1-\left(\tfrac{x}{x-1}\right)^{\frac{1}{N}}\omega^{j+2} & & \\ & & \ddots & \\ & & & 1-\left(\tfrac{x}{x-1}\right)^{\frac{1}{N}}\omega^{j+N}\end{pmatrix}^j$$

Hence we have

$$\det\left(D\left(\mathsf{X}\omega\left(\tfrac{x}{x-1}\right)^{\frac{1}{N}}\right)\right) = \prod_{j=1}^{N-1}\prod_{i=1}^{N}\left(1-\left(\tfrac{x}{x-1}\right)^{\frac{1}{N}}\omega^{j+i}\right)^j = \prod_{j=1}^{N-1}\left(1-\left(\left(\tfrac{x}{x-1}\right)^{\frac{1}{N}}\omega^{j}\right)^N\right)^j$$
$$= \prod_{j=1}^{N-1}\left(1-\left(\tfrac{x}{x-1}\right)\right)^j = \prod_{j=1}^{N-1}\left(\tfrac{1}{1-x}\right)^j = \left(\tfrac{1}{1-x}\right)^{\frac{N(N-1)}{2}}$$

Since $\mathsf{U} = -\mathsf{M}(X\otimes \mathrm{Id}_{\mathcal{V}})\mathsf{M}^{-1} = -\mathsf{M}(\mathsf{X}\otimes\mathrm{Id}_{\mathbb{C}^N}\otimes\mathrm{Id}_{\mathcal{V}})\mathsf{M}^{-1}$, we have

$$\det\left(D\left(\mathsf{U}\omega\left(\tfrac{x}{1-x}\right)^{\frac{1}{N}}\right)\right) = \det\left(D\left(\mathsf{X}\omega\left(\tfrac{x}{x-1}\right)^{\frac{1}{N}}\right)\right)^{N^3} = (1-x)^{-\frac{N^4(N-1)}{2}}.$$

Therefore, we have

$$\det\left(\Psi_x(\mathsf{U})^N\right) = \det\left(\psi_{x,0}^N(1-x)^{1-N}D(1)D\left((1-x)^{-\frac{1}{N}}\right)^{-1}D\left(\mathsf{U}\omega\left(\tfrac{x}{1-x}\right)^{\frac{1}{N}}\right)^{-1}\right)$$
$$= \left(\psi_{x,0}^N(1-x)^{1-N}D(1)D\left((1-x)^{-\frac{1}{N}}\right)^{-1}\right)^{N^4}\det\left(D\left(\mathsf{U}\omega\left(\tfrac{x}{1-x}\right)^{\frac{1}{N}}\right)\right)^{-1}$$
$$= \left(\psi_{x,0}^N(1-x)^{1-N}D(1)D\left((1-x)^{-\frac{1}{N}}\right)^{-1}(1-x)^{\frac{N-1}{2}}\right)^{N^4}$$
$$= \left(\psi_{x,0}^N(1-x)^{\frac{1-N}{2}}D(1)D\left((1-x)^{-\frac{1}{N}}\right)^{-1}\right)^{N^4}.$$

So if $\psi_{x,0}^N = (1-x)^{\frac{N-1}{2}}D\left((1-x)^{-\frac{1}{N}}\right)D(1)^{-1}$ then $\det\left(\Psi_x(\mathsf{U})^N\right)=1$. □

From now on, for all $x\in\mathbb{R}_{\neq 0,1}$, we assume that $\psi_{x,0}$ satisfies (2.6) and is such that $\det\left(\Psi_x(\mathsf{U})\right)=1$.

The inverse of $\Psi_x(\mathsf{U})$ that we will denote $\bar{\Psi}_x(\mathsf{U})$ satisfies

$$\frac{\bar{\Psi}_x(\mathsf{U})}{\bar{\Psi}_x(\omega^{-1}\mathsf{U})} = (1-x)^{\frac{1}{N}} - x^{\frac{1}{N}}\mathsf{U} \qquad \text{and} \qquad \Psi_x(\mathsf{U})\bar{\Psi}_x(\mathsf{U}) = 1.$$

Setting

$$\bar{\Psi}_x(\mathsf{U}) = \sum_{\alpha\in\mathbb{Z}_N} \bar{\psi}_{x,\alpha}(-\mathsf{U})^\alpha$$

we find, in a similar way, that for all $\alpha \in \mathbb{Z}_N$

$$\bar{\psi}_{x,\alpha} = \bar{\psi}_{x,0} \left(\tfrac{x}{x-1}\right)^{\frac{\alpha}{N}} \omega^\alpha w\left((1-x)^{-\frac{1}{N}}\Big|\alpha\right) \tag{2.7}$$

where $\bar{\psi}_{x,0} \in \mathbb{C}^*$ is chosen according to $\psi_{x,0} \in \mathbb{C}^*$. In order to make this choice, we use a formula similar to the formula (2.4). We will show this formula and the following Lemma in Chapter 2.

$$\bar{\Psi}_x(\mathsf{U})^N = \bar{\psi}_{x,0}^N N^{-N}(-x)^{-\frac{N-1}{2}} \frac{D(1)^3}{D\left((1-x)^{\frac{1}{N}}\right) D\left(\mathsf{U}^{-1}\left(\frac{1-x}{x}\right)^{\frac{1}{N}}\right)} \tag{2.8}$$

LEMMA 2.3. *There exists $\alpha \in \mathbb{Z}_N$ such that for any $x \in \mathbb{R}_{\neq 0,1}$ and any operator $\mathsf{U} \in \mathcal{A}^{\otimes 2}$ such that $\mathsf{U}^N = -\operatorname{Id}_{\mathcal{V}^{\otimes 2}}$ we have*

$$\bar{\psi}_{x,0} = \omega^\alpha \varrho\left((1-x)^{\frac{1}{N}}\right)\psi_{x,0}^{-1} \quad \Rightarrow \quad \Psi_x(\mathsf{U})\bar{\Psi}_x(\mathsf{U}) = 1 \tag{2.9}$$

where

$$\varrho(x) = N^{-1}\frac{1-x^N}{1-x}. \tag{2.10}$$

From now on, for all $x \in \mathbb{R}_{\neq 0,1}$, we assume that $\bar{\psi}_{x,0}$ satisfies (2.9).

REMARK 2.4. The fact that our invariant is defined up to multiplication by an integer power of ω is independent from the fact that we have determined $\bar{\psi}_{x,0}$ up to multiplication by an integer power of ω. In other words, even if we had exactly determined $\bar{\psi}_{x,0}$, our invariant would still be defined up to multiplication by an integer power of ω. This will be explained in Section 5.

The second operator valued function, is defined by

$$L(\mathsf{U},\mathsf{V}) = \frac{1}{N}\sum_{\alpha,\beta\in\mathbb{Z}_N} \omega^{-\alpha\beta}\mathsf{U}^\alpha \otimes \mathsf{V}^\beta.$$

where $\mathsf{U},\mathsf{V} \in \mathcal{A}$ satisfy $\mathsf{U}^N = \mathsf{V}^N = \operatorname{Id}_{\mathcal{V}}$.

LEMMA 2.5. *Let $\alpha, \beta \in \mathbb{Z}_N$, we have*

$$L(\omega^\alpha \mathsf{U}, \omega^\beta \mathsf{V}) = L(\mathsf{U},\mathsf{V})\omega^{\alpha\beta}\mathsf{U}^\beta \otimes \mathsf{V}^\alpha.$$

PROOF.

$$\begin{aligned}
L(\omega^{\alpha}\mathsf{U},\omega^{\beta}\mathsf{V}) &= \frac{1}{N}\sum_{\gamma,\delta\in\mathbb{Z}_N}\omega^{-\gamma\delta+\alpha\gamma+\beta\delta}\mathsf{U}^{\gamma}\otimes\mathsf{V}^{\delta}\\
&= \frac{1}{N}\sum_{\gamma,\delta\in\mathbb{Z}_N}\omega^{-\gamma(\delta-\alpha)+\beta\delta}\mathsf{U}^{\gamma}\otimes\mathsf{V}^{\delta-\alpha+\alpha} = \frac{1}{N}\sum_{\gamma,\delta\in\mathbb{Z}_N}\omega^{-\gamma\delta+\beta(\delta+\alpha)}\mathsf{U}^{\gamma}\otimes\mathsf{V}^{\delta}\mathsf{V}^{\alpha}\\
&= \frac{1}{N}\sum_{\gamma,\delta\in\mathbb{Z}_N}\omega^{-(\gamma-\beta)\delta+\alpha\beta}\mathsf{U}^{\gamma-\beta+\beta}\otimes\mathsf{V}^{\delta}\mathsf{V}^{\alpha} = \frac{1}{N}\sum_{\gamma,\delta\in\mathbb{Z}_N}\omega^{-\gamma\delta}\mathsf{U}^{\gamma}\otimes\mathsf{V}^{\delta}\omega^{\alpha\beta}\mathsf{U}^{\beta}\otimes\mathsf{V}^{\alpha}\\
&= L(\mathsf{U},\mathsf{V})\omega^{\alpha\beta}\mathsf{U}^{\beta}\otimes\mathsf{V}^{\alpha}.
\end{aligned}$$

□

LEMMA 2.6. *If* $\mathsf{U}\in\mathcal{A}$ *is an operator such that* $\mathsf{U}^N=\mathrm{Id}_{\mathcal{V}}$, *then*

$$\det\left(L(\mathsf{U},X)\right)=1. \tag{2.11}$$

PROOF. We consider the operators of $\mathcal{A}$ in their matrix form in the basis $\{v_\alpha\}_{\alpha\in\mathbb{Z}_N^2}$ of $\mathcal{V}$. For any $\beta\in\mathbb{Z}_N$, we compute

$$X^{\beta}=\mathsf{X}^{\beta}\otimes\mathrm{Id}_{\mathbb{C}^N}=\begin{pmatrix}\boxed{\mathrm{Id}_{\mathbb{C}^N}} & & & \\ & \boxed{\omega^{\beta}\,\mathrm{Id}_{\mathbb{C}^N}} & & \\ & & \ddots & \\ & & & \boxed{\omega^{\beta(N-1)}\,\mathrm{Id}_{\mathbb{C}^N}}\end{pmatrix}.$$

Thereby, for any $\alpha\in\mathbb{Z}_N$, we have

$$\begin{aligned}
\sum_{\beta\in\mathbb{Z}_N}\omega^{-\alpha\beta}X^{\beta} &= \sum_{\beta\in\mathbb{Z}_N}\begin{pmatrix}\boxed{\omega^{-\alpha\beta}\,\mathrm{Id}_{\mathbb{C}^N}} & & & \\ & \boxed{\omega^{\beta(1-\alpha)}\,\mathrm{Id}_{\mathbb{C}^N}} & & \\ & & \ddots & \\ & & & \boxed{\omega^{\beta(N-1-\alpha)}\,\mathrm{Id}_{\mathbb{C}^N}}\end{pmatrix}\\
&= N\begin{pmatrix}\boxed{\delta_{\alpha,0}\,\mathrm{Id}_{\mathbb{C}^N}} & & & \\ & \boxed{\delta_{\alpha,1}\,\mathrm{Id}_{\mathbb{C}^N}} & & \\ & & \ddots & \\ & & & \boxed{\delta_{\alpha,N-1}\,\mathrm{Id}_{\mathbb{C}^N}}\end{pmatrix}.
\end{aligned}$$

Let $\mathsf{M}\in\mathcal{A}^{\otimes 2}$ be the permutation matrix such that for all $\mathsf{V},\mathsf{W}\in\mathcal{A}$ we have

$$\mathsf{W}\otimes\mathsf{V}=\mathsf{M}\mathsf{V}\otimes\mathsf{W}\mathsf{M}^{-1}.$$

Then we compute

$$\mathsf{M}L(\mathsf{U},X)\mathsf{M}^{-1} = \frac{1}{N}\sum_{\alpha,\beta\in\mathbb{Z}_N}\omega^{-\alpha\beta}X^{\beta}\otimes\mathsf{U}^{\alpha} = \sum_{\alpha\in\mathbb{Z}_N}\left(\frac{1}{N}\sum_{\beta\in\mathbb{Z}_N}\omega^{-\alpha\beta}X^{\beta}\otimes\mathsf{U}^{\alpha}\right)$$

$$= \sum_{\alpha\in\mathbb{Z}_N}\begin{pmatrix} \boxed{\delta_{\alpha,0}\operatorname{Id}_{\mathbb{C}^N}\otimes\mathsf{U}^{\alpha}} & & & \\ & \boxed{\delta_{\alpha,1}\operatorname{Id}_{\mathbb{C}^N}\otimes\mathsf{U}^{\alpha}} & & \\ & & \ddots & \\ & & & \boxed{\delta_{\alpha,N-1}\operatorname{Id}_{\mathbb{C}^N}\otimes\mathsf{U}^{\alpha}} \end{pmatrix}$$

$$= \begin{pmatrix} \boxed{\operatorname{Id}_{\mathbb{C}^N}\otimes\operatorname{Id}_{\mathbb{C}^N}} & & & \\ & \boxed{\operatorname{Id}_{\mathbb{C}^N}\otimes\mathsf{U}} & & \\ & & \ddots & \\ & & & \boxed{\operatorname{Id}_{\mathbb{C}^N}\otimes\mathsf{U}^{N-1}} \end{pmatrix}.$$

Finally we have

$$\det\left(L(\mathsf{U},X)\right) = \det\left(\mathsf{M}L(\mathsf{U},X)\mathsf{M}^{-1}\right) = \det\left(\mathsf{U}^{\frac{N(N-1)}{2}}\right) = 1.$$

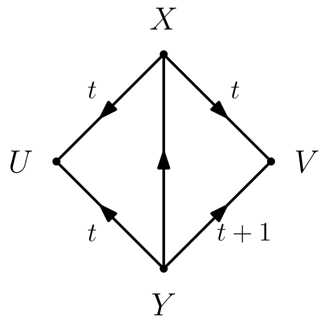
□

2.3. The operator $S(x)$**.** The operator $S(x)$, where $x\in\mathbb{R}_{\neq 0,1}$, that we are now going to define is central for our theory. Indeed, the bases for the multiplicity spaces as well as the $6j$-symbols, which are key ingredients to construct our invariant, will be expressed through this operator.

NOTATION 2.7. Let U and V be two operators such that $\mathsf{U}\mathsf{V}=\omega^{\alpha}\mathsf{V}\mathsf{U}$, where $\alpha\in\mathbb{Z}_N$. We will write this relation in the following way

if $\alpha=0$ if $\alpha=1$ if $\alpha=-1$ if $\alpha\in\mathbb{Z}_N\setminus\{0,\pm1\}$

Note that in the case of operators X,Y,U and V, we have the following relations



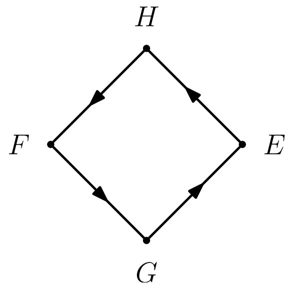
We define, for any $x \in \mathbb{R}_{\neq 0,1}$, the following invertible operator valued function

$$S(x) = \Psi_x(E)\Psi_x(F)\Psi_x(G)\Psi_x(H)L(U^tV, X)$$

where

$$E = -Y_1^{-1}X_1Y_2, \quad F = U_1^{-1}X_1^{t+1}V_2E^{-1}, \quad G = U_2V_2^{-1}F, \quad H = U_1V_1^{-1}E. \tag{2.12}$$

and the subscripts show how the operators are embedded in $\mathcal{A}^{\otimes 2}$. These operators have the following commutation relations



Note that E, F, G and H satisfy the condition of Lemma 2.3. Thereby, using Lemma 2.6, we see that the operator $S(x)$ is invertible since $\det\left(S(x)^N\right) = 1$. Moreover, since

$$L(U^tV, X)^{-1} = L(U^{-t}V^{-1}, X),$$

we have

$$S(x)^{-1} = L(U^{-t}V^{-1}, X)\bar{\Psi}_x(H)\bar{\Psi}_x(G)\bar{\Psi}_x(F)\bar{\Psi}_x(E).$$

PROPOSITION 2.8. *For any admissible pair $(p, q) \in (\mathbb{R}_{\neq 0})^2$, the following equation is satisfied*

$$(\pi_p \otimes \pi_q)\Delta(a) = S\left(\tfrac{q}{p+q}\right)(\pi_{p+q}(a) \otimes \mathrm{Id}_{\mathcal{A}})S\left(\tfrac{q}{p+q}\right)^{-1}, \quad \forall a \in A_{\omega,t} \tag{2.13}$$

PROOF. Since $e_2e_1 = a_1 + a_2$, it is enough to check equation (2.13) for $a \in \{k_1, e_1, a_1, a_2\}$. In the following computaions, we set $x = \frac{q}{p+q} \in \mathbb{R}_{\neq 0,1}$ and we will thus write $\frac{q-xq}{x}$ instead of p and $\frac{q}{x}$ instead of $p+q$.

<u>For $a = k_1$:</u> Since $XU^tV = \omega^{-1}U^tVX$, we have, by Lemma 2.5,

$$L(U^tV, X)X_1 = \frac{1}{N}\sum_{\alpha,\beta\mathbb{Z}_N} \omega^{-\alpha\beta}X_1(\omega U_1^tV_1)^\alpha X_2^\beta = X_1L(\omega U^tV, X)$$
$$= X_1X_2L(U^tV, X).$$

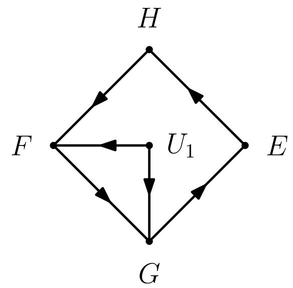
Hence, since X_1X_2 commutes with E, F, G and H, we have, using (1.8)

$$\begin{aligned} S(x)(\pi_{\frac{q}{x}}(k_1)\otimes \mathrm{Id}_{\mathcal{A}}) &= \Psi_x(E)\Psi_x(F)\Psi_x(G)\Psi_x(H)L(U^tV,X)X_1 \\ &= \Psi_x(E)\Psi_x(F)\Psi_x(G)\Psi_x(H)X_1X_2L(U^tV,X) \\ &= X_1X_2\Psi_x(E)\Psi_x(F)\Psi_x(G)\Psi_x(H)L(U^tV,X) \\ &= (\pi_{\frac{q-xq}{x}}\otimes \pi_q)\Delta(k_1)S(x). \end{aligned}$$

<u>For $a = e_1$:</u> Since Y_1 commutes with F, G, H and $U_1^tV_1$ and that $Y_1E = \omega EY_1$ we have, using (1.8) and equation (2.1)

$$\begin{aligned} S(x)(\pi_{\frac{q}{x}}(e_1)\otimes \mathrm{Id}_{\mathcal{A}}) &= \Psi_x(E)\Psi_x(F)\Psi_x(G)\Psi_x(H)L(U^tV,X)\left(\tfrac{q}{x}\right)^{\frac{1}{N}}Y_1 \\ &= \left(\tfrac{q}{x}\right)^{\frac{1}{N}}Y_1\Psi_x(\omega^{-1}E)\Psi_x(F)\Psi_x(G)\Psi_x(H)L(U^tV,X) \\ &= \left(\tfrac{q}{x}\right)^{\frac{1}{N}}Y_1\left((1-x)^{\frac{1}{N}}-x^{\frac{1}{N}}E\right)S(x) = \left(\left(\tfrac{q-xq}{x}\right)^{\frac{1}{N}}Y_1+q^{\frac{1}{N}}X_1Y_2\right)S(x) \\ &= (\pi_{\frac{q-xq}{x}}\otimes \pi_q)\Delta(e_1)S(x). \end{aligned}$$

<u>For $a = a_1$:</u> Since U_1 commutes with $U_1^tV_1$ and has the following commutation relations



we have,

$$\begin{aligned} S(x)(\pi_{\frac{q}{x}}(a_1)\otimes \mathrm{Id}_{\mathcal{A}}) &= \Psi_x(E)\Psi_x(F)\Psi_x(G)\Psi_x(H)L(U^tV,X)\left(\tfrac{q^2}{2x^2}\right)^{\frac{1}{N}}U_1 \\ &= \left(\tfrac{q^2}{2x^2}\right)^{\frac{1}{N}}U_1\Psi_x(E)\Psi_x(\omega^{-1}F)\Psi_x(\omega^{-1}G)\Psi_x(H)L(U^tV,X). \end{aligned}$$

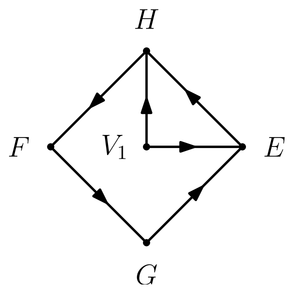
Now, using equality (2.1), we compute

$$\begin{aligned} \Psi_x(E)\Psi_x(\omega^{-1}F)\Psi_x(\omega^{-1}G) &= \Psi_x(E)\Psi_x(\omega^{-1}F)\left((1-x)^{\frac{1}{N}}-x^{\frac{1}{N}}G\right)\Psi_x(G) \\ &= \Psi_x(E)\left((1-x)^{\frac{1}{N}}\Psi_x(\omega^{-1}F)-x^{\frac{1}{N}}G\Psi_x(F)\right)\Psi_x(G) \\ &= \left((1-x)^{\frac{1}{N}}\left((1-x)^{\frac{1}{N}}-x^{\frac{1}{N}}F\right)\Psi_x(E)-x^{\frac{1}{N}}G\Psi_x(\omega^{-1}E)\right)\Psi_x(F)\Psi_x(G) \\ &= \left((1-x)^{\frac{2}{N}}-(x-x^2)^{\frac{1}{N}}F-x^{\frac{1}{N}}G\left((1+x)^{\frac{1}{N}}-x^{\frac{1}{N}}E\right)\right)\Psi_x(E)\Psi_x(F)\Psi_x(G) \\ &= \left((1-x)^{\frac{2}{N}}+x^{\frac{2}{N}}GE-(x-x^2)^{\frac{1}{N}}(F+G)\right)\Psi_x(E)\Psi_x(F)\Psi_x(G). \end{aligned}$$

Therefore, we finally have, using (2.12) and (1.17)

$$\begin{aligned}
&S(x)(\pi_{\frac{q}{x}}(a_1)\otimes \mathrm{Id}_{\mathcal{V}})\\
&= \left(\tfrac{q^2}{2x^2}\right)^{\frac{1}{N}} U_1\Psi_x(E)\Psi_x(\omega^{-1}F)\Psi_x(\omega^{-1}G)\Psi_x(H)L(U^tV,X)\\
&= \left(\tfrac{q^2}{2x^2}\right)^{\frac{1}{N}} U_1\left((1-x)^{\frac{2}{N}}+x^{\frac{2}{N}}GE-(x-x^2)^{\frac{1}{N}}(F+G)\right)S(x)\\
&= \left(\left(\tfrac{1}{2}(\tfrac{q-xq}{x})^2\right)^{\frac{1}{N}}U_1+\left(\tfrac{1}{2}q^2\right)^{\frac{1}{N}}U_1GE-\left(\tfrac{q^2-xq^2}{2x}\right)^{\frac{1}{N}}U_1(F+G)\right)S(x)\\
&= \left(\left(\tfrac{1}{2}(\tfrac{q-xq}{x})^2\right)^{\frac{1}{N}}U_1+\left(\tfrac{1}{2}q^2\right)^{\frac{1}{N}}X_1^{t+1}U_2-\left(\tfrac{q^2-xq^2}{2x}\right)^{\frac{1}{N}}X_1^tY_1(U_2+V_2)Y_2^{-1}\right)S(x)\\
&= (\pi_{\frac{q-xq}{x}}\otimes\pi_q)\left(a_1\otimes 1+k_1^{t+1}\otimes a_1+k_1^te_1\otimes e_2\right)S(x)=(\pi_{\frac{q-xq}{x}}\otimes\pi_q)\Delta(a_1)S(x).
\end{aligned}$$

<u>For $a=a_2$:</u> Since V_1 commutes with $U_1^tV_1$ and has the following commutation relations



we have,

$$\begin{aligned}
S(x)(\pi_{\frac{q}{x}}(a_2)\otimes \mathrm{Id}_{\mathcal{V}}) &= \Psi_x(E)\Psi_x(F)\Psi_x(G)\Psi_x(H)L(U^tV,X)\left(\tfrac{q^2}{2x^2}\right)^{\frac{1}{N}}V_1\\
&= \left(\tfrac{q^2}{2x^2}\right)^{\frac{1}{N}}V_1\Psi_x(\omega^{-1}E)\Psi_x(F)\Psi_x(G)\Psi_x(\omega^{-1}H)L(U^tV,X)\\
&= \left(\tfrac{q^2}{2x^2}\right)^{\frac{1}{N}}V_1\Psi_x(\omega^{-1}E)\Psi_x(F)\Psi_x(\omega^{-1}H)\Psi_x(G)L(U^tV,X).
\end{aligned}$$

Now, using equality (2.1), we compute

$$\begin{aligned}
\Psi_x(F)\Psi_x(\omega^{-1}H) &= \Psi_x(F)\left((1-x)^{\frac{1}{N}}-x^{\frac{1}{N}}H\right)\Psi_x(H)\\
&= \left((1-x)^{\frac{1}{N}}\Psi_x(F)-x^{\frac{1}{N}}H\Psi_x(\omega^{-1}F)\right)\Psi_x(H)\\
&= \left((1-x)^{\frac{1}{N}}\Psi_x(F)-x^{\frac{1}{N}}H\left((1-x)^{\frac{1}{N}}-x^{\frac{1}{N}}F\right)\Psi_x(F)\right)\Psi_x(H)\\
&= \left((1-x)^{\frac{1}{N}}+x^{\frac{2}{N}}HF-(x-x^2)^{\frac{1}{N}}H\right)\Psi_x(F)\Psi_x(H).
\end{aligned}$$

Hence, using equality (2.1) again, we have

$$\begin{aligned}
&\Psi_x(\omega^{-1}E)\Psi_x(F)\Psi_x(\omega^{-1}H)\\
&= \Big((1-x)^{\frac{1}{N}}\Big((1-x)^{\frac{1}{N}} - x^Y 1NE\Big) + x^{\frac{2}{N}}HF - (x-x^2)^{\frac{1}{N}}H\Big)\,\Psi_x(E)\Psi_x(F)\Psi_x(H)\\
&\qquad\qquad = \Big((1-x)^{\frac{2}{N}} + x^{\frac{2}{N}}HF - (x-x^2)^{\frac{1}{N}}(E+H)\Big)\,\Psi_x(E)\Psi_x(F)\Psi_x(H).
\end{aligned}$$

Therefore, using (2.12), (1.17) and the fact that $\Psi_x(H)\Psi_x(G) = \Psi_x(G)\Psi_x(H)$, we finally have

$$\begin{aligned}
&S(x)(\pi_{\frac{q}{x}}(a_2)\otimes \mathrm{Id}_{\mathcal{V}})\\
&= \left(\tfrac{q^2}{2x^2}\right)^{\frac{1}{N}} V_1\Psi_x(\omega^{-1}E)\Psi_x(F)\Psi_x(\omega^{-1}H)\Psi_x(G)L(U^tV,X)\\
&= \left(\tfrac{q^2}{2x^2}\right)^{\frac{1}{N}} V_1\Big((1-x)^{\frac{2}{N}} + x^{\frac{2}{N}}HF - (x-x^2)^{\frac{1}{N}}(E+H)\Big)\,S(x)\\
&= \bigg(\left(\tfrac{1}{2}(\tfrac{q-xq}{x})^2\right)^{\frac{1}{N}} V_1 + \left(\tfrac{1}{2}q^2\right)^{\frac{1}{N}} V_1HF - \left(\tfrac{q^2-xq^2}{2x}\right)^{\frac{1}{N}} V_1(E+H)\bigg)\,S(x)\\
&= \bigg(\left(\tfrac{1}{2}(\tfrac{q-xq}{x})^2\right)^{\frac{1}{N}} V_1 + \left(\tfrac{1}{2}q^2\right)^{\frac{1}{N}} X_1^{t+1}V_2 - \left(\tfrac{q^2-xq^2}{2x}\right)^{\frac{1}{N}} (U_1+V_1)Y_1^{-1}X_1Y_2\bigg)\,S(x)\\
&= (\pi_{\frac{q-xq}{x}}\otimes\pi_q)\left(a_2\otimes 1 + k_1^{t+1}\otimes a_2 + e_2k_1\otimes e_1\right)S(x) = (\pi_{\frac{q-xq}{x}}\otimes\pi_q)\Delta(a_2)S(x).
\end{aligned}$$

□

From now on, for technical reasons, we will use the following particular basis of $\mathcal{V}$ instead of the canonical basis $\{v_\alpha\}_{\alpha\in\mathbb{Z}_N^2}$.

LEMMA 2.9. *The set* $\{u_\alpha\}_{\alpha\in\mathbb{Z}_N^2}\subset\mathcal{V}$ *where*

$$u_\alpha = \sum_{\beta\in\mathbb{Z}_N} \omega^{-\frac{1}{2}t\beta(\beta+1)+\beta\left(\alpha_2-\alpha_1+\frac{1}{2}\right)} v_{(\alpha_1,\beta)},$$

for all $\alpha\in\mathbb{Z}_N^2$, *is a basis of* $\mathcal{V}$. *Its dual* $\{\bar{u}_\alpha\}_{\alpha\in\mathbb{Z}_N^2}\subset\mathcal{V}^*$ *is given, for all* $\alpha\in\mathbb{Z}_N^2$, *by*

$$\bar{u}_\alpha = \frac{1}{N}\sum_{\beta\in\mathbb{Z}_N} \omega^{\frac{1}{2}t\beta(\beta+1)-\beta(\alpha_2-\alpha_1+\frac{1}{2})}\bar{v}_{(\alpha_1,\beta)},$$

PROOF. For any $\alpha\in\mathbb{Z}_N^2$ we have

$$v_\alpha = \frac{1}{N}\omega^{\frac{1}{2}t\alpha_2(\alpha_2+1)+\alpha_2\left(\alpha_1-\frac{1}{2}\right)}\sum_{\beta\in\mathbb{Z}_N}\omega^{-\alpha_2\beta}u_{(\alpha_1,\beta)}$$

and

$$\bar{v}_\alpha = \omega^{-\frac{1}{2}t\alpha_2(\alpha_2+1)-\alpha_2\left(\alpha_1-\frac{1}{2}\right)}\sum_{\beta\in\mathbb{Z}_N}\omega^{\alpha_2\beta}\bar{u}_{(\alpha_1,\beta)}.$$

This shows that $\{u_\alpha\}_{\alpha\in\mathbb{Z}_N^2}$, and $\{\bar{u}_\alpha\}_{\alpha\in\mathbb{Z}_N^2}$, are generating sets of $\mathcal{V}$ and $\mathcal{V}^*$ respectively. Hence these sets are bases since their cardinality is the same as the dimension of $\mathcal{V}$ and $\mathcal{V}^*$.

A straightforward computation shows that for all $\alpha, \beta \in \mathbb{Z}_N^2$ we have

$$\bar{u}_\beta u_\alpha = \delta_{\alpha,\beta}$$

which implies that $\{u_\alpha\}_{\alpha\in\mathbb{Z}_N^2}$, and $\{\bar{u}_\alpha\}_{\alpha\in\mathbb{Z}_N^2}$ are dual bases. □

The basis $\{u_\alpha\}_{\alpha\in\mathbb{Z}_N^2}$ satisfies the following relations.

LEMMA 2.10. *For all $\alpha \in \mathbb{Z}_N^2$ and all $m \in \mathbb{Z}_N$*

$$X^m u_\alpha = \omega^{-m\alpha_1} u_\alpha, \quad Y^m u_\alpha = u_{(\alpha_1+m,\alpha_2+m)},$$

$$U^m u_\alpha = \omega^{-\frac{1}{2}m(m-1)(t+1)-mt\alpha_1} u_{(\alpha_1-mt,\alpha_2)},$$

$$V^m u_\alpha = \omega^{-\frac{1}{2}m(m-1)(t+1)-m\left(t\alpha_1+\alpha_2+\frac{1}{2}\right)} u_{(\alpha_1-mt,\alpha_2)}.$$

In particular, we have for all $\alpha \in \mathbb{Z}_N^2$ and all $m \in \mathbb{Z}_N$

$$(UV^{-1})^m u_\alpha = \omega^{m(\alpha_2+\frac{1}{2})} u_\alpha, \quad (U^tV)^m u_\alpha = \omega^{\frac{1}{2}m(m-1)+m\left(\alpha_1-\alpha_2+\frac{1}{2}t\right)} u_{(\alpha_1+m,\alpha_2)}.$$

PROOF. For any $\alpha \in \mathbb{Z}_N^2$ we have, by the equalities (1.7),

$$\begin{aligned} Xv_\alpha = \omega^{-\alpha_1} v_\alpha, \quad Yv_\alpha &= v_{(\alpha_1+1,\alpha_2)}, \\ Uv_\alpha &= \omega^{-t(\alpha_1-\alpha_2)} v_{(\alpha_1-t,\alpha_2)}, \\ Vv_\alpha &= \omega^{-(t+1)\alpha_1} v_{(\alpha_1-t,\alpha_2+1)}. \end{aligned}$$

Then we can easily derive the following equalities

$$\begin{aligned} UV^{-1}v_\alpha &= \omega^{\alpha_1+t\alpha_2} v_{(\alpha_1,\alpha_2-1)}, \\ U^tVv_\alpha &= \omega^{-(t+1)(\alpha_2+\frac{1}{2})} v_{(\alpha_1+1,\alpha_2+1)}. \end{aligned}$$

Now we determine the action of these operators on the basis u_α for any $\alpha \in \mathbb{Z}_N^2$. For X, we have

$$\begin{aligned} Xu_\alpha = \sum_{\beta\in\mathbb{Z}_N} & \omega^{-\frac{1}{2}t\beta(\beta+1)+\beta(\alpha_2-\alpha_1+\frac{1}{2})} Xv_{(\alpha_1,\beta)} \\ &= \sum_{\beta\in\mathbb{Z}_N} \omega^{-\frac{1}{2}t\beta(\beta+1)+\beta(\alpha_2-\alpha_1+\frac{1}{2})} \omega^{-\alpha_1} v_{(\alpha_1,\beta)} = \omega^{-\alpha_1} u_\alpha. \end{aligned}$$

For Y, we have

$$\begin{aligned} Yu_\alpha = \sum_{\beta\in\mathbb{Z}_N} \omega^{-\frac{1}{2}t\beta(\beta+1)+\beta(\alpha_2-\alpha_1+\frac{1}{2})} Y v_{(\alpha_1,\beta)} \\ = \sum_{\beta\in\mathbb{Z}_N} \omega^{-\frac{1}{2}t\beta(\beta+1)+\beta(\alpha_2-\alpha_1+\frac{1}{2})} v_{(\alpha_1+1,\beta)} \\ = \sum_{\beta\in\mathbb{Z}_N} \omega^{-\frac{1}{2}t\beta(\beta+1)+\beta\left((\alpha_2+1)-(\alpha_1+1)+\frac{1}{2}\right)} v_{(\alpha_1+1,\beta)} = u_{(\alpha_1+1,\alpha_2+1)}. \end{aligned}$$

For U, we have

$$\begin{aligned} Uu_\alpha = \sum_{\beta\in\mathbb{Z}_N} \omega^{-\frac{1}{2}t\beta(\beta+1)+\beta(\alpha_2-\alpha_1+\frac{1}{2})} U v_{(\alpha_1,\beta)} \\ = \sum_{\beta\in\mathbb{Z}_N} \omega^{-\frac{1}{2}t\beta(\beta+1)+\beta(\alpha_2-\alpha_1+\frac{1}{2})} \omega^{-t(\alpha_1-\beta)} v_{(\alpha_1-t,\beta)} \\ = \omega^{-t\alpha_1} \sum_{\beta\in\mathbb{Z}_N} \omega^{-\frac{1}{2}t\beta(\beta+1)+\beta\left(\alpha_2-(\alpha_1-t)+\frac{1}{2}\right)} v_{(\alpha_1-t,\beta)} = \omega^{-t\alpha_1} u_{(\alpha_1-t,\alpha_2)}. \end{aligned}$$

For V, we have

$$\begin{aligned} Vu_\alpha = \sum_{\beta\in\mathbb{Z}_N} \omega^{-\frac{1}{2}t\beta(\beta+1)+\beta(\alpha_2-\alpha_1+\frac{1}{2})} V v_{(\alpha_1,\beta)} \\ = \sum_{\beta\in\mathbb{Z}_N} \omega^{-\frac{1}{2}t\beta(\beta+1)+\beta(\alpha_2-\alpha_1+\frac{1}{2})} \omega^{-(t+1)\alpha_1} v_{(\alpha_1-t,\beta+1)} \\ = \omega^{-(t\alpha_1+\alpha_2+\frac{1}{2})} \sum_{\beta\in\mathbb{Z}_N} \omega^{-\frac{1}{2}t(\beta+1)(\beta+2)+(\beta+1)\left(\alpha_2-(\alpha_1-t)+\frac{1}{2}\right)} v_{(\alpha_1-t,\beta+1)} \\ = \omega^{-(t\alpha_1+\alpha_2+\frac{1}{2})} u_{(\alpha_1-t,\alpha_2)}. \end{aligned}$$

For UV^{-1}, we have

$$\begin{aligned} UV^{-1}u_\alpha = \sum_{\beta\in\mathbb{Z}_N} \omega^{-\frac{1}{2}t\beta(\beta+1)+\beta(\alpha_2-\alpha_1+\frac{1}{2})} UV^{-1} v_{(\alpha_1,\beta)} \\ = \sum_{\beta\in\mathbb{Z}_N} \omega^{-\frac{1}{2}t\beta(\beta+1)+\beta(\alpha_2-\alpha_1+\frac{1}{2})} \omega^{\alpha_1+t\beta} v_{(\alpha_1,\beta-1)} \\ = \omega^{\alpha_2+\frac{1}{2}} \sum_{\beta\in\mathbb{Z}_N} \omega^{-\frac{1}{2}t(\beta-1)\beta+(\beta-1)\left(\alpha_2-\alpha_1+\frac{1}{2}\right)} v_{(\alpha_1,\beta-1)} = \omega^{\alpha_2+\frac{1}{2}} u_\alpha. \end{aligned}$$

Finally, for U^tV, we have

$$\begin{aligned} U^tVu_\alpha &= \sum_{\beta\in\mathbb{Z}_N} \omega^{-\frac{1}{2}t\beta(\beta+1)+\beta\left(\alpha_2-\alpha_1+\frac{1}{2}\right)} U^tVv_{(\alpha_1,\beta)} \\ &= \sum_{\beta\in\mathbb{Z}_N} \omega^{-\frac{1}{2}t\beta(\beta+1)+\beta\left(\alpha_2-\alpha_1+\frac{1}{2}\right)} \omega^{-(t+1)\left(\beta+\frac{1}{2}\right)} v_{(\alpha_1+1,\beta+1)} \\ &= \omega^{\alpha_1-\alpha_2+\frac{1}{2}t} \sum_{\beta\in\mathbb{Z}_N} \omega^{-\frac{1}{2}t(\beta+1)(\beta+2)+(\beta+1)\left(\alpha_2-(\alpha_1+1)+\frac{1}{2}\right)} v_{(\alpha_1+1,\beta+1)} \\ &= \omega^{\alpha_1-\alpha_2+\frac{1}{2}t} u_{(\alpha_1+1,\alpha_2)}. \end{aligned}$$

The Lemma follows easily from the previous equalities. □

Now we can give an explicit expression of matrix elements of $S(x)$ and its inverse using two functions defined in [**9**]. Let us recall what are these functions. The first one is defined by

$$f(x,y|z) = \sum_{m=0}^{N-1} \frac{w(x|m)}{w(y|m)} z^m, \tag{2.14}$$

where $x, y, z \in \mathbb{C}$ are such that

$$\left\{x^N, y^N\right\} \subset \mathbb{C}\setminus\{1\} \quad \text{and} \quad z^N = \frac{1-x^N}{1-y^N}. \tag{2.15}$$

The last condition provides periodicity on (2.14) on the variable m of period N.

