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Applications of Path Expansions to Statistical Mechanics

Panis, Romain

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Applications of Path Expansions to Statistical Mechanics

THÈSE

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par

Romain Panis

de
Paris (France)

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*À Papy,
à Tonton Guy*



On a sunny forest day, a rabbit sat outside his burrow, typing away on an old typewriter. A passing fox stopped to ask, "What are you working on?"

"My thesis," replied the rabbit.

"And what's it about?" the fox inquired.

"It's about how rabbits eat foxes," said the rabbit.

The fox scoffed. "That's absurd! Rabbits don't eat foxes."

"Sure they do, and I can prove it. Come with me," the rabbit said.

The fox, curious and skeptical, followed the rabbit into his burrow. After a few minutes, the rabbit reemerged alone and resumed typing.

Inside the burrow, the scene was clear: a pile of fox bones in one corner, and a satisfied lion in the other, leisurely picking his teeth.

It doesn't matter what you choose for a thesis topic.

What truly matters is who you have for a thesis advisor.

Remerciements

Il est de tradition de commencer par remercier les personnes qui, à leur manière, ont influencé l'écriture des pages qui vont suivre.

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¹I should probably also mention that Trish taught me humility during a particularly challenging FIFA session in London—an experience that will likely haunt me for years.

²Special thanks to Gady for sharing the rabbit tale!

more!

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Mon parcours mathématique s'est progressivement dessiné sous l'influence de certaines rencontres précieuses. Je tiens d'abord à remercier Thomas Chene pour ses conseils éclairés. Je remercie ensuite mes professeurs de classe préparatoire, et plus spécifiquement Stéphane Komilikis, Eddy Routin et Serge Dupont. J'ai eu le privilège de forger et cultiver mon amour pour les Mathématiques et la Physique à vos côtés. Votre bienveillance et votre pédagogie exceptionnelles sont un véritable modèle, et j'espère sincèrement réussir à le suivre. Enfin, je remercie Raphaël Cerf et Nicolas Curien qui ont tout les deux grandement influencé mes goûts mathématiques.

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Et enfin, merci à Mathilde de me permettre d'écrire la ligne que je chéris le plus dans ce manuscrit. Le texte qui suit est empreint de l'inépuisable tendresse avec laquelle tu m'as soutenu pendant ces quatre années. Quel bonheur de grandir à tes côtés !

Résumé

Cette thèse porte sur l'étude de phénomènes en haute dimension en physique statistique. Nous étudions divers *modèles* présentant une *transition de phase*, c'est-à-dire qui subissent une transformation radicale de leurs propriétés macroscopiques lorsque l'un de leurs paramètres physiques, tel que la température, varie au-delà d'une valeur *critique*. Notre fil conducteur est le *modèle d'Ising*. Nous sommes particulièrement intéressés par le comportement proche du *point critique*. Une prédiction issue de la physique théorique est que le comportement de ces modèles se simplifie drastiquement lorsque la dimension de l'espace ambiant est suffisamment élevée. Nous proposons une analyse rigoureuse de cette observation reposant sur des *expansions en chemin* à la fois classiques et nouvelles, qui procurent une interprétation géométrique de certaines quantités clés. Cette thèse est à l'interface entre physique statistique, combinatoire et probabilités. Elle est divisée en trois parties.

La première partie concerne l'étude générale du *régime de champ moyen* en physique statistique. Nous développons une nouvelle perspective plus probabiliste que les précédentes approches et appliquons nos méthodes à deux modèles : la *marche faiblement auto-évitanes* et la *percolation de Bernoulli étendue* (i.e. à portée longue mais finie). Ces modèles contiennent un paramètre *perturbatif* dans leur définition et constituent donc un cadre naturel pour développer nos techniques.

Dans la deuxième partie, nous portons notre attention sur l'étude du modèle d'Ising. En utilisant la *représentation en courants aléatoires*, nous obtenons certaines propriétés fines du modèle lorsque la dimension est suffisamment grande. Pour appréhender le comportement d'un modèle près du point critique, il est pertinent de calculer ses *exposants critiques*. Nous calculons deux nouveaux exposants critiques du modèle en dimensions $d \geq 4$. De plus, nous caractérisons la *limite d'échelle* du modèle critique en dimensions $d > 4$. Ce second résultat nécessite le développement d'une nouvelle expansion en chemins étroitement liée à la représentation en courants aléatoires. Enfin, nous considérons des modèles d'Ising à *longue portée*. Leur comportement est particulièrement complexe et fascinant. Nous présentons une analyse de leur régime de champ moyen.

La troisième et dernière partie est consacrée à une version *non-bornée* du modèle d'Ising appelée le *modèle φ^4* . Nous développons une nouvelle représentation géométrique du modèle appelée *représentation en courants emmêlés aléatoires*. Cet objet dispose de certaines propriétés combinatoires pouvant être utilisées pour analyser le modèle φ^4 . Notre principal résultat est une preuve du fait que la transition de phase du modèle est *continue* (ou de *second ordre*) en dimensions $d \geq 3$.

J'espère que vous prendrez autant de plaisir à lire ces lignes que j'en ai pris à les écrire. Bonne lecture !

Abstract

This thesis explores high-dimensional phenomena in statistical mechanics. We examine various models that undergo a *phase transition*, i.e. a significant change in a macroscopic property that occurs when a physical parameter, such as the temperature, varies beyond its *critical value*. Our guiding thread is the celebrated *Ising model*. We are particularly interested in the behaviour of these models *near* and *at criticality*. Our motivation stems from a prediction in physics suggesting a drastic *simplification* in the behaviour of these models when the dimensionality of the ambient space is sufficiently large. To conduct this study, we use both known and novel *path expansions* which offer geometric interpretations of key quantities. This thesis is at the interface between statistical mechanics, combinatorics, and probabilities. It is divided into three parts.

The first part is devoted to the general study of the *mean-field regime* in statistical mechanics. We develop a novel perspective which is more probabilistic than previous approaches. We apply our techniques to the *weakly self-avoiding walk model* and to *spread-out Bernoulli percolation*. Given the presence of a natural *perturbative* parameter in their definition, these models are a natural testing ground to develop our approach.

The second part focuses on the Ising model. Using the so-called *random current representation*, we derive various fine properties in large enough dimensions. One way to study the properties of a model near or at criticality is through the computation of the so-called *critical exponents*. We compute two previously unknown critical exponents in dimensions $d \geq 4$. Moreover, we characterise the *scaling limit* of the critical model in dimensions $d > 4$. This second result involves the development of a novel path expansion of the model that is closely related to the random current representation. Finally, we consider *long-range* Ising models. Their behaviour is notably intricate and fascinating. We provide an analysis of their mean-field regime.

The last part is concerned with an unbounded version of the Ising model called the φ^4 *model*. We develop a novel geometric representation of the model called the *random tangled current representation*. This object enjoys a number of combinatorial symmetries that can be used to analyse the φ^4 model. Our main result is a proof that the model undergoes a *continuous* (or *second-order*) phase transition in dimensions $d \geq 3$.

I hope you will find as much pleasure in reading the following pages as I did in writing them. *Bonne lecture !*

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Chapter 1

Introduction

This chapter is independent of the rest of the thesis. It provides a general overview of statistical mechanics and of some of the main questions in the field, written for a broad audience of mathematicians. The primary goal is not to achieve completeness or strict rigour, but to offer a gentle introduction to the key concepts presented in this thesis.

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1.1 What is a phase transition?

If you heat water sufficiently, it starts to boil and turns into vapour. This simple experiment of our everyday life is perhaps the most common instance of a *phase transition*. It is even a *discontinuous* phase transition (or a *first order* phase transition) in the sense that the *density* of water varies discontinuously when its state changes¹.

A *phase transition* is a transformation in a physical system composed of many particles where a change occurs between different *states* or *phases* due to variations in external conditions like temperature, pressure, or other parameters. This change is marked by an abrupt transformation in one or more physical properties of the system. In the case of water, this physical property is the density.

¹When liquid water turns into vapour, its density drops by a factor of 1500.

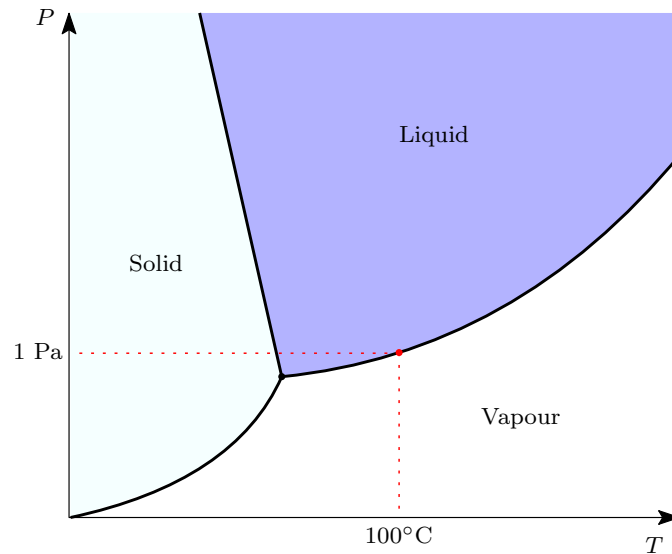


Figure 1.1: The phase diagram of water. Different states can be observed when modifying the temperature T and the pressure P . The bold black lines correspond to the *critical lines* that separate two different states of water. They merge at the so-called *triple* point of water. The red dot corresponds to the exact moment water starts boiling under normal conditions of pressure.

After observing a phase transition, it is rather natural to try to provide some explanation: *what mechanisms trigger this state transition?* Imagine an experimental physicist who has just discovered that liquid water can turn into vapour and who wants to understand why. One of the first steps to take is to repeat the experiment multiple times, each time slightly changing the relevant parameters of the problem. One may try to vary more parameters than just the temperature, and for instance also modify the pressure. The result of this investigation can be summarised in the *phase diagram* of water, see Figure 1.1. This experimental picture fully describes which state of water one expects to observe given a choice of temperature T and pressure P .

This first step, although completely satisfactory for everyday purposes, is not sufficiently compelling for our physicist. In particular, it does not explain what happens at a microscopic level to justify the change of state of water.

The next stage is to propose a mathematical description of the experiment. However, we immediately encounter a technical hurdle: a glass of water typically consists of 10^{25} molecules² of water H_2O which interact with each other in a (very) complicated way. Understanding how the *microscopic* structure of water governs the *macroscopic* behaviour of the system is therefore very challenging both in physics and mathematics.

The idea is to introduce a *model*, that is, a *drastic* simplification of the real picture designed to make the computations more tractable and (hopefully) to provide a satisfying explanation for the observed phenomenon. Efficiently modeling systems containing a large number of

²There are about 10 moles of water in 200mL. This gives $10 \times N_{\text{Avogadro}} = 10 \times 6.10^{23} \approx 10^{25}$ molecules of water.

particles (molecules in this case) is the main objective of *statistical mechanics*.

There have been many attempts to model water and describe why it starts boiling at 100°C, but none of them has provided an acceptable answer so far. The mathematical understanding of this fundamental characteristic of water remains unresolved. Are there any other phase transitions that can be observed in nature? The answer is yes, and the following example will serve as the central illustration for this thesis. It is related to *magnets* but it is not part of our everyday experience. However, it plays a key historical role in the study of phase transitions.

In 1895, the French physicist Pierre Curie³ [70] noticed⁴ that magnets lose their defining property once heated to a specific *critical temperature* T_c (now called the *Curie temperature*). Additionally, he observed that this critical temperature depends on the material, see Table 1.1 below. The exact value of T_c is not so relevant for us. What matters the most is the common underlying phenomenon that magnets cannot retain their “strength” or *spontaneous magnetisation* above a certain temperature. We say that the magnet moves from a *ferromagnetic* phase to a *paramagnetic* phase when its temperature exceeds T_c . See Figure 1.4 for an illustration of the phase diagram in this context.

<i>Material</i>	<i>Curie temperature T_c</i>
Iron (Fe)	770°C
Nickel (Ni)	354°C
Neodymium ($\text{Nd}_2\text{Fe}_{14}\text{B}$)	310 – 400°C
Cobalt (Co)	1130°C
Gadolinium (Gd)	19°C
Dysprosium (Dy)	−185.2°C

Table 1.1: The Curie temperature T_c of various materials. The most frequent magnets are made of Iron and Nickel. Their high Curie temperature explains why we typically never observe their paramagnetic phase in usual conditions.

A few years later, Pierre Weiss⁵ [246] proposed a physical explanation of the phenomenon observed by Curie. He introduced an assumption called the *mean-field approximation* (see Section 1.3 for a more detailed discussion) which gave rise to the introduction of the *Curie–Weiss* model. In this model, a magnet consists of tiny constituents called *dipoles* that are themselves small magnets. The key feature of the model is that there is “no geometry⁶”: dipoles can move freely in the magnet. Although mathematically interesting, this model failed to provide a sufficiently accurate description of the phenomenon.

³Pierre Curie (1859-1906) received the Nobel Prize in Physics in 1903, sharing the honor with his wife, Marie Curie (1867-1934), and Henri Becquerel (1852-1908), for their pioneering work in the discovery of radioactivity.

⁴In fact, this phenomenon was already observed much earlier by William Gilbert [108]. To quote him “Iron filings, after being heated for a long time, are attracted by a loadstone, yet not so strongly or from so great a distance as when not heated. A loadstone loses some of its virtue by too great a heat; for its humour is set free, whence its peculiar nature is marred.” We also mention another related work published in 1832 by the French physicist Claude Pouillet (see [77] and references therein).

⁵Pierre Weiss (1865-1940) was a French physicist who specialised in the study of magnetism.

⁶Mathematically, this is seen through the fact that the Curie–Weiss model is defined on the complete graph.

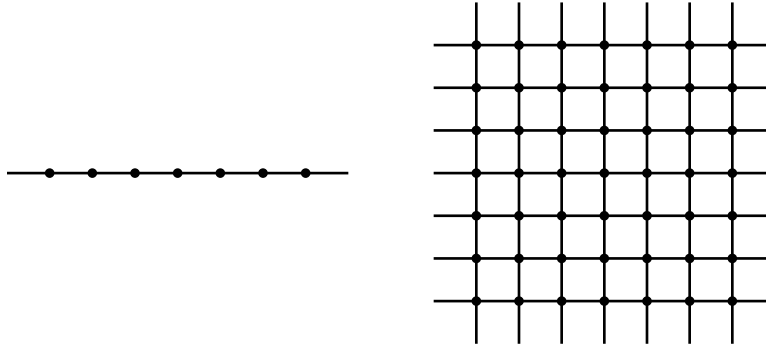


Figure 1.2: An illustration of the lattice $\mathbb{Z}^d$ when $d = 1$ (on the left) and $d = 2$ (on the right). In Lenz’s model, the dipoles are constrained to live on the vertices of these lattices.

The German physicist Wilhelm Lenz⁷ [176] pushed Weiss’ ideas further and postulated a *crystal*-like structure for the dipoles, meaning that they should be geometrically constrained by the material and cannot move freely. A simple crystal-like structure is given by the hypercubic lattice $\mathbb{Z}^d$ (where $d \geq 1$ is the dimension), see Figure 1.2. The second assumption that Lenz made is that the dipoles can be represented by their magnetic *spin* which should orient into two directions only: $+1$ or -1 (upwards or downwards). Despite these simplifications, Lenz was unable to justify why a magnet loses its strength when sufficiently heated. A major caveat in his paper was that he did not specify how the spins interact with each other.

Lenz proposed to one of his PhD students, Ernst Ising⁸, to pursue the study he initiated. In his famous work [155], Ising postulated a simple form for the interaction between spins. He also made two major assumptions: (i) spins can only interact directly with their closest neighbours, (ii) they are constrained to live on the linear chain $\mathbb{Z}$. With these assumptions, he proved that the system never magnetises, regardless of the temperature. In particular, there is no ferromagnetic phase. This led him to postulate that the same should occur in any other dimension, like for instance the physical dimension $d = 3$. The ferromagnetism did not seem to be explicable.

This negative result sidelined the Lenz–Ising model for a few years. Even more unfavourably, convincing alternative suggestions were made to model ferromagnetism. Werner Heisenberg⁹ [141] introduced a more complex version of the model, called the *classical Heisenberg model*. In this model, Heisenberg kept Lenz’s assumption of a crystal-like structure but allowed magnetic spins to rotate freely in all directions, meaning that they live on the two-dimensional sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. He further introduced a much more compelling model, the *quantum Heisenberg model*, in which the *classical* spins are replaced by the *quantum* spins of the electrons. In short, Heisenberg tried to explain magnetism¹⁰ with the interaction of the

⁷Wilhelm Lenz (1888-1957) is mostly known for his invention of the Ising model and for his application of the Laplace–Runge–Lenz vector to the (old) quantum mechanical treatment of hydrogen-like atoms.

⁸Ernst Ising (1900-1998) was a German physicist.

⁹Werner Heisenberg (1901-1976) is one of the founders of quantum mechanics. He was awarded the Nobel Prize in Physics in 1933.

¹⁰On the mathematical level, it is still an open question to show that Heisenberg’s quantum model of ferromagnetism undergoes a phase transition. Very recent progress was made in the study of a simplified model called the *interchange process*, see [242, 90].

spin *angular momentum* of the electrons, while in the Lenz–Ising model, only the magnetic moment plays a role.

However, in 1936, Rudolf Peierls¹¹ [208], who did not believe in Ising’s conjecture, revived what he came to call the *Ising model* by providing a mathematical justification for the existence of a phase transition in this model in dimensions $d \geq 2$. His argument (now commonly referred to as *Peierls’ argument*) became one of the most robust tools used in statistical mechanics.

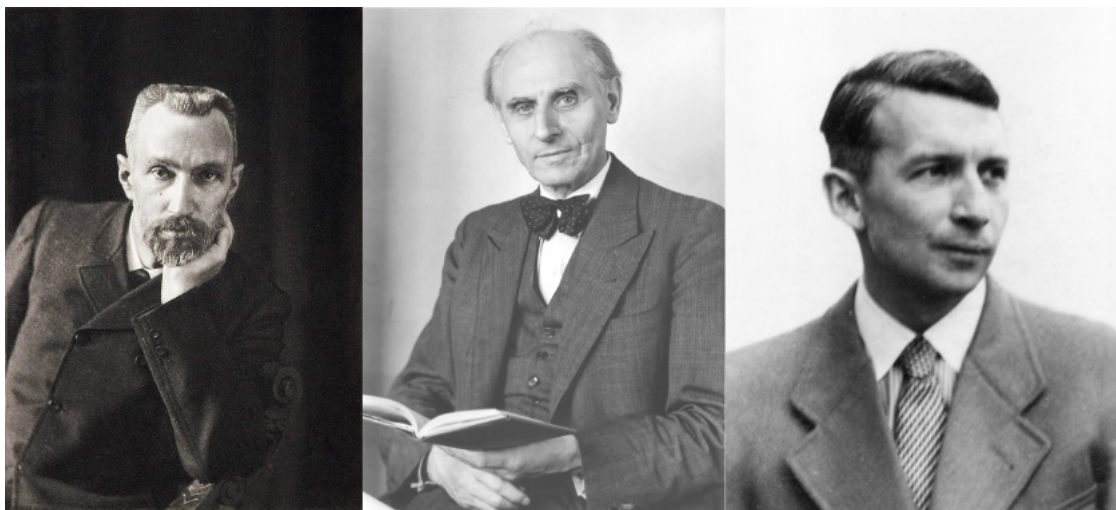


Figure 1.3: From left to right: Pierre Curie (1859-1906), Wilhelm Lenz (1888-1957), and Ernst Ising (1900-1998)¹².

This short story is only the beginning of the study of the Ising model. Ever since Peierls’ seminal paper, thousands of articles have been published on the subject. Sadly, soon after his thesis defense, demotivated by the poor reception of his result, and pushed away from Germany because of his Jewish descent, Ising stopped research and became a professor in physics at Bradley University (United States). He only learnt very late that his name was associated to one of the most iconic models in statistical mechanics.

As it turns out, the Ising model is not very good to describe magnetism¹³ and its *classical* definition might be the main reason. However, as we will see below, it remains extremely significant in physics because it serves as a simple example of a model with a rich critical behaviour. It is therefore a natural test-ground to develop new (technical) tools. On the mathematical side, it is a challenge to provide a fully rigorous description of the model and many natural questions remain unanswered. The history of this model is captivating and we refer to the beautiful review [77] for a more complete perspective on the subject.

¹¹Rudolph Peierls (1907-1995) was a German-born British physicist. He is mostly known for his pivotal role in *Tube Alloys*, Britain’s nuclear program, as well as in the subsequent *Manhattan Project*.

¹²Credits for Lenz and Ising respectively: Fachbereich Physik der Universität Hamburg, AIP Emilio Segre Visual Archives.

¹³It seems to be better designed to model the so-called *cooperative phenomena*.

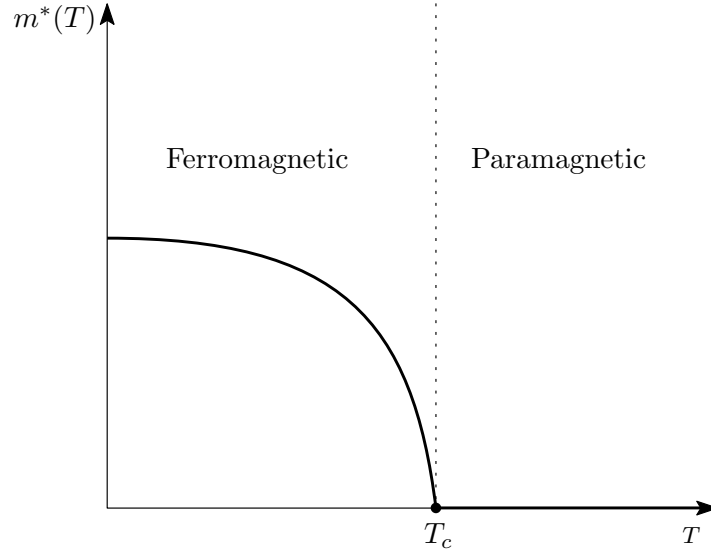


Figure 1.4: An illustration of the phase diagram of a magnet. The spontaneous magnetisation is denoted by $m^*(T)$. It represents the *strength* of a magnet. As the temperature varies and increases, $m^*(T)$ decreases and eventually reaches zero (at T_c). Compared to the phase transition of water, this phase transition is *continuous* (or of *second order*) as the spontaneous magnetisation varies continuously with T .

1.2 The main protagonists

In this section, we provide a (light) mathematical description of the main protagonists of this thesis. We begin with a more precise and rigorous description of the Ising model.

1.2.1 The Ising model

Definition. In order to cook up a definition of the model, we need some physical intuition. We follow Lenz's simplifications and assume that the spins live on a lattice and that they can take two values $+1$ or -1 . Each spin configuration carries an energy which reflects the fact that neighbouring spins prefer to be aligned (similarly to magnets which attract themselves).

Mathematically, we assume that we work on a finite subset Λ of $\mathbb{Z}^d$ with $d \geq 1$. An element $\sigma \in \{-1, 1\}^\Lambda$ is called a *spin configuration*. Following Ising's suggestion, we associate to each configuration σ an energy (or *Hamiltonian*)

$$H_\Lambda(\sigma) := - \sum_{\substack{\{x,y\} \subset \Lambda \\ x \sim y}} \sigma_x \sigma_y, \quad (1.2.1)$$

where $x \sim y$ means that x and y are neighbours in $\mathbb{Z}^d$, i.e. that $\|x - y\| = 1$ where $\|\cdot\|$ is the ℓ^2 norm on $\mathbb{R}^d$. A slightly more general (and physically more relevant) definition consists in allowing long-range *interactions*. Given $J = (J_{x,y})_{x,y \in \mathbb{Z}^d}$ with $J_{x,y} \geq 0$, we may define a generalised energy of a configuration σ

$$H_{\Lambda,J}(\sigma) := - \sum_{\{x,y\} \subset \Lambda} J_{x,y} \sigma_x \sigma_y. \quad (1.2.2)$$

Notice that we recover (1.2.1) from (1.2.2) by setting $J_{x,y} = \mathbb{1}_{x \sim y}$. We refer to this particular choice as the *nearest-neighbour ferromagnetic*¹⁴ (n.n.f) model. Once we know the energy of a configuration, we follow the fundamental principles of statistical mechanics [40] and choose to construct a model of *random* configurations. That is, we construct a probability measure on $\{-1, 1\}^\Lambda$. Rather than trying to understand exactly how each spin behaves, we focus on the *typical* behaviour of a configuration.

The most natural choice is to weight a configuration σ according to Boltzman's weight: given a temperature $T > 0$, the probability of a configuration σ is

$$\mu_{\Lambda, J, T}[\sigma] := \frac{\exp(-\frac{1}{k_B T} H_{\Lambda, J}(\sigma))}{Z_{\Lambda, J, T}}, \quad (1.2.3)$$

where

$$Z_{\Lambda, J, T} := \sum_{\sigma \in \{-1, 1\}^\Lambda} \exp(-\frac{1}{k_B T} H_{\Lambda, J}(\sigma))$$

is the so-called *partition function* of the model that guarantees that $\mu_{\Lambda, J, T}$ is a probability measure, and $k_B \approx 1,38 \cdot 10^{-23} \text{J} \cdot \text{K}^{-1}$ is the *Boltzmann constant*. For compactness, below we write

$$\beta := \frac{1}{k_B T}$$

and parametrise all the relevant quantities in terms of β (the *inverse temperature*) rather than T . The model introduced above is the “ferromagnetic Ising model with free boundary condition on Λ ”.

The configurations that are the most likely, called *ground states*, are those carrying the smallest energy. It is not hard to check that these configurations correspond to all spins being aligned in the +1 or −1 direction. However, there are many $(2^{|\Lambda|})$ possible configurations. This creates a competition between *entropy* and *energy* which makes the model harder to analyse.

When the temperature goes to 0 (or β goes to infinity), the model favours more and more *ordered* configurations, i.e. configurations with spins aligned in the same direction. On the opposite, when T goes to infinity (or β goes to 0), the Boltzman weight becomes less sensitive to the energy of a configuration, causing the system to become *disordered*, with spins aligning (independently of each other) in random directions and no global ordering emerging.

How to define the phase transition of the Ising model? For the sake of simplicity we now restrict our attention to the nearest-neighbour ferromagnetic case. A convenient way to formulate the existence of a phase transition is through the introduction of *boundary conditions*. Given a configuration $\tau \in \{-1, 0, 1\}^{\mathbb{Z}^d}$, we may modify (1.2.1) to model a situation in which all the spins outside Λ are given by τ . More precisely, for a configuration σ in Λ , introduce,

$$H_\Lambda^\tau(\sigma) := H_\Lambda(\sigma) - \sum_{\substack{x \in \Lambda \\ y \notin \Lambda \\ x \sim y}} \sigma_x \tau_y.$$

As above, this allows one to construct a measure $\mu_{\Lambda, \beta}^\tau$ on $\{-1, 1\}^\Lambda$. When $\tau \equiv 0$ we recover the free measure $\mu_{\Lambda, \beta}$. When $\tau \equiv 1$ (resp. $\tau \equiv -1$) we write $\mu_{\Lambda, \beta}^\tau = \mu_{\Lambda, \beta}^+$ (resp. $\mu_{\Lambda, \beta}^-$). The

¹⁴Here the term ferromagnetic refers to the positivity of $J_{x,y}$.

measure $\mu_{\Lambda,\beta}^+$ (resp. $\mu_{\Lambda,\beta}^-$) corresponds to the “ferromagnetic Ising model on Λ with *plus* (resp. *minus*) boundary condition”. Classical *correlation inequalities* [116, 117] permit to define the infinite volume¹⁵ weak limit

$$\mu_{\beta}^{\#} := \lim_{\Lambda \nearrow \mathbb{Z}^d} \mu_{\Lambda,\beta}^{\#}$$

for $\# \in \{-, 0, +\}$. The *spontaneous magnetisation* at inverse temperature $\beta > 0$ is defined by the averaged value of the spin at the origin under the measure μ_{β}^+ ,

$$m^*(\beta) := \mu_{\beta}^+[\sigma_0].$$

When $d \geq 2$, the model undergoes a phase transition in the following sense [208]: there exists a critical parameter $\beta_c \in (0, \infty)$ satisfying

$$m^*(\beta) \begin{cases} > 0 & \text{if } \beta > \beta_c, \\ = 0 & \text{if } \beta < \beta_c. \end{cases}$$

The spontaneous magnetisation vanishes when the parameter β is sufficiently small, i.e. the temperature is sufficiently high. This corresponds to a phase where the spin at the origin cannot retain the information of the boundary condition in the infinite volume limit. In other words, the disorder created by the thermal agitation makes the spin of the origin unable to interact with the boundary.

It is a very natural question to ask what happens at the critical point. This question was answered using very different techniques when $d = 2$ [255, 247] and $d \geq 3$ [12, 10]. In both cases, it was proved that

$$m^*(\beta_c) = 0.$$

Combined with the right-continuity¹⁶ of the magnetisation, this implies that the phase transition is *continuous* (as illustrated in Figure 1.4).

There is an alternative way to construct the measure μ_{β}^+ that is physically more relevant. Following the experiment of Curie [70], we can assume that our system is plunged in an *external* magnetic field h . Mathematically speaking, this corresponds to modifying (1.2.1) as follows: for $h \in \mathbb{R}$, set

$$\begin{aligned} H_{\Lambda,h}(\sigma) &:= H_{\Lambda}(\sigma) - h \sum_{x \in \Lambda} \sigma_x, \\ Z_{\Lambda,\beta,h} &:= \sum_{\sigma \in \{-1,1\}^{\Lambda}} \exp(-\beta H_{\Lambda,h}(\sigma)). \end{aligned}$$

Again, we may define a probability measure $\mu_{\Lambda,\beta,h}$ with this Hamiltonian and construct the infinite volume limit $\mu_{\beta,h}$. Under this measure, spins are constrained to align in the direction of the sign of the external magnetic field h to minimise the energy. Moreover, for all $\beta > 0$,

$$m(\beta, h) := \mu_{\beta,h}[\sigma_0] \begin{cases} > 0 & \text{if } h > 0, \\ < 0 & \text{if } h < 0. \end{cases}$$

¹⁵More precisely, $\mu_{\beta}^{\#}$ is a measure on $\{-1, 1\}^{\mathbb{Z}^d}$ equipped with the cylinder σ -algebra of events.

¹⁶This classical fact follows from Griffiths' inequalities which yield that the sequence of functions $(\beta \mapsto \mu_{\Lambda_n,\beta}^+[\sigma_0])_{n \geq 1}$, where $\Lambda_n := [-n, n]^d \cap \mathbb{Z}^d$, is a decreasing sequence of increasing continuous functions.

But what happens when we take $h > 0$ and let $h \searrow 0$? Does the magnetisation persist? The answer is given by the following (weak) limit:

$$\lim_{h \searrow 0} \mu_{\beta, h} = \mu_{\beta}^+.$$

As a consequence, the system retains a spontaneous magnetisation when $h \searrow 0$ if $\beta > \beta_c$, and it does not if $\beta < \beta_c$. This shows that the physical and the mathematical definition of the spontaneous magnetisation coincide.

We have reached a robust definition of the phase transition of the Ising model. This paves the way for much more difficult questions: *how to study the (near-) critical behaviour of the model? What are the characteristics of the phase transition of the Ising model? How does the dimension of the underlying lattice affect the critical picture? How is this phase transition different from the one of any other model of statistical mechanics?* We propose one approach to study these questions, which involves the central notion of *critical exponents*.

From now on, we write averages with respect to the measure $\mu_{\Lambda, \beta}^{\#}[\cdot]$ (where $\#$ is a boundary condition) with brackets $\langle \cdot \rangle_{\Lambda, \beta}^{\#}$.

Critical exponents. A fundamental concept in physics is the idea that thermodynamic quantities of physical systems near or at criticality tend to simplify when expressed in terms of the parameters of the system. A major progress was made by Benjamin Widom [248] who proposed that these quantities follow power laws in this regime. In the case of the Ising model, we have seen that the two natural parameters are the inverse temperature β and the external magnetic field h . Widom's hypothesis is related to the existence of the so-called *critical exponents* of the model. Postponing for a moment all the definitions, we list a few examples:

$$\begin{aligned} c_H(\beta) &= (\beta_c - \beta)^{-\alpha+o(1)} && \text{as } \beta \nearrow \beta_c, \\ m^*(\beta) &= (\beta - \beta_c)_+^{\beta+o(1)} && \text{as } \beta \searrow \beta_c, \\ \chi(\beta) &= (\beta_c - \beta)^{-\gamma+o(1)} && \text{as } \beta \nearrow \beta_c, \\ m(\beta_c, h) &= h^{1/\delta+o(1)} && \text{as } h \searrow 0, \\ \langle \sigma_0 \sigma_x \rangle_{\beta_c} &= \|x\|^{-(d-2)-\eta+o(1)} && \text{as } \|x\| \rightarrow \infty, \\ \xi(\beta) &= (\beta_c - \beta)^{-\nu+o(1)} && \text{as } \beta \nearrow \beta_c, \end{aligned}$$

where $o(1)$ goes to 0 as the parameter goes to its limit point. All these exponents are expected to exist when $d \geq 2$, see Table 1.2.

The *free energy*¹⁷ $f(\beta, h)$ is defined as follows,

$$f(\beta, h) := -\frac{1}{\beta} \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \log Z_{\Lambda_n, \beta, h},$$

where $\Lambda_n := [-n, n]^d \cap \mathbb{Z}^d$ and where $|\Lambda_n|$ denotes the cardinality of Λ_n . The *specific heat* $c_H(\beta)$ is then defined as

$$c_H(\beta) := -\beta^3 \frac{\partial^2 f}{\partial \beta^2}(\beta, 0).$$

¹⁷This is Gibbs' definition of the free energy. Alternatively, Helmholtz defines the free energy as the Legendre transform of Gibbs' free energy.

<i>Dimension</i>	α	β	γ	δ	η	ν
$d = 2$	0	$\frac{1}{8}$	$\frac{7}{4}$	15	$\frac{1}{4}$	1
$d = 3$	≈ 0.1101	≈ 0.3264	≈ 1.2371	≈ 4.7898	≈ 0.03629	≈ 0.6299
$d \geq 4$	0	$\frac{1}{2}$	1	3	0	$\frac{1}{2}$

Table 1.2: Conjectured values of the main critical exponents of the Ising model on $\mathbb{Z}^d$ with $d \geq 2$. For $d = 2$, all the exponents have been computed thanks to exact integrability [200, 161, 255], and the proof of conformal invariance [233, 66] which relates the critical scaling limit to the *Schramm–Loewner evolution* SLE(3) and the *conformal loop ensemble* CLE(3). When $d = 3$, the existence of the exponents is still open. The above conjectured values have been reported using conformal bootstrap [88, 89, 166]. When $d \geq 4$, the critical exponents take a *classical* or *mean-field* value. We discuss this in Section 1.3. The exponent α was obtained in [234], γ was computed in [4, 14], β and δ were computed in [12], ν and η were derived in [83].

The spontaneous magnetisation $m^*(\beta)$ and the magnetisation $m(\beta, h)$ were defined above. Additionally, we have the relation

$$m(\beta, h) = -\frac{\partial f}{\partial h}(\beta, h).$$

The *susceptibility* $\chi(\beta)$ is defined for $\beta > 0$ by

$$\chi(\beta) := \sum_{x \in \mathbb{Z}^d} \langle \sigma_0 \sigma_x \rangle_\beta.$$

Again, we note that the following relation holds

$$\chi(\beta) = \frac{\partial m}{\partial h}(\beta, 0).$$

The critical exponent η is related to the decay of the model's *two-point function* $\langle \sigma_0 \sigma_x \rangle_{\beta_c}$ at criticality as $\|x\|$ tends to infinity. It is sometimes called the *anomalous dimension*. The *correlation length* $\xi(\beta)$ is defined¹⁸ for $\beta > 0$ by

$$\xi(\beta) := \lim_{n \rightarrow \infty} \left(-\frac{1}{n} \log \langle \sigma_0 \sigma_{n\mathbf{e}_1} \rangle_\beta \right)^{-1}, \quad (1.2.4)$$

where $\mathbf{e}_1$ is the unit vector of $\mathbb{R}^d$ whose first coordinate is equal to 1. That is,

$$\langle \sigma_0 \sigma_{n\mathbf{e}_1} \rangle_\beta = \exp \left(-\frac{n}{\xi(\beta)} (1 + o(1)) \right), \quad (1.2.5)$$

where $o(1)$ tends to 0 as n tends to infinity. The existence of the limit in (1.2.4) follows classically by sub-additivity. Although not completely transparent from the mathematical definition, the correlation length corresponds to the typical length above which the system

¹⁸Here we define the correlation length in the $\mathbf{e}_1$ direction. Given any vector $s \in \mathbb{S}^{d-1}$, we may also define the correlation length in the s direction. The limit then obtained generally differs from $\xi(\beta)$.

feels that it is subcritical. Previously, we defined the phase transition in the infinite volume, which is not “physical”, nor is the notion of an exact critical point. The existence of this critical length-scale allows one to observe critical features in a finite volume and in the near-critical regime. This observation is used in Figure 1.5 to sample the “critical” Ising model.

It is not clear that the point where the susceptibility and the correlation length first diverge coincides¹⁹ with β_c . This is related to the *sharpness* of the phase transition of the Ising model. We discuss this question in Chapter 2.

Understanding the phase transition of a model often boils down to deriving these critical exponents. After defining them, a natural question arises: *how many degrees of freedom exist in the Ising model, and more generally in any model of statistical mechanics?* More explicitly, are there any *relations* between these exponents that would allow to derive some of them out of the knowledge of others? The physicists Essam, Fisher, and Widom [92, 96, 248] proposed connections between these exponents referred as *scaling relations*. To give an example, it is predicted that in dimensions $d = 2, 3, 4$ the critical exponents should satisfy (in a wide generality that goes beyond the Ising model, see [79]),

$$2 - \eta = \frac{\gamma}{\nu} = d \frac{\delta - 1}{\delta + 1}, \quad \nu d = 2 - \alpha = 2\beta + \gamma = \beta(\delta + 1).$$

One can verify that these relations are (approximately when $d = 3$) satisfied with the values of Table 1.2. Above the dimension $d = 4$, these relations do not depend on d anymore. The dependency in the dimension of these relations is a phenomenon called *hyperscaling*. It reflects how dimensionality affects the critical behaviour of the model. It is expected that hyperscaling disappears in dimensions $d > 4$, meaning that the dimension does not affect the critical exponents anymore. This will be discussed in Section 1.3.

The Ising model played a key role in the discovery of these scaling relations. This is due to the fact that it qualifies as the simplest example of a theoretical model which does not exhibit mean-field behaviour (in dimensions 2 and 3), i.e. a model for which the critical exponents do not take their mean-field (or classical) values. At the time of this discovery, justifying mathematically the existence of these critical exponents was a widely open question. Clear heuristics came later with the development of the *renormalization group method*, see Section 1.3.

To conclude this introduction of the Ising model, we (very) briefly survey some known results.

The case $d = 2$. The two-dimensional Ising model enjoys a number of combinatorial symmetries which help its study. This is best illustrated by the breakthrough computation of Lars Onsager²⁰ [200] who relied on the transfer matrix formalism to obtain an exact expression of the free energy of the model,

$$-\beta f(\beta) = \log 2 + \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} \log \left[\cosh(2\beta)^2 - \sinh(2\beta)(\cos \theta_1 + \cos \theta_2) \right] d\theta_1 d\theta_2. \quad (1.2.6)$$

This impressive computation implies that the model is *exactly solvable* or *integrable*. Onsager also proposed an exact formula for the spontaneous magnetisation which was later rigourously

¹⁹However, by classical correlation inequalities (see [76]), we know that $\chi(\beta)$ and $\xi(\beta)$ are infinite above β_c .

²⁰Lars Onsager (1903-1976) was a Norwegian theoretical chemist. He was awarded the Nobel Prize in Chemistry in 1968.

justified by Chen-Ning Yang²¹ [255]: for $\beta > \beta_c$,

$$m^*(\beta) = \left(1 - \frac{1}{\sinh(2\beta)^4}\right)^{\frac{1}{8}}.$$

Moreover, let us mention that the value of β_c is known in that setup:

$$\beta_c = \frac{\log(1 + \sqrt{2})}{2}.$$

Using *discrete holomorphicity*, Smirnov [233] and Chelkak and Smirnov [66] proved *conformal invariance* of the model. This breakthrough work paved the way for the analysis of the model from the point of view of the *conformal field theory* as introduced by Belavin, Polyakov, and Zamolodchikov [31]. In particular, the critical scaling limit of the model relates to the so-called *Schramm–Loewner Evolution* SLE(3) (see [64]) and the *Conformal Loop Ensemble* CLE(3) (see [33, 34, 106]).

The case $d = 3$. Quite surprisingly, the dimension for which the model is the least understood corresponds to the *physical* dimension $d = 3$. None of the critical exponents are rigorously known to exist. However, the recent *conformal bootstrap* approach provides many predictions under several strong hypotheses (such as conformal invariance of the scaling limit). See [88, 89, 166]. In particular, absence of mean-field behaviour is still an open problem (see however [215]), and the only information available for the critical exponents is a set of (mean-field) bounds (conditioned on the existence of the critical exponents):

$$\beta \leq \frac{1}{2}, \quad \gamma \geq 1, \quad 0 \leq \eta \leq \frac{1}{2}.$$

The bound $\eta \geq 0$ is a classical consequence of *reflection positivity* (see [104, 103, 36]) which, by means of the *infrared bound* implies, for all dimensions $d \geq 3$, the existence of $C = C(d) > 0$ such that for all $x \in \mathbb{Z}^d \setminus \{0\}$,

$$\langle \sigma_0 \sigma_x \rangle_{\beta_c} \leq \frac{C}{\|x\|^{d-2}}. \quad (1.2.7)$$

The case $d \geq 4$. In sufficiently high dimensions, the behaviour of the Ising model is expected to simplify in the sense that the approximation suggested by Weiss becomes closer to the truth. Heuristically, when d is large enough, the geometry of the underlying lattice stops playing a role and the model *feels* like it is defined on the complete graph. Physics predictions suggest that this simplification occurs above the *upper-critical dimension* $d_c = 4$. We refer to Section 1.3 for more information.

1.2.2 The φ^4 model

Definition. The φ^4 model is defined similarly as the Ising model. It corresponds to its *unbounded* version, in the sense that spins are no longer constrained to take values in $\{-1, 1\}$

²¹Chen-Ning Yang (1922-) is one of the most influential theoretical physicists of the second half of the twentieth century. His name is ubiquitous in mathematical physics: from Yang–Baxter’s equation, Yang–Mills’ theory, Lee–Yang’s theory, etc. He received the Nobel Prize in Physics in 1957.

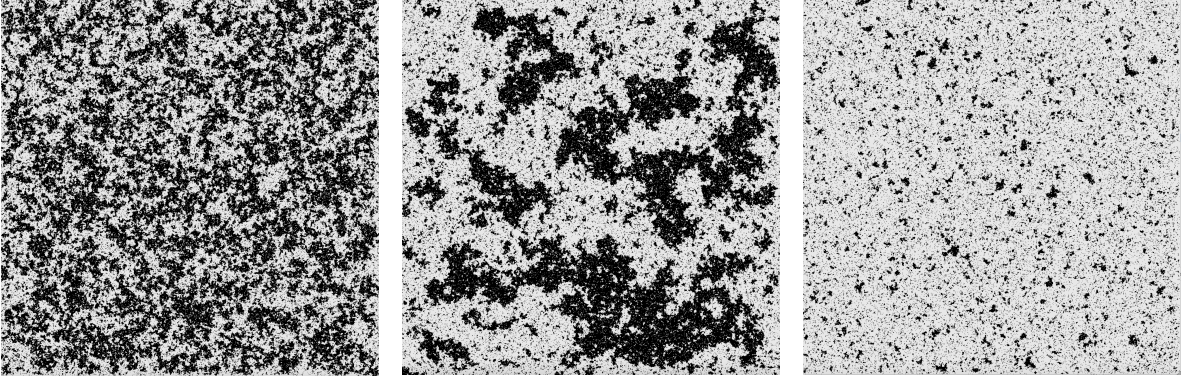


Figure 1.5: Samples of the nearest-neighbour Ising model in a box of size 1024×1024 . White (resp. black) dots correspond to a $+1$ (resp. -1) spin. From left to right: $\beta < \beta_c$, $\beta \approx \beta_c$, and $\beta > \beta_c$. In the subcritical regime $\beta < \beta_c$, spins are disordered: $+1$ and -1 spins are balanced. In the supercritical regime $\beta > \beta_c$, spins are ordered and tend to align in the $+1$ direction (with small fluctuations). In the (near) critical regime $\beta \approx \beta_c$, *fractal* structures (related to SLE(3) and CLE(3) curves) emerge and the picture is much more complex (and richer). The simulations are due to Raphaël Cerf.

but can vary freely in $\mathbb{R}$ according to a *quartic* potential. The physical intuition underlying the definition of the model is the same as for the Ising model. We therefore turn to the mathematical definition.

Again, we work on a finite subset Λ of $\mathbb{Z}^d$ with $d \geq 1$. For $g > 0$ and $a \in \mathbb{R}$, let $\rho_{g,a}$ be the measure on $\mathbb{R}$ defined by

$$d\rho_{g,a}(t) = \frac{1}{z_{g,a}} e^{-gt^4 - at^2} dt, \quad (1.2.8)$$

where $z_{g,a} = \int_{\mathbb{R}} e^{-gt^4 - at^2} dt$ is a normalisation constant. For an external magnetic field $h \in \mathbb{R}$, we define the Hamiltonian

$$H_{\Lambda,h}(\varphi) := - \sum_{\substack{\{x,y\} \subset \Lambda \\ x \sim y}} \varphi_x \varphi_y - h \sum_{x \in \Lambda} \varphi_x, \quad \forall \varphi \in \mathbb{R}^\Lambda.$$

The n.n.f φ^4 model on Λ with free boundary condition, at inverse temperature $\beta > 0$, and external magnetic field h , is the measure $\langle \cdot \rangle_{\Lambda,\beta,h}$ defined via the expectation values

$$\langle F(\varphi) \rangle_{\Lambda,\beta,h} := \frac{1}{Z_{\Lambda,\beta,h}^{\varphi^4}} \int_{\varphi \in \mathbb{R}^\Lambda} F(\varphi) e^{-\beta H_{\Lambda,h}(\varphi)} d\rho_{\Lambda,g,a}(\varphi)$$

where $d\rho_{\Lambda,g,a}(\varphi) = \prod_{x \in \Lambda} d\rho_{g,a}(\varphi_x)$, $F : \mathbb{R}^\Lambda \rightarrow \mathbb{R}$ is a bounded measurable function and $Z_{\Lambda,\beta,h}^{\varphi^4}$ is the partition function, i.e. the normalisation constant chosen such that $\langle 1 \rangle_{\Lambda,\beta,h} = 1$. For simplicity, we omit²² the definition of general boundary conditions which is more technical in this setup due to the unboundedness of the spins. See Part III for more information.

²²As we saw with the Ising model, this is not necessary to define the phase transition of the model.

One may again define the infinite volume limit of the model. In particular, the spontaneous magnetisation is given by

$$m^*(\beta) := \lim_{h \searrow 0} \lim_{\Lambda \nearrow \mathbb{Z}^d} \langle \varphi_0 \rangle_{\Lambda, \beta, h}.$$

When $d \geq 2$, the model undergoes a phase transition for the vanishing of $m^*(\beta)$: there exists $\beta_c \in (0, \infty)$ such that $m^*(\beta) > 0$ if $\beta > \beta_c$ and $m^*(\beta) = 0$ if $\beta < \beta_c$. Moreover, one may define the critical exponents $(\alpha, \beta, \delta, \gamma, \eta, \nu)$ similarly as above.

While the φ^4 model serves as a model for ferromagnetism, its primary purpose lies in its connections to *Euclidean field theory* and *quantum field theory*.

Euclidean field theory and constructive quantum field theory. Constructive Euclidean field theory aims at building random distributions on $\mathbb{R}^d$, with a particular focus on interacting, or non-*trivial*, field theories. This contrasts with the study of Gaussian fields, which have a trivial correlation function structure in the sense that it is entirely determined by their two-point function via Wick's law. We now briefly describe this program.

A natural attempt to build field theories is to try to define a measure on the set of functions $\mathbb{R}^d \rightarrow \mathbb{R}$ whose averages are given, for a functional $F : \mathbb{R}^{\mathbb{R}^d} \rightarrow \mathbb{R}$, by

$$\langle F(\Phi) \rangle = \frac{1}{Z} \int F(\Phi) \exp(-H(\Phi)) \prod_{x \in \mathbb{R}^d} d\Phi_x, \quad (1.2.9)$$

with

$$H(\Phi) := (\Phi, A\Phi) + \int_{\mathbb{R}^d} P(\Phi(x)) dx,$$

where $(\Phi, A\Phi)$ is a positive definite (and reflection positive) quadratic form, and where P is an even polynomial with a strictly positive leading coefficient, whose term of order $\Phi(x)^{2k}$ can be interpreted as representing k -particle interactions. To fix the ideas, one may think of the quadratic form given by: for $K_1, K_2 > 0$,

$$(\Phi, A\Phi) := \int_{\mathbb{R}^d} [K_1 |\nabla \Phi(x)|^2 + K_2 |\Phi(x)|^2] dx. \quad (1.2.10)$$

Among the candidates of functionals F for which (1.2.9) is intended, we find the so-called *smeared averages* defined as follows. If $f \in \mathcal{C}_0(\mathbb{R}^d)$ (the set of continuous and compactly supported functions on $\mathbb{R}^d$), define

$$\mathcal{T}_f(\Phi) := \int_{\mathbb{R}^d} f(x) \Phi(x) dx.$$

The expected values under $\langle \cdot \rangle$ of products of smeared observables take the form

$$\left\langle \prod_{i=1}^n \mathcal{T}_{f_i}(\Phi) \right\rangle = \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \left(\prod_{i=1}^n f_i(x_i) \right) \mathcal{S}_n(x_1, \dots, x_n) dx_1 \cdots dx_n,$$

where the family $(\mathcal{S}_n)_{n \geq 1}$ is referred to as the *Schwinger functions* of the Euclidean field theory provided by $\langle \cdot \rangle$. These functions characterise the field theory and can be viewed (in a distributional sense) as

$$\mathcal{S}_n(x_1, \dots, x_n) = \left\langle \prod_{i=1}^n \Phi(x_i) \right\rangle.$$

The simplest class of Euclidean fields is given by the Gaussian fields. They correspond to choosing only quadratic terms in the definition of H above²³. The choice (1.2.10) leads to the definition of the *Gaussian free field*. As we mentioned above, Gaussian fields are characterised by their simplified correlation structure: the 2-point Schwinger function determines entirely the field theory through Wick’s law: for all $n \geq 1$,

$$\mathcal{S}_{2n}(x_1, \dots, x_{2n}) = \sum_{\pi} \prod_{i=1}^n \mathcal{S}_2(x_{\pi(2i-1)}, x_{\pi(2i)}),$$

where π ranges over pairings of $\{1, \dots, 2n\}$. This simple structure corresponds to a *non-interacting* field from the physics perspective. Gaussian fields are thus referred to as *trivial*.

It is natural to try to build an interacting Euclidean field theory by adding a quartic term $g\Phi^4$ (with $g > 0$) in the definition of H . This is the continuous analogue of the φ^4 model.

Due to the lack of a natural Lebesgue measure on infinite dimensional spaces, (1.2.9) is ill-defined. However, it is still possible to make sense of it using a pair of *ultraviolet* (short distance) and *infrared* (long distance) cutoffs. By doing so, we recover the *discrete* φ^4 model described above. Removing these cutoffs (especially the former one) is in general very difficult because of the notorious problem of *ultraviolet divergences*, see [113]. It is however possible to overcome this technicality by introducing cut-off dependent counter-terms.

Highlights of this approach include the rigorous construction of the φ^4 measure, with infrared cutoff, in dimension two by Nelson [195], and in dimension three by Glimm and Jaffe [110]. These works were later extended to the infinite volume limit [188, 113].

A few years later, Aizenman [4] and Fröhlich [102] showed that any field constructed as a scaling limit of Ising or φ^4 models, from the single phase regime (i.e. $\beta \leq \beta_c$ and $h = 0$), in dimension $d > 4$, is a Gaussian field. The same result was very recently extended to the (*marginal*) four-dimensional case by the breakthrough work of Aizenman and Duminil-Copin [9], thus confirming a prediction from physics. These triviality results are rather negative from the perspective of physics. However, they play a central role in mathematics by showcasing new powerful methods to analyse non-integrable models. These methods will be discussed in Chapter 2. Let us also mention that building on the so-called *random walk representation* of the φ^4 model, Brydges, Fröhlich, and Sokal [50, 51] established rigorously the non-triviality of the weakly-coupled φ_2^4 and φ_3^4 theories constructed as subsequential limits of the (single phase) discrete φ^4 model.

Constructive Euclidean field theory is also closely related to *constructive quantum field theory* (CQFT). Properly defining what a *quantum field theory* (QFT) is would go beyond the scope of this modest introduction. In short, QFT is a fundamental framework in theoretical physics that combines classical field theory, special relativity, and quantum mechanics. It is essential for understanding the behaviour of fundamental forces and particles, providing (for instance) the mathematical and conceptual foundation for the *Standard Model* of particle physics.

The American mathematical physicist Arthur Wightman [249] proposed to define QFTs as operator valued distributions satisfying a certain set of axioms. Wightman’s *reconstruction* theorem allows one to recover QFTs from the collection of its corresponding correlation functions. The Osterwalder–Schrader theorem [201, 202] provides a way to build these correlation

²³We stress however, that even in this “simpler case”, one cannot immediately make sense of (1.2.9) since the integral in the definition of H is in general almost surely divergent.

functions through analytic continuations of the Schwinger functions of random distributions defined over a well-chosen Euclidean space. The correspondence between QFT and Euclidean field theory is obtained through a *Wick rotation*. We refer to [113] for a more accurate description of the CQFT point of view.

Relation to the Ising model and universality. There is an easy way to relate the φ^4 model to the Ising model. Indeed, the weak limit

$$\frac{\delta_{-1} + \delta_1}{2} = \lim_{g \rightarrow \infty} \frac{1}{z_{g,-2g}} e^{-g(\varphi^2-1)^2+g} d\varphi \quad (1.2.11)$$

allows one to realise the Ising model as a weak limit of φ^4 models with coupling g tending to infinity.

Conversely, the φ^4 model can be recovered from a certain near-critical scaling limit of a collection of Curie–Weiss models, i.e Ising models on the complete graph. Let us clarify this further. Fix $\Lambda \subset \mathbb{Z}^d$ finite. For $N \geq 1$, we let K_N be the complete graph on N vertices. We define an Ising model $\langle \cdot \rangle_{\Lambda, \beta}^{(N)}$ on $\{-1, 1\}^{\Lambda \times K_N}$ by the Hamiltonian²⁴

$$H_{\Lambda, \beta}^{(N)}(\sigma^{(N)}) := -\beta \sum_{\substack{x, y \in \Lambda \\ x \sim y}} \varphi_x^{(N)} \varphi_y^{(N)} - d_N \sum_{x \in \Lambda} \sum_{n, m=1}^N \sigma_{x, n}^{(N)} \sigma_{x, m}^{(N)},$$

where $\beta > 0$, $\sigma^{(N)} = (\sigma_{x, n}^{(N)})_{x \in \Lambda, 1 \leq n \leq N}$ is a spin configuration on $\Lambda \times K_N$, $\varphi^{(N)}$ is given by

$$\varphi_x^{(N)} := c_N \sum_{n=1}^N \sigma_{x, n}^{(N)},$$

and c_N, d_N are explicit coupling constants²⁵. Griffiths and Simon [224] showed that the limiting law of the family of averages $(\varphi_x^{(N)})_{x \in \Lambda}$ when N tends to infinity is the φ^4 model on Λ . To give a concrete example, if $x, y \in \Lambda$,

$$\lim_{N \rightarrow \infty} \langle \varphi_x^{(N)} \varphi_y^{(N)} \rangle_{\Lambda, \beta}^{(N)} = \langle \varphi_x \varphi_y \rangle_{\Lambda, \beta}.$$

This can be seen as a central limit type result for dependent random variables.

The relation between the Ising model and the φ^4 model is anticipated to be much richer: the physics prediction is that their critical exponents coincide. This is related to the deep hypothesis of *universality*. This hypothesis refers to the remarkable phenomenon where systems with vastly different microscopic details exhibit the same macroscopic behaviour near their critical points. A justification of universality is provided by renormalization group heuristics, as explained in Section 1.3.

²⁴Note that this time we included β in the definition of the Hamiltonian.

²⁵It turns out that to recover in the limit $N \rightarrow \infty$ the φ^4 measure with quartic potential $\rho_{g, a}$, one may set $c_N = (12g)^{-1/4} N^{-3/4}$ and $d_N = N^{-1}(1 - 2a(12gN)^{-1/2})$. The choice of d_N corresponds to a *near-critical* Curie–Weiss model. For more details, see Chapter 12.

1.2.3 Bernoulli percolation

Our last example is arguably the simplest model in statistical mechanics. *Bernoulli percolation* was introduced by Broadbent and Hammersley [45] to model the propagation of fluid (like a gas or a liquid) in a porous medium (like a porous stone, see Figure 1.6). We give a quick introduction to the model. For more information, we refer to the superb monograph [121].

Definition Again, physical intuition helps us to build a model. We make two essential simplifications.

First, we view a porous medium as a collection of *tubes* through which a fluid can flow. A priori, the structure of these tubes can be anything. To simplify the problem, we constrain these tubes to live on the edges of a graph. We keep our previous example of the hypercubic lattice $\mathbb{Z}^d$ for which the set of edges $\mathbb{E}$ consists of pairs of nearest-neighbour vertices, that is

$$\mathbb{E} := \left\{ \{x, y\} : x, y \in \mathbb{Z}^d, x \sim y \right\}. \quad (1.2.12)$$

Second, on the basis that it is impossible to predict the arrangement of the tubes within the porous medium, we use probability theory and view tube arrangements as random. The simplest way to build a random tube configuration is by declaring each edge of $\mathbb{E}$ to be *open* (i.e. fluid can go through) or *closed* independently of each other. Mathematically, this amounts to tossing a (biased) coin for each edge and opening or closing the edge depending on the result.

The model we obtain is a model of random *subgraphs* of $\mathbb{Z}^d$. To be more precise, it is defined as follows: fix $p \in [0, 1]$, each edge of the hypercubic lattice is kept with probability p and erased with probability $1 - p$, independently of each other. This procedure generates a *percolation configuration* $\omega_p = (\omega_p(e))_{e \in \mathbb{E}}$ where the random variables $\omega_p(e)$ are independent and identically distributed and follow a Bernoulli law of parameter p . The law of ω_p is denoted $\mathbb{P}_p$. The random subgraph then obtained is $(\mathbb{Z}^d, \{e \in \mathbb{E} : \omega_p(e) = 1\})$. For simplicity we identify this random graph with the percolation configuration ω_p . We give an illustration in Figure 1.7.

This model is called *bond* Bernoulli percolation on $\mathbb{Z}^d$. A slightly different model, called *site* Bernoulli percolation, is obtained by opening and closing sites rather than edges.



Figure 1.6: An example of a natural porous medium: open-cell ceramic.

Phase transition For a fluid to flow through the porous medium, there must be a set of open edges forming a macroscopic connected component. This motivates one to study the emergence of an infinite connected component (or *cluster*) in the random percolation configuration distributed according to $\mathbb{P}_p$. For $p \in [0, 1]$, we let

$$\Psi(p) := \mathbb{P}_p[\text{There exists an infinite cluster}].$$

To give some intuition, as p increases, the *density* of open edges increases and the random graph ω_p becomes more and more connected since the connected components become larger and larger. Peierl's argument allows one to prove that when $d \geq 2$, the model undergoes a phase transition for the existence of an infinite cluster. Namely, there exists $p_c = p_c(d) \in (0, 1)$ such that:

- For all $p < p_c$, $\Psi(p) = 0$. This means that *almost surely* there is no infinite cluster.
- For all $p > p_c$, $\Psi(p) = 1$. This means that almost surely there exists an infinite cluster. Moreover, this infinite cluster is almost surely unique, as proved in [15] (alternative approaches are given in [105, 57]).

In dimension $d = 1$ there is no phase transition and $p_c(1) = 1$. This is comparable to the situation observed for the Ising model.

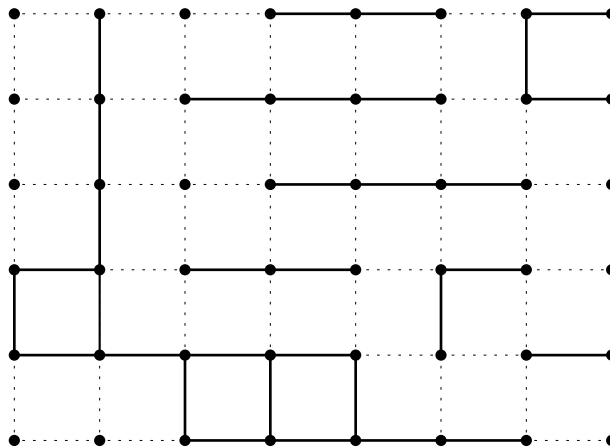


Figure 1.7: A configuration of bond percolation on a subset of the square lattice $\mathbb{Z}^2$. Open edges are the bold black lines and closed edges are the thin black dotted lines.

Regardless of the behaviour at p_c , the function Ψ has a discontinuity at the critical point. This hints that it might not be the proper *order parameter* to consider. We will rather²⁶ study the function θ defined for $p \in [0, 1]$ by

$$\theta(p) := \mathbb{P}_p[0 \text{ belongs to an infinite cluster}].$$

This function is exactly the *density* of the unique cluster and satisfies $\theta(p) > 0$ when $p > p_c$. It vanishes when $p < p_c$.

²⁶Here, we have fixed our origin to be 0. This is fine by the model's invariance under translation of the lattice $\mathbb{Z}^d$.

A natural question to ask is what happens at p_c . In the case $d = 2$, this was answered by Kesten [165]. He found that $\theta(p_c) = 0$, i.e. there is almost surely no infinite cluster at the critical point p_c . Moreover, he computed the value of the critical point, showing that

$$p_c(2) = \frac{1}{2}.$$

Simulations of two-dimensional Bernoulli percolation in the regimes $p < p_c$, $p = p_c$, and $p > p_c$ are given in Figure 1.8.

Using the *lace expansion*, which we describe in Section 1.3, it is possible to prove that $\theta(p_c) = 0$ when $d > 10$, see [134, 132, 100]. The same result is expected to hold when $3 \leq d \leq 10$. Proving the absence of infinite cluster at the critical point for these intermediate dimensions is one of the biggest open problems in statistical mechanics.

Critical exponents Widom's prediction also applies to the case of Bernoulli percolation, and we may define the critical exponents $(\alpha, \beta, \gamma, \delta, \eta, \nu)$ as for the Ising model. Below, we write $\{0 \leftrightarrow x\}$ for the event that 0 and x are in the same connected component. We also let $\mathcal{C}(0)$ denote the cluster of the origin, that is

$$\mathcal{C}(0) := \{x \in \mathbb{Z}^d : 0 \leftrightarrow x\}.$$

Postponing the definitions, the main critical exponents are given by:

$$\begin{aligned} f''(p) &= (p_c - p)^{-\alpha+o(1)} && \text{as } p \nearrow p_c, \\ \theta(p) &= (p - p_c)_+^{\beta+o(1)} && \text{as } p \searrow p_c, \\ \chi(p) &= (p_c - p)^{-\gamma+o(1)} && \text{as } p \nearrow p_c, \\ m(p_c, h) &= h^{1/\delta+o(1)} && \text{as } h \searrow 0, \\ \mathbb{P}_{p_c}[0 \leftrightarrow x] &= \|x\|^{-(d-2)-\eta+o(1)} && \text{as } \|x\| \rightarrow \infty, \\ \xi(p) &= (p_c - p)^{-\nu+o(1)} && \text{as } p \nearrow p_c, \end{aligned}$$

where $o(1)$ goes to 0 as the parameter goes to its limit point. The expected values of these exponents are presented in Table 1.3.

In the context of Bernoulli percolation, the specific heat is replaced by the second derivative of the free energy $f(p)$ given by

$$f(p) := \mathbb{E}_p[|\mathcal{C}(0)|^{-1}].$$

The susceptibility is the expected size of the cluster of the origin, that is

$$\chi(p) := \mathbb{E}_p[|\mathcal{C}(0)|] = \sum_{x \in \mathbb{Z}^d} \mathbb{P}_p[0 \leftrightarrow x].$$

It is not completely clear how to define the “magnetisation” $m(p, h)$ in the case of Bernoulli percolation. We will define it as

$$m(p, h) := 1 - \sum_{n \geq 0} \mathbb{P}_p[|\mathcal{C}(0)| = n] e^{-hn} = 1 - \mathbb{E}_p[\exp(-h|\mathcal{C}(0)|)].$$

<i>Dimension</i>	α	β	γ	δ	η	ν
$d = 2$	$-\frac{2}{3}$	$\frac{5}{36}$	$\frac{43}{18}$	$\frac{91}{5}$	$\frac{5}{24}$	$\frac{4}{3}$
$d = 3$	≈ -0.64	≈ 0.429	≈ 1.78	≈ 5.16	≈ -0.03	≈ 0.88
$d = 4$	≈ -0.75	≈ 0.658	≈ 1.430	≈ 3.175	≈ -0.084	≈ 0.686
$d = 5$	≈ -0.870	≈ 0.8454	≈ 1.1792	≈ 2.3952	≈ -0.0547	≈ 0.5739
$d \geq 6$	-1	1	1	2	0	$\frac{1}{2}$

Table 1.3: Conjectured values of the main critical exponents of Bernoulli percolation on $\mathbb{Z}^d$ with $d \geq 2$. For $d = 2$, all the exponents have been computed in the case of site percolation on the triangular lattice using Smirnov’s breakthrough proof of Cardy’s formula [232]. This result and other subsequent works allowed to relate the critical scaling limit of the model to SLE(6) and CLE(6). By universality, it is conjectured that the critical exponents should take the same values for bond percolation on the square lattice. When $3 \leq d \leq 5$, the critical exponents were estimated using the conjectured connection of percolation to the so-called φ^3 model. Many attempts were made to propose numerical estimates on these exponents. Here, we follow the recent work [42]. Interestingly, η is expected to take negative values when $3 \leq d \leq 5$. This means that the model cannot satisfy the infrared bound of (1.2.7). In sufficiently large dimensions, the critical exponents are expected to take their mean-field value. The critical exponents $\beta, \gamma, \delta, \eta$ were successfully derived when $d > 10$ using the *lace expansion*, see [16, 134, 23, 132, 100]. The conjectured value of the upper-critical dimension of the model is $d_c = 6$. Let us mention the work of Chayes and Chayes [61], in which it is shown that β and γ cannot simultaneously take their mean-field value when $3 \leq d \leq 5$. More precisely, one has that $d\nu \geq 1 + 2\beta$ (where d is the dimension) if the exponents exist.

It is possible to give a probabilistic interpretation of this quantity in terms of a *ghost field*. The following representation is due to Aizenman and Barsky [7]. The idea is to *augment* our probability space by adding an independent site Bernoulli percolation of a well-chosen parameter. More precisely, define a family of independent and identically distributed random variables $(g_x)_{x \in \mathbb{Z}^d}$ that follow a Bernoulli law of parameter $1 - e^{-h}$. That is, denoting by $\mathbb{P}_h^{\text{site}}$ their joint law,

$$\mathbb{P}_h^{\text{site}}[g_x = 1] = 1 - e^{-h} = 1 - \mathbb{P}_h^{\text{site}}[g_x = 0].$$

Let $\mathbb{P}$ denote the law of a pair (ω_p, g) where $\omega_p \sim \mathbb{P}_p$ and $g \sim \mathbb{P}_h^{\text{site}}$ are independent. Then, one can check that

$$\mathbb{P}[\mathcal{C}(0) \cap \{x \in \mathbb{Z}^d : g_x = 1\} \neq \emptyset] = m(p, h).$$

In words, this allows one to interpret $m(p, h)$ as the probability that 0 connects to the *ghost field* $\{x \in \mathbb{Z}^d : g_x = 1\}$ in ω_p .

Finally, the correlation length $\xi(p)$ is defined similarly as in the case of the Ising model:

$$\xi(p) := \lim_{n \rightarrow \infty} \left(-\frac{1}{n} \log \mathbb{P}_p[0 \leftrightarrow n\mathbf{e}_1] \right)^{-1}.$$

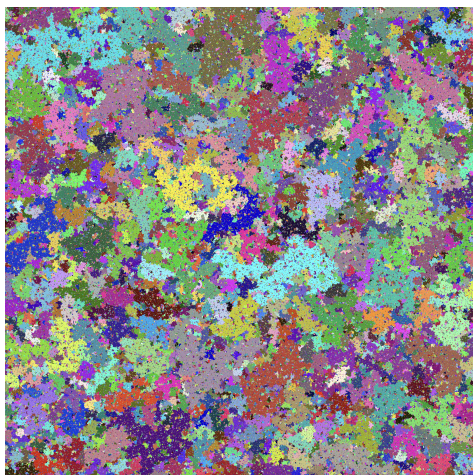
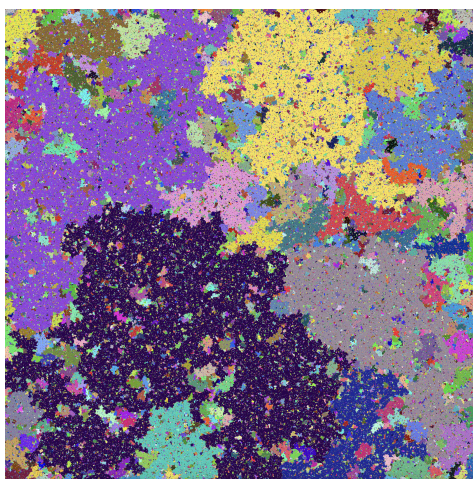
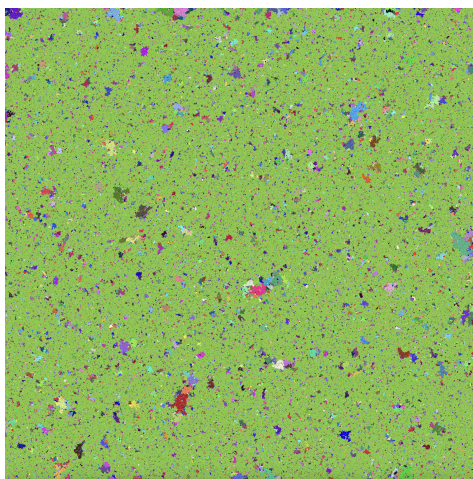
(a) $p = 0.49$ (b) $p = \frac{1}{2}$ (c) $p = 0.51$

Figure 1.8: Samples of planar bond Bernoulli percolation in a large box in three regimes: $p < p_c$, $p = p_c$, and $p > p_c$. Different colours represent different clusters. In the subcritical phase $p < p_c$, all clusters are tiny, whereas in the supercritical phase $p > p_c$ one giant cluster emerges (in green). At criticality, fractal structures described by SLE(6) and CLE(6) arise. The simulations are due to Raphaël Cerf.

1.3 Mean-field regime

Rigorously computing the critical exponents of a model is in general very complicated as it involves in a subtle way the geometry of the lattices on which they are defined. A striking observation was made in the case of models defined on the hypercubic lattice $\mathbb{Z}^d$: above an *upper-critical dimension*²⁷ d_c , the geometry ceases to play a role and the critical exponents take an easier form, equalling those obtained on a Cayley tree (or *Bethe lattice*) or on the complete graph. The regime $d \geq d_c$ forms²⁸ the *mean-field regime* of a model.

The goal of this last section is twofold. First, we give some historical background on mean-field theory and explain the physical heuristics behind it. In particular, we explain what is meant by *mean-field approximation* and how one can guess the value of d_c . The most efficient theoretical framework to study these questions is the so-called *renormalization group*, which we briefly describe. This is also a nice opportunity to elaborate on the concept of *universality*. Second, we give an overview of the rigorous results regarding the mean-field behaviour of the aforementioned models. Once again, the Ising model will serve as a guiding example.

1.3.1 Mean-field theory and Ginzburg's criterion

This section is non-rigorous. The aim is to provide a physical understanding of the mean-field regime in the context of the Ising model.

Weiss' approximation. Mean-field theory was initiated with Weiss' attempt to explain Curie's experiment. In his seminal paper [246], Weiss introduced the so-called *mean-field approximation*. Although not fully rigorous in the case of the hypercubic lattice, this assumption provides some insights on the value of the mean-field exponents presented in Table 1.2.

We go back to the definition of the Hamiltonian of the nearest-neighbour Ising model provided in (1.2.1). Again, we assume that we work on a finite set $\Lambda \subset \mathbb{Z}^d$. Let x be in the interior of Λ . The total energy coming from the interaction with a spin σ_x can be written as

$$-\sum_{y \sim x} \sigma_x \sigma_y = -2d\sigma_x \cdot \frac{1}{2d} \sum_{y \sim x} \sigma_y.$$

The contribution of σ_x to the total Hamiltonian is therefore obtained from the interaction of σ_x with the *local average*

$$\frac{1}{2d} \sum_{y \sim x} \sigma_y.$$

The mean-field approximation²⁹ (or *Weiss' hypothesis*) consists in assuming that these *local*

²⁷There is also a *lower-critical dimension* which corresponds to the dimension below which no phase transition occurs. In the case of the n.n.f Ising model it is equal to 2.

²⁸The mean-field regime of a model is often defined to be the dimensions $d > d_c$. This choice is based on the validity of the mean-field approximation. As we will see below, the mean-field approximation fails at $d = d_c$. However, it only *marginally* fails in the sense that critical exponents still take their mean-field values but additional logarithmic corrections need to be handled.

²⁹A further simplification is to assume that each spin interacts with all the others. This corresponds to defining the model on the complete graph K_N where N is the number of spins. By doing so, one may render these computations rigorous. For more information, we refer to [101, Chapter 2].

averages can be approximated by the *global* average

$$\frac{1}{|\Lambda|} \sum_{x \in \Lambda} \sigma_x.$$

This naive approach is built on the idea that “long-range correlations do not contribute much”. We do not question the validity of the hypothesis yet and rather show how it allows us to compute many relevant quantities. Let $\beta > 0$ and $h \in \mathbb{R}$, and denote by M the average

$$M := \left\langle \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \sigma_x \right\rangle_{\Lambda, \beta, h}.$$

A quick computation gives,

$$\mu_{\Lambda, \beta, h}[\sigma_x = \pm 1 \mid (\sigma_y)_{y \in \Lambda \setminus \{x\}}] = \frac{1}{Z} \exp \left(\pm \left(\beta \sum_{y \sim x} \sigma_y + h \right) \right),$$

where $Z := 2 \cosh(\beta \sum_{y \sim x} \sigma_y + h)$. The mean-field approximation allows one to replace local averages by M in the formula above. This implies that

$$\mu_{\Lambda, \beta, h}[\sigma_0 = \pm 1] = \frac{1}{2 \cosh(2d\beta M + h)} \exp(\pm(2d\beta M + h)),$$

so that,

$$M = \tanh(2d\beta M + h). \quad (1.3.1)$$

From there, one may recover mean-field critical exponents. Assuming (1.3.1) holds in the infinite volume limit, we get that in any dimension where the above approximation holds,

$$m(\beta, h) = \tanh(2d\beta m(\beta, h) + h). \quad (1.3.2)$$

Taking $h \searrow 0$ thus gives

$$m^*(\beta) = \tanh(2d\beta m^*(\beta)). \quad (1.3.3)$$

From (1.3.3) and some (easy) analysis, one may prove that $m^*(\beta)$ must vanish when $\beta < \frac{1}{2d}$, and may take a strictly positive value when $\beta > \frac{1}{2d}$. As a result, under the mean-field approximation,

$$\beta_c = \frac{1}{2d}. \quad (1.3.4)$$

Notice that (1.3.3) also implies that $m^*(\beta)$ tends to 0 as β tends to β_c . Thus, using a Taylor expansion, we get that as $\beta \searrow \beta_c$,

$$m^*(\beta) = 2d\beta m^*(\beta) - \frac{1}{3}(2d\beta m^*(\beta))^3 + O([2d\beta m^*(\beta)]^5),$$

which readily implies that³⁰

$$m^*(\beta) \approx \left(\frac{3(\beta - \beta_c)}{4d^2\beta^3} \right)^{1/2},$$

and thus that $\beta = \frac{1}{2}$. Using (1.3.2) for $\beta = \beta_c$ and sending h to 0, we may prove (again with a Taylor expansion) that $\delta = 3$.

³⁰We used (1.3.4) to argue that $2d\beta = 2d(\beta - \beta_c + \beta_c) = 1 + 2d(\beta - \beta_c)$.

One may also compute the critical exponent γ . Fixing $\beta < \beta_c$, one has that

$$\chi(\beta) = \left. \frac{\partial m(\beta, h)}{\partial h} \right|_{h=0} = \left. \frac{1 - \tanh^2(2d\beta m(\beta, h))}{1 - 2d\beta(1 - \tanh^2(2d\beta m(\beta, h)))} \right|_{h=0} = \frac{1}{1 - 2d\beta} \stackrel{(1.3.4)}{=} \frac{1}{1 - \beta/\beta_c}.$$

With these computations, we recover some of the mean-field exponents of Table 1.2. However, the main setback of this approach is that it does not provide any information on the value of the upper-critical dimension d_c : *what is the domain of validity of the mean-field approximation?* As it turns out, Weiss' hypothesis is never true on the hypercubic lattice. However, it is *almost* true for sufficiently large dimensions. The so-called *Landau theories* provide some insights into this question.

Landau theory and Ginzburg's criterion. This short paragraph is by no means a rigorous introduction to Landau³¹ theories. For more information, we refer to the monographs [95, 210].

Landau theory (or Ginzburg–Landau theory) for the Ising model postulates that the Legendre transform of the free energy F is *analytic* in β and $m^*(\beta)$. By imposing the known qualitative behaviour (namely the vanishing of $m^*(\beta)$ at the critical point), one recovers the mean-field critical exponents. Surprisingly, this approach accurately predicts the expected behavior for $d > 4$. However, Landau theory falls short for $d < 4$, as demonstrated by *Ginzburg's criterion*.

Similarly to Weiss' mean-field approximation, Landau theories are constructed on the assumption that the fluctuations of the order parameter (in that case the spin field σ) are negligible compared to the value of the order parameter near criticality. In our case, these fluctuations are measured by the truncated correlation function

$$\langle \sigma_0; \sigma_x \rangle_\beta := \langle \sigma_0 \sigma_x \rangle_\beta - m^*(\beta)^2.$$

Ginzburg's criterion may be (informally) rephrased as follows: for $\beta > \beta_c$ “close” to β_c , for “typical” $x \in \mathbb{Z}^d$,

$$\langle \sigma_0 \sigma_x \rangle_\beta - m^*(\beta)^2 \ll m^*(\beta)^2.$$

Here the word “typical” refers to the values of $\|x\|$ which are of order the correlation length³² $\xi(\beta)$. Landau theory predicts that for $x \in \mathbb{Z}^d \setminus \{0\}$,

$$\frac{ce^{-C\|x\|/\xi(\beta)}}{\|x\|^{d-2}} \leq \langle \sigma_0; \sigma_x \rangle_\beta \leq \frac{Ce^{-c\|x\|/\xi(\beta)}}{\|x\|^{d-2}},$$

where $0 < c < C < \infty$ are two constants. As a consequence, for a typical $x \in \mathbb{Z}^d$, $\langle \sigma_0; \sigma_x \rangle_\beta \approx \xi(\beta)^{2-d}$. Combined with Widom's scaling assumption, Ginzburg's criterion can be rephrased as,

$$(\beta - \beta_c)^{\nu(d-2)} \ll (\beta - \beta_c)^{2\beta} \quad \text{when } \beta \searrow \beta_c.$$

Equivalently,

$$d > 2 + \frac{2\beta}{\nu}.$$

³¹Lev D. Landau (1908–1968) was a Soviet physicist. He was awarded the Nobel Prize in Physics in 1962 for his development of a mathematical theory of superfluidity.

³²Above, we only defined the correlation length in the regime $\beta < \beta_c$. We may extend the definition to all values of $\beta > 0$ by replacing $\langle \sigma_0 \sigma_{ne_1} \rangle_\beta$ with $\langle \sigma_0; \sigma_{ne_1} \rangle_\beta$ in (1.2.4).

If the critical exponents assume their mean-field values, one has $\beta = \nu = \frac{1}{2}$, and Ginzburg's criterion is equivalent to the requirement that $d > 4$. In particular, we see that fluctuations are *relevant* when $d < 4$ and mean-field theory cannot be applied there.

Despite its efficiency in determining the value of the critical exponents in the mean-field regime, Landau's theory does not explain why the mean-field exponents are *exact* in dimensions $d \geq 4$. The renormalization group method is the proper framework for this agenda.

1.3.2 Renormalization group theory

“Wilson has provided scientists with the single most relevant tool for understanding the basis of physics” Leo P. Kadanoff

The *renormalization group* (RG) theory is one of the most prolific developments of the twentieth century in theoretical physics. A key contributor to its advancement, Kenneth G. Wilson, received the Nobel Prize in 1982 for his outstanding contributions to the formulation of the theory. Providing a full description of RG theory would go far beyond the scope of this introduction. For a more comprehensive understanding, we recommend the fascinating review by Fisher [98] and the references therein.

Let us recall some context and explain the state of statistical mechanics in the first half of the twentieth century. Again, our guiding thread will be the Ising model. A physically relevant quantity introduced above is the two-point function $\langle \sigma_0 \sigma_x \rangle_\beta$. At criticality, it is predicted to observe a power-law decay of the form

$$\langle \sigma_0 \sigma_x \rangle_{\beta_c} = \frac{1}{\|x\|^{d-2+\eta+o(1)}},$$

where $o(1)$ tends to 0 as $\|x\|$ tends to infinity, and where $\eta \in \mathbb{R}$ is the so-called anomalous dimension. Before the 1940s, all the “classical” theories of Curie–Weiss or Landau–Ginzburg predicted that η vanishes. This hypothesis typically arises under the assumption that only the fluctuations coming from “smaller scales” play a significant role. This however does not provide any insights on the cases $d = 2, 3$ for which it is expected that $\eta > 0$. What is missing is a mathematical framework which allows for non-classical critical exponents to emerge.

The first major breakthrough comes with the mathematical *tour de force* of Lars Onsager [200] involving the computation of the free energy of the two-dimensional Ising model, see (1.2.6). This notable result was later complemented by Onsager and Kaufman's joint proof that $\eta = \frac{1}{4}$ for the same model, see [161]. These methods, although extremely powerful, rely on a special feature of the model called *integrability* and are not expected to hold in more general setups³⁴.

It became evident that a new mathematical framework was necessary to develop an extensive understanding of critical phenomena. Let us pause for a second and list all the requirements one would like such a theory to fulfill:

- (a) It should explain the existence of critical exponents and in particular allow for the emergence of non-classical values.

³³Credits for Kadanoff and Wilson respectively: National Science & Technology Medals Foundation, Sol Goldberg Cornell University Photography.

³⁴For instance, this approach breaks down when considering the Ising model on $\mathbb{Z}^3$.

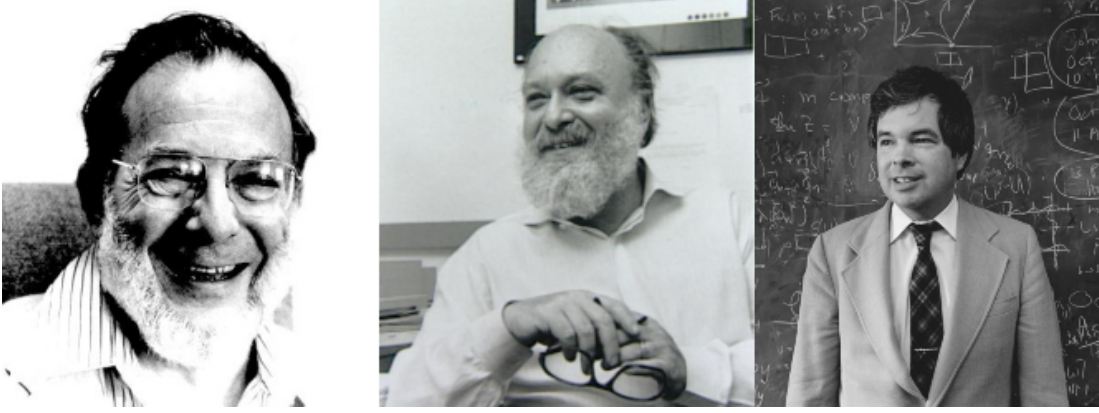


Figure 1.9: From left to right: Michael Fisher (1931-2021), Leo P. Kadanoff (1937-2015), and Kenneth G. Wilson (1936-2013)³³.

- (b) It should enable the derivation of scaling relations and elucidate the concept of hyperscaling (in particular explain why it vanishes when $d > 4$).
- (c) It should help the understanding of the concept of universality.

The renormalization group method provides a conceptual and computational framework which allows physicists and mathematicians to meet these requirements. Its development began with the seminal work of Leo P. Kadanoff.

Kadanoff's block spin renormalization. Kadanoff [159] envisioned a unified treatment of Ising type models with the introduction of the *block-spin renormalization*. The idea is fairly simple: one maps a critical or near-critical system onto itself by reducing the effective number of degrees of freedom. In the case of the Ising model, this can be performed by coarse-graining the lattice $\mathbb{Z}^d$ with blocks of a certain size, and assigning the value of the majority spin to the block, see Figure 1.10 for an illustration. One notable result obtained by Kadanoff is the following hyperscaling relation

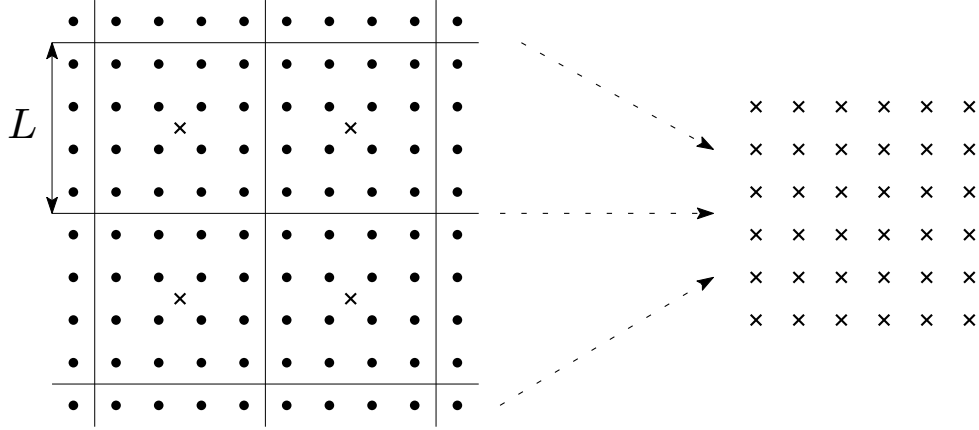
$$\nu d = 2 - \alpha,$$

which was known to hold for $d = 2$ and for other integrable models when $d < 4$. However, it fails when $d > 4$ which hints that something must be missing in this picture.

The main fault in Kadanoff's reasoning is to assume that the system maps onto itself, meaning that an Ising model is mapped to an Ising model. Upon detailed analysis, one finds that iterations of Kadanoff's renormalization map make $2k$ -spins interactions appear in the *effective* Hamiltonian (i.e. terms of the form $J_K \prod_{i \in K} \sigma_i$ for $|K| = 2k$ and some coupling J_K). Thus, this scheme is too simplistic and new ideas are required.

Wilson's approach. The major progress in the development of the renormalization group theory came a few years later in the breakthrough works of Wilson [250, 251]. Briefly, Wilson analysed a lattice φ^4 model with potential

$$V^{(0)}(\varphi) = V^{(0)}(\varphi; g_0, a_0) := g_0 \varphi^4 + a_0 \varphi^2.$$



$$x \Rightarrow x' = x/L$$

Figure 1.10: An illustration of Kadanoff's block spin renormalization for the Ising model. The dots correspond to Ising spins. They are divided into blocks of size L^d . Each such block contains a majority spin denoted by a cross. Kadanoff's assumption is that after one step of the renormalization the Ising model of parameters (β, h) on dots is mapped to an Ising model of parameters (β', h') on crosses.

In his picture, small effective spins are averaged over blocks of size 2 to obtain a new renormalized effective spin. With sufficiently many assumptions, Wilson derived an explicit non-linear integral recursion relation that mapped the ℓ -times renormalized potential $V^{(\ell)}$ to a new renormalized potential $V^{(\ell+1)}$. This relation explicitly allowed for the emergence of complexities at the level of the potential. Namely³⁵,

$$V^{(\ell)}(\varphi) = V^{(\ell)}(\varphi; g_\ell, a_\ell, z_\ell, u_\ell, \dots) = g_\ell \varphi^4 + a_\ell \varphi^2 + z_\ell \varphi(\Delta \varphi) + u_\ell \varphi^6 + \dots$$

To put it differently, under one step of the renormalization, a φ^4 model is *not* mapped to a φ^4 model, but rather to a more or less relevant perturbation of the φ^4 model. Wilson showed that the evolution of this dynamical system is governed by a finite-dimensional part consisting of *relevant* and *marginal* variables, together with an infinite-dimensional *irrelevant* part. In other words, one can restrict their attention to a finite number of parameters rather than the full family. The term “irrelevant” refers to the fact that these parameters *contract* (or vanish) under successive applications of the RG map. Relevant and marginal parameters do not contract and they may be distinguished by the critical exponents associated with their behaviour. If this exponent allows for an algebraic blow-up of the parameter in the vicinity of the fixed point of the dynamic, we say that the variable is relevant. If not, the variable is marginal. Blow-up can still occur, but at a more controlled rate. Using a computer, Wilson

³⁵Here Δ refers to the *discrete Laplacian* on $\mathbb{Z}^d$. More precisely, for $x \in \mathbb{Z}^d$,

$$(\Delta \varphi)_x := \sum_{y \sim x} (\varphi_y - \varphi_x).$$

gave predictions on the critical exponents in three dimensions: in particular, he obtained the non-classical value $\nu \approx 0.61$ (one could compare it to the now predicted value for the Ising model, see Table 1.2).

Another fundamental idea of Wilson is to view successive applications of the renormalization group map as a continuous *flow* in an infinite-dimensional space of Hamiltonians $\mathfrak{H}$, see Figure 1.11. This point of view provides a justification of the concept of *universality* in statistical mechanics.

Fixed points and universality. A striking feature of Wilson’s approach is the observation that models sharing the same relevant and marginal parameters, and differing at the level of their irrelevant parameters, may be mapped to similar *fixed points* of the RG map. These fixed points correspond to different *universality classes* of critical behaviour, which attract all the systems that are eventually mapped onto the corresponding flows. The renormalization group method thus explains a previous observation that, in the vicinity of the critical point, the behaviour of a system becomes essentially independent of its microscopic details³⁶.

In the case of the Ising model, it is predicted that in dimensions $d \geq 4$ there are exactly two renormalization group fixed points: the *massless Gaussian free field* and the *white noise*. The dimension $d = 4$ is characterised by the emergence of marginal variables that are not enough to breakdown the mean-field behaviour but that lead to logarithmic corrections of the quantities of interest³⁷. As d decreases, relevant variables appear, the number of fixed points increases, and non-Gaussian fixed points can be found. The renormalization group method is also an efficient setup to treat this non-classical regime through the so-called $(d_c - \varepsilon)$ -*expansions*.

Going beyond the mean-field regime: $(d_c - \varepsilon)$ -expansions. For the Ising model, the upper-critical dimension is $d_c = 4$. This dimension corresponds to a *borderline* case in which marginally relevant parameters emerge at the level of the RG map. It is interesting to ask how critical exponent evolves “close to but below d_c ”. In their famous paper “*Critical phenomena in 3.99 dimensions*”, Wilson and Fisher [252] applied the renormalization group method in dimension $d = 4 - \varepsilon$ and obtained striking estimates on several critical exponents:

$$\gamma = 1 + \frac{\varepsilon}{6} + O(\varepsilon^2), \quad \nu = \frac{1}{2} + \frac{\varepsilon}{12} + O(\varepsilon^2), \quad \eta = \frac{\varepsilon^2}{54} + O(\varepsilon^3).$$

Quite surprisingly, their result provides estimates consistent with other predictions when applied to $\varepsilon = 4 - 3 = 1$. Of course, this expansion is not rigorous as our models are not defined in non-integer dimensions. There is however a rigorous way to tune the *effective*

³⁶As a consequence, one may study simple models (like the Ising model) to understand the (near)-critical properties of much more realistic models which belong to the same universality class.

³⁷Let us give an example. For the Ising model one can show (see [4]) that when $d > 4$, for $\beta < \beta_c$,

$$\chi(\beta) \asymp (1 - \beta/\beta_c)^{-1},$$

where $\asymp$ means that the ratio of the quantities is bounded away from 0 and infinity by two constants that are independent of β . When $d = 4$, it is expected that for $\beta < \beta_c$,

$$\chi(\beta) \asymp \frac{|\log(1 - \beta/\beta_c)|^{1/3}}{1 - \beta/\beta_c}.$$

dimension of a model to non-integers. The idea is to consider a long-range Ising model with interaction $(J_{x,y})_{x,y \in \mathbb{Z}^d}$ given by

$$J_{x,y} = \frac{1}{\|x - y\|^{d+\alpha}} \quad (1.3.5)$$

for some $\alpha \in (0, 2)$. Fisher, Ma, and Nickel [99] identified that the upper-critical dimension of these models satisfies $d_c(\alpha) = 2\alpha$. As a result, if $d \in \{1, 2, 3\}$, it is possible to build a model of effective dimension $d = d_c - \varepsilon$ by setting $\alpha = \frac{d+\varepsilon}{2}$. Using that observation and RG theory, they derived the following non-classical values of various critical exponents:

$$\gamma = 1 + \frac{\varepsilon}{3\alpha} + O(\varepsilon^2), \quad \nu = \frac{\gamma}{\alpha} + O(\varepsilon^2), \quad \eta = 2 - \alpha + O(\varepsilon^2).$$

Some of these results can be made rigorous. We discuss this point below.

Conclusion. The renormalization group approach provides the means to carry out otherwise intractable calculations of the critical exponents, and enlightens the understanding of the concept of universality classes by realising them as domains of attraction or fixed points for a well-chosen dynamical system in the space of Hamiltonians.

1.3.3 Rigorous study of the mean-field regime

We have seen what the mean-field regime is and what tools physicists have developed to conduct its systematic study. From the perspective of mathematics, rigorously implementing these methods is still a challenge. However, various approaches exist. We now review the most prominent strategies.

1.3.3.1 Geometric representations and differential inequalities

A notable approach to studying the critical regime of a model is to provide *geometric representations* to the main quantities of interest. Remarkably, connections between spin systems and random walks were elucidated in the seminal work of Symanzik [239], see [95] for more information. In the case of the Ising model, it is possible to expand the partition function in such a way that we obtain a geometric interpretation of the two-point function $\langle \sigma_0 \sigma_x \rangle_\beta$ in terms of loops, percolation configurations, or paths. However, one representation, that is dear to us, plays a pivotal role in the study of the mean-field regime of the Ising model. The *random current representation* expresses various quantities of interest in terms of weighted sums over *currents*, which are functions $\mathbf{n} : \mathbb{E} \rightarrow \mathbb{N}$ (recall that $\mathbb{E}$ was defined in (1.2.12)). We postpone the proper introduction of this fascinating object to Chapter 2.

A fruitful idea in statistical mechanics is to derive differential inequalities relating the various thermodynamic quantities of interest. Two of the first such inequalities were obtained in the seminal work of Aizenman³⁸ [4]. There, using a powerful combinatorial property of random currents called the *switching lemma* (see Chapter 2), he derived (a weak³⁹ version of) the following differential inequalities:

$$\left(1 - \frac{B(\beta)}{\chi(\beta)}\right) \frac{2d\chi(\beta)^2}{1 + 2d\beta_c B(\beta)} \leq \frac{\partial \chi}{\partial \beta} \leq 2d\chi(\beta)^2, \quad \forall \beta < \beta_c, \quad (1.3.6)$$

³⁸The upper bound in (1.3.11) was derived in an earlier work by Glimm and Jaffe [111].

³⁹The stronger form (1.3.6) was derived by Aizenman and Graham [14].

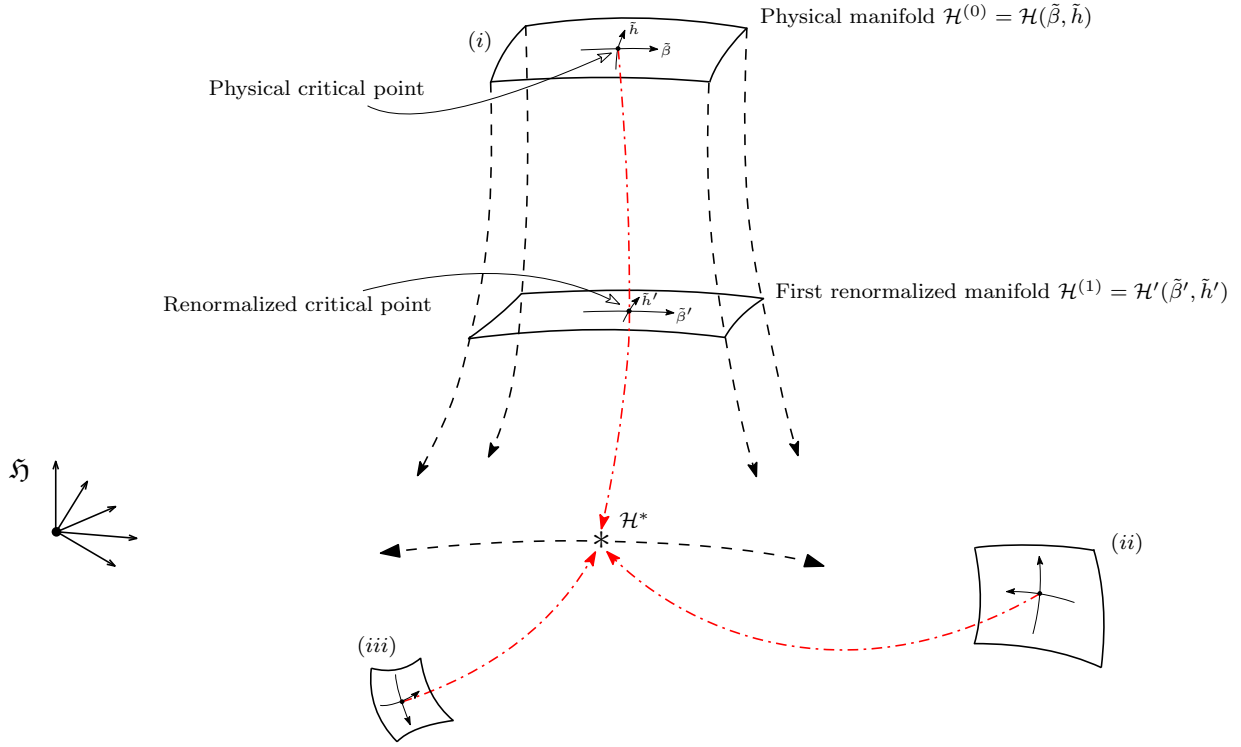


Figure 1.11: An illustration of the renormalization group flow introduced by Wilson. Initial *physical* manifolds are labelled (i), (ii), and (iii). One could for instance think of different four-dimensional Ising-type models: (i) the n.n.f Ising model, (ii) a next-nearest-neighbour ferromagnetic Ising model with interaction $J_{x,y} = \mathbb{1}_{1 \leq |x-y| \leq 2}$, (iii) the n.n.f φ^4 model. The models initially “live” in the infinite dimensional space of Hamiltonians $\mathfrak{H}$. The dashed and dotted red lines correspond to critical trajectories, i.e. trajectories followed by successive applications of a proper RG map that preserves the criticality. In this picture, all critical trajectories end at a unique fixed manifold $\mathcal{H}^*$. This is reminiscent of the fact that the models (i)–(iii) are expected to lie in the same universality class. The dashed lines correspond to off-critical trajectories. In greater generality, the space may contain more fixed points and distinct universality classes.

where $B(\beta)$ is the so-called *bubble diagram* (see Figure 1.12 for an illustration)

$$B(\beta) := \sum_{x \in \mathbb{Z}^d} \langle \sigma_0 \sigma_x \rangle_\beta^2. \quad (1.3.7)$$

A striking feature of (1.3.6) is that under the assumption that $B(\beta_c) < \infty$ (an assumption sometimes referred to as the *bubble condition*), one recovers that for $\beta < \beta_c$,

$$\chi(\beta) \asymp (1 - \beta/\beta_c)^{-1},$$

and hence that $\gamma = 1$ in that case. These differential inequalities thus identify a sufficient condition for mean-field exponents to appear. This condition is also relevant in the study of the mean-field behaviour of the critical exponents β and δ . Indeed, Aizenman, Barsky, and Fernández [8] derived the following inequalities involving the magnetisation $m(\beta, h)$:

$$m(\beta, h) \leq \tanh(\beta h) \frac{\partial m}{\partial(\beta h)} + m(\beta, h)^3 + \beta m(\beta, h)^2 \frac{\partial m}{\partial \beta}, \quad \forall \beta, h > 0 \quad (1.3.8)$$

$$\frac{\partial m^2}{\partial \beta} \geq c_0, \quad \forall \beta > \beta_c. \quad (1.3.9)$$

Integrated versions of (1.3.8) and (1.3.9) provide the mean-field bounds

$$\beta \leq \frac{1}{2}, \quad \delta \geq 3. \quad (1.3.10)$$

It is possible to complement (1.3.8) and (1.3.9) with another pair of differential inequalities involving the susceptibility $\chi(\beta, h)$ defined by⁴⁰

$$\chi(\beta, h) := \sum_{x \in \mathbb{Z}^d} \langle \sigma_0; \sigma_x \rangle_{\beta, h}.$$

Aizenman and Fernández [12] proved that, for all $\beta, h > 0$

$$\frac{(1 - B(\beta)/\chi(\beta, h))_+}{900B(\beta)^3} \tanh(\beta h) \chi(\beta, h)^4 \leq \left| \frac{\partial \chi}{\partial(\beta h)} \right| \leq 64d\beta \left(1 + \frac{1}{2d\beta\chi(\beta, h)} \right) \frac{m(\beta, h)^4}{[\tanh(\beta h)]^3} \quad (1.3.11)$$

Once again, the bubble diagram appears in the differential inequality. One can show (with some work) that an integrated version of (1.3.11) gives, under the bubble condition, that

$$\beta \geq \frac{1}{2}, \quad \delta \leq 3.$$

Combined with (1.3.10), this yields that β and δ assume their mean-field values as soon as $B(\beta_c) < \infty$.

One natural question to ask is whether one can check that the bubble condition is satisfied or not. Thanks to the infrared bound (1.2.7) (which implies that $\eta \geq 0$), one obtains that this condition is satisfied⁴¹ when $d > 4$.

⁴⁰Note that $\chi(\beta, 0) = \chi(\beta)$.

⁴¹In dimensions $d \leq 4$, the bubble-diagram is expected to diverge. It is only “barely” diverging at the upper-critical dimension in the sense that its finite volume version blows up logarithmically (in the volume). This allows one to prove that the critical exponents β, δ , and γ are still mean-field when $d = 4$.

We do not resist mentioning another series of rigorous work elucidating the mean-field nature of the critical Ising model when $d \geq 4$. In two independent papers, Aizenman [4] and Fröhlich [102] established the *triviality* of the scaling limits⁴² of the Ising model in dimensions $d > 4$ (in the single phase regime). While Aizenman relies on the random current representation, Fröhlich builds on a closely related path expansion of the model called the *Brydges–Fröhlich–Spencer* (BFS) expansion, introduced in [47]. In a recent beautiful work, Aizenman and Duminil-Copin [9] managed to extend this result to the marginal case $d = 4$. We refer to Chapter 2 for more information.

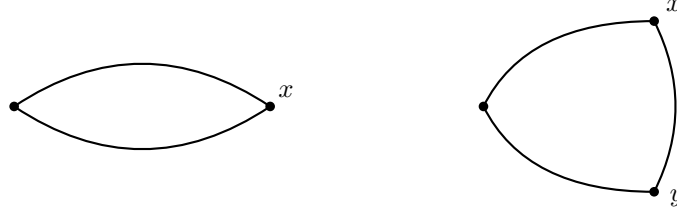


Figure 1.12: On the left: a diagrammatic representation of the bubble diagram $B(\beta)$ defined in (1.3.7). Each line represents a two-point function. Only the summed variables are labelled. On the right: a diagrammatic representation of the triangle diagram $\nabla(p)$ defined in (1.3.19).

1.3.3.2 Rigorous renormalization group

A rigorous implementation of the renormalization group approach involves finding a proper RG map and establishing that the RG flow of finitely many parameters contains all the information about the critical exponents. This scheme is in general difficult to implement.

As it turns out, the most natural testing ground for developing rigorous renormalization group techniques is the *weakly-coupled* φ^4 model⁴³ in dimension $d = 4$. Here, “weakly-coupled” refers to the fact that the constant g appearing in (1.2.8) is chosen to be “small enough”. In some sense, this choice corresponds to initiating the renormalization group at a model which is “close” to the Gaussian fixed point.

First rigorous implementations of the RG method in that context appeared in the mid-1980s: in two independent works, Gawedzki and Kupiainen [107], and Feldman, Magnen, Rivasseau, and Sénéor [94] obtained the $\|x\|^{-2}$ decay for the critical two-point function. Logarithmic corrections for the susceptibility were derived by Hara and Tasaki [131, 139].

In a recent series of work, Bauerschmidt, Brydges, and Slade developed a general method for implementing Wilson’s strategy [52, 53, 28, 54, 55]. The following notable results were derived using this perspective on RG:

- **Logarithmic corrections** [25]: if $d = 4$ and g is sufficiently small in (1.2.8), there exists $A > 0$ such that for $\beta < \beta_c$,

$$\chi(\beta) = \frac{A |\log(1 - \beta/\beta_c)|^{1/3}}{1 - \beta/\beta_c} (1 + o(1)),$$

⁴²In the context of RG, these results are related to the fact that the only fixed points of the RG map are Gaussian when $d \geq 4$.

⁴³In view of (1.2.11), one may understand the Ising model as a *strongly* coupled φ^4 model.

where $o(1)$ tends to 0 as β tends to β_c .

- **Critical two-point function** [231]: if $d = 4$ and g is sufficiently small in (1.2.8), there exists $C > 0$ such that for $x \in \mathbb{Z}^d \setminus \{0\}$,

$$\langle \varphi_0 \varphi_x \rangle_{\beta_c} = \frac{C}{\|x\|^2} (1 + o(1)),$$

where $o(1)$ tends to 0 as $\|x\|$ tends to infinity.

- **$(d_c - \varepsilon)$ -expansion** [227]: if $d \in \{1, 2, 3\}$ and J is chosen as a “spread-out version” of (1.3.5) with $\alpha = \frac{d+\varepsilon}{2}$ and $\varepsilon > 0$ sufficiently small, then

$$\gamma = 1 + \frac{\varepsilon}{3\alpha} + O(\varepsilon^2).$$

Let us observe that rigorous renormalization group seems to be restricted (so far) to models for which a *perturbative* parameter is available.

1.3.3.3 The lace expansion

We conclude this review with a short introduction to the *lace expansion method* introduced by Brydges and Spencer [49] in 1985. The full power of this approach was clarified in the seminal works of Hara and Slade [225, 134, 136]. In the case of the Ising model, the lace expansion was successfully derived by Sakai [216, 218]. For more information and an introduction to the theory, we refer to the monograph [226] and the paper [39].

We begin with some useful definitions. If $f, g : \mathbb{Z}^d \rightarrow \mathbb{R}$, the convolution $f * g$ is defined for $x \in \mathbb{Z}^d$ by

$$(f * g)(x) := \sum_{y \in \mathbb{Z}^d} f(y)g(x - y).$$

Letting $\mathbb{T}^d = (-\pi, \pi]^d$, the Fourier transform of f , denoted $\hat{f}$, is given for $k \in \mathbb{T}^d$ by

$$\hat{f}(k) := \sum_{x \in \mathbb{Z}^d} f(x)e^{ik \cdot x}.$$

Moreover, one can recover f from $\hat{f}$ using an inverse Fourier transform. Namely, for $x \in \mathbb{Z}^d$,

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \hat{f}(k)e^{-ik \cdot x} dk.$$

Let $D : \mathbb{Z}^d \rightarrow [0, 1]$ be defined by

$$D(x) := \frac{1}{2d} \mathbf{1}_{\|x\|=1}.$$

The *lattice Green function* $\mathbb{G}$ is the solution, vanishing at infinity, of the *convolution* equation

$$(\delta_0 - D) * \mathbb{G} = \delta_0, \tag{1.3.12}$$

where $\delta_0(x) = \mathbf{1}_{x=0}$. Thus, $\mathbb{G}$ is the *deconvolution* of $\delta_0 - D$. Taking the Fourier transform of (1.3.12) yields

$$\hat{\mathbb{G}} = \frac{1}{1 - \hat{D}}, \tag{1.3.13}$$

so that

$$\mathbb{G}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{e^{-ik \cdot x}}{1 - \hat{D}(k)} dk.$$

An alternative probabilistic definition of $\mathbb{G}$ is given by

$$\mathbb{G}(x) = \mathbb{E}_0 \left[\sum_{k \geq 0} \mathbb{1}_{X_k = x} \right],$$

where $\mathbb{P}_0$ is the law of the *simple random walk* $(X_k)_{k \geq 0}$ started at 0. In words, $\mathbb{G}(x)$ is the expected number of visits to x by a simple random walk started at the origin. It is a classical result (see for instance [245]) that for $d \geq 3$,

$$\mathbb{G}(x) = \frac{a_d}{\|x\|^{d-2}} (1 + o(1)),$$

where $o(1)$ tends to 0 as $\|x\|$ tends to infinity, and where

$$a_d := \frac{1}{2} \pi^{-d/2} \Gamma(\frac{d}{2} - 1).$$

It is expected (from renormalization group heuristics) that the two-point function of the critical Ising model obeys similar asymptotics when $d \geq 4$: there exists $C = C(d) > 0$ such that,

$$\langle \sigma_0 \sigma_x \rangle_{\beta_c} = \frac{C}{\|x\|^{d-2}} (1 + o(1)),$$

where $o(1)$ tends to 0 as $\|x\|$ tends to infinity. It seems natural to try to compare the two-point function of the Ising model to $\mathbb{G}$. One way to do so is to obtain a convolution equation satisfied by $\langle \sigma_0 \sigma_x \rangle_\beta$. The lace expansion method consists in deriving such a convolution equation. In some cases, it is possible to *deconvolve* it to recover Green function asymptotics. We briefly describe the general strategy of lace expansion approaches. Below, we let $S_\beta(x) := \langle \sigma_0 \sigma_x \rangle_\beta$.

Step 1 Convolution equation: The first step is to derive a convolution equation for the two-point function. In the case of the Ising model, this expansion takes the form

$$S_\beta = \Pi_\beta + (\Pi_\beta * [2d \tanh(\beta)] D * S_\beta), \quad (1.3.14)$$

where

$$\Pi_\beta(x) := \sum_{j \geq 0} (-1)^{j+1} \pi_\beta^{(j)}(x),$$

and where the $\pi_\beta^{(j)}$ are some explicit positive weights. This equation may be obtained using an *inclusion-exclusion* approach. Taking the Fourier transform of (1.3.14), we obtain

$$\hat{S}_\beta = \frac{\hat{\Pi}_\beta}{1 - \hat{\Pi}_\beta[2d \tanh(\beta)] \hat{D}}. \quad (1.3.15)$$

Comparing (1.3.13) to (1.3.15), we see that if $\hat{\Pi}_\beta$ is “reasonable”, we may expect to recover an asymptotic expression for S_{β_c} that is similar to that of $\mathbb{G}$. Note also that it is not clear that (1.3.14) converges for all $\beta \leq \beta_c$.

Step 2 Bootstrap argument: Once a convolution equation has been derived, the idea is to use it to improve an *a priori* estimate on the two-point function. For instance, it is useful to begin by assuming that for some $\beta < \beta_c$ and some $C > 0$, one has, for all $x \in \mathbb{Z}^d$,

$$S_\beta(x) \leq 2C\mathbb{G}(x). \quad (1.3.16)$$

This strategy is called a *bootstrap argument*, and was initially introduced in the work of Slade [225]. It can be split into three sub-steps.

Step 2.1 Obtain diagrammatic estimates on the $\pi_\beta^{(j)}$: Equations (1.3.14) and (1.3.15), as well as the assumption (1.3.16), suggest trying to estimate Π_β in terms of the two-point function S_β . This technical step involves intricate diagrams controlling the $\pi_\beta^{(j)}$, whose form is model-dependent. In the case of the Ising model, it is possible to obtain the following estimate [216, Proposition 1.2]: there exist $C_1, C_2 > 0$ which depend on C and d such that for all $x \in \mathbb{Z}^d$,

$$|\Pi_\beta(x) - \delta_0(x)| \leq \frac{C_1}{d-4} + \frac{C_2(1 - \delta_0(x))}{\|x\|^{d+2+\rho}}, \quad (1.3.17)$$

where $\rho = 2(d-4)$.

Step 2.2 Improving the a priori estimate: The second step of the bootstrap argument consists of injecting (1.3.17) in the convolution equation (1.3.14) to improve the initial estimate on the two-point function. Namely, we obtain (for a good choice of C) that, for all $x \in \mathbb{Z}^d$,

$$S_\beta(x) \leq C\mathbb{G}(x).$$

Step 2.3 Conclusion: Thanks to exponential decay below criticality, one can prove that the function

$$f : \beta \mapsto \sup_{x \in \mathbb{Z}^d} \frac{S_\beta(x)}{\mathbb{G}(x)}$$

is continuous over $[0, \beta_c)$ and satisfies $f(0) = 1$. By the two preceding sub-steps, for a well suited $C > 0$ and any $\beta < \beta_c$, if $f(\beta) \leq 2C$ then $f(\beta) \leq C$. By continuity, this forces

$$f(\beta) \leq C \quad \forall \beta < \beta_c.$$

As a consequence, for every $\beta < \beta_c$ and every $x \in \mathbb{Z}^d$,

$$S_\beta(x) \leq C\mathbb{G}(x). \quad (1.3.18)$$

The monotone convergence theorem then allows us to extend the above bound to $\beta = \beta_c$.

