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Conformal symmetries of FRW accelerating cosmologies

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Abstract

We show that any accelerating Friedmann–Robertson–Walker (FRW) cosmology with equation of state $w < -1/3$ (and therefore not only a de Sitter stage with $w = -1$) exhibits three-dimensional conformal symmetry on future constant-time hypersurfaces if the bulk theory is invariant under bulk conformal Killing vectors. We also offer an alternative derivation of this result in terms of conformal Killing vectors and show that long wavelength comoving curvature perturbations of the perturbed FRW metric are just conformal Killing motions of the FRW background. We then extend the boundary conformal symmetry to the bulk for accelerating cosmologies. Our findings indicate that one can easily generate perturbations of scalar fields which are not only scale invariant, but also fully conformally invariant on super-Hubble scales. Measuring a scale-invariant power spectrum for the cosmological perturbation does not automatically imply that the universe went through a de Sitter stage.

1. Introduction

It has been appreciated for some time now that symmetries play a crucial role in characterizing the properties of the cosmological perturbations generated during a de Sitter stage [1]. Indeed, the de Sitter isometry group acts like conformal group on $\mathbb{R}^3$ when the fluctuations are

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on super-Hubble scales. As a consequence, the correlators of scalar fields, other than the inflaton, are constrained by conformal invariance as the SO(1, 4) isometry of the de Sitter background is realized as conformal symmetry of the flat $\mathbb{R}^3$ sections [2–6]. The fact that the de Sitter isometry group acts as conformal group on the three-dimensional Euclidean space on super-Hubble scales can be also used to characterize the correlators involving the inflaton and vector fields [7]. In the case in which the inflationary perturbations originate from the inflaton itself, one can find conformal consistency relations among the inflationary correlators [8–14]. Finally, consistency relations involving the soft limit of the $(n+1)$-correlator functions of matter and galaxy overdensities have also be found by investigating the symmetries enjoyed by the Newtonian equations of motion of the non-relativistic dark matter fluid coupled to gravity [15,16].

In this paper we wish to make the simple, yet relevant, observation that any accelerating Friedmann–Robertson–Walker (FRW) cosmology with equation of state $w < -1/3$ (and therefore not only a de Sitter stage with $w = -1$) exhibits three-dimensional conformal symmetry on super-Hubble scales. This is because for an accelerating universe the future constant-time boundary possesses ten conformal Killing vectors which form an SO(1, 4) algebra, precisely the three-dimensional conformal algebra. Of course, only for exact de Sitter space these ten conformal Killing vectors are actually isometries, whereas for all the other cases only translations and rotations are isometries. Our observation implies that if one constructs a theory invariant under translations, rotations, dilations and special conformal transformations (the generators of the three-dimensional conformal field group in $\mathbb{R}^3$) and coupled it to any accelerating FRW metric, then the full theory is automatically SO(1, 4) invariant and therefore conformal invariant at the future boundary. This means that correlators of the super-Hubble fluctuations of a scalar field must satisfy such a three-dimensional conformal field symmetry. By using the set of available conformal Killing vectors of FRW cosmologies one can also show that for a perturbed FRW universe, the long wavelength perturbations may be interpreted as conformal Killing motions of the FRW background. This allows to extend the consistency relations found in Refs. [15,16] to the relativistic case [17]. We also show how one can extend the conformal symmetry to the full FRW bulk for accelerating cosmologies, following what is done in the AdS/CFT correspondence. In such a case the conformal symmetry is also an isometry of the FRW metric.

The paper is organized as follows. In Section 2 we show that the accelerating FRW cosmologies on the future constant-time hypersurface exhibits three-dimensional conformal symmetry by embedding their geometry in five-dimensional Minkowski space–time; we also study an application of our findings, by showing that conformal invariance of scalar perturbations in an accelerating FRW universe can be obtained even away from de Sitter. In Section 3 we offer an alternative explanation which make use of conformal Killing vectors to show that accelerating FRW cosmologies on the future constant-time hypersurface exhibits three-dimensional conformal symmetry and make use of these conformal Killing transformations to comment about the perturbed FRW cosmology in Section 4. In Section 5, we extend the boundary conformal symmetry to the bulk of accelerating FRW cosmologies, again presenting an explicit example based on a scalar field. Finally, Section 6 presents conclusions.

2. Future boundary conformal symmetry of FRW accelerating cosmologies

In this section we wish to make use of the embedding of the FRW space–time into five-dimensional hyperboloids to show that FRW cosmology exhibits three-dimensional conformal symmetry on the future constant-time hypersurfaces. First, we start with the well-known case of de Sitter.

2.1. Five-dimensional hyperboloids and conformal symmetry of de Sitter

It is well-known that in Euclidean three-dimensional space $\mathbb{R}^3$ conformal invariance is connected to the symmetry group SO(1, 4) in the same way that conformal invariance in a four-dimensional Minkowski space–time is connected to the SO(2, 4) group. As SO(1, 4) is the isometry group of de Sitter space–time, one may conclude that during a de Sitter stage of the evolution of the universe the isometry group acts as conformal group on $\mathbb{R}^3$ when the fluctuations of a given quantum scalar field are on super-Hubble scales. In such a regime, the SO(1, 4) isometry of the de Sitter background is realized as conformal symmetry of the flat $\mathbb{R}^3$ constant-time hypersurfaces sections and correlators of such fields are constrained by conformal invariance [2,4–6]. To show this explicitly, we can remind some of the basic geometrical and algebraic properties of de Sitter space–time and group [18].

The four-dimensional de Sitter space–time of constant radius H^{-1} is described by the hyperboloid

$$\eta_{AB}X^A X^B = -X_0^2 + X_i^2 + X_5^2 = \frac{1}{H^2} \quad (i = 1, 2, 3), \tag{2.1}$$

embedded in five-dimensional Minkowski space–time $\mathbb{M}^{1,4}$ with coordinates X^A and flat metric $\eta_{AB} = \mathrm{diag}(-1, 1, 1, 1, 1)$. A particular parametrization of the de Sitter hyperboloid is provided by

$$\begin{aligned} X^0 &= \frac{1}{2H}\left(H\tau - \frac{1}{H\tau}\right) - \frac{1}{2}\frac{\vec{x}^2}{\tau}, \\ X^i &= \frac{x^i}{H\tau}, \\ X^5 &= -\frac{1}{2H}\left(H\tau + \frac{1}{H\tau}\right) + \frac{1}{2}\frac{\vec{x}^2}{\tau}, \end{aligned} \tag{2.2}$$

which satisfies Eq. (2.1) as it can be easily checked. The de Sitter metric is the induced metric on the hyperboloid from the five-dimensional ambient Minkowski space–time

$$\mathrm{d}s_5^2 = \eta_{AB}\mathrm{d}X^A\mathrm{d}X^B. \tag{2.3}$$

For the particular parametrization (2.2), for example, we find the familiar expression of the four-dimensional de Sitter metric with conformal time τ and scale factor $a(\tau) = -1/H\tau$

$$\mathrm{d}s_{\mathrm{dS}}^2 = \frac{1}{H^2\tau^2}\left(-\mathrm{d}\tau^2 + \mathrm{d}\vec{x}^2\right). \tag{2.4}$$