The second function is a generalisation of the previous one, namely

$$F\left(\begin{matrix} x & u \\ y & v \end{matrix}\middle| z\right) = \sum_{m=0}^{N-1} \frac{w(x|m)w(u|m)}{w(y|m)w(v|m)} z^m, \tag{2.16}$$

where $x, y, u, v, z \in \mathbb{C}$ are such that

$$\left\{x^N, y^N, u^N, v^N\right\} \subset \mathbb{C}\setminus\{1\} \quad \text{and} \quad z^N = \frac{(1-x^N)(1-u^N)}{(1-y^N)(1-v^N)}. \tag{2.17}$$

The last condition provides periodicity on (2.16) on the variable m of period N, just as for the function $f(x,y|z)$.

PROPOSITION 2.11. *For all $x \in \mathbb{R}_{\neq 0,1}$ and all $\alpha, \beta, \mu, \nu \in \mathbb{Z}_N^2$, set*

$$r = (t+1)(\nu_1 - \beta_1 + \mu_2 - \alpha_2) \textit{ and } s = (t+1)(\nu_1 - \beta_1) + t(\mu_2 - \alpha_2).$$

Then we have

$$\begin{aligned}\Big\langle \bar{u}_\mu \otimes \bar{u}_\nu \Big| S(x) \Big| u_\alpha \otimes u_\beta \Big\rangle &= \delta_{\mu_1+\nu_1,\alpha_1}\delta_{\mu_2+\nu_2,\alpha_2+\beta_2} \\ &\times \omega^{-\frac{1}{2}(t+1)(\alpha_2-\mu_2)(\alpha_2-\mu_2-t-2((t+1)\alpha_1-\beta_1+\nu_2+1))} \\ &\times \omega^{-\frac{1}{2}(t+1)\{\beta_1(\beta_1-1+2(t\alpha_1+\alpha_2-\nu_2))+\nu_1((t+1)\nu_1+2(\mu_1+\nu_2+1))\}} \\ &\times \psi_{x,0}^2\psi_{x,r}\psi_{x,s} f\left((1-x)^{\frac{1}{N}},(1-x)^{-\frac{1}{N}}\omega^{-r-1}\Big|(x-1)^{\frac{1}{N}}\omega^{\beta_2+\frac{1}{2}}\right) \\ &\times F\left(\begin{matrix}(1-x)^{\frac{1}{N}} & (x-1)^{-\frac{1}{N}}\omega^{\beta_2-r-\frac{1}{2}} \\ (1-x)^{-\frac{1}{N}}\omega^{-s-1} & (x-1)^{\frac{1}{N}}\omega^{\beta_2-\frac{1}{2}}\end{matrix}\Bigg| -\omega^{\alpha_2+\frac{1}{2}}\right)\end{aligned}$$

and

$$\begin{aligned}\Big\langle \bar{u}_\mu \otimes \bar{u}_\nu \Big| S(x)^{-1} \Big| u_\alpha \otimes u_\beta \Big\rangle &= \delta_{\mu_1,\alpha_1+\beta_1}\delta_{\mu_2+\nu_2,\alpha_2+\beta_2} \\ &\times \omega^{-\frac{1}{2}(t+1)(\alpha_2-\mu_2)(\alpha_2-\mu_2-t-2(\alpha_1+t\mu_1+\nu_2+1))} \\ &\times \omega^{\frac{1}{2}(t+1)\{\beta_1((t-1)\beta_1+2(t\mu_1+\nu_2+1))+\nu_1(\nu_1-1-2((t+1)\mu_1-t\mu_2-\nu_2-t))\}} \\ &\times \bar{\psi}_{x,0}^2\bar{\psi}_{x,r}\bar{\psi}_{x,s} f\left((1-x)^{-\frac{1}{N}},(1-x)^{\frac{1}{N}}\omega^{-r-1}\Big|(x-1)^{-\frac{1}{N}}\omega^{\nu_2+r+\frac{1}{2}}\right) \\ &\times F\left(\begin{matrix}(1-x)^{-\frac{1}{N}} & (x-1)^{\frac{1}{N}}\omega^{\nu_2-r-\frac{1}{2}} \\ (1-x)^{\frac{1}{N}}\omega^{-s-1} & (x-1)^{-\frac{1}{N}}\omega^{-\nu_2-\frac{1}{2}}\end{matrix}\Bigg| -\omega^{\alpha_2+\frac{1}{2}}\right),\end{aligned}$$

2.4. Basis for the multiplicity spaces. For all $x \in \mathbb{R}_{\neq 0,1}$, we define

$$e_\alpha(x) = S(x)(\mathrm{Id}_{\mathcal{V}} \otimes u_\alpha) \quad \bar{e}_\alpha(x) = (\mathrm{Id}_{\mathcal{V}} \otimes \bar{u}_\alpha)S(x)^{-1}. \tag{2.18}$$

If $(p,q) \in (\mathbb{R}_{\neq 0})^2$ is an admissible pair, then by equation (2.13),

$$e_\alpha\left(\tfrac{q}{p+q}\right) \in \mathcal{H}_{p,q} \quad \text{and} \quad \bar{e}_\alpha\left(\tfrac{q}{p+q}\right) \in \bar{\mathcal{H}}_{p,q}, \quad \forall \alpha \in \mathbb{Z}_N^2.$$

Moreover, $\left\{e_\alpha\left(\frac{q}{p+q}\right)\right\}_{\alpha\in\mathbb{Z}_N^2}$ and $\left\{\bar{e}_\alpha\left(\frac{q}{p+q}\right)\right\}_{\alpha\in\mathbb{Z}_N^2}$ form dual basis of $\mathcal{H}_{p,q}$ and $\bar{\mathcal{H}}_{p,q}$ respectively, where the duality is reflected by the relations

$$\bar{e}_\beta\left(\tfrac{q}{p+q}\right) e_\alpha\left(\tfrac{q}{p+q}\right) = \delta_{\alpha,\beta}\,\mathrm{Id}_{\mathcal{V}}, \tag{2.19}$$

and

$$\sum_{\alpha\in\mathbb{Z}_N^2} e_\alpha\left(\tfrac{q}{p+q}\right)\bar{e}_\alpha\left(\tfrac{q}{p+q}\right) = \mathrm{Id}_{\mathcal{V}} \otimes \mathrm{Id}_{\mathcal{V}} \tag{2.20}$$

Remark 2.12. As we have just shown, the non-trivial elements of the multiplicity spaces $\mathcal{H}_{p,q}$ and $\bar{\mathcal{H}}_{p,q}$ depend only on $\frac{q}{p+q} \in \mathbb{R}_{\neq 0,1}$. We will call this property *the scaling invariance property* of the multiplicity spaces. In order to keep track of the difference between $\mathcal{H}_{p,q}$ and $\mathcal{H}_{\lambda p,\lambda q}$ and the difference between $\bar{\mathcal{H}}_{p,q}$ and $\bar{\mathcal{H}}_{\lambda p,\lambda q}$

respectively, where $\lambda \in \mathbb{R}_{\neq 0}$, we define, for all admissible pair $(p,q) \in (\mathbb{R}_{\neq 0})^2$, the following isomorphism

$$h_{p,q} : \mathcal{V} \to \mathcal{H}_{p,q} \quad \text{and} \quad \bar{h}_{p,q} : \mathcal{V}^* \to \bar{\mathcal{H}}_{p,q}$$

by

$$h_{p,q}(u_\alpha) = e_\alpha \left(\tfrac{q}{p+q}\right), \qquad \bar{h}_{p,q}(\bar{u}_\alpha) = \bar{e}_\alpha \left(\tfrac{q}{p+q}\right), \quad \forall \alpha \in \mathbb{Z}_N^2$$

2.5. The Ψ-system.

Definition 2.13. A Ψ-system in the category of $A_{\omega,t}$-modules consists of

(1) a distinguished set of simple objects $\{V_i\}_{i\in I}$ such that $\mathrm{Hom}(V_i, V_j) = 0$ for all $i \neq j$,
(2) an involution, $I \to I$, $i \mapsto i^*$
(3) two families of morphisms $\{b_i : \mathbb{C} \to V_i \otimes V_{i^*}\}_{i\in I}$ and $\{d_i : V_i \otimes V_{i^*} \to \mathbb{C}\}_{i\in I}$ such that for all $i \in I$
$$(\mathrm{Id}_{V_i} \otimes d_{i^*})(b_i \otimes \mathrm{Id}_{V_i}) = \mathrm{Id}_{V_i} = (d_i \otimes \mathrm{Id}_{V_i})(\mathrm{Id}_{V_i} \otimes b_{i^*}) \tag{2.21}$$
(4) Let $H_k^{i,j}$ and $H_{i,j}^k$ be $\mathrm{Hom}_{\mathcal{A}_{\omega,t}}(V_k, V_i \otimes V_j)$ and $\mathrm{Hom}_{\mathcal{A}_{\omega,t}}(V_i \otimes V_j, V_k)$ respectively. For any $i, j \in I$ such that $H_k^{i,j} \neq 0$ for some $k \in I$, the identity morphism $\mathrm{Id}_{V_i \otimes V_j}$ is in the image of the linear map
$$\begin{array}{ccc} \bigoplus_{k\in I} H_k^{i,j} \otimes H_{i,j}^k & \longrightarrow & \mathrm{End}_{\mathcal{A}_{\omega,t}}(V_i \otimes V_j) \\ x \otimes y & \longmapsto & x \circ y \end{array}$$

Note that $\mathbb{C}$ is provided with an $\mathcal{A}_{\omega,t}$-module structure through the counit

$$\varepsilon : \mathcal{A}_{\omega,t} \to \mathrm{End}(\mathbb{C}) \cong \mathbb{C}.$$

If the $\mathcal{A}_{\omega,t}$-module structure of V_i and V_{i^*} is provided by $\pi_i : \mathcal{A}_{\omega,t} \to \mathrm{End}(V_i)$ and $\pi_{i^*} : \mathcal{A}_{\omega,t} \to \mathrm{End}(V_{i^*})$ respectively, then the morphisms b_i and d_i satisfy

$$b_p(k)\varepsilon(a) = (\pi_p \otimes \pi_{p^*})\Delta(a) b_p(k)$$
$$\varepsilon(a) d_p(u \otimes v) = d_p(\pi_p \otimes \pi_{p^*})\Delta(a)(u \otimes v)$$

for all $a \in \mathcal{A}_{\omega,t}$, all $k \in \mathbb{C}$ and all $u \otimes v \in V_i \otimes V_{i^*}$.

In our case, the set I is $\mathbb{R}_{\neq 0}$ and the involution is given by $p^* = -p$.

Lemma 2.14. *For any $p \in \mathbb{R}_{\neq 0}$, the morphism d_p is a solution of the following system of homogeneous linear equations*

$$\begin{aligned} d_p &= d_p(X \otimes X) \\ d_p &= d_p\left(XY^{-1} \otimes Y\right) \\ d_p &= d_p\left(V^{-1} \otimes X^{-(t+1)} U\right) \\ d_p &= \omega^{-1} d_p\left(UV^{-1} \otimes UV^{-1}\right). \end{aligned}$$

PROOF. By definition, for all $a \in \mathcal{A}_{\omega,t}$ we have

$$\varepsilon(a)d_p = d_p(\pi_p \otimes \pi_{-p})\Delta(a). \tag{2.22}$$

If $a = k_1$, we have $\varepsilon(k_1) = 1$ and $(\pi_p \otimes \pi_{-p})\Delta(k_1) = X \otimes X$. Hence, equality (2.22) becomes

$$d_p = d_p(X \otimes X). \tag{2.23}$$

If $a = e_1$, we have $\varepsilon(e_1) = 0$ and

$$(\pi_p \otimes \pi_{-p})\Delta(e_1) = p^{\frac{1}{N}} Y \otimes \mathrm{Id}_{\mathcal{A}} + (-p)^{\frac{1}{N}} X \otimes Y.$$

Hence, equality (2.22) is equivalent to

$$0 = d_p(Y \otimes \mathrm{Id}_{\mathcal{A}} - X \otimes Y) \Leftrightarrow d_p(Y \otimes \mathrm{Id}_{\mathcal{A}}) = dp(X \otimes Y)$$

which leads us to

$$d_p = d_p(XY^{-1} \otimes Y). \tag{2.24}$$

If $a = e_2$, we have $\varepsilon(e_2) = 0$ and

$$(\pi_p \otimes \pi_{-p})\Delta(e_2) = \left(\tfrac{1}{2}p\right)^{\frac{1}{N}} Z \otimes \mathrm{Id}_{\mathcal{A}} + \left(-\tfrac{1}{2}p\right)^{\frac{1}{N}} X^t \otimes Z,$$

where $Z = (U + V)Y^{-1}$. A similar computation to the previous one leads us to the following equality

$$d_p = d_p(X^t Z^{-1} \otimes Z) \tag{2.25}$$

If $a = a_1$, we have $\varepsilon(a_1) = 0$ and

$$(\pi_p \otimes \pi_{-p})\Delta(a_1) = \left(\tfrac{1}{2}p^2\right)^{\frac{1}{N}} U \otimes \mathrm{Id}_{\mathcal{A}} + \left(\tfrac{1}{2}p^2\right)^{\frac{1}{N}} X^{t+1} \otimes U + \left(-\tfrac{1}{2}p^2\right)^{\frac{1}{N}} X^t Y \otimes Z,$$

Therefore, using equality (2.22) we compute

$$\begin{aligned}
0 = d_p(U \otimes \mathrm{Id}_{\mathcal{A}} + X^{t+1} \otimes U - X^t Y \otimes Z) &= d_p(U \otimes \mathrm{Id}_{\mathcal{A}}) + d_p(X^{t+1} \otimes U) - d_p(X^t Y \otimes Z) \\
&\overset{(2.25)}{=} d_p(U \otimes \mathrm{Id}_{\mathcal{A}}) + d_p(X^{t+1} \otimes U) - d_p(ZY \otimes \mathrm{Id}_{\mathcal{A}}) \\
&= d_p(U \otimes \mathrm{Id}_{\mathcal{A}}) + d_p(X^{t+1} \otimes U) - d_p(U \otimes \mathrm{Id}_{\mathcal{A}}) - d_p(V \otimes \mathrm{Id}_{\mathcal{A}}) \\
&= d_p(X^{t+1} \otimes U) - d_p(V \otimes \mathrm{Id}_{\mathcal{A}}) \overset{(2.23)}{=} d_p(\mathrm{Id}_{\mathcal{A}} \otimes X^{-(t+1)}U) - d_p(V \otimes \mathrm{Id}_{\mathcal{A}}),
\end{aligned}$$

which leads to

$$d_p = d_p(V^{-1} \otimes X^{-(t+1)}U). \tag{2.26}$$

Finally, by reconsidering the last computation, we get

$$\begin{aligned}0 &= d_p(U\otimes \mathrm{Id}_{\mathcal{A}} + X^{t+1}\otimes U - X^tY\otimes Z)\\ &= d_p(U\otimes \mathrm{Id}_{\mathcal{A}}) + d_p(X^{t+1}\otimes U) - \omega^{-t}d_p(YX^t\otimes Z)\\ &\overset{(2.24)}{=} d_p(U\otimes \mathrm{Id}_{\mathcal{A}}) + d_p(X^{t+1}\otimes U) - \omega^{-t}d_p(X^{t+1}\otimes YZ)\\ &= d_p(U\otimes \mathrm{Id}_{\mathcal{A}}) + d_p(X^{t+1}\otimes U) - d_p(X^{t+1}\otimes U) - \omega d_p(X^{t+1}\otimes V)\\ &= d_p(U\otimes \mathrm{Id}_{\mathcal{A}}) - \omega d_p(X^{t+1}\otimes V) \overset{(2.23)}{=} d_p(U\otimes \mathrm{Id}_{\mathcal{A}}) - \omega d_p(\mathrm{Id}_{\mathcal{A}}\otimes X^{-(t+1)}V),\end{aligned}$$

which leads to

$$d_p = d_p(U^{-1}\otimes X^{-(t+1)}V). \tag{2.27}$$

From there, we use equality (2.26) to get

$$d_p(U^{-1}\otimes X^{-(t+1)}V) = d_p(V^{-1}\otimes X^{-(t+1)}U),$$

which is equivalent to

$$d_p = \omega^{-1}d_p(UV^{-1}\otimes UV^{-1}) \tag{2.28}$$

□

LEMMA 2.15. *The duality morphisms $b_p : \mathbb{C}\to V_p\otimes V_{-p}$ and $d_p : V_p\otimes V_{-p}\to\mathbb{C}$ are given by*

$$b_p(1) = \sum_{\alpha,\beta\in\mathbb{Z}_N^2} b_{p,\alpha,\beta}u_\alpha\otimes u_\beta$$

where

$$b_{p,\alpha,\beta} = \delta_{\alpha,-\beta}\omega^{\frac{1}{2}\left((t+1)(\alpha_1-\alpha_2)(\alpha_1-\alpha_2+2)+\alpha_1^2+\alpha_2\right)}, \tag{2.29}$$

and

$$d_p(u_\alpha\otimes u_\beta) = \delta_{\alpha,-\beta}\omega^{-\frac{1}{2}\left((t+1)(\alpha_1-\alpha_2)(\alpha_1-\alpha_2-2)+\alpha_1^2-\alpha_2\right)}. \tag{2.30}$$

PROOF. First we compute d_p using Lemma 2.14 and Lemma 2.10.

We start with the first and the last equality of the Lemma 2.14. We have

$$d_p(u_\alpha\otimes u_\beta) = d_p(X\otimes X)(u_\alpha\otimes u_\beta) = \omega^{-\alpha_1-\beta_1}d_p(u_\alpha\otimes u_\beta)$$

and

$$d_p(u_\alpha\otimes u_\beta) = \omega^{-1}d_p(UV^{-1}\otimes UV^{-1})(u_\alpha\otimes u_\beta) = \omega^{\alpha_2+\beta_2}d_p(u_\alpha\otimes u_\beta).$$

Hence we have

$$d_p(u_\alpha\otimes u_\beta) = \delta_{\alpha,-\beta}d_p(u_\alpha\otimes u_\beta). \tag{2.31}$$

We consider the third equality of the Lemma 2.14

$$d_p = d_p(V^{-1}\otimes X^{-(t+1)}U).$$

Since $X, U, V \in \mathcal{A}$ are invertible, this equality is equivalent to

$$d_p = d_p \left(V^{-1} \otimes X^{-(t+1)} U\right)^{t+1}$$

A straightforward computation shows that

$$\left(V^{-1} \otimes X^{-(t+1)} U\right)^{t+1} = \omega^{\frac{1}{2}} \left(V^{-(t+1)} \otimes X^{-t} U^{t+1}\right),$$

hence, the following equality holds true

$$d_p = \omega^{\frac{1}{2}} d_p \left(V^{-(t+1)} \otimes X^{-t} U^{t+1}\right).$$

Therefore, using Lemma 2.10, we compute

$$\begin{aligned}
d_p(u_\alpha \otimes u_\beta) &= \delta_{\alpha,-\beta} d_p(u_\alpha \otimes u_\beta) = \delta_{\alpha,-\beta} \omega^{\frac{1}{2}} d_p \left(V^{-(t+1)} \otimes X^{-t} U^{t+1}\right)(u_\alpha \otimes u_\beta) \\
&= \delta_{\alpha,-\beta} \omega^{\frac{1}{2}} d_p \left(V^{-(t+1)} \otimes X^{-t} U^{t+1}\right)(u_\alpha \otimes u_{-\alpha}) \\
&= \delta_{\alpha,-\beta} \omega^{-(t+2)\alpha_1 + (t+1)\alpha_2 + \frac{3}{2}t + 2} d_p\big(u_{(\alpha_1 - 1, \alpha_2)} \otimes u_{(-\alpha_1 + 1, -\alpha_2)}\big) \\
&= \delta_{\alpha,-\beta} \omega^{-(t+2)\frac{1}{2}\alpha_1(\alpha_1+1) + \alpha_1\left((t+1)\alpha_2 + \frac{3}{2}t + 2\right)} d_p\big(u_{(\alpha_1 - \alpha_1, \alpha_2)} \otimes u_{(-\alpha_1 + \alpha_1, -\alpha_2)}\big) \\
&= \delta_{\alpha,-\beta} \omega^{-\frac{1}{2}(t+2)\alpha_1^2 + (t+1)\alpha_1(\alpha_2+1)} d_p\big(u_{(0,\alpha_2)} \otimes u_{(0,-\alpha_2)}\big). \qquad (2.32)
\end{aligned}$$

Now we use the second equality of Lemma 2.14 in the following way

$$\begin{aligned}
d_p\big(u_{(0,\alpha_2)} \otimes u_{(0,-\alpha_2)}\big) &= d_p(XY^{-1} \otimes Y)\big(u_{(0,\alpha_2)} \otimes u_{(0,-\alpha_2)}\big) \\
&= \omega d_p\big(u_{(-1,\alpha_2-1)} \otimes u_{(1,-\alpha_2+1)}\big) \overset{(2.32)}{=} \omega^{-(t+1)\alpha_2 - \frac{1}{2}t} d_p\big(u_{(0,\alpha_2-1)} \otimes u_{(0,-\alpha_2+1)}\big) \\
&= \omega^{-\frac{1}{2}(t+1)\alpha_2(\alpha_2+1) - \frac{1}{2}t\alpha_2} d_p\big(u_{(0,\alpha_2-\alpha_2)} \otimes u_{(0,-\alpha_2+\alpha_2)}\big) \\
&= \omega^{-\frac{1}{2}(t+1)\alpha_2^2 - \frac{1}{2}(2t+1)\alpha_2} d_p\big(u_{(0,0)} \otimes u_{(0,0)}\big). \qquad (2.33)
\end{aligned}$$

Using equalities (2.32) and (2.33), we get

$$\begin{aligned}
d_p(u_\alpha \otimes u_\beta) &= \delta_{\alpha,-\beta} \omega^{-\frac{1}{2}(t+2)\alpha_1^2 + (t+1)\alpha_1(\alpha_2+1)} d_p\big(u_{(0,\alpha_2)} \otimes u_{(0,-\alpha_2)}\big) \\
&= \delta_{\alpha,-\beta} \omega^{-\frac{1}{2}(t+2)\alpha_1^2 + (t+1)\alpha_1(\alpha_2+1)} \omega^{-\frac{1}{2}(t+1)\alpha_2^2 - \frac{1}{2}(2t+1)\alpha_2} d_p\big(u_{(0,0)} \otimes u_{(0,0)}\big) \\
&= \delta_{\alpha,-\beta} \omega^{-\frac{1}{2}\left((t+1)(\alpha_1-\alpha_2)(\alpha_1-\alpha_2-2) + \alpha_1^2 - \alpha_2\right)} d_p\big(u_{(0,0)} \otimes u_{(0,0)}\big).
\end{aligned}$$

Finally, we set $d_p(u_{(0,0)} \otimes u_{(0,0)}) = 1$. The formula for b_p is easily computed using formula (2.21). □

THEOREM 2.16. *In the category of $\mathcal{A}_{\omega,t}$-modules, the set of objects $\{V_p\}_{p \in \mathbb{R}_{\neq 0}}$ with the involution $p^* = -p$ and the duality morphisms defined in the Lemma 2.15 is a Ψ-system.*

PROOF. By definition, we have to check the following three points :

(1) $\mathrm{Hom}(V_p, V_q) = 0$ for all $p \neq q$,

(2) The morphisms $\{b_p : \mathbb{C} \to V_p \otimes V_{-p}\}_{p\in\mathbb{R}_{\neq 0}}$ and $\{d_p : V_p \otimes V_{-p} \to \mathbb{C}\}_{p\in\mathbb{R}_{\neq 0}}$ satisfy

$$(\mathrm{Id}_{\mathcal{V}} \otimes d_{-p})(b_p \otimes \mathrm{Id}_{\mathcal{V}}) = \mathrm{Id}_{\mathcal{V}} = (d_p \otimes \mathrm{Id}_{\mathcal{V}})(\mathrm{Id}_{\mathcal{V}} \otimes b_{-p}), \quad \forall p \in \mathbb{R}_{\neq 0} \tag{2.34}$$

(3) If (p, q) is admissible, then $\mathrm{Id}_{V_p\otimes V_q}$ is in the image of the linear map

$$\mathcal{H}_{p,q} \otimes \bar{\mathcal{H}}_{p,q} \to \mathrm{End}(V_p \otimes V_q), \quad x \otimes y \mapsto xy.$$

Point (1) is clear by Schur's lemma, point (2) is straightforward using Lemma 2.15 and point (3) is given by formula (2.20). ☐

3. Operators in the space of multiplicities $\mathcal{H}$

We consider the vector space $\mathcal{H}$, called the *space of multiplicities*, defined by

$$\mathcal{H} = \bigoplus_{\substack{(p,q)\in(\mathbb{R}_{\neq 0})^2 \\ \text{admissible}}} \mathcal{H}_{p,q} \oplus \bar{\mathcal{H}}_{p,q}.$$

In this section, we are going to determine the key operators in $\mathrm{End}(\mathcal{H})$ that will allow us to extend the Ψ-system defined in Theorem 2.16 into a $\hat{\Psi}$-system.

3.1. The standard operators. Let us first recall that the *Mobius group* is defined as the group $\mathrm{PGL}(2, \mathbb{C})$ acting on $\mathbb{C} \cup \{\infty\}$ as follows

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(x) = \begin{cases} \dfrac{ax+b}{cx+d} & if \quad x \in \mathbb{C} \\ \dfrac{a}{c} & if \quad x = \infty. \end{cases}$$

The elements of the Mobius group are called *Mobius transformation*.

DEFINITION 3.1. We say that $f \in \mathrm{End}(\mathcal{H})$ is a *standard operator* if f is invertible and if for all admissible pair $(p, q) \in (\mathbb{R}_{\neq 0})^2$, there exists an admissible pair $(r, s) \in (\mathbb{R}_{\neq 0})^2$ and a Mobius transformation $M \in \mathrm{PGL}(2, \mathbb{C})$ satisfying the following two conditions :

(1)

$$f(\mathcal{H}_{p,q}) = \mathcal{H}_{r,s} \text{ and } f(\bar{\mathcal{H}}_{p,q}) = \bar{\mathcal{H}}_{r,s}, \tag{3.1}$$

or

$$f(\mathcal{H}_{p,q}) = \bar{\mathcal{H}}_{r,s}, \text{ and } f(\bar{\mathcal{H}}_{p,q}) = \mathcal{H}_{r,s}, \tag{3.2}$$

(2)

$$M\left(\tfrac{q}{p+q}\right) = \tfrac{s}{r+s} \tag{3.3}$$

The scaling invariance property of the multiplicity spaces extends to the standard operators in the following sense : if $f \in \mathrm{End}(\mathcal{H})$ is a standard operator, then for all $\alpha, \beta \in \mathbb{Z}_N^2$, there exists functions

$$f_{\alpha,\beta}, \bar{f}_{\alpha,\beta} : \mathbb{R}_{\neq 0,1} \to \mathbb{C}$$

such that

$$\begin{aligned} f h_{p,q}(u_\alpha) &= \sum_{\beta \in \mathbb{Z}_N^2} f_{\alpha,\beta}\left(\tfrac{q}{p+q}\right) h_{r,s}(\bar{u}_\beta), \\ f \bar{h}_{p,q}(\bar{u}_\alpha) &= \sum_{\beta \in \mathbb{Z}_N^2} \bar{f}_{\alpha,\beta}\left(\tfrac{q}{p+q}\right) h_{r,s}(u_\beta), \end{aligned} \tag{3.4}$$

if f satisfies (3.1) and

$$\begin{aligned} f h_{p,q}(u_\alpha) &= \sum_{\beta \in \mathbb{Z}_N^2} f_{\alpha,\beta}\left(\tfrac{q}{p+q}\right) \bar{h}_{r,s}(\bar{u}_\beta), \\ f \bar{h}_{p,q}(\bar{u}_\alpha) &= \sum_{\beta \in \mathbb{Z}_N^2} \bar{f}_{\alpha,\beta}\left(\tfrac{q}{p+q}\right) h_{r,s}(u_\beta), \end{aligned} \tag{3.5}$$

if f satisfies (3.2).

Definition 3.2. An operator $f \in \mathrm{End}(\mathcal{H})$ is *grading-preserving* if for all admissible pairs $(p, q) \in (\mathbb{R}_{\neq 0})^2$ we have

$$f(\mathcal{H}_{p,q}) \subset \mathcal{H}_{p,q} \quad \text{and} \quad f(\bar{\mathcal{H}}_{p,q}) \subset \bar{\mathcal{H}}_{p,q}.$$

Clearly, the invertible grading-preserving operators are standard.

Let

$$\pi_{p,q} : \mathcal{H} \to \mathcal{H}_{p,q}, \quad \bar{\pi}_{p,q} : \mathcal{H} \to \bar{\mathcal{H}}_{p,q}$$

be the obvious projections. We provide $\mathcal{H}$ with a symmetric bilinear pairing $\langle , \rangle : \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$ by

$$\langle u, v \rangle = \sum_{\substack{(p,q) \in (\mathbb{R}_{\neq 0})^2 \\ \text{admissible}}} \left(\langle \bar{\pi}_{p,q}(u) \pi_{p,q}(v) \rangle + \langle \bar{\pi}_{p,q}(v) \pi_{p,q}(u) \rangle \right)$$

for any $u, v \in \mathcal{H}$.

Definition 3.3. A *transpose* of $f \in \mathrm{End}(\mathcal{H})$ is a map $f^* \in \mathrm{End}(\mathcal{H})$ such that $\langle fu, v \rangle = \langle u, f^* v \rangle$ for all $u, v \in \mathcal{H}$. We say that $f \in \mathrm{End}(\mathcal{H})$ is *symmetric* if $f^* = f$.

Since $\mathcal{H}_{p,q}$ and $\bar{\mathcal{H}}_{p,q}$ are dual vector spaces, the transpose f^* of $f \in \mathrm{End}(\mathcal{H})$, if it exists, is unique and $(f^*)^* = f$.

If $f \in \mathrm{End}(\mathcal{H})$ is standard, the equalities (3.1) and (3.2) ensure that f^* exists. Moreover, in that case, f^* is also standard.

3.2. The operators A and B and their transpose. We now define the operators $A, B \in \mathrm{End}(\mathcal{H})$ by

$$Au = \sum_{\substack{(p,q)\in(\mathbb{R}_{\neq 0})^2 \\ \text{admissible}}} (\mathrm{Id}_{\mathcal{V}} \otimes \bar{\pi}_{p,q}(u))(b_{-p} \otimes \mathrm{Id}_{\mathcal{V}}) + (d_{-p} \otimes \mathrm{Id}_{\mathcal{V}})(\mathrm{Id}_{\mathcal{V}} \otimes \pi_{p,q}(u)), \tag{3.6}$$

$$Bu = \sum_{\substack{(p,q)\in(\mathbb{R}_{\neq 0})^2 \\ \text{admissible}}} (\bar{\pi}_{p,q}(u) \otimes \mathrm{Id}_{\mathcal{V}})(\mathrm{Id}_{\mathcal{V}} \otimes b_q) + (\mathrm{Id}_{\mathcal{V}} \otimes d_q)(\pi_{p,q}(u) \otimes \mathrm{Id}_{\mathcal{V}}). \tag{3.7}$$

For each $u \in \mathcal{H}$, there are only finitely many non-zero terms in these sums, since u has only finitely many non-zero components $\pi_{p,q}(u)$ and $\bar{\pi}_{p,q}(u)$.

Using (2.21), one can easily prove that the operators A and B are involutive (also see [**5**, Lemma 3]). Hence, from their definition, we clearly have the following equalities

$$\begin{aligned} A(\mathcal{H}_{p,q}) &= \bar{\mathcal{H}}_{-p,p+q}, \quad A(\bar{\mathcal{H}}_{p,q}) = \mathcal{H}_{-p,p+q}, \\ B(\mathcal{H}_{p,q}) &= \bar{\mathcal{H}}_{p+q,-q}, \quad B(\bar{\mathcal{H}}_{p,q}) = \mathcal{H}_{p+q,-q}. \end{aligned} \tag{3.8}$$

Moreover, A and B are both standard operators. Indeed, we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left(\tfrac{q}{p+q}\right) = \tfrac{p+q}{q} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \left(\tfrac{q}{p+q}\right) = \tfrac{-q}{p}. \tag{3.9}$$

The equalities (3.8) ensure us that we have the following for A^* and B^*

$$\begin{aligned} A^*(\mathcal{H}_{p,q}) &= \bar{\mathcal{H}}_{-p,p+q}, \quad A^*(\bar{\mathcal{H}}_{p,q}) = \mathcal{H}_{-p,p+q}, \\ B^*(\mathcal{H}_{p,q}) &= \bar{\mathcal{H}}_{p+q,-q}, \quad B^*(\bar{\mathcal{H}}_{p,q}) = \mathcal{H}_{p+q,-q}. \end{aligned} \tag{3.10}$$

The equalities (3.8) and (3.10) ensure us that for all $\alpha, \beta \in \mathbb{Z}_N^2$ there exists functions

$$A_{\alpha,\beta}, \bar{A}_{\alpha,\beta}, A^*_{\alpha,\beta}, \bar{A}^*_{\alpha,\beta} : \mathbb{R}_{\neq 0,1} \to \mathbb{C}$$

and

$$B_{\alpha,\beta}, \bar{B}_{\alpha,\beta}, B^*_{\alpha,\beta}, \bar{B}^*_{\alpha,\beta} : \mathbb{R}_{\neq 0,1} \to \mathbb{C}$$

such that for all admissible pairs $(p, q) \in (\mathbb{R}_{\neq 0})^2$ we have

$$\begin{aligned} Ah_{p,q}(u_\alpha) &= \sum_{\beta\in\mathbb{Z}_N^2} A_{\alpha,\beta}\left(\tfrac{q}{p+q}\right) \bar{h}_{-p,p+q}(\bar{u}_\beta), \\ A\bar{h}_{p,q}(\bar{u}_\alpha) &= \sum_{\beta\in\mathbb{Z}_N^2} \bar{A}_{\alpha,\beta}\left(\tfrac{q}{p+q}\right) h_{-p,p+q}(u_\beta), \\ A^*h_{p,q}(u_\alpha) &= \sum_{\beta\in\mathbb{Z}_N^2} A^*_{\alpha,\beta}\left(\tfrac{q}{p+q}\right) \bar{h}_{-p,p+q}(\bar{u}_\beta), \\ A^*\bar{h}_{p,q}(\bar{u}_\alpha) &= \sum_{\beta\in\mathbb{Z}_N^2} \bar{A}^*_{\alpha,\beta}\left(\tfrac{q}{p+q}\right) h_{-p,p+q}(u_\beta). \end{aligned}$$

and

$$\begin{aligned} Bh_{p,q}(u_\alpha) &= \sum_{\beta\in\mathbb{Z}_N^2} B_{\alpha,\beta}\left(\tfrac{q}{p+q}\right)\bar{h}_{p+q,-q}(\bar{u}_\beta),\\ B\bar{h}_{p,q}(\bar{u}_\alpha) &= \sum_{\beta\in\mathbb{Z}_N^2} \bar{B}_{\alpha,\beta}\left(\tfrac{q}{p+q}\right)h_{p+q,-q}(u_\beta),\\ B^*h_{p,q}(u_\alpha) &= \sum_{\beta\in\mathbb{Z}_N^2} B^*_{\alpha,\beta}\left(\tfrac{q}{p+q}\right)\bar{h}_{p+q,-q}(\bar{u}_\beta),\\ B^*\bar{h}_{p,q}(\bar{u}_\alpha) &= \sum_{\beta\in\mathbb{Z}_N^2} \bar{B}^*_{\alpha,\beta}\left(\tfrac{q}{p+q}\right)h_{p+q,-q}(u_\beta). \end{aligned}$$

We define

$$\epsilon_N = \begin{cases} 1 & if \quad N = 1 \mod 4 \\ i & if \quad N = 3 \mod 4 \end{cases}$$

The following Proposition will be proved in Chapter 2.