Step 3 Deconvolution: Using (1.3.18), it is possible to derive estimates on Π_{β_c} (using Step 2.1) and to make sense of (1.3.14) for all $\beta \leq \beta_c$. From there various strategies have been proposed to deconvolve (1.3.14). Hara's *Gaussian lemma* [132] provides a sufficient condition to recover Gaussian estimates, but its proof is a little opaque. In a recent work, Liu and Slade [181] proposed an elementary alternative based on simple Fourier analysis and Hölder's inequality.

The above strategy has to be taken with a pinch of salt. One major limitation of the lace expansion lies in the derivation of Step 2.2 where it is often required to identify a *small* parameter in the problem. In the case of the Ising model, Sakai's results are limited to the situation where $(d-4)^{-1} \ll 1$. Finally, let us mention that unlike renormalization group, the lace expansion is not suited to studying a model at its upper-critical dimension.

1.3.3.4 What about Bernoulli percolation?

The above results have analogous forms in the case of Bernoulli percolation. It is possible to obtain a set of differential inequalities that are comparable to (1.3.6)–(1.3.9). In fact, a number of results were obtained in parallel for the Ising model and Bernoulli percolation in the 1980s. To give only two examples,

$$\begin{aligned} [1 - (\nabla(p) - 1)]2d\chi(p)^2 &\leq \frac{\partial\chi}{\partial p} \leq 2d\chi(p)^2, & \forall p < p_c, \\ m(p, h) &\leq h \frac{\partial m}{\partial h} + m(p, h)^2 + pm(p, h) \frac{\partial m}{\partial p}, & \forall p \in (0, 1), \forall h > 0, \end{aligned}$$

where $\nabla(p)$ is the so-called *triangle diagram* (see Figure 1.12 for an illustration) defined by

$$\nabla(p) := \sum_{x, y \in \mathbb{Z}^d} \mathbb{P}_p[0 \leftrightarrow x] \mathbb{P}_p[x \leftrightarrow y] \mathbb{P}_p[y \leftrightarrow 0]. \quad (1.3.19)$$

The identification of the triangle diagram as a key quantity in the study of the mean-field regime of the model is due to Aizenman and Newman [16]. Under the *triangle condition* $\nabla(p_c) < \infty$, it is possible to recover mean-field values for the critical exponents β , δ , and γ , see [16, 7, 23].

A rigorous implementation of renormalization group methods for Bernoulli percolation is not yet available. On the basis of the deep connection of the model to the φ^3 model, one can however non-rigourously derive the values of some critical exponents using RG methods, see for instance [42]. No rigorous connections⁴⁴ between φ^3 and Bernoulli percolation exist. Nevertheless, hints of a φ^3 structure can be obtained in mathematical works: the tree graph bounds of [16] have vertices of degree 3, and so do the lace expansion diagrams of [134].

Finally, lace expansion has been applied in the context of Bernoulli percolation to derive finiteness of the triangle diagram in sufficiently large dimensions in the seminal work of Hara and Slade [134]. Since then, the techniques have been optimised⁴⁵ [133, 132, 100] to reach mean-field behaviour in dimensions $d > 10$. It is a major challenge in statistical mechanics to develop new tools to cover the full predicted mean-field regime of Bernoulli percolation.

Conclusion Mathematicians are yet to catch up with physicists regarding the study of the mean-field regime. Known methods are often restricted to perturbative situations (e.g. the weakly-coupled φ^4 model) or specific setups for which additional tools are available (e.g. reflection positivity).

⁴⁴The very definition of the φ^3 model is problematic as the function $e^{-\varphi^3}$ is not integrable on $\mathbb{R}$.

⁴⁵A fundamental feature of these proofs is the fact that they are often computer assisted.

1.4 Organisation of the thesis

Our goal is twofold. First, we propose a novel approach to the study of the mean-field regime of models of statistical mechanics. Second, we use existing geometric representations (in particular, the random current representation) to provide some rigorous answers to the conjectural picture described above. Let us provide a brief overview of the content of each chapter.

Chapter 2 is an introduction to the random current representation of the Ising model. Our aim is to illustrate the power of geometric representations in statistical mechanics. We present some of the main results derived using random currents: sharpness of the phase transition, continuity of the magnetisation, triviality of the scaling limits in dimensions $d \geq 4$, etc.

Part I: Alternative approach to the study of the mean-field regime in statistical mechanics. Chapters 3 and 4 form the first part of this thesis. We develop new techniques to study the mean-field regime of weakly self-avoiding walks and Bernoulli percolation in sufficiently large dimensions. These chapters are based on joint works with Hugo Duminil-Copin.

In Chapter 3, we consider the weakly self-avoiding walk model on $\mathbb{Z}^d$ with $d > 4$. This model has received significant attention over the past forty years, as it serves as a natural testbed to develop new tools for the study of the mean-field regime in statistical mechanics. Indeed, its definition includes a parameter λ that can be chosen to be small, and which plays the role of a *perturbative* parameter. The lace expansion method has been successfully applied to this model to prove mean-field behaviour in dimensions $d > 4$. Using a completely novel approach which builds on a “one-step lace expansion” and on classical random walk results, we prove up-to-constant estimates for the (near) critical two-point function in the full-space and in the half-space.

In Chapter 4, we apply a similar strategy to establish mean-field behaviour of sufficiently *spread-out* Bernoulli percolation in dimensions $d > 6$. Here, the term “spread-out” refers to the fact that we allow for long (but finite) range edges between points at distance $\leq L$. The quantity L^{-1} plays the role of a perturbative parameter. This model has been successfully studied using lace expansion techniques by Hara and Slade. Our proof is very similar to the one for the weakly self-avoiding walk model presented in Chapter 3. However, the long edges require particular care.

Part II: The Ising model. Chapters 5 to 9 contain a more elaborate study of the mean-field regime of the Ising model. They build on the techniques presented in Chapter 2.

In Chapter 5, we construct the so-called *incipient infinite cluster* (IIC) for the Fortuin–Kasteleyn representation of the n.n.f Ising model in dimensions $d \geq 3$. The two-dimensional counterpart of this object played a pivotal role in the study of the critical behaviour of the planar random-cluster model. We also construct the IIC of the random current model. We then improve the near-critical understanding of the susceptibility $\chi(\beta)$ and show that in dimensions $d \geq 5$, there exists $A = A(d)$ such that, for $\beta < \beta_c$,

$$\chi(\beta) = \frac{A}{1 - \beta/\beta_c}(1 + o(1)), \quad (1.4.1)$$

where $o(1)$ tends to 0 as β tends to β_c . Additionally, we relate A to the IIC of the *double* random current.

Chapters 6 and 7 are joint works with Hugo Duminil-Copin. In Chapter 6, we obtain new lower bounds on the (near) critical two-point function of the n.n.f Ising model in dimensions $d \geq 3$. These bounds complement the so-called infrared bound. In particular, we obtain the mean-field critical exponents $\eta = 0$ and $\nu = \frac{1}{2}$ in dimensions $d \geq 4$. When $d = 3$, we obtain that, if it exists, then $\eta \leq \frac{1}{2}$. Moreover, we extend these results to the setup of the n.n.f φ^4 model. In Chapter 7, we build on the results on Chapter 6 and establish the existence of the scaling limit of the critical Ising model in dimensions $d \geq 5$. We identify the limit to be the *massless Gaussian free field* on $\mathbb{R}^d$. This confirms a prediction from physics and extends a previous series of works by Sakai who derived a similar result in sufficiently large dimensions using the lace expansion. Our proof relies on the introduction of a novel *effective random walk* that relates the two-point function of the model at two different parameters $\beta' < \beta$.

Chapter 8 is devoted to the study of the *torus plateau phenomenon* in the setup of the n.n.f Ising model in dimensions $d \geq 5$. It is a joint work with Yucheng Liu and Gordon Slade. The torus plateau refers to the observation that, near criticality, the periodic boundary condition *boosts* the two-point function of the model in such a way that above a certain scale it *levels* and stabilises at a *plateau* (hence stops decreasing). As a consequence, the torus susceptibility adopts an unusual growth that is much faster than that of the finite volume susceptibility on $\mathbb{Z}^d$. Also, we obtain the non-Gaussianity of the scaling limit of the average spin on the torus. This may seem surprising in view of the triviality results mentioned in Section 1.3.

In Chapter 9, we study long-range Ising models on $\mathbb{Z}^d$ with $d \geq 1$. As explained in Section 1.3, it is possible to choose the interaction in such a way that the *effective* upper-critical dimension of the model matches the ambient dimension. This means that we may expect to recover some mean-field features in dimensions $d \in \{1, 2, 3\}$. Our main result is the extension of the breakthrough work of Aizenman and Duminil-Copin to the setup of models of effective dimensions $d_{\text{eff}} \geq 4$.

Chapter 10 is a joint work with Trishen S. Gunaratnam. We investigate subcritical long-range Ising models on $\mathbb{Z}^d$. It is well-known that the scaling limit of such models is the *white noise*. We study the fluctuations of the spin field around the white noise and relate them to explicit *fractional* Gaussian free fields.

Part III: The φ^4 model. Chapters 11 and 12 form the last part of this thesis. They focus on the study of the discrete φ^4 model.

Chapter 11 is a joint work with Trishen S. Gunaratnam, Christoforos Panagiotis, and Franco Severo. We develop a new geometric representation of the φ^4 model called the *random tangled current representation*. This geometric representation is analogous to the random current representation of the Ising model. In particular, we prove a *switching lemma* that provides geometric interpretations of ratios and differences of correlation functions. This combinatorial tool allows us to extend the analysis of Gibbs measures previously achieved by Raoufi in the setup of the Ising model. As a byproduct of our results, we derive the continuity of the magnetisation of the n.n.f φ^4 model in dimensions $d \geq 3$.

Chapter 12 is a joint work with Dmitrii Krachun and Christoforos Panagiotis. We study the scaling limit of the single and double random current on the complete graph. We use this analysis to fully characterise the so-called *tangling probabilities* appearing in the statement

of the switching lemma for φ^4 . This provides an interesting interpretation of the switching lemma as a *Wick rule* for non-Gaussian fields.

Chapter 2

The random current representation of the Ising model

Geometric representations (and in particular path expansions) of the models of study play a central role in this thesis. In order to illustrate the full power of these tools, we introduce the so-called *random current representation* of the Ising model. This geometric representation will be used extensively in Part II. Its main power lies in a combinatorial tool called the *switching lemma*, which provides percolation interpretations of differences and ratios of correlation functions. Random currents were initially introduced by Griffiths, Hurst and Sherman [120] to prove concavity of the magnetisation with an external magnetic field. However, the full probabilistic power attached to this geometric representation flourished in the seminal paper of Aizenman [4] and the many subsequent works.

The objective of this chapter is not to provide a full survey of random currents but rather to display some major achievements its use has led to. We refer to the lecture notes [76] and to Part II for more information.

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2.1 Setup

We introduce random currents in a general setup. Below, $\mathcal{G} = (V, \mathcal{E})$ is a fixed locally finite and countable infinite graph. We will sometimes specify to the case of the hypercubic lattice $\mathbb{Z}^d$ with edge-set $\mathbb{E} := \{\{x, y\} : |x - y|_1 = 1\}$, where $|\cdot|_1$ denotes the ℓ^1 norm on $\mathbb{R}^d$.

Let G be a finite subset of V . We define the Ising model on G as follows. Fix $J = (J_{x,y})_{\{x,y\} \subset G}$ a family¹ of positive real numbers and $\mathbf{h} = (\mathbf{h}_x)_{x \in V} \in \mathbb{R}^V$ a *non-homogeneous* external magnetic field. For $\sigma \in \{-1, 1\}^G$, introduce the *Hamiltonian*

$$H_{G,J,\mathbf{h}}(\sigma) := - \sum_{\{x,y\} \subset G} J_{x,y} \sigma_x \sigma_y - \sum_{x \in G} \sigma_x \mathbf{h}_x. \quad (2.1.1)$$

The associated *finite volume Gibbs equilibrium state* $\langle \cdot \rangle_{G,J,\beta,\mathbf{h}}$ at inverse temperature $\beta \geq 0$ to be the probability measure is defined, for each $F : \{\pm 1\}^G \rightarrow \mathbb{R}$, by

$$\langle F \rangle_{G,J,\beta,\mathbf{h}} := \frac{1}{Z(G, J, \beta, \mathbf{h})} \sum_{\sigma \in \{\pm 1\}^G} F(\sigma) \exp(-\beta H_{G,J,\mathbf{h}}(\sigma)), \quad (2.1.2)$$

where

$$Z(G, J, \beta, \mathbf{h}) := \sum_{\sigma \in \{\pm 1\}^G} \exp(-\beta H_{G,J,\mathbf{h}}(\sigma)) \quad (2.1.3)$$

is the *partition function* of the model.

It is possible to encode any boundary condition $\tau \in \{-1, 0, 1\}^V$ in the above by setting, for $x \in G$,

$$\mathbf{h}_x := \sum_{y \notin G} J_{x,y} \tau_y. \quad (2.1.4)$$

In particular, in the case of the *plus* boundary condition, we let

$$\mathbf{h}_x^+ := \sum_{y \notin G} J_{x,y}. \quad (2.1.5)$$

For convenience, we often omit writing J in the above notations. When $\mathbf{h} \equiv 0$, we also drop it from the notations. For $A \subset G$ we write

$$\sigma_A := \prod_{x \in A} \sigma_x. \quad (2.1.6)$$

Remark 2.1.1. The above definition corresponds to the Ising model on the *complete* graph $(G, \mathcal{P}_2(G))$ where $\mathcal{P}_2(G) := \{\{x, y\} : x, y \in G, x \neq y\}$. Below, it will be useful to allow for restricted set of edges in the definition. For $E \subset \mathcal{P}_2(G)$ a set of edges, we denote by $\langle \cdot \rangle_{E,\beta,\mathbf{h}}$ the Ising measure on G with pair interaction restricted to elements of E in the definition of the Hamiltonian. In particular, one has $\langle \cdot \rangle_{\mathcal{P}_2(G),\beta,\mathbf{h}} = \langle \cdot \rangle_{G,\beta,\mathbf{h}}$.

2.2 The expansion

Let G be a finite subset of V .

¹The particular choice $J_{x,y} = \mathbb{1}_{\{\{x,y\} \in E\}}$ corresponds to the nearest-neighbour ferromagnetic Ising model.

2.2.1 Definitions

Definition 2.2.1 (Griffiths' ghost). We define $G_{\mathfrak{g}}$ to be the set $G \cup \{\mathfrak{g}\}$. The additional vertex $\mathfrak{g}$ is called the *ghost*. It naturally appears in the presence of an external magnetic field $\mathbf{h}$. By convention, we will always assume that the ghost vertex carries a +1 spin, i.e. $\sigma_{\mathfrak{g}} = 1$.

Definition 2.2.2. A *current* (or *current configuration*) $\mathbf{n}$ on $G_{\mathfrak{g}}$ is an integer valued function defined on the set $\mathcal{P}_2(G_{\mathfrak{g}}) := \{\{x, y\}, x, y \in G_{\mathfrak{g}}\}$. We write $\mathbf{n}_{x,y} = \mathbf{n}_{xy} = \mathbf{n}(\{x, y\})$ for the value of a current $\mathbf{n}$ on the pair $\{x, y\}$. We denote by $\Omega_{G_{\mathfrak{g}}}$ the set of currents on $G_{\mathfrak{g}}$. The set of *sources* of $\mathbf{n}$, denoted by $\partial\mathbf{n}$, is defined as

$$\partial\mathbf{n} := \left\{x \in G : \sum_{y \in G_{\mathfrak{g}}} \mathbf{n}_{xy} \text{ is odd}\right\}. \quad (2.2.1)$$

Note that by convention we do not include $\mathfrak{g}$ in $\partial\mathbf{n}$. We also define the weight

$$w_{\beta, \mathbf{h}}(\mathbf{n}) := \prod_{\{x, y\} \subset G} \frac{(\beta J_{x,y})^{\mathbf{n}_{xy}}}{\mathbf{n}_{xy}!} \prod_{x \in G} \frac{(\beta \mathbf{h}_x)^{\mathbf{n}_{x\mathfrak{g}}}}{\mathbf{n}_{x\mathfrak{g}}!}. \quad (2.2.2)$$

Remark 2.2.3. In view of Remark 2.1.1, for $E \subset \mathcal{P}_2(G)$, we define Ω_E to be the set of currents of E .

The trace of a current $\mathbf{n}$ induces a percolation configuration $\hat{\mathbf{n}} := (\mathbf{1}_{\mathbf{n}_{xy} > 0})_{\{x, y\} \in \mathcal{P}_2(G_{\mathfrak{g}})}$. As it turns out, this percolation configuration will play a very important role later on. This motivates the following terminology.

Definition 2.2.4. Let $\mathbf{n} \in \Omega_{G_{\mathfrak{g}}}$ and $x, y \in G_{\mathfrak{g}}$.

- (i) We say that x is connected to y in $\mathbf{n}$ and write $x \xleftrightarrow{\mathbf{n}} y$, if there exists a sequence of points $x_0 = x, x_1, \dots, x_m = y$ such that $\mathbf{n}_{x_i x_{i+1}} > 0$ for $0 \leq i \leq m-1$.
- (ii) The cluster of x , denoted by $\mathbf{C}_{\mathbf{n}}(x)$, is the set of points connected to x in $\mathbf{n}$.

A current $\mathbf{n}$ with source set $\partial\mathbf{n} = \{x, y\}$ can be seen as a path between x and y together with a collection of “loops”. We give an illustration in Figure 2.1.

There is a way to expand the correlation functions of the Ising model that links them to currents. Indeed, using the following expansions

$$\exp(\beta J_{x,y} \sigma_x \sigma_y) = \sum_{\mathbf{n}_{xy} \geq 0} \frac{(\beta J_{x,y} \sigma_x \sigma_y)^{\mathbf{n}_{xy}}}{\mathbf{n}_{xy}!}, \quad \exp(\beta \mathbf{h}_x \sigma_x) = \sum_{\mathbf{n}_{x\mathfrak{g}} \geq 0} \frac{(\beta \mathbf{h}_x \sigma_x)^{\mathbf{n}_{x\mathfrak{g}}}}{\mathbf{n}_{x\mathfrak{g}}!} \quad (2.2.3)$$

we obtain that

$$Z(G, \beta, \mathbf{h}) = \sum_{\mathbf{n} \in \Omega_{G_{\mathfrak{g}}}} w_{\beta, \mathbf{h}}(\mathbf{n}) \sum_{\sigma \in \{\pm 1\}^G} \prod_{x \in G} \sigma_x^{\sum_{y \in G_{\mathfrak{g}}} \mathbf{n}_{xy}}. \quad (2.2.4)$$

The sign-flip symmetry implies that

$$\sum_{\sigma \in \{\pm 1\}^G} \prod_{x \in G} \sigma_x^{\sum_{y \in G_{\mathfrak{g}}} \mathbf{n}_{xy}} = 2^{|G|} \mathbf{1}_{\{\partial\mathbf{n} = \emptyset\}}, \quad (2.2.5)$$

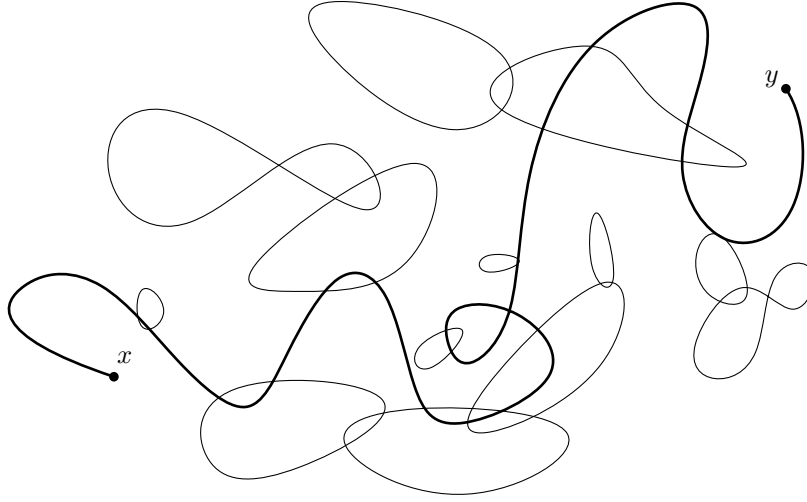


Figure 2.1: An illustration of the trace of a current configuration $\mathbf{n}$ with $\partial\mathbf{n} = \{x, y\}$. The bold line represents the *backbone* of $\mathbf{n}$, which is an exploration of one of the paths from x to y , see Section 2.3.

which leads to

$$Z(G, \beta, \mathbf{h}) = 2^{|G|} \sum_{\partial\mathbf{n}=\emptyset} w_{\beta, \mathbf{h}}(\mathbf{n}). \quad (2.2.6)$$

More generally, the correlation functions are given by, for $A \subset G$,

$$\langle \sigma_A \rangle_{G, \beta} = \frac{\sum_{\partial\mathbf{n}=A} w_{\beta, \mathbf{h}}(\mathbf{n})}{\sum_{\partial\mathbf{n}=\emptyset} w_{\beta, \mathbf{h}}(\mathbf{n})}. \quad (2.2.7)$$

Remark 2.2.5. One can recover Griffiths' *first* inequality: if $\mathbf{h}_x \geq 0$ for all $x \in G$ then the weights $w_{\beta, \mathbf{h}}(\mathbf{n})$ are positive and by (2.2.7),

$$\langle \sigma_A \rangle_{G, \beta, \mathbf{h}} \geq 0. \quad (2.2.8)$$

The formula in (2.2.7) involves a ratio of sums of currents with different sources conditions. We do not have a probabilistic interpretation yet. The next result is a fundamental tool to manipulate currents. It will provide a general recipe to give probabilistic interpretation to (differences and ratios of) correlation functions.

2.2.2 The switching lemma

The following combinatorial result first appeared in [120], but the probabilistic picture attached to it was developed in [4].

To give the most general statement, we consider $H_1 = (V_1, E_1), H_2 = (V_2, E_2)$ **subgraphs** of $(G_{\mathfrak{g}}, \mathcal{P}_2(G_{\mathfrak{g}}))$. We also assume that H_2 is a subgraph of H_1 . If $\mathbf{n}$ is a current defined on a subset H_2 , we can extend it to a current on H_1 by setting $\mathbf{n}_{xy} = 0$ for all edges in $E_1 \setminus E_2$. This extension is still denoted $\mathbf{n}$. Given a current $\mathbf{n} \in \Omega_{E_1}$, we can also consider its restriction $\mathbf{n}|_{E_2}$ to edges in E_2 .

Lemma 2.2.6 (Switching lemma). *For any $A \subset V_1, B \subset V_2$ and any function F from the set of currents into $\mathbb{R}$,*

$$\begin{aligned} & \sum_{\substack{\mathbf{n}_1 \in \Omega_{E_1} : \partial \mathbf{n}_1 = A \\ \mathbf{n}_2 \in \Omega_{E_2} : \partial \mathbf{n}_2 = B}} F(\mathbf{n}_1 + \mathbf{n}_2) w_{\beta, \mathbf{h}}(\mathbf{n}_1) w_{\beta, \mathbf{h}}(\mathbf{n}_2) \\ &= \sum_{\substack{\mathbf{n}_1 \in \Omega_{E_1} : \partial \mathbf{n}_1 = A \Delta B \\ \mathbf{n}_2 \in \Omega_{E_2} : \partial \mathbf{n}_2 = \emptyset}} F(\mathbf{n}_1 + \mathbf{n}_2) w_{\beta, \mathbf{h}}(\mathbf{n}_1) w_{\beta, \mathbf{h}}(\mathbf{n}_2) \mathbb{1}\{(\mathbf{n}_1 + \mathbf{n}_2)|_{E_2} \in \mathcal{F}_B^{E_2}\} \end{aligned} \quad (2.2.9)$$

where $A \Delta B = (A \cup B) \setminus (A \cap B)$ is the symmetric difference of sets and $\mathcal{F}_B^{E_2}$ is the subset of Ω_{E_2} defined by

$$\mathcal{F}_B^{E_2} := \{\mathbf{n} \in \Omega_{E_2} : \exists \mathbf{m} \leq \mathbf{n}, \partial \mathbf{m} = B\}. \quad (2.2.10)$$

Proof. Notice that if $\mathbf{m} = \mathbf{n}_1 + \mathbf{n}_2$ and $\mathbf{n} = \mathbf{n}_2$, then

$$w_{\beta, \mathbf{h}}(\mathbf{n}_1) w_{\beta, \mathbf{h}}(\mathbf{n}_2) = w_{\beta, \mathbf{h}}(\mathbf{m}) \begin{bmatrix} \mathbf{m} \\ \mathbf{n} \end{bmatrix}, \quad (2.2.11)$$

where

$$\begin{bmatrix} \mathbf{m} \\ \mathbf{n} \end{bmatrix} := \prod_{xy \in E_1} \binom{\mathbf{m}_{xy}}{\mathbf{n}_{xy}}. \quad (2.2.12)$$

The equality we want to prove is then equivalent to

$$\sum_{\substack{\mathbf{m} \in \Omega_{E_1} \\ \partial \mathbf{m} = A \Delta B}} F(\mathbf{m}) w_{\beta, \mathbf{h}}(\mathbf{m}) \sum_{\substack{\mathbf{n} \leq \mathbf{m} : \partial \mathbf{n} = B \\ \mathbf{n} \in \Omega_{E_2}}} \begin{bmatrix} \mathbf{m} \\ \mathbf{n} \end{bmatrix} = \sum_{\substack{\mathbf{m} \in \Omega_{E_1} \\ \partial \mathbf{m} = A \Delta B}} F(\mathbf{m}) w_{\beta, \mathbf{h}}(\mathbf{m}) \mathbb{1}\{\mathbf{m}|_{E_2} \in \mathcal{F}_B^{E_2}\} \sum_{\substack{\mathbf{n} \leq \mathbf{m} : \partial \mathbf{n} = \emptyset \\ \mathbf{n} \in \Omega_{E_2}}} \begin{bmatrix} \mathbf{m} \\ \mathbf{n} \end{bmatrix}. \quad (2.2.13)$$

It is thus sufficient to prove that for every $\mathbf{m} \in \Omega_{E_1}$ with $\partial \mathbf{m} = A \Delta B$,

$$\sum_{\substack{\mathbf{n} \leq \mathbf{m} : \partial \mathbf{n} = B \\ \mathbf{n} \in \Omega_{E_2}}} \begin{bmatrix} \mathbf{m} \\ \mathbf{n} \end{bmatrix} = \mathbb{1}\{\mathbf{m}|_{E_2} \in \mathcal{F}_B^{E_2}\} \sum_{\substack{\mathbf{n} \leq \mathbf{m} : \partial \mathbf{n} = \emptyset \\ \mathbf{n} \in \Omega_{E_2}}} \begin{bmatrix} \mathbf{m} \\ \mathbf{n} \end{bmatrix}. \quad (2.2.14)$$

To prove this equality, observe that if $\mathbf{n}$ is a current, it induces a multigraph $\mathcal{N}$: if $xy \in E_1$, we put $\mathbf{n}_{xy}$ edges between x and y in $\mathcal{N}$. In this representation, $\partial \mathcal{N}$ is the set of vertices of odd degree. This representation is very useful in our setup as (2.2.14) rewrites

$$|\{\mathcal{N} \subset \mathcal{M} : \partial \mathcal{N} = B\}| = \mathbb{1}\{\mathbf{m}|_{E_2} \in \mathcal{F}_B^{E_2}\} |\{\mathcal{N} \subset \mathcal{M} : \partial \mathcal{N} = \emptyset\}|. \quad (2.2.15)$$

If $\mathbb{1}\{\mathbf{m}|_{E_2} \in \mathcal{F}_B^{E_2}\} = 0$, then the left term vanishes. We can then assume that $\mathbb{1}\{\mathbf{m}|_{E_2} \in \mathcal{F}_B^{E_2}\} = 1$. In this case, we denote by $\mathcal{K}_0$ a subgraph of $\mathcal{M}$ such that $\partial \mathcal{K}_0 = B$ ($\mathcal{K}_0$ is a multigraph on H). The map $\mathcal{N} \mapsto \mathcal{N} \Delta \mathcal{K}_0$ is a bijection (an involution) between $\{\mathcal{N} \subset \mathcal{M} : \partial \mathcal{N} = B\}$ and $\{\mathcal{N} \subset \mathcal{M} : \partial \mathcal{N} = \emptyset\}$, which yields the result. $\square$

As we just saw, if $\mathcal{K} \subset \mathcal{M}$, the crux of the proof lies in the fact that the map $\mathcal{N} \mapsto \mathcal{N} \Delta \mathcal{K}$ is an involution from $\{\mathcal{N} \subset \mathcal{M} : \partial \mathcal{N} = A\}$ to $\{\mathcal{N} \subset \mathcal{M} : \partial \mathcal{N} = A \Delta \partial \mathcal{K}\}$. This leads to the slightly more general formulation of the above result, called the *switching principle*.

Lemma 2.2.7 (Switching principle). *For any multigraph $\mathcal{M}$ with vertex set G , any $A \subset G$, and any function f of a current,*

$$\sum_{\substack{\mathcal{N} \subset \mathcal{M} \\ \partial \mathcal{N} = A}} f(\mathcal{N}) = \mathbf{1}\{\exists \mathcal{K} \subset \mathcal{M} : \partial \mathcal{K} = A\} \sum_{\substack{\mathcal{N} \subset \mathcal{M} \\ \partial \mathcal{N} = \emptyset}} f(\mathcal{N} \Delta \mathcal{K}). \quad (\text{SP})$$

2.2.3 First applications

We now turn to some easy applications of the switching lemma. All we did so far was purely deterministic. The next definition provides a probabilistic picture.

Definition 2.2.8 (Probability measure on Ω_{G_g}). Given $\mathbf{h} = (\mathbf{h}_x)_{x \in V}$ a non-homogeneous positive external field and $A \subset G_g$, we define a probability measure on Ω_{G_g} by the formula

$$\mathbf{P}_{G_g, \beta, \mathbf{h}}^A[\mathbf{n}] := \frac{2^{|G|} w_\beta(\mathbf{n}) \mathbf{1}\{\partial \mathbf{n} = A\}}{\langle \sigma_A \rangle_{G, \beta, \mathbf{h}} Z(G, \beta, \mathbf{h})} = \frac{\mathbf{1}\{\partial \mathbf{n} = A\} w_{\beta, \mathbf{h}}(\mathbf{n})}{\sum_{\partial \mathbf{m} = A} w_{\beta, \mathbf{h}}(\mathbf{m})} \quad \forall \mathbf{n} \in \Omega_{G_g}, \quad (2.2.16)$$

and we denote $\mathbf{E}_{G_g, \beta, \mathbf{h}}^A$ the expectation with respect to $\mathbf{P}_{G_g, \beta, \mathbf{h}}^A$. When $A = \{x, y\}$ is a pair, we write $\mathbf{P}_{G_g, \beta, \mathbf{h}}^{xy}$ instead of $\mathbf{P}_{G_g, \beta, \mathbf{h}}^{\{x, y\}}$. If $A_1, \dots, A_n \subset G_g$, we write

$$\mathbf{P}_{G_g, \beta, \mathbf{h}}^{A_1, \dots, A_n} := \mathbf{P}_{G_g, \beta, \mathbf{h}}^{A_1} \otimes \dots \otimes \mathbf{P}_{G_g, \beta, \mathbf{h}}^{A_n}. \quad (2.2.17)$$

We introduce a useful notation for the *partition function* of the above measures by setting for $\mathcal{E} \subset \Omega_{G_g}^n$,

$$Z_{G_g, \beta, \mathbf{h}}^{A_1, \dots, A_n}[\mathcal{E}] := \sum_{\partial \mathbf{n}_1 = A_1, \dots, \partial \mathbf{n}_n = A_n} w_{\beta, \mathbf{h}}(\mathbf{n}_1) \dots w_{\beta, \mathbf{h}}(\mathbf{n}_n) \mathbf{1}\{(\mathbf{n}_1, \dots, \mathbf{n}_n) \in \mathcal{E}\}. \quad (2.2.18)$$

When $\mathbf{h} \equiv 0$, we write $\mathbf{P}_{G, \beta}^A$ instead of $\mathbf{P}_{G_g, \beta, 0}^A$; and in the case of the *plus* boundary condition with no additional external magnetic field we write $\mathbf{P}_{G_g, \beta}^A$ instead of $\mathbf{P}_{G_g, \beta, \mathbf{h}^+}^A$.

We now turn to immediate consequences of the switching lemma of Lemma 2.2.6.

Proposition 2.2.9. *Let G be a finite subset of V . Let $\beta \geq 0$ and $\mathbf{h} = (\mathbf{h}_x)_{x \in V} \in \mathbb{R}_+^V$.*

(i) *For $A, B \subset G_g$,*

$$\frac{\langle \sigma_A \rangle_{G, \beta, \mathbf{h}} \langle \sigma_B \rangle_{G, \beta, \mathbf{h}}}{\langle \sigma_A \sigma_B \rangle_{G, \beta, \mathbf{h}}} = \mathbf{P}_{G_g, \beta, \mathbf{h}}^{A \Delta B, \emptyset}[\mathbf{n}_1 + \mathbf{n}_2 \in \mathcal{F}_B] \quad (2.2.19)$$

and in particular, for $x, y, u \in G$,

$$\frac{\langle \sigma_x \sigma_u \rangle_{G, \beta, \mathbf{h}} \langle \sigma_u \sigma_y \rangle_{G, \beta, \mathbf{h}}}{\langle \sigma_x \sigma_y \rangle_{G, \beta, \mathbf{h}}} = \mathbf{P}_{G_g, \beta, \mathbf{h}}^{xy, \emptyset}[x \xrightarrow{\mathbf{n}_1 + \mathbf{n}_2} u]. \quad (2.2.20)$$

(ii) *For $x, y \in G_g$,*

$$\langle \sigma_x \sigma_y \rangle_{G, \beta, \mathbf{h}}^2 = \mathbf{P}_{G_g, \beta, \mathbf{h}}^{\emptyset, \emptyset}[x \xrightarrow{\mathbf{n}_1 + \mathbf{n}_2} y]. \quad (2.2.21)$$

In particular this formula gives a nice interpretation of the so-called bubble diagram $B_{G, \beta, \mathbf{h}}(\cdot)$ of the Ising model,

$$B_{G, \beta, \mathbf{h}}(x) := \sum_{u \in G} \langle \sigma_x \sigma_u \rangle_{G, \beta, \mathbf{h}}^2 = \mathbf{E}_{G_g, \beta, \mathbf{h}}^{\emptyset, \emptyset}[\mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_2}(x)]. \quad (2.2.22)$$

Proof. We begin with (i). It is sufficient to prove (2.2.19) since (2.2.20) corresponds to $A = \{x, u\}$ and $B = \{u, y\}$. By Lemma 2.2.6, one has

$$Z_{G_{\mathfrak{g}}, \beta, \mathbf{h}}^{A, B} = \sum_{\substack{\partial \mathbf{n}_1 = A \Delta B \\ \partial \mathbf{n}_2 = \emptyset}} w_{\beta, \mathbf{h}}(\mathbf{n}_1) w_{\beta, \mathbf{h}}(\mathbf{n}_2) \mathbf{1}\{\mathbf{n}_1 + \mathbf{n}_2 \in \mathcal{F}_B\} = Z_{G_{\mathfrak{g}}, \beta, \mathbf{h}}^{A \Delta B, \emptyset}[\mathbf{n}_1 + \mathbf{n}_2 \in \mathcal{F}_B]. \quad (2.2.23)$$

Using Definition 2.2.8 one gets

$$\langle \sigma_A \rangle_{G, \beta, \mathbf{h}} \langle \sigma_B \rangle_{G, \beta, \mathbf{h}} = \langle \sigma_A \sigma_B \rangle_{G, \beta, \mathbf{h}} \mathbf{P}_{G_{\mathfrak{g}}, \beta, \mathbf{h}}^{A \Delta B, \emptyset}[\mathbf{n}_1 + \mathbf{n}_2 \in \mathcal{F}_B]. \quad (2.2.24)$$

We turn to the proof of (ii). This is a direct consequence of (i) with $A = B = \{x, y\}$. To obtain the formula for the bubble diagram, simply notice that

$$|\mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_2}(x)| = \sum_{u \in G} \mathbf{1}\{u \in \mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_2}(x)\}. \quad (2.2.25)$$

□

The switching lemma provides an easy proof of Griffiths' second inequality.

Proposition 2.2.10 (Griffiths' second inequality). *Let $A, B \subset G$, $\beta \geq 0$, and $\mathbf{h} = (\mathbf{h}_x)_{x \in V} \in \mathbb{R}_+^V$. Then,*

$$\langle \sigma_A \sigma_B \rangle_{G, \beta, \mathbf{h}} \geq \langle \sigma_A \rangle_{G, \beta, \mathbf{h}} \langle \sigma_B \rangle_{G, \beta, \mathbf{h}}. \quad (2.2.26)$$

Proof. Using (2.2.19) we get

$$1 - \frac{\langle \sigma_A \rangle_{G, \beta, \mathbf{h}} \langle \sigma_B \rangle_{G, \beta, \mathbf{h}}}{\langle \sigma_A \sigma_B \rangle_{G, \beta, \mathbf{h}}} = \mathbf{P}_{G_{\mathfrak{g}}, \beta, \mathbf{h}}^{A \Delta B, \emptyset}[\mathbf{n}_1 + \mathbf{n}_2 \notin \mathcal{F}_B] \geq 0, \quad (2.2.27)$$

which yields the result. □

We will need a notation for the *truncated* correlation functions: if $A, B \subset G$,

$$\langle \sigma_A; \sigma_B \rangle_{G, \beta, \mathbf{h}} := \langle \sigma_A \sigma_B \rangle_{G, \beta, \mathbf{h}} - \langle \sigma_A \rangle_{G, \beta, \mathbf{h}} \langle \sigma_B \rangle_{G, \beta, \mathbf{h}}. \quad (2.2.28)$$

Note that $\langle \sigma_A; \sigma_B \rangle_{G, \beta, \mathbf{h}} \geq 0$ by Proposition 2.2.10.

Proposition 2.2.11. *Let $A \subset G$, $x, y \in G$, $\beta \geq 0$, and $\mathbf{h} = (\mathbf{h}_x)_{x \in V} \in \mathbb{R}_+^V$. Then,*

$$\frac{\partial}{\partial \beta} \langle \sigma_A \rangle_{G, \beta, \mathbf{h}} \geq 0, \quad \frac{\partial}{\partial J_{x, y}} \langle \sigma_A \rangle_{G, J, \beta, \mathbf{h}} \geq 0, \quad \frac{\partial}{\partial \mathbf{h}_x} \langle \sigma_A \rangle_{G, \beta, \mathbf{h}} \geq 0. \quad (2.2.29)$$

In particular, if $\mathbf{h} = \mathbf{h}^+$,

$$\frac{\partial}{\partial \beta} \langle \sigma_x \rangle_{G, \beta}^+ \geq 0. \quad (2.2.30)$$

Proof. An easy computation gives

$$\frac{\partial}{\partial \beta} \langle \sigma_A \rangle_{G, \beta, \mathbf{h}} = \sum_{\{x, y\} \subset G} J_{x, y} \langle \sigma_A; \sigma_x \sigma_y \rangle_{G, \beta, \mathbf{h}} + \sum_{x \in G} \mathbf{h}_x \langle \sigma_A; \sigma_x \rangle_{G, \beta, \mathbf{h}}, \quad (2.2.31)$$

$$\frac{\partial}{\partial J_{x, y}} \langle \sigma_A \rangle_{G, J, \beta, \mathbf{h}} = \langle \sigma_A; \sigma_x \sigma_y \rangle_{G, J, \beta, \mathbf{h}}, \quad (2.2.32)$$

$$\frac{\partial}{\partial \mathbf{h}_x} \langle \sigma_A \rangle_{G, \beta, \mathbf{h}} = \langle \sigma_A; \sigma_x \rangle_{G, \beta, \mathbf{h}}. \quad (2.2.33)$$

The proof follows readily from Proposition 2.2.10. □

Remark 2.2.12. A classical consequence of the above proposition is the construction of infinite volume limits. Indeed, given a positive external magnetic field $\mathbf{h}$, the (weak) limit

$$\lim_{G \nearrow V} \langle \cdot \rangle_{G,\beta,\mathbf{h}} =: \langle \cdot \rangle_{\beta,\mathbf{h}} \quad (2.2.34)$$

exists by monotonicity.

An other interesting application of the switching lemma is the following easy proof of Simon's inequality [223]. Below, we will also give a proof of Lieb's improvement [178] of this inequality. We will need the following definition.

A *separating surface* between two sites x and y is a set $S \subset G$ such that any J -path (meaning that the path takes jumps on the support of J) between x and y must intersect S .

Proposition 2.2.13 (Simon's inequality). *Assume that $\mathbf{h} \equiv 0$. Let $x, y \in G$. Let $S \subset G$ be a separating surface between x and y . Then,*

$$\langle \sigma_x \sigma_y \rangle_{G,\beta} \leq \sum_{u \in S} \langle \sigma_x \sigma_u \rangle_{G,\beta} \langle \sigma_u \sigma_y \rangle_{G,\beta}. \quad (2.2.35)$$

Proof. Using Proposition 2.2.9,

$$\sum_{u \in S} \frac{\langle \sigma_x \sigma_u \rangle_{G,\beta} \langle \sigma_u \sigma_y \rangle_{G,\beta}}{\langle \sigma_x \sigma_y \rangle_{G,\beta}} = \sum_{u \in S} \mathbf{P}_{G,\beta}^{xy,\emptyset}[x \xrightarrow{\mathbf{n}_1+\mathbf{n}_2} u] = \mathbf{E}_{G,\beta}^{xy,\emptyset}[|\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x) \cap S|]. \quad (2.2.36)$$

However, a sample if $(\mathbf{n}_1, \mathbf{n}_2)$ is sampled according to $\mathbf{P}_{G,\beta}^{xy,\emptyset}$, then there must be a path from x to y in $\mathbf{n}_1 + \mathbf{n}_2$. Since S is a separating surface between x and y ,

$$\mathbf{E}_{G,\beta}^{xy,\emptyset}[|\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x) \cap S|] \geq 1, \quad (2.2.37)$$

which yields the inequality. $\square$

2.3 The backbone representation

We introduce another geometric representation of the Ising model called the *backbone* representation (also called the *random walk* representation). As we saw, a current $\mathbf{n}$ on a finite set G with sources $\partial \mathbf{n} = \{x, y\}$ must have at least one path from x to y in the percolation configuration induced by $\mathbf{n}$. The backbone of $\mathbf{n}$ is an exploration of one of these paths. This expansion was introduced by Aizenman [4]. We will follow the exposition of Aizenman and Fernández [12, Section 4]. It is closely related to the so-called *BFS expansion* introduced by Brydges, Fröhlich, and Spencer [47]. For more information on these random walk representations, we refer to the monograph [95].

In what follows we work with a ferromagnetic Ising model on $G \subset V$ with non homogeneous and positive external field $\mathbf{h} = (\mathbf{h}_x)_{x \in V}$. We start with some definitions.

A *bond* is pair of vertices in G . A *step* from u to v is an oriented bond uv . We fix an ordering $\prec$ of the steps that satisfies the following property: for each vertex $u \in G$, the earliest step coming from u is $u\mathbf{g}$. To each step uv , we associate the set of *cancelled bonds* made of the bond $\{u, v\}$ itself and all the bonds $\{u, w\}$ with $uw \prec uv$. With this convention, a lattice step uv with $u, v \in G$, cancels, beside itself, the step $u\mathbf{g}$ and potentially other steps in G . The step $u\mathbf{g}$ only cancels itself.

Definition 2.3.1. A sequence of steps is said to be *consistent* if no step of the sequence uses a bond cancelled by a previous step. If γ is a consistent sequence of steps, we denote by $\bar{\gamma}$ the set of all the bonds its steps cancel. We will say that a bond b *belongs* to γ if one of the two steps associated with b appears in γ . A *path* or a *walk* from x to y is a sequence of steps $\{(x_i, x_{i+1}) : 0 \leq i \leq n-1\}$ with $x_0 = x$ and $x_n = y$. Given γ_1 and γ_2 , the composition $\gamma_1 \circ \gamma_2$ is the sequence of steps obtained by traversing first γ_1 and then γ_2 . The *lattice sources* $\partial\gamma$ of γ is the set of vertices of G covered by an odd number of steps of γ .

We can now define the backbone of a current configuration $\mathbf{n}$. Recall that we have fixed an ordering $\prec$ of all the steps.

Definition 2.3.2. Given a current configuration $\mathbf{n}$ and $x \in \partial\mathbf{n}$, the x -*backbone* of $\mathbf{n}$ is the path $\Gamma_x(\mathbf{n}) = \{(x_i, x_{i+1}), 0 \leq i \leq n-1\}$ determined by the following properties:

- (1) $x_0 = x$ and the first step xx_1 is the earliest one of all the steps emerging from x with $\mathbf{n}_{xx_1}$ odd.
- (2) Each step $x_i x_{i+1}$ is the earliest of all steps emerging from x_i that has not been cancelled by previous steps, and for which $\mathbf{n}_{x_i x_{i+1}}$ is odd.
- (3) The path stops when one of the following occurs: either it reaches a source, or it reaches the ghost.

If more lattice sources are present, we have several candidates for backbones. For a set of sources $A \subset G$, we will define the A -backbone as follows. Below, we assume that we have fixed any ordering on the vertices of V .

Definition 2.3.3. For a configuration $\mathbf{n}$ with $A \subset \partial\mathbf{n}$, the A -*backbone* of $\mathbf{n}$ is the sequence of paths $\Gamma_A(\mathbf{n}) = \Gamma_1 \circ \dots \circ \Gamma_s$ with $s \leq |A|$ defined as follows :

- (1) Pick the earliest point in A , say x , and take $\Gamma_1 = \Gamma_x(\mathbf{n})$. Note that Γ_1 may end at $\mathbf{g}$.
- (2) To determine Γ_i once $\Gamma_1, \dots, \Gamma_{i-1}$ are found, consider a system deprived of all the bonds cancelled by the sequence $\Gamma_1 \circ \dots \circ \Gamma_{i-1}$ and remove from A all the vertices already visited by such sequence. Then repeat (1).
- (3) Continue until the set A is exhausted.

If $A = \partial\mathbf{n}$, we let $\Gamma(\mathbf{n}) := \Gamma_A(\mathbf{n})$ denote the *backbone* of $\mathbf{n}$.

By definition, $\bar{\Gamma}(\mathbf{n})$ is determined by $\Gamma(\mathbf{n})$ and the ordering $\prec$, and similarly $\Gamma(\mathbf{n})$ can be determined from $\bar{\Gamma}(\mathbf{n})$ by selecting odd edges in $\bar{\Gamma}(\mathbf{n})$ according to $\prec$. Also, the current $\mathbf{n} \setminus \bar{\Gamma}(\mathbf{n})$, defined to be the restriction of $\mathbf{n}$ to $\bar{\Gamma}(\mathbf{n})^c$, is sourceless.

With these definitions in hand, we can expand the correlation functions in terms of this backbone. Indeed, for a set $A \subset G$,

$$\langle \sigma_A \rangle_{G, \beta, \mathbf{h}} = \sum_{\partial\gamma=A} \rho_{\beta, \mathbf{h}}(\gamma) \quad (2.3.1)$$

where, for a sequence of steps γ ,

$$\rho_{\beta, \mathbf{h}}(\gamma) := \mathbf{1}\{\gamma \text{ is consistent}\} \sum_{\partial\mathbf{n}=\partial\gamma} \frac{w_{\beta, \mathbf{h}}(\mathbf{n})}{Z_{G, \beta, \mathbf{h}}^\emptyset} \mathbf{1}\{\Gamma(\mathbf{n}) = \gamma\}. \quad (2.3.2)$$

Note that if $\partial\gamma = A$,

$$\rho_{\beta, \mathbf{h}}(\gamma) = \langle \sigma_A \rangle_{G, \beta, \mathbf{h}} \mathbf{P}_{G_{\mathbf{g}}, \beta, \mathbf{h}}^A[\Gamma(\mathbf{n}) = \gamma]. \quad (2.3.3)$$

It is possible to obtain more explicit expressions for $\rho_{\beta, \mathbf{h}}(\gamma)$. First, notice that for a consistent sequence of steps γ , the current configurations $\mathbf{n}$ contributing to $\rho_{\beta, \mathbf{h}}(\gamma)$ are exactly those which satisfy:

- (i) $\mathbf{n}$ is odd on all bonds in γ ,
- (ii) $\mathbf{n}$ is even on all bonds in $\bar{\gamma} \setminus \gamma$,
- (iii) $\mathbf{n}$ restricted to γ^c or $\bar{\gamma}^c$ is sourceless.

With this in mind, one has (with the convention that $J_{x\mathbf{g}} = \mathbf{h}_x$ for $x \in G$)

$$\rho_{\beta, \mathbf{h}}(\gamma) = \mathbb{1}\{\gamma \text{ is consistent}\} \prod_{b \in \gamma} \sinh(\beta J_b) \prod_{b \in \bar{\gamma} \setminus \gamma} \cosh(\beta J_b) \sum_{\mathbf{n} \in \Omega_{\bar{\gamma}^c} : \partial \mathbf{n} = \emptyset} \frac{w_{\beta, \mathbf{h}}(\mathbf{n})}{Z_{G_{\mathbf{g}}, \beta, \mathbf{h}}^{\emptyset}} \quad (2.3.4)$$

which can also be written²

$$\rho_{\beta, \mathbf{h}}(\gamma) = \mathbb{1}\{\gamma \text{ is consistent}\} \prod_{b \in \gamma} \tanh(\beta J_b) \cdot \left(\prod_{b \in \bar{\gamma}} \cosh(\beta J_b) \cdot \frac{Z_{\bar{\gamma}^c, \beta, \mathbf{h}}^{\emptyset}}{Z_{G_{\mathbf{g}}, \beta, \mathbf{h}}^{\emptyset}} \right) \quad (2.3.5)$$

$$= \mathbb{1}\{\gamma \text{ is consistent}\} \prod_{b \in \gamma} \tanh(\beta J_b) \mathbf{P}_{G_{\mathbf{g}}, \beta, \mathbf{h}}^{\emptyset}[\mathbf{n} \text{ even on } \bar{\gamma}]. \quad (2.3.6)$$

We now list some useful properties of the weights $\rho_{\beta, \mathbf{h}}(\gamma)$.

Proposition 2.3.4. (a) *For any sequence of steps γ ,*

$$\rho_{\beta, \mathbf{h}}(\gamma) \leq \mathbb{1}\{\gamma \text{ is consistent}\} \prod_{b \in \gamma} \tanh(\beta J_b). \quad (2.3.7)$$

(b) *Let E be a set of bonds. If $\gamma \cap E = \emptyset$, then*

$$\rho_{\beta, \mathbf{h}}(\gamma) \leq \rho_{E^c, \beta, \mathbf{h}}(\gamma), \quad (2.3.8)$$

where $\rho_{E^c, \beta, \mathbf{h}}(\gamma)$ is the backbone weight of γ in the system depleted of the bonds in E .

(c) *If γ does not visit ghosts,*

$$\rho_{\beta, \mathbf{h}}(\gamma) \leq \rho_{\beta, \mathbf{h} \equiv 0}(\gamma). \quad (2.3.9)$$

(d) **Composition law** *If $\gamma_1 \circ \gamma_2$ is a consistent sequence of steps,*

$$\rho_{\beta, \mathbf{h}}(\gamma_1 \circ \gamma_2) = \rho_{\beta, \mathbf{h}}(\gamma_1) \rho_{\bar{\gamma}_1^c, \beta, \mathbf{h}}(\gamma_2). \quad (2.3.10)$$

²For a set of bonds E , we write $Z_{E^c, \beta, \mathbf{h}}^{\emptyset}$ for the partition function of sourceless currents whose value is set to 0 on E .

(e) **Dichotomy** For any pair of sequences of steps (γ_1, γ_2) , one has:

$$\rho_{\beta, \mathbf{h}}(\gamma_1 \circ \gamma_2) \geq \rho_{\beta, \mathbf{h}}(\gamma_1) \rho_{\beta, \mathbf{h}}(\gamma_2) \quad (2.3.11)$$

if $\gamma_1 \circ \gamma_2$ is consistent, and

$$\rho_{\beta, \mathbf{h}}(\gamma_1 \circ \gamma_2) = 0 \quad (2.3.12)$$

otherwise.

Proof. (a) This is a direct consequence of (2.3.6).

(b) We obtain an upper bound on $\rho_{\beta, \mathbf{h}}(\gamma)$ by replacing the event “ $\mathbf{n}$ is even on $\bar{\gamma}$ ” in (2.3.6) by “ $\mathbf{n}$ is even on $\bar{\gamma} \setminus E$ ”. We get,

$$\rho_{\beta, \mathbf{h}}(\gamma) \leq \mathbb{1}\{\gamma \text{ is consistent}\} \prod_{b \in \gamma} \tanh(\beta J_b) \cdot \prod_{b \in \bar{\gamma} \setminus E} \cosh(\beta J_b) \cdot \frac{Z_{(\bar{\gamma} \setminus E)^c, \beta, \mathbf{h}}^\emptyset}{Z_{G_{\mathfrak{g}}, \beta, \mathbf{h}}^\emptyset}. \quad (2.3.13)$$

Then, using (2.3.5),

$$\rho_{E^c, \beta, \mathbf{h}}(\gamma) = \mathbb{1}\{\gamma \text{ is consistent}\} \prod_{b \in \gamma} \tanh(\beta J_b) \prod_{b \in \bar{\gamma} \setminus E} \cosh(\beta J_b) \cdot \frac{Z_{\bar{\gamma}^c \setminus E, \beta, \mathbf{h}}^\emptyset}{Z_{E^c, \beta, \mathbf{h}}^\emptyset}. \quad (2.3.14)$$

Hence,

$$\begin{aligned} \rho_{\beta, \mathbf{h}}(\gamma) &\leq \rho_{E^c, \beta, \mathbf{h}}(\gamma) \frac{Z_{E^c, \beta, \mathbf{h}}^\emptyset}{Z_{(\bar{\gamma}^c \setminus E), \beta, \mathbf{h}}^\emptyset} \frac{Z_{(\bar{\gamma} \setminus E)^c, \beta, \mathbf{h}}^\emptyset}{Z_{G_{\mathfrak{g}}, \beta, \mathbf{h}}^\emptyset} \\ &= \rho_{E^c, \beta, \mathbf{h}}(\gamma) \frac{Z_{E^c, \beta, \mathbf{h}}^\emptyset}{Z_{(E^c \setminus \bar{\gamma}), \beta, \mathbf{h}}^\emptyset} \frac{Z_{E \cup (E^c \setminus \bar{\gamma}), \beta, \mathbf{h}}^\emptyset}{Z_{G_{\mathfrak{g}}, \beta, \mathbf{h}}^\emptyset}. \end{aligned} \quad (2.3.15)$$

The claim is therefore a consequence of the following inequality: for E, E' two disjoint families of bonds,

$$\frac{Z_{E^c, \beta, \mathbf{h}}^\emptyset}{Z_{G_{\mathfrak{g}}, \beta, \mathbf{h}}^\emptyset} \leq \frac{Z_{E'^c, \beta, \mathbf{h}}^\emptyset}{Z_{E \cup E'^c, \beta, \mathbf{h}}^\emptyset}. \quad (2.3.16)$$

Using the relation between the partition function of the Ising model and the partition function for currents, we can interpret the left-hand side as the change in the partition function produced by turning off the bounds in E (which means setting $J_b = 0$ for $b \in E$). The right-hand side is the change when one turns off the bounds of E in a system after the bounds of $(E \cup E')^c$ have already been turned off. Thus, (2.3.16) is equivalent to (recall the convention that $\sigma_{\mathfrak{g}} = 1$)

$$\frac{1}{\left\langle \exp \left(\beta \sum_{\{x, y\} \in E} J_{x, y} \sigma_x \sigma_y \right) \right\rangle_{E^c, \beta, \mathbf{h}}} \leq \frac{1}{\left\langle \exp \left(\beta \sum_{\{x, y\} \in E} J_{x, y} \sigma_x \sigma_y \right) \right\rangle_{E', \beta, \mathbf{h}}}, \quad (2.3.17)$$

which is a consequence of Griffiths' second inequality obtained Proposition 2.2.11.

(c) Simply set $E = \{\{x, \mathfrak{g}\}, x \in G\}$ in (b).

- (d) If $\gamma_1 \circ \gamma_2$ is a consistent path, then $\bar{\gamma}_1 \cap \gamma_2 = \emptyset$. Thus, every current configuration $\mathbf{n}$ contributing to $\rho_{\beta, \mathbf{h}}(\gamma_1 \circ \gamma_2)$ can be uniquely written as $\mathbf{n} = \mathbf{n}_1 + \mathbf{n}_2$ with $\mathbf{n}_1$ supported on $\bar{\gamma}_1$ and $\mathbf{n}_2$ supported on $\bar{\gamma}_1^c$. Moreover, $\partial \mathbf{n}_1 = \partial \gamma_1$ and $\partial \mathbf{n}_2 = \partial \gamma_2$. Since $\mathbf{n}_1$ and $\mathbf{n}_2$ have disjoint supports,

$$w_{\beta, \mathbf{h}}(\mathbf{n}) = w_{\beta, \mathbf{h}}(\mathbf{n}_1)w_{\beta, \mathbf{h}}(\mathbf{n}_2). \quad (2.3.18)$$

We obtain the result by conditioning on $\mathbf{n}_1$.

- (e) If $\gamma_1 \circ \gamma_2$ is consistent then the result follows from (d) and (b). If not, then the equality is trivial. $\square$

The backbone representation of the Ising model is convenient to derive random walk type bounds expressed in terms of the correlation functions of the model. The following proposition is a very useful example of such an estimate.

Proposition 2.3.5 (Chain rule for the backbone). *Assume that $\mathbf{h} \equiv 0$. Let $x, y, u, v \in G$. Then,*

$$\mathbf{P}_{G, \beta}^{xy}[\Gamma(\mathbf{n}) \text{ passes through } u \text{ first and then through } v] \leq \frac{\langle \sigma_x \sigma_u \rangle_{G, \beta} \langle \sigma_u \sigma_v \rangle_{G, \beta} \langle \sigma_v \sigma_y \rangle_{G, \beta}}{\langle \sigma_x \sigma_y \rangle_{G, \beta}}. \quad (2.3.19)$$

In particular, if $u = v$,

$$\mathbf{P}_{G, \beta}^{xy}[\Gamma(\mathbf{n}) \text{ passes through } u] \leq \frac{\langle \sigma_x \sigma_u \rangle_{G, \beta} \langle \sigma_u \sigma_y \rangle_{G, \beta}}{\langle \sigma_x \sigma_y \rangle_{G, \beta}}. \quad (2.3.20)$$

Proof. Write,

$$\langle \sigma_x \sigma_y \rangle_{G, \beta} \mathbf{P}_{G, \beta}^{xy}[\Gamma(\mathbf{n}) \text{ passes through } u \text{ first and then through } v] \quad (2.3.21)$$

$$= \sum_{\gamma: x \rightarrow u \rightarrow v \rightarrow y} \rho_{\beta}(\gamma) \quad (2.3.22)$$

$$= \sum_{\substack{\gamma_1: x \rightarrow u \\ \gamma_2: u \rightarrow v \\ \gamma_3: v \rightarrow y}} \rho_{\beta}(\gamma_1 \circ \gamma_2 \circ \gamma_3) \quad (2.3.23)$$

$$= \sum_{\substack{\gamma_1: x \rightarrow u \\ \gamma_2: u \rightarrow v \\ \gamma_3: v \rightarrow y}} \rho_{\beta}(\gamma_1) \rho_{\bar{\gamma}_1^c, \beta}(\gamma_2) \rho_{\overline{\gamma_1 \circ \gamma_2^c}, \beta}(\gamma_3) \quad (2.3.24)$$

$$= \sum_{\substack{\gamma_1: x \rightarrow u \\ \gamma_2: u \rightarrow v}} \rho_{\beta}(\gamma_1) \rho_{\bar{\gamma}_1^c, \beta}(\gamma_2) \langle \sigma_v \sigma_y \rangle_{\overline{\gamma_1 \circ \gamma_2^c}, \beta} \quad (2.3.25)$$

$$\leq \langle \sigma_x \sigma_u \rangle_{G, \beta} \langle \sigma_u \sigma_v \rangle_{G, \beta} \langle \sigma_v \sigma_y \rangle_{G, \beta}, \quad (2.3.26)$$

where we used Proposition 2.3.4 (d) and Proposition 2.2.10. $\square$

2.4 Sharpness of the phase transition

In this section, $\mathcal{G} = (V, E)$ is a locally-finite, countable, vertex transitive graph. We assume it contains a fixed origin $0 \in V$. For $n \geq 0$, we let

$$\Lambda_n := \{x \in V : d(0, x) \leq n\}, \quad (2.4.1)$$

where d is the graph distance. We assume that J is invariant with respect to some transitively acting group, i.e. there exists a group Γ of automorphisms acting transitively on V such that: for all $\gamma \in \Gamma$, for all $x, y \in V$,

$$J_{\gamma(x), \gamma(y)} = J_{x, y}. \quad (2.4.2)$$

Additionally, we assume that J is ferromagnetic and *locally finite*, i.e. $|J| := \sum_{x \in V} J_{0, x} < \infty$. Finally, we say that J is *finite-range* if there exists $R > 0$ such that, for all $x, y \in V$, $J_{x, y} = 0$ whenever $d(x, y) > R$.

Let G be a finite subset of V . Fix $\beta, h > 0$. In this section, $\langle \cdot \rangle_{G, \beta, h}$ denotes the measure obtained with the homogeneous external magnetic field³ $\mathbf{h} \equiv \frac{h}{\beta}$. The *magnetisation* is defined for $\beta > 0$ and $h \in \mathbb{R}$ by

$$m(\beta, h) := \langle \sigma_0 \rangle_{\beta, h}. \quad (2.4.3)$$

Recall that the spontaneous magnetisation is defined as $m^*(\beta) = \lim_{h \searrow 0} m(\beta, h)$. The following result was first proved by Aizenman, Barsky and Fernández [8]. The proof presented here is due to Duminil-Copin and Tassion [84].

Theorem 2.4.1 (Sharpness of the phase transition). *Assume that we work with the setup described at the beginning of the section. Then,*

(i) For $\beta > \beta_c$,

$$m^*(\beta) \geq \sqrt{\frac{\beta^2 - \beta_c^2}{\beta^2}}. \quad (2.4.4)$$

(ii) For $\beta < \beta_c$,

$$\chi(\beta) = \sum_{x \in V} \langle \sigma_0 \sigma_x \rangle_{\beta}^+ < \infty. \quad (2.4.5)$$

(iii) Assume that J is finite range. Then, for any $\beta < \beta_c$, there exists $c = c(\beta) > 0$ such that for all $x \in V$,

$$\langle \sigma_0 \sigma_x \rangle_{\beta} \leq \exp(-cd(0, x)). \quad (2.4.6)$$

Remark 2.4.2. Exponential decay of correlation functions in the subcritical regime disappears (in general) when working with long-range interactions. See for instance [199, 20, 21].

The following quantity will play a crucial role in the proof. For $\beta > 0$ and a finite subset S of V containing 0, let

$$\varphi_{\beta}(S) := \sum_{\substack{x \in S \\ y \notin S}} \beta J_{x, y} \langle \sigma_0 \sigma_x \rangle_{S, \beta}. \quad (2.4.7)$$

Define,

$$\tilde{\beta}_c := \sup \{ \beta \geq 0 : \varphi_{\beta}(S) < 1 \text{ for some finite } S \subset V \text{ containing } 0 \}. \quad (2.4.8)$$

We will establish that (i), (ii) and (iii) hold with $\tilde{\beta}_c$ in place of β_c . This will immediately give that $\beta_c = \tilde{\beta}_c$ and the result. We proceed in two steps.

³This convention may seem a little bit strange but will be more convenient in the computations.

- 1) The quantity $\varphi_\beta(S)$ appears naturally in the derivative of (a finite-volume approximation of) $m(\beta, h)$. More precisely, we obtain a finite volume version of the following differential inequality: for $\beta, h > 0$

$$\frac{\partial m}{\partial \beta}(\beta, h) \geq \frac{2}{\beta} \cdot \inf_{S \ni 0} \varphi_\beta(S) \cdot (1 - m(\beta, h)^2). \quad (2.4.9)$$

This yields that for $\beta > \tilde{\beta}_c$ and $h > 0$,

$$m(\beta, h) \geq \sqrt{\frac{\beta^2 - \tilde{\beta}_c^2}{\beta^2}}, \quad (2.4.10)$$

and taking the limit $h \searrow 0$,

$$m^*(\beta) \geq \sqrt{\frac{\beta^2 - \tilde{\beta}_c^2}{\beta^2}}, \quad (2.4.11)$$

which gives $\beta_c \leq \tilde{\beta}_c$.

- We then prove (ii) and (iii) using an improved version of Simon's inequality. More precisely, we show that if $S \subset V$ is a finite set containing 0, then,

$$\chi(\beta) \leq \frac{|S|}{1 - \varphi_\beta(S)}, \quad (2.4.12)$$

which readily implies that $\beta_c \geq \tilde{\beta}_c$.

We start with the proof of (i). We begin with some notations. For $S \subset G$ two finite subsets of V , introduce the measure $\langle \cdot \rangle_{S, \beta, h}^G$ obtained from $\langle \cdot \rangle_{G, \beta, h}$ by setting all the coupling constants $J_{x, y}$ with x or $y \in G \setminus S$ to be equal to 0. Notice that if $S = G$ then one has $\langle \cdot \rangle_{S, \beta, h}^G = \langle \cdot \rangle_{G, \beta, h}$. If S is fixed and $G \nearrow V$, then $\langle \cdot \rangle_{S, \beta, h}^G \rightarrow \langle \cdot \rangle_{S, \beta, h}$.

For $S \subset G$, and $\mathbf{n}$ a current on $S_{\mathfrak{g}}$, define

$$w_S(\mathbf{n}) := \prod_{\{x, y\} \subset S_{\mathfrak{g}}} \frac{(\beta J_{x, y})^{\mathbf{n}_{xy}}}{\mathbf{n}_{xy}!}. \quad (2.4.13)$$

Note that for $A \subset S$,

$$\langle \sigma_A \rangle_{S, \beta, h}^G = \frac{\sum_{\partial \mathbf{n} \cap S = A} w_S(\mathbf{n})}{\sum_{\partial \mathbf{n} = \emptyset} w_S(\mathbf{n})}. \quad (2.4.14)$$

When $S = G$ we write $w(\mathbf{n})$ instead of $w_G(\mathbf{n})$. The proof of (i) reduces to the following lemma.

Lemma 2.4.3. *Let $\beta, h > 0$ and $G \subset V$ finite containing 0. Then,*

$$\frac{\partial}{\partial \beta} \langle \sigma_0 \rangle_{G, \beta, h}^2 \geq 2c(G, \beta, h) \left[\frac{1}{\beta} \cdot \inf_{S \ni 0} \varphi_\beta(S) \cdot (1 - \langle \sigma_0 \rangle_{G, \beta, h}^2) - \varepsilon(G, \beta, h) \right] \quad (2.4.15)$$

where

$$c(G, \beta, h) := \min_{y \in G} \frac{\langle \sigma_0 \rangle_{G, \beta, h}}{\langle \sigma_y \rangle_{G, \beta, h}}, \quad \varepsilon(G, \beta, h) := \sum_{\substack{x \in G \\ y \in V \setminus G}} J_{x, y} \left(\langle \sigma_0 \sigma_x \rangle_{G, \beta, h} - \langle \sigma_0 \rangle_{G, \beta, h} \langle \sigma_x \rangle_{G, \beta, h} \right). \quad (2.4.16)$$

Proof of Theorem 2.4.1 (i) for $\beta > \tilde{\beta}_c$. Let $\beta > \tilde{\beta}_c$. Apply Lemma 2.4.3 to $G = \Lambda_n$ with $n \geq 1$. Integrating between β and $\tilde{\beta}_c$, letting $n \rightarrow \infty$, and $h \searrow 0$, we obtain (i) and $\beta_c \leq \tilde{\beta}_c$ thanks to the following observations.

Griffiths' inequality implies that for $n \geq 1$,

$$\frac{\langle \sigma_0 \rangle_{\Lambda_n, \beta, h}}{\langle \sigma_0 \rangle_{\Lambda_{2n}, \beta, h}} \leq c(\Lambda_n, \beta, h) \leq 1. \quad (2.4.17)$$

As a result,

$$c(\Lambda_n, \beta, h) \xrightarrow{n \rightarrow \infty} 1. \quad (2.4.18)$$

We also have that

$$\varepsilon(\Lambda_n, \beta, h) \xrightarrow{n \rightarrow \infty} 0. \quad (2.4.19)$$

Indeed, using [120], the magnetisation $\langle \sigma_0 \rangle_{G, \beta, h}$ is a concave function of h . Hence, for $h > 0$,

$$\sum_{x \in G} \left(\langle \sigma_0 \sigma_x \rangle_{G, \beta, h} - \langle \sigma_0 \rangle_{G, \beta, h} \langle \sigma_x \rangle_{G, \beta, h} \right) = \frac{\partial}{\partial h} \langle \sigma_0 \rangle_{G, \beta, h} \leq \frac{\langle \sigma_0 \rangle_{G, \beta, h} - \langle \sigma_0 \rangle_{G, \beta, 0}}{h} \leq \frac{1}{h}. \quad (2.4.20)$$

Thus, we get that for $n, k \geq 0$,

$$\sum_{\substack{x \in \Lambda_n \\ y \in V \setminus \Lambda_{n+k}}} J_{x,y} \left(\langle \sigma_0 \sigma_x \rangle_{\Lambda_{n+k}, \beta, h} - \langle \sigma_0 \rangle_{\Lambda_{n+k}, \beta, h} \langle \sigma_x \rangle_{\Lambda_{n+k}, \beta, h} \right) \leq \frac{1}{h} \sum_{x \in V \setminus \Lambda_k} J_{0,x}. \quad (2.4.21)$$

A new application of the result of [120] yields

$$\begin{aligned} \sum_{\substack{x \in \Lambda_{n+k} \setminus \Lambda_n \\ y \in \mathbb{Z}^d \setminus \Lambda_{n+k}}} J_{x,y} \left(\langle \sigma_0 \sigma_x \rangle_{\Lambda_{n+k}, \beta, h} - \langle \sigma_0 \rangle_{\Lambda_{n+k}, \beta, h} \langle \sigma_x \rangle_{\Lambda_{n+k}, \beta, h} \right) \\ \leq \frac{\langle \sigma_0 \rangle_{\Lambda_{n+k}, \beta, h} - \langle \sigma_0 \rangle_{\Lambda_{n+k}, \beta, \mathbf{h}}}{h} \sum_{x \in \mathbb{Z}^d} J_{0,x}, \end{aligned} \quad (2.4.22)$$

where $\mathbf{h}$ is the non-homogeneous external magnetic field equal to h in Λ_n , and to 0 in $\Lambda_{n+k} \setminus \Lambda_n$. The right-hand side of (2.4.22) goes to 0 as n goes to infinity. Additionally taking k to infinity yields (2.4.19). $\square$

Proof of Lemma 2.4.3. Let $\beta, h > 0$ and $G \subset V$ finite. Then⁴,

$$\frac{\partial}{\partial \beta} \langle \sigma_0 \rangle_{G, \beta, h} = \sum_{\{x,y\} \subset G} J_{x,y} \left(\langle \sigma_0 \sigma_x \sigma_y \rangle_{G, \beta, h} - \langle \sigma_0 \rangle_{G, \beta, h} \langle \sigma_x \sigma_y \rangle_{G, \beta, h} \right). \quad (2.4.23)$$

Using (2.2.19), we obtain

$$\frac{\partial}{\partial \beta} \langle \sigma_0 \rangle_{G, \beta, h} = \frac{1}{(Z_{G, \beta, h}^\emptyset)^2} \sum_{\{x,y\} \subset G} J_{x,y} \sum_{\substack{\partial \mathbf{n}_1 = \{0\} \Delta \{x,y\} \\ \partial \mathbf{n}_2 = \emptyset}} w(\mathbf{n}_1) w(\mathbf{n}_2) \mathbf{1}\{0 \not\leftrightarrow \mathbf{g}\}. \quad (2.4.24)$$

Now, if $\mathbf{n}_1$ and $\mathbf{n}_2$ are two currents with $\partial \mathbf{n}_1 = \{0\} \Delta \{x, y\}$ and $\partial \mathbf{n}_2 = \emptyset$, and 0 and $\mathbf{g}$ are not connected in $\mathbf{n}_1 + \mathbf{n}_2$, then exactly one of the following holds:

⁴Recall that $\mathbf{h} \equiv \frac{h}{\beta}$.

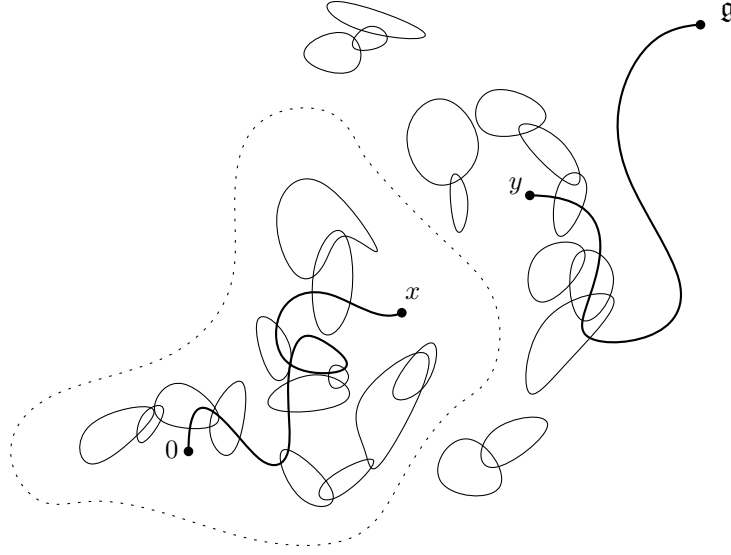


Figure 2.2: An illustration of $\widehat{\mathbf{n}_1 + \mathbf{n}_2}$ for $(\mathbf{n}_1, \mathbf{n}_2)$ contributing to $\delta_{x,y}$. The bold line represents the backbone of $\mathbf{n}_1$. The dotted line represents the boundary of the cluster of 0.

- 1) 0 is connected to x in $\mathbf{n}_1 + \mathbf{n}_2$ and y is connected to $\mathfrak{g}$ in $\mathbf{n}_1 + \mathbf{n}_2$,
- 2) 0 is connected to y in $\mathbf{n}_1 + \mathbf{n}_2$ and x is connected to $\mathfrak{g}$ in $\mathbf{n}_1 + \mathbf{n}_2$.

By symmetry, we get

$$\frac{\partial}{\partial \beta} \langle \sigma_0 \rangle_{G, \beta, h} = \frac{1}{(Z_{G, \beta, h}^\emptyset)^2} \sum_{x, y \in G} J_{x, y} \delta_{x, y}, \quad (2.4.25)$$

where

$$\delta_{x, y} = \sum_{\substack{\partial \mathbf{n}_1 = \{0\} \Delta \{x, y\} \\ \partial \mathbf{n}_2 = \emptyset}} w(\mathbf{n}_1) w(\mathbf{n}_2) \mathbb{1} \{0 \xleftrightarrow{\mathbf{n}_1 + \mathbf{n}_2} \mathfrak{g}, 0 \xleftrightarrow{\mathbf{n}_1 + \mathbf{n}_2} x, y \xleftrightarrow{\mathbf{n}_1 + \mathbf{n}_2} \mathfrak{g}\}. \quad (2.4.26)$$

An illustration of configurations contributing to $\delta_{x,y}$ is given in Figure 2.2. Given two currents $\mathbf{n}_1$ and $\mathbf{n}_2$, and $z \in \{0, \mathfrak{g}\}$, define $\mathcal{S}_z$ to be the set of vertices in $G_{\mathfrak{g}}$ that are **not connected** to z in $\mathbf{n}_1 + \mathbf{n}_2$. We compute $\delta_{x,y}$ by summing over the different possible values for $\mathcal{S}_0$:

$$\begin{aligned} \delta_{x, y} &= \sum_{S \subset G_{\mathfrak{g}}} \sum_{\substack{\partial \mathbf{n}_1 = \{0\} \Delta \{x, y\} \\ \partial \mathbf{n}_2 = \emptyset}} w(\mathbf{n}_1) w(\mathbf{n}_2) \mathbb{1} \{0 \xleftrightarrow{\mathbf{n}_1 + \mathbf{n}_2} \mathfrak{g}, 0 \xleftrightarrow{\mathbf{n}_1 + \mathbf{n}_2} x, y \xleftrightarrow{\mathbf{n}_1 + \mathbf{n}_2} \mathfrak{g}\} \\ &= \sum_{\substack{S \subset G_{\mathfrak{g}} \\ y, g \in S \\ 0, x \notin S}} \sum_{\substack{\partial \mathbf{n}_1 = \{0\} \Delta \{x, y\} \\ \partial \mathbf{n}_2 = \emptyset}} w(\mathbf{n}_1) w(\mathbf{n}_2) \mathbb{1} \{\mathcal{S}_0 = S, y \xleftrightarrow{\mathbf{n}_1 + \mathbf{n}_2} \mathfrak{g}\}. \end{aligned}$$

Let $S \subset G_{\mathfrak{g}}$. If $\mathcal{S}_0 = S$, then the currents $\mathbf{n}_1$ and $\mathbf{n}_2$ vanish on every $\{u, v\}$ with $v \in S$ and $u \notin S$. Thus, for $i = 1, 2$, we have the following decomposition for $\mathbf{n}_i$,

$$\mathbf{n}_i = \mathbf{n}_i^S + \mathbf{n}_i^{G_{\mathfrak{g}} \setminus S}, \quad (2.4.27)$$

where $\mathbf{n}_i^S$ (resp. $\mathbf{n}_i^{G_g \setminus S}$) denotes the restriction of $\mathbf{n}_i$ to bonds $\{u, v\}$ with $u, v \in S$ (resp. $u, v \in G_g \setminus S$). Moreover, $\partial \mathbf{n}_i^S = \partial \mathbf{n}_i \cap S$ and $\partial \mathbf{n}_i^{G_g \setminus S} = \mathbf{n}_i \cap (G_g \setminus S)$. Using this observation, we obtain

$$\delta_{x,y} = \sum_{\substack{S \subset G_g \\ y, g \in S \\ 0, x \notin S}} \sum_{\substack{\mathbf{n}_1, \mathbf{n}_2 \in \Omega_{G_g \setminus S} \\ \partial \mathbf{n}_1 = \{0\} \Delta \{x\} \\ \partial \mathbf{n}_2 = \emptyset}} w_{G_g \setminus S}(\mathbf{n}_1) w_{G_g \setminus S}(\mathbf{n}_2) \langle \sigma_y \rangle_{S, \beta, h}^G \left(\sum_{\mathbf{n} \in \Omega_{S_g} : \partial \mathbf{n} = \emptyset} w_S(\mathbf{n}) \right)^2 \mathbb{1}\{\mathcal{S}_0 = S\}. \quad (2.4.28)$$

Multiplying the above expression by $\langle \sigma_y \rangle_{G, \beta, h}$, and using that $\langle \sigma_y \rangle_{G, \beta, h} \geq \langle \sigma_y \rangle_{S, \beta, h}^G$ (by Proposition 2.2.11), we find

$$\begin{aligned} \langle \sigma_y \rangle_{G, \beta, h} \delta_{x,y} &\geq \sum_{\substack{S \subset G_g \\ y, g \in S \\ 0, x \notin S}} \sum_{\substack{\mathbf{n}_1, \mathbf{n}_2 \in \Omega_{G_g \setminus S} \\ \partial \mathbf{n}_1 = \{0\} \Delta \{x\} \\ \partial \mathbf{n}_2 = \emptyset}} w_{G_g \setminus S}(\mathbf{n}_1) w_{G_g \setminus S}(\mathbf{n}_2) \left(\langle \sigma_y \rangle_{S, \beta, h}^G \sum_{\partial \mathbf{n} = \emptyset} w_S(\mathbf{n}) \right)^2 \mathbb{1}\{\mathcal{S}_0 = S\} \\ &= \sum_{\substack{S \subset G_g \\ y, g \in S \\ 0, x \notin S}} \sum_{\substack{\mathbf{n}_1, \mathbf{n}_2 \in \Omega_{G_g} \\ \partial \mathbf{n}_1 = \{0\} \Delta \{x\} \Delta \{y\} \\ \partial \mathbf{n}_2 = \{y\}}} w(\mathbf{n}_1) w(\mathbf{n}_2) \mathbb{1}\{\mathcal{S}_0 = S\} \\ &= \sum_{\substack{S \subset G_g \\ y, g \in S \\ 0, x \notin S}} \sum_{\substack{\mathbf{n}_1, \mathbf{n}_2 \in \Omega_{G_g} \\ \partial \mathbf{n}_1 = \{0\} \Delta \{x\} \\ \partial \mathbf{n}_2 = \emptyset}} w(\mathbf{n}_1) w(\mathbf{n}_2) \mathbb{1}\{\mathcal{S}_0 = S, y \xleftrightarrow{\mathbf{n}_1 + \mathbf{n}_2} g\} \\ &= \sum_{\substack{\partial \mathbf{n}_1 = \{0\} \Delta \{x\} \\ \partial \mathbf{n}_2 = \emptyset}} w(\mathbf{n}_1) w(\mathbf{n}_2) \mathbb{1}\{0 \not\xleftrightarrow{\mathbf{n}_1 + \mathbf{n}_2} g, y \xleftrightarrow{\mathbf{n}_1 + \mathbf{n}_2} g\} \end{aligned}$$

where we used the switching lemma in the third line. We now sum over the possible values of $\mathcal{S}_g$,

$$\begin{aligned} \langle \sigma_y \rangle_{G, \beta, h} \delta_{x,y} &\geq \sum_{S \subset G} \sum_{\substack{\partial \mathbf{n}_1 = \{0\} \Delta \{x\} \\ \partial \mathbf{n}_2 = \emptyset}} w(\mathbf{n}_1) w(\mathbf{n}_2) \mathbb{1}\{\mathcal{S}_g = S, 0 \not\xleftrightarrow{\mathbf{n}_1 + \mathbf{n}_2} g, y \xleftrightarrow{\mathbf{n}_1 + \mathbf{n}_2} g\} \\ &= \sum_{\substack{S \subset G \\ 0, x \in S \\ y \notin S}} \sum_{\substack{\partial \mathbf{n}_1 = \{0\} \Delta \{x\} \\ \partial \mathbf{n}_2 = \emptyset}} w(\mathbf{n}_1) w(\mathbf{n}_2) \mathbb{1}\{\mathcal{S}_g = S\}. \end{aligned}$$

Using the same ideas as above, if $\mathcal{S}_g = S$ then the currents $\mathbf{n}_1$ and $\mathbf{n}_2$ must vanish on each $\{u, v\}$ with $u \in S$ and $v \notin S$. Thus, for $i = 1, 2$, we still have the decomposition

$$\mathbf{n}_i = \mathbf{n}_i^S + \mathbf{n}_i^{G_g \setminus S}. \quad (2.4.29)$$

This observation yields

$$\langle \sigma_y \rangle_{G, \beta, h} \delta_{x,y} \geq \sum_{\substack{S \subset G \\ 0, x \in S \\ y \notin S}} \sum_{\substack{\mathbf{n}_1, \mathbf{n}_2 \in \Omega_{G_g} \\ \partial \mathbf{n}_1 = \emptyset \\ \partial \mathbf{n}_2 = \emptyset}} w(\mathbf{n}_1) w(\mathbf{n}_2) \langle \sigma_0 \sigma_x \rangle_{S, \beta, 0} \mathbb{1}\{\mathcal{S}_g = S\}. \quad (2.4.30)$$

By plugging this last inequality into (2.4.25), we get

$$\begin{aligned} \frac{\partial}{\partial \beta} \langle \sigma_0 \rangle_{G, \beta, h}^2 &= 2 \langle \sigma_0 \rangle_{G, \beta, h} \frac{\partial}{\partial \beta} \langle \sigma_0 \rangle_{G, \beta, h} \\ &\geq \frac{2}{(Z_{G, \beta, h}^\emptyset)^2} \inf_{y \in G} \frac{\langle \sigma_0 \rangle_{G, \beta, h}}{\langle \sigma_y \rangle_{G, \beta, h}} \sum_{\substack{S \subset G \\ 0 \in S}} \sum_{x \in S} \sum_{y \in G \setminus S} \sum_{\substack{\mathbf{n}_1, \mathbf{n}_2 \in \Omega_{G_g} \\ \partial \mathbf{n}_1 = \emptyset \\ \partial \mathbf{n}_2 = \emptyset}} w(\mathbf{n}_1) w(\mathbf{n}_2) J_{x, y} \langle \sigma_0 \sigma_x \rangle_{S, \beta} \mathbb{1}\{\mathcal{S}_g = S\}. \end{aligned} \quad (2.4.31)$$

Notice that

$$\sum_{\substack{x \in S \\ y \in G \setminus S}} J_{x, y} \langle \sigma_0 \sigma_x \rangle_{S, \beta, 0} \geq \frac{1}{\beta} \varphi_\beta(S) - \sum_{\substack{x \in S \\ y \in V \setminus G}} J_{x, y} \langle \sigma_0 \sigma_x \rangle_{S, \beta}. \quad (2.4.32)$$

Thus, noticing that

$$\sum_{\substack{S \subset G \\ 0 \in S}} \sum_{\substack{x \in S \\ y \in V \setminus G}} J_{x, y} \langle \sigma_0 \sigma_x \rangle_{S, \beta} \frac{1}{(Z_{G, \beta, h}^\emptyset)^2} \sum_{\substack{\mathbf{n}_1, \mathbf{n}_2 \in \Omega_{G_g} \\ \partial \mathbf{n}_1 = \emptyset \\ \partial \mathbf{n}_2 = \emptyset}} w(\mathbf{n}_1) w(\mathbf{n}_2) \mathbb{1}\{\mathcal{S}_g = S\} = \varepsilon(G, \beta, h), \quad (2.4.33)$$

we get

$$\begin{aligned} \frac{\partial}{\partial \beta} \langle \sigma_0 \rangle_{G, \beta, h}^2 &\geq 2c(G, \beta, h) \left[\frac{1}{\beta} \sum_{\substack{S \subset G \\ 0 \in S}} \frac{\varphi_\beta(S)}{(Z_{G, \beta, h}^\emptyset)^2} \sum_{\substack{\mathbf{n}_1, \mathbf{n}_2 \in \Omega_{G_g} \\ \partial \mathbf{n}_1 = \emptyset \\ \partial \mathbf{n}_2 = \emptyset}} w(\mathbf{n}_1) w(\mathbf{n}_2) \mathbb{1}\{\mathcal{S}_g = S\} - \varepsilon(G, \beta, h) \right] \\ &\geq 2c(G, \beta, h) \\ &\quad \times \left[\frac{1}{\beta} \cdot \inf_{S \ni 0} \varphi_\beta(S) \cdot \frac{\sum_{\partial \mathbf{n}_1 = \partial \mathbf{n}_2 = \emptyset} w(\mathbf{n}_1) w(\mathbf{n}_2) \left(1 - \mathbb{1}\{0 \xrightarrow{\mathbf{n}_1 + \mathbf{n}_2} g\} \right)}{\sum_{\partial \mathbf{n}_1 = \partial \mathbf{n}_2 = \emptyset} w(\mathbf{n}_1) w(\mathbf{n}_2)} - \varepsilon(G, \beta, h) \right] \\ &= 2c(G, \beta, h) \left[\frac{1}{\beta} \cdot \inf_{S \ni 0} \varphi_\beta(S) \cdot \left(1 - \langle \sigma_0 \rangle_{G, \beta, h}^2 \right) - \varepsilon(G, \beta, h) \right], \end{aligned} \quad (2.4.34)$$

where we applied the switching lemma one more time in the last equality. $\square$

We now turn to the proof of (ii) and (iii). The following result is an improvement of Proposition 2.2.13 due to Lieb [178].

Lemma 2.4.4 (Improved Simon's inequality). *Let S be a finite subset of V containing 0. For every $z \notin S$,*

$$\langle \sigma_0 \sigma_z \rangle_\beta^+ \leq \sum_{x \in S} \sum_{y \notin S} \langle \sigma_0 \sigma_x \rangle_{S, \beta} \beta J_{x, y} \langle \sigma_y \sigma_z \rangle_\beta^+. \quad (2.4.35)$$

Proof. We use the notion of backbone developed in the preceding section. We need some notations. Let G be a finite subset of V containing S . For a consistent sequence of steps γ with endpoints $\partial \gamma = \{x, y\}$, we set

$$\rho_S^G(\gamma) := \frac{\sum_{\partial \mathbf{n} = \{x, y\}} w_S(\mathbf{n}) \mathbb{1}\{\Gamma(\mathbf{n}) = \gamma\}}{\sum_{\partial \mathbf{n} = \emptyset} w_S(\mathbf{n})}. \quad (2.4.36)$$

Let $\gamma = (\{x_k, x_{k+1}\})_{0 \leq k \leq n-1}$ be a consistent sequence of steps from 0 to z with $z \notin S$. Define k to be the first integer for which $x_{k+1} \notin S$. The following occurs: γ goes from 0 to x_k staying in S_g , then it takes the step $x_k x_{k+1}$, and then finally γ goes from x_{k+1} to z in G_g . Thus, using Proposition 2.3.4,

$$\begin{aligned} \langle \sigma_0 \sigma_z \rangle_{G, \beta, h} &= \sum_{\gamma: 0 \rightarrow z} \rho_G^G(\gamma) \\ &\leq \sum_{x \in S} \sum_{y \notin S} \sum_{\substack{\gamma_1: 0 \rightarrow x \\ \gamma_2: x \rightarrow y \\ \gamma_3: y \rightarrow z}} \rho_S^G(\gamma_1) \rho_{\{x, y\}}^G(\gamma_2) \rho_{G \setminus \overline{\gamma_1 \circ \gamma_2}}^G(\gamma_3) \\ &\leq \sum_{x \in S} \sum_{y \notin S} \langle \sigma_0 \sigma_x \rangle_{S, \beta, h}^G \langle \sigma_x \sigma_y \rangle_{\{x, y\}, \beta, h}^G \langle \sigma_y \sigma_z \rangle_{G, \beta, h}. \end{aligned}$$

The proof follows by taking the limits $G \nearrow V$, letting $h \searrow 0$, and noticing that,

$$\langle \sigma_x \sigma_y \rangle_{\{x, y\}, \beta, h}^G \xrightarrow[G \nearrow V]{} \langle \sigma_x \sigma_y \rangle_{\{x, y\}, \beta, h}, \quad (2.4.37)$$

and

$$\langle \sigma_x \sigma_y \rangle_{\{x, y\}, \beta, h} \xrightarrow[h \searrow 0]{} \langle \sigma_x \sigma_y \rangle_{\{x, y\}, \beta} = \tanh(\beta J_{x, y}) \leq \beta J_{x, y}. \quad (2.4.38)$$

□

Proof of Theorem 2.4.1 (ii)–(iii). We begin with (ii). Let $\beta < \tilde{\beta}_c$. Fix a finite set S containing 0 such that $\varphi_\beta(S) < 1$. Define,

$$\tilde{\chi}_n(\beta) := \sup_{u \in V} \sum_{v \in \Lambda_n} \langle \sigma_u \sigma_v \rangle_\beta^+. \quad (2.4.39)$$

For $u \in V$, we let $\tau_u \in \Gamma$ be an automorphism of $\mathcal{G}$ that sends 0 to u . Then, using Lemma 2.4.4 and the model's invariance under Γ , we get

$$\begin{aligned} \sum_{v \in \Lambda_n} \langle \sigma_u \sigma_v \rangle_\beta^+ &= \sum_{v \in \tau_u(S) \cap \Lambda_n} \langle \sigma_u \sigma_v \rangle_\beta^+ + \sum_{v \in \Lambda_n \setminus \tau_u(S)} \langle \sigma_u \sigma_v \rangle_\beta^+ \\ &\leq |S| + \sum_{v \in \Lambda_n \setminus \tau_u(S)} \sum_{x \in \tau_u(S)} \sum_{y \notin \tau_u(S)} \tanh(\beta J_{x, y}) \langle \sigma_u \sigma_x \rangle_{\tau_u(S), \beta, 0} \langle \sigma_y \sigma_v \rangle_\beta^+ \\ &= |S| + \sum_{x \in \tau_u(S)} \sum_{y \notin \tau_u(S)} \tanh(\beta J_{x, y}) \langle \sigma_u \sigma_x \rangle_{\tau_u(S), \beta, 0} \left(\sum_{v \in \Lambda_n \setminus \tau_u(S)} \langle \sigma_y \sigma_v \rangle_\beta^+ \right) \\ &\leq |S| + \tilde{\chi}_n(\beta) \sum_{x \in \tau_u(S)} \sum_{y \notin \tau_u(S)} \tanh(\beta J_{x, y}) \langle \sigma_u \sigma_x \rangle_{\tau_u(S), \beta, 0} \\ &= |S| + \tilde{\chi}_n(\beta) \sum_{x \in S} \sum_{y \notin S} \tanh(\beta J_{x, y}) \langle \sigma_0 \sigma_x \rangle_{S, \beta, 0} \\ &= |S| + \tilde{\chi}_n(\beta) \varphi_\beta(S). \end{aligned}$$

Optimising in u , we get

$$\tilde{\chi}_n(\beta) \leq |S| + \tilde{\chi}_n(\beta) \varphi_\beta(S), \quad (2.4.40)$$

so that

$$\tilde{\chi}_n(\beta) \leq \frac{|S|}{1 - \varphi_\beta(S)}. \quad (2.4.41)$$

Letting $n \rightarrow \infty$ above, we get that for $\beta < \tilde{\beta}_c$

$$\chi(\beta) \leq \frac{|S|}{1 - \varphi_\beta(S)} < \infty. \quad (2.4.42)$$

This yields $\tilde{\beta}_c \leq \beta_c$. Indeed, for $\beta > \beta_c$,

$$\chi(\beta) = \sum_{x \in V} \langle \sigma_0 \sigma_x \rangle_\beta^+ \geq \sum_{x \in V} m^*(\beta)^2 = \infty. \quad (2.4.43)$$

Combined with the previous results, this gives $\tilde{\beta}_c = \beta_c$.

Finally, we prove (iii). Let R be the range of J . Let $\beta < \beta_c$. Let S be such that $\varphi_\beta(S) < 1$ and L be such that $S \subset \Lambda_{L-R}$. By Lemma 2.4.4, for any z with $d(0, z) > L$,

$$\begin{aligned} \langle \sigma_0 \sigma_z \rangle_\beta^+ &\leq \sum_{x \in S} \sum_{y \notin S} \tanh(\beta J_{x,y}) \langle \sigma_0 \sigma_x \rangle_{S, \beta, 0} \langle \sigma_y \sigma_z \rangle_\beta^+ \\ &\leq \varphi_\beta(S) \cdot \max_{y \in \Lambda_L} \langle \sigma_y \sigma_z \rangle_\beta^+. \end{aligned}$$

The proof follows by iterating the above procedure $\lfloor d(0, z)/L \rfloor$ times. $\square$

Remark 2.4.5. The preceding proof also gives that

$$\sum_{x \in V} \langle \sigma_0 \sigma_x \rangle_{\beta_c}^+ = \infty. \quad (2.4.44)$$

Indeed, we have $\varphi_{\beta_c}(\Lambda_n) \geq 1$ for all $n \geq 0$. Using monotonicity in boundary conditions, we have that

$$\langle \sigma_0 \sigma_x \rangle_{\beta_c}^+ \geq \langle \sigma_0 \sigma_x \rangle_{\beta_c} \geq \langle \sigma_0 \sigma_x \rangle_{\Lambda_n, \beta_c, 0} \quad (2.4.45)$$

so that

$$\left(\sum_{y \in V} \beta_c J_{0,y} \right) \cdot \sum_{x \in V} \langle \sigma_0 \sigma_x \rangle_{\beta_c}^+ \geq \sum_{n \geq 0} \varphi_{\beta_c}(\Lambda_n) = \infty. \quad (2.4.46)$$

2.5 Continuity of the phase transition

In this section, $\mathcal{G} = (V, E)$ is a locally-finite, countable, vertex transitive, and *amenable* graph⁵ that contains a fixed origin 0. Let d be the graph distance and Γ be a subgroup of the automorphism group that acts transitively on V . As in the preceding section, we assume that J is ferromagnetic, locally finite, and invariant under Γ . Additionally, we require that J is *aperiodic*, that is: for all $x \in V$, there exist $n \geq 1$ and $0 = x_0, \dots, x_n = x$ such that

$$J_{0,x_1} \dots J_{x_{n-1},x_n} > 0. \quad (2.5.1)$$

We are interested in the behaviour of the spontaneous magnetisation at criticality. It is classical⁶ that m^* is right-continuous at β_c . Therefore, m^* is continuous at the critical point if and only if $m^*(\beta_c) = 0$. This corresponds to a *continuous* (or *second order*) phase transition. The following criterion for continuity was derived in [10].

⁵This means that $\inf_{S \subset V, |S| < \infty} \frac{|\partial_{\text{ext}} S|}{|S|} = 0$, where $\partial S_{\text{ext}} := \{x \in V \setminus S : \exists y \in S, xy \in E\}$.

⁶Indeed, m^* is the limit of a decreasing sequence of (finite volume, hence) continuous functions.

Theorem 2.5.1. *Assume that we work with the setup described at the beginning of the section. If*

$$\lim_{d(x,y) \rightarrow \infty} \langle \sigma_x \sigma_y \rangle_{\beta_c} = 0, \quad (2.5.2)$$

then

$$m^*(\beta_c) = 0. \quad (2.5.3)$$

One can rephrase the result as follows: as soon as the critical Ising measure with free boundary conditions has no long-range order, then so does the $+$ Gibbs measure. In fact, in such cases, one also has that⁷ $\langle \cdot \rangle_{\beta_c} = \langle \cdot \rangle_{\beta_c}^+$.

One can verify the above criterion in the case of the nearest-neighbor Ising model on $\mathbb{Z}^d$ in dimensions $d \geq 3$. Indeed, thanks to the *infrared bound* (see [104, 103]) and the *Messenger–Miracle–Solé inequalities* [191]: there exists $C = C(d) > 0$ such that for all $x \in \mathbb{Z}^d \setminus \{0\}$,

$$\langle \sigma_0 \sigma_x \rangle_{\beta_c} \leq \frac{C}{|x|^{d-2}}, \quad (2.5.4)$$

where $|\cdot|$ in the ℓ^∞ norm on $\mathbb{R}^d$.

Corollary 2.5.2. *Let $d \geq 3$. Then, if J is the nearest-neighbour ferromagnetic interaction,*

$$m^*(\beta_c) = 0. \quad (2.5.5)$$

Remark 2.5.3. The corresponding result for the two-dimensional case relies on integrability [255] or planarity [247].

We now turn to the proof of Theorem 2.5.1. The first step involves constructing infinite volume limits of the current measures introduced above.

The measure $\mathbf{P}_{G,\beta}^\emptyset$ on Ω_G , where G is a finite subset of V , induces a measure $\mathbf{P}_{G,\beta}^{\text{free}}$ on Ω_V . The same applies for the measure $\mathbf{P}_{G,\beta}^\emptyset$ with plus boundary conditions, and the induced measure is denoted $\mathbf{P}_{G,\beta}^+$. Recall that $\Lambda_n := \{x \in V : d(0, x) \leq n\}$.

Proposition 2.5.4. *Let $\beta > 0$. There exist two measures $\mathbf{P}_\beta^{\text{free}}$ and $\mathbf{P}_\beta^+$ on Ω_V such that:*

(P1) *For any event $\mathcal{A}$ depending on finitely many edges,*

$$\lim_{n \rightarrow \infty} \mathbf{P}_{\Lambda_n, \beta}^{\text{free}}[\mathcal{A}] = \mathbf{P}_\beta^{\text{free}}[\mathcal{A}], \quad \lim_{n \rightarrow \infty} \mathbf{P}_{\Lambda_n, \beta}^+[\mathcal{A}] = \mathbf{P}_\beta^+[\mathcal{A}]. \quad (2.5.6)$$

(P2) *The measures $\mathbf{P}_\beta^{\text{free}}$ and $\mathbf{P}_\beta^+$ are invariant under the automorphisms of Γ .*

(P3) *The measures $\mathbf{P}_\beta^{\text{free}}$ and $\mathbf{P}_\beta^+$ are mixing (hence ergodic) with respect to Γ .*

Proof. The proof is essentially the same for $+$ and free boundary conditions. In what follows we fix $\# \in \{+, \text{free}\}$. We denote the set of edges of $G \subset V$ by $\mathcal{P}_2(G)$ (i.e the set of pairs $\{x, y\} \subset G$).

⁷Indeed, it is a consequence of the following general result: for $(\beta, h) \in \mathbb{R}^+ \times \mathbb{R}$, if $\langle \sigma_0 \rangle_{\beta, h}^+ = \langle \sigma_0 \rangle_{\beta, h}^-$ then $\langle \cdot \rangle_{\beta, h}^+ = \langle \cdot \rangle_{\beta, h}^-$. This result can be found in [101, Theorem 3.28].

Let us start with the proof of **(P1)**. Notice that the distribution of the random currents simplifies into a product measure when conditioned on the parity variables $\mathbf{r} = (\mathbf{r}_{xy})_{\{x,y\} \in \mathcal{P}_2(V)}$, with

$$\mathbf{r}_{xy} = (-1)^{\mathbf{n}_{xy}}. \quad (2.5.7)$$

The conditional distribution of $\mathbf{n}$, given $\mathbf{r}$, is simply the product measure of independent Poisson measures of mean values $\beta J_{x,y}$, and conditioned on the corresponding parity. Thus, to prove the convergence, it suffices to establish convergence of the law of the parity variables. For a set of edges E , define the events

$$\mathcal{C}_E = \{\mathbf{r}_{xy} = 1, \forall \{x,y\} \in E\}, \quad (2.5.8)$$

$$\mathcal{C}_E^{(0)} = \{\mathbf{n}_{xy} = 0, \forall \{x,y\} \in E\}. \quad (2.5.9)$$

We prove that $\mathbf{P}_{\Lambda_n, \beta}^\#(\mathcal{C}_E)$ converges (as n goes to infinity) for any finite set of edges E .

Write $\Omega_n^\# = \Omega_{\Lambda_n}$ for $\# = \text{free}$ and $\Omega_n^\# = \Omega_{\Lambda_n \cup \{\mathfrak{g}\}}$ for $\# = +$. Choose n large enough so that $E \subset \mathcal{P}_2(\Lambda_n)$. Then,

$$\mathbf{P}_{\Lambda_n, \beta}^\#[\mathcal{C}_E] = \frac{\sum_{\mathbf{n} \in \Omega_n^\# : \partial \mathbf{n} = \emptyset} w_\beta(\mathbf{n}) \mathbb{1}\{\mathcal{C}_E\}}{\sum_{\mathbf{n} \in \Omega_n^\# : \partial \mathbf{n} = \emptyset} w(\mathbf{n})} \quad (2.5.10)$$

$$= \frac{\sum_{\mathbf{n} \in \Omega_n^\# : \partial \mathbf{n} = \emptyset} w_\beta(\mathbf{n}) \mathbb{1}\{\mathcal{C}_E^{(0)}\}}{\sum_{\mathbf{n} \in \Omega_n^\# : \partial \mathbf{n} = \emptyset} w(\mathbf{n})} \prod_{\{x,y\} \in E} \cosh(\beta J_{x,y}) \quad (2.5.11)$$

$$= \frac{Z^\#(\Lambda_n(E), \beta)}{Z^\#(\Lambda_n, \beta)} \prod_{\{x,y\} \in E} \cosh(\beta J_{x,y}) \quad (2.5.12)$$

where $\Lambda_n(E)$ is the graph of vertex set Λ_n and of edge set $\mathcal{P}_2(\Lambda_n) \setminus E$, and where $Z^\#$ denotes the partition function of the Ising model with $\#$ boundary condition. Now, one has

$$\frac{Z^\#(\Lambda_n(E), \beta)}{Z^\#(\Lambda_n, \beta)} = \langle \exp(-\beta K_E) \rangle_{\Lambda_n, \beta}^\# \quad (2.5.13)$$

where

$$K_E = \sum_{\{x,y\} \in E} J_{x,y} \sigma_x \sigma_y. \quad (2.5.14)$$

Hence,

$$\mathbf{P}_{\Lambda_n, \beta}^\#[\mathcal{C}_E] = \langle \exp(-\beta K_E) \rangle_{\Lambda_n, \beta}^\# \prod_{\{x,y\} \in E} \cosh(\beta J_{x,y}). \quad (2.5.15)$$

The convergence of the above expression follows directly from the convergence of the correlation functions as n tends to infinity (expand the exponential in (2.5.15)). To conclude, it suffices to notice that the events $\mathcal{C}_E$, with E ranging over finite sets of edges, span the algebra of events expressible in terms of finite collections of the binary variables of $\mathbf{r}$. This fact, and the above observation that the probability distribution of the random current $\mathbf{n}$ conditioned on $\mathbf{r}$ does not depend on n , imply the existence of $\mathbf{P}_\beta^\#$.

We now prove **(P2)**. Fix $x \in V$. Let $\gamma_x \in \Gamma$ be an automorphism satisfying $\gamma_x(0) = x$. It suffices to notice that the limit $\lim_{n \rightarrow \infty} \mathbf{P}_{\Lambda_n, \beta}^\#[\mathcal{C}_E]$ is unchanged if the sequence $(\Lambda_n)_{n \geq 1}$ is

replaced by the sequence $(\gamma_x \Lambda_n)_{n \geq 1}$. This follows from invariance under automorphisms of J , (2.5.15), and the fact that $\langle \cdot \rangle_{\gamma_x \Lambda_n, \beta}^\#$ converges weakly to $\langle \cdot \rangle_\beta^\#$.

Finally, we prove (P3). Every automorphism invariant event can be approximated by events depending on a finite number of edges. Thus, it is sufficient to prove that for any events A and B depending on a finite number of edges,

$$\lim_{d(0,x) \rightarrow \infty} \mathbf{P}_\beta^\# [A \cap \gamma_x B] = \mathbf{P}_\beta^\# [A] \mathbf{P}_\beta^\# [B]. \quad (2.5.16)$$

Thanks to the conditional independence of the $\mathbf{n}_{xy}$ given the parity variables $\mathbf{r}_{xy}$, the requirements can be further simplified to the proof that for any two finite sets of edges E and F ,

$$\lim_{d(0,x) \rightarrow \infty} \mathbf{P}_\beta^\# [\mathcal{C}_{E \cup \gamma_x F}] = \mathbf{P}_\beta^\# [\mathcal{C}_E] \mathbf{P}_\beta^\# [\mathcal{C}_F]. \quad (2.5.17)$$

Choosing x “large” enough so that $E \cap \gamma_x F = \emptyset$ and using (2.5.15) we get that

$$\frac{\mathbf{P}_\beta^\# [\mathcal{C}_{E \cup \gamma_x F}]}{\mathbf{P}_\beta^\# [\mathcal{C}_E] \mathbf{P}_\beta^\# [\mathcal{C}_F]} = \frac{\langle \exp(-\beta K_E - \beta K_{\gamma_x F}) \rangle_\beta^\#}{\langle \exp(-\beta K_E) \rangle_\beta^\# \langle \exp(-\beta K_F) \rangle_\beta^\#}. \quad (2.5.18)$$

Ergodicity then follows from the statement that the above ratio tends to 1. This statement holds as a consequence of the mixing property of the Gibbs states $\langle \cdot \rangle_\beta^\#$ when restricted to functions which are invariant under global spin flip. The precise statement is presented in Lemma 2.5.5 below. $\square$

Lemma 2.5.5 ([10, Proposition A.1]). *For any $\beta > 0$, the measure $\langle \cdot \rangle_\beta^\#$ is ergodic and mixing in its restriction to the σ -algebra of even events. In other words, for any pair of functions f, g from spin configurations to $\mathbb{R}$ with $f(-\sigma) = f(\sigma)$ and $g(-\sigma) = g(\sigma)$, the following limit exists and satisfies:*

$$\lim_{d(0,x) \rightarrow \infty} \langle f(\sigma)(g \circ \gamma_x)(\sigma) \rangle_\beta^\# = \langle f(\sigma) \rangle_\beta^\# \langle g(\sigma) \rangle_\beta^\#. \quad (2.5.19)$$

Define $\mathbf{P}_\beta$ to be the law of $\widehat{\mathbf{n}_1 + \mathbf{n}_2}$ with $(\mathbf{n}_1, \mathbf{n}_2)$ distributed according to $\mathbf{P}_\beta^{\text{free},+} := \mathbf{P}_\beta^{\text{free}} \otimes \mathbf{P}_\beta^+$. Denote by $\mathbf{E}_\beta$ the expectation with respect to $\mathbf{P}_\beta$. It follows from Proposition 2.5.4 that $\mathbf{P}_\beta$ is invariant and ergodic with respect to elements of Γ . The following proposition is a key step of the proof. It extends the so-called *Burton–Keane* argument to the highly dependent percolation measure $\mathbf{P}_\beta$.

Theorem 2.5.6. *For any $\beta > 0$, there exists at most one infinite cluster $\mathbf{P}_\beta$ -almost surely.*

The proof of this theorem is essentially the same as the one of [57] used for (independent) Bernoulli percolation. The finite energy property used in [57] is replaced here by *insertion tolerance*.

Lemma 2.5.7 (Insertion tolerance). *Consider the map Φ_N from percolation configurations onto itself opening all edges $\{x, y\}$ in Λ_N with $J_{x,y} > 0$. Let $N > 0$, then there exists $c = c(N, J, \beta) > 0$ such that for any event $\mathcal{E}$,*

$$\mathbf{P}_\beta[\Phi_N(\mathcal{E})] \geq c \mathbf{P}_\beta[\mathcal{E}]. \quad (2.5.20)$$

Proof. It is sufficient to consider events $\mathcal{E}$ depending on a finite number of edges. Let $\mathbf{P}_{\Lambda_n, \beta}$ be the law of $\widehat{\mathbf{n}_1 + \mathbf{n}_2}$ with $(\mathbf{n}_1, \mathbf{n}_2)$ distributed according to $\mathbf{P}_{\Lambda_n, \beta}^{\text{free}, +} := \mathbf{P}_{\Lambda_n, \beta}^{\text{free}} \otimes \mathbf{P}_{\Lambda_n, \beta}^+$. Proposition 2.5.4 shows that $\mathbf{P}_{\Lambda_n, \beta}$ converges weakly to $\mathbf{P}_\beta$. The lemma reduces to showing the existence of $c = c(N, J, \beta)$ with $\mathbf{P}_{\Lambda_n, \beta}$ in place of $\mathbf{P}_\beta$ and c not depending on $n > N$.

Define a map $\Psi_N : (\Omega_{\Lambda_n})^2 \rightarrow (\Omega_{\Lambda_n})^2$ by $\Psi_N(\mathbf{n}_1, \mathbf{n}_2)(x, y) = (0, 2)$ if $(\mathbf{n}_1(x, y), \mathbf{n}_2(x, y)) = (0, 0)$, $J_{x, y} > 0$ and $x, y \in \Lambda_N$; and $\Psi_N(\mathbf{n}_1, \mathbf{n}_2)(x, y) = (\mathbf{n}_1(x, y), \mathbf{n}_2(x, y))$ otherwise.

Fix $\omega \in \{0, 1\}^{\mathcal{P}_2(\Lambda_n)}$ and let $\Omega_2(\omega) = \{(\mathbf{n}_1, \mathbf{n}_2) \in (\Omega_{\Lambda_n})^2 : \widehat{\mathbf{n}_1 + \mathbf{n}_2} = \omega\}$. The set $\Psi_N(\Omega_2(\omega))$ is obtained from $\Omega_2(\omega)$ by changing the value of the current $\mathbf{n}_2$ on edges $\{x, y\}$ with $\omega_{x, y} = 0$ from 0 to 2. We have that

$$\begin{aligned}
\mathbf{P}_{\Lambda_n, \beta}[\Phi_N(\mathcal{E})] &= \sum_{\omega' \in \Phi_N(\mathcal{E})} \mathbf{P}_{\Lambda_n, \beta}[\omega'] \\
&= \sum_{\omega \in \mathcal{E}} \frac{1}{|\Phi_N^{-1}(\{\Phi_N(\omega)\})|} \mathbf{P}_{\Lambda_n, \beta}[\Phi_N(\omega)] \\
&\geq \frac{1}{2^{|\Lambda_N|^2}} \sum_{\omega \in \mathcal{E}} \mathbf{P}_{\Lambda_n, \beta}[\Phi_N(\omega)] \\
&= \frac{1}{2^{|\Lambda_N|^2}} \sum_{\omega \in \mathcal{E}} \mathbf{P}_{\Lambda_n, \beta}^{\text{free}, +}[\Psi_N(\Omega_2(\omega))] \\
&\geq \frac{1}{2^{|\Lambda_N|^2}} \sum_{\omega \in \mathcal{E}} \left(\prod_{\{x, y\} \subset \Lambda_N : \omega_{\{x, y\}} = 0} \frac{(\beta J_{x, y})^2}{2} \right) \mathbf{P}_{\Lambda_n, \beta}^{\text{free}, +}[\Omega_2(\omega)] \\
&\geq c \sum_{\omega \in \mathcal{E}} \mathbf{P}_{\Lambda_n, \beta}^{\text{free}, +}[\Omega_2(\omega)] \\
&= c \mathbf{P}_{\Lambda_n, \beta}[\mathcal{E}]
\end{aligned}$$

where $c = c(N, J, \beta)$ does not depend on n . $\square$

Proof of Theorem 2.5.6. For $\ell \in \mathbb{N} \cup \{\infty\}$, let $\mathcal{E}_\ell$ be the event that there exist exactly ℓ disjoint infinite clusters. Our goal is to show that

$$\mathbf{P}_\beta[\mathcal{E}_\ell] = 0 \tag{2.5.21}$$

for $\ell \geq 2$. Note that the ergodicity of $\mathbf{P}_\beta$ yields that

$$\mathbf{P}_\beta[\mathcal{E}_\ell] \in \{0, 1\} \tag{2.5.22}$$

for all ℓ . The following proof is a variation of the Burton–Keane argument known for Bernoulli percolation.

We start by constructing a convenient sequence of finite domains $(B_N)_N$ as follows. For each $x \in V$, consider a sequence of vertices $\pi_{0, x} = (u_0 = 0, u_1, \dots, u_k = x)$ with $J_{u_i, u_{i+1}} > 0$ for all $0 \leq i \leq k-1$, which exists by the irreducibility assumption. We then consider the domains $B_N := \bigcup_{x \in B_N} \pi_x$.