This metric is invariant under the infinitesimal coordinate transformations

$$\begin{aligned} &\delta\tau = \lambda\tau, \qquad \delta_D x^i = \lambda x^i, \\ &\delta\tau = -2\tau\,\vec{b}\cdot\vec{x}, \qquad \delta_K x^i = -2x^i(\vec{b}\cdot\vec{x}) + \left(\vec{x}^2 - \tau^2\right)b^i, \end{aligned} \tag{2.5}$$

which together with rotations and translations form the de Sitter isometry group. Let us now focus on the limit $H\tau \ll 1$ or, equivalently, we look at future constant-time hypersurfaces $\tau \to 0$. For a generic scalar field this is equivalent at focusing onto those fluctuations which are already outside the Hubble radius. The parametrization (2.2) turns out then to be

$$X^0 = -\frac{1}{2H^2\tau} - \frac{1}{2}\frac{\vec{x}^2}{\tau},$$
$$X^i = \frac{x^i}{H\tau},$$
$$X^5 = -\frac{1}{2H^2\tau} + \frac{1}{2}\frac{\vec{x}^2}{\tau} \tag{2.6}$$

and we may easily check that the hyperboloid has been degenerated to the hypercone

$$-X_0^2 + X_i^2 + X_5^2 = 0. \tag{2.7}$$

We identify points $X^A \equiv \lambda X^A$ (which turns the cone (2.7) into a projective space). As a result, τ in the denominator of the X^A can be ignored due to projectivity condition. The conformal group SO(1, 4) acts linearly on X^A, but it induces the (non-linear) conformal transformations $x_i \to x_i'$ with

$$x_i \to x_i' = a_i + M_i{}^j x_j, \tag{2.8}$$
$$x_i' = \lambda x_i, \tag{2.9}$$
$$x_i' = \frac{x_i + b_i\vec{x}^2}{1 + 2\vec{b}\cdot\vec{x} + b^2\vec{x}^2}. \tag{2.10}$$

One recognizes the conformal group acting on Euclidean $\mathbb{R}^3$ with coordinates x_i.

For future use, we notice that they can be written also in terms of inversion

$$\text{(inversion)} \times \text{(translation)} \times \text{(inversion)}, \quad \text{(inversion)} : x_i \to x_i' = \frac{x_i}{\vec{x}^2}, \tag{2.11}$$

We should also specify how τ transforms under the conformal group. This can be determined either by the fact that τ is a coordinate, or by the induced metric on the cone, which turns out to be

$$ds^2 = \frac{d\vec{x}^2}{H^2\tau^2}. \tag{2.12}$$

In both ways one finds that τ transforms as

$$\tau \to \tau' = \lambda\tau, \tag{2.13}$$
$$\tau \to \tau' = \frac{\tau}{\vec{x}^2}, \tag{2.14}$$

under dilations and inversions, respectively. As a result, the symmetry of the constant future time hypersurfaces $\tau \to 0$ is the three-dimensional conformal group generated by the transformations (2.8), (2.9) and (2.10) augmented by the τ transformations (2.13) and (2.14). Fluctuations of a generic scalar field (as well as tensor mode) must satisfy this symmetry on super-Hubble scales.

2.2. *Five-dimensional hyperboloids and conformal symmetry of FRW accelerating cosmologies*

A generic FRW universe can similarly be embedded in the five-dimensional Minkowski space–time $\mathbb{M}^{1,4}$ with coordinates X^A and flat metric $\eta_{AB} = \text{diag}(-1, 1, 1, 1, 1)$. Indeed, it is easy to verify that the induced metric on the five-dimensional hyperboloid is now determined by the parametrization (we present it only for the spatially flat FRW cosmology)

$$X^0 = \frac{1}{2\tau_0} a(t)\left(\vec{x}^2 + \tau_0^2\right) + \frac{1}{\tau_0}\int^t \frac{\mathrm{d}t'}{2\dot{a}(t')},$$
$$X^i = a(t)x^i,$$
$$X^5 = \frac{1}{2\tau_0} a(t)\left(\vec{x}^2 - \tau_0^2\right) + \frac{1}{\tau_0}\int^t \frac{\mathrm{d}t'}{2\dot{a}(t')}, \tag{2.15}$$

in $\mathbb{M}^{1,4}$, where dots indicate differentiation with respect to the cosmic time t and τ_0 is a constant. The metric (2.3) becomes the familiar FRW for spatially flat sections

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}\vec{x}^2, \tag{2.16}$$

where $a(t)$ is the scale factor. It is also straightforward to determine the hypersurface which is embedded, as the de Sitter hyperboloid, in $\mathbb{M}^{1,4}$. It is

$$-X_0^2 + X_1^2 + X_2^2 + X_3^3 + X_5^2 = (X_5 + X_0) f(t), \tag{2.17}$$

where

$$f(t) = \int^t \frac{\mathrm{d}t'}{\dot{a}(t')} = f_0\, a^{2+3w} \tag{2.18}$$

and f_0 is a constant. Note that de Sitter is a particular case of this general embedding.

If the energy density of the universe is dominated by a single fluid at a given epoch with a constant equation of state w, the scale factor is $a(t) \sim t^{2/3(1+w)}$. This is the case we will employ below for illustrating purposes although the discussion is also valid for a general $a(t)$. In terms of the conformal time $\mathrm{d}\tau = \mathrm{d}t/a$ we have then (for $w \neq -\frac{1}{3}$ and we set $a_0 = a(\tau_0) = 1$)

$$a = (\tau/\tau_0)^{-q}, \qquad H\tau = -\frac{q}{a}, \qquad q = -\frac{2}{1+3w}, \tag{2.19}$$

where $H = \dot{a}/a$. Correspondingly, we find that

$$f = -\frac{\tau_0^2}{q}\left(-\frac{q}{H\tau}\right)^{2+3w}. \tag{2.20}$$

Note that the range of q is such that

$$q \geq 1 \quad \Longleftrightarrow \quad -1 \leq w < -\frac{1}{3},$$
$$q \leq -\frac{1}{2} \quad \Longleftrightarrow \quad -\frac{1}{3} < w \leq 1. \tag{2.21}$$

In conformal time the embedding equations turn out to be

$$X^0 = -\frac{1}{2}\frac{\tau_0}{q}\left\{\left(-\frac{q}{H\tau}\right)^{2+3w} + \frac{q}{H\tau}\right\} - \frac{1}{2}\frac{q}{H\tau}\frac{\vec{x}^2}{\tau_0},$$
$$X^i = -q\frac{x^i}{H\tau},$$
$$X^5 = -\frac{1}{2}\frac{\tau_0}{q}\left\{\left(-\frac{q}{H\tau}\right)^{2+3w} - \frac{q}{H\tau}\right\} + \frac{1}{2}\frac{q}{H\tau}\frac{\vec{x}^2}{\tau_0}. \tag{2.22}$$

Now the similarity with Eq. (2.6) is obvious. In particular, it can be easily checked that the embedding coordinates (2.22) for $H\tau \ll 1$ turn out to be

$$X^0 = -\frac{1}{2}\frac{\tau_0}{H\tau} - \frac{1}{2}\frac{q}{H\tau}\frac{\vec{x}^2}{\tau_0},$$
$$X^i = -q\frac{x^i}{H\tau}, \quad (w < -1/3)$$
$$X^5 = \frac{1}{2}\frac{\tau_0}{H\tau} - \frac{1}{2}\frac{q}{H\tau}\frac{\vec{x}^2}{\tau_0}, \tag{2.23}$$

and

$$X^0 = \frac{1}{2}\frac{\tau_0}{q}\left(-\frac{q}{H\tau}\right)^{2+3w} - \frac{1}{2}\frac{q}{H\tau}\frac{\vec{x}^2}{\tau_0},$$
$$X^i = -q\frac{x^i}{H\tau}, \quad (w > -1/3)$$
$$X^5 = \frac{1}{2}\frac{\tau_0}{q}\left(-\frac{q}{H\tau}\right)^{2+3w} - \frac{1}{2}\frac{q}{H\tau}\frac{\vec{x}^2}{\tau_0}. \tag{2.24}$$