PROPOSITION 3.4. *For all $x\in\mathbb{R}_{\neq 0,1}$ and all $\alpha,\beta\in\mathbb{Z}_N^2$ we have*

$$\begin{aligned} A_{\alpha,\beta}(x) \equiv& \epsilon_N^2 x^{\frac{2(N-1)}{N}}\frac{\delta_{\alpha_2,-\beta_2}}{N}\omega^{-\frac12(t(\alpha_1-\alpha_2-\beta_1)(\alpha_1-\alpha_2-\beta_1-1)-\alpha_1(\alpha_1-3)-(t+1)\alpha_2-\beta_1(\beta_1+2t+1))},\\ \bar{A}_{\alpha,\beta}(x) \equiv& \epsilon_N^{-2} x^{\frac{2(N-1)}{N}}\delta_{\alpha_2,-\beta_2}\omega^{\frac12(t(\beta_1+\alpha_2-\alpha_1)(\beta_1+\alpha_2-\alpha_1-1)+\beta_1(\beta_1-3)-(t+1)\alpha_2+\alpha_1(\alpha_1+2t+1))},\\ A^*_{\alpha,\beta}(x) \equiv& \epsilon_N^2 x^{-\frac{2(N-1)}{N}}\frac{\delta_{\alpha_2,-\beta_2}}{N}\omega^{-\frac12(t(\beta_1+\alpha_2-\alpha_1)(\beta_1+\alpha_2-\alpha_1-1)+\beta_1(\beta_1-3)-(t+1)\alpha_2+\alpha_1(\alpha_1+2t+1))},\\ \bar{A}^*_{\alpha,\beta}(x) \equiv& \epsilon_N^{-2} x^{-\frac{2(N-1)}{N}}\delta_{\alpha_2,-\beta_2}\omega^{\frac12(t(\alpha_1-\alpha_2-\beta_1)(\alpha_1-\alpha_2-\beta_1-1)-\alpha_1(\alpha_1-3)-(t+1)\alpha_2-\beta_1(\beta_1+2t+1))}, \end{aligned}$$

and

$$\begin{aligned} B_{\alpha,\beta}(x) \equiv& (1-x)^{\frac{2(N-1)}{N}}\delta_{\alpha,-\beta}\omega^{-\frac12\left((t+1)(\alpha_1-\alpha_2)(\alpha_1-\alpha_2-2)+\alpha_1^2-\alpha_2\right)},\\ \bar{B}_{\alpha,\beta}(x) \equiv& (1-x)^{\frac{2(N-1)}{N}}\delta_{\alpha,-\beta}\omega^{\frac12\left((t+1)(\alpha_1-\alpha_2)(\alpha_1-\alpha_2+2)+\alpha_1^2+\alpha_2\right)},\\ B^*_{\alpha,\beta}(x) \equiv& (1-x)^{-\frac{2(N-1)}{N}}\delta_{\alpha,-\beta}\omega^{-\frac12\left((t+1)(\alpha_1-\alpha_2)(\alpha_1-\alpha_2+2)+\alpha_1^2+\alpha_2\right)},\\ \bar{B}^*_{\alpha,\beta}(x) \equiv& (1-x)^{\frac{-2(N-1)}{N}}\delta_{\alpha,-\beta}\omega^{\frac12\left((t+1)(\alpha_1-\alpha_2)(\alpha_1-\alpha_2-2)+\alpha_1^2-\alpha_2\right)}. \end{aligned}$$

3.3. The operators L, R and C. The operators L, R and C are defined as follows

$$L = A^*A, \quad R = B^*B, \quad C = (AB)^3 \in \mathrm{End}(\mathcal{H}). \tag{3.11}$$

The operators L, R and C are clearly invertible and by the equalities (3.8) and (3.10), we easily see that they are grading-preserving. Hence, these operators are also standard.

Moreover, these operators are symmetric. It is clear for L and R. For C, we use ([**5**, Lemma 5]) which states that

$$(ABA)^* = BAB.$$

Hence we have

$$C = (AB)^3 = ABABAB = ABA(ABA)^*.$$

Now we can determine these operators using the functions

$$L_{\alpha,\beta}, \bar{L}_{\alpha,\beta}, R_{\alpha,\beta}, \bar{R}_{\alpha,\beta}, C_{\alpha,\beta}, \bar{C}_{\alpha,\beta} : \mathbb{R}_{\neq 0,1} \to \mathbb{C}$$

satisfying

$$\begin{aligned}
Lh_{p,q}(u_\alpha) &= \sum_{\beta\in\mathbb{Z}_N^2} L_{\alpha,\beta}\left(\tfrac{q}{p+q}\right) h_{p,q}(u_\beta),\\
L\bar{h}_{p,q}(\bar{u}_\alpha) &= \sum_{\beta\in\mathbb{Z}_N^2} \bar{L}_{\alpha,\beta}\left(\tfrac{q}{p+q}\right) \bar{h}_{p,q}(\bar{u}_\beta),\\
Rh_{p,q}(u_\alpha) &= \sum_{\beta\in\mathbb{Z}_N^2} R_{\alpha,\beta}\left(\tfrac{q}{p+q}\right) h_{p,q}(u_\beta),\\
R\bar{h}_{p,q}(\bar{u}_\alpha) &= \sum_{\beta\in\mathbb{Z}_N^2} \bar{R}_{\alpha,\beta}\left(\tfrac{q}{p+q}\right) \bar{h}_{p,q}(\bar{u}_\beta),\\
Ch_{p,q}(u_\alpha) &= \sum_{\beta\in\mathbb{Z}_N^2} C_{\alpha,\beta}\left(\tfrac{q}{p+q}\right) h_{p,q}(u_\beta),\\
C\bar{h}_{p,q}(\bar{u}_\alpha) &= \sum_{\beta\in\mathbb{Z}_N^2} \bar{C}_{\alpha,\beta}\left(\tfrac{q}{p+q}\right) \bar{h}_{p,q}(\bar{u}_\beta),
\end{aligned}$$

for all admissible pairs $(p,q) \in (\mathbb{R}_{\neq 0})^2$.

Proposition 3.5. *For all $x \in \mathbb{R}_{\neq 0,1}$ and all $\alpha,\beta \in \mathbb{Z}_N^2$ we have*

$$\begin{aligned}
L_{\alpha,\beta}(x) \equiv& x^{\frac{4(N-1)}{N}} \delta_{\alpha_1+2t,\beta_1}\delta_{\alpha_2,\beta_2}\omega^{2t\alpha_1+\alpha_2-4t-3},\\
\bar{L}_{\alpha,\beta}(x) \equiv& x^{\frac{4(N-1)}{N}} \delta_{\alpha_1-2t,\beta_1}\delta_{\alpha_2,\beta_2}\omega^{2t\alpha_1+\alpha_2+1},\\
R_{\alpha,\beta}(x) \equiv& (1-x)^{\frac{4(N-1)}{N}} \delta_{\alpha,\beta}\omega^{2(t+1)(\alpha_1-\alpha_2)+\alpha_2},\\
\bar{R}_{\alpha,\beta}(x) \equiv& (1-x)^{\frac{4(N-1)}{N}} \delta_{\alpha,\beta}\omega^{2(t+1)(\alpha_1-\alpha_2)+\alpha_2},\\
C_{\alpha,\beta}(x) \equiv& \delta_{\alpha,\beta}\omega^{-(2t+1)\alpha_2},\\
\bar{C}_{\alpha,\beta}(x) \equiv& \delta_{\alpha,\beta}\omega^{-(2t+1)\alpha_2}.
\end{aligned}$$

Proof. Using Proposition 3.4, a straightforward computation leads to the results. ☐

Remark 3.6. Note that the fact that the operator C is non trivial implies that the category of $\mathcal{A}_{\omega,t}$-modules is non-pivotal.

3.4. The operators $L^{\frac{1}{2}}, R^{\frac{1}{2}}$ and $C^{\frac{1}{2}}$. Now we can fix square roots of the operators L, R and C. These are the key operators mentioned in the beginning of this section that will allow us to extend our Ψ-system into a $\hat{\Psi}$-system.

PROPOSITION 3.7. *The square roots of the operators L, R and C can be chosen to be, respectively, the grading-preserving operators given, for all $x \in \mathbb{R}_{\neq 0,1}$ and all $\alpha, \beta \in \mathbb{Z}_N^2$, by*

$$\begin{aligned}
L^{\frac{1}{2}}_{\alpha,\beta}(x) \equiv& x^{\frac{2(N-1)}{N}} \delta_{\alpha_1+t,\beta_1} \delta_{\alpha_2,\beta_2} \omega^{\frac{1}{2}(2t\alpha_1+\alpha_2-3t-2)},\\
\bar{L}^{\frac{1}{2}}_{\alpha,\beta}(x) \equiv& x^{\frac{2(N-1)}{N}} \delta_{\alpha_1-t,\beta_1} \delta_{\alpha_2,\beta_2} \omega^{\frac{1}{2}(2t\alpha_1+\alpha_2-t)},\\
R^{\frac{1}{2}}_{\alpha,\beta}(x) \equiv& (1-x)^{\frac{2(N-1)}{N}} \delta_{\alpha,\beta} \omega^{(t+1)(\alpha_1-\alpha_2)+\frac{1}{2}\alpha_2},\\
\bar{R}^{\frac{1}{2}}_{\alpha,\beta}(x) \equiv& (1-x)^{\frac{2(N-1)}{N}} \delta_{\alpha,\beta} \omega^{(t+1)(\alpha_1-\alpha_2)+\frac{1}{2}\alpha_2},\\
C^{\frac{1}{2}}_{\alpha,\beta}(x) \equiv& \delta_{\alpha,\beta} \omega^{-(t+\frac{1}{2})\alpha_2},\\
\bar{C}^{\frac{1}{2}}_{\alpha,\beta}(x) \equiv& \delta_{\alpha,\beta} \omega^{-\left(t+\frac{1}{2}\right)\alpha_2}.
\end{aligned}$$

PROOF. Let us write for any $L = L_0 L_1$ as a product of commuting operators L_0 and L_1 such that, for any admissible pair $(p, q) \in (\mathbb{R}_{\neq 0})^2$ we have

$$L_0|_{\mathcal{H}_{p,q} \oplus \bar{\mathcal{H}}_{p,q}} \equiv \left(\tfrac{q}{p+q}\right)^{\frac{4(N-1)}{N}} \quad \text{and} \quad L_1^N|_{\mathcal{H}} = \mathrm{Id}_{\mathcal{H}}\,.$$

Since N is odd, we set $L^{\frac{1}{2}} = L_0^{\frac{1}{2}} L_1^{\frac{N+1}{2}}$. We do similarly for $R^{\frac{1}{2}}$ and $C^{\frac{1}{2}}$. □

REMARK 3.8. In [**5**, equation (35)] the square root of L is defined by

$$L^{\frac{1}{2}} = BAR^{-\frac{1}{2}}AB \tag{3.12}$$

and it is shown that it implies that $(L^{\frac{1}{2}})^2 = L$. Although $L^{\frac{1}{2}}$ has not been defined this way in our case, a straightforward computation shows that equality (3.12) holds true.

The next Proposition shows that the operator

$$\mathfrak{q} = R^{\frac{1}{2}} B L^{\frac{1}{2}} B L^{-\frac{1}{2}} C^{-\frac{1}{2}},$$

acts as a scalar on

$$\check{\mathcal{H}} = \bigoplus_{\substack{(p,q)\in(\mathbb{R}_{\neq 0})^2\\ \text{admissible}}} \mathcal{H}_{p,q} \quad \text{and} \quad \hat{\mathcal{H}} = \bigoplus_{\substack{(p,q)\in(\mathbb{R}_{\neq 0})^2\\ \text{admissible}}} \bar{\mathcal{H}}_{p,q}.$$

This property is central to define our invariant.

PROPOSITION 3.9. *For all $x \in \mathbb{R}_{\neq 0,1}$ and all $\alpha \in \mathbb{Z}_N^2$, the operator*

$$\mathfrak{q} = R^{\frac{1}{2}} B L^{\frac{1}{2}} B L^{-\frac{1}{2}} C^{-\frac{1}{2}}$$

is given by

$$\mathfrak{q}e_\alpha(x) = \omega^{\frac{1}{2}} e_\alpha(x) \quad \text{and} \quad \mathfrak{q}\bar{e}_\alpha(x) = \omega^{-\frac{1}{2}} \bar{e}_\alpha(x).$$

PROOF. A straightforward computation leads to the results. □

4. A $\hat{\Psi}$-system in the category of $\mathcal{A}_{\omega,t}$-modules

Now that we have shown that there is a Ψ-system in the category of $\mathcal{A}_{\omega,t}$-modules, we can construct its associated $6j$-symbols. As we have already said, the operator $S(x)$ will have an essential role in this construction.

With the square roots $L^{\frac{1}{2}}, R^{\frac{1}{2}}$ and $C^{\frac{1}{2}}$ chosen in the previous section, we are going to show that the Ψ-system defined in Theorem 2.16 extends to a $\hat{\Psi}$-system.

4.1. The $6j$-symbols. For any $x \in \mathbb{R}_{\neq 0,1}$, we define the algebra morphism

$$\Delta_x : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$$

by

$$\Delta_x(a) = S(x)(a \otimes \mathrm{Id}_{\mathcal{V}})S(x)^{-1}, \tag{4.1}$$

We also consider the following function

$$\begin{array}{rccc} * : & (\mathbb{R}_{\neq 0,1})^2 & \longrightarrow & \mathbb{R}_{\neq 0,1} \\ & (x,y) & \longmapsto & \dfrac{y - xy}{1 - xy} \end{array}$$

This function is well defined only for pairs $(x, y) \in (\mathbb{R}_{\neq 0,1})^2$ such that $x \neq y^{-1}$.

DEFINITION 4.1. We say that a pair $(x, y) \in (\mathbb{R}_{\neq 0,1})^2$ is *compatible* if $x \neq y^{-1}$.

LEMMA 4.2. *For all compatible pairs $(x, y) \in (\mathbb{R}_{\neq 0,1})^2$, we have*

$$(\Delta_{x*y} \otimes \mathrm{Id}_{\mathcal{V}})\, \Delta_{xy} = (\mathrm{Id}_{\mathcal{V}} \otimes \Delta_x)\, \Delta_y \tag{4.2}$$

PROOF. If $(x, y) \in (\mathbb{R}_{\neq 0,1})^2$ is a compatible pair, then Δ_{xy} and Δ_{x*y} are well defined functions. For any $x \in \mathbb{R}_{\neq 0,1}$, Proposition 2.8 implies the following equalities

$$\begin{aligned} &\Delta_x(X) = X_1 X_2, \quad \Delta_x(Y) = (1-x)^{\frac{1}{N}} Y_1 + x^{\frac{1}{N}} X_1 Y_2 \\ &\Delta_x(U) = (1-x)^{\frac{2}{N}} U_1 + x^{\frac{2}{N}} X_1^{t+1} U_2 + (x - x^2)^{\frac{1}{N}} X_1^t Y_1 Z_2, \\ &\Delta_x(V) = (1-x)^{\frac{2}{N}} V_1 + x^{\frac{2}{N}} X_1^{t+1} V_2 + (x - x^2)^{\frac{1}{N}} Z_1 X_1 Y_2, \end{aligned}$$

where $Z = (U + V)Y^{-1}$. Using these equalities, a straightforward computation shows that equation (4.2) holds true for $a \in \{X, Y, U, V\}$. This ends the proof since $\{X, Y, U, V\}$ is a generating set of $\mathcal{A}$.

□

Let $(x, y) \in (\mathbb{R}_{\neq 0,1})^2$ be compatible. Using (4.1), we have for all $a \in \mathcal{A}$

$$(\Delta_{x*y} \otimes \mathrm{Id}_{\mathcal{V}})\, \Delta_{xy}(a) = S_{12}(x*y)S_{13}(xy)(a \otimes \mathrm{Id}_{\mathcal{V}} \otimes \mathrm{Id}_{\mathcal{V}})\left(S_{12}(x*y)S_{13}(xy)\right)^{-1}$$

and

$$(\mathrm{Id}_{\mathcal{V}} \otimes \Delta_x)\, \Delta_y(a) = S_{23}(x)S_{12}(y)(a \otimes \mathrm{Id}_{\mathcal{V}} \otimes \mathrm{Id}_{\mathcal{V}})\left(S_{23}(x)S_{12}(y)\right)^{-1}.$$

Therefore, by Lemma 4.2, the following equality holds in $\mathcal{A}^{\otimes 3}$ for all $a \in \mathcal{A}$

$$\begin{aligned}(S_{23}(x)S_{12}(y))^{-1}\, &S_{12}(x*y)S_{13}(xy)(a \otimes \mathrm{Id}_{\mathcal{V}} \otimes \mathrm{Id}_{\mathcal{V}}) \\ &= (a \otimes \mathrm{Id}_{\mathcal{V}} \otimes \mathrm{Id}_{\mathcal{V}})\,(S_{23}(x)S_{12}(y))^{-1}\, S_{12}(x*y)S_{13}(xy)\end{aligned}$$

Since the center of $\mathcal{A}$ is trivial, the former equality implies the existence of an element $T(x, y) \in \mathcal{A}^{\otimes 2}$ such that

$$(S_{23}(x)S_{12}(y))^{-1}\, S_{12}(x*y)S_{13}(xy)T_{23}(x,y) = \mathrm{Id}_{\mathcal{V}} \otimes \mathrm{Id}_{\mathcal{V}} \otimes \mathrm{Id}_{\mathcal{V}}\,.$$

Hence we have

$$S_{23}(x)S_{12}(y) = S_{12}(x*y)S_{13}(xy)T_{23}(x,y). \tag{4.3}$$

DEFINITION 4.3. The operator $T(x, y) \in \mathcal{A}^{\otimes 2}$ and its inverse are called $6j$-*symbols.*

REMARK 4.4. The operators $T(x, y)$ and $T(x, y)^{-1}$ correspond to the $6j$-symbols (positive and negative respectively) defined in [**5**, p.13] in the following way : for $p, q, r \in \mathbb{R}_{\neq 0}$ such that $x = \frac{r}{q+r}$ and $y = \frac{q+r}{p+q+r}$, we have

$$(\bar{v} \otimes \bar{u})T(x,y)(v \otimes u) = \begin{Bmatrix} p & q & p+q \\ r & p+q+r & q+r \end{Bmatrix} (\bar{u} \otimes \bar{v} \otimes u \otimes v) \in \mathbb{C}$$

where

$$\bar{u} \otimes \bar{v} \otimes u \otimes v \in \bar{\mathcal{H}}_{p+q,r} \otimes \bar{\mathcal{H}}_{p,q} \otimes \mathcal{H}_{q,r} \otimes \mathcal{H}_{p,q+r}$$

and

$$(\bar{u}' \otimes \bar{v}')T(x,y)^{-1}(u' \otimes v') = \begin{Bmatrix} p & q & p+q \\ r & p+q+r & q+r \end{Bmatrix}^{-} (\bar{u}' \otimes \bar{v}' \otimes u' \otimes v') \in \mathbb{C}$$

where

$$\bar{u}' \otimes \bar{v}' \otimes u' \otimes v' \in \bar{\mathcal{H}}_{p,q+r} \otimes \bar{\mathcal{H}}_{q,r} \otimes \mathcal{H}_{p,q} \otimes \mathcal{H}_{p+q,r}.$$

Thus, $T(x, y)$ and $T(x, y)^{-1}$ are interpreted as elements of $\mathrm{End}(\mathcal{H}^{\otimes 2})$, or more precisely,

$$\begin{aligned} T(x,y) &: \mathcal{H}_{p,q+r} \otimes \mathcal{H}_{q,r} \longrightarrow \mathcal{H}_{p,q} \otimes \mathcal{H}_{p+q,r}, \\ T(x,y)^{-1} &: \mathcal{H}_{p,q} \otimes \mathcal{H}_{p+q,r} \longrightarrow \mathcal{H}_{p,q+r} \otimes \mathcal{H}_{q,r}. \end{aligned}$$

4.2. Determination of the $6j$-symbols. The following Theorem is the key to determine the $6j$-symbols $T(x,y)$. For the sake of simplicity, our statement is adapted to our context. It is therefore slightly different from the original one.

THEOREM 4.5 ([**4**], p.5). *Let $(x,y)\in(\mathbb{R}_{\neq 0,1})^2$ be compatible and U,V be two operators such that $\mathsf{UV}=\omega^{-1}\mathsf{VU}$ and $\mathsf{U}^N=\mathsf{V}^N=-1$, then*

$$\Psi_x(\mathsf{U})\Psi_y(\mathsf{V})=\Psi_{x*y}(\mathsf{V})\Psi_{xy}(-\mathsf{VU})\Psi_{y*x}(\mathsf{U}).$$

LEMMA 4.6. *Let $\mathsf{U},\mathsf{V}\in\mathcal{A}$ be such that $\mathsf{U}^N=\mathsf{V}^N=1$ and $\mathsf{UV}=\omega\mathsf{VU}$. Then*

$$L_{23}(\mathsf{U},\mathsf{V})L_{12}(\mathsf{U},\mathsf{V})=L_{12}(\mathsf{U},\mathsf{V})L_{13}(\mathsf{U},\mathsf{V})L_{23}(\mathsf{U},\mathsf{V}).$$

PROOF. For all $i\in\mathbb{Z}_N$ we have

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}_N}\left(\frac{1}{N}\sum_{j\in\mathbb{Z}_N}\omega^{-ij}\mathsf{V}^j\right)\left(\frac{1}{N}\sum_{l\in\mathbb{Z}_N}\omega^{-kl}\mathsf{V}^l\right)\\
&\quad=\frac{1}{N^2}\sum_{l,k\in\mathbb{Z}_N}\omega^{-kl}\left(\sum_{j\in\mathbb{Z}_N}\omega^{-ij}\mathsf{V}^{j+l}\right)=\frac{1}{N^2}\sum_{l,k\in\mathbb{Z}_N}\omega^{l(i-k)}\left(\sum_{j\in\mathbb{Z}_N}\omega^{-ij}\mathsf{V}^j\right)\\
&\quad=\left(\sum_{j\in\mathbb{Z}_N}\omega^{-ij}\mathsf{V}^j\right)\frac{1}{N^2}\sum_{l,k\in\mathbb{Z}_N}\omega^{l(i-k)}=\left(\sum_{j\in\mathbb{Z}_N}\omega^{-ij}\mathsf{V}^j\right)\frac{1}{N^2}\sum_{k\in\mathbb{Z}_N}N\delta_{i,k}\\
&\quad=\frac{1}{N}\sum_{j\in\mathbb{Z}_N}\omega^{-ij}\mathsf{V}^j.
\end{aligned}$$

Using the identity above and Lemma 2.5, we easily make the following computation

$$\begin{aligned}
L_{23}(\mathsf{U},\mathsf{V})L_{12}(\mathsf{U},\mathsf{V})&=\frac{1}{N}\sum_{i,j\in\mathbb{Z}_N}\omega^{-ij}\mathsf{U}_2^i\mathsf{V}_3^jL_{12}(\mathsf{U},\mathsf{V})\\
&=\frac{1}{N}\sum_{i,j\in\mathbb{Z}_N}\omega^{-ij}L_{12}(\mathsf{U},\omega^i\mathsf{V})\mathsf{U}_2^i\mathsf{V}_3^j=L_{12}(\mathsf{U},\mathsf{V})\frac{1}{N}\sum_{i,j\in\mathbb{Z}_N}\omega^{-ij}\mathsf{U}_1^i\mathsf{U}_2^i\mathsf{V}_3^j\\
&=L_{12}(\mathsf{U},\mathsf{V})\sum_{i\in\mathbb{Z}_N}\mathsf{U}_1^i\mathsf{U}_2^i\sum_{k\in\mathbb{Z}_N}\left(\frac{1}{N}\sum_{j\in\mathbb{Z}_N}\omega^{-ij}\mathsf{V}_3^j\right)\left(\frac{1}{N}\sum_{l\in\mathbb{Z}_N}\omega^{-kl}\mathsf{V}_3^l\right)\\
&=L_{12}(\mathsf{U},\mathsf{V})\sum_{i,k\in\mathbb{Z}_N}\mathsf{U}_1^i\mathsf{U}_2^k\left(\frac{1}{N}\sum_{j\in\mathbb{Z}_N}\omega^{-ij}\mathsf{V}_3^j\right)\left(\frac{1}{N}\sum_{l\in\mathbb{Z}_N}\omega^{-kl}\mathsf{V}_3^l\right)\\
&=L_{12}(\mathsf{U},\mathsf{V})\left(\frac{1}{N}\sum_{i,j\in\mathbb{Z}_N}\omega^{-ij}\mathsf{U}_1^i\mathsf{V}_3^j\right)\left(\frac{1}{N}\sum_{k,l\in\mathbb{Z}_N}\omega^{-kl}\mathsf{U}_2^k\mathsf{V}_3^l\right)\\
&=L_{12}(\mathsf{U},\mathsf{V})L_{13}(\mathsf{U},\mathsf{V})L_{23}(\mathsf{U},\mathsf{V}).
\end{aligned}$$

□

Now we show that $T(x,y)=S(y*x)$ is a solution of the equation (4.3).

REMARK 4.7. Setting $z = y * x$ and using Remark 4.4, we have $z = \frac{pr}{(p+q)(q+r)}$,

$$(\bar{v} \otimes \bar{u})S(z)(v \otimes u) = \begin{Bmatrix} p & q & p+q \\ r & p+q+r & q+r \end{Bmatrix} (\bar{u} \otimes \bar{v} \otimes u \otimes v) \in \mathbb{C}$$

where

$$\bar{u} \otimes \bar{v} \otimes u \otimes v \in \bar{\mathcal{H}}_{p+q,r} \otimes \bar{\mathcal{H}}_{p,q} \otimes \mathcal{H}_{q,r} \otimes \mathcal{H}_{p,q+r}$$

and

$$(\bar{u}' \otimes \bar{v}')S(z)^{-1}(u' \otimes v') = \begin{Bmatrix} p & q & p+q \\ r & p+q+r & q+r \end{Bmatrix}^{-} (\bar{u}' \otimes \bar{v}' \otimes u' \otimes v') \in \mathbb{C}$$

where

$$\bar{u}' \otimes \bar{v}' \otimes u' \otimes v' \in \bar{\mathcal{H}}_{p,q+r} \otimes \bar{\mathcal{H}}_{q,r} \otimes \mathcal{H}_{p,q} \otimes \mathcal{H}_{p+q,r}.$$

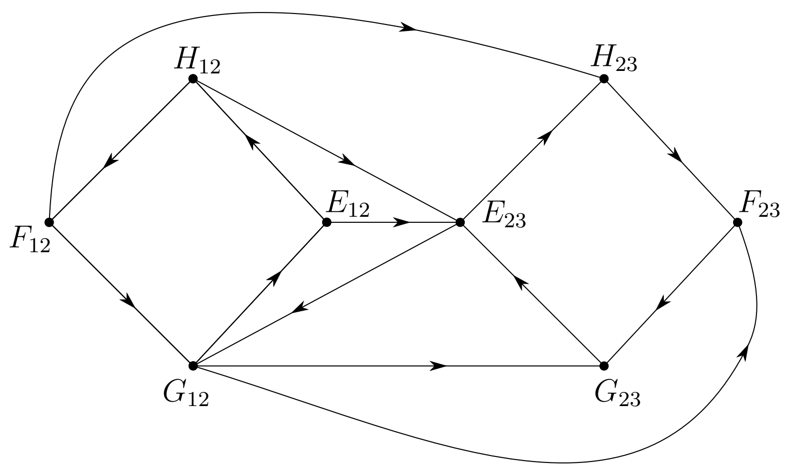
Thus, $S(z)$ and $S(z)^{-1}$ are interpreted as operators with the following source and target spaces

$$S(z) : \mathcal{H}_{p,q+r} \otimes \mathcal{H}_{q,r} \longrightarrow \mathcal{H}_{p,q} \otimes \mathcal{H}_{p+q,r}, \tag{4.4}$$

$$S(z)^{-1} : \mathcal{H}_{p,q} \otimes \mathcal{H}_{p+q,r} \longrightarrow \mathcal{H}_{p,q+r} \otimes \mathcal{H}_{q,r}. \tag{4.5}$$

THEOREM 4.8. $S_{23}(x)S_{12}(y) = S_{12}(x * y)S_{13}(xy)S_{23}(y * x)$.

PROOF. The following commutation relations hold true



Using Lemma 2.5 and Lemma 4.6 for $\mathsf{U} = U^t V$, we see that the proposition is equivalent to the following equality

$$\begin{aligned} &\Psi_x(E_{23})\Psi_x(F_{23})\Psi_x(G_{23})\Psi_x(H_{23})\Psi_y(E_{12})\Psi_y(F_{12})\Psi_y(G_{12})\Psi_y(H_{12}) \\ &\quad = \Psi_{x*y}(E_{12})\Psi_{x*y}(F_{12})\Psi_{x*y}(G_{12})\Psi_{x*y}(H_{12})\Psi_{xy}(X_2E_{13})\Psi_{xy}(X_2^tF_{13}) \\ &\quad \times \Psi_{xy}(X_2^tG_{13})\Psi_{xy}(X_2H_{13})\Psi_{y*x}(E_{23})\Psi_{y*x}(F_{23})\Psi_{y*x}(G_{23})\Psi_{y*x}(H_{23}). \end{aligned} \tag{4.6}$$

Since

$$X_2E_{13} = -\,E_{12}E_{23}, \qquad X_2^tF_{13} = -\,G_{12}F_{23},$$
$$X_2^tG_{13} = -\,G_{12}G_{23}, \qquad X_2H_{13} = -\,H_{12}E_{23},$$

the former equality is equivalent to

$$\begin{aligned}
&\Psi_x(E_{23})\Psi_x(F_{23})\Psi_x(G_{23})\Psi_x(H_{23})\Psi_y(E_{12})\Psi_y(F_{12})\Psi_y(G_{12})\Psi_y(H_{12})\\
&\quad = \Psi_{x*y}(E_{12})\Psi_{x*y}(F_{12})\Psi_{x*y}(G_{12})\Psi_{x*y}(H_{12})\Psi_{xy}(-E_{12}E_{23})\Psi_{xy}(-G_{12}F_{23})\\
&\quad\ \times \Psi_{xy}(-G_{12}G_{23})\Psi_{xy}(-H_{12}E_{23})\Psi_{y*x}(E_{23})\Psi_{y*x}(F_{23})\Psi_{y*x}(G_{23})\Psi_{y*x}(H_{23}).
\end{aligned}$$

On the left hand side, we can apply Theorem 4.5 to E_{12} and E_{23}. Indeed, E_{12} commutes with F_{23}, G_{23} and H_{23} and $E_{23}E_{12} = \omega^{-1}E_{12}E_{23}$. This gives

$$\begin{aligned}
&\Psi_x(E_{23})\Psi_x(F_{23})\Psi_x(G_{23})\Psi_x(H_{23})\Psi_y(E_{12})\Psi_y(F_{12})\Psi_y(G_{12})\Psi_y(H_{12})\\
&\quad = \Psi_{x*y}(E_{12})\Psi_{xy}(-E_{12}E_{23})\Psi_{y*x}(E_{23})\Psi_x(F_{23})\Psi_x(G_{23})\Psi_x(H_{23})\Psi_y(F_{12})\\
&\qquad\qquad \times \Psi_y(G_{12})\Psi_y(H_{12}).
\end{aligned}$$

On the right hand side, since $E_{12}E_{23}$ commutes with F_{12}, G_{12} and H_{12}, we have

$$\begin{aligned}
&\Psi_{x*y}(E_{12})\Psi_{x*y}(F_{12})\Psi_{x*y}(G_{12})\Psi_{x*y}(H_{12})\Psi_{xy}(-E_{12}E_{23})\Psi_{xy}(-G_{12}F_{23})\\
&\quad \times \Psi_{xy}(-G_{12}G_{23})\Psi_{xy}(-H_{12}E_{23})\Psi_{y*x}(E_{23})\Psi_{y*x}(F_{23})\Psi_{y*x}(G_{23})\Psi_{y*x}(H_{23})\\
&\quad = \Psi_{x*y}(E_{12})\Psi_{xy}(-E_{12}E_{23})\Psi_{x*y}(F_{12})\Psi_{x*y}(G_{12})\Psi_{x*y}(H_{12})\Psi_{xy}(-G_{12}F_{23})\\
&\qquad \times \Psi_{xy}(-G_{12}G_{23})\Psi_{xy}(-H_{12}E_{23})\Psi_{y*x}(E_{23})\Psi_{y*x}(F_{23})\Psi_{y*x}(G_{23})\Psi_{y*x}(H_{23})
\end{aligned}$$

Hence, (4.6) is equivalent to

$$\begin{aligned}
&\Psi_{y*x}(E_{23})\Psi_x(F_{23})\Psi_x(G_{23})\Psi_x(H_{23})\Psi_y(F_{12})\Psi_y(G_{12})\Psi_y(H_{12})\\
&\quad = \Psi_{x*y}(F_{12})\Psi_{x*y}(G_{12})\Psi_{x*y}(H_{12})\Psi_{xy}(-G_{12}F_{23})\Psi_{xy}(-G_{12}G_{23})\\
&\qquad \times \Psi_{xy}(-H_{12}E_{23})\Psi_{y*x}(E_{23})\Psi_{y*x}(F_{23})\Psi_{y*x}(G_{23})\Psi_{y*x}(H_{23})
\end{aligned} \tag{4.7}$$

On the left hand side of (4.7), we have

$$\begin{aligned}
&\Psi_{y*x}(E_{23})\Psi_x(F_{23})\Psi_x(G_{23})\underline{\Psi_x(H_{23})\Psi_y(F_{12})}\Psi_y(G_{12})\Psi_y(H_{12})\\
=&\Psi_{y*x}(E_{23})\Psi_x(F_{23})\Psi_x(G_{23})\Psi_{x*y}(F_{12})\underline{\Psi_{xy}(-F_{12}H_{23})}\Psi_{y*x}(H_{23})\Psi_y(G_{12})\Psi_y(H_{12})\\
=&\Psi_{y*x}(E_{23})\Psi_x(F_{23})\Psi_x(G_{23})\underline{\Psi_{x*y}(F_{12})}\Psi_{xy}(-E_{23}G_{12})\underline{\Psi_{y*x}(H_{23})}\Psi_y(G_{12})\Psi_y(H_{12})\\
=&\Psi_{x*y}(F_{12})\Psi_{y*x}(E_{23})\Psi_x(F_{23})\underline{\Psi_x(G_{23})\Psi_{xy}(-E_{23}G_{12})\Psi_y(G_{12})}\Psi_y(H_{12})\Psi_{y*x}(H_{23})\\
=&\Psi_{x*y}(F_{12})\underline{\Psi_{y*x}(E_{23})\Psi_x(F_{23})\Psi_{xy}(-E_{23}G_{12})\Psi_{x*y}(G_{12})}\Psi_{xy}(-G_{12}G_{23})\\
&\qquad\qquad \times \Psi_{y*x}(G_{23})\Psi_y(H_{12})\Psi_{y*x}(H_{23})\\
=&\Psi_{x*y}(F_{12})\Psi_x(F_{23})\Psi_y(G_{12})\Psi_x(E_{23})\Psi_{xy}(-G_{12}G_{23})\underline{\Psi_{y*x}(G_{23})\Psi_y(H_{12})}\Psi_{y*x}(H_{23})\\
=&\Psi_{x*y}(F_{12})\Psi_x(F_{23})\Psi_y(G_{12})\Psi_x(E_{23})\Psi_{xy}(-G_{12}G_{23})\Psi_y(H_{12})\Psi_{y*x}(G_{23})\Psi_{y*x}(H_{23})
\end{aligned}$$

where we successively

(1) applied theorem 4.5 to H_{23} and F_{12},

(2) used the equality $F_{12}H_{23} = E_{23}G_{12}$,
(3) used the fact that F_{12} commutes with E_{23}, F_{23} and G_{23}, and the fact that H_{23} commutes with G_{12} and H_{12},
(4) used the fact that G_{23} commutes with $E_{23}G_{12}$, and applied Theorem 4.5 to G_{23} and G_{12},
(5) used the fact that E_{23} commutes with F_{23}, and applied Theorem 4.5 to E_{23} and G_{12},
(6) used the fact that G_{23} commutes with H_{12}.

On the right hand side of (4.7), we have

$$\begin{aligned}
&\Psi_{x*y}(F_{12})\Psi_{x*y}(G_{12})\underline{\Psi_{x*y}(H_{12})\Psi_{xy}(-G_{12}F_{23})\Psi_{xy}(-G_{12}G_{23})}\\
&\qquad\qquad \times \underline{\Psi_{xy}(-H_{12}E_{23})\Psi_{y*x}(E_{23})}\Psi_{y*x}(F_{23})\Psi_{y*x}(G_{23})\Psi_{y*x}(H_{23})\\
=&\Psi_{x*y}(F_{12})\underline{\Psi_{x*y}(G_{12})\Psi_{xy}(-G_{12}F_{23})\Psi_{xy}(-G_{12}G_{23})}\\
&\qquad\qquad \times \underline{\Psi_{x}(E_{23})\Psi_{y}(H_{12})\Psi_{y*x}(F_{23})}\Psi_{y*x}(G_{23})\Psi_{y*x}(H_{23})\\
=&\Psi_{x*y}(F_{12})\Psi_{x}(F_{23})\Psi_{y}(G_{12})\underline{\Psi_{xy}(-G_{12}G_{23})\Psi_{x}(E_{23})}\Psi_{y}(H_{12})\Psi_{y*x}(G_{23})\Psi_{y*x}(H_{23})\\
=&\Psi_{x*y}(F_{12})\Psi_{x}(F_{23})\Psi_{y}(G_{12})\Psi_{x}(E_{23})\Psi_{xy}(-G_{12}G_{23})\Psi_{y}(H_{12})\Psi_{y*x}(G_{23})\Psi_{y*x}(H_{23}).
\end{aligned}$$

where we successively

(1) used the fact that H_{12} commutes with $G_{12}F_{23}$ and $G_{12}G_{23}$, and applied Theorem 4.5 to H_{12} and E_{23},
(2) used the fact that F_{23} commutes with H_{12} E_{23} and $G_{12}G_{23}$, and applied Theorem 4.5 to G_{12} and F_{23},
(3) used the fact that E_{23} commutes with $G_{12}G_{23}$.