We first establish that $\mathbf{P}_\beta[\mathcal{E}_\ell] = 0$ for $\ell \in \mathbb{N} \setminus \{0, 1\}$. Let $\ell \in \mathbb{N}$ with $\ell \geq 2$ and assume by contradiction that $\mathbf{P}_\beta[\mathcal{E}_\ell] = 1$. Let $\mathcal{I}_N = \mathcal{I}_N(\ell) \subset \mathcal{E}_\ell$ be the event that the ℓ infinite clusters intersect Λ_N . Observe that $\mathbf{P}_\beta[\mathcal{I}_N] \xrightarrow{N \rightarrow \infty} 1$, so in particular there exists N such that $\mathbf{P}_\beta[\mathcal{I}_N] > 0$. By Proposition 2.5.7, we have

$$\mathbf{P}_\beta[\Phi_N(\mathcal{I}_N)] > 0. \tag{2.5.23}$$

However, one can easily see that $\Phi_N(\mathcal{I}_N) \subset \mathcal{E}_1$, and therefore $\mathbf{P}_\beta[\mathcal{E}_1] > 0$, which is a contradiction with $\mathbf{P}_\beta[\mathcal{E}_\ell] = 1$, as desired.

Now we establish $\mathbf{P}_\beta[\mathcal{E}_\infty] = 0$. Assume by contradiction that $\mathbf{P}_\beta[\mathcal{E}_\infty] = 1$. Let $\mathcal{J}_N$ be the event that at least three distinct infinite clusters intersect B_N . Note that $\mathbf{P}_\beta[\mathcal{J}_N] \xrightarrow{N \rightarrow \infty} 1$. Hence there is some N , fixed throughout, such that

$$\mathbf{P}_\beta[\mathcal{J}_N] > 0. \quad (2.5.24)$$

We introduce the following notion of *coarse trifurcation*. Let $\mathbf{CT}_0$ denote the set of $\mathbf{n}$ such that:

- $\hat{\mathbf{n}}_{xy} > 0$ for all edges $xy \in B_N$.
- The cluster of the origin $\mathbf{C}(0)$ breaks into at least three distinct infinite clusters by removing edges in B_N .

One can easily verify that $\Phi_N(\mathcal{J}_N) \subset \mathbf{CT}_0$, hence by Proposition 2.5.7 we have

$$\mathbf{P}_\beta[\mathbf{CT}_0] > 0. \quad (2.5.25)$$

Furthermore, note that by Γ -invariance, we have $\mathbf{P}_\beta[\mathbf{CT}_0] = \mathbf{P}_\beta[\mathbf{CT}_x]$ for all $x \in V$, where $\mathbf{CT}_x = \gamma_x \mathbf{CT}_0$, and $\gamma_x \in \Gamma$ is an automorphism such that $\gamma_x(0) = x$. A vertex $x \in V$ is called a coarse trifurcation if $\mathbf{CT}_x$ occurs.

For each $x \in V$ denote $B_N(x) = \gamma_x B_N$. Let $n \in \mathbb{N}$ and define $S_n \subset \Lambda_n$ to be a set of (sparse) vertices such that $B_N(x) \cap B_N(y) = \emptyset$ for all distinct $x, y \in S_n$, and S_n is maximal with respect to this property. Then

$$|S_n| \geq \frac{|\Lambda_n|}{|\Lambda_K|}, \quad (2.5.26)$$

where $K > N$ is large enough so that $B_N \subset \Lambda_K$. Let $T_n = T(S_n)$ denote the set of coarse trifurcations in S_n , and L_n denote the set of open edges between Λ_n and $V \setminus \Lambda_n$. There is a natural way of constructing a forest where every leaf corresponds to an element of L_n , while every element of T_n corresponds to a vertex of degree at least three.

To see this, define a new (random) graph H as follows. For $t \in T_n$, we identify all vertices of $B_N(t)$ into a single vertex $v(t)$. The vertex set of H is then given by the union of the vertices in $V \setminus \bigcup_{t \in T_n} B_N(t)$ together with the vertices $\{v(t) : t \in T_n\}$. The edge set of H is the natural edge set induced from $\hat{\mathbf{n}}$ under the equivalence relation defining $v(t)$. Note that for all $t \in T_n$, $v(t)$ is a trifurcation in H , i.e. the cluster of $v(t)$ in H splits into at least 3 infinite clusters when we remove $v(t)$. We now restrict H to the graph H' which is induced by the edges with at least one endpoint in $\Pi(\Lambda_n)$, where Π is the natural map $G \mapsto H$. Take a spanning forest of H' and successively delete vertices of degree 1 from the spanning forest whose pre-image lies Λ_n ; note that this does not affect the vertices $v(t)$, which still have degree at least 3. Moreover, the only leaves of the resulting graph correspond to elements of L_n .

Therefore, we have

$$|T_n| \leq |L_n|. \quad (2.5.27)$$

On one hand, by (2.5.25) and Γ -invariance,

$$\mathbf{E}_\beta^{\text{free},+}[|T_n|] = |S_n| \mathbf{P}_\beta[\mathbf{CT}_0] \geq c(N) |\Lambda_n|, \quad (2.5.28)$$

for some constant $c(N) > 0$. On the other hand, for every $m > 0$ we have, by Γ -invariance,

$$\mathbf{E}_\beta^{\text{free},+}[|L_n|] = \sum_{\substack{x \in \Lambda_n \\ y \in V \setminus \Lambda_n}} \mathbf{P}_\beta^{\text{free},+}[(\mathbf{n}_1 + \mathbf{n}_2)_{xy} > 0] \quad (2.5.29)$$

$$\leq \sum_{x \in \partial_{\text{in}}^m \Lambda_n} 2|J| + \sum_{\substack{x \in \Lambda_n \setminus \partial_{\text{in}}^m \Lambda_n \\ y \in V, d(x,y) \geq m}} \mathbf{P}_\beta^{\text{free},+}[(\mathbf{n}_1 + \mathbf{n}_2)_{xy} > 0] \quad (2.5.30)$$

$$\leq 2|\partial_{\text{in}}^m \Lambda_n||J| + 2|\Lambda_n| \sum_{y \in V, d(0,y) \geq m} J_{0,y} \quad (2.5.31)$$

where $\partial_{\text{in}}^m \Lambda_n := \{x \in \Lambda_n : d(x, V \setminus \Lambda_n) < m\}$. Notice that $|\partial_{\text{in}}^m \Lambda_n| \leq |\Lambda_n||\partial_{\text{ext}} \Lambda_n|$. Since $|J| < \infty$, $c_m := \sum_{y \in V, d(0,y) \geq m} J_{0,y} \rightarrow 0$ as $m \rightarrow \infty$. Altogether, we have

$$\mathbf{E}_\beta^{\text{free},+}[|L_n|] \leq 2|\Lambda_n||\partial_{\text{ext}} \Lambda_n||J| + 2c_m|\Lambda_n|. \quad (2.5.32)$$

Combining (2.5.27), (2.5.28) and (2.5.32), we obtain

$$c(N) \leq 2|\Lambda_n| \frac{|\partial_{\text{ext}} \Lambda_n|}{|\Lambda_n|} |J| + 2c_m. \quad (2.5.33)$$

Letting $n \rightarrow \infty$ and then $m \rightarrow \infty$, we find that $c(N) \leq 0$, thus obtaining the desired contradiction. $\square$

Theorem 2.5.8. *Let $\beta \leq \beta_c$. If $\langle \sigma_0 \sigma_x \rangle_\beta \rightarrow 0$ as $d(0, x)$ goes to infinity, then*

$$\mathbf{P}_\beta[0 \leftrightarrow \infty] = 0. \quad (2.5.34)$$

Proof. Let $n \geq 0$ and $x, y \in \Lambda_n$. Using the switching lemma,

$$\begin{aligned} \mathbf{P}_{\Lambda_n, \beta}^{\text{free},+}[x \xrightarrow{\mathbf{n}_1 + \mathbf{n}_2} y \text{ in } \Lambda_n] &= \frac{\sum_{\substack{\partial \mathbf{n}_1 = \emptyset \\ \partial \mathbf{n}_2 = \emptyset}} w_\beta(\mathbf{n}_1) w_\beta(\mathbf{n}_2) \mathbb{1}\{x \leftrightarrow y \text{ in } \Lambda_n\}}{\sum_{\substack{\partial \mathbf{n}_1 = \emptyset \\ \partial \mathbf{n}_2 = \emptyset}} w_\beta(\mathbf{n}_1) w_\beta(\mathbf{n}_2)} \\ &= \frac{\sum_{\substack{\partial \mathbf{n}_1 = \{x, y\} \\ \partial \mathbf{n}_2 = \{x, y\}}} w_\beta(\mathbf{n}_1) w_\beta(\mathbf{n}_2)}{\sum_{\substack{\partial \mathbf{n}_1 = \emptyset \\ \partial \mathbf{n}_2 = \emptyset}} w_\beta(\mathbf{n}_1) w_\beta(\mathbf{n}_2)}. \end{aligned}$$

Thus,

$$\mathbf{P}_{\Lambda_n, \beta}^{\text{free},+}[x \xrightarrow{\mathbf{n}_1 + \mathbf{n}_2} y \text{ in } \Lambda_n] = \langle \sigma_x \sigma_y \rangle_{\Lambda_n, \beta} \langle \sigma_x \sigma_y \rangle_{\Lambda_n, \beta}^+ \leq \langle \sigma_x \sigma_y \rangle_{\Lambda_n, \beta}. \quad (2.5.35)$$

The right-hand side tends to $\langle \sigma_x \sigma_y \rangle_\beta$ as n tends to infinity. Proposition 2.5.4 yields that the left-hand side converges to $\mathbf{P}_\beta[x \leftrightarrow y]$ (the percolation event does not depend on finitely many edges but one can first modify it by requiring that connection occurs in a box of a given size N and then let N tend to infinity). As a result,

$$\mathbf{P}_\beta[x \leftrightarrow y] \leq \langle \sigma_x \sigma_y \rangle_\beta. \quad (2.5.36)$$

Now, let $B \subset V$ be a finite set. Applying the Cauchy–Schwarz inequality to the random variable $X = \sum_{x \in B} \mathbb{1}\{x \leftrightarrow \infty\}$ leads to

$$(|B| \mathbf{P}_\beta[0 \leftrightarrow \infty])^2 = \mathbf{E}_\beta[X]^2 \leq \mathbf{E}_\beta[X^2] = \sum_{x, y \in B} \mathbf{P}_\beta[x, y \leftrightarrow \infty]. \quad (2.5.37)$$

Using Theorem 2.5.6,

$$(|B|\mathbf{P}_\beta[0 \leftrightarrow \infty])^2 \leq \sum_{x,y \in B} \mathbf{P}_\beta[x, y \leftrightarrow \infty] \leq \sum_{x,y \in B} \mathbf{P}_\beta[x \leftrightarrow y]. \quad (2.5.38)$$

Combining (2.5.36) and (2.5.38), we get

$$\mathbf{P}_\beta[0 \leftrightarrow \infty]^2 \leq \frac{1}{|B|^2} \sum_{x,y \in B} \langle \sigma_x \sigma_y \rangle_\beta. \quad (2.5.39)$$

Optimising over B leads to

$$\mathbf{P}_\beta[0 \leftrightarrow \infty] = 0. \quad (2.5.40)$$

□

We can now prove Theorem 2.5.1.

Proof of Theorem 2.5.1. Fix a pair of vertices x and y and $n \geq 1$. Applying the switching lemma again,

$$\langle \sigma_x \sigma_y \rangle_{\Lambda_n, \beta}^+ - \langle \sigma_x \sigma_y \rangle_{\Lambda_n, \beta} = \frac{\sum_{\substack{\partial \mathbf{n}_1 = \emptyset \\ \partial \mathbf{n}_2 = \{x, y\}}} w_\beta(\mathbf{n}_1) w_\beta(\mathbf{n}_2) (1 - \mathbb{1}\{x \leftrightarrow y \text{ in } \Lambda_n\})}{\sum_{\substack{\partial \mathbf{n}_1 = \emptyset \\ \partial \mathbf{n}_2 = \emptyset}} w_\beta(\mathbf{n}_1) w_\beta(\mathbf{n}_2)}. \quad (2.5.41)$$

Notice that since $\partial(\mathbf{n}_1 + \mathbf{n}_2) = \{x, y\}$, x and y must be connected to each other. Since this connection does not happen in Λ_n , it must be that x and y are both connected to $\mathbf{g}$. This yields

$$\langle \sigma_x \sigma_y \rangle_{\Lambda_n, \beta}^+ - \langle \sigma_x \sigma_y \rangle_{\Lambda_n, \beta} \leq \frac{\sum_{\substack{\partial \mathbf{n}_1 = \emptyset \\ \partial \mathbf{n}_2 = \{x, y\}}} w_\beta(\mathbf{n}_1) w_\beta(\mathbf{n}_2) \mathbb{1}\{x \leftrightarrow \mathbf{g}\}}{\sum_{\substack{\partial \mathbf{n}_1 = \emptyset \\ \partial \mathbf{n}_2 = \emptyset}} w_\beta(\mathbf{n}_1) w_\beta(\mathbf{n}_2)}. \quad (2.5.42)$$

We now estimate the sum in the numerator. The idea is to compare it to a sum in which the source condition of $(\mathbf{n}_1, \mathbf{n}_2)$ is changed to $\partial \mathbf{n}_1 = \partial \mathbf{n}_2 = \emptyset$.

Fix a sequence of vertices $x = x_0, \dots, x_m = y$ with $J_{x_i, x_{i+1}} > 0$ for $0 \leq i < m$ (this sequence exists by irreducibility). Fix n large enough so that $x_i \in \Lambda_n$ for all $i \leq m$. Consider the map $\mathbf{f}$ defined as follows: $\mathbf{f}(\mathbf{n}_1, \mathbf{n}_2) = (\mathbf{n}_1, \mathbf{n}'_2)$ where

$$\mathbf{n}'_2(e) = \begin{cases} \mathbf{n}_2(e) - 1 & \text{if there exists } 0 \leq i < m \text{ such that } e = \{x_i, x_{i+1}\} \text{ and } \mathbf{n}_2(e) > 0, \\ 1 & \text{if there exists } 0 \leq i < m \text{ such that } e = \{x_i, x_{i+1}\} \text{ and } \mathbf{n}_2(e) = 0, \\ \mathbf{n}_2(e) & \text{otherwise.} \end{cases}$$

If $(\mathbf{n}_1, \mathbf{n}_2)$ contributes to the numerator of (2.5.42), then $x \xleftrightarrow{\mathbf{n}_1 + \mathbf{n}'_2} \mathbf{g}$ and $\partial \mathbf{n}_1 = \partial \mathbf{n}'_2 = \emptyset$. As a result,

$$\sum_{\substack{\partial \mathbf{n}_1 = \emptyset \\ \partial \mathbf{n}_2 = \{x, y\}}} w_\beta(\mathbf{n}_1) w_\beta(\mathbf{n}_2) \mathbb{1}\{x \leftrightarrow \mathbf{g}\} \leq C(x, y) \sum_{\substack{\partial \mathbf{n}_1 = \emptyset \\ \partial \mathbf{n}'_2 = \emptyset}} w_\beta(\mathbf{n}_1) w_\beta(\mathbf{n}'_2) \mathbb{1}\{x \leftrightarrow \mathbf{g}\}, \quad (2.5.43)$$

where

$$C(x, y) := 2^m \prod_{j=0}^{m-1} \max \left\{ \frac{1}{\beta J_{x_j, y}}, \beta J_{x_j, y} \right\}. \quad (2.5.44)$$

Inserting this in (2.5.42), we find that

$$\langle \sigma_x \sigma_y \rangle_{\Lambda_{n,\beta}}^+ - \langle \sigma_x \sigma_y \rangle_{\Lambda_{n,\beta}} \leq C(x, y) \mathbf{P}_{\Lambda_{n,\beta}}^{\text{free},+} [x \xleftrightarrow{\mathbf{n}_1 + \mathbf{n}_2} \mathfrak{g}]. \quad (2.5.45)$$

Taking the limit⁸ $n \rightarrow \infty$, which exists by Proposition 2.5.4, we obtain

$$0 \leq \langle \sigma_x \sigma_y \rangle_{\beta}^+ - \langle \sigma_x \sigma_y \rangle_{\beta} \leq C(x, y) \mathbf{P}_{\beta} [x \leftrightarrow \infty]. \quad (2.5.46)$$

We now consider $\beta \leq \beta_c$. Using Theorem 2.5.8 and using the fact that $\mathbf{P}_{\beta}$ is invariant under shifts, we get

$$\mathbf{P}_{\beta} [x \leftrightarrow \infty] = 0. \quad (2.5.47)$$

Thus, using the Griffiths' inequality,

$$0 \leq m^*(\beta)^2 \leq \langle \sigma_0 \sigma_x \rangle_{\beta}^+ = \langle \sigma_0 \sigma_x \rangle_{\beta} \quad (2.5.48)$$

for any $x \in V$ and any $\beta \leq \beta_c$. The assumption that $\langle \sigma_0 \sigma_x \rangle_{\beta_c}$ goes to 0 as $d(0, x)$ goes to infinity leads to the result. $\square$

2.6 Triviality of the scaling limits of the critical Ising model

In this section, we restrict ourselves to the case of the nearest-neighbour ferromagnetic Ising model on $\mathcal{G} = \mathbb{Z}^d$. Similar results can be obtained for more general (in particular long-range) models. We refer to Chapter 9 for more information.

As explained in Section 1.3, it is expected that the upper critical dimension of the Ising model is equal to four. From the perspective of the renormalization group method, this implies that in dimensions $d > 4$, successive (critical) renormalizations should converge to the *Gaussian fixed point*, and that critical exponents should take their mean-field value. The same behaviour is expected at $d = 4$. However, there, the mean-field approximation fails because of the emergence of logarithmic divergences in the renormalization group analysis. Therefore, obtaining the result is expected to be much harder.

Establishing the full convergence of the critical model is in general extremely hard. We already saw before that rigourously implementing the renormalization group method is challenging and restricted to situations involving a *perturbative* parameter. However, it is possible to recover the *qualitative* result predicted by renormalization group, i.e. that the scaling limit (under a certain sense described below) is a Gaussian field if it exists. Again, the random current representation of the Ising model plays a crucial role in the proof.

Being able to achieve such a result in a non-integrable setup and without using RG methods is a challenge in mathematics. From the physics perspective, these triviality result represent a barrier in the program of constructive quantum field theory. More information on this can be found in Chapter 1.

The first item on our agenda is to explain what one means by *scaling limit* of the Ising model.

⁸Here the event $\{x \xleftrightarrow{\mathbf{n}_1 + \mathbf{n}_2} \mathfrak{g}\}$ is not local. However, for a fixed $k \geq 0$ and n large enough it is included in the event $\{0 \xleftrightarrow{\mathbf{n}_1 + \mathbf{n}_2} \Lambda_k^c\}$ which is “essentially” local since $|J| < \infty$.

We look at the joint distribution of the *smeared observables*⁹ given for $\beta > 0$ and $L \geq 1$ by

$$T_{f,L,\beta}(\sigma) := \frac{1}{\sqrt{\Sigma_L(\beta)}} \sum_{x \in \mathbb{Z}^d} f\left(\frac{x}{L}\right) \sigma_x, \quad (2.6.2)$$

where f ranges over the set $\mathcal{C}_0(\mathbb{R}^d)$ of continuous, real valued, and compactly supported functions, and where

$$\Sigma_L(\beta) := \left\langle \left(\sum_{x \in \Lambda_L} \sigma_x \right)^2 \right\rangle_\beta = \sum_{x,y \in \Lambda_L} \langle \sigma_x \sigma_y \rangle_\beta, \text{ with } \Lambda_L := [-L, L]^d \cap \mathbb{Z}^d. \quad (2.6.3)$$

Definition 2.6.1. A discrete system as above is said to converge in distribution to a scaling limit if the collection of random variables $(T_{f,L,\beta}(\sigma))_{f \in \mathcal{C}_0(\mathbb{R}^d)}$ converges in distribution (in the sense of finite dimensional distributions) as L goes to infinity. Using Kolmogorov's extension theorem and the separability of $\mathcal{C}_0(\mathbb{R}^d)$, we can represent any scaling limit as a random field.

The following result, proved independently by Aizenman [4] and Fröhlich [102], implies that any subsequential scaling limit from the single phase regime (i.e. $\beta \leq \beta_c$ and $h = 0$) (in the sense of Definition 2.6.1) has to be Gaussian.

Theorem 2.6.2 (Triviality of the Ising model in dimensions $d \geq 5$, [4, 102]). *Let $d \geq 5$. There exist $C, \alpha > 0$ such that for every $\beta \leq \beta_c$, every $L \leq \xi(\beta)$, every $f \in \mathcal{C}_0(\mathbb{R}^d)$ and every $z \in \mathbb{R}$,*

$$\left| \langle \exp(z T_{f,L,\beta}(\sigma)) \rangle_\beta - \exp\left(\frac{z^2}{2} \langle T_{f,L,\beta}(\sigma)^2 \rangle_\beta\right) \right| \leq \exp\left(\frac{z^2}{2} \langle T_{f,L,\beta}(\sigma)^2 \rangle_\beta\right) \frac{C \|f\|_\infty^4 r_f^\alpha z^4}{L^{d-4}} \quad (2.6.4)$$

where $\|f\|_\infty = \sup_{x \in \mathbb{R}^d} |f(x)|$ and $r_f = \max\{\max\{r \geq 0, \exists x \in \mathbb{R}^d, |x| = r, f(x) \neq 0\}, 1\}$.

The preceding result yields no information about the nature of the scaling limit when $d = 4$. However, in a recent breakthrough work, Aizenman and Duminil-Copin [9] managed to settle this long-standing open problem. They obtained the following statement, which we will not prove, but for which we will provide some heuristics.

Theorem 2.6.3 (Triviality of the Ising model in dimension $d = 4$, [9]). *Let $d = 4$. There exist $c, C, \alpha > 0$ such that for every $\beta \leq \beta_c$, every $L \leq \xi(\beta)$, every $f \in \mathcal{C}_0(\mathbb{R}^d)$ and every $z \in \mathbb{R}$,*

$$\left| \langle \exp(z T_{f,L,\beta}(\sigma)) \rangle_\beta - \exp\left(\frac{z^2}{2} \langle T_{f,L,\beta}(\sigma)^2 \rangle_\beta\right) \right| \leq \exp\left(\frac{z^2}{2} \langle T_{f,L,\beta}(\sigma)^2 \rangle_\beta\right) \frac{C \|f\|_\infty^4 r_f^\alpha z^4}{(\log L)^c}. \quad (2.6.5)$$

Let us finally mention that these results extend (using the so-called *Griffith–Simon approximation*) to the discrete φ^4 model. This different point of view is more suited to discuss the consequences of triviality from the perspective of constructive Euclidean field theory. We refer to [4, 9, 204] for more information.

⁹Note that for $f = \mathbb{1}_{[-1,1]^d}$,

$$\langle T_{f,L,\beta}(\sigma)^2 \rangle_\beta = 1, \quad (2.6.1)$$

and more generally for $f \neq 0$, one has $0 < c_f \leq \langle T_{f,L,\beta}(\sigma)^2 \rangle_\beta \leq C_f < \infty$, which means that the following quantity is bounded away from 0 and infinity by constants that only depend on f . This indicates that this is the scaling that is the most likely to yield interesting limits.

2.6.1 A sufficient condition for triviality

Establishing Gaussianity of the scaling limits of the model can be achieved by showing that Wick's law is asymptotically satisfied for properly rescaled observables of the model. This requires studying the deviation of all $2n$ -point functions from their "Gaussian counterpart". In the case of the Ising model, a simple criterion has been established by Newman [196]. We begin with two definitions.

Definition 2.6.4 (Ursell functions). Let $n \geq 2$ and $\beta > 0$. Define Ursell's n -point function U_n^β at inverse temperature β by: for every $x_1, \dots, x_n \in \mathbb{Z}^d$,

$$U_n^\beta(x_1, \dots, x_n) = \sum_{\mathcal{P}} (-1)^{|\mathcal{P}|-1} (|\mathcal{P}| - 1)! \prod_{A \in \mathcal{P}} \langle \sigma_A \rangle, \quad (2.6.6)$$

where the sum ranges over all partitions $\mathcal{P}$ of $\{1, \dots, n\}$. In particular, when $n = 4$,

$$\begin{aligned} U_4^\beta(x_1, x_2, x_3, x_4) &= \langle \sigma_{x_1} \sigma_{x_2} \sigma_{x_3} \sigma_{x_4} \rangle_\beta - \langle \sigma_{x_1} \sigma_{x_2} \rangle_\beta \langle \sigma_{x_3} \sigma_{x_4} \rangle_\beta \\ &\quad - \langle \sigma_{x_1} \sigma_{x_3} \rangle_\beta \langle \sigma_{x_2} \sigma_{x_4} \rangle_\beta - \langle \sigma_{x_1} \sigma_{x_4} \rangle_\beta \langle \sigma_{x_2} \sigma_{x_3} \rangle_\beta. \end{aligned} \quad (2.6.7)$$

Hence, U_4^β represents the *deviation* of the four-point function of the model to Wick's law.

Definition 2.6.5 (Renormalised coupling constant). The *renormalised coupling constant* is defined for $\beta < \beta_c$ by

$$g(\beta) := \frac{1}{\chi(\beta)^2 \xi(\beta)^d} \sum_{x, y, z \in \mathbb{Z}^d} |U_4^\beta(0, x, y, z)|. \quad (2.6.8)$$

Theorem 2.6.6 (Sufficient condition for Gaussianity [196]). *Let $d \geq 2$. If*

$$\lim_{\beta \nearrow \beta_c} g(\beta) = 0, \quad (2.6.9)$$

then any scaling limit of the critical Ising model is trivial.

The above result can be interpreted as follows: asymptotic validity of Wick's law for the properly rescaled four-point function entails the same result for higher order correlation functions. Similar results have been obtained in the context of Wigner chaos, see for instance [162].

Aizenman [4] obtained a comparable but more explicit result. We begin by introducing some useful notations.

If $n \geq 1$ and $x_1, \dots, x_{2n} \in \mathbb{Z}^d$, let

$$\mathcal{G}_{2n}^\beta(x_1, \dots, x_{2n}) := \sum_{\pi} \prod_{i=1}^n \langle \sigma_{x_{\pi(2i-1)}} \sigma_{x_{\pi(2i)}} \rangle_\beta, \quad (2.6.10)$$

where π ranges over pairing permutations of $\{1, \dots, 2n\}$. The deviation of the model's $2n$ -point function from Wick's law is denoted by

$$\mathcal{D}_{2n}^\beta(x_1, \dots, x_{2n}) := \langle \sigma_{x_1} \dots \sigma_{x_{2n}} \rangle_\beta - \mathcal{G}_{2n}^\beta(x_1, \dots, x_{2n}). \quad (2.6.11)$$

In particular,

$$\mathcal{D}_4^\beta = U_4^\beta. \quad (2.6.12)$$

The next result provides a geometric interpretation of U_4^β in terms of random currents.

Proposition 2.6.7 ([4, Proposition 5.1]). *Let $d \geq 3$ and $\beta \leq \beta_c$. For all $x_1, x_2, x_3, x_4 \in \mathbb{Z}^d$,*

$$U_4^\beta(x_1, x_2, x_3, x_4) = -2\langle \sigma_{x_1} \sigma_{x_2} \rangle_\beta \langle \sigma_{x_3} \sigma_{x_4} \rangle_\beta \mathbf{P}_\beta^{x_1 x_2, x_3 x_4}[\mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_2}(x_1) \cap \mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_2}(x_3) \neq \emptyset] \quad (2.6.13)$$

$$= -2\langle \sigma_{x_1} \sigma_{x_2} \sigma_{x_3} \sigma_{x_4} \rangle_\beta \mathbf{P}_\beta^{x_1 x_2 x_3 x_4, \emptyset}[x_2, x_3, x_4 \in \mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_2}(x_1)]. \quad (2.6.14)$$

Proof. Let Λ be a finite subset of $\mathbb{Z}^d$. We let $U_4^{\Lambda, \beta}$ be the Ursell four-point function of the Ising model on Λ . We prove the result in finite volume and extend it to the infinite volume limit by taking the limit $\Lambda \nearrow \mathbb{Z}^d$. Using the switching lemma,

$$\begin{aligned} U_4^{\Lambda, \beta}(x_1, x_2, x_3, x_4) &= \frac{Z_{\Lambda, \beta}^{x_1 x_2 x_3 x_4, \emptyset} - Z_{\Lambda, \beta}^{x_1 x_2, x_3 x_4} - Z_{\Lambda, \beta}^{x_1 x_3, x_2 x_4} - Z_{\Lambda, \beta}^{x_1 x_4, x_2 x_3}}{Z_{\Lambda, \beta}^{\emptyset, \emptyset}} \\ &= \frac{1}{Z_{\Lambda, \beta}^{\emptyset, \emptyset}} \sum_{\substack{\partial \mathbf{n}_1 = \{x_1, x_2, x_3, x_4\} \\ \partial \mathbf{n}_2 = \emptyset}} w_\beta(\mathbf{n}_1) w_\beta(\mathbf{n}_2) \left(1 - \mathbb{1}\{x_3 \xleftrightarrow{\mathbf{n}_1 + \mathbf{n}_2} x_4\} - \mathbb{1}\{x_2 \xleftrightarrow{\mathbf{n}_1 + \mathbf{n}_2} x_4\} \right. \\ &\quad \left. - \mathbb{1}\{x_2 \xleftrightarrow{\mathbf{n}_1 + \mathbf{n}_2} x_3\} \right) \\ &= \frac{1}{Z_{\Lambda, \beta}^{\emptyset, \emptyset}} \sum_{\substack{\partial \mathbf{n}_1 = \{x_1, x_2, x_3, x_4\} \\ \partial \mathbf{n}_2 = \emptyset}} w_\beta(\mathbf{n}_1) w_\beta(\mathbf{n}_2) (-2) \mathbb{1}\{x_1, x_2, x_3, x_4 \in \mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_2}(x_1)\} \\ &= \frac{1}{Z_{\Lambda, \beta}^{\emptyset, \emptyset}} \sum_{\substack{\partial \mathbf{n}_1 = \{x_1, x_2\} \\ \partial \mathbf{n}_2 = \{x_3, x_4\}}} w_\beta(\mathbf{n}_1) w_\beta(\mathbf{n}_2) (-2) \mathbb{1}\{x_1, x_2, x_3, x_4 \in \mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_2}(x_1)\}, \end{aligned} \quad (2.6.15)$$

which yields the result. $\square$

Remark 2.6.8. Let us assume that random currents behave like random walks. Then, we expect that in dimensions $d > 4$, two independent random currents will typically never intersect. Hence, the probability in the definition of U_4^β should be “small” (although, for now, $\mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_2}(x_1)$ and $\mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_2}(x_3)$ are not independent). Remarkably, this heuristic is valid when $d > 4$. However, the dimension $d = 4$ is the *critical dimension* for the intersection of two independent random walks (see for instance [172]), which hints that it may also be critical for random currents. This makes the analysis much more intricate and explains why it took forty years to extend the triviality result to the four-dimensional setup.

Remark 2.6.9. Note that we also proved the non-trivial fact that $U_4^\beta \leq 0$ which is the so-called *Lebowitz inequality* [173].

The following result expresses $\mathcal{D}_{2n}^\beta$ in terms of U_4^β .

Theorem 2.6.10 ([4, Proposition 12.1]). *Let $d \geq 2$, $n \geq 2$, and $\beta > 0$. Introduce for $x_1, \dots, x_{2n} \in \mathbb{Z}^d$,*

$$\mathcal{R}_{2n}^\beta(x_1, \dots, x_{2n}) := \sum_{1 \leq i < j < k < \ell \leq 2n} |U_4^\beta(x_i, x_j, x_k, x_\ell)| \mathcal{G}_{2n-4}^\beta(\dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, \widehat{x_k}, \dots, \widehat{x_\ell}, \dots) \quad (2.6.16)$$

where the notation $\widehat{u}$ indicates that the site u is omitted. Then,

$$\frac{2}{n(n-1)} \mathcal{R}_{2n}^\beta(x_1, \dots, x_{2n}) \leq |\mathcal{D}_{2n}^\beta(x_1, \dots, x_{2n})| \leq \frac{3}{2} \mathcal{R}_{2n}^\beta(x_1, \dots, x_{2n}). \quad (2.6.17)$$

Proof. For convenience, we drop β from the notations. We begin by proving the upper bound. Using the backbone representation one gets that

$$\langle \sigma_{x_1} \dots \sigma_{x_{2n}} \rangle = \sum_{\pi \in \text{Pair}(\{1, \dots, 2n\})} \sum_{\gamma \in \Omega_1(\pi) \circ \dots \circ \Omega_n(\pi)} \rho(\gamma), \quad (2.6.18)$$

where $\text{Pair}(\{1, \dots, 2n\})$ is the collection of pairings of $\{1, \dots, 2n\}$, each pairing being represented by a permutation π such that for all $1 \leq k < \ell \leq n$, $\pi(k) < \pi(\ell)$ and $\pi(k) < \pi(k+n)$, the k -th pairing being $\{\pi(k), \pi(k+n)\}$, and where

$$\Omega_k(\pi) := \Omega(\{x_{\pi(k)}, x_{\pi(k+n)}\}) \quad (2.6.19)$$

is the set of sequences of steps from $x_{\pi(k)}$ to $x_{\pi(k+n)}$. The notation “ $\circ$ ” refers to the additional requirement that these sequences of steps are consistent. More precisely, the set $\Omega_1(\pi) \circ \dots \circ \Omega_n(\pi)$ is the set of sequences of steps γ such that $\partial\gamma = \{x_1, \dots, x_{2n}\}$ with $\gamma = \gamma_1 \circ \dots \circ \gamma_n$ where $\gamma_i \in \Omega_i(\pi)$. Using Proposition 2.3.4, one deduces that

$$\begin{aligned} \langle \sigma_{x_1} \dots \sigma_{x_{2n}} \rangle &\geq \sum_{\pi \in \text{Pair}(\{1, \dots, 2n\})} \sum_{\gamma_1 \circ \dots \circ \gamma_n \in \Omega_1(\pi) \circ \dots \circ \Omega_n(\pi)} \rho(\gamma_1) \dots \rho(\gamma_n) \\ &= \sum_{\pi} \left(\sum_{(\gamma_1, \dots, \gamma_n) \in \Omega_1(\pi) \times \dots \times \Omega_n(\pi)} \rho(\gamma_1) \dots \rho(\gamma_n) - \sum_{\substack{(\gamma_1, \dots, \gamma_n) \in \Omega_1(\pi) \times \dots \times \Omega_n(\pi) \\ \text{not all consistent}}} \rho(\gamma_1) \dots \rho(\gamma_n) \right) \\ &\geq \mathcal{G}_{2n}(x_1, \dots, x_{2n}) - \sum_{1 \leq a < b \leq n} \sum_{\pi} \sum_{\substack{(\gamma_1, \dots, \gamma_n) \in \Omega_1(\pi) \times \dots \times \Omega_n(\pi) \\ (\gamma_a, \gamma_b) \text{ not consistent}}} \rho(\gamma_1) \dots \rho(\gamma_n). \end{aligned} \quad (2.6.20)$$

If we arrange (2.6.20) in terms of the sources of γ_a and γ_b , and use Newman’s inequality [196] which states that for all $n \geq 1$,

$$\langle \sigma_{x_1} \dots \sigma_{x_{2n}} \rangle \leq \mathcal{G}_{2n}(x_1, \dots, x_{2n}), \quad (2.6.21)$$

we get

$$\begin{aligned} 0 &\leq -\mathcal{D}_{2n}(x_1, \dots, x_{2n}) \\ &\leq \sum_{1 \leq j < k < \ell < m \leq 2n} \sum_{\pi \in \text{Pair}(\{j, k, \ell, m\})} \sum_{\substack{(\gamma_1, \gamma_2) \in \Omega_1(\pi) \times \Omega_2(\pi) \\ \text{not consistent}}} \rho(\gamma_1) \rho(\gamma_2) \cdot \widehat{\mathcal{G}_{2n-4}}(x_j, x_k, x_\ell, x_m) \end{aligned} \quad (2.6.22)$$

where we wrote

$$\widehat{\mathcal{G}_{2n-4}}(x_j, x_k, x_\ell, x_m) := \mathcal{G}_{2n-4}(\dots, \widehat{x_j}, \dots, \widehat{x_k}, \dots, \widehat{x_\ell}, \dots, \widehat{x_m}, \dots). \quad (2.6.23)$$

The upper bound then reduces to the following bound:

$$\sum_{\pi \in \text{Pair}(\{j, k, \ell, m\})} \sum_{\substack{(\gamma_1, \gamma_2) \in \Omega_1(\pi) \times \Omega_2(\pi) \\ \text{not consistent}}} \rho(\gamma_1) \rho(\gamma_2) \leq \frac{3}{2} |U_4(x_j, x_k, x_\ell, x_m)|. \quad (2.6.24)$$

One may prove this inequality by giving a probabilistic interpretation to the left-hand side of (2.6.24). For a sequence of steps γ , write $\mathbf{C}(\gamma)$ the set of vertices it visits. One may write

$$\begin{aligned} & \sum_{\pi \in \text{Pair}(\{j,k,\ell,m\})} \sum_{\substack{(\gamma_1, \gamma_2) \in \Omega_1(\pi) \times \Omega_2(\pi) \\ \text{not consistent}}} \rho(\gamma_1) \rho(\gamma_2) \\ &= \langle \sigma_{x_j} \sigma_{x_k} \rangle \langle \sigma_{x_\ell} \sigma_{x_m} \rangle \mathbf{P}^{x_j x_k, x_\ell x_m}[(\Gamma(\mathbf{n}_1), \Gamma(\mathbf{n}_2)) \text{ not consistent}] + (k \Leftrightarrow \ell) + (k \Leftrightarrow m), \end{aligned} \quad (2.6.25)$$

where $(k \Leftrightarrow \ell)$ (resp. $(k \Leftrightarrow m)$) refers to the same term with k and ℓ (resp. m) permuted.

If a pair $(\Gamma(\mathbf{n}_1), \Gamma(\mathbf{n}_2))$ is not consistent then $\mathbf{C}(\Gamma(\mathbf{n}_1)) \cap \mathbf{C}(\Gamma(\mathbf{n}_2)) \neq \emptyset$. This implies that $\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x_j) \cap \mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x_\ell) \neq \emptyset$. Thus,

$$\mathbf{P}^{x_j x_k, x_\ell x_m}[(\Gamma(\mathbf{n}_1), \Gamma(\mathbf{n}_2)) \text{ not consistent}] \leq \mathbf{P}^{x_j x_k, x_\ell x_m}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x_j) \cap \mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x_\ell) \neq \emptyset] \quad (2.6.26)$$

so that by Proposition 2.6.7

$$\langle \sigma_{x_j} \sigma_{x_k} \rangle \langle \sigma_{x_\ell} \sigma_{x_m} \rangle \mathbf{P}_\beta^{x_j x_k, x_\ell x_m}[(\Gamma(\mathbf{n}_1), \Gamma(\mathbf{n}_2)) \text{ not consistent}] \leq \frac{1}{2} |U_4(x_j, x_k, x_\ell, x_m)|. \quad (2.6.27)$$

This inequality is true for any of the three terms in (2.6.25) so that we recover (2.6.24).

We now prove the lower bound. One can check¹⁰ that the following identity is true:

$$\mathcal{G}_{2n}(x_1, \dots, x_{2n}) = \frac{2}{n(n-1)} \sum_{1 \leq j < k < \ell < m \leq 2n} \mathcal{G}_4(x_j, x_k, x_\ell, x_m) \widehat{\mathcal{G}_{2n-4}}(x_j, x_k, x_\ell, x_m). \quad (2.6.28)$$

Using this identity, we write

$$\begin{aligned} & \langle \sigma_{x_1} \dots \sigma_{x_{2n}} \rangle = \\ & \frac{2}{n(n-1)} \sum_{1 \leq j < k < \ell < m \leq 2n} \sum_{\substack{\pi \in \text{Pair}(\{j,k,\ell,m\}) \\ \pi' \in \text{Pair}(\{\dots, \widehat{j}, \dots, \widehat{k}, \dots, \widehat{\ell}, \dots, \widehat{m}, \dots\})}} \sum_{\gamma \in \Omega_1(\pi) \circ \Omega_2(\pi) \circ \Omega_1(\pi') \circ \dots \circ \Omega_{n-2}(\pi')} \rho(\gamma). \end{aligned} \quad (2.6.29)$$

For each $\gamma_1 \circ \gamma_2 \in \Omega_1(\pi) \circ \Omega_2(\pi)$,

$$\begin{aligned} & \sum_{\pi' \in \text{Pair}(\{\dots, \widehat{j}, \dots, \widehat{k}, \dots, \widehat{\ell}, \dots, \widehat{m}, \dots\})} \sum_{\substack{\gamma \in \Omega_1(\pi') \circ \dots \circ \Omega_{n-2}(\pi') \\ \gamma \text{ consistent with } \gamma_1 \circ \gamma_2}} \rho(\gamma_1 \circ \gamma_2 \circ \gamma) \\ &= \rho(\gamma_1 \circ \gamma_2) \sum_{\partial \gamma = \{1, \dots, 2n\} \setminus \{j,k,\ell,m\}} \rho_{\overline{\gamma_1 \circ \gamma_2}^c}(\gamma) \\ &= \rho(\gamma_1 \circ \gamma_2) \left\langle \prod_{u \in \{1, \dots, 2n\} \setminus \{j,k,\ell,m\}} \sigma_u \right\rangle_{\overline{\gamma_1 \circ \gamma_2}^c} \\ &\leq \rho(\gamma_1 \circ \gamma_2) \left\langle \prod_{u \in \{1, \dots, 2n\} \setminus \{j,k,\ell,m\}} \sigma_u \right\rangle, \end{aligned} \quad (2.6.30)$$

¹⁰Use the fact that the number of pairings of $\{1, \dots, 2n\}$ is $\prod_{k=1}^n (2n - 2k + 1)$.

where we used Proposition 2.2.11 in the last inequality. Plugging (2.6.30) in (2.6.29) we get that

$$\langle \sigma_{x_1} \dots \sigma_{x_{2n}} \rangle \leq \frac{2}{n(n-1)} \sum_{1 \leq j < k < \ell < m \leq 2n} \langle \sigma_{x_j} \sigma_{x_k} \sigma_{x_\ell} \sigma_{x_m} \rangle \left\langle \prod_{u \in \{1, \dots, 2n\} \setminus \{j, k, \ell, m\}} \sigma_u \right\rangle. \quad (2.6.31)$$

Using once again Newman's inequality,

$$\langle \sigma_{x_1} \dots \sigma_{x_{2n}} \rangle \leq \frac{2}{n(n-1)} \sum_{1 \leq j < k < \ell < m \leq 2n} \langle \sigma_{x_j} \sigma_{x_k} \sigma_{x_\ell} \sigma_{x_m} \rangle \widehat{\mathcal{G}_{2n-4}}(x_j, x_k, x_\ell, x_m). \quad (2.6.32)$$

Subtracting (2.6.32) to (2.6.28) yields the result. $\square$

Remark 2.6.11. We provide a quick proof of Newman's inequality due to Aizenman [4]. We prove by induction that for all $n \geq 1$, for all $x_1, \dots, x_{2n} \in \mathbb{Z}^d$, one has

$$\langle \sigma_{x_1} \dots \sigma_{x_{2n}} \rangle_\beta \leq \mathcal{G}_{2n}(x_1, \dots, x_{2n}). \quad (2.6.33)$$

Using the switching lemma, we get

$$\langle \sigma_{x_1} \dots \sigma_{x_{2n}} \rangle_\beta \leq \sum_{k=2}^{2n} \langle \sigma_{x_1} \sigma_{x_k} \rangle_\beta \left\langle \prod_{u \in \{1, \dots, 2n\} \setminus \{1, k\}} \sigma_u \right\rangle_\beta. \quad (2.6.34)$$

Indeed,

$$\begin{aligned} \frac{1}{\langle \sigma_{x_1} \dots \sigma_{x_{2n}} \rangle_\beta} \sum_{k=2}^{2n} \langle \sigma_{x_1} \sigma_{x_k} \rangle_\beta \left\langle \prod_{u \in \{1, \dots, 2n\} \setminus \{1, k\}} \sigma_u \right\rangle_\beta \\ = \mathbf{E}_\beta^{\{x_1, \dots, x_{2n}\}, \emptyset} [|\mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_2}(x_1) \cap \{x_2, \dots, x_{2n}\}|] \geq 1. \end{aligned} \quad (2.6.35)$$

Iterating this procedure yields (2.6.33).

2.6.2 Triviality in dimensions $d > 4$

The proof presented here is due to Aizenman [4]. It relies on the derivation of the *tree diagram bound* (see Proposition 2.6.12) which bounds Ursell's four-point function in terms of correlation functions. This bound is enough to show the vanishing of the renormalized coupling constant $g(\beta)$.

We then provide a proof of Theorem 2.6.2, which gives a quantitative version of this triviality result. It relies on a *sliding-scale infrared bound* obtained in [9].

The driving idea of these proofs is the comparison of the intersection properties of two independent random currents with those of two independent random walks, which typically "avoid" each other in dimensions $d > 4$.

Proposition 2.6.12 (Tree diagram bound). *Let $d \geq 2$ and $\beta < \beta_c$. For all $x_1, x_2, x_3, x_4 \in \mathbb{Z}^d$,*

$$0 \leq -U_4^\beta(x_1, x_2, x_3, x_4) \leq 2 \sum_{y \in \mathbb{Z}^d} \langle \sigma_{x_1} \sigma_y \rangle_\beta \langle \sigma_{x_2} \sigma_y \rangle_\beta \langle \sigma_{x_3} \sigma_y \rangle_\beta \langle \sigma_{x_4} \sigma_y \rangle_\beta. \quad (2.6.36)$$

Proof. The first inequality is a consequence of Proposition 2.6.7. For the second one, we study the probability appearing in the expression of U_4^β and write

$$\begin{aligned} \mathbf{P}_\beta^{x_1x_2,x_3x_4}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x_1) \cap \mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x_3) \neq \emptyset] \\ = \mathbf{P}_\beta^{x_1x_2,x_3x_4,\emptyset}[x_3 \in \mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x_1)] \\ \leq \mathbf{P}_\beta^{x_1x_2,x_3x_4,\emptyset}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_3}(x_1) \cap \mathbf{C}_{\mathbf{n}_2}(x_3) \neq \emptyset] \\ \leq \mathbf{P}_\beta^{x_1x_2,x_3x_4,\emptyset,\emptyset}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_3}(x_1) \cap \mathbf{C}_{\mathbf{n}_2+\mathbf{n}_4}(x_3) \neq \emptyset] \end{aligned} \quad (2.6.37)$$

$$\leq \sum_{y \in \mathbb{Z}^d} \mathbf{P}_\beta^{x_1x_2,x_3x_4,\emptyset,\emptyset}[y \in \mathbf{C}_{\mathbf{n}_1+\mathbf{n}_3}(x_1) \cap \mathbf{C}_{\mathbf{n}_2+\mathbf{n}_4}(x_3)]. \quad (2.6.38)$$

where the first inequality is explained below, and where we used that

$$\{\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_3}(x_1) \cap \mathbf{C}_{\mathbf{n}_2}(x_3) \neq \emptyset\} \subset \{\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_3}(x_1) \cap \mathbf{C}_{\mathbf{n}_2+\mathbf{n}_4}(x_3) \neq \emptyset\} \quad (2.6.39)$$

in the second inequality. Before explaining how to obtain the first inequality, we conclude the proof. Using (2.6.38), the independence of $\mathbf{n}_1 + \mathbf{n}_3$ and $\mathbf{n}_2 + \mathbf{n}_4$, and (2.2.19), one gets

$$\mathbf{P}_\beta^{x_1x_2,x_3x_4}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x_1) \cap \mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x_3) \neq \emptyset] \leq \sum_{y \in \mathbb{Z}^d} \frac{\langle \sigma_{x_1} \sigma_y \rangle_\beta \langle \sigma_y \sigma_{x_2} \rangle_\beta}{\langle \sigma_{x_1} \sigma_{x_2} \rangle_\beta} \frac{\langle \sigma_{x_3} \sigma_y \rangle_\beta \langle \sigma_y \sigma_{x_4} \rangle_\beta}{\langle \sigma_{x_3} \sigma_{x_4} \rangle_\beta} \quad (2.6.40)$$

which is enough to conclude using Proposition 9.4.7. It remains to show that

$$\mathbf{P}_\beta^{x_1x_2,x_3x_4}[x_3 \in \mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x_1)] \leq \mathbf{P}_\beta^{x_1x_2,x_3x_4,\emptyset}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_3}(x_1) \cap \mathbf{C}_{\mathbf{n}_2}(x_3) \neq \emptyset]. \quad (2.6.41)$$

To prove this inequality we first work on a finite subset Λ of $\mathbb{Z}^d$. Going to the complementary events,

$$\begin{aligned} \mathbf{P}_{\Lambda,\beta}^{x_1x_2,x_3x_4}[x_3 \notin \mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x_1)] \\ = \mathbf{P}_{\Lambda,\beta}^{x_1x_2,x_3x_4}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x_1) \cap \mathbf{C}_{\mathbf{n}_2}(x_3) = \emptyset] \\ = \sum_{\substack{S,T \subset \Lambda \\ x_3 \notin S \\ T \cap S = \emptyset}} \mathbf{P}_{\Lambda,\beta}^{x_1x_2,x_3x_4}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x_1) = S, \mathbf{C}_{\mathbf{n}_2}(x_3) = T] \\ = \sum_{\substack{S,T \subset \Lambda \\ x_3 \notin S \\ T \cap S = \emptyset}} \frac{Z_{S,\beta}^{x_1x_2,\emptyset}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x_1) = S] Z_{T,\beta}^{x_3x_4}[\mathbf{C}_{\mathbf{n}}(x_3) = T] Z_{\Lambda \setminus S,\beta}^\emptyset Z_{\Lambda \setminus (S \cup T),\beta}^\emptyset}{Z_{\Lambda,\beta}^{x_1x_2,x_3x_4}}. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbf{P}_{\Lambda,\beta}^{x_1x_2,x_3x_4,\emptyset}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_3}(x_1) \cap \mathbf{C}_{\mathbf{n}_2}(x_3) = \emptyset] \\ = \sum_{\substack{S,T \subset \Lambda \\ x_3 \notin S \\ T \cap S = \emptyset}} \frac{Z_{S,\beta}^{x_1x_2,\emptyset}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(x_1) = S] Z_{T,\beta}^{x_3x_4}[\mathbf{C}_{\mathbf{n}}(x_3) = T] (Z_{\Lambda \setminus S,\beta}^\emptyset)^2 Z_{\Lambda \setminus T,\beta}^\emptyset}{Z_{\Lambda,\beta}^{x_1x_2,x_3x_4,\emptyset}}. \end{aligned} \quad (2.6.42)$$

To prove (2.6.41) it is then sufficient to show that for $S, T \subset \Lambda$ disjoint,

$$\frac{Z_{\Lambda \setminus S \cup T, \beta}^{\emptyset}}{Z_{\Lambda \setminus S, \beta}^{\emptyset}} \geq \frac{Z_{\Lambda \setminus T, \beta}^{\emptyset}}{Z_{\Lambda, \beta}^{\emptyset}}. \quad (2.6.43)$$

We use the same method as in the proof of Proposition 2.3.4. Namely, (2.6.43) is equivalent to

$$\left\langle \exp \left(\sum_{\substack{\{x,y\} \subset \Lambda \\ x \in T \text{ or } y \in T}} \tilde{J}_{x,y} \sigma_x \sigma_y \right) \right\rangle_{\Lambda \setminus T \cup S, \beta}^{-1} \geq \left\langle \exp \left(\sum_{\substack{\{x,y\} \subset \Lambda \\ x \in T \text{ or } y \in T}} J_{x,y} \sigma_x \sigma_y \right) \right\rangle_{\Lambda \setminus T, \beta}^{-1}, \quad (2.6.44)$$

where $\tilde{J}_{x,y} = J_{x,y}$ everywhere except one edges xy with $x \in S$ and $y \in T$ where it is equal to 0. Notice that Proposition 2.2.11 yields

$$\left\langle \exp \left(\sum_{\substack{\{x,y\} \subset \Lambda \\ x \in T \text{ or } y \in T}} J_{x,y} \sigma_x \sigma_y \right) \right\rangle_{\Lambda \setminus T, \beta} \geq \left\langle \exp \left(\sum_{\substack{\{x,y\} \subset \Lambda \\ x \in T \text{ or } y \in T}} J_{x,y} \sigma_x \sigma_y \right) \right\rangle_{\Lambda \setminus T \cup S, \beta} \quad (2.6.45)$$

$$\geq \left\langle \exp \left(\sum_{\substack{\{x,y\} \subset \Lambda \\ x \in T \text{ or } y \in T}} \tilde{J}_{x,y} \sigma_x \sigma_y \right) \right\rangle_{\Lambda \setminus T \cup S, \beta}, \quad (2.6.46)$$

which therefore implies (2.6.41) and the result. $\square$

Before turning to the proof of Theorem 2.6.2, we show how Proposition 2.6.12 implies the vanishing of the renormalised coupling constant.

We recall two classical consequences of reflection positivity: the *infrared bound* (see [104, 103]) which yields a useful bound on the model's two-point function and the *Messenger–Miracle–Solé* (MMS) *inequalities* (see [191]) which provide monotonicity properties of the two-point function.

Theorem 2.6.13 (Infrared bound). *Let $d \geq 3$. There exists $C = C(d) > 0$ such that for all $\beta \leq \beta_c$, for all $x \in \mathbb{Z}^d \setminus \{0\}$,*

$$\langle \sigma_0 \sigma_x \rangle_{\beta} \leq \frac{C}{|x|^{d-2}}, \quad (2.6.47)$$

where $|\cdot|$ is the infinite norm on $\mathbb{R}^d$.

For $1 \leq j \leq d$, we let $\mathbf{e}_j$ be the unit vector of $\mathbb{R}^d$ with j -th coordinate equal to 1. Let also $|\cdot|_1$ denote the ℓ^1 norm on $\mathbb{R}^d$.

Theorem 2.6.14 (Monotonicity of the two-point function). *Let $d \geq 1$ and $\beta > 0$. Then,*

- (i) *for all $1 \leq j \leq d$, the sequence $(\langle \sigma_0 \sigma_{k\mathbf{e}_j} \rangle_{\beta})_{k \geq 0}$ is decreasing,*
- (ii) *for $x \in \mathbb{Z}^d$,*

$$\langle \sigma_0 \sigma_{(|x|, 0_{\perp})} \rangle_{\beta} \geq \langle \sigma_0 \sigma_x \rangle_{\beta} \geq \langle \sigma_0 \sigma_{(|x|_1, 0_{\perp})} \rangle_{\beta}, \quad (2.6.48)$$

where $0_{\perp} \in \mathbb{Z}^{d-1}$ is the null vector. In particular, for all $x, y \in \mathbb{Z}^d$ with $d|x| \leq |y|$,

$$\langle \sigma_0 \sigma_x \rangle_{\beta} \geq \langle \sigma_0 \sigma_y \rangle_{\beta}. \quad (2.6.49)$$

The infrared bound provides a bound on the susceptibility $\chi(\beta)$ in terms of the correlation length $\xi(\beta)$.

Corollary 2.6.15 (Upper bound on the susceptibility). *Let $d \geq 3$. There exists $C = C(d) > 0$ such that for all $\beta < \beta_c$,*

$$\chi(\beta) = \sum_{x \in \mathbb{Z}^d} \langle \sigma_0 \sigma_x \rangle_\beta \leq C(\xi(\beta) \log \xi(\beta))^2. \quad (2.6.50)$$

Proof. The constants C_i only depend on d . Let $\beta < \beta_c$ and K to be fixed. Using the infrared bound,

$$\sum_{|x| \leq K\xi(\beta) \log \xi(\beta)} \langle \sigma_0 \sigma_x \rangle_\beta \leq C_1(K\xi(\beta) \log \xi(\beta))^2. \quad (2.6.51)$$

Then, a classical consequence of subadditivity is that

$$\langle \sigma_0 \sigma_{n\mathbf{e}_1} \rangle_\beta \leq e^{-n/\xi(\beta)}. \quad (2.6.52)$$

Moreover, the Messager–Miracle–Solé inequalities [191] give that

$$\max_{|x|=n} \langle \sigma_0 \sigma_x \rangle_\beta = \langle \sigma_0 \sigma_{n\mathbf{e}_1} \rangle_\beta. \quad (2.6.53)$$

Combining the last two displayed equations gives

$$\sum_{|x| > K\xi(\beta) \log \xi(\beta)} \langle \sigma_0 \sigma_x \rangle_\beta \leq C_2 \sum_{k \geq K\xi(\beta) \log \xi(\beta)} k^{d-1} e^{-k/\xi(\beta)} \leq C_3 \quad (2.6.54)$$

if $K > d$. This gives the result. $\square$

Recall that the renormalized coupling constant $g(\beta)$ is defined for $\beta < \beta_c$ by

$$g(\beta) = -\frac{1}{\chi(\beta)^2 \xi(\beta)^d} \sum_{x,y,z \in \mathbb{Z}^d} U_4^\beta(0, x, y, z). \quad (2.6.55)$$

Theorem 2.6.16. *Let $d > 4$. Then,*

$$g(\beta) \xrightarrow[\beta \nearrow \beta_c]{} 0. \quad (2.6.56)$$

Proof. Using Proposition 2.6.12 we immediatly get that

$$0 \leq g(\beta) \leq 2 \frac{\chi(\beta)^4}{\chi(\beta)^2 \xi(\beta)^d}. \quad (2.6.57)$$

Then, using Corollary 2.6.15,

$$g(\beta) \leq 2 \frac{C^2(\xi(\beta) \log \xi(\beta))^4}{\xi(\beta)^d} = 2C^2 \xi(\beta)^{4-d} (\log \xi(\beta))^2, \quad (2.6.58)$$

which implies the result when $d > 4$ since $\xi(\beta)$ diverges when β tends to β_c . $\square$

We now turn to the proof of Theorem 2.6.2. The proof essentially relies on Theorem 2.6.10 and Proposition 2.6.12. We also need an additional tool called the *sliding-scale infrared bound*. Before stating it, we introduce a notation: if $\beta > 0$ and $\ell \geq 1$,

$$\chi_\ell(\beta) := \sum_{x \in \Lambda_\ell} \langle \sigma_0 \sigma_x \rangle_\beta. \quad (2.6.59)$$

Theorem 2.6.17 (Sliding-scale infrared bound [9, Theorem 5.6]). *Let $d \geq 3$. There exists $C = C(d) > 0$ such that for every $\beta \leq \beta_c$, and every $L \geq \ell \geq 1$,*

$$\frac{\chi_L(\beta)}{L^2} \leq C \frac{\chi(\ell)}{\ell^2}. \quad (2.6.60)$$

Proof of Theorem 2.6.2. Again, the constants C_i only depend on d . Let $f \in \mathcal{C}_0(\mathbb{R}^d)$, $\beta \leq \beta_c$, $L \leq \xi(\beta)$ and $z \in \mathbb{R}$. Recall that $r_f \geq 1$. Using Theorem 2.6.10 one gets for $n \geq 2$ (the inequality being trivial for $n = 0, 1$),

$$\left| \left\langle T_{f,L,\beta}(\sigma)^{2n} \right\rangle_\beta - \frac{(2n)!}{2^n n!} \left\langle T_{f,L,\beta}(\sigma)^2 \right\rangle_\beta^n \right| \leq \frac{3}{2} (2n)^4 \left\langle T_{|f|,L,\beta}(\sigma)^{2n-4} \right\rangle_\beta \|f\|_\infty^4 S(\beta, L, f) \quad (2.6.61)$$

where

$$S(\beta, L, f) := \sum_{x_1, x_2, x_3, x_4 \in \Lambda_{r_f L}} \frac{|U_4^\beta(x_1, x_2, x_3, x_4)|}{\Sigma_L(\beta)^2}. \quad (2.6.62)$$

Multiplying by $\frac{z^{2n}}{(2n)!}$ and summing (2.6.61) over n , one gets

$$\left| \left\langle \exp(z T_{f,L,\beta}(\sigma)) \right\rangle_\beta - \exp\left(\frac{z^2}{2} \langle T_{f,L,\beta}(\sigma)^2 \rangle_\beta\right) \right| \leq C_1 z^4 \exp\left(\frac{z^2}{2} \langle T_{|f|,L,\beta}(\sigma)^2 \rangle_\beta\right) \|f\|_\infty^4 S(\beta, L, f). \quad (2.6.63)$$

We will prove a bound of the type

$$S(\beta, L, f) \leq \frac{C_2 r_f^{\alpha(d)}}{L^{d-4}}. \quad (2.6.64)$$

We start by applying the tree diagram bound of Proposition 2.6.12 to obtain

$$S(\beta, L, f) \leq 2 \sum_{\substack{x \in \mathbb{Z}^d \\ x_1, x_2, x_3, x_4 \in \Lambda_{r_f L}}} \frac{\langle \sigma_x \sigma_{x_1} \rangle_\beta \langle \sigma_x \sigma_{x_2} \rangle_\beta \langle \sigma_x \sigma_{x_3} \rangle_\beta \langle \sigma_x \sigma_{x_4} \rangle_\beta}{\Sigma_L(\beta)^2}. \quad (2.6.65)$$

Splitting the above sum, one gets

$$\frac{S(\beta, L, f)}{2} \leq \underbrace{\sum_{\substack{x \in \Lambda_{dr_f L} \\ x_1, x_2, x_3, x_4 \in \Lambda_{r_f L}}} (\dots)}_{(1)} + \underbrace{\sum_{\substack{x \notin \Lambda_{dr_f L} \\ x_1, x_2, x_3, x_4 \in \Lambda_{r_f L}}} (\dots)}_{(2)}. \quad (2.6.66)$$

Bound on (1). The first term can be written

$$\sum_{\substack{x \in \Lambda_{dr_f L} \\ x_1, x_2, x_3, x_4 \in \Lambda_{r_f L}}} \frac{\langle \sigma_x \sigma_{x_1} \rangle_\beta \langle \sigma_x \sigma_{x_2} \rangle_\beta \langle \sigma_x \sigma_{x_3} \rangle_\beta \langle \sigma_x \sigma_{x_4} \rangle_\beta}{\Sigma_L(\beta)^2} = \sum_{x \in \Lambda_{dr_f L}} \frac{\left(\sum_{y \in \Lambda_{r_f L}} \langle \sigma_x \sigma_y \rangle_\beta \right)^4}{\Sigma_L(\beta)^2}.$$

Noticing that for $x \in \Lambda_{dr_f L}$,

$$\sum_{y \in \Lambda_{r_f L}} \langle \sigma_x \sigma_y \rangle_\beta \leq \chi_{2dr_f L}(\beta), \quad (2.6.67)$$

one gets

$$(1) \leq C_1 \frac{|\Lambda_{dr_f L}| \chi_{2dr_f L}(\beta)^4}{\Sigma_L(\beta)^2}. \quad (2.6.68)$$

We can use Theorem 2.6.17 to bound $\chi_{2dr_f L}(\beta)$ in terms of $\chi_L(\beta)$; and the following inequality,

$$\chi_L(\beta) \leq C_2 L^{-d} \Sigma_L(\beta), \quad (2.6.69)$$

to obtain that

$$(1) \leq C_3 r_f^{8+d} \frac{\chi_L(\beta)^2}{L^d}. \quad (2.6.70)$$

Using the infrared bound,

$$\chi_L(\beta) \leq C_4 L^2, \quad (2.6.71)$$

so that

$$(1) \leq C_5 r_f^{8+d} L^{4-d}. \quad (2.6.72)$$

Bound on (2). For $x \notin \Lambda_{dr_f L}(\beta)$ and $x_i \in \Lambda_{r_f L}$,

$$\langle \sigma_x \sigma_{x_i} \rangle_\beta \leq \frac{C_6}{|x|^d} \chi_{|x|/d}(\beta) \leq \frac{C_7}{|x|^{d-2} L^2} \chi_L(\beta), \quad (2.6.73)$$

where the first inequality follows from Theorem 2.6.14 and the second one from Theorem 2.6.17. Bounding the terms indexed by x_1 and x_2 in the sum using (2.6.73) and the other two using the infrared bound, we get

$$\begin{aligned} (2) &\leq C_8 \sum_{x \notin \Lambda_{dr_f L}} \sum_{x_1, \dots, x_4 \in \Lambda_{r_f L}} \frac{\chi_L(\beta)^2}{\Sigma_L(\beta)^2 |x|^{2d-4} L^4} \frac{1}{|x - x_3|^{d-2}} \frac{1}{|x - x_4|^{d-2}} \\ &\leq C_9 r_f^{2d+4} L^{2d} \frac{\chi_L(\beta)^2}{\Sigma_L(\beta)^2} \sum_{x \notin \Lambda_{dr_f L}} \frac{1}{|x|^{2d-4}} \\ &\leq C_{10} r_f^{8+d} L^{4-d}, \end{aligned}$$

where we used the inequality $|x - x_i| \geq (d-1)r_f L$ in the second line, and (2.6.69) in the third line. The proof follows readily. $\square$

2.6.3 The marginal case $d = 4$

When $d = 4$, the tree diagram bound of Proposition 2.6.12 is too rough to conclude. It is possible to improve on this result by following once again the analogy with random walks.

It is a well-known result that $d = 4$ is the critical dimension in terms of intersection for the simple random walk, meaning that two independent (simple) random walks with starting and ending points at mutual distance L , will intersect with probability $O(1/\log L)$ while the expected number of points in the intersection will typically be of order $\Omega(1)$ (see [172, Chapter 10]). This shows that when two independent (simple) random walks in dimension four intersect, they do so a logarithmic number of times.

Transposing this idea in the realm of random currents suggests that the probability that two independent random currents with sources at mutual distance at least L intersect, but “not so many times”, should decay as $O(1/(\log L)^c)$ for some $c > 0$.

Let us clarify further. Recall from (2.6.37) that for all $x, y, z, t \in \mathbb{Z}^4$,

$$U_4^{\beta_c}(x, y, z, t) \leq 2\langle \sigma_x \sigma_y \rangle_{\beta_c} \langle \sigma_z \sigma_t \rangle_{\beta_c} \mathbf{P}_{\beta_c}^{xy,zt,\emptyset,\emptyset}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_3}(x) \cap \mathbf{C}_{\mathbf{n}_2+\mathbf{n}_4}(z) \neq \emptyset]. \quad (2.6.74)$$

Denoting $\mathcal{I} := \mathbf{C}_{\mathbf{n}_1+\mathbf{n}_3}(x) \cap \mathbf{C}_{\mathbf{n}_2+\mathbf{n}_4}(z)$, the tree diagram bound follows from Markov's inequality:

$$\mathbf{P}_{\beta_c}^{xy,zt,\emptyset,\emptyset}[|\mathcal{I}| \geq 1] \leq \mathbf{E}_{\beta_c}^{xy,zt,\emptyset,\emptyset}[|\mathcal{I}|] = \sum_{u \in \mathbb{Z}^4} \frac{\langle \sigma_x \sigma_u \rangle_{\beta_c} \langle \sigma_u \sigma_y \rangle_{\beta_c} \langle \sigma_z \sigma_u \rangle_{\beta_c} \langle \sigma_u \sigma_t \rangle_{\beta_c}}{\langle \sigma_x \sigma_y \rangle_{\beta_c} \langle \sigma_z \sigma_t \rangle_{\beta_c}}. \quad (2.6.75)$$

However, when $d = 4$, we expect that $\mathbf{E}_{\beta_c}^{xy,zt,\emptyset,\emptyset}[|\mathcal{I}|] = O(1)$. The comparison with random walks suggests that one can write, for some well-chosen $c > 0$ and for L being the minimal distance between x, y, z, t ,

$$\mathbf{P}_{\beta_c}^{xy,zt,\emptyset,\emptyset}[|\mathcal{I}| \geq 1] \leq \mathbf{P}_{\beta_c}^{xy,zt,\emptyset,\emptyset}[1 \leq |\mathcal{I}| \leq (\log L)^c] + \frac{1}{(\log L)^c} \mathbf{E}_{\beta_c}^{xy,zt,\emptyset,\emptyset}[|\mathcal{I}|]. \quad (2.6.76)$$

The core of the result of Aizenman and Duminil-Copin [9] is an analysis of the probability $\mathbf{P}_{\beta_c}^{xy,zt,\emptyset,\emptyset}[1 \leq |\mathcal{I}| \leq (\log L)^c]$. This will be described in detail in Chapter 9.

To cut a long story short, their argument relies on the following two steps.

- 1) First, working under the assumption that $|\mathcal{I}| \geq 1$, it is possible to re-center the problem at a given intersection point. Therefore, we can reduce the analysis to the study of the probability

$$\mathbf{P}_{\beta_c}^{uy,ut,\emptyset,\emptyset}[|\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_3}(u) \cap \mathbf{C}_{\mathbf{n}_2+\mathbf{n}_4}(u)| \leq (\log L')^c], \quad (2.6.77)$$

where L' is the minimum between $|u - y|$ and $|u - t|$. For a well-chosen sequence $(\ell_k)_{k \geq 0}$, Aizenman and Duminil-Copin obtain the existence of an absolute constant $c > 0$ such that

$$\min_{k: \ell_{k+1} \leq L} \mathbf{P}_{\beta_c}^{uy,ut,\emptyset,\emptyset}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_3}(u) \cap \mathbf{C}_{\mathbf{n}_2+\mathbf{n}_4}(u) \cap \text{Ann}(\ell_k, \ell_{k+1}) \neq \emptyset] \geq c, \quad (2.6.78)$$

where $\text{Ann}(n, m) := \Lambda_m \setminus \Lambda_{n-1}$.

- 2) Second, they prove a *mixing property* for the currents. This allows one to view the events

$$\{\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_3}(u) \cap \mathbf{C}_{\mathbf{n}_2+\mathbf{n}_4}(u) \cap \text{Ann}(\ell_{2k}, \ell_{2k+1}) \neq \emptyset\} \quad (2.6.79)$$

as being “almost” independent. Classical probabilistic considerations then allow to derive a bound of the form

$$\mathbf{P}_{\beta_c}^{uy, ut, \emptyset, \emptyset}[|\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_3}(u) \cap \mathbf{C}_{\mathbf{n}_2+\mathbf{n}_4}(u)| \leq (\log L')^c] \leq o(1), \quad (2.6.80)$$

where $o(1)$ tends to 0 as L' tends to infinity. This $o(1)$ is explicit and can be chosen to be $B_L(\beta_c)^{-c}$ for some small $c > 0$, where

$$B_L(\beta_c) := \sum_{x \in \Lambda_L} \langle \sigma_0 \sigma_x \rangle_\beta^2 \quad (2.6.81)$$

is the *truncated* bubble diagram¹¹.

2.6.4 Non-triviality when $d = 2$

In dimension two, exact integrability is sufficient to conclude that the scaling limit of the critical n.n.f Ising model is not Gaussian. However, one can recover this qualitative result with a simple argument.

Proposition 2.6.18 (Lower bound on $|U_4^\beta|$ [4, Proposition 8.1]). *Let $d = 2$ and $\beta \leq \beta_c$. Let $x \in \mathbb{N}$. Let $x_1 = -x_3 = (x, 0)$ and $x_2 = -x_4 = (0, x)$. Then,*

$$|U_4^\beta(x_1, x_2, x_3, x_4)| \geq \langle \sigma_{x_1} \sigma_{x_2} \rangle_\beta \langle \sigma_{x_2} \sigma_{x_4} \rangle_\beta. \quad (2.6.82)$$

Proof. Let Γ_{x_2, x_4} be the set of curves in $\mathbb{R}^2 \setminus \{x_2, x_4\}$ which are homotopic to the segment $[x_1, x_3]$. We let

$$E_1 := \{x_2, x_4 \notin \mathbf{C}_{\mathbf{n}_1}(x_1), \text{ and any path linking } x_1 \text{ and } x_3 \\ \text{along neighboring sites in } \mathbf{C}_{\mathbf{n}_1}(x_1) \text{ is in } \Gamma_{x_2, x_4}\},$$

and

$$E_2 := \{x_2, x_4 \notin \mathbf{C}_{\mathbf{n}_1}(x_1), \text{ and no path linking } x_1 \text{ and } x_3 \\ \text{along neighboring sites in } \mathbf{C}_{\mathbf{n}_1}(x_1) \text{ is in } \Gamma_{x_2, x_4}\}.$$

Also write

$$P_1 := \mathbf{P}_\beta^{x_1 x_3, x_2 x_4}[E_1], \quad P_2 := \mathbf{P}_\beta^{x_1 x_3, x_2 x_4}[E_2]. \quad (2.6.83)$$

It is clear that

$$P_1 + P_2 \leq 1. \quad (2.6.84)$$

An analysis of the two cases in which $\mathbf{C}_{\mathbf{n}_1}(x_1) \cap \mathbf{C}_{\mathbf{n}_2}(x_2) = \emptyset$ (see Figure 2.3), and invariance under $\pi/2$ rotations yield that

$$\mathbf{P}_\beta^{x_1 x_3, x_2 x_4}[\mathbf{C}_{\mathbf{n}_1}(x_1) \cap \mathbf{C}_{\mathbf{n}_2}(x_2) = \emptyset] \leq 2P_1 P_2 = \frac{(P_1 + P_2)^2 - (P_1 - P_2)^2}{2} \leq \frac{1}{2}. \quad (2.6.85)$$

¹¹Recall that when $d = 4$, we expect the critical two-point function of the Ising model to be of order $|x|^{-2}$. This implies that $B_L(\beta_c)$ is of order $\log L$.

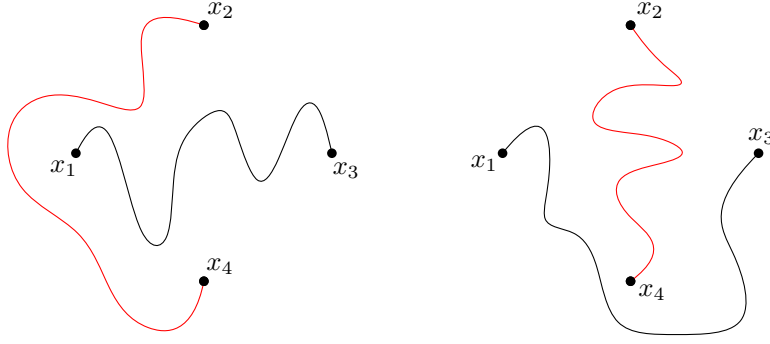


Figure 2.3: An illustration of the possible non-intersecting current configurations $(\mathbf{n}_1, \mathbf{n}_2)$. Each situation has a probability at most $P_1 P_2$ by invariance under $\pi/2$ rotations.

Using Proposition 2.6.7,

$$\begin{aligned} |U_4^\beta(x_1, x_2, x_3, x_4)| &\geq 2\langle \sigma_{x_1} \sigma_{x_3} \rangle_\beta \langle \sigma_{x_2} \sigma_{x_4} \rangle_\beta \mathbf{P}_\beta^{x_1 x_3, x_2 x_4} [\mathbf{C}_{\mathbf{n}_1}(x_1) \cap \mathbf{C}_{\mathbf{n}_2}(x_2) \neq \emptyset] \\ &\geq \langle \sigma_{x_1} \sigma_{x_3} \rangle_\beta \langle \sigma_{x_2} \sigma_{x_4} \rangle_\beta, \end{aligned}$$

which yields the result. $\square$

Remark 2.6.19. This result suggests that in the continuum limit, $U_4^\beta \neq 0$.

Before stating the next result, we need some notations. If $\sigma > 0$, the correlation length of order σ is defined, for $\beta < \beta_c$ by,

$$\xi_\sigma(\beta) := \left(\frac{1}{\chi(\beta)} \sum_{x \in \mathbb{Z}^d} |x|^\sigma \langle \sigma_0 \sigma_x \rangle_\beta \right)^{1/\sigma}. \quad (2.6.86)$$

The renormalised coupling constant of order σ is then defined by

$$g_\sigma(\beta) := -\frac{1}{\chi(\beta) \xi_\sigma(\beta)^d} \sum_{x, y, z \in \mathbb{Z}^d} U_4^\beta(0, x, y, z). \quad (2.6.87)$$

Again, Newman's criterion [196] establishes that any scaling limit of the critical model is Gaussian if and only if g_σ vanishes at criticality.

Finally, introduce ν_1 and Δ the critical exponents defined as follows:

$$\xi_1(\beta) = (\beta_c - \beta)^{-\nu_1 + o(1)}, \quad \sum_{x, y, z \in \mathbb{Z}^d} U_4^\beta(0, x, y, z) = (\beta_c - \beta)^{-\gamma - 2\Delta + o(1)}, \quad (2.6.88)$$

where $o(1)$ tends to 0 as β tends to β_c .

The following result extends the non-Gaussianity result to finite-range models on $\mathbb{Z}^2$. This can be viewed as a universality-type result.

Proposition 2.6.20 (Hyperscaling in two dimensions, [4, Proposition 8.2]). *Fix an Ising model on $\mathbb{Z}^2$ with a finite-range interaction matrix J which has all the translation and reflection invariance of $\mathbb{Z}^2$. For all $\beta < \beta_c$,*

$$g_1(\beta) \geq \frac{1}{2R^2}, \quad (2.6.89)$$

where $R = \max\{|x| : x \in \mathbb{Z}^d, J_{0,x} \neq 0\}$ is the range of interaction. In particular, one has the following hyperscaling relation ($d = 2$):

$$d\nu_1 - 2\Delta + \gamma = 0. \quad (2.6.90)$$

We will need the following geometrical lemma.

Lemma 2.6.21. *Let $R > 0$. Let $A = \{x_1, \dots, x_\ell\}$ and $B = \{y_1, \dots, y_k\}$ be two sequences of points in $\mathbb{Z}^2$ satisfying the following property:*

$$\max_{1 \leq i \leq \ell-1} |x_{i+1} - x_i| \leq R, \quad \max_{1 \leq j \leq k-1} |y_{j+1} - y_j| \leq R. \quad (2.6.91)$$

Then, if $A + B := \{x + y : x \in A, y \in B\}$, one has

$$|A + B| \geq \frac{|(x_\ell - x_1, 0) \wedge (y_k - y_1, 0)|}{R^2}, \quad (2.6.92)$$

where $\wedge$ is the cross product¹² on $\mathbb{R}^3$.

Proof of Proposition 2.6.20. Let $\beta < \beta_c$ and drop it from the notations. Using Proposition 2.6.7,

$$\sum_{x,y,z \in \mathbb{Z}^2} |U_4(0, x, y, z)| = \sum_{x,y,z \in \mathbb{Z}^2} |U_4(0, x, y, y+z)| \quad (2.6.93)$$

$$\geq 2 \sum_{x,z \in \mathbb{Z}^2} \langle \sigma_0 \sigma_x \rangle \langle \sigma_0 \sigma_z \rangle \sum_{y \in \mathbb{Z}^2} \mathbf{P}^{0x,y(y+z)} [\mathbf{C}_{\mathbf{n}_1}(0) \cap \mathbf{C}_{\mathbf{n}_2}(y) \neq \emptyset]. \quad (2.6.94)$$

Now, notice that $\mathbf{C}_{\mathbf{n}_1}(0)$ (resp. $\mathbf{C}_{\mathbf{n}_2}(0)$) necessarily contains a path from 0 to x (resp. from 0 to z) that satisfies the condition of Lemma 2.6.21. Hence, using translation invariance and Lemma 2.6.21,

$$\sum_{y \in \mathbb{Z}^2} \mathbf{P}^{0x,y(y+z)} [\mathbf{C}_{\mathbf{n}_1}(0) \cap \mathbf{C}_{\mathbf{n}_2}(y) \neq \emptyset] = \sum_{y \in \mathbb{Z}^2} \mathbf{P}^{0x,0z} [\mathbf{C}_{\mathbf{n}_1}(0) \cap (y + \mathbf{C}_{\mathbf{n}_2}(0)) \neq \emptyset] \quad (2.6.95)$$

$$= \mathbf{E}^{0x,0z} [|\mathbf{C}_{\mathbf{n}_1}(0) - \mathbf{C}_{\mathbf{n}_2}(0)|] \quad (2.6.96)$$

$$\geq \frac{|(x, 0) \wedge (z, 0)|}{R^2}. \quad (2.6.97)$$

As a result, using the reflection symmetries,

$$\sum_{x,y,z \in \mathbb{Z}^2} |U_4(0, x, y, z)| \geq 2 \sum_{x,z \in \mathbb{Z}^2} \langle \sigma_0 \sigma_x \rangle \langle \sigma_0 \sigma_z \rangle \frac{(x_1 z_2 - x_2 z_1) \operatorname{sgn}(x_1) \operatorname{sgn}(z_2)}{R^2} \quad (2.6.98)$$

$$= 2 \sum_{x,z \in \mathbb{Z}^d} \langle \sigma_0 \sigma_x \rangle \langle \sigma_0 \sigma_z \rangle \frac{|x_1| |z_2|}{R^2} \quad (2.6.99)$$

$$\geq \frac{\chi^2 \xi_1^2}{2R^2}. \quad (2.6.100)$$

¹²We embed the two-dimensional vectors $x_\ell - x_1$ and $y_k - y_1$ in $\mathbb{R}^3$ by setting their third coordinate to 0.

The hyperscaling relation follows from the lower bound then obtained on g_1 , together with the complementary upper bound (see [4, Proposition 15.1])

$$g_{d/2}(\beta) \leq \frac{12\pi^{d/2}}{\Gamma\left(\frac{d}{2} + 1\right)}. \quad (2.6.101)$$

□

Part I

Alternative approach to the study of the mean-field regime in statistical mechanics

Chapter 3

An alternative approach for the mean-field behaviour of weakly self-avoiding walks in dimensions $d > 4$

We propose a new way of deriving mean-field exponents for the weakly self-avoiding walk model in dimensions $d > 4$. Among other results, we obtain up-to-constant estimates for the full-space and half-space two-point functions in the critical and near-critical regimes.

This chapter is based on the preprint “*An alternative approach for the mean-field behaviour of weakly self-avoiding walks in dimensions $d > 4$* ” [82]. It is a joint work with Hugo Duminil-Copin.

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3.1 Introduction

One of the main challenges of statistical mechanics consists in understanding the (near) critical behaviour of diverse lattice models. Among other things, one may for instance compute

the models' *critical exponents*. Conducting this task is in general extremely difficult as it involves in a subtle way both the special features of the models and the geometry of the graphs on which they are defined.

A striking observation was made in the case of models defined on the hypercubic lattice $\mathbb{Z}^d$: above a so-called *upper-critical dimension* d_c , the geometry ceases to play a role and the critical exponents take an easier form, equalling those obtained on a Cayley tree (or *Bethe lattice*) or the complete graph. The regime $d > d_c$ is called the *mean-field* regime of a model. Noteworthy methods such as the *lace expansion* [49] or the *rigorous renormalisation group method* [25, 26, 27, 29] have emerged to carry out the analysis of the mean-field regime. However, a limitation of these methods lies in their predominantly *perturbative* nature, which is reflected in the necessity of exhibiting a small parameter in the model. The Weakly Self-Avoiding Walk (WSAW) model includes such a small parameter in its definition, making it a natural testing ground for developing the analysis of the mean-field regime.

Lace expansion was successfully applied to derive the mean-field behaviour of the WSAW model in dimensions $d > 4$: in [49], Gaussian limit laws were exhibited for the displacement of the n -steps WSAW, while in [132, 39, 48, 228, 229], estimates of the two-point function were obtained.

The WSAW model is also a toy model to study (*strictly*) self-avoiding walks. Using the lace expansion, Slade [225] extended the results of [49] to the setup of the self-avoiding walk model in sufficiently large dimensions. This restriction was later removed by Hara and Slade [136] who optimised the techniques to obtain mean-field behaviour of the SAW model in dimensions $d > 4$. Let us mention that the lace expansion was also applied to a variety of models of statistical mechanics including Bernoulli percolation [134, 133, 132, 100], lattice trees and animals [135], the Ising model [216, 218], and the φ^4 model [217, 48]. For more information on the lace expansion approach, we refer to the monograph [226].

In this chapter, we provide an alternative argument to obtain mean-field bounds on the two-point function of the weakly self-avoiding walk in dimensions $d > 4$. This technique extends to a number of other models after suitable modifications (see e.g. [81] for the example of percolation), but we choose to stick to the case of the WSAW model to present the method in its simplest context.

Notations. Consider the hypercubic lattice $\mathbb{Z}^d$ and let $y \sim z$ denote the fact that y and z are neighbours in $\mathbb{Z}^d$. Set $\mathbf{e}_j$ to be the unit vector with j -th coordinate equal to 1. Write x_j for the j -th coordinate of x , and denote its ℓ^∞ norm by $|x| := \max\{|x_j| : 1 \leq j \leq d\}$. Set $\Lambda_n := \{x \in \mathbb{Z}^d : |x| \leq n\}$ and for $x \in \mathbb{Z}^d$, $\Lambda_n(x) := \Lambda_n + x$. Also, set $\mathbb{H}_n := -n\mathbf{e}_1 + \mathbb{H}$, where $\mathbb{H} := \mathbb{Z}_+ \times \mathbb{Z}^{d-1} = \{0, 1, \dots\} \times \mathbb{Z}^{d-1}$. Finally, let ∂S be the boundary of the set S , given by the vertices in S with at least one neighbour outside S .

3.1.1 Definitions and statements of the results

Let $\lambda \in (0, 1)$. Since λ will be fixed for the whole article, we omit it from the notations. Let $\mathcal{W}$ be the set of finite paths in $\mathbb{Z}^d$. Let $|\gamma|$ be the number of edges of $\gamma \in \mathcal{W}$. For $\gamma = (\gamma(0), \dots, \gamma(|\gamma|)) \in \mathcal{W}$, introduce the weight

$$\rho(\gamma) := \prod_{0 \leq s < t \leq |\gamma|} (1 - \lambda \mathbb{1}_{\gamma(s) = \gamma(t)}). \quad (3.1.1)$$

For a set $\Lambda \subset \mathbb{Z}^d$, $\beta \geq 0$, and $x, y \in \Lambda$, define the *two-point function* (in Λ) by

$$G_\beta^\Lambda(x, y) := \sum_{\gamma: x \rightarrow y \subset \Lambda} \beta^{|\gamma|} \rho(\gamma), \quad (3.1.2)$$

where $x \rightarrow y$ means that γ starts at x and ends at y (in particular, if $x = y$, we count the walk of length zero). When $\Lambda = \mathbb{Z}^d$, we omit it from the notation and simply write $G_\beta(x, y)$.

Let β_c be the critical value for the finiteness of the *susceptibility*

$$\chi(\beta) := \sum_{x \in \mathbb{Z}^d} G_\beta(0, x), \quad (3.1.3)$$

defined by the formula

$$\beta_c = \beta_c(\lambda) := \sup \left\{ \beta \geq 0 : \chi(\beta) < \infty \right\}. \quad (3.1.4)$$

It is easy to obtain (see [226, 30]) that $\beta_c \in [(2d)^{-1}, \mu_c(d)^{-1}]$, where $\mu_c(d)$ is the connective constant of $\mathbb{Z}^d$. For $\beta < \beta_c$, the two-point function decays exponentially fast in the distance. A convenient quantity helping to monitor the rate of decay is the *sharp length* L_β defined below (see also [84, 204, 83] for a study of this quantity in the context of Bernoulli percolation and the Ising model). For $\beta \geq 0$ and $S \subset \mathbb{Z}^d$, let

$$\varphi_\beta(S) := \sum_{\substack{y \in S \\ z \notin S \\ y \sim z}} G_\beta^S(0, y) \beta. \quad (3.1.5)$$

The sharp length L_β is defined by

$$L_\beta := \inf \{k \geq 1 : \varphi_\beta(\Lambda_k) \leq 1/e\}. \quad (3.1.6)$$

Exponential decay of the two-point function guarantees that L_β is finite for $\beta < \beta_c$, and it is easy to prove that it is infinite¹ for $\beta = \beta_c$ (see [223, 84]).

We now state our first main result, which provides uniform upper bounds on the full-space and half-space two-point functions.

Theorem 3.1.1 (Upper bounds). *Let $d > 4$. There exist $C, \lambda_0 > 0$ such that for every $\lambda < \lambda_0$, every $\beta \leq \beta_c$, and every $x \in \mathbb{Z}^d \setminus \{0\}$,*

$$G_\beta(0, x) \leq \frac{C}{|x|^{d-2}} \exp(-|x|/2L_\beta), \quad (3.1.7)$$

$$G_\beta^{\mathbb{H}}(0, x) \leq \frac{C}{|x_1|^{d-1}} \exp(-|x_1|/2L_\beta). \quad (3.1.8)$$

A near-critical upper bound was derived using the lace expansion in [229] (see also [179] for the case of the strictly self-avoiding walk model). There, L_β is replaced by $C_0(\beta_c - \beta)^{-1/2}$ for some $C_0 > 0$. In fact, we will prove that the two quantities are within a multiplicative constant of each other, see Corollary 3.1.4. The second main theorem of this article provides lower bounds matching the bounds in Theorem 3.1.1.

¹The existence of k such that $\varphi_{\beta_c}(\Lambda_k) < 1$ would imply that $\chi(\beta_c) \leq \frac{|\Lambda_k|}{1 - \varphi_{\beta_c}(\Lambda_k)}$ by Lemma 3.1.5 and the strategy of [84]. In particular, this yields $\chi(\beta_c) < \infty$ which is impossible, see e.g [30, (4.6)].

Theorem 3.1.2 (Lower bounds). *Let $d > 4$. There exist $c, C, \lambda_0 > 0$ such that for every $\lambda < \lambda_0$, every $\frac{1}{2d} \leq \beta \leq \beta_c$, and every $x \in \mathbb{Z}^d \setminus \{0\}$,*

$$G_\beta(0, x) \geq \frac{c}{|x|^{d-2}} \exp(-C|x|/L_\beta), \quad (3.1.9)$$

$$G_\beta^{\mathbb{H}}(0, x) \geq \frac{c}{|x|^{d-1}} \exp(-C|x|/L_\beta) \quad \text{provided that } x_1 = |x|. \quad (3.1.10)$$

The bounds of Theorems 3.1.1 and 3.1.2 are expected to hold also at the upper-critical dimension $d = 4$ (with potential logarithmic corrections in the exponential). In the case of the critical full-space estimate, this has been successfully derived in [26].

A direct consequence of Theorem 3.1.1 is the finiteness at criticality of the so-called *bubble diagram*, which plays a central role in the study of the mean-field regime of the WSAW model, see e.g. [43, 226, 30].

Corollary 3.1.3 (Finiteness of the bubble diagram). *Let $d > 4$. There exists $\lambda_0 > 0$ such that for every $\lambda < \lambda_0$,*

$$B(\beta_c) := \sum_{x \in \mathbb{Z}^d} G_{\beta_c}(0, x)^2 < \infty. \quad (3.1.11)$$

We now describe how to recover the mean-field behaviour of the *susceptibility* (defined in (3.1.3)) and the *correlation length* ξ_β defined for $\beta < \beta_c$ by²

$$\xi_\beta^{-1} := \lim_{n \rightarrow \infty} -\frac{1}{n} \log G_\beta(0, n\mathbf{e}_1). \quad (3.1.12)$$

Corollary 3.1.4. *Let $d > 4$. There exist $c, C, \lambda_0 > 0$ such that for every $\lambda < \lambda_0$ and every $\frac{1}{2d} \leq \beta < \beta_c$,*

$$c(\beta_c - \beta)^{-1} \leq \chi(\beta) \leq C(\beta_c - \beta)^{-1}, \quad (3.1.13)$$

$$c(\beta_c - \beta)^{-1/2} \leq \xi_\beta \leq C(\beta_c - \beta)^{-1/2}, \quad (3.1.14)$$

$$c(\beta_c - \beta)^{-1/2} \leq L_\beta \leq C(\beta_c - \beta)^{-1/2}. \quad (3.1.15)$$

Proof. Let $d > 4$ and λ_0 be given by Corollary 3.1.3. It is classical (see e.g. [30, Sections 4.1–4.2]) that for $\beta < \beta_c$,

$$\frac{\chi(\beta)^2}{B(\beta)} \leq \frac{d\chi}{d\beta} \leq \chi(\beta)^2. \quad (3.1.16)$$

Combined with Corollary 3.1.3, which gives that $B(\beta) \leq B(\beta_c) < \infty$, (3.1.16) readily implies (3.1.13).

The bounds (3.1.14) and (3.1.15) are obtained using (3.1.13), and Theorems 3.1.1 and 3.1.2 twice. More precisely, using Theorems 3.1.1 and 3.1.2 together with (3.1.12) we obtain that

$$\xi_\beta \asymp L_\beta \quad (3.1.17)$$

for $\frac{1}{2d} \leq \beta < \beta_c$, where $\asymp$ means that the ratio of the quantities is bounded away from 0 and ∞ by two constants that are independent of β . Then, using again Theorems 3.1.1 and 3.1.2 to bound $\chi(\beta)$, we obtain

$$\chi(\beta) \asymp L_\beta^2 \quad (3.1.18)$$

for $\frac{1}{2d} \leq \beta < \beta_c$, which also gives $\chi(\beta) \asymp \xi_\beta^2$ by (3.1.17). The proofs of (3.1.14) and (3.1.15) follow readily from these observations and from (3.1.13). $\square$

²The limit is shown to exist by a classical subadditivity argument, see [187, Chapter 4].

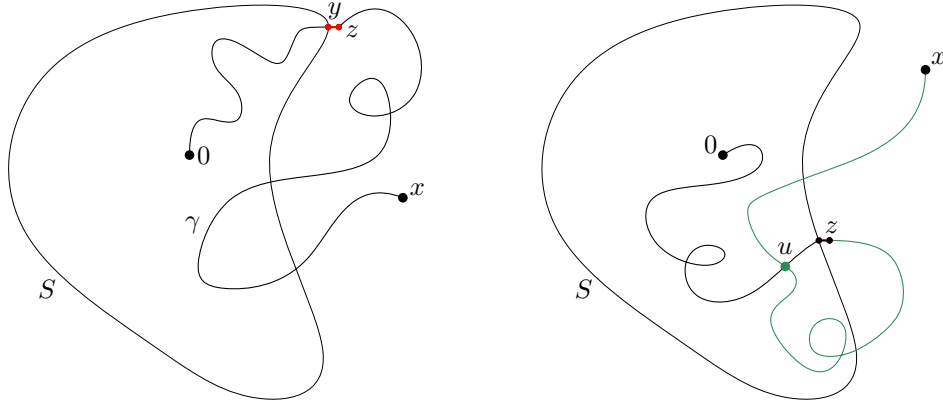


Figure 3.1: A depiction of Lemma 3.1.5. On the left we illustrate how we decompose a path γ contributing to $G_\beta^\Lambda(0, x)$. The first edge leaving S is represented in red. On the right we illustrate a configuration in which the portions of γ from 0 to z (in black) and from z to x (in green) interact through an intersection inside S . This situation contributes to the *error* term in (3.1.20).

3.1.2 The fundamental inequalities

A crucial role will be played by the following two inequalities, see Figure 3.1 for an illustration. For completeness, we include the proof of this (classical) statement in the appendix.

Lemma 3.1.5. *Let $d \geq 2$. For $0 < \beta < \beta_c$, $0 \in S \subset \Lambda$ with $\Lambda \subset \mathbb{Z}^d$, and $x \in \Lambda$,*

$$G_\beta^\Lambda(0, x) \leq G_\beta^S(0, x) + \sum_{\substack{y \in S \\ z \in \Lambda \setminus S \\ y \sim z}} G_\beta^S(0, y) \beta G_\beta^\Lambda(z, x), \quad (3.1.19)$$

$$G_\beta^\Lambda(0, x) \geq G_\beta^S(0, x) + \sum_{\substack{y \in S \\ z \in \Lambda \setminus S \\ y \sim z}} G_\beta^S(0, y) \beta G_\beta^\Lambda(z, x) - \lambda \sum_{u \in S} E_\beta^{S, \Lambda}(u) G_\beta^\Lambda(u, x), \quad (3.1.20)$$

where

$$E_\beta^{S, \Lambda}(u) := \sum_{\substack{y \in S \\ z \in \Lambda \setminus S \\ y \sim z}} G_\beta^S(0, u) G_\beta^S(u, y) \beta G_\beta^\Lambda(z, u). \quad (3.1.21)$$

The quantity

$$E_\beta^{S, \Lambda} := \sum_{u \in S} E_\beta^{S, \Lambda}(u) \quad (3.1.22)$$

will be referred to as the *error amplitude*. This quantity will be shown to be finite when $d > 4$, which is responsible for the restriction on the dimension in this chapter (see (3.2.20)). Controlling the size of this error amplitude will be crucial to the argument.

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3.2 Proof of Theorem 3.1.1

We will use a bootstrap argument (following the original idea from [225]) and prove that an a priori estimate on the two-point function can be improved for sufficiently small λ . One original feature of our proof is that the key quantities we track in the bootstrap involve the half-space rather than the full-space. This contrasts for instance with lace expansion, in which all arguments require the full-space two-point function.

The idea will be to observe that the two inequalities of Lemma 3.1.5 provide good control on $\varphi_\beta(\mathbb{H}_n)$, which can be interpreted as an averaged (or ℓ^1) estimate on the half-space two-point function at distance n . The point-wise (or ℓ^∞) half-space estimate (3.1.8) will follow from a *regularity* property which allows one to compare two-point functions ending at close points. Finally, we will deduce the full-space estimate from the half-space one.

To implement this scheme, we introduce the following quantity β^* .

Definition 3.2.1. Fix $d > 4$ and $\lambda \in (0, 1)$. Let $C > 0$ to be fixed later³. Define $\beta^* = \beta^*(C, \lambda)$ to be the largest real number in $[0, \beta_c]$ such that for every $\beta < \beta^*$,

$$\varphi_\beta(\mathbb{H}_n) < 1 + \frac{1}{2d} \quad \forall n \geq 0, \quad (\ell_\beta^1)$$

$$G_\beta^{\mathbb{H}_n}(0, x) < \frac{C}{n^{d-1}} \quad \forall n \geq 1, \forall x \in \partial\mathbb{H}_n. \quad (\ell_\beta^\infty)$$

The $\frac{1}{2d}$ is quite arbitrary in (ℓ_β^1) . In fact, we could take any number in $(0, \frac{1}{2d}]$. Note that $\beta^* \geq \frac{1}{2d}$ when C exceeds a large enough constant $\mathbf{C}_{\text{RW}} = \mathbf{C}_{\text{RW}}(d) > 0$, as a bound by the corresponding random walk quantities implies⁴ that the estimates are true at $\beta = \frac{1}{2d}$.

Our goal is to show that β^* is in fact equal to β_c provided that C and λ are properly chosen. We proceed in three steps. First, we show that the bound on $\varphi_\beta(\mathbb{H}_n)$ can be improved when $\beta < \beta^*$. Second, we control the gradient of the two-point function. Third, we use the fact that the two-point function does not fluctuate too much (thanks to the second point) to obtain an improved ℓ^∞ estimate. From these improvements, we obtain that β^* cannot be strictly smaller than β_c , since in this case the improved estimates would remain true for β slightly larger than β^* , which would contradict the definition of β^* .

3.2.1 Improving the bound on $\varphi_\beta(\mathbb{H}_n)$

This section is the crucial step of our strategy: from the bounds (ℓ_β^1) and (ℓ_β^∞) , we obtain a bound on $\varphi_\beta(\mathbb{H}_n)$ that involves the parameter λ . For small λ , this bound is an improvement

³One may think that C will be chosen first to be very large, and then λ small enough.

⁴More precisely, $\varphi_\beta(\mathbb{H}_n)$ and $G_\beta^{\mathbb{H}_n}(0, x)$ are maximal when $\lambda = 0$, which corresponds to the simple random walk. When $\beta = \frac{1}{2d}$, we get $\varphi_{\beta, \lambda=0}(\mathbb{H}_n) = 1$ for all $n \geq 0$ as a consequence of Markov's property. Moreover, classical random walk estimates (see [172]) imply the existence of $\mathbf{C}_{\text{RW}} = \mathbf{C}_{\text{RW}}(d) > 0$ such that for all $n \geq 1$ and for all $x \in \partial\mathbb{H}_n$, one has $G_{\beta, \lambda=0}^{\mathbb{H}_n}(0, x) < \mathbf{C}_{\text{RW}}/n^{d-1}$. Hence, choosing $C > \mathbf{C}_{\text{RW}}$ implies that for all $\lambda \in (0, 1)$, one has $\beta^*(C, \lambda) \geq \frac{1}{2d}$.

on (ℓ_β^1) . Recall that $\mathbf{C}_{\text{RW}}$ satisfies the following property: for every $C > \mathbf{C}_{\text{RW}}$ and every $\lambda \in (0, 1)$, $\beta^*(C, \lambda) \geq \frac{1}{2d}$.

Proposition 3.2.2 (Improving the bound on $\varphi_\beta(\mathbb{H}_n)$). *Fix $d > 4$ and $C > \mathbf{C}_{\text{RW}}$. There exists $K = K(C, d) < \infty$ such that for every $\lambda \in (0, 1)$, every $\beta < \beta^*(C, \lambda)$, and every $n \geq 0$,*

$$\varphi_\beta(\mathbb{H}_n) < 1 + K\lambda, \quad (3.2.1)$$

$$\sup\{\varphi_\beta(B) : B \in \mathcal{B}\} \leq 1 + K\lambda, \quad (3.2.2)$$

where $\mathcal{B}$ is the set of blocks, that is $\mathcal{B} := \{(\prod_{i=1}^d [a_i, b_i]) \cap \mathbb{Z}^d : \forall 1 \leq i \leq d, a_i \leq 0 \leq b_i\}$.

In the rest of Section 3.2.1, we fix $\lambda \in (0, 1)$ and drop it from the notations. We start with a number of elementary bounds on the two-point function in the bulk and in half-space induced by the assumption that $\beta < \beta^*$.

Lemma 3.2.3. *Fix $d > 4$ and $C > 0$. For every $\beta < \beta^*(C)$,*

$$G_\beta(0, x) \leq \frac{2C}{|x|^{d-2}} \quad \forall x \in \mathbb{Z}^d \setminus \{0\}, \quad (3.2.3)$$

$$G_\beta^{\mathbb{H}_n}(0, x) \leq \frac{2C(k+1)}{(n-k)^{d-1}} \quad \forall x \in \partial\mathbb{H}_{n-k} \text{ with } 1 \leq k < n, \quad (3.2.4)$$

$$\sum_{x \in \partial\mathbb{H}_{n-k}} G_\beta^{\mathbb{H}_n}(0, x) \beta \leq 4(k+1) \quad \text{with } n, k \geq 0. \quad (3.2.5)$$

Proof. Let us start with (3.2.3). Without loss of generality, assume that $x_1 = |x|$. Consider a path from 0 to x . If the path is not included in $\mathbb{H}$, decompose it according to the first edge yz where z is a left-most point; see Figure 3.2. This yields

$$G_\beta(0, x) \leq G_\beta^{\mathbb{H}}(0, x) + \sum_{n \geq 0} \sum_{\substack{y \in \mathbb{H}_n \\ z \notin \mathbb{H}_n \\ y \sim z}} G_\beta^{\mathbb{H}_n}(0, y) \beta G_\beta^{\mathbb{H}_{n+1}}(z, x) \quad (3.2.6)$$

$$\stackrel{(\ell_\beta^\infty)}{\leq} \frac{C}{|x|^{d-1}} + \sum_{n \geq 0} \varphi_\beta(\mathbb{H}_n) \frac{C}{(|x| + n + 1)^{d-1}} \stackrel{(\ell_\beta^1)}{\leq} \frac{C(1 + \frac{1+(2d)^{-1}}{d-2})}{|x|^{d-2}} \leq \frac{2C}{|x|^{d-2}}, \quad (3.2.7)$$

where in the penultimate inequality, we used that $\sum_{n \geq \alpha} \frac{1}{(n+1)^{d-1}} \leq \frac{1}{(d-2)\alpha^{d-2}}$ for every $\alpha \geq 1$.

We turn to (3.2.4). Let $1 \leq k < n$. Pick $x \in \partial\mathbb{H}_{n-k}$. To bound $G_\beta^{\mathbb{H}_n}(x, 0) = G_\beta^{\mathbb{H}_n}(0, x)$, decompose the path from x to 0 in the same fashion as above (see Figure 3.2) to get

$$G_\beta^{\mathbb{H}_n}(x, 0) \leq G_\beta^{\mathbb{H}_{n-k}}(x, 0) + \sum_{j=1}^k \sum_{\substack{y \in \mathbb{H}_{n-j} \\ z \notin \mathbb{H}_{n-j} \\ y \sim z}} G_\beta^{\mathbb{H}_{n-j}}(x, y) \beta G_\beta^{\mathbb{H}_{n-j+1}}(z, 0) \quad (3.2.8)$$

$$\stackrel{(\ell_\beta^\infty)}{\leq} \frac{C}{(n-k)^{d-1}} + \sum_{j=1}^k \left(\sum_{\substack{y \in \mathbb{H}_{n-j} \\ z \notin \mathbb{H}_{n-j} \\ y \sim z}} G_\beta^{\mathbb{H}_{n-j}}(x, y) \beta \right) \frac{C}{(n-j+1)^{d-1}} \quad (3.2.9)$$

$$\stackrel{(\ell_\beta^1)}{\leq} \frac{C(1 + (2d)^{-1})(k+1)}{(n-k)^{d-1}} \leq \frac{2C(k+1)}{(n-k)^{d-1}}. \quad (3.2.10)$$

For the proof of (3.2.5), consider the same decomposition (3.2.8) and sum it over $x \in \partial\mathbb{H}_{n-k}$, then use (ℓ_β^1) twice instead of (ℓ_β^1) and (ℓ_β^∞) . When⁵ $1 \leq k \leq n$, this gives

$$\sum_{x \in \partial\mathbb{H}_{n-k}} G_\beta^{\mathbb{H}_n}(0, x) \stackrel{(3.2.8)}{\leq} \frac{\varphi_\beta(\mathbb{H}_{n-k})}{\beta} + \sum_{j=1}^k \varphi_\beta(\mathbb{H}_{k-j}) \frac{\varphi_\beta(\mathbb{H}_{n-j+1})}{\beta} \quad (3.2.11)$$

$$\stackrel{(\ell_\beta^1)}{\leq} \frac{1}{\beta} \left((1 + (2d)^{-1}) + k(1 + (2d)^{-1})^2 \right) \leq \frac{4(k+1)}{\beta}. \quad (3.2.12)$$

When $k > n$ we adapt (3.2.8) and get

$$G_\beta^{\mathbb{H}_n}(x, 0) \leq G_\beta^{\mathbb{H}}(x, 0) + \sum_{j=1}^n \sum_{\substack{y \in \mathbb{H}_{n-j} \\ z \notin \mathbb{H}_{n-j} \\ y \sim z}} G_\beta^{\mathbb{H}_{n-j}}(x, y) \beta G_\beta^{\mathbb{H}_{n-j+1}}(z, 0). \quad (3.2.13)$$

Summing (3.2.13) over $x \in \partial\mathbb{H}_{n-k}$, we obtain similarly,

$$\sum_{x \in \partial\mathbb{H}_{n-k}} G_\beta^{\mathbb{H}_n}(0, x) \leq \frac{4(n+1)}{\beta} \leq \frac{4(k+1)}{\beta}. \quad (3.2.14)$$

This concludes the proof. $\square$

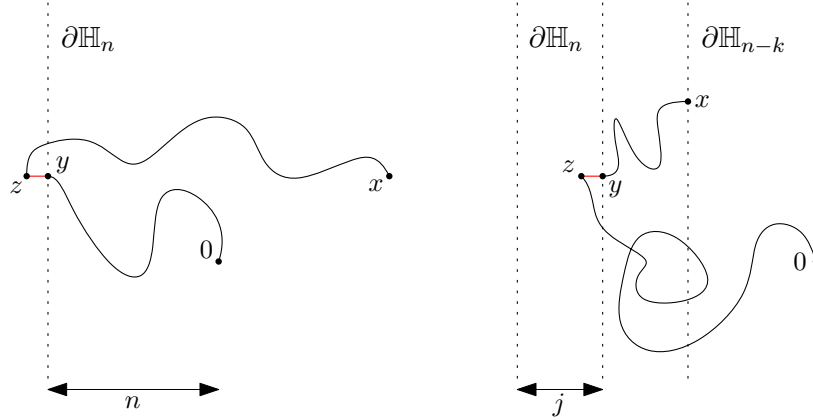


Figure 3.2: On the left, an illustration of the decomposition of the path used in the proof of (3.2.3). The red edge yz is the earliest edge satisfying that z is a left-most point of the path. On the right, an illustration of the decomposition of the path used in the proof of (3.2.4).

We now turn to the estimate of the error amplitude in (3.1.20) when $\beta < \beta^*$.

Lemma 3.2.4 (Bounding the error amplitude). *Fix $d > 4$ and $C > \mathbf{C}_{\text{RW}}$. There exists $K = K(C, d) > 0$ such that for every $\beta < \beta^*(C)$, and every $n \geq 0$,*

$$E_\beta^{\mathbb{H}_n, \mathbb{Z}^d} \leq K, \quad (3.2.15)$$

$$\sup \{E_\beta^{B, \mathbb{Z}^d} : B \in \mathcal{B}\} \leq K. \quad (3.2.16)$$

⁵We used that (3.2.8) is also valid when $k = n$.

Proof. The second inequality follows from the first one (by changing K) since for every $B \in \mathcal{B}$,

$$E_\beta^{B, \mathbb{Z}^d} \leq 2d \max \{E_\beta^{\mathbb{H}_k, \mathbb{Z}^d} : k \geq 0\}. \quad (3.2.17)$$

For the first inequality, we notice that $E_\beta^{\mathbb{H}_n, \mathbb{Z}^d}$ is increasing in β , so that it is sufficient to prove the bound for $\beta \geq \frac{1}{4d}$ (recall that since $C > \mathbf{C}_{\text{RW}}$, one has $\beta^* \geq \frac{1}{2d}$). The previous lemma (more specifically (3.2.3) and (3.2.5)) and (ℓ_β^1) give

$$E_\beta^{\mathbb{H}_n, \mathbb{Z}^d} = \sum_{k \geq 0} \sum_{u \in \partial \mathbb{H}_{n-k}} \sum_{\substack{y \in \mathbb{H}_n \\ z \notin \mathbb{H}_n \\ y \sim z}} G_\beta^{\mathbb{H}_n}(0, u) G_\beta^{\mathbb{H}_n}(u, y) \beta G_\beta(z, u) \quad (3.2.18)$$

$$\leq \sum_{k \geq 0} \left(\sum_{u \in \partial \mathbb{H}_{n-k}} G_\beta^{\mathbb{H}_n}(0, u) \right) \cdot \left(1 + \frac{1}{2d}\right) \cdot \frac{2C}{(k+1)^{d-2}} \quad (3.2.19)$$

$$\leq \sum_{k \geq 0} \frac{4(k+1)}{\beta} \cdot \left(1 + \frac{1}{2d}\right) \cdot \frac{2C}{(k+1)^{d-2}} \quad (3.2.20)$$

which is finite and depends only on C, d as soon as $d > 4$ and $\beta \geq \frac{1}{4d}$. $\square$

We are now in a position to prove Proposition 3.2.2.

Proof of Proposition 3.2.2. Summing (3.1.20) for $S = \mathbb{H}_n$ and $\Lambda = \mathbb{Z}^d$ over every $x \in \mathbb{Z}^d$ gives

$$\varphi_\beta(\mathbb{H}_n) \chi(\beta) - \lambda \chi(\beta) E_\beta^{\mathbb{H}_n, \mathbb{Z}^d} \leq \chi(\beta), \quad (3.2.21)$$

where we used that $G_\beta^{\mathbb{H}_n}(0, x) \geq 0$ for all $x \in \mathbb{Z}^d$. Dividing by $\chi(\beta)$ and using Lemma 3.2.4, we obtain

$$\varphi_\beta(\mathbb{H}_n) \leq 1 + \lambda E_\beta^{\mathbb{H}_n, \mathbb{Z}^d} \leq 1 + K\lambda. \quad (3.2.22)$$

The same reasoning, with (3.1.20) applied to $S = B \in \mathcal{B}$ and $\Lambda = \mathbb{Z}^d$ yields

$$\varphi_\beta(B) \leq 1 + \lambda E_\beta^{B, \mathbb{Z}^d} \stackrel{(3.2.16)}{\leq} 1 + K\lambda, \quad (3.2.23)$$

which concludes the proof. $\square$

3.2.2 Control of the gradient

Proposition 3.2.2 implies an ℓ^1 -type bound on $G_\beta^{\mathbb{H}_n}$ which involves the quantity λ and which is better than (ℓ_β^1) . The following regularity estimate, which will be the goal of this section, will later allow us to convert this ℓ^1 bound into an improved ℓ^∞ bound.

Proposition 3.2.5 (Regularity estimate at mesoscopic scales). *Fix $d > 4$ and $C > \mathbf{C}_{\text{RW}}$. For every $\eta > 0$, there exists $\delta = \delta(\eta, d) \in (0, 1/2)$ and $\lambda_0 = \lambda_0(\eta, C, d) > 0$ such that for every $\lambda < \lambda_0$, every $\beta < \beta^*(C, \lambda)$, every integer n with $\lfloor \delta n \rfloor \geq 6$, every $\Lambda \supset \Lambda_{3n}$, every $x \notin \Lambda_{3n}$,*

$$\max \{|G_\beta^\Lambda(u, x) - G_\beta^\Lambda(v, x)| : u, v \in \Lambda_{\lfloor \delta n \rfloor} \cap 2\mathbb{Z}^d\} \leq \eta \max \{G_\beta^\Lambda(w, x) : w \in \Lambda_{3n}\}. \quad (3.2.24)$$

Remark 3.2.6. We will see later that the assumption $u, v \in 2\mathbb{Z}^d$ is not necessary (see Corollary 3.3.4), but we will not need this stronger result in this section.

We start with a lemma. Let $\Lambda_n^+ := \{x \in \Lambda_n : x_1 > 0\}$ and $H := \{v \in \mathbb{Z}^d : v_1 = 0\}$.

Lemma 3.2.7. *Fix $d > 4$ and $C > \mathbf{C}_{\text{RW}}$. Assume that $\lambda \leq \frac{1}{2dK}$, where $K = K(C, d)$ is given by Proposition 3.2.2. Let $n \geq 12$. For every $v \in \Lambda_k^+$ with $6 \leq k \leq n/2$ and every $\beta < \beta^*(C, \lambda)$,*

$$\sum_{\substack{y \in \Lambda_n^+ \\ z \notin \Lambda_n^+ \cup H \\ y \sim z}} G_\beta^{\Lambda_n^+}(v, y) \beta \leq 4 \left(\frac{2k}{n} \right)^c, \quad (3.2.25)$$

where $c := \frac{|\log(1 - \frac{1}{4d^2})|}{2 \log 2}$.