One can now easily verify that for the embedding coordinates (2.23), the hyperboloid degenerates to the cone

$$-X_0^2 + X_i^2 + X_5^2 = 0. \tag{2.25}$$

This does not happen for the embedding coordinates (2.24). Therefore, we conclude that FRW cosmology exhibits three-dimensional conformal symmetry on the future constant-time hypersurfaces not only during a de Sitter ($w = -1$) phase, but in fact for any accelerating ($w < -1/3$) phase. The induced metric on the cone (2.25) is

$$\mathrm{d}s^2 = \left(\frac{\tau_0}{\tau}\right)^{2q} \mathrm{d}\vec{x}^2. \tag{2.26}$$

This metric is invariant under space translations and rotations

$$x_i \to x_i' = a_i + M_i^{\ j} x_j, \tag{2.27}$$

as well as under dilations

$$\tau \to \tau' = \lambda^z \tau, \qquad x_i' = \lambda x_i, \tag{2.28}$$

and inversions

$$\tau \to \tau' = \frac{\tau}{|\vec{x}|^{2z}}, \qquad x_i \to x_i' = \frac{x_i}{\vec{x}^2}, \tag{2.29}$$

where

$$z = \frac{1}{q}. \tag{2.30}$$

Alternatively, we can write the infinitesimal dilations D and special conformal transformations K_i that leave the induced metric invariant, act as

$$\delta_D x_i = \lambda x_i, \qquad \delta_D \tau = z\lambda\tau, \qquad \delta_{D,K_i}\tau_0 = 0,$$
$$\delta_K x_i = -2x_i(\vec{b}\cdot\vec{x}) + b_i\left(-\tau^2 + \vec{x}^2\right), \qquad \delta_K \tau = -2q\tau(\vec{b}\cdot\vec{x}). \tag{2.31}$$

This is nothing but the Lifshitz conformal group acting on Euclidean $\mathbb{R}^3$ with coordinates x_i. We conclude that the symmetry of the future constant-time hypersurfaces $\tau \to 0$ is the three-dimensional conformal group generated by the transformations (2.27), (2.28), and (2.29). We should also note that although the above discussion has been concentrated in the single fluid case with constant equation of state, it applies to any FRW background with a generic $a(t)$. In this case, the cone and the corresponding conformal symmetry is recovered in the $a(t) \gg 1$ limit.

2.3. Conformal invariance of scalar perturbations in an accelerating FRW universe with boundary conformal symmetry

Let us now show explicitly how the fluctuations of a scalar field can be rendered conformal invariant on super-Hubble scales thanks what we have learned in the previous section. Consider the action of a free scalar field $\sigma(\tau, \vec{x})$ in a FRW background

$$S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\, g^{\mu\nu}\,\partial_\mu\sigma\,\partial_\nu\sigma. \tag{2.32}$$

In a de Sitter background this theory is conformal invariant for invariant σ. However, for an FRW metric of the form

$$\mathrm{d}s^2 = \frac{-\mathrm{d}\tau^2 + \mathrm{d}\vec{x}^2}{(\tau/\tau_0)^{2q}}, \tag{2.33}$$

the action (2.32) is written as

$$S = \frac{1}{2}\int \mathrm{d}^3x\mathrm{d}\tau\left(\frac{\tau_0}{\tau}\right)^{2q}\left[\sigma'^2 - (\nabla\sigma)^2\right], \tag{2.34}$$

where the primes denote differentiation with respect to the conformal time τ. Clearly, the action (2.32) is not invariant under the transformations (2.28) and (2.29). We can restore conformal invariance by considering instead the action

$$S = \frac{1}{2}\int \mathrm{d}^3x\mathrm{d}\tau\, I(\tau)\left(\frac{\tau_0}{\tau}\right)^{2q}\left[\sigma'^2 - (\nabla\sigma)^{2/q}\right], \tag{2.35}$$

where $I(\tau)$ is an appropriate function of conformal time. It is easy to verify that $I(\tau)$ should be chosen as

$$I(\tau) = \left(\frac{-\tau}{\tau_0}\right)^{1-q}. \tag{2.36}$$

Such coupling can be easily obtained, for instance, in a power-law inflationary model as follows [19]. In power-law inflation [20] the inflaton potential is (we work with the Planckian mass set to unity)

$$V(\phi) = V_0 e^{-\sqrt{2s}\phi}, \tag{2.37}$$

where s is some positive coefficient. The solution to the Einstein equations are

$$a(\tau) = (-\tau/\tau_0)^{\frac{1}{s-1}}, \qquad \phi = \frac{\sqrt{2s}}{s-1}\ln(-\tau/\tau_0) + \phi_0, \tag{2.38}$$

with the constant ϕ_0 given by

$$\phi_0 = \frac{1}{\sqrt{2s}} \ln\left(\frac{V_0(s-1)^2}{2(s-3)\tau_0^2}\right). \tag{2.39}$$

Therefore, we may express $I(\tau)$ as

$$I(\tau) = (-\tau/\tau_0)^{1-q} = e^{\sqrt{\frac{s}{2}}(\phi-\phi_0)}, \tag{2.40}$$

where $s = (q-1)/q$. Thus, during a power-law inflationary phase, a Lifshitz model non-minimally coupled to the inflaton with action

$$S = \frac{1}{2}\int \mathrm{d}^3x\mathrm{d}\tau \left(\frac{\tau}{\tau_0}\right)^{-3q+1} \left\{\sigma'^2 - (\nabla\sigma)^{2/q}\right\}, \tag{2.41}$$

is conformal invariant if we assume that

$$\sigma(\tau,\vec{x}) \to \sigma'(\tau,\vec{x}), \quad \text{with } \sigma'\left(\tau',\vec{x}'\right) = \sigma(\tau,\vec{x}). \tag{2.42}$$

To find the conformal dimension of the field σ on super-Hubble scales, we should look for time-dependent solutions for $\sigma_\tau = \sigma(\tau)$. The scalar σ_τ satisfies the equation

$$\sigma_\tau'' + \frac{1-3q}{\tau}\sigma_\tau' = 0, \tag{2.43}$$

from where we find that

$$\sigma_\tau = C_1 + C_2\tau^{3q} \tag{2.44}$$

It is easy to see, by using

$$q = -\frac{2}{1+3w}, \qquad s = \frac{q-1}{q} = \frac{3}{2}(1+w), \tag{2.45}$$

that the exponent of τ above is always positive for acceleration

$$q > 0 \quad \Longleftrightarrow \quad w < -\frac{1}{3}. \tag{2.46}$$

Therefore, the constant mode dominates in Eq. (2.44) as $\tau \to 0$, giving rise to an exactly scale-invariant power spectrum

$$\langle \sigma_{\vec{k}_1}\sigma_{\vec{k}_2}\rangle \sim \frac{(2\pi)^3}{k^3}\delta^{(3)}(\vec{k}_1+\vec{k}_2). \tag{2.47}$$

Similarly, for deceleration $q < 0$, the constant mode dominates again, now at the boundary $\tau \to \infty$, leading similarly to a scale-invariant spectrum.