□

4.3. The $\hat{\Psi}$-system. Let $S(z)$ be as described in Remark 4.7. The Ψ-system of Theorem 2.16 extends to a $\hat{\Psi}$-system if the following equations holds true

$$C_1^{\frac{1}{2}}C_2^{\frac{1}{2}}S(z) = S(z)C_1^{\frac{1}{2}}C_2^{\frac{1}{2}} \tag{4.8}$$

$$L_1^{\frac{1}{2}}R_2^{\frac{1}{2}}S(z) = S(z)L_1^{\frac{1}{2}}R_2^{\frac{1}{2}} \tag{4.9}$$

$$R_1^{\frac{1}{2}}R_2^{\frac{1}{2}}S(z) = S(z)R_1^{\frac{1}{2}}C_2^{\frac{1}{2}} \tag{4.10}$$

$$L_2^{\frac{1}{2}}S(z) = C_1^{\frac{1}{2}}S(z)L_1^{\frac{1}{2}}L_2^{\frac{1}{2}} \tag{4.11}$$

THEOREM 4.9. *The Ψ-system of Theorem 2.16 extends to a $\hat{\Psi}$-system with $L^{\frac{1}{2}}, R^{\frac{1}{2}}$ and $C^{\frac{1}{2}}$ given in Proposition 3.7.*

PROOF. By defintion of a Ψ-system, these equations hold true without the square roots (see equations (33a), (33b), (33c) and (33d) of [**5**]). Therefore, it is easy to see that they also hold true for $L_1^{\frac{1}{2}}, R_1^{\frac{1}{2}}$ and $C_1^{\frac{1}{2}}$. □

5. The charged 6j-symbols and the proof of Theorem 3.1

5.1. Determination of the charged 6j-symbols.

DEFINITION 5.1. For any $z \in \mathbb{R}_{\neq 0,1}$ and $a, c \in \frac{1}{2}\mathbb{Z}$ we define the *charged 6j-symbols* by

$$S(z|a,c) = \mathfrak{q}_2^{-4ac} R_2^c R_1^{-a} S(z) L_2^{-a} R_2^{-c} \tag{5.1}$$

$$S(z|a,c)^{-1} = \mathfrak{q}_1^{4ac} R_2^{-c} L_2^{-a} S(z)^{-1} R_1^{-a} R_2^c \tag{5.2}$$

Explicitly we have

$$\left\langle \bar{u}_\mu \otimes \bar{u}_\nu \middle| S(z|a,c) \middle| u_\alpha \otimes u_\beta \right\rangle = \left\langle \bar{u}_\mu \otimes \bar{u}_\nu \middle| S(z) \middle| u_\alpha \otimes u_{(\beta_1 - 2ta, \beta_2)} \right\rangle \left(z^a (1-z)^c\right)^{\frac{4(1-N)}{N}}$$
$$\times\, \omega^{a(2c-2t\beta_1-\beta_2)+2(t+1)\{a(a+\mu_1)+c(\beta_1-\nu_1)\}+(2t+1)\{a(\mu_2+1)+c(\beta_2-\nu_2)\}} \tag{5.3}$$

$$\left\langle \bar{u}_\mu \otimes \bar{u}_\nu \middle| S(z|a,c)^{-1} \middle| u_\alpha \otimes u_\beta \right\rangle = \left\langle \bar{u}_\mu \otimes \bar{u}_{(\nu_1+2ta,\nu_2)} \middle| S(z)^{-1} \middle| u_\alpha \otimes u_\beta \right\rangle \left(z^a (1-z)^c\right)^{\frac{4(1-N)}{N}}$$
$$\times\, \omega^{a(2c-2t\nu_1-\nu_2)+2(t+1)\{a(a-\alpha_1)+c(\beta_1-\nu_1)\}+(2t+1)\{a(\alpha_2+1)+c(\nu_2-\beta_2)\}} \tag{5.4}$$

where $\langle \bar{u}_\mu \otimes \bar{u}_\nu | S(z) | u_\alpha \otimes u_\beta \rangle$ and $\langle \bar{u}_\mu \otimes \bar{u}_\nu | S(z)^{-1} | u_\alpha \otimes u_\beta \rangle$ are given by Proposition 2.11.

5.2. The symmetry relations of the charged 6j-symbols. The symmetry relations of the charged 6j-symbols are expressed with the following symmetric operators $\mathsf{A} = AL^{-\frac{1}{2}}$ and $\mathsf{B} = BR^{-\frac{1}{2}}$ in $\mathrm{End}(\mathcal{H})$. For any $\alpha, \beta \in \mathbb{Z}_N^2$, we consider the functions

$$\mathsf{A}_{\alpha,\beta}, \bar{\mathsf{A}}_{\alpha,\beta}, \mathsf{B}_{\alpha,\beta}, \bar{\mathsf{B}}_{\alpha,\beta} : \mathbb{R}_{\neq 0,1} \to \mathbb{C}$$

satisfying

$$\mathsf{A} h_{p,q}(u_\alpha) = \sum_{\beta \in \mathbb{Z}_N^2} \mathsf{A}_{\alpha,\beta}\left(\tfrac{q}{p+q}\right) h_{p,q}(u_\beta),$$
$$\mathsf{A} \bar{h}_{p,q}(\bar{u}_\alpha) = \sum_{\beta \in \mathbb{Z}_N^2} \bar{\mathsf{A}}_{\alpha,\beta}\left(\tfrac{q}{p+q}\right) \bar{h}_{p,q}(\bar{u}_\beta),$$
$$\mathsf{B} h_{p,q}(u_\alpha) = \sum_{\beta \in \mathbb{Z}_N^2} \mathsf{B}_{\alpha,\beta}\left(\tfrac{q}{p+q}\right) h_{p,q}(u_\beta),$$
$$\mathsf{B} \bar{h}_{p,q}(\bar{u}_\alpha) = \sum_{\beta \in \mathbb{Z}_N^2} \bar{\mathsf{B}}_{\alpha,\beta}\left(\tfrac{q}{p+q}\right) \bar{h}_{p,q}(\bar{u}_\beta),$$

for all admissible pairs $(p,q) \in (\mathbb{R}_{\neq 0})^2$. A straightforward computation, using Lemmas 3.4 and 3.7, leads us to the followings results.

LEMMA 5.2. *The operators* $\mathsf{A}, \mathsf{B} \in \mathrm{End}(\mathcal{H})$ *are symmetric involutions. For all* $\alpha, \beta \in \mathbb{Z}_N^2$, *the numbers*

$$\mathsf{A}_{\alpha,\beta} \equiv \epsilon_N^2 \frac{\delta_{\alpha_2,-\beta_2}}{N} \omega^{-\frac{1}{2}\left(t(\alpha_1-\alpha_2-\beta_1)^2+\alpha_1(\alpha_1+t-1)+\beta_1(\beta_1+t-1)-1\right)}, \tag{5.5}$$

$$\bar{\mathsf{A}}_{\alpha,\beta} \equiv \epsilon_N^{-2} \delta_{\alpha_2,-\beta_2} \omega^{\frac{1}{2}\left(t(\alpha_1-\alpha_2-\beta_1)^2+\alpha_1(\alpha_1+t-1)+\beta_1(\beta_1+t-1)-1\right)}, \tag{5.6}$$

$$\mathsf{B}_{\alpha,\beta} \equiv \delta_{\alpha,-\beta} \omega^{-\frac{1}{2}\left((t+1)(\alpha_1-\alpha_2)^2+\alpha_1^2\right)}, \tag{5.7}$$

$$\bar{\mathsf{B}}_{\alpha,\beta} \equiv \delta_{\alpha,-\beta} \omega^{\frac{1}{2}\left((t+1)(\alpha_1-\alpha_2)^2+\alpha_1^2\right)}, \tag{5.8}$$

satisfy the following equalities for all $x \in \mathbb{R}_{\neq 0,1}$

$$\mathsf{A}_{\alpha,\beta} = \mathsf{A}_{\alpha,\beta}(x), \quad \bar{\mathsf{A}}_{\alpha,\beta} = \bar{\mathsf{A}}_{\alpha,\beta}(x), \quad \mathsf{B}_{\alpha,\beta}(x) = \mathsf{B}_{\alpha,\beta}(x), \quad \bar{\mathsf{B}}_{\alpha,\beta} = \bar{\mathsf{B}}_{\alpha,\beta}(x).$$

Using the previous Lemma, Remark 4.7 and Proposition 3.9, the following Proposition is a direct adaptation of the formulas (50), (51) and (52) of [**5**].

PROPOSITION 5.3 (The symmetry relations). *The charged 6j-symbols verify the following symmetry relations*

$$\left\langle \bar{u}_\alpha \otimes \bar{u}_\nu \middle| S(z|a,c) \middle| u_\mu \otimes u_\beta \right\rangle \equiv \omega^a \sum_{\alpha',\mu' \in \mathbb{Z}_N^2} \mathsf{A}_{\alpha,\alpha'} \bar{\mathsf{A}}_{\mu,\mu'} \left\langle \bar{u}_{\mu'} \otimes \bar{u}_\nu \middle| S\left(\tfrac{z}{z-1}|a,b\right)^{-1} \middle| u_{\alpha'} \otimes u_\beta \right\rangle \tag{5.9}$$

$$\left\langle \bar{u}_\alpha \otimes \bar{u}_\mu \middle| S(z|a,c) \middle| u_\beta \otimes u_\nu \right\rangle \equiv \omega^{-c} \sum_{\alpha',\nu' \in \mathbb{Z}_N^2} \bar{\mathsf{A}}_{\nu,\nu'} \mathsf{B}_{\alpha,\alpha'} \left\langle \bar{u}_\mu \otimes \bar{u}_{\nu'} \middle| S\left(z^{-1}|b,c\right)^{-1} \middle| u_{\alpha'} \otimes u_\beta \right\rangle \tag{5.10}$$

$$\left\langle \bar{u}_\mu \otimes \bar{u}_\beta \middle| S(z|a,c) \middle| u_\alpha \otimes u_\nu \right\rangle \equiv \omega^a \sum_{\beta',\nu' \in \mathbb{Z}_N^2} \mathsf{B}_{\beta,\beta'} \bar{\mathsf{B}}_{\nu,\nu'} \left\langle \bar{u}_\mu \otimes \bar{u}_{\nu'} \middle| S\left(\tfrac{z}{z-1}|a,b\right)^{-1} \middle| u_\alpha \otimes u_{\beta'} \right\rangle \tag{5.11}$$

5.3. Proof of Theorem 3.1.

PROOF OF THEOREM 3.1. The statement of our Theorem is an adaptation of the Theorem 29 of [**5**]. In our case, the former can be expressed as follows.

Suppose that there exists a scalar $\tilde{q} \in \mathbb{C}$ *such that*

$$\mathfrak{q} = \tilde{q}\check{\mathcal{H}} \otimes \tilde{q}^{-1}\hat{\mathcal{H}} \in \mathrm{End}(\mathcal{H}).$$

Then up to multiplication by integer powers of $\tilde{q}$, *the state sum* $\mathsf{K}_N(\mathcal{T}, \mathcal{L}, \Phi, c)$ *depends only on the isotopy class of* L *in* M *and the cohomology classes* $[\Phi] \in H^1(M, \mathbb{R})$ *and* $[c] \in H^1(M, \mathbb{Z}/2\mathbb{Z})$ *(and does not depend on the choice of* Φ *and* c *in their cohomology classes, the* H*-triangulation* $(\mathcal{T}, \mathcal{L})$ *of* (M, L)*, and the ordering of the vertices of* $\mathcal{T}$*).*

By Proposition 3.9 we have

$$\mathfrak{q} = \omega^{\frac{1}{2}}\check{\mathcal{H}} \otimes \omega^{-\frac{1}{2}}\hat{\mathcal{H}} \in \mathrm{End}(\mathcal{H}).$$

This ends the proof since $\omega^{\frac{1}{2}} = \omega^{\frac{N+1}{2}}$ is an integer power of ω. □

CHAPTER 2

Determination of $S(x)$, A and B

In this Chapter we will explicitly determine the operators $S(x)$, A and B. But first, we determine the operator $\bar{\Psi}_x(\mathsf{U})$ by proving the Lemma 2.3.

1. Determination of $\bar{\Psi}_x(\mathsf{U})$

1.1. Proof of formulae (2.4) and (2.8). We start by proving the formulae (2.4) and (2.8)

$$\Psi_x(\mathsf{U})^N = \psi_{x,0}^N (1-x)^{1-N} \frac{D(1)}{D\left((1-x)^{-\frac{1}{N}}\right) D\left(\mathsf{U}\omega\left(\frac{x}{1-x}\right)^{\frac{1}{N}}\right)}$$

$$\bar{\Psi}_x(\mathsf{U})^N = \bar{\psi}_{x,0}^N N^{-N} (-x)^{-\frac{N-1}{2}} \frac{D(1)^3}{D\left((1-x)^{\frac{1}{N}}\right) D\left(\mathsf{U}^{-1}\left(\frac{1-x}{x}\right)^{\frac{1}{N}}\right)}.$$

We recall that for any $x \in \mathbb{C}$ such that $x^N \neq 1$ and any $m \in \{0, \cdots, N-1\} \subset \mathbb{Z}$ we had defined

$$w(x|m) = \prod_{j=1}^{m} \frac{1}{1 - x\omega^j}.$$

Following [**9**] this function can be defined for any $m \in \mathbb{Z}$ by

$$w(x|0) = 1, \quad \frac{w(x|m)}{w(x|m-1)} = \frac{1}{1 - x\omega^m}.$$

Note that $w(x|m)$ is not periodic on the variable m. Nevertheless, the following formula holds true for any $m, n \in \mathbb{Z}_{\geq 0}$

$$w(x|m+n) = w(x|m)w(x\omega^m|n). \tag{1.1}$$

The following Lemma shows another identity for $w(x|m)$.

LEMMA 1.1. *For all $m \in \{0, \ldots, N-1\} \subset \mathbb{Z}$ and all $x \in \mathbb{C}^*$ such that $x^N \neq 1$, we have*

$$w(x|N-m) = \frac{(-x)^m}{1-x^N} \omega^{-\frac{1}{2}m(m-1)} w((x\omega)^{-1}|m)^{-1}.$$

PROOF. Let $m \in \{0, \ldots, N-1\} \subset \mathbb{Z}$. Since $N-m \in \{0, \ldots, N-1\} \subset \mathbb{Z}$, we have

$$w(x|N-m) = \prod_{j=1}^{N-m} \frac{1}{1-x\omega^j}.$$

Then, we compute

$$\begin{aligned} w(x|N-m) &= \prod_{j=1}^{N-m} \frac{1}{1-x\omega^j} = \prod_{j=1}^{N} \frac{1}{1-x\omega^j} \prod_{j=N-m+1}^{N} (1-x\omega^j) \\ &= \frac{1}{1-x^N} \prod_{j=0}^{m-1} (1-x\omega^{-j}) = \frac{(-x)^m}{1-x^N} \omega^{-\frac{1}{2}m(m-1)} \prod_{j=0}^{m-1} (1-x^{-1}\omega^j) \\ &= \frac{(-x)^m}{1-x^N} \omega^{-\frac{1}{2}m(m-1)} \prod_{j=1}^{m} (1-(x\omega)^{-1}\omega^j) = \frac{(-x)^m}{1-x^N} \omega^{-\frac{1}{2}m(m-1)} w((x\omega)^{-1}|m)^{-1}. \end{aligned}$$

□

We also recall that for $x, y, z \in \mathbb{C}$ are such that

$$\left\{x^N, y^N\right\} \subset \mathbb{C} \setminus \{1\} \quad \text{and} \quad z^N = \frac{1-x^N}{1-y^N}. \tag{1.2}$$

we had defined

$$f(x, y|z) = \sum_{m=0}^{N-1} \frac{w(x|m)}{w(y|m)} z^m. \tag{1.3}$$

Note that the conditions (1.2) provide periodicity on the variable m of period N.

The following formula was shown in [**9**, C.7] and will play a key role in the sequel

$$f(x, y|z)^N = (y\omega)^{\frac{N(N-1)}{2}} \frac{D(1)\, D\left(\frac{y\omega}{x}\right) D\left(\frac{x}{yz}\right)}{D\left(\frac{1}{x}\right) D(y\omega)\, D\left(\frac{\omega}{z}\right)} \tag{1.4}$$

where

$$D(x) = \prod_{j=1}^{N-1} (1-x\omega^j)^j.$$

LEMMA 1.2. *For any $x \in \mathbb{R}_{\neq 0,1}$ we have*

$$\frac{D(x\omega)}{D(x)} = \frac{(1-x)^N}{1-x^N} \tag{1.5}$$

$$D(x)D(x^{-1}) = D(1)^2 x^{-\frac{N(N-1)}{2}} \varrho(x)^N \tag{1.6}$$

where

$$\varrho(x) = N^{-1} \frac{1-x^N}{1-x}$$

Proof. Consider $x \in \mathbb{R}_{\neq 0,1}$. The equality (1.5) is proven by the following computation

$$D(x\omega) = \prod_{j=1}^{N-1}(1-x\omega^{j+1})^j = \prod_{j=2}^{N}(1-x\omega^j)^{j-1} = (1-x)^{N-1}\prod_{j=1}^{N-1}(1-x\omega^j)^{j-1}$$
$$\frac{(1-x)^{N-1}}{\prod_{j=1}^{N-1}(1-x\omega^j)}\prod_{j=1}^{N-1}(1-x\omega^j)^j = \frac{(1-x)^{N-1}(1-x)}{1-x^N}D(x) = \frac{(1-x)^N}{1-x^N}D(x).$$

In order to show equality (1.6) we consider a particular evaluation of the function $f(x,y|z)$, namely

$$f\left(x,\tfrac{x}{\omega}|\omega\right).$$

On the one hand, it was shown in [**9**, A.13] that

$$f\left(x,\tfrac{x}{\omega}|\omega\right) = N\frac{1-x^{-1}}{1-x^{-N}}.$$

Hence we have

$$f\left(x,\tfrac{x}{\omega}|\omega\right) = N\frac{1-x^{-1}}{1-x^{-N}} = Nx^{N-1}\frac{1-x}{1-x^N} = x^{N-1}\varrho(x)^{-1}. \tag{1.7}$$

On the other hand, using equality (1.4), we have

$$f\left(x,\tfrac{x}{\omega}|\omega\right)^N = \left(\tfrac{x}{\omega}\omega\right)^{\frac{N(N-1)}{2}}\frac{D\left(1\right)D\left(\frac{\frac{x}{\omega}\omega}{x}\right)D\left(\frac{x}{\frac{x}{\omega}\omega}\right)}{D\left(\frac{1}{x}\right)D\left(\frac{x}{\omega}\omega\right)D\left(\frac{\omega}{\omega}\right)} = x^{\frac{N(N-1)}{2}}\frac{D(1)^2}{D\left(x\right)D\left(x^{-1}\right)} \tag{1.8}$$

Finally, from the two last equalities we get the following one

$$x^{N(N-1)}\varrho(x)^{-N} = x^{\frac{N(N-1)}{2}}\frac{D(1)^2}{D\left(x\right)D\left(x^{-1}\right)} \tag{1.9}$$

wich leads to the result. □

We recall that for any operator $\mathsf{U} \in \mathcal{A}^{\otimes 2}$ such that $\mathsf{U}^N = -\operatorname{Id}_{\mathcal{V}^{\otimes 2}}$ and any $x \in \mathbb{R}_{\neq 0,1}$ such that $x^N \neq 1$ we had defined

$$\Psi_x(\mathsf{U}) = \sum_{m\in\mathbb{Z}_N}\psi_{x,m}(-\mathsf{U})^m \quad\text{and}\quad \bar{\Psi}_x(\mathsf{U}) = \sum_{m\in\mathbb{Z}_N}\bar{\psi}_{x,m}(-\mathsf{U})^m$$

where

$$\psi_{x,m} = \psi_{x,0}x^{\frac{m}{N}}\omega^{\frac{1}{2}m(m+1)}w((1-x)^{\frac{1}{N}}|m)$$

and

$$\bar{\psi}_{x,m} = \bar{\psi}_{x,0}\left(\frac{x}{x-1}\right)^{\frac{m}{N}}\omega^m w((1-x)^{-\frac{1}{N}}|m).$$

Note that $\psi_{x,m}$ and $\bar{\psi}_{x,m}$ are periodic on the variable m of period N.

LEMMA 1.3. *Let* $\mathsf{U} \in \mathcal{A}^{\otimes 2}$ *be an operator such that* $\mathsf{U}^N = -\operatorname{Id}_{\mathcal{V}^{\otimes 2}}$ *and* $x \in \mathbb{R}_{\neq 0,1}$. *Then we have*

$$\Psi_x(\mathsf{U}) = \psi_{x,0} f\left(0, (1-x)^{-\frac{1}{N}}\omega^{-1}\middle|\mathsf{U}^{-1}\left(\tfrac{1-x}{x}\right)^{\frac{1}{N}}\right), \tag{1.10}$$

$$\bar{\Psi}_x(\mathsf{U}) = \bar{\psi}_{x,0} f\left((1-x)^{-\frac{1}{N}}, 0\middle|\mathsf{U}\left(\tfrac{x}{1-x}\right)^{\frac{1}{N}}\omega\right). \tag{1.11}$$

PROOF. We start by proving the equality (1.11). We have

$$\begin{aligned}
\bar{\Psi}_x(\mathsf{U}) = \sum_{m\in\mathbb{Z}_N} \bar{\psi}_{x,m}(-\mathsf{U})^m = \bar{\psi}_{x,0} \sum_{m\in\mathbb{Z}_N} \left(\frac{x}{x-1}\right)^{\frac{m}{N}} \omega^m w((1-x)^{-\frac{1}{N}}|m)(-\mathsf{U})^m \\
= \bar{\psi}_{x,0} \sum_{m\in\mathbb{Z}_N} w((1-x)^{-\frac{1}{N}}|m)\left(\mathsf{U}\left(\frac{x}{1-x}\right)^{\frac{1}{N}}\omega\right)^m \\
= \bar{\psi}_{x,0} f\left((1-x)^{-\frac{1}{N}}, 0\middle|\mathsf{U}\left(\tfrac{x}{1-x}\right)^{\frac{1}{N}}\omega\right)
\end{aligned}$$

where the last equality is justified by the fact that we have

$$\left(\mathsf{U}\left(\frac{x}{1-x}\right)^{\frac{1}{N}}\omega\right)^N = \left(1-(1-x)^{-\frac{1}{N}}\right)^N \operatorname{Id}_{\mathcal{V}^{\otimes 2}}.$$

For the equality (1.10), we start by computing, for any $m \in \{0, \dots, N-1\} \subset \mathbb{Z}$

$$\begin{aligned}
\psi_{x,m} &= \psi_{x,0} x^{\frac{m}{N}} \omega^{\frac{1}{2}m(m+1)} w((1-x)^{\frac{1}{N}}|m) \\
&= \psi_{x,0} x^{\frac{m}{N}} \omega^{\frac{1}{2}m(m+1)} \frac{\left(-(1-x)^{-\frac{1}{N}}\omega^{-1}\right)^m}{1-(1-x)^{-\frac{N}{N}}} \omega^{-\frac{1}{2}m(m-1)} w((1-x)^{-\frac{1}{N}}\omega^{-1}|N-m)^{-1} \\
&= \psi_{x,0}\left(\frac{x}{x-1}\right)^{\frac{m}{N}}\left(\frac{x-1}{x}\right)^{\frac{N}{N}} w((1-x)^{-\frac{1}{N}}\omega^{-1}|N-m)^{-1} \\
&= \psi_{x,0}\left(\frac{x-1}{x}\right)^{\frac{N-m}{N}} w((1-x)^{-\frac{1}{N}}\omega^{-1}|N-m)^{-1}.
\end{aligned}$$

Hence we have

$$\begin{aligned}\Psi_x(\mathsf{U}) &= \sum_{m\in\mathbb{Z}_N} \psi_{x,m}(-\mathsf{U})^m \\ &= \psi_{x,0}\sum_{m\in\mathbb{Z}_N}\left(\frac{x-1}{x}\right)^{\frac{N-m}{N}} w((1-x)^{-\frac{1}{N}}\omega^{-1}|N-m)^{-1}(-\mathsf{U}^{-1})^{N-m} \\ &= \psi_{x,0}\sum_{m\in\mathbb{Z}_N} w((1-x)^{-\frac{1}{N}}\omega^{-1}|N-m)^{-1}\left(-\mathsf{U}^{-1}\left(\frac{x-1}{x}\right)^{\frac{1}{N}}\right)^{N-m} \\ &= \psi_{x,0}\sum_{m\in\mathbb{Z}_N}\frac{1}{w((1-x)^{-\frac{1}{N}}\omega^{-1}|N-m)}\left(\mathsf{U}^{-1}\left(\frac{1-x}{x}\right)^{\frac{1}{N}}\right)^{N-m} \\ &= \psi_{x,0} f\left(0,(1-x)^{-\frac{1}{N}}\omega^{-1}\middle|\mathsf{U}^{-1}\left(\tfrac{1-x}{x}\right)^{\frac{1}{N}}\right)\end{aligned}$$

where the last equality is justified by the fact that we have

$$\left(\mathsf{U}^{-1}\left(\frac{1-x}{x}\right)^{\frac{1}{N}}\right)^N = \frac{1}{1-\left((1-x)^{-\frac{1}{N}}\omega^{-1}\right)^N}\,\mathrm{Id}_{\mathcal{V}^{\otimes 2}}\,.$$

□

Lemma 1.4. *We have*

$$f(0,y|z)^N = y^{N(N-1)}\frac{D(1)}{D(y\omega)D\left(\frac{\omega}{z}\right)} \tag{1.12}$$

$$f(x,0|z)^N = N^{-N}\left(\frac{x}{z}\right)^{\frac{N(N-1)}{2}}\frac{D\left(1\right)^3}{D\left(\frac{1}{x}\right)D\left(\frac{\omega}{z}\right)}. \tag{1.13}$$

Proof. Consider $x,y,z\in\mathbb{C}$ such that

$$\left\{x^N,y^N\right\}\subset\mathbb{C}\setminus\{1\}\quad\text{and}\quad z^N=\frac{1-x^N}{1-y^N}.$$

We start by showing equality (1.13). We are going to use equality (1.6) of Lemma 1.2 to transform equality (1.4). Indeed, we have

$$D\left(\frac{x}{yz}\right) = \left(\frac{yz}{x}\right)^{-\frac{N(N-1)}{2}}\frac{D(1)^2\varrho\left(\frac{yz}{x}\right)^N}{D\left(\frac{yz}{x}\right)}$$

hence we have

$$f(x,y|z)^N = (y\omega)^{\frac{N(N-1)}{2}} \frac{D(1)\, D\left(\frac{y\omega}{x}\right) D\left(\frac{x}{yz}\right)}{D\left(\frac{1}{x}\right) D(y\omega)\, D\left(\frac{\omega}{z}\right)}$$

$$= \left(\frac{x}{z}\right)^{\frac{N(N-1)}{2}} \frac{D(1)^3\, D\left(\frac{y\omega}{x}\right) \varrho\left(\frac{yz}{x}\right)^N}{D\left(\frac{1}{x}\right) D(y\omega)\, D\left(\frac{\omega}{z}\right) D\left(\frac{yz}{x}\right)}.$$

Therefore, if $y = 0$ in the previous equality, we get, since $\varrho(0) = N^{-1}$ and $D(0) = 1$,

$$f(x,0|z)^N = N^{-N} \left(\frac{x}{z}\right)^{\frac{N(N-1)}{2}} \frac{D(1)^3}{D\left(\frac{1}{x}\right) D\left(\frac{\omega}{z}\right)} \tag{1.14}$$

Now we show equality (1.12). Again, we are going to transform equality (1.4). Using equality (1.6) we have

$$D\left(\frac{1}{x}\right) = x^{-\frac{N(N-1)}{2}} \frac{D(1)^2 \varrho(x)^N}{D(x)} \tag{1.15}$$

$$D\left(\frac{y\omega}{x}\right) = \left(\frac{x}{y\omega}\right)^{-\frac{N(N-1)}{2}} \frac{D(1)^2 \varrho\left(\frac{x}{y\omega}\right)^N}{D\left(\frac{x}{y\omega}\right)} \tag{1.16}$$

Hence, using equalities (1.15) and (1.16) we get

$$f(x,y|z)^N = (y\omega)^{\frac{N(N-1)}{2}} \frac{D(1)\, D\left(\frac{y\omega}{x}\right) D\left(\frac{x}{yz}\right)}{D\left(\frac{1}{x}\right) D(y\omega)\, D\left(\frac{\omega}{z}\right)}$$

$$= y^{N(N-1)} \frac{D(1) D\left(\frac{x}{yz}\right) D(x) \varrho\left(\frac{x}{y\omega}\right)^N}{D(y\omega) D\left(\frac{\omega}{z}\right) D\left(\frac{x}{y\omega}\right) \varrho(x)^N}$$

Therefore, if $x = 0$ in the previous equality, we get, since $\varrho(0) = N^{-1}$ and $D(0) = 1$,

$$f(0,y|z)^N = y^{N(N-1)} \frac{D(1)}{D(y\omega) D\left(\frac{\omega}{z}\right)}.$$

□

PROOF OF FORMULAE (2.4) AND (2.8). Using the last two Lemmas, a straightforward computation leads to the result. □

1.2. Proof of Lemma 2.3. We recall the statement of Lemma 2.3.

There exists $\alpha \in \mathbb{Z}_N$ *such that for any* $x \in \mathbb{R}_{\neq 0,1}$ *and any operator* $\mathsf{U} \in \mathcal{A}^{\otimes 2}$ *such that* $\mathsf{U}^N = -\,\mathrm{Id}_{\mathcal{V}^{\otimes 2}}$ *we have*

$$\bar{\psi}_{x,0} = \omega^{\alpha} \varrho\left((1-x)^{\frac{1}{N}}\right) \psi_{x,0}^{-1} \quad \Rightarrow \quad \Psi_x(\mathsf{U}) \bar{\Psi}_x(\mathsf{U}) = \mathrm{Id}_{\mathcal{V}^{\otimes 2}}\,.$$

Proof of Lemma 2.3. By definition, the product $\Psi_x(\mathsf{U})\bar{\Psi}_x(\mathsf{U})$ is equal to the identity up to a non zero constant factor in $\mathbb{C}$. We are going to determine this factor up to multiplication by an integer power of ω.

We set $a = (1-x)^{\frac{1}{N}}$ and $b = \mathsf{U}\left(\frac{x}{1-x}\right)^{\frac{1}{N}}$ and we will write $1-b$ instead of $\mathrm{Id}_{\mathcal{A}^{\otimes 2}} - b$ in the sequel. Then, equalities (1.10) and (1.11) can be written as follow

$$\begin{aligned}\Psi_x(\mathsf{U}) &= \psi_{x,0} f\left(0, a^{-1}\omega^{-1}\middle| b^{-1}\right),\\ \bar{\Psi}_x(\mathsf{U}) &= \bar{\psi}_{x,0} f\left(a, 0\middle| b\omega\right).\end{aligned}$$

Using equalities (1.5) and (1.6) of Lemma 1.2, we compute

$$\begin{aligned}\Psi_x(\mathsf{U})^N\bar{\Psi}_x(\mathsf{U})^N &= \psi_{x,0}^N\bar{\psi}_{x,0}^N N^{-N}\left(a^3 b\right)^{-\frac{N(N-1)}{2}}\frac{D(1)^4}{D(a^{-1})D(a)D(b^{-1})D(b\omega)}\\ &= \psi_{x,0}^N\bar{\psi}_{x,0}^N N^{-N}\left(a^3 b\right)^{-\frac{N(N-1)}{2}} a^{\frac{N(N-1)}{2}}\frac{1-b^N}{(1-b)^N}\frac{D(1)^2}{\varrho(a)^N D(b^{-1})D(b)}\\ &= \psi_{x,0}^N\bar{\psi}_{x,0}^N N^{-N}\left(a^2 b\right)^{-\frac{N(N-1)}{2}} b^{\frac{N(N-1)}{2}}\frac{1-b^N}{(1-b)^N}\frac{1}{\varrho(a)^N\varrho(b)^N}\\ &= \psi_{x,0}^N\bar{\psi}_{x,0}^N N^{-N} a^{-N(N-1)}\frac{1-b^N}{(1-b)^N}N^N\frac{(1-b)^N}{(1-b^N)^N}\frac{1}{\varrho(a)^N}\\ &= \frac{\psi_{x,0}^N\bar{\psi}_{x,0}^N}{\left(a^N(1-b^N)\right)^{N-1}\varrho(a)^N}.\end{aligned}$$

Since

$$a^N(1-b^N) = (1-x)\left(1+\frac{x}{1-x}\right)\mathrm{Id}_{\mathcal{V}^{\otimes 2}} = \mathrm{Id}_{\mathcal{V}^{\otimes 2}}$$

hence

$$\Psi_x(\mathsf{U})^N\bar{\Psi}_x(\mathsf{U})^N = \frac{\psi_{x,0}^N\bar{\psi}_{x,0}^N}{\varrho(a)^N}\mathrm{Id}_{\mathcal{V}^{\otimes 2}}.$$

□

2. Determination of $S(x)$ and of its inverse

Let us recall that we had defined the following generalisation of the function $f(x,y|z)$, namely

$$F\begin{pmatrix} x & u \\ y & v \end{pmatrix}\!\!\left| z\right) = \sum_{m=0}^{N-1}\frac{w(x|m)w(u|m)}{w(y|m)w(v|m)}z^m, \tag{2.1}$$

where $x, y, u, v, z \in \mathbb{C}$ are such that

$$\left\{x^N, y^N, u^N, v^N\right\} \subset \mathbb{C}\setminus\{1\} \quad \text{and} \quad z^N = \frac{(1-x^N)(1-u^N)}{(1-y^N)(1-v^N)}. \tag{2.2}$$

The conditions (2.2) provide periodicity on the variable m of period N.