Proof. Define (n_ℓ) by $n_0 := n$ and then $n_{\ell+1} := \lfloor (n_\ell - 1)/2 \rfloor$. We prove by induction that for every $\ell \geq 0$ and $v \in \Lambda_{n_\ell}^+$,

$$\sum_{\substack{y \in \Lambda_n^+ \\ z \notin \Lambda_n^+ \cup H \\ y \sim z}} G_\beta^{\Lambda_n^+}(v, y) \beta \leq \left(1 - \frac{1}{4d^2}\right)^\ell \left(1 + \frac{1}{2d}\right). \quad (3.2.26)$$

The case $\ell = 0$ follows from Proposition 3.2.2 and from the assumption made on λ . Let us transfer the estimate from ℓ to $\ell + 1$. Fix $v \in \Lambda_{n_{\ell+1}}^+$. Let $B := \Lambda_{v_1-1}(v)$, which is included in $\Lambda_{n_\ell}^+$ and has one of its faces at distance 1 from H . By symmetry, we have that

$$\sum_{\substack{r \in B \\ s \notin B \cup H \\ r \sim s}} G_\beta^B(v, r) \beta \leq \frac{2d-1}{2d} \varphi_\beta(B) \stackrel{(3.2.2)}{\leq} \left(1 - \frac{1}{2d}\right) (1 + \lambda K) \leq \left(1 - \frac{1}{2d}\right) \left(1 + \frac{1}{2d}\right) = 1 - \frac{1}{4d^2}. \quad (3.2.27)$$

Lemma 3.1.5 (applied to $S = B$ and $\Lambda = \Lambda_n^+$) and the induction hypothesis imply that

$$\sum_{\substack{y \in \Lambda_n^+ \\ z \notin \Lambda_n^+ \cup H \\ y \sim z}} G_\beta^{\Lambda_n^+}(v, y) \beta \leq \sum_{\substack{y \in \Lambda_n^+ \\ z \notin \Lambda_n^+ \cup H \\ y \sim z}} \left(\sum_{\substack{r \in B \\ s \notin B \cup H \\ r \sim s}} G_\beta^B(v, r) \beta G_\beta^{\Lambda_n^+}(s, y) \right) \beta \quad (3.2.28)$$

$$= \sum_{\substack{r \in B \\ s \notin B \cup H \\ r \sim s}} G_\beta^B(v, r) \beta \left(\sum_{\substack{y \in \Lambda_n^+ \\ z \notin \Lambda_n^+ \cup H \\ y \sim z}} G_\beta^{\Lambda_n^+}(s, y) \beta \right) \quad (3.2.29)$$

$$\leq \left(1 - \frac{1}{4d^2}\right)^\ell \left(1 + \frac{1}{2d}\right) \sum_{\substack{r \in B \\ s \notin B \cup H \\ r \sim s}} G_\beta^B(v, r) \beta \quad (3.2.30)$$

$$\stackrel{(3.2.27)}{\leq} \left(1 - \frac{1}{4d^2}\right)^{\ell+1} \left(1 + \frac{1}{2d}\right). \quad (3.2.31)$$

This concludes the proof of the induction. Now, if $k \leq n/2$, one has⁶ $k \leq n_{\lfloor \alpha/2 \rfloor}$ where

⁶Indeed, by definition $n_\ell \geq \frac{n}{2^\ell} - \sum_{k=0}^{\ell-1} \frac{3}{2^k} \geq \frac{n}{2^\ell} - 6$ for all $\ell \geq 0$. Hence, for $\ell = \lfloor \alpha/2 \rfloor$ with $\alpha = \log_2(\frac{n}{2k})$,

$$n_\ell \geq \frac{n}{2^{\alpha/2}} - 6 = \sqrt{2k}\sqrt{n} - 6 \geq 2k - 6 \geq k,$$

where we used that $n \geq 2k$ and $k \geq 6$.

$\alpha = \log_2(\frac{n}{2k})$. Hence, by (3.2.26), if $v \in \Lambda_k^+$,

$$\sum_{\substack{y \in \Lambda_n^+ \\ z \notin \Lambda_n^+ \cup H \\ y \sim z}} G_\beta^{\Lambda_n^+}(v, y) \beta \leq \left(1 - \frac{1}{4d^2}\right)^{\lfloor \frac{1}{2} \log_2(\frac{n}{2k}) \rfloor} \left(1 + \frac{1}{2d}\right) \leq \left(\frac{2k}{n}\right)^c \left(1 + \frac{1}{2d}\right)^2 \leq 4 \left(\frac{2k}{n}\right)^c, \quad (3.2.32)$$

where $c = \frac{|\log(1 - \frac{1}{4d^2})|}{2 \log 2}$, and where we used that $(1 - \frac{1}{4d^2})^{-1} \leq 1 + \frac{1}{2d} \leq 2$ for $d \geq 1$. $\square$

Proof of Proposition 3.2.5. It suffices to prove the statement when u and v differ in one coordinate only as the general case follows by summing increments over coordinates. By rotating and translating, we may consider $u = k\mathbf{e}_1$ and $v = -k\mathbf{e}_1$ belong to $\Lambda_{\lfloor \delta n \rfloor} \cap (2\mathbb{Z}^d)$ for δ to be fixed, and later replace the maximum in Λ_{2n} by the maximum in $\Lambda_{3n} \supset \Lambda_{2n + \lfloor \delta n \rfloor}$.

Consider the sets $A := \Lambda_n^+$ and $B := -\Lambda_n^+$. Applying Lemma 3.1.5 twice as well as Lemma 3.2.4 gives

$$G_\beta^\Lambda(u, x) \leq \sum_{\substack{y \in A \\ z \notin A \\ y \sim z}} G_\beta^A(u, y) \beta G_\beta^\Lambda(z, x), \quad (3.2.33)$$

$$G_\beta^\Lambda(v, x) \geq \sum_{\substack{y \in B \\ z \notin B \\ y \sim z}} G_\beta^B(v, y) \beta G_\beta^\Lambda(z, x) - K\lambda \max\{G_\beta^\Lambda(w, x) : w \in B\}, \quad (3.2.34)$$

where in (3.2.34) we used that,

$$\sum_{u \in B} E_\beta^{B, \Lambda}(u) G_\beta^\Lambda(u, x) \leq E_\beta^{B, \Lambda} \cdot \max\{G_\beta^\Lambda(w, x) : w \in B\} \quad (3.2.35)$$

$$\stackrel{(3.2.16)}{\leq} K \max\{G_\beta^\Lambda(w, x) : w \in B\}. \quad (3.2.36)$$

We take the difference and use that when $z \in H$, the corresponding terms in the two sums cancel each other, see Figure 3.3. Assume that $\lambda \leq \frac{1}{2dK}$ and $\lfloor \delta n \rfloor \geq 6$. Lemma 3.2.7 applied to $k = \lfloor \delta n \rfloor$ and n gives

$$G_\beta^\Lambda(u, x) - G_\beta^\Lambda(v, x) \leq \sum_{\substack{y \in A \\ z \notin A \cup H \\ y \sim z}} G_\beta^A(u, y) \beta G_\beta^\Lambda(z, x) + K\lambda \max\{G_\beta^\Lambda(w, x) : w \in B\} \quad (3.2.37)$$

$$\leq \left(4(2\delta)^c + K\lambda\right) \max\{G_\beta^\Lambda(w, x) : w \in \Lambda_{2n}\}. \quad (3.2.38)$$

The proof follows by choosing $\delta = \delta(\eta, d) > 0$ and $\lambda_0 = \lambda_0(\eta, C, d) > 0$ small enough. $\square$

3.2.3 Wrapping up the proof of Theorem 3.1.1

We start by showing how to use Proposition 3.2.5 to turn the improved ℓ^1 estimate of $G_\beta^{\mathbb{H}_n}$ given by Proposition 3.2.2 into an improved ℓ^∞ bound.

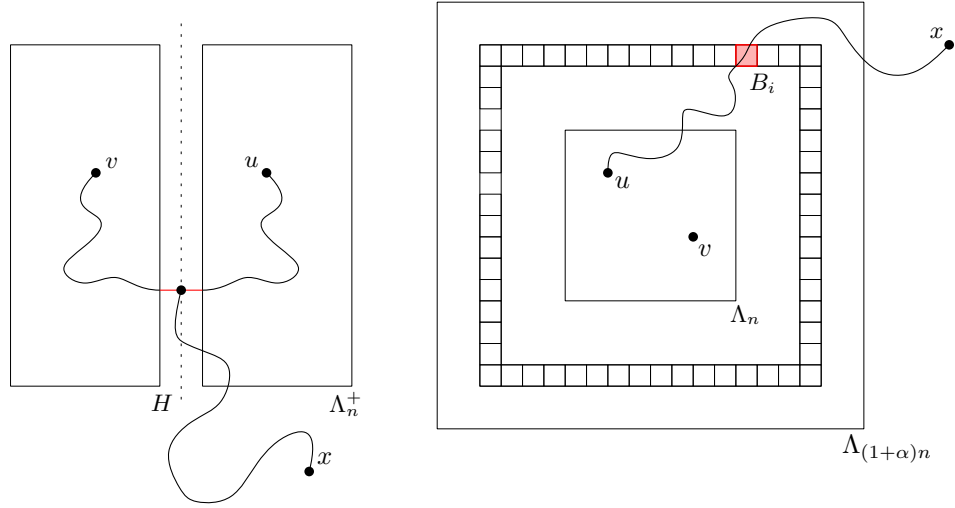


Figure 3.3: On the left, an illustration of the pairing used in the proof of Proposition 3.2.5. Since u (resp. v) is close to H , a path started from u will most likely touch H if it exits Λ_n^+ . On the right, an illustration of the proof of Proposition 3.3.2.

Proposition 3.2.8 (Improving the bound on $G_{\beta}^{\mathbb{H}^n}$). *Fix $d > 4$. There exist $C = C(d)$, $\lambda_0 = \lambda_0(d) > 0$ such that, for every $\lambda < \lambda_0$, and every $\beta < \beta^*(C, \lambda)$,*

$$G_{\beta}^{\mathbb{H}^n}(0, x) \leq \frac{C}{2n^{d-1}} \quad \forall n \geq 1, \forall x \in \partial\mathbb{H}_n. \quad (3.2.39)$$

Proof. Let $\eta > 0$ and $C > \mathbf{C}_{\text{RW}}$ to be fixed. By monotonicity of $G_{\beta}^{\mathbb{H}^n}(0, x)$ in β , it suffices to prove the result for $\beta \in [\frac{1}{4d}, \beta^*)$. Let $\delta = \delta(\eta, d)$ and $\lambda_0 = \lambda_0(\eta, C, d)$ be provided by Proposition 3.2.5. Assume first that $\frac{\delta n}{6}$ is an integer (otherwise simply round the number) and that $\frac{\delta n}{6} \geq 6$. Set

$$V_n = V_n(\delta) := \{x \in \Lambda_{\delta n/6} : x_1 = 0, x_j \text{ even for } j \geq 2\}. \quad (3.2.40)$$

Proposition 3.2.5 applied to $n/6$ gives that for every $\beta < \beta^*$, every $x \in \partial\mathbb{H}_n$ and $y \in V_n$,

$$G_{\beta}^{\mathbb{H}^n}(0, x - y) = G_{\beta}^{\mathbb{H}^n}(y, x) \quad (3.2.41)$$

$$\geq G_{\beta}^{\mathbb{H}^n}(0, x) - \eta \max\{G_{\beta}^{\mathbb{H}^n}(v, x) : v \in \Lambda_{n/2}\}. \quad (3.2.42)$$

Fix $x \in \partial\mathbb{H}_n$. Averaging the last displayed equation over $y \in V_n$ and using that $\beta \geq \frac{1}{4d}$ gives

$$\frac{8d}{|V_n|} \geq \frac{4d(1 + (2d)^{-1})}{|V_n|} \stackrel{(\ell_{\beta}^1)}{\geq} \frac{\varphi_{\beta}(\mathbb{H}_n)}{\beta|V_n|} \geq \frac{1}{|V_n|} \sum_{y \in V_n} G_{\beta}^{\mathbb{H}^n}(0, x - y) \geq G_{\beta}^{\mathbb{H}^n}(0, x) - \eta \cdot \frac{C}{(n/2)^{d-1}}, \quad (3.2.43)$$

where in the last inequality we used (ℓ_{β}^{∞}) to argue that

$$\max\{G_{\beta}^{\mathbb{H}^n}(v, x) : v \in \Lambda_{n/2}\} \leq \frac{C}{(n/2)^{d-1}}. \quad (3.2.44)$$

If η is chosen so that $\eta \leq 1/2^{d+1}$, then $C = C(\delta, d)$ is chosen large enough, we find that

$$\sup_{x \in \partial \mathbb{H}_n} G_{\beta}^{\mathbb{H}_n}(0, x) \leq \frac{C}{2n^{d-1}} \quad (3.2.45)$$

provided n is large enough (i.e. $\frac{\delta n}{6} \geq 6$).

To treat the small values of n , we observe that for all $n \geq 1$, for all $x \in \partial \mathbb{H}_n$, $G_{\beta}^{\mathbb{H}_n}(0, x) \leq \varphi_{\beta}(\mathbb{H}_n) \leq 1 + \frac{1}{2d} \leq 2$. Thus, if we additionally require that $C \geq 4(36\delta^{-1})^{d-1}$, we ensure that for all $1 \leq n \leq 36\delta^{-1}$,

$$\sup_{x \in \partial \mathbb{H}_n} G^{\mathbb{H}_n}(0, x) \leq \frac{C}{2n^{d-1}}. \quad (3.2.46)$$

This concludes the proof. $\square$

Proposition 3.2.9. *Fix $d > 4$. There exist $C = C(d)$, $\lambda_0 = \lambda_0(d) > 0$, and $K = K(C, d) > 0$ such that for every $\lambda < \lambda_0$,*

$$G_{\beta_c}(0, x) \leq \frac{2C}{|x|^{d-2}} \quad \forall x \in \mathbb{Z}^d \setminus \{0\}, \quad (3.2.47)$$

$$G_{\beta_c}^{\mathbb{H}}(0, x) \leq \frac{C}{|x_1|^{d-1}} \quad \forall x \in \mathbb{H} \setminus \partial \mathbb{H}, \quad (3.2.48)$$

$$\varphi_{\beta_c}(\mathbb{H}_n) \leq 1 + K\lambda \quad \forall n \geq 0, \quad (3.2.49)$$

$$\sup\{\varphi_{\beta_c}(B) : B \in \mathcal{B}\} \leq 1 + K\lambda, \quad (3.2.50)$$

$$E_{\beta_c}^{\mathbb{H}_n, \mathbb{Z}^d} \leq K \quad \forall n \geq 0, \quad (3.2.51)$$

$$\sup\{E_{\beta_c}^{B, \mathbb{Z}^d} : B \in \mathcal{B}\} \leq K. \quad (3.2.52)$$

Proof. We let C, λ_0 be given by Proposition 3.2.8. This choice gives (still by Proposition 3.2.8) that for every $\lambda < \lambda_0$ and every $\beta < \beta^*(C, \lambda)$,

$$G_{\beta}^{\mathbb{H}_n}(0, x) \leq \frac{C}{2n^{d-1}} \quad \forall n \geq 1, \forall x \in \partial \mathbb{H}_n. \quad (3.2.53)$$

Let $K = K(C, d)$ be given by Proposition 3.2.2. We potentially decrease the value of λ_0 (which does not affect the value of C) and require that $K\lambda_0 < \frac{1}{4d}$. Proposition 3.2.2 implies that for every $\lambda < \lambda_0$ and every $\beta < \beta^*(C, \lambda)$,

$$\varphi_{\beta}(\mathbb{H}_n) \leq 1 + K\lambda \leq 1 + \frac{1}{4d} \quad \forall n \geq 0. \quad (3.2.54)$$

The monotone convergence theorem implies that (3.2.53)–(3.2.54) still hold at β^* . We will use (3.2.53)–(3.2.54) to prove that for this choice of C, λ_0 : for all $\lambda < \lambda_0$,

$$\beta^* = \beta^*(C, \lambda) = \beta_c. \quad (3.2.55)$$

Let $\lambda < \lambda_0$. Assume by contradiction that $\beta^* = \beta^*(C, \lambda) < \beta_c$. Exponential decay of the two-point function below β_c implies that for every $\beta < \beta_c$, there exists $c(\beta), C(\beta) > 0$ such that for all $n \geq 1$, for all $x \in \partial \mathbb{H}_n$,

$$G_{\beta}^{\mathbb{H}_n}(0, x) \leq C(\beta)e^{-c(\beta)|x|}. \quad (3.2.56)$$

Let β^{**} be any number in (β^*, β_c) . By monotonicity of all the quantities in β and (3.2.56), there exists $N \geq 0$ such that for every $\beta < \beta^{**}$, one has

$$\varphi_\beta(\mathbb{H}_n) < 1 + \frac{1}{2d} \quad \forall n \geq N, \quad G_\beta^{\mathbb{H}_n}(0, x) < \frac{C}{n^{d-1}} \quad \forall n \geq 1, \forall x \in \partial\mathbb{H}_n \setminus \Lambda_N. \quad (3.2.57)$$

Now, (3.2.57), the validity of (3.2.53)–(3.2.54) at β^* , together with the continuity (below β_c) of the maps $\beta \mapsto \max\{\varphi_\beta(\mathbb{H}_n) : 0 \leq n \leq N-1\}$ and $\beta \mapsto \max\{G_\beta^{\mathbb{H}_n}(0, x) : x \in \Lambda_N \cap \partial\mathbb{H}_n, n \geq 1\}$ yield the existence of $\beta \in (\beta^*, \beta^{**})$ such that

$$\varphi_\beta(\mathbb{H}_n) < 1 + \frac{1}{2d} \quad \forall n \geq 0, \quad G_\beta^{\mathbb{H}_n}(0, x) < \frac{C}{n^{d-1}} \quad \forall n \geq 1, \forall x \in \partial\mathbb{H}_n. \quad (3.2.58)$$

This clearly contradicts the definition of β^* and therefore implies that $\beta^* = \beta_c$.

For the proofs of (3.2.47)–(3.2.52), we use Lemmata 3.2.3–3.2.4 and Proposition 3.2.2 to show that the bounds hold for $\beta < \beta_c$. We extend these bounds at β_c using once again the monotone convergence theorem. $\square$

Remark 3.2.10. The bound $2d\beta_c = \varphi_{\beta_c}(\{0\}) \leq 1 + K\lambda$ gives $\beta_c \leq \frac{1}{2d}(1 + K\lambda)$, which complements the bound $\beta_c \geq \frac{1}{2d}$.

We are now in a position to prove Theorem 3.1.1.

Proof of Theorem 3.1.1. Let C, λ_0 be given by Proposition 3.2.9 and let $\lambda < \lambda_0$. Let $A = A(d) \geq 1$ be such that

$$t^{d-1} \leq Ae^{t/2} \quad \forall t \geq 1. \quad (3.2.59)$$

Proof of (3.1.7). First, if $\beta \leq \beta_c$, Proposition 3.2.9 implies that for $x \in \mathbb{Z}^d \setminus \{0\}$ with $|x| \leq L_\beta$,

$$G_\beta(0, x) \leq \frac{2C}{|x|^{d-2}} \leq \frac{2eC}{|x|^{d-2}} e^{-|x|/L_\beta}. \quad (3.2.60)$$

We now turn to the case of $\beta < \beta_c$ and $|x| > L_\beta$ (recall that $L_{\beta_c} = \infty$). Iterating (3.1.19) $k := \lfloor |x|/L_\beta \rfloor - 1$ times with S being translates of Λ_{L_β} and $\Lambda = \mathbb{Z}^d$, we get that⁷

$$G_\beta(0, x) \leq \varphi_\beta(\Lambda_{L_\beta})^k \max\{G_\beta(y, x) : y \notin \Lambda_{L_\beta}(x)\} \quad (3.2.61)$$

$$\stackrel{(3.2.47)}{\leq} e^{-k} \frac{2C}{L_\beta^{d-2}} \leq \frac{2e^2 C}{L_\beta^{d-2}} e^{-|x|/L_\beta} \stackrel{(3.2.59)}{\leq} \frac{2e^2 AC}{|x|^{d-2}} e^{-|x|/2L_\beta}. \quad (3.2.62)$$

Proof of (3.1.8). Again, if $\beta \leq \beta_c$, Proposition 3.2.9 implies that for $x \in \mathbb{H}$ with $0 < x_1 \leq L_\beta$,

$$G_\beta^{\mathbb{H}}(0, x) \leq \frac{C}{|x_1|^{d-1}} \leq \frac{eC}{|x_1|^{d-1}} e^{-|x_1|/L_\beta}. \quad (3.2.63)$$

⁷Recall that by definition of L_β , one has $\varphi_\beta(\Lambda_{L_\beta}) \leq e^{-1}$.

Turning to the case $\beta < \beta_c$ and $|x_1| > L_\beta$, iterating (3.1.19) $\ell := \lfloor |x_1|/L_\beta \rfloor - 1$ times with S being translates of Λ_{L_β} (starting with $\Lambda_{L_\beta}(x)$) and $\Lambda = \mathbb{H}$, we obtain

$$G_\beta^{\mathbb{H}}(0, x) = G_\beta^{\mathbb{H}}(x, 0) \leq \varphi_\beta(\Lambda_{L_\beta})^\ell \max\{G_\beta^{\mathbb{H}}(0, y) : y_1 > L_\beta\} \stackrel{(3.2.48)}{\leq} \frac{e^2 C}{L_\beta^{d-1}} e^{-|x_1|/L_\beta} \quad (3.2.64)$$

$$\stackrel{(3.2.59)}{\leq} \frac{e^2 AC}{|x_1|^{d-1}} e^{-|x_1|/2L_\beta} \quad (3.2.65)$$

Gathering (3.2.60), (3.2.62), (3.2.63), and (3.2.65), we obtained: for all $\beta \leq \beta_c$, for all $x \in \mathbb{Z}^d \setminus \{0\}$,

$$G_\beta(0, x) \leq \frac{2e^2 AC}{|x|^{d-2}} \exp(-|x|/2L_\beta), \quad (3.2.66)$$

$$G_\beta^{\mathbb{H}}(0, x) \leq \frac{2e^2 AC}{|x_1|^{d-1}} \exp(-|x_1|/2L_\beta). \quad (3.2.67)$$

This concludes the proof. $\square$

3.3 Proof of Theorem 3.1.2

For $\varepsilon \in (0, 1)$, introduce the quantity

$$L_\beta(\varepsilon) := \inf\{k \geq 1 : \varphi_\beta(\Lambda_k) \leq 1 - \varepsilon\}. \quad (3.3.1)$$

This new correlation length only differs from L_β by a constant, as stated in the next lemma, but it will be more convenient for the rest of the proof.

Lemma 3.3.1 (Comparison between L_β and $L_\beta(\varepsilon)$). *Fix $d > 4$ and $\varepsilon \in (0, 1 - e^{-1})$. Let λ_0 be given by Proposition 3.2.9. There exists $C(\varepsilon, \lambda_0, d) \in (0, \infty)$ such that, for every $\lambda < \lambda_0$, and every $\beta < \beta_c$,*

$$L_\beta(\varepsilon) \leq L_\beta \leq C(\varepsilon, \lambda_0) L_\beta(\varepsilon). \quad (3.3.2)$$

Proof. The first inequality is clear. We turn to the second one. Assume that $L_\beta - 1 \geq k(L_\beta(\varepsilon) + 1)$. Similarly to (3.2.61), if we iterate (3.1.19) k times with S being translates of $\Lambda_{L_\beta(\varepsilon)}$ and $\Lambda = \Lambda_{L_\beta-1}$, we obtain that: for every $x \in \partial\Lambda_{L_\beta-1}$,

$$G_\beta^\Lambda(0, x)\beta \leq \sum_{\substack{u_1 \in \Lambda_{L_\beta(\varepsilon)} \\ v_1 \notin \Lambda_{L_\beta(\varepsilon)} \\ u_1 \sim v_1}} G_\beta^\Lambda(0, u_1)\beta \dots \sum_{\substack{u_k \in \Lambda_{L_\beta(\varepsilon)}(v_{k-1}) \\ v_k \notin \Lambda_{L_\beta(\varepsilon)}(v_{k-1}) \\ u_k \sim v_k}} G_\beta^\Lambda(v_{k-1}, u_k)\beta G_\beta^\Lambda(v_k, x)\beta. \quad (3.3.3)$$

Summing the above displayed equation over $x \in \partial\Lambda_{L_\beta-1}$ and $y \sim x$ with $y \notin \Lambda_{L_\beta-1}$ gives

$$\frac{1}{e} \leq \varphi_\beta(\Lambda_{L_\beta-1}) \leq \varphi_\beta(\Lambda_{L_\beta(\varepsilon)})^k \max\{\varphi_\beta(\Lambda_{L_\beta-1}(x)) : x \in \Lambda_{L_\beta-1}\} \leq (1 - \varepsilon)^k (1 + K\lambda), \quad (3.3.4)$$

where K is given by Proposition 3.2.9. As a consequence, there exists $C(\varepsilon, \lambda_0, d) < \infty$ such that $L_\beta \leq C(\varepsilon, \lambda_0, d) L_\beta(\varepsilon)$. $\square$

Note that the assumption $n \leq L_\beta(\varepsilon) - 1$ implies an ℓ^1 -type lower bound on the half-space two-point function (see (3.3.28)). The core of this section will be to turn this estimate into a point-wise estimate for the half-space two-point function. The corresponding lower bound for the full-space two-point function will follow readily. The proof is organised in two steps: we begin by proving a regularity estimate, and then we use it to get the theorem.

3.3.1 A Harnack-type estimate

We start with another regularity estimate relating the minimum to the maximum of the two-point function in a domain.

Proposition 3.3.2 (Harnack-type estimate at macroscopic scales). *Fix $d > 4$ and $\alpha > 0$. There exists $C_{\text{RW}} = C_{\text{RW}}(\alpha, d) > 0$ and for every $\eta > 0$, there exist $\lambda_0 = \lambda_0(\eta, \alpha, d)$, $\varepsilon_0 = \varepsilon_0(\eta, d) > 0$ small enough and $N_0 = N_0(\eta, \alpha, d)$ large enough such that the following holds. For every $\lambda < \lambda_0$, every $\varepsilon < \varepsilon_0$, every $N_0 \leq n \leq 6L_\beta(\varepsilon)$, every $\frac{1}{2d} \leq \beta \leq \beta_c$, and every $x \notin \Lambda_{(1+\alpha)n} \subset \Lambda$,*

$$\max\{G_\beta^\Lambda(u, x) : u \in \Lambda_n\} \leq C_{\text{RW}} \min\{G_\beta^\Lambda(u, x) : u \in \Lambda_n\} + \eta \max\{G_\beta^\Lambda(u, x) : u \in \Lambda_{(1+\alpha)n}\}. \quad (3.3.5)$$

We will derive Proposition 3.3.2 using classical random walk estimates. We start with a lemma which is useful to go around the parity assumption in Proposition 3.2.5.

Lemma 3.3.3. *Fix $d > 4$ and $\eta > 0$. There exists $\lambda_0 > 0$ such that the following holds. For every $\ell \geq 1$, there exists $L = L(\eta, \ell, d) \geq \ell$ such that for every $\lambda < \lambda_0$, every $\frac{1}{2d} \leq \beta \leq \beta_c$, every set $\Lambda \supset \Lambda_L$, and every $x \notin \Lambda_L$,*

$$\max\{|G_\beta^\Lambda(u, x) - G_\beta^\Lambda(v, x)| : u, v \in \Lambda_\ell\} \leq \eta \max\{G_\beta^\Lambda(w, x) : w \in \Lambda_L\}. \quad (3.3.6)$$

Proof. Fix $\eta > 0$. Consider the simple random walk (X_n) defined by

$$\mathbb{P}_u[X_1 = v] := \frac{1}{2d} \mathbb{1}_{v \sim u}. \quad (3.3.7)$$

Let τ be the hitting time of $\mathbb{Z}^d \setminus \Lambda_{L-1}$. We will use two a priori estimates on the random walk and the stopping time that can be easily obtained from classical random walk analysis⁸ (see [172]): for every $\ell > 0$, there exist $C_{\text{RW}} = C_{\text{RW}}(d)$ and $L = L(\eta, \ell, d)$ such that for some $\varepsilon = \varepsilon(\eta, \ell, L, d) > 0$ and every $\varphi \in [1 - \varepsilon, 1 + \varepsilon]$,

$$\mathbb{E}_u \left[\sum_{s=0}^{\tau} \varphi^s \right] \leq C_{\text{RW}} \quad \forall u \in \Lambda_\ell, \quad (3.3.8)$$

$$\mathbb{E}_u[\varphi^\tau \mathbb{1}_{X_\tau=z}] \leq (1 + \eta) \mathbb{E}_v[\varphi^\tau \mathbb{1}_{X_\tau=z}] \quad \forall u, v \in \Lambda_\ell, \forall z \in \partial\Lambda_L. \quad (3.3.9)$$

From now on, we let λ_0 small enough be given by Proposition 3.2.9 and (to the cost of diminishing λ_0) we additionally require that $K\lambda_0 \leq \varepsilon$ where K is given by the same proposition. Let $\lambda < \lambda_0$ and $\beta \leq \beta_c$. By the assumption $\beta \geq \frac{1}{2d}$ and Proposition 3.2.9, we find

$$1 \leq 2d\beta = \varphi_\beta(\{0\}) \leq 1 + K\lambda \leq 1 + \varepsilon. \quad (3.3.10)$$

⁸For the first inequality (3.3.8), simply observe that every L^2 steps there is a probability $c > 0$ of exiting the box, so as soon as $\varepsilon \ll L^{-2}$, the estimate follows easily from a Laplace transform estimate. Then, (3.3.9) follows from Harnack's inequality for the exit probabilities by choosing L large enough.

Below, introduce the shorthand notation $\varphi := \varphi_\beta(\{0\})$. We let $u, v \in \Lambda_\ell$ and $x \notin \Lambda_L$. Iterating the two bounds of Lemma 3.1.5 (with S a singleton) until time τ and using Lemma 3.2.4 gives

$$G_\beta^\Lambda(u, x) \stackrel{(3.1.19)}{\leq} \mathbb{E}_u[\varphi^\tau G_\beta^\Lambda(X_\tau, x)], \quad (3.3.11)$$

$$G_\beta^\Lambda(v, x) \stackrel{(3.1.20)}{\geq} \mathbb{E}_v[\varphi^\tau G_\beta^\Lambda(X_\tau, x)] - K\lambda \mathbb{E}_v\left[\sum_{s=0}^{\tau} \varphi^s\right] \max\{G_\beta^\Lambda(w, x) : w \in \Lambda_L\} \quad (3.3.12)$$

$$\stackrel{(3.3.8)}{\geq} \mathbb{E}_v[\varphi^\tau G_\beta^\Lambda(X_\tau, x)] - K\lambda C_{\text{RW}} \max\{G_\beta^\Lambda(w, x) : w \in \Lambda_L\}. \quad (3.3.13)$$

Taking the difference of (3.3.11) and (3.3.13) gives

$$G_\beta^\Lambda(u, x) - G_\beta^\Lambda(v, x) \quad (3.3.14)$$

$$\leq \mathbb{E}_u[\varphi^\tau G_\beta^\Lambda(X_\tau, x)] - \mathbb{E}_v[\varphi^\tau G_\beta^\Lambda(X_\tau, x)] + C_{\text{RW}}K\lambda \max\{G_\beta^\Lambda(w, x) : w \in \Lambda_L\} \quad (3.3.15)$$

$$\stackrel{(3.3.9)}{\leq} \eta \mathbb{E}_v[\varphi^\tau G_\beta^\Lambda(X_\tau, x)] + C_{\text{RW}}K\lambda \max\{G_\beta^\Lambda(w, x) : w \in \Lambda_L\} \quad (3.3.16)$$

$$\stackrel{(3.3.13)}{\leq} \eta G_\beta^\Lambda(v, x) + C_{\text{RW}}K\lambda(1 + \eta) \max\{G_\beta^\Lambda(w, x) : w \in \Lambda_L\}. \quad (3.3.17)$$

The proof follows by choosing λ_0 even smaller. $\square$

Corollary 3.3.4 (Regularity estimate at mesoscopic scales without the parity assumption). *Fix $d > 4$. For every $\eta > 0$, there exist $\delta = \delta(\eta, d) \in (0, 1/2)$, $\lambda_0 = \lambda_0(\eta, d) > 0$, and $L = L(\eta, d) > 0$ such that for every $\lambda < \lambda_0$, every $\beta \leq \beta_c$, every $n \geq L$, every $\Lambda \supset \Lambda_{3n}$, every $x \notin \Lambda_{3n}$,*

$$\max\{|G_\beta^\Lambda(u, x) - G_\beta^\Lambda(v, x)| : u, v \in \Lambda_{\lfloor \delta n \rfloor}\} \leq \eta \max\{G_\beta^\Lambda(w, x) : w \in \Lambda_{3n}\}. \quad (3.3.18)$$

Proof. Take λ_0 small enough such that Proposition 3.2.9 and Lemma 3.3.3 hold. By continuity, it is sufficient to prove the result for $\beta < \beta_c$. The proof follows from applying Proposition 3.2.5 with $\eta/2$ and Lemma 3.3.3 with $\eta/2$ and $\ell = 1$. $\square$

We now turn to the proof of Proposition 3.3.2. The idea of the proof is the same as in Lemma 3.3.3, except we introduce a well-chosen rescaled random walk to replace the simple random walk.

Proof of Proposition 3.3.2. Fix $M := \lfloor \alpha n/6 \rfloor$. Set $m := \lfloor \delta M \rfloor$, where $\delta = \delta(\eta, A, d)$ is chosen in such a way that Corollary 3.3.4 is valid with η/A where $A > 0$ will be taken large enough below. Also, assume that n is large enough so that $m \geq L$ where L is again provided by Corollary 3.3.4 (this means $n \geq 6\alpha^{-1}\delta^{-1}L$). Let $B_1, \dots, B_s$ be boxes of size m that are disjoint and covering the annulus $\Lambda_{n+M+m} \setminus \Lambda_{n+M}$, see Figure 3.3. Consider the random walk (X'_n) defined by

$$\mathbb{P}'_u[X'_1 = v] := \frac{\mathbb{1}_{v \notin \Lambda_m(u)}}{\varphi_\beta(\Lambda_m)} \sum_{\substack{w \in \Lambda_m(u) \\ w \sim v}} G_\beta^{\Lambda_m(u)}(u, w) \beta. \quad (3.3.19)$$

Let τ be the hitting time of $\mathbb{Z}^d \setminus \Lambda_{n+M}$. We will use two a priori estimates on the random walk and the stopping time τ , that can be easily obtained from classical random walk analysis (see [172]) : there exists a constant $C_{\text{RW}} = C_{\text{RW}}(\alpha, d)$ (note that it does not depend on m) such that for $\varepsilon = \varepsilon(\alpha, \eta, d) > 0$ small enough and every $\varphi \in [1 - \varepsilon, 1 + \varepsilon]$,

$$\mathbb{E}'_u \left[\sum_{s=0}^{\tau} \varphi^s \right] \leq C_{\text{RW}} \quad \forall u \in \Lambda_n, \quad (3.3.20)$$

$$\mathbb{E}'_u [\varphi^\tau \mathbf{1}_{X'_\tau \in B_i}] \leq C_{\text{RW}} \mathbb{E}'_v [\varphi^\tau \mathbf{1}_{X'_\tau \in B_i}] \quad \forall u, v \in \Lambda_n, \forall i \leq s. \quad (3.3.21)$$

We let λ_0 small enough be given by Proposition 3.2.9 and (to the cost of diminishing λ_0) we additionally require that $K\lambda_0 \leq \varepsilon$ where K is given by the same proposition. Also, consider $u, v \in \Lambda_n$. By the assumption $6m < n \leq 6L_\beta(\varepsilon)$ and Proposition 3.2.9, we find

$$1 - \varepsilon \leq \varphi_\beta(\Lambda_m) \leq 1 + K\lambda \leq 1 + \varepsilon. \quad (3.3.22)$$

Corollary 3.3.4 gives that for every $i \leq s$,

$$\begin{aligned} \mathbb{E}'_u [\varphi^\tau \mathbf{1}_{X'_\tau \in B_i} G_\beta^\Lambda(X_\tau, x)] &\leq \mathbb{E}'_u [\varphi^\tau \mathbf{1}_{X'_\tau \in B_i}] \min\{G_\beta^\Lambda(w, x) : w \in B_i\} \\ &\quad + \frac{\eta}{A} \mathbb{E}'_u [\varphi^\tau \mathbf{1}_{X'_\tau \in B_i}] \max\{G_\beta^\Lambda(w, x) : w \in \Lambda_{n+4M+m}\}, \end{aligned} \quad (3.3.23)$$

where we used that $B_i + \Lambda_{3M} \subset \Lambda_{n+4M+m}$. Moreover, by (3.3.21),

$$\mathbb{E}'_u [\varphi^\tau \mathbf{1}_{X'_\tau \in B_i}] \min\{G_\beta^\Lambda(w, x) : w \in B_i\} \leq C_{\text{RW}} \mathbb{E}'_v [\varphi^\tau \mathbf{1}_{X'_\tau \in B_i} G_\beta^\Lambda(X_\tau, x)]. \quad (3.3.24)$$

Since $n + 4M + m \leq (1 + \alpha)n$, the end of the proof follows the same lines as in Lemma 3.3.3: iterating this time (3.1.19) and (3.1.20) with S being the box of size m until the stopping time τ , and using (3.3.20),

$$G_\beta^\Lambda(u, x) \leq C_{\text{RW}} G_\beta^\Lambda(v, x) + C_{\text{RW}} \left(K\lambda + \frac{\eta}{A} \right) \max\{G_\beta^\Lambda(w, x) : w \in \Lambda_{(1+\alpha)n}\}. \quad (3.3.25)$$

The proof follows from fixing $A = 2C_{\text{RW}}$, choosing λ_0 small enough so that $C_{\text{RW}}K\lambda_0 \leq \eta/2$, and setting $N_0 := 6\alpha^{-1}\delta^{-1}L$. $\square$

3.3.2 Conclusion

We start by lower bounding the half-space two-point function at scale below $6L_\beta(\varepsilon)$ (for some technical reasons we will need a multiple of $L_\beta(\varepsilon)$ later). Let

$$A_n := \{x \in \mathbb{Z}^d : x_1 = |x| = n\}. \quad (3.3.26)$$

Lemma 3.3.5. *Fix $d > 4$. There exist $c, \varepsilon_0, \lambda_0 > 0$ such that for every $\lambda < \lambda_0$, every $\varepsilon < \varepsilon_0$, every $\frac{1}{2d} \leq \beta \leq \beta_c$, and every $x \in \mathbb{H}$ with $x_1 = |x| \leq 6L_\beta(\varepsilon)$,*

$$G_\beta^\mathbb{H}(0, x) \geq \frac{c}{|x|^{d-1}}. \quad (3.3.27)$$

Proof. Let $\varepsilon \in (0, \frac{1}{2})$ to be fixed later. First, by invariance under translation, there exists $c_0 = c_0(d) > 0$ such that, for every $n \leq L_\beta(\varepsilon) - 1$,

$$\frac{1}{|A_n|} \sum_{y \in A_n} G_\beta^{\mathbb{H}}(0, y) \geq \frac{1}{|A_n|} \sum_{y \in \partial \Lambda_n: y_1 = -n} G_\beta^{\Lambda_n}(0, y) \geq \frac{\varphi_\beta(\Lambda_n)}{2d\beta|A_n|} \geq \frac{c_0}{n^{d-1}}. \quad (3.3.28)$$

We now want to turn this averaged bound into a point-wise one. Let $\eta > 0$ to be chosen sufficiently small. Let $\lambda_0, \varepsilon_0, N_0$ be given by Proposition 3.3.2 applied to $\alpha = \frac{1}{12}$ and η . Let $\lambda < \lambda_0$ and $\varepsilon < \varepsilon_0$. Consider $x \in A_N$ with $N \leq 6L_\beta(\varepsilon)$ and set $n := \lfloor N/6 \rfloor - 1 \leq L_\beta(\varepsilon) - 1$. Assume first that $N \geq N_0$, i.e. $n \geq 6N_0$. Proposition 3.3.2 (as illustrated in Figure 3.4) implies that for every $y \in A_n$,

$$C_{\text{RW}} G_\beta^{\mathbb{H}}(0, x) \geq G_\beta^{\mathbb{H}}(0, y) - \eta \max\{G_\beta^{\mathbb{H}}(0, z) : z_1 \geq n/2\}. \quad (3.3.29)$$

Averaging over $y \in A_n$, using the lower bound (3.3.28) and the upper bound (3.1.7) gives

$$C_{\text{RW}} G_\beta^{\mathbb{H}}(0, x) \geq \frac{c_0}{n^{d-1}} - \eta \cdot \frac{C}{(n/2)^{d-1}}. \quad (3.3.30)$$

Choosing $\eta = \eta(c_0, C, d)$ such that $\eta C 2^{d-1} \leq \frac{c_0}{2}$ (which affects how small λ_0, ε_0 have to be and how large N_0 is) implies the lower bound when $|x| \geq N_0$. The proof follows readily by choosing c_1 small enough such that

$$\min\{G_{1/(2d)}^{\mathbb{H}}(0, x) : x_1 = |x| \leq N_0\} \geq \frac{c_1}{N_0^{d-1}}, \quad (3.3.31)$$

and setting $c := c_0 \wedge c_1$. □

We now turn to the full-plane lower bound below scale $5L_\beta(\varepsilon)$.

Lemma 3.3.6. *Fix $d > 4$. There exist $c, \varepsilon_0, \lambda_0 > 0$ such that for every $\lambda < \lambda_0$, every $\varepsilon < \varepsilon_0$, every $\frac{1}{2d} \leq \beta \leq \beta_c$, and every $|x| \leq 5L_\beta(\varepsilon)$,*

$$G_\beta(0, x) \geq \frac{c}{|x|^{d-2}}. \quad (3.3.32)$$

Proof. Let λ_0, ε_0 be given by Lemma 3.3.5. We will choose λ_0 even smaller below. Let $\lambda < \lambda_0$ and $\varepsilon < \varepsilon_0$. By symmetry, we may consider $x \in A_n$, where $n = |x|$. Lemma 3.1.5 applied to $S = \mathbb{H}_k$ and $\Lambda = \mathbb{H}_{k+1}$ gives that

$$\begin{aligned} G_\beta(0, x) &= G_\beta^{\mathbb{H}}(0, x) + \sum_{k \geq 0} \left(G_\beta^{\mathbb{H}_{k+1}}(0, x) - G_\beta^{\mathbb{H}_k}(0, x) \right) \\ &\geq \sum_{k \geq 0} \sum_{\substack{y \in \mathbb{H}_k, \\ z \notin \mathbb{H}_k, y \sim z}} G_\beta^{\mathbb{H}_k}(0, y) \beta G_\beta^{\mathbb{H}_{k+1}}(z, x) - \lambda \sum_{k \geq 0} \sum_{u \in \mathbb{H}_k} E_\beta^{\mathbb{H}_k, \mathbb{H}_{k+1}}(u) G_\beta^{\mathbb{H}_{k+1}}(u, x). \end{aligned} \quad (3.3.33)$$

Looking at the first sum, we see that

$$\begin{aligned} \sum_{k \geq 0} \sum_{\substack{y \in \mathbb{H}_k, \\ z \notin \mathbb{H}_k, y \sim z}} G_\beta^{\mathbb{H}_k}(0, y) \beta G_\beta^{\mathbb{H}_{k+1}}(z, x) &\geq \sum_{k=0}^{n \wedge (L_\beta(\varepsilon)-1)} \frac{1}{2d} \varphi_\beta(\Lambda_k) \min\{G_\beta^{\mathbb{H}_{k+1}}(z, x) : -z_1 = |z| = k+1\} \\ &\geq \sum_{k=0}^{n \wedge (L_\beta(\varepsilon)-1)} \frac{1}{2d} (1 - \varepsilon) \frac{c}{(n+k)^{d-1}} \\ &\geq \frac{c_1}{n^{d-2}}, \end{aligned}$$

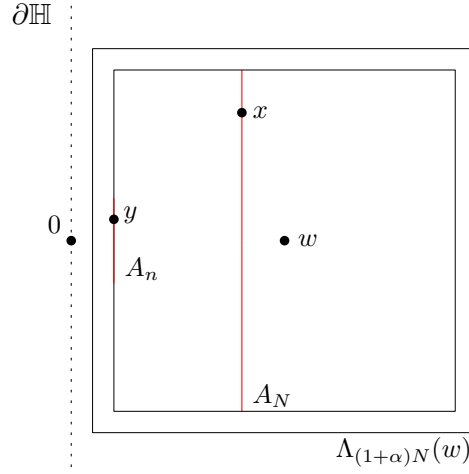


Figure 3.4: An illustration of the application of Proposition 3.3.2 in the proof of Lemma 3.3.5. We wish to argue that $G_\beta^{\mathbb{H}}(0, y)$ and $G_\beta^{\mathbb{H}}(0, x)$ are of the same order. The red segments represent the sets A_n and A_N . The boxes are centered at $w = \frac{7N}{6}\mathbf{e}_1$. The parameter α is chosen so that $\Lambda_{(1+\alpha)N}(w) \subset \{z \in \mathbb{H}, z_1 \geq n/2\}$, i.e. $\alpha = \frac{1}{12}$.

where $c_1 = c_1(d) > 0$, and where we restricted our attention to special positions of y and used both the bound $\varphi_\beta(\Lambda_k) \geq 1 - \varepsilon$ provided by $k \leq n \wedge (L_\beta(\varepsilon) - 1)$ and the lower bound from Lemma 3.3.5 (note that $n + n \wedge L_\beta(\varepsilon) \leq 6L_\beta(\varepsilon)$ for $n \leq 5L_\beta(\varepsilon)$).

Turning to the “error” term in (3.3.33), we take the position of u into account and use the upper bound (3.1.8) to get

$$\begin{aligned} & \sum_{k \geq 0} \sum_{p \geq 0} \sum_{u \in \partial \mathbb{H}_{k-p}} \sum_{\substack{y \in \mathbb{H}_k \\ z \notin \mathbb{H}_k \\ y \sim z}} G_\beta^{\mathbb{H}_k}(0, u) G_\beta^{\mathbb{H}_k}(u, y) \beta G_\beta^{\mathbb{H}_{k+1}}(z, u) G_\beta^{\mathbb{H}_{k+1}}(u, x) \\ & \leq \sum_{k \geq 0} \sum_{p \geq 0} \frac{C}{(p+1)^{d-1}} \cdot \varphi_\beta(\mathbb{H}_p) \cdot \sum_{u \in \partial \mathbb{H}_{k-p}} G_\beta^{\mathbb{H}_k}(0, u) G_\beta^{\mathbb{H}_{k+1}}(u, x). \end{aligned} \quad (3.3.34)$$

Using Proposition 3.2.9, we have that $\varphi_\beta(\mathbb{H}_p) \leq 1 + K\lambda$. We bound (3.3.34) differently according to the value of p .

First, use (3.1.7) to get

$$\sum_{p \geq (n+k)/2} \frac{C(1+K\lambda)}{(p+1)^{d-1}} \sum_{u \in \partial \mathbb{H}_{k-p}} G_\beta^{\mathbb{H}_k}(0, u) G_\beta^{\mathbb{H}_{k+1}}(u, x) \leq \frac{C_1}{(n+k)^{d-1}} \sum_{u \in \mathbb{Z}^d} G_\beta(0, u) G_\beta(u, x) \quad (3.3.35)$$

$$\leq \frac{C_2}{(n+k)^{d-1}}, \quad (3.3.36)$$

where $C_1, C_2 > 0$ only depend on C and d .

Second, turning to the contribution coming from $p < \frac{n+k}{2}$, we write

$$\max_{u \in \partial \mathbb{H}_{k-p}} G_\beta^{\mathbb{H}_{k+1}}(u, x) \stackrel{(3.2.4)}{\leq} \frac{2^d C(p+2)}{(n+k)^{d-1}}, \quad \sum_{u \in \partial \mathbb{H}_{k-p}} G_\beta^{\mathbb{H}_k}(0, u) \stackrel{(3.2.5)}{\leq} \frac{4(p+1)}{\beta}, \quad (3.3.37)$$

where we used that for $u \in \partial\mathbb{H}_{k-p}$, one has $|u_1 - x_1| = n + k - p \geq \frac{n+k}{2}$. Hence,

$$\sum_{p < (n+k)/2} \frac{C(1+K\lambda)}{(p+1)^{d-1}} \sum_{u \in \partial\mathbb{H}_{k-p}} G_{\beta}^{\mathbb{H}_k}(0, u) G_{\beta}^{\mathbb{H}_{k+1}}(u, x) \leq C_3 \sum_{p \leq \frac{n+k}{2}} \frac{1}{(p+1)^{d-3}} \frac{1}{(n+k)^{d-1}} \quad (3.3.38)$$

$$\leq \frac{C_4}{(n+k)^{d-1}}, \quad (3.3.39)$$

where $C_3, C_4 > 0$ only depend on C and d . Combining (3.3.36) and (3.3.39) gives that

$$\begin{aligned} \sum_{k \geq 0} \sum_{p \geq 0} \sum_{u \in \partial\mathbb{H}_{k-p}} \sum_{\substack{y \in \mathbb{H}_k \\ z \notin \mathbb{H}_k \\ y \sim z}} G_{\beta}^{\mathbb{H}_k}(0, u) G_{\beta}^{\mathbb{H}_k}(u, y) \beta G_{\beta}^{\mathbb{H}_{k+1}}(z, u) G_{\beta}^{\mathbb{H}_{k+1}}(u, x) &\leq \sum_{k \geq 0} \frac{C_5}{(n+k)^{d-1}} \\ &\leq \frac{C_6}{n^{d-2}}, \end{aligned} \quad (3.3.40)$$

where C_5, C_6 only depend on C and d . The proof follows by choosing λ_0 small enough so that $\lambda_0 C_6 < c_1/2$, and setting $c = c_1/2$. $\square$

We are now in a position to prove Theorem 3.1.2.

Proof of Theorem 3.1.2. Let $\lambda_0, \varepsilon_0 > 0$ be such that the previous two lemmata apply. We will (potentially) choose them even smaller below. Let $\lambda < \lambda_0$ and $\varepsilon < \varepsilon_0$.

Set $L'_{\beta} := L_{\beta}(\varepsilon)$. If $\beta = \beta_c$ then $L_{\beta} = L'_{\beta} = \infty$ and the Lemmata 3.3.5 and 3.3.6 are sufficient to conclude. We therefore assume $\beta < \beta_c$. By Lemma 3.3.1 it suffices to prove the lower bounds with L'_{β} instead of L_{β} in the exponential.

We already have the corresponding lower bounds for $|x| \leq 5L'_{\beta}$. Let us turn to the general case. We focus on the full-space estimate but the half-space follows from a similar proof. Let $c > 0$ be given by Lemma 3.3.6. Introduce, for $k \geq 0$,

$$m_k := \min\{G_{\beta}(0, x) : x \in \Lambda_{(k+1)L'_{\beta}-1}\}, \quad (3.3.41)$$

and $S_k := \{x \in \mathbb{Z}^d : kL'_{\beta} \leq |x| < (k+1)L'_{\beta}\}$. We prove by induction for $k \geq 5$ that for some $c_1 > 0$,

$$m_k \geq \frac{c}{(5L'_{\beta})^{d-2}} c_1^{k-5}. \quad (3.3.42)$$

For $k = 5$, it is simply (3.3.27). We now assume that $k \geq 6$. Applying⁹ Lemma 3.1.5 (twice) and Lemma 3.2.4, there exists $c_0 = c_0(d) > 0$ such that for every $x \in S_k$,

$$G_{\beta}(0, x) \geq 2c_0 m_{k-1} - 2K\lambda M_2(x) \quad (3.3.46)$$

⁹Indeed, without loss of generality we can assume that $x_i \geq 0$ for all $1 \leq i \leq d$. Let $\Lambda := \Lambda_{L'_{\beta}-1}$ and $H_1 := \partial\Lambda \cap \{u \in \mathbb{Z}^d : u_2 = -(L'_{\beta}-1), u_1 \leq 0, u_i \leq 0, \forall 3 \leq i \leq d\}$. Apply a first time (3.1.20) and Lemma 3.2.4 to get

$$G_{\beta}(0, x) \geq \sum_{\substack{y \in (\Lambda \cap H_1 + x) \\ z \notin (\Lambda + x) \\ y \sim z}} G_{\beta}^{\Lambda+x}(x, y) \beta G_{\beta}(z, 0) - K\lambda M_1(x). \quad (3.3.43)$$

Then, letting $H_2 := \partial\Lambda \cap \{u \in \mathbb{Z}^d : u_1 = -(L'_{\beta}-1), u_i \leq 0, \forall 2 \leq i \leq d\}$, a new application of (3.1.20) and

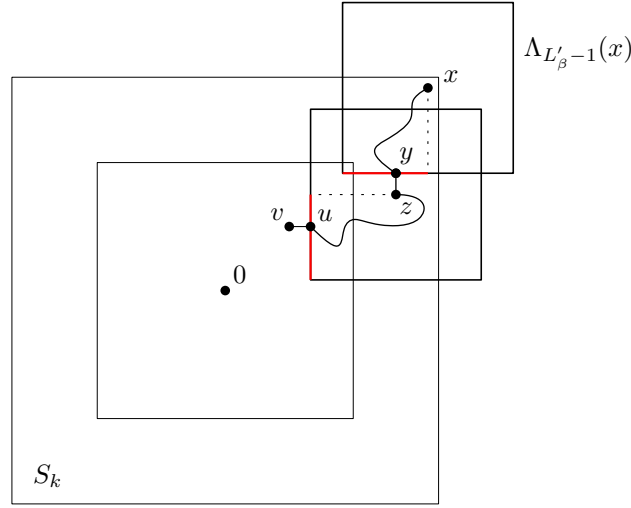


Figure 3.5: An illustration of the proof of (3.3.46). The situation depicted here is somehow the *least* favorable. The sets H_1 and H_2 are the red bold lines. With the two successive applications of Lemma 3.1.5, we “drive” the point x towards the box $\Lambda_{kL'_\beta-1}$. This provides a recurrence relation between m_k and m_{k-1} .

where for $\ell \geq 1$,

$$M_\ell(x) := \max\{G_\beta(0, y) : y \in \Lambda_{\ell L'_\beta}(x)\}. \quad (3.3.47)$$

If $x \in S_k$ is such that $2K\lambda M_2(x) \leq G_\beta(0, x)$, then (3.3.46) gives

$$G_\beta(0, x) \geq c_0 m_{k-1}. \quad (3.3.48)$$

Moreover, when $x \in \Lambda_{kL'_\beta-1}$,

$$G_\beta(0, x) \geq m_{k-1} \geq c_0 m_{k-1}. \quad (3.3.49)$$

As a consequence, we find $m_k \geq c_0 m_{k-1}$ and therefore the induction hypothesis (with $c_1 = c_0$), except if there is $x \in S_k$ such that $2K\lambda M_2(x) > G_\beta(0, x)$. We show below that this is in fact impossible by proceeding by contradiction.

Let $\eta < \min\{1/(4C_{RW}), c/(4C5^{d-2})\}$ and $\lambda_0 < \eta/K$. Also, (potentially) decrease ε_0 so that Proposition 3.3.2 holds true for this η and for $\alpha = 1$.

Lemma 3.2.4 gives,

$$G_\beta(0, x) \geq \sum_{\substack{y \in (\Lambda \cap H_1 + x) \\ z \notin (\Lambda + x) \\ z \sim y}} G_\beta^{\Lambda+x}(x, y) \beta \sum_{\substack{u \in (\Lambda \cap H_2 + z) \\ v \notin (\Lambda + z) \\ u \sim v}} G_\beta^{\Lambda+z}(z, u) \beta G_\beta(v, 0) - K\lambda M_2(x) - K\lambda M_1(x). \quad (3.3.44)$$

Now, by construction v as above must lie in $\Lambda_{kL'_\beta-1}$ (see Figure 3.5) and therefore, by symmetry and by the definition of L'_β ,

$$G_\beta(0, x) \geq \left(\frac{1}{2d} \frac{1}{2^{d-1}} (1 - \varepsilon) \right)^2 m_{k-1} - 2K\lambda M_2(x), \quad (3.3.45)$$

and (3.3.46) follows from setting $2c_0 := (\frac{1}{2d} \frac{1}{2^{d-1}} (1 - \varepsilon))^2$.

For ℓ such that $0 \notin \Lambda_{(\ell+4)L'_\beta}(x)$, Proposition 3.3.2 applied to all boxes of size $2L'_\beta$ centered on sites in $\Lambda_{\ell L'_\beta}(x)$ gives

$$M_{\ell+2}(x) \leq C_{\text{RW}} M_\ell(x) + \eta M_{\ell+4}(x). \quad (3.3.50)$$

Yet, the choices of η and λ_0 , as well as the assumption that

$$M_0(x) = G_\beta(0, x) < 2K\lambda M_2(x) \quad (3.3.51)$$

imply recursively that $M_\ell(x) \leq 2\eta M_{\ell+2}(x)$ as long as $0 \notin \Lambda_{(\ell+2)L'_\beta}(x)$. In particular, if $L := \lfloor |x|/L'_\beta \rfloor - 3$, we obtain that

$$\frac{c}{(5L'_\beta)^{d-2}} \stackrel{(3.3.32)}{\leq} m_5 \leq M_L(x) \leq 2\eta M_{L+2}(x) \stackrel{(3.2.47)}{\leq} 2\eta \frac{2C}{(L'_\beta)^{d-2}}. \quad (3.3.52)$$

The choice of η leads to a contradiction, therefore concluding the proof. $\square$

3.A Appendix: proof of Lemma 3.1.5

Write γ as the concatenation $\gamma_1 \circ (yz) \circ \gamma_2$, where (yz) is the first edge exiting S (with the convention that $y \in S$ and $z \notin S$). The structure of the weights implies that

$$\rho(\gamma_1)\rho(\gamma_2) - \lambda\rho(\gamma_1)\rho(\gamma_2) \sum_{\substack{0 \leq i \leq |\gamma_1| \\ 1 \leq j \leq |\gamma_2|}} \mathbb{1}_{\gamma_1(i)=\gamma_2(j)} \leq \rho(\gamma_1 \circ (yz) \circ \gamma_2) \leq \rho(\gamma_1)\rho(\gamma_2), \quad (\text{A.1})$$

where we used that

$$1 - \lambda \sum_{\substack{0 \leq i \leq |\gamma_1| \\ 1 \leq j \leq |\gamma_2|}} \mathbb{1}_{\gamma_1(i)=\gamma_2(j)} \leq \prod_{\substack{0 \leq i \leq |\gamma_1| \\ 1 \leq j \leq |\gamma_2|}} (1 - \lambda \mathbb{1}_{\gamma_1(i)=\gamma_2(j)}). \quad (\text{A.2})$$

After observing that $\gamma_1 \subset S$, resumming the right-hand side of (A.1) gives the upper bound of (3.1.19).

For the lower bound, note that conditioning on the common value u of $\gamma_1(i)$ and $\gamma_2(j)$, the “error” is controlled by

$$\lambda \sum_{u \in S} \sum_{\substack{y \in S \\ z \notin S \\ y \sim z}} \sum_{n, m \geq 0} \beta^{n+m+1} \sum_{\substack{0 \leq i \leq n \\ 1 \leq j \leq m}} \sum_{\substack{\gamma_1: 0 \rightarrow y \subset S \\ \gamma_2: z \rightarrow x \subset \Lambda}} \rho(\gamma_1)\rho(\gamma_2) \mathbb{1}_{\gamma_1(i)=u, |\gamma_1|=n} \mathbb{1}_{\gamma_2(j)=u, |\gamma_2|=m}. \quad (\text{A.3})$$

We further split γ_1 into $\gamma'_1 : 0 \rightarrow u$ and $\gamma''_1 : u \rightarrow t$ and use that $\rho(\gamma_1) \leq \rho(\gamma'_1)\rho(\gamma''_1)$, and do the same for γ_2 . Note that by definition, $|\gamma'_1| = i$ and $|\gamma''_1| = n - i$. Similarly, $|\gamma'_2| = j$ and $|\gamma''_2| = m - j$. The Cauchy product implies that

$$\sum_{n \geq 0} \sum_{i=0}^n \beta^n \sum_{\substack{\gamma'_1: 0 \rightarrow u \subset S \\ \gamma''_1: u \rightarrow y \subset S}} \rho(\gamma'_1) \mathbb{1}_{|\gamma'_1|=i} \rho(\gamma''_1) \mathbb{1}_{|\gamma''_1|=n-i} = G_\beta^S(0, u) G_\beta^S(u, y). \quad (\text{A.4})$$

Similarly,

$$\sum_{m \geq 0} \sum_{j=1}^m \beta^m \sum_{\substack{\gamma'_2: z \rightarrow u \subset \Lambda \\ \gamma''_2: u \rightarrow x \subset \Lambda}} \rho(\gamma'_2) \mathbb{1}_{|\gamma'_1|=j} \rho(\gamma''_2) \mathbb{1}_{|\gamma''_2|=m-j} \leq G_\beta^B(z, u) G_\beta^B(u, x). \quad (\text{A.5})$$

This yields

$$G_\beta^\Lambda(0, x) - \sum_{\substack{y \in S \\ z \notin S \\ y \sim z}} G_\beta^S(0, y) \beta G_\beta^\Lambda(z, x) \leq \lambda \sum_{u \in S} \sum_{\substack{y \in S \\ z \notin S \\ y \sim z}} G_\beta^S(0, u) G_\beta^S(u, y) \beta G_\beta^\Lambda(z, u) G_\beta^\Lambda(u, x), \quad (\text{A.6})$$

and concludes the proof.

Chapter 4

An alternative approach for the mean-field behaviour of spread-out Bernoulli percolation in dimensions $d > 6$

We propose a new way of deriving mean-field exponents for sufficiently spread-out Bernoulli percolation in dimensions $d > 6$. We obtain up-to-constant bounds for the full-space and half-space two-point functions in the critical and near-critical regimes.

This chapter is based on the preprint “*An alternative approach for the mean-field behaviour of spread-out Bernoulli percolation in dimensions $d > 6$* ” [81]. It is a joint work with Hugo Duminil-Copin.

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4.1 Introduction

4.1.1 Motivation

Grasping the (near-)critical behaviour of lattice models is one of the key challenges in statistical mechanics. A possible approach involves determining the models' *critical exponents*. Performing this task is typically very difficult as it involves the unique characteristics of the models and the geometry of the graphs on which they are constructed.

A significant observation was made for models defined on the hypercubic lattice $\mathbb{Z}^d$: beyond the *upper-critical dimension* d_c , the influence of geometry disappears, and the critical exponents simplify, matching those found on a Cayley tree (or *Bethe lattice*) or on the complete graph. The regime $d > d_c$ forms the *mean-field* regime of a model.

Prominent techniques such as the *lace expansion* [49] and the *rigorous renormalisation group method* [25, 26, 27, 29] have been developed to analyse the mean-field regime. However, a drawback of these approaches is their predominantly *perturbative* nature, necessitating the identification of a small parameter within the model. It has been established, using lace expansion, that in several contexts [134, 216, 132, 217, 100, 218] this small parameter can be taken to be $\frac{1}{d-d_c}$, meaning that mean-field behaviour was recovered in these setups in dimensions $d \gg d_c$.

In the example of the nearest-neighbour (meaning that bonds are pairs of vertices separated by unit Euclidean distance) Bernoulli percolation, mean-field behaviour was established in dimensions $d > 10$ [134, 132, 100]. This leaves a gap to fill in order to reach the expected upper critical dimension of the model $d_c = 6$. It is however possible to provide rigorous arguments to identify d_c by introducing an additional perturbative parameter in the model. In the *spread-out* Bernoulli percolation model, the bonds are pairs of vertices separated by distance between 1 and L , where L is taken to be sufficiently large. According to the *universality* conjecture, the critical exponents of these two models should match. This makes spread-out Bernoulli percolation a natural testing ground to develop the analysis of the mean-field regime of Bernoulli percolation.

Lace expansion was successfully applied to study various spread-out models in statistical mechanics, including Bernoulli percolation [134, 133], lattice trees and animals [135], the Ising model [216, 218], and even some long-range versions of the aforementioned examples [67, 68]. Much more information on the lace expansion approach can be found in [226].

In this paper, we provide an alternative argument to obtain mean-field upper bounds on the two-point function of sufficiently spread-out Bernoulli percolation in dimensions $d > 6$. This technique extends to a number of other (spread-out) models after relevant modifications. In a companion paper [82], we provide a treatment of the weakly self-avoiding walk model. However, the strategy developed there does not apply *mutatis mutandis* to the setup of spread-out Bernoulli percolation, and the presence of long finite range interactions requires additional care.

Notations. Consider the hypercubic lattice $\mathbb{Z}^d$. Set $\mathbf{e}_j$ to be the unit vector with j -th coordinate equal to 1. Write x_j for the j -th coordinate of x and denote its ℓ^∞ norm by $|x| := \max\{|x_j| : 1 \leq j \leq d\}$. Set $\Lambda_n := \{x \in \mathbb{Z}^d : |x| \leq n\}$ and for $x \in \mathbb{Z}^d$, $\Lambda_n(x) := \Lambda_n + x$. Also, set $\mathbb{H}_n := -n\mathbf{e}_1 + \mathbb{H}$, where $\mathbb{H} := \mathbb{Z}_+ \times \mathbb{Z}^{d-1} = \{0, 1, \dots\} \times \mathbb{Z}^{d-1}$. For a set S of vertices, introduce $S^* := S \setminus \{0\}$. Let ∂S be the boundary of the set S given by the vertices in S with one neighbour outside S . Finally, define the set of *generalised blocks* of $\mathbb{Z}^d$,

$$\mathcal{B} := \left\{ \left(\prod_{i=1}^d [a_i, b_i] \right) \cap \mathbb{Z}^d \text{ such that } -\infty \leq a_i \leq 0 \leq b_i \leq \infty \text{ for } 1 \leq i \leq d \right\}. \quad (4.1.1)$$

If $\Lambda \subset \mathbb{Z}^d$ and $x, y \in \Lambda$, we denote by $\{x \xleftrightarrow{\Lambda} y\}$ the event that x and y are connected by a path of open edges in Λ . If A and B are two percolation events, we write $A \circ B$ for the event of *disjoint* occurrence of A and B , i.e. the event that there exist two disjoint sets I and J of edges such that the configuration restricted to I (resp. J) is sufficient to decide that A (resp. B) occurs.

4.1.2 Definitions and statement of the results

Let $L \geq 1$. Since L is fixed for the whole article, we omit it from the notations. We consider the Bernoulli percolation measure $\mathbb{P}_\beta$ such that for every $u, v \in \mathbb{Z}^d$,

$$p_{uv, \beta} := \mathbb{P}_\beta[uv \text{ is open}] = 1 - \exp(-\beta J_{uv}) = 1 - \mathbb{P}_\beta[uv \text{ is closed}], \quad (4.1.2)$$

where $J_{uv} = c_L \mathbb{1}_{1 \leq |u-v| \leq L}$, and c_L is a normalization constant which guarantees that $|J| := \sum_{x \in \mathbb{Z}^d} J_{0,x} = 1$ (i.e. $c_L = (|\Lambda_L| - 1)^{-1}$). We also let $p_\beta := 1 - e^{-\beta c_L}$. Much more general choices can be made for J (see e.g. [134, 152, 133]) but we restrict our attention to the above for simplicity.

We are interested in the model's two-point function which is, for $\Lambda \subset \mathbb{Z}^d$, the probability $\mathbb{P}_\beta[x \xleftrightarrow{\Lambda} y]$. When $\Lambda = \mathbb{Z}^d$, we simply write $\mathbb{P}_\beta[x \leftrightarrow y] = \mathbb{P}_\beta[x \xleftrightarrow{\mathbb{Z}^d} y]$. It is well known that Bernoulli percolation undergoes a phase transition for the existence of an infinite cluster at some parameter $\beta_c \in (0, \infty)$. Moreover, for $\beta < \beta_c$, $\mathbb{P}_\beta[x \leftrightarrow y]$ decays exponentially fast in $|x - y|$, see [16, 190, 7, 84].

Our main result is a near-critical estimate of the two-point function in the full-space and in the half-space $\mathbb{H}$. It is convenient to use as a correlation length the *sharp length* L_β defined below (see also [84, 204, 83] for a study of this quantity in the context of the Ising model). For $\beta \geq 0$ and $S \subset \mathbb{Z}^d$, let

$$\varphi_\beta(S) := \sum_{\substack{y \in S \\ z \notin S}} \mathbb{P}_\beta[0 \xleftrightarrow{S} y] p_{yz, \beta}. \quad (4.1.3)$$

The sharp length L_β is defined as follows:

$$L_\beta := \inf\{k \geq 1 : \varphi_\beta(\Lambda_k) \leq 1/e^2\}. \quad (4.1.4)$$

Exponential decay of the two-point function yields that L_β is finite for $\beta < \beta_c$. It is infinite for $\beta = \beta_c$; see [223].

We now turn to our main result, which provides uniform near-critical upper bounds on the full-space and half-space two-point functions.

Theorem 4.1.1. *Let $d > 6$. There exist $C, L_0 > 0$ such that for every $L \geq L_0$ and every $\beta \leq \beta_c$ for which $L_\beta \geq L$,*

$$\mathbb{P}_\beta[0 \leftrightarrow x] \leq \frac{C}{L^d} \left(\frac{L}{L \vee |x|} \right)^{d-2} \exp(-|x|/L_\beta) \quad \forall x \in (\mathbb{Z}^d)^*, \quad (4.1.5)$$

$$\mathbb{P}_\beta[0 \overset{\mathbb{H}}{\leftrightarrow} x] \leq \frac{C}{L^d} \left(\frac{L}{L \vee |x_1|} \right)^{d-1} \exp(-|x_1|/L_\beta) \quad \forall x \in \mathbb{H}^*. \quad (4.1.6)$$

Using the lace expansion, it is possible to obtain exact asymptotics of the full-space critical two-point function [133, 182]: if $d > 6$, there exist $D, L_0 > 0$ such that if $L \geq L_0$,

$$\mathbb{P}_{\beta_c}[0 \leftrightarrow x] = \frac{A}{L^2 |x|_2^{d-2}} (1 + o(1)), \quad (4.1.7)$$

where $|\cdot|_2$ is the Euclidean norm on $\mathbb{R}^d$ and $o(1)$ tends to 0 as $|x|_2$ tends to infinity. However, the best uniform (in x) upper bound that this method yield is the following: if $d > 6$, there exist $\epsilon, C > 0$ such that for all L large enough and $\beta \leq \beta_c$,

$$\mathbb{P}_\beta[0 \leftrightarrow x] \leq \frac{C}{L^{2-\epsilon} |x|^{d-2}} \quad \forall x \in (\mathbb{Z}^d)^*. \quad (4.1.8)$$

Hence, Theorem 4.1.1 is an improvement of the bound obtained using the lace expansion.

Half-space up-to-constants critical estimates have been obtained in [59] using the lace expansion. Near-critical estimates have also been derived in [60, 154]. In these papers, L_β is replaced by $(\beta_c - \beta)^{-1/2}$. We will prove that the two quantities are within a multiplicative constant of each other, see Corollary 4.1.4. The second main theorem of this chapter provides lower bounds matching the bounds in Theorem 4.1.1. We introduce,

$$\beta_0 := \inf\{\beta \geq 0 : \varphi_\beta(\{0\}) = 1\}. \quad (4.1.9)$$

Theorem 4.1.2. *Let $d > 6$. There exist c, L_0 such that for every $L \geq L_0$, every $\beta_0 \leq \beta \leq \beta_c$, and every $x \in \mathbb{Z}^d \setminus \{0\}$ with $|x| \leq cL_\beta$,*

$$\mathbb{P}_\beta[0 \leftrightarrow x] \geq \frac{c}{L^d} \left(\frac{L}{L \vee |x|} \right)^{d-2}, \quad (4.1.10)$$

$$\mathbb{P}_\beta[0 \overset{\mathbb{H}}{\leftrightarrow} x] \geq \frac{c}{L^d} \left(\frac{L}{L \vee |x_1|} \right)^{d-1} \quad \text{provided that } x_1 = |x|. \quad (4.1.11)$$

A direct consequence of Theorem 4.1.1 is the finiteness at criticality of the so-called *triangle diagram*, which plays a central role in the study of the mean-field regime of Bernoulli percolation, see [16, 134, 23, 226]. In particular, its finiteness implies that various critical exponents exist and take their mean-field value.

Corollary 4.1.3 (Finiteness of the triangle diagram). *Let $d > 6$. There exists $L_0 = L_0(d) > 0$ such that for every $L \geq L_0$,*

$$\nabla(\beta_c) := \sum_{x, y \in \mathbb{Z}^d} \mathbb{P}_{\beta_c}[0 \leftrightarrow x] \mathbb{P}_{\beta_c}[x \leftrightarrow y] \mathbb{P}_{\beta_c}[y \leftrightarrow 0] < \infty. \quad (4.1.12)$$

Finally, we describe how to recover the mean-field behaviour of the *susceptibility* defined by

$$\chi(\beta) := \sum_{x \in \mathbb{Z}^d} \mathbb{P}_\beta[0 \leftrightarrow x], \quad (4.1.13)$$

and the *correlation length* ξ_β defined by

$$\xi_\beta^{-1} := \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}_\beta[0 \leftrightarrow n\mathbf{e}_1]. \quad (4.1.14)$$

Corollary 4.1.4. *Let $d > 6$. There exist c, C, L_0 such that for every $L \geq L_0$ and every $\beta_0 \leq \beta < \beta_c$,*

$$c(\beta_c - \beta)^{-1} \leq \chi(\beta) \leq C(\beta_c - \beta)^{-1}, \quad (4.1.15)$$

$$c(\beta_c - \beta)^{-1/2} \leq L_\beta \leq C(\beta_c - \beta)^{-1/2}. \quad (4.1.16)$$

Moreover,

$$\xi_\beta = (\beta_c - \beta)^{-1/2+o(1)} \quad (4.1.17)$$

where $o(1)$ tends to 0 as β tends to β_c .

Proof. Let $d > 6$. Let L_0 be given by Corollary 4.1.3. It is classical (see [16]) that $\nabla(\beta_c) < \infty$ implies the existence of $c, C > 0$ such that, for $\beta < \beta_c$,

$$c(\beta_c - \beta)^{-1} \leq \chi(\beta) \leq C(\beta_c - \beta)^{-1}. \quad (4.1.18)$$

Combining Theorems 4.1.1 and 4.1.2, we additionally obtain that $\chi(\beta) \asymp L_\beta^2$ (where $\asymp$ means that the ratio of the quantities is bounded away from 0 and ∞ by two constants that are independent of β). This gives (4.1.15)–(4.1.16).

It remains to observe that $L_\beta = \xi_\beta^{1+o(1)}$. We use the same proof as in [83, Theorem 1.6]: combining (4.1.5) and (4.1.14) readily implies $\xi_\beta \lesssim L_\beta$; using [121, Proposition 6.47] implies that $\varphi_\beta(\Lambda_n) \lesssim n^{d-1}e^{-n/\xi_\beta}$ which entails $L_\beta \lesssim \xi_\beta \log \xi_\beta$ and the result. $\square$

4.1.3 Strategy of the proof

Recall the van den Berg–Kesten (BK) inequality (see [121, Section 2.3]) stating that for two increasing events (i.e. events that are stable by the action of opening edges) A and B ,

$$\mathbb{P}_\beta[A \circ B] \leq \mathbb{P}_\beta[A]\mathbb{P}_\beta[B]. \quad (\text{BK})$$

The strategy of the proof is the same as in [82]. The starting point is the following inequality, which is a classical consequence of the BK inequality.

Lemma 4.1.5. *Let $d \geq 2$, $\beta > 0$, $o \in S \subset \Lambda$, and $x \in \Lambda$. Then,*

$$\mathbb{P}_\beta[o \overset{\Lambda}{\leftrightarrow} x] \leq \mathbb{P}_\beta[o \overset{S}{\leftrightarrow} x] + \sum_{\substack{y \in S \\ z \in \Lambda \setminus S}} \mathbb{P}_\beta[o \overset{S}{\leftrightarrow} y] p_{yz, \beta} \mathbb{P}_\beta[z \overset{\Lambda}{\leftrightarrow} x]. \quad (4.1.19)$$

A similar statement in the context of the Ising model is often referred to as the Simon–Lieb inequality, see [223, 178]. For completeness, we include the proof immediately.

Proof. Considering an open self-avoiding path from o to x gives

$$\{o \xleftrightarrow{\Lambda} x\} \setminus \{o \xleftrightarrow{S} x\} \subset \bigcup_{\substack{y \in S \\ z \in \Lambda \setminus S}} \{o \xleftrightarrow{S} y\} \circ \{yz \text{ is open}\} \circ \{z \xleftrightarrow{\Lambda} x\}, \quad (4.1.20)$$

which implies the result by applying (BK). $\square$

For $\beta > 0$ and $k \geq 0$, define

$$\psi_\beta(\mathbb{H}_k) := \sum_{x \in (\partial \mathbb{H}_k)^*} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_k} x]. \quad (4.1.21)$$

We will use a bootstrap argument (following the original idea from [225]) and prove that an *a priori* estimate on the half-space two-point function can be improved for sufficiently large L . The idea will be to observe that the Simon–Lieb type inequality (4.1.19) can be coupled with a reversed version of the inequality — see Lemma 4.2.1 below — to provide good control over $\psi_\beta(\mathbb{H}_k)$, which can be interpreted as an ℓ^1 estimate on the half-space two-point function at distance k . The point-wise, or ℓ^∞ , half-space bound will follow from a *regularity* estimate which allows one to compare two-point functions for points that are close. Finally, we will deduce the full-space estimate from the half-space one. The improvement of the *a priori* estimates will be permitted by classical random walk computations involving the spread-out random walk defined as follows.

Definition 4.1.6. Define the random walk $(X_k^x)_{k \geq 0}$ started at $x \in \mathbb{Z}^d$ and of law $\mathbb{P}_x^{\text{RW}}$ given by the step distribution:

$$\mathbb{P}_x^{\text{RW}}[X_1^x = y] := J_{xy}. \quad (4.1.22)$$

To implement the scheme described in the previous paragraph, we introduce the following parameter β^* .

Definition 4.1.7. Fix $d > 6$, $\mathbf{C} > 1$, and $L \geq 1$. We define $\beta^* = \beta^*(\mathbf{C}, L)$ to be the largest real number in $[0, 2 \wedge \beta_c]$ such that for every $\beta < \beta^*$,

$$\begin{aligned} \psi_\beta(\mathbb{H}_n) &< \frac{\mathbf{C}}{L} & \forall n \geq 0, & (\ell_\beta^1) \\ \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}} x] &< \frac{\mathbf{C}}{L^d} \left(\frac{L}{L \vee |x_1|} \right)^{d-1} & \forall x \in \mathbb{H}^*. & (\ell_\beta^\infty) \end{aligned}$$

It might be surprising to require $\beta^* \leq 2$. However, we will see that $\beta_c = 1 + O(L^{-d})$ as L goes to infinity so this bound is a posteriori harmless. Let us mention that the lace expansion allows one to obtain an estimate of the form $\beta_c = 1 + C_1 L^{-d} + O(L^{-(d+1)})$ for an explicit $C_1 > 0$, see [150].

It could also be, a priori, that $\beta^*(\mathbf{C}, L) = 0$. However, we will see in the lemma below that if $\mathbf{C}$ exceeds a large enough constant one may compare $\beta^*(\mathbf{C}, L)$ to β_0 defined by

$$\beta_0 := \min\{\beta \geq 0 : \varphi_\beta(\{0\}) = 1\} \quad (4.1.23)$$

and obtain that $\beta^*(\mathbf{C}, L) \geq 1$.

Lemma 4.1.8. *We have $1 \leq \beta_0 \leq 2$ and for $\mathbf{C}$ large enough, $\beta^*(\mathbf{C}, L) \geq \beta_0$ for every L .*

Proof. Let us first prove that $1 \leq \beta_0 \leq 2$. Using that $|J| = 1$, we get,

$$1 = \varphi_{\beta_0}(\{0\}) = \sum_{x \in \mathbb{Z}^d} (1 - e^{-\beta_0 J_{0x}}) \leq \beta_0 \sum_{x \in \mathbb{Z}^d} J_{0x} = \beta_0 |J| = \beta_0. \quad (4.1.24)$$

Using that for $x \geq 0$, $1 - e^{-x} \geq x - \frac{x^2}{2}$, we get that

$$\varphi_2(\{0\}) \geq (|\Lambda_L| - 1)(2c_L - 2c_L^2) = 2 - 2c_L > 1, \quad (4.1.25)$$

which implies that $\beta_0 \leq 2$.

We now turn to the proof of $\beta^*(\mathbf{C}, L) \geq \beta_0$. Iterating (4.1.19) with S a singleton and $\Lambda = \mathbb{H}$ gives for $x \neq 0$,

$$\mathbb{P}_{\beta_0}[0 \xleftrightarrow{\mathbb{H}} x] \leq \mathbb{E}_0 \left[\sum_{\ell < \tau} \mathbb{1}_{X_\ell = x} \right], \quad (4.1.26)$$

where τ is the exit time of $\mathbb{H}$, that is $\tau = \inf\{\ell \geq 1 : X_\ell \notin \mathbb{H}\}$ (and more generally, τ_k is the exit time of $\mathbb{H}_k$). Classical random walk estimates (see [172]) give the existence of $C_{\text{RW}} = C_{\text{RW}}(d) > 0$ such that: for every $L \geq 1$ and every $x \in \mathbb{H}^*$,

$$\mathbb{E}_0 \left[\sum_{\ell < \tau} \mathbb{1}_{X_\ell = x} \right] \leq \frac{C_{\text{RW}}}{L^d} \left(\frac{L}{L \vee |x_1|} \right)^{d-1}. \quad (4.1.27)$$

The two previous displayed equations imply that

$$\mathbb{P}_{\beta_0}[0 \xleftrightarrow{\mathbb{H}} x] \leq \frac{C_{\text{RW}}}{L^d} \left(\frac{L}{L \vee |x_1|} \right)^{d-1}. \quad (4.1.28)$$

Moreover, Markov's property gives that

$$\varphi^{\text{RW}}(\mathbb{H}_k) := \sum_{\substack{y \in \mathbb{H}_k \\ z \notin \mathbb{H}_k}} \mathbb{E}_0 \left[\sum_{\ell < \tau_k} \mathbb{1}_{X_\ell = y} \right] J_{yz} = 1.$$

Since $p_{\beta, xy} \leq \beta J_{xy}$, we have by (4.1.26) that $\beta_0 \varphi^{\text{RW}}(\mathbb{H}_k) \geq \varphi_{\beta_0}(\mathbb{H}_k)$. Using Lemma 4.3.2, we obtain that $\varphi_{\beta_0}(\mathbb{H}_k) \frac{64e^2}{L} \geq \psi_{\beta_0}(\mathbb{H}_k)$. Collecting all the above, this gives

$$\psi_{\beta_0}(\mathbb{H}_k) \leq \beta_0 \frac{64}{L} \leq \frac{128e^2}{L} \leq \frac{\mathbf{C}_{\text{RW}}}{L} \quad (4.1.29)$$

if we set $\mathbf{C}_{\text{RW}} := C_{\text{RW}} \vee 128e^2$.

Hence, from (4.1.26)–(4.1.29) and the monotonicity of the two-point function in β , we have obtained that for every $\mathbf{C} > \mathbf{C}_{\text{RW}}$ and every $L \geq 1$, one has $\beta^*(\mathbf{C}, L) \geq \beta_0$. $\square$

Additionally, we note (this will be useful below) that for all $\beta \leq \beta^*$,

$$(|\Lambda_L| - 1)p_\beta \leq \beta^* \leq 2. \quad (4.1.30)$$

Organisation. Our goal is to show that β^* is in fact equal to β_c provided that $\mathbf{C}$ and L are large enough. The proof goes in three steps:

- First, we show that we can improve the ℓ^1 bound. We do so by relating $\psi_\beta(\mathbb{H}_n)$ to $\varphi_\beta(\mathbb{H}_n)$ (see Lemma 4.3.2) and then obtaining a bound on $\varphi_\beta(\mathbb{H}_n)$ when $\beta < \beta^*$ that becomes better and better when L is chosen larger and larger.
- Second, we control the gradient of the two-point function (see Proposition 4.3.6) and use this estimate to turn the improved ℓ^1 bound into an improved ℓ^∞ bound.
- Third, we show that these improvements imply that β^* cannot be strictly smaller than β_c (we will also derive that $\beta_c < 2$), since otherwise the improved estimates would remain true for β slightly larger than β^* , which would contradict the definition of β^* .

The three steps are implemented in Section 4.3. Before this, we describe the reversed Simon–Lieb type inequality in Section 4.2, and explain how one may bound an “error” term appearing in this reversed inequality.

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4.2 A reversed Simon–Lieb type inequality

4.2.1 The inequality

A crucial role will be played by the following inequality, which can be viewed as a *reversed* Simon–Lieb inequality.

Lemma 4.2.1. *Let $d \geq 2$, $0 < \beta < \beta_c$, $o \in S \subset \Lambda$, and $x \in \Lambda$. Then,*

$$\mathbb{P}_\beta[o \xleftrightarrow{\Lambda} x] \geq \mathbb{P}_\beta[o \xleftrightarrow{S} x] + \sum_{\substack{y \in S \\ z \in \Lambda \setminus S}} \mathbb{P}_\beta[o \xleftrightarrow{S} y] p_{yz, \beta} \mathbb{P}_\beta[z \xleftrightarrow{\Lambda} x] - E_\beta(S, \Lambda, o, x), \quad (4.2.1)$$

where

$$E_\beta(S, \Lambda, o, x) := \sum_{u, v \in S} \sum_{\substack{y \in S \\ z \in \Lambda \setminus S}} \mathbb{P}_\beta[o \xleftrightarrow{S} u] \mathbb{P}_\beta[u \xleftrightarrow{S} y] p_{yz, \beta} \mathbb{P}_\beta[z \xleftrightarrow{\Lambda} v] \mathbb{P}_\beta[u \xleftrightarrow{\Lambda} v] \mathbb{P}_\beta[v \xleftrightarrow{\Lambda} x] \quad (4.2.2)$$

$$+ \sum_{\substack{u \in S \\ v \in \Lambda}} \sum_{\substack{y, s \in S \\ z, t \in \Lambda \setminus S \\ yz \neq st}} \mathbb{P}_\beta[o \xleftrightarrow{S} u] \mathbb{P}_\beta[u \xleftrightarrow{S} y] \mathbb{P}_\beta[u \xleftrightarrow{S} s] p_{yz, \beta} p_{st, \beta} \mathbb{P}_\beta[z \xleftrightarrow{\Lambda} v] \mathbb{P}_\beta[t \xleftrightarrow{\Lambda} v] \mathbb{P}_\beta[v \xleftrightarrow{\Lambda} x]. \quad (4.2.3)$$

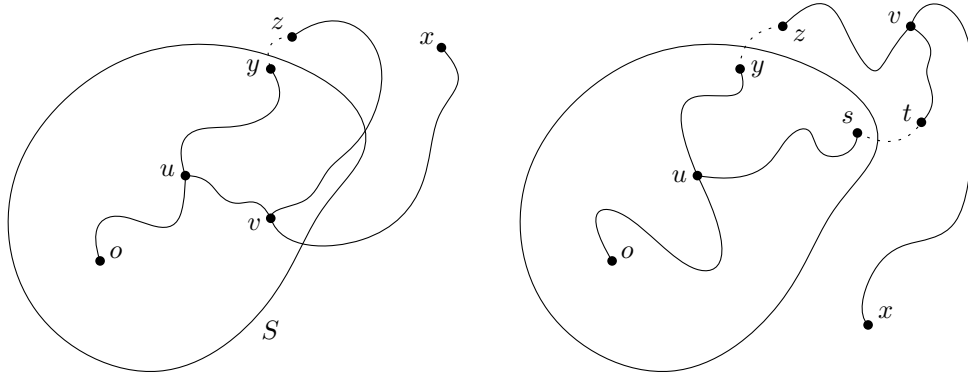


Figure 4.1: An illustration of the different terms contributing to $E_\beta(S, \Lambda, o, x)$. The dotted lines represent the open edges leaving S . From left to right, we illustrate (4.2.2) and (4.2.3).

Before diving into the proof of this lemma, let us recall an elementary yet important *tree-bound inequality* from [16, Proposition 4.1]: for every $0, a, b \in S$,

$$\mathbb{P}[o \xleftrightarrow{S} a, b] \leq \sum_{u \in S} \mathbb{P}[o \xleftrightarrow{S} u] \mathbb{P}[u \xleftrightarrow{S} a] \mathbb{P}[u \xleftrightarrow{S} b]. \quad (4.2.4)$$

This bound directly follows from the BK inequality and the observation that for o to be connected to both a and b in S , there must exist $u \in S$ such that the following event occurs

$$\{o \xleftrightarrow{S} u\} \circ \{u \xleftrightarrow{S} a\} \circ \{u \xleftrightarrow{S} b\}. \quad (4.2.5)$$

Proof. We fix β and drop it from notation (e.g. we write p_{xy} instead of $p_{xy,\beta}$). Let

$$\mathcal{N} := \sum_{\substack{y \in S \\ z \in \Lambda \setminus S}} \mathbf{1}\{o \xleftrightarrow{S} y, yz \text{ is open}, z \xleftrightarrow{\mathcal{C}^c} x\}, \quad (4.2.6)$$

where $\mathcal{C}$ is the (vertex) cluster of o in S . Clearly, $\{\mathcal{N} \geq 1\} \subset \{o \xleftrightarrow{\Lambda} x\} \setminus \{o \xleftrightarrow{S} x\}$. Since for $t \in \mathbb{N}$, $\mathbf{1}_{t \geq 1} \geq 2t - t^2$, we find

$$\mathbb{P}[o \xleftrightarrow{\Lambda} x] - \mathbb{P}[o \xleftrightarrow{S} x] \geq \mathbb{P}[\mathcal{N} \geq 1] \geq 2\mathbb{E}[\mathcal{N}] - \mathbb{E}[\mathcal{N}^2] = \mathbb{E}[\mathcal{N}] - (\mathbb{E}[\mathcal{N}^2] - \mathbb{E}[\mathcal{N}]). \quad (4.2.7)$$

Lower bound on $\mathbb{E}[\mathcal{N}]$. Write

$$\mathbb{E}[\mathcal{N}] = \sum_{\substack{y \in S \\ z \in \Lambda \setminus S}} \sum_{C \ni y} \mathbb{P}[C = \mathcal{C}] p_{yz} \mathbb{P}[z \xleftrightarrow{\mathcal{C}^c} x]. \quad (4.2.8)$$

Using [16, Proposition 5.2], we find that

$$\mathbb{P}[z \xleftrightarrow{\Lambda} x] - \mathbb{P}[z \xleftrightarrow{\mathcal{C}^c} x] \leq \sum_{v \in C} \mathbb{P}[\mathcal{A}(z, v)] \mathbb{P}[v \xleftrightarrow{\Lambda} x], \quad (4.2.9)$$

where $\mathcal{A}(z, v)$ is the event that z and v are connected by a path whose only element in C is the last vertex. Combining (4.2.8) and (4.2.9) with the fact that $C \subset S$ yields

$$\sum_{\substack{y \in S \\ z \in \Lambda \setminus S}} \mathbb{P}[o \xrightarrow{S} y] p_{yz} \mathbb{P}[z \xrightarrow{\Lambda} x] - \mathbb{E}[\mathcal{N}] \leq \sum_{\substack{y \in S \\ z \in \Lambda \setminus S}} \sum_{v \in S} \mathbb{P}[\{o \xrightarrow{S} y, v\} \circ \{z \xrightarrow{\Lambda} v\}] p_{yz} \mathbb{P}[v \xrightarrow{\Lambda} x]. \quad (4.2.10)$$

Using (BK) and (4.2.4) yields,

$$\mathbb{P}[\{o \xrightarrow{S} y, v\} \circ \{z \xrightarrow{\Lambda} v\}] \leq \mathbb{P}[o \xrightarrow{S} y, v] \mathbb{P}[z \xrightarrow{\Lambda} v] \quad (4.2.11)$$

$$\leq \sum_{u \in S} \mathbb{P}[o \xrightarrow{S} u] \mathbb{P}[u \xrightarrow{S} v] \mathbb{P}[u \xrightarrow{S} y] \mathbb{P}[z \xrightarrow{\Lambda} v]. \quad (4.2.12)$$

Altogether, we obtained,

$$\begin{aligned} \mathbb{E}[\mathcal{N}] &\geq \sum_{\substack{y \in S \\ z \in \Lambda \setminus S}} \mathbb{P}[o \xrightarrow{S} u] p_{yz} \mathbb{P}[z \xrightarrow{\Lambda} x] \\ &\quad - \sum_{u, v \in S} \sum_{\substack{y \in S \\ z \in \Lambda \setminus S}} \mathbb{P}[o \xrightarrow{S} u] \mathbb{P}[u \xrightarrow{S} y] \mathbb{P}[u \xrightarrow{S} v] p_{yz} \mathbb{P}[z \xrightarrow{\Lambda} v] \mathbb{P}[v \xrightarrow{\Lambda} x]. \end{aligned} \quad (4.2.13)$$

Upper bound on $\mathbb{E}[\mathcal{N}^2] - \mathbb{E}[\mathcal{N}]$. Notice that,

$$\mathbb{E}[\mathcal{N}^2] = \mathbb{E}[\mathcal{N}] + \sum_{\substack{y, s \in S \\ z, t \in \Lambda \setminus S \\ yz \neq st}} \mathbb{P}[o \xrightarrow{S} y, yz \text{ is open}, z \xrightarrow{C^c} x, o \xrightarrow{S} s, st \text{ is open}, t \xrightarrow{C^c} x]. \quad (4.2.14)$$

Using the same techniques as above gives

$$\mathbb{E}[\mathcal{N}^2] - \mathbb{E}[\mathcal{N}] \leq \sum_{\substack{y, s \in S \\ z, t \in \Lambda \setminus S \\ yz \neq st}} \mathbb{P}[o \xrightarrow{S} y, s] p_{yz} p_{st} \mathbb{P}[z, t \xrightarrow{\Lambda} x] \quad (4.2.15)$$

$$\leq \sum_{\substack{u \in S \\ v \in \Lambda}} \sum_{\substack{y, s \in S \\ z, t \in \Lambda \setminus S \\ yz \neq st}} \mathbb{P}[o \xrightarrow{S} u] \mathbb{P}[u \xrightarrow{S} y] \mathbb{P}[u \xrightarrow{S} s] p_{yz} p_{st} \mathbb{P}[z \xrightarrow{\Lambda} v] \mathbb{P}[t \xrightarrow{\Lambda} v] \mathbb{P}[v \xrightarrow{\Lambda} x] \quad (4.2.16)$$

where in the second line we used (4.2.4) twice.

Conclusion. The proof follows by combining (4.2.7), (4.2.13), and (4.2.16). $\square$

4.2.2 The error amplitude: definition and bound

For a set S , introduce the *error amplitude*

$$\begin{aligned} E_\beta(S) := & \sum_{\substack{u,v,y \in S \\ z \notin S}} \mathbb{P}_\beta[o \xleftrightarrow{S} u] \mathbb{P}_\beta[u \xleftrightarrow{S} y] p_{yz,\beta} \mathbb{P}_\beta[z \leftrightarrow v] \mathbb{P}_\beta[u \leftrightarrow v] \\ & + \sum_{\substack{u,y,s \in S \\ v \in \mathbb{Z}^d \\ z,t \notin S \\ yz \neq st}} \mathbb{P}_\beta[o \xleftrightarrow{S} u] \mathbb{P}_\beta[u \xleftrightarrow{S} y] \mathbb{P}_\beta[u \xleftrightarrow{S} s] p_{yz,\beta} p_{st,\beta} \mathbb{P}_\beta[z \leftrightarrow v] \mathbb{P}_\beta[t \leftrightarrow v]. \end{aligned} \quad (4.2.17)$$

The main goal of this section is to prove the following bound, which provides a control of the error amplitude in terms of $\mathbf{C}$ and L when $\beta < \beta^*(\mathbf{C}, L)$. It is the place where the assumption that $d > 6$ plays a fundamental role.

Proposition 4.2.2 (Control of the error amplitude). *Fix $d > 6$, $\mathbf{C} > 1$, and $L \geq 1$. There exists $K = K(\mathbf{C}, d) > 0$ such that for every $\beta < \beta^*(\mathbf{C}, L)$ and every $B \in \mathcal{B}$,*

$$E_\beta(B) \leq \frac{K}{L^d}. \quad (4.2.18)$$

As a warm-up, we derive some elementary computations obtained under the assumption that $\beta < \beta^*$. Our first estimate is a bound on the full-space two-point function.

Lemma 4.2.3 (Bound on the full-space two-point function). *Fix $d > 6$, $\mathbf{C} > 1$, and $L \geq 1$. For every $\beta < \beta^*(\mathbf{C}, L)$,*

$$\mathbb{P}_\beta[0 \leftrightarrow x] \leq \frac{3\mathbf{C}^2}{L^d} \left(\frac{L}{L \vee |x|} \right)^{d-2} \quad \forall x \in (\mathbb{Z}^d)^*. \quad (4.2.19)$$

The proof consists in decomposing a self-avoiding path with respect to the first vertex on it minimizing the first coordinate, and then use the half-space estimates provided by the assumption $\beta < \beta^*$.

Proof. Without loss of generality, assume that $x_1 = |x|$. If the connection to x is not included in $\mathbb{H}$, decompose an open self-avoiding path connecting 0 to x according to the first left-most point z on it (see Figure 4.2) to get

$$\mathbb{P}_\beta[0 \leftrightarrow x] \stackrel{(\text{BK})}{\leq} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}} x] + \sum_{n \geq 1} \sum_{z \in \partial \mathbb{H}_n} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} z] \mathbb{P}_\beta[z \xleftrightarrow{\mathbb{H}_n} x] \quad (4.2.20)$$

$$\stackrel{(\ell_\beta^\infty)}{\leq} \frac{\mathbf{C}}{L^d} \left(\frac{L}{|x| \vee L} \right)^{d-1} + \sum_{n \geq 1} \psi_\beta(\mathbb{H}_n) \frac{\mathbf{C}}{L^d} \left(\frac{L}{L \vee (|x| + n)} \right)^{d-1} \quad (4.2.21)$$

$$\begin{aligned} &= \frac{\mathbf{C}}{L^d} \left(\frac{L}{|x| \vee L} \right)^{d-1} + \sum_{n=1}^{L-|x|} \psi_\beta(\mathbb{H}_n) \frac{\mathbf{C}}{L^d} + \sum_{n \geq L-|x|+1} \psi_\beta(\mathbb{H}_n) \frac{\mathbf{C}}{L} \frac{1}{(|x| + n)^{d-1}} \\ &\quad (4.2.22) \end{aligned}$$

$$\stackrel{(\ell_\beta^1)}{\leq} \frac{\mathbf{C}(1+2\mathbf{C})}{L^d} \left(\frac{L}{|x| \vee L} \right)^{d-2}, \quad (4.2.23)$$

where on the last inequality, we used that $\sum_{n \geq \alpha} \frac{1}{(n+1)^{d-1}} \leq \frac{1}{(d-2)\alpha^{d-2}}$ for $\alpha \geq 1$. The proof follows readily. $\square$

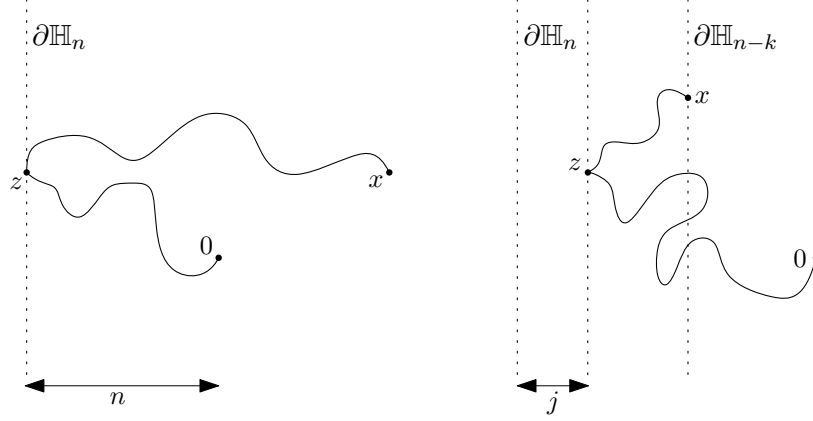


Figure 4.2: On the left, an illustration of the decomposition of a self-avoiding path connecting 0 to x used in the proof of (4.2.19). On the right, a similar decomposition used in the proof of (4.2.25) in the case $x_1 < 0$.

For a generalised block $B \in \mathcal{B}$ and $k \geq 0$, we define the k -boundary of B to be

$$\partial^k B := \{x \in B : d(x, \partial B) = k\}, \quad (4.2.24)$$

where $d(x, \partial B) := \inf\{|x - b| : b \in \partial B\}$. In particular, $\partial^0 B = \partial B$. We now turn to ℓ^∞ and ℓ^1 estimates for points living on the k -boundary of a generalised block B .

Lemma 4.2.4 (ℓ^∞ and ℓ^1 bounds at distance k from the boundary). *Fix $d > 6$, $\mathbf{C} > 1$, and $L \geq 1$. Fix $B = (\prod_{i=1}^d [a_i, b_i]) \cap \mathbb{Z}^d \in \mathcal{B}$, with $a_i \leq 0 \leq b_i$. For every $\beta < \beta^*(\mathbf{C}, L)$,*

$$\sup_{\substack{x \in (\partial^k B)^* \\ x_i \in \{a_i + k, b_i - k\}}} \mathbb{P}_\beta[0 \xleftrightarrow{B} x] \leq \frac{\mathbf{C}^2(k+L)}{L^{d+1}} \left(\frac{L}{L \vee |x_i|} \right)^{d-1} \quad \forall 1 \leq i \leq d, \forall k \geq 0, \quad (4.2.25)$$

$$\sum_{x \in (\partial^k B)^*} \mathbb{P}_\beta[0 \xleftrightarrow{B} x] \leq \frac{2d\mathbf{C}^2(k+L)}{L^2} \quad \forall k \geq 0. \quad (4.2.26)$$

The proof consists in a simple decomposition with respect to the first vertex on the path minimizing the first coordinate.

Proof. We prove each point separately.

Proof of (4.2.25). Without loss of generality, assume that $i = 1$ and that $x \in (\partial^k B)^*$ is such that $x_1 = a_1 + k$. Let $n = -a_1$ so that $x_1 = k - n$. Notice that

$$\mathbb{P}_\beta[0 \xleftrightarrow{B} x] \leq \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x]. \quad (4.2.27)$$

We distinguish two cases: $x_1 < 0$ and $x_1 \geq 0$. We begin with the former case, i.e. $k < n$. To bound $\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x]$, decompose an open self-avoiding path connecting 0 to x according to its first left-most point z (see Figure 4.2) to get

$$\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x] \stackrel{(\text{BK})}{\leq} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_{n-k}} x] + \sum_{j=0}^{k-1} \sum_{z \in \partial \mathbb{H}_{n-j}} \mathbb{P}_\beta[z \xleftrightarrow{\mathbb{H}_{n-j}} x] \mathbb{P}_\beta[z \xleftrightarrow{\mathbb{H}_{n-j}} 0] \quad (4.2.28)$$

$$\stackrel{(\ell_\beta^\infty)}{\leq} \frac{\mathbf{C}}{L^d} \left(\frac{L}{L \vee (n-k)} \right)^{d-1} + \sum_{j=0}^{k-1} \sum_{z \in \partial \mathbb{H}_{n-j}} \frac{\mathbf{C}}{L^d} \left(\frac{L}{L \vee (n-j)} \right)^{d-1} \mathbb{P}_\beta[z \xleftrightarrow{\mathbb{H}_{n-j}} x] \quad (4.2.29)$$

$$\stackrel{(\ell_\beta^1)}{\leq} \left(\frac{\mathbf{C}}{L^d} + \frac{\mathbf{C}^2 k}{L^{d+1}} \right) \left(\frac{L}{L \vee (n-k)} \right)^{d-1} = \frac{\mathbf{C}^2(k+L)}{L^{d+1}} \left(\frac{L}{L \vee |x_1|} \right)^{d-1}, \quad (4.2.30)$$

where on the third line we used translation invariance and (ℓ_β^1) to write that

$$\sum_{z \in \partial \mathbb{H}_{n-j}} \mathbb{P}_\beta[z \xleftrightarrow{\mathbb{H}_{n-j}} x] = \psi_\beta(\mathbb{H}_{k-j}) \left(\leq \frac{\mathbf{C}}{L} \right). \quad (4.2.31)$$

We turn to the case $x_1 \geq 0$, i.e. $k \geq n$. Again, to bound $\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x]$, we decompose an open self-avoiding path connecting 0 to x according to its first left-most point z . We obtain

$$\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x] \stackrel{(\text{BK})}{\leq} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}} x] + \sum_{j=0}^{n-1} \sum_{z \in \partial \mathbb{H}_{n-j}} \mathbb{P}_\beta[z \xleftrightarrow{\mathbb{H}_{n-j}} x] \mathbb{P}_\beta[z \xleftrightarrow{\mathbb{H}_{n-j}} 0] \quad (4.2.32)$$

$$\stackrel{(\ell_\beta^\infty)}{\leq} \frac{\mathbf{C}}{L^d} \left(\frac{L}{L \vee |x_1|} \right)^{d-1} + \sum_{j=0}^{n-1} \sum_{z \in \partial \mathbb{H}_{n-j}} \frac{\mathbf{C}}{L^d} \left(\frac{L}{L \vee (|x_1| + n - j)} \right)^{d-1} \mathbb{P}_\beta[z \xleftrightarrow{\mathbb{H}_{n-j}} 0] \quad (4.2.33)$$

$$\stackrel{(\ell_\beta^1)}{\leq} \frac{\mathbf{C}^2(n+L)}{L^{d+1}} \left(\frac{L}{L \vee |x_1|} \right)^{d-1} \leq \frac{\mathbf{C}^2(k+L)}{L^{d+1}} \left(\frac{L}{L \vee |x_1|} \right)^{d-1}. \quad (4.2.34)$$

Proof of (4.2.26). This time, notice that

$$\sum_{x \in (\partial^k B)^*} \mathbb{P}_\beta[0 \xleftrightarrow{B} x] \leq \sum_{i=1}^d \sum_{\substack{x \in (\partial^k B)^* \\ x_i \in \{a_i + k, b_i - k\}}} \mathbb{P}_\beta[0 \xleftrightarrow{B} x] \quad (4.2.35)$$

$$\leq (2d) \cdot \sup_{n \geq 0} \sum_{x \in (\partial \mathbb{H}_{n-k})^*} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x]. \quad (4.2.36)$$

We let $n \geq 0$ and bound the sum appearing in (4.2.36). We consider the same decomposition as in (4.2.28), but use (ℓ_β^1) twice instead of (ℓ_β^1) and (ℓ_β^∞) . We first look at the case $0 \leq k \leq n-1$. Summing (4.2.28) (which holds for that range of k) over $x \in \partial \mathbb{H}_{n-k}$ gives

$$\sum_{x \in \partial \mathbb{H}_{n-k}} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x] \stackrel{(\ell_\beta^1)}{\leq} \frac{\mathbf{C}}{L} + \frac{k \mathbf{C}^2}{L^2}. \quad (4.2.37)$$

We similarly get that for $k = n$ (here it is important to omit 0),

$$\sum_{x \in (\partial \mathbb{H})^*} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x] \leq \frac{\mathbf{C}}{L} + \frac{n\mathbf{C}^2}{L^2}. \quad (4.2.38)$$

The case $k > n$ is handled similarly, using this time (4.2.32) instead of (4.2.28). $\square$

We are now equipped to prove the main result of this section on the error amplitude. This computation is lengthy. Since it simply consists in plugging the previous estimates in the expression for the error, it may be skipped in a first reading.

Proof of Proposition 4.2.2. We fix $B = (\prod_{i=1}^d [a_i, b_i]) \cap \mathbb{Z}^d \in \mathcal{B}$ and assume that $B \neq \mathbb{Z}^d$ (otherwise one has $E_\beta(\mathbb{Z}^d) = 0$ by definition). We fix $\beta < \beta^*$ and drop it from the notations. Introduce the quantities $\mathcal{E}(1)$ and $\mathcal{E}(2)$ corresponding to the sums in (4.2.17) on the first and second line respectively. Since $E_\beta(B) = \mathcal{E}(1) + \mathcal{E}(2)$, it suffices to bound these two terms separately.

Bound on $\mathcal{E}(1)$. We take the position of u into account and write

$$\mathcal{E}(1) = \sum_{\ell \geq 0} \sum_{u \in \partial^\ell B} \mathbb{P}[0 \xleftrightarrow{B} u] \sum_{\substack{y \in B \\ z \notin B}} \mathbb{P}[u \xleftrightarrow{B} y] p_{yz} \sum_{v \in B} \mathbb{P}[u \leftrightarrow v] \mathbb{P}[z \leftrightarrow v]. \quad (4.2.39)$$

Using (4.2.19) and Proposition 4.A.1, there exists $C_1 = C_1(\mathbf{C}, d) > 0$ such that for all $\ell \geq 0$ and all u and z contributing to (4.2.39),

$$\sum_{v \in \mathbb{Z}^d} \mathbb{P}[u \leftrightarrow v] \mathbb{P}[z \leftrightarrow v] \leq \frac{1}{L^4} \frac{C_1}{(L \vee \ell)^{d-4}}, \quad (4.2.40)$$

where we used that $|z - u| \geq \ell + 1$.

Let $u \in \partial^\ell B$. Recall from (4.1.30) that for $\beta < \beta^*$, one has $(|\Lambda_L| - 1)p \leq 2$. Thus,

$$\sum_{\substack{y \in B \\ z \notin B}} \mathbb{P}[u \xleftrightarrow{B} y] p_{yz} \leq (|\Lambda_L| - 1)p \sum_{k=0}^{L-1} \sum_{y \in \partial^k B} \mathbb{P}[u \xleftrightarrow{B} y] \stackrel{(4.2.26)}{\leq} 2(4d\mathbf{C}^2 + 1), \quad (4.2.41)$$

where in the last line we took into account the possibility of having $y = u$.

Finally, using again (4.2.26),

$$\sum_{u \in \partial^\ell B} \mathbb{P}[0 \xleftrightarrow{B} u] \leq \mathbf{1}_{0 \in \partial^\ell B} + \frac{2d\mathbf{C}^2(\ell + L)}{L^2} \leq \mathbf{1}_{0 \in \partial^\ell B} + \frac{4d\mathbf{C}^2(L \vee \ell)}{L^2}. \quad (4.2.42)$$

Plugging (4.2.40)–(4.2.42) in (4.2.39), we found $C_2 = C_2(\mathbf{C}, d) > 0$ such that,

$$\mathcal{E}(1) \leq \frac{C_2}{L^d} + \frac{C_2}{L^6} \sum_{\ell \geq 0} \frac{1}{(L \vee \ell)^{d-5}}, \quad (4.2.43)$$

where the sum converges when $d > 6$.

Bound on $\mathcal{E}(2)$. Again, we take the position of u into account,

$$\mathcal{E}(2) = \sum_{\ell \geq 0} \sum_{u \in \partial^\ell B} \mathbb{P}[0 \xleftrightarrow{B} u] \sum_{\substack{y, s \in B \\ z, t \notin B \\ yz \neq st}} \mathbb{P}[u \xleftrightarrow{B} y] \mathbb{P}[u \xleftrightarrow{B} s] p_{yz} p_{st} \sum_{v \in \mathbb{Z}^d} \mathbb{P}[z \leftrightarrow v] \mathbb{P}[t \leftrightarrow v]. \quad (4.2.44)$$

Using (4.2.19) and Proposition 4.A.1, there exists $C_3 = C_3(\mathbf{C}, d) > 0$ such that for all $z, t \in \mathbb{Z}^d$,

$$\sum_{v \in \mathbb{Z}^d} \mathbb{P}[z \leftrightarrow v] \mathbb{P}[t \leftrightarrow v] \leq C_3 \left(\mathbb{1}_{z=t} + \mathbb{1}_{z \neq t} \frac{1}{L^4} \frac{1}{(L \vee |z-t|)^{d-4}} \right). \quad (4.2.45)$$

We plug (4.2.45) in (4.2.44) and consider the two contributions coming from $z = t$ and $z \neq t$.

In the case $z \neq t$, we first notice that

$$\begin{aligned} & \frac{1}{L^4} \sum_{\ell \geq 0} \sum_{u \in \partial^\ell B} \mathbb{P}[0 \xleftrightarrow{B} u] \sum_{\substack{y, s \in B \\ z, t \notin B \\ |z-t| \geq \ell+1}} \mathbb{P}[u \xleftrightarrow{B} y] \mathbb{P}[u \xleftrightarrow{B} s] \frac{p_{yz} p_{st}}{(L \vee |z-t|)^{d-4}} \\ & \stackrel{(4.2.41)}{\leq} \frac{1}{L^4} \sum_{\ell \geq 0} \frac{[2(4d\mathbf{C}^2 + 1)]^2}{(L \vee \ell)^{d-4}} \sum_{u \in \partial^\ell B} \mathbb{P}[0 \xleftrightarrow{B} u] \\ & \stackrel{(4.2.42)}{\leq} \frac{C_4}{L^d} + \frac{C_4}{L^6} \sum_{\ell \geq 0} \frac{1}{(L \vee \ell)^{d-5}}, \end{aligned} \quad (4.2.46)$$

where $C_4 = C_4(\mathbf{C}, d) > 0$, and where the sum (4.2.46) is finite when $d > 6$. Then, by (4.2.25) we find that for $u \in \partial^\ell B$ and $w \in \cup_{k=0}^{L-1} \partial^k B$ with $u \neq w$,

$$\mathbb{P}[u \xleftrightarrow{B} w] \leq \frac{2\mathbf{C}^2}{L^d} \left(\frac{L}{(\ell - L) \vee L} \right)^{d-1}. \quad (4.2.47)$$

Moreover, there exists $C_5 = C_5(d) > 0$ such that for fixed y, z as above,

$$\sum_{\substack{t \in \Lambda_\ell(z) \setminus \{z\} \cap B^c \\ s \in B}} p_{st} \frac{1}{(L \vee |z-t|)^{d-4}} \leq 2(2d) \sum_{t \in \Lambda_\ell \setminus \{0\} \cap [-L, L] \times [-\ell, \ell]^{d-1}} \frac{1}{(L \vee |t|)^{d-4}} \quad (4.2.48)$$

$$\leq C_5(L \cdot \ell^3), \quad (4.2.49)$$

where we used again that $p(|\Lambda_L| - 1) \leq 2$. Combining (4.2.47) and (4.2.49), (4.2.45), and using (4.2.26) once again, we get

$$\begin{aligned} & \frac{1}{L^4} \sum_{\ell \geq 0} \sum_{u \in \partial^\ell B} \mathbb{P}[0 \xleftrightarrow{B} u] \sum_{\substack{y, s \in B \\ z, t \notin B \\ |z-t| \leq \ell}} \mathbb{P}[u \xleftrightarrow{B} y] \mathbb{P}[u \xleftrightarrow{B} s] \frac{p_{yz} p_{st}}{(L \vee |z-t|)^{d-4}} \\ & \leq \frac{C_6}{L^d} + \frac{1}{L^4} \frac{2\mathbf{C}^2 C_5}{L^d} 2(4d\mathbf{C}^2 + 1) \sum_{\ell \geq 0} \left(\frac{L}{(\ell - L) \vee L} \right)^{d-1} L \ell^3 \sum_{u \in \partial^\ell B} \mathbb{P}[0 \xleftrightarrow{B} u] \\ & \leq \frac{C_7}{L^d}, \end{aligned} \quad (4.2.50)$$

where $C_6, C_7 > 0$ only depend on $\mathbf{C}$ and d , and where the first term on the second line is due to the case $u = y = s$.