We may also consider a massive scalar. In this case we should add a mass term to the action (2.35). However, a pure mass term will spoil conformal symmetry under (2.28) and (2.29) and one way to restore it is to consider instead

$$S = \frac{1}{2}\int \frac{\mathrm{d}^3x\mathrm{d}\tau}{(\tau/\tau_0)^{4q}}\left\{I(\tau)\left(\frac{\tau_0}{\tau}\right)^{-2q}\left[\sigma'^2 - (\nabla\sigma)^{2/q}\right] - m^2 J(\tau)\sigma^2\right\}, \tag{2.48}$$

where

$$J(\tau) = (-\tau/\tau_0)^{q-1} = e^{-\sqrt{\frac{s}{2}}(\phi-\phi_0)}. \tag{2.49}$$

In this case, the action (2.48) is explicitly written as

$$S = \frac{1}{2}\int \mathrm{d}^3x\,\mathrm{d}\tau\left\{\left(\frac{\tau}{\tau_0}\right)^{-3q+1}\left[\sigma'^2 - (\nabla\sigma)^{2/q}\right] - m^2\left(\frac{\tau}{\tau_0}\right)^{-3q-1}\sigma^2\right\}, \tag{2.50}$$

and the corresponding field equation for a time-dependent field σ_τ is

$$\sigma_\tau'' + \frac{1-3q}{\tau}\sigma_\tau' + \frac{m^2\tau_0^2}{\tau^2}\sigma_\tau = 0. \tag{2.51}$$

The solution to this equation turns out to be

$$\sigma_\tau = \hat{C}_1\tau^{\Delta_-} + \hat{C}_2\tau^{\Delta_+}, \tag{2.52}$$

where

$$\Delta_\pm = \frac{3q}{2}\left(1 \pm \sqrt{1 - \frac{4m^2\tau_0^2}{9q^2}}\right). \tag{2.53}$$

It is easy to verify that during acceleration ($q > 0$) at the $\tau \to 0$ boundary,

$$\sigma_\tau = \hat{C}_1\tau^{\Delta_-} \tag{2.54}$$

dominates, which give rise a two-point function

$$\langle\sigma_{\vec{k}_1}\sigma_{\vec{k}_2}\rangle \sim \frac{(2\pi)^3}{k^{3-\Delta_-}}\delta^{(3)}(\vec{k}_1 + \vec{k}_2). \tag{2.55}$$

For $m\tau_0 \ll 1$, we find that

$$\Delta_- \approx \frac{m^2\tau_0^2}{3q} \tag{2.56}$$

and therefore we get from (2.55) an almost scale invariant red-shifted spectrum.

Similarly, for deceleration, ($q < 0$) at the $\tau \to \infty$ boundary, the solution is again given by (2.54) and the corresponding two-point function by (2.55). However, in this case, for $m^2\tau_0^2 \ll 1$ we have $\Delta_- < 0$ and therefore the spectrum is almost scale invariant, but blue-shifted this time.

Our findings imply that one cannot rule out power-law inflation [20] as a possible model for inflation. It is a common lore that power-law inflation is in a bad shape in the light of the recent Planck data [21]: a potential like (2.37) predicts a spectral index for the scalar perturbations $n_s = 1 + 2s/(s-1)$ and a tensor-to-scalar ratio $r = 16s$; the current 95% C.L. range on n_s implies $10^{-2} \lesssim s \lesssim 1/40$, which in turn implies $0.16 \lesssim r \lesssim 0.43$. This clashes with the 95% C.L. Planck bound $r \lesssim 0.12$. All these arguments are correct only if one assumes that the cosmological perturbations are associated with the perturbations of the inflaton field itself. Relaxing this hypothesis, our results show that a curvaton-like field coupled to the inflaton field via the action (2.41) acquires a scale-invariant power spectrum on super-Hubble scales and the tensor-to-scalar ratio can easily respect the Planck upper bound by choosing V_0 appropriately small.

3. Conformal Killing vectors and FRW accelerating cosmologies

In this subsection, we wish to present an alternative derivation of the fact that any accelerating FRW universe enjoys the three-dimensional conformal group on the future constant-time hypersurface. Let us consider the set of conformal motions on any spatially FRW background, i.e. the group of motions generated by vector fields $\mathbf{X} = X^\alpha\partial_\alpha$ satisfying the equation

$$\mathcal{L}_{\mathbf{X}} g_{\alpha\beta} = \partial_\gamma g_{\alpha\beta} X^\gamma + g_{\alpha\gamma}\partial_\beta X^\gamma + g_{\gamma\beta}\partial_\alpha X^\gamma = 2\phi(x) g_{\alpha\beta}. \tag{3.1}$$

The vectors $\mathbf{X}$ are called the conformal Killing vectors (CKVs) and include as special cases Killing vectors ($\phi = 0$), dilation vectors ($\partial_\alpha \phi = 0$) or special conformal vectors ($\partial^2_{\alpha\beta}\phi = 0$).

The set of CKV on an FRW background can be found, since FRW is conformal to Minkowski space–time, by using the fact that conformally related spaces have the same set of CKVs. Indeed, for two metrics $g_{\alpha\beta}$ and $\gamma_{\alpha\beta} = \rho^2 g_{\alpha\beta}$, if $g_{\alpha\beta}$ satisfies Eq. (3.1), then the metric $\gamma_{\alpha\beta}$ satisfies [22]

$$\mathcal{L}_{\mathbf{X}} \gamma_{\alpha\beta} = 2\,\psi\,\gamma_{\alpha\beta}, \tag{3.2}$$

where

$$\psi = \frac{1}{2\rho^2}\mathbf{X}\big(\rho^2\big) + \phi. \tag{3.3}$$

The CKVs of four-dimensional Minkowski space–time are

$$\begin{aligned} &\mathbf{P}_\alpha = \partial_\alpha, \qquad \mathbf{M}_{\alpha\beta} = x_\alpha\partial_\beta - x_\beta\partial_\alpha, \\ &\mathbf{D} = x^\alpha\partial_\alpha, \qquad \mathbf{K}_\alpha = 2x_\alpha\mathbf{D} - x_\beta x^\beta \mathbf{P}_\alpha. \end{aligned} \tag{3.4}$$

Of these $\mathbf{P}_\alpha$ (translations) and $\mathbf{M}_{\alpha\beta}$ (rotations) are Killing vectors ($\phi = 0$) and generate isometries of Minkowski space–time whereas, D is the dilation ($\phi = 1$) and $\mathbf{K}_\alpha$ generates the special conformal transformations ($\phi = 2x^a$). These vectors satisfy the SO(2, 4) algebra defined by the non-zero commutation relations

$$\begin{aligned} &[\mathbf{M}_{\alpha\beta}, \mathbf{M}_{\gamma\delta}] = \tau_{\alpha\delta}\mathbf{M}_{\beta\gamma} + \tau_{\beta\gamma}\mathbf{M}_{\alpha\delta} - \tau_{\alpha\gamma}\mathbf{M}_{\beta\delta} - \tau_{\beta\delta}\mathbf{M}_{\alpha\gamma}, \\ &[\mathbf{P}_\alpha, \mathbf{M}_{\beta\gamma}] = \tau_{\alpha\beta}\mathbf{P}_\gamma - \tau_{\alpha\gamma}\mathbf{P}_\alpha, \qquad [\mathbf{K}_\alpha, \mathbf{M}_{\beta\gamma}] = \tau_{\alpha\beta}\mathbf{K}_\gamma - \tau_{\alpha\gamma}\mathbf{K}_\alpha, \\ &[\mathbf{P}_\alpha, \mathbf{K}_\beta] = 2(\tau_{\alpha\beta}\mathbf{D} - 2\mathbf{M}_{\alpha\beta}), \qquad [\mathbf{D}, \mathbf{K}_\alpha] = \mathbf{K}_\alpha, \qquad [\mathbf{D}, \mathbf{P}_\alpha] = -\mathbf{P}_\alpha. \end{aligned} \tag{3.5}$$