The Proposition 2.11 follows directly from the two following Lemmas

LEMMA 2.1. *For all $\alpha, \beta, \mu, \nu \in \mathbb{Z}_N^2$ and all $x \in \mathbb{R}_{\neq 0,1}$, we have*

$$\begin{aligned}\Big\langle \bar{u}_\mu \otimes \bar{u}_\nu \Big| S(x) \Big| u_\alpha \otimes u_\beta \Big\rangle &= \delta_{\mu_1+\nu_1,\alpha_1}\delta_{\mu_2+\nu_2,\alpha_2+\beta_2} \\ &\times \omega^{-\frac{1}{2}(t+1)(\alpha_2-\mu_2)(\alpha_2-\mu_2-t-2((t+1)\alpha_1-\beta_1+\nu_2+1))} \\ &\times \omega^{-\frac{1}{2}(t+1)\{\beta_1(\beta_1-1+2(t\alpha_1+\alpha_2-\nu_2))+\nu_1((t+1)\nu_1+2(\mu_1+\nu_2+1))\}} \\ &\times \sum_{m,n\in\mathbb{Z}_N} \psi_{x,m}\psi_{x,(t+1)(\nu_1-\beta_1+\mu_2-\alpha_2)-m}\psi_{x,n}\psi_{x,(t+1)(\nu_1-\beta_1)+t(\mu_2-\alpha_2)-n} \\ &\times \omega^{-\frac{1}{2}(m-n)^2+m\beta_2+n\alpha_2}\end{aligned}$$

and

$$\begin{aligned}\Big\langle \bar{u}_\mu \otimes \bar{u}_\nu \Big| S(x)^{-1} \Big| u_\alpha \otimes u_\beta \Big\rangle &= \delta_{\mu_1,\alpha_1+\beta_1}\delta_{\mu_2+\nu_2,\alpha_2+\beta_2} \\ &\times \omega^{-\frac{1}{2}(t+1)(\alpha_2-\mu_2)(\alpha_2-\mu_2-t-2(\alpha_1+t\mu_1+\nu_2+1))} \\ &\times \omega^{\frac{1}{2}(t+1)\{\beta_1((t-1)\beta_1+2(t\mu_1+\nu_2+1))+\nu_1(\nu_1-1-2((t+1)\mu_1-t\mu_2-\nu_2-t))\}} \\ &\times \sum_{m,n\in\mathbb{Z}_N} \bar{\psi}_{x,m}\bar{\psi}_{x,(t+1)(\nu_1-\beta_1+\mu_2-\alpha_2)-m}\bar{\psi}_{x,n}\bar{\psi}_{x,(t+1)(\nu_1-\beta_1)+t(\mu_2-\alpha_2)-n} \\ &\times \omega^{\frac{1}{2}(m-n)^2+m\nu_2+n\mu_2}.\end{aligned}$$

LEMMA 2.2. *For all $x \in \mathbb{R}_{\neq 0,1}$ and all $m, n, r, s, \alpha, \beta \in \mathbb{Z}_N$ we have*

$$\begin{aligned}\sum_{m,n\in\mathbb{Z}_N} \psi_{x,m}\psi_{x,r-m}\psi_{x,n}\psi_{x,s-n}\omega^{-\frac{1}{2}(m-n)^2+m\beta+n\alpha} &= \psi_{x,0}^2\psi_{x,r}\psi_{x,s} \\ &\times f\left((1-x)^{\frac{1}{N}}, (1-x)^{-\frac{1}{N}}\omega^{-r-1}\Big|(x-1)^{\frac{1}{N}}\omega^{\beta+\frac{1}{2}}\right) \\ &\times F\left(\begin{matrix}(1-x)^{\frac{1}{N}} & (x-1)^{-\frac{1}{N}}\omega^{\beta-r-\frac{1}{2}} \\ (1-x)^{-\frac{1}{N}}\omega^{-s-1} & (x-1)^{\frac{1}{N}}\omega^{\beta-\frac{1}{2}}\end{matrix}\Bigg| -\omega^{\alpha+\frac{1}{2}}\right)\end{aligned}$$

and

$$\begin{aligned}\sum_{m,n\in\mathbb{Z}_N} \bar{\psi}_{x,m}\bar{\psi}_{x,r-m}\bar{\psi}_{x,n}\bar{\psi}_{x,s-n}\omega^{\frac{1}{2}(m-n)^2+m\beta+n\alpha} &= \bar{\psi}_{x,0}^2\bar{\psi}_{x,r}\bar{\psi}_{x,s} \\ &\times f\left((1-x)^{-\frac{1}{N}}, (1-x)^{\frac{1}{N}}\omega^{-r-1}\Big|(x-1)^{-\frac{1}{N}}\omega^{\beta+r+\frac{1}{2}}\right) \\ &\times F\left(\begin{matrix}(1-x)^{-\frac{1}{N}} & (x-1)^{\frac{1}{N}}\omega^{\beta-r-\frac{1}{2}} \\ (1-x)^{\frac{1}{N}}\omega^{-s-1} & (x-1)^{-\frac{1}{N}}\omega^{-\beta-\frac{1}{2}}\end{matrix}\Bigg| -\omega^{s-r+\alpha+\frac{1}{2}}\right).\end{aligned}$$

2.1. Proof of Lemma 2.1.

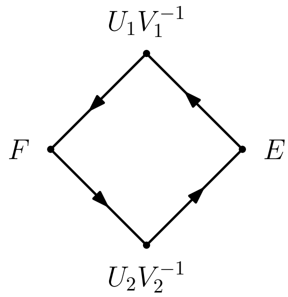
LEMMA 2.3. *For all $\alpha, \beta, \mu, \nu \in \mathbb{Z}_N^2$ we have*

$$\left\langle \bar{u}_\mu \otimes \bar{u}_\nu \middle| S(x) \middle| u_\alpha \otimes u_\beta \right\rangle = \sum_{\substack{m,n,\\ r,s\in\mathbb{Z}_N}} (-1)^r \psi_{x,m}\psi_{x,s-m}\psi_{x,n}\psi_{x,r+s-n} \times \omega^{-\frac{1}{2}(m-n)^2+m\beta_2+n\alpha_2} \left\langle \bar{u}_\mu \otimes \bar{u}_\nu \middle| E^r(EF)^s L(U^tV, X) \middle| u_\alpha \otimes u_\beta \right\rangle$$

and

$$\left\langle \bar{u}_\mu \otimes \bar{u}_\nu \middle| S(x)^{-1} \middle| u_\alpha \otimes u_\beta \right\rangle = \sum_{\substack{m,n,\\ r,s\in\mathbb{Z}_N}} (-1)^r \bar{\psi}_{x,m}\bar{\psi}_{x,s-m}\bar{\psi}_{x,n}\bar{\psi}_{x,r+s-n} \times \omega^{\frac{1}{2}(m-n)^2+m\nu_2+n\mu_2} \left\langle \bar{u}_\mu \otimes \bar{u}_\nu \middle| L(U^{-t}V^{-1}, X) E^r(EF)^s \middle| u_\alpha \otimes u_\beta \right\rangle$$

PROOF. Note that the following commutation relation holds



Since $G = U_2V_2^{-1}F$ and $H = U_1V_1^{-1}E$, we have

$$\begin{aligned}
\Psi_x(E)\Psi_x(F)\Psi_x(G)\Psi_x(H) &= \sum_{\substack{n_1,n_2,\\ n_3,n_4\in\mathbb{Z}_N}} \left(\prod_{i=1}^4 (-1)^{n_i}\psi_{x,n_i}\right) E^{n_1}F^{n_2}G^{n_3}H^{n_4} \\
&= \sum_{\substack{n_1,n_2,\\ n_3,n_4\in\mathbb{Z}_N}} \left(\prod_{i=1}^4 (-1)^{n_i}\psi_{x,n_i}\right) \omega^{-\frac{1}{2}(n_3(n_3+1)+n_4(n_4+1))} \\
&\quad \times E^{n_1}F^{n_2}F^{n_3}(U_2V_2^{-1})^{n_3}E^{n_4}(U_1V_1^{-1})^{n_4} \\
&= \sum_{\substack{n_1,n_2,\\ n_3,n_4\in\mathbb{Z}_N}} \left(\prod_{i=1}^4 (-1)^{n_i}\psi_{x,n_i}\right) \omega^{-\frac{1}{2}(n_3(n_3+1)+n_4(n_4+1))+n_3n_4} \\
&\quad \times E^{n_1}F^{n_2}F^{n_3}E^{n_4}(U_1V_1^{-1})^{n_4}(U_2V_2^{-1})^{n_3} \\
&= \sum_{\substack{n_1,n_2,\\ n_3,n_4\in\mathbb{Z}_N}} \left(\prod_{i=1}^4 (-1)^{n_i}\psi_{x,n_i}\right) \omega^{-\frac{1}{2}\left((n_3-n_4)^2+(n_3+n_4)\right)} \\
&\quad \times E^{n_1+n_4-(n_2+n_3)}(EF)^{n_2+n_3}(U_1V_1^{-1})^{n_4}(U_2V_2^{-1})^{n_3}.
\end{aligned}$$

Setting $r = n_1 + n_4 - (n_2 + n_3), s = n_2 + n_3, m = n_3$ and $n = n_4$, we find

$$\Psi_x(E)\Psi_x(F)\Psi_x(G)\Psi_x(H) = \sum_{\substack{r,s,\\ m,n\in\mathbb{Z}_N}} (-1)^{r+2s}\psi_{x,m}\psi_{x,s-m}\psi_{x,n}\psi_{x,r+s-n} \times \omega^{-\frac{1}{2}\left((m-n)^2+(m+n)\right)} E^r(EF)^s(U_1V_1^{-1})^n(U_2V_2^{-1})^m.$$

Since UV^{-1} commutes with U^tV and X and using Lemmas 2.10, we compute

$$\begin{aligned}\left\langle \bar{u}_\mu\otimes\bar{u}_\nu \middle| S(x) \middle| u_\alpha\otimes u_\beta\right\rangle &= \sum_{\substack{r,s,\\ m,n\in\mathbb{Z}_N}} (-1)^r\psi_{x,m}\psi_{x,s-m}\psi_{x,n}\psi_{x,r+s-n}\omega^{-\frac{1}{2}\left((m-n)^2+(m+n)\right)}\\ &\quad\times\left\langle \bar{u}_\mu\otimes\bar{u}_\nu \middle| E^r(EF)^sL(U^tV,X)(U_1V_1^{-1})^n(U_2V_2^{-1})^m \middle| u_\alpha\otimes u_\beta\right\rangle\\ &= \sum_{\substack{r,s,\\ m,n\in\mathbb{Z}_N}} (-1)^r\psi_{x,m}\psi_{x,s-m}\psi_{x,n}\psi_{x,r+s-n}\omega^{-\frac{1}{2}\left((m-n)^2+(m+n)\right)}\\ &\quad\times\omega^{m\beta_2+n\alpha_2+\frac{1}{2}(m+n)}\left\langle \bar{u}_\mu\otimes\bar{u}_\nu \middle| E^r(EF)^sL(U^tV,X) \middle| u_\alpha\otimes u_\beta\right\rangle\\ &= \sum_{\substack{r,s,\\ m,n\in\mathbb{Z}_N}} (-1)^r\psi_{x,m}\psi_{x,s-m}\psi_{x,n}\psi_{x,r+s-n}\\ &\quad\times\omega^{-\frac{1}{2}(m-n)^2+m\beta_2+n\alpha_2}\left\langle \bar{u}_\mu\otimes\bar{u}_\nu \middle| E^r(EF)^sL(U^tV,X) \middle| u_\alpha\otimes u_\beta\right\rangle.\end{aligned}$$

The second formula is computed in a similar way. □

LEMMA 2.4. *For all $\alpha,\beta,\mu,\nu\in\mathbb{Z}_N^2$ and all $r,s\in\mathbb{Z}_N$ we have*

$$\begin{aligned}\left\langle \bar{u}_\mu\otimes\bar{u}_\nu \middle| E^r(EF)^sL(U^tV,X) \middle| u_\alpha\otimes u_\beta\right\rangle &= \delta_{r,\alpha_2-\mu_2}\delta_{ts,\beta_1-\nu_1+r}\delta_{\mu_1+\nu_1,\alpha_1}\delta_{\mu_2+\nu_2,\alpha_2+\beta_2}\\ &\quad\times(-1)^{\alpha_2-\mu_2}\omega^{-\frac{1}{2}(t+1)(\alpha_2-\mu_2)(\alpha_2-\mu_2-t-2((t+1)\alpha_1-\beta_1+\nu_2+1))}\\ &\quad\times\omega^{-\frac{1}{2}(t+1)\{\beta_1(\beta_1-1+2(t\alpha_1+\alpha_2-\nu_2))+\nu_1((t+1)\nu_1+2(\mu_1+\nu_2+1))\}}\end{aligned}$$

and

$$\begin{aligned}\left\langle \bar{u}_\mu\otimes\bar{u}_\nu \middle| L(U^{-t}V^{-1},X)E^r(EF)^s \middle| u_\alpha\otimes u_\beta\right\rangle &= \delta_{r,\alpha_2-\mu_2}\delta_{ts,\beta_1-\nu_1+r}\delta_{\mu_1,\alpha_1+\beta_1}\delta_{\mu_2+\nu_2,\alpha_2+\beta_2}\\ &\quad\times(-1)^{\alpha_2-\mu_2}\omega^{-\frac{1}{2}(t+1)(\alpha_2-\mu_2)(\alpha_2-\mu_2-t-2(\alpha_1+t\mu_1+\nu_2+1))}\\ &\quad\times\omega^{\frac{1}{2}(t+1)\{\beta_1((t-1)\beta_1+2(t\mu_1+\nu_2+1))+\nu_1(\nu_1-1-2((t+1)\mu_1-t\mu_2-\nu_2-t))\}}.\end{aligned}$$

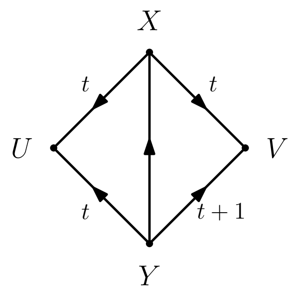
Proof. Using Lemma 2.10 we do the following computation

$$\Big\langle \bar{u}_\mu \otimes \bar{u}_\nu \Big| L(U^t V, X) \Big| u_\alpha \otimes u_\beta \Big\rangle = \frac{1}{N} \sum_{i,j\in\mathbb{Z}_N} \omega^{-ij} \Big\langle \bar{u}_\mu \Big| (U^t V)^i \Big| u_\alpha \Big\rangle \Big\langle \bar{u}_\nu \Big| X^j \Big| u_\beta \Big\rangle$$
$$= \frac{1}{N} \sum_{i,j\in\mathbb{Z}_N} \omega^{-j(i+\beta_1)} \omega^{\frac{1}{2}i(i-1)+i(\alpha_1-\alpha_2+\frac{1}{2}t)} \delta_{\mu_1,\alpha_1+i} \delta_{\mu_2,\alpha_2} \delta_{\nu,\beta}$$
$$= \frac{1}{N} \sum_{i\in\mathbb{Z}_N} N \delta_{i,-\beta_1} \omega^{\frac{1}{2}i(i-1)+i(\alpha_1-\alpha_2+\frac{1}{2}t)} \delta_{\mu_1,\alpha_1+i} \delta_{\mu_2,\alpha_2} \delta_{\nu,\beta}$$
$$= \omega^{\frac{1}{2}\beta_1(\beta_1+1)-\beta_1(\alpha_1-\alpha_2+\frac{1}{2}t)} \delta_{\mu_1,\alpha_1-\beta_1} \delta_{\mu_2,\alpha_2} \delta_{\nu,\beta}.$$

Since $U^{-t}V^{-1} = (U^t V)^{-1}$, a similar computation leads us to the following equality

$$\Big\langle \bar{u}_\mu \otimes \bar{u}_\nu \Big| L(U^{-t} V^{-1}, X) \Big| u_\alpha \otimes u_\beta \Big\rangle = \omega^{-\frac{1}{2}\nu_1(\nu_1+1)+\nu_1(\mu_1-\mu_2+\frac{1}{2}t)} \delta_{\mu_1,\alpha_1+\beta_1} \delta_{\mu_2,\alpha_2} \delta_{\nu,\beta}.$$

We recall that $E = -Y_1^{-1} X_1 Y_2$ and $F = U_1^{-1} X_1^{t+1} V_2 E^{-1}$ and that we have the following commutation relations



Then, for any $r, s \in \mathbb{Z}_N$, we easily see that

$$E^r = (-1)^r \omega^{\frac{1}{2}r(r-1)} Y_1^{-r} X_1^r Y_2^r$$

and

$$(EF)^s = \left(U_1^{-1} X_1^{t+1} V_2\right)^s = \omega^{\frac{1}{2}s(s-1)} U_1^{-s} X_1^{s(t+1)} V_2^s.$$

Using Lemma 2.10 again, we find

$$\Big\langle \bar{u}_\mu \otimes \bar{u}_\nu \Big| E^r \Big| u_\alpha \otimes u_\beta \Big\rangle = (-1)^r \omega^{\frac{1}{2}r(r-1)-r\alpha_1} \delta_{\mu_1,\alpha_1-r} \delta_{\mu_2,\alpha_2-r} \delta_{\nu_1,\beta_1+r} \delta_{\nu_2,\beta_2+r}$$
$$= (-1)^r \omega^{\frac{1}{2}r(r-1)-r\alpha_1} \delta_{r,\alpha_1-\mu_1} \delta_{r,\nu_1-\beta_1} \delta_{\mu_1-\mu_2,\alpha_1-\alpha_2} \delta_{\nu_1-\nu_2,\beta_1-\beta_2}$$

and

$$\Big\langle \bar{u}_\mu \otimes \bar{u}_\nu \Big| (EF)^s \Big| u_\alpha \otimes u_\beta \Big\rangle = \omega^{s^2(t+\frac{1}{2})-s(\mu_1+t\nu_1+\nu_2+1)} \delta_{\mu_1,\alpha_1+st} \delta_{\mu_2,\alpha_2} \delta_{\nu_1,\beta_1-st} \delta_{\nu_2,\beta_2}$$
$$= \omega^{s^2(t+\frac{1}{2})-s(\mu_1+t\nu_1+\nu_2+1)} \delta_{st,\mu_1-\alpha_1} \delta_{\mu_1-\alpha_1,\beta_1-\nu_1} \delta_{\mu_2,\alpha_2} \delta_{\nu_2,\beta_2}$$
$$= \omega^{s^2(t+\frac{1}{2})-s(\mu_1+t\nu_1+\nu_2+1)} \delta_{s,(t+1)(\alpha_1-\mu_1)} \delta_{\mu_1-\alpha_1,\beta_1-\nu_1} \delta_{\mu_2,\alpha_2} \delta_{\nu_2,\beta_2}.$$

Using the previous equalities, a straightforward computation leads to the result. □

PROOF OF LEMMA 2.1. A straightforward computation using the previous two Lemmas. □

2.2. Proof of Lemma 2.2.

LEMMA 2.5. *For any $m, s \in \{0, \ldots, N-1\}$ we have*

$$\sum_{m=0}^{N-1} \psi_{x,m}\psi_{x,s-m} = \psi_{x,0}\psi_{x,s} \sum_{m=0}^{N-1} \omega^{\frac{1}{2}m(m+1)}(x-1)^{\frac{m}{N}} \frac{w((1-x)^{\frac{1}{N}}|m)}{w((1-x)^{-\frac{1}{N}}\omega^{-s-1}|m)}$$

and

$$\sum_{m=0}^{N-1} \bar{\psi}_{x,m}\bar{\psi}_{x,s-m} = \bar{\psi}_{x,0}\bar{\psi}_{x,s} \sum_{m=0}^{N-1} \omega^{-\frac{1}{2}m(m-1)+sm}(x-1)^{-\frac{m}{N}} \frac{w((1-x)^{-\frac{1}{N}}|m)}{w((1-x)^{\frac{1}{N}}\omega^{-s-1}|m)}$$

PROOF. Let m, s be an integer in $\{0, \cdots, N-1\}$. Since $\psi_{x,m}$ and $\bar{\psi}_{x,m}$ are periodic we compute, using equality (1.1) and Lemma 1.1,

$$\begin{aligned}
\sum_{m=0}^{N-1} \psi_{x,m}\psi_{x,s-m} &= \sum_{m=0}^{N-1} \psi_{x,m}\psi_{x,s+N-m} = \sum_{m=0}^{N-1} \psi_{x,0}\omega^{\frac{1}{2}(m(m+1)}x^{\frac{m}{N}}w((1-x)^{\frac{1}{N}}|m) \\
&\quad \times \psi_{x,0}\omega^{\frac{1}{2}(s+N-m)(s+N-m+1)}x^{\frac{s+N-m}{N}}w((1-x)^{\frac{1}{N}}|s+N-m) \\
&= \psi_{x,0}^2 \sum_{m=0}^{N-1} \omega^{\frac{1}{2}s(s+1)-sm+m^2}x^{\frac{s+N}{N}}w((1-x)^{\frac{1}{N}}|m)w((1-x)^{\frac{1}{N}}|s)w((1-x)^{\frac{1}{N}}\omega^s|N-m) \\
&= \psi_{x,0}^2 x^{\frac{s}{N}}\omega^{\frac{1}{2}s(s+1)}w((1-x)^{\frac{1}{N}}|s)\sum_{m=0}^{N-1}\omega^{-sm+m^2}xw((1-x)^{\frac{1}{N}}|m)w((1-x)^{\frac{1}{N}}\omega^s|N-m) \\
&= \psi_{x,0}\psi_{x,s}\sum_{m=0}^{N-1}\omega^{-sm+m^2}xw((1-x)^{\frac{1}{N}}|m)\frac{(x-1)^{\frac{m}{N}}\omega^{sm}}{1-(1-x)^{\frac{N}{N}}}\omega^{-\frac{1}{2}m(m-1)}w((1-x)^{-\frac{1}{N}}\omega^{-s-1}|m)^{-1} \\
&= \psi_{x,0}\psi_{x,s}\sum_{m=0}^{N-1}\omega^{\frac{1}{2}m(m+1)}(x-1)^{\frac{m}{N}}\frac{w((1-x)^{\frac{1}{N}}|m)}{w((1-x)^{-\frac{1}{N}}\omega^{-s-1}|m)}
\end{aligned}$$

and

$$\sum_{m=0}^{N-1}\bar{\psi}_{x,m}\bar{\psi}_{x,s-m}=\sum_{m=0}^{N-1}\bar{\psi}_{x,m}\bar{\psi}_{x,s+N-m}=\sum_{m=0}^{N-1}\bar{\psi}_{x,0}\left(\frac{x}{x-1}\right)^{\frac{m}{N}}\omega^{m}w((1-x)^{-\frac{1}{N}}|m)$$
$$\times\,\bar{\psi}_{x,0}\left(\frac{x}{x-1}\right)^{\frac{s+N-m}{N}}\omega^{s+N-m}w((1-x)^{-\frac{1}{N}}|s+N-m)$$
$$=\bar{\psi}_{x,0}^{2}\sum_{m=0}^{N-1}\left(\frac{x}{x-1}\right)^{\frac{s+N}{N}}\omega^{s}w((1-x)^{-\frac{1}{N}}|m)w((1-x)^{-\frac{1}{N}}|s)w((1-x)^{-\frac{1}{N}}\omega^{s}|N-m)$$
$$=\bar{\psi}_{x,0}^{2}\left(\frac{x}{x-1}\right)^{\frac{s}{N}}\omega^{s}w((1-x)^{-\frac{1}{N}}|s)\sum_{m=0}^{N-1}\frac{x}{x-1}w((1-x)^{-\frac{1}{N}}|m)w((1-x)^{-\frac{1}{N}}\omega^{s}|N-m)$$
$$=\bar{\psi}_{x,0}\bar{\psi}_{x,s}\sum_{m=0}^{N-1}\frac{x}{x-1}w((1-x)^{-\frac{1}{N}}|m)\frac{(x-1)^{-\frac{m}{N}}\omega^{sm}}{1-(1-x)^{-\frac{N}{N}}}\omega^{-\frac{1}{2}m(m-1)}w((1-x)^{\frac{1}{N}}\omega^{-s-1}|m)^{-1}$$
$$=\bar{\psi}_{x,0}\bar{\psi}_{x,s}\sum_{m=0}^{N-1}\omega^{-\frac{1}{2}m(m-1)+sm}(x-1)^{-\frac{m}{N}}\frac{w((1-x)^{-\frac{1}{N}}|m)}{w((1-x)^{\frac{1}{N}}\omega^{-s-1}|m)}$$

□

LEMMA 2.6. *Let $m\in\{0,\dots,N-1\}$ and $x,y,z\in\mathbb{C}$ satisfying the conditions (1.2) and $x\neq 0$. Then we have*

$$\frac{f(x,y|z\omega^{m})}{f(x,y|z)}=x^{-m}\frac{w(\frac{yz}{x}|m)}{w(\frac{z}{\omega}|m)} \tag{2.3}$$

PROOF. From equality (A.6) of [**9**], for all $x,y,z\in\mathbb{C}$ satisfying (1.2) we have

$$\frac{f(x,y|z\omega)}{f(x,y|z)}=\frac{1-z}{x-yz\omega}=x^{-1}\frac{1-\frac{z}{\omega}\omega}{1-\frac{yz}{x}\omega}.$$

Hence, for any $m\in\{0,\dots,N-1\}$, we have

$$\frac{f(x,y|z\omega^{m})}{f(x,y|z)}=\prod_{j=1}^{m}\frac{f(x,y|z\omega^{j})}{f(x,y|z\omega^{j-1})}=\prod_{j=1}^{m}x^{-1}\frac{1-\frac{z}{\omega}\omega^{j}}{1-\frac{yz}{x}\omega^{j}}=x^{-m}\frac{w(\frac{yz}{x}|m)}{w(\frac{z}{\omega}|m)}.$$

□

PROOF OF LEMMA 2.2. Using Lemma 2.5 we compute

$$\begin{aligned}
&\sum_{m,n=0}^{N-1} \psi_{x,m}\psi_{x,s-m}\psi_{x,n}\psi_{x,r-n}\omega^{-\frac{1}{2}(m-n)^2+m\beta+n\alpha} \\
&\quad= \psi_{x,0}^2\psi_{x,s}\psi_{x,r}\sum_{m,n=0}^{N-1}\omega^{\frac{1}{2}\left(m(m+1)+n(n+1)-(m-n)^2\right)+m\beta+n\alpha}(x-1)^{\frac{m+n}{N}} \\
&\qquad\times \frac{w((1-x)^{\frac{1}{N}}|m)}{w((1-x)^{-\frac{1}{N}}\omega^{-s-1}|m)}\frac{w((1-x)^{\frac{1}{N}}|n)}{w((1-x)^{-\frac{1}{N}}\omega^{-r-1}|n)} \\
&\quad= \psi_{x,0}^2\psi_{x,s}\psi_{x,r}\sum_{m,n=0}^{N-1}\left((x-1)^{\frac{1}{N}}\omega^{n+\beta+\frac{1}{2}}\right)^m\frac{w((1-x)^{\frac{1}{N}}|m)}{w((1-x)^{-\frac{1}{N}}\omega^{-s-1}|m)} \\
&\qquad\times\left((x-1)^{\frac{1}{N}}\omega^{\alpha+\frac{1}{2}}\right)^n\frac{w((1-x)^{\frac{1}{N}}|n)}{w((1-x)^{-\frac{1}{N}}\omega^{-r-1}|n)}.
\end{aligned}$$

We note that the conditions (1.2) are satisfied since for any $\beta, n \in \{0,\ldots,N-1\}$ we have

$$\left((x-1)^{\frac{1}{N}}\omega^{n+\beta+\frac{1}{2}}\right)^N = \frac{1-\left((1-x)^{\frac{1}{N}}\right)^N}{1-\left((1-x)^{-\frac{1}{N}}\omega^{-1}\right)^N}.$$

Therefore we can use the function $f(x,y|z)$ in the computation.

$$\begin{aligned}
&\sum_{m,n=0}^{N-1} \psi_{x,m}\psi_{x,s-m}\psi_{x,n}\psi_{x,r-n}\omega^{-\frac{1}{2}(m-n)^2+m\beta+n\alpha} \\
&\quad= \psi_{x,0}^2\psi_{x,s}\psi_{x,r}\sum_{m,n=0}^{N-1}\left((x-1)^{\frac{1}{N}}\omega^{n+\beta+\frac{1}{2}}\right)^m\frac{w((1-x)^{\frac{1}{N}}|m)}{w((1-x)^{-\frac{1}{N}}\omega^{-s-1}|m)} \\
&\qquad\times\left((x-1)^{\frac{1}{N}}\omega^{\alpha+\frac{1}{2}}\right)^n\frac{w((1-x)^{\frac{1}{N}}|n)}{w((1-x)^{-\frac{1}{N}}\omega^{-r-1}|n)} \\
&\quad= \psi_{x,0}^2\psi_{x,s}\psi_{x,r}\sum_{n=0}^{N-1} f\left((1-x)^{\frac{1}{N}},(1-x)^{-\frac{1}{N}}\omega^{-s-1}\middle|(x-1)^{\frac{1}{N}}\omega^{n+\beta+\frac{1}{2}}\right) \\
&\qquad\times\left((x-1)^{\frac{1}{N}}\omega^{\alpha+\frac{1}{2}}\right)^n\frac{w((1-x)^{\frac{1}{N}}|n)}{w((1-x)^{-\frac{1}{N}}\omega^{-r-1}|n)}.
\end{aligned}$$

Now using Lemma 2.6 we have

$$\sum_{m,n=0}^{N-1} \psi_{x,m}\psi_{x,s-m}\psi_{x,n}\psi_{x,r-n}\omega^{-\frac{1}{2}(m-n)^2+m\beta+n\alpha}$$
$$= \psi_{x,0}^2\psi_{x,s}\psi_{x,r}\sum_{n=0}^{N-1} f\left((1-x)^{\frac{1}{N}},(1-x)^{-\frac{1}{N}}\omega^{-s-1}\middle|(x-1)^{\frac{1}{N}}\omega^{n+\beta+\frac{1}{2}}\right)$$
$$\times\left((x-1)^{\frac{1}{N}}\omega^{\alpha+\frac{1}{2}}\right)^n\frac{w((1-x)^{\frac{1}{N}}|n)}{w((1-x)^{-\frac{1}{N}}\omega^{-r-1}|n)}$$
$$= \psi_{x,0}^2\psi_{x,s}\psi_{x,r}\sum_{n=0}^{N-1} f\left((1-x)^{\frac{1}{N}},(1-x)^{-\frac{1}{N}}\omega^{-s-1}\middle|(x-1)^{\frac{1}{N}}\omega^{\beta+\frac{1}{2}}\right)$$
$$\times\left((1-x)^{\frac{1}{N}}\right)^{-n}\frac{w((x-1)^{-\frac{1}{N}}\omega^{\beta-s-\frac{1}{2}}|n)}{w((x-1)^{\frac{1}{N}}\omega^{\beta-\frac{1}{2}}|n)}\left((x-1)^{\frac{1}{N}}\omega^{\alpha+\frac{1}{2}}\right)^n\frac{w((1-x)^{\frac{1}{N}}|n)}{w((1-x)^{-\frac{1}{N}}\omega^{-r-1}|n)}$$
$$= \psi_{x,0}^2\psi_{x,s}\psi_{x,r}f\left((1-x)^{\frac{1}{N}},(1-x)^{-\frac{1}{N}}\omega^{-s-1}\middle|(x-1)^{\frac{1}{N}}\omega^{\beta+\frac{1}{2}}\right)$$
$$\times\sum_{n=0}^{N-1}\left(-\omega^{\alpha+\frac{1}{2}}\right)\frac{w((1-x)^{\frac{1}{N}}|n)w((x-1)^{-\frac{1}{N}}\omega^{\beta-s-\frac{1}{2}}|n)}{w((1-x)^{-\frac{1}{N}}\omega^{-r-1}|n)w((x-1)^{\frac{1}{N}}\omega^{\beta-\frac{1}{2}}|n)}.$$

We note that the conditions (2.2) are satisfied since for any $\alpha,\beta,r,s\in\{0,\dots,N-1\}$ we have

$$\left(-\omega^{\alpha+\frac{1}{2}}\right)^N=\frac{\left(1-\left((1-x)^{\frac{1}{N}}\right)^N\right)\left(1-\left((x-1)^{-\frac{1}{N}}\omega^{\beta-s-\frac{1}{2}}\right)^N\right)}{\left(1-\left((1-x)^{-\frac{1}{N}}\omega^{-r-1}\right)^N\right)\left(1-\left((x-1)^{\frac{1}{N}}\omega^{\beta-\frac{1}{2}}\right)^N\right)}$$

Therefore we can use the function $F\left(\begin{smallmatrix} x & u\\ y & v\end{smallmatrix}\middle|z\right)$ in the computation. Hence we finally find

$$\sum_{m,n=0}^{N-1} \psi_{x,m}\psi_{x,s-m}\psi_{x,n}\psi_{x,r-n}\omega^{-\frac{1}{2}(m-n)^2+m\beta+n\alpha}=\psi_{x,0}^2\psi_{x,s}\psi_{x,r}$$
$$\times f\left((1-x)^{\frac{1}{N}},(1-x)^{-\frac{1}{N}}\omega^{-s-1}\middle|(x-1)^{\frac{1}{N}}\omega^{\beta+\frac{1}{2}}\right)$$
$$\times F\left(\begin{matrix}(1-x)^{\frac{1}{N}} & (x-1)^{-\frac{1}{N}}\omega^{\beta-s-\frac{1}{2}}\\ (1-x)^{-\frac{1}{N}}\omega^{-r-1} & (x-1)^{\frac{1}{N}}\omega^{\beta-\frac{1}{2}}\end{matrix}\middle|-\omega^{\alpha+\frac{1}{2}}\right).$$

The second formula is computed in a similar way.

□

3. Determination of the operators A and B up to a factor

The following Lemma shows that we can completely determine A and B in terms of the functions $A_{\alpha,\beta}$ and $B_{\alpha,\beta}$.

LEMMA 3.1. *For all $x \in \mathbb{R}_{\neq 0,1}$ and all $\alpha, \beta \in \mathbb{Z}_N^2$, the following equalities hold true*

$$\begin{array}{ll} A^*_{\alpha,\beta}(x) = A_{\beta,\alpha}(x^{-1}), & B^*_{\alpha,\beta}(x) = B_{\beta,\alpha}\left(\frac{x}{x-1}\right), \\ 1 = \sum_{\beta\in\mathbb{Z}_N^2} A_{\alpha,\beta}(x)\bar{A}^*_{\alpha,\beta}(x), & 1 = \sum_{\beta\in\mathbb{Z}_N^2} B_{\alpha,\beta}(x)\bar{B}^*_{\alpha,\beta}(x), \\ 1 = \sum_{\beta\in\mathbb{Z}_N^2} A_{\beta,\alpha}(x^{-1})\bar{A}_{\alpha,\beta}(x), & 1 = \sum_{\beta\in\mathbb{Z}_N^2} B_{\beta,\alpha}\left(\frac{x}{x-1}\right)\bar{B}_{\alpha,\beta}(x). \end{array}$$

PROOF. First, we have

$$A_{\alpha,\beta}(x) = \left\langle e_\beta\left(x^{-1}\right), Ae_\alpha(x)\right\rangle = \left\langle A^* e_\beta\left(x^{-1}\right), e_\alpha(x)\right\rangle = A^*_{\beta,\alpha}\left(x^{-1}\right). \quad (3.1)$$

Hence we have

$$A^*_{\alpha,\beta}(x) = A_{\beta,\alpha}\left(x^{-1}\right).$$

Secondly, from the equality $A^2 = \mathrm{Id}_{\mathcal{H}}$, we have

$$\begin{aligned} 1 &= \langle \bar{e}_\alpha(x), e_\alpha(x)\rangle = \left\langle \bar{e}_\alpha(x), A^2 e_\alpha(x)\right\rangle = \langle A^*\bar{e}_\alpha(x), Ae_\alpha(x)\rangle \\ &= \sum_{\beta,\gamma\in\mathbb{Z}_N^2} A_{\alpha,\beta}(x)\bar{A}^*_{\alpha,\gamma}(x)\left\langle e_\gamma\left(x^{-1}\right), \bar{e}_\beta\left(x^{-1}\right)\right\rangle = \sum_{\beta\in\mathbb{Z}_N^2} A_{\alpha,\beta}(x)\bar{A}^*_{\alpha,\beta}(x). \quad (3.2) \end{aligned}$$

Thus, we have

$$1 = \sum_{\beta\in\mathbb{Z}_N^2} A_{\alpha,\beta}(x)\bar{A}^*_{\alpha,\beta}(x).$$

Finally, we also have

$$\bar{A}^*_{\alpha,\beta}(x) = \left\langle \bar{e}_\beta\left(x^{-1}\right), A^*\bar{e}_\alpha(x)\right\rangle = \left\langle A\bar{e}_\beta\left(x^{-1}\right), \bar{e}_\alpha(x)\right\rangle = \bar{A}_{\beta,\alpha}\left(x^{-1}\right). \quad (3.3)$$

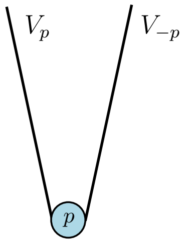
Therefore we have

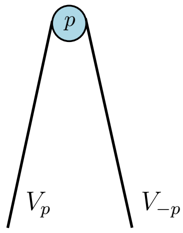
$$1 = \sum_{\beta\in\mathbb{Z}_N^2} A_{\beta,\alpha}(x^{-1})\bar{A}_{\alpha,\beta}(x).$$

By doing similar computations to (3.1),(3.2) and (3.3) with B, we easily show the last three formulae. □

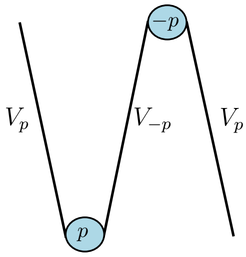
3.1. Graphical notation. Following [**5**], the morphisms of the category of $\mathcal{A}_{\omega,t}$-modules will be represented by plane diagrams to be read from the bottom to the top. The diagrams are made of (non-oriented) arcs colored by objects of the category and of boxes or circles colored by morphisms of the category.

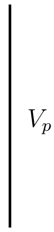
The graphical representation of the duality morphism is as follows

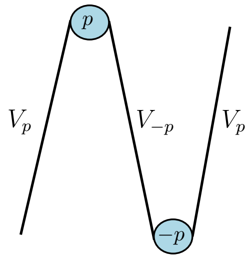
b_p = 

, d_p = 

Therefore, the equality (2.21) is graphically given by

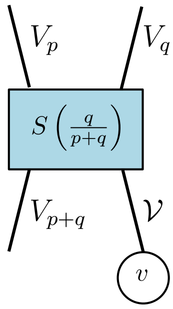


= 

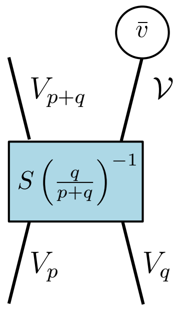
= 

Let $v \in \mathcal{V}$ and $\bar{v} \in \mathcal{V}^*$. Then $h_{p,q}(v) \in \mathcal{H}_{p,q}$ and $\bar{h}_{p,q}(\bar{v}) \in \bar{\mathcal{H}}_{p,q}$ are graphically represented in the following manner

$$h_{p,q}(v) = S\left(\frac{q}{p+q}\right)(\mathrm{Id}_{\mathcal{V}} \otimes v) =$$

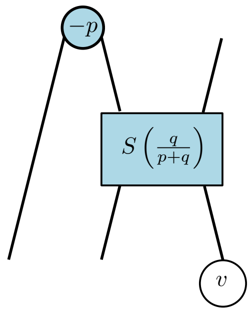


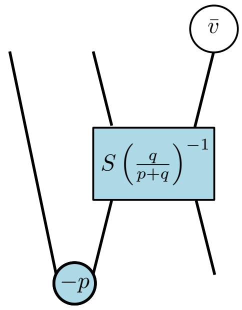
$$\bar{h}_{p,q}(\bar{v}) = (\mathrm{Id}_{\mathcal{V}} \otimes \bar{v})\, S\left(\frac{q}{p+q}\right)^{1} =$$

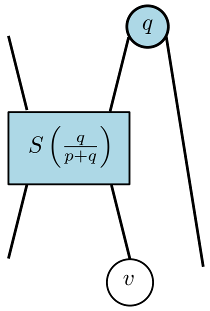


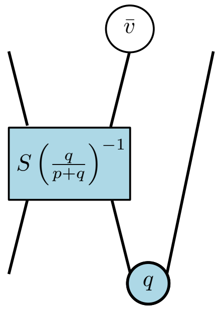
From now on, we will omit the labellings of the $\mathcal{A}_{\omega,t}$-modules and the multiplicity spaces since we can easily recover this information.