In the case $z = t$, we must have $y \neq s$ and therefore $y \neq u$ or $s \neq u$. We get,

$$\begin{aligned}
& \sum_{\ell \geq 0} \sum_{u \in \partial^\ell B} \mathbb{P}[0 \xleftrightarrow{B} u] \sum_{\substack{y, s \in B \\ y \neq s \\ z \notin B}} \mathbb{P}[u \xleftrightarrow{B} y] \mathbb{P}[u \xleftrightarrow{B} s] p_{yz} p_{sz} \\
& \stackrel{(4.2.47)}{\leq} 2p(|\Lambda_L| - 1) \frac{2\mathbf{C}^2}{L^d} \sum_{\ell \geq 0} \left(\frac{L}{L \vee (\ell - L)} \right)^{d-1} \sum_{u \in \partial^\ell B} \mathbb{P}[0 \xleftrightarrow{B} u] \sum_{\substack{y \in B \\ z \notin B}} \mathbb{P}[u \xleftrightarrow{B} y] p_{yz} \\
& \stackrel{(4.2.41)}{\leq} \frac{16\mathbf{C}^2(4d\mathbf{C}^2 + 1)}{L^d} \sum_{\ell \geq 0} \left(\frac{L}{L \vee (\ell - L)} \right)^{d-1} \sum_{u \in \partial^\ell B} \mathbb{P}[0 \xleftrightarrow{B} u] \\
& \stackrel{(4.2.42)}{\leq} \frac{C_8}{L^d} \sum_{\ell \geq 0} \left(\frac{L}{L \vee (\ell - L)} \right)^{d-1} \left(\delta_n(\ell) + \frac{\mathbf{C}^2(\ell + L)}{L^2} \right) \leq \frac{C_9}{L^d}, \tag{4.2.51}
\end{aligned}$$

where $n \geq 0$ is such that $0 \in \partial^n B$, where we used again that $p(|\Lambda_L| - 1) \leq 2$, and where $C_8, C_9 > 0$ only depend on $\mathbf{C}$ and d .

Combining (4.2.51), (4.2.46), and (4.2.50) gives

$$\mathcal{E}(2) \leq \frac{C_{10}}{L^d} \tag{4.2.52}$$

for some $C_{10} = C_{10}(\mathbf{C}, d) > 0$. □

To conclude this section, one last estimate (whose proof is postponed to the appendix) is derived similarly as Proposition 4.2.2. The interest of this result may be unclear for now but will become much more explicit in Section 4.3.

Lemma 4.2.5. *Fix $d > 6$, $\mathbf{C} > 1$, and $L \geq 1$. There exists $D = D(\mathbf{C}, d) > 0$ such that the following holds. For every $\beta < \beta^*(\mathbf{C}, L)$, every $n > 12L$, and every $x \in \partial \mathbb{H}_n$,*

$$\max_{w \in \Lambda_{n/2}} \max_{\substack{B \in \mathcal{B} \\ B \subset \Lambda_{n/2} \\ B \ni w}} E_\beta(B, \mathbb{H}_n, w, x) \leq \frac{D}{L^{d-2}} \cdot \frac{\mathbf{C}}{Ln^{d-1}}. \tag{4.2.53}$$

4.3 Proof of Theorem 4.1.1

In this section, we prove Theorem 4.1.1. As explained in Section 4.1.3, we will show that for $\mathbf{C}$ and L large enough, one has $\beta^*(\mathbf{C}, L) = \beta_c$. We begin with an improvement of the ℓ^1 bound.

4.3.1 Improving the ℓ^1 bound

The main goal of this section is the following proposition.

Proposition 4.3.1 (Improvement of the ℓ^1 bound). *Fix $d > 6$. There exists $\mathbf{C}_1 = \mathbf{C}_1(d) > 0$ such that the following holds true: for every $\mathbf{C} > \mathbf{C}_1$, there exists $L_1 = L_1(\mathbf{C}, d) > 0$ such that for every $L \geq L_1$ and every $\beta < \beta^*(\mathbf{C}, L)$,*

$$\psi_\beta(\mathbb{H}_n) \leq \frac{128e^2}{L} \quad \forall n \geq 0. \quad (4.3.1)$$

Note that for $\mathbf{C} > 128e^2$, the previous bound is an improvement on the a priori bound (ℓ_β^1) provided by the assumption $\beta < \beta^*$. To prove the proposition, we start by relating $\psi_\beta(\mathbb{H}_n)$ to $\varphi_\beta(\mathbb{H}_n)$.

Lemma 4.3.2 (Bounding $\psi_\beta(\mathbb{H}_n)$ in terms of $\varphi_\beta(\mathbb{H}_n)$). *Let $d > 6$, $\mathbf{C} > 1$, and $L \geq 1$. For all $\beta < \beta^*(\mathbf{C}, L)$,*

$$\psi_\beta(\mathbb{H}_n) \leq \frac{64e^2}{L} \varphi_{\beta \vee \beta_0}(\mathbb{H}_n) \quad \forall n \geq 0, \quad (4.3.2)$$

where we recall that β_0 was defined in (4.1.23).

Before proving this lemma, let us explain how it implies the proposition. Lemma 4.3.2 suggests to find a bound on $\varphi_\beta(\mathbb{H}_n)$ for $n \geq 0$. The following proposition is the crucial step of our strategy: from Proposition 4.2.2, we obtain a bound on $\varphi_\beta(B)$ for $B \in \mathcal{B}$ that involves the range L . For large L , this bound will provide an improvement on (ℓ_β^1) thanks to Lemma 4.3.2.

Proposition 4.3.3. *Fix $d > 6$, $\mathbf{C} > 1$, and $L \geq 1$. There exists $K = K(\mathbf{C}, d) > 0$ such that for every $\beta \leq \beta^*(\mathbf{C}, L)$ and every $B \in \mathcal{B}$,*

$$\varphi_\beta(B) \leq 1 + \frac{K}{L^d}. \quad (4.3.3)$$

Proof. Thanks to the monotone convergence theorem, it suffices to prove the result for $\beta < \beta^*(\mathbf{C}, L)$. Fix $\beta < \beta^*(\mathbf{C}, L)$ and $B \in \mathcal{B}$. Observe that

$$\sum_{x \in \mathbb{Z}^d} E_\beta(B, \mathbb{Z}^d, 0, x) = \chi(\beta) E_\beta(B), \quad (4.3.4)$$

where $\chi(\beta) := \sum_{x \in \mathbb{Z}^d} \mathbb{P}_\beta[0 \leftrightarrow x] < \infty$ because $\beta < \beta_c$ [190, 7] (see also [84]). Hence, summing (4.2.1) (with $S = B$ and $\Lambda = \mathbb{Z}^d$) over every $x \in \mathbb{Z}^d$, and using that $\mathbb{P}_\beta[0 \xrightarrow{B} x] \geq 0$, gives

$$\chi(\beta) \geq \varphi_\beta(B) \chi(\beta) - E_\beta(B) \chi(\beta). \quad (4.3.5)$$

The result follows by dividing by $\chi(\beta)$ and using Proposition 4.2.2. $\square$

Remark 4.3.4. It would be interesting to prove that such a bound holds for every finite set B containing 0 and not only for $B \in \mathcal{B}$.

We are now in a position to prove the main result of this section.

Proof of Proposition 4.3.1. Let $\mathbf{C} > \mathbf{C}_{\text{RW}}$ so that for every $L \geq 1$, one has $\beta^*(\mathbf{C}, L) \geq \beta_0$. Fix L_1 such that $\frac{K}{L_1^d} \leq 1$, where K is provided by Proposition 4.3.3. Fix $L \geq L_1$ and $n \geq 0$. Combining Lemma 4.3.2 and Proposition 4.3.3 gives

$$\psi_\beta(\mathbb{H}_n) \leq \frac{64e^2}{L} \varphi_{\beta \vee \beta_0}(\mathbb{H}_n) \leq \frac{64e^2}{L} \varphi_{\beta^*(\mathbf{C}, L)}(\mathbb{H}_n) \leq \frac{64e^2}{L} \left(1 + \frac{K}{L^d}\right) \leq \frac{128e^2}{L}. \quad (4.3.6)$$

□

To conclude the proof of Proposition 4.3.1, it remains to prove Lemma 4.3.2.

Proof of Lemma 4.3.2. By monotonicity, it suffices to prove the result for $\beta_0 \leq \beta < \beta^*$. Fix $n \geq 0$ and $\beta_0 \leq \beta < \beta^*$. Drop β from the notations. Let $S_k := \mathbb{H}_n \setminus \mathbb{H}_{n-k-1}$. If $x \in (\partial \mathbb{H}_n)^*$, Lemma 4.1.5 with $S = \{x\}$ and $\Lambda = \mathbb{H}_n$ gives

$$\mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} x] \leq \sum_{y \in S_L} p_{xy} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} y]. \quad (4.3.7)$$

Summing (4.3.7) over $x \in (\partial \mathbb{H}_n)^*$ implies

$$\psi(\mathbb{H}_n) \leq \sum_{\substack{x \in (\partial \mathbb{H}_n)^* \\ y \in S_L}} p_{xy} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} y] \leq \frac{p|\Lambda_L|}{2L+1} \sum_{y \in S_L} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} y], \quad (4.3.8)$$

Our aim is now to compare $p|\Lambda_L| \sum_{y \in S_L} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} y]$ to $\varphi_\beta(\mathbb{H}_n)$. Such a comparison is not completely straightforward since $|\{z \notin \mathbb{H}_n : |z - y| \leq L\}|$ can be of much smaller order than $|\Lambda_L|$ for y in $S_L \setminus S_{L/2}$. The idea will be to argue that for $y \in \mathbb{H}_n$, if the event $\{0 \xleftrightarrow{\mathbb{H}_n} y\}$ occurs, then y typically connects to a point z in $S_{L/2}$. This will allow to bound $\sum_{y \in S_L} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} y]$ in terms of $\sum_{z \in S_{L/2}} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} z]$ and to conclude.

For $y \in S_L$, let

$$A_y := S_{L/2} \cap \Lambda_L(y) \setminus \{z \in S_{L/2} : z \xleftrightarrow{\mathbb{H}_n} y\} \quad (4.3.9)$$

be the set of points in the slab $S_{L/2}$ which are within a distance L of y , but not connected to z in $\mathbb{H}_n$, see Figure 4.3. Observe that

$$\frac{1}{8} |\Lambda_L| \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} y, |A_y| \geq \frac{1}{8} |\Lambda_L|] \leq \mathbb{E}[\mathbf{1}\{0 \xleftrightarrow{\mathbb{H}_n} y\} |A_y|] = \sum_{z \in S_{L/2} \cap \Lambda_L(y)} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} y, z \in A_y]. \quad (4.3.10)$$

Moreover, by choosing $z \in A_y$ (note that z is not in the cluster of y in $\mathbb{H}_n$) and opening the edge yz while closing all other edges emanating from z , we get that for $z \in S_{L/2} \cap \Lambda_L(y)$,

$$p \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} y, z \in A_y] \leq e^\beta \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} y, yz \text{ open}, E_z], \quad (4.3.11)$$

where E_z be the event that there is *exactly one* open edge emanating from z (we used that $\mathbb{P}[zu \text{ closed}, \forall u \neq y] \geq e^{-\beta c_L(|\Lambda_L| - 2)} \geq e^{-\beta}$). As a consequence,

$$\sum_{y \in S_L} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} y] = \sum_{y \in S_L} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} y, |A_y| < \frac{1}{8} |\Lambda_L|] + \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} y, |A_y| \geq \frac{1}{8} |\Lambda_L|] \quad (4.3.12)$$

$$\leq \sum_{y \in S_L} \sum_{z \in S_{L/2} \cap \Lambda_L(y)} \frac{8}{|\Lambda_L|} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} y, z] + \frac{8e^\beta}{p|\Lambda_L|} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} y, yz \text{ open}, E_z], \quad (4.3.13)$$

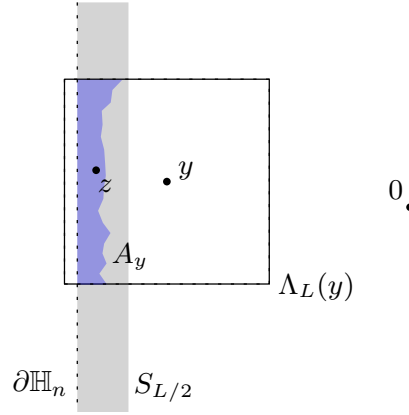


Figure 4.3: An illustration of the proof of Lemma 4.3.2. The grey region represents $S_{L/2}$. The blue region is the (random) set A_y . It is disconnected from y in $\mathbb{H}_n$.

where on the second line we used (4.3.10)–(4.3.11), and that

$$\mathbb{1}\{|A_y| < \frac{1}{8}|\Lambda_L|\} = \mathbb{1}\{|S_{L/2} \cap \Lambda_L(y) \setminus A_y| > \frac{1}{8}|\Lambda_L|\} \leq \frac{8}{|\Lambda_L|} \sum_{z \in S_{L/2} \cap \Lambda_L(y)} \mathbb{1}\{z \xleftrightarrow{\mathbb{H}_n} y\}. \quad (4.3.14)$$

Thus, (4.3.13) yields

$$\sum_{y \in S_L} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} y] \leq \sum_{z \in S_{L/2}} \sum_{y \in S_L \cap \Lambda_L(z)} \frac{8}{|\Lambda_L|} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} z] + \frac{8e^\beta}{p|\Lambda_L|} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} z, yz \text{ open}, E_z] \quad (4.3.15)$$

$$\leq 8 \left(1 + \frac{e^\beta}{p|\Lambda_L|}\right) \sum_{z \in S_{L/2}} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} z]. \quad (4.3.16)$$

To conclude, we notice that

$$p \frac{1}{8} |\Lambda_L| \sum_{z \in S_{L/2}} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} z] \leq \sum_{\substack{z \in S_{L/2} \\ t \notin \mathbb{H}_n}} p_{zt} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} z] \leq \varphi_\beta(\mathbb{H}_n), \quad (4.3.17)$$

and

$$1 \leq |\Lambda_L|p \leq \beta^* \frac{|\Lambda_L|}{|\Lambda_L| - 1} \leq 4 \quad (4.3.18)$$

when $\beta_0 \leq \beta \leq \beta^* \leq 2$. These observations, combined with (4.3.8) and (4.3.16), imply that

$$\psi(\mathbb{H}_n) \leq \frac{8(p|\Lambda_L| + e^2)}{2L + 1} \sum_{z \in S_{L/2}} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} z] \leq \frac{8(4 + e^2)}{2L + 1} \frac{8}{p|\Lambda_L|} \varphi(\mathbb{H}_n) \leq \frac{64e^2}{L} \varphi(\mathbb{H}_n). \quad (4.3.19)$$

□

A refined version of Proposition 4.3.3. As we conclude this section, we would like to highlight a supplementary result — building on Proposition 4.3.3 — which will be useful in the next section. We begin with some notations. Let $\Lambda_n^+ := \{x \in \Lambda_n : x_1 > 0\}$. Also, define the slabs $H = H(L) := \{v \in \mathbb{Z}^d : |v_1| \leq L\}$ and $H^- = H^-(L) := \{v \in H : v_1 \leq 0\}$.

Lemma 4.3.5. Fix $d > 6$ and $\mathbf{C} > 1$. There exists $L_2 = L_2(\mathbf{C}, d) > 0$ such that for every $L \geq L_2$, every $\beta < \beta^*(\mathbf{C}, L)$, every $n \geq 12L$, and every $v \in \Lambda_k^+$ with $6L \leq k \leq n/2$,

$$\sum_{\substack{y \in \Lambda_n^+ \\ z \notin \Lambda_n^+ \cup H}} \mathbb{P}_\beta[v \xleftrightarrow{\Lambda_n^+} y] p_{yz, \beta} \leq 4 \left(\frac{2k}{n} \right)^c, \quad (4.3.20)$$

where $c = c(d) := \frac{|\log(1 - \frac{1}{4d^2})|}{2 \log 2}$.

Roughly speaking, Lemma 4.3.5 formalises the fact that when a generalised block B is not centred and 0 is close to ∂B , most of the sum in $\varphi_\beta(B)$ comes from edges yz on the side of B that is the closest to 0. The proof is very similar to that of [82, Lemma 2.7], but the presence of jumps of range L requires some extra care.

Proof. Let L_2 be such that $\frac{K}{L_2^2} \leq \frac{1}{2d}$. Fix $L \geq L_2$. Define $(n_\ell)_{\ell \geq 0}$ by $n_0 = n$ and then $n_{\ell+1} = \lfloor (n_\ell - L)/2 \rfloor \vee 0$. We proceed by induction by proving that for every $\ell \geq 0$ and $v \in \Lambda_{n_\ell}^+$,

$$\sum_{\substack{y \in \Lambda_n^+ \\ z \notin \Lambda_n^+ \cup H}} \mathbb{P}_\beta[v \xleftrightarrow{\Lambda_n^+} y] p_{yz, \beta} \leq \left(1 - \frac{1}{4d^2}\right)^\ell \left(1 + \frac{1}{2d}\right). \quad (4.3.21)$$

The case $\ell = 0$ follows from Proposition 4.3.3 and the assumption on L . We now transfer the estimate from ℓ to $\ell + 1$. Fix $v \in \Lambda_{n_{\ell+1}}^+$. Let $B := \Lambda_{v_1-1}(v)$, which is included in $\Lambda_{n_\ell}^+$ and has one of its faces included in H , see Figure 4.4. By symmetry and Proposition 4.3.3,

$$\sum_{\substack{r \in B \\ s \notin B \cup H^-}} \mathbb{P}_\beta[v \xleftrightarrow{B} r] p_{rs, \beta} \leq \frac{2d-1}{2d} \varphi_\beta(B) \leq \left(1 - \frac{1}{2d}\right) \left(1 + \frac{K}{L^d}\right) \leq 1 - \frac{1}{4d^2}. \quad (4.3.22)$$

We deduce from Lemma 4.1.5 and the induction hypothesis that

$$\begin{aligned} \sum_{\substack{y \in \Lambda_n^+ \\ z \notin \Lambda_n^+ \cup H}} \mathbb{P}_\beta[v \xleftrightarrow{\Lambda_n^+} y] p_{yz, \beta} &\leq \sum_{\substack{y \in \Lambda_n^+ \\ z \notin \Lambda_n^+ \cup H}} \left(\sum_{\substack{r \in B \\ s \notin B \cup H^-}} \mathbb{P}_\beta[v \xleftrightarrow{B} r] p_{rs, \beta} \mathbb{P}_\beta[s \xleftrightarrow{\Lambda_n^+} y] \right) p_{yz, \beta} \\ &= \sum_{\substack{r \in B \\ s \notin B \cup H^-}} \mathbb{P}_\beta[v \xleftrightarrow{B} r] \left(\sum_{\substack{y \in \Lambda_n^+ \\ z \notin \Lambda_n^+ \cup H}} \mathbb{P}_\beta[s \xleftrightarrow{\Lambda_n^+} y] p_{yz, \beta} \right) p_{rs, \beta} \\ &\leq \left(1 - \frac{1}{4d^2}\right)^\ell \left(1 + \frac{1}{2d}\right) \sum_{\substack{r \in B \\ s \notin B \cup H^-}} \mathbb{P}_\beta[v \xleftrightarrow{B} r] p_{rs, \beta} \\ &\stackrel{(4.3.22)}{\leq} \left(1 - \frac{1}{4d^2}\right)^{\ell+1} \left(1 + \frac{1}{2d}\right). \end{aligned}$$

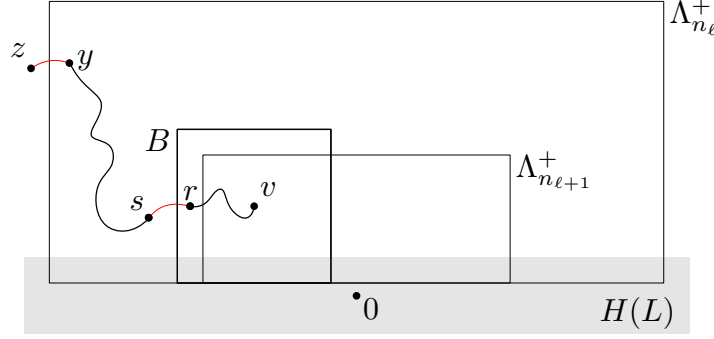


Figure 4.4: An illustration of the proof of Lemma 4.3.5. For convenience, we have rotated the picture by $\pi/2$. The light grey region corresponds to H . The bold black line corresponds to an open self-avoiding path connecting v to y . The red lines correspond to long open edges. The sequence $(n_\ell)_{\ell \geq 0}$ has been constructed in such a way that s remains in Λ_{n_ℓ} .

This concludes the induction. Now, let $k \leq n/2$. Then¹, $k \leq n_{\lfloor m/2 \rfloor}$ where $m := \log_2(\frac{n}{2k})$. As a consequence, by (4.3.21), if $v \in \Lambda_k^+$,

$$\sum_{\substack{y \in \Lambda_n^+ \\ z \notin \Lambda_n^+ \cup H}} \mathbb{P}_\beta[v \xleftrightarrow{\Lambda_n^+} y] p_{yz, \beta} \leq \left(1 - \frac{1}{4d^2}\right)^{(\alpha/2)-1} \left(1 + \frac{1}{2d}\right) \leq 4 \left(\frac{2k}{n}\right)^2, \quad (4.3.23)$$

where $c = \frac{|\log(1 - \frac{1}{4d^2})|}{2 \log 2}$, and where we used that for $d \geq 1$, $(1 - \frac{1}{4d^2})^{-1} \leq 1 + \frac{1}{2d} \leq 2$. $\square$

4.3.2 Improving the ℓ^∞ bound

Proposition 4.3.1 implies an improved ℓ^1 -type bound on the half-space two-point function (when $\mathbf{C}, L$ are large). We will use this improvement to deduce an improved ℓ^∞ bound on the half-space two-point function. The following estimate is a key step of the proof.

Proposition 4.3.6 (Control of the gradient). *Fix $d > 6$ and $\eta > 0$. There exist $\delta = \delta(\eta, d) \in (0, 1/2)$ and $\mathbf{C}_3 = \mathbf{C}_3(\eta, d) > 0$ such that the following holds true: for every $\mathbf{C} > \mathbf{C}_3$, there exists $L_3 = L_3(\eta, \mathbf{C}, d)$ such that*

$$|\mathbb{P}_\beta[u \xleftrightarrow{\mathbb{H}_n} x] - \mathbb{P}_\beta[v \xleftrightarrow{\mathbb{H}_n} x]| \leq \eta \frac{\mathbf{C}}{L^d} \left(\frac{L}{L \vee n}\right)^{d-1} \quad (4.3.24)$$

for every $L \geq L_3$, $\beta < \beta^*(\mathbf{C}, L)$, $n \geq 0$, $x \in \partial \mathbb{H}_n$, and $u, v \in \Lambda_{\lfloor \delta n \rfloor \vee L} \setminus \{x\}$.

With this regularity estimate, one may improve (ℓ_β^∞) .

¹We notice that for $\ell \geq 0$, one has $n_\ell \geq \frac{n}{2^\ell} - \sum_{k=0}^{\ell-1} \frac{L+2}{2^k} \geq \frac{n}{2^\ell} - 6L$. Hence, for $\ell = \lfloor m/2 \rfloor$ with $m = \log_2(\frac{n}{2k})$, we get

$$n_\ell \geq \frac{n}{2^{m/2}} - 6L = \sqrt{2k}\sqrt{n} - 6L \geq 2k - 6L \geq k,$$

where we used that $n \geq 2k$ and $k \geq 6L$.

Corollary 4.3.7 (Improvement of the ℓ^∞ bound). *Fix $d > 6$. There exists $\mathbf{C}_4 = \mathbf{C}_4(d) > 0$ such that the following holds. For every $\mathbf{C} > \mathbf{C}_4$, there exists $L_4 = L_4(\mathbf{C}, d)$ such that for every $L \geq L_4$ and every $\beta < \beta^*(\mathbf{C}, L)$,*

$$\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x] \leq \frac{\mathbf{C}}{2L^d} \left(\frac{L}{L \vee n} \right)^{d-1} \quad \forall n \geq 0, \forall x \in (\partial\mathbb{H}_n)^*. \quad (4.3.25)$$

Proof. Fix $d > 6$. Set $\eta = \frac{1}{4}$. Let $\mathbf{C}_1$ be given by Proposition 4.3.1 and $\delta = \delta(\eta, d)$, $\mathbf{C}_3 = \mathbf{C}_3(\eta, d)$ be given by Proposition 4.3.6. Let $\mathbf{C} > \mathbf{C}_1 \vee \mathbf{C}_3$. It will be taken larger below. Let also $L \geq L_3$ where $L_3 = L_3(\eta, \mathbf{C}, d)$ is given by Proposition 4.3.6. Let $n \geq 0$ and $x \in (\partial\mathbb{H}_n)^*$. Set $V_n := \{y \in \Lambda_{[\delta n] \vee L} \setminus \{x\} : y_1 = 0\}$. Proposition 4.3.6 gives that for every $\beta < \beta^*(\mathbf{C}, L)$ and every $y \in V_n$,

$$\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x - y] = \mathbb{P}_\beta[y \xleftrightarrow{\mathbb{H}_n} x] \geq \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x] - \eta \frac{\mathbf{C}}{L^d} \left(\frac{L}{L \vee n} \right)^{d-1}.$$

Averaging over $y \in V_n$ gives

$$\frac{\psi(\mathbb{H}_n)}{|V_n|} \geq \frac{1}{|V_n|} \sum_{y \in V_n} \mathbb{P}_\beta[y \xleftrightarrow{\mathbb{H}_n} x] \geq \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x] - \eta \frac{\mathbf{C}}{L^d} \left(\frac{L}{L \vee n} \right)^{d-1}. \quad (4.3.26)$$

Proposition 4.3.1 implies that for $L \geq L_1 \vee L_3$ (where $L_1 = L_1(\mathbf{C}, d)$ is provided by Proposition 4.3.1),

$$\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x] \leq \frac{1}{|V_n|} \frac{128e^2}{L} + \eta \frac{\mathbf{C}}{L^d} \left(\frac{L}{L \vee n} \right)^{d-1}. \quad (4.3.27)$$

Since $\eta = \frac{1}{4}$, choose $\mathbf{C} > \mathbf{C}_4 := (4 \frac{128e^2}{\delta^{d-1}}) \vee \mathbf{C}_1 \vee \mathbf{C}_3$ and $L_4 := L_1 \vee L_3$ to conclude. $\square$

The remainder of this section is devoted to the proof of Proposition 4.3.6. The following lemma is an iterated version of Lemmata 4.1.5 and 4.2.1.

Lemma 4.3.8. *Fix $d > 6$, $\mathbf{C} > 1$, and $L \geq 1$. Then, for every $\beta < \beta^*(\mathbf{C}, L)$, every $n \geq 0$, every $x \in \partial\mathbb{H}_n$, every $u \in \mathbb{H}_n$, every $T \geq 1$,*

$$\begin{aligned} 0 &\leq \left(\sum_{t=0}^T \varphi^t \mathbb{P}_u^{\text{RW}}[X_t^u = x, \tau_n^u > t] + \varphi^T \mathbb{E}_u^{\text{RW}} \left[\mathbb{P}_\beta[X_T^u \xleftrightarrow{\mathbb{H}_n} x] \mathbf{1}_{\tau_n^u > T, X_T^u \neq x} \right] \right) - \mathbb{P}_\beta[u \xleftrightarrow{\mathbb{H}_n} x] \\ &\leq \left(\sum_{t=0}^{T-1} \varphi^t \right) \max_{w \in \Lambda_{(T-1)L}(u) \cap \mathbb{H}_n} E_\beta(\{w\}, \mathbb{H}_n, w, x), \end{aligned} \quad (4.3.28)$$

where $\tau_n^u := \inf\{k \geq 1 : X_k^u \notin \mathbb{H}_n\}$ and $\varphi := \varphi_\beta(\{0\})$.

Proof. One application of Lemmata 4.1.5 and 4.2.1 with $S = \{u\}$ and $\Lambda = \mathbb{H}_n$ gives,

$$0 \leq \left(\mathbf{1}_{u=x} + \varphi \mathbb{E}_u^{\text{RW}} \left[\mathbb{P}_\beta[X_1^u \xleftrightarrow{\mathbb{H}_n} x] \mathbf{1}_{\tau_n^u > 1} \right] \right) - \mathbb{P}_\beta[u \xleftrightarrow{\mathbb{H}_n} x] \leq E_\beta(\{u\}, \mathbb{H}_n, u, x). \quad (4.3.29)$$

The term in the middle of (4.3.29) can be rewritten

$$\left(\sum_{t=0}^1 \varphi^t \mathbb{P}_u^{\text{RW}}[X_t^u = x, \tau_n^u > t] + \varphi \mathbb{E}_u^{\text{RW}} \left[\mathbb{P}_\beta[X_1^u \xleftrightarrow{\mathbb{H}_n} x] \mathbf{1}_{\tau_n^u > 1, X_1^u \neq x} \right] \right) - \mathbb{P}_\beta[u \xleftrightarrow{\mathbb{H}_n} x]. \quad (4.3.30)$$

The proof follows by iterating the application of the two lemmata (successively with $S = \{X_1^u\}, \{X_2^u\}, \dots, \{X_{T-1}^u\}$ and $\Lambda = \mathbb{H}_n$) and by noticing that, for $k \leq T-1$,

$$\mathbb{E}_u^{\text{RW}}[E_\beta(\{X_k^u\}, \mathbb{H}_n, X_k^u, x)] \leq \max_{w \in \Lambda_{kL}(u) \cap \mathbb{H}_n} E_\beta(\{w\}, \mathbb{H}_n, w, x) \quad (4.3.31)$$

$$\leq \max_{w \in \Lambda_{(T-1)L}(u) \cap \mathbb{H}_n} E_\beta(\{w\}, \mathbb{H}_n, w, x). \quad (4.3.32)$$

□

Proof of Proposition 4.3.6. Fix $\eta > 0$. We let $\delta = \delta(\eta, d) > 0$ to be chosen later (think of it as small). Also, let $\mathbf{C} > 1$ and $L \geq 1$ to be taken large enough. Let $\beta < \beta^*(\mathbf{C}, L)$, $n \geq 0$, and $x \in \partial \mathbb{H}_n$. We distinguish two cases: $\lfloor \delta n \rfloor \vee L = \lfloor \delta n \rfloor$, or $\lfloor \delta n \rfloor \vee L = L$. The first case fixes how small δ has to be, while the second case fixes how large $\mathbf{C}$ and L have to be. For simplicity, below, we omit integer rounding.

Case $n \geq \delta^{-1}L$. We split the proof in two depending on whether the distance between u and v is smaller than $2L$ or not.

Assume first that $u, v \in \Lambda_{n/3}$ are such that $|u - v| \leq 2L$. Let $T \geq 1$ to be fixed and assume that $\delta \leq \frac{1}{6T}$, so that $n \geq 6TL$. We set $\varphi := \varphi_\beta(\{0\})$. By Lemma 4.3.8,

$$\mathbb{P}_\beta[u \xleftrightarrow{\mathbb{H}_n} x] \leq \varphi^T \mathbb{E}_u^{\text{RW}}[\mathbb{P}_\beta[X_T^u \xleftrightarrow{\mathbb{H}_n} x]], \quad (4.3.33)$$

$$\mathbb{P}_\beta[v \xleftrightarrow{\mathbb{H}_n} x] \geq \varphi^T \mathbb{E}_v^{\text{RW}}[\mathbb{P}_\beta[X_T^v \xleftrightarrow{\mathbb{H}_n} x]] - \left(\sum_{t=0}^{T-1} \varphi^t \right) \max_{w \in \Lambda_{n/3+TL}} E_\beta(\{w\}, \mathbb{H}_n, w, x), \quad (4.3.34)$$

where we used that since $n \geq 6TL$, one has (if $w \in \{u, v\}$) $\mathbb{1}_{\tau_n^w > T} = 1$ and also $\mathbb{1}_{X_k^w \neq x} = 1$ for all $k \leq T$. A random-walk estimate² implies the existence of $T = T(\eta, d)$ large enough such that X_T^u and X_T^v can be coupled to coincide with probability larger than $1 - \eta/2^{d+2}$. This implies

$$\begin{aligned} \mathbb{E}_u^{\text{RW}}[\mathbb{P}_\beta[X_T^u \xleftrightarrow{\mathbb{H}_n} x]] - \mathbb{E}_v^{\text{RW}}[\mathbb{P}_\beta[X_T^v \xleftrightarrow{\mathbb{H}_n} x]] &\leq \frac{\eta}{2^{d+2}} \max \left\{ \mathbb{P}_\beta[w \xleftrightarrow{\mathbb{H}_n} x] : w \in \Lambda_{n/3+TL} \right\} \\ &\leq \frac{\eta}{2^{d+2}} \max \left\{ \mathbb{P}_\beta[w \xleftrightarrow{\mathbb{H}_n} x] : w \in \Lambda_{n/2} \right\} \\ &\stackrel{(\ell_\beta^\infty)}{\leq} \frac{\eta}{8} \frac{\mathbf{C}}{Ln^{d-1}}. \end{aligned} \quad (4.3.35)$$

Lemma 4.2.5 gives

$$\max_{w \in \Lambda_{n/3+TL}} E_\beta(\{w\}, \mathbb{H}_n, w, x) \leq \frac{D}{L^{d-2}} \frac{\mathbf{C}}{Ln^{d-1}}. \quad (4.3.36)$$

²Let us sketch a proof. Let $\kappa > 0$ to be chosen small enough in terms of η . The idea is to begin by coupling the two random walks X^u and X^v in such a way that they get at a distance $\leq 1 \vee \kappa L$ after $t = t(\eta, \kappa, d)$ steps with probability at least $\eta/2^{d+3}$. This follows from classical results (see for instance [243] and references therein) since $|u - v| \leq 2L$. Conditional on this event, the uniform laws on $\Lambda_L(X_t^u) \setminus \{X_t^u\}$ and $\Lambda_L(X_t^v) \setminus \{X_t^v\}$ are at distance (in total variation) of order $O(\kappa)$. Considering the best coupling of these two laws permits to construct X_{t+1}^u and X_{t+1}^v in such a way that $\mathbb{P}[X_{t+1}^u \neq X_{t+1}^v] \leq O(\kappa) + \eta/2^{d+3}$. It remains to choose κ small enough so that $O(\kappa) \leq \eta/2^{d+3}$.

Furthermore, by Proposition 4.3.3, $\varphi \leq 1 + \frac{K}{L^d}$. Hence, we may choose $L_3 = L_3(\eta, T, \mathbf{C}, d)$ large enough such that for $L \geq L_3$, $\frac{D}{L^{d-2}} \sum_{t=0}^{T-1} \varphi^t \leq \eta/4$ and $\varphi^T \leq 2$. Plugging (4.3.35) and (4.3.36) in the difference of (4.3.33) and (4.3.34) gives

$$\mathbb{P}_\beta[u \xleftrightarrow{\mathbb{H}_n} x] - \mathbb{P}_\beta[v \xleftrightarrow{\mathbb{H}_n} x] \leq \frac{\eta}{2} \frac{\mathbf{C}}{Ln^{d-1}}. \quad (4.3.37)$$

We turn to the general case $u, v \in \Lambda_{\delta n}$. We prove the result when u and v differ by one coordinate only and assume that the difference of the coordinate is even as the latter assumption can be relaxed by using the previous estimate when u and v are at distance 1 (which is less than $2L$), and the former by summing increments over coordinates. By rotating and translating (here we will use that $\delta n + n/3 \leq n/2$), we may therefore consider $u = k\mathbf{e}_1$ and $v = -k\mathbf{e}_1$ with $k \leq \delta n$.

Recall that for $\ell \geq 1$, $\Lambda_\ell^+ = \{x \in \Lambda_\ell : x_1 > 0\}$. Consider the sets $B^+ := \Lambda_{n/3}^+$ and $B^- := -\Lambda_{n/3}^+$. By Lemmata 4.1.5 and 4.2.1,

$$\mathbb{P}_\beta[u \xleftrightarrow{\mathbb{H}_n} x] \leq \sum_{\substack{y \in B^+ \\ z \notin B^+}} \mathbb{P}_\beta[u \xleftrightarrow{B^+} y] p_{yz, \beta} \mathbb{P}_\beta[z \xleftrightarrow{\mathbb{H}_n} x], \quad (4.3.38)$$

$$\mathbb{P}_\beta[v \xleftrightarrow{\mathbb{H}_n} x] \geq \sum_{\substack{y \in B^- \\ z \notin B^-}} \mathbb{P}_\beta[v \xleftrightarrow{B^-} y] p_{yz, \beta} \mathbb{P}_\beta[z \xleftrightarrow{\mathbb{H}_n} x] - E_\beta(B^-, \mathbb{H}_n, v, x). \quad (4.3.39)$$

Now, when $z \in H(L) = \{v \in \mathbb{Z}^d : |v_1| \leq L\}$, we may associate every pair (y, z) in the sum in (4.3.38) with the pair (y', z') symmetric with respect to the hyperplane $\{u \in \mathbb{Z}^d : u_1 = 0\}$ in the sum in (4.3.39), see Figure 4.5. By doing so, we notice that z and z' are within a distance $2L$ of each other. Hence, if $L \geq L_3$, for such pairs (y, z) and (y', z') , by (4.3.37),

$$|\mathbb{P}_\beta[z \xleftrightarrow{\mathbb{H}_n} x] - \mathbb{P}_\beta[z' \xleftrightarrow{\mathbb{H}_n} x]| \leq \frac{\eta}{2} \frac{\mathbf{C}}{Ln^{d-1}} \quad (4.3.40)$$

Plugging this estimate in the difference of (4.3.38) and (4.3.39), and then invoking Lemma 4.3.5 with $k = \delta n$ and $n/3$ (to the cost of potentially increasing L again), gives³

$$\mathbb{P}_\beta[u \xleftrightarrow{\mathbb{H}_n} x] - \mathbb{P}_\beta[v \xleftrightarrow{\mathbb{H}_n} x] \quad (4.3.41)$$

$$\begin{aligned} &\leq \frac{\eta}{2} \frac{\mathbf{C}}{Ln^{d-1}} \varphi_\beta(B^+) + \frac{\mathbf{C}}{L(n/2)^{d-1}} \sum_{\substack{y \in B^+ \\ z \notin B^+ \cup H}} \mathbb{P}_\beta[u \xleftrightarrow{B^+} y] p_{yz, \beta} + E_\beta(B^-, \Lambda, v, x) \\ &\leq \frac{\mathbf{C}}{Ln^{d-1}} \left[\frac{\eta}{2} \left(1 + \frac{K}{L^d}\right) + 2^{d-1} 4(6\delta)^c + \frac{D}{L^{d-2}} \right], \end{aligned} \quad (4.3.42)$$

where we used (ℓ_β^∞) in the second line, and Proposition 4.3.3 and Lemma 4.2.5 in the third line. The proof follows by choosing δ small enough and L_3 large enough so that

$$\frac{\eta}{2} \left(1 + \frac{K}{L_3^d}\right) + 2^{d-1} 4(6\delta)^c + \frac{D}{L_3^{d-2}} \leq \eta. \quad (4.3.43)$$

³Here, we used that if $d(z, B^- \cup B^+) \leq L (\leq \frac{n}{6T} \leq \frac{n}{6})$, then

$$\mathbb{P}_\beta[z \xleftrightarrow{\mathbb{H}_n} x] \leq \max\{\mathbb{P}_\beta[w \xleftrightarrow{\mathbb{H}_n} x] : w \in \Lambda_{n/3+L}\} \leq \max\{\mathbb{P}_\beta[w \xleftrightarrow{\mathbb{H}_n} x] : w \in \Lambda_{n/2}\} \stackrel{(\ell_\beta^\infty)}{\leq} \frac{\mathbf{C}}{L(n/2)^{d-1}}.$$

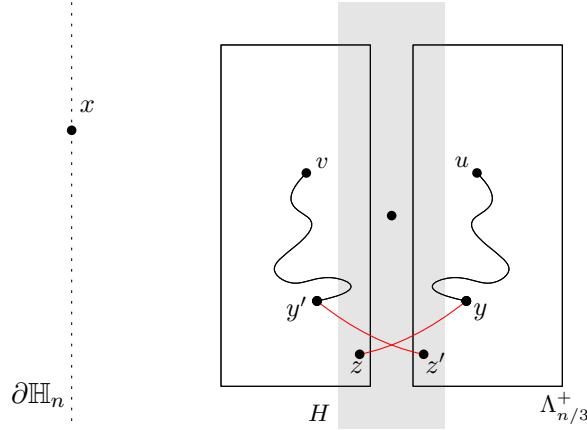


Figure 4.5: An illustration of the pairing used in the proof of Proposition 4.3.6. The red path corresponds to a long open edge “jumping” outside $\Lambda_{n/3}^+$ (resp. $-\Lambda_{n/3}^+$). The grey region corresponds to H . Since u (resp. v) is close to $\{u \in \mathbb{Z}^d : u_1 = 0\}$, a connection from u (resp. v) to x will likely enter H if it exits $\Lambda_{n/3}^+$ (resp. $-\Lambda_{n/3}^+$), as seen in Lemma 4.3.5.

Case $n \leq \delta^{-1}L$. The value of δ is now fixed by the previous case. Here, we will prove that both $\mathbb{P}_\beta[u \xleftrightarrow{\mathbb{H}_n} x]$ and $\mathbb{P}_\beta[v \xleftrightarrow{\mathbb{H}_n} x]$ can be bounded efficiently by a random walk estimate (they will end up being both smaller than $\frac{1}{2}\eta \frac{\mathbf{C}}{L^d} (L/(L \vee n))^{d-1}$).

Consider $u \in \Lambda_L$ not equal to x . We will choose $\mathbf{C}$ large enough, which will force to take L_3 even larger. Let $T \geq 1$ to be fixed. Using Lemma 4.3.8,

$$\mathbb{P}_\beta[u \xleftrightarrow{\mathbb{H}_n} x] \leq \varphi^T \mathbb{E}_u^{\text{RW}}[\mathcal{M}] + \varphi^T \mathbb{E}_u^{\text{RW}}[\mathbb{P}_\beta[X_T \xleftrightarrow{\mathbb{H}} x] \mathbb{1}_{\tau_n^u > T, X_T^u \neq x}], \quad (4.3.44)$$

where $\mathcal{M} := |\{1 \leq t \leq T \wedge (\tau_n^u - 1) : X_t^u = x\}|$. Classical random walk estimates (see [172]) give the existence of $C_{\text{RW}} = C_{\text{RW}}(\delta, d) > 0$ such that

$$\mathbb{E}_u^{\text{RW}}[\mathcal{M}] \leq \frac{C_{\text{RW}}}{L^d} \left(\frac{L}{L \vee n} \right)^{d-1}, \quad (4.3.45)$$

$$\mathbb{P}_u^{\text{RW}}[\tau > T] \leq \frac{C_{\text{RW}}}{T^{(d-1)/2}}. \quad (4.3.46)$$

Choose T large enough such that $2C_{\text{RW}}T^{-(d-1)/2} \leq \delta^{d-1}\eta/4$ and L such that $(1+K/L^d)^T \leq 2$. Plugging (4.3.45) and (4.3.46) in (4.3.44) for this choice of T gives

$$\mathbb{P}_\beta[u \xleftrightarrow{\mathbb{H}_n} x] \leq \frac{C_{\text{RW}}}{L^d} \left(\frac{L}{L \vee n} \right)^{d-1} + \frac{\eta}{4} \frac{\mathbf{C}}{L(L/\delta)^{d-1}} \quad (4.3.47)$$

$$\leq \frac{C_{\text{RW}}}{L^d} \left(\frac{L}{L \vee n} \right)^{d-1} + \frac{\eta \mathbf{C}}{4L} \left(\frac{L}{L \vee n} \right)^{d-1}, \quad (4.3.48)$$

where we used (ℓ_β^∞) to argue that $\mathbb{P}_\beta[X_T \xleftrightarrow{\mathbb{H}} x] \mathbb{1}_{\tau_n^u > T, X_T^u \neq x} \leq \frac{\mathbf{C}}{L^d}$. Finally, choosing $\mathbf{C} > 4C_{\text{RW}}/\eta$ gives

$$\mathbb{P}_\beta[u \xleftrightarrow{\mathbb{H}_n} x] \leq \frac{1}{2}\eta \frac{\mathbf{C}}{L^d} \left(\frac{L}{L \vee n} \right)^{d-1} \quad (4.3.49)$$

and concludes the proof. $\square$

4.3.3 Conclusion

We are now in a position to prove the following proposition.

Proposition 4.3.9. *Fix $d > 6$. There exist $C, L_0 > 0$ such that for every $L \geq L_0$,*

$$\beta_c \leq 1 + \frac{C}{L^d}, \quad (4.3.50)$$

$$\varphi_{\beta_c}(B) \leq 1 + \frac{C}{L^d} \quad \forall B \in \mathcal{B}, \quad (4.3.51)$$

$$E_{\beta_c}(B) \leq \frac{C}{L^d} \quad \forall B \in \mathcal{B}, \quad (4.3.52)$$

$$\psi_{\beta_c}(\mathbb{H}_n) \leq \frac{C}{L} \quad \forall n \geq 0, \quad (4.3.53)$$

$$\mathbb{P}_{\beta_c}[0 \leftrightarrow x] \leq \frac{C}{L^d} \left(\frac{L}{L \vee |x|} \right)^{d-2} \quad \forall x \in (\mathbb{Z}^d)^*, \quad (4.3.54)$$

$$\mathbb{P}_{\beta_c}[0 \overset{\mathbb{H}}{\longleftrightarrow} x] \leq \frac{C}{L^d} \left(\frac{L}{L \vee |x_1|} \right)^{d-1} \quad \forall x \in \mathbb{H}^*. \quad (4.3.55)$$

Proof. Let $\mathbf{C} > 1$ and $L \geq 1$ to be fixed. By Proposition 4.3.1 and Corollary 4.3.7, we find that if $\mathbf{C}$ and L are large enough, for every $\beta < \beta^*(\mathbf{C}, L)$,

$$\psi_\beta(\mathbb{H}_n) \leq \frac{\mathbf{C}}{2L} \quad \forall n \geq 0, \quad (4.3.56)$$

$$\mathbb{P}_\beta[0 \overset{\mathbb{H}_n}{\longleftrightarrow} x] \leq \frac{\mathbf{C}}{2L^d} \left(\frac{L}{L \vee n} \right)^{d-1} \quad \forall n \geq 0, \forall x \in (\partial\mathbb{H}_n)^*. \quad (4.3.57)$$

These bounds are still true at $\beta = \beta^*$ thanks to the monotone convergence theorem. We will use (4.3.56)–(4.3.57) to show by contradiction that, for this choice of $\mathbf{C}$ and L , one has $\beta^*(\mathbf{C}, L) = 2 \wedge \beta_c$.

Let us now assume by contradiction that $\beta^* < 2 \wedge \beta_c$. Consider $\beta^{**} \in (\beta^*, 2 \wedge \beta_c)$. Exponential decay of correlations when $\beta < \beta_c$ [190, 7] (see also [84]) implies the existence of $c, C > 0$ (which depend on β^{**}) such that, for every $n \geq 0$ and every $x \in \partial\mathbb{H}_n$,

$$\mathbb{P}_{\beta^{**}}[0 \overset{\mathbb{H}_n}{\longleftrightarrow} x] \leq \mathbb{P}_{\beta^{**}}[0 \leftrightarrow x] \leq Ce^{-c|x|}. \quad (4.3.58)$$

By monotonicity of the quantities of interest in β and (4.3.58), there exists $N = N(\beta^{**}, \mathbf{C}, L)$ such that, for all $\beta \leq \beta^{**}$,

$$\psi_\beta(\mathbb{H}_n) = \sum_{x \in (\partial\mathbb{H}_n)^*} \mathbb{P}_\beta[0 \overset{\mathbb{H}_n}{\longleftrightarrow} x] < \frac{\mathbf{C}}{L} \quad \forall n \geq N, \quad (4.3.59)$$

$$\mathbb{P}_\beta[0 \overset{\mathbb{H}_n}{\longleftrightarrow} x] < \frac{\mathbf{C}}{L^d} \left(\frac{L}{L \vee n} \right)^{d-1} \quad \forall n \geq 0, \forall x \in \partial\mathbb{H}_n \setminus \Lambda_N. \quad (4.3.60)$$

Now, the continuity (below β_c) of the maps $\beta \mapsto \max\{\varphi_\beta(\mathbb{H}_n) : 0 \leq n \leq N-1\}$ and $\beta \mapsto \max\{\mathbb{P}_\beta[0 \overset{\mathbb{H}_n}{\longleftrightarrow} x] : x \in \Lambda_N \cap \partial\mathbb{H}_n, n \geq 0\}$ together with (4.3.56)–(4.3.57) gives the

existence of $\beta \in (\beta^*, \beta^{**})$ such that

$$\psi_\beta(\mathbb{H}_n) = \sum_{x \in (\partial\mathbb{H}_n)^*} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x] < \frac{\mathbf{C}}{L} \quad \forall n \leq N-1, \quad (4.3.61)$$

$$\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x] < \frac{\mathbf{C}}{L^d} \left(\frac{L}{L \vee n} \right)^{d-1} \quad \forall n \geq 0, \forall x \in \partial\mathbb{H}_n \cap \Lambda_N. \quad (4.3.62)$$

Combining (4.3.59)–(4.3.62), we obtained the existence of $\beta \in (\beta^*, 2 \wedge \beta_c)$ such that (ℓ_β^1) and (ℓ_β^∞) hold. This contradicts the definition of β^* .

From all of this, we obtain that $\beta^* = 2 \wedge \beta_c$ and that in addition the properties hold until $2 \wedge \beta_c$. Proposition 4.3.3 implies the right bound on $\varphi_\beta(B)$ and Lemma 4.2.2 implies the right bound on $E_\beta(B)$, for every $B \in \mathcal{B}$, and every $\beta < \beta^*$. A new application of the monotone convergence theorem implies the bounds at β^* .

It remains to show that $\beta_c < 2$ or simply that $\beta^* < 2$ for L large enough. To do so, notice that

$$|\Lambda_L|(1 - e^{-\beta^* c_L}) = \varphi_{\beta^*}(\{0\}) \leq 1 + \frac{K}{L^d} \quad (4.3.63)$$

implies that for L large enough,

$$\beta^* \leq 1 + \frac{2K}{L^d}. \quad (4.3.64)$$

This concludes the proof by choosing L so large that $\frac{2K}{L^d} < 1$. $\square$

We conclude this section by proving Theorem 4.1.1.

Proof of Theorem 4.1.1. Fix $d > 6$. Let C, L_0 be given by Proposition 4.3.9 and let $L \geq L_0$. Let $A = A(d) \geq 1$ be such that

$$t^{d-1} \leq Ae^t \quad \forall t \geq 1. \quad (4.3.65)$$

Proof of (4.1.5). First, if $\beta \leq \beta_c$, Proposition 4.3.9 implies that for $x \in \mathbb{Z}^d \setminus \{0\}$ with $|x| \leq L_\beta$,

$$\mathbb{P}_\beta[0 \leftrightarrow x] \leq \frac{C}{L^d} \left(\frac{L}{L \vee |x|} \right)^{d-2} \leq \frac{e^2 C}{L^d} \left(\frac{L}{L \vee |x|} \right)^{d-2} e^{-2|x|/L_\beta}. \quad (4.3.66)$$

We now turn to the case of $\beta < \beta_c$ with $L_\beta \geq L$ and $|x| > L_\beta$ (recall that $L_{\beta_c} = \infty$). Iterating (4.1.19) $k := \lfloor \frac{|x|}{L_\beta} \rfloor - 2$ times with S being translates of Λ_{L_β} and $\Lambda = \mathbb{Z}^d$, we get (recall that by definition of L_β , one has $\varphi_\beta(\Lambda_{L_\beta}) \leq e^{-2}$)

$$\mathbb{P}_\beta[0 \leftrightarrow x] \leq \varphi_\beta(\Lambda_{L_\beta})^k \max \{ \mathbb{P}_\beta[y \leftrightarrow x] : y \notin \Lambda_{L_\beta}(x) \} \quad (4.3.67)$$

$$\stackrel{(4.3.54)}{\leq} e^{-2k} \frac{C}{L^2 L_\beta^{d-2}} \leq \frac{e^6 C}{L^2 L_\beta^{d-2}} e^{-2|x|/L_\beta} \stackrel{(4.3.65)}{\leq} \frac{e^6 AC}{L^2 |x|^{d-2}} e^{-|x|/L_\beta}. \quad (4.3.68)$$

Proof of (4.1.6). Again, if $\beta \leq \beta_c$, Proposition 4.3.9 implies that for $x \in \mathbb{H}^*$ with $x_1 \leq L_\beta$,

$$\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}} x] \leq \frac{C}{L^d} \left(\frac{L}{L \vee |x_1|} \right)^{d-1} \leq \frac{e^2 C}{L^d} \left(\frac{L}{L \vee |x_1|} \right)^{d-1} e^{-2|x_1|/L_\beta}. \quad (4.3.69)$$

Turning to the case of $\beta < \beta_c$ with $L_\beta \geq L$ and $|x_1| > L_\beta$, iterating (4.1.19) $\ell := \lfloor |x_1|/L_\beta \rfloor - 2$ times with S being translates of Λ_{L_β} (starting with $\Lambda_{L_\beta}(x)$) and $\Lambda = \mathbb{H}$, we obtain

$$\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}} x] = \mathbb{P}_\beta[x \xleftrightarrow{\mathbb{H}} 0] \leq \varphi_\beta(\Lambda_{L_\beta})^\ell \max\{\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}} y : y_1 > L_\beta]\} \stackrel{(4.3.55)}{\leq} \frac{e^6 C}{L L_\beta^{d-1}} e^{-2|x_1|/L_\beta} \quad (4.3.70)$$

$$\stackrel{(4.3.65)}{\leq} \frac{e^6 A C}{L |x_1|^{d-1}} e^{-|x_1|/L_\beta} \quad (4.3.71)$$

Gathering (4.3.66)–(4.3.69) and (4.3.71), we obtained: for all $\beta \leq \beta_c$ such that $L_\beta \geq L$, for all $x \in \mathbb{Z}^d \setminus \{0\}$,

$$\mathbb{P}_\beta[0 \leftrightarrow x] \leq \frac{e^6 A C}{L^d} \left(\frac{L}{L \vee |x|} \right)^{d-2} \exp(-|x|/L_\beta), \quad (4.3.72)$$

$$\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}} x] \leq \frac{e^6 A C}{L^d} \left(\frac{L}{L \vee |x_1|} \right)^{d-1} \exp(-|x_1|/L_\beta). \quad (4.3.73)$$

This concludes the proof. $\square$

4.4 Lower bounds

This section is devoted to the proof of Theorem 4.1.2. We follow the strategy of [82]. Fix $d > 6$. We let C and L_0 be given by Proposition 4.3.9.

4.4.1 Preliminaries

We begin the section by stating two alternative versions of Proposition 4.3.6 which can be proved similarly.

Lemma 4.4.1 (Regularity estimate at microscopic scales). *For every $\eta > 0$, there exist $A_1 = A_1(\eta) > 0$ and $L_1 = L_1(\eta, A_1) \geq L_0$ large enough such that for every $L \geq L_1$, every $\beta \leq \beta_c$, every $n \geq A_1 L$, every $\Lambda \supset \Lambda_{2n}$, every $X \subset \Lambda \setminus \Lambda_{2n}$, every $u, v \in \Lambda_n$ with $|u - v| \leq 3L$,*

$$\left| \sum_{x \in X} \mathbb{P}_\beta[u \xleftrightarrow{\Lambda} x] - \mathbb{P}_\beta[v \xleftrightarrow{\Lambda} x] \right| \leq \max_{w, w' \in \Lambda_{2n}} \left(\eta \sum_{x \in X} \mathbb{P}_\beta[w \xleftrightarrow{\Lambda} x] + A_1 \sum_{x \in X} E_\beta(\{w'\}, \Lambda, w', x) \right). \quad (4.4.1)$$

Proof. We prove the result for $X = \{x\}$, but the general case follows similarly. Set $\varphi := \varphi_\beta(\{0\})$. Let $T \geq 1$ to be fixed, $n \geq 2TL$, and assume $u, v \in \Lambda_n$ with $|u - v| \leq 3L$.

Iterating (4.1.19)–(4.2.1) T times with S a singleton and Λ gives

$$\mathbb{P}_\beta[u \xleftrightarrow{\Lambda} x] \leq \varphi^T \mathbb{E}_{\text{RW},u,\beta} [\mathbb{P}_\beta[X_T \xleftrightarrow{\Lambda} x]], \quad (4.4.2)$$

$$\mathbb{P}_\beta[v \xleftrightarrow{\Lambda} x] \geq \varphi^T \mathbb{E}_{\text{RW},v,\beta} [\mathbb{P}_\beta[X_T \xleftrightarrow{\Lambda} x]] - \left(\sum_{t=0}^{T-1} \varphi^t \right) \max_{w \in \Lambda_{n+TL}} E_\beta(\{w\}, \Lambda, w, x). \quad (4.4.3)$$

A random-walk estimate (similar as the one used in the proof of Proposition 4.3.6) implies that for every $\eta > 0$, there exists $T = T(\eta, d)$ large enough such that the random walks X_T^u and X_T^v can be coupled to coincide with probability larger than $1 - \eta/2$. This implies

$$\mathbb{E}_{\text{RW},u,\beta} [\mathbb{P}_\beta[X_T \xleftrightarrow{\Lambda} x]] - \mathbb{E}_{\text{RW},v,\beta} [\mathbb{P}_\beta[X_T \xleftrightarrow{\Lambda} x]] \leq \frac{\eta}{2} \max \left\{ \mathbb{P}_\beta[w \xleftrightarrow{\Lambda} x] : w \in \Lambda_{n+TL} \right\}. \quad (4.4.4)$$

Furthermore, since $\varphi \leq 1 + \frac{C}{L^d}$, we may choose $L_1 = L_1(\eta, T, C, d)$ large enough such that for $L \geq L_1$, $\varphi^T \leq 2$. The result follows by plugging these estimates and (4.4.4) in the difference of (4.4.2) and (4.4.3), and setting $A_1 := 2T$. $\square$

It is possible to combine Lemmata 4.3.5 and 4.4.1 together with the strategy used in the proof of Proposition 4.3.6 to obtain the following general regularity estimate at mesoscopic scales.

Proposition 4.4.2 (Regularity estimate at mesoscopic scales). *For every $\eta > 0$, there exist $\delta = \delta(\eta, d) \in (0, 1/2)$, $A = A(\eta, d)$, and $L_2 = L_2(\eta, A) \geq L_0$ such that for every $L \geq L_2$, every $\beta \leq \beta_c$, every $n \geq AL$, every $\Lambda \supset \Lambda_{3n}$, every $X \subset \Lambda \setminus \Lambda_{3n}$, and every $u, v \in \Lambda_{[\delta n]}$,*

$$\left| \sum_{x \in X} \mathbb{P}_\beta[u \xleftrightarrow{\Lambda} x] - \mathbb{P}_\beta[v \xleftrightarrow{\Lambda} x] \right| \leq \eta \max_{w \in \Lambda_{3n}} \sum_{x \in X} \mathbb{P}_\beta[w \xleftrightarrow{\Lambda} x] \quad (4.4.5)$$

$$+ A \max_{w \in \Lambda_{3n}} \max_{\substack{S \in \mathcal{B} \\ S \subset \Lambda_{3n} \\ S \ni w}} \sum_{x \in X} E_\beta(S, \Lambda, w, x). \quad (4.4.6)$$

We conclude these preliminaries with an easy estimate which is a ℓ^1 version of Lemma 4.2.5.

Lemma 4.4.3. *Let $A > 0$ and $L \geq L_0$. There exists $D_1 > 0$ such that, for every $\beta \leq \beta_c$,*

$$\max_{w \in \Lambda_{AL}} \sum_{x \in \partial \mathbb{H}_n} E_\beta(\{w\}, \mathbb{H}_n, w, x) \leq \frac{D_1}{L} \frac{C}{L^d} \quad \forall n > 2AL. \quad (4.4.7)$$

Proof. By definition, if $w \in \Lambda_{AL}$,

$$\sum_{x \in \partial \mathbb{H}_n} E_\beta(\{w\}, \mathbb{H}_n, w, x) = \sum_{x \in \partial \mathbb{H}_n} \sum_{z \in \mathbb{H}_n \setminus \{v\}} p_{wz,\beta} \mathbb{P}_\beta[z \xleftrightarrow{\mathbb{H}_n} w] \mathbb{P}_\beta[w \xleftrightarrow{\mathbb{H}_n} x] \quad (4.4.8)$$

$$+ \sum_{x \in \partial \mathbb{H}_n} \sum_{v \in \mathbb{H}_n} \sum_{\substack{z, t \in \mathbb{H}_n \\ z \neq t}} p_{wz,\beta} p_{wt,\beta} \mathbb{P}_\beta[z \xleftrightarrow{\mathbb{H}_n} v] \mathbb{P}_\beta[t \xleftrightarrow{\mathbb{H}_n} v] \mathbb{P}_\beta[v \xleftrightarrow{\mathbb{H}_n} x] \quad (4.4.9)$$

$$=: (I) + (II). \quad (4.4.10)$$

Bound on (I). We notice that by Proposition 4.3.9,

$$(I) \leq \frac{C}{L} \sum_{z \in \mathbb{H}_n \setminus \{v\}} p_{wz, \beta} \mathbb{P}_\beta[z \xleftrightarrow{\mathbb{H}_n} w] \leq \frac{C}{L} \frac{C}{L^d} (|\Lambda_L| - 1) p_\beta \leq \frac{2C}{L} \frac{C}{L^d}. \quad (4.4.11)$$

Bound on (II). The contribution for coming from $v \notin \partial \mathbb{H}_n$ is bounded by

$$\sum_{v \in \mathbb{H}_{n-1}} \sum_{\substack{z, t \in \mathbb{H}_n \\ z \neq t}} p_{wz, \beta} p_{wt, \beta} \mathbb{P}_\beta[z \leftrightarrow v] \mathbb{P}_\beta[t \leftrightarrow v] \left(\sum_{x \in \partial \mathbb{H}_n} \mathbb{P}_\beta[v \xleftrightarrow{\mathbb{H}_n} x] \right) \leq \frac{C}{L} E_\beta(\{w\}) \leq \frac{C^2}{L^{d+1}}, \quad (4.4.12)$$

where we used Proposition 4.3.9 in the last inequality. Then, one more use of Proposition 4.3.9 gives

$$\sum_{v \in \partial \mathbb{H}_n} \sum_{\substack{z, t \in \mathbb{H}_n \\ z \neq t}} p_{wz, \beta} p_{wt, \beta} \mathbb{P}_\beta[z \xleftrightarrow{\mathbb{H}_n} v] \mathbb{P}_\beta[t \xleftrightarrow{\mathbb{H}_n} v] \left(\sum_{x \in \partial \mathbb{H}_n} \mathbb{P}_\beta[v \xleftrightarrow{\mathbb{H}_n} x] \right) \quad (4.4.13)$$

$$\leq \left(1 + \frac{C}{L}\right) \sum_{v \in \partial \mathbb{H}_n} \sum_{\substack{z, t \in \mathbb{H}_n \\ z \neq t}} p_{wz, \beta} p_{wt, \beta} \mathbb{P}_\beta[z \xleftrightarrow{\mathbb{H}_n} v] \mathbb{P}_\beta[t \xleftrightarrow{\mathbb{H}_n} v] \quad (4.4.14)$$

$$\leq \left(1 + \frac{C}{L}\right) \frac{C}{L^d} \sum_{v \in \partial \mathbb{H}_n} \sum_{t \in \mathbb{H}_n} p_{wz, \beta} p_{wt, \beta} \mathbb{P}_\beta[t \xleftrightarrow{\mathbb{H}_n} v] \quad (4.4.15)$$

$$\leq \left(1 + \frac{C}{L}\right) \frac{C}{L^d} \frac{C}{L} \left(\sum_{z \in \mathbb{Z}^d} p_{0z, \beta} \right)^2 \quad (4.4.16)$$

$$\leq \frac{4C^2}{L^{d+1}} \left(1 + \frac{C}{L}\right) \leq \frac{8C^3}{L^{d+1}}, \quad (4.4.17)$$

where in the second inequality we used the fact that $n > 2AL$ (which ensures that $z, t \neq v$). The proof follows readily. $\square$

4.4.2 Lower bound on $\psi_\beta(\mathbb{H}_n)$

We start with our basic estimate for this section. It is a lower bound corresponding to the upper bound on $\psi_\beta(\mathbb{H}_n)$ obtained in the previous section. Recall that β_0 is such that $\varphi_{\beta_0}(\{0\}) = 1$. Introduce for $n, k \geq 1$,

$$\psi_\beta^{[k]}(\mathbb{H}_n) := \sum_{\substack{x \in \partial \mathbb{H}_n \\ |x| \leq k}} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x]. \quad (4.4.18)$$

Lemma 4.4.4. *There exists $c > 0$ and $L_3 \geq L_0$ such that for every $L \geq L_3$, every $\beta_0 \leq \beta \leq \beta_c$ and every $1 \leq n \leq L_\beta - 1$,*

$$\psi_\beta(\mathbb{H}_n) \geq \psi_\beta^{[n]}(\mathbb{H}_n) \geq \frac{c}{L}. \quad (4.4.19)$$

Proof. The first inequality is clear, we therefore focus on the second one. Let $\eta > 0$ to be fixed. We divide the proof between the case $n > 4A_1L$ and $n \leq 4A_1L$, where $A_1 = A_1(\eta)$ is provided by Lemma 4.4.1. We begin with the former case as it is the most interesting one.

Case $n > 4A_1L$ We begin by noticing that

$$\varphi_\beta(\Lambda_n) \leq 2d \sum_{\substack{y \in \Lambda_n \\ y_1 \in \{n-L+1, \dots, n\} \\ z \notin \mathbb{H}_n}} \mathbb{P}_\beta[0 \xleftrightarrow{\Lambda_n} y] p_{yz, \beta} \quad (4.4.20)$$

$$\leq 2d \sum_{k=0}^{L-1} \sum_{\substack{y \in \partial \mathbb{H}_{n-k} \\ |y| \leq n \\ z \notin \mathbb{H}_n}} p_{yz, \beta} \sum_{\ell=0}^k \sum_{\substack{u \in \partial \mathbb{H}_{n-\ell} \\ |u| \leq n}} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_{n-\ell}} u] \mathbb{P}_\beta[u \xleftrightarrow{\mathbb{H}_{n-\ell}} y] \quad (4.4.21)$$

$$\leq 2d(1 + 2dC^2) \sum_{\ell=0}^{L-1} \psi_\beta^{[n]}(\mathbb{H}_{n-\ell}), \quad (4.4.22)$$

where we used (4.2.26) in the last inequality. Since $n > 4A_1L$, we may apply Lemma 4.4.1 and Proposition 4.3.9 to get for L large enough,

$$\sum_{\ell=0}^{L-1} \psi_\beta^{[n]}(\mathbb{H}_{n-\ell}) \leq L\psi_\beta^{[n]}(\mathbb{H}_n) + \eta C + A_1L \max_{w \in \Lambda_{2A_1L}} \sum_{x \in \partial \mathbb{H}_n} E_\beta(\{w\}, \mathbb{H}_n, w, x). \quad (4.4.23)$$

Now, using Lemma 4.4.3 and the fact that $n > 4A_1L$,

$$\max_{w \in \Lambda_{2A_1L}} \sum_{x \in \partial \mathbb{H}_n} E_\beta(\{w\}, \mathbb{H}_n, w, x) \leq \frac{D_1}{L} \frac{C}{L^d} \leq \frac{\eta}{A_1L} \quad (4.4.24)$$

if L is large enough. Observe that since $n \leq L_\beta - 1$, one has $\varphi_\beta(\Lambda_n) \geq 1/e^2$. Combining all the previous displayed equations and considering η small enough (which only influences how big A_1 is, and therefore how big L should be taken) concludes the proof.

Case $n \leq 4A_1L$ Set $\varphi := \varphi_\beta(\{0\})$. Summing over $x \in \partial \mathbb{H}_n$ with $|x| \leq n$ and $t \leq T$ the t -th iteration of (4.2.1) with S being a singleton and $\Lambda = \mathbb{H}_n$ gives

$$\psi_\beta^{[n]}(\mathbb{H}_n) \geq \mathbb{E}_0^{\text{RW}}[\varphi^{\tau_\partial} \mathbf{1}_{\tau_\partial < T \wedge \tau}] - \left(\sum_{t=0}^{T-1} \varphi^t \right) \max_{w \in \mathbb{H}_n} \sum_{x \in \partial \mathbb{H}_n} E_\beta(\{w\}, \mathbb{H}_n, w, x), \quad (4.4.25)$$

where τ_∂ and τ are respectively the hitting times of $\{x \in \partial \mathbb{H}_n : |x| \leq n\}$ and the complement of $\mathbb{H}_n$. Using a simple random-walk estimate and the fact that $\beta \geq \beta_0$, we get that for T large enough (in terms of A_1), there exists $c_{\text{RW}} = c_{\text{RW}}(A_1, d) > 0$ such that

$$\mathbb{E}_0^{\text{RW}}[\varphi^{\tau_\partial} \mathbf{1}_{\tau_\partial < T \wedge \tau}] \geq \mathbb{P}_0^{\text{RW}}[\tau_\partial < T \wedge \tau] \geq \frac{c_{\text{RW}}}{L}. \quad (4.4.26)$$

Reproducing⁴ the argument of Lemma 4.4.3, we get $C_1 = C_1(C, d) > 0$ such that

$$\max_{w \in \mathbb{H}_n} \sum_{x \in \partial \mathbb{H}_n} E_\beta(\{w\}, \mathbb{H}_n, w, x) \leq \frac{C_1}{L^d} \quad (4.4.27)$$

⁴We obtain a bound with a diminished power of L because z and t might simultaneously be in $\partial \mathbb{H}_n$ in (4.4.13).

By Proposition 4.3.9, $\varphi \leq 1 + \frac{C}{L^d}$. We take L large enough so that $\sum_{t=0}^{T-1} \varphi^t \leq 2T$. Gathering the previous displayed equations and using (4.3.53) gives

$$\psi_\beta^{[n]}(\mathbb{H}_n) \geq \frac{c_{\text{RW}}}{L} - \frac{2TC_1}{L^d}. \quad (4.4.28)$$

It remains to take L large enough as a function of T and C to conclude. $\square$

4.4.3 A Harnack-type estimate

We will need to turn average estimates into pointwise ones. We therefore show another regularity estimate. For $\varepsilon > 0$, introduce the quantity

$$L_\beta(\varepsilon) := \inf\{n \geq 1 : \varphi_\beta(\Lambda_n) \leq 1 - \varepsilon\}. \quad (4.4.29)$$

We already proved in the preceding chapter that $L_\beta(\varepsilon) \asymp L_\beta$, see Lemma 3.3.1.

Proposition 4.4.5 (Regularity estimate at macroscopic scales). *For every $\alpha > 0$, there exists $C_{\text{RW}} = C_{\text{RW}}(\alpha) > 0$ such that for every $\eta > 0$, there exist A and $L_4 \geq L_0$ large enough, and $\varepsilon_0 > 0$ small enough such that the following holds. For every $L \geq L_4$, every $\varepsilon < \varepsilon_0$, every $n \leq 7L_\beta(\varepsilon) - 1$ with $n \geq AL$, every $\beta \leq \beta_c$, and every $y \notin \Lambda_{(1+\alpha)n} \subset \Lambda$,*

$$\begin{aligned} \max_{x \in \Lambda_n} \mathbb{P}_\beta[x \overset{\Lambda}{\longleftrightarrow} y] &\leq C_{\text{RW}} \min_{x \in \Lambda_n} \mathbb{P}_\beta[x \overset{\Lambda}{\longleftrightarrow} y] + \eta \max_{x \in \Lambda_{(1+\alpha)n}} \mathbb{P}_\beta[x \overset{\Lambda}{\longleftrightarrow} y] \\ &\quad + A \max_{u \in \Lambda_{(1+\alpha)n}} \max_{\substack{S \in \mathcal{B} \\ S \subset \Lambda_{(1+\alpha)n} \\ S \ni u}} E_\beta(S, \Lambda, u, y). \end{aligned} \quad (4.4.30)$$

The idea of the proof is to introduce a well-chosen rescaled random-walk and to observe that its exit probabilities do not drastically depend on the start of the walk. Combined with Proposition 4.4.2, this will enable us to conclude.

Proof. Fix $m = \lfloor \alpha n / 7 \rfloor$. Let $\eta > 0$ to be fixed later. Let $\delta = \delta(\eta)$, $A = A(\eta)$, and $L_2 = L_2(\eta)$ be given by Proposition 4.4.2. Set $k = \lfloor \delta m \rfloor$, where $\delta = \delta(\eta, d)$. Additionally assume that $n \geq (7\alpha^{-1}AL) \vee (7\alpha^{-1}\delta^{-1}L)$ so that $m \geq AL$ and $k \geq L$. Consider the random walk (X_n^u) defined by

$$\mathbb{P}_{\text{RW}, u, \beta}^{(k)}[X_1 = v] := \frac{\mathbb{1}_{v \notin \Lambda_k(u)}}{\varphi_\beta(\Lambda_k)} \sum_{\substack{w \in \Lambda_k(u), \\ w \sim v}} \mathbb{P}_\beta[u \overset{\Lambda_k(u)}{\longleftrightarrow} w] p_{wv, \beta}. \quad (4.4.31)$$

Note that this random walk does jumps at distance at most $k + L$. Let τ be the hitting time of $\mathbb{Z}^d \setminus \Lambda_{n+m}$. Let $B_1, \dots, B_s$ be the two layers of boxes of size k , centred at b_i , that are disjoint and covering $\Lambda_{n+m+4k} \setminus \Lambda_{n+m}$, see Figure 4.6.

We will use two a priori estimates on the random walk and the stopping time, that can be easily obtained from classical random walk analysis: there exist $C_{\text{RW}}(\alpha)$ and $\varepsilon_0 = \varepsilon_0(\alpha, \eta) > 0$ such that for every $\varphi \in [1 - \varepsilon_0, 1 + \varepsilon_0]$,

$$\mathbb{E}_{\text{RW}, x, \beta}^{(k)} \left[\sum_{s=0}^{\tau} \varphi^s \right] \leq C_{\text{RW}} \quad \forall x \in \Lambda_n, \quad (4.4.32)$$

$$\mathbb{E}_{\text{RW}, x, \beta}^{(k)} [\varphi^\tau \mathbb{1}_{X_\tau \in B_i}] \leq C_{\text{RW}} \mathbb{E}_{\text{RW}, x', \beta}^{(k)} [\varphi^\tau \mathbb{1}_{X_\tau \in B_i}] \quad \forall x, x' \in \Lambda_n, \forall i \leq s. \quad (4.4.33)$$

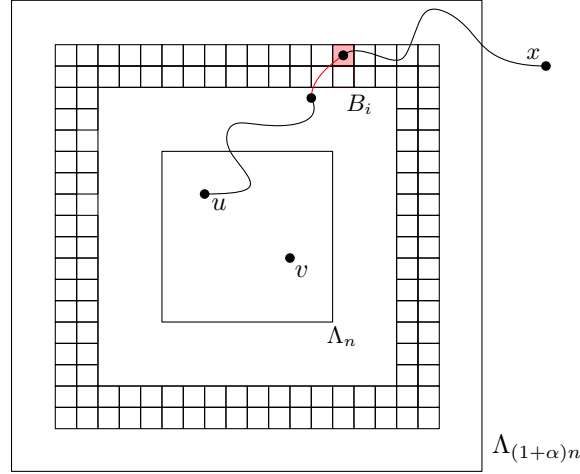


Figure 4.6: An illustration of the proof of Proposition 4.4.5.

From now on, we assume that $\varepsilon < \varepsilon_0$. By Proposition 4.3.9, $\varphi_\beta(\Lambda_k) \leq 1 + \frac{C}{L^d}$. We thus fix $L_4 = L_4(\varepsilon_0)$ large enough that for $L \geq L_4$, $\frac{C}{L^d} \leq \varepsilon_0$. By the assumption $n \leq 7L_\beta(\varepsilon) - 1$, we find

$$1 - \varepsilon \leq \varphi_\beta(\Lambda_k) \leq 1 + \varepsilon_0. \quad (4.4.34)$$

Below, introduce the short-hand notation $\varphi := \varphi_\beta(\Lambda_k)$. Iterating the two bounds (4.1.19)–(4.2.1) until the hitting time τ gives

$$\mathbb{P}_\beta[x' \xleftrightarrow{\Lambda} y] \leq \mathbb{E}_{\text{RW}, x', \beta}^{(k)} \left[\varphi^\tau \mathbb{P}_\beta[X_\tau \xleftrightarrow{\Lambda} y] \right], \quad (4.4.35)$$

$$\begin{aligned} \mathbb{P}_\beta[x \xleftrightarrow{\Lambda} y] &\geq \mathbb{E}_{\text{RW}, x, \beta}^{(k)} \left[\varphi^\tau \mathbb{P}_\beta[X_\tau \xleftrightarrow{\Lambda} y] \right] - \mathbb{E}_{\text{RW}, x, \beta}^{(k)} \left[\sum_{s=0}^{\tau} \varphi^s \right] \max_{w \in \Lambda_{n+m}} E_\beta(\Lambda_k(w), \Lambda, w, y) \\ &\stackrel{(4.4.32)}{\geq} \mathbb{E}_{\text{RW}, x, \beta}^{(k)} \left[\varphi^\tau \mathbb{P}_\beta[X_\tau \xleftrightarrow{\Lambda} y] \right] - C_{\text{RW}} \max_{w \in \Lambda_{n+m}} E_\beta(\Lambda_k(w), \Lambda, w, y). \end{aligned} \quad (4.4.36)$$

Proposition 4.4.2 gives that for every i ,

$$\begin{aligned} \max_{u \in B_i} \mathbb{P}_\beta[u \xleftrightarrow{\Lambda} y] &\leq \min_{u \in B_i} \mathbb{P}_\beta[u \xleftrightarrow{\Lambda} y] + \eta \max_{u \in \Lambda_{3m}(b_i)} \mathbb{P}_\beta[u \xleftrightarrow{\Lambda} y] \\ &\quad + A \max_{u \in \Lambda_{3m}(b_i)} \max_{\substack{S \in \mathcal{B} \\ S \subset \Lambda_{3m}(b_i) \\ S \ni u}} E_\beta(S, \Lambda, u, y). \end{aligned} \quad (4.4.37)$$

Combining this estimate with (4.4.33) implies that for every i ,

$$\begin{aligned} \mathbb{E}_{\text{RW}, x', \beta}^{(k)} \left[\varphi^\tau \mathbb{1}_{X_\tau \in B_i} \mathbb{P}_\beta[X_\tau \xleftrightarrow{\Lambda} y] \right] &\leq C_{\text{RW}} \mathbb{E}_{\text{RW}, x, \beta}^{(k)} \left[\varphi^\tau \mathbb{1}_{X_\tau \in B_i} \mathbb{P}_\beta[X_\tau \xleftrightarrow{\Lambda} y] \right] \\ &\quad + \eta \mathbb{E}_{\text{RW}, x', \beta}^{(k)} \left[\varphi^\tau \mathbb{1}_{X_\tau \in B_i} \right] \max_{u \in \Lambda_{3m}(b_i)} \mathbb{P}_\beta[u \xleftrightarrow{\Lambda} y] \\ &\quad + A \max_{u \in \Lambda_{3m}(b_i)} \max_{\substack{S \in \mathcal{B} \\ S \subset \Lambda_{3m}(b_i) \\ S \ni u}} E_\beta(S, \Lambda, u, y). \end{aligned} \quad (4.4.38)$$

Since X_τ belongs to some B_i , the previous estimate together with (4.4.35) and (4.4.36) gives⁵

$$\begin{aligned} \mathbb{P}_\beta[x' \xleftrightarrow{\Lambda} y] &\leq C_{\text{RW}} \mathbb{P}_\beta[x \xleftrightarrow{\Lambda} y] + \eta C_{\text{RW}} \max_{u \in \Lambda_{n+7m}} \mathbb{P}_\beta[u \xleftrightarrow{\Lambda} y] \\ &\quad + (A + C_{\text{RW}}^2) \max_{u \in \Lambda_{n+7m}} \max_{\substack{S \in \mathcal{B} \\ S \subset \Lambda_{n+7m} \\ S \ni u}} E_\beta(S, \Lambda, u, y). \end{aligned} \quad (4.4.39)$$

(We also used one more time (4.4.32) to get that $\mathbb{E}_{x'}[\varphi^\tau] \leq C_{\text{RW}}$.) It remains to notice that $n + 7m \leq n + \alpha n$, and to pick⁶ $\eta = \eta(d)$ small enough, and then L_4 large enough. $\square$

4.4.4 Proof of Theorem 4.1.2

To shorten the notations, we write $L'_\beta := L_\beta(\varepsilon)$. We start by proving a lower bound on the half-space two-point function at scale below $6L'_\beta$ (for some technical reason we will need this multiplicative factor later). Let

$$A_n := \{x \in \mathbb{Z}^d : x_1 = |x| = n\}. \quad (4.4.40)$$

Lemma 4.4.6. *There exist $c, \varepsilon_0 > 0$ and $L_5 \geq L_0$ such that for every $L \geq L_5$, every $\varepsilon < \varepsilon_0$, every $\beta_0 \leq \beta \leq \beta_c$, and every $x \in \mathbb{H}$ with $x_1 = |x| \leq 6L'_\beta$,*

$$\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}} x] \geq \frac{c}{L^d} \left(\frac{L}{L \vee |x|} \right)^{d-1}. \quad (4.4.41)$$

Proof. First, by translation invariance, for every $n \leq L'_\beta - 1$,

$$\frac{1}{|A_n|} \sum_{y \in A_n} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}} y] = \frac{1}{|A_n|} \psi_\beta^{[n]}(\mathbb{H}_n) \geq \frac{c}{Ln^{d-1}}, \quad (4.4.42)$$

where $c > 0$ and where we used Lemma 4.4.4. We want to turn this average estimate into a point wise one.

Fix $x \in A_N$ with $N \leq 6L'_\beta$ and set $n := \lfloor N/6 \rfloor - 1 \leq L'_\beta - 1$. Let $\eta > 0$ to be fixed. Let $C_{\text{RW}}, A, \varepsilon_0 > 0$ be given by Proposition 4.4.5 with $\alpha = \frac{1}{12}$, η , and $\Lambda = \mathbb{H}$. We consider two cases according to how large n is.

Case $n \geq AL$ In this case, we can apply Proposition 4.4.5 (we refer to Figure 3.4 of the preceding chapter for an illustration) to get, for all $y \in A_n$,

$$\begin{aligned} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}} y] &\leq C_{\text{RW}} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}} x] + \eta \max \left\{ \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}} w] : w_1 \geq n/2 \right\} \\ &\quad + A \max \left\{ E_\beta(S, \mathbb{H}, w, 0) : w_1 \geq n/2, S \in \mathcal{B}, S \subset \Lambda_{13N/12}(\frac{7N}{6}\mathbf{e}_1), S \ni w \right\}, \end{aligned} \quad (4.4.43)$$

Combining the above display with (4.4.42), the upper bound from (4.3.55), and (a minor generalisation of) Lemma 4.2.5 yields

$$C_{\text{RW}} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}} x] \geq \frac{c}{Ln^{d-1}} - \eta \frac{2^{d-1}C}{Ln^{d-1}} - \frac{AD}{L^{d-2}} \frac{C}{Ln^{d-1}}. \quad (4.4.44)$$

Choosing η small enough and then L large enough concludes the proof in that case.

⁵We used that $n + m + 2k + 3m \leq n + 7m$.

⁶Note that it was fundamental that C_{RW} was depending on α, d only.

Case $n < AL$ To handle the small values of n , we repeat the random walk argument used in the proof of Lemma 4.4.4. We omit the details. $\square$

Lemma 4.4.7. *There exist $c, \varepsilon_0 > 0$ and $L_6 \geq L_0$ such that for every $L \geq L_6$, every $\varepsilon < \varepsilon_0$, every $\beta_0 \leq \beta \leq \beta_c$, and every $x \in \Lambda_{5L'_\beta}$,*

$$\mathbb{P}_\beta[0 \leftrightarrow x] \geq \frac{c}{L^d} \left(\frac{L}{L \vee |x|} \right)^{d-2}. \quad (4.4.45)$$

Proof. We fix $\beta \in [\beta_0, \beta_c]$ and drop it from the notations. Let L_5, ε_0 be given by Lemma 4.4.6. We will choose L_6 even larger below. Let $L \geq L_5$ and $\varepsilon < \varepsilon_0$. By symmetry, we may consider $x \in A_n$, where $n = |x|$. We begin by assuming that $n \geq 2L$. By (4.2.1) applied to $S = \mathbb{H}_k$ and $\Lambda = \mathbb{H}_{k+1}$,

$$\begin{aligned} \mathbb{P}[0 \leftrightarrow x] &= \mathbb{P}[0 \xleftrightarrow{\mathbb{H}} x] + \sum_{k \geq 0} \left(\mathbb{P}[0 \xleftrightarrow{\mathbb{H}_{k+1}} x] - \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_k} x] \right) \\ &\geq \sum_{k \geq 0} \sum_{\substack{y \in \mathbb{H}_k, \\ z \in \mathbb{H}_{k+1} \setminus \mathbb{H}_k}} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_k} y] p_{yz} \mathbb{P}[z \xleftrightarrow{\mathbb{H}_{k+1}} x] - \sum_{k \geq 0} E(\mathbb{H}_k, \mathbb{H}_{k+1}, 0, x). \end{aligned} \quad (4.4.46)$$

Looking at the first sum, we see that

$$\sum_{k \geq 0} \sum_{\substack{y \in \mathbb{H}_k, \\ z \in \mathbb{H}_{k+1} \setminus \mathbb{H}_k}} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_k} y] p_{yz} \mathbb{P}[z \xleftrightarrow{\mathbb{H}_{k+1}} x] \quad (4.4.47)$$

$$\geq \sum_{k \geq 0} \sum_{\substack{y \in \mathbb{H}_k, \\ z \in \mathbb{H}_{k+1} \setminus \mathbb{H}_k}} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_k} y] p_{yz} \min\{\mathbb{P}[z \xleftrightarrow{\mathbb{H}_{k+1}} x] : -z_1 = |z| = k+1\} \quad (4.4.48)$$

$$\stackrel{(4.4.41)}{\geq} \sum_{k=0}^{n \wedge (L'_\beta - 1)} \sum_{\substack{y \in \mathbb{H}_k, \\ z \in \mathbb{H}_{k+1} \setminus \mathbb{H}_k}} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_k} y] p_{yz} \frac{c}{L} \frac{1}{(n+k+1)^{d-1}} \quad (4.4.49)$$

$$\geq \sum_{k=0}^{n \wedge (L'_\beta - 1)} \frac{1}{2dL} \varphi(\Lambda_k) \frac{c}{L(n+k+1)^{d-1}} \quad (4.4.50)$$

$$\geq \frac{c_1}{L^2 n^{d-2}}, \quad (4.4.51)$$

where $c_1 = c_1(d) > 0$ and where we used the bound $\varphi(\Lambda_k) \geq 1 - \varepsilon$ provided by the condition $k \leq n \wedge (L'_\beta - 1)$. It remains to control the “error” term in (4.4.46). We proceed as in the previous chapter. Notice that

$$\sum_{k \geq 0} E(\mathbb{H}_k, \mathbb{H}_{k+1}, 0, x) = (I) + (II), \quad (4.4.52)$$

where

$$(I) := \sum_{k \geq 0} \sum_{u, v \in \mathbb{H}_k} \sum_{\substack{y \in \mathbb{H}_k, \\ z \in \mathbb{H}_{k+1} \setminus \mathbb{H}_k}} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_k} u] \mathbb{P}[u \xleftrightarrow{\mathbb{H}_k} y] p_{yz} \mathbb{P}[z \xleftrightarrow{\mathbb{H}_{k+1}} v] \mathbb{P}[u \xleftrightarrow{\mathbb{H}_{k+1}} v] \mathbb{P}[v \xleftrightarrow{\mathbb{H}_{k+1}} x], \quad (4.4.53)$$

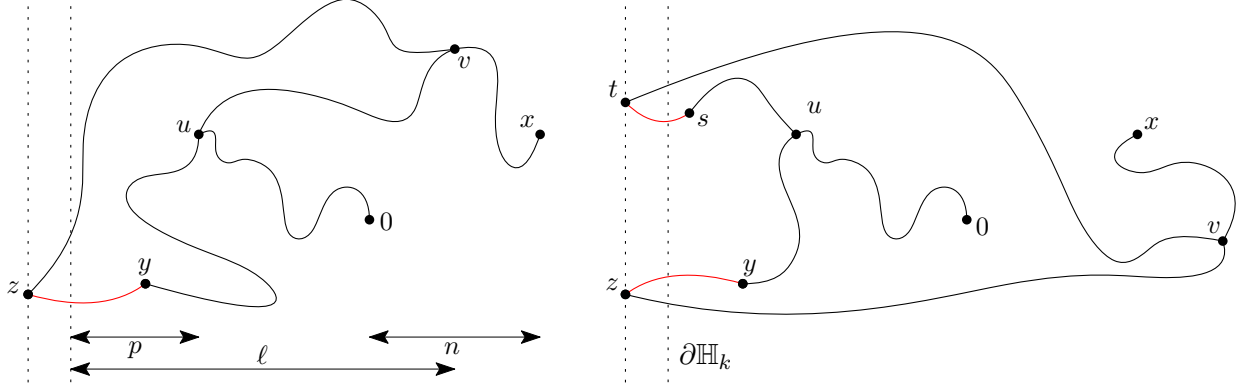


Figure 4.7: On the left, a diagrammatic representation of (I). On the right, a diagrammatic representation of (II). The red lines corresponds to long edges jumping out of $\mathbb{H}_k$.

$$\begin{aligned}
 (II) := & \sum_{k \geq 0} \sum_{\substack{u \in \mathbb{H}_k \\ v \in \mathbb{H}_{k+1}}} \sum_{\substack{y, s \in \mathbb{H}_k \\ z, t \in \mathbb{H}_{k+1} \setminus \mathbb{H}_k}} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_k} u] \mathbb{P}[u \xleftrightarrow{\mathbb{H}_k} y] \mathbb{P}[u \xleftrightarrow{\mathbb{H}_k} s] p_{yz} p_{st} \\
 & \times \mathbb{P}[z \xleftrightarrow{\mathbb{H}_{k+1}} v] \mathbb{P}[t \xleftrightarrow{\mathbb{H}_{k+1}} v] \mathbb{P}[v \xleftrightarrow{\mathbb{H}_{k+1}} x]. \quad (4.4.54)
 \end{aligned}$$

See Figure 4.7 for a diagrammatic representation of (I) and (II). To control (I), we take the position of v into account and write

$$(I) = \sum_{k \geq 0} \sum_{\ell \geq 0} \sum_{\substack{u \in \mathbb{H}_k \\ v \in \partial \mathbb{H}_{k-\ell} \\ y \in \mathbb{H}_k \\ z \in \partial \mathbb{H}_{k+1}}} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_k} u] \mathbb{P}[u \xleftrightarrow{\mathbb{H}_k} y] p_{yz} \mathbb{P}[z \xleftrightarrow{\mathbb{H}_{k+1}} v] \mathbb{P}[u \xleftrightarrow{\mathbb{H}_{k+1}} v] \mathbb{P}[v \xleftrightarrow{\mathbb{H}_{k+1}} x] \quad (4.4.55)$$

$$\leq \frac{C_1}{L} \sum_{k \geq 0} \sum_{\ell \geq 0} \frac{C}{L^d} \left(\frac{L}{L \vee (\ell + 1)} \right)^{d-1} \sum_{\substack{u \in \mathbb{H}_k \\ v \in \partial \mathbb{H}_{k-\ell}}} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_k} u] \mathbb{P}[u \xleftrightarrow{\mathbb{H}_{k+1}} v] \mathbb{P}[v \xleftrightarrow{\mathbb{H}_{k+1}} x], \quad (4.4.56)$$

where we used Proposition 4.3.9 to bound $\mathbb{P}[z \xleftrightarrow{\mathbb{H}_{k+1}} v]$, and where $C_1 = C_1(d) > 0$ was obtained using (4.2.26) (and the fact that $\beta_c \leq 2$) to argue that

$$\sum_{\substack{y \in \mathbb{H}_k \\ z \in \partial \mathbb{H}_{k+1}}} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_k} y] p_{yz} \leq \frac{|\Lambda_L| p}{2L + 1} \sum_{m=0}^L \sum_{y \in \partial \mathbb{H}_{k-m}} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_k} y] \leq \frac{C_1}{L}. \quad (4.4.57)$$

We bound the ℓ -dependent term in (4.4.56) differently according to the value of ℓ . First, there

exist $C_2, C_3 > 0$ (which only depend on d) such that

$$\sum_{\ell \geq (n+k)/2} \frac{C}{L^d} \left(\frac{L}{L \vee (\ell+1)} \right)^{d-1} \sum_{\substack{u \in \mathbb{H}_k \\ v \in \partial \mathbb{H}_{k-\ell}}} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_k} u] \mathbb{P}[u \xleftrightarrow{\mathbb{H}_{k+1}} v] \mathbb{P}[v \xleftrightarrow{\mathbb{H}_{k+1}} x] \quad (4.4.58)$$

$$\leq \frac{C_2}{L(n+k)^{d-1}} \sum_{u, v \in \mathbb{Z}^d} \mathbb{P}[0 \leftrightarrow u] \mathbb{P}[u \leftrightarrow v] \mathbb{P}[v \leftrightarrow x] \quad (4.4.59)$$

$$\leq \frac{C_3}{L(n+k)^{d-1}} \frac{1}{L^d}, \quad (4.4.60)$$

where we used that, since $x \neq 0$,

$$\sum_{u, v \in \mathbb{Z}^d} \mathbb{P}[0 \leftrightarrow u] \mathbb{P}[u \leftrightarrow v] \mathbb{P}[v \leftrightarrow x] = O(L^{-d}). \quad (4.4.61)$$

Then, if $\ell < (n+k)/2$,

$$\max_{v \in \partial \mathbb{H}_{k-\ell}} \mathbb{P}[v \xleftrightarrow{\mathbb{H}_{k+1}} x] \leq \frac{C_4(\ell+L)}{L^2(n+k)^{d-1}} \quad (4.4.62)$$

where $C_4 = C_4(d) > 0$ where we used Lemma 4.2.4. Moreover, by Proposition 4.A.1, if $v \in \partial \mathbb{H}_{k-\ell}$ with $\ell < (n+k)/2$,

$$\mathbb{1}_{v \neq 0} \sum_{u \in \mathbb{H}_k} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_k} u] \mathbb{P}[u \xleftrightarrow{\mathbb{H}_{k+1}} v] = O(L^{-d}). \quad (4.4.63)$$

The last two displayed equations give that

$$\begin{aligned} \sum_{\ell < (n+k)/2} \frac{C}{L^d} \left(\frac{L}{L \vee (\ell+1)} \right)^{d-1} \sum_{\substack{u \in \mathbb{H}_k \\ v \in \partial \mathbb{H}_{k-\ell}}} \mathbb{1}_{v \neq 0} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_k} u] \mathbb{P}[u \xleftrightarrow{\mathbb{H}_{k+1}} v] \mathbb{P}[v \xleftrightarrow{\mathbb{H}_{k+1}} x] \\ \leq \frac{C_5}{L(n+k)^{d-1}} \frac{1}{L^d}, \end{aligned} \quad (4.4.64)$$

where $C_5 = C_5(d) > 0$. Finally, if $0 = v$ in (4.4.56), then $k = \ell$ and the contribution is bounded by

$$\frac{C_1}{L} \sum_{\ell \geq 0} \frac{C}{L^d} \left(\frac{L}{L \vee (\ell+1)} \right)^{d-1} \sum_{u \in \mathbb{H}_\ell} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_\ell} u] \mathbb{P}[u \xleftrightarrow{\mathbb{H}_{\ell+1}} 0] \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_{\ell+1}} x] \leq \frac{C_6}{L^2 n^{d-2}} \frac{1}{L}, \quad (4.4.65)$$

where $C_6 = C_6(d) > 0$, and where we used Proposition 4.3.9. Combining (4.4.56), (4.4.60), (4.4.64), and (4.4.65) gives the existence of $C_7 = C_7(d) > 0$ such that,

$$(I) \leq \frac{C_7}{L^2 n^{d-2}} \frac{1}{L}. \quad (4.4.66)$$

One may show similarly that for some $C_8 = C_8(d) > 0$,

$$(II) \leq \frac{C_8}{L^2 n^{d-2}} \frac{1}{L}. \quad (4.4.67)$$

Choosing L large enough so that $\frac{C_7}{L} + \frac{C_8}{L} \leq \frac{c_1}{2}$ and using (4.4.51) we obtained

$$\mathbb{P}[0 \leftrightarrow x] \geq \frac{c_1}{2L^d} \left(\frac{L}{L \vee |x|} \right)^{d-2}. \quad (4.4.68)$$

If $n < 2L$, we use Lemma 4.4.6. The proof follows by choosing $c > 0$ small enough. $\square$

Proof of Theorem 4.1.2. Theorem 4.1.2 immediately follows from the two preceding lemmata. $\square$

4.A A convolution estimate

Proposition 4.A.1. *Let $d > 4$ and $L \geq 1$. Let $f : \mathbb{Z}^d \rightarrow \mathbb{R}^+$. Assume that there exists $\mathbf{C} > 0$ such that for all $x \in \mathbb{Z}^d$,*

$$f(x) \leq \mathbf{1}_{x=0} + \frac{\mathbf{C}}{L^d} \left(\frac{L}{L \vee |x|} \right)^{d-2}. \quad (4.A.1)$$

Then, there exists $A = A(\mathbf{C}, d) > 0$ (independent of L) such that, for all $x \in \mathbb{Z}^d$,

$$(f * f)(x) := \sum_{y \in \mathbb{Z}^d} f(y)f(x-y) \leq A \left(\mathbf{1}_{x=0} + \mathbf{1}_{x \neq 0} \frac{1}{L^4} \frac{1}{(L \vee |x|)^{d-4}} \right). \quad (4.A.2)$$

Proof. Below, the constants C_i depend on $\mathbf{C}$ and d only. The case $x = 0$ follows from the hypothesis that $d > 4$. We turn to the case $x \neq 0$, which we split into two subcases.

Case $|x| \leq 2L$. We write

$$\sum_{y \in \Lambda_{4L} \setminus \{0, x\}} f(y)f(x-y) \stackrel{(4.A.1)}{\leq} |\Lambda_{4L}| \frac{\mathbf{C}^2}{L^{2d}} \leq \frac{C_1}{L^d}. \quad (4.A.3)$$

Moreover,

$$\sum_{y \notin \Lambda_{4L}} f(y)f(x-y) \stackrel{(4.A.1)}{\leq} \frac{C_2}{L^4} \sum_{|u| \geq 2L} \frac{1}{|u|^{2d-4}} \leq \frac{C_3}{L^d}. \quad (4.A.4)$$

The proof in this case follows from the two last displayed equations and the fact that $f(0)f(x) \leq \frac{C_4}{L^d}$.

Case $|x| > 2L$. We start by noticing that

$$\sum_{y \in \Lambda_{|x|/2}(0) \cup \Lambda_{|x|/2}(x)} f(y)f(x-y) \stackrel{(4.A.1)}{\leq} \frac{C_5}{L^2|x|^{d-2}} \sum_{y \in \Lambda_{|x|/2}(0)} f(y) \stackrel{(4.A.1)}{\leq} \frac{C_6}{L^2|x|^{d-2}} \frac{|x|^2}{L^2} = \frac{C_6}{L^4|x|^{d-4}}. \quad (4.A.5)$$

Then,

$$\sum_{y \notin \Lambda_{|x|/2}(0) \cup \Lambda_{|x|/2}(x)} f(y)f(x-y) \stackrel{(4.A.1)}{\leq} \frac{C_7}{L^4} \sum_{|u| \geq |x|/2} \frac{1}{|u|^{2d-4}} \leq \frac{C_8}{L^4|x|^{d-4}}. \quad (4.A.6)$$

The proof follows readily. $\square$

4.B Proof of Lemma 4.2.5

Fix $d > 6$, $\mathbf{C} > 1$, and $L \geq 1$. Also, fix $\beta < \beta^*(\mathbf{C}, L)$ and drop it from the notations. Let $n > 12L$, $x \in \partial\mathbb{H}_n$, $w \in \Lambda_{n/2}$ and $B \in \mathcal{B}$ included in $\Lambda_{n/2}$ and containing w . By definition,

$$E(B, \mathbb{H}_n, w, x) = \sum_{\substack{u \in B \\ v \in B}} \sum_{\substack{y \in B \\ z \in \mathbb{H}_n \setminus B}} \mathbb{P}[w \xleftrightarrow{B} u] \mathbb{P}[u \xleftrightarrow{B} y] p_{yz} \mathbb{P}[z \xleftrightarrow{\mathbb{H}_n} v] \mathbb{P}[u \xleftrightarrow{\mathbb{H}_n} v] \mathbb{P}_\beta[v \xleftrightarrow{\mathbb{H}_n} x] \quad (4.B.1)$$

$$+ \sum_{\substack{u \in B \\ v \in \mathbb{H}_n}} \sum_{\substack{y, s \in B \\ z, t \in \mathbb{H}_n \setminus B \\ yz \neq st}} \mathbb{P}[w \xleftrightarrow{B} u] \mathbb{P}[u \xleftrightarrow{B} y] \mathbb{P}_\beta[u \xleftrightarrow{B} s] p_{yz} p_{st} \mathbb{P}[z \xleftrightarrow{\mathbb{H}_n} v] \mathbb{P}[t \xleftrightarrow{\mathbb{H}_n} v] \mathbb{P}[v \xleftrightarrow{\mathbb{H}_n} x]. \quad (4.B.2)$$

We split $E(B, \mathbb{H}_n, w, x)$ into a contribution coming from $v \in \mathbb{H}_{3n/4}$ that we denote (I) , and a contribution coming from $v \in \mathbb{H}_n \setminus \mathbb{H}_{3n/4}$ that we denote (II) .

Bound on (I) . Notice that

$$\max_{v \in \mathbb{H}_{3n/4}} \mathbb{P}[v \xleftrightarrow{\mathbb{H}_n} x] \stackrel{(\ell_\beta^\infty)}{\leq} \frac{\mathbf{C}}{L} \frac{1}{(n/4)^{d-1}}. \quad (4.B.3)$$

Recall the definition of the error amplitude $E(B)$. By monotonicity and Proposition 4.2.2,

$$(I) \leq E(B) \cdot 4^{d-1} \frac{\mathbf{C}}{Ln^{d-1}} \leq \frac{4^{d-1}K}{L^d} \frac{\mathbf{C}}{Ln^{d-1}}. \quad (4.B.4)$$

Bound on (II) . We turn to the contribution for $v \in \mathbb{H}_n \setminus \mathbb{H}_{3n/4}$. This contribution is only coming from (4.B.2). Notice that z, t contribute if they are at a distance at most L from B , that is $z, t \in \Lambda_{n/2+L} \subset \Lambda_{n/2+n/12}$. If $p \in \{0, \dots, n/4 - 1\}$, $v \in \partial\mathbb{H}_{n-p}$, and z, t are as above, then $|z - v|, |t - v| \geq n/6$ and

$$\mathbb{P}[z \xleftrightarrow{\mathbb{H}_n} v] \mathbb{P}[t \xleftrightarrow{\mathbb{H}_n} v] \stackrel{(4.2.19)}{\leq} \frac{9\mathbf{C}^4}{L^4} \frac{6^{2d-4}}{n^{2d-4}}. \quad (4.B.5)$$

Moreover,

$$\sum_{v \in \mathbb{H}_{n-p}} \mathbb{P}[v \xleftrightarrow{\mathbb{H}_n} x] = \delta_0(p) + \psi_\beta(\mathbb{H}_p) \stackrel{(\ell_\beta^1)}{\leq} \delta_0(p) + \frac{\mathbf{C}}{L}. \quad (4.B.6)$$

For a fixed $u \in B$, Proposition 4.3.3 gives

$$\sum_{\substack{y, s \in B \\ y \neq s \\ z, t \in \mathbb{H}_n \setminus B \\ z \neq t}} \mathbb{P}[u \xleftrightarrow{B} y] \mathbb{P}[u \xleftrightarrow{B} s] p_{yz} p_{st} \leq \varphi(B - u)^2 \leq \left(1 + \frac{K}{L^d}\right)^2. \quad (4.B.7)$$

Finally, we use (4.2.19) to get $C_1 = C_1(\mathbf{C}, d) > 0$ such that

$$\sum_{u \in S} \mathbb{P}[w \xleftrightarrow{B} u] \leq \sum_{u \in \Lambda_{n/2}} \mathbb{P}[w \leftrightarrow u] \leq C_1 n^2. \quad (4.B.8)$$

Putting all the previous displayed equations together and using that $n > 12L$, we obtain $C_2 = C_2(\mathbf{C}, d) > 0$ such that

$$(II) \leq \frac{C_2}{L^4 n^{d-6}} \frac{\mathbf{C}}{L n^{d-1}} \leq \frac{C_2 12^{d-6}}{L^{d-2}} \frac{\mathbf{C}}{L n^{d-1}}. \quad (4.B.9)$$

The result follows from setting $D := 4^{d-1}K + C_2 12^{d-6}$.

Part II

The Ising model

Chapter 5

The incipient infinite cluster of the FK-Ising model in dimensions $d \geq 3$ and the susceptibility of the high-dimensional Ising model

We consider the critical FK-Ising measure ϕ_{β_c} on $\mathbb{Z}^d$ with $d \geq 3$. We construct the measure $\phi^\infty := \lim_{|x| \rightarrow \infty} \phi_{\beta_c}[\cdot \mid 0 \leftrightarrow x]$ and prove it satisfies $\phi^\infty[0 \leftrightarrow \infty] = 1$. This corresponds to the natural candidate for the *incipient infinite cluster* measure of the FK-Ising model. Our proof uses a result of Lupu and Werner (Electron. Commun. Probab., 2016) that relates the FK-Ising model to the random current representation of the Ising model, together with a *mixing property* of random currents recently established by Aizenman and Duminil-Copin (Ann. Math., 2021).

We then study the susceptibility $\chi(\beta)$ of the nearest-neighbour Ising model on $\mathbb{Z}^d$. When $d > 4$, we improve a previous result of Aizenman (Comm. Math. Phys., 1982) to obtain the existence of $A > 0$ such that, for $\beta < \beta_c$,

$$\chi(\beta) = \frac{A}{1 - \beta/\beta_c}(1 + o(1)),$$

where $o(1)$ tends to 0 as β tends to β_c . Additionally, we relate the constant A to the incipient infinite cluster of the double random current.

This chapter is based on the preprint “*The incipient infinite cluster of the FK-Ising model in dimensions $d \geq 3$ and the susceptibility of the high-dimensional Ising model*” [205].

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5.1 Introduction

In the context of independent bond percolation on the hypercubic lattice $\mathbb{Z}^d$ in dimensions $d \geq 2$, it is believed¹ that there is no infinite cluster at the critical threshold p_c . However, the expected size of the cluster of the origin (or *susceptibility*) is infinite (see [16]). This implies that, from the perspective of an observer at the origin, increasingly large (but finite) clusters emerge as larger length scales are considered. The *incipient infinite cluster* (IIC) was introduced in the physics literature [19, 177, 236] as a way to analyze how this extensive structure “emerges” at the critical point. It can be viewed as a description of the not fully materialised infinite cluster. As explained in [5], a rigorous definition of this object is not clear, and one may obtain different results depending on the chosen approach.

Various methods have been suggested to construct the IIC. In a seminal paper, Kesten [163] proposed two ways of constructing the IIC in the case of planar Bernoulli percolation. One consists in conditioning the critical measure $\mathbb{P}_{p_c}$ on the one-arm event $\{0 \leftrightarrow \Lambda_n^c\}$, where $\Lambda_n = [-n, n] \cap \mathbb{Z}^d$, and to send n to infinity. The other one consists in conditioning 0 to lie in an infinite cluster in the supercritical measure $\mathbb{P}_p$ with $p > p_c$, and to send the parameter p to p_c . Both approaches were shown to converge (weakly) to the same percolation measure, which additionally satisfies almost surely the property that 0 lies in an infinite cluster. The cluster of the origin under this measure is a natural candidate for the IIC. Chayes, Chayes, and Durrett [62] proposed a different construction based on the introduction of a well-chosen *non-homogeneous* bond parameter p . Subsequent works by Járai [157, 158] provided alternative but equivalent constructions of Kesten’s infinite incipient cluster, thus confirming the robustness of the two dimensional IIC for independent bond percolation.

Building on the development of the *lace expansion* [49, 134, 135, 136], van der Hofstad and Járai [149] proposed a construction of the IIC in the high-dimensional setup (i.e. $d > 10$, see [134, 132, 100]). In their approach, they constructed the limit of the sequence of measures $\mathbb{P}_{p_c}[\cdot | 0 \leftrightarrow x]$ as $|x| \rightarrow \infty$. They also proposed a *subcritical* construction of the measure which is complementary to Kesten’s supercritical definition. These formulations are equivalent to Kesten’s when $d = 2$. An additional construction, involving conditioning on the one-arm event, was successfully achieved in [144] using the computation of the one-arm exponent

¹The absence of infinite cluster at the critical point was successfully proved when $d = 2$ [165], and when $d > 10$ [134, 132, 100]. Proving the corresponding result when $3 \leq d \leq 10$ is one of the main open problems in percolation theory.

performed by Kozma and Nachmias [168]. Additionally, the lace expansion methods were used to construct the IIC in the setup of oriented percolation [148].

Fine properties of the IIC were derived using the aforementioned techniques: sub-diffusivity of the random-walk on the IIC has been studied in [164, 22, 167], the connection to invasion percolation has been explored in [158, 149], and the relation of the scaling limit of the high-dimensional IIC to the super-Brownian excursion has been investigated in [137, 138].

Removing independency makes the problem more intricate. In the setup of the planar *random cluster model* (or *Fortuin–Kasteleyn (FK) percolation*) with cluster weight $1 < q \leq 4$ ($q = 1$ corresponds to Bernoulli percolation), the Russo–Seymour–Welsh (RSW) theory [214, 221] is sufficient to extend Kesten’s argument. See [163, 158, 24]. In higher dimensions, the lace expansion (crucial to build the IIC for Bernoulli percolation) has only been developed for $q \in \{1, 2\}$. The choice $q = 2$ corresponds to the so-called *FK-Ising model*. However, the lace expansion of the Ising model is developed at the level of the spin model and very little information is available on the percolation side. In particular, it is not clear how to extend the approach of [149]. Nevertheless, other tools are available for the analysis of the case $q = 2$.

The goal of this chapter is twofold. First, we study the existence of the IIC measure of the FK-Ising model (or *random-cluster model* with cluster weight $q = 2$) in dimensions $d \geq 3$. Using the relation of the FK-Ising model to the *random current representation* of the Ising model established in [186] (see also [11]), together with the mixing property recently obtained in [9], we construct in two different ways a natural candidate for the IIC measure of the FK-Ising model in dimensions $d \geq 3$. Our definitions are similar to the ones suggested in [149]. Along the way, we also construct the corresponding object for the percolation measure induced by random currents.

Second, we study the high-dimensional susceptibility of the Ising model and obtain its exact asymptotics as $\beta \nearrow \beta_c$, i.e. we prove that the limit $\lim_{\beta \nearrow \beta_c} \chi(\beta)(1 - \beta/\beta_c)$ exists. This improves a result of Aizenman [4], where up-to-constants estimates were derived. We then relate this limit to the IIC measure of the *double* random current. Again, our proof relies on the mixing property of [9].

We stress that our proofs do not rely on the lace expansion.

5.1.1 Definition of the model

The Ising model is one of the most studied models in statistical mechanics (see [77] for a review). It is formally defined as follows: given $\Lambda \subset \mathbb{Z}^d$ finite, $\tau \in \{-1, 0, +1\}^{\mathbb{Z}^d}$, and an inverse temperature $\beta \geq 0$, construct a probability measure on $\{-1, +1\}^\Lambda$ according to the formula

$$\langle F(\sigma) \rangle_{\Lambda, \beta}^\tau := \frac{1}{Z_{\Lambda, \beta}^\tau} \sum_{\sigma \in \{-1, +1\}^\Lambda} F(\sigma) \exp \left(\beta \sum_{\substack{x, y \in \Lambda \\ x \sim y}} \sigma_x \sigma_y + \beta \sum_{\substack{x \in \Lambda \\ y \notin \Lambda \\ x \sim y}} \sigma_x \tau_y \right), \quad (5.1.1)$$

where $F : \{-1, +1\}^\Lambda \rightarrow \mathbb{R}$, $Z_{\Lambda, \beta}^\tau$ is the *partition function* of the model which guarantees that $\langle 1 \rangle_{\Lambda, \beta}^\tau = 1$, and $x \sim y$ means that $|x - y|_2 = 1$ (where $|\cdot|_2$ is the ℓ^2 norm on $\mathbb{R}^d$). This definition corresponds to the *nearest-neighbour ferromagnetic Ising model* on Λ with *boundary condition* τ . When $\tau \equiv 1$ (resp $\tau \equiv 0$), we write $\langle \cdot \rangle_{\Lambda, \beta}^\tau = \langle \cdot \rangle_{\Lambda, \beta}^+$ (resp. $\langle \cdot \rangle_{\Lambda, \beta}$).