Similarly, a generic FRW metric can always be written in conformally flat form as

$$ds^2 = a(\tau)^2\big(-\mathrm{d}\tau^2 + \mathrm{d}\vec{x}^2\big) \tag{3.6}$$

Therefore, the vectors $\mathbf{X} = (\mathbf{P}_\alpha, \mathbf{M}_{\alpha\beta}, \mathbf{K}_\alpha, \mathbf{D})$ are CKV for (3.6) with conformal factor

$$\psi = \frac{1}{2a^2}\mathbf{X}\big(a^2\big) + \phi \tag{3.7}$$

Let us now consider a particular FRW metric in conformal time

$$\mathrm{d}s^2 = \left(\frac{\tau_0}{\tau}\right)^{2q}\big(-\mathrm{d}\tau^2 + \mathrm{d}\vec{x}^2\big), \tag{3.8}$$

which also includes as special cases Minkowski ($q = 0$) and de Sitter ($q = 1$). The CKVs and the corresponding conformal factor (ϕ for Minkowski and ψ for de Sitter or generic FRW) are given in the following table

	Minkowski	de Sitter	Generic FRW
$\mathbf{P}_0$	0	$-\frac{1}{\tau}$	$-q\frac{1}{\tau}$
$\mathbf{P}_i$	0	0	0
$\mathbf{M}_{ij}$	0	0	0
$\mathbf{M}_{0i}$	0	$\frac{x_i}{\tau}$	$q\frac{x_i}{\tau}$
$\mathbf{D}$	1	0	$q-1$
$\mathbf{K}_0$	2τ	$\frac{\tau^2+\vec{x}^2}{\tau}$	$\frac{(2q-1)\tau^2+\vec{x}^2}{\tau}$
$\mathbf{K}_i$	$2x_i$	0	$2(q-1)x_i$

(3.9)

It is easy to see that the metric has a boundary at $\tau \to 0$ for $q > 0$ and a boundary at $\tau \to \infty$ for $q < 0$. In other words, accelerating backgrounds posses a boundary at $\tau \to 0$ whereas for decelerating ones the boundary is at $\tau \to \infty$. At the boundary, the FRW CKVs can be projected into tangential and normal to the boundary. It is easy to verify that the tangential vectors turn out to be

$$\hat{\mathbf{P}}_i = \partial_i, \qquad \hat{\mathbf{M}}_{ij} = x_i \partial_j - x_i \partial_j,$$
$$\hat{\mathbf{D}} = x^i \partial_i, \qquad \hat{\mathbf{K}}_i = 2x_i \hat{\mathbf{D}} - x_i x^j \hat{\mathbf{P}}_i, \quad (i, j = 1, 2, 3). \tag{3.10}$$

We conclude that for an accelerating universe the $\tau \to 0$ boundary possess ten CKVs which form an SO(1, 4) algebra, precisely the three-dimensional conformal algebra. Only for de Sitter space $q = 1$, however, these ten CKVs are actually isometries, whereas for all the other cases ($q < 1$) only $\hat{\mathbf{P}}_i$ and $\hat{\mathbf{M}}_{ij}$ generate isometries. Therefore, if a theory is invariant under the generators $\hat{\mathbf{P}}_i, \hat{\mathbf{M}}_{ij}, \hat{\mathbf{D}}$ and $\hat{\mathbf{K}}_i$ in the bulk of an accelerating FRW cosmology, it will be automatically SO(1, 4) invariant, i.e. conformal invariant at the boundary $\tau \to 0$.

4. Perturbed FRW universe and conformal symmetries

As an application of what we have described in the previous section, let us consider now a perturbed FRW metric and write the corresponding metric in, say, the Newtonian gauge

$$\mathrm{d}s^2 = \left(\frac{\tau_0}{\tau}\right)^{2q} \left[-(1 + 2\Phi)\mathrm{d}\tau^2 + (1 - 2\Phi)\mathrm{d}\vec{x}^2\right], \tag{4.1}$$

where $\Phi(\tau, \vec{x})$ is the gravitational potential. We restrict ourselves to long wavelength part of the perturbations, indicated by Φ_L, which are still in the linear regime. More in particular, we make the following assumptions: *i*) the long wavelength gravitational potential does not evolve in time and *ii*) we consider such perturbations only up to first order both in Φ_L and its gradient $\partial_i \Phi_L$. The first condition is automatically attained during the matter-dominated (MD) period while in the radiation-dominated (RD) period it is valid thanks to the second condition, as $\dot{\Phi}_L = \mathcal{O}(c_s^2 \tau^2 \nabla^2 \Phi_L)$, where $c_s \simeq 1/\sqrt{3}$ is the sound speed of the perturbations (notice that during the matter-dominated period $c_s \simeq 0$ and therefore Φ_L is constant even on very small scales). Notice that the gravitational potential Φ_L is related to the comoving curvature perturbation ζ_L whose value is set by inflation, $\Phi_L(\mathrm{MD}) = 9/10\Phi_L(\mathrm{RD}) = 3/5\zeta_L$. Notice also that the following considerations hold also during inflation, i.e. during a period of quasi-de Sitter expansion.

Our goal is to show that the long wavelength perturbations (in the Newtonian gauge) are just (projected) conformal Killing motions of the FRW background. Let us first perform a direct computation and write the metric (4.1) as

$$\mathrm{d}s^2 = \left(\frac{\tau_0}{\tau}\right)^{2q} (1 + 2\Phi_L)\left[-\mathrm{d}\tau^2 + (1 - 4\Phi_L)\mathrm{d}\vec{x}^2\right], \tag{4.2}$$

and perform an infinitesimal dilation transformation followed by an infinitesimal special conformal transformation on the coordinates x^i

$$y^i = x^i(1 + \lambda) - 2(\vec{x} \cdot \vec{b})x^i + b^i \vec{x}^2. \tag{4.3}$$

Since this is a three-dimensional conformal transformation in flat $\mathbb{R}^3$, it implies that

$$\mathrm{d}\vec{y}^2 = (1 + 2\lambda - 4\vec{b} \cdot \vec{x})\,\mathrm{d}\vec{x}^2. \tag{4.4}$$