Thus, the operators A and B are represented in the following graphical form. For each $v \in \mathcal{V}$, $\bar{v} \in \mathcal{V}^*$ and each admissible pair $(p, q) \in \mathbb{R}^2_{\neq 0}$, we have

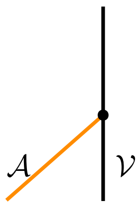
$Ah_{p,q}(v) =$ 

, $A\bar{h}_{p,q}(\bar{v}) =$ 

$Bh_{p,q}(v) =$ 

, $B\bar{h}_{p,q}(\bar{v}) =$ 

Finally, the action $\mathcal{A} \times \mathcal{V} \to \mathcal{V}$ of $\mathcal{A}$ on $\mathcal{V}$ will be represented in the following way



3.2. Linear forms $A_{p,q}$ and $B_{p,q}$. Let $(p, q) \in \mathbb{R}^2_{\neq 0}$ be an admissible pair. We define two linear forms

$$A_{p,q} : \mathcal{V}^{\otimes 2} \to \mathbb{C} \qquad \text{and} \qquad B_{p,q} : \mathcal{V}^{\otimes 2} \to \mathbb{C}$$

by

$$A_{p,q}(u \otimes v) = \langle h_{-p,p+q}(u), Ah_{p,q}(v) \rangle \,, \tag{3.4}$$

$$B_{p,q}(u \otimes v) = \langle Bh_{p,q}(u), h_{p+q,-q}(v) \rangle \,, \tag{3.5}$$

where $u, v \in \mathcal{V}$. In particular, for any $\alpha, \beta \in \mathbb{Z}^2_N$ we have the following equalities

$$A_{\alpha,\beta}\left(\tfrac{q}{p+q}\right) = A_{p,q}\left(u_\beta \otimes u_\alpha\right), \tag{3.6}$$

$$B_{\alpha,\beta}\left(\tfrac{q}{p+q}\right) = B_{p,q}\left(u_\alpha \otimes u_\beta\right). \tag{3.7}$$

Indeed, we have

$$\begin{aligned} Ah_{p,q}(u_\alpha) &= \sum_{\beta\in\mathbb{Z}_N^2} \langle (Ah_{p,q}(u_\alpha))\, h_{-p,p+q}(u_\beta)\rangle\, \bar{h}_{-p,p+q}(\bar{u}_\beta) \\ &= \sum_{\beta\in\mathbb{Z}_N^2} \langle h_{-p,p+q}(u_\beta), Ah_{p,q}(u_\alpha)\rangle\, \bar{h}_{-p,p+q}(\bar{u}_\beta) = \sum_{\beta\in\mathbb{Z}_N^2} A_{p,q}\,(u_\beta\otimes u_\alpha)\, \bar{h}_{-p,p+q}(\bar{u}_\beta) \end{aligned}$$

and

$$\begin{aligned} Bh_{p,q}(u_\alpha) &= \sum_{\beta\in\mathbb{Z}_N^2} \langle (Bh_{p,q}(u_\alpha))\, h_{p+q,-q}(u_\beta)\rangle\, \bar{h}_{p+q,-q}(\bar{u}_\beta) \\ &= \sum_{\beta\in\mathbb{Z}_N^2} \langle Bh_{p,q}(u_\alpha), h_{p+q,-q}(u_\beta)\rangle\, \bar{h}_{p+q,-q}(\bar{u}_\beta) = \sum_{\beta\in\mathbb{Z}_N^2} B_{p,q}\,(u_\alpha\otimes u_\beta)\, \bar{h}_{p+q,-q}(\bar{u}_\beta). \end{aligned}$$

In the next Lemma we will show that each of the linear forms $A_{p,q}$ and $B_{p,q}$ satisfy an equation involving the algebra morphisms $\Delta'_x, \Delta''_x : \mathcal{A}\to\mathcal{A}^{\otimes 2}$ defined by

$$\Delta'_x(a) = S(x)^{-1}(\mathrm{Id}_{\mathcal{V}}\otimes a)S(x), \tag{3.8}$$

$$\Delta''_x(a) = PS(x)^{-1}(\mathrm{Id}_{\mathcal{V}}\otimes a)S(x)P, \tag{3.9}$$

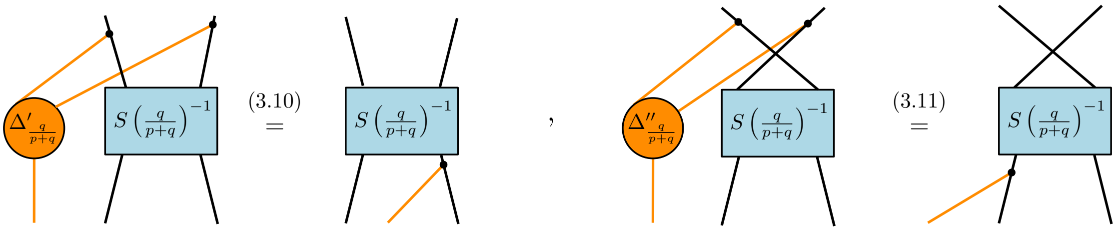
where $x\in\mathbb{R}_{\neq 0,1}$, $a\in\mathcal{A}$ and where the flip P permutes the first and the second tensor factors of $\mathcal{V}^{\otimes 2}$.

These equations will help us to compute $A_{\alpha,\beta}\left(\frac{q}{p+q}\right)\in\mathbb{C}$ and $B_{\alpha,\beta}\left(\frac{q}{p+q}\right)\in\mathbb{C}$ for all $\alpha,\beta\in\mathbb{Z}_N^2$, up to a factor depending on $\frac{q}{p+q}$. In order to do so, we write the equations (3.8) and (3.9) in the following way

$$\Delta'_x(a)S(x)^{-1} = S(x)^{-1}(\mathrm{Id}_{\mathcal{V}}\otimes a), \tag{3.10}$$

$$\Delta''_x(a)PS(x)^{-1} = PS(x)^{-1}(a\otimes \mathrm{Id}_{\mathcal{V}}) \tag{3.11}$$

For an admissible pair $(p,q)\in\mathbb{R}^2_{\neq 0}$, we draw these actions in the following manner



LEMMA 3.2. *Let $(p,q)\in\mathbb{R}^2_{\neq 0}$ be an admissible pair. Then the linear forms $A_{p,q}$ and $B_{p,q}$ satisfy the following equations*

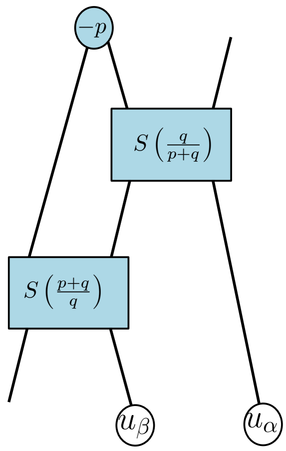
$$a\otimes A_{p,q} = (\mathrm{Id}_{\mathcal{V}}\otimes A_{p,q})\left(\Delta'_{\frac{p+q}{q}}\otimes \mathrm{Id}_{\mathcal{V}}\right)\Delta'_{\frac{q}{p+q}}(a), \quad \forall a\in\mathcal{A}, \tag{3.12}$$

$$B_{p,q}\otimes a = (B_{p,q}\otimes \mathrm{Id}_{\mathcal{V}})\left(\mathrm{Id}_{\mathcal{V}}\otimes\Delta''_{\frac{-q}{p}}\right)\Delta''_{\frac{q}{p+q}}(a), \quad \forall a\in\mathcal{A}. \tag{3.13}$$

PROOF. For all $\alpha, \beta \in \mathbb{Z}_N^2$, we have

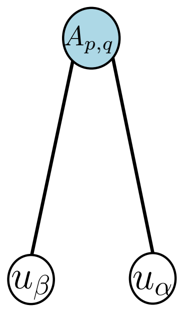
$$A_{p,q}(u_\beta \otimes v_\alpha) = \langle h_{-p,p+q}(u_\beta), Ah_{p,q}(u_\alpha)\rangle = \langle (Ah_{p,q}(u_\alpha))\, h_{-p,p+q}(u_\beta)\rangle$$

But $(Ah_{p,q}(u_\alpha))\, h_{-p,p+q}(u_\beta) \in \mathrm{End}(V_q)$ is graphically represented by

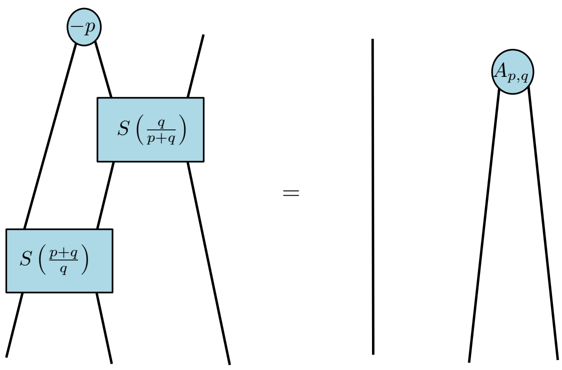


Hence, setting

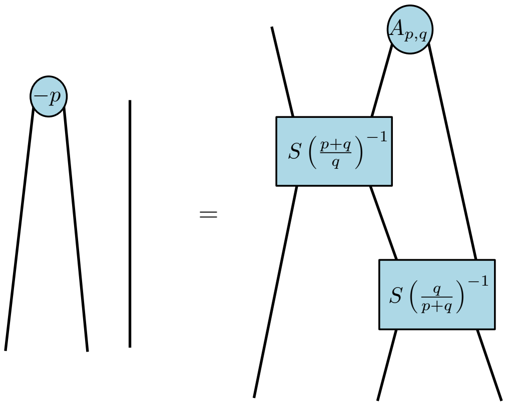
$$A_{p,q}\left(u_\beta \otimes u_\alpha\right) \quad =$$



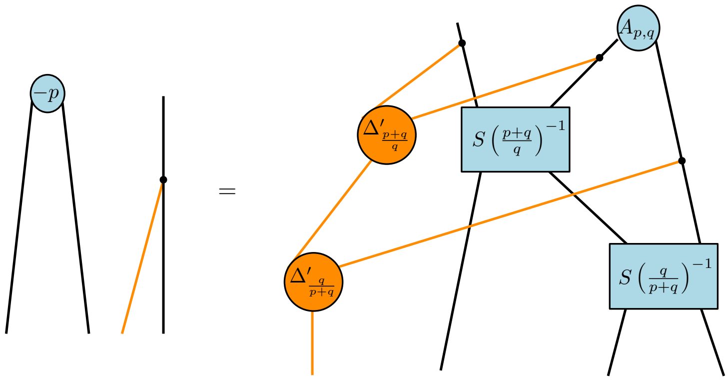
we get to the following equality



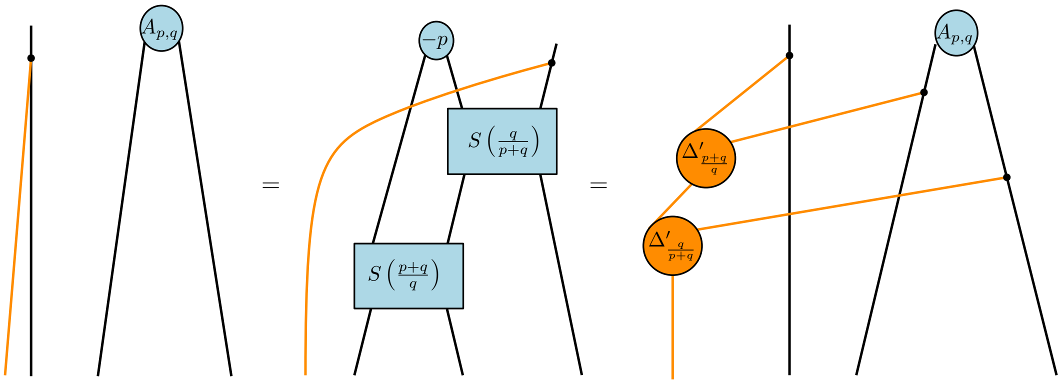
By multiplying this equality by $\left(S\left(\frac{p+q}{q}\right)^{-1} \otimes \mathrm{Id}_{\mathcal{V}}\right)\left(\mathrm{Id}_{V_{-p}} \otimes S\left(\frac{q}{p+q}\right)^{-1}\right)$, we have



Then, using (3.10) two times, we have



By multiplying this equality by $\left(\mathrm{Id}_{V_{-p}} \otimes S\left(\frac{q}{p+q}\right)\right)\left(S\left(\frac{p+q}{q}\right) \otimes \mathrm{Id}_{\mathcal{V}}\right)$, we obtain



Hence, for all $a \in \mathcal{A}$, the left-hand side and the right-hand side of the last equality can be written as

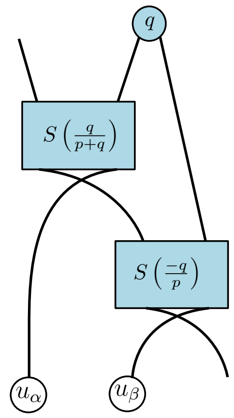
$$a \otimes A_{p,q} = (\mathrm{Id}_{V_q} \otimes A_{p,q}) \left(\Delta'_{\frac{p+q}{q}} \otimes \mathrm{Id}_{\mathcal{V}}\right) \Delta'_{\frac{q}{p+q}}(a), \quad \forall a \in \mathcal{A},$$

which is equivalent to (3.12) since, V_q is, as a vector space, equal to $\mathcal{V}$.

We do a similar argument to show (3.13). For all $\alpha, \beta \in \mathbb{Z}_N^2$, we have

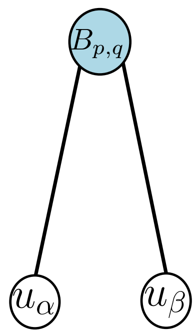
$$B_{p,q}(u_\alpha \otimes u_\beta) = \langle Bh_{p,q}(u_\alpha), h_{p+q,-q}(u_\beta)\rangle = \langle (Bh_{p,q}(u_\alpha))\, h_{p+q,-q}(u_\beta)\rangle$$

But $(Bh_{p,q}(u_\alpha))\, h_{p+q,-q}(u_\beta) \in \mathrm{End}(V_p)$ can graphically represented by

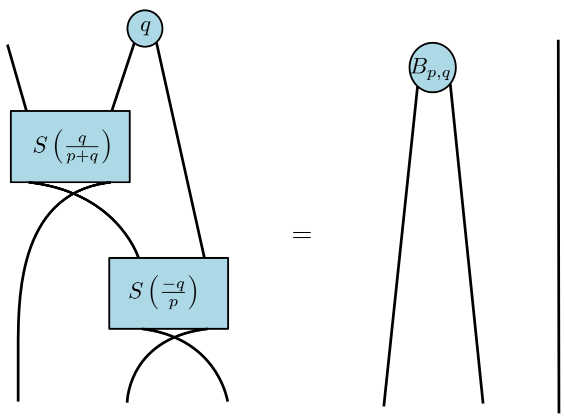


Setting,

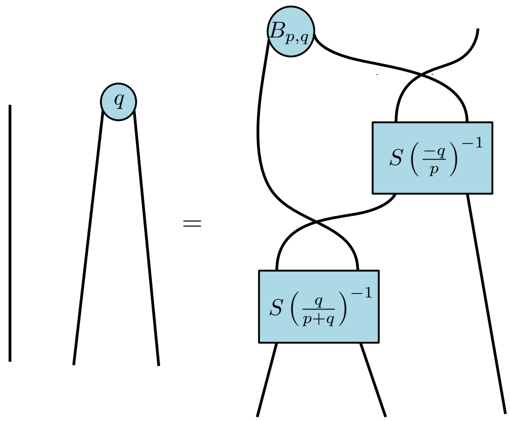
$$B_{p,q}\left(u_\alpha \otimes u_\beta\right) =$$



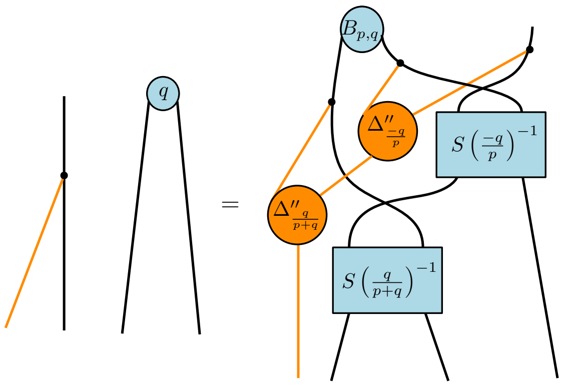
we get to the following equality



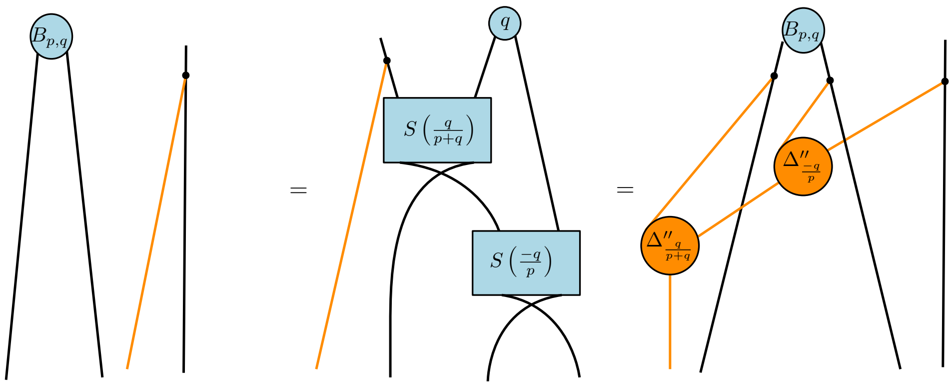
By multiplying this equality by $\left(\mathrm{Id}_{\mathcal{V}} \otimes PS\left(\frac{-q}{p}\right)^{-1}\right)\left(PS\left(\frac{q}{p+q}\right)^{-1} \otimes \mathrm{Id}_{V_{-q}}\right)$, we have



Then, using (3.11) two times, we have



By multiplying this equality by $\left(S\left(\frac{q}{p+q}\right)P \otimes \mathrm{Id}_{V_{-p}}\right)\left(\mathrm{Id}_{\mathcal{V}} \otimes S\left(\frac{-q}{p}\right)P\right)$, we obtain



Hence, for all $a \in \mathcal{A}$, the left-hand side and the right-hand side of the last equality can be written as

$$B_{p,q} \otimes a = \left(B_{p,q} \otimes \mathrm{Id}_{V_p}\right)\left(\mathrm{Id}_{\mathcal{V}} \otimes \Delta''_{\frac{-q}{p}}\right)\Delta''_{\frac{q}{p+q}}(a), \quad \forall a \in \mathcal{A}.$$

which is equivalent to (3.13) since, V_p is, as a vector space, equal to $\mathcal{V}$. $\square$

3.3. Determination of A up to a factor. In order to determine $A_{\alpha,\beta}$ for all $\alpha, \beta \in \mathbb{Z}_N^2$, we need to find an optimal set of generators of $\mathcal{A}$ with respect to Δ'_x in the sense that their images are given by simplest possible expressions in terms of themselves.

LEMMA 3.3. *The elements*

$$X' = U^{-1}V^{t+1}, \quad Y' = X^tYU^{-1}, \quad U' = X^{t+1}V^{-1}, \quad V' = X^{t+1}U^{-1} \tag{3.14}$$

generate $\mathcal{A}$ and have the following properties

$$\begin{aligned}
&\Delta'_x(X') = X'_1X'_2, \quad \Delta'_x(Y') = x^{\frac{1}{N}}Y'_1 + (1-x)^{\frac{1}{N}}X'_1Y'_2\\
&\Delta'_x(U') = x^{\frac{2}{N}}U'_1 + (1-x)^{\frac{2}{N}}(X'_1)^{t+1}U'_2 + (x-x^2)^{\frac{1}{N}}(X'_1)^tY'_1Z'_2,\\
&\Delta'_x(V') = x^{\frac{2}{N}}V'_1 + (1-x)^{\frac{2}{N}}(X'_1)^{t+1}V'_2 + (x-x^2)^{\frac{1}{N}}Z'_1X'_1Y'_2,
\end{aligned}$$

where $Z' = (U' + V')(Y')^{-1}$.

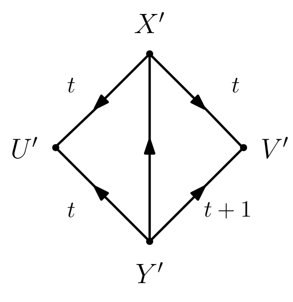
REMARK 3.4. It directly follows form this Lemma that

$$\Delta'_x(Z') = x^{\frac{1}{N}}Z'_1 + (1-x)^{\frac{1}{N}}(X'_1)^tZ'_2.$$

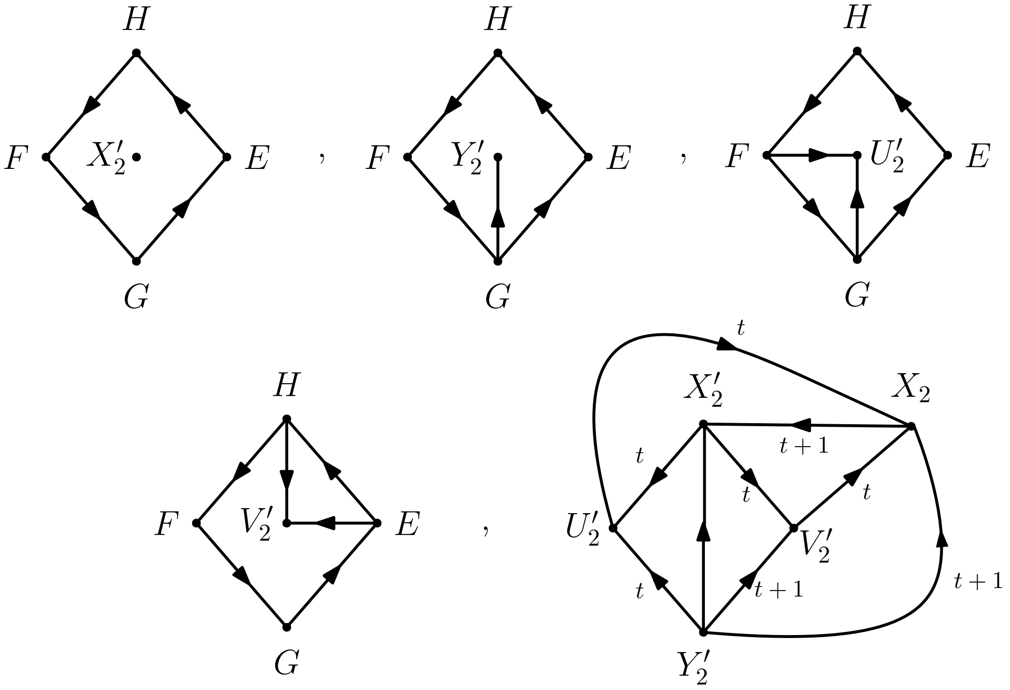
PROOF. $\mathcal{A}$ is clearly generated by the set $\{X', Y', U', V'\}$ since

$$\begin{aligned}
&X = \omega^{t+\frac{1}{2}}(X')^{-1}(U')^{-(t+1)}V', && Y = \omega^{t-\frac{1}{2}}(X')^{-1}Y'(U')^{-(t+1)},\\
&U = (X')^{-(t+1)}(U')^{-t}(V')^t, && V = (X')^{-(t+1)}(U')^{-(t+1)}(V')^{t+1}.
\end{aligned}$$

Note that we have the following commutation relations



The formulae for Δ'_x are computed in the same way as we did in the proof of the Proposition 2.8. Indeed, we have the following commutation relations



which lead to the result. □

LEMMA 3.5. *The operator $A_{p,q}$ is a solution of the following system of homogeneous linear equations*

$$\begin{aligned}
A_{p,q} &= A_{p,q}(X' \otimes X') \\
A_{p,q} &= A_{p,q}\left(X'(Y')^{-1} \otimes Y'\right) \\
A_{p,q} &= A_{p,q}\left((X')^{t+1}(V')^{-1} \otimes U'\right) \\
A_{p,q} &= \omega^{-1} A_{p,q}\left(U'(V')^{-1} \otimes U'(V')^{-1}\right)
\end{aligned}$$

PROOF. By Lemma 3.3, we can easily apply the equation (3.12) to X', Y', Z' and V'.

For X', we have

$$X' \otimes A_{p,q} = (\mathrm{Id}_{\mathcal{V}} \otimes A_{p,q})\left(\Delta'_{\frac{p+q}{q}} \otimes \mathrm{Id}_{\mathcal{V}}\right)\Delta'_{\frac{q}{p+q}}(X') = (\mathrm{Id}_{\mathcal{V}} \otimes A_{p,q})X'_1 X'_2 X'_3.$$

This implies that

$$A_{p,q} = A_{p,q}(X' \otimes X'). \tag{3.15}$$

For Y' we have

$$\begin{aligned}
Y'\otimes A_{p,q} &= (\mathrm{Id}_{\mathcal{V}}\otimes A_{p,q})\left(\Delta'_{\frac{p+q}{q}}\otimes \mathrm{Id}_{\mathcal{V}}\right)\Delta'_{\frac{q}{p+q}}(Y')\\
&= (\mathrm{Id}_{\mathcal{V}}\otimes A_{p,q})\left(\Delta'_{\frac{p+q}{q}}\otimes \mathrm{Id}_{\mathcal{V}}\right)\left(\left(\tfrac{q}{p+q}\right)^{\frac{1}{N}}Y'_1+\left(\tfrac{p}{p+q}\right)^{\frac{1}{N}}X'_1Y'_2\right)\\
&= (\mathrm{Id}_{\mathcal{V}}\otimes A_{p,q})\left(Y'_1+\left(\tfrac{p}{p+q}\right)^{\frac{1}{N}}X'_1\,(X'_2Y'_3-Y'_2)\right)\\
&= Y'\otimes A_{p,q}+\left(\tfrac{p}{p+q}\right)^{\frac{1}{N}}(X'\otimes A_{p,q})\,(X'_2Y'_3-Y'_2)
\end{aligned}$$

Therefore, we have

$$A_{p,q}(Y'\otimes \mathrm{Id}_{\mathcal{V}}) = A_{p,q}(X'\otimes Y'). \tag{3.16}$$

With a computation similar to the previous one, we find for Z' the following equality

$$A_{p,q}(Z'\otimes \mathrm{Id}_{\mathcal{V}}) = A_{p,q}((X')^t\otimes Z'). \tag{3.17}$$

For V', we first compute

$$\begin{aligned}
&\left(\Delta'_{\frac{p+q}{q}}\otimes \mathrm{Id}_{\mathcal{V}}\right)\Delta'_{\frac{q}{p+q}}(V')\\
&\quad= \left(\Delta'_{\frac{p+q}{q}}\otimes \mathrm{Id}_{\mathcal{V}}\right)\left(\left(\tfrac{q}{p+q}\right)^{\frac{2}{N}}V'_1+\left(\tfrac{p}{p+q}\right)^{\frac{2}{N}}(X'_1)^{t+1}V'_2+\left(\tfrac{pq}{(p+q)^2}\right)^{\frac{1}{N}}Z'_1X'_1Y'_2\right)\\
&= V'_1+\left(\tfrac{p}{p+q}\right)^{\frac{1}{N}}Z'_1X'_1\,(X'_2Y'_3-Y'_2)+\left(\tfrac{p}{p+q}\right)^{\frac{2}{N}}(X'_1)^{t+1}\left(V'_2+(X'_2)^{t+1}V'_3-Z'_2X'_2Y'_3\right)
\end{aligned}$$

Hence, we have

$$\begin{aligned}
V'\otimes A_{p,q} &= (\mathrm{Id}_{\mathcal{V}}\otimes A_{p,q})\left(\Delta'_{\frac{p+q}{q}}\otimes \mathrm{Id}_{\mathcal{V}}\right)\Delta'_{\frac{q}{p+q}}(V')\\
&= V'\otimes A_{p,q}+\left(\tfrac{p}{p+q}\right)^{\frac{1}{N}}(Z'X'\otimes A_{p,q})\,\underline{\left(X'_2Y'_3-Y'_2\right)}+\\
&\qquad\left(\tfrac{p}{p+q}\right)^{\frac{2}{N}}((X')^{t+1}\otimes A_{p,q})\left(V'_2+(X'_2)^{t+1}V'_3-Z'_2X'_2Y'_3\right)\\
&\overset{(3.16)}{=} V'\otimes A_{p,q}+\left(\tfrac{p}{p+q}\right)^{\frac{2}{N}}((X')^{-t}\otimes A_{p,q})\left(V'_2+(X'_2)^{t+1}V'_3-Z'_2X'_2Y'_3\right)
\end{aligned}$$

from which we deduce that

$$A_{p,q}(V'\otimes \mathrm{Id}_{\mathcal{V}})+A_{p,q}((X')^{t+1}\otimes V') = A_{p,q}(Z'X'\otimes Y'). \tag{3.18}$$

Using (3.17) on the right hand side, we also have

$$A_{p,q}(Z'X'\otimes Y') = A_{p,q}((X')^{t+1}\otimes Z'Y') = A_{p,q}((X')^{t+1}\otimes U')+A_{p,q}((X')^{t+1}\otimes V')$$

which leads to

$$A_{p,q}(V'\otimes \mathrm{Id}_{\mathcal{V}}) = A_{p,q}((X')^{t+1}\otimes U'). \tag{3.19}$$

But if we use (3.16) on the right hand instead, we have

$$A_{p,q}(Z'X'\otimes Y') \overset{(3.16)}{=} A_{p,q}(Y'(X')^{-1}Z'X'\otimes \mathrm{Id}_{\mathcal{V}}) = \omega^{-(t+1)}A_{p,q}(Y'Z'\otimes \mathrm{Id}_{\mathcal{V}})$$
$$= \omega^{-1}A_{p,q}(U'\otimes \mathrm{Id}_{\mathcal{V}}) + A_{p,q}(V'\otimes \mathrm{Id}_{\mathcal{V}})$$

which leads to

$$A_{p,q}((X')^{t+1}\otimes V') = \omega^{-1}A_{p,q}(U'\otimes \mathrm{Id}_{\mathcal{V}}). \tag{3.20}$$

Finally, by (3.19) and (3.20), we have

$$A_{p,q}(V'(X')^{-(t+1)}\otimes (U')^{-1}) = A_{p,q} = \omega^{-1}A_{p,q}(U'(X')^{-(t+1)}\otimes (V')^{-1}).$$

which leads to

$$A_{p,q} = \omega^{-1}A_{p,q}\left(U'(V')^{-1}\otimes U'(V')^{-1}\right).$$

□

LEMMA 3.6. *For all $x\in\mathbb{R}_{\neq 0,1}$ and all $\alpha,\beta\in\mathbb{Z}_N^2$ we have*

$$A_{\alpha,\beta}(x) = \mathfrak{a}_{\alpha,\beta}\,[\![A,x]\!]$$

where $[\![A,]\!] = A_{(0,0),(0,0)}:\mathbb{R}_{\neq 0,1}\to\mathbb{C}$ and

$$\mathfrak{a}_{\alpha,\beta} = \delta_{\alpha_2,-\beta_2}\omega^{-\frac{1}{2}(t(\alpha_1-\alpha_2-\beta_1)(\alpha_1-\alpha_2-\beta_1-1)-\alpha_1(\alpha_1-3)-(t+1)\alpha_2-\beta_1(\beta_1+2t+1))}.$$

PROOF. Consider $\alpha,\beta\in\mathbb{Z}_N^2$ and let $(p,q)\in(\mathbb{R}_{\neq 0})^2$ be an admissible pair such that $x=\frac{q}{p+q}$. By equality (3.6), it is enough to show that

$$A_{p,q}(u_\beta\otimes u_\alpha) = \mathfrak{a}_{\alpha,\beta}A_{p,q}(u_{(0,0)}\otimes u_{(0,0)}).$$

We show this later equality by using Lemma 3.5 and Lemma 2.10.

We start with the last equality of the Lemma 3.5. We have

$$A_{p,q}(u_\beta\otimes u_\alpha) = \omega^{-1}A_{p,q}(UV^{-1}\otimes UV^{-1})(u_\beta\otimes u_\alpha) = \omega^{\alpha_2+\beta_2}A_{p,q}(u_\beta\otimes u_\alpha).$$

Hence we have

$$A_{p,q}(u_\beta\otimes u_\alpha) = \delta_{\alpha_2,-\beta_2}A_{p,q}(u_\beta\otimes u_\alpha). \tag{3.21}$$

We consider the third equality of the Lemma 3.5

$$A_{p,q} = A_{p,q}((V')^{-1}\otimes (X')^{-(t+1)}U').$$

Since $X',U',V'\in\mathcal{A}$ are invertible, this equality is equivalent to

$$A_{p,q} = A_{p,q}\left((V')^{-1}\otimes (X')^{-(t+1)}U'\right)^{-(t+1)}.$$

A straightforward computation shows that

$$\left((V')^{-1}\otimes (X')^{-(t+1)}U'\right)^{-(t+1)} = \left(UX^{-(t+1)}\otimes (U^{-1}V^{t+1})^{-(t+1)}X^{t+1}V^{-1}\right)^{-(t+1)}$$
$$= \omega^{-t-\frac{1}{2}}\left(X^tU^{-(t+1)}\otimes X^{-t}(UV^{-1})^{-t}\right),$$

hence, the following equality holds true

$$A_{p,q} = \omega^{-t-\frac{1}{2}} A_{p,q}\left(X^t U^{-(t+1)} \otimes X^{-t}(UV^{-1})^{-t}\right).$$

Therefore, using Lemma 2.10, we compute

$$\begin{aligned}
A_{p,q}(u_\alpha \otimes u_\beta) &= \delta_{\alpha_2,-\beta_2} A_{p,q}(u_\alpha \otimes u_\beta)\\
&= \delta_{\alpha_2,-\beta_2}\omega^{-t-\frac{1}{2}} A_{p,q}\left(X^t U^{-(t+1)} \otimes X^{-t}(UV^{-1})^{-t}\right)(u_\alpha \otimes u_\beta)\\
&= \delta_{\alpha_2,-\beta_2}\omega^{-(t+1)\alpha_1+t(\alpha_2+\beta_1)+1} A_{p,q}\big(u_{(\alpha_1-1,\alpha_2)} \otimes u_{(\beta_1,-\alpha_2)}\big)\\
&= \delta_{\alpha_2,-\beta_2}\omega^{-(t+1)\frac{1}{2}\alpha_1(\alpha_1+1)+\alpha_1(t(\alpha_2+\beta_1)+1)} A_{p,q}\big(u_{(\alpha_1-\alpha_1,\alpha_2)} \otimes u_{(\beta_1,-\alpha_2)}\big)\\
&= \delta_{\alpha_2,-\beta_2}\omega^{-\frac{1}{2}(t+1)\alpha_1^2+\alpha_1\left(t(\alpha_2+\beta_1)-\frac{1}{2}(t-1)\right)} A_{p,q}\big(u_{(0,\alpha_2)} \otimes u_{(\beta_1,-\alpha_2)}\big). \qquad (3.22)
\end{aligned}$$

The first equality of Lemma 3.5 is equivalent to

$$A_{p,q} = A_{p,q}((X')^t \otimes (X')^t).$$

Since $(X')^t = (U^t V)^{-1}$, we use Lemma 2.10 and the previous equality to compute

$$\begin{aligned}
A_{p,q}\big(u_{(0,\alpha_2)} \otimes u_{(\beta_1,-\alpha_2)}\big) &= A_{p,q}\left((U^tV)^{-1} \otimes (U^tV)^{-1}\right)\big(u_{(0,\alpha_2)} \otimes u_{(\beta_1,-\alpha_2)}\big)\\
&= \omega^{-\beta_1-t+2} A_{p,q}\big(u_{(-1,\alpha_2)} \otimes u_{(\beta_1-1,-\alpha_2)}\big)\\
&\overset{(3.22)}{=} \omega^{-\beta_1-t+2}\omega^{-\frac{1}{2}(t+1)-\left(t(\alpha_2+\beta_1-1)-\frac{1}{2}(t-1)\right)} A_{p,q}\big(u_{(0,\alpha_2)} \otimes u_{(\beta_1-1,-\alpha_2)}\big)\\
&= \omega^{-(t+1)(\beta_1-1)-t(\alpha_2+1)} A_{p,q}\big(u_{(0,\alpha_2)} \otimes u_{(\beta_1-1,-\alpha_2)}\big)\\
&= \omega^{-\frac{1}{2}(t+1)\beta_1(\beta_1-1)-t\beta_1(\alpha_2+1)} A_{p,q}\big(u_{(0,\alpha_2)} \otimes u_{(\beta_1-\beta_1,-\alpha_2)}\big)\\
&= \omega^{-\frac{1}{2}(t+1)\beta_1^2-\beta_1\left(t\alpha_2+\frac{1}{2}(t-1)\right)} A_{p,q}\big(u_{(0,\alpha_2)} \otimes u_{(0,-\alpha_2)}\big). \qquad (3.23)
\end{aligned}$$

Using Lemma 2.10, a straightforward computation shows that for all $\alpha, \beta \in \mathbb{Z}_N^2$ we have

$$\begin{aligned}
&\left(X'(Y')^{-1} \otimes Y'\right)(u_\alpha \otimes u_\beta) = \left(U^{-1}V^{t+1}UY^{-1}X^{-t} \otimes X^tYU^{-1}\right)(u_\alpha \otimes u_\beta)\\
&= \left(V^{t+1}Y^{-1}X^{-t} \otimes X^tYU^{-1}\right)(u_\alpha \otimes u_\beta) = \omega^{(t+1)(\alpha_1-\alpha_2)}\big(u_{(\alpha_1,\alpha_2-1)} \otimes u_{(\beta_1+t+1,\beta_2+1)}\big).
\end{aligned}$$

Hence, using the second equality of Lemma 3.5 we have

$$\begin{aligned}
A_{p,q}\big(u_{(0,\alpha_2)} \otimes u_{(0,-\alpha_2)}\big) &= A_{p,q}\left(X'(Y')^{-1} \otimes Y'\right)\big(u_{(0,\alpha_2)} \otimes u_{(0,-\alpha_2)}\big)\\
&= \omega^{-(t+1)\alpha_2} A_{p,q}\big(u_{(0,\alpha_2-1)} \otimes u_{(t+1,-\alpha_2+1)}\big)\\
&\overset{(3.23)}{=} \omega^{-(t+1)\alpha_2}\omega^{-\frac{1}{2}(t+1)^3-(t+1)\left(t(\alpha_2-1)+\frac{1}{2}(t-1)\right)} A_{p,q}\big(u_{(0,\alpha_2-1)} \otimes u_{(0,-\alpha_2+1)}\big)\\
&= \omega^{-t\alpha_2+\frac{1}{2}(t+1)} A_{p,q}\big(u_{(0,\alpha_2-1)} \otimes u_{(0,-\alpha_2+1)}\big)\\
&= \omega^{-\frac{1}{2}t\alpha_2(\alpha_2+1)+\frac{1}{2}(t+1)\alpha_2} A_{p,q}\big(u_{(0,\alpha_2-\alpha_2)} \otimes u_{(0,-\alpha_2+\alpha_2)}\big)\\
&= \omega^{-\frac{1}{2}t\alpha_2^2+\frac{1}{2}\alpha_2} A_{p,q}\big(u_{(0,0)} \otimes u_{(0,0)}\big).
\end{aligned}$$

□

PROPOSITION 3.7. *For all* $\alpha, \beta \in \mathbb{Z}_N^2$, *we have*

$$\bar{A}_{\alpha,\beta}(x) = \frac{1}{N}\mathfrak{a}_{\beta,\alpha}^{-1} \left[\!\left[A, x^{-1}\right]\!\right]^{-1} \tag{3.24}$$

$$A^*_{\alpha,\beta}(x) = \mathfrak{a}_{\beta,\alpha} \left[\!\left[A, x^{-1}\right]\!\right] \tag{3.25}$$

$$\bar{A}^*_{\alpha,\beta}(x) = \frac{1}{N}\mathfrak{a}_{\alpha,\beta}^{-1} \left[\!\left[A, x\right]\!\right]^{-1} \tag{3.26}$$

PROOF. The result follows from the previous Lemma and Lemma 3.1. □

3.4. Determination of B up to a factor. As we did for $A_{\alpha,\beta}$, we need to find an optimal set of generators of $\mathcal{A}$ with respect to Δ''_x in order to determine $B_{\alpha,\beta}$ for all $\alpha, \beta \in \mathbb{Z}_N^2$.