It is a well-known fact (see [116, 117]) that the measures $\langle \cdot \rangle_{\Lambda, \beta}^+$ and $\langle \cdot \rangle_{\Lambda, \beta}$ admit weak limits when $\Lambda \nearrow \mathbb{Z}^d$. We respectively denote them by $\langle \cdot \rangle_\beta^+$ and $\langle \cdot \rangle_\beta$. When $d \geq 2$, the model

undergoes a *phase transition* for the vanishing of the *magnetisation* at a critical parameter $\beta_c \in (0, \infty)$ defined by

$$\beta_c := \inf \left\{ \beta \geq 0, m^*(\beta) := \langle \sigma_0 \rangle_\beta^+ > 0 \right\}. \quad (5.1.2)$$

The phase transition is *continuous* in the sense that $m^*(\beta_c) = 0$, see [12, 247, 10, 76].

The Ising model is classically related to a (dependent) percolation model: the *FK-Ising model*, which we now define. For more information we refer to the monograph [122] or the lecture notes [76].

Define $\mathbb{E} := \{\{x, y\} : x \sim y, x, y \in \mathbb{Z}^d\}$. Let $G = (V, E)$ be a finite subgraph of $\mathbb{Z}^d$. Fix $\xi \in \{0, 1\}^{\mathbb{E} \setminus E}$. For a fixed percolation configuration $\omega \in \{0, 1\}^E$, we let $k^\xi(\omega)$ be the number of connected components of the graph with vertex set $\mathbb{Z}^d$ and edge set $\omega \vee \xi$ that intersect G . The FK-Ising model at inverse temperature $\beta \geq 0$ with boundary condition ξ is the measure on $\{0, 1\}^E$ defined by

$$\phi_{G, \beta, \xi}^\xi[\omega] := \frac{2^{k^\xi(\omega)} \prod_{\{x, y\} \in E} (e^{2\beta} - 1)^{\omega(\{x, y\})}}{\mathbf{Z}_{G, \beta, \xi}^{\text{FK}}}, \quad (5.1.3)$$

where $\mathbf{Z}_{G, \beta, \xi}^{\text{FK}}$ is the partition function of the model, which ensures that $\phi_{G, \beta, \xi}^\xi$ is a probability measure. We let $\phi_{G, \beta}^1$ and $\phi_{G, \beta}^0$ respectively denote the cases $\xi \equiv 1$ and $\xi \equiv 0$. It is classical that the Ising model and the FK-Ising model can be coupled together. This leads to the following formulas: for $x, y \in G$,

$$\langle \sigma_x \sigma_y \rangle_{G, \beta} = \phi_{G, \beta}^0[x \leftrightarrow y], \quad \langle \sigma_x \sigma_y \rangle_{G, \beta}^+ = \phi_{G, \beta}^1[x \leftrightarrow y], \quad \langle \sigma_x \rangle_{G, \beta}^+ = \phi_{G, \beta}^1[0 \leftrightarrow \partial G], \quad (5.1.4)$$

where ∂G is the vertex boundary of G defined by $\partial G := \{x \in G : \exists y \notin G, x \sim y\}$. Once again, it is possible to construct (weak) limits of these measures when $G \nearrow \mathbb{Z}^d$. We denote them by ϕ_β^1 and ϕ_β^0 . In particular, $m^*(\beta) = \phi_\beta^1[0 \leftrightarrow \infty]$. The FK-Ising model undergoes a phase transition for the existence of an infinite cluster at the parameter β_c introduced above. That is, an alternative definition of β_c is

$$\beta_c = \inf \left\{ \beta \geq 0 : \phi_\beta^1[0 \leftrightarrow \infty] > 0 \right\}. \quad (5.1.5)$$

It is known (see [37, 10, 211]) that $\phi_\beta^1 = \phi_\beta^0$ for all $\beta > 0$. We denote the common measure ϕ_β . The relation described above ensures that there is no infinite cluster at criticality: $\phi_{\beta_c}[0 \leftrightarrow \infty] = 0$. As for Bernoulli percolation, defining a notion of critical measure conditioned on the event $\{0 \leftrightarrow \infty\}$ is therefore not clear.

5.1.2 Construction of the incipient infinite cluster

The construction of the IIC for the FK-Ising model when $d = 2$ is a classical consequence of RSW theory, see [78] and references therein. Our first result provides two equivalent constructions of the IIC measure in dimensions $d \geq 3$. We begin by introducing some useful terminology. Let $\mathcal{F}_0$ be the set of cylinder events on $\{0, 1\}^{\mathbb{E}}$, i.e. the set of *local* events that are measurable with respect to the state of finitely many edges. Let $\mathcal{F}$ be the σ -algebra generated by $\mathcal{F}_0$.

Theorem 5.1.1. *Let $d \geq 3$. For all $E \in \mathcal{F}_0$, the limit*

$$\lim_{|x| \rightarrow \infty} \phi_{\beta_c}[E \mid 0 \leftrightarrow x] \quad (5.1.6)$$

exists independently of the manner in which x goes to infinity. We denote it by $\phi^\infty[E]$. Moreover, ϕ^∞ extends to a probability measure on $\mathcal{F}$, that we still denote ϕ^∞ , and which satisfies

$$\phi^\infty[0 \leftrightarrow \infty] = 1. \quad (5.1.7)$$

Note that the above measure is not translation invariant. The cluster of the origin under the measure ϕ^∞ is a natural candidate for the IIC of the FK-Ising model in dimensions $d \geq 3$. We now provide a *subcritical* definition of ϕ^∞ that is inspired by [149].

For $\beta < \beta_c$, we define a probability measure $\mathbb{Q}_\beta$ on $\{0, 1\}^\mathbb{E}$ by setting for $E \in \mathcal{F}$,

$$\mathbb{Q}_\beta[E] := \frac{1}{\chi(\beta)} \sum_{x \in \mathbb{Z}^d} \phi_\beta[E, 0 \leftrightarrow x], \quad (5.1.8)$$

where $\chi(\beta) := \sum_{x \in \mathbb{Z}^d} \langle \sigma_0 \sigma_x \rangle_\beta = \sum_{x \in \mathbb{Z}^d} \phi_\beta[0 \leftrightarrow x]$ is the *susceptibility*.

Theorem 5.1.2. *Let $d \geq 3$. For all $E \in \mathcal{F}_0$, the limit*

$$\lim_{\beta \nearrow \beta_c} \mathbb{Q}_\beta[E] \quad (5.1.9)$$

exists. We denote it by $\mathbb{Q}_{\beta_c}[E]$. Moreover, one has $\mathbb{Q}_{\beta_c}[E] = \phi^\infty[E]$.

It is not clear how to construct the IIC with Kesten's approach. We discuss this issue in Section 5.1.4.

Our strategy to prove these results is very similar to the one employed in [149]: we use a *mixing property* to argue that for a local event E , the quantity $\phi_{\beta_c}[E \mid 0 \leftrightarrow x]$ “does not depend much” on x when x is sufficiently far away from the support of E . We are not able to directly prove such a mixing property at the level of the FK-Ising model, but we circumvent this difficulty by using an intermediate model related to the Ising model: the *random current model*. We introduce this classical geometric representation of the Ising model in Section 5.2. In the recent breakthrough work of Aizenman and Duminil-Copin [9], a mixing property of the random current measure is derived and plays a pivotal role. In the same paper, the authors suggest that their result may be used to construct the IIC for the FK-Ising model in all dimensions $d \geq 3$. Although their mixing property is only proved in dimensions $d \geq 4$, they suggest a possible route (which relies on [10]) to extend it to the three-dimensional case. We provide a full argument in Section 5.5, see also Theorem 5.2.4 for a statement. With this result, one may construct the IIC measure of the random current measure, see Theorem 5.3.1.

The construction of the IIC for the FK-Ising measure then follows from an observation of Lupu and Werner [186] (see also [11, Theorem 3.2]) that identifies the law $\phi_{\beta_c}[\cdot \mid 0 \leftrightarrow x]$ with that of a percolation configuration induced by a single current configuration on $\mathbb{Z}^d$ with sources $\{0, x\}$ *sprinkled* by an independent (well-chosen) Bernoulli percolation. We refer to Proposition 5.2.5 for a precise statement.

5.1.3 Susceptibility of the high-dimensional Ising model

We recall that the susceptibility is defined for $\beta < \beta_c$ by

$$\chi(\beta) := \sum_{x \in \mathbb{Z}^d} \langle \sigma_0 \sigma_x \rangle_{\beta_c}. \quad (5.1.10)$$

It is well known that the nearest-neighbor Ising model introduced above is *reflection positive*, see [104, 103, 36]. This property provides two tools that are fundamental in the study of the Ising model: the *infrared bound* [104, 103] and the Messenger–Miracle–Solé (MMS) inequalities [191]. These tools combined together imply the existence of $C = C(d) > 0$ such that for all $\beta \leq \beta_c$, and all $x \in \mathbb{Z}^d \setminus \{0\}$,

$$\langle \sigma_0 \sigma_x \rangle_{\beta} \leq \frac{C}{|x|^{d-2}}, \quad (5.1.11)$$

see [204] for more details. When $d > 4$, a matching lower bound (with a different constant) was recently derived [83].

In his seminal paper, Aizenman [4] used (5.1.11) to obtain the existence of $C > 0$ such that for all $\beta < \beta_c$,

$$\frac{(2d\beta_c)^{-1}}{1 - \beta/\beta_c} \leq \chi(\beta) \leq \frac{C}{1 - \beta/\beta_c}. \quad (5.1.12)$$

In fact, for every $\varepsilon > 0$, provided that β is sufficiently close to β_c , C can be chosen² [14] as

$$C = \frac{1 + (2d\beta_c)B(\beta_c)}{2d\beta_c}(1 + \varepsilon), \quad (5.1.13)$$

where

$$B(\beta) := \sum_{x \in \mathbb{Z}^d} \langle \sigma_0 \sigma_x \rangle_{\beta}^2 \quad (5.1.14)$$

is the so-called *bubble diagram*. Note that it is finite when $d > 4$ by (5.1.11). Our next result is a strengthening of (5.1.12).

Theorem 5.1.3. *Let $d > 4$. There exists $A = A(d) > 0$ such that, for $\beta < \beta_c$,*

$$\chi(\beta) = \frac{A}{1 - \beta/\beta_c}(1 + o(1)), \quad (5.1.15)$$

where $o(1)$ tends to 0 as β tends to β_c .

This result is new, even in the setup where the lace expansion can be applied³. We write the asymptotic in terms of β/β_c rather than $\beta_c - \beta$ as it makes a constant $A(d)$ appear, which should tend to 1 as d tends to ∞ . This can be justified by looking at the Curie–Weiss model (which corresponds to $d = \infty$), for which it is known that $A = 1$ (see for instance [101, Chapter 2]). Below, we provide a proof of this convergence result.

²In fact, it is possible to extract from their methods a slightly better bound, where the bubble is replaced by an *open* bubble diagram. See discussion below Theorem 5.1.4 and Remark 5.4.4.

³The existence of A has been derived using the lace expansion for the self-avoiding walk model, see for instance [69]. To the best of our knowledge, the exact asymptotic of the susceptibility is not known for Bernoulli percolation. It would be interesting to see whether it is related the IIC as in the case of the FK-Ising model.

In fact, it is possible to relate the constant A obtained in Theorem 5.1.3 to the IIC of a *double* random current. Let $\mathbf{e}_1$ be the unit vector with first coordinate equal to 1. Let $\mathbf{P}^{0\infty, \mathbf{e}_1\infty}$ be the IIC measure defined in Theorem 5.3.1. Under this measure, 0 lies in an infinite cluster made of an infinite path emerging from the origin and a collection of finite loops attached to this path. Using the terminology introduced in Section 5.2, the *sources* of $\mathbf{n}_1$ can be viewed as “ $\partial\mathbf{n}_1 = \{0, \infty\}$ ”. A similar observation can be made for $\mathbf{e}_1$. Hence, under this measure, two infinite structures emerge from two neighbouring points of $\mathbb{Z}^d$. The next result relates A to the probability that these clusters avoid each other. If $(\mathbf{n}_1, \mathbf{n}_2) \sim \mathbf{P}^{0\infty, \mathbf{e}_1\infty}$, we let $\mathbf{C}(0)$ (resp. $\mathbf{C}(\mathbf{e}_1)$) be the (infinite) cluster of 0 (resp. $\mathbf{e}_1$) in the percolation configuration induced by $\mathbf{n}_1 + \mathbf{n}_2$.

Theorem 5.1.4. *Let $d > 4$. Let A be the constant of Theorem 5.1.3. Then,*

$$A^{-1} = (2d\beta_c) \cdot \mathbf{P}^{0\infty, \mathbf{e}_1\infty}[\mathbf{C}(0) \cap \mathbf{C}(\mathbf{e}_1) = \emptyset]. \quad (5.1.16)$$

We obtain Theorems 5.1.3 and 5.1.4 using the probabilistic interpretation of the derivative of the susceptibility obtained in [4], see (5.4.4). This result already makes the probability of occurrence of the event $\{\mathbf{C}(0) \cap \mathbf{C}(\mathbf{e}_1) = \emptyset\}$ under the following measure appear:

$$\mathbf{Q}_\beta[\cdot] := \frac{1}{\chi(\beta)^2} \sum_{x, y \in \mathbb{Z}^d} \langle \sigma_0 \sigma_x \rangle_\beta \langle \sigma_{\mathbf{e}_1} \sigma_y \rangle_\beta \mathbf{P}_\beta^{0x, \mathbf{e}_1 y}[\cdot], \quad (5.1.17)$$

where $\mathbf{P}_\beta^{0x, \mathbf{e}_1 y}$ samples independent currents $\mathbf{n}_1$ and $\mathbf{n}_2$ with respective source sets $\partial\mathbf{n}_1 = \{0, x\}$ and $\partial\mathbf{n}_2 = \{\mathbf{e}_1, y\}$. In Section 5.3, we prove that the measure $\mathbf{Q}_\beta$ converges weakly as $\beta \nearrow \beta_c$ to the natural candidate for the IIC measure of the double random current. We also prove that convergence holds for *local* events. However, the event of interest is not local. Our main contribution is a *localization* of the event $\{\mathbf{C}(0) \cap \mathbf{C}(\mathbf{e}_1) = \emptyset\}$ which relies on the assumption $d > 4$. See Lemma 5.4.3 for a precise statement.

Limit of $A(d)$ as d tends to infinity Together with the results of [14], we can prove (see Remark 5.4.4) that

$$\mathbf{P}^{0\infty, \mathbf{e}_1\infty}[\mathbf{C}(0) \cap \mathbf{C}(\mathbf{e}_1)] \geq \frac{1}{1 + (2d\beta_c)B^{(\mathbf{e}_1)}(\beta_c)}, \quad (5.1.18)$$

where

$$B^{(\mathbf{e}_1)}(\beta) := \sum_{x \in \mathbb{Z}^d} \langle \sigma_0 \sigma_x \rangle_\beta \langle \sigma_{\mathbf{e}_1} \sigma_x \rangle_\beta \quad (5.1.19)$$

is an *open* bubble diagram. The lace expansion analysis of the Ising model (see [216, 218]) implies that

$$\lim_{d \rightarrow \infty} 2d\beta_c = 1, \quad \lim_{d \rightarrow \infty} B^{(\mathbf{e}_1)}(\beta_c) = 0. \quad (5.1.20)$$

Together with (5.1.18), this gives

$$\lim_{d \rightarrow \infty} \mathbf{P}^{0\infty, \mathbf{e}_1\infty}[\mathbf{C}(0) \cap \mathbf{C}(\mathbf{e}_1) = \emptyset] = 1. \quad (5.1.21)$$

Plugging (5.1.20) and (5.1.21) in Theorem 5.1.4 yields

$$\lim_{d \rightarrow \infty} A(d) = 1. \quad (5.1.22)$$

5.1.4 Open problems

To conclude this introduction, we list a few questions which naturally arise from the above results.

5.1.4.1 Other definitions of the IIC

In view of the above, it is natural to ask whether Kesten's approach can be used to construct the IIC of the FK-Ising model. This leads to the following questions.

Open problem 1. *Let $d \geq 3$. Show that one may define the measure ϕ^∞ by conditioning on the one-arm event:*

$$\phi^\infty = \lim_{n \rightarrow \infty} \phi_{\beta_c}[\cdot \mid 0 \leftrightarrow \partial\Lambda_n]. \quad (5.1.23)$$

In [144] this construction was successfully achieved in the context of high-dimensional Bernoulli percolation using the computation of the one-arm exponent⁴ performed in [168] (to be more precise, they only need a lower bound on this exponent). To the best of our knowledge, a lower bound on this exponent, even in the mean-field regime, is not known (see [130] for an upper bound).

Open problem 2. *Let $d \geq 3$. Show that one may define the measure ϕ^∞ from the supercritical regime:*

$$\phi^\infty = \lim_{\beta \searrow \beta_c} \phi_\beta[\cdot \mid 0 \leftrightarrow \infty]. \quad (5.1.24)$$

The proof of Theorem 5.1.2 relies on the near-critical version of the mixing property of [9]. Such a near-critical tool is not available on the supercritical side and this seems to be the main obstacle to following this approach.

5.1.4.2 Behaviour of the susceptibility when $d = 4$

In view of Theorem 5.1.3, it is natural to ask what is the behaviour of the susceptibility when $d = 4$, where mean-field behaviour still holds. Renormalization group analysis and the universality hypothesis (see for instance [25] for the corresponding result in the setup of the weakly-coupled φ^4 model) suggest the following behaviour.

Conjecture 1. *Let $d = 4$. There exists $A > 0$ such that*

$$\chi(\beta) = \frac{A |\log(1 - \beta/\beta_c)|^{1/3}}{1 - \beta/\beta_c} (1 + o(1)), \quad (5.1.25)$$

where $o(1)$ tends to 0 as β tends to β_c .

The best result regarding this conjecture was obtained in [14] where the authors obtained the existence of $C > 0$ such that for all $\beta < \beta_c$,

$$\frac{(2d\beta_c)^{-1}}{1 - \beta/\beta_c} \leq \chi(\beta) \leq \frac{C |\log(1 - \beta/\beta_c)|}{1 - \beta/\beta_c}. \quad (5.1.26)$$

⁴The one-arm exponent ρ of Bernoulli percolation is defined by $\mathbb{P}_{p_c}[0 \leftrightarrow \partial\Lambda_n] \asymp n^{-1/\rho}$. It is known to be equal to $1/2$ when $d > 10$.

In the four-dimensional case, Theorem 5.1.4 suggests that the divergence of $\chi(\beta)(1 - \beta/\beta_c)$ as β tends to β_c is related to the fact that $\mathbf{P}^{0\infty, \mathbf{e}_1\infty}[\mathbf{C}(0) \cap \mathbf{C}(\mathbf{e}_1) = \emptyset] = 0$. This hints that $d = 4$ is (just like for random walks) the *critical dimension* for the intersection of two neighbouring random current IICs. As we will see in the proof of Theorem 5.1.4, when $d > 5$ it is possible to relate the susceptibility to $\mathbf{P}^{0\infty, \mathbf{e}_1\infty}[\mathbf{C}(0) \cap \mathbf{C}(\mathbf{e}_1) \cap \Lambda_k = \emptyset] + o(1)$ where $o(1)$ is uniform in β and tends to 0 as k tends to infinity. When $d = 4$, this “ $o(1)$ ” is not uniform in β anymore. One potential idea to tackle this problem is to choose $k = k(\beta)$ as a function of β . The following choice seems to be a natural candidate for $k(\beta)$. We recall that $\xi(\beta)$ is the so-called *correlation length*, and is defined by

$$\xi(\beta)^{-1} := \lim_{n \rightarrow \infty} -\frac{1}{n} \log \langle \sigma_0 \sigma_{n\mathbf{e}_1} \rangle_\beta. \quad (5.1.27)$$

Open problem 3. When $d = 4$, relate the asymptotic behaviour of the susceptibility to the probability of avoidance of two neighbouring random current IICs up to distance $\xi(\beta)^C$, where $C \geq 1$.

5.2 The random current representation

We begin by introducing the random current representation of the Ising model. For more information we refer to the lecture notes [76].

5.2.1 Definition and main properties

Let Λ be a finite subset of $\mathbb{Z}^d$.

Definition 5.2.1. A *current* $\mathbf{n}$ on Λ is a function defined on the edge-set $E(\Lambda) := \{\{x, y\} : x, y \in \Lambda \text{ and } x \sim y\}$ and taking its values in $\mathbb{N} = \{0, 1, \dots\}$. We let Ω_Λ be the set of currents on Λ . The set of *sources* of $\mathbf{n}$, denoted by $\partial\mathbf{n}$, is defined as

$$\partial\mathbf{n} := \left\{ x \in \Lambda : \sum_{y \sim x} \mathbf{n}_{x,y} \text{ is odd} \right\}. \quad (5.2.1)$$

We also define

$$w_\beta(\mathbf{n}) := \prod_{\{x,y\} \in E(\Lambda)} \frac{\beta^{\mathbf{n}_{x,y}}}{\mathbf{n}_{x,y}!}. \quad (5.2.2)$$

It is classical [4, 76] that the correlation functions of the Ising model are related to currents: for $S \subset \Lambda$, if $\sigma_S := \prod_{x \in S} \sigma_x$,

$$\langle \sigma_S \rangle_{\Lambda, \beta} = \frac{\sum_{\partial\mathbf{n}=S} w_\beta(\mathbf{n})}{\sum_{\partial\mathbf{n}=\emptyset} w_\beta(\mathbf{n})}. \quad (5.2.3)$$

The trace of a current $\mathbf{n}$ naturally induces a percolation configuration $(\mathbb{1}_{\mathbf{n}_{x,y} > 0})_{\{x,y\} \in E(\Lambda)}$ on Λ . As it turns out, the connectivity properties of this percolation model play an important role in the analysis of the Ising model. This motivates the following terminology.

Definition 5.2.2. Let $\mathbf{n} \in \Omega_\Lambda$ and $x, y \in \Lambda$.

- (i) We say that x is *connected* to y in $\mathbf{n}$ and write $x \xrightarrow{\mathbf{n}} y$, if there is a sequence of points $x_0 = x, x_1, \dots, x_m = y$ such that $\mathbf{n}_{x_i, x_{i+1}} > 0$ for $0 \leq i \leq m-1$.

(ii) The *cluster* of x , denoted by $\mathbf{C}_{\mathbf{n}}(x)$, is the set of points connected to x in $\mathbf{n}$.

The main interest of the random current representation lies in the following combinatorial tool, called the *switching lemma*, which was first introduced in [120]. The probabilistic picture attached to it was developed in [4].

Lemma 5.2.3 (Switching Lemma). *For every $S_1, S_2 \subset \Lambda$ and every function F from Ω_Λ into $\mathbb{R}$,*

$$\begin{aligned} \sum_{\substack{\mathbf{n}_1 \in \Omega_\Lambda: \partial \mathbf{n}_1 = S_1 \\ \mathbf{n}_2 \in \Omega_\Lambda: \partial \mathbf{n}_2 = S_2}} F(\mathbf{n}_1 + \mathbf{n}_2) w_\beta(\mathbf{n}_1) w_\beta(\mathbf{n}_2) \\ = \sum_{\substack{\mathbf{n}_1 \in \Omega_\Lambda: \partial \mathbf{n}_1 = S_1 \Delta S_2 \\ \mathbf{n}_2 \in \Omega_\Lambda: \partial \mathbf{n}_2 = \emptyset}} F(\mathbf{n}_1 + \mathbf{n}_2) w_\beta(\mathbf{n}_1) w_\beta(\mathbf{n}_2) \mathbb{1}_{(\mathbf{n}_1 + \mathbf{n}_2) \in \mathcal{F}_{S_2}}, \end{aligned} \quad (5.2.4)$$

where $S_1 \Delta S_2 = (S_1 \cup S_2) \setminus (S_1 \cap S_2)$ is the symmetric difference of sets and $\mathcal{F}_{S_2}$ is given by

$$\mathcal{F}_{S_2} = \{\mathbf{n} \in \Omega_\Lambda, \exists \mathbf{m} \leq \mathbf{n}, \partial \mathbf{m} = S_2\}. \quad (5.2.5)$$

If $S \subset \Lambda$, define a probability measure $\mathbf{P}_{\Lambda, \beta}^S$ on Ω_Λ as follows: for every $\mathbf{n} \in \Omega_\Lambda$,

$$\mathbf{P}_{\Lambda, \beta}^S[\mathbf{n}] := \mathbb{1}_{\partial \mathbf{n} = S} \frac{w_\beta(\mathbf{n})}{Z_{\Lambda, \beta}^S}, \quad (5.2.6)$$

where $Z_{\Lambda, \beta}^S := \sum_{\partial \mathbf{n} = S} w_\beta(\mathbf{n})$ is a normalisation constant. Moreover, for $S_1, \dots, S_k \subset \Lambda$, define

$$\mathbf{P}_{\Lambda, \beta}^{S_1, \dots, S_k} := \mathbf{P}_{\Lambda, \beta}^{S_1} \otimes \dots \otimes \mathbf{P}_{\Lambda, \beta}^{S_k}. \quad (5.2.7)$$

When $S = \{x, y\}$, we will write $\mathbf{P}_{\Lambda, \beta}^{xy}$ instead of $\mathbf{P}_{\Lambda, \beta}^{\{x, y\}}$. If $\mathcal{E} \subset \Omega_\Lambda$, we will also write $Z_{\Lambda, \beta}^S[\mathcal{E}] := \sum_{\partial \mathbf{n} = S} w_\beta(\mathbf{n}) \mathbb{1}_{\mathbf{n} \in \mathcal{E}}$. As proved in [10], if S is a finite even (i.e. $|S|$ is even) subset of $\mathbb{Z}^d$, the sequence of probability measures $(\mathbf{P}_{\Lambda, \beta}^S)_{\Lambda \subset \mathbb{Z}^d}$ admits a weak limit as $\Lambda \nearrow \mathbb{Z}^d$ that we denote by $\mathbf{P}_\beta^S$.

The following mixing statement will be the main tool to construct the incipient infinite cluster.

Theorem 5.2.4 (Mixing property of currents). *Let $d \geq 3$ and $s \geq 1$. Let $\varepsilon > 0$. For every $n \geq 1$, there exist $N \geq n$ and $\beta(\varepsilon) < \beta_c$ such that, for every $1 \leq t \leq s$, every $\beta \in [\beta(\varepsilon), \beta_c]$, every $x_i \in \Lambda_n$ and $y_i \notin \Lambda_N$ ($i \leq t$), and every events E and F depending on the restriction of $(\mathbf{n}_1, \dots, \mathbf{n}_s)$ to edges with endpoints within Λ_n and outside Λ_N respectively,*

$$\left| \mathbf{P}_\beta^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}[E \cap F] - \mathbf{P}_\beta^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}[E] \mathbf{P}_\beta^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}[F] \right| \leq \varepsilon. \quad (5.2.8)$$

Furthermore, for every $x'_1, \dots, x'_t \in \Lambda_n$ and $y'_1, \dots, y'_t \notin \Lambda_N$, we have that

$$\left| \mathbf{P}_\beta^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}[E] - \mathbf{P}_\beta^{x'_1 y'_1, \dots, x'_t y'_t, \emptyset, \dots, \emptyset}[E] \right| \leq \varepsilon, \quad (5.2.9)$$

$$\left| \mathbf{P}_\beta^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}[F] - \mathbf{P}_\beta^{x'_1 y'_1, \dots, x'_t y'_t, \emptyset, \dots, \emptyset}[F] \right| \leq \varepsilon. \quad (5.2.10)$$

We will only use (5.2.9). However, for sake of completeness, we state and prove the stronger result.

Theorem 5.2.4 was obtained by Aizenman and Duminil-Copin [9] to derive the *marginal triviality* of the scaling limits of Ising and φ^4 systems in dimension four. Their result is even stronger in the sense that it is *quantitative*. However, their argument only works in dimensions $d \geq 4$. This is mainly due to the fact that the infrared bound (5.1.11) is not sharp when $d = 3$. As they mention (see discussion below [9, Theorem 6.4]), it is possible to extend their strategy to the three-dimensional case to the cost of losing the quantitative statement. We provide a full proof in Section 5.5, which heavily relies on the fact that $m^*(\beta_c) = 0$.

5.2.2 Coupling with the FK-Ising model

The following result can be found in [11, Theorem 3.2] and originates from the observation of [186].

Proposition 5.2.5. *Let $G = (V, E)$ be a subgraph of $\mathbb{Z}^d$ and let $S \subset G$ be a finite even subset of G . Let $\beta > 0$. Let $\mathbf{n}$ be distributed according to $\mathbf{P}_{G,\beta}^S$. Let $(\omega_e)_{e \in E}$ be an independent Bernoulli percolation with parameter $1 - \exp(-\beta)$. For each $e \in E$, we define*

$$\eta_e := \max(\mathbf{1}_{\mathbf{n}_e > 0}, \omega_e). \quad (5.2.11)$$

Then, the law of η is exactly $\phi_{G,\beta}^0[\cdot \mid \mathcal{F}_S]$, where $\mathcal{F}_S$ is the event introduced in (5.2.5). In particular, if $\mathcal{A}$ is an increasing event,

$$\mathbf{P}_{G,\beta}^S[\mathcal{A}] \leq \phi_{G,\beta}^0[\mathcal{A} \mid \mathcal{F}_S]. \quad (5.2.12)$$

We will apply the above statement to the special cases $S = \emptyset$ or $S = \{x, y\}$ for $x, y \in G$. In the former case the law of η is $\phi_{G,\beta}^0$, while in the latter case it is $\phi_{G,\beta}^0[\cdot \mid x \leftrightarrow y]$, i.e. the FK-Ising measure conditioned on the event that x is connected to y . Finally, when $G = \mathbb{Z}^d$, by the results recalled above, $\phi_{G,\beta}^0 = \phi_\beta$.

5.3 The incipient infinite cluster

In this section, we construct the IIC measure for both random currents and the FK-Ising model.

5.3.1 Construction of the IIC measure for random currents

We will prove the following result.

Theorem 5.3.1 (The IIC measure of the random current). *Let $d \geq 3$ and $s \geq 1$. Let $1 \leq t \leq s$. For all $x_1, \dots, x_t \in \mathbb{Z}^d$ there exists a measure $\mathbf{P}^{x_1\infty, \dots, x_t\infty, \emptyset, \dots, \emptyset}$ (where $\emptyset$ appears $s - t$ times) on $\Omega_{\mathbb{Z}^d}$ such that, for all local events $\mathcal{A} \subset (\Omega_{\mathbb{Z}^d})^s$,*

$$\lim_{|y_1|, \dots, |y_t| \rightarrow \infty} \mathbf{P}_{\beta_c}^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}[\mathcal{A}] = \mathbf{P}^{x_1\infty, \dots, x_t\infty, \emptyset, \dots, \emptyset}[\mathcal{A}], \quad (5.3.1)$$

regardless of the manner in which $y_1, \dots, y_t$ are sent to infinity. Moreover, one also has,

$$\lim_{\beta \nearrow \beta_c} \frac{1}{\chi(\beta)^t} \sum_{y_1, \dots, y_t \in \mathbb{Z}^d} \langle \sigma_{x_1} \sigma_{y_1} \rangle_\beta \dots \langle \sigma_{x_t} \sigma_{y_t} \rangle_\beta \mathbf{P}_\beta^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}[\mathcal{A}] = \mathbf{P}^{x_1 \infty, \dots, x_t \infty, \emptyset, \dots, \emptyset}[\mathcal{A}]. \quad (5.3.2)$$

Finally, the measure $\mathbf{P}^{x_1 \infty, \dots, x_t \infty, \emptyset, \dots, \emptyset}$ satisfies,

$$\mathbf{P}^{x_1 \infty, \dots, x_t \infty, \emptyset, \dots, \emptyset}[x_i \xrightarrow{\mathbf{n}_i} \infty, \forall 1 \leq i \leq t] = 1. \quad (5.3.3)$$

Proof. We let $\mathcal{A}$ be a local event. Let $n \geq 1$ such that $\mathcal{A}$ is measurable in terms of the edges with endpoints within Λ_n .

Let $\varepsilon > 0$. Using Theorem 5.2.4, we find N large enough such that for all $y_1, y'_1, \dots, y_t, y'_t \notin \Lambda_N$,

$$\left| \mathbf{P}_{\beta_c}^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}[\mathcal{A}] - \mathbf{P}_{\beta_c}^{x_1 y'_1, \dots, x_t y'_t, \emptyset, \dots, \emptyset}[\mathcal{A}] \right| \leq \varepsilon. \quad (5.3.4)$$

This shows that the sequence $(\mathbf{P}_{\beta_c}^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}[\mathcal{A}])_{y_1, \dots, y_t}$ is Cauchy⁵ and thus admits a limit that we denote by $\mathbf{P}^{x_1 \infty, \dots, x_t \infty, \emptyset, \dots, \emptyset}[\mathcal{A}]$. This allows one to define the measure $\mathbf{P}^{x_1 \infty, \dots, x_t \infty, \emptyset, \dots, \emptyset}$ on the cylinder σ -algebra of $\Omega_{\mathbb{Z}^d}$.

We turn to the proof of (5.3.2). Let $\varepsilon > 0$. By Theorem 5.2.4 we have the existence of N large enough and $\beta(\varepsilon)$ close enough to β_c , such that for all $\beta \in [\beta(\varepsilon), \beta_c]$, for all $y_1, y'_1, \dots, y_t, y'_t \notin \Lambda_N$,

$$\left| \mathbf{P}_\beta^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}[\mathcal{A}] - \mathbf{P}_\beta^{x_1 y'_1, \dots, x_t y'_t, \emptyset, \dots, \emptyset}[\mathcal{A}] \right| \leq \frac{\varepsilon}{3}. \quad (5.3.5)$$

Using (5.3.1), we can find $z_1, \dots, z_n \notin \Lambda_N$ such that

$$\left| \mathbf{P}_{\beta_c}^{x_1 z_1, \dots, x_t z_t, \emptyset, \dots, \emptyset}[\mathcal{A}] - \mathbf{P}^{x_1 \infty, \dots, x_t \infty, \emptyset, \dots, \emptyset}[\mathcal{A}] \right| \leq \frac{\varepsilon}{3}. \quad (5.3.6)$$

Finally, we use [10, Theorem 2.3] to argue that the map $\beta \mapsto \mathbf{P}_\beta^{x_1 z_1, \dots, x_t z_t, \emptyset, \dots, \emptyset}[\mathcal{A}]$ is continuous at β_c . This yields the existence of $\beta_1 \in [\beta(\varepsilon), \beta_c]$ such that for all $\beta \in [\beta_1, \beta_c]$,

$$\left| \mathbf{P}_\beta^{x_1 z_1, \dots, x_t z_t, \emptyset, \dots, \emptyset}[\mathcal{A}] - \mathbf{P}_{\beta_c}^{x_1 z_1, \dots, x_t z_t, \emptyset, \dots, \emptyset}[\mathcal{A}] \right| \leq \frac{\varepsilon}{3}. \quad (5.3.7)$$

Combining the last three displayed equations yields: for all $\beta \in [\beta_1, \beta_c]$, for all $y_1, \dots, y_t \notin \Lambda_N$,

$$\left| \mathbf{P}_\beta^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}[\mathcal{A}] - \mathbf{P}^{x_1 \infty, \dots, x_t \infty, \emptyset, \dots, \emptyset}[\mathcal{A}] \right| \leq \varepsilon. \quad (5.3.8)$$

In particular,

$$\limsup_{\beta \nearrow \beta_c} \frac{1}{\chi(\beta)^t} \sum_{y_1, \dots, y_t \in \mathbb{Z}^d} \left(\prod_{i=1}^t \langle \sigma_{x_i} \sigma_{y_i} \rangle_\beta \right) \mathbf{P}_\beta^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}[\mathcal{A}] \leq \varepsilon + \mathbf{P}^{x_1 \infty, \dots, x_t \infty, \emptyset, \dots, \emptyset}[\mathcal{A}], \quad (5.3.9)$$

where we used the fact that $\lim_{\beta \nearrow \beta_c} \frac{\chi_N(\beta)}{\chi(\beta)} = 0$, where $\chi_N(\beta) := \sum_{x \in \Lambda_N} \langle \sigma_0 \sigma_x \rangle_\beta$. Since (5.3.9) holds for any $\varepsilon > 0$ this gives one inequality. The other inequality follows similarly. $\square$

⁵By this, we mean that for any enumerations $r^{(i)}$, $1 \leq i \leq t$, of $\mathbb{Z}^d$, the sequence $(\mathbf{P}_{\beta_c}^{x_1 r_n^{(1)}, \dots, x_t r_n^{(t)}, \emptyset, \dots, \emptyset}[\mathcal{A}])_{n \geq 1}$ is Cauchy, and all the limits are the same.

The known properties of the spin model shed some light on the IIC of the random current. Below we state a (short) non-exhaustive list of properties one may easily derive in the case of the measure $\mathbf{P}^{0\infty, \emptyset}$.

Proposition 5.3.2. *Let $d \geq 4$. The measure $\mathbf{P}^{0\infty, \emptyset}$ constructed in Theorem 5.3.1 satisfies the following properties:*

- (i) *The cluster $\mathbf{C}(0)$ of 0 under $\mathbf{P}^{0\infty, \emptyset}$ is one ended almost surely.*
- (ii) *For any $y \in \mathbb{Z}^d$, one has*

$$\mathbf{P}^{0\infty, \emptyset}[0 \xleftrightarrow{\mathbf{n}_1 + \mathbf{n}_2} y] = \langle \sigma_0 \sigma_y \rangle_{\beta_c}. \quad (5.3.10)$$

Moreover, there exist $c, C > 0$ such that for every $y \in \mathbb{Z}^d$ with $|y| \geq 2$,

$$\frac{C}{|y|^{d-2}} \geq \mathbf{P}^{0\infty, \emptyset}[0 \xleftrightarrow{\mathbf{n}_1 + \mathbf{n}_2} y] \geq \begin{cases} \frac{c}{|y|^{d-2}} & \text{if } d \geq 5, \\ \frac{c}{|y|^2 \log |y|} & \text{if } d = 4. \end{cases} \quad (5.3.11)$$

We postpone the proof of this result to Section 5.6. The property (ii) above suggests that, when $d \geq 4$, the IIC of the measure $\mathbf{P}^{0\infty, \emptyset}$ is a two-dimensional object. We can easily prove that the above result also holds when $d = 3$ under the assumption⁶ that $\lim_{|x| \rightarrow \infty} |x| \langle \sigma_0 \sigma_x \rangle_{\beta_c} = 0$.

5.3.2 Construction of the IIC measure for the FK-Ising model

Theorems 5.1.1 and 5.1.2 are simple consequences of Proposition 5.2.5 and Theorem 5.3.1.

Proof of Theorem 5.1.1. Let ω be a Bernoulli percolation of parameter $1 - \exp(-\beta_c)$, and write $\mathbb{P}_{1-\exp(-\beta_c)}$ for its law. Let $E \in \mathcal{F}_0$ be measurable in terms of edges in Λ_n . By Proposition 5.2.5, for all $x \in \mathbb{Z}^d$,

$$\left(\mathbf{P}_{\beta_c}^{0x} \otimes \mathbb{P}_{1-\exp(-\beta_c)} \right) [(\mathbf{1}_{\mathbf{n}_e > 0})_{e \in \mathbb{E}} \vee \omega \in E] = \phi_{\beta_c}[E \mid 0 \leftrightarrow x]. \quad (5.3.12)$$

Conditioning on the restriction of ω to the edges in Λ_n ,

$$\sum_{\omega_n \in \{0,1\}^{E(\Lambda_n)}} \mathbb{P}_{1-\exp(-\beta_c)}[\omega_n] \mathbf{P}_{\beta_c}^{0x}[(\mathbf{1}_{\mathbf{n}_e > 0})_{e \in \mathbb{E}} \vee \omega_n \in E] = \phi_{\beta_c}[E \mid 0 \leftrightarrow x]. \quad (5.3.13)$$

For a fixed configuration ω_n , the event $\{(\mathbf{1}_{\mathbf{n}_e > 0})_{e \in \mathbb{E}} \vee \omega_n \in E\}$ is local and hence by Theorem 5.3.1 one has

$$\begin{aligned} \lim_{|x| \rightarrow \infty} \sum_{\omega_n \in \{0,1\}^{E(\Lambda_n)}} \mathbb{P}_{1-\exp(-\beta_c)}[\omega_n] \mathbf{P}_{\beta_c}^{0x}[(\mathbf{1}_{\mathbf{n}_e > 0})_{e \in \mathbb{E}} \vee \omega_n \in E] \\ = \sum_{\omega_n \in \{0,1\}^{E(\Lambda_n)}} \mathbb{P}_{1-\exp(-\beta_c)}[\omega_n] \mathbf{P}^{0\infty}[(\mathbf{1}_{\mathbf{n}_e > 0})_{e \in \mathbb{E}} \vee \omega_n \in E] \\ = \left(\mathbf{P}^{0\infty} \otimes \mathbb{P}_{1-\exp(-\beta_c)} \right) [(\mathbf{1}_{\mathbf{n}_e > 0})_{e \in \mathbb{E}} \vee \omega \in E]. \end{aligned} \quad (5.3.14)$$

⁶In fact, we even expect $\lim_{|x| \rightarrow \infty} |x|^{1+\varepsilon} \langle \sigma_0 \sigma_x \rangle_{\beta_c} = 0$ for some $\varepsilon > 0$. This is related to the fact that the critical exponent η of the two-point function is conjectured to satisfy $\eta > 0$ in the 3D Ising model.

Plugging (5.3.14) in (5.3.13), it follows that $\lim_{|x| \rightarrow \infty} \phi_{\beta_c}[E \mid 0 \leftrightarrow x]$ exists and satisfies

$$\left(\mathbf{P}^{0\infty} \otimes \mathbb{P}_{1-\exp(-\beta_c)}\right)[(\mathbf{1}_{\mathbf{n}_e > 0})_{e \in \mathbb{E}} \vee \omega \in E] = \lim_{|x| \rightarrow \infty} \phi_{\beta_c}[E \mid 0 \leftrightarrow x]. \quad (5.3.15)$$

Since this holds for every $E \in \mathcal{F}_0$, the proof follows by defining ϕ^∞ to be the law of a percolation configuration sampled by $\mathbf{P}^{0\infty}$ and sprinkled by an independent Bernoulli percolation of parameter $1 - \exp(-\beta_c)$. $\square$

Proof of Theorem 5.1.2. Using (5.1.4) and Proposition 5.2.5, for every $\beta < \beta_c$, for every $E \in \mathcal{F}_0$,

$$\mathbb{Q}_\beta[E] = \frac{1}{\chi(\beta)} \sum_{x \in \mathbb{Z}^d} \langle \sigma_0 \sigma_x \rangle_\beta \phi_\beta[E \mid 0 \leftrightarrow x] \quad (5.3.16)$$

$$= \frac{1}{\chi(\beta)} \sum_{x \in \mathbb{Z}^d} \langle \sigma_0 \sigma_x \rangle_\beta \left(\mathbf{P}_\beta^{0x} \otimes \mathbb{P}_{1-\exp(-\beta)} \right) [(\mathbf{1}_{\mathbf{n}_e > 0})_{e \in \mathbb{E}} \vee \omega \in E] \quad (5.3.17)$$

$$= \left(\frac{1}{\chi(\beta)} \sum_{x \in \mathbb{Z}^d} \langle \sigma_0 \sigma_x \rangle_\beta \mathbf{P}_\beta^{0x} \right) \otimes \mathbb{P}_{1-\exp(-\beta)} [(\mathbf{1}_{\mathbf{n}_e > 0})_{e \in \mathbb{E}} \vee \omega \in E]. \quad (5.3.18)$$

Once again, if E is measurable in terms of edges in Λ_n , for every $\omega_n \in \{0, 1\}^{E(\Lambda_n)}$, Theorem 5.3.1 implies the following convergence,

$$\lim_{\beta \nearrow \beta_c} \frac{1}{\chi(\beta)} \sum_{x \in \mathbb{Z}^d} \langle \sigma_0 \sigma_x \rangle_\beta \mathbf{P}_\beta^{0x} [(\mathbf{1}_{\mathbf{n}_e > 0})_{e \in \mathbb{E}} \vee \omega_n \in E] = \mathbf{P}^{0\infty} [(\mathbf{1}_{\mathbf{n}_e > 0})_{e \in \mathbb{E}} \vee \omega_n \in E]. \quad (5.3.19)$$

Thus, as above,

$$\lim_{\beta \nearrow \beta_c} \mathbb{Q}_\beta[E] = \left(\mathbf{P}^{0\infty} \otimes \mathbb{P}_{1-\exp(-\beta_c)} \right) [(\mathbf{1}_{\mathbf{n}_e > 0})_{e \in \mathbb{E}} \vee \omega \in E] = \phi^\infty[E], \quad (5.3.20)$$

where we used Theorem 5.1.1 (or more specifically (5.3.15)) in the last equality. $\square$

5.4 Near-critical behaviour of the susceptibility when $d > 4$: proof of Theorem 5.1.3

We now turn to the study of the high-dimensional susceptibility of the Ising model. We begin by recalling a classical consequence of the switching lemma.

5.4.1 Geometric interpretation of $\frac{d\chi^{-1}}{d\beta}$

Set $J_{u,v} = \mathbb{1}_{|u-v|_2=1}$ and let $|J| := \sum_{x \in \mathbb{Z}^d} J_{0,x} = 2d$. The following computation goes back to [4]. We start by writing

$$-\frac{d\chi^{-1}(\beta)}{d\beta} = \frac{1}{2} \frac{1}{\chi(\beta)^2} \sum_{y \in \mathbb{Z}^d} \sum_{u,v \in \mathbb{Z}^d} J_{u,v} \langle \sigma_0 \sigma_y; \sigma_u \sigma_v \rangle_\beta. \quad (5.4.1)$$

Now, notice that $\langle \sigma_0 \sigma_y; \sigma_u \sigma_v \rangle_\beta = \langle \sigma_0 \sigma_u \rangle_\beta \langle \sigma_v \sigma_y \rangle_\beta + \langle \sigma_0 \sigma_v \rangle_\beta \langle \sigma_u \sigma_y \rangle_\beta + U_4^\beta(0, y, u, v)$, where $U_4^\beta(0, y, u, v)$ is Ursell's four-point function and is defined by

$$U_4^\beta(0, y, u, v) := \langle \sigma_0 \sigma_y \sigma_u \sigma_v \rangle_\beta - \langle \sigma_0 \sigma_y \rangle_\beta \langle \sigma_u \sigma_v \rangle_\beta - \langle \sigma_0 \sigma_u \rangle_\beta \langle \sigma_y \sigma_v \rangle_\beta - \langle \sigma_0 \sigma_v \rangle_\beta \langle \sigma_y \sigma_u \rangle_\beta. \quad (5.4.2)$$

A classical consequence of the switching lemma [4] is the following representation of U_4^β :

$$U_4^\beta(0, y, u, v) = -2 \langle \sigma_0 \sigma_u \rangle_\beta \langle \sigma_v \sigma_y \rangle_\beta \mathbf{P}_\beta^{0u,vy}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(u) \cap \mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(v) \neq \emptyset]. \quad (5.4.3)$$

Hence, using (5.4.1), (5.4.3), and the symmetries of the model,

$$\begin{aligned} -\frac{d\chi^{-1}(\beta)}{d\beta} &= |J| - \frac{1}{\chi(\beta)^2} \sum_{y,u,v \in \mathbb{Z}^d} J_{u,v} \langle \sigma_0 \sigma_u \rangle_\beta \langle \sigma_v \sigma_y \rangle_\beta \mathbf{P}_\beta^{0u,vy}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(u) \cap \mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(v) \neq \emptyset] \\ &= \frac{1}{\chi(\beta)^2} \sum_{y,u,v \in \mathbb{Z}^d} J_{u,v} \langle \sigma_0 \sigma_u \rangle_\beta \langle \sigma_v \sigma_y \rangle_\beta \mathbf{P}_\beta^{0u,vy}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(u) \cap \mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(v) = \emptyset] \\ &= \frac{1}{\chi(\beta)^2} \sum_{x,y \in \mathbb{Z}^d} \sum_{\substack{w \in \mathbb{Z}^d \\ |w|_2=1}} \langle \sigma_0 \sigma_x \rangle_\beta \langle \sigma_w \sigma_y \rangle_\beta \mathbf{P}_\beta^{0x,wy}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(0) \cap \mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(w) = \emptyset] \\ &= \frac{2d}{\chi(\beta)^2} \sum_{x,y \in \mathbb{Z}^d} \langle \sigma_0 \sigma_x \rangle_\beta \langle \sigma_{\mathbf{e}_1} \sigma_y \rangle_\beta \mathbf{P}_\beta^{0x,\mathbf{e}_1 y}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(0) \cap \mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(\mathbf{e}_1) = \emptyset]. \end{aligned} \quad (5.4.4)$$

See Figure 5.1 for an illustration. One of the contributions of this chapter is a precise analysis of the probability appearing in (5.4.4). The main difficulty with this lies in the lack of independence of $\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(0)$ and $\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(\mathbf{e}_1)$, which is usually helpful for the purpose of using the switching lemma. To lighten the notations below, we will write for an event $\mathcal{E}$,

$$\bar{\mathbf{P}}_\beta^{(\ell)}[\mathcal{E}] := \frac{1}{\chi(\beta)^2} \sum_{\substack{|x| \geq \ell \\ |y| \geq \ell}} \langle \sigma_0 \sigma_x \rangle_\beta \langle \sigma_{\mathbf{e}_1} \sigma_y \rangle_\beta \mathbf{P}_\beta^{0x,\mathbf{e}_1 y}[\mathcal{E}]. \quad (5.4.5)$$

This quantity corresponds to averaging the probability of occurrence of an event $\mathcal{E}$ with respect to the position of the sources.

Remark 5.4.1. In (5.4.4), the contribution of x or y in Λ_ℓ is bounded by

$$4d \frac{\chi_\ell(\beta)}{\chi(\beta)}. \quad (5.4.6)$$

This can be made arbitrarily small by choosing β close enough to β_c . As a result, as we approach β_c , it is enough to study $\bar{\mathbf{P}}_\beta^{(\ell)}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(0) \cap \mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(\mathbf{e}_1) = \emptyset]$. This justifies the notation introduced in (5.4.5).

Definition 5.4.2 (Local event of avoidance). Let $k \geq 1$. We introduce the local event

$$\mathcal{A}_k := \left\{ \mathbf{n} \in \Omega_{\mathbb{Z}^d}, \mathbf{C}_{\mathbf{n}|_{\Lambda_k}}(0) \cap \mathbf{C}_{\mathbf{n}|_{\Lambda_k}}(\mathbf{e}_1) = \emptyset \right\}, \quad (5.4.7)$$

or, in other words, the event that the clusters of 0 and $\mathbf{e}_1$ in the restriction of $\mathbf{n}$ to $\Lambda_k(0)$ do not intersect.

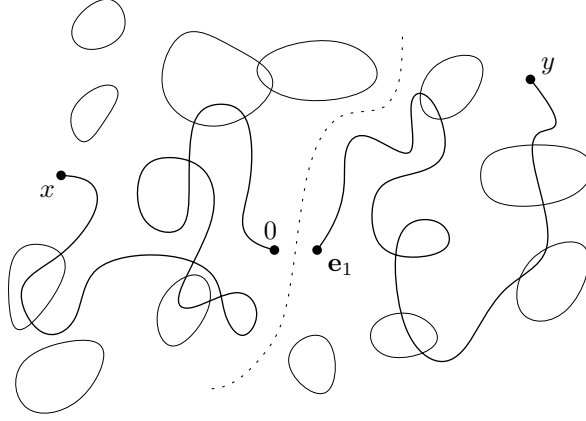


Figure 5.1: A configuration $\mathbf{n}$ which satisfies the event appearing in (5.4.4). The dotted line highlights the fact that 0 and $\mathbf{e}_1$ are not connected in the percolation configuration induced by $\mathbf{n}$.

The following lemma formalises the fact that the event appearing in (5.4.4) is essentially local around 0 and $\mathbf{e}_1$. This is the main technical step of the proof.

Lemma 5.4.3. *There exists $C = C(d), \eta > 0$ such that, for all $\beta \leq \beta_c$, for all $k \geq 1$ and $\ell \geq 2k$,*

$$\left| \bar{\mathbf{P}}_{\beta}^{(\ell)}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(0) \cap \mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(\mathbf{e}_1) = \emptyset] - \bar{\mathbf{P}}_{\beta}^{(\ell)}[\mathcal{A}_k] \right| \leq Ck^{-\eta}. \quad (5.4.8)$$

Before proving Lemma 5.4.3, we show how to conclude the proof of Theorems 5.1.3 and 5.1.4.

Proof of Theorems 5.1.3 and 5.1.4. Using Theorem 5.3.1,

$$\lim_{\beta \nearrow \beta_c} \frac{1}{\chi(\beta)^2} \sum_{x,y \in \mathbb{Z}^d} \langle \sigma_0 \sigma_x \rangle_{\beta} \langle \sigma_{\mathbf{e}_1} \sigma_y \rangle_{\beta} \mathbf{P}_{\beta}^{0x, \mathbf{e}_1 y}[\mathcal{A}_k] = \mathbf{P}^{0\infty, \mathbf{e}_1 \infty}[\mathcal{A}_k]. \quad (5.4.9)$$

Let $\varepsilon > 0$. Combining (5.4.4), Lemma 5.4.3 and (5.4.9), if β is sufficiently close to β_c and $\ell \geq 2k$,

$$-\frac{d\chi^{-1}(\beta)}{d\beta} \leq 4d \frac{\chi_{\ell}(\beta)}{\chi(\beta)} + 2d \cdot \mathbf{P}^{0\infty, \mathbf{e}_1 \infty}[\mathcal{A}_k] + \frac{C}{k^{\eta}} + \varepsilon. \quad (5.4.10)$$

Using Remark 5.4.1, we may choose β even closer to β_c to get

$$-\frac{d\chi^{-1}(\beta)}{d\beta} \leq 2d \cdot \mathbf{P}^{0\infty, \mathbf{e}_1 \infty}[\mathcal{A}_k] + \frac{C}{k^{\eta}} + 2\varepsilon. \quad (5.4.11)$$

so that, integrating between β and β_c and taking β to β_c yields,

$$\limsup_{\beta \nearrow \beta_c} [\chi(\beta)(1 - \beta/\beta_c)]^{-1} \leq (2d\beta_c) \cdot \mathbf{P}^{0\infty, \mathbf{e}_1 \infty}[\mathcal{A}_k] + \frac{C}{k^{\eta}} + 2\varepsilon. \quad (5.4.12)$$

Letting $k \rightarrow \infty$ and $\varepsilon \rightarrow 0$,

$$\limsup_{\beta \nearrow \beta_c} [\chi(\beta)(1 - \beta/\beta_c)]^{-1} \leq (2d\beta_c) \cdot \mathbf{P}^{0\infty, \mathbf{e}_1 \infty}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(0) \cap \mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(\mathbf{e}_1) = \emptyset]. \quad (5.4.13)$$

Similarly, if β is close enough to β_c , $\bar{\mathbf{P}}_\beta^{(\ell)}[\mathcal{A}_k] \geq \mathbf{P}^{0\infty, \mathbf{e}_1\infty}[\mathcal{A}_k] - \frac{\varepsilon}{2d}$ so that

$$-\frac{d\chi^{-1}(\beta)}{d\beta} \geq 2d \cdot \bar{\mathbf{P}}_\beta^{(\ell)}[\mathcal{A}_k] - \frac{C}{k^\eta} \geq 2d \cdot \mathbf{P}^{0\infty, \mathbf{e}_1\infty}[\mathcal{A}_k] - \frac{C}{k^\eta} - \varepsilon. \quad (5.4.14)$$

We deduce again that

$$\liminf_{\beta \nearrow \beta_c} [\chi(\beta)(1 - \beta/\beta_c)]^{-1} \geq (2d\beta_c) \cdot \mathbf{P}^{0\infty, \mathbf{e}_1\infty}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(0) \cap \mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(\mathbf{e}_1) = \emptyset]. \quad (5.4.15)$$

The proof follows readily $\square$

Remark 5.4.4. In fact, the above proof also shows that

$$\lim_{\beta \nearrow \beta_c} \left| \frac{d\chi^{-1}(\beta)}{d\beta} \right| = 2d \cdot \mathbf{P}^{0\infty, \mathbf{e}_1\infty}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(0) \cap \mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(\mathbf{e}_1) = \emptyset]. \quad (5.4.16)$$

Using [14, (3.4)], this gives

$$\mathbf{P}^{0\infty, \mathbf{e}_1\infty}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(0) \cap \mathbf{C}_{\mathbf{n}_1+\mathbf{n}_2}(\mathbf{e}_1) = \emptyset] \geq \frac{1}{1 + (2d\beta_c)B(\beta_c)}. \quad (5.4.17)$$

However, it is possible to obtain a slightly better bound, replacing $B(\beta_c)$ by $B^{(\mathbf{e}_1)}(\beta_c)$ (defined in (5.1.19)). We briefly explain how for the sake of completeness. The lower bound on $\left| \frac{d\chi^{-1}(\beta)}{d\beta} \right|$ is based on an improved bound on U_4^β [14, (4.1')]: for $x_1, x_2, x_3, x_4 \in \mathbb{Z}^d$,

$$\begin{aligned} |U_4^\beta(x_1, x_2, x_3, x_4)| &\leq \sum_{u,v \in \mathbb{Z}^d} J_{u,v} \langle \sigma_{x_4} \sigma_v \rangle_\beta \langle \sigma_{x_2} \sigma_v \rangle_\beta (\beta J_{u,v}) \frac{\partial \langle \sigma_{x_1} \sigma_{x_3} \rangle_\beta}{\partial (\beta J_{u,v})} \\ &\quad + \langle \sigma_{x_1} \sigma_{x_2} \rangle_\beta \langle \sigma_{x_1} \sigma_{x_3} \rangle_\beta \langle \sigma_{x_1} \sigma_{x_4} \rangle_\beta + \langle \sigma_{x_3} \sigma_{x_1} \rangle_\beta \langle \sigma_{x_3} \sigma_{x_2} \rangle_\beta \langle \sigma_{x_3} \sigma_{x_4} \rangle_\beta. \end{aligned} \quad (5.4.18)$$

As in (5.4.1) and below, and using Lebowitz's inequality $U_4^\beta \leq 0$, we write

$$\left| \frac{d\chi^{-1}(\beta)}{d\beta} \right| = |J| - \frac{1}{2} \frac{1}{\chi(\beta)^2} \sum_{x,y,z \in \mathbb{Z}^d} J_{y,z} |U_4^\beta(0, y, x, z)|. \quad (5.4.19)$$

Injecting (5.4.18) in (5.4.19),

$$\left| \frac{d\chi^{-1}(\beta)}{d\beta} \right| \geq |J| - \frac{1}{2} \frac{1}{\chi(\beta)^2} \sum_{x,y,z,u,v \in \mathbb{Z}^d} J_{y,z} \langle \sigma_z \sigma_v \rangle_\beta \langle \sigma_y \sigma_v \rangle_\beta (\beta J_{u,v}) \frac{\partial \langle \sigma_0 \sigma_x \rangle_\beta}{\partial (\beta J_{u,v})} \quad (5.4.20)$$

$$- \frac{1}{2} \frac{1}{\chi(\beta)^2} \sum_{x,y,z \in \mathbb{Z}^d} J_{y,z} \langle \sigma_0 \sigma_x \rangle_\beta \langle \sigma_0 \sigma_y \rangle_\beta \langle \sigma_0 \sigma_z \rangle_\beta \quad (5.4.21)$$

$$- \frac{1}{2} \frac{1}{\chi(\beta)^2} \sum_{x,y,z \in \mathbb{Z}^d} J_{y,z} \langle \sigma_x \sigma_0 \rangle_\beta \langle \sigma_x \sigma_y \rangle_\beta \langle \sigma_x \sigma_z \rangle_\beta \quad (5.4.22)$$

$$=: |J| - (I) - (II) - (III). \quad (5.4.23)$$

Now, using translation invariance⁷

$$(I) = \frac{1}{2} \frac{1}{\chi(\beta)^2} \sum_{x,u,v \in \mathbb{Z}^d} (\beta J_{u,v}) \frac{\partial \langle \sigma_0 \sigma_x \rangle_\beta}{\partial (\beta J_{u,v})} \left(\sum_{y \in \mathbb{Z}^d} \sum_{|w|_2=1} \langle \sigma_y \sigma_v \rangle_\beta \langle \sigma_{y+w} \sigma_v \rangle_\beta \right) \quad (5.4.24)$$

$$= (2d\beta) B^{(\mathbf{e}_1)}(\beta) \left| \frac{d\chi^{-1}(\beta)}{d\beta} \right|, \quad (5.4.25)$$

where we recall that $B^{(\mathbf{e}_1)}(\beta) = \sum_{x \in \mathbb{Z}^d} \langle \sigma_0 \sigma_x \rangle_\beta \langle \sigma_{\mathbf{e}_1} \sigma_x \rangle_\beta$. Moreover, using the Cauchy–Schwarz inequality

$$(II) + (III) \leq \frac{|J|B(\beta)}{\chi(\beta)}. \quad (5.4.26)$$

Plugging (5.4.25) and (5.4.26) in (5.4.23) gives

$$\left| \frac{d\chi^{-1}(\beta)}{d\beta} \right| \geq \frac{2d}{1 + (2d\beta_c)B^{(\mathbf{e}_1)}(\beta_c)} \left(1 - \frac{B(\beta_c)}{\chi(\beta)} \right). \quad (5.4.27)$$

Since $B(\beta_c) < \infty$ by (5.1.11), one gets

$$\lim_{\beta \nearrow \beta_c} \left| \frac{d\chi^{-1}(\beta)}{d\beta} \right| \geq \frac{2d}{1 + (2d\beta_c)B^{(\mathbf{e}_1)}(\beta_c)}, \quad (5.4.28)$$

which immediately yields the improvement on (5.4.17) by (5.4.16).

5.4.2 Proof of Lemma 5.4.3

The rest of the section is devoted to the proof of the *localization* result of Lemma 5.4.3. We will use the notion of *backbone* of a current introduced in [8, Section 4] (see also [11, 204]). A current $\mathbf{n}$ on a $\Lambda \subset \mathbb{Z}^d$ finite with sources $\partial \mathbf{n} = \{x, y\}$ has at least one path from x to y in the percolation configuration induced by $\mathbf{n}$. The backbone of $\mathbf{n}$ is an exploration of one such path. We fix an arbitrary ordering $\prec$ of the edges of $E(\Lambda)$.

Definition 5.4.5. Let $\mathbf{n} \in \Omega_\Lambda$ with $\partial \mathbf{n} = \{x, y\}$. The *backbone* of $\mathbf{n}$, denoted $\Gamma(\mathbf{n})$, is the unique oriented and edge self-avoiding path from x to y , supported on edges with $\mathbf{n}$ odd, that is minimal for $\prec$. The backbone $\Gamma(\mathbf{n}) = \{x_i x_{i+1} : 0 \leq i < k\}$ is obtained via the following exploration:

- (i) Set $x_0 = x$. The first edge $x_0 x_1$ of $\Gamma(\mathbf{n})$ is the first of all the edges e emerging from x_0 for which $\mathbf{n}_e$ is odd. Thus, all edges $e \ni x_0$ satisfying $e \prec x_0 x_1$ and $e \neq x_0 x_1$ are explored and have $\mathbf{n}_e$ even.
- (ii) Each edge $x_i x_{i+1}$, $i \geq 1$, is the first of all edges e emerging from x_i that have not been explored previously, and for which $\mathbf{n}_e$ is odd.
- (iii) The exploration stops when it reaches a vertex from which there are no more odd or non-explored edges available. This must happen at $x_k = y$.

Let $\bar{\Gamma}(\mathbf{n})$ denote the set of explored edges, i.e. $\Gamma(\mathbf{n})$ together with all explored even edges.

⁷In [14], the authors apply the Cauchy–Schwarz inequality and get $B(\beta)$ instead of $B^{(\mathbf{e}_1)}(\beta)$.

By definition, $\bar{\Gamma}(\mathbf{n})$ is determined by $\Gamma(\mathbf{n})$ and the ordering $\prec$, and similarly $\Gamma(\mathbf{n})$ can be determined from $\bar{\Gamma}(\mathbf{n})$ by choosing odd edges in $\bar{\Gamma}(\mathbf{n})$ according to $\prec$. The current $\mathbf{n} \setminus \bar{\Gamma}(\mathbf{n})$, defined to be the restriction of $\mathbf{n}$ to the complement of $\bar{\Gamma}(\mathbf{n})$, is sourceless. Moreover, we can write

$$\langle \sigma_x \sigma_y \rangle_{\Lambda, \beta} = \sum_{\gamma: x \rightarrow y} \rho_{\Lambda}(\gamma), \quad (5.4.29)$$

where for a path $\gamma: x \rightarrow y$,

$$\rho_{\Lambda}(\gamma) := \langle \sigma_x \sigma_y \rangle_{\beta} \mathbf{P}_{\Lambda, \beta}^{xy}[\Gamma(\mathbf{n}) = \gamma]. \quad (5.4.30)$$

When $\Lambda = \mathbb{Z}^d$, we write $\rho(\gamma) = \rho_{\Lambda}(\gamma)$. We will use the following useful property of backbones, often referred to as the *chain rule* (see [8]).

Proposition 5.4.6 (Chain rule for the backbone). *Let $\Lambda \subset \mathbb{Z}^d$. Let $x, y, u, v \in \Lambda$. Then,*

$$\mathbf{P}_{\Lambda, \beta}^{xy}[\Gamma(\mathbf{n}) \text{ passes through } u \text{ first and then through } v] \leq \frac{\langle \sigma_x \sigma_u \rangle_{\Lambda, \beta} \langle \sigma_u \sigma_v \rangle_{\Lambda, \beta} \langle \sigma_v \sigma_y \rangle_{\Lambda, \beta}}{\langle \sigma_x \sigma_y \rangle_{\Lambda, \beta}}. \quad (5.4.31)$$

In particular, if $u = v$,

$$\mathbf{P}_{\Lambda, \beta}^{xy}[\Gamma(\mathbf{n}) \text{ passes through } u] \leq \frac{\langle \sigma_x \sigma_u \rangle_{\Lambda, \beta} \langle \sigma_u \sigma_y \rangle_{\Lambda, \beta}}{\langle \sigma_x \sigma_y \rangle_{\Lambda, \beta}}. \quad (5.4.32)$$

Proof of Lemma 5.4.3. Clearly $\{\mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_2}(0) \cap \mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_2}(\mathbf{e}_1) = \emptyset\} \subset \mathcal{A}_k$. It then suffices to analyse the event $\mathcal{B} := \mathcal{A}_k \cap \{\mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_2}(0) \cap \mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_2}(\mathbf{e}_1) \neq \emptyset\}$. Note that if the two clusters intersect, there must be a path from $\mathbf{e}_1$ to $\Gamma(\mathbf{n}_1)$ that does not use any edge of $\bar{\Gamma}(\mathbf{n}_1)$, except maybe for the last edge:

$$\{\mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_2}(0) \cap \mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_2}(\mathbf{e}_1) \neq \emptyset\} \subset \bigcup_{\substack{u \in \mathbb{Z}^d \\ v \in \mathbb{Z}^d \\ u \sim v}} \{u \in \Gamma(\mathbf{n}_1)\} \cap \{\mathbf{e}_1 \xrightarrow{\mathbf{n}_1 + \mathbf{n}_2 \setminus \bar{\Gamma}(\mathbf{n}_1)} v\} \cap \{(\mathbf{n}_1 + \mathbf{n}_2)_{u,v} > 0\}. \quad (5.4.33)$$

Let $\mathcal{B}_{u,v} := \mathcal{B} \cap \{u \in \Gamma(\mathbf{n}_1)\} \cap \{\mathbf{e}_1 \xrightarrow{\mathbf{n}_1 + \mathbf{n}_2 \setminus \bar{\Gamma}(\mathbf{n}_1)} v\} \cap \{(\mathbf{n}_1 + \mathbf{n}_2)_{u,v} > 0\}$. Let $m = k^\delta$ for some small $\delta > 0$ to be fixed.

• We first look at the case $u \notin \Lambda_m$, see Figure 5.2 for an illustration. Exploring $\Gamma(\mathbf{n}_1)$, and forgetting about $\mathcal{B} \cap \{(\mathbf{n}_1 + \mathbf{n}_2)_{u,v} > 0\}$, we get⁸

$$\langle \sigma_0 \sigma_x \rangle_{\beta} \langle \sigma_{\mathbf{e}_1} \sigma_y \rangle_{\beta} \mathbf{P}_{\beta}^{0x, \mathbf{e}_1 y}[\mathcal{B}_{u,v}] \leq \sum_{\substack{\gamma: 0 \rightarrow x \\ \gamma \ni u}} \rho(\gamma) \langle \sigma_{\mathbf{e}_1} \sigma_y \rangle_{\beta} \mathbf{P}_{\bar{\gamma}^c, \mathbb{Z}^d, \beta}^{\emptyset, \mathbf{e}_1 y}[\mathbf{e}_1 \xrightarrow{\mathbf{n}_1 + \mathbf{n}_2 \setminus \bar{\gamma}} v] \quad (5.4.34)$$

$$= \sum_{\substack{\gamma: 0 \rightarrow x \\ \gamma \ni u}} \rho(\gamma) \langle \sigma_{\mathbf{e}_1} \sigma_v \rangle_{\bar{\gamma}^c, \beta} \langle \sigma_v \sigma_y \rangle_{\beta} \quad (5.4.35)$$

$$\leq \langle \sigma_{\mathbf{e}_1} \sigma_v \rangle_{\beta} \langle \sigma_v \sigma_y \rangle_{\beta} \sum_{\substack{\gamma_1: 0 \rightarrow u \\ \gamma_2: u \rightarrow x}} \rho(\gamma_1 \circ \gamma_2) \quad (5.4.36)$$

$$\leq \langle \sigma_{\mathbf{e}_1} \sigma_v \rangle_{\beta} \langle \sigma_v \sigma_y \rangle_{\beta} \langle \sigma_0 \sigma_u \rangle_{\beta} \langle \sigma_u \sigma_x \rangle_{\beta}, \quad (5.4.37)$$

⁸A similar argument is used in the proof of [204, Lemma 6.13].

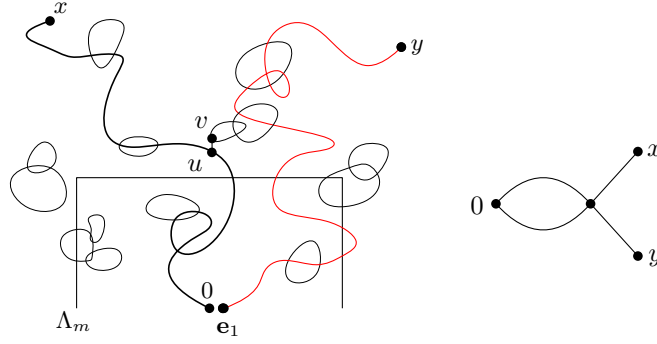


Figure 5.2: On the left, an illustration of a configuration realizing $\mathcal{B}_{u,v}$ for $u \notin \Lambda_m$. The backbone of $\mathbf{n}_1$ (resp. $\mathbf{n}_2$) is the black (resp. red) bold line. A string of loops connects $\Gamma(\mathbf{n}_2)$ to $\Gamma(\mathbf{n}_1)$ in $\mathbf{n}_1 + \mathbf{n}_2$ without using any edges of $\bar{\Gamma}(\mathbf{n}_1)$. On the right, a diagrammatic representation of the bound obtained in (5.4.37).

where we used the switching lemma on the second line, Griffiths' inequality on the third line, and Proposition 5.4.6 on the last line. As a result, using the infrared bound (5.1.11), there exists $C_1 = C_1(d) > 0$ such that

$$\frac{1}{\chi(\beta)^2} \sum_{|x|, |y| \geq \ell} \sum_{u \notin \Lambda_m} \sum_{v \sim u} \langle \sigma_0 \sigma_x \rangle_\beta \langle \sigma_{\mathbf{e}_1} \sigma_y \rangle_\beta \mathbf{P}_\beta^{0x, \mathbf{e}_1 y}[\mathcal{B}_{u,v}] \leq \sum_{u \notin \Lambda_m} \sum_{v \sim u} \langle \sigma_0 \sigma_u \rangle_\beta \langle \sigma_{\mathbf{e}_1} \sigma_u \rangle_\beta \leq \frac{C_1}{m^{d-4}}. \quad (5.4.38)$$

• We now turn to the case $u \in \Lambda_m$. We introduce an additional intermediate scale $M = m^A \leq k$ with $A > 0$ large enough⁹ to be fixed. Notice that under the occurrence of $\mathcal{B}_{u,v}$, any path connecting $\mathbf{e}_1$ to v in the complement of $\bar{\Gamma}(\mathbf{n}_1)$ has to¹⁰ exit the box. We claim that if $\mathcal{B}_{u,v}$ occurs then one of the following events occurs (see Figure 5.3),

- $\mathcal{G}_1$: the backbone $\Gamma(\mathbf{n}_2)$ makes two successive crossings of $\text{Ann}(M, k)$,
- $\mathcal{G}_2$: the event $\mathcal{F}_1$ does not occur and there exist $s \in \partial\Lambda_M$ and $t \in \partial\Lambda_m$ such that $s \xleftrightarrow{\mathbf{n}_1 + \mathbf{n}_2 \setminus (\bar{\Gamma}(\mathbf{n}_1) \cup \bar{\Gamma}(\mathbf{n}_2))} t$.

Indeed, explore a path from $\mathbf{e}_1$ to v in $\mathbf{n}_1 + \mathbf{n}_2 \setminus \bar{\Gamma}(\mathbf{n}_1)$. As mentioned above, this path must reach $\partial\Lambda_M$ a last time at a vertex s before reaching v . If $\mathcal{G}_1$ does not occur, then s must be connected to v without using any edges of $\bar{\Gamma}(\mathbf{n}_2)$. If not, there would be a path inside Λ_M connecting s to both v and a point¹¹ of $\Gamma(\mathbf{n}_2)$. Since $(\mathbf{n}_1 + \mathbf{n}_2)_{u,v} > 0$ this would create a connection between 0 and $\mathbf{e}_1$ that is entirely included in Λ_k , which is impossible by assumption.

We begin with a bound of $\mathcal{G}_1$. A similar bound was already used in [9, (4.21)]. By the

⁹How large A can be taken will depend on how small δ is chosen.

¹⁰This is where the event $\{(\mathbf{n}_1 + \mathbf{n}_2)_{u,v} > 0\}$ becomes useful in the definition of $\mathcal{B}_{u,v}$.

¹¹We are using the fact that any edge of $\bar{\Gamma}(\mathbf{n}_2)$ is incident to a vertex of $\Gamma(\mathbf{n}_2)$.

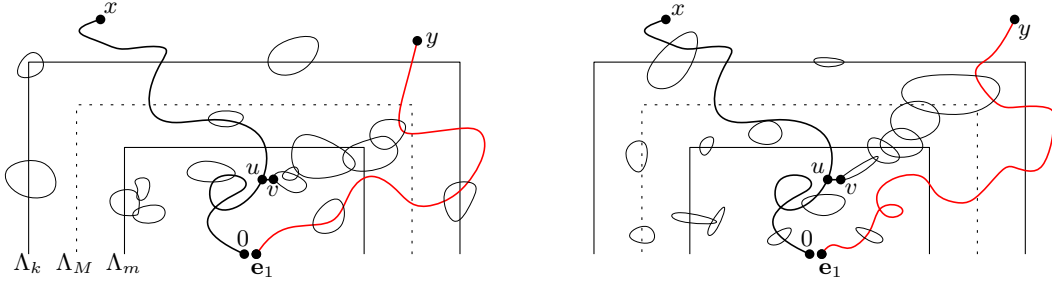


Figure 5.3: An illustration of the events $\mathcal{G}_1$ and $\mathcal{G}_2$. The backbone of $\mathbf{n}_1$ (resp. $\mathbf{n}_2$) is the black (resp. red) bold line. On the left, $\Gamma(\mathbf{n}_2)$ crosses $\text{Ann}(M, k)$ twice and a string of loops creates a connection between the clusters of 0 and $\mathbf{e}_1$. On the right, $\Gamma(\mathbf{n}_2)$ only crosses $\text{Ann}(M, k)$ once. This forces the existence of a crossing of $\text{Ann}(m, M)$ in $\mathbf{n}_1 + \mathbf{n}_2 \setminus \bar{\Gamma}(\mathbf{n}_1) \cup \bar{\Gamma}(\mathbf{n}_2)$.

chain rule of Proposition 5.4.6,

$$\langle \sigma_0 \sigma_x \rangle_\beta \langle \sigma_{\mathbf{e}_1} \sigma_y \rangle_\beta \mathbf{P}_\beta^{0x, \mathbf{e}_1 y}[\mathcal{G}_1] = \langle \sigma_0 \sigma_x \rangle_\beta \langle \sigma_{\mathbf{e}_1} \sigma_y \rangle_\beta \mathbf{P}_\beta^{\mathbf{e}_1 y}[\mathcal{G}_1] \quad (5.4.39)$$

$$\leq \langle \sigma_0 \sigma_x \rangle_\beta \sum_{\substack{a \in \partial \Lambda_k \\ b \in \partial \Lambda_M}} \langle \sigma_{\mathbf{e}_1} \sigma_a \rangle_\beta \langle \sigma_a \sigma_b \rangle_\beta \langle \sigma_b \sigma_y \rangle_\beta. \quad (5.4.40)$$

Using once again the infrared bound (5.1.11), there exists $C_2 = C_2(d) > 0$ such that,

$$\bar{\mathbf{P}}_\beta^{(\ell)}[\mathcal{G}_1] \leq \sum_{\substack{a \in \partial \Lambda_k \\ b \in \partial \Lambda_M}} \langle \sigma_{\mathbf{e}_1} \sigma_a \rangle_\beta \langle \sigma_a \sigma_b \rangle_\beta \leq C_2 \frac{k^{d-1} M^{d-1}}{k^{2d-4}} = C_2 \frac{M^{d-1}}{k^{d-3}} = C_2 \frac{k^{(d-1)\delta A}}{k^{d-3}}. \quad (5.4.41)$$

We turn to the bound of $\mathcal{G}_2$. Write,

$$\langle \sigma_0 \sigma_x \rangle_\beta \langle \sigma_{\mathbf{e}_1} \sigma_y \rangle_\beta \mathbf{P}_\beta^{0x, \mathbf{e}_1 y}[\mathcal{G}_2] \leq \sum_{\substack{s \in \partial \Lambda_M \\ t \in \partial \Lambda_m}} \langle \sigma_0 \sigma_x \rangle_\beta \langle \sigma_{\mathbf{e}_1} \sigma_y \rangle_\beta \mathbf{P}_\beta^{0x, \mathbf{e}_1 y}[s \xleftarrow{\mathbf{n}_1 + \mathbf{n}_2 \setminus (\bar{\Gamma}(\mathbf{n}_1) \cup \bar{\Gamma}(\mathbf{n}_2))} t]. \quad (5.4.42)$$

We claim that $\mathbf{P}_\beta^{0x, \mathbf{e}_1 y}[C(s, t)] \leq \langle \sigma_s \sigma_t \rangle_\beta^2$ where $C(s, t) := \{s \xleftarrow{\mathbf{n}_1 + \mathbf{n}_2 \setminus (\bar{\Gamma}(\mathbf{n}_1) \cup \bar{\Gamma}(\mathbf{n}_2))} t\}$. Indeed, going to partition functions¹²

$$Z_\beta^{0x, \mathbf{e}_1 y}[C(s, t)] = \sum_{\substack{\gamma_1: 0 \rightarrow x \\ \gamma_2: \mathbf{e}_1 \rightarrow y}} Z_\beta^{0x}[\Gamma(\mathbf{n}_1) = \gamma_1] Z_\beta^{\mathbf{e}_1 y}[\Gamma(\mathbf{n}_2) = \gamma_2] \mathbf{P}_{\bar{\gamma}_1^c, \bar{\gamma}_2^c, \beta}^{\emptyset, \emptyset}[C(s, t)]. \quad (5.4.43)$$

Now, write for $\mathbf{n}_1 \in \Omega_{\bar{\gamma}_1^c}$ and $\mathbf{n}_2 \in \Omega_{\bar{\gamma}_2^c}$

$$\mathbf{n}_1 = \mathbf{k}_1 + \mathbf{m}_1, \quad \mathbf{k}_1(e) = \mathbf{n}_1(e) \mathbf{1}\{e \in \bar{\gamma}_2\} \quad (e \in \mathbb{E}), \quad (5.4.44)$$

$$\mathbf{n}_2 = \mathbf{k}_2 + \mathbf{m}_2, \quad \mathbf{k}_2(e) = \mathbf{n}_2(e) \mathbf{1}\{e \in \bar{\gamma}_1\} \quad (e \in \mathbb{E}), \quad (5.4.45)$$

¹²For full disclosure, this step would require working in a finite volume Λ and then taking the limit $\Lambda \nearrow \mathbb{Z}^d$. We omit this detail here as it does not affect the argument.

so that $\mathbf{m}_1$ and $\mathbf{m}_2$ are both currents on $(\bar{\gamma}_1 \cup \bar{\gamma}_2)^c$. Then,

$$Z_{\bar{\gamma}_1^c, \bar{\gamma}_2^c, \beta}^{\emptyset, \emptyset}[C(s, t)] = \sum_{\mathbf{k}_1, \mathbf{k}_2} w_\beta(\mathbf{k}_1) w_\beta(\mathbf{k}_2) \sum_{\substack{\partial \mathbf{m}_1 = \partial \mathbf{k}_1 \\ \partial \mathbf{m}_2 = \partial \mathbf{k}_2}} w_\beta(\mathbf{m}_1) w_\beta(\mathbf{m}_2) \mathbb{1}\{C(s, t)\} \quad (5.4.46)$$

$$= \sum_{\mathbf{k}_1, \mathbf{k}_2} w_\beta(\mathbf{k}_1) w_\beta(\mathbf{k}_2) \sum_{\substack{\partial \mathbf{m}_1 = \partial \mathbf{k}_1 \Delta \{s, t\} \\ \partial \mathbf{m}_2 = \partial \mathbf{k}_2 \Delta \{s, t\}}} w_\beta(\mathbf{m}_1) w_\beta(\mathbf{m}_2) \mathbb{1}\{C(s, t)\} \quad (5.4.47)$$

$$= Z_{\bar{\gamma}_1^c, \bar{\gamma}_2^c, \beta}^{st, st}[C(s, t)] \leq Z_{\bar{\gamma}_1^c, \bar{\gamma}_2^c, \beta}^{st, st}, \quad (5.4.48)$$

where on the second line we used a slightly more general version of the switching lemma, called the *switching principle*, see [11, Lemma 2.1]. As a result,

$$\mathbf{P}_{\bar{\gamma}_1^c, \bar{\gamma}_2^c, \beta}^{\emptyset, \emptyset}[C(s, t)] \leq \langle \sigma_s \sigma_t \rangle_{\bar{\gamma}_1^c, \beta} \langle \sigma_s \sigma_t \rangle_{\bar{\gamma}_2^c, \beta}. \quad (5.4.49)$$

Collecting the above work and using Griffiths' inequality yields

$$\mathbf{P}_\beta^{0x, \mathbf{e}_1 y}[C(s, t)] \leq \sum_{\substack{\gamma_1: 0 \rightarrow x \\ \gamma_2: \mathbf{e}_1 \rightarrow y}} \mathbf{P}_\beta^{0x}[\Gamma(\mathbf{n}_1) = \gamma_1] \mathbf{P}_\beta^{\mathbf{e}_1 y}[\Gamma(\mathbf{n}_2) = \gamma_2] \langle \sigma_s \sigma_t \rangle_{\bar{\gamma}_1^c, \beta} \langle \sigma_s \sigma_t \rangle_{\bar{\gamma}_2^c, \beta} \leq \langle \sigma_s \sigma_t \rangle_\beta^2. \quad (5.4.50)$$

Injecting (5.4.50) in (5.4.42) and averaging over x and y gives,

$$\bar{\mathbf{P}}_\beta^{(\ell)}[\mathcal{G}_2] \leq \sum_{\substack{s \in \partial \Lambda_M \\ t \in \partial \Lambda_m}} \langle \sigma_s \sigma_t \rangle_\beta^2 \leq C_3 \frac{M^{d-1} m^{d-1}}{M^{2d-4}} = C_3 \frac{k^{\delta(d-1)}}{k^{A\delta(d-3)}}. \quad (5.4.51)$$

Combining (5.4.41) and (5.4.51) gives $C_4 = C_4(d)$, $C_5 = C_5(d) > 0$ such that

$$\frac{1}{\chi(\beta)^2} \sum_{|x|, |y| \geq \ell} \sum_{u \in \Lambda_m} \sum_{v \sim u} \langle \sigma_0 \sigma_x \rangle_\beta \langle \sigma_{\mathbf{e}_1} \sigma_y \rangle_\beta \mathbf{P}_\beta^{0x, \mathbf{e}_1 y}[\mathcal{B}_{u, v}] \leq C_4 m^d \left(\frac{k^{\delta A}}{k^{d-3}} + \frac{k^{\delta(d-1)}}{k^{A\delta(d-3)}} \right) \leq C_5 k^{-\eta}, \quad (5.4.52)$$

for some $\eta > 0$ sufficiently small, provided δ is small enough and A is large enough. The proof follows readily. $\square$

5.5 Proof of the mixing property

In this section, we prove Theorem 5.2.4. We follow the strategy of [9]. The argument only requires a modification in one of the steps (namely [9, Lemma 6.7]), but we provide a full proof for sake of completeness. One essential ingredient to the proof will be the following result.

Theorem 5.5.1 (Absence of infinite cluster at β_c , [10]). *Let $d \geq 3$. Then,*

$$\phi_{\beta_c}[0 \leftrightarrow \infty] = 0. \quad (5.5.1)$$

We begin by recalling a fundamental result: the existence of *regular* scales.

Definition 5.5.2 (Regular scales). Fix $c, C > 0$. An annular region $\text{Ann}(n/2, 8n)$ is said to be (c, C) -*regular* if the following properties hold:

(P1) for every $x, y \in \text{Ann}(n/2, 8n)$, $\langle \sigma_0 \sigma_y \rangle_\beta \leq C \langle \sigma_0 \sigma_x \rangle_\beta$,

(P2) for every $x, y \in \text{Ann}(n/2, 8n)$, $|\langle \sigma_0 \sigma_x \rangle_\beta - \langle \sigma_0 \sigma_y \rangle_\beta| \leq \frac{C|x-y|}{|x|} \langle \sigma_0 \sigma_x \rangle_\beta$,

(P3) $\chi_{2n}(\beta) - \chi_n(\beta) \geq c\chi_n(\beta)$,

(P4) for every $x \in \Lambda_n$ and $y \notin \Lambda_{Cn}$, $\langle \sigma_0 \sigma_y \rangle_\beta \leq \frac{1}{2} \langle \sigma_0 \sigma_x \rangle_\beta$.

A scale k is said to be *regular* if $n = 2^k$ is such that $\text{Ann}(n/2, 8n)$ is (c, C) -regular, a vertex $x \in \mathbb{Z}^d$ is said to be *in a regular scale* if it belongs to an annulus $\text{Ann}(n, 2n)$ with $n = 2^k$ and k a regular scale.

Theorem 5.5.3 (Existence of regular scales, [9, Theorem 5.12]). *Let $d \geq 3$. Let $\gamma > 2$. There exist $\mathbf{c}, \mathbf{C} > 0$ such that, for every $\beta \leq \beta_c$, and every $1 \leq n^\gamma \leq N \leq \xi(\beta)$, there are at least $\mathbf{c} \log_2 \left(\frac{N}{n} \right)$ $(\mathbf{c}, \mathbf{C})$ -regular scales between n and N .*

We are now equipped to prove Theorem 5.2.4. Below, $\mathbf{c}, \mathbf{C}$ are given by Theorem 5.5.3.

Let $d \geq 3$ and $\varepsilon > 0$. Fix $n \geq 1$, $s \geq 1$, and t satisfying $1 \leq t \leq s$. Let $\beta \leq \beta_c$ to be taken sufficiently close to β_c , and let $N \geq n$ to be taken large enough. Introduce integers m, M such that $n \leq m \leq M \leq N$.

For $\mathbf{x} = (x_1, \dots, x_t)$ and $\mathbf{y} = (y_1, \dots, y_t)$, we define

$$\mathbf{P}_\beta^{\mathbf{xy}} := \mathbf{P}_\beta^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}, \quad \mathbf{P}_\beta^{\mathbf{xy}, \emptyset} := \mathbf{P}_\beta^{\mathbf{xy}} \otimes \mathbf{P}_\beta^{\emptyset, \dots, \emptyset}, \quad (5.5.2)$$

where $\mathbf{P}_\beta^{\emptyset, \dots, \emptyset}$ is the law of a sum of s independent sourceless currents that we denote by $(\mathbf{n}'_1, \dots, \mathbf{n}'_s)$. We also let $\mathbf{E}_\beta^{\mathbf{xy}}$ and $\mathbf{E}_\beta^{\mathbf{xy}, \emptyset}$ be the expectations with respect to these measures. If $p \geq 1$, define for $y \notin \Lambda_{2dp}$,

$$\mathbb{A}_y(p) := \left\{ u \in \text{Ann}(p, 2p) : \forall x \in \Lambda_{p/d}, \langle \sigma_x \sigma_y \rangle_\beta \leq \left(1 + \mathbf{C} \frac{|x-u|}{|y|} \right) \langle \sigma_u \sigma_y \rangle_\beta \right\}. \quad (5.5.3)$$

Note that if y is in a regular scale, then $\mathbb{A}_y(p) = \text{Ann}(p, 2p)$. Moreover, the MMS inequalities ensure that this set is not empty, see [9, Remark 6.5].

Let $\mathfrak{K}$ be the set of regular scales k between m and $M/2$ with the property that for all $k < k' \in \mathfrak{K}$, $2^{k'} \geq C_0 2^k$. By Theorem 5.5.3, we may choose β sufficiently close to β_c and m, M, N large enough so that $\mathfrak{K} \neq \emptyset$. Introduce $\mathbf{U} := \prod_{i=1}^t \mathbf{U}_i$, where

$$\mathbf{U}_i := \frac{1}{|\mathfrak{K}|} \sum_{k \in \mathfrak{K}} \frac{1}{A_{x_i, y_i}(2^k)} \sum_{u \in \mathbb{A}_{y_i}(2^k)} \mathbb{1}\{u \xleftrightarrow{\mathbf{n}_i + \mathbf{n}'_i} x_i\}, \quad (5.5.4)$$

and,

$$a_{x,y}(u) := \frac{\langle \sigma_x \sigma_u \rangle_\beta \langle \sigma_u \sigma_y \rangle_\beta}{\langle \sigma_x \sigma_y \rangle_\beta}, \quad A_{x,y}(p) := \sum_{u \in \mathbb{A}_y(p)} a_{x,y}(u). \quad (5.5.5)$$

Using the switching lemma, we have that $\mathbf{E}_\beta^{\mathbf{xy}, \emptyset}[\mathbf{U}] = 1$. The following concentration inequality is a consequence of the definition of $\mathbb{A}_{y_i}(2^k)$ and the properties of regular scales, and is still valid when $d = 3$ (see [9, Proposition 6.6]). The assumption on m/n and N/M below is justified by the definition of $\mathbb{A}_y(p)$.

Lemma 5.5.4 (Concentration of $\mathbf{U}$). *Assume that $x_i \in \Lambda_n$ and $y_i \notin \Lambda_N$ for all $1 \leq i \leq t$. Additionally, assume that $m \geq 2dn$ and $N \geq 2dM$. Then, there exists $C = C(d, s) > 0$ such that,*

$$\mathbf{E}_\beta^{\mathbf{xy}, \emptyset}[(\mathbf{U} - 1)^2] \leq \frac{C^2}{|\mathfrak{K}|}. \quad (5.5.6)$$

We work under the assumptions of Lemma 5.5.4. Using the Cauchy–Schwarz inequality together with Lemma 5.5.4,

$$\left| \mathbf{P}_\beta^{\mathbf{xy}}[E \cap F] - \mathbf{E}_\beta^{\mathbf{xy}, \emptyset}[\mathbf{U} \cdot \mathbb{1}\{(\mathbf{n}_1, \dots, \mathbf{n}_s) \in E \cap F\}] \right| \leq \left(\mathbf{E}_\beta^{\mathbf{xy}, \emptyset}[(\mathbf{U} - 1)^2] \right)^{\frac{1}{2}} \leq \frac{C}{\sqrt{|\mathfrak{K}|}}. \quad (5.5.7)$$

We now introduce the proper event which will allow us to “decouple” the events E and F .

Definition 5.5.5. Let $\mathbf{u} = (u_1, \dots, u_t)$ with $u_i \in \text{Ann}(m, M)$ for every i . The event $\mathcal{G}(u_1, \dots, u_t) = \mathcal{G}(\mathbf{u})$ is defined as follows: for every $i \leq s$, there exists $\mathbf{k}_i \leq \mathbf{n}_i + \mathbf{n}'_i$ such that $\mathbf{k}_i = 0$ on Λ_n , $\mathbf{k}_i = \mathbf{n}_i + \mathbf{n}'_i$ outside Λ_N , $\partial \mathbf{k}_i = \{u_i, y_i\}$ for $i \leq t$, and $\partial \mathbf{k}_i = \emptyset$ for $t < i \leq s$.

By the switching principle (see [11, Lemma 2.1]), one has

$$\begin{aligned} \mathbf{P}_\beta^{\mathbf{xy}, \emptyset} \left[\{(\mathbf{n}_1, \dots, \mathbf{n}_s) \in E \cap F\} \cap \{u_i \xleftrightarrow{\mathbf{n}_i + \mathbf{n}'_i} y_i, \forall 1 \leq i \leq t\} \cap \mathcal{G}(\mathbf{u}) \right] \\ = \left(\prod_{i=1}^t a_{x_i, y_i}(u_i) \right) \mathbf{P}_\beta^{\mathbf{xu}, \mathbf{uy}} [(\mathbf{n}_1, \dots, \mathbf{n}_s) \in E, (\mathbf{n}'_1, \dots, \mathbf{n}'_s) \in F, \mathcal{G}(\mathbf{u})]. \end{aligned} \quad (5.5.8)$$

The identity,

$$\begin{aligned} \mathbf{P}_\beta^{\mathbf{xu}, \mathbf{uy}} [(\mathbf{n}_1, \dots, \mathbf{n}_s) \in E, (\mathbf{n}'_1, \dots, \mathbf{n}'_s) \in F] \\ = \mathbf{P}_\beta^{\mathbf{xu}} [(\mathbf{n}_1, \dots, \mathbf{n}_s) \in E] \mathbf{P}_\beta^{\mathbf{uy}} [(\mathbf{n}'_1, \dots, \mathbf{n}'_s) \in F] \end{aligned} \quad (5.5.9)$$

motivates us to prove that under $\mathbf{P}_\beta^{\mathbf{xu}, \mathbf{uy}}$, the event $\mathcal{G}(\mathbf{u})$ occurs with high probability. This is the technical step in the proof of [9] that requires some work to be extended to the three-dimensional case. Recall that $n \leq m \leq M \leq N$.

Lemma 5.5.6. *We keep the assumptions of Lemma 5.5.4. Assume $\frac{\beta_c}{2} \leq \beta \leq \beta_c$. For any fixed value $\alpha = m/M$, there exist $m, N > 0$ large enough (and which only depend on $\varepsilon, \alpha, n, s, d$) such that, for every $\mathbf{u}$ with $u_i \in \mathbb{A}_{y_i}(2^{k_i})$ with $k_i \in \mathfrak{K}$ for $1 \leq i \leq t$,*

$$\mathbf{P}_\beta^{\mathbf{xy}, \emptyset} \left[\{u_i \xleftrightarrow{\mathbf{n}_i + \mathbf{n}'_i} y_i, \forall 1 \leq i \leq t\} \cap \mathcal{G}(\mathbf{u}) \right] \left(\prod_{i=1}^t a_{x_i, y_i}(u_i) \right)^{-1} = \mathbf{P}_\beta^{\mathbf{xu}, \mathbf{uy}} [\mathcal{G}(\mathbf{u})^c] \leq \frac{\varepsilon}{2}. \quad (5.5.10)$$

Proof. Below, the constants C_i only depend on d and s . The equality follows from an application of the switching lemma, we thus focus on the inequality. Write $\mathcal{G}(\mathbf{u}) = \cap_{1 \leq i \leq s} G_i$ (where the definition of G_i is implicit). Then, $H_i^c \cap F_i^c \subset G_i$ where,

$$H_i := \{\text{Ann}(M, N) \text{ is crossed by a cluster in } \mathbf{n}_i\}, \quad (5.5.11)$$

and

$$F_i := \{\text{Ann}(n, m) \text{ is crossed by a cluster in } \mathbf{n}'_i\}. \quad (5.5.12)$$

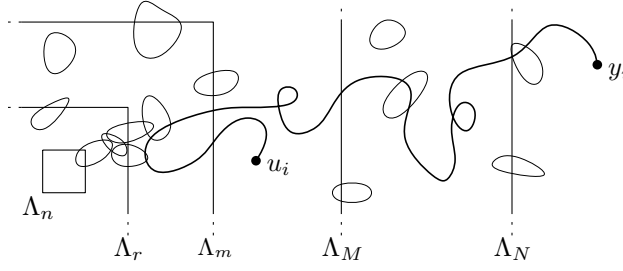


Figure 5.4: An illustration of a configuration $\mathbf{n}'_i$ contributing to the event F_i introduced in (5.5.12). The backbone of $\mathbf{n}'_i$ is the bold black line. Here, $\Gamma(\mathbf{n}'_i)$ does not cross $\text{Ann}(r, m)$ but $\mathbf{n}'_i$ crosses $\text{Ann}(n, m)$ thanks to a string of loops crossing $\text{Ann}(n, r)$.

Indeed, if $H_i^c \cap F_i^c$ occurs, we may define $\mathbf{k}_i$ as the sum of the restriction of $\mathbf{n}_i$ to the clusters intersecting Λ_N^c and the restriction of $\mathbf{n}'_i$ to the clusters intersecting Λ_m^c . Introduce an intermediate scale $n \leq r \leq m$. Using a union bound,

$$\mathbf{P}_\beta^{\mathbf{xu}, \mathbf{uy}}[\mathcal{G}(\mathbf{u})^c] \leq \sum_{i=1}^t \left(\mathbf{P}_\beta^{x_i u_i}[H_i] + \mathbf{P}_\beta^{u_i y_i}[F_i] \right) + (s-t) \left(\mathbf{P}_\beta^\emptyset[H_s] + \mathbf{P}_\beta^\emptyset[F_s] \right). \quad (5.5.13)$$

We progressively fix r, m, N large enough (recall that $m/M = \alpha$ is fixed).

Bound on $\mathbf{P}_\beta^{u_i y_i}[F_i]$ Fix $1 \leq i \leq t$. Notice that

$$\mathbf{P}_\beta^{u_i y_i}[F_i] \leq \mathbf{P}_\beta^{u_i y_i}[\Gamma(\mathbf{n}'_i) \text{ crosses } \text{Ann}(r, m)] + \mathbf{P}_\beta^{u_i y_i}[\mathbf{n}'_i \setminus \bar{\Gamma}(\mathbf{n}'_i) \text{ crosses } \text{Ann}(n, r)], \quad (5.5.14)$$

see Figure 5.4. Now, using the chain rule (Proposition 5.4.6),

$$\mathbf{P}_\beta^{u_i y_i}[\Gamma(\mathbf{n}'_i) \text{ crosses } \text{Ann}(r, m)] \leq \sum_{v \in \partial \Lambda_r} \frac{\langle \sigma_{u_i} \sigma_v \rangle_\beta \langle \sigma_v \sigma_{y_i} \rangle_\beta}{\langle \sigma_{u_i} \sigma_{y_i} \rangle_\beta} \quad (5.5.15)$$

$$\leq C_1 \frac{r^2}{m} \max_{v \in \partial \Lambda_r} \frac{\langle \sigma_v \sigma_{y_i} \rangle_\beta}{\langle \sigma_{u_i} \sigma_{y_i} \rangle_\beta} \quad (5.5.16)$$

$$\leq C_2 \frac{r^2}{m}, \quad (5.5.17)$$

where we used the fact that $u \in \mathbb{A}_{y_i}(2^{k_i})$ on the third line.

Then, arguing as in (5.4.50) (see also the proof of [204, Lemma 6.3]),

$$\mathbf{P}_\beta^{u_i y_i}[\mathbf{n}'_i \setminus \bar{\Gamma}(\mathbf{n}'_i) \text{ crosses } \text{Ann}(n, r)] = \sum_{\gamma: u_i \rightarrow y_i} \mathbf{P}_\beta^{u_i y_i}[\Gamma(\mathbf{n}'_i) = \gamma] \mathbf{P}_{\bar{\gamma}^c, \beta}^\emptyset[\partial \Lambda_n \xleftrightarrow{\mathbf{n}'_i} \partial \Lambda_r]. \quad (5.5.18)$$

However, by Proposition 5.2.5, one has

$$\mathbf{P}_{\bar{\gamma}^c, \beta}^\emptyset[\partial \Lambda_n \xleftrightarrow{\mathbf{n}'_i} \partial \Lambda_r] \leq \phi_{\bar{\gamma}^c, \beta_c}^0[\partial \Lambda_n \leftrightarrow \partial \Lambda_r] \quad (5.5.19)$$

$$\leq \phi_{\beta_c}[\partial \Lambda_n \leftrightarrow \partial \Lambda_r], \quad (5.5.20)$$

where we additionally used monotonicity properties of the FK-Ising measure (see [122, 76]). Using Theorem 5.5.1, we find that for r large enough, one has

$$\phi_{\beta_c}[\partial\Lambda_n \leftrightarrow \partial\Lambda_r] \leq \frac{\varepsilon}{8s}. \quad (5.5.21)$$

Combining this observation and (5.5.17), we obtain that for m large enough,

$$\mathbf{P}_\beta^{u_i y_i}[F_i] \leq \frac{\varepsilon}{4s}. \quad (5.5.22)$$

Bound on $\mathbf{P}_\beta^\emptyset[H_s]$ Using Proposition 5.2.5,

$$\mathbf{P}_\beta^\emptyset[H_s] \leq \phi_{\beta_c}[\partial\Lambda_n \leftrightarrow \partial\Lambda_m]. \quad (5.5.23)$$

Using Theorem 5.5.1 and choosing m large enough (and such that (5.5.22) holds) therefore yields

$$\mathbb{P}_\beta^\emptyset[H_s] \leq \frac{\varepsilon}{4s} \quad (5.5.24)$$

Bound on $\mathbf{P}_\beta^{x_i u_i}[H_i]$ Fix $1 \leq i \leq t$. Fix r, m such that the above bounds holds. Notice that this fixes the value of M by assumption. Still using Proposition 5.2.5, one has

$$\mathbf{P}_\beta^{x_i u_i}[H_i] = \phi_\beta[H_i \mid x_i \leftrightarrow u_i] \leq \max_{w, v \in \Lambda_m} \phi_{\beta_c/2}[w \leftrightarrow v]^{-1} \cdot \phi_{\beta_c}[\partial\Lambda_M \leftrightarrow \partial\Lambda_N], \quad (5.5.25)$$

which can be made smaller than $\frac{\varepsilon}{4s}$ by using Theorem 5.5.1 and choosing $N > 0$ large enough.

Bound on $\mathbf{P}_\beta^\emptyset[F_s]$ We notice that

$$\mathbf{P}_\beta^\emptyset[F_s] \leq \phi_{\beta_c}[\partial\Lambda_M \leftrightarrow \partial\Lambda_N], \quad (5.5.26)$$

which we bound similarly by $\frac{\varepsilon}{4s}$ for N large enough.

Plugging the above bounds in (5.5.13), we obtain

$$\mathbf{P}_\beta^{\mathbf{x}, \mathbf{u}, \mathbf{y}}[\mathcal{G}(\mathbf{u})^c] \leq \sum_{i=1}^t \left(\frac{\varepsilon}{4s} + \frac{\varepsilon}{4s} \right) + (s-t) \left(\frac{\varepsilon}{4s} + \frac{\varepsilon}{4s} \right) \leq \frac{\varepsilon}{2}, \quad (5.5.27)$$

which concludes the proof. $\square$

We can now conclude.

Proof of Theorem 5.2.4. Introduce the weights $\delta(\mathbf{u}, \mathbf{x}, \mathbf{y})$ defined by

$$\delta(\mathbf{u}, \mathbf{x}, \mathbf{y}) := \mathbb{1}\{\exists(k_1, \dots, k_t) \in \mathfrak{K}^t, \mathbf{u} \in \mathbb{A}_{y_1}(2^{k_1}) \times \dots \times \mathbb{A}_{y_t}(2^{k_t})\} \prod_{i=1}^t \frac{a_{x_i, y_i}(u_i)}{|\mathfrak{K}| A_{x_i, y_i}(2^{k_i})}. \quad (5.5.28)$$

Notice that

$$\sum_{\substack{(k_1, \dots, k_t) \in \mathfrak{K}^t \\ \mathbf{u} \in \mathbb{A}_{y_1}(2^{k_1}) \times \dots \times \mathbb{A}_{y_t}(2^{k_t})}} \delta(\mathbf{u}, \mathbf{x}, \mathbf{y}) = 1. \quad (5.5.29)$$

We abbreviate the above sum as $\sum_{\mathbf{u}}$. Assume that $m/M = \alpha$. Let $\alpha > 0$ to be chosen small enough, and let $m = m(\alpha)$, $N = N(\alpha)$ be given by Lemma 5.5.6.

Let $\mathbf{x} = (x_1, \dots, x_t) \in (\Lambda_n)^t$ and $\mathbf{y} = (y_1, \dots, y_t) \in (\Lambda_N^c)^t$. Equation (5.5.7) together with Lemma 5.5.6 yield

$$\left| \mathbf{P}_\beta^{\mathbf{xy}}[E \cap F] - \sum_{\mathbf{u}} \delta(\mathbf{u}, \mathbf{x}, \mathbf{y}) \mathbf{P}_\beta^{\mathbf{xu}}[E] \mathbf{P}_\beta^{\mathbf{uy}}[F] \right| \leq \frac{C}{\sqrt{|\mathcal{K}|}} + \frac{\varepsilon}{2}. \quad (5.5.30)$$

We now use Theorem 5.5.3 to argue that if β is sufficiently close to β_c and if M/m is large enough (which affects how large N is), then $|\mathfrak{K}| \geq 2\varepsilon^{-2}C$. This gives for every $\mathbf{y} \in (\Lambda_N^c)^t$,

$$\left| \mathbf{P}_\beta^{\mathbf{xy}}[E \cap F] - \sum_{\mathbf{u}} \delta(\mathbf{u}, \mathbf{x}, \mathbf{y}) \mathbf{P}_\beta^{\mathbf{xu}}[E] \mathbf{P}_\beta^{\mathbf{uy}}[F] \right| \leq \varepsilon. \quad (5.5.31)$$

Proof of (5.2.9) We begin by proving (5.2.9) when y_i, y'_i are in regular scales (but not necessarily the same ones). Applying the above inequality once for $\mathbf{y}$ and once for $\mathbf{y}'$ with the event E and $F = \Omega_{\mathbb{Z}^d}$,

$$\left| \mathbf{P}_\beta^{\mathbf{xy}}[E] - \mathbf{P}_\beta^{\mathbf{xy}'}[E] \right| \leq \left| \sum_{\mathbf{u}} (\delta(\mathbf{u}, \mathbf{x}, \mathbf{y}) - \delta(\mathbf{u}, \mathbf{x}, \mathbf{y}')) \mathbf{P}_\beta^{\mathbf{xu}}[E] \right| + 2\varepsilon. \quad (5.5.32)$$

Since all the y_i, y'_i are in regular scales, one has $\mathbb{A}_{y_i}(2^{k_i}) = \mathbb{A}_{y'_i}(2^{k_i}) = \text{Ann}(2^{k_i}, 2^{k_i+1})$. Moreover, using Property (P2) of regular scales, there exists $C_1 = C_1(s, d) > 0$ such that

$$|\delta(\mathbf{u}, \mathbf{x}, \mathbf{y}) - \delta(\mathbf{u}, \mathbf{x}, \mathbf{y}')| \leq C_1 \left(\frac{M}{N} \right) \delta(\mathbf{u}, \mathbf{x}, \mathbf{y}) \leq \varepsilon \delta(\mathbf{u}, \mathbf{x}, \mathbf{y}), \quad (5.5.33)$$

if N is large enough. Indeed, in this configuration $\delta(\mathbf{u}, \mathbf{x}, \mathbf{y})$ and $\delta(\mathbf{u}, \mathbf{x}, \mathbf{y}')$ are both close to

$$\prod_{i \leq t} \frac{\langle \sigma_{x_i} \sigma_{u_i} \rangle}{|\mathfrak{K}| \sum_{v_i \in \text{Ann}(2^{k_i}, 2^{k_i+1})} \langle \sigma_{x_i} \sigma_{v_i} \rangle}. \quad (5.5.34)$$

This gives that for every $\mathbf{y}, \mathbf{y}' \in (\Lambda_N^c)^t$ having all coordinates in regular scales,

$$\left| \mathbf{P}_\beta^{\mathbf{xy}}[E] - \mathbf{P}_\beta^{\mathbf{xy}'}[E] \right| \leq 3\varepsilon. \quad (5.5.35)$$

Note that we can increase m, M, N , while maintaining the value M/m , and choose β even closer to β_c , to also obtain (using the exact same argument): for every $\mathbf{z}, \mathbf{z}' \in (\Lambda_m^c)^t$ having all coordinates in regular scales,

$$\left| \mathbf{P}_\beta^{\mathbf{xz}}[E] - \mathbf{P}_\beta^{\mathbf{xz}'}[E] \right| \leq 3\varepsilon. \quad (5.5.36)$$

We now consider $\mathbf{z} = (z_1, \dots, z_t)$ with $z_i \in \text{Ann}(m, M)$ in a regular scale. Also, pick $\mathbf{y}$ on which we do not assume anything. We have,

$$\begin{aligned} \left| \mathbf{P}_\beta^{\mathbf{xy}}[E] - \mathbf{P}_\beta^{\mathbf{xz}}[E] \right| &= \left| \mathbf{P}_\beta^{\mathbf{xy}}[E] - \sum_{\mathbf{u}} \delta(\mathbf{u}, \mathbf{x}, \mathbf{y}) \mathbf{P}_\beta^{\mathbf{xu}}[E] \right| \\ &\leq \left| \mathbf{P}_\beta^{\mathbf{xy}}[E] - \sum_{\mathbf{u}} \delta(\mathbf{u}, \mathbf{x}, \mathbf{y}) \mathbf{P}_\beta^{\mathbf{xu}}[E] \right| + 3\varepsilon \\ &\leq 4\varepsilon, \end{aligned}$$

where on the second line we used (5.5.36) with $(\mathbf{y}, \mathbf{y}') = (\mathbf{u}, \mathbf{z})$, and in the third line we used (5.5.31) with $F = \Omega_{\mathbb{Z}^d}$. This gives (5.2.9).

Proof of (5.2.10) The same argument works for (5.2.10) for every $\mathbf{x}, \mathbf{x}', \mathbf{y}$, noticing that for every regular $\mathbf{u}$ for which $\delta(\mathbf{u}, \mathbf{x}, \mathbf{y}) \neq 0$, using once again (P2),

$$|\delta(\mathbf{u}, \mathbf{x}, \mathbf{y}) - \delta(\mathbf{u}, \mathbf{x}', \mathbf{y})| \leq C_2 \left(\frac{n}{m} \right) \delta(\mathbf{u}, \mathbf{x}, \mathbf{y}) \leq \varepsilon \delta(\mathbf{u}, \mathbf{x}, \mathbf{y}) \quad (5.5.37)$$

if we take m sufficiently large.

Proof of (5.2.8) To obtain (5.2.8) we repeat the same line of reasoning. We start by applying (5.5.31). Then, we replace each $\mathbf{P}_\beta^{\mathbf{x}\mathbf{u}}[E]$ by $\mathbf{P}_\beta^{\mathbf{x}\mathbf{y}}[E]$ using (5.5.36). Similarly, we replace $\mathbf{P}_\beta^{\mathbf{u}\mathbf{y}}[F]$ by $\mathbf{P}_\beta^{\mathbf{x}\mathbf{y}}[F]$. The proof follows readily. $\square$

5.6 Proof of Proposition 5.3.2

In this last section, we use the techniques introduced in Section 5.5 to prove Proposition 5.3.2.

Proof of Proposition 5.3.2. We begin with (i). The argument follows from the fact that the IIC of the measure of interest can be described by the union of the backbone and a collection of finite loops. If the IIC has two ends, then there must be an infinite path which does not use any edges of the backbone. Fix $y \in \mathbb{Z}^d$, $n \geq 1$ and $|x| \gg 1$. Reasoning again as in (5.4.49),

$$\mathbf{P}_{\beta_c}^{0x, \emptyset} [y \xleftrightarrow{\mathbf{n}_1 + \mathbf{n}_2 \setminus \bar{\Gamma}(\mathbf{n}_1)} \partial\Lambda_n] \leq \sum_{u \in \partial\Lambda_n} \langle \sigma_u \sigma_y \rangle_{\beta_c}^2 \leq \frac{C_3}{n^{d-3}}. \quad (5.6.1)$$

Hence, by Theorem 5.3.1,

$$\mathbf{P}^{0\infty, \emptyset} [y \xleftrightarrow{\mathbf{n}_1 + \mathbf{n}_2 \setminus \bar{\Gamma}(\mathbf{n}_1)} \partial\Lambda_n] \leq \frac{C_3}{n^{d-3}}, \quad (5.6.2)$$

which immediately yields,

$$\mathbf{P}^{0\infty, \emptyset} [y \xleftrightarrow{\mathbf{n}_1 + \mathbf{n}_2 \setminus \bar{\Gamma}(\mathbf{n}_1)} \infty] = 0 \quad (5.6.3)$$

and the result.