Expanding the long wavelength part of the gravitational potential around an arbitrary point $\vec{x}_0$ (which we choose to set as the origin of the coordinates),

$$\Phi_L(\vec{x}) \simeq \Phi_L(\vec{0}) + \partial_i \Phi_L(\vec{0})\, x^i + \mathcal{O}\big(\partial_i \partial_j \Phi_L(\vec{0})\big), \tag{4.5}$$

and choosing

$$\lambda = -2\Phi_L(\vec{0}), \qquad b^i = \partial^i \Phi_L(\vec{0}), \tag{4.6}$$

we can recast the metric in the form

$$\mathrm{d}s^2 = \tau^2 \left(\frac{\tau_0}{\tau}\right)^{2q} (1 + 2\Phi_L) \left(\frac{-\mathrm{d}\tau^2 + \mathrm{d}\vec{y}^2}{\tau^2}\right), \tag{4.7}$$

which is conformal to the de Sitter metric. Since following infinitesimal transformation

$$z^i = y^i(1+\alpha) - 2(\vec{y}\cdot\vec{a})y^i + a^i\big(-\tau^2 + \vec{y}^2\big), \qquad \eta = \tau(1 + \alpha - 2\vec{y}\cdot\vec{a}), \tag{4.8}$$

is an isometry of the de Sitter metric, and choosing

$$\alpha = \frac{1}{1-q}\Phi_L(\vec{0}), \qquad a^i = \frac{1}{2(q-1)}\partial^i \Phi_L(\vec{0}), \tag{4.9}$$

we can recast the metric (3.8) into the form

$$\mathrm{d}s^2 = \tau^2 \left(\frac{\tau_0}{\tau}\right)^{2q} \left(\frac{-\mathrm{d}\tau^2 + \mathrm{d}\vec{z}^2}{\tau^2}\right), \tag{4.10}$$

which is the background FRW metric. In terms of the original coordinates in (4.1), the transformation (4.8) is written as

$$\begin{aligned} z^i &= x^i \left(1 + \frac{2q-1}{1-q}\Phi_L(\vec{0})\right) + \frac{1}{2(q-1)}\big\{-\tau^2 + (2q-1)\vec{x}^2\big\}\partial^i \Phi_L(\vec{0}) \\ &\quad + \frac{1-2q}{q-1}\big(\vec{x}\cdot\vec{\nabla}\Phi_L(\vec{0})\big)x^i \\ &= x^i \left(1 + \frac{2q-1}{1-q}\Phi_L(\vec{x})\right) + \frac{1}{2(q-1)}\big\{-\tau^2 + (2q-1)\vec{x}^2\big\}\partial^i \Phi_L(\vec{x}), \\ \eta &= \tau\left(1 + \frac{1}{1-q}\Phi(\vec{0}) + \frac{1}{1-q}\vec{x}\cdot\nabla\Phi_L(\vec{0})\right) = \tau\left(1 + \frac{1}{1-q}\Phi_L(\vec{x})\right). \end{aligned} \tag{4.11}$$

Note that during single-field inflation, where the comoving curvature perturbation is generated by an inflaton field $\phi(\tau, \vec{x})$ with potential $V(\phi)$ and one has a period of quasi-de Sitter, the parameter q is given by $q = (1+\epsilon)$, being $\epsilon = (1 - \mathcal{H}'/\mathcal{H}^2)$ one of the slow-roll parameters. However, the change of coordinates is not singular in the de Sitter limit of tiny ϵ as, in the Newtonian gauge, $\Phi_L(\vec{x}) = \epsilon\mathcal{H}\delta\phi(\vec{x})/\phi'$. It is easy to see that the transformations (4.11) in this case reduces to (2.5), i.e. they are just an isometry of the background and therefore leave the metric invariant.

For a matter dominated era $q = -2$, so that the conformal transformation that eliminates (or, equivalently, generates) long wavelength comoving perturbations, is

$$\begin{aligned} z^i &= x^i \left(1 - \frac{5}{3}\Phi_L(\vec{x})\right) + \left(\frac{1}{6}\tau^2 + \frac{5}{6}\vec{x}^2\right)\partial^i \Phi_L(\vec{x}), \\ \eta &= \tau\left(1 + \frac{1}{3}\Phi(\vec{x})\right). \end{aligned} \tag{4.12}$$

These relativistic transformations generalize the non-relativistic ones found in Refs. [15,16] and have been recently derived in Ref. [17]. They allow to derive relativistic consistency relations involving the soft limit of the $(n+1)$-correlator functions of matter overdensities and the primordial gravitational potential [17]. These consistency relations, as pointed out in Refs. [15,17], are valid not only for dark matter, but also for halos and galaxy number density, and might be used to test theories of modified gravity.

We wish to show now that the long wavelength comoving curvature perturbations in the Newtonian gauge are indeed generated by CKVs. The reason behind this is that long wavelength perturbations are linear in the spatial coordinates as can be seen from Eq. (4.5). As a result, they can be generated by dilations and special conformal transformations. In particular, it is straightforward to verify that

$$\mathcal{L}_{\xi_1} g_{ij} = (1 + 2\lambda - 4\vec{b}\cdot\vec{x}) g_{ij}, \qquad \mathcal{L}_{\xi_1} g_{00} = 0, \qquad \mathcal{L}_{\xi_1} g_{0i} = 0, \tag{4.13}$$

where ξ_1 is the vector

$$\xi_1 = (1 + 2\lambda) x^i \partial_i + \left\{ b^i \vec{x}^2 - 2(\vec{b}\cdot\vec{x}) \right\} \partial_i . \tag{4.14}$$

This means that ξ_1 is nothing else than a linear combinations of the CKVs tangent to the $\tau =$ const. hypersurfaces. Indeed, ξ_1 can be written as

$$\xi_1 = (1 + 2\lambda) \Pi(\mathbf{D}) - b^i \Pi(\mathbf{K}_i), \tag{4.15}$$

where $\Pi(\mathbf{D}), \Pi(\mathbf{K}_i)$ are the projections of the CKV along the $\tau =$ const. hypersurfaces. Therefore, ξ_1 generates the factor $(1 - 4\Phi_L)$ in front of the spatial part of the metric (4.2). Then it is easy to verify by a simple inspection of the table (3.9), that the conformal factor $(1 + 2\Phi_L)$ is generated simply by the CKV ξ_2

$$\xi_2 = \frac{s-1}{2s} (1 - \lambda) \mathbf{D} + \frac{s-1}{2s} b^i \mathbf{K}_i, \tag{4.16}$$

since we have

$$\mathcal{L}_{\xi_2} g_{\mu\nu} = (1 - \lambda + 2\vec{b}\cdot\vec{x}) g_{\mu\nu}. \tag{4.17}$$

Therefore, long wavelength comoving curvature perturbations of the FRW metric are just conformal Killing motions of the FRW background.

For the sake of completeness, let us offer another, direct derivation of the transformation (4.12). Consider the metric (2.33) and make the change of coordinates $\tau \to \tau + \eta(\tau, \vec{x})$ and $x_i \to x^i + \xi_i(\tau, \vec{x})$. Under this transformation, the metric (2.33) changes to

$$\begin{aligned} \mathrm{d}s^2 = \left(\frac{\tau_0}{\tau}\right)^{2q} \Bigg\{ & -\left(1 - 2q\frac{\eta}{\tau} + \partial_\tau \eta\right) \mathrm{d}\tau^2 + \left(1 - 2q\frac{\eta}{\tau}\right) \mathrm{d}\vec{x}^2 + 2\partial_j \xi_i \mathrm{d}x^i \mathrm{d}x^j \\ & + 2(\partial_\tau \xi_i - \partial_i \eta) \mathrm{d}\tau \mathrm{d}x^i \Bigg\}. \end{aligned} \tag{4.18}$$

This is of the form (4.1) if the following conditions are satisfied

$$\partial_i \eta = \partial_\tau \xi_i, \tag{4.19}$$

$$\partial_\tau \eta - q\frac{\eta}{\tau} = \Phi_L, \tag{4.20}$$

$$\partial_i \xi_j + \partial_j \xi_i = 2\delta_{ij} \left(q\frac{\eta}{\tau} - \Phi_L \right). \tag{4.21}$$

The solution of Eq. (4.19) is

$$\eta = \frac{1}{1-q}\tau\,\Phi_L + \tau^q c(\vec{x}), \tag{4.22}$$

so that Eq. (4.20) gives

$$\xi_i = \frac{\tau^2}{2(1-q)}\partial_i\Phi_L + \frac{\tau^{q+1}}{q+1}\partial_i c(\vec{x}) + f_i(\vec{x}). \tag{4.23}$$