LEMMA 3.8. *The elements*

$$X'' = XU^{t+1}V^{-1}, \quad Y'' = YV^{-1}, \quad U'' = U^{-1}, \quad V'' = V^{-1} \tag{3.27}$$

generate $\mathcal{A}$ *and have the following properties*

$$\begin{aligned}
\Delta''_x(X'') &= (U''_1)^{-(t+1)}V''_1(X''_1)^{-1}X''_2,\\
\Delta''_x(Y'') &= (1-x)^{\frac{1}{N}}Y''_2 + x^{\frac{1}{N}}Y''_1(U''_1)^{-(t+1)}(X''_1)^{-1}X''_2\\
\Delta''_x(U'') &= (1-x)^{\frac{2}{N}}U''_2 + x^{\frac{2}{N}}(U''_1)^{-t}(V''_1)^{t}(X''_1)^{-(t+1)}(X''_2)^{t+1}\\
&\quad + (x-x^2)^{\frac{1}{N}}(X''_1)^{-t}(V''_1)^{t}Z''_1(X''_2)^{t}Y''_2\\
\Delta''_x(V'') &= (1-x)^{\frac{2}{N}}V''_2 + x^{\frac{2}{N}}(U''_1)^{-(t+1)}(V''_1)^{t+1}(X''_1)^{-(t+1)}(X''_2)^{t+1}\\
&\quad + (x-x^2)^{\frac{1}{N}}Y''_1(U''_1)^{-(t+1)}(X''_1)^{-1}Z''_2X''_2
\end{aligned}$$

where $Z'' = (U'' + V'')(Y'')^{-1}$.

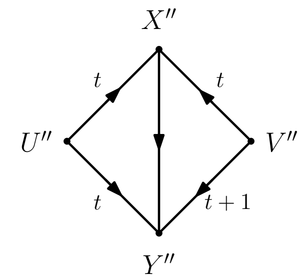
REMARK 3.9. It directly follows form this Lemma that

$$\Delta''_x(Z'') = (1-x)^{\frac{1}{N}}Z''_2 + x^{\frac{1}{N}}(X''_1)^{-t}(V''_1)^{t}Z''_1(X''_2)^{t} \tag{3.28}$$

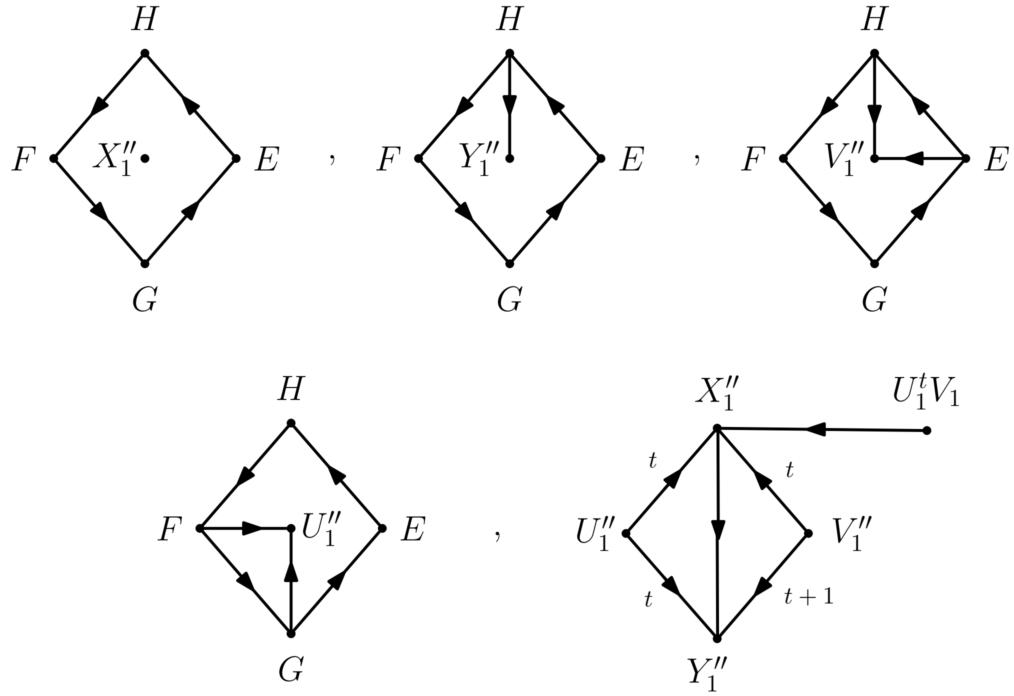
PROOF. We clearly have $\mathcal{A} = \langle X'', Y'', U'', V''\rangle$ since

$$\begin{aligned}
X &= X''(V'')^{-1}(U'')^{t+1}, & Y &= Y''(V'')^{-1},\\
U &= (U'')^{-1}, & V &= (V'')^{-1}.
\end{aligned}$$

Note that we have the following commutation relations



The formulae for Δ''_x are computed in the same way as we did in the proof of the Proposition 2.8. Indeed, we have the following commutation relations



which lead to the result. ☐

LEMMA 3.10. *The operator $B_{p,q}$ is a solution of the following system of homogeneous linear equations*

$$\begin{aligned} B_{p,q} &= B_{p,q}(X \otimes X) \\ B_{p,q} &= B_{p,q}\left(XY^{-1} \otimes Y\right) \\ B_{p,q} &= B_{p,q}\left(V^{-1} \otimes X^{-(t+1)}U\right) \\ B_{p,q} &= \omega^{-1} B_{p,q}\left(UV^{-1} \otimes UV^{-1}\right) \end{aligned}$$

PROOF. By Lemma 3.8, we can easily apply the equation (3.12) to X'', Y'', Z'' and U''.

Since $X^{-1} = (U'')^{-(t+1)}V''(X'')^{-1}$, we have for X'',

$$B_{p,q} \otimes X'' = (B_{p,q} \otimes \mathrm{Id}_{\mathcal{V}}) \left(\mathrm{Id}_{\mathcal{V}} \otimes \Delta''_{\frac{-q}{p}}\right) \Delta''_{\frac{q}{p+q}}(X'') = (B_{p,q} \otimes \mathrm{Id}_{\mathcal{V}}) X_1^{-1} X_2^{-1} X_3''.$$

This implies that

$$B_{p,q} = B_{p,q}(X \otimes X). \tag{3.29}$$

Since $Y''(U'')^{-(t+1)}(X'')^{-1} = YX^{-1}$, we have for Y''

$$\begin{aligned} B_{p,q} \otimes Y'' &= (B_{p,q} \otimes \mathrm{Id}_{\mathcal{V}}) \left(\mathrm{Id}_{\mathcal{V}} \otimes \Delta''_{\frac{-q}{p}}\right) \Delta''_{\frac{q}{p+q}}(Y'') \\ &= (B_{p,q} \otimes \mathrm{Id}_{\mathcal{V}}) \left(\mathrm{Id}_{\mathcal{V}} \otimes \Delta''_{\frac{-q}{p}}\right) \left(\left(\tfrac{p}{p+q}\right)^{\frac{1}{N}} Y_2'' + \left(\tfrac{q}{p+q}\right)^{\frac{1}{N}} Y_1 X_1^{-1} X_2''\right) \\ &= (B_{p,q} \otimes \mathrm{Id}_{\mathcal{V}}) \left(Y_3'' - \left(\tfrac{q}{p+q}\right)^{\frac{1}{N}} Y_2 X_2^{-1} X_3'' + \left(\tfrac{q}{p+q}\right)^{\frac{1}{N}} Y_1 X_1^{-1} X_2^{-1} X_3''\right) \\ &= (B_{p,q} \otimes \mathrm{Id}_{\mathcal{V}}) \left(Y_3'' + \left(\tfrac{q}{p+q}\right)^{\frac{1}{N}} \left(Y_1 X_1^{-1} - Y_2\right) X_2^{-1} X_3''\right) \\ &= B_{p,q} \otimes Y'' + \left(\tfrac{q}{p+q}\right)^{\frac{1}{N}} (B_{p,q} \otimes X'') \left(\left(Y_1 X_1^{-1} - Y_2\right) X_2^{-1}\right). \end{aligned}$$

Therefore, we have

$$B_{p,q}(\mathrm{Id}_{\mathcal{V}} \otimes Y) = B_{p,q}(YX^{-1} \otimes \mathrm{Id}_{\mathcal{V}}). \tag{3.30}$$

With a computation similar to the previous one, we find for Z'' the following equality

$$B_{p,q}(\mathrm{Id}_{\mathcal{V}} \otimes (X'')^{-t}(V'')^t Z'') = B_{p,q}((X'')^{-t}(V'')^t Z'' \otimes X^{-t}). \tag{3.31}$$

For U'', we use $X^{-1} = (U'')^{-(t+1)}V''(X'')^{-1}$, $Y''(U'')^{-(t+1)}(X'')^{-1} = YX^{-1}$ to compute first

$$\begin{aligned} &\left(\mathrm{Id}_{\mathcal{V}} \otimes \Delta''_{\frac{-q}{p}}\right) \Delta''_{\frac{q}{p+q}}(U'') \\ &\quad = \left(\mathrm{Id}_{\mathcal{V}} \otimes \Delta''_{\frac{-q}{p}}\right) \Bigg(\left(\tfrac{p}{p+q}\right)^{\frac{2}{N}} U_2'' + \left(\tfrac{q}{p+q}\right)^{\frac{2}{N}} (U_1'')^{-t}(V_1'')^t (X_1'')^{-(t+1)} (X_2'')^{t+1} \\ &\qquad\qquad + \left(\tfrac{pq}{(p+q)^2}\right)^{\frac{1}{N}} (X_1'')^{-t}(V_1'')^t Z_1'' (X_2'')^t Y_2''\Bigg) \\ &\quad = U_3'' - \left(\tfrac{q}{p+q}\right)^{\frac{1}{N}} \left((X_2'')^{-t}(V_2'')^t Z_2'' - (X_1'')^{-t}(V_1'')^t Z_1'' X_2^{-t}\right) (X_3'')^t Y_3'' \\ &\quad + \left(\tfrac{q}{p+q}\right)^{\frac{2}{N}} \Big((U_2'')^{-t}(V_2'')^t (X_2'')^{-(t+1)} + (U_1'')^{-t}(V_1'')^t (X_1'')^{-(t+1)} X_2^{-(t+1)} - \\ &\qquad\qquad (X_1'')^{-t}(V_1'')^t Z_1'' X_2^{-t} Y_2 X_2^{-1}\Big) (X_3'')^t. \end{aligned}$$

Hence, using (3.31), the equation

$$B_{p,q} \otimes U'' = (B_{p,q} \otimes \mathrm{Id}_{\mathcal{V}}) \left(\mathrm{Id}_{\mathcal{V}} \otimes \Delta''_{\frac{-q}{p}}\right) \Delta''_{\frac{q}{p+q}}(U'') \tag{3.32}$$

is equivalent to the following one

$$\begin{aligned}0 = {} & B_{p,q}\left((U_2'')^{-t}(V_2'')^t(X_2'')^{-(t+1)}\right) + \\ & B_{p,q}\left((U_1'')^{-t}(V_1'')^t(X_1'')^{-(t+1)}X_2^{-(t+1)}\right) - B_{p,q}\left(\underline{(X_1'')^{-t}(V_1'')^t Z_1'' X_2^{-t}}Y_2 X_2^{-1}\right) \\ \overset{(3.31)}{=} {} & B_{p,q}\left((U_2'')^{-t}(V_2'')^t(X_2'')^{-(t+1)}\right) + B_{p,q}\left((U_1'')^{-t}(V_1'')^t(X_1'')^{-(t+1)}X_2^{-(t+1)}\right) - \\ & B_{p,q}\left((X_2'')^{-t}(V_2'')^t Z_2'' Y_2 X_2^{-1}\right)\end{aligned}$$

A straightforward computation shows that we have the following equalities

$$\begin{aligned}(U'')^{-t}(V'')^t(X'')^{-(t+1)} = {} & \omega^{\frac{1}{2}(t-1)}X^{-(t+1)}V \\ (X'')^{-t}(V'')^t Z'' Y X^{-1} = {} & \omega^{\frac{1}{2}(t-1)}\left(X^{-(t+1)}U + X^{-(t+1)}V\right).\end{aligned}$$

Therefore (3.32) is equivalent to

$$\begin{aligned}0 = {} & B_{p,q}\left(\underline{X_1^{-(t+1)}}V_1\underline{X_2^{-(t+1)}}\right) - B_{p,q}\left(X_2^{-(t+1)}U_2\right) \\ \overset{(3.29)}{=} {} & B_{p,q}(V_1) - B_{p,q}\left(X_2^{-(t+1)}U_2\right).\end{aligned}$$

Hence we have

$$B_{p,q}(V \otimes \mathrm{Id}_{\mathcal{V}}) = B_{p,q}\left(\mathrm{Id}_{\mathcal{V}} \otimes X^{-(t+1)}U\right). \tag{3.33}$$

In order to find the last equation, we do a slightly different computation from the previous one. We have seen that (3.32) is equivalent to

$$\begin{aligned}0 = {} & B_{p,q}\left((U_2'')^{-t}(V_2'')^t(X_2'')^{-(t+1)}\right) + \\ & B_{p,q}\left((U_1'')^{-t}(V_1'')^t(X_1'')^{-(t+1)}X_2^{-(t+1)}\right) - B_{p,q}\left((X_1'')^{-t}(V_1'')^t Z_1'' X_2^{-t}Y_2 X_2^{-1}\right) \\ = {} & B_{p,q}\left((U_2'')^{-t}(V_2'')^t(X_2'')^{-(t+1)}\right) + B_{p,q}\left((U_1'')^{-t}(V_1'')^t(X_1'')^{-(t+1)}X_2^{-(t+1)}\right) - \\ & \omega^t B_{p,q}\left((X_1'')^{-t}(V_1'')^t Z_1''\underline{Y_2}X_2^{-(t+1)}\right) \\ \overset{(3.30)}{=} {} & B_{p,q}\left((U_2'')^{-t}(V_2'')^t(X_2'')^{-(t+1)}\right) + B_{p,q}\left((U_1'')^{-t}(V_1'')^t(X_1'')^{-(t+1)}X_2^{-(t+1)}\right) - \\ & \omega^t B_{p,q}\left(Y_1 X_1^{-1}(X_1'')^{-t}(V_1'')^t Z_1'' X_2^{-(t+1)}\right).\end{aligned}$$

A straightforward computation shows that

$$\omega^t Y_1 X_1^{-1}(X_1'')^{-t}(V_1'')^t Z_1'' X_2^{-(t+1)} = \omega^{\frac{1}{2}(t-1)}X_1^{-(t+1)}\left(V_1 + \omega^{-1}U_1\right)X_2^{-(t+1)}.$$

Hence, (3.32) is equivalent to

$$\begin{aligned}0 &= B_{p,q}\left((U_2'')^{-t}(V_2'')^{t}(X_2'')^{-(t+1)}\right) + B_{p,q}\left((U_1'')^{-t}(V_1'')^{t}(X_1'')^{-(t+1)}X_2^{-(t+1)}\right) - \\ &\qquad \omega^{\frac{1}{2}(t-1)}B_{p,q}X_1^{-(t+1)}\left(V_1+\omega^{-1}U_1\right)X_2^{-(t+1)} \\ &= \omega^{\frac{1}{2}(t-1)}B_{p,q}\left(X_2^{-(t+1)}V_2\right) + \omega^{\frac{1}{2}(t-1)}B_{p,q}\left(X_1^{-(t+1)}V_1X_2^{-(t+1)}\right) - \\ &\qquad \omega^{\frac{1}{2}(t-1)}B_{p,q}X_1^{-(t+1)}\left(V_1+\omega^{-1}U_1\right)X_2^{-(t+1)} \\ &= \omega^{\frac{1}{2}(t-1)}\left(B_{p,q}\left(X_2^{-(t+1)}V_2\right) - \omega^{-1}B_{p,q}\left(\underline{X_1^{-(t+1)}}U_1\underline{X_2^{-(t+1)}}\right)\right) \\ &\overset{(3.29)}{=} \omega^{\frac{1}{2}(t-1)}\left(B_{p,q}\left(X_2^{-(t+1)}V_2\right) - \omega^{-1}B_{p,q}\left(U_1\right)\right)\end{aligned}$$

which leads to

$$\omega^{-1}B_{p,q}(U\otimes \mathrm{Id}_{\mathcal{V}}) = B_{p,q}(\mathrm{Id}_{\mathcal{V}}\otimes X^{-(t+1)}V) \tag{3.34}$$

Using (3.33) and (3.34), we finally have

$$B_{p,q}(V\otimes U^{-1}X^{t+1}) = \omega^{-1}B_{p,q}(U\otimes V^{-1}X^{t+1})$$

which implies that

$$B_{p,q} = \omega^{-1}B_{p,q}(UV^{-1}\otimes UV^{-1}). \tag{3.35}$$

□

LEMMA 3.11. *For all $x\in\mathbb{R}_{\neq 0,1}$ and all $\alpha,\beta\in\mathbb{Z}_N^2$ we have*

$$B_{\alpha,\beta}(x) = \mathfrak{b}_{\alpha,\beta}\,[\![B,x]\!]$$

where $[\![B,]\!] = B_{(0,0),(0,0)} : \mathbb{R}_{\neq 0,1}\to\mathbb{C}$ and

$$\mathfrak{b}_{\alpha,\beta} = \delta_{\alpha,-\beta}\,\omega^{-\frac{1}{2}\left((t+1)(\alpha_1-\alpha_2)(\alpha_1-\alpha_2-2)+\alpha_1^2-\alpha_2\right)}.$$

PROOF. Consider $\alpha,\beta\in\mathbb{Z}_N^2$ and let $(p,q)\in(\mathbb{R}_{\neq 0})^2$ be an admissible pair such that $x=\frac{q}{p+q}$. By equality (3.7), it is enough to show that

$$B_{p,q}(u_\alpha\otimes u_\beta) = \mathfrak{b}_{\alpha,\beta}B_{p,q}(u_{(0,0)}\otimes u_{(0,0)}).$$

Since the system of Lemma 3.10 is the same as the one of Lemma 2.14, the computation of $\mathfrak{b}_{\alpha,\beta}$ is the same as the one we did for d_p in Lemma 2.15.

□

PROPOSITION 3.12. *For all $\alpha,\beta\in\mathbb{Z}_N^2$ and all $x\in\mathbb{R}_{\neq 0,1}$, we have*

$$\bar{B}_{\alpha,\beta}(x) = \mathfrak{b}_{\beta,\alpha}^{-1}\left[\!\left[B,\tfrac{x}{x-1}\right]\!\right]^{-1} \tag{3.36}$$

$$B^*_{\alpha,\beta}(x) = \mathfrak{b}_{\beta,\alpha}\left[\!\left[B,\tfrac{x}{x-1}\right]\!\right] \tag{3.37}$$

$$\bar{B}^*_{\alpha,\beta}(x) = \mathfrak{b}_{\alpha,\beta}^{-1}\,[\![B,x]\!]^{-1} \tag{3.38}$$

PROOF. The results follows from the previous Lemma and Lemma 3.1. □

4. Determination of $[\![B, x]\!]$

LEMMA 4.1. *For all $x \in \mathbb{R}_{\neq 0,1}$ we have*

$$\bar{\psi}^N_{\frac{x}{x-1},0}\psi^{-N}_{x,0} = (1-x)^{-\frac{N-1}{2}}$$

PROOF. We recall that

$$\psi^N_{x,0} = (1-x)^{\frac{N-1}{2}} D\left((1-x)^{-\frac{1}{N}}\right) D(1)^{-1},$$
$$\bar{\psi}^N_{x,0} = \varrho\left((1-x)^{\frac{1}{N}}\right)^N \psi^{-N}_{x,0}.$$

Therefore, using equality (1.6) of Lemma 1.2, we compute

$$\bar{\psi}^N_{\frac{x}{x-1},0}\psi^{-N}_{x,0} = \varrho\left((1-x)^{-\frac{1}{N}}\right)^N \psi^{-N}_{\frac{x}{x-1},0}\psi^{-N}_{x,0} = \frac{\varrho\left((1-x)^{-\frac{1}{N}}\right)^N D(1)^2}{D\left((1-x)^{\frac{1}{N}}\right) D\left((1-x)^{-\frac{1}{N}}\right)}$$
$$= \frac{\varrho\left((1-x)^{-\frac{1}{N}}\right)^N D(1)^2}{D(1)^2(1-x)^{\frac{N-1}{2}}\varrho\left((1-x)^{-\frac{1}{N}}\right)^N} = (1-x)^{-\frac{N-1}{2}}$$

□

LEMMA 4.2. *For all $x \in \mathbb{R}_{\neq 0,1}$ we have*

$$\left\langle \bar{u}_0 \otimes \bar{u}_0 \middle| S\left(\tfrac{x}{x-1}\right)^{-1} \middle| u_0 \otimes u_0 \right\rangle \equiv (1-x)^{-\frac{2(N-1)}{N}} \left\langle \bar{u}_0 \otimes \bar{u}_0 \middle| S(x) \middle| u_0 \otimes u_0 \right\rangle$$

PROOF. By Proposition 2.11 we have

$$\left\langle \bar{u}_0 \otimes \bar{u}_0 \middle| S(x) \middle| u_0 \otimes u_0 \right\rangle = \psi^4_{x,0}$$
$$\times f\left((1-x)^{\frac{1}{N}}, (1-x)^{-\frac{1}{N}}\omega^{-1} \middle| (x-1)^{\frac{1}{N}}\omega^{\frac{1}{2}}\right)$$
$$\times F\left(\begin{matrix} (1-x)^{\frac{1}{N}} & (x-1)^{-\frac{1}{N}}\omega^{-\frac{1}{2}} \\ (1-x)^{-\frac{1}{N}}\omega^{-1} & (x-1)^{\frac{1}{N}}\omega^{-\frac{1}{2}} \end{matrix} \middle| -\omega^{\frac{1}{2}}\right)$$

and

$$\left\langle \bar{u}_0 \otimes \bar{u}_0 \middle| S\left(\tfrac{x}{x-1}\right)^{-1} \middle| u_0 \otimes u_0 \right\rangle = \bar{\psi}^4_{\frac{x}{x-1},0}$$
$$\times f\left((1-x)^{\frac{1}{N}}, (1-x)^{-\frac{1}{N}}\omega^{-1} \middle| (x-1)^{\frac{1}{N}}\omega^{\frac{1}{2}}\right)$$
$$\times F\left(\begin{matrix} (1-x)^{\frac{1}{N}} & (x-1)^{-\frac{1}{N}}\omega^{-\frac{1}{2}} \\ (1-x)^{-\frac{1}{N}}\omega^{-1} & (x-1)^{\frac{1}{N}}\omega^{-\frac{1}{2}} \end{matrix} \middle| -\omega^{\frac{1}{2}}\right).$$

Therefore, by the previous Lemma, we have

$$\left\langle \bar{u}_0 \otimes \bar{u}_0 \middle| S\left(\tfrac{x}{x-1}\right)^{-1} \middle| u_0 \otimes u_0 \right\rangle = \bar{\psi}^4_{\frac{x}{x-1},0}\psi^{-4}_{x,0}\left\langle \bar{u}_0 \otimes \bar{u}_0 \middle| S(x) \middle| u_0 \otimes u_0 \right\rangle$$
$$\equiv (1-x)^{-\frac{2(N-1)}{N}} \left\langle \bar{u}_0 \otimes \bar{u}_0 \middle| S(x) \middle| u_0 \otimes u_0 \right\rangle.$$

□

PROPOSITION 4.3. *For all* $x \in \mathbb{R}_{\neq 0,1}$ *we have* $[\![B,x]\!] \equiv (1-x)^{\frac{2(N-1)}{N}}$.

PROOF. Let $(p,q) \in (\mathbb{R}_{\neq 0})^2$ be an admissible pair such that $x = \frac{q}{p+q}$. On the one hand, using Lemmas 3.11 and 4.2 and the fact that

$$\bar{h}_{p+q,-q}(\bar{u}_0) = (\mathrm{Id}_{\mathcal{V}} \otimes \bar{u}_0) S\left(\tfrac{x}{x-1}\right)^{-1},$$

we have

$$\left\langle \bar{u}_0 \middle| Bh_{p,q}(u_0) \middle| u_0 \otimes u_0 \right\rangle = \sum_{\alpha \in \mathbb{Z}_N^2} [\![B,x]\!]\, \mathfrak{b}_{0,\alpha} \left\langle \bar{u}_0 \middle| \bar{h}_{p+q,-q}(\bar{u}_\alpha) \middle| u_0 \otimes u_0 \right\rangle$$
$$= [\![B,x]\!] \left\langle \bar{u}_0 \middle| \bar{h}_{p+q,-q}(\bar{u}_0) \middle| u_0 \otimes u_0 \right\rangle = [\![B,x]\!] \left\langle \bar{u}_0 \otimes \bar{u}_0 \middle| S\left(\tfrac{x}{x-1}\right)^{-1} \middle| u_0 \otimes u_0 \right\rangle$$
$$\equiv [\![B,x]\!]\, (1-x)^{-\frac{2(N-1)}{N}} \left\langle \bar{u}_0 \otimes \bar{u}_0 \middle| S(x) \middle| u_0 \otimes u_0 \right\rangle.$$

On the other hand, by the definition (3.7) of B and that

$$h_{p,q}(u_0) = S(x)(\mathrm{Id}_{\mathcal{V}} \otimes u_0),$$

we have

$$Bh_{p,q}(u_0) = (\mathrm{Id}_{\mathcal{V}} \otimes d_q)\left(S(x) \otimes \mathrm{Id}_{\mathcal{V}}\right)(\mathrm{Id}_{\mathcal{V}} \otimes u_0 \otimes \mathrm{Id}_{\mathcal{V}}).$$

Therefore, we compute

$$\left\langle \bar{u}_0 \middle| Bh_{p,q}(u_0) \middle| u_0 \otimes u_0 \right\rangle = \left\langle \bar{u}_0 \middle| (\mathrm{Id}_{\mathcal{V}} \otimes d_q)\left(S(x) \otimes \mathrm{Id}_{\mathcal{V}}\right) \middle| u_0 \otimes u_0 \otimes u_0 \right\rangle$$
$$= \sum_{\alpha,\beta,\gamma \in \mathbb{Z}_N^2} \left\langle \bar{u}_0 \middle| (\mathrm{Id}_{\mathcal{V}} \otimes d_q) \middle| u_\alpha \otimes u_\beta \otimes u_\gamma \right\rangle \left\langle \bar{u}_\alpha \otimes \bar{u}_\beta \otimes \bar{u}_\gamma \middle| \left(S(x) \otimes \mathrm{Id}_{\mathcal{V}}\right) \middle| u_0 \otimes u_0 \otimes u_0 \right\rangle$$
$$= \sum_{\beta \in \mathbb{Z}_N^2} d_q(u_\beta \otimes u_0) \left\langle \bar{u}_0 \otimes \bar{u}_\beta \middle| S(x) \middle| u_0 \otimes u_0 \right\rangle \overset{(2.30)}{=} \left\langle \bar{u}_0 \otimes \bar{u}_0 \middle| S(x) \middle| u_0 \otimes u_0 \right\rangle.$$

Since $\left\langle \bar{u}_0 \otimes \bar{u}_0 \middle| S(x) \middle| u_0 \otimes u_0 \right\rangle \neq 0$ we conclude that $[\![B,x]\!] \equiv (1-x)^{\frac{2(N-1)}{N}}$. □

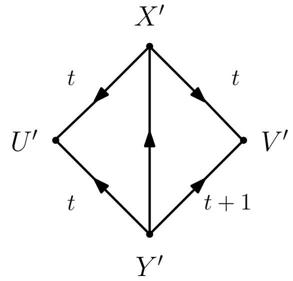
5. Determination of $[\![A,x]\!]$

The computation of $[\![A,x]\!]$ is more tricky than the one we did for $[\![B,x]\!]$. Indeed, we first need to compute A on another basis of the multiplicity spaces.

5.1. Another basis for the multiplicity spaces. We consider the following elements of $\mathcal{A}$

$$X' = U^{-1}V^{t+1}, \quad Y' = X^tYU^{-1}, \quad U' = X^{t+1}V^{-1}, \quad V' = X^{t+1}U^{-1}. \tag{5.1}$$

We have shown in Lemma 3.3 that they generate $\mathcal{A}$ and that they have the same commutation relations than the set $\{X, Y, U, V\}$, namely



This allows us to consider the element $\Omega \in \mathcal{A}$ which satisfies

$$X' = \Omega X \Omega^{-1}, \quad Y' = \Omega Y \Omega^{-1}, \quad U' = \Omega U \Omega^{-1}, \quad V' = \Omega V \Omega^{-1}. \tag{5.2}$$

Proposition 5.1. *For any admissible pair $(p, q) \in (\mathbb{R}_{\neq 0})^2$, the following equation is satisfied for all $a \in A_{\omega,t}$*

$$(\pi_p \otimes \pi_q)\Delta(a) = S'\left(\tfrac{p}{p+q}\right)(\mathrm{Id}_{\mathcal{V}} \otimes \pi_{p+q}(a))S'\left(\tfrac{p}{p+q}\right)^{-1}, \tag{5.3}$$

where $S'\left(\frac{p}{p+q}\right) = \Omega_1^{-1}\Omega_2^{-1}S\left(\frac{p}{p+q}\right)^{-1}\Omega_1\Omega_2$.

Proof. By Lemma 3.3 and equalities 5.2, we have for all $a \in \mathcal{A}_{\omega,t}$

$$\Omega_1\Omega_2(\pi_p \otimes \pi_q)\Delta(a)\Omega_1^{-1}\Omega_2^{-1} = \Delta'_{\frac{p}{p+q}}\left(\Omega\pi_{p+q}(a)\Omega^{-1}\right)$$

Therefore we have

$$\begin{aligned}(\pi_p \otimes \pi_q)\Delta(a) &= \Omega_1^{-1}\Omega_2^{-1}\Delta'_{\frac{p}{p+q}}\left(\Omega\pi_{p+q}(a)\Omega^{-1}\right)\Omega_1\Omega_2 \\ &= \Omega_1^{-1}\Omega_2^{-1}S\left(\tfrac{p}{p+q}\right)^{-1}\left(\mathrm{Id}_{\mathcal{V}} \otimes \Omega\pi_{p+q}(a)\Omega^{-1}\right)S\left(\tfrac{p}{p+q}\right)\Omega_1\Omega_2 \\ &= \Omega_1^{-1}\Omega_2^{-1}S\left(\tfrac{p}{p+q}\right)^{-1}\Omega_1\Omega_2\left(\mathrm{Id}_{\mathcal{V}} \otimes \pi_{p+q}(a)\right)\Omega_1^{-1}\Omega_2^{-1}S\left(\tfrac{p}{p+q}\right)\Omega_1\Omega_2 \\ &= S'\left(\tfrac{p}{p+q}\right)(\mathrm{Id}_{\mathcal{V}} \otimes \pi_{p+q}(a))S'\left(\tfrac{p}{p+q}\right)^{-1}\end{aligned}$$

□

For all $x \in \mathbb{R}_{\neq 0,1}$, we define

$$e'_\alpha(x) = S'(x)(u_\alpha \otimes \mathrm{Id}_{\mathcal{V}}) \quad \text{and} \quad \bar{e}'_\alpha(x) = (\bar{u}_\alpha \otimes \mathrm{Id}_{\mathcal{V}})S'(x)^{-1}. \tag{5.4}$$

If $(p, q) \in (\mathbb{R}_{\neq 0})^2$ is an admissible pair, then by equation (5.3),

$$e'_\alpha\left(\tfrac{p}{p+q}\right) \in \mathcal{H}_{p,q} \quad \text{and} \quad \bar{e}'_\alpha\left(\tfrac{p}{p+q}\right) \in \bar{\mathcal{H}}_{p,q}, \quad \forall \alpha \in \mathbb{Z}_N^2.$$

Moreover, $\left\{e'_\alpha\left(\frac{p}{p+q}\right)\right\}_{\alpha\in\mathbb{Z}_N^2}$ and $\left\{\bar{e}'_\alpha\left(\frac{p}{p+q}\right)\right\}_{\alpha\in\mathbb{Z}_N^2}$ form dual bases of $\mathcal{H}_{p,q}$ and $\bar{\mathcal{H}}_{p,q}$ respectively, where the duality is reflected by the relations

$$\bar{e}'_\beta\left(\tfrac{p}{p+q}\right)e'_\alpha\left(\tfrac{p}{p+q}\right)=\delta_{\alpha,\beta}\operatorname{Id}_{\mathcal{V}},\tag{5.5}$$

and

$$\sum_{\alpha\in\mathbb{Z}_N^2}e'_\alpha\left(\tfrac{p}{p+q}\right)\bar{e}'_\alpha\left(\tfrac{p}{p+q}\right)=\operatorname{Id}_{\mathcal{V}}\otimes\operatorname{Id}_{\mathcal{V}}\tag{5.6}$$

In order to keep track of the difference between $\mathcal{H}_{p,q}$ and $\mathcal{H}_{\lambda p,\lambda q}$ and $\bar{\mathcal{H}}_{p,q}$ and $\bar{\mathcal{H}}_{\lambda p,\lambda q}$ respectively, where $\lambda\in\mathbb{R}_{\neq 0}$, we define, for all admissible pairs $(p,q)\in(\mathbb{R}_{\neq 0})^2$, the following isomorphism

$$h'_{p,q}:\mathcal{V}\to\mathcal{H}_{p,q}\quad\text{and}\quad\bar{h}'_{p,q}:\mathcal{V}^*\to\bar{\mathcal{H}}_{p,q}$$

by

$$h'_{p,q}(u_\alpha)=e'_\alpha\left(\tfrac{p}{p+q}\right),\qquad\bar{h}'_{p,q}(\bar{u}_\alpha)=\bar{e}'_\alpha\left(\tfrac{p}{p+q}\right),\quad\forall\alpha\in\mathbb{Z}_N^2.$$

We are now going to express how the operator A acts on the basis $\left\{h'_{p,q}(u_\alpha)\right\}_{\alpha\in\mathbb{Z}_N^2}$ of $\mathcal{H}_{p,q}$. In order to do so, we introduce an analogue of the linear forms $A_{p,q}$ and $B_{p,q}$.

5.2. The linear form $A'_{p,q}$. For any admissible pair $(p,q)\in(\mathbb{R}_{\neq 0})^2$, we define the linear form $A'_{p,q}:\mathcal{V}^{\otimes 2}\to\mathbb{C}$ by

$$A'_{p,q}(u_\beta\otimes u_\alpha)=\langle h'_{-p,p+q}(u_\beta),Ah'_{p,q}(u_\alpha)\rangle$$

By definition, for any $\alpha,\beta\in\mathbb{Z}_N^2$ we have

$$Ah'_{p,q}(u_\alpha)=\sum_{\beta\in\mathbb{Z}_N^2}A'_{p,q}(u_\beta\otimes u_\alpha)\bar{h}'_{-p,p+q}(u_\beta)$$

In order to determine $A'_{p,q}$, we define for any $x\in\mathbb{R}_{\neq 0,1}$, as we did for $A_{p,q}$ and $B_{p,q}$, the algebra morphism $\Delta^\circ_x;\mathcal{A}\to\mathcal{A}^{\otimes 2}$ by

$$\Delta^\circ_x(a)=PS'(x)^{-1}(\operatorname{Id}_{\mathcal{V}}\otimes a)S'(x)P\tag{5.7}$$

Using the graphical calculus, we find that the following equality holds true for any admissible pair $(p,q)\in(\mathbb{R}_{\neq 0})^2$ and any $a\in\mathcal{A}$

$$a\otimes A'_{p,q}=\left(\operatorname{Id}_{\mathcal{V}}\otimes A'_{p,q}\right)\left(\Delta^\circ_{\frac{-p}{q}}\otimes\operatorname{Id}_{\mathcal{V}}\right)\Delta^\circ_{\frac{p}{p+q}}(a)\tag{5.8}$$

It turns out that This equality will help us to determine $A'_{p,q}$.

LEMMA 5.2. *For any $x\in\mathbb{R}_{\neq 0,1}$ we have*

$$\Delta^\circ_x(X')=X'_1X_2,\quad\Delta^\circ_x(Y')=x^{\frac{1}{N}}Y'_1+(1-x)^{\frac{1}{N}}\omega^{-t+\frac{1}{2}}X'_1Y_2$$
$$\Delta^\circ_x(U')=x^{\frac{2}{N}}U'_1+(1-x)^{\frac{2}{N}}(X'_1)^{t+1}U_2+(x-x^2)^{\frac{1}{N}}\omega^{t-\frac{1}{2}}(X'_1)^tY'_1Z_2,$$
$$\Delta^\circ_x(V')=x^{\frac{2}{N}}V'_1+(1-x)^{\frac{2}{N}}(X'_1)^{t+1}V_2+(x-x^2)^{\frac{1}{N}}\omega^{-t+\frac{1}{2}}Z'_1X'_1Y_2,$$

and

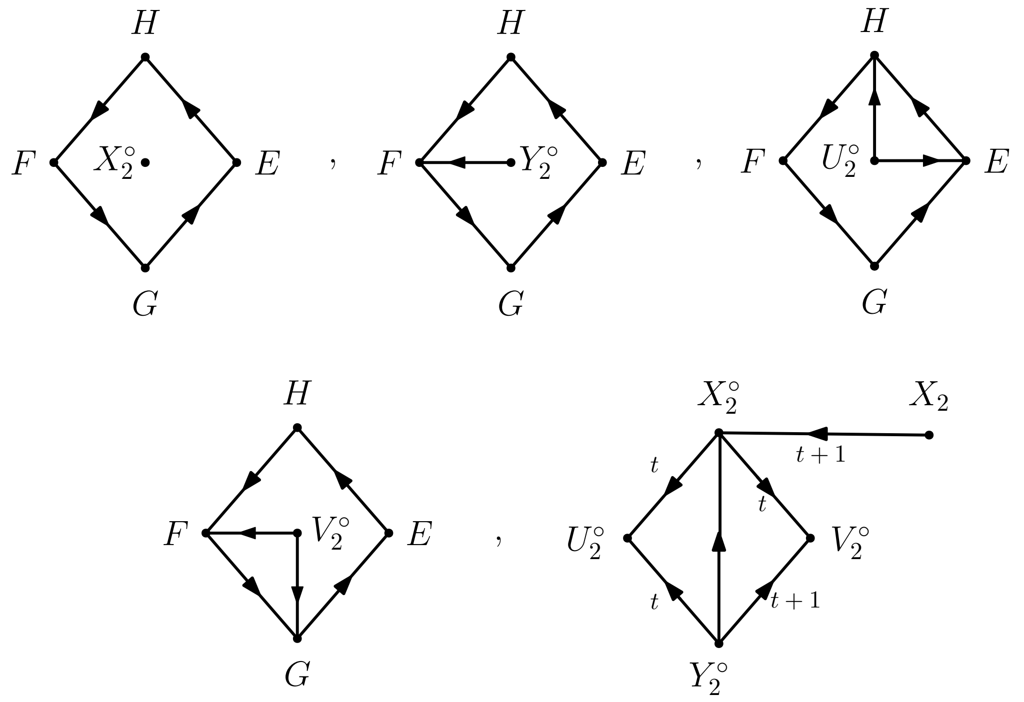
$$\Delta_x^{\circ}(Z') = x^{\frac{1}{N}} Z_1' + (1-x)^{\frac{1}{N}} \omega^{t-\frac{1}{2}} (X_1')^t Z_2$$

where $Z' = (U' + V')(Y')^{-1}$.