We turn to (ii). Notice that by the switching lemma, if $x, y \in \mathbb{Z}^d$,

$$\mathbf{P}_{\beta_c}^{0x, \emptyset} [0 \leftrightarrow y] = \frac{\langle \sigma_0 \sigma_y \rangle_{\beta_c} \langle \sigma_y \sigma_x \rangle_{\beta_c}}{\langle \sigma_0 \sigma_x \rangle_{\beta_c}}. \quad (5.6.4)$$

The gradient estimate [9, Proposition 5.9] gives

$$\lim_{|x| \rightarrow \infty} \frac{\langle \sigma_y \sigma_x \rangle_{\beta_c}}{\langle \sigma_0 \sigma_x \rangle_{\beta_c}} = 1, \quad (5.6.5)$$

which seems to imply (5.3.10) by Theorem 5.3.1. However, the event $\{0 \leftrightarrow y\}$ is not local¹³, so we need to work a little to conclude. Let $2|y| \leq m \leq M$ to be taken large enough. We follow the argument used in the proof of Lemma 5.5.6 and notice that

$$\mathbf{P}_{\beta_c}^{0x, \emptyset} [\{0 \xleftrightarrow{\Lambda_M} y\}^c \cap \{0 \leftrightarrow y\}] \leq \mathbf{P}_{\beta_c}^{0x} [\mathcal{H}_1] + \mathbf{P}_{\beta_c}^{0x, \emptyset} [\mathcal{H}_2], \quad (5.6.6)$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ are two events defined as follows:

¹³For each fixed x , this event can be approximated by local events using the fact that the measure $\mathbf{P}_{\beta_c}^{0x, \emptyset}$ does not percolate, see [10]. Here we need a version of that statement which is uniform in x . This is why the qualitative result of [10] is not enough when $d = 3$.

- $\mathcal{H}_1$: the backbone $\Gamma(\mathbf{n}_1)$ makes two successive crossings of $\text{Ann}(m, M)$,
- $\mathcal{H}_2$: the event $\mathcal{H}_1$ does not occur and there exists $s \in \Lambda_m$ such that $y \xleftarrow{\mathbf{n}_1 + \mathbf{n}_2 \setminus \bar{\Gamma}(\mathbf{n}_1)} s$.

Using the chain rule for backbones and (5.1.11),

$$\mathbf{P}_{\beta_c}^{0x}[\mathcal{H}_1] \leq \sum_{\substack{u \in \partial\Lambda_m \\ v \in \partial\Lambda_M}} \frac{\langle \sigma_0 \sigma_v \rangle_{\beta_c} \langle \sigma_v \sigma_u \rangle_{\beta_c} \langle \sigma_u \sigma_x \rangle_{\beta_c}}{\langle \sigma_0 \sigma_x \rangle_{\beta_c}} \leq \max_{u \in \Lambda_m} \frac{\langle \sigma_u \sigma_x \rangle_{\beta_c}}{\langle \sigma_0 \sigma_x \rangle_{\beta_c}} \cdot C_1 \frac{m^{d-1}}{M^{d-3}}, \quad (5.6.7)$$

where $C_1 = C_1(d) > 0$. Moreover, a similar reasoning as in (5.4.49) yields the existence of $C_2 = C_2(d) > 0$ such that,

$$\mathbf{P}_{\beta_c}^{0x, \emptyset}[\mathcal{H}_2] \leq \sum_{u \in \partial\Lambda_m} \langle \sigma_u \sigma_y \rangle_{\beta_c}^2 \leq \frac{C_2}{m^{d-3}}, \quad (5.6.8)$$

where we used (5.1.11) in the last inequality. Taking $m = M^{1/4}$ and using the gradient estimate [9, Proposition 5.9] again,

$$\limsup_{M \rightarrow \infty} \limsup_{|x| \rightarrow \infty} \mathbf{P}_{\beta_c}^{0x, \emptyset}[\{0 \xleftrightarrow{\Lambda_M} y\}^c] = 0. \quad (5.6.9)$$

The proof of (5.3.10) follows readily. Moreover, (5.3.11) follows from (5.1.11) and [83, Theorem 1.4].

□

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Chapter 6

New lower bounds for the (near) critical Ising and φ^4 models' two-point functions

We study the nearest-neighbour Ising and φ^4 models on $\mathbb{Z}^d$ with $d \geq 3$ and obtain new lower bounds on their two-point functions at (and near) criticality. Together with the classical infrared bound, these bounds turn into up-to constant estimates when $d \geq 5$. When $d = 4$, we obtain an “almost” sharp lower bound corrected by a logarithmic factor. As a consequence of these results, we show that $\eta = 0$ and $\nu = 1/2$ when $d \geq 4$, where η is the critical exponent associated with the decay of the model’s two-point function at criticality and ν is the critical exponent of the correlation length $\xi(\beta)$. When $d = 3$, we improve previous results and obtain that $\eta \leq 1/2$. As a byproduct of our proofs, we also derive the blow-up at criticality of the so-called bubble diagram when $d = 3, 4$.

This chapter is based on the preprint “*New lower bounds for the (near) critical Ising and φ^4 models’ two-point functions*” [83]. It is a joint work with Hugo Duminil-Copin.

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6.1 Introduction

We are interested in two ferromagnetic and real-valued spin models on $\mathbb{Z}^d$ that are amongst the most studied in statistical mechanics: the Ising model and the (discrete) φ^4 model. These models are formally defined as follows: given $\Lambda \subset \mathbb{Z}^d$ finite, $\mathcal{S} \subset \mathbb{R}$, a probability measure ρ on $\mathcal{S}$, and an inverse temperature $\beta \geq 0$, we define a probability measure on $\mathcal{S}^\Lambda$ according to the formula

$$\langle F(\tau) \rangle_{\Lambda, \rho, \beta} := \frac{1}{Z_{\Lambda, \rho, \beta}} \int_{\mathcal{S}^\Lambda} F(\tau) \exp \left(\beta \sum_{\substack{x, y \in \Lambda \\ x \sim y}} \tau_x \tau_y \right) \prod_{x \in \Lambda} d\rho(\tau_x), \quad (6.1.1)$$

where $F : \mathcal{S}^\Lambda \rightarrow \mathbb{R}$, $Z_{\Lambda, \rho, \beta}$ is the *partition function* of the model ensuring that $\langle 1 \rangle_{\Lambda, \rho, \beta} = 1$, and $x \sim y$ means that $|x - y|_2 = 1$ (where $|\cdot|_2$ is the ℓ^2 norm on $\mathbb{R}^d$). The Ising model corresponds to choosing $\mathcal{S} = \{-1, 1\}$ and $\rho(dx) = \frac{1}{2}(\delta_{-1} + \delta_1)$ above; while the φ^4 model corresponds to choosing $\mathcal{S} = \mathbb{R}$ and $\rho = \rho_{g, a}$ where $g > 0$, $a \in \mathbb{R}$, and

$$\rho(dt) = \rho_{g, a}(dt) = \frac{e^{-gt^4 - at^2} dt}{\int_{\mathbb{R}} e^{-gs^4 - as^2} ds}. \quad (6.1.2)$$

In the case of the Ising model (resp. the φ^4 model), we write σ (resp. φ) instead of τ for the spin (resp. field) variable. We drop the subscript ρ in the above notations.

It is a well-known fact (see [116]) that the measures $\langle \cdot \rangle_{\Lambda, \beta}$ admit a weak limit when $\Lambda \nearrow \mathbb{Z}^d$. We denote it by $\langle \cdot \rangle_\beta$. When $d \geq 2$, the model undergoes a (second-order) phase transition (see [12, 10, 126]) at a critical parameter $\beta_c = \beta_c(\rho) \in (0, \infty)$ that can be defined as follows (see [8])

$$\beta_c := \inf \left\{ \beta \geq 0 : \chi(\beta) := \sum_{x \in \mathbb{Z}^d} \langle \tau_0 \tau_x \rangle_\beta = \infty \right\}. \quad (6.1.3)$$

The connection between the Ising model and the φ^4 model is anticipated to be highly intricate, with both models believed to fall within the same universality class. Renormalisation group arguments, as detailed in [119, 160] and the recent book [29], suggest that many of their properties, including *critical exponents*, precisely coincide at their respective critical points. Griffiths and Simon [224] laid the groundwork for these profound connections by demonstrating that the φ^4 model emerges as a specific near-critical scaling limit of a collection of mean-field Ising models. This allows for a rigorous transfer of numerous useful properties from the Ising model, such as correlation inequalities, to the φ^4 model. Conversely, the Ising model can be derived as a limit of the φ^4 model using the following (weak) convergence:

$$\frac{\delta_{-1} + \delta_1}{2} = \lim_{g \rightarrow \infty} \frac{e^{-g(\varphi^2 - 1)^2 + g} d\varphi}{\int_{\mathbb{R}} e^{-gs^4 + 2gs^2} ds}. \quad (6.1.4)$$

The upper critical dimension of these models should be equal to 4 meaning that they should exhibit *mean-field behaviour* above dimension $d \geq 5$, and a *marginal* (i.e. with potential logarithmic corrections) mean-field behaviour in dimension $d = 4$. In such setups, the expected behaviour of the models undergoes a notable simplification, where lattice geometry ceases to exert a significant influence. A prominent approach to investigating the mean-field regime involves the computation of the critical exponents of the model. Despite substantial progress, determining these exponents systematically in the mean-field regime has remained

a challenging task. Noteworthy methods such as the *lace expansion* and the *renormalisation group method* have emerged as powerful alternatives, showcasing significant successes. In the realm of spin models on $\mathbb{Z}^d$, their success covers the weakly-coupled [107, 25, 231, 29, 192], high-dimensional [216, 217, 218], or sufficiently spread-out [216, 67, 217, 68, 218] setups. Other milestones toward the study of the mean-field regime of these models include the proofs of triviality of the scaling limits at criticality in dimension $d \geq 5$ independently obtained by Aizenman [4] and Fröhlich [102], together with the recent work of Aizenman and Duminil-Copin [9] which establishes the corresponding result in dimension $d = 4$ (see also [204] for a treatment of models of *effective* dimension $d_{\text{eff}} \geq 4$).

The situation in dimension $d = 3$ is more complicated and much less well understood. Obtaining a rigorous understanding of the critical 3d Ising model remains one of the main challenges of statistical mechanics. In the physics literature, recent progress has been made regarding this problem using the so-called *conformal bootstrap*, see [88, 89, 215].

In this work, we study the critical exponent related to the decay of the model's two-point function at criticality when $d \geq 3$. It is expected that at criticality the two-point function of the above models decays algebraically, thus justifying the introduction of the critical exponent η defined as follows: for all $x \in \mathbb{Z}^d \setminus \{0\}$,

$$\langle \tau_0 \tau_x \rangle_{\beta_c} = \frac{1}{|x|^{d-2+\eta-o(1)}}, \quad (6.1.5)$$

where $|\cdot|$ denotes the infinite norm on $\mathbb{R}^d$ and $o(1)$ is a quantity tending to zero as $|x|$ tends to infinity. Since these models are reflection positive (see [104, 103, 36]), the *infrared bound* and the Messenger–Miracle–Solé (MMS) inequalities¹ [191] provide the existence of $C = C(d) > 0$ such that for all $\beta \leq \beta_c$, and all $x \in \mathbb{Z}^d \setminus \{0\}$,

$$\langle \tau_0 \tau_x \rangle_{\beta} \leq \frac{C}{|x|^{d-2}}. \quad (6.1.6)$$

Moreover, as a consequence of the Simon–Lieb inequality [223, 178], Simon proved the existence of $c = c(d) > 0$ such that for all $x \in \mathbb{Z}^d \setminus \{0\}$,

$$\langle \tau_0 \tau_x \rangle_{\beta_c} \geq \frac{c}{|x|^{d-1}}. \quad (6.1.7)$$

These two results show that (if it exists) the critical exponent η satisfies $0 \leq \eta \leq 1$. We now survey existing results regarding the computation of η .

In dimension $d = 2$, the transfer matrix formalism leads to exact solutions of the Ising model [200, 161, 255]. This formalism was later used by Wu [254, 189] to demonstrate that $\eta = 1/4$. Much more information was obtained on the two-point function through the proof of conformal invariance of Smirnov [233], see for instance [65] for a proof of conformal invariance of the spin correlations.

¹To be more precise, it relies on the following simple consequences of the MMS inequalities which hold for all $\beta > 0$ (see [191] or e.g. [9, Proposition 5.1]): (i) the sequence $(\langle \tau_0 \tau_{k\mathbf{e}_1} \rangle_{\beta})_{k \geq 0}$ is decreasing; (ii) for any $x \in \mathbb{Z}^d$, one has

$$\langle \tau_0 \tau_{(|x|_1, 0_{\perp})} \rangle_{\beta} \leq \langle \tau_0 \tau_x \rangle_{\beta} \leq \langle \tau_0 \tau_{(|x|, 0_{\perp})} \rangle_{\beta},$$

where $|\cdot|_1$ denotes the ℓ^1 norm on $\mathbb{R}^d$.

When $d = 3$, the results are confined to the bounds $0 \leq \eta \leq 1$. However, the conformal bootstrap method provides precise predictions [166] on the value of η , with the most accurate numerical estimate being $\eta \approx 0.0362978(20)$.

When $d \geq 4$, the mean-field regime can be derived. Lace expansion methods [216, 217, 218] were applied in very large dimension to obtain, not only that for both models $\eta = 0$, but also exact asymptotics for the critical two-point function, showing that it is equivalent (at large scales) to $A/|x|_2^{d-2}$ (where $A > 0$ is a model-dependent constant). For the case of the weakly-coupled φ^4 model (i.e. with small coupling g), lace expansion was successfully implemented for $d \geq 5$ in [48]. The renormalisation group method was applied up to dimension $d = 4$ to obtain exact asymptotics in this setup, see [231]. Away from these perturbative regimes, the best bounds on η are the ones given by (6.1.6) and (6.1.7).

In this chapter, we obtain new (near) critical lower bounds on the two-point functions of these models when $d \geq 3$. In particular, we establish a sharp lower bound when $d \geq 5$, and an almost sharp lower bound (with a logarithmic correction) when $d = 4$, showing that $\eta = 0$ in these dimensions. We also obtain a new bound on η when $d = 3$: if it exists, $\eta \leq 1/2$. When $d \geq 4$, these results allow us to compute the critical exponent ν associated with the blow-up near criticality of the correlation lengths $L(\beta)$, $\xi(\beta)$, and $\xi_p(\beta)$ for $p > 0$ (see (6.1.9), (6.1.19), and (6.1.20) below). To the best of our knowledge, this result is new in our setup. Let us mention that the exponents η and ν are usually more complicated to derive than the exponents $\alpha, \beta, \delta, \gamma$ (for the second derivative of the free energy, the magnetisation, and the susceptibility, see [4, 14, 12, 25]) as they depend on the graph metric rather than intrinsic distances. Although only stated in the case of the Ising and the φ^4 models, we will show that these results extend for measures ρ that belong to the Griffiths–Simon class of measures, see Section 6.3.1.

6.1.1 The main theorem: a new inequality for two-point functions

Before stating the main results of this work, we introduce the distance below which, for $\beta < \beta_c$, the model recovers critical features. The following distance, called the *sharp length*, was introduced in [84, 83].

Definition 6.1.1 (Sharp length). Let ρ correspond to either the Ising or the φ^4 model. Let $\beta > 0$. Let S be a finite subset of $\mathbb{Z}^d$ containing 0 and set

$$\varphi_{\rho,\beta}(S) := \beta \sum_{\substack{x \in S \\ y \notin S, y \sim x}} \langle \tau_0 \tau_x \rangle_{S,\rho,\beta}. \quad (6.1.8)$$

Define the sharp length by

$$L(\beta) = L(\rho, \beta) := \inf \left\{ k \geq 1 : \exists S \subset \mathbb{Z}^d, 0 \in S, \text{diam}(S) \leq 2k, \varphi_{\rho,\beta}(S) < 1/2 \right\}, \quad (6.1.9)$$

where $\text{diam}(S) := \max\{|x - y|, x, y \in S\}$.

With this definition at hand, the Simon–Lieb inequality together with the infrared bound (6.1.6) yield the following result: there exist $c, C > 0$ such that, if $\beta \leq \beta_c$ and $x \in \mathbb{Z}^d \setminus \{0\}$,

$$\langle \tau_0 \tau_x \rangle_\beta \leq C \left(\frac{1}{|x| \wedge L(\beta)} \right)^{d-2} \exp \left(-c \frac{|x|}{L(\beta)} \right). \quad (6.1.10)$$

Our main result is the following inequality. If $n \geq 1$, introduce the hyperplane $\mathbb{H}_n := \{x \in \mathbb{Z}^d, x_1 = n\}$ and denote by $\mathcal{R}_n$ the orthogonal reflection with respect to $\mathbb{H}_n$. Also introduce the box $\Lambda_n := [-n, n]^d \cap \mathbb{Z}^d$.

Theorem 6.1.2. *Let $d \geq 3$. There exist $c_0, N_0 > 0$ such that for all $\beta \leq \beta_c$ and for all $N_0 \leq n \leq L(\beta)$,*

$$\beta \sum_{\substack{x, y \in \Lambda_n \\ y \sim x}} \left(\langle \tau_0 \tau_x \rangle_\beta - \langle \tau_0 \tau_{\mathcal{R}_n(x)} \rangle_\beta \right) \langle \tau_y \tau_{\mathcal{R}_n(y)} \rangle_\beta \geq c_0. \quad (6.1.11)$$

In fact, the bound will take a slightly different form for the φ^4 model, which does not affect its applications, see Proposition 6.3.3 for details.

We briefly explain the strategy of proof of the above inequality. We prove the result for the Ising model first and then extend it to the φ^4 model (and in fact, to all models in the Griffiths–Simon class of measures) using the Griffiths–Simon approximation (see Proposition 9.2.2). Focusing now on the Ising model, the usual way of obtaining a lower bound on the model’s two-point function at criticality goes by noticing that $\varphi_{\beta_c}(\Lambda_n) \geq 1$, and additionally using Griffiths’ inequality together with the MMS inequalities. Our method is a refinement of this inequality which consists in using the above observation for a well-chosen—random—set S instead of Λ_n . The randomness of this set is obtained through the use of the *random current representation* of the Ising model (see Section 6.2.1). The result will follow from an appropriate application of the switching principle to a reflected current, see Section 6.2.2. Such reflected currents have been used in [11] to derive a Messenger–Miracle–Solé inequality without using reflection positivity.

The regularity properties provided by reflection positivity (see [9, Section 5] or [204, Section 3]) allow one to turn the inequality of Theorem 6.1.2 into a pointwise lower bound on the two-point function that is valid for all dimensions $d \geq 3$. Let $\mathbf{e}_i$ be the unit vector of i -th coordinate equal to 1.

Theorem 6.1.3. *Let $d \geq 3$. There exist $c_1, N_1 > 0$ such that for all $\beta \leq \beta_c$ and for all $N_1 \leq n \leq L(\beta)$,*

$$\langle \tau_0 \tau_{n\mathbf{e}_1} \rangle_\beta \geq \frac{c_1}{\chi_{4n}(\beta) + n^{d-2} \sum_{1 \leq k \leq 2n} k \langle \tau_0 \tau_{k\mathbf{e}_1} \rangle_\beta}. \quad (6.1.12)$$

Proof. Let $c_0, N_0 > 0$ be given by Theorem 6.1.2. Let $\beta \leq \beta_c$ and $n \geq N_0$. The constants $C_i > 0$ below only depend on the dimension. Divide the sum on the left-hand side of (6.1.11) according to whether $-n \leq x_1 \leq n/2$ or $n/2 < x_1 \leq n$. By the MMS inequalities,

$$\sum_{\substack{x, y \in \Lambda_n \\ x_1 \leq n/2 \\ y \sim x}} \left(\langle \tau_0 \tau_x \rangle_\beta - \langle \tau_0 \tau_{\mathcal{R}_n(x)} \rangle_\beta \right) \langle \tau_y \tau_{\mathcal{R}_n(y)} \rangle_\beta \leq \sum_{\substack{x, y \in \Lambda_n \\ x_1 \leq n/2 \\ y \sim x}} \langle \tau_0 \tau_x \rangle_\beta \langle \tau_y \tau_{\mathcal{R}_n(y)} \rangle_\beta \leq \chi_n(\beta) \langle \tau_0 \tau_{(n/4)\mathbf{e}_1} \rangle_\beta. \quad (6.1.13)$$

Then, using the spectral representation of these models and the MMS inequalities, we obtain the following gradient estimate²

$$\langle \tau_0 \tau_x \rangle_\beta - \langle \tau_0 \tau_{\mathcal{R}_n(x)} \rangle_\beta \leq C_1 \frac{|x - \mathcal{R}_n(x)|}{n} \langle \tau_0 \tau_{(n/4)\mathbf{e}_1} \rangle_\beta. \quad (6.1.14)$$

²The gradient estimate is obtained similarly as in [9, Proposition 5.9]: using the spectral representation of

Using (6.1.14), we get

$$\begin{aligned} \sum_{\substack{x, y \in \Lambda_n \\ x_1 > n/2 \\ y \sim x}} \left(\langle \tau_0 \tau_x \rangle_\beta - \langle \tau_0 \tau_{\mathcal{R}_n(x)} \rangle_\beta \right) \langle \tau_y \tau_{\mathcal{R}_n(y)} \rangle_\beta &\leq C_2 \frac{\langle \tau_0 \tau_{(n/4)\mathbf{e}_1} \rangle_\beta}{n} \sum_{\substack{x \in \Lambda_n \\ x_1 \geq n/2}} |x - \mathcal{R}_n(x)| \langle \tau_x \tau_{\mathcal{R}_n(x)} \rangle_\beta \\ &\leq C_3 \langle \tau_0 \tau_{(n/4)\mathbf{e}_1} \rangle_\beta n^{d-2} \sum_{1 \leq k \leq n/2} k \langle \tau_0 \tau_{k\mathbf{e}_1} \rangle_\beta. \end{aligned} \quad (6.1.15)$$

Plugging (6.1.13) and (6.1.15) in (6.1.11) yields

$$\frac{c_1}{\beta} \leq C_4 \langle \tau_0 \tau_{(n/4)\mathbf{e}_1} \rangle_\beta \left(\chi_n(\beta) + n^{d-2} \sum_{1 \leq k \leq n/2} k \langle \tau_0 \tau_{k\mathbf{e}_1} \rangle_\beta \right), \quad (6.1.16)$$

from which the proof follows readily. $\square$

6.1.2 Applications

We now list a number of applications and we include their (short) derivation.

Lower bounds for the spin-spin correlations To begin, the infrared bound (6.1.6) allows one to obtain a more explicit formulation of Theorem 6.1.3. It can be interpreted as the fact that $\eta = 0$ when $d \geq 4$.

Theorem 6.1.4 (Pointwise lower bound in dimension $d \geq 4$). *Let $d \geq 4$. There exists $c = c(d) > 0$ such that for all $\beta \leq \beta_c$ and for all $x \in \mathbb{Z}^d$ with $2 \leq |x| \leq L(\beta)$,*

$$\langle \tau_0 \tau_x \rangle_\beta \geq \begin{cases} \frac{c}{|x|^{d-2}} & \text{if } d \geq 5, \\ \frac{c}{|x|^2 \log |x|} & \text{if } d = 4. \end{cases} \quad (6.1.17)$$

Proof. Let $c_1, N_1 > 0$ be given by Theorem 6.1.3. Let $\beta \leq \beta_c$ and $N_1 \leq n \leq L(\beta)$. Using the infrared bound (6.1.6) gives

$$\chi_{4n}(\beta) + n^{d-2} \sum_{1 \leq k \leq 2n} k \langle \tau_0 \tau_{k\mathbf{e}_1} \rangle_\beta \leq C_1 \left(n^2 + n^{d-2} \sum_{k=1}^{2n} \frac{1}{k^{d-3}} \right) \leq \begin{cases} C_2 n^{d-2} & \text{if } d \geq 5, \\ C_3 n^2 \log n & \text{if } d = 4. \end{cases} \quad (6.1.18)$$

The proof follows readily by Theorem 6.1.3, the MMS inequalities, and by choosing $c > 0$ small enough (in particular to include the case $|x| \leq N_1$). $\square$

the Ising model, for all $x_\perp \in \mathbb{Z}^{d-1}$, there exists a measure $\mu = \mu(x_\perp, \beta)$ on $[0, 1]$ such that for all $k \geq 1$,

$$\langle \tau_0 \tau_{(k, x_\perp)} \rangle_\beta = \int_0^1 \lambda^k d\mu(\lambda).$$

The above display implies that

$$\langle \tau_0 \tau_{(k, x_\perp)} \rangle_\beta - \langle \tau_0 \tau_{(k+1, x_\perp)} \rangle_\beta = \frac{1}{k} \int_0^1 k \lambda^k (1 - \lambda) d\mu(\lambda).$$

The estimate (6.1.14) follows by telescoping and replacing $k\lambda^k(1 - \lambda)$ by $C\lambda^{k/2}$ for some constant $C > 0$.

When $d = 3$ we do not obtain a more explicit pointwise lower bound, but we still improve on the existing bound on η .

Theorem 6.1.5. *Let $d = 3$. If the critical exponent η exists, it satisfies $\eta \leq \frac{1}{2}$.*

Proof. The proof follows by plugging the estimate provided by the existence of η in (6.1.12). $\square$

Behaviour of the correlation lengths Besides the sharp length $L(\beta)$, there are other natural typical lengths that one may define. The following quantity, called the *correlation length*, is well defined for $\beta < \beta_c$

$$\xi(\beta) := - \lim_{n \rightarrow \infty} \left(\frac{1}{n} \log \langle \tau_0 \tau_{n\mathbf{e}_1} \rangle_\beta \right)^{-1}. \quad (6.1.19)$$

For $p > 0$, one may also define

$$\xi_p(\beta) := \left(\frac{1}{\chi(\beta)} \sum_{x \in \mathbb{Z}^d} |x|^p \langle \tau_0 \tau_x \rangle_\beta \right)^{1/p}. \quad (6.1.20)$$

It is expected that there exist ν_1, ν_2 , and $\nu_3 > 0$ such that for $\beta < \beta_c$

$$L(\beta) = (\beta_c - \beta)^{-\nu_1/2+o(1)}, \quad \xi(\beta) = (\beta_c - \beta)^{-\nu_2/2+o(1)}, \quad \xi_p(\beta) = (\beta_c - \beta)^{-\nu_3/2+o(1)}, \quad (6.1.21)$$

where $o(1)$ tends to zero as β tends to β_c , and where ν_3 may depend on $p > 0$. The relation between these quantities is not clear a priori. Theorem 6.1.4 allows one to compute these exponents when $d \geq 4$.

Theorem 6.1.6. *Let $d \geq 4$. Let $p > 0$. Then, for $\beta < \beta_c$,*

$$L(\beta) = (\beta_c - \beta)^{-1/2+o(1)}, \quad (6.1.22)$$

$$\xi(\beta) = (\beta_c - \beta)^{-1/2+o(1)}, \quad (6.1.23)$$

$$\xi_p(\beta) = (\beta_c - \beta)^{-1/2+o(1)}, \quad (6.1.24)$$

where $o(1)$ tends to 0 as β tends to β_c .

Proof. By [4, 14] we know that $\chi(\beta) = (\beta_c - \beta)^{-1+o(1)}$. Theorem 6.1.4 and (6.1.10) yield that $\chi(\beta) = L(\beta)^{2+o(1)}$. Combined with the estimate on $\chi(\beta)$ this yields (6.1.22). The estimate (6.1.24) follows by similar arguments. We obtain (6.1.23) by proving that $L(\beta) = \xi(\beta)^{1+o(1)}$. Using (6.1.10) yields that $\xi(\beta) \leq C_1 L(\beta)$ for some $C_1 = C_1(d) > 0$. Moreover, by classical sub-additivity arguments, one has that $\langle \tau_0 \tau_{n\mathbf{e}_1} \rangle_\beta \leq \exp(-n/\xi(\beta))$. Together with the MMS inequalities, this yields

$$\varphi_\beta(\Lambda_n) \leq C_2 n^{d-1} e^{-n/\xi(\beta)}, \quad (6.1.25)$$

which implies that $L(\beta) \leq C_3 \xi(\beta) \log \xi(\beta)$. $\square$

Remark 6.1.7. When $d \geq 5$, there exists $c_1, C_1 > 0$ such that for all $\beta < \beta_c$, $c(\beta_c - \beta)^{-1} \leq \chi(\beta) \leq C(\beta_c - \beta)^{-1}$, see [4]. This observation improves (6.1.22) and yields the existence of $c_2, C_2 > 0$ such that for all $\beta < \beta_c$, $c_2(\beta_c - \beta)^{-1/2} \leq L(\beta) \leq C_2(\beta_c - \beta)^{-1/2}$. Combined with (6.1.10) and Theorem 6.1.4, we obtain the following near-critical estimate on the two-point function: there exist $c, C > 0$ such that for all $\beta < \beta_c$,

$$\langle \tau_0 \tau_x \rangle_\beta \leq \frac{C}{|x|^{d-2}} e^{-c(\beta_c - \beta)^{1/2}|x|}, \quad x \in \mathbb{Z}^d \setminus \{0\}, \quad (6.1.26)$$

$$\langle \tau_0 \tau_x \rangle_\beta \geq \frac{c}{|x|^{d-2}}, \quad 1 \leq |x| \leq L(\beta). \quad (6.1.27)$$

Such estimates have been essential for the study of the *torus plateau* in various models of statistical mechanics [230, 154, 179], and are used in [180] to establish it in the case of the Ising model in dimension $d \geq 5$.

Divergence of the bubble diagram As described by Aizenman in his seminal work [4], one may define the *bubble diagram*

$$B(\beta) := \sum_{x \in \mathbb{Z}^d} \langle \tau_0 \tau_x \rangle_\beta^2, \quad (6.1.28)$$

whose finiteness at criticality allows one to establish that some critical exponents exist and take their mean-field values. The condition $B(\beta_c) < \infty$ also implies triviality of the critical scaling limits of the model, see [204, Theorem D.2]. The infrared bound yields that this condition is satisfied for $d \geq 5$. The estimates for $d = 2$ imply that the bubble diagram diverges. In the following result, we complete the picture by proving that the bubble diagram diverges for $d = 3, 4$.

Theorem 6.1.8 (Divergence of the bubble diagram). *Let $d = 3, 4$. Then, $B(\beta_c) = \infty$.*

Proof. We proceed by contradiction and assume that $B(\beta_c)$ is finite. Using Cauchy–Schwarz inequality one has

$$\sum_{\substack{x, y \in \Lambda_n \\ x_1 \leq n/2 \\ y \sim x}} \langle \tau_0 \tau_x \rangle_{\beta_c} \langle \tau_y \tau_{\mathcal{R}_n(y)} \rangle_{\beta_c} \leq C_1 \sqrt{B(\beta_c)} \sqrt{B_{\geq n/2}(\beta_c)}, \quad (6.1.29)$$

where $B_{\geq k}(\beta)$ is the sum of the squares of spin-spin correlations for $x \notin \Lambda_k$. Since $B(\beta_c) < \infty$, the sum on the left-hand side of (6.1.29) tends to 0 as n tends to infinity. Using Theorem 6.1.2 together with the MMS inequalities, there exists $c_1 > 0$ such that for n large enough,

$$c_1 \leq C_2 n^{d-2} \langle \tau_0 \tau_{(n/4)\mathbf{e}_1} \rangle_{\beta_c} \sum_{1 \leq k \leq n} k \langle \tau_0 \tau_{k\mathbf{e}_1} \rangle_{\beta_c}. \quad (6.1.30)$$

Assume first that $d = 4$. Using the Cauchy–Schwarz inequality and the MMS inequalities once more gives

$$\sum_{1 \leq k \leq n} k \langle \tau_0 \tau_{k\mathbf{e}_1} \rangle_{\beta_c} \leq C_3 \sqrt{\log n} \sqrt{\sum_{1 \leq k \leq n} k^3 \langle \tau_0 \tau_{k\mathbf{e}_1} \rangle_{\beta_c}^2} \leq C_4 \sqrt{\log n} \sqrt{B(\beta_c)}. \quad (6.1.31)$$

Putting the two previous displayed equations together gives

$$\langle \tau_0 \tau_{(n/4)\mathbf{e}_1} \rangle_{\beta_c} \geq \frac{c_2}{n^{d-2} \sqrt{\log n}}. \quad (6.1.32)$$

Using the MMS inequalities we obtain that a similar bound holds for all $x \in \mathbb{Z}^d$. We find that $B(\beta_c) = \infty$, which is a contradiction.

If $d = 3$, using the Cauchy–Schwarz inequality as in (6.1.31) gives

$$\sum_{1 \leq k \leq n} k \langle \tau_0 \tau_{k\mathbf{e}_1} \rangle_{\beta_c} \leq C_5 \sqrt{n} \sqrt{B(\beta_c)}, \quad (6.1.33)$$

and hence, using (6.1.30) again,

$$\langle \tau_0 \tau_{(n/4)\mathbf{e}_1} \rangle_{\beta_c} \geq \frac{c_3}{n^{3/2}}. \quad (6.1.34)$$

This contradicts, once again, the finiteness of the bubble diagram. $\square$

Remark 6.1.9. It is possible to use Theorem 6.1.3 to obtain a quantitative estimate on the blow-up of the bubble diagram when $d = 3, 4$: more precisely, there exists $c = c(d) > 0$ such that, for all $\beta \leq \beta_c$, for all $1 \leq n \leq L(\beta)$,

$$B_n(\beta) \geq \begin{cases} c\sqrt{\log n} & \text{if } d = 3, \\ c\sqrt{\log \log n} & \text{if } d = 4. \end{cases} \quad (6.1.35)$$

Let $u_k := \langle \tau_0 \tau_{k\mathbf{e}_1} \rangle_{\beta}$. By Theorem 6.1.3, for all $1 \leq j \leq L(\beta)$,

$$\frac{u_j \chi_{4j}(\beta)}{j} + u_j j^{d-3} \sum_{k=1}^{2j} k u_k \geq \frac{c_1}{j}. \quad (6.1.36)$$

When j belongs to a *regular scale* (see [9, Definition 5.11]), one has $\chi_{4j}(\beta) \leq C_0 j^d u_j$. Assume first that $d = 3$. Summing (6.1.36) for $1 \leq j \leq n$ which belongs to a regular scale with $n \leq L(\beta)$ and using the abundance of regular scales (see [9, Theorem 5.12]),

$$C_0 \sum_{j=1}^n j^2 u_j^2 + \sum_{j=1}^n u_j \sum_{k=1}^{2j} k u_k \geq c_2 \log n; \quad (6.1.37)$$

Introduce $a_j := \sum_{k=1}^{2j} k^2 u_k^2$. Applying the Cauchy–Schwarz inequality twice and using the fact that $(a_k)_{k \geq 1}$ is increasing,

$$\sum_{j=1}^n u_j \sum_{k=1}^{2j} k u_k \leq C_1 \sum_{j=1}^n u_j \sqrt{j} \sqrt{a_j} \leq C_2 \sqrt{\sum_{j=1}^n \frac{a_j}{j}} \sqrt{a_{n/2}} \leq C_3 a_n \sqrt{\log n}. \quad (6.1.38)$$

Plugging (6.1.38) into (6.1.37) gives

$$a_n \geq c_3 \sqrt{\log n}. \quad (6.1.39)$$

Using the MMS inequalities,

$$a_n \leq C_4 B_{dn}(\beta). \quad (6.1.40)$$

We obtained that when $d = 3$, there exists $c_4 > 0$ such that for all $n \leq L(\beta)$,

$$B_n(\beta) \geq c_4 \sqrt{\log n}. \quad (6.1.41)$$

It is possible to use a similar argument when $d = 4$. Replace a_j with $b_j := \sum_{k=1}^{2j} k^3 u_k^2$ and rewrite (6.1.36) as

$$\frac{u_j \chi_{4j}(\beta)}{j \log j} + \frac{u_j j^2}{\log j} \sum_{k=1}^{2j} k u_k \geq \frac{c_1}{j \log j}. \quad (6.1.42)$$

This yields the following: when $d = 4$, there exists $c_5 > 0$ such that for all $n \leq L(\beta)$,

$$B_n(\beta) \geq c_5 \sqrt{\log \log n}. \quad (6.1.43)$$

6.2 Proof of Theorem 6.1.2 for the Ising model

As mentioned above, the proof builds on the random current representation of the Ising model. We briefly recall this classical expansion and refer to [76] for a more detailed introduction.

6.2.1 The random current representation

Let Λ be a finite subset of $\mathbb{Z}^d$.

Definition 6.2.1. A *current* $\mathbf{n}$ on Λ is a function defined on the set $E(\Lambda) := \{\{x, y\}, x, y \in \Lambda \text{ and } x \sim y\}$ and taking its values in $\mathbb{N} = \{0, 1, \dots\}$. We let Ω_Λ be the set of currents on Λ . The set of *sources* of $\mathbf{n}$, denoted by $\partial \mathbf{n}$, is defined as

$$\partial \mathbf{n} := \left\{ x \in \Lambda : \sum_{y \sim x} \mathbf{n}_{x,y} \text{ is odd} \right\}. \quad (6.2.1)$$

We also set

$$w_\beta(\mathbf{n}) := \prod_{\substack{\{x,y\} \subset \Lambda \\ x \sim y}} \frac{\beta^{\mathbf{n}_{x,y}}}{\mathbf{n}_{x,y}!}. \quad (6.2.2)$$

If $A \subset \Lambda$, define a probability measure $\mathbf{P}_{\Lambda,\beta}^A$ on Ω_Λ as follows: for a current $\mathbf{n}$ on Λ ,

$$\mathbf{P}_{\Lambda,\beta}^A[\mathbf{n}] := \mathbf{1}_{\partial \mathbf{n} = A} \frac{w_\beta(\mathbf{n})}{Z_{\Lambda,\beta}^A}, \quad (6.2.3)$$

where $Z_{\Lambda,\beta}^A := \sum_{\partial \mathbf{n} = A} w_\beta(\mathbf{n})$ is a normalisation constant. If $\mathcal{E} \subset \Omega_\Lambda$, we will also write $Z_{\Lambda,\beta}^A[\mathcal{E}] := \sum_{\partial \mathbf{n} = A} w_\beta(\mathbf{n}) \mathbf{1}_{\mathbf{n} \in \mathcal{E}}$. As proved in [10], if A is a finite (even) subset of $\mathbb{Z}^d$, the sequence of probability measures $(\mathbf{P}_{\Lambda,\beta}^A)_{\Lambda \subset \mathbb{Z}^d}$ admits a weak limit as $\Lambda \nearrow \mathbb{Z}^d$ that we denote by $\mathbf{P}_\beta^A$.

It is possible to expand the correlation functions of the Ising model to relate them to currents: for $A \subset \Lambda$, if $\sigma_A := \prod_{x \in A} \sigma_x$ we get

$$\langle \sigma_A \rangle_{\Lambda,\beta} = \frac{\sum_{\partial \mathbf{n} = A} w_\beta(\mathbf{n})}{\sum_{\partial \mathbf{n} = \emptyset} w_\beta(\mathbf{n})}. \quad (6.2.4)$$

A current $\mathbf{n}$ with empty source set can be seen as the edge count of a multigraph obtained as a union of loops. Adding sources to a current configuration comes down to adding a collection of paths connecting pairwise the sources. We will identify $\mathbf{n}$ with its multigraph $\mathcal{N}$ in which the vertex set is Λ and where there are exactly $\mathbf{n}_{x,y}$ edges between x and y . We will write $w_\beta(\mathcal{N}) := w_\beta(\mathbf{n})$ and $\partial\mathcal{N} := \partial\mathbf{n}$.

As it turns out, connectivity properties of the multigraph induced by a current will play a crucial role in the analysis of the underlying Ising model. For a multigraph $\mathcal{N}$ on Λ and $x, y \in \Lambda$, we write $x \xleftrightarrow{\mathcal{N}} y$ if x is connected to y in $\mathcal{N}$. More generally, if $A, B \subset \Lambda$, we write $A \xleftrightarrow{\mathcal{N}} B$ if there exists $x \in A$ and $y \in B$ such that $x \xleftrightarrow{\mathcal{N}} y$.

The main interest of the above expansion lies in the following result called the *switching principle*, see [11, Lemma 2.1]. This combinatorial result first appeared in [120] to prove the concavity of the magnetisation of an Ising model with a positive external field, but the probabilistic picture attached to it flourished in [4].

Lemma 6.2.2 (Switching principle). *For any multigraphs $\mathcal{M}, \mathcal{K}$ with vertex set Λ , any $A \subset \Lambda$, and any function f of a current: if $\mathcal{K} \subset \mathcal{M}$, then*

$$\sum_{\substack{\mathcal{N} \subset \mathcal{M} \\ \partial\mathcal{N} = A}} f(\mathcal{N}) = \sum_{\substack{\mathcal{N} \subset \mathcal{M} \\ \partial\mathcal{N} = A \Delta \partial\mathcal{K}}} f(\mathcal{N} \Delta \mathcal{K}). \quad (6.2.5)$$

6.2.2 Proof of the theorem

We now turn to the proof of Theorem 6.1.2 in the case of the Ising model. We will follow [11] and consider “folded” single currents on $\mathbb{Z}^d$. We partition the edge-set $E(\mathbb{Z}^d)$ of $\mathbb{Z}^d$ into three disjoint sets:

$$\begin{aligned} E_-(\mathbb{H}_n) &:= \left\{ \{u, v\} \in E(\mathbb{Z}^d) \text{ s.t. at least one endpoint is strictly on the left of } \mathbb{H}_n \right\}, \\ E_+(\mathbb{H}_n) &:= \left\{ \{u, v\} \in E(\mathbb{Z}^d) \text{ s.t. at least one endpoint is strictly on the right of } \mathbb{H}_n \right\}, \\ E_0(\mathbb{H}_n) &:= E(\mathbb{Z}^d) \setminus (E_-(\mathbb{H}_n) \cup E_+(\mathbb{H}_n)). \end{aligned}$$

Decompose $\mathbf{n} \in \Omega_{\mathbb{Z}^d}$ into its restrictions $\mathbf{n}_-$, $\mathbf{n}_+$ and $\mathbf{n}_0$ to the above three subsets of $E(\mathbb{Z}^d)$. For the purpose of later applying the switching principle, it is convenient to consider the multigraph representation of these objects. Consider the multigraph $\mathcal{M}_n$ obtained by taking the union of the multigraph $\mathcal{N}_-$ associated with $\mathbf{n}_-$ and the reflection $\mathcal{R}_n(\mathcal{N}_+)$ of the multigraph $\mathcal{N}_+$ associated with $\mathbf{n}_+$. In this context, the switching principle yields the following result that we will use several times below.

Lemma 6.2.3 (Switching principle for reflected currents). *Let $\beta > 0$. Let $\Lambda \subset \mathbb{Z}^d$ be finite and symmetric under $\mathcal{R}_n$. Assume that $A \subset \Lambda$ and $x \in \Lambda$ are strictly on the left of $\mathbb{H}_n$. Then,*

$$Z_{\Lambda, \beta}^A[x \xleftrightarrow{\mathcal{M}_n} \mathbb{H}_n] = Z_{\Lambda, \beta}^{A \Delta \{x, \mathcal{R}_n(x)\}}[x \xleftrightarrow{\mathcal{M}_n} \mathbb{H}_n]. \quad (6.2.6)$$

Proof. By definition,

$$Z_{\Lambda, \beta}^A[x \xleftrightarrow{\mathcal{M}_n} \mathbb{H}_n] = \sum_{\mathbf{n}_0} w_\beta(\mathbf{n}_0) \sum_{\substack{\mathbf{n}_-, \mathbf{n}_+ \\ \partial(\mathbf{n}_- + \mathbf{n}_+) = \partial\mathbf{n}_0 \Delta A}} \mathbb{1}_{x \xleftrightarrow{\mathcal{M}_n} \mathbb{H}_n} w_\beta(\mathbf{n}_-) w_\beta(\mathbf{n}_+). \quad (6.2.7)$$

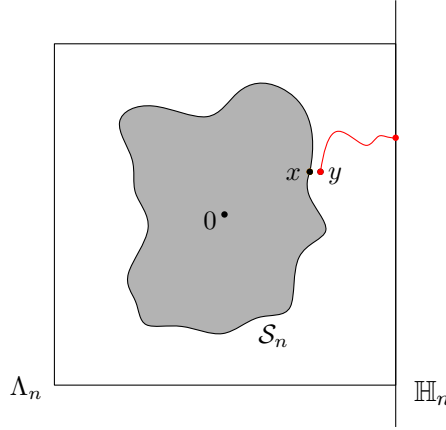


Figure 6.1: The set $\mathcal{S}_n$ is represented in grey. On its boundary, a point x (in black) is next to a point y (in red) which connects to $\mathbb{H}_n$ in $\mathcal{M}_n$, i.e. which does not belong to $\mathcal{S}_n(+\mathbf{e}_1)$.

When it exists, call $\Gamma = \Gamma(\mathcal{M}_n, x)$ the smallest path connecting x to $\mathbb{H}_n$ in $\mathcal{M}_n$ (according to some fixed ordering). In order to be in the setup of application of Lemma 6.2.2, we condition on the sources of $\mathbf{n}_-$ and $\mathbf{n}_+$ as well as on Γ and adopt the multigraph notations for the currents. Together with (6.2.7) we obtain,

$$\begin{aligned}
 Z_{\Lambda, \beta}^A[x \xleftrightarrow{\mathcal{M}_n} \mathbb{H}_n] &= \sum_{\mathbf{n}_0} w_{\beta}(\mathbf{n}_0) \sum_{D \subset \mathbb{H}_n \cap \Lambda} \sum_{\substack{\alpha \in \mathbb{H}_n \\ \gamma: x \rightarrow \alpha}} \sum_{\substack{\partial \mathbf{n}_- = D \Delta \partial \mathbf{n}_0 \Delta A \\ \partial \mathbf{n}_+ = D}} w_{\beta}(\mathbf{n}_-) w_{\beta}(\mathbf{n}_+) \mathbb{1}_{\Gamma=\gamma} \\
 &= \sum_{\mathbf{n}_0} w_{\beta}(\mathbf{n}_0) \sum_{D \subset \mathbb{H}_n \cap \Lambda} \sum_{\gamma: x \rightarrow \alpha} \sum_{\partial \mathcal{M}_n = \partial \mathbf{n}_0 \Delta A} w_{\beta}(\mathcal{M}_n) \mathbb{1}_{\Gamma=\gamma} \sum_{\substack{\mathcal{N} \subset \mathcal{M}_n \\ \partial \mathcal{N} = D}} 1 \\
 &= \sum_{\mathbf{n}_0} w_{\beta}(\mathbf{n}_0) \sum_{D \subset \mathbb{H}_n \cap \Lambda} \sum_{\gamma: x \rightarrow \alpha} \sum_{\partial \mathcal{M}_n = \partial \mathbf{n}_0 \Delta A} w_{\beta}(\mathcal{M}_n) \mathbb{1}_{\Gamma=\gamma} \sum_{\substack{\mathcal{N} \subset \mathcal{M}_n \\ \partial \mathcal{N} = D \Delta \{x, \alpha\}}} 1
 \end{aligned}$$

where we used the switching principle from Lemma 6.2.2 in the third line. As a result, we obtained,

$$\begin{aligned}
 Z_{\Lambda, \beta}^A[x \xleftrightarrow{\mathcal{M}_n} \mathbb{H}_n] &= \sum_{\mathbf{n}_0} w_{\beta}(\mathbf{n}_0) \sum_{D \subset \mathbb{H}_n \cap \Lambda} \sum_{\substack{\partial \mathbf{n}_- = D \Delta \partial \mathbf{n}_0 \Delta A \Delta \{x, \alpha\} \\ \partial \mathbf{n}_+ = D \Delta \{\mathcal{R}_n(x), \alpha\}}} w_{\beta}(\mathbf{n}_-) w_{\beta}(\mathbf{n}_+) \mathbb{1}_{x \xleftrightarrow{\mathcal{M}_n} \mathbb{H}_n} \\
 &= \sum_{\mathbf{n}_0} w_{\beta}(\mathbf{n}_0) \sum_{\substack{\mathbf{n}_-, \mathbf{n}_+ \\ \partial(\mathbf{n}_- + \mathbf{n}_+) = \partial \mathbf{n}_0 \Delta A \Delta \{x, \mathcal{R}_n(x)\}}} w_{\beta}(\mathbf{n}_-) w_{\beta}(\mathbf{n}_+) \mathbb{1}_{x \xleftrightarrow{\mathcal{M}_n} \mathbb{H}_n} \\
 &= Z_{\Lambda, \beta}^{A \Delta \{x, \mathcal{R}_n(x)\}}[x \xleftrightarrow{\mathcal{M}_n} \mathbb{H}_n],
 \end{aligned}$$

which concludes the proof. $\square$

The above quantities were all defined with respect to the direction $\mathbf{e}_1$. In order to generalise these objects, let us write $\mathcal{M}_n(+\mathbf{e}_1) = \mathcal{M}_n$, $\mathcal{R}_n(+\mathbf{e}_1) = \mathcal{R}_n$, $\mathbb{H}_n(+\mathbf{e}_1) = \mathbb{H}_n$, and define

$$\mathcal{S}_n(+\mathbf{e}_1) := \left\{ x \in \Lambda_n, x \xleftrightarrow{\mathcal{M}_n(+\mathbf{e}_1)} \mathbb{H}_n(+\mathbf{e}_1) \right\}. \quad (6.2.8)$$

Similarly, define $\mathbb{H}_n(\pm \mathbf{e}_i)$, $\mathcal{R}_n(\pm \mathbf{e}_i)$, $\mathcal{M}_n(\pm \mathbf{e}_i)$, and $\mathcal{S}_n(\pm \mathbf{e}_i)$ for $1 \leq i \leq d$ in the other $2d - 1$ directions. Let (see Figure 6.1)

$$\mathcal{S}_n := \bigcap_{1 \leq i \leq d} (\mathcal{S}_n(+\mathbf{e}_i) \cap \mathcal{S}_n(-\mathbf{e}_i)). \quad (6.2.9)$$

Note that by definition, $\mathcal{S}_n \subset \Lambda_{n-1}$. Recall the definitions of $\mathbf{P}_\beta^\emptyset$ and $\varphi_\beta(S)$ from above.

By definition of $L(\beta)$, if $0 \leq n \leq L(\beta)$ and $S \subset \Lambda_n$, then $\varphi_\beta(S) \geq 1/2$. As a result, if $1 \leq n \leq L(\beta)$, then

$$\frac{1}{2} \mathbf{P}_\beta^\emptyset[0 \in \mathcal{S}_n] \leq \mathbf{E}_\beta^\emptyset[\mathbb{1}_{0 \in \mathcal{S}_n} \varphi_\beta(\mathcal{S}_n)]. \quad (6.2.10)$$

The first step is to observe that the left-hand side in (6.2.10) is bounded away from 0.

Lemma 6.2.4. *There exist $c, N_0 > 0$ such that for $\beta \leq \beta_c$ and $n \geq N_0$,*

$$\mathbf{P}_\beta^\emptyset[0 \in \mathcal{S}_n] \geq c. \quad (6.2.11)$$

Proof. Using a union bound and the symmetries of $\mathbf{P}_\beta^\emptyset$,

$$\mathbf{P}_\beta^\emptyset[0 \notin \mathcal{S}_n] \leq \sum_{i=1}^d (\mathbf{P}_\beta^\emptyset[0 \notin \mathcal{S}_n(+\mathbf{e}_i)] + \mathbf{P}_\beta^\emptyset[0 \notin \mathcal{S}_n(-\mathbf{e}_i)]) = 2d \mathbf{P}_\beta^\emptyset[0 \notin \mathcal{S}_n(+\mathbf{e}_1)]. \quad (6.2.12)$$

Fix for a moment $\Lambda \subset \mathbb{Z}^d$ finite, symmetric under $\mathcal{R}_n$, and containing Λ_{4n} . The switching principle for reflected currents (Lemma 6.2.3) implies

$$Z_{\Lambda, \beta}^\emptyset[0 \notin \mathcal{S}_n(+\mathbf{e}_1)] = Z_{\Lambda, \beta}^\emptyset[0 \xleftrightarrow{\mathcal{M}_n} \mathbb{H}_n] = \sum_{\mathbf{n} \in \Omega_\Lambda, \partial \mathbf{n} = \emptyset} w_\beta(\mathbf{n}) \mathbb{1}_{0 \xleftrightarrow{\mathcal{M}_n} \mathbb{H}_n} = Z_{\Lambda, \beta}^{\{0, \mathcal{R}_n(0)\}}. \quad (6.2.13)$$

Dividing by $Z_{\Lambda, \beta}^\emptyset$, and using monotonicity of the two-point function in Λ , we find that for $n \geq 1$,

$$\mathbf{P}_{\Lambda, \beta}^\emptyset[0 \xleftrightarrow{\mathcal{M}_n} \mathbb{H}_n] \leq \langle \sigma_0 \sigma_{2n\mathbf{e}_1} \rangle_\beta. \quad (6.2.14)$$

The proof follows readily from the infrared bound (6.1.6) and the limit $\Lambda \nearrow \mathbb{Z}^d$. $\square$

Combining (6.2.10) and Lemma 6.2.4, we obtain that for some $c, N_0 > 0$, if $N_0 \leq n \leq L(\beta)$,

$$\frac{c}{2\beta} \leq \mathbf{E}_\beta^\emptyset \left[\mathbb{1}_{0 \in \mathcal{S}_n} \sum_{\substack{x \in \mathcal{S}_n \\ y \sim x, y \notin \mathcal{S}_n}} \langle \sigma_0 \sigma_x \rangle_{\mathcal{S}_n, \beta} \right]. \quad (6.2.15)$$

Notice that if $x \in \mathcal{S}_n$, $y \sim x$ and $y \notin \mathcal{S}_n$, then there exist $1 \leq i \leq d$ and $\varepsilon \in \{\pm 1\}$ such that $y \notin \mathcal{S}_n(\varepsilon \mathbf{e}_i)$. As a result,

$$\begin{aligned} \mathbb{1}_{0 \in \mathcal{S}_n} \sum_{\substack{x \in \mathcal{S}_n \\ y \sim x, y \notin \mathcal{S}_n}} \langle \sigma_0 \sigma_x \rangle_{\mathcal{S}_n, \beta} &\leq \mathbb{1}_{0 \in \mathcal{S}_n} \sum_{i=1}^d \sum_{\varepsilon \in \{\pm 1\}} \sum_{\substack{x \in \mathcal{S}_n \\ y \sim x, y \notin \mathcal{S}_n(\varepsilon \mathbf{e}_i)}} \langle \sigma_0 \sigma_x \rangle_{\mathcal{S}_n, \beta} \\ &\leq \mathbb{1}_{0 \in \mathcal{S}_n} \sum_{i=1}^d \sum_{\varepsilon \in \{\pm 1\}} \sum_{\substack{x \in \mathcal{S}_n(\varepsilon \mathbf{e}_i) \cap \Lambda_{n-1} \\ y \sim x, y \notin \mathcal{S}_n(\varepsilon \mathbf{e}_i)}} \langle \sigma_0 \sigma_x \rangle_{\mathcal{S}_n(\varepsilon \mathbf{e}_i), \beta}, \end{aligned}$$

where we used that $\mathcal{S}_n \subset \mathcal{S}_n(\varepsilon \mathbf{e}_i) \cap \Lambda_{n-1}$ together with Griffiths' inequality on the second line. Using the symmetries of $\mathbf{P}_\beta^\theta$ and (6.2.15), we get

$$\frac{c}{4d\beta} \leq \mathbf{E}_\beta^\theta \left[\mathbb{1}_{0 \in \mathcal{S}_n^1} \sum_{\substack{x \in \mathcal{S}_n^1 \\ y \sim x, y \in \Lambda_n}} \mathbb{1}_{y \xrightarrow{\mathcal{M}_n} \mathbb{H}_n} \langle \sigma_0 \sigma_x \rangle_{\mathcal{S}_n^1, \beta} \right], \quad (6.2.16)$$

where $\mathcal{S}_n^1 := \mathcal{S}_n(+\mathbf{e}_1) \cap \Lambda_{n-1}$. Theorem 6.1.2 follows from (6.2.16) and Lemma 6.2.5 below.

Lemma 6.2.5. *For $\beta \leq \beta_c$,*

$$\mathbf{E}_\beta^\theta \left[\mathbb{1}_{0 \in \mathcal{S}_n^1} \sum_{\substack{x \in \mathcal{S}_n^1 \\ y \sim x, y \in \Lambda_n}} \mathbb{1}_{y \xrightarrow{\mathcal{M}_n} \mathbb{H}_n} \langle \sigma_0 \sigma_x \rangle_{\mathcal{S}_n^1, \beta} \right] \leq \sum_{\substack{x, y \in \Lambda_n \\ y \sim x}} (\langle \sigma_0 \sigma_x \rangle_\beta - \langle \sigma_0 \sigma_{\mathcal{R}_n(x)} \rangle_\beta) \langle \sigma_y \sigma_{\mathcal{R}_n(y)} \rangle_\beta. \quad (6.2.17)$$

Proof. Fix $\Lambda \subset \mathbb{Z}^d$ finite, symmetric under $\mathcal{R}_n$, which contains Λ_{4n} . Observe that

$$\mathbf{E}_{\Lambda, \beta}^\theta \left[\mathbb{1}_{0 \in \mathcal{S}_n^1} \sum_{\substack{x \in \mathcal{S}_n^1 \\ y \sim x, y \in \Lambda_n}} \mathbb{1}_{y \xrightarrow{\mathcal{M}_n} \mathbb{H}_n} \langle \sigma_0 \sigma_x \rangle_{\mathcal{S}_n^1, \beta} \right] = \sum_{\substack{x, y \in \Lambda_n \\ y \sim x}} \mathbf{E}_{\Lambda, \beta}^\theta \left[\mathbb{1}_{0, x \in \mathcal{S}_n^1, y \xrightarrow{\mathcal{M}_n} \mathbb{H}_n} \langle \sigma_0 \sigma_x \rangle_{\mathcal{S}_n^1, \beta} \right]. \quad (6.2.18)$$

Set $\mathcal{G}_n := \{x \in \Lambda, x \text{ or } \mathcal{R}_n(x) \xrightarrow{\mathcal{M}_n} \mathbb{H}_n\}$. For $x \in \Lambda_n$ and $y \sim x$ with $y \in \Lambda_n$, write

$$\begin{aligned} \mathbf{E}_{\Lambda, \beta}^\theta \left[\mathbb{1}_{0, x \in \mathcal{S}_n^1, y \xrightarrow{\mathcal{M}_n} \mathbb{H}_n} \langle \sigma_0 \sigma_x \rangle_{\mathcal{S}_n^1, \beta} \right] &\leq \mathbf{E}_{\Lambda, \beta}^\theta \left[\mathbb{1}_{0, x \in \mathcal{G}_n, y \xrightarrow{\mathcal{M}_n} \mathbb{H}_n} \langle \sigma_0 \sigma_x \rangle_{\mathcal{G}_n, \beta} \right] \\ &= \sum_{\substack{S \subset \Lambda \\ S \ni 0, x \\ S \not\ni y}} \mathbf{E}_{\Lambda, \beta}^\theta \left[\mathbb{1}_{\mathcal{G}_n = S} \frac{Z_{S, \beta}^{\{0, x\}}}{Z_{S, \beta}^\theta} \right]. \end{aligned} \quad (6.2.19)$$

Notice that

$$Z_{\Lambda, \beta}^\theta[\mathcal{G}_n = S] = Z_{S, \beta}^\theta Z_{\Lambda \setminus S, \beta}^\theta[\mathcal{G}_n = S] \quad \text{and} \quad Z_{S, \beta}^{\{0, x\}} Z_{\Lambda \setminus S, \beta}^\theta[\mathcal{G}_n = S] = Z_{\Lambda, \beta}^{\{0, x\}}[\mathcal{G}_n = S]. \quad (6.2.20)$$

As a result,

$$\mathbf{E}_{\Lambda, \beta}^\theta \left[\mathbb{1}_{\mathcal{G}_n = S} \frac{Z_{S, \beta}^{\{0, x\}}}{Z_{S, \beta}^\theta} \right] = \frac{Z_{\Lambda, \beta}^\theta[\mathcal{G}_n = S]}{Z_{\Lambda, \beta}^\theta} \frac{Z_{S, \beta}^{\{0, x\}}}{Z_{S, \beta}^\theta} = \frac{Z_{\Lambda, \beta}^{\{0, x\}}[\mathcal{G}_n = S]}{Z_{\Lambda, \beta}^\theta}. \quad (6.2.21)$$

We rewrite the right-hand side of (6.2.19) as

$$\sum_{\substack{S \subset \Lambda \\ S \ni 0, x \\ S \not\ni y}} \frac{Z_{\Lambda, \beta}^{\{0, x\}}[\mathcal{G}_n = S]}{Z_{\Lambda, \beta}^\theta} = \frac{Z_{\Lambda, \beta}^{\{0, x\}}[0, x \xrightarrow{\mathcal{M}_n} \mathbb{H}_n, y \xrightarrow{\mathcal{M}_n} \mathbb{H}_n]}{Z_{\Lambda, \beta}^\theta}. \quad (6.2.22)$$

Now, we analyse $Z_{\Lambda, \beta}^{\{0, x\}}[0, x \xrightarrow{\mathcal{M}_n} \mathbb{H}_n, y \xrightarrow{\mathcal{M}_n} \mathbb{H}_n]$ by conditioning on the set $\mathcal{C}_n(0)$ defined as the union of the cluster of 0 in $\mathcal{M}_n$ and its reflection with respect to $\mathbb{H}_n$. Write,

$$Z_{\Lambda, \beta}^{\{0, x\}}[0, x \xrightarrow{\mathcal{M}_n} \mathbb{H}_n, y \xrightarrow{\mathcal{M}_n} \mathbb{H}_n] = \sum_{\substack{S \cap \mathbb{H}_n = \emptyset \\ S \ni 0, x \\ S \not\ni y}} Z_{\Lambda, \beta}^{\{0, x\}}[\mathcal{C}_n(0) = S, y \xrightarrow{\mathcal{M}_n} \mathbb{H}_n]. \quad (6.2.23)$$

Finally,

$$Z_{\Lambda,\beta}^{\{0,x\}}[\mathcal{C}_n(0) = S, y \xleftarrow{\mathcal{M}_n} \mathbb{H}_n] = Z_{\Lambda,\beta}^{\{0,x\}}[\mathcal{C}_n(0) = S] \frac{Z_{\Lambda \setminus S, \beta}^{\emptyset}[y \xleftarrow{\mathcal{M}_n} \mathbb{H}_n]}{Z_{\Lambda \setminus S, \beta}^{\emptyset}}. \quad (6.2.24)$$

Applying the switching principle for reflected currents (Lemma 6.2.3),

$$Z_{\Lambda \setminus S, \beta}^{\emptyset}[y \xleftarrow{\mathcal{M}_n} \mathbb{H}_n] = Z_{\Lambda \setminus S, \beta}^{\{y, \mathcal{R}_n(y)\}}. \quad (6.2.25)$$

Moreover, using one last time Lemma 6.2.3,

$$\sum_{\substack{S \cap \mathbb{H}_n = \emptyset \\ S \ni 0, x}} Z_{\Lambda, \beta}^{\{0,x\}}[\mathcal{C}_n(0) = S] = Z_{\Lambda, \beta}^{\{0,x\}}[0, x \xleftarrow{\mathcal{M}_n} \mathbb{H}_n] = Z_{\Lambda, \beta}^{\{0,x\}} - Z_{\Lambda, \beta}^{\{0, \mathcal{R}_n(x)\}}. \quad (6.2.26)$$

Putting (6.2.23)-(6.2.26) in (6.2.22) and using Griffiths' inequality to replace $\Lambda \setminus S$ by Λ , we get

$$\mathbf{E}_{\Lambda, \beta}^{\emptyset} \left[\mathbf{1}_{0, x \in S_n^1, y \xleftarrow{\mathcal{M}_n} \mathbb{H}_n} \langle \sigma_0 \sigma_x \rangle_{S_n^1, \beta} \right] \leq \left(\langle \sigma_0 \sigma_x \rangle_{\Lambda, \beta} - \langle \sigma_0 \sigma_{\mathcal{R}_n(x)} \rangle_{\Lambda, \beta} \right) \langle \sigma_y \sigma_{\mathcal{R}_n(y)} \rangle_{\Lambda, \beta}. \quad (6.2.27)$$

We conclude by taking $\Lambda \nearrow \mathbb{Z}^d$. $\square$

6.3 Proof of Theorem 6.1.2 for the φ^4 model

As explained in the introduction, the discrete φ^4 model can be recovered from taking the limit of well-chosen Ising models. In fact, with this procedure we may recover more models which we now describe.

6.3.1 The Griffiths–Simon class of measures

Definition 6.3.1 (The Griffiths–Simon class of measures). A Borel measure ρ on $\mathbb{R}$ falls into the *Griffiths–Simon (GS) class* of measures if it satisfies one of the following conditions:

- (i) there exists an integer $N \geq 1$, a renormalisation constant $Z > 0$, $(J_{i,j})_{1 \leq i,j \leq N} \in (\mathbb{R}^+)^{N^2}$, and $(Q_n)_{1 \leq n \leq N} \in \mathbb{R}^N$ such that for every $F : \mathbb{R} \rightarrow \mathbb{R}^+$ bounded and measurable,

$$\int_{\mathbb{R}} F(\tau) d\rho(\tau) = \frac{1}{Z} \sum_{\sigma \in \{\pm 1\}^N} F\left(\sum_{n=1}^N Q_n \sigma_n\right) \exp\left(\sum_{i,j=1}^N J_{i,j} \sigma_i \sigma_j\right), \quad (6.3.1)$$

- (ii) the measure ρ can be obtained as a weak limit of probability measures of the above type, and it is of sub-Gaussian growth: for some $\alpha > 2$,

$$\int_{\mathbb{R}} e^{|\tau|^\alpha} d\rho(\tau) < \infty. \quad (6.3.2)$$

Measures that fulfill condition (i) are described as being of the “Ising type”.

The following result was proved in [224] (see also [169]).

Proposition 6.3.2. *Let $g > 0$ and $a \in \mathbb{R}$. The probability measure $\rho_{g,a}$ on $\mathbb{R}$ given by*

$$d\rho_{g,a}(\varphi) = \frac{1}{z_{g,a}} e^{-g\varphi^4 - a\varphi^2} d\varphi, \quad (6.3.3)$$

where $z_{g,a}$ is a renormalisation constant, belongs to the GS class.

6.3.2 Random current representation of Ising-type models in the GS class

Fix ρ in the GS class of the Ising-type, and $\beta > 0$. The measure $\langle \cdot \rangle_{\Lambda, \rho, \beta}$ can be represented as an Ising measure³ on $\Lambda_N := \Lambda \times K_N$ that we denote by $\langle \cdot \rangle_{\Lambda_N, \rho, \beta}$. In that case, we identify τ_x with averages of the form

$$\sum_{i=1}^N Q_i \sigma_{(x,i)}, \quad (6.3.4)$$

where $Q_i \geq 0$ for $1 \leq i \leq N$. For $x \in \mathbb{Z}^d$, we will denote $B_x := \{(x, i), 1 \leq i \leq N\}$. This point of view allows one to use the random current representation. We introduce a measure $\mathbf{P}_{\Lambda, \rho, \beta}^{xy}$ on $\Omega_{\Lambda \times K_N}$ which we define in the following two steps procedure:

- first, sample two integers $1 \leq i, j \leq N$ with probability given by

$$\frac{Q_i Q_j \langle \sigma_{(x,i)} \sigma_{(y,j)} \rangle_{\Lambda_N, \rho, \beta}}{\langle \tau_x \tau_y \rangle_{\Lambda, \rho, \beta}}, \quad (6.3.5)$$

- then, sample a current according to the “standard” current measure $\mathbf{P}_{\Lambda_N, \rho, \beta}^{\{(x,i), (y,j)\}}$ introduced above.

As before, it is possible to define the infinite volume version of the above measure. We will denote it $\mathbf{P}_{\rho, \beta}^{xy}$. Samples of $\mathbf{P}_{\rho, \beta}^{xy}$ are random currents with *random sources* in B_x and B_y . The interest of this measure lies in the fact that it allows one to derive bounds on connection probabilities that are formulated in terms of the correlation functions of the field variable τ .

6.3.3 Proof of the theorem for Ising-type models in the GS class

We restate the result, as it is slightly different than for the Ising model (notice the introduction of the variable y').

Proposition 6.3.3. *Let $d \geq 3$ and let ρ be a measure of the Ising type in the GS class. There exist $c, N_0 > 0$ such that, for all $\beta \leq \beta_c(\rho)$ and for all $N_0 \leq n \leq L(\beta, \rho)$,*

$$c \leq \beta \sum_{\substack{x \in \Lambda_n \\ y \sim x \\ y' \sim y}} \left(\langle \tau_0 \tau_x \rangle_{\rho, \beta} - \langle \tau_0 \tau_{\mathcal{R}_n(x)} \rangle_{\rho, \beta} \right) \langle \tau_y \tau_{\mathcal{R}_n(y')} \rangle_{\rho, \beta}. \quad (6.3.6)$$

We begin with some notations: for a subset $\mathcal{E}$ of $\Omega_{\Lambda \times K_N}$, set

$$Z_{\Lambda, \rho, \beta}^{\emptyset}[\mathcal{E}] := Z_{\Lambda_N, \rho, \beta}^{\emptyset}[\mathcal{E}] \text{ and } Z_{\Lambda, \rho, \beta}^{\{u,v\}}[\mathcal{E}] = \sum_{i,j=1}^N Q_i Q_j Z_{\Lambda_N, \rho, \beta}^{\{(u,i), (v,j)\}}[\mathcal{E}]. \quad (6.3.7)$$

Note that

$$\mathbf{P}_{\Lambda, \rho, \beta}^{\{u,v\}}[\mathcal{E}] = \sum_{i,j=1}^N \frac{Q_i Q_j \langle \sigma_{(u,i)} \sigma_{(v,j)} \rangle_{\Lambda_N, \rho, \beta}}{\langle \tau_u \tau_v \rangle_{\Lambda, \rho, \beta}} \mathbf{P}_{\Lambda_N, \rho, \beta}^{\{(u,i), (v,j)\}}[\mathcal{E}] = \frac{Z_{\Lambda, \rho, \beta}^{\{u,v\}}[\mathcal{E}]}{Z_{\Lambda, \rho, \beta}^{uv}}, \quad (6.3.8)$$

³In this measure, if $x \sim y$, the spins (x, i) and (y, j) interact through the coupling constant $\beta Q_i Q_j$.

and

$$\frac{Z_{\Lambda, \rho, \beta}^{\{u, v\}}}{Z_{\Lambda, \rho, \beta}^{\emptyset}} = \langle \tau_u \tau_v \rangle_{\Lambda, \rho, \beta}. \quad (6.3.9)$$

Finally, we write $\mathbb{H}_n(N) := \mathbb{H}_n \times K_N$.

We start by the switching principle in this context.

Lemma 6.3.4 (Switching principle for reflected currents of Ising-type models in the GS class). *Let $\Lambda \subset \mathbb{Z}^d$ finite and symmetric with respect to $\mathcal{R}_n$. Let $x \in \Lambda^-$. Then,*

- (i) $Z_{\Lambda, \rho, \beta}^{\emptyset}[B_x \xrightarrow{\mathcal{M}_n} \mathbb{H}_n(N)] \leq \beta \sum_{x' \sim x} Z_{\Lambda, \rho, \beta}^{\{x, \mathcal{R}_n(x')\}},$
- (ii) $Z_{\Lambda, \rho, \beta}^{\{0, x\}}[\partial \mathbf{n} \cap B_x \xrightarrow{\mathcal{M}_n} \mathbb{H}_n(N)] = Z_{\Lambda, \rho, \beta}^{\{0, \mathcal{R}_n(x)\}}$

Proof. For (i), we follow the strategy employed in [9, Lemma A.7]. Write,

$$\begin{aligned} Z_{\Lambda, \rho, \beta}^{\emptyset}[B_x \xrightarrow{\mathcal{M}_n} \mathbb{H}_n(N)] &\leq \sum_{i=1}^N Z_{\Lambda_N, \rho, \beta}^{\emptyset}[(x, i) \xrightarrow{\mathcal{M}_n} \mathbb{H}_n(N)] \\ &\leq \sum_{x' \sim x} \sum_{i, j=1}^N (\beta Q_i Q_j) Z_{\Lambda_N, \rho, \beta}^{\{(x, i), (x', j)\}}[(x', j) \xrightarrow{\mathcal{M}_n} \mathbb{H}_n(N)] \\ &\leq \sum_{x' \sim x} \sum_{i, j=1}^N (\beta Q_i Q_j) Z_{\Lambda_N, \rho, \beta}^{\{(x, i), (\mathcal{R}_n(x'), j)\}}, \end{aligned}$$

where on the second line we used the fact that any path going out of B_x has to visit $B_{x'}$ for some $x' \sim x$, and on the third line we used⁴ Lemma 6.2.3.

For (ii), we essentially reproduce the argument of Lemma 6.2.3 and omit the proof. $\square$

With this result, we obtain Proposition 6.3.3 by copying the argument used above. We will modify the notations above to make them adapted to our setup. We now define,

$$\mathcal{S}_n(+\mathbf{e}_1) := \left\{ x \in \Lambda_n, B_x \xrightarrow{\mathcal{M}_n(+\mathbf{e}_1)} \mathbb{H}_n(+\mathbf{e}_1) \times K_N \right\}, \quad (6.3.10)$$

and modify accordingly the definitions of $\mathcal{S}_n$ and $\mathcal{S}_n^1$.

Proof of Proposition 6.3.3. With the help of the previous lemma, the proof is basically the same as for the Ising case. The only modification lies in the statement and in the proof of the adaptation of Lemma 6.2.5, which reads as follows. $\square$

Lemma 6.3.5. *For $\beta \leq \beta_c(\rho)$,*

$$\begin{aligned} \mathbb{E}_{\rho, \beta}^{\emptyset} \left[\mathbb{1}_{0 \in \mathcal{S}_n^1} \sum_{\substack{x \in \mathcal{S}_n^1 \\ y \sim x, y \in \Lambda_n}} \mathbb{1}_{B_y \xrightarrow{\mathcal{M}_n} \mathbb{H}_n(N)} \langle \tau_0 \tau_x \rangle_{\mathcal{S}_n^1, \rho, \beta} \right] \\ \leq \beta \sum_{\substack{x, y \in \Lambda_n \\ y \sim x \\ y' \sim y}} \left(\langle \tau_0 \tau_x \rangle_{\rho, \beta} - \langle \tau_0 \tau_{\mathcal{R}_n(x)} \rangle_{\rho, \beta} \right) \langle \tau_y \tau_{\mathcal{R}_n(y')} \rangle_{\rho, \beta}. \end{aligned} \quad (6.3.11)$$

⁴For full disclosure, we actually used a straightforward generalisation of Lemma 6.2.3 to graphs of the form $\Lambda \times K_N$ that are symmetric with respect to $\mathcal{R}_n$, in the sense that for all $x \in \Lambda$ and all $1 \leq i \leq N$, $(x, i) \in \Lambda_N$ if and only if $\mathcal{R}_n((x, i)) := (\mathcal{R}_n(x), i) \in \Lambda_N$.

Proof. Fix $\Lambda \subset \mathbb{Z}^d$ finite, symmetric under $\mathcal{R}_n$, which contains Λ_{4n} . Observe that

$$\mathbb{E}_{\Lambda, \rho, \beta}^\emptyset \left[\mathbb{1}_{0 \in S_n^1} \sum_{\substack{x \in S_n^1 \\ y \sim x, y \in \Lambda_n}} \mathbb{1}_{B_y \xleftarrow{\mathcal{M}_n} \mathbb{H}_n(N)} \langle \tau_0 \tau_x \rangle_{S_n^1, \rho, \beta} \right] = \sum_{\substack{x, y \in \Lambda_n \\ y \sim x}} \mathbb{E}_{\Lambda, \beta}^\emptyset \left[\mathbb{1}_{0, x \in S_n^1, B_y \xleftarrow{\mathcal{M}_n} \mathbb{H}_n(N)} \langle \tau_0 \tau_x \rangle_{S_n^1, \beta} \right]. \quad (6.3.12)$$

For $x \in \Lambda_n$ and $y \sim x$ with $y \in \Lambda_n$, write

$$\begin{aligned} \mathbb{E}_{\Lambda, \beta}^\emptyset \left[\mathbb{1}_{0, x \in S_n^1, B_y \xleftarrow{\mathcal{M}_n} \mathbb{H}_n(N)} \langle \tau_0 \tau_x \rangle_{S_n^1, \rho, \beta} \right] &\leq \mathbb{E}_{\Lambda, \beta}^\emptyset \left[\mathbb{1}_{B_0, B_x \subset \mathcal{G}_n, B_y \xleftarrow{\mathcal{M}_n} \mathbb{H}_n(N)} \langle \tau_0 \tau_x \rangle_{\mathcal{G}_n, \beta} \right] \\ &= \sum_{\substack{S \subset \Lambda \times K_N \\ S \supset B_0, B_x \\ S \not\supset B_y}} \mathbb{E}_{\Lambda, \rho, \beta}^\emptyset \left[\mathbb{1}_{\mathcal{G}_n = S} \frac{Z_{S, \rho, \beta}^{\{0, x\}}}{Z_{S, \rho, \beta}^\emptyset} \right], \end{aligned} \quad (6.3.13)$$

where⁵ $\mathcal{G}_n := \{(x, i) \in \Lambda_N, (x, i) \text{ or } (\mathcal{R}_n(x), i) \xleftarrow{\mathcal{M}_n} \mathbb{H}_n(N)\}$. Notice that,

$$Z_{\Lambda, \rho, \beta}^\emptyset [\mathcal{G}_n = S] = Z_{S, \rho, \beta}^\emptyset Z_{\Lambda_N \setminus S, \rho, \beta}^\emptyset [\mathcal{G}_n = S], \quad (6.3.14)$$

and $Z_{S, \rho, \beta}^{\{0, x\}} Z_{\Lambda_N \setminus S, \rho, \beta}^\emptyset [\mathcal{G}_n = S] = Z_{\Lambda_N, \rho, \beta}^{\{0, x\}} [\mathcal{G}_n = S]$. As a result,

$$\mathbb{E}_{\Lambda, \rho, \beta}^\emptyset \left[\mathbb{1}_{\mathcal{G}_n = S} \frac{Z_{S, \rho, \beta}^{\{0, x\}}}{Z_{S, \rho, \beta}^\emptyset} \right] = \frac{Z_{\Lambda, \rho, \beta}^{\{0, x\}} [\mathcal{G}_n = S]}{Z_{\Lambda, \rho, \beta}^\emptyset}. \quad (6.3.15)$$

The right-hand side of (6.3.13) rewrites

$$\sum_{\substack{S \subset \Lambda \times K_N \\ S \supset B_0, B_x \\ S \not\supset B_y}} \frac{Z_{\Lambda, \rho, \beta}^{\{0, x\}} [\mathcal{G}_n = S]}{Z_{\Lambda, \rho, \beta}^\emptyset} = \frac{Z_{\Lambda, \rho, \beta}^{\{0, x\}} [B_0, B_x \xleftarrow{\mathcal{M}_n} \mathbb{H}_n(N), B_y \xleftarrow{\mathcal{M}_n} \mathbb{H}_n(N)]}{Z_{\Lambda, \rho, \beta}^\emptyset}. \quad (6.3.16)$$

Now, we analyse $Z_{\Lambda, \rho, \beta}^{\{0, x\}} [B_0, B_x \xleftarrow{\mathcal{M}_n} \mathbb{H}_n(N), B_y \xleftarrow{\mathcal{M}_n} \mathbb{H}_n(N)]$ by conditioning on the cluster of $B_0 \cup B_x$ in $\mathcal{M}_n$ that we denote by $\mathcal{C}_n(0, x)$. Write,

$$\begin{aligned} Z_{\Lambda, \rho, \beta}^{\{0, x\}} [B_0, B_x \xleftarrow{\mathcal{M}_n} \mathbb{H}_n(N), B_y \xleftarrow{\mathcal{M}_n} \mathbb{H}_n(N)] \\ = \sum_{S \cap \mathbb{H}_n(N) = \emptyset, S \supset B_0, B_x} Z_{\Lambda, \rho, \beta}^{\{0, x\}} [\mathcal{C}_n(0, x) = S, B_y \xleftarrow{\mathcal{M}_n} \mathbb{H}_n(N)]. \end{aligned} \quad (6.3.17)$$

Finally,

$$Z_{\Lambda, \rho, \beta}^{\{0, x\}} [\mathcal{C}_n(0, x) = S, B_y \xleftarrow{\mathcal{M}_n} \mathbb{H}_n(N)] = Z_{\Lambda, \rho, \beta}^{\{0, x\}} [\mathcal{C}_n(0, x) = S] \frac{Z_{\Lambda \setminus S, \rho, \beta}^\emptyset [B_y \xleftarrow{\mathcal{M}_n} \mathbb{H}_n(N)]}{Z_{\Lambda \setminus S, \rho, \beta}^\emptyset}. \quad (6.3.18)$$

⁵There is a small abuse of notation here since the correlations of τ were technically only defined on subsets of $\mathbb{Z}^d$ not subsets of $\mathbb{Z}^d \times K_N$. We solve this issue by replacing τ_u with $\sum_{i=1}^N \sigma_{(u, i)}$, and by writing, for $G_N \subset \Lambda_N$ containing B_u and B_v , $Z_{G_N, \rho, \beta}^{\{u, v\}} := \sum_{i, j=1}^N Q_i Q_j Z_{G_N, \rho, \beta}^{\{(u, i), (v, j)\}}$.

Applying Lemma 6.3.4 gives

$$Z_{\Lambda \setminus S, \rho, \beta}^{\emptyset}[B_y \xleftrightarrow{\mathcal{M}_n} \mathbb{H}_n(N)] \leq \beta \sum_{y' \sim y} Z_{\Lambda \setminus S, \beta}^{\{y, \mathcal{R}_n(y')\}}. \quad (6.3.19)$$

Moreover, using Lemma 6.3.4 one more time on the last line,

$$\begin{aligned} \sum_{S \cap \mathbb{H}_n = \emptyset, S \supset B_0, B_x} Z_{\Lambda, \rho, \beta}^{\{0, x\}}[C_n(0) = S] &= Z_{\Lambda, \rho, \beta}^{\{0, x\}}[B_0, B_x \xleftrightarrow{\mathcal{M}_n} \mathbb{H}_n(N)] \\ &\leq Z_{\Lambda, \rho, \beta}^{\{0, x\}} - Z_{\Lambda, \rho, \beta}^{\{0, x\}}[\partial \mathbf{n} \cap B_x \xleftrightarrow{\mathcal{M}_n} \mathbb{H}_n(N)] \\ &= Z_{\Lambda, \rho, \beta}^{\{0, x\}} - Z_{\Lambda, \rho, \beta}^{\{0, \mathcal{R}_n(x)\}}. \end{aligned}$$

Collecting the above work and taking the limit as $\Lambda \nearrow \mathbb{Z}^d$, we get the result. $\square$

6.3.4 Proof of the theorem for general models in the GS class

We now turn to the extension of Proposition 6.3.3 to all models in the GS class (and in particular to the φ^4 model). It suffices to show that the inequality is stable under weak limits $\rho_k \rightarrow \rho$ for $(\rho_k)_{k \geq 1}$ a sequence of Ising-type measures in the GS class. The same kind of difficulty appeared in [9, 204] and we will essentially import the tools used there. The extension of the result follows readily from this proposition.

Proposition 6.3.6 ([204, Proposition 8.7]). *Let $d \geq 3$. Let ρ be a measure in the GS class. Let $(\rho_k)_{k \geq 1}$ be a sequence of measures of the Ising type in the GS class that converges weakly to ρ . Then,*

- (i) $\liminf \beta_c(\rho_k) \geq \beta_c(\rho)$,
- (ii) for every $\beta > 0$, $\liminf L'(\rho_k, \beta) \geq L(\rho, \beta)$ (where L' is defined similarly as L with $1/2$ replaced by $1/4$),
- (iii) for every $\beta < \beta_c(\rho)$, for every $x, y \in \mathbb{Z}^d$,

$$\lim_{k \rightarrow \infty} \langle \tau_x \tau_y \rangle_{\rho_k, \beta} = \langle \tau_x \tau_y \rangle_{\rho, \beta}. \quad (6.3.20)$$

The above statement was initially proved for $d \geq 4$ and the argument for (iii) used in [204] does not extend to $d = 3$. However, it is possible to solve this issue by noticing that for $\beta < \beta_c(\rho)$, for k large enough, if $u \in \mathbb{Z}^d$,

$$\langle \tau_0 \tau_u \rangle_{\rho_k, \beta} \leq C \exp \left(-c \frac{|u|}{2L(\rho, \beta)} \right). \quad (6.3.21)$$

Indeed, this follows directly from (6.1.10) and the observation that $\limsup L(\rho_k, \beta) \leq L(\rho, \beta)$.

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Chapter 7

The scaling limit of the critical Ising model on $\mathbb{Z}^d$ in dimensions $d > 4$ is the Gaussian free field

We study the nearest-neighbour Ising model on $\mathbb{Z}^d$ in dimensions $d > 4$. We obtain the exact asymptotic behaviour of the model's critical two-point function. More precisely, we show that there exists $D = D(d) > 0$ such that

$$\langle \sigma_0 \sigma_x \rangle_{\beta_c} = \frac{D}{\|x\|^{d-2}} (1 + o(1)),$$

where $\|\cdot\|$ is the standard Euclidean norm on $\mathbb{R}^d$, and where $o(1)$ tends to 0 as $\|x\|$ tends to infinity. Combined with the triviality result of Aizenman (Comm. Math. Phys., 1982) and Fröhlich (Nucl. Phys. B, 1982), this implies convergence of the (properly) renormalised spin field to the standard Gaussian free field on $\mathbb{R}^d$. Our proof relies on a novel path expansion of the model which provides an interpretation of the critical two-point function as the Green function of an *effective* random walk.

This chapter is based on the work in preparation “The scaling limit of the critical Ising model on $\mathbb{Z}^d$ in dimensions $d > 4$ is the Gaussian free field” [80]. It is a joint work with Hugo Duminil-Copin.

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7.1 Introduction

7.1.1 Definitions and statement of the results

We consider the Ising model on the hypercubic lattice $\mathbb{Z}^d$. It is formally defined as follows: given $\Lambda \subset \mathbb{Z}^d$ finite and an inverse temperature $\beta \geq 0$, construct a probability measure on $\{-1, 1\}^\Lambda$ according to the formula

$$\langle F(\sigma) \rangle_{\Lambda, \beta} := \frac{1}{\mathbf{Z}_{\Lambda, \beta}} \sum_{\sigma \in \{-1, 1\}^\Lambda} F(\sigma) \exp(-H_{\Lambda, \beta}(\sigma)), \quad (7.1.1)$$

where $H_{\Lambda, \beta}$ is the *Hamiltonian* defined by

$$H_{\Lambda, \beta}(\sigma) := -\beta \sum_{\substack{x, y \in \Lambda \\ x \sim y}} \sigma_x \sigma_y, \quad (7.1.2)$$

$F : \{-1, 1\}^\Lambda \rightarrow \mathbb{R}$, $\mathbf{Z}_{\Lambda, \beta}$ is the *partition function* of the model which guarantees that $\langle 1 \rangle_{\Lambda, \beta} = 1$, and $x \sim y$ means that $\|x - y\| = 1$ (where $\|\cdot\|$ is the ℓ^2 norm on $\mathbb{R}^d$). It is well-known (see [116, 117]) that the measure $\langle \cdot \rangle_{\Lambda, \beta}$ admits a weak limit when $\Lambda \nearrow \mathbb{Z}^d$. We denote it by $\langle \cdot \rangle_\beta$. When $d \geq 2$, the model undergoes a *phase transition* at a critical parameter $\beta_c \in (0, \infty)$ defined by

$$\beta_c := \inf \left\{ \beta \geq 0 : \chi(\beta) := \sum_{x \in \mathbb{Z}^d} \langle \sigma_0 \sigma_x \rangle_\beta = \infty \right\}. \quad (7.1.3)$$

We restrict our attention to the dimensions $d > 4$. This is the *mean-field* regime of the model. We are interested in the large scale behaviour of the model's critical *two-point function* $\langle \sigma_0 \sigma_x \rangle_{\beta_c}$.

Asymptotic behaviour of the critical two-point function. The nearest-neighbour Ising model is *reflection positive*, see [104, 103, 36]. This property provides two tools that are fundamental in its study: the *infrared bound* [104, 103] and the Messager–Miracle-Solé (MMS) inequalities [191]. Combined together, these tools imply the existence of $C > 0$ such that for all $\beta \leq \beta_c$, and all $x \in \mathbb{Z}^d \setminus \{0\}$,

$$\langle \sigma_0 \sigma_x \rangle_\beta \leq \frac{C}{|x|^{d-2}}, \quad (7.1.4)$$

where $|\cdot|$ is the ℓ^∞ norm on $\mathbb{R}^d$, see [204] for more details. It is also possible to obtain a *near-critical* version of (7.1.4). If S be a finite subset of $\mathbb{Z}^d$ containing 0, let

$$\varphi_\beta(S) := \beta \sum_{\substack{x \in S \\ y \notin S \\ x \sim y}} \langle \sigma_0 \sigma_x \rangle_{S, \beta}. \quad (7.1.5)$$

The *sharp length* L_β is defined by

$$L_\beta := \inf \left\{ k \geq 1 : \exists S \subset \mathbb{Z}^d, \text{rad}(S) \leq 2k, \varphi_\beta(S) < \frac{1}{2} \right\}, \quad (7.1.6)$$

where $\text{rad}(S) := \max\{|x - y|, x, y \in S\}$. Using the Simon–Lieb inequality [223, 178], we obtain the existence of $c, C > 0$ such that for every $\beta \leq \beta_c$, every $x \in \mathbb{Z}^d \setminus \{0\}$,

$$\langle \sigma_0 \sigma_x \rangle_{\beta_c} \leq \frac{C}{|x|^{d-2}} e^{-c|x|/L_\beta}. \quad (7.1.7)$$

When $d > 4$, a lower bound was recently derived in [83]: there exists $c = c(d) > 0$ such that, for all $\beta \leq \beta_c$, and all $1 \leq |x| \leq L_\beta$,

$$\langle \sigma_0 \sigma_x \rangle_\beta \geq \frac{c}{|x|^{d-2}}. \quad (7.1.8)$$

Using the *lace expansion*, Sakai [216, 218] derived exact asymptotics of the critical two-point function in sufficiently large dimensions.

The first result of this chapter is an extension of Sakai’s result to all dimensions $d > 4$.

Theorem 7.1.1. *Let $d > 4$. There exists $D = D(d) > 0$ such that,*

$$\langle \sigma_0 \sigma_x \rangle_{\beta_c} = \frac{D}{\|x\|^{d-2}} (1 + o(1)), \quad (7.1.9)$$

where $o(1)$ tends to 0 as $\|x\|$ tends to infinity.

Scaling limit of the critical model. In two independent papers, Aizenman [4] and Fröhlich [102] proved that any scaling limit of a critical Ising model in dimensions $d > 4$ is a Gaussian field. Before giving a more explicit version of their result, we begin by explaining what we mean by “scaling limit” of the model.

We consider the joint distribution of the *smearred observables*¹ given for $\beta > 0$ and $L \geq 1$ by

$$T_{f,L,\beta}(\sigma) := \frac{1}{\sqrt{\Sigma_L(\beta)}} \sum_{x \in \mathbb{Z}^d} f\left(\frac{x}{L}\right) \sigma_x, \quad (7.1.10)$$

where f ranges over the set $\mathcal{C}_c^\infty(\mathbb{R}^d)$ of infinitely differentiable, real-valued, and compactly supported functions, and where

$$\Sigma_L(\beta) := \left\langle \left(\sum_{x \in \Lambda_L} \sigma_x \right)^2 \right\rangle_\beta = \sum_{x,y \in \Lambda_L} \langle \sigma_x \sigma_y \rangle_\beta, \text{ with } \Lambda_L := [-L, L]^d \cap \mathbb{Z}^d. \quad (7.1.11)$$

Definition 7.1.2. A discrete system as above is said to converge in distribution to a scaling limit if the collection of random variables $(T_{f,L,\beta}(\sigma))_{f \in \mathcal{C}_c^\infty(\mathbb{R}^d)}$ converges in distribution (in the sense of finite dimensional distributions) as L goes to infinity.

¹Note that for $f = \mathbf{1}_{[-1,1]^d}$,

$$\langle T_{f,L,\beta}(\sigma)^2 \rangle_\beta = 1.$$

Using Kolmogorov's extension theorem and the separability of $\mathcal{C}_c^\infty(\mathbb{R}^d)$, we can represent any scaling limit as a random field. Using [9, Theorem 5.6], we may obtain a more quantitative version of the result of Aizenman and Fröhlich (see Chapter 2 for more information).

Theorem 7.1.3. *Let $d > 4$. There exist $C, \alpha > 0$ such that for every $L \geq 1$, every $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and every $z \in \mathbb{R}$,*

$$\left| \langle \exp(z T_{f,L,\beta_c}(\sigma)) \rangle_{\beta_c} - \exp\left(\frac{z^2}{2} \langle T_{f,L,\beta_c}(\sigma)^2 \rangle_{\beta_c}\right) \right| \leq \exp\left(\frac{z^2}{2} \langle T_{|f|,L,\beta_c}(\sigma)^2 \rangle_{\beta_c}\right) \frac{C \|f\|_\infty^4 r_f^\alpha z^4}{L^{d-4}} \quad (7.1.12)$$

where $\|f\|_\infty = \sup_{x \in \mathbb{R}^d} |f(x)|$ and $r_f = \max\{\max\{r \geq 0, \exists x \in \mathbb{R}^d, |x| = r, f(x) \neq 0\}, 1\}$.

Thanks to Theorem 7.1.3, one may establish the convergence of the critical Ising model in the sense of Definition 7.1.2 by computing the limit as L tends to infinity of the covariance matrix of $(T_{f,L,\beta_c}(\sigma))_{f \in \mathcal{C}_c^\infty(\mathbb{R}^d)}$. As it turns out, this computation is a simple consequence of Theorem 7.1.1.

Theorem 7.1.4. *Let $d > 4$. Let $f, g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. Then,*

$$\lim_{L \rightarrow \infty} \langle T_{f,L,\beta_c}(\sigma) T_{g,L,\beta_c}(\sigma) \rangle_{\beta_c} = \mathbf{C} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) g(y) \|x - y\|^{2-d} dx dy, \quad (7.1.13)$$

where

$$\mathbf{C} := \left(\int_{[-1,1]^d} \int_{[-1,1]^d} \|x - y\|^{2-d} dx dy \right)^{-1} \quad (7.1.14)$$

The above result is enough to characterise the scaling limit of the model in the sense of Definition 7.1.2. We begin by recalling the definition of the standard Gaussian free field on $\mathbb{R}^d$.

Definition 7.1.5. The *standard Gaussian free field* is a stochastic process $W = \{W(f) : f \in \mathcal{C}_c^\infty(\mathbb{R}^d)\}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that: i) each $W(f)$ is a centred, Gaussian random variable; ii) the mapping $f \mapsto W(f)$ is linear; and, iii) the covariance of W is given by,

$$\mathbb{E}[W(f)W(g)] = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x)g(y) \frac{\mathbf{c}_d}{\|x - y\|^{d-2}} dx dy, \quad (7.1.15)$$

where

$$\mathbf{c}_d := \frac{\pi^{-d/2}}{4} \Gamma\left(\frac{d}{2} - 1\right). \quad (7.1.16)$$

We are now in a position to state the main result of this chapter.

Theorem 7.1.6 (Scaling limit of the Ising model). *Let $d \geq 5$. The scaling limit of the Ising model under appropriate rescaling is the standard Gaussian free field on $\mathbb{R}^d$.*

Notations. Set $\mathbf{e}_j$ to be the unit vector with j -th coordinate equal to 1. Write x_j for the j -th coordinate of x . Set $\Lambda_n := \{x \in \mathbb{Z}^d : |x| \leq n\}$ and for $x \in \mathbb{Z}^d$, $\Lambda_n(x) := \Lambda_n + x$. If $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, let $[x] := ([x_1], \dots, [x_d])$.

If $(a_n)_{n \geq 0}, (b_n)_{n \geq 0} \in (\mathbb{R}_+^*)^{\mathbb{N}}$, we write $a_n \gtrsim b_n$ (resp. $a_n \asymp b_n$) if there exists $C_1 = C_1(d)$ (resp. $C_1 = C_1(d), C_2 = C_2(d) > 0$) such that for all $n \geq 1$, $b_n \leq C_1 a_n$ (resp. $C_1 a_n \leq b_n \leq C_2 b_n$). We also use Landau's formalism and write $a_n = O(b_n)$ (resp. $a_n = o(b_n)$) if there exists $C = C(d) > 0$ such that for all $n \geq 1$, $a_n \leq C b_n$ (resp. $\lim_{n \rightarrow \infty} a_n/b_n = 0$).

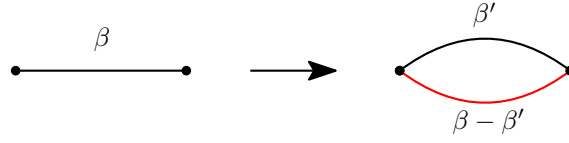


Figure 7.1: An illustration of the duplicated edges of the graph $\mathbb{Z}_{\beta', \beta}^d$.

7.1.2 Strategy of the proof

Our proof is based on a new *path expansion* of the two-point function of the model which provides a random walk interpretation of the difference

$$\langle \sigma_0 \sigma_x \rangle_\beta - \langle \sigma_0 \sigma_x \rangle_{\beta'}, \quad (7.1.17)$$

where $\beta' < \beta$. This expansion is essentially a *backbone expansion* on an augmented version of the hypercubic lattice $\mathbb{Z}^d$, denoted $\mathbb{Z}_{\beta', \beta}^d$, in which all edges are duplicated, carrying respectively an interaction β' and $\beta - \beta'$ in the associated Ising model. See Figure 7.1.

The idea is to apply the above expansion to $\beta = \beta_c$ and $\beta' = \beta(x)$ where $\beta(x)$ is chosen in such a way that

$$L_{\beta(x)} = o(\|x\|), \quad L_{\beta(x)} \gg \frac{\|x\|}{\log \|x\|}. \quad (7.1.18)$$

The choice of $\beta(x)$ will become more transparent in the proof. This allows us to show that $\langle \sigma_0 \sigma_x \rangle_{\beta_c}$ is *almost-harmonic* for a well-chosen *effective random walk*. Namely, we prove that for all $x \in \mathbb{Z}^d$, there exists a step distribution $(p_x(y))_{y \in \mathbb{Z}^d}$ (i.e positive and which sums to 1) such that

$$\langle \sigma_0 \sigma_x \rangle_{\beta_c} = (1 + o(1)) \sum_{y \in \mathbb{Z}^d} p_x(y) \langle \sigma_y \sigma_x \rangle_{\beta_c}, \quad (7.1.19)$$

where $o(1)$ tends to 0 as $\|x\|$ tends to infinity. Due to the x -dependency of the step distribution $(p_x(y))_{y \in \mathbb{Z}^d}$, it is not obvious how to turn this almost-harmonicity into the Gaussian asymptotic behaviour of the Green function of a random walk. Nevertheless, we may remove this x -dependency with the help of the mixing property of [9]. Then, standard random walk analysis allows us to prove that for all $x, y \in \mathbb{R}^d \setminus \{0\}$,

$$\lim_{n \rightarrow \infty} \frac{\langle \sigma_0 \sigma_{[nx]} \rangle_{\beta_c}}{\langle \sigma_0 \sigma_{[ny]} \rangle_{\beta_c}} = \frac{\|y\|^{d-2}}{\|x\|^{d-2}}. \quad (7.1.20)$$

This is not quite the result we want but it contains already most of the interesting information. The last technical step of the proof consists in using (7.1.20) together with the classical Messager–Miracle-Solé inequalities [191] to prove that the following limit exists

$$\lim_{n \rightarrow \infty} n^{d-2} \langle \sigma_0 \sigma_{n\mathbf{e}_1} \rangle_{\beta_c}. \quad (7.1.21)$$

Theorem 7.1.1 then follows from (7.1.20) applied to $y = \mathbf{e}_1$.

7.2 Random currents and backbone exploration

In this section only, we let $d \geq 1$. Fix Λ be a finite subset of $\mathbb{Z}^d$. We begin by recalling some definitions from Chapter 2.

Definition 7.2.1. A *current* $\mathbf{n}$ on Λ is a function defined on the set $E(\Lambda) := \{\{x, y\}, x, y \in \Lambda \text{ and } x \sim y\}$ and taking its values in $\mathbb{N} = \{0, 1, \dots\}$. We let Ω_Λ be the set of currents on Λ . The set of *sources* of $\mathbf{n}$, denoted by $\partial\mathbf{n}$, is defined as

$$\partial\mathbf{n} := \left\{x \in \Lambda : \sum_{y \sim x} \mathbf{n}_{x,y} \text{ is odd}\right\}. \quad (7.2.1)$$

We also set

$$w_\beta(\mathbf{n}) := \prod_{\substack{\{x,y\} \subset \Lambda \\ x \sim y}} \frac{\beta^{\mathbf{n}_{x,y}}}{\mathbf{n}_{x,y}!}. \quad (7.2.2)$$

If $S \subset \Lambda$, define a probability measure $\mathbf{P}_{\Lambda,\beta}^S$ on Ω_Λ as follows: for every $\mathbf{n} \in \Omega_\Lambda$,

$$\mathbf{P}_{\Lambda,\beta}^S[\mathbf{n}] := \mathbb{1}_{\partial\mathbf{n}=S} \frac{w_\beta(\mathbf{n})}{Z_{\Lambda,\beta}^S}, \quad (7.2.3)$$

where $Z_{\Lambda,\beta}^S := \sum_{\partial\mathbf{n}=S} w_\beta(\mathbf{n})$ is a normalisation constant. Moreover, for $S_1, \dots, S_k \subset \Lambda$, define

$$\mathbf{P}_{\Lambda,\beta}^{S_1, \dots, S_k} := \mathbf{P}_{\Lambda,\beta}^{S_1} \otimes \dots \otimes \mathbf{P}_{\Lambda,\beta}^{S_k}. \quad (7.2.4)$$

When $S = \{x, y\}$, we will write $\mathbf{P}_{\Lambda,\beta}^{xy}$ instead of $\mathbf{P}_{\Lambda,\beta}^{\{x,y\}}$. If $\mathcal{E} \subset \Omega_\Lambda$, we will also write $Z_{\Lambda,\beta}^S[\mathcal{E}] := \sum_{\partial\mathbf{n}=S} w_\beta(\mathbf{n}) \mathbb{1}_{\mathbf{n} \in \mathcal{E}}$. As proved in [10], if S is a finite even (i.e. $|S|$ is even) subset of $\mathbb{Z}^d$, the sequence of probability measures $(\mathbf{P}_{\Lambda,\beta}^S)_{\Lambda \subset \mathbb{Z}^d}$ admits a weak limit as $\Lambda \nearrow \mathbb{Z}^d$ that we denote by $\mathbf{P}_\beta^S$.

Lemma 7.2.2 (Switching Lemma). *For every $S_1, S_2 \subset \Lambda$ and every function F from Ω_Λ into $\mathbb{R}$,*

$$\begin{aligned} & \sum_{\substack{\mathbf{n}_1 \in \Omega_\Lambda : \partial\mathbf{n}_1 = S_1 \\ \mathbf{n}_2 \in \Omega_\Lambda : \partial\mathbf{n}_2 = S_2}} F(\mathbf{n}_1 + \mathbf{n}_2) w_\beta(\mathbf{n}_1) w_\beta(\mathbf{n}_2) \\ &= \sum_{\substack{\mathbf{n}_1 \in \Omega_\Lambda : \partial\mathbf{n}_1 = S_1 \Delta S_2 \\ \mathbf{n}_2 \in \Omega_\Lambda : \partial\mathbf{n}_2 = \emptyset}} F(\mathbf{n}_1 + \mathbf{n}_2) w_\beta(\mathbf{n}_1) w_\beta(\mathbf{n}_2) \mathbb{1}_{(\mathbf{n}_1 + \mathbf{n}_2) \in \mathcal{F}_{S_2}}, \end{aligned} \quad (7.2.5)$$

where $S_1 \Delta S_2 = (S_1 \cup S_2) \setminus (S_1 \cap S_2)$ is the symmetric difference of sets and $\mathcal{F}_{S_2}$ is given by

$$\mathcal{F}_{S_2} = \{\mathbf{n} \in \Omega_\Lambda : \exists \mathbf{m} \leq \mathbf{n}, \partial\mathbf{m} = S_2\}. \quad (7.2.6)$$

Backbone exploration revisited. Let $0 < \beta' < \beta$. We consider the graph $\mathbb{Z}_{\beta',\beta}^d$ of vertex set $\mathbb{Z}^d$ and edge set given by

$$\{(xy)_{\beta'} : x \sim y\} \cup \{(xy)_{\beta-\beta'} : x \sim y\}. \quad (7.2.7)$$

In words, $\mathbb{Z}_{\beta',\beta}^d$ is a modification of the hypercubic lattice $\mathbb{Z}^d$ in which each edge xy of $\mathbb{Z}^d$ comes in two copies: $(xy)_{\beta'}$ and $(xy)_{\beta-\beta'}$, see Figure 7.1. We let $\mathbb{Z}_{\beta'}^d \subset \mathbb{Z}_{\beta',\beta}^d$ be the subgraph of $\mathbb{Z}_{\beta',\beta}^d$ of edge-set $\{(xy)_{\beta'} : x \sim y\}$ and define similarly $\mathbb{Z}_{\beta-\beta'}^d$. We denote by $\Lambda_{\beta',\beta}$ the restriction of $\mathbb{Z}_{\beta',\beta}^d$ to Λ . It is possible to embed our preceding definition of the Ising model in this duplicated system by writing for $\sigma \in \{-1, 1\}^\Lambda$,

$$H_{\Lambda,\beta}(\sigma) = -\beta' \sum_{\substack{x,y \in \Lambda \\ x \sim y}} \sigma_x \sigma_y - (\beta - \beta') \sum_{\substack{x,y \in \Lambda \\ x \sim y}} \sigma_x \sigma_y. \quad (7.2.8)$$

Additionally, one may perform the random current expansion of the Ising model on this duplicated system. We call $\Omega_{\beta',\beta-\beta'}^\Lambda$ the set of currents on $\Lambda_{\beta',\beta}$. If $\Lambda = \mathbb{Z}^d$, we just write $\Omega_{\beta',\beta}$.

We will use the backbone representation of the model. In this setup, the backbone exploration introduced in Chapter 2 needs to be clarified.

Fix an ordering $\prec$ of the edges of $\mathbb{Z}^d$. It induces an ordering of the edges of $\mathbb{Z}_{\beta',\beta}^d$, that we still denote $\prec$, by requiring that for all edges $xy \prec uv$ of $\mathbb{Z}^d$, $(xy)_{\beta-\beta'} \prec (xy)_{\beta'} \prec (uv)_{\beta-\beta'} \prec (uv)_{\beta'}$. In words, to compare two edges of $\mathbb{Z}_{\beta',\beta}^d$, we first compare their projections on the edges of $\mathbb{Z}^d$. If these projections are equal, we choose by convention the copy carrying the interaction $\beta - \beta'$ to be the smallest edge.

Consider a current $\mathbf{n}$ on $\mathbb{Z}_{\beta',\beta}^d$ with $\partial \mathbf{n} = \{x, y\}$. The current $\mathbf{n}$ can be decomposed into $\mathbf{n} = \mathbf{n}^{\beta'} + \mathbf{n}^{\beta-\beta'}$ with $\mathbf{n}^{\beta'}$ (resp. $\mathbf{n}^{\beta-\beta'}$) supported on edges of the form $(xy)_{\beta'}$ (resp. $(xy)_{\beta-\beta'}$). The backbone $\Gamma(\mathbf{n}) = \{(x_i x_{i+1})_{\beta_i} : 0 \leq i \leq n-1, \beta_i \in \{\beta', \beta - \beta'\}\}$ with $x_0 = x$ and $x_n = y$ is an edge self-avoiding path living on $\mathbb{Z}_{\beta',\beta}^d$ determined by the following exploration:

- (1) $x_0 = x$ and $(x_0 x_1)_{\beta_0}$ is the earliest edge leaving x_0 such that $\mathbf{n}_{x_0 x_1}^{\beta_0}$ is odd. For all $x_0 u \prec x_0 x_1$ with $u \neq x_1$, erase all pairs $(x_0 u)_{\beta'}$, $(x_0 u)_{\beta-\beta'}$. Additionally, if $\beta_0 = \beta'$ erase $(x_0 x_1)_{\beta'}$ and $(x_0 x_1)_{\beta-\beta'}$; otherwise, just erase $(x_0 x_1)_{\beta-\beta'}$.
- (2) If $1 \leq i \leq n-1$, choose $(x_i x_{i+1})_{\beta_i}$ to be the earliest of all steps emerging from x_i that have not been cancelled by previous steps, and for which $\mathbf{n}_{x_i x_{i+1}}^{\beta_i}$ is odd. Erase all pairs $(x_i u)_{\beta'}$, $(x_i u)_{\beta-\beta'}$ for $x_i u \preceq x_i x_{i+1}$ with $u \neq x_{i+1}$. Additionally, if $\beta_i = \beta'$ erase $(x_i x_{i+1})_{\beta'}$ and $(x_i x_{i+1})_{\beta-\beta'}$; otherwise, just erase $(x_i x_{i+1})_{\beta-\beta'}$.
- (3) Stop the exploration at a site from which there are no more non-cancelled bonds with odd flux number available. This must happen at a source of $\mathbf{n}$.

In words, our exploration prioritises the $\beta - \beta'$ edges. This will be extremely important in what follows. See Figure 7.2 for an illustration.

With this exploration in hand, we may extend the terminology and the notations introduced in Chapter 2. In particular, we let $\bar{\Gamma}(\mathbf{n})$ be the union of the edges visited by the backbone together with the ones erased during the exploration. We also keep the notion of consistency of a sequence of steps.

We make one additional very important observation: if $\gamma \subset \mathbb{Z}_{\beta'}^d$, then $\bar{\gamma}$ contains the two copies of all the edges it contains. It can therefore be identified with a subset of edges of $\mathbb{Z}^d$.

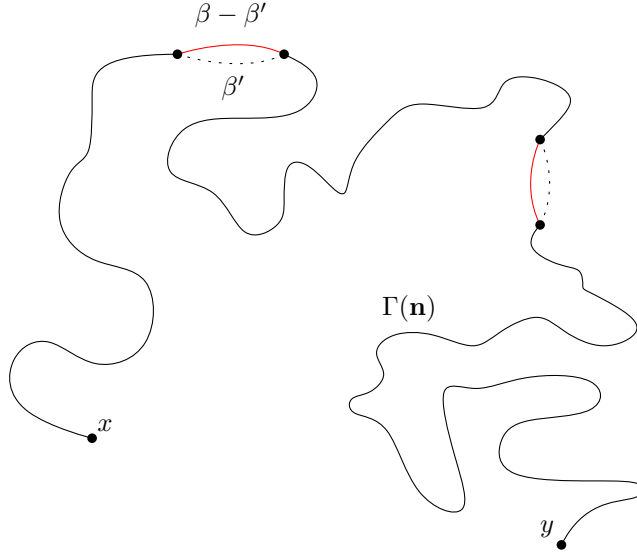


Figure 7.2: An illustration of a backbone $\Gamma(\mathbf{n})$ for $\mathbf{n} \in \Omega_{\beta', \beta}$. The red lines correspond to edges explored in $\mathbf{n}^{\beta - \beta'}$.

Let γ be a finite consistent sequence of steps on $\mathbb{Z}_{\beta', \beta}^d$. Let $\partial\gamma$ be its endpoints. Define the weight

$$\rho_{\beta', \beta}^\Lambda(\gamma) = \frac{1}{Z} \sum_{\substack{\mathbf{n} \in \Omega_{\beta', \beta - \beta'}^\Lambda \\ \partial\mathbf{n} = \partial\gamma}} w(\mathbf{n}) \mathbf{1}_{\Gamma(\mathbf{n}) = \gamma}, \quad (7.2.9)$$

where $w(\mathbf{n}) = w_{\beta'}(\mathbf{n}^{\beta'}) w_{\beta - \beta'}(\mathbf{n}^{\beta - \beta'})$ and $Z = \sum_{\partial\mathbf{n} = \emptyset} w(\mathbf{n})$. One has,

$$\langle \sigma_x \sigma_y \rangle_{\Lambda, \beta} = \sum_{\gamma: x \rightarrow y} \rho_{\beta', \beta}^\Lambda(\gamma), \quad (7.2.10)$$

where the sum is supported on paths $\gamma : x \rightarrow y$ living on $\mathbb{Z}_{\beta', \beta}^d$ and which are consistent. Thanks to the result of [10], it is possible to define infinite volume versions of the above quantities. In particular, we write $\rho_{\beta', \beta} := \rho_{\beta', \beta}^{\mathbb{Z}^d}$. The following quantity will appear several times in the analysis,

$$G_{\beta', \beta}(x) := \sum_{\gamma: 0 \rightarrow x \subset \mathbb{Z}_{\beta'}^d} \rho_{\beta', \beta}(\gamma), \quad (x \in \mathbb{Z}^d). \quad (7.2.11)$$

Unsurprisingly, $G_{\beta', \beta}$ can be compared to the model's two-point function.

Lemma 7.2.3. *Let $0 < \beta' < \beta$ and $x \in \mathbb{Z}^d$. Then,*

$$G_{\beta', \beta}(x) \leq \langle \sigma_0 \sigma_x \rangle_{\beta'}. \quad (7.2.12)$$

Proof. We simply use [8, Proposition 4.4 (e)]. □

7.3 Weak convergence to the Gaussian free field

For $x, y \in \mathbb{R}^d$, with $x \neq y$, let

$$G(x, y) := \frac{\mathbf{c}_d}{\|x - y\|^{d-2}}, \quad (7.3.1)$$

where we recall that $\mathbf{c}_d = \frac{1}{4}\pi^{-d/2}\Gamma(\frac{d}{2} - 1)$. The goal of this section is to prove the following result.

Theorem 7.3.1. *Let $d > 4$. Fix $x, y \in \mathbb{R}^d \setminus \{0\}$. Then,*

$$\lim_{n \rightarrow \infty} \frac{\langle \sigma_0 \sigma_{[nx]} \rangle_{\beta_c}}{\langle \sigma_0 \sigma_{[ny]} \rangle_{\beta_c}} = \frac{G(0, x)}{G(0, y)}. \quad (7.3.2)$$

7.3.1 A Simon–Lieb type inequality

We use the exploration of Section 7.2 to obtain a Simon–Lieb type inequality and its reverse.