Finally the last equation (4.21) turns out to be

$$2\frac{\tau^{q+1}}{q+1}\partial_i\partial_j c + \partial_i f_j + \partial_j f_i = 2\frac{2q-1}{1-q}\Phi_L\delta_{ij} + 2q\tau^{q-1}c\delta_{ij}. \tag{4.24}$$

Therefore,

$$c = 0, \qquad \partial_i f_j + \partial_j f_i = 2\frac{2q-1}{1-q}\Phi_i\delta_{ij} \tag{4.25}$$

and thus f_i are just conformal Killing vectors. The solution to Eq. (4.25) as we have seen in Section 3 is

$$f_i = 2a^j x_j x_i - \vec{x}^2 a_i + \lambda x_i, \tag{4.26}$$

where, for the present case,

$$a_i = \frac{2q-1}{2(1-q)}\partial_i\Phi_L(\vec{0}), \qquad \lambda = \frac{2q-1}{1-q}\Phi_L(\vec{0}). \tag{4.27}$$

Then, it is straightforward to verify that

$$\eta = \frac{1}{1-q}\tau\Phi_L(\vec{0}), \qquad \xi_i = \frac{2q-1}{1-q}x_i\Phi_L(\vec{0}) + \frac{1}{2(1-q)}\left\{\tau^2 - (2q-1)x^2\right\}\partial_i\Phi(\vec{0}), \tag{4.28}$$

so that we get the result (4.11). Note that for a de Sitter space, $q = 1$ and the transformation is singular. The reason is that the integration of Eq. (4.19) for the $q = 1$ case gives instead

$$\eta = \Phi_L(\vec{0})\tau\log\tau + \tau c(\vec{x}), \tag{4.29}$$

which leads to

$$\xi_i = \partial_i\Phi_L(\vec{0})\left(\frac{1}{2}\tau^2\log\tau - \frac{\tau^2}{4}\right) + \frac{\tau^2}{2}\partial_i c(\vec{x}) + g_i(\vec{x}). \tag{4.30}$$

Eq. (4.21) is then written as

$$2\frac{\tau^2}{2}\partial_i\partial_j c + \partial_i g_j + \partial_j g_i = 2\left\{\Phi_L(\vec{0})(\log\tau - 1) + c\right\}\delta_{ij}, \tag{4.31}$$

which does not have a solution. Long wavelength perturbations in de Sitter cannot be generated in this way. This is consistent with the fact that there are no scalar perturbations in de Sitter background. This is also consistent with the fact that conformal Killing motions in de Sitter space–time are just isometries and therefore long wavelength perturbations cannot be generated.

5. Bulk conformal symmetry of FRW accelerating cosmologies

In the previous section we have seen that the future boundary of any FRW accelerating cosmology enjoys conformal invariance for bulk theories that are invariant under the bulk CKVs. On the other hand, it is well known that in the AdS/CFT correspondence as well as in any holographic theory, the symmetries of the boundary theory correspond to isometries of the bulk. Translating this in the present context, one should expect that since the boundary theory enjoys conformal invariance, the isometries of the bulk of an accelerating FRW is the full conformal group. However, the metric (3.8) is clearly not invariant under the three-dimensional conformal group. We may write it as

$$\mathrm{d}s^2 = \left(\frac{\tau^s}{\tau_0}\right)^{\frac{2}{s-1}} \left(\frac{-\mathrm{d}\tau^2 + \mathrm{d}\vec{x}^2}{\tau^2}\right) = \Omega^2\, \mathrm{d}s^2_{\mathrm{dS}}, \quad s = 1 - \frac{1}{q}, \tag{5.1}$$

that is in a conformally de Sitter form where $\mathrm{d}s^2_{\mathrm{dS}}$ is the de Sitter metric and

$$\Omega^2 = \left(\frac{\tau^s}{\tau_0}\right)^{\frac{2}{s-1}}. \tag{5.2}$$

Then, since de Sitter space–time has the three-dimensional conformal group SO(1, 4) as isometry group, the failure of the FRW metric to have the same symmetries is due to the conformal factor Ω. Therefore, to implement SO(1, 4) invariance for an FRW background, we have to demand that

$$\delta_D \Omega = 0, \qquad \delta_{K_i} \Omega = 0, \tag{5.3}$$

under infinitesimal dilations D and special conformal transformations K_i. One possibility is the following. Consider the action

$$S = \int \mathrm{d}^4x\, \sqrt{-g} \left(\frac{1}{2}\varphi^2 R + 3\partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2}\varphi^2 \partial_\mu \phi \partial^\mu \phi - \varphi^4 V_0 e^{-\sqrt{2s}\,\phi} \right), \tag{5.4}$$

where we have introduced an additional field (compensator or Stuckelberg field) φ. Note that the kinetic term for the field φ has opposite sign and seems to describe a ghost [23]. However, φ can be fixed at will as the theory is invariant under conformal rescalings

$$g_{\mu\nu}(x) \to e^{2\omega(x)} g_{\mu\nu}(x), \qquad \varphi(x) \to e^{-\omega(x)} \varphi(x), \qquad \phi(x) \to \phi(x). \tag{5.5}$$

Indeed, the action (5.5) can be written in terms of the invariant metric $\varphi^2 g_{\mu\nu}$ and a solution is provided by

$$\begin{aligned} \mathrm{d}s^2 &= g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = \varphi^{-2} \left(\frac{\tau}{\tau_0}\right)^{\frac{2}{s-1}} \left(-\mathrm{d}\tau^2 + \mathrm{d}\vec{x}^2\right), \\ \phi &= \sqrt{\frac{2}{s}} \frac{1}{s-1} \ln\left(\frac{\tau^s}{\tau_0}\right) + \widetilde{\phi}_0, \end{aligned} \tag{5.6}$$

where

$$\widetilde{\phi}_0 = \frac{1}{\sqrt{2s}} \ln\left(\frac{V_0 (s-1)^2}{2(s-3)}\right). \tag{5.7}$$

In fact, we may express (5.6) as

$$\mathrm{d}s^2 = \varphi^{-2} e^{\sqrt{2s}(\phi-\widetilde{\phi}_0)}\left(\frac{-\mathrm{d}\tau^2+\mathrm{d}\vec{x}^2}{\tau^2}\right), \qquad \phi = \sqrt{\frac{2}{s}}\frac{1}{s-1}\ln\left(\frac{\tau^s}{\tau_0}\right)+\widetilde{\phi}_0. \tag{5.8}$$

Fixing the value of φ corresponds to choosing the conformal frame and removes the freedom (5.5). Interesting gauges are the $\varphi = 1$ gauge which is just the Einstein frame and gives the solution (2.38). Another gauge is

$$\varphi = e^{\sqrt{\frac{s}{2}}(\phi-\widetilde{\phi}_0)}, \qquad \mathrm{d}s^2 = \frac{-\mathrm{d}\tau^2+\mathrm{d}\vec{x}^2}{\tau^2}, \qquad \phi = \sqrt{\frac{2}{s}}\frac{1}{s-1}\ln\left(\frac{\tau^s}{\tau_0}\right)+\widetilde{\phi}_0, \tag{5.9}$$

which gives a de Sitter background metric. The metric in Eq. (5.6) can be written as

$$\mathrm{d}s^2 = \widetilde{\Omega}^2\left(\frac{-\mathrm{d}\tau^2+\mathrm{d}\vec{x}^2}{\tau^2}\right), \qquad \widetilde{\Omega}^2 = \varphi^2\frac{\tau_0^{2q}}{\tau^{2(q-1)}} = \varphi^2\Omega^2, \tag{5.10}$$

and $\widetilde{\Omega}^2$ reduces to (5.2) in the $\varphi = 1$ gauge. Now, under a conformal transformation $\widetilde{\Omega}$ satisfies

$$\delta_D\widetilde{\Omega} = 0, \qquad \delta_{K_i}\widetilde{\Omega} = 0. \tag{5.11}$$