Proof. The formulae for Δ_x° are computed in the same way as we did in the proof of the Proposition 2.8. Since we have

$$\begin{aligned} \Omega X' \Omega^{-1} &= \omega^{t+\frac{1}{2}} X^{-1} U^{-(t+1)} V, & \Omega Y' \Omega^{-1} &= \omega^{-t} X^{-1} Y U^{-(t+1)}, \\ \Omega U' \Omega^{-1} &= X^{-(t+1)} U^{-t} V^t, & \Omega V' \Omega^{-1} &= X^{-(t+1)} U^{-(t+1)} V^{t+1}, \end{aligned}$$

and also have the following commutation relations, where we use the notation $a^{\circ} = \Omega a \Omega^{-1}$ for $a \in \{X', Y', U', V'\}$,



a straightforward computation leads to the result. □

Lemma 5.3. *The operator* $A'_{p,q}$ *is a solution of the following system of homogeneous linear equations*

$$\begin{aligned} A'_{p,q} &= A'_{p,q}(X \otimes X) \\ A'_{p,q} &= A'_{p,q}\left(XY^{-1} \otimes Y\right) \\ A'_{p,q} &= A'_{p,q}\left(V^{-1} \otimes X^{-(t+1)} U\right) \\ A'_{p,q} &= \omega^{-1} A'_{p,q}\left(UV^{-1} \otimes UV^{-1}\right) \end{aligned}$$

Proof. Using the equality (5.8) and the previous Lemma, the proof is the same as the one of Lemmas 3.5 and 3.10. □

Since the system for $A'_{p,q}$ is the same as the one for $B_{p,q}$, we deduce that for any $\alpha,\beta\in\mathbb{Z}_N^2$ we have

$$A'_{p,q}(u_\beta\otimes u_\alpha)=\mathfrak{b}_{\alpha,\beta}A'_0\left(\tfrac{p}{p+q}\right) \tag{5.9}$$

where

$$\mathfrak{b}_{\alpha,\beta}=\delta_{\alpha,-\beta}\omega^{-\frac{1}{2}\left((t+1)(\alpha_1-\alpha_2)(\alpha_1-\alpha_2-2)+\alpha_1^2-\alpha_2\right)} \tag{5.10}$$

and $A'_0\left(\frac{p}{p+q}\right)=A'_{p,q}(u_{(0,0)}\otimes u_{(0,0)})$.

LEMMA 5.4. *For any $x\in\mathbb{R}_{\neq 0,1}$, we have $A'_0(x)\equiv x^{\frac{2(N-1)}{N}}$.*

PROOF. Let $(p,q)\in(\mathbb{R}_{\neq 0})^2$ be an admissible pair and $x=\frac{p}{p+q}$.
On the one hand, using equality (5.9) and the fact that

$$\bar{h}'_{-p,p+q}(\bar{u}_0)=(\bar{u}_0\otimes\mathrm{Id}_\mathcal{V})S'\left(\tfrac{x}{x-1}\right)^{-1},$$

we have

$$\begin{aligned}\left\langle\bar{u}_0\,\middle|Ah'_{p,q}(u_0)\middle|u_0\otimes u_0\right\rangle&=\sum_{\alpha\in\mathbb{Z}_N^2}A'_0(x)\mathfrak{b}_{0,\alpha}\left\langle\bar{u}_0\,\middle|\bar{h}'_{-p,p+q}(\bar{u}_\alpha)\middle|u_0\otimes u_0\right\rangle\\&=A'_0(x)\left\langle\bar{u}_0\,\middle|\bar{h}'_{-p,p+q}(\bar{u}_0)\middle|u_0\otimes u_0\right\rangle=A'_0(x)\left\langle\bar{u}_0\otimes\bar{u}_0\middle|S'\left(\tfrac{x}{x-1}\right)^{-1}\middle|u_0\otimes u_0\right\rangle.\end{aligned}$$

On the other hand, by the definition (3.7) of A and that

$$h'_{p,q}(u_0)=S'(x)(u_0\otimes\mathrm{Id}_\mathcal{V}),$$

we have

$$Ah'_{p,q}(u_0)=(d_{-p}\otimes\mathrm{Id}_\mathcal{V})\,(\mathrm{Id}_\mathcal{V}\otimes S'(x))\,(\mathrm{Id}_\mathcal{V}\otimes u_0\otimes\mathrm{Id}_\mathcal{V}).$$

Hence, we compute

$$\begin{aligned}&\left\langle\bar{u}_0\,\middle|Ah'_{p,q}(u_0)\middle|u_0\otimes u_0\right\rangle=\left\langle\bar{u}_0\middle|(d_{-p}\otimes\mathrm{Id}_\mathcal{V})\,(\mathrm{Id}_\mathcal{V}\otimes S'(x))\,\middle|u_0\otimes u_0\otimes u_0\right\rangle\\&=\sum_{\alpha,\beta,\gamma\in\mathbb{Z}_N^2}\left\langle\bar{u}_0\middle|(d_{-p}\otimes\mathrm{Id}_\mathcal{V})\middle|u_\alpha\otimes u_\beta\otimes u_\gamma\right\rangle\left\langle\bar{u}_\alpha\otimes\bar{u}_\beta\otimes\bar{u}_\gamma\middle|\,(S(x)\otimes\mathrm{Id}_\mathcal{V})\,\middle|u_0\otimes u_0\otimes u_0\right\rangle\\&\quad=\sum_{\beta\in\mathbb{Z}_N^2}d_{-p}(u_0\otimes u_\beta)\left\langle\bar{u}_\beta\otimes\bar{u}_0\middle|S'(x)\middle|u_0\otimes u_0\right\rangle\overset{(2.30)}{=}\left\langle\bar{u}_0\otimes\bar{u}_0\middle|S'(x)\middle|u_0\otimes u_0\right\rangle.\end{aligned}$$

Therefore, we have

$$\left\langle\bar{u}_0\otimes\bar{u}_0\middle|S'(x)\middle|u_0\otimes u_0\right\rangle=A'_0(x)\left\langle\bar{u}_0\otimes\bar{u}_0\middle|S'\left(\tfrac{x}{x-1}\right)^{-1}\middle|u_0\otimes u_0\right\rangle. \tag{5.11}$$

Since we have

$$\begin{aligned}&\Omega^{-1}X'\Omega=\omega^{t+\frac{1}{2}}X^{-1}U^{-(t+1)}V, &&\Omega^{-1}Y'\Omega=\omega^{t-\frac{1}{2}}X^{-1}YU^{-(t+1)},\\&\Omega^{-1}U'\Omega=X^{-(t+1)}U^{-t}V^{t}, &&\Omega^{-1}V'\Omega=X^{-(t+1)}U^{-(t+1)}V^{t+1},\end{aligned} \tag{5.12}$$

we can compute $\left\langle \bar{u}_\mu \otimes \bar{u}_\nu \middle| S'(x) \middle| u_\alpha \otimes u_\beta \right\rangle$ and $\left\langle \bar{u}_\mu \otimes \bar{u}_\nu \middle| S'(x)^{-1} \middle| u_\alpha \otimes u_\beta \right\rangle$, for any $\alpha, \beta, \mu, \nu \in \mathbb{Z}_N^2$, as we did for $S(x)$ and $S(x)^{-1}$. In particular, we have the following equalities

$$\left\langle \bar{u}_0 \otimes \bar{u}_0 \middle| S'(x) \middle| u_0 \otimes u_0 \right\rangle = \left\langle \bar{u}_0 \otimes \bar{u}_0 \middle| S(x)^{-1} \middle| u_0 \otimes u_0 \right\rangle$$
$$\left\langle \bar{u}_0 \otimes \bar{u}_0 \middle| S'\left(\tfrac{x}{x-1}\right)^{-1} \middle| u_0 \otimes u_0 \right\rangle = \left\langle \bar{u}_0 \otimes \bar{u}_0 \middle| S\left(\tfrac{x}{x-1}\right) \middle| u_0 \otimes u_0 \right\rangle.$$

But, by Lemma 4.2 we have

$$\left\langle \bar{u}_0 \otimes \bar{u}_0 \middle| S\left(\tfrac{x}{x-1}\right) \middle| u_0 \otimes u_0 \right\rangle \equiv (1-x)^{-\frac{2(N-1)}{N}} \left\langle \bar{u}_0 \otimes \bar{u}_0 \middle| S(x)^{-1} \middle| u_0 \otimes u_0 \right\rangle$$

We use this last equality in equality (5.11) to end the proof. □

5.3. Change of basis. For all $x \in \mathbb{R}_{\neq 0,1}$, the bases $\{e_\alpha(1-x)\}_{\alpha \in \mathbb{Z}_N^2}$ and $\{e'_\alpha(x)\}_{\alpha \in \mathbb{Z}_N^2}$ lie in the same multiplicity space. Therefore, for all $\alpha, \beta \in \mathbb{Z}_N^2$, we can write

$$e'_\alpha(x) = \sum_{\beta \in \mathbb{Z}_N^2} c_{\alpha,\beta}(x) e_\beta(1-x) \tag{5.13}$$

where $c_{\alpha,\beta}(x) \in \mathbb{C}$ is given by

$$c_{\alpha,\beta}(x) = \langle \bar{e}_\beta(1-x) e'_\alpha(x) \rangle \overset{(2.18)}{=} \langle (\mathrm{Id}_\mathcal{V} \otimes \bar{u}_\beta)\, S(1-x)^{-1} S'(x) (u_\alpha \otimes \mathrm{Id}_\mathcal{V}) \rangle \tag{5.14}$$

In the next proposition, we are going to show that the choices we have made for $\psi_{x,0}$ and $\bar{\psi}_{x,0}$ (see (2.6) and (2.9)) makes $c_{\alpha,\beta}(x)$ independent on x. More precisely, we will show that the product $S(1-x)^{-1}S'(x)$ does not depend on x. In order to do so, we need the following Lemma.

LEMMA 5.5. *For all $x \in \mathbb{R}_{\neq 0,1}$ and all $\mathsf{U} \in \mathcal{A}$ such that $\mathsf{U}^N = -1$, the operator*

$$\bar{\Psi}_{1-x}(\omega^{-1}\mathsf{U}^{-1}) \bar{\Psi}_x(\mathsf{U})$$

does not depend on x. Moreover, if we write

$$\Phi(\mathsf{U}) = \bar{\Psi}_{1-x}(\omega^{-1}\mathsf{U}^{-1}) \bar{\Psi}_x(\mathsf{U}),$$

we have

$$\frac{\Phi(\omega \mathsf{U})}{\Phi(\mathsf{U})} = -\omega \mathsf{U} \tag{5.15}$$

PROOF. It is enough to show that the N-th power of $\bar{\Psi}_{1-x}(\mathsf{U})\bar{\Psi}_x(\mathsf{U})$ does not depend on x. Hence, using equalities (2.4) and (2.8), we have

$$\bar{\Psi}_{1-x}(\omega^{-1}\mathsf{U}^{-1})^N \bar{\Psi}_x(\mathsf{U})^N = \bar{\psi}_{1-x,0}^N \bar{\psi}_{x,0}^N N^{-2N} \left(x(1-x)\right)^{-\frac{N-1}{2}} D(1)^6 \times$$
$$D\left(x^{\frac{1}{N}}\right)^{-1} D\left((1-x)^{\frac{1}{N}}\right)^{-1} D\left(\mathsf{U}^{-1}\left(\tfrac{1-x}{x}\right)^{\frac{1}{N}}\right)^{-1} D\left(\omega \mathsf{U}\left(\tfrac{x}{1-x}\right)^{\frac{1}{N}}\right)^{-1}$$

Considering the last factor and using Lemma 1.2, we easily compute

$$D\left(\omega\mathsf{U}\left(\tfrac{x}{1-x}\right)^{\frac{1}{N}}\right)^{-1} = N^N\left(x(x-1)\right)^{\frac{N-1}{2}} D(1)^{-2} D\left(\mathsf{U}^{-1}\left(\tfrac{1-x}{x}\right)^{\frac{1}{N}}\right)$$

Hence we have

$$\bar{\Psi}_{1-x}(\omega^{-1}\mathsf{U}^{-1})^N\bar{\Psi}_x(\mathsf{U})^N = \bar{\psi}^N_{1-x,0}\bar{\psi}^N_{x,0}N^{-N}D(1)^4 D\left(x^{\frac{1}{N}}\right)^{-1} D\left((1-x)^{\frac{1}{N}}\right)^{-1}\mathrm{Id}_{\mathcal{V}}\,.$$

Therefore, the choices we have made for $\psi_{x,0}$ and $\bar{\psi}_{x,0}$ (see (2.6) and (2.9)) give

$$\bar{\Psi}_{1-x}(\omega^{-1}\mathsf{U}^{-1})^N\bar{\Psi}_x(\mathsf{U})^N = (-1)^{\frac{N-1}{2}}N^{-N}D(1)^2\,\mathrm{Id}_{\mathcal{V}}\,.$$

Finally, by definition of $\bar{\Psi}_x(\mathsf{U})$, we have

$$\begin{aligned}\frac{\Phi(\omega\mathsf{U})}{\Phi(\mathsf{U})} &= \frac{\bar{\Psi}_{1-x}(\omega^{-2}\mathsf{U}^{-1})\bar{\Psi}_x(\omega\mathsf{U})}{\bar{\Psi}_{1-x}(\omega^{-1}\mathsf{U}^{-1})\bar{\Psi}_x(\mathsf{U})} = \frac{\bar{\Psi}_{1-x}(\omega^{-2}\mathsf{U}^{-1})\bar{\Psi}_x(\omega\mathsf{U})}{\bar{\Psi}_{1-x}(\omega\omega^{-2}\mathsf{U}^{-1})\bar{\Psi}_x(\mathsf{U})}\\ &= \frac{(1-x)^{\frac{1}{N}} - x^{\frac{1}{N}}\omega\mathsf{U}}{x^{\frac{1}{N}} - (1-x)^{\frac{1}{N}}\omega^{-1}\mathsf{U}^{-1}} = \frac{(1-x)^{\frac{1}{N}} - x^{\frac{1}{N}}\omega\mathsf{U}}{(-\omega^{-1}\mathsf{U}^{-1})\left((1-x)^{\frac{1}{N}} - x^{\frac{1}{N}}\omega\mathsf{U}\right)} = -\omega\mathsf{U}\end{aligned}$$

□

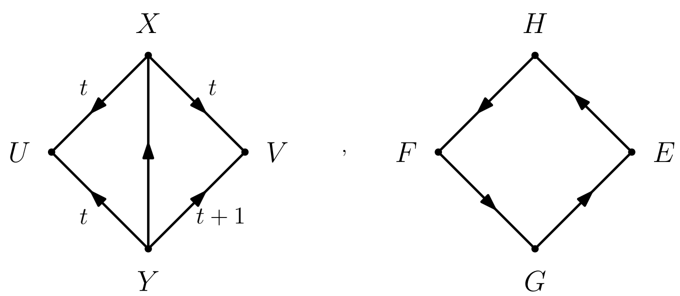
PROPOSITION 5.6. *For all $x \in \mathbb{R}_{\neq 0,1}$ there exists numbers $\{c_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{Z}^2_N} \subset \mathbb{C}$ such that*

$$e'_\alpha(x) = \sum_{\beta\in\mathbb{Z}^2_N} c_{\alpha,\beta}e_\beta(1-x). \tag{5.16}$$

PROOF. By equality (5.14) it is enough to show that $S(1-x)^{-1}S'(x)$ does not depend on x. Let us recall that

$$E = -Y_1^{-1}X_1Y_2,\quad F = U_1^{-1}X_1^{t+1}V_2E^{-1},\quad G = U_2V_2^{-1}F,\quad H = U_1V_1^{-1}E,$$

and that we have the following commutation relations



We start by the following computation

$$\begin{aligned}
\Omega_1\Omega_2\bar{\Psi}_{1-x}(E)\Omega_1^{-1}\Omega_2^{-1}L(U^tV,X)^{-1} &= \bar{\Psi}_{1-x}\left(-(Y_1')^{-1}X_1'Y_2'\right)L(U^tV,X)^{-1} \\
&= L(U^tV,X)^{-1}\bar{\Psi}_{1-x}\left(-L(U^tV,X)(Y_1')^{-1}X_1'Y_2'L(U^tV,X)^{-1}\right) \\
&= L(U^tV,X)^{-1}\bar{\Psi}_{1-x}\left(-(Y_1')^{-1}X_1'Y_2'L(\omega^{-t}U^tV,\omega^{-t-1}X)L(U^tV,X)^{-1}\right) \\
&\overset{\text{Lemma 2.5}}{=} L(U^tV,X)^{-1}\bar{\Psi}_{1-x}\left(-(Y_1')^{-1}X_1'Y_2'\omega^{t(t+1)}(U_1^tV_1)^{-t-1}X_2^{-t}\right) \\
&= L(U^tV,X)^{-1}\bar{\Psi}_{1-x}\left(\omega^{-1}G^{-1}\right). \qquad (5.17)
\end{aligned}$$

By doing similar computation we get

$$\Omega_1\Omega_2\bar{\Psi}_{1-x}(F)\Omega_1^{-1}\Omega_2^{-1}L(U^tV,X)^{-1} = L(U^tV,X)^{-1}\bar{\Psi}_{1-x}\left(\omega^{-1}H^{-1}\right), \qquad (5.18)$$

$$\Omega_1\Omega_2\bar{\Psi}_{1-x}(G)\Omega_1^{-1}\Omega_2^{-1}L(U^tV,X)^{-1} = L(U^tV,X)^{-1}\bar{\Psi}_{1-x}\left(\omega^{-1}U_2V_2^{-1}H^{-1}\right) \qquad (5.19)$$

and

$$\Omega_1\Omega_2\bar{\Psi}_{1-x}(H)\Omega_1^{-1}\Omega_2^{-1}L(U^tV,X)^{-1} = L(U^tV,X)^{-1}\bar{\Psi}_{1-x}\left(\omega^{-1}U_1V_1^{-1}G^{-1}\right). \qquad (5.20)$$

Hence we have

$$\begin{aligned}
S(1-x)^{-1}S'(x) &= L(U^tV,X)^{-1}\bar{\Psi}_{1-x}(H)\,\bar{\Psi}_{1-x}(G)\,\bar{\Psi}_{1-x}(F)\,\bar{\Psi}_{1-x}(E) \\
&\quad\times \Omega_1^{-1}\Omega_2^{-1}L(U^tV,X)^{-1}\underline{\bar{\Psi}_x(H)\,\bar{\Psi}_x(G)}\bar{\Psi}_x(F)\,\bar{\Psi}_x(E)\,\Omega_1\Omega_2 \\
&= L(U^tV,X)^{-1}\bar{\Psi}_{1-x}(H)\,\bar{\Psi}_{1-x}(G)\,\bar{\Psi}_{1-x}(F)\,\Omega_1^{-1}\Omega_2^{-1} \\
&\quad\times \underline{\Omega_1\Omega_2\bar{\Psi}_{1-x}(E)\,\Omega_1^{-1}\Omega_2^{-1}L(U^tV,X)^{-1}\bar{\Psi}_x(G)}\bar{\Psi}_x(H)\,\bar{\Psi}_x(F)\,\bar{\Psi}_x(E)\,\Omega_1\Omega_2 \\
&= L(U^tV,X)^{-1}\bar{\Psi}_{1-x}(H)\,\bar{\Psi}_{1-x}(G)\,\bar{\Psi}_{1-x}(F)\,\Omega_1^{-1}\Omega_2^{-1} \\
&\quad\times L(U^tV,X)^{-1}\underline{\Phi(G)\bar{\Psi}_x(H)}\bar{\Psi}_x(F)\,\bar{\Psi}_x(E)\,\Omega_1\Omega_2 \\
&= L(U^tV,X)^{-1}\bar{\Psi}_{1-x}(H)\,\bar{\Psi}_{1-x}(G)\,\Omega_1^{-1}\Omega_2^{-1} \\
&\quad\times \underline{\Omega_1\Omega_2\bar{\Psi}_{1-x}(F)\,\Omega_1^{-1}\Omega_2^{-1}L(U^tV,X)^{-1}\bar{\Psi}_x(H)}\Phi(G)\bar{\Psi}_x(F)\,\bar{\Psi}_x(E)\,\Omega_1\Omega_2 \\
&= L(U^tV,X)^{-1}\bar{\Psi}_{1-x}(H)\,\bar{\Psi}_{1-x}(G)\,\Omega_1^{-1}\Omega_2^{-1} \\
&\quad\times L(U^tV,X)^{-1}\Phi(H)\Phi(G)\bar{\Psi}_x(F)\,\bar{\Psi}_x(E)\,\Omega_1\Omega_2 \\
&= L(U^tV,X)^{-1}\bar{\Psi}_{1-x}(H)\,\Omega_1^{-1}\Omega_2^{-1}\underline{\Omega_1\Omega_2\bar{\Psi}_{1-x}(G)\,\Omega_1^{-1}\Omega_2^{-1}L(U^tV,X)^{-1}} \\
&\quad\times \Phi(H)\Phi(G)\bar{\Psi}_x(F)\,\bar{\Psi}_x(E)\,\Omega_1\Omega_2 \\
&= L(U^tV,X)^{-1}\Omega_1^{-1}\Omega_2^{-1}\underline{\Omega_1\Omega_2\bar{\Psi}_{1-x}(H)\,\Omega_1^{-1}\Omega_2^{-1}L(U^tV,X)^{-1}} \\
&\quad\times \bar{\Psi}_{1-x}\left(\omega^{-1}U_2V_2^{-1}H^{-1}\right)\Phi(H)\Phi(G)\bar{\Psi}_x(F)\,\bar{\Psi}_x(E)\,\Omega_1\Omega_2 \\
&= L(U^tV,X)^{-1}\Omega_1^{-1}\Omega_2^{-1}L(U^tV,X)^{-1}\bar{\Psi}_{1-x}\left(\omega^{-1}U_1V_1^{-1}G^{-1}\right) \\
&\quad\times \bar{\Psi}_{1-x}\left(\omega^{-1}U_2V_2^{-1}H^{-1}\right)\Phi(H)\Phi(G)\bar{\Psi}_x(F)\,\bar{\Psi}_x(E)\,\Omega_1\Omega_2. \qquad (5.21)
\end{aligned}$$

Now we also have

$$\bar{\Psi}_{1-x}\left(\omega^{-1}U_2V_2^{-1}H^{-1}\right)\Phi(H)\Phi(G)=\Phi(H)\Phi(G)\bar{\Psi}_{1-x}\left(\omega^{-1}F^{-1}\right).$$

Indeed, using equality (5.15) we have

$$\begin{aligned}\bar{\Psi}_{1-x}\left(\omega^{-1}U_2V_2^{-1}H^{-1}\right)\Phi(H)&=\Phi(H)\bar{\Psi}_{1-x}\left(\Phi(H)^{-1}\omega^{-1}U_2V_2^{-1}H^{-1}\Phi(H)\right)\\&=\Phi(H)\bar{\Psi}_{1-x}\left(\Phi(H)^{-1}\Phi(\omega H)\omega^{-1}U_2V_2^{-1}H^{-1}\right)\\&\overset{(5.15)}{=}\Phi(H)\bar{\Psi}_{1-x}\left((-\omega H)\omega^{-1}U_2V_2^{-1}H^{-1}\right)=\Phi(H)\bar{\Psi}_{1-x}\left(-\omega^{-1}U_2V_2^{-1}\right),\end{aligned}$$

and

$$\begin{aligned}\bar{\Psi}_{1-x}\left(-\omega^{-1}U_2V_2^{-1}\right)\Phi(G)&=\Phi(G)\bar{\Psi}_{1-x}\left(\Phi(G)^{-1}(-\omega^{-1}U_2V_2^{-1})\Phi(G)\right)\\&=\Phi(G)\bar{\Psi}_{1-x}\left(\Phi(G)^{-1}\Phi(\omega^{-1}G)(-\omega^{-1}U_2V_2^{-1})\right)\\&\overset{(5.15)}{=}\Phi(G)\bar{\Psi}_{1-x}\left(G^{-1})\omega^{-1}U_2V_2^{-1}\right)=\Phi(G)\bar{\Psi}_{1-x}\left(\omega^{-1}F^{-1}\right).\end{aligned}$$

A similar computation gives

$$\bar{\Psi}_{1-x}\left(\omega^{-1}U_1V_1^{-1}G^{-1}\right)\Phi(H)\Phi(G)=\Phi(H)\Phi(G)\bar{\Psi}_{1-x}\left(\omega^{-1}E^{-1}\right).$$

Therefore (5.21) becomes

$$\begin{aligned}S(1-x)^{-1}S'(x)&=L(U^tV,X)^{-1}\Omega_1^{-1}\Omega_2^{-1}L(U^tV,X)^{-1}\bar{\Psi}_{1-x}\left(\omega^{-1}U_1V_1^{-1}G^{-1}\right)\\&\quad\times\underline{\bar{\Psi}_{1-x}\left(\omega^{-1}U_2V_2^{-1}H^{-1}\right)\Phi(H)\Phi(G)}\bar{\Psi}_x(F)\,\bar{\Psi}_x(E)\,\Omega_1\Omega_2\\&=L(U^tV,X)^{-1}\Omega_1^{-1}\Omega_2^{-1}L(U^tV,X)^{-1}\underline{\bar{\Psi}_{1-x}\left(\omega^{-1}U_1V_1^{-1}G^{-1}\right)\Phi(H)\Phi(G)}\\&\quad\times\bar{\Psi}_{1-x}\left(\omega^{-1}F^{-1}\right)\bar{\Psi}_x(F)\,\bar{\Psi}_x(E)\,\Omega_1\Omega_2\\&=L(U^tV,X)^{-1}\Omega_1^{-1}\Omega_2^{-1}L(U^tV,X)^{-1}\Phi(H)\Phi(G)\underline{\bar{\Psi}_{1-x}\left(\omega^{-1}E^{-1}\right)\Omega(F)}\bar{\Psi}_x(E)\,\Omega_1\Omega_2\\&=L(U^tV,X)^{-1}\Omega_1^{-1}\Omega_2^{-1}L(U^tV,X)^{-1}\Phi(H)\Phi(G)\Omega(F)\,\bar{\Psi}_{1-x}\left(\omega^{-1}E^{-1}\right)\bar{\Psi}_x(E)\,\Omega_1\Omega_2\\&=L(U^tV,X)^{-1}\Omega_1^{-1}\Omega_2^{-1}L(U^tV,X)^{-1}\Phi(H)\Phi(G)\Phi(F)\Phi(E)\Omega_1\Omega_2\end{aligned}$$

which ends the proof. □

5.4. Determination of $[\![A,x]\!]$.

PROPOSITION 5.7. *For all $x\in\mathbb{R}_{\neq 0,1}$, we have $[\![A,x]\!]\equiv\epsilon_N^2N^{-1}x^{\frac{2(N-1)}{N}}$.*

PROOF. Let $(p,q)\in(\mathbb{R}_{\neq 0})^2$ be an admissible pair and $x=\frac{q}{p+q}\in\mathbb{R}_{\neq 0,1}$. Using Lemma 5.4 and equalities (5.16) and (5.10) we compute

$$\begin{aligned}[\![A,x]\!]=\langle h_{-p,p+q}(u_0),Ah_{p,q}(u_0)\rangle&=\sum_{\alpha,\beta\in\mathbb{Z}_N^2}c_{0,\alpha}c_{0,\beta}\langle h'_{-p,p+q}(u_0),Ah'_{p,q}(u_0)\rangle\\&=A'_0(1-x)\sum_{\alpha,\beta\in\mathbb{Z}_N^2}c_{0,\alpha}c_{0,\beta}\mathfrak{b}_{\alpha,\beta}\equiv x^{\frac{2(N-1)}{N}}\sum_{\alpha\in\mathbb{Z}_N^2}c_{0,\alpha}c_{0,-\alpha}\mathfrak{b}_{\alpha,-\alpha}.\end{aligned}$$

Thereby, we need to compute $c_{0,\alpha}$. In order to do so, we use the last equality of the previous Lemma, namely

$$S(1-x)^{-1}S'(x) = L(U^tV,X)^{-1}\Omega_1^{-1}\Omega_2^{-1}L(U^tV,X)^{-1}\Phi(H)\Phi(G)\Phi(F)\Phi(E)\Omega_1\Omega_2.$$

Hence we have

$$c_{0,\alpha} = \left\langle \bar{u}_0 \otimes \bar{u}_\alpha \middle| S(1-x)^{-1}S'(x) \middle| u_0 \otimes u_0 \right\rangle$$

$$= \left\langle \bar{u}_0\otimes\bar{u}_\alpha \middle| L(U^tV,X)^{-1}\Omega_1^{-1}\Omega_2^{-1}L(U^tV,X)^{-1}\Phi(H)\Phi(G)\Phi(F)\Phi(E)\Omega_1\Omega_2 \middle| u_0\otimes u_0 \right\rangle.$$

Let us write

$$\Phi(\mathsf{U}) = \sum_{m\in\mathbb{Z}_N} \varphi_m(-\mathsf{U})^m$$

where $\varphi_m \in \mathbb{C}$, for any operator $\mathsf{U} \in \mathcal{A}^{\otimes 2}$ such that $\mathsf{U}^N = -\operatorname{Id}_{\mathcal{V}^{\otimes 2}}$. Using (5.15), we easily see that for any $m \in \mathbb{Z}_N$ we have $\varphi_m = \varphi_0\omega^{-\frac{1}{2}m(m-1)}$. Thus, using equalities (5.12) and by doing similar computation we did for $S(x)$ and $S(x)^{-1}$, we find

$$c_{0,\alpha} = \left\langle \bar{u}_0 \otimes \bar{u}_\alpha \middle| S(1-x)^{-1}S'(x) \middle| u_0 \otimes u_0 \right\rangle = \delta_{0,\alpha_1(t+2)}\delta_{0,\alpha_2}\varphi_0^4 \sum_{m,n\in\mathbb{Z}_N} \omega^{-\frac{1}{2}(m+n)^2}.$$

Since $(t+2)^2 = 3(t+1) \mod N$ and $N \notin 3\mathbb{N}$, $t+2$ is invertible in $\mathbb{Z}_N$, we have

$$c_{0,\alpha} = \delta_{0,\alpha}\varphi_0^4 \sum_{m,n\in\mathbb{Z}_N} \omega^{-\frac{1}{2}(m+n)^2} = \delta_{0,\alpha}\varphi_0^4 N \sum_{s\in\mathbb{Z}_N} \omega^{-\frac{1}{2}s^2}.$$

Because $\sum_{s\in\mathbb{Z}_N} \omega^{-\frac{1}{2}s^2}$ is the Gauss sum $g(-\frac{(N-1)}{2};N) = \pm\epsilon_N N^{\frac{1}{2}}$ (see [**3**] for details), we conclude that

$$c_{0,\alpha} = \pm\delta_{0,\alpha}\varphi_0^4\epsilon_N N^{\frac{3}{2}}.$$

Using the following Lemma, we finally get

$$[\![A,x]\!] \equiv \epsilon_N^2 N^{-1} x^{\frac{2(N-1)}{N}}.$$

□

LEMMA 5.8. *The following equality holds true*

$$\varphi_0^4 \equiv N^{-2}$$

Using this Lemma, we have

$$c_{0,\alpha} = \pm\delta_{0,\alpha}\epsilon_N N^{-\frac{1}{2}},$$

PROOF. Let U be an operator such that $U^N = -1$. By definition we have

$$\Phi(\mathsf{U}) = \sum_{s\in\mathbb{Z}_N} \varphi_0\omega^{-\frac{1}{2}s(s-1)}(-\mathsf{U})^s$$

and

$$\Phi(\mathsf{U}) = \bar{\Psi}_{1-x}(\omega^{-1}\mathsf{U}^{-1})\bar{\Psi}_x(-\mathsf{U}) = \sum_{s\in\mathbb{Z}_N}\sum_{m\in\mathbb{Z}_N} \bar{\psi}_{1-x,m-s}\bar{\psi}_{x,m}\omega^{s-m}(-\mathsf{U})^s.$$

This implies in particular that

$$\varphi_0 = \sum_{m\in\mathbb{Z}_N} \bar{\psi}_{x,m}\bar{\psi}_{1-x,m}\omega^{-m}.$$

Using (2.7) we have for any $m \in \mathbb{Z}_N$

$$\bar{\psi}_{x,m}\bar{\psi}_{1-x,m}\omega^{-m} = \bar{\psi}_{x,0}\bar{\psi}_{1-x,0}w\left((1-x)^{-\frac{1}{N}}|m\right)w\left(x^{-\frac{1}{N}}|m\right)\omega^m.$$

Since

$$\frac{\left(1-(1-x)^{-1}\right)\left(1-x^{-1}\right)}{(1-0^N)(1-0^N)} = \omega^N,$$

we can use the function $F\left(\begin{smallmatrix} x & u \\ y & v\end{smallmatrix}\middle| z\right)$ and we have

$$\varphi_0 = \bar{\psi}_{x,0}\bar{\psi}_{1-x,0}F\left(\begin{matrix}(1-x)^{-\frac{1}{N}} & x^{-\frac{1}{N}} \\ 0 & 0\end{matrix}\middle|\omega\right).$$

Now we use formula (C.8) of [**9**] which is

$$F\left(\begin{matrix} x & u \\ y & v\end{matrix}\middle| z\right)^N = u^{\frac{N(N-1)}{2}}\frac{D(1)D\left(\frac{xu}{yv\omega}\right)D(x\omega)D\left(\frac{y\omega}{u}\right)D\left(\frac{v\omega}{u}\right)}{D(y\omega)D(v\omega)D\left(\frac{x}{y}\right)D\left(\frac{x}{v}\right)D\left(\frac{1}{u}\right)}.$$

Using Lemma 1.2 on the factor $D\left(\frac{xu}{yv\omega}\right)$, $D\left(\frac{x}{y}\right)$, $D\left(\frac{x}{v}\right)$ and $D(x\omega)$ we get

$$F\left(\begin{matrix}(1-x)^{-\frac{1}{N}} & x^{-\frac{1}{N}} \\ 0 & 0\end{matrix}\middle|\omega\right)^N = D(1)D\left(x^{\frac{1}{N}}\right)^{-1}D\left((1-x)^{\frac{1}{N}}\right)^{-1}.$$

The equalities (2.6) and (2.9) then imply that

$$\begin{aligned}\varphi_0^N = \bar{\psi}_{x,0}^N\bar{\psi}_{1-x,0}^N F&\left(\begin{matrix}(1-x)^{-\frac{1}{N}} & x^{-\frac{1}{N}} \\ 0 & 0\end{matrix}\middle|\omega\right)^N \\ &= \bar{\psi}_{x,0}^N\bar{\psi}_{1-x,0}^N D(1)D\left(x^{\frac{1}{N}}\right)^{-1}D\left((1-x)^{\frac{1}{N}}\right)^{-1} = D(1)^{-1}.\end{aligned}$$

Finally, we show that

$$D(1)^2 = (-1)^{\frac{N(N-1)}{2}}N^N.$$

For any $x \in \mathbb{C}\setminus\{0\}$ and we compute

$$\begin{aligned}D(x) = \prod_{j=1}^{N-1}(1-x\omega^j)^j &= \prod_{j=1}^{N-1}(-x\omega^j)^j(1-x^{-1}\omega^{-j})^j \\ &= (-1)^{\frac{N(N-1)}{2}}\omega^{\frac{N(N-1)(2N-1)}{6}}x^{\frac{N(N-1)}{2}}\prod_{j=1}^{N-1}(1-x^{-1}\omega^j)^{N-j} \\ &= (-1)^{\frac{N(N-1)}{2}}\omega^{\frac{N(N-1)(2N-1)}{6}}x^{\frac{N(N-1)}{2}}\left(\frac{1-x^{-N}}{1-x^{-1}}\right)^N\prod_{j=1}^{N-1}(1-x^{-1}\omega^j)^{-j} \\ &= (-1)^{\frac{N(N-1)}{2}}\omega^{\frac{N(N-1)(2N-1)}{6}}x^{-\frac{N(N-1)}{2}}\left(\frac{1-x^{N}}{1-x}\right)^N D(x^{-1})^{-1} \\ &= (-1)^{\frac{N(N-1)}{2}}\omega^{\frac{N(N-1)(2N-1)}{6}}N^N x^{-\frac{N(N-1)}{2}}\varrho(x)^N D(x^{-1})^{-1}.\end{aligned}$$

Since N is odd and $N \notin 3\mathbb{N}$ that implies that $\frac{(N-1)(2N-1)}{6}$ is an integer. Hence $\omega^{\frac{N(N-1)(2N-1)}{6}} = 1$ and we get

$$D(x)D(x^{-1}) = (-1)^{\frac{N(N-1)}{2}}N^N x^{-\frac{N(N-1)}{2}}\varrho(x)^N.$$

By Lemma 1.2 we have

$$D(x)D(x^{-1}) = D(1)^2 x^{-\frac{N(N-1)}{2}}\varrho(x)^N$$

which ends the proof. □

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