For $\beta < \beta_c$ and $x, y \in \mathbb{Z}^d$, we let

$$q_{\beta, \beta_c}^x(y) := \sum_{u \sim y} \sum_{\gamma: 0 \rightarrow u \subset \mathbb{Z}_{\beta'}^d} \rho_{\beta, \beta_c}(\gamma \circ (uy)_{\beta_c - \beta}) \frac{\langle \sigma_y \sigma_x \rangle_{(\bar{\gamma} \cap \Lambda_k(y))^c, \beta_c}^{(uy)_\beta}}{\langle \sigma_y \sigma_x \rangle_{\beta_c}}, \quad (7.3.3)$$

where from the observation made above, we identify $\bar{\gamma}$ with a collection of edges in $\mathbb{Z}^d$ and $\langle \cdot \rangle_{(\bar{\gamma} \cap \Lambda_k(u))^c, \beta_c}^{(uy)_\beta}$ is the measure where we have set to 0 the interaction on edges of $\bar{\gamma}$ which lie in $\Lambda_k(u)$, and set the interaction to β on the edge uy .

Lemma 7.3.2. *There exists $C = C(d) > 0$ such that for $y \in \mathbb{Z}^d$,*

$$q_{\beta, \beta_c}^x(y) \leq (\beta_c - \beta) \sum_{u \sim y} \langle \sigma_0 \sigma_u \rangle_\beta. \quad (7.3.4)$$

Proof. We use the properties of the backbone obtained in Chapter 2. We write, for $\gamma \subset \mathbb{Z}_{\beta'}^d$,

$$\rho_{\beta, \beta_c}(\gamma \circ (uy)_{\beta_c - \beta}) \leq \rho_{\beta, \beta_c}(\gamma) \rho_{\beta', \beta}^{\{(uy)_{\beta_c - \beta}\}}((uy)_{\beta_c - \beta}) = \rho_{\beta, \beta_c}(\gamma) \tanh(\beta_c - \beta). \quad (7.3.5)$$

The proof follows readily from Griffiths' inequality and Lemma 7.2.3. $\square$

Let $\beta < \beta_c$. Define the truncated bubble diagram,

$$B_k(\beta) := \sum_{x \in \Lambda_k} \langle \sigma_0 \sigma_x \rangle_\beta^2. \quad (7.3.6)$$

Proposition 7.3.3 (Simon–Lieb type inequality). *Let $d > 4$ and $\beta < \beta_c$. Let $k \geq 1$. There exists $C = C(d) > 0$ such that for all $x \in \mathbb{Z}^d \setminus \{0\}$,*

$$\langle \sigma_0 \sigma_x \rangle_{\beta_c} \leq G_{\beta, \beta_c}(x) + \sum_{y \in \mathbb{Z}^d} q_{\beta, \beta_c}^x(y) \langle \sigma_y \sigma_x \rangle_{\beta_c}, \quad (7.3.7)$$

and

$$\langle \sigma_0 \sigma_x \rangle_{\beta_c} \geq G_{\beta, \beta_c}(x) + \sum_{y \in \mathbb{Z}^d} q_{\beta, \beta_c}^x(y) \langle \sigma_y \sigma_x \rangle_{\beta_c} - \frac{C[B(\beta_c) - B_k(\beta_c)]}{|x|^{d-2}}. \quad (7.3.8)$$

Proof. We may write for $x \in \mathbb{Z}^d \setminus \{0\}$,

$$\langle \sigma_0 \sigma_x \rangle_{\beta_c} = G_{\beta, \beta_c}(x) + \sum_{y \in \mathbb{Z}^d} \sum_{u \sim y} \sum_{\substack{\gamma_1: 0 \rightarrow u \subset \mathbb{Z}_{\beta'}^d \\ \gamma_2: y \rightarrow x}} \rho_{\beta, \beta_c}(\gamma_1(u, y) \circ \gamma_2) \mathbb{1}_{\overline{\gamma_1(u, y)} \cap \gamma_2 = \emptyset}, \quad (7.3.9)$$

where $\gamma_1(u, y) = \gamma_1 \circ (uy)_{\beta_c - \beta}$ is the concatenation of γ_1 with the copy of the edge uy that lies in $\mathbb{Z}_{\beta_c - \beta}^d$, and where $\overline{\gamma_1(u, y)} \cap \gamma_2 = \emptyset$ means that γ_2 does not use any edge in $\overline{\gamma_1(u, y)} = \overline{\gamma_1} \cup \{(uy)_{\beta_c - \beta}\}$. Notice that by [12, Proposition 4.4 (b)]

$$\rho_{\beta, \beta_c}(\gamma_1(u, y) \circ \gamma_2) \mathbb{1}_{\overline{\gamma_1(u, y)} \cap \gamma_2 = \emptyset} = \rho_{\beta, \beta_c}(\gamma_1(u, y)) \rho_{\beta, \beta_c}^{\overline{\gamma_1(u, y)}^c}(\gamma_2) \mathbb{1}_{\overline{\gamma_1(u, y)} \cap \gamma_2 = \emptyset}, \quad (7.3.10)$$

where $\rho_{\beta, \beta_c}^{\overline{\gamma_1(u, y)}^c}$ is the backbone weight in the system depleted of the edges lying in $\overline{\gamma_1(u, y)}^c$. Using Griffiths' inequality we get

$$\sum_{\gamma_2: y \rightarrow x} \rho_{\beta, \beta_c}^{\overline{\gamma_1(u, y)}^c}(\gamma_2) \mathbb{1}_{\overline{\gamma_1(u, y)} \cap \gamma_2 = \emptyset} = \langle \sigma_y \sigma_x \rangle_{\overline{\gamma_1}^c, \beta_c}^{(uy)_\beta} \leq \langle \sigma_y \sigma_x \rangle_{(\overline{\gamma_1} \cap \Lambda_k(y))^c, \beta_c}^{(uy)_\beta}, \quad (7.3.11)$$

The upper bound follows readily.

For the lower bound, we use [12, Proposition 4.4 (e)] to get that if $\gamma_2 \cap \overline{\gamma_1(u, y)} = \emptyset$,

$$\rho_{\beta, \beta_c}^{\overline{\gamma_1(u, y)}^c}(\gamma_2) \geq \rho_{\beta, \beta_c}^{(\overline{\gamma_1(u, y)} \cap \Lambda_k(y))^c}(\gamma_2). \quad (7.3.12)$$

Writing

$$\mathbb{1}_{\overline{\gamma_1(u, y)} \cap \gamma_2 = \emptyset} = \mathbb{1}_{\overline{\gamma_1(u, y)} \cap \Lambda_k(y) \cap \gamma_2 = \emptyset} - \mathbb{1}_{\overline{\gamma_1(u, y)} \cap \Lambda_k(y) \cap \gamma_2 = \emptyset, \overline{\gamma_1(u, y)} \cap \Lambda_k(y)^c \cap \gamma_2 \neq \emptyset}, \quad (7.3.13)$$

and using (7.3.9) again, we get

$$\langle \sigma_0 \sigma_x \rangle_{\beta_c} - G_{\beta, \beta_c}(x) = \sum_{y \in \mathbb{Z}^d} q_{\beta, \beta_c}^x(y) \langle \sigma_y \sigma_x \rangle_{\beta_c} - E(x), \quad (7.3.14)$$

where

$$E(x) := \sum_{y \in \mathbb{Z}^d} \sum_{u \sim y} \sum_{\substack{\gamma_1: 0 \rightarrow u \subset \mathbb{Z}_{\beta'}^d \\ \gamma_2: y \rightarrow x}} \rho_{\beta, \beta_c}(\gamma_1(u, y) \circ \gamma_2) \mathbb{1}_{\overline{\gamma_1(u, y)} \cap \Lambda_k(y) \cap \gamma_2 = \emptyset, \overline{\gamma_1(u, y)} \cap \Lambda_k(y)^c \cap \gamma_2 \neq \emptyset}. \quad (7.3.15)$$

We then conclude using the lemma below. $\square$

Lemma 7.3.4. *We keep the assumptions of Proposition 7.3.3. Then, there exists $C = C(d) > 0$ such that,*

$$E(x) \leq C[B(\beta_c) - B_k(\beta_c)]|x|^{2-d}. \quad (7.3.16)$$

Proof. Notice that, using Lemmas 7.2.3 and 7.3.2,

$$E(x) \leq \sum_{\substack{y \in \mathbb{Z}^d \\ u \sim y}} \sum_{\substack{v \in \mathbb{Z}^d \\ |y-v| > k}} \langle \sigma_0 \sigma_v \rangle_\beta \langle \sigma_v \sigma_u \rangle_\beta (\beta_c - \beta) \langle \sigma_y \sigma_v \rangle_{\beta_c} \langle \sigma_v \sigma_x \rangle_{\beta_c}. \quad (7.3.17)$$

Now, using [83], we get $C_1 = C_1(d) > 0$ such that

$$(\beta_c - \beta) \leq C_1 L_\beta^{-2}. \quad (7.3.18)$$

Using (7.1.7) together with (7.3.18), we obtain $C_2, C_3 > 0$ which only depend on d such that,

$$\begin{aligned} E(x) &\leq C_2 [B(\beta_c) - B_k(\beta_c)] L_\beta^{-2} \sum_{v \in \mathbb{Z}^d} |v|^{2-d} \exp(-c|v|/L_\beta) |x - v|^{2-d} \\ &\leq C_3 [B(\beta_c) - B_k(\beta_c)] |x|^{2-d}. \end{aligned}$$

This concludes the proof. $\square$

7.3.2 The effective random walk

In the last section, we were able to show that the two-point function at β_c is “quasi”-harmonic for the Laplacian associated to the random-walk generated by the family $(q_{\beta, \beta_c}^x(u))_{u \in \mathbb{Z}^d}$. However, since these weights depend on x , this interpretation is not very helpful. To make this intuition work we proceed as follows.

1. We start by fixing appropriate values for β, k in terms of x .
2. We show that it is possible to truncate the family $(q_{\beta, \beta_c}^x(u))_{u \in \mathbb{Z}^d}$, thus making the renormalised step distribution finite-range.
3. We show that for our choice of parameters, $q_{\beta, \beta_c}^x(u)$ does not depend (too much) on x . This step relies on the mixing property obtained in [9].

7.3.2.1 Step 1: choosing β, k

Let $n \geq 1$. We now assume that $|x| \geq n$. We choose $\beta = \beta_n := \inf\{\beta' < \beta_c : L(\beta') \geq \ell_n\}$, where

$$\ell_n := \frac{n}{(\log n)^\eta}, \quad (7.3.19)$$

with $\eta > 0$ a parameter to be fixed later. We choose $k = n^\varepsilon$, where $\varepsilon \in (0, 1)$ will also be fixed later. We write $\delta\beta(n) = \beta_c - \beta(n)$. As observed above, using [4] and [83] we obtain $c = c(d), C = C(d) > 0$ such that

$$c\ell_n^{-2} \leq \delta\beta(n) \leq C\ell_n^{-2}. \quad (7.3.20)$$

7.3.2.2 Step 2: truncation

Proposition 7.3.5. *With the above choices, for any $\delta \in (0, \eta)$, there exist $c, C > 0$ such that,*

$$\sum_{y \notin \Lambda_{\ell_n(\log n)^\delta}} q_{\beta, \beta_c}^x(y) \langle \sigma_y \sigma_x \rangle_{\beta_c} \leq C \exp(-c(\log n)^\delta) \langle \sigma_0 \sigma_x \rangle_{\beta_c}. \quad (7.3.21)$$

Proof. The constant $C_i, c_i > 0$ only depend on d . By (7.1.7) and (7.1.8),

$$\langle \sigma_0 \sigma_x \rangle_\beta \leq \frac{C_1}{|x|^{d-2}} \exp(-c_1(\log n)^\eta) \leq C_2 \langle \sigma_0 \sigma_x \rangle_{\beta_c} \exp(-c_1(\log n)^\eta). \quad (7.3.22)$$

Also, by Lemma 7.3.2 and (7.3.20), $q_{\beta, \beta_c}^x(y) \leq \delta \beta(n) \sum_{u \sim y} \langle \sigma_0 \sigma_u \rangle_\beta \leq C_2 \ell_n^{-2} \sum_{u \sim y} \langle \sigma_0 \sigma_u \rangle_\beta$.

Divide the sum on the left-hand side of (7.3.21) into two contributions for $u \in \Lambda_{|x|/2}(x)$ and its complement. Using the above observations and (7.1.7), we find that

$$\begin{aligned} \sum_{y \in \Lambda_{|x|/2}(x)} q_{\beta, \beta_c}^x(y) \langle \sigma_y \sigma_x \rangle_{\beta_c} &\leq C_2 \ell_n^{-2} \sum_{y \in \Lambda_{|x|/2}(x)} \sum_{u \sim y} \langle \sigma_0 \sigma_u \rangle_\beta \langle \sigma_y \sigma_x \rangle_{\beta_c} \\ &\leq C_3 \langle \sigma_0 \sigma_x \rangle_{\beta_c} (|x|/\ell_n)^2 \exp(-c_1 |x|/\ell_n) \\ &\leq C_4 \langle \sigma_0 \sigma_x \rangle_{\beta_c} \exp(-c_2 (\log n)^\eta), \end{aligned}$$

and,

$$\begin{aligned} \sum_{y \notin \Lambda_{\ell_n(\log n)^\delta} \cup \Lambda_{|x|/2}(x)} q_{\beta, \beta_c}^x(y) \langle \sigma_y \sigma_x \rangle_{\beta_c} &\leq C_2 \ell_n^{-2} \sum_{y \notin \Lambda_{\ell_n(\log n)^\delta} \cup \Lambda_{|x|/2}(x)} \sum_{u \sim y} \langle \sigma_0 \sigma_u \rangle_\beta \langle \sigma_y \sigma_x \rangle_{\beta_c} \\ &\leq C_5 \langle \sigma_0 \sigma_x \rangle_{\beta_c} \ell_n^{-2} \sum_{y \notin \Lambda_{\ell_n(\log n)^\delta}} |u|^{2-d} e^{-c_3 |u|/\ell_n} \\ &\leq C_6 \langle \sigma_0 \sigma_x \rangle_{\beta_c} \ell_n^{-2} \sum_{k \geq \ell_n(\log n)^\delta} k e^{-c_3 k/\ell_n} \\ &\leq C_7 \langle \sigma_0 \sigma_x \rangle_{\beta_c} \exp(-c_4 (\log n)^\delta). \end{aligned}$$

The proof follows readily. $\square$

Recall that $k = n^\varepsilon$. Observe that by (7.1.4)

$$B(\beta_c) - B_k(\beta_c) \leq C k^{-(d-4)}. \quad (7.3.23)$$

Combined with Propositions 7.3.3 and 7.3.5, (7.3.23) yields the following result.

Proposition 7.3.6. *Let $d \geq 5$. Let $\delta \in (0, \eta)$. There exist $c, C > 0$ such that for all n large enough, for $\beta = \beta_n$ and $|x| \geq n$,*

$$\begin{aligned} (1 - C \exp(-c(\log n)^\delta)) \sum_{y \in \Lambda_{\ell_n(\log n)^\delta}} q_{\beta, \beta_c}^x(y) \langle \sigma_y \sigma_x \rangle_{\beta_c} \\ \leq \langle \sigma_0 \sigma_x \rangle_{\beta_c} \leq (1 + C \exp(-c(\log n)^\delta)) \sum_{y \in \Lambda_{\ell_n(\log n)^\delta}} q_{\beta, \beta_c}^x(y) \langle \sigma_y \sigma_x \rangle_{\beta_c}. \end{aligned}$$

7.3.2.3 Step 3: mixing argument

We fix $y \in \Lambda_{\ell_n(\log n)^\delta}$. Using the switching lemma, we may rewrite the ratio of correlation functions appearing in the formula

$$q_{\beta, \beta_c}^x(y) = \sum_{u \sim y} \sum_{\gamma: 0 \rightarrow u \subset \mathbb{Z}_{\beta'}^d} \rho_{\beta, \beta_c}(\gamma \circ (uy)_{\beta_c - \beta}) \frac{\langle \sigma_y \sigma_x \rangle_{(\bar{\gamma} \cap \Lambda_{n^\varepsilon}(y))^c, \beta_c}^{(uy)_\beta}}{\langle \sigma_y \sigma_x \rangle_{\beta_c}} \quad (7.3.24)$$

as

$$\frac{\langle \sigma_y \sigma_x \rangle_{(\bar{\gamma} \cap \Lambda_{n^\varepsilon}(y))^c, \beta_c}^{(uy)_\beta}}{\langle \sigma_y \sigma_x \rangle_{\beta_c}} = \mathbf{P}_{(\bar{\gamma} \cap \Lambda_{n^\varepsilon}(y))^c, \mathbb{Z}^d, \beta_c}^{\emptyset, yx} [y \longleftrightarrow x \text{ in } (\mathbf{n}_1 + \mathbf{n}_2)_{|(\bar{\gamma} \cap \Lambda_{n^\varepsilon}(y))^c}], \quad (7.3.25)$$

where in the above current measure, the first current has weight β over the edge uy , and where for a set of edges E we defined $\mathbf{n}_E$ to be the restriction of a current $\mathbf{n}$ to edges in E . We will do an additional step in the rewriting of q_{β, β_c}^x . Let $\mathcal{W}(0, y)$ be the set of $\gamma \subset \mathbb{Z}_\beta^d \cup \mathbb{Z}_{\beta_c - \beta}^d$ that are consistent, connect 0 to y , and only use edges in $\mathbb{Z}_{\beta'}^d$ except the last edge which lies in $\mathbb{Z}_{\beta_c - \beta}^d$. Then, we obtain

$$q_{\beta, \beta_c}^x(y) = \langle \sigma_0 \sigma_y \rangle_{\beta_c} \mathbf{E}_{\beta_c}^{0y} \left[\mathbf{1}_{\Gamma \in \mathcal{W}(0, y)} \mathbf{P}_{(\overline{\Gamma^-} \cap \Lambda_{n^\varepsilon}(y))^c, \mathbb{Z}^d, \beta_c}^{\emptyset, yx} [y \longleftrightarrow x \text{ in } (\mathbf{n}_1 + \mathbf{n}_2)_{|(\overline{\Gamma} \cap \Lambda_{n^\varepsilon}(y))^c}] \right], \quad (7.3.26)$$

where Γ^- is the path minus its last step.

Definition 7.3.7. Let $x, y \in \mathbb{Z}^d$. For a consistent path $\gamma : 0 \rightarrow y$ and $\mathbf{n}_1 \in \Omega_{(\overline{\gamma^-} \cap \Lambda_{n^\varepsilon}(y))^c}$, define the event $\mathcal{A}(y, x; \gamma, \mathbf{n}_1)$ by

$$\mathcal{A}(y, x; \gamma, \mathbf{n}_1) := \left\{ \mathbf{n}_2 \in \Omega_{\mathbb{Z}^d} : y \longleftrightarrow x \text{ in } (\mathbf{n}_1 + \mathbf{n}_2)_{|(\overline{\gamma^-} \cap \Lambda_{n^\varepsilon}(y))^c} \right\}. \quad (7.3.27)$$

Also, for $A > 0$, define the local (around y) event $\mathcal{B}_A(y, x; \gamma, \mathbf{n}_1)$ by,

$$\mathcal{B}_A(y, x; \gamma, \mathbf{n}_1) := \left\{ \mathbf{n}_2 \in \Omega_{\mathbb{Z}^d} : \begin{array}{l} \text{the restriction of } \mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_2}(y) \text{ to edges in } \Lambda_{n^{A\varepsilon}}(y) \\ \text{connects to } \partial \Lambda_{n^{A\varepsilon}}(y) \text{ in } (\overline{\gamma^-} \cap \Lambda_{n^\varepsilon}(y))^c \end{array} \right\}.$$

We also write $\mathfrak{A}(y, x; \gamma) := \{(\mathbf{n}_1, \mathbf{n}_2) \in \Omega_{(\overline{\gamma^-} \cap \Lambda_{n^\varepsilon}(y))^c} \times \Omega_{\mathbb{Z}^d} : \mathbf{n}_2 \in \mathcal{A}(y, x; \gamma, \mathbf{n}_1)\}$ and define similarly $\mathfrak{B}_A(y, x; \gamma)$. With these definitions, notice that (7.3.26) rewrites

$$q_{\beta, \beta_c}^x(y) = \langle \sigma_0 \sigma_y \rangle_{\beta_c} \mathbf{E}_{\beta_c}^{0y} \left[\mathbf{1}_{\Gamma \in \mathcal{W}(0, y)} \mathbf{P}_{(\overline{\Gamma^-} \cap \Lambda_{n^\varepsilon}(y))^c, \mathbb{Z}^d, \beta_c}^{\emptyset, yx} [\mathfrak{A}(y, x; \Gamma)] \right] \quad (7.3.28)$$

$$= \langle \sigma_0 \sigma_y \rangle_{\beta_c} \mathbf{E}_{\beta_c}^{0y} \left[\mathbf{1}_{\Gamma \in \mathcal{W}(0, y)} \mathbf{E}_{(\overline{\Gamma^-} \cap \Lambda_{n^\varepsilon}(y))^c, \beta_c}^{\emptyset} [\mathbf{P}_{\mathbb{Z}^d, \beta_c}^{yx} [\mathcal{A}(y, x; \Gamma, \mathbf{n}_1)]] \right] \quad (7.3.29)$$

The following Proposition is inspired by [9] and provides a way to “replace” $\mathcal{A}$ by $\mathcal{B}_A$ for A large enough. The argument below also works when $d = 4$.

Proposition 7.3.8 (Localization procedure). *Let $d \geq 4$. There exist $A > 0$ large enough and $\kappa = \kappa(\varepsilon, A) > 0$ such that, uniformly in Γ sampled according to $\mathbf{E}_{\beta_c}^{0y}$,*

$$\begin{aligned} \mathbf{P}_{(\overline{\Gamma^-} \cap \Lambda_{n^\varepsilon}(y))^c, \mathbb{Z}^d, \beta_c}^{\emptyset, yx} [\mathfrak{A}(y, x; \Gamma)] &= \mathbf{P}_{(\overline{\Gamma^-} \cap \Lambda_{n^\varepsilon}(y))^c, \mathbb{Z}^d, \beta_c}^{\emptyset, yx} [\mathfrak{B}_A(y, x; \Gamma)] + O(n^{-\kappa}) \\ &= \mathbf{E}_{(\overline{\Gamma^-} \cap \Lambda_{n^\varepsilon}(y))^c, \beta_c}^{\emptyset} [\mathbf{P}_{\mathbb{Z}^d, \beta_c}^{yx} [\mathcal{B}_A(y, x; \Gamma, \mathbf{n}_1)]] + O(n^{-\kappa}). \end{aligned}$$

Proof. Clearly $\mathfrak{A}(y, x; \Gamma) \subset \mathfrak{B}_A(y, x; \Gamma)$. If $\mathfrak{B}_A(y, x; \Gamma)$ occurs but not $\mathfrak{A}(y, x; \Gamma)$ then one of the two following situations must occur:

- (i) $(\mathbf{n}_1 + \mathbf{n}_2) \setminus \overline{\Gamma(\mathbf{n}_2)}$ connects $\partial \Lambda_{n^{10\varepsilon}}(y)$ to $\partial \Lambda_{n^{A\varepsilon}}(y)$ in $(\overline{\Gamma} \cap \Lambda_{n^\varepsilon}(y))^c$,
- (ii) $\Gamma(\mathbf{n}_2)$ does a “zig-zag” from $\partial \Lambda_{n^\varepsilon}(y)$ to $\partial \Lambda_{n^{10\varepsilon}}(y)$.

Calling $\mathcal{F}_1$ the first event and $\mathcal{F}_2$ the second. The “chain rule” for the backbone (see [204, 205]) yields

$$\mathbf{P}_{(\Gamma^- \cap \Lambda_{n^\varepsilon}(y))^c, \mathbb{Z}^d, \beta_c}^{\emptyset, yx}[\mathcal{F}_2] \leq \sum_{\substack{v \in \partial \Lambda_{n^\varepsilon}(y) \\ w \in \partial \Lambda_{n^{10\varepsilon}}(y)}} \frac{\langle \sigma_y \sigma_w \rangle_{\beta_c} \langle \sigma_w \sigma_v \rangle_{\beta_c} \langle \sigma_v \sigma_x \rangle_{\beta_c}}{\langle \sigma_y \sigma_x \rangle_{\beta_c}} \leq C_1 \frac{n^{\varepsilon(d-1)} n^{10\varepsilon(d-1)}}{n^{20\varepsilon(d-2)}}. \quad (7.3.30)$$

Notice that $\varepsilon(d-1) + 10\varepsilon(d-1) - 20\varepsilon(d-2) < 0$ if $d \geq 4$. Hence, for some $c_1 = c_1(\varepsilon) > 0$,

$$\mathbf{P}_{(\Gamma^- \cap \Lambda_{n^\varepsilon}(y))^c, \mathbb{Z}^d, \beta_c}^{\emptyset, yx}[\mathcal{F}_2] \leq C_1 n^{-c_1}. \quad (7.3.31)$$

Furthermore,

$$\mathbf{P}_{(\Gamma^- \cap \Lambda_{n^\varepsilon}(y))^c, \mathbb{Z}^d, \beta_c}^{\emptyset, yx}[\mathcal{F}_2] \leq \sum_{\substack{v \in \partial \Lambda_{n^{10\varepsilon}}(y) \\ w \in \partial \Lambda_{n^{A\varepsilon}}(y)}} \langle \sigma_v \sigma_w \rangle_{\beta_c}^2 \leq C_2 \frac{n^{10\varepsilon(d-1)} n^{A\varepsilon(d-1)}}{n^{2A\varepsilon(d-2)}}. \quad (7.3.32)$$

It is easy to check that taking $A > 11$ yields the existence of $c_2 = c_2(A, \varepsilon) > 0$ such that,

$$\mathbf{P}_{(\Gamma^- \cap \Lambda_{n^\varepsilon}(y))^c, \mathbb{Z}^d, \beta_c}^{\emptyset, yx}[\mathcal{F}_2] \leq C_2 n^{-c_2}. \quad (7.3.33)$$

This concludes the proof. $\square$

The mixing statement of [9] allows us to obtain the following result.

Proposition 7.3.9. *There exists $\varepsilon_0 = \varepsilon(A)$, $C = C(d) > 0$ such that if $\varepsilon < \varepsilon_0$ the following holds. For all $y \in \Lambda_{\ell_n(\log n)^\delta}$, and given samples Γ and $\mathbf{n}_1$ as above, there exists a random variable $c_A(y; \Gamma, \mathbf{n}_1)$ such that*

$$\left| \mathbb{P}_{\mathbb{Z}^d, \beta_c}^{yx}[\mathcal{B}_A(y, x; \Gamma, \mathbf{n}_1)] - c_A(y; \Gamma, \mathbf{n}_1) \right| \leq \frac{C}{\sqrt{\log n}}. \quad (7.3.34)$$

Remark 7.3.10. By the results of [205],

$$c_A(y; \Gamma, \mathbf{n}_1) := \mathbf{P}^{y\infty}[\mathcal{B}_A(y, x; \Gamma, \mathbf{n}_1)]. \quad (7.3.35)$$

We now define, for $y \in \mathbb{Z}^d$,

$$q_{\beta, \beta_c}^\infty(y) := \langle \sigma_0 \sigma_y \rangle_{\beta_c} \mathbf{E}_{\beta_c}^{0y} \left[\mathbf{1}_{\Gamma \in \mathcal{W}(0, y)} \mathbf{E}_{(\Gamma^- \cap \Lambda_{n^\varepsilon}(y))^c, \beta_c}^\emptyset [c_A(y; \Gamma, \mathbf{n}_1)] \right] \mathbf{1}_{y \in \Lambda_{\ell_n(\log n)^\delta}} \quad (7.3.36)$$

As an immediate corollary of Lemma 7.3.2 and Propositions 7.3.8 and 7.3.9, we get,

Corollary 7.3.11. *There exists $C = C(d) > 0$ such that, for all $y \in \Lambda_{\ell_n(\log n)^\delta}$,*

$$\left| q_{\beta, \beta_c}^x(y) - q_{\beta, \beta_c}^\infty(y) \right| \leq \frac{C}{\sqrt{\log n}} \delta\beta(n) \sum_{u \sim y} \langle \sigma_0 \sigma_u \rangle_\beta. \quad (7.3.37)$$

Putting all the pieces together, we obtained the following result.

Proposition 7.3.12. *Let $d > 4$. There exists $C > 0$ such that for all n large enough, for all $|x| \geq n$,*

$$\begin{aligned} \left(1 - \frac{C}{\sqrt{\log n}}\right) \sum_{y \in \Lambda_{\ell_n(\log n)^\delta}} q_{\beta, \beta_c}^\infty(y) \langle \sigma_y \sigma_x \rangle_{\beta_c} \\ \leq \langle \sigma_0 \sigma_x \rangle_{\beta_c} \leq \left(1 + \frac{C}{\sqrt{\log n}}\right) \sum_{y \in \Lambda_{\ell_n(\log n)^\delta}} q_{\beta, \beta_c}^\infty(y) \langle \sigma_y \sigma_y \rangle_{\beta_c}. \end{aligned} \quad (7.3.38)$$

Proof. It suffices to observe that

$$\delta\beta(n) \sum_{y \in \Lambda_{\ell_n(\log n)^\delta}} \sum_{u \sim y} \langle \sigma_0 \sigma_u \rangle_\beta \langle \sigma_y \sigma_x \rangle_{\beta_c} \leq C_1 \langle \sigma_0 \sigma_x \rangle_{\beta_c}. \quad (7.3.39)$$

□

Corollary 7.3.13 (Bound on the mass). *There exists $C > 0$ such that*

$$1 - \frac{C}{\sqrt{\log n}} \leq \sum_{y \in \Lambda_{\ell_n(\log n)^\delta}} q_{\beta, \beta_c}^\infty(y) \leq 1 + \frac{C}{\sqrt{\log n}}. \quad (7.3.40)$$

Proof. Summing (7.3.38) $x \in \text{Ann}(n, M)$ for $M \geq n$, dividing by $\chi_M(\beta_c)$, and taking $M \rightarrow \infty$ yields,

$$\left(1 - \frac{C}{\sqrt{\log n}}\right) \sum_{y \in \Lambda_{\ell_n(\log n)^\delta}} q_{\beta, \beta_c}^\infty(y) \leq 1, \quad (7.3.41)$$

and

$$\left(1 + \frac{C}{\sqrt{\log n}}\right) \sum_{y \in \Lambda_{\ell_n(\log n)^\delta}} q_{\beta, \beta_c}^\infty(y) \geq 1. \quad (7.3.42)$$

The proof follows readily. □

7.3.2.4 The good random walk

Definition 7.3.14 (The good random walk). Let $n \geq 1$. Let $d > 4$. Define a finite-range step distribution $(p_n(y))_{y \in \mathbb{Z}^d}$ by the formula,

$$p_n(y) := \frac{q_{\beta, \beta_c}^\infty(y)}{Z_{\beta, \beta_c}}, \quad \forall y \in \mathbb{Z}^d, \quad (7.3.43)$$

where $Z_{\beta, \beta_c} := \sum_{y \in \mathbb{Z}^d} q_{\beta, \beta_c}^\infty(y)$. Call $\mathbb{P}_n$ the law of the random walk $(X_k)_{k \geq 0}$ started at 0 of step distribution given by $(p_n(u))_{u \in \mathbb{Z}^d}$. Call also $\mathbb{E}_n$ the expectation with respect to $\mathbb{P}_n$. We also let $\mathbb{V}_n(X_1) := \mathbb{E}_n[\|X_1\|^2]$.

We now prove some useful properties of the random walk $(X_k)_{k \geq 0}$.

Proposition 7.3.15. *There exists $c_i, C_i > 0$ such that, for all n sufficiently large,*

$$(i) \text{ (Mean-square displacement) } c_1 \ell_n^2 \leq \mathbb{V}_n \leq C_1 \ell_n^2,$$

(ii) (Non-lazyness) $\mathbb{P}_n[X_1 \notin \Lambda_{c_2 \ell_n}] \geq c_2$,

(iii) (Anti-concentration inequality) $\sup_{y \in \mathbb{Z}^d} \mathbb{P}_n[X_m \in \Lambda_{c_3 \ell_n}(y)] \leq \frac{C_3}{m^{d/2}}$.

Proof. We begin with (i). For the upper bound, we write

$$\sum_{y \in \mathbb{Z}^d} \|y\|^2 p_n(y) \leq \frac{\delta \beta(n)}{Z_{\beta, \beta_c}} \sum_{y \in \mathbb{Z}^d} \sum_{u \sim y} \|y\|^2 \langle \sigma_0 \sigma_u \rangle_\beta. \quad (7.3.44)$$

We conclude using (7.1.7). For the lower bound, use (7.3.42), and the fact that $q_{\beta, \beta_c}^\infty(y) \leq C \ell_n^{-2} \sum_{u \sim y} \langle \sigma_0 \sigma_u \rangle_\beta$ to get for any $\epsilon > 0$,

$$\begin{aligned} 1 - o_n(1) &\leq \sum_{y \in \Lambda_{\ell_n(\log n)^\delta}} q_{\beta, \beta_c}^\infty(y) \leq \sum_{y \notin \Lambda_{\ell_n}} q_{\beta, \beta_c}^\infty(y) + C \ell_n^{-2} \sum_{y \in \Lambda_{\ell_n}} \sum_{u \sim y} \langle \sigma_0 \sigma_u \rangle_\beta \\ &\leq \sum_{y \notin \Lambda_{\ell_n}} q_{\beta, \beta_c}^\infty(y) + O(\epsilon^2). \end{aligned}$$

This implies that $\mathbb{P}_n[|X_1| > \epsilon \ell_n] \geq 1 - o_n(1) - O(\epsilon^2)$ and also yields (ii) for sufficiently small ϵ .

We now turn to (iii). Using Theorem 3 of [93] we get the existence of $C_1 = C_1(d) > 0$ such that, if $c > 0$, for $m \geq 0$,

$$\sup_{y \in \mathbb{Z}^d} \mathbb{P}_n[X_m \in \Lambda_{c \ell_n}(y)] \leq \frac{C_1(c \ell_n + 1)^d}{(m(c \ell_n + 1)^2(1 - Q(X_1; \Lambda_{c \ell_n})))^{d/2}} \quad (7.3.45)$$

where

$$Q(X_1; \Lambda_{c \ell_n}) := \sup_{x \in \mathbb{Z}^d} \mathbb{P}_n[X_1 \in \Lambda_{c \ell_n}(x)]. \quad (7.3.46)$$

Choose $c_3 := c_2/4$ where $c_2 > 0$ is given by (ii). Then, one can check that

$$Q(X_1, \Lambda_{c_3 \ell_n}) \leq (1/2) \wedge 1 - c_2. \quad (7.3.47)$$

The proof follows readily. $\square$

Using the above definition and Corollary 7.3.13, we obtained,

Proposition 7.3.16 (Renormalisation inequality). *Let $d > 4$. There exists $C > 0$ such that for all n large enough, for all $|x| \geq n$,*

$$\begin{aligned} \left(1 - \frac{C}{\sqrt{\log n}}\right) \sum_{y \in \Lambda_{\ell_n(\log n)^\delta}} p_n(y) \langle \sigma_y \sigma_x \rangle_{\beta_c} \\ \leq \langle \sigma_0 \sigma_x \rangle_{\beta_c} \leq \left(1 + \frac{C}{\sqrt{\log n}}\right) \sum_{y \in \Lambda_{\ell_n(\log n)^\delta}} p_n(y) \langle \sigma_y \sigma_x \rangle_{\beta_c}. \end{aligned}$$

Remark 7.3.17. We may actually assume only that $|x| \geq n/(\log n)^\epsilon$ for a sufficiently small ϵ .

Now, assume that $\delta < \eta/2$. Let $d_n := \ell_n(\log n)^{2\delta}$. Also, we define

$$\mathbb{S}_k := \{u \in \mathbb{Z}^d : \|u\| \leq k\}. \quad (7.3.48)$$

Corollary 7.3.18. *Let $\mathbb{P}_n$ be the law of the finite-range aperiodic random-walk $(X_k)_{k \geq 0}$ started at 0 of step distribution given by $(p_n(y))_{y \in \mathbb{Z}^d}$. Let $\mathbb{A}_n(x) \subset \mathbb{Z}^d$ be the set centered at x , defined by $\mathbb{A}_n(x) := \{u \in \mathbb{Z}^d, \ell_n(\log n)^{2\delta} - 2\ell_n(\log n)^\delta \leq \|x - u\| \leq \ell_n(\log n)^{2\delta}\}$, and define $\tau_n := \inf\{k \geq 1, X_k \in \mathbb{A}_n(x)\} = \inf\{k \geq 1, X_k \in \mathbb{S}_{d_n}(x)\}$. Then, there exists $C > 0$ such that for all $N \geq 1$,*

$$\begin{aligned} \left(1 - \frac{C}{\sqrt{\log n}}\right)^N \sum_{k=1}^N \mathbb{E}_n [\mathbf{1}_{\tau_n=k} \langle \sigma_{X_k} \sigma_x \rangle_{\beta_c}] &\leq \langle \sigma_0 \sigma_x \rangle_{\beta_c} \\ &\leq \left(1 + \frac{C}{\sqrt{\log n}}\right)^N \left(\sum_{k=1}^N \mathbb{E}_n [\mathbf{1}_{\tau_n=k} \langle \sigma_{X_k} \sigma_x \rangle_{\beta_c}] \right. \\ &\quad \left. + \mathbb{E}_n [\langle \sigma_{X_N} \sigma_x \rangle_{\beta_c} \mathbf{1}_{\tau_n > N}] \right). \end{aligned}$$

Proof. We used the fact that we can iterate Proposition 7.3.16 N times as long as the target point x satisfies $\|X_k - x\| \geq n/(\log n)^\epsilon$ (for $\epsilon > 0$ sufficiently small in terms of δ and η) for all $k \geq N$ (this guarantees that the mixing argument stays valid). $\square$

We will take² $\mathbf{c}n \leq \|x\| \leq \mathbf{C}n$ for $\mathbf{c}, \mathbf{C} > 0$ and fixed (the constants below will depend on $\mathbf{c}$ and $\mathbf{C}$) and $N = (\log n)^{1/2-\zeta}$ (for $\zeta > 0$ small to be fixed) above so that $(1 \pm C(\log n)^{-1/2})^N \rightarrow 1$ as $n \rightarrow \infty$. Moreover, for such a choice of N we know that X_N should typically be located at a (Euclidean) distance from 0 which is $\sqrt{N \mathbb{V}_n(X_1)} \approx n(\log n)^{1/4-\zeta/2-\eta}$. In particular, it is going to be extremely unlikely that $\|X_N\|$ remains smaller than $n(\log n)^{1/8}$. We make this much more precise in what follows.

We now fix $\delta = \eta/4$. Hence,

$$d_n = \frac{n}{(\log n)^{\eta/2}}, \quad \text{and} \quad \ell_n(\log n)^\delta = \frac{n}{(\log n)^{3\eta/4}}. \quad (7.3.49)$$

Lemma 7.3.19. *There exists a choice of $\eta, \zeta > 0$ such that as $n \rightarrow \infty$, one has*

$$\mathbb{E}_n [\langle \sigma_{X_N} \sigma_x \rangle_{\beta_c} \mathbf{1}_{\tau_n > N}] \leq \mathbb{E}_n [\langle \sigma_{X_N} \sigma_x \rangle_{\beta_c} \mathbf{1}_{\|X_N - x\| > d_n}] = o(\langle \sigma_0 \sigma_x \rangle_{\beta_c}). \quad (7.3.50)$$

Proof. One may write,

$$\begin{aligned} \mathbb{E}_n [\langle \sigma_{X_N} \sigma_x \rangle_{\beta_c} \mathbf{1}_{\|X_N - x\| > d_n}] &\leq \max_{y \in \mathbb{S}_{n(\log n)^{1/8}} \setminus \mathbb{S}_{d_n}} \langle \sigma_0 \sigma_y \rangle_{\beta_c} \cdot \mathbb{P}_n [d_n < \|X_N - x\| \leq n(\log n)^{1/8}] \\ &\quad + \max_{y \notin \mathbb{S}_{n(\log n)^{1/8}}} \langle \sigma_0 \sigma_y \rangle_{\beta_c}. \end{aligned}$$

²This would require modifying slightly the definition of ℓ_n . We omit the details.

Note that by the infrared bound (7.1.4), one has

$$\max_{y \in \mathbb{S}_{n(\log n)^{1/8} \setminus \mathbb{S}_{d_n}}} \langle \sigma_0 \sigma_y \rangle_{\beta_c} \leq \frac{C_1}{d_n^{d-2}} \leq C_2 \frac{(\log n)^{(d-2)\eta/2}}{n^{d-2}}, \quad (7.3.51)$$

and,

$$\max_{y \notin \mathbb{S}_{n(\log n)^{1/8}}} \langle \sigma_0 \sigma_y \rangle_{\beta_c} \leq C_3 \frac{1}{n^{d-2}(\log n)^{(d-2)/8}}. \quad (7.3.52)$$

Using Proposition 7.3.15 (iii), we get that for some $C_4 > 0$

$$\mathbb{P}_n \left[d_n < \|X_N - x\| \leq n(\log n)^{1/8} \right] \leq C_4 \left(\frac{n(\log n)^{1/8}}{\ell_n} \right)^d \frac{1}{N^{d/2}} = C_4 \frac{(\log n)^{d\eta + \zeta d/2}}{(\log n)^{d/4}}. \quad (7.3.53)$$

Hence, by choosing $\eta, \zeta > 0$ small enough, we get for $C_5, C_6 > 0$

$$\mathbb{E}_n \left[\langle \sigma_{X_N} \sigma_x \rangle_{\beta_c} \mathbf{1}_{\|X_N - x\| > d_n} \right] \leq C_5 \frac{1}{n^{d-2}(\log n)^{(d-2)/8}} \leq C_6 \frac{\langle \sigma_0 \sigma_x \rangle_{\beta_c}}{(\log n)^{(d-2)/8}}, \quad (7.3.54)$$

where we used (7.1.8) in the last inequality. $\square$

Hence, Corollary 7.3.18 rewrites, for n large enough, uniformly in $cn \leq |x| \leq Cn$,

$$(1 - o(1)) \mathbb{E}_n [\mathbf{1}_{\tau_n \leq N} \langle \sigma_{X_{\tau_n}} \sigma_x \rangle_{\beta_c}] \leq \langle \sigma_0 \sigma_x \rangle_{\beta_c} \leq (1 + o(1)) \mathbb{E}_n [\mathbf{1}_{\tau_n \leq N} \langle \sigma_{X_{\tau_n}} \sigma_x \rangle_{\beta_c}]. \quad (7.3.55)$$

Moreover, the $o(1)$ can be chosen to be $O((\log n)^{-\kappa})$ for $\kappa > 0$ sufficiently small. In fact, using Lemma 7.3.20 below, we can also write

$$(1 - o(1)) \mathbb{E}_n [\mathbf{1}_{\tau_n < \infty} \langle \sigma_{X_{\tau_n}} \sigma_x \rangle_{\beta_c}] \leq \langle \sigma_0 \sigma_x \rangle_{\beta_c} \leq (1 + o(1)) \mathbb{E}_n [\mathbf{1}_{\tau_n < \infty} \langle \sigma_{X_{\tau_n}} \sigma_x \rangle_{\beta_c}]. \quad (7.3.56)$$

Lemma 7.3.20. *One has, for η small enough and n large enough,*

$$\mathbb{E}_n [\mathbf{1}_{N < \tau_n < \infty} \langle \sigma_{X_{\tau_n}} \sigma_x \rangle_{\beta_c}] = o(\langle \sigma_0 \sigma_x \rangle_{\beta_c}). \quad (7.3.57)$$

Proof. Use Proposition 7.3.15 (iii) to get

$$\mathbb{P}_n [N < \tau_n < \infty] \leq \sum_{k \geq N+1} \mathbb{P}_n [X_k \in \mathbb{A}_n(x)] \leq C_1 \frac{|\mathbb{A}_n(x)|}{\ell_n^d} \frac{1}{N^{d/2-1}}. \quad (7.3.58)$$

Moreover, $|\mathbb{A}_n(x)|/\ell_n^d = (\log n)^{2\delta}$, from which we conclude using the definition of N above, and the fact that

$$\langle \sigma_{X_{\tau_n}} \sigma_x \rangle_{\beta_c} \leq C_2 (\log n)^{(d-2)\eta/2} \langle \sigma_0 \sigma_x \rangle_{\beta_c}. \quad (7.3.59)$$

$\square$

Let us explain what we did above. Let $S_{\beta_c}(x) := \langle \sigma_0 \sigma_x \rangle_{\beta_c}$. Proposition 7.3.16 gives that $(S_{\beta_c}(x - X_k))_{1 \leq k \leq N \wedge \tau_n}$ is “almost” a martingale started at $S_{\beta_c}(x)$. What we obtained above is

$$\mathbb{E}_n [S_{\beta_c}(x)] = (1 + o(1)) \mathbb{E}_n [\mathbf{1}_{\tau_n < \infty} S_{\beta_c}(x - X_{\tau_n})], \quad (7.3.60)$$

which is exactly like an application of the stopping time theorem to the (almost) martingale $(S_{\beta_c}(x - X_k))_{k \geq 0}$.

In fact we obtained the following result,

Proposition 7.3.21. *One has,*

$$\langle \sigma_0 \sigma_y \rangle_{\beta_c} = (1 + o_{\|y\|}(1)) \mathbb{E}_{\|y\|} \left[\mathbf{1}_{\tau_{\|y\|} < N(y)} \langle \sigma_{X_{\tau_{\|y\|}}} \sigma_y \rangle_{\beta_c} \right], \quad (7.3.61)$$

where $N(y) = (\log \|y\|)^{1/2-\zeta}$ with ζ small enough, and where $o_{\|y\|}(1)$ can be chosen to be $O((\log \|y\|)^{-\kappa})$ for $\kappa > 0$ small enough.

Below, we shall see the consequences of this “almost-harmonicity”.

7.3.3 Asymptotic rotational invariance of the two-point function

In this section we study the first consequence of Proposition 7.3.21: the two-point function is asymptotically rotationally invariant. More precisely, we will prove

Proposition 7.3.22. *Let $d > 4$. Then, one has,*

$$\lim_{\|x\| \rightarrow \infty} \frac{\langle \sigma_0 \sigma_x \rangle_{\beta_c}}{\langle \sigma_0 \sigma_{\|x\| \mathbf{e}_1} \rangle_{\beta_c}} = 1. \quad (7.3.62)$$

In order to prove the above result, we will proceed as follows: first, we couple the random walk $(X_k)_{k \geq 0}$ with a Brownian motion B using KMT theory. Then, we use that if τ is the stopping time for the hitting of a sphere located far away from the origin, then $B(\tau)$ is asymptotically uniformly distributed on the surface of the sphere. Combined with the gradient estimates of [9] and Proposition 7.3.21, this will be enough to get Proposition 7.3.22.

7.3.3.1 Coupling with a Brownian motion

In this subsection, we take $x \in \mathbb{Z}^d$. We write $\|x\| = n$ (and omit integer rounding) and we keep the notations introduced above. We start by gathering some nice properties of the effective walk.

Lemma 7.3.23 (The covariance matrix is scalar). *Let $\Sigma_n = (\mathbb{E}_n[X_1^{(i)} X_1^{(j)}])_{1 \leq i, j \leq d}$ be the covariance matrix of the increment of the walk under $\mathbb{P}_n$. Then,*

$$\Sigma_n = \frac{\mathbb{V}_n}{d} \cdot \mathbb{I}_d. \quad (7.3.63)$$

Proof. Using the symmetries of $\mathbb{P}_n$, we get that for $1 \leq i \neq j \leq d$,

$$\mathbb{E}_n [X_1^{(i)} X_1^{(j)}] = 0, \quad (7.3.64)$$

and that for all $1 \leq i \leq d$,

$$\mathbb{V}_n(X_1) = \mathbb{E}_n [\|X_1\|^2] = d \mathbb{E}_n [(X_1 \cdot \mathbf{e}_i)^2] = d \mathbb{E}_n [(X_1^{(1)})^2]. \quad (7.3.65)$$

This yields the result. $\square$

Lemma 7.3.24 (Uniform bound on exponential moments). *There exist $c_1, C_1 > 0$ such that, for all n large enough, for all $|t| \leq c_1$,*

$$\mathbb{E}_n [\exp(t \|\Sigma_n^{-1/2} X_1\|)] \leq C_1. \quad (7.3.66)$$

Proof. The proof is based on similar arguments as those used above. There exist $c_i, C_i > 0$ (which only depend on d) such that

$$\mathbb{P}_n[\|X_1\| \geq k\mathbb{V}_n] \leq C_1 \cdot \delta\beta(n) \cdot \sum_{\|u\| \geq k\mathbb{V}_n} \langle \sigma_0 \sigma_u \rangle_\beta \leq \frac{C_2}{\ell_n^d} \sum_{\|u\| \geq k\mathbb{V}_n} e^{-c_1(\|u\|/\ell_n)} \leq C_3 e^{-c_2 k}. \quad (7.3.67)$$

This gives the result. $\square$

We now use a generalisation of the KMT coupling obtained by Zaitsev [256]. We let $\xi_i := \Sigma_n^{-1/2}(X_i - X_{i-1})$ for $i \geq 1$. Let $S_k = \sum_{i=1}^k \xi_i$.

Theorem 7.3.25 (Multidimensional KMT coupling). *Let $K > 0$. There exists a coupling $\mathbf{P}_n$ of a Brownian motion $(B_t)_{t \geq 0}$ (started at the origin) on $\mathbb{R}^d$, and of $(\xi_i)_{1 \leq i \leq K}$, together with constants $c, C, T_0 > 0$, such that for $t \geq T_0$,*

$$\mathbf{P} \left[\max_{1 \leq k \leq K} \|S_k - B(k)\| \geq t \log K \right] \leq C e^{-ct \log K}. \quad (7.3.68)$$

Proof. Apply [256, Theorem 1.3] to show that the coupling $\mathbf{P}_n$ exists together with constants c_1, c_2 which only depend on d , such that

$$\mathbf{E}_n \left[\exp \left(c_1 \max_{1 \leq k \leq K} \|S_k - B(k)\| \right) \right] \leq \exp(c_2 \log K), \quad (7.3.69)$$

and conclude with an exponential Markov. $\square$

We can apply it in our case with $K = N = (\log n)^{1/2-\zeta}$. We introduce some notations.

$$T_n^{(S)}(x) := \inf \left\{ k \geq 1 : S_k \in \Sigma_n^{-1/2} \mathbb{S}_{d_n}(x) \right\}, \quad (7.3.70)$$

$$T_n^{(B)}(x) := \inf \left\{ s \geq 0 : B(s) \in \Sigma_n^{-1/2} \mathbb{S}_{d_n}(x) \right\}. \quad (7.3.71)$$

Note that standard Brownian computations (use that $\|B(t) + x\|^{2-d}$ is a martingale) give

$$\mathbf{P}_n \left[T_n^{(B)}(x) < \infty \right] = \left(\frac{d_n}{\|x\|} \right)^{d-2}. \quad (7.3.72)$$

In the following, we show that up to a small error, we can replace the random walk X by a Brownian motion in the expectation which appears in Proposition 7.3.21. We start with a lemma.

Lemma 7.3.26. *If $t_0 > 0$ is large enough,*

$$\langle \sigma_0 \sigma_x \rangle_{\beta_c} = (1 + o_{\|x\|}(1)) \mathbf{E}_n \left[\mathbb{1}_{T_n^{(S)}(x) \leq N} \mathbb{1}_{A_n(t_0)} \langle \sigma_{\Sigma_n^{1/2} S_{T_n^{(S)}(y)}} \sigma_x \rangle_{\beta_c} \right], \quad (7.3.73)$$

where,

$$A_n(t_0) := \left\{ \max_{1 \leq k \leq N} \|S_k - B(k)\| \leq t_0 \log \log n \right\}. \quad (7.3.74)$$

Proof. Just use the KMT coupling to argue that for $t_0 > 0$ large enough,

$$\mathbf{P}_n[A_n(t_0)^c] \leq (\log n)^{-ct_0}. \quad (7.3.75)$$

The proof follows from choosing $t_0 > 0$ large enough so that

$$\frac{1}{d_n^{d-2}} (\log n)^{-ct_0} = o(n^{2-d}). \quad (7.3.76)$$

□

We now fix t_0 as above. We define $d_n^\pm := d_n \pm 2t_0 \ell_n \log \log n$. The next step is to replace S by B above. Define,

$$T_n^{(B),\pm}(x) := \inf \left\{ s \geq 0 : B(s) \in \Sigma_n^{-1/2} \mathbb{S}_{d_n^\pm}(x) \right\}. \quad (7.3.77)$$

To lighten the notations, we drop x from the notation of the stopping times.

Proposition 7.3.27. *One has,*

$$\begin{aligned} (1 - o(1)) \mathbf{E}_n \left[\mathbf{1}_{T_n^{(B),-} \leq N} \langle \sigma_{\Sigma_n^{1/2} B(T_n^{(B),-})} \sigma_x \rangle_{\beta_c} \right] &\leq \mathbf{E}_n \left[\mathbf{1}_{T_n^{(S)} \leq N} \langle \sigma_{\Sigma_n^{1/2} S_{T_n^{(S)}}} \sigma_x \rangle_{\beta_c} \right] \\ &\leq (1 + o(1)) \mathbf{E}_n \left[\mathbf{1}_{T_n^{(B),+} \leq N} \langle \sigma_{\Sigma_n^{1/2} B(T_n^{(B),+})} \sigma_x \rangle_{\beta_c} \right] \end{aligned}$$

Proof. We first use that whenever $\mathbf{1}_{A_n(t_0)} \mathbf{1}_{T_n^{(S)} \leq N} = 1$, one has,

$$T_n^{(B),+} \leq T_n^{(S)} \leq T_n^{(B),-} \leq \infty. \quad (7.3.78)$$

We have essentially two situations: either $T_n^{(B),-} - T_n^{(B),+}$ is “not too large” (this will be quantified below), in which case we expect to have that $\|B(T_n^{(B),\pm}) - S(T_n^{(S)})\| = o((\log n)^{2\delta})$ with high probability; or this difference is large but this is not likely to happen, see Figure 7.3. Indeed, if you start a Brownian motion close to a very large sphere, it will either hit it fast or never hit it.

Claim 1. Let $\epsilon \in (0, \delta)$. Let

$$\begin{aligned} \mathcal{A} := & \left\{ T_n^{(S)} \leq N \right\} \cap A_n(t_0) \cap \left\{ |T_n^{(B),-} - T_n^{(B),+}| \leq (\log n)^{2\delta+\epsilon} \right\} \cap \mathcal{C} \\ & \cap \left\{ \max_{T_n^{(B),+} \leq t \leq T_n^{(B),-} + (\log n)^{2\delta+\epsilon}} \|B(t) - B(T_n^{(B),+})\| \leq (\log n)^{\delta+\epsilon} \right\}, \end{aligned}$$

where

$$\mathcal{C} := \left\{ \forall 1 \leq j \leq N, \max_{s \in [j-1, j]} \|B(s) - B(j)\| \leq 2\sqrt{t_0 \log \log n} \right\}, \quad (7.3.79)$$

Then, if $\mathcal{A}$ occurs, one has $\|S(T_n^{(S)}) - B(T_n^{(B),\pm})\| \leq 2(\log n)^{\delta+\epsilon}$, and in particular

$$\|S(T_n^{(S)}) - B(T_n^{(B)})\| \leq 3(\log n)^{\delta+\epsilon}. \quad (7.3.80)$$

Proof of Claim 1. This is almost straightforward. If $\{|T_n^{(B),-} - T_n^{(B),+}| \leq (\log n)^{2\delta+\epsilon}\}$ occurs, then $T_n^{(S)} \in [T_n^{(B),+}, T_n^{(B),+} + (\log n)^{2\delta+\epsilon}]$. Hence, when $\mathcal{A}$ occurs,

$$\begin{aligned} \|S(T_n^{(S)}) - B(T_n^{(B),\pm})\| &\leq \|S(T_n^{(S)}) - B(T_n^{(S)})\| \\ &+ \max_{T_n^{(B),+} \leq t \leq T_n^{(B),+} + (\log n)^{2\delta+\epsilon}} \|B(t) - B(T_n^{(B),+})\| \leq t_0 \log \log n + (\log n)^{\delta+\epsilon}. \end{aligned}$$

□

Using that $B(t + T_n^{(B),+}) - B(T_n^{(B),+})$ is a Brownian motion on $[0, (\log n)^{2\delta+\epsilon}]$, we get (by standard computations),

$$\begin{aligned} \mathbf{P}_n \left[\max_{t \in [0, (\log n)^{2\delta+\epsilon}]} \|B(t + T_n^{(B),+}) - B(T_n^{(B),+})\| \geq (\log n)^{\delta+\epsilon} \right] \\ \leq C \mathbb{P} \left[\|B^{(1)}\|_{\infty, [0,1]} \geq c(\log n)^{\epsilon/2} \right], \end{aligned}$$

where $B^{(1)}$ is a one dimensional standard Brownian motion. The last probability is bounded by $Ce^{-c(\log n)^\epsilon}$. By a similar computation,

$$\mathbf{P}_n[\mathcal{C}^c] \leq (\log n)^{-ct_0}. \quad (7.3.81)$$

The following claim is straightforward.

Claim 2. Define the event

$$\mathcal{B} := \{T_n^{(S)} \leq N\} \cap \{|T_n^{(B),-} - T_n^{(B),+}| > (\log n)^{2\delta+\epsilon}\}. \quad (7.3.82)$$

Denoting by $\tilde{B}(t) := B(t + T_n^{(B),+})$, we have that

$$\mathcal{B} \subset \{T_n^{(B),+} < \infty\} \cap \{T_n^{(\tilde{B}),-} \geq (\log n)^{2\delta+\epsilon}\}. \quad (7.3.83)$$

Hence, when $\mathcal{B}$ occurs, we end up with the situation of a Brownian motion started at distance r from a disk of radius R (with $r \ll R$), and we look at the probability that it does not touch the disk before waiting for a long time. This is a rather unlikely event as suggested by Figure 7.3.

Let us fix some notations. Write $r = r_n = 4t_0 \log \log n$, $R = R_n = (\log n)^{2\delta} - r/2$, and $T = (\log n)^{2\delta+\epsilon}$. We look at the disk $\mathbb{D}_R$ centered at 0 of radius R . We start a Brownian motion B (of law $\mathbb{P}$) at Euclidean distance $R+r$ from the origin. Let $\tau_{r,R}$ be the first time B hits $\mathbb{D}_R$. Then, using the above analysis and the (strong) Markov Property,

$$\mathbf{P}_n[\mathcal{B}] \leq \mathbf{P}_n[T_n^{(\tilde{B}),-} \geq T, T_n^{(B),+} < \infty] \leq \mathbb{P}[\tau_{r,R} \geq T] \mathbf{P}_n[T_n^{(B),+} < \infty]. \quad (7.3.84)$$

We now import a result from [129, Theorem 4.1] (applied with $\nu = \frac{d-2}{2}$) which exactly estimates $\mathbb{P}[\tau_{r,R} \geq T]$. Using standard Brownian computations,

$$\mathbb{P}[\tau_{r,R} \geq T] = \mathbb{P}[\tau_{r,R} = \infty] + \mathbb{P}[T \leq \tau_{r,R} < \infty] = 1 - \left(\frac{R}{R+r} \right)^{d-2} + \mathbb{P}[T \leq \tau_{r,R} < \infty]. \quad (7.3.85)$$

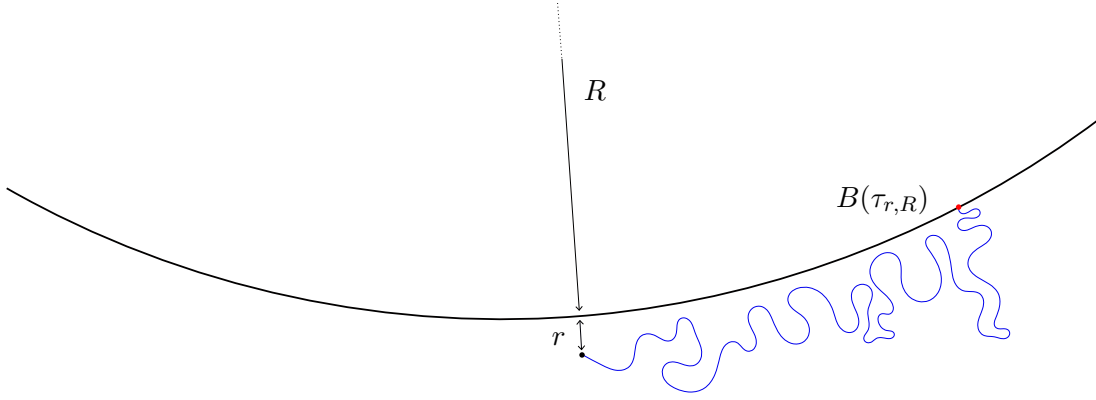


Figure 7.3: An illustration of how unlikely the event $\{T \leq \tau_{r,R} < \infty\}$ is, when $R \gg r$. The Brownian motion (in blue), started at distance r from a Euclidean disk $\mathbb{D}_R$ of radius R will most likely hit $\mathbb{D}_R$ in a “short” time.

Notice that there exists $C > 0$ such that,

$$1 - \left(\frac{R}{R+r} \right)^{d-2} \leq C \frac{\log \log n}{(\log n)^{2\delta}}. \quad (7.3.86)$$

Using [129, Theorem 4.1],

$$\mathbb{P}[T \leq \tau_{r,R} < \infty] \leq \frac{C}{T^{(d-2)/2}} \leq \frac{C}{(\log n)^{(d-2)\delta}}. \quad (7.3.87)$$

Putting all the pieces together, we proved that for some $c(\delta) > 0$,

$$\mathbf{P}_n[\mathcal{B}] \leq \frac{C}{(\log n)^{c(\delta)}} \mathbf{P}_n[T_n^{(B),+} < \infty] = \frac{C}{(\log n)^{c(\delta)}} \left(\frac{d_n}{\|x\|} \right)^{d-2}. \quad (7.3.88)$$

We conclude with a last claim.

Claim 3. If $\mathcal{A}$ occurs, one has, for some $C > 0$,

$$\left| \langle \sigma_{\Sigma_n^{1/2} S_{T_n^{(S)}}} \sigma_x \rangle_{\beta_c} - \langle \sigma_{\Sigma_n^{1/2} B(T_n^{(B),\pm})} \sigma_x \rangle_{\beta_c} \right| \leq \frac{C}{(\log n)^{\delta-\epsilon}} \langle \sigma_{\Sigma_n^{1/2} B(T_n^{(B),\pm})} \sigma_x \rangle_{\beta_c}. \quad (7.3.89)$$

Proof of Claim 3. The result follows from the gradient property [9, Proposition 5.9]. $\square$

The proof of the proposition follows readily from the above claims. $\square$

We now make two additional observations.

Lemma 7.3.28. *One has,*

- (i) $\mathbf{P}_n[N \leq T_n^{(B),\pm}(x) < \infty] = o(\mathbf{P}_n[T_n^{(B),\pm}(x) < \infty]),$
- (ii) $\mathbf{P}_n[T_n^{(B),+}(x) < \infty] - \mathbf{P}_n[T_n^{(B),-}(x) < \infty] = o(\mathbf{P}_n[T_n^{(B)}(x) < \infty]).$

Proof. Use [129, Theorem 4.1] again. $\square$

The above analysis yields the following result.

Proposition 7.3.29. *Let $\mathbb{P}_0$ be the law of a standard BM started at 0. Then,*

$$\langle \sigma_0 \sigma_x \rangle_{\beta_c} = (1 + o(1)) \mathbb{E}_0 \left[\mathbf{1}_{T_n^{(B)}(x) < \infty} \langle \sigma_{\Sigma_n^{1/2} B(T_n^{(B)}(x))} \sigma_x \rangle_{\beta_c} \right]. \quad (7.3.90)$$

The next step consists in using the results of [244] to argue that conditionally on the event $\{T_n^{(B)} < \infty\}$, the distribution of $\Sigma_n^{1/2} B(T_n^{(B)}(x))$ is essentially uniform on $\mathbb{S}_{d_n}(x)$.

Notice that,

$$\mathbb{E}_0 \left[\mathbf{1}_{T_n^{(B)}(x) < (\log n)^{3\eta/4}} \langle \sigma_{\Sigma_n^{1/2} B(T_n^{(B)}(x))} \sigma_x \rangle_{\beta_c} \right] \leq \mathbb{P}_0[T_n^{(B)}(x) < (\log n)^{3\eta/4}] \frac{C}{d_n^{d-2}}. \quad (7.3.91)$$

Classical arguments yield that

$$\mathbb{P}_0[T_n^{(B)}(x) < (\log n)^{3\eta/4}] \leq \mathbb{P}_0[\|B^{(1)}\|_{\infty, [0, (\log n)^{3\eta/4}]} \geq (1/2)(\log n)^\eta] \quad (7.3.92)$$

$$= \mathbb{P}_0[\|B^{(1)}\|_{\infty, [0, 1]} \geq (1/2)(\log n)^{5\eta/8}] \quad (7.3.93)$$

$$\leq C \exp(-c(\log n)^{5\eta/4}), \quad (7.3.94)$$

so that

$$\mathbb{E}_0 \left[\mathbf{1}_{T_n^{(B)}(x) < (\log n)^{3\eta/4}} \langle \sigma_{\Sigma_n^{1/2} B(T_n^{(B)}(x))} \sigma_x \rangle_{\beta_c} \right] = o(\langle \sigma_0 \sigma_x \rangle_{\beta_c}). \quad (7.3.95)$$

Now, write,

$$\langle \sigma_0 \sigma_x \rangle_{\beta_c} = (1 + o(1)) \int_{(\log n)^{3\eta/4}}^{\infty} q_{d_n}(x, t) \mathbb{E}_x \left[\langle \sigma_{\Sigma_n^{1/2} B(T_n^{(B)}(0))} \sigma_0 \rangle_{\beta_c} \mid T_n^{(B)}(0) = t \right] dt, \quad (7.3.96)$$

where $\mathbb{E}_x$ is the law of a BM started at x , and where $q_{d_n}(x, t)$ is the density of the hitting time $T_n^{(B)}(0)$ with respect to the Lebesgue measure. The scaling properties of the BM yield that

$$q_{d_n}(x, t) = \frac{1}{d_n^2} q_1 \left(\frac{x}{d_n}, \frac{t}{d_n^2} \right), \quad (7.3.97)$$

where $q_1 \left(\frac{x}{d_n}, \frac{t}{d_n^2} \right)$ is density of the hitting time of $\mathbb{S}_1$ by a BM started at x/d_n . Moreover,

$$\mathbb{E}_x \left[\langle \sigma_{\Sigma_n^{1/2} B(T_n^{(B)}(0))} \sigma_0 \rangle_{\beta_c} \mid T_n^{(B)}(0) = t \right] \quad (7.3.98)$$

$$= \int_{\partial \mathbb{S}_{(\log n)^{\eta/2}}} \langle \sigma_{\Sigma_n^{1/2} \xi} \sigma_0 \rangle_{\beta_c} \frac{\mathbb{P}_x[B(t) \in d\xi \mid T_n^{(B)}(0) = t]}{d\mu(\xi)} d\mu_{(\log n)^{\eta/2}}(\xi) \quad (7.3.99)$$

$$= \int_{\partial \mathbb{S}_1} \langle \sigma_{\Sigma_n^{1/2} d_n \xi} \sigma_0 \rangle_{\beta_c} \frac{\mathbb{P}_{\mathbf{e}_x(\log n)^{\eta/2}}[B(t) \in d\xi \mid T_n^{(B)}(0) = t]}{d\mu_1(\xi)} d\mu_1(\xi), \quad (7.3.100)$$

where $d\mu_\ell$ denotes the uniform measure on the $(d-1)$ -dimensional sphere $\partial \mathbb{S}_\ell$. Finally, by [244, Remark 2]: for $t \geq (\log n)^{3\eta/4}$,

$$\frac{\mathbb{P}_{\mathbf{e}_x(\log n)^{\eta/2}}[B(t) \in d\xi \mid T_n^{(B)}(0) = t]}{d\mu_1(\xi)} = 1 + O((\log n)^{-\eta/4}). \quad (7.3.101)$$

Combining the above displayed equation with (7.3.96) we obtain

$$\langle \sigma_0 \sigma_x \rangle_{\beta_c} = (1 + o(1))(1 + O((\log n)^{-\eta/4})) \int_{(\log n)^{3\eta/4}}^{\infty} q_{d_n}(x, t) dt. \quad (7.3.102)$$

Proof of Proposition 7.3.22. Applying (7.3.102) to x and $\|x\|e_1$, we obtain that

$$\frac{\langle \sigma_0 \sigma_x \rangle_{\beta_c}}{\langle \sigma_0 \sigma_{\|x\|e_1} \rangle_{\beta_c}} = (1 + o(1)), \quad (7.3.103)$$

where $o(1)$ tends to 0 as $\|x\|$ tends to infinity. $\square$

7.3.4 Proof of Theorem 7.3.1

We are now in a position to prove the main result of this section. Use Proposition 7.3.29 and the asymptotic rotational invariance given by Proposition 7.3.22 to argue that

$$\frac{\langle \sigma_0 \sigma_{[nx]} \rangle_{\beta_c}}{\langle \sigma_0 \sigma_{[ny]} \rangle_{\beta_c}} = (1 + o(1)) \frac{\mathbb{P}_0[T(\mathbb{S}_{d_{\|[nx]\|}}([nx]) < \infty]}{\mathbb{P}_0[T(\mathbb{S}_{d_{\|[ny]\|}}([ny]) < \infty]}. \quad (7.3.104)$$

As a result, if $x, y \in \mathbb{R}^d \setminus \{0\}$,

$$\lim_{n \rightarrow \infty} \frac{\langle \sigma_0 \sigma_{[nx]} \rangle_{\beta_c}}{\langle \sigma_0 \sigma_{[ny]} \rangle_{\beta_c}} = \frac{G(0, x)}{G(0, y)}. \quad (7.3.105)$$

7.4 Full convergence to the GFF

7.4.1 Convergence along the e_1 direction

For $x \in \mathbb{Z}^d$, let $S_\beta(x) := \langle \sigma_0 \sigma_x \rangle_\beta$. For $\beta < \beta_c$ and $p \in \mathbb{R}^d$, define

$$\hat{S}_\beta(p) := \sum_{x \in \mathbb{Z}^d} e^{ip \cdot x} S_\beta(x). \quad (7.4.1)$$

Introduce for $\beta < \beta_c$ and $t \in [-\pi, \pi]$,

$$f_\beta(t) := \mathcal{E}_1(t) \hat{S}_\beta(t, 0, \dots, 0), \quad (7.4.2)$$

where $\mathcal{E}_1(t) = 2(1 - \cos t)$. Note that this function is increasing in $|t|$ on $[-\pi, \pi]$ by [9, Proposition 5.4]. In fact, it is an even function so it suffices to study it on $[0, \pi]$.

Proposition 7.4.1 (Boundedness of f_β). *There exists $C = C(d) > 0$ such that, for all $\beta < \beta_c$, for all $t \in (-\pi, \pi]$,*

$$0 \leq f_\beta(t) \leq C. \quad (7.4.3)$$

Proof. This is a simple consequence of the infrared bound (and more specifically of [204, Proposition 3.7]). $\square$

Corollary 7.4.2. *There exists an increasing sequence $(\beta_k)_{k \geq 1}$, and a function $\mathfrak{f} : (-\pi, \pi] \rightarrow [0, C]$ which is increasing in $|t|$, such that $\beta_k \rightarrow \beta_c$ and for all $t \in [-\pi, \pi]$,*

$$\lim_{k \rightarrow \infty} f_{\beta_k}(t) = \mathfrak{f}(t). \quad (7.4.4)$$

In words, $(f_{\beta_k})_{k \geq 1}$ converges simply to $\mathfrak{f}$ on $[-\pi, \pi]$.

Proof. This is a direct consequence of Helly's selection theorem. $\square$

Corollary 7.4.3. *The limit*

$$\lim_{L \rightarrow \infty} \mathfrak{f}(1/L) \quad (7.4.5)$$

exists and belongs to $[0, C]$. We denote it $\hat{\mathbf{C}}$.

Our next goal is to come up with a more explicit expression for $\mathfrak{f}$. In order to do so, we will prove the following result.

Proposition 7.4.4. *There exists $C_1 = C_1(d) > 0$ such that, for all $\beta < \beta_c$, for all $C \geq 1$,*

$$\left| \mathcal{E}_1(1/L) \hat{S}_\beta(1/L, 0, \dots, 0) - \mathcal{E}_1(1/L) \sum_{|x| \leq CL} e^{ix_1/L} S_\beta(x) \right| \leq \frac{C_1}{C}. \quad (7.4.6)$$

The proof relies on a proper application of the MMS inequalities. We postpone it to the next subsection. As an immediate corollary of Proposition 7.4.4, we obtain,

Corollary 7.4.5. *One has for $L \geq 1$, for all $C \geq 1$,*

$$\mathfrak{f}(1/L) = L^d \mathcal{E}_1(1/L) S_{\beta_c}(L\mathbf{e}_1) \left(\frac{1}{L^d} \sum_{|x| \leq CL} e^{ix_1/L} \frac{S_{\beta_c}(x)}{S_{\beta_c}(L\mathbf{e}_1)} \right) + O(1/C). \quad (7.4.7)$$

We are now in a position to prove the main result of this section.

Theorem 7.4.6. *There exists $\mathbf{C} = \mathbf{C}(d) > 0$ such that*

$$\lim_{L \rightarrow \infty} L^{d-2} S_{\beta_c}(L\mathbf{e}_1) = \mathbf{C}. \quad (7.4.8)$$

Proof. Apply Corollary 7.4.5 to get that for all $C \geq 1$, as $L \rightarrow \infty$,

$$\mathfrak{f}(1/L) = (1 + o_L(1)) L^{d-2} S_{\beta_c}(L\mathbf{e}_1) \int_{[-C, C]^d} e^{i[tL]/L} \frac{S_{\beta_c}([t_1 L])}{S_{\beta_c}(L\mathbf{e}_1)} dt + O(1/C). \quad (7.4.9)$$

Using the dominated convergence theorem together with Theorem 7.3.1, and Corollary 7.4.3, we get that for all $C \geq 1$, as $L \rightarrow \infty$,

$$(1 + o_L(1)) \hat{\mathbf{C}} = L^{d-2} S_{\beta_c}(L\mathbf{e}_1) (1 + o_L(1)) \int_{[-C, C]^d} e^{it_1} \|t\|^{2-d} dt + O(1/C). \quad (7.4.10)$$

Taking $C \rightarrow \infty$ and applying Proposition 7.A.1, we get that $(L^{d-2} S_{\beta_c}(L\mathbf{e}_1))_{L \geq 1}$ converges and

$$\lim_{L \rightarrow \infty} L^{d-2} S_{\beta_c}(L\mathbf{e}_1) = \hat{\mathbf{C}} \cdot \mathbf{c}_d. \quad (7.4.11)$$

$\square$

Combining the above result with Theorem 7.3.1 yields a more explicit version of Theorem 7.1.1.

Theorem 7.4.7 (Asymptotic for the two-point function). *Let $d > 4$. Then, as $\|x\| \rightarrow \infty$,*

$$\langle \sigma_0 \sigma_x \rangle_{\beta_c} = (1 + o(1)) \frac{\hat{\mathbf{C}} \cdot \mathbf{c}_d}{\|x\|^{d-2}}. \quad (7.4.12)$$

7.4.2 Proof of Proposition 7.4.4

We start by fixing $\beta < \beta_c$. Note that for $L \geq 1$,

$$\mathcal{E}_1(1/L) \hat{S}_\beta(1/L, 0, \dots, 0) = \sum_{k \in \mathbb{Z}} e^{ik/L} (2H_k(\beta) - H_{k+1}(\beta) - H_{k-1}(\beta)), \quad (7.4.13)$$

where

$$H_k(\beta) := \sum_{x_\perp \in \mathbb{Z}^{d-1}} S_\beta((k, x_\perp)). \quad (7.4.14)$$

The first step is to prove that the contribution of this sum coming from values of $k \geq CL + 1$ is $O(1/C)$. For that, we start by rewriting the above sum.

Lemma 7.4.8. *Let $u_k(\beta) := 2H_k(\beta) - H_{k+1}(\beta) - H_{k-1}(\beta)$ and $s_k(L) := \sum_{j=CL}^k e^{ij/L}$. Then,*

$$\sum_{A \geq k \geq CL+1} e^{ik/L} u_k(\beta) = s_A(L) u_A(\beta) - s_{CL}(L) u_{CL+1}(\beta) + \sum_{k=CL}^{A-1} s_k(L) (u_k(\beta) - u_{k+1}(\beta)). \quad (7.4.15)$$

Moreover, we can take $A = \infty$ above since $\lim_{A \rightarrow \infty} u_A(\beta) = 0$ and $|s_A(L)| \leq C_1 L$ for all A . As a result,

$$\sum_{k \geq CL+1} e^{ik/L} u_k(\beta) = -s_{CL}(L) u_{CL+1}(\beta) + \sum_{k=CL}^{\infty} s_k(L) (u_k(\beta) - u_{k+1}(\beta)). \quad (7.4.16)$$

It is clear (see observation below) that $u_k(\beta) \leq 0$. Next, we notice that

$$\left| \sum_{k \geq CL+1} e^{ik/L} u_k(\beta) \right| \leq \sup_{k \geq CL} |s_k(L)| \left(|u_{CL+1}(\beta)| + \sum_{k \geq CL} |u_k(\beta) - u_{k+1}(\beta)| \right). \quad (7.4.17)$$

We can make the following observations: first, by standard computations, there exists $C_1 > 0$ such that

$$\sup_{k \geq CL} |s_k(L)| \leq C_1 L. \quad (7.4.18)$$

Second, one has that for all $k \geq 1$,

$$u_k(\beta) \leq u_{k+1}(\beta). \quad (7.4.19)$$

This second observation is a direct consequence of the spectral representation of the Ising model (see for instance [9]). Indeed, for each $x_\perp \in \mathbb{Z}^{d-1}$, one has the existence of a Borel measure $\mu_{x_\perp, \beta}$ on $[0, 1]$, such that the contribution of $x_\perp$ to $u_k(\beta)$ can be written,

$$2S_\beta((k, x_\perp)) - S_\beta((k+1, x_\perp)) - S_\beta((k-1, x_\perp)) = - \int_{[0,1]} \lambda^{k-1} (1-\lambda)^2 d\mu(\lambda), \quad (7.4.20)$$

and, similarly, the contribution of $x_\perp$ to $u_{k+1}(\beta)$ takes the form

$$2S_\beta((k+1, x_\perp)) - S_\beta((k+2, x_\perp)) - S_\beta((k, x_\perp)) = - \int_{[0,1]} \lambda^k (1-\lambda)^2 d\mu(\lambda). \quad (7.4.21)$$

Hence, we may write,

$$u_k(\beta) - u_{k+1}(\beta) = \sum_{x_\perp \in \mathbb{Z}^{d-1}} \int_{[0,1]} -\lambda^{k-1} (1-\lambda)^3 d\mu_{x_\perp, \beta}(\lambda) \leq 0. \quad (7.4.22)$$

With these two observations, we find that

$$\left| \sum_{k \geq CL+1} e^{ik/L} u_k(\beta) \right| \leq C_1 L \left(-u_{CL+1}(\beta) - u_{CL}(\beta) \right). \quad (7.4.23)$$

As a result, the first step will follow from this next proposition.

Proposition 7.4.9. *There exists $C_1 > 0$ such that for all $\beta < \beta_c$, for all $k \geq 1$,*

$$0 \leq -u_k(\beta) \leq \frac{C_1}{k}. \quad (7.4.24)$$

We write

$$u_k(\beta) = u_k^{(1)}(\beta) + u_k^{(2)}(\beta), \quad (7.4.25)$$

where

$$u_k^{(1)}(\beta) := \sum_{|x_\perp| \leq k} 2S_\beta((k, x_\perp)) - S_\beta((k-1, x_\perp)) - S_\beta((k+1, x_\perp)), \quad (7.4.26)$$

and

$$u_k^{(2)}(\beta) := \sum_{|x_\perp| > k} 2S_\beta((k, x_\perp)) - S_\beta((k-1, x_\perp)) - S_\beta((k+1, x_\perp)). \quad (7.4.27)$$

We start with bounding $u_k^{(1)}(\beta)$. This can be done using the spectral representation.

Lemma 7.4.10. *There exists $C_1 > 0$ such that, for $k \geq 1$,*

$$|u_k^{(1)}(\beta)| \leq \frac{C_1}{k}. \quad (7.4.28)$$

Proof. Using (7.4.20), we find that for $|x_\perp| \leq k$,

$$\left| 2S_\beta((k, x_\perp)) - S_\beta((k-1, x_\perp)) - S_\beta((k+1, x_\perp)) \right| \leq \frac{C_1}{k^2} \langle \sigma_0 \sigma_{(k/2)\mathbf{e}_1} \rangle_\beta \leq \frac{C_2}{k^d}. \quad (7.4.29)$$

□

We are left with the terms corresponding to $|x_\perp| > k$ and $k \geq CL$. By symmetry,

$$\begin{aligned} 0 \leq -u_k^{(2)}(\beta) &= \sum_{|x_\perp| > k} S_\beta((k-1, x_\perp)) + S_\beta((k+1, x_\perp)) - 2S_\beta((k, x_\perp)) \\ &= 2(d-1) \sum_{\substack{m > k \\ x^\# \in \mathbb{Z}^{d-2} \\ |x^\#| \leq m}} \left(S_\beta((k-1, m, x^\#)) - S_\beta((k, m, x^\#)) \right) \\ &\quad - \left(S_\beta((k, m, x^\#)) - S_\beta((k+1, m, x^\#)) \right). \end{aligned} \quad (7.4.30)$$

We rely on the inequalities obtained in [191].

Lemma 7.4.11. *Let $m > k$ and $x^\# \in \mathbb{Z}^{d-2}$. Then,*

$$S_\beta((k-1, m, x^\#)) - S_\beta((k, m, x^\#)) \leq S_\beta((k, m-1, x^\#)) - S_\beta((k, m, x^\#)). \quad (7.4.31)$$

Proof. We use [191, Theorem 1] with the diagonal plane $\mathcal{R}$ of coordinates $x_2 = x_1 - k + m$ to get

$$S_\beta((k-1, m, x^\#)) \leq S_\beta(\mathcal{R}((k-1, m, x^\#))) = S_\beta((k, m-1, x^\#)). \quad (7.4.32)$$

□

Lemma 7.4.12. *Let $m > k$ and $x^\# \in \mathbb{Z}^{d-2}$. Then,*

$$S_\beta((k, m, x^\#)) - S_\beta((k+1, m, x^\#)) \geq S_\beta((k, m+1, x^\#)) - S_\beta((k, m+2, x^\#)). \quad (7.4.33)$$

Proof. The proof follows from two successive applications of the gradient monotonicity property given by [191]. First, we prove that

$$\begin{aligned} S_\beta((k, m, x^\#)) - S_\beta((k+1, m, x^\#)) \\ \geq S_\beta((k-1, m+1, x^\#)) - S_\beta((k-1, m+2, x^\#)). \end{aligned} \quad (7.4.34)$$

This is done (see Figure 7.4) by choosing a reflection plane $\overline{\mathcal{R}}$ of equation $x_1 = k + 1/2$, and then $\mathcal{R}^*$ diagonal reflection plane of equation $x_2 = x_1 + (m + 1 - k)$. Technically, this case is not included in [191] where they only consider situations in which the two reflection planes are of the same type (i.e. both orthogonal to one of the coordinate axes, are both “diagonal”). However, the extension of their argument is easy.

An application of the correlation inequality of [191] then yields, for $x = (k, m, x^\#)$ as above,

$$\langle \sigma_0 \sigma_x \rangle_\beta - \langle \sigma_0 \sigma_{\overline{\mathcal{R}}(x)} \rangle_\beta \geq \langle \sigma_0 \sigma_{\mathcal{R}^*(x)} \rangle_\beta - \langle \sigma_0 \sigma_{\mathcal{R}^*(\overline{\mathcal{R}}(x))} \rangle_\beta, \quad (7.4.35)$$

which in turn implies (7.4.34).

Then, we apply this inequality once again, this time with the reflection planes $\overline{\mathcal{P}}$ and $\mathcal{P}^*$, of respective equations, $x_2 = m + 1 + 1/2$ and $x_1 = k - 1/2$, to get for $z = (k-1, m+1, x^\#)$,

$$\langle \sigma_0 \sigma_z \rangle_\beta - \langle \sigma_0 \sigma_{\overline{\mathcal{P}}(z)} \rangle_\beta \geq \langle \sigma_0 \sigma_{\mathcal{P}^*(z)} \rangle_\beta - \langle \sigma_0 \sigma_{\mathcal{P}^*(\overline{\mathcal{P}}(z))} \rangle_\beta, \quad (7.4.36)$$

which is equivalent to

$$S_\beta((k-1, m+1, x^\#)) - S_\beta((k-1, m+2, x^\#)) \geq S_\beta((k, m+1, x^\#)) - S_\beta((k, m+2, x^\#)). \quad (7.4.37)$$

This concludes the proof. □

What we proved above essentially says that a gradient in the $\mathbf{e}_1$ direction can be compared to a gradient in the $\mathbf{e}_2$ direction. When considering points $x \in \mathbb{Z}^d$ for which $|x| = x_2$, this allows us to obtain the following estimate.

Lemma 7.4.13. *There exists $C_1 > 0$ such that, for all $\beta \leq \beta_c$, for all $m > k$ and $|x^\#| \leq m$,*

$$\left(S_\beta((k-1, m, x^\#)) - S_\beta((k, m, x^\#)) \right) - \left(S_\beta((k, m, x^\#)) - S_\beta((k+1, m, x^\#)) \right) \leq \frac{C_1}{m^d}. \quad (7.4.38)$$

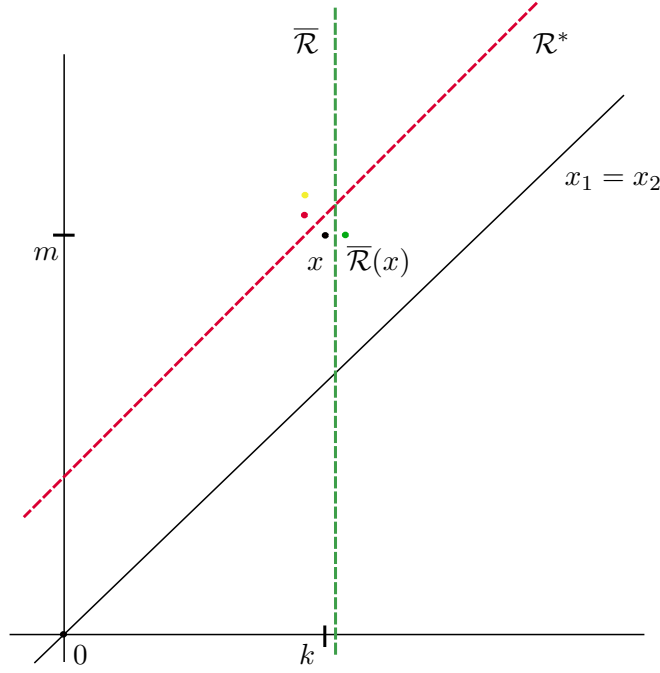


Figure 7.4: Illustration of the reflection planes used to prove (7.4.34). The reflection plane $\bar{\mathcal{R}}$ (resp. $\mathcal{R}^*$) is represented in the green (resp. red) dotted line. The points 0 and $x = (k, m, x^\#)$ are on the same side of each hyperplane. We represented $\bar{\mathcal{R}}(x)$ in green, $\mathcal{R}^*(x)$ in red, and $\mathcal{R}^*(\bar{\mathcal{R}}(x))$ in yellow.

Proof. We apply Lemmas 7.4.11 and 7.4.12 to get

$$\begin{aligned} & \left(S_\beta((k-1, m, x^\#)) - S_\beta((k, m, x^\#)) \right) - \left(S_\beta((k, m, x^\#)) - S_\beta((k+1, m, x^\#)) \right) \\ & \leq \left(S_\beta((k, m-1, x^\#)) - S_\beta((k, m, x^\#)) \right) - \left(S_\beta((k, m+1, x^\#)) - S_\beta((k, m+2, x^\#)) \right). \end{aligned} \quad (7.4.39)$$

Then, write

$$\begin{aligned} & \left(S_\beta((k, m-1, x^\#)) - S_\beta((k, m, x^\#)) \right) - \left(S_\beta((k, m+1, x^\#)) - S_\beta((k, m+2, x^\#)) \right) \\ & = \sum_{p=m-1}^m \left(S_\beta((k, p, x^\#)) - S_\beta((k, p+1, x^\#)) \right) - \left(S_\beta((k, p+1, x^\#)) - S_\beta((k, p+2, x^\#)) \right), \end{aligned} \quad (7.4.40)$$

and use once again the spectral representation to obtain the existence of $C > 0$ such that for $p \in \{m-1, m\}$ and $k, x^\#$ as above,

$$\left(S_\beta((k, p, x^\#)) - S_\beta((k, p+1, x^\#)) \right) - \left(S_\beta((k, p+1, x^\#)) - S_\beta((k, p+2, x^\#)) \right) \leq \frac{C}{m^d}. \quad (7.4.41)$$

□

Combining the above result with the upper bound on $|u_k^{(2)}(\beta)|$ given in (7.4.30), we obtain,

Lemma 7.4.14. *There exists $C_1 > 0$ such that, for $k \geq 1$,*

$$|u_k^{(2)}(\beta)| \leq \frac{C_1}{k}. \quad (7.4.42)$$

This proves Proposition 7.4.9. To obtain Proposition 7.4.4, we need to handle contributions coming from $x = (x_1, x_\perp) \in \mathbb{Z}^d$ with $|x_1| \leq CL$ and $|x_\perp| > CL$. This corresponds to

$$\sum_{0 \leq k \leq CL} e^{ik/L} v_k^{(CL)}(\beta), \quad (7.4.43)$$

where $v_k^{(p)}(\beta) := \sum_{|x_\perp| > p} (2S_\beta((k, x_\perp)) - S_\beta((k+1, x_\perp)) - S_\beta((k-1, x_\perp)))$.

Introducing $r_p(L) := \sum_{j=0}^p e^{ij/L}$, we have that

$$\sum_{0 \leq k \leq CL} e^{ik/L} v_k^{(p)}(\beta) = v_0^{(p)}(\beta) - v_1^{(p)}(\beta) + r_{CL}(L) v_{CL}^{(p)}(\beta) + \sum_{k=1}^{CL-1} r_k(L) (v_k^{(p)}(\beta) - v_{k+1}^{(p)}(\beta)). \quad (7.4.44)$$

Using the same methods as above, one could show that

$$v_{k+1}^{(p)}(\beta) \geq v_k^{(p)}(\beta). \quad (7.4.45)$$

Hence,

$$\begin{aligned} \left| \sum_{0 \leq k \leq CL} e^{ik/L} v_k^{(CL)}(\beta) \right| &\leq |v_0^{(CL)}(\beta)| + |v_1^{(CL)}(\beta)| \\ &+ C_1 L |v_{CL}^{(CL)}(\beta)| + C_1 L \sum_{k=1}^{CL-1} (v_{k+1}^{(CL)}(\beta) - v_k^{(CL)}(\beta)) \leq C_2 L \sup_{0 \leq p \leq CL} |v_p^{(CL)}(\beta)|. \end{aligned} \quad (7.4.46)$$

where we used that $|r_p(L)| \leq C_1 L$ for all p . The next lemma is sufficient to conclude.

Lemma 7.4.15. *There exists $C_1 > 0$ such that, for all $\beta < \beta_c$, for all $p \geq k$,*

$$v_k^{(p)} \leq \frac{C_1}{p}. \quad (7.4.47)$$

Proof. The proof follows the exact same lines of Lemma 7.4.14. $\square$

Putting all the pieces together, we obtained that there exists $C_1 > 0$ such that, for all $\beta < \beta_c$, for all $C, L \geq 1$,

$$\left| \sum_{x \notin \Lambda_{CL}} e^{ix_1/L} (2S_\beta(x) - S_\beta(x + \mathbf{e}_1) - S_\beta(x - \mathbf{e}_1)) \right| \leq \frac{C_1}{C}. \quad (7.4.48)$$

This yields Proposition 7.4.4.

7.A Fourier transform of Riesz potentials

For $x \in \mathbb{R}^d \setminus \{0\}$, let

$$G(x) = G(0, x) := \frac{\mathbf{c}_d}{\|x\|^{d-2}}, \quad (7.A.1)$$

where

$$\mathbf{c}_d := \frac{\pi^{-d/2}}{4} \Gamma\left(\frac{d}{2} - 1\right). \quad (7.A.2)$$

Proposition 7.A.1 (Fourier transform of G [115]). *One has, for $\xi \in \mathbb{R}^d \setminus \{0\}$,*

$$\hat{G}(\xi) = \|2\pi\xi\|^{-2}, \quad (7.A.3)$$

where

$$\hat{G}(\xi) := \int_{\mathbb{R}^d} e^{-2i\pi x \cdot \xi} G(x) dx. \quad (7.A.4)$$

In particular,

$$\int_{\mathbb{R}^d} e^{ix_1} \|x\|^{2-d} dx = \mathbf{c}_d^{-1} \hat{G}\left(-\frac{1}{2\pi}(1, 0, \dots, 0)\right) = \mathbf{c}_d^{-1}. \quad (7.A.5)$$

Chapter 8

The torus plateau for the high-dimensional Ising model

We consider the Ising model on a d -dimensional discrete torus of volume r^d , in dimensions $d > 4$ and for large r , in the vicinity of the infinite-volume critical point β_c . We prove that for $\beta = \beta_c - \text{const } r^{-d/2}$ (with a suitable constant) the susceptibility is bounded above and below by multiples of $r^{d/2}$, and that the two-point function has a “plateau” in the sense that it decays like $|x|^{-(d-2)}$ when $|x|$ is small relative to the volume but for larger $|x|$ it levels off to a constant value of order $r^{-d/2}$. We also prove that at $\beta = \beta_c - \text{const } r^{-d/2}$ the renormalised coupling constant is nonzero, which implies a non-Gaussian limit for the average spin.

This chapter is based on the preprint “*The torus plateau for the high-dimensional Ising model*” [180]. It is a joint work with Yucheng Liu and Gordon Slade.

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8.1 Introduction and main results

8.1.1 Introduction

The Ising model has been studied for a century as a fundamental example of a phase transition in statistical mechanics. We are interested here in the finite-size scaling of the Ising model on a finite box in the Euclidean lattice $\mathbb{Z}^d$ in dimensions $d > 4$, particularly in the case of periodic boundary conditions which make the box into a torus.

For $d \geq 1$ and an integer $r \geq 3$, we write Λ_r for the discrete box $[-r/2, r/2)^d \cap \mathbb{Z}^d$ of volume r^d , and write $\mathbb{T}_r = (\mathbb{Z}/r\mathbb{Z})^d$ for the d -dimensional discrete torus of period r . We study large r , and seek estimates that are uniform in large r . For convenience, we often identify a point $x \in \mathbb{T}_r$ with its representative in Λ_r .

Notation. We write $f \lesssim g$ to denote the existence of a constant $C > 0$ such that $f \leq Cg$, write $f \asymp g$ to mean $f \lesssim g$ and $g \lesssim f$, and write $f \sim g$ to denote $\lim f/g = 1$. To avoid division by zero, for $x \in \mathbb{R}^d$ we write $\|x\| = 1 \vee \|x\|_\infty$.

Let $\mathcal{G} = (V, E)$ be a finite graph with vertex set V and edge set E . The Ising model on $\mathcal{G}$ is a family of probability measures on spin configurations $\sigma : V \rightarrow \{-1, +1\}$, defined using the Hamiltonian

$$H^{\mathcal{G}}(\sigma) = - \sum_{xy \in E} \sigma_x \sigma_y. \quad (8.1.1)$$

The expectation of a function F of spin configurations, at inverse temperature $\beta > 0$, is defined by

$$\langle F \rangle_{\beta}^{\mathcal{G}} = \frac{1}{Z_{\beta}^{\mathcal{G}}} \sum_{\sigma} F(\sigma) e^{-\beta H^{\mathcal{G}}(\sigma)}, \quad (8.1.2)$$

where $Z_{\beta}^{\mathcal{G}} = \sum_{\sigma} e^{-\beta H^{\mathcal{G}}(\sigma)}$ is the partition function. Free boundary conditions (FBC) on the box correspond to $\mathcal{G} = \Lambda_r$ with edges connecting nearest-neighbours in Λ_r . Periodic boundary conditions (PBC) correspond to $\mathcal{G} = \mathbb{T}_r$ with edges connecting nearest-neighbours in $\mathbb{T}_r$, which include all edges of Λ_r and additional edges that join opposite sides of Λ_r . The *two-point functions* are defined by

$$\tau_{\beta}^{\Lambda_r}(x, y) = \langle \sigma_x \sigma_y \rangle_{\beta}^{\Lambda_r}, \quad \tau_{\beta}^{\mathbb{T}_r}(x, y) = \langle \sigma_x \sigma_y \rangle_{\beta}^{\mathbb{T}_r}, \quad (8.1.3)$$

and the finite-volume *susceptibilities* are defined by

$$\chi^{\Lambda_r}(\beta) = \sum_{x \in \Lambda_r} \tau_{\beta}^{\Lambda_r}(0, x), \quad \chi^{\mathbb{T}_r}(\beta) = \sum_{x \in \mathbb{T}_r} \tau_{\beta}^{\mathbb{T}_r}(0, x). \quad (8.1.4)$$

It follows from the Griffiths inequalities that $\tau_{\beta}^{\Lambda_r}$ is nonnegative and increasing in r , so that the infinite-volume two-point function $\tau_{\beta}(x, y) = \lim_{r \rightarrow \infty} \tau_{\beta}^{\Lambda_r}(x, y)$ and infinite-volume susceptibility $\chi(\beta) = \lim_{r \rightarrow \infty} \chi^{\Lambda_r}(\beta)$ exist in $[0, \infty]$. Moreover, it is known that there is a critical inverse temperature β_c , below which $\tau_{\beta}(x, y)$ decays exponentially as $|x - y| \rightarrow \infty$ and $\chi(\beta)$ is finite. The susceptibility is infinite at β_c , and from [4] we know that for $d > 4$ and $\beta < \beta_c$,

$$\chi(\beta) \asymp \frac{1}{\beta_c - \beta}. \quad (8.1.5)$$

An essential ingredient in our analysis is the very recent result from [83] that, for $d > 4$, there are positive constants c_0, C_0 such that for all $\beta \leq \beta_c$,

$$\tau_\beta(0, x) \leq \frac{C_0}{\|x\|^{d-2}} e^{-c_0(\beta_c - \beta)^{1/2} \|x\|_\infty} \quad (x \in \mathbb{Z}^d), \quad (8.1.6)$$

$$\tau_\beta(0, x) \geq \frac{c_0}{\|x\|^{d-2}} \quad (\|x\|_\infty \leq c_0(\beta_c - \beta)^{-1/2}). \quad (8.1.7)$$

For the spread-out Ising model in dimensions $d > 4$, or for the nearest-neighbour model in sufficiently high dimensions, more precise asymptotics for the critical two-point function have been proved using the lace expansion [216].

The influence of boundary conditions on finite-size critical behaviour in dimensions $d \geq 4$ has been studied extensively in the physics literature for Ising and related models, e.g., [184, 253, 185, 257, 35, 72, 73]. In particular, it has been observed numerically and via physics scaling arguments that, for $d > 4$,

$$\tau_{\beta_c}^{\Lambda_r}(0, x) \asymp \frac{1}{\|x\|^{d-2}}, \quad \tau_{\beta_c}^{\mathbb{T}_r}(0, x) \asymp \frac{1}{\|x\|^{d-2}} + \frac{1}{r^{d/2}}, \quad \chi^{\Lambda_r}(\beta_c) \asymp r^2, \quad \chi^{\mathbb{T}_r}(\beta_c) \asymp r^{d/2}. \quad (8.1.8)$$

(The claim for $\tau_{\beta_c}^{\Lambda_r}(0, x)$ is for x not too close to the boundary of the box.) The constant term $r^{-d/2}$ in $\tau_{\beta_c}^{\mathbb{T}_r}(0, x)$ is referred to as the *plateau*, and it is responsible for the larger susceptibility for PBC compared to FBC.

In recent years, the plateau for PBC has been proven to exist for simple random walk in dimensions $d > 2$ [230, 73], self-avoiding walk for $d > 4$ [230, 179], and percolation for $d > 6$ [151, 154]. A general theory of the effect of FBC vs PBC in the setting of the weakly-coupled hierarchical n -component $|\varphi|^4$ model in dimensions $d \geq 4$ was developed in [192, 207], and we comment further on this below.

For the Ising model in dimensions $d > 4$, the absence of a plateau for FBC was proven in [58]. For PBC, incomplete steps were taken in [206] towards proving existence of the plateau. In this chapter, we build on the method of [206] to prove (8.1.8) for PBC in dimensions $d > 4$. We also prove the non-vanishing of the renormalised coupling constant under PBC, indicating a non-Gaussian limit for the average spin. This can be contrasted with the Gaussian limit under FBC obtained in [4, 102, 9, 204].

We emphasise that we do not use the lace expansion in any way, and our results do not require a small parameter. The input from the lace expansion used to establish plateau theorems for self-avoiding walk and percolation in [151, 230, 154, 179] is replaced here by the near-critical bounds (8.1.6)–(8.1.7).

8.1.2 Main results

Our first theorem indicates that for β in an interval slightly below β_c , the torus and infinite-volume two-point functions are related by a plateau term $\chi(\beta)/r^d$.

Theorem 8.1.1. *Let $d > 4$. There are constants $c_i > 0$, depending on d but not on β , such that*

$$\tau_\beta^{\mathbb{T}_r}(0, x) \leq \tau_\beta(0, x) + c_1 \frac{\chi(\beta)}{r^d} \quad (8.1.9)$$

for all $r \geq 3$, all $\beta \in (0, \beta_c)$, and all $x \in \mathbb{T}_r$, and

$$\tau_{\beta}^{\mathbb{T}_r}(0, x) \geq \tau_{\beta}(0, x) + c_2 \frac{\chi(\beta)}{r^d} \quad (8.1.10)$$

for all $r \geq 3$, all $\beta \in [\beta_c - c_3 r^{-2}, \beta_c - c_4 r^{-d/2}]$ and all $\|x\|_{\infty} \geq c_5$. Moreover, the constant c_3 is given by $c_3 = \frac{4}{9} c_0^2$ with c_0 the constant of (8.1.7).

Note that for the lower bound, the interval of β values is nonempty when r is sufficiently large, since $d > 4$. In this case, we write

$$\beta_* = \beta_*(r) = \beta_c - c_4 r^{-d/2}. \quad (8.1.11)$$

At β_* , we have $\chi(\beta_*) \asymp r^{d/2}$ by (8.1.5), so (8.1.9)–(8.1.10) identify a plateau term in $\tau_{\beta_*}^{\mathbb{T}_r}$ of order $r^{-d/2}$. The monotonicity in β then implies that the plateau term persists for larger β , in particular for β_c .

Corollary 8.1.2. *Let $d > 4$ and r be large. Then*

$$\tau_{\beta_*}^{\mathbb{T}_r}(0, x) \asymp \|x\|^{-(d-2)} + r^{-d/2} \quad (8.1.12)$$

for all $x \in \mathbb{T}_r$. In particular, $\tau_{\beta}^{\mathbb{T}_r}(0, x) \gtrsim \|x\|^{-(d-2)} + r^{-d/2}$ for all $\beta \geq \beta_*$.

Summation of (8.1.12) over $x \in \mathbb{T}_r$ immediately gives the following bounds on the susceptibility, because $\sum_{x \in \mathbb{T}_r} \|x\|^{-(d-2)} = O(r^2) = o(r^{d/2})$ in dimensions $d > 4$.

Corollary 8.1.3. *For $d > 4$ and r large,*

$$\chi^{\mathbb{T}_r}(\beta_*) \asymp r^{d/2}. \quad (8.1.13)$$

In particular, $\chi^{\mathbb{T}_r}(\beta) \gtrsim r^{d/2}$ for all $\beta \geq \beta_*$.

Let $S_r = r^{-d} \sum_{x \in \mathbb{T}_r} \sigma_x$ denote the average spin on the torus. We define the *renormalised coupling constant* by

$$g^{\mathbb{T}_r}(\beta) = - \frac{\langle S_r^4 \rangle_{\beta}^{\mathbb{T}_r} - 3(\langle S_r^2 \rangle_{\beta}^{\mathbb{T}_r})^2}{(\langle S_r^2 \rangle_{\beta}^{\mathbb{T}_r})^2}. \quad (8.1.14)$$

By the Lebowitz inequality, the above numerator is non-positive, so $g^{\mathbb{T}_r}(\beta) \geq 0$. By the second Griffiths inequality, $\langle S_r^4 \rangle_{\beta}^{\mathbb{T}_r} \geq (\langle S_r^2 \rangle_{\beta}^{\mathbb{T}_r})^2$, so $g^{\mathbb{T}_r}(\beta) \leq 2$. The following theorem shows that $g^{\mathbb{T}_r}(\beta_*)$ is bounded away from zero. This indicates a non-Gaussian limit for the average field at $\beta = \beta_*$: for a Gaussian random variable the numerator in (8.1.14) is zero. This is in contrast to the situation at (and below) β_c with free boundary conditions [4, 102, 9, 204], where the limit is Gaussian. We do not have a proof that $g^{\mathbb{T}_r}$ is increasing in β , but see (8.1.20) below.

Theorem 8.1.4. *Let $d > 4$. There is a constant $c_g > 0$ such that for all r large,*

$$0 < c_g \leq g^{\mathbb{T}_r}(\beta_*) \leq 2. \quad (8.1.15)$$

Our lower bounds on the susceptibility in Corollary 8.1.3, on the two-point function in Corollary 8.1.2, and on the renormalised coupling constant in Theorem 8.1.4, are claimed in [206], but the arguments put forth in [206] do not constitute a proof. Nevertheless, a useful strategy is proposed in [206], and we build upon that strategy in our work.

8.1.3 Conjectured asymptotic formulas

A precise and detailed account of the effect of boundary conditions on finite-size scaling is given in [192, 207], via a rigorous renormalisation group analysis. The results of [192, 207] are proved for the weakly-coupled n -component $|\varphi|^4$ model on the d -dimensional hierarchical lattice for $d \geq 4$ and for all $n \geq 1$, but, on the basis of universality, the results obtained there for the 1-component model are conjectured to apply also to all 1-component spin models in the universality class of the nearest-neighbour Ising model. An antecedent of the conjectures goes back to [56] (see also [258, Chapter 32]). The conjectures present predictions for a specific and universal non-Gaussian limit for the average field, and precise asymptotic amplitudes for the susceptibility, the plateau, and the renormalised coupling constant, throughout a critical window of width $r^{-d/2}$ (with a logarithmic correction for $d = 4$) centred at β_c . We define the *window scale* and the *large-field scale*, respectively, by

$$w_r = \begin{cases} a_4(\log r)^{-1/6}r^{-2} & (d = 4) \\ a_d r^{-d/2} & (d > 4), \end{cases} \quad h_r = \begin{cases} b_4(\log r)^{1/4}r^{-1} & (d = 4) \\ b_d r^{-d/4} & (d > 4), \end{cases} \quad (8.1.16)$$

with suitably chosen constants $a_d, b_d > 0$. We also define the *universal profile* $f : \mathbb{R} \rightarrow (0, \infty)$ by

$$f(s) = \int_{\mathbb{R}} x^2 d\sigma_s, \quad d\sigma_s = \frac{e^{-\frac{1}{4}x^4 + \frac{s}{2}x^2} dx}{\int_{\mathbb{R}} e^{-\frac{1}{4}x^4 + \frac{s}{2}x^2} dx}. \quad (8.1.17)$$

The function f is strictly monotone increasing, and satisfies $f(s) \sim |s|^{-1}$ as $s \rightarrow -\infty$ and $f(s) \sim s$ as $s \rightarrow \infty$ (see [192, Section 1.5.2] where $f_1(s) = f(-s)$). The following conjecture proposes a precise non-Gaussian limit for the average field S_r , and a precise role for the universal profile f in the critical finite-size scaling of the susceptibility, the two-point function, and the renormalised coupling constant for the Ising model. The conjecture is a special case ($n = 1$) of conjectures from [192, 207] for n -component models for all $n \geq 0$ ($n = 0$ corresponds to the self-avoiding walk). The profile f also arises in the Curie–Weiss model; see [87, Theorem 3.2].

Conjecture 2 (Non-Gaussian limit and universal profile). *For $d \geq 4$, $s \in \mathbb{R}$, and $\beta = \beta_c + sw_r$, as $r \rightarrow \infty$ the rescaled average field converges in distribution to a real-valued random variable with the non-Gaussian distribution $d\sigma_s$:*

$$h_r^{-1} S_r \Rightarrow d\sigma_s \quad \text{under } \langle \cdot \rangle_{\beta_c + sw_r}^{\mathbb{T}_r}. \quad (8.1.18)$$

Moments of $h_r^{-1} S_r$ also converge to moments of $d\sigma_s$. In particular,

$$\chi^{\mathbb{T}_r}(\beta_c + sw_r) = r^d \langle S_r^2 \rangle_{\beta_c + sw_r}^{\mathbb{T}_r} \sim r^d h_r^2 f(s) = b_d^2 f(s) \times \begin{cases} (\log r)^{1/2} r^2 & (d = 4) \\ r^{d/2} & (d > 4), \end{cases} \quad (8.1.19)$$

$$\lim_{r \rightarrow \infty} g^{\mathbb{T}_r}(\beta_c + sw_r) = 3 - \frac{\int_{\mathbb{R}} x^4 d\sigma_s}{(\int_{\mathbb{R}} x^2 d\sigma_s)^2}. \quad (8.1.20)$$

For $x \in \mathbb{T}_r$, the plateau is asymptotically the susceptibility divided by the volume:

$$\tau_{\beta_c + sw_r}^{\mathbb{T}_r}(0, x) - \tau_{\beta_c}(0, x) \sim h_r^2 f(s) = b_d^2 f(s) \times \begin{cases} (\log r)^{1/2} r^{-2} & (d = 4) \\ r^{-d/2} & (d > 4). \end{cases} \quad (8.1.21)$$

As observed in [192, Section 1.5.3], the right-hand side of (8.1.20) is strictly monotone increasing, and has limits 0 and 2 as $s \rightarrow -\infty$ and $s \rightarrow +\infty$ respectively. Thus, assuming (8.1.20), the renormalised coupling constant varies monotonically over its possible range $[0, 2]$ as the window is traversed. At the infinite-volume critical point, (8.1.20) gives

$$\lim_{r \rightarrow \infty} g^{\mathbb{T}_r}(\beta_c) = 3 - 4 \left(\frac{\Gamma(5/4)}{\Gamma(3/4)} \right)^2 = 0.81156 \dots \quad (d \geq 4). \quad (8.1.22)$$

The results of [192, 207] further suggest the following conjecture for the Ising model with FBC in dimensions $d \geq 4$.

Conjecture 3 (Free boundary conditions). *There is an effective critical point $\beta_{c,r}$ for the Ising model with free boundary conditions on the box Λ_r , which obeys*

$$\beta_{c,r} = \beta_c + v_r, \quad v_r \asymp \begin{cases} (\log r)^{1/3} r^{-2} & (d = 4) \\ r^{-2} & (d > 4), \end{cases} \quad (8.1.23)$$

such that the statements of Conjecture 2 holds as stated when β_c is replaced by $\beta_{c,r}$. In particular, we conjecture that the plateau phenomenon also occurs for free boundary conditions at the effective critical point.

The shift scale v_r exceeds the window scale w_r , so β_c is *not* inside the conjectured scaling window for FBC. This is consistent with theorems proving a Gaussian limit at β_c under FBC [4, 102, 9, 192, 207].

Closely related conjectures for percolation are discussed in Appendix 8.C. Our analysis is based on the random current representation for the Ising model, which gives a geometric representation for correlation functions with similarities to bond percolation. In view of this parallel with percolation, the following intuitive picture of the percolation plateau provides intuition more generally.

Origin of the plateau. For spread-out independent Bernoulli bond percolation on the torus $\mathbb{T}_r$ in dimensions $d > 6$, it is proved in [154] that, in a window of width $r^{-d/3}$ containing the infinite-volume critical point p_c , the two-point function behaves as $|x|^{-(d-2)} + r^{-2d/3}$ and the susceptibility (expected cluster size) behaves as $r^{d/3}$. Also, it has been proved for sufficiently high dimensions (presumably true for all $d > 6$) that at p_c the largest cluster on $\mathbb{T}_r$ is of order $r^{2d/3}$, whereas with FBC the largest cluster has size r^4 ; see [143, Theorem 13.5], [143, Theorem 13.22], [5]. For PBC, if we assume that two points 0 and x in $\mathbb{T}_r$ are connected to a largest cluster (in particular connected to each other) with probability $(r^{2d/3}/r^d)^2$, we find a plateau term of the correct order $r^{-2d/3}$. For FBC, a similar computation gives a probability $(r^4/r^d)^2 = r^{-(d-2)+(d-6)}$, which is negligible compared to $\tau_{p_c}(0, x) \asymp |x|^{-(d-2)}$. This rough computation is consistent with a plateau for PBC and no plateau for FBC.

The equality of powers $r^{2d/3}$ and r^4 when $d = 6$ indicates a special role for dimension 6. No plateau is predicted for $d < 6$, or indeed for other models such as spin systems below their upper critical dimension. For percolation in dimensions below 6, hyperscaling plays a role [121]. Hyperscaling reflects the dimension-dependent manner in which critical clusters are constrained to fit into space. The distinction between a plateau above the critical dimension, and no plateau below the critical dimension, may be understood intuitively as due to the

fact that in high dimensions a critical cluster that “wraps around” discovers new vertices, whereas in low dimensions the wrapping mostly encounters vertices already present in the cluster without wrapping. In [72], the role of wrapping above the upper critical dimension is studied numerically in related settings.

8.1.4 Structure of proof

We now state five propositions and prove our main results subject to these propositions. The proof of Theorem 8.1.1 mirrors the general strategy used for self-avoiding walk in [230, 179] and for percolation in [154]. However, there are model-specific elements in the proof of these propositions. For the Ising model, we will use (8.1.6)–(8.1.7) as well as the random current representation [120, 4].

We observe that τ_β and $\tau_\beta^{\mathbb{T}_r}$ are translation invariant (but $\tau_\beta^{\Lambda_r}$ is not), so we write $\tau_\beta(x) = \tau_\beta(0, x)$ and $\tau_\beta^{\mathbb{T}_r}(x) = \tau_\beta^{\mathbb{T}_r}(0, x)$ from now on. For $x \in \mathbb{Z}d$, we define

$$T_\beta^{(r)}(x) = \sum_{u \in \mathbb{Z}d} \tau_\beta(x + ru). \quad (8.1.24)$$

Proposition 8.1.5. *Let $