In other words, the metric (5.1) does not look invariant under conformal transformations simply because is written in the particular conformal frame $\varphi = 1$, the Einstein frame. In fact, under a conformal transformation, one should transform φ as well before setting $\varphi = 1$, thus rendering manifest the bulk conformal symmetry of FRW accelerating cosmologies. Notice that the transformation of the conformon field φ may be traded with the transformation of the dimensional quantity τ_0. This is not strange as τ_0 may be thought as a scalar field being directly linked to the initial value ϕ_0 of the vacuum expectation value of the inflaton field, see Eq. (5.6).

5.1. Conformal invariance of scalar perturbations in an accelerating FRW universe with bulk conformal symmetry

Let us now make use of what we have just learnt and try to construct a conformally-invariant action for a scalar field which will deliver a scale-invariant spectrum on super-Hubble scales.

Let us consider now a scalar field $\sigma(\tau,\vec{x})$, the dynamics of which is determined by the action

$$S = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\varphi^2 I^2(\phi)g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma, \tag{5.12}$$

where $I(\phi)$ represents a coupling to the inflaton. Note that (5.12) is invariant under the transformation (5.5). Using (5.8), the action (5.12) is independent of the conformal frame as the φ factors drop out and we get

$$S = -\frac{1}{2}\int \mathrm{d}^3x\mathrm{d}\tau I^2(\phi)e^{-\sqrt{2s}(\phi-\widetilde{\phi}_0)}\frac{\tau_0^2}{\tau^2}\left\{\sigma'^2-(\nabla\sigma)^2\right\} \tag{5.13}$$

Then, for

$$I(\phi) = e^{-\sqrt{\frac{s}{2}}(\phi-\widetilde{\phi}_0)}, \tag{5.14}$$

the action (5.12) is explicitly written as

$$S = \frac{1}{2}\int \mathrm{d}^3x\mathrm{d}\tau\frac{\tau_0^2}{\tau^2}\left\{\sigma'^2-(\nabla\sigma)^2\right\} \tag{5.15}$$

Clearly, (5.15) is conformal invariant, if we assume that

$$\sigma(\tau,\vec{x})\to\sigma'(\tau,\vec{x}),\quad \text{with } \sigma'\big(\tau',\vec{x}'\big)=\sigma(\tau,\vec{x}). \tag{5.16}$$

To find the conformal dimension of the field σ on super-Hubble scales, we should look as usual for time-dependent solutions for $\sigma_\tau=\sigma(\tau)$ to the equations of motion

$$\sigma_\tau''-\frac{2}{\tau}\sigma_\tau'=0. \tag{5.17}$$

Then we find that

$$\sigma_\tau=\sigma_0+\sigma_1\tau^3, \tag{5.18}$$

so that on super-Hubble scales we will have the constant mode

$$\sigma(\tau,\vec{x})=\sigma_0(\vec{x}). \tag{5.19}$$

The conformal dimension of $\sigma_0(\vec{x})$ is then $\Delta_{\sigma_0}=0$ and therefore, we will have an exact scale-invariant spectrum

$$\langle\sigma_{\vec{k}_1}\sigma_{\vec{k}_2}\rangle\sim\frac{(2\pi)^3}{k_1^3}\delta^{(3)}(\vec{k}_1+\vec{k}_2). \tag{5.20}$$

The corresponding conformal weight of the initial field $\sigma(\tau,\vec{x})$ can be easily deduced from the transformations of the constant-time τ_0.

Note that, the above findings are valid for any FRW geometry, including decelerating one. However, in the latter case, as the boundary is at $\tau\to\infty$, the mode that dominates is the σ_1 which has conformal dimension $\Delta_{\sigma_1}=3$. As a result, the two-point correlator is given in this case by

$$\langle\sigma_{\vec{k}_1}\sigma_{\vec{k}_2}\rangle\sim(2\pi)^3\,k_1^3\,\delta^{(3)}(\vec{k}_1+\vec{k}_2)\quad\text{(deceleration)}. \tag{5.21}$$

Finally, we may also consider massive scalars described by

$$S=-\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\varphi^2I^2(\phi)\big(g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma+m^2\varphi^2\sigma^2\big). \tag{5.22}$$

This action is again invariant under the transformation (5.5). By using (5.8), the action (5.12) is also independent of the conformal frame (φ factors again drop out) and it is explicitly written as

$$S=\frac{1}{2}\int \mathrm{d}^3x\mathrm{d}\tau\, I^2(\phi)e^{-\sqrt{2s}(\phi-\tilde{\phi}_0)}\left\{\frac{\tau_0^2}{\tau^2}\big(\sigma'^2-(\nabla\sigma)^2\big)-m^2\sigma^2\right\}. \tag{5.23}$$

Thus, (5.14) reduces (5.23) to

$$S=\frac{1}{2}\int \mathrm{d}^3x\mathrm{d}\tau\left\{\frac{\tau_0^2}{\tau^2}\big(\sigma'^2-(\nabla\sigma)^2\big)-m^2\sigma^2\right\}, \tag{5.24}$$

which is the standard action of a massive scalar on a de Sitter background. The two-point correlator is in this case

$$\langle\sigma_{\vec{k}_1}\sigma_{\vec{k}_2}\rangle\sim\frac{(2\pi)^3}{k_1^{3-2\Delta}}\delta^{(3)}(\vec{k}_1+\vec{k}_2),\qquad \Delta=\frac{3}{2}\left(1-\sqrt{1-\frac{4}{9}m^2\tau_0^2}\right). \tag{5.25}$$

We stress that the above results are frame independent as the conformon field disappears from the action.

6. Conclusions

In this paper we have shown that any accelerating FRW cosmology enjoys a three-dimensional Lifshitz conformal symmetry on the future constant-time hypersurface if the bulk theory is invariant under bulk CKVs. It has been recently pointed out that flat FRW cosmologies and higher-dimensional hyperscaling-violating geometries can be connected by analytic continuation [24]. It would be interesting to investigate this connection further in the light of our results. We have also shown that the boundary conformal symmetry can be extended to the bulk of FRW accelerating cosmologies. This is reminiscent of the so-called generalized conformal symmetry which was proposed as an extension of the conformal symmetry of (boundary) D3-branes to general (bulk) Dp-brane systems [25].

Our results imply that one can construct a theory of a free scalar field in an accelerating FRW cosmology whose perturbations will be scale-invariant and, in fact, even conformal invariant on super-Hubble scales. On general grounds this means that measuring a scale-invariant power spectrum for the cosmological perturbation does not imply that the universe went necessarily through a de Sitter stage. Indeed, it suffices to have a curvaton-like field [26–28] appropriately coupled to the any accelerating FRW cosmology and its perturbations will be scale-invariant. Furthermore, its three-point correlator will respect the full three-dimensional conformal symmetry on super-Hubble scales and therefore it will be enhanced in the squeezed configuration [4,5].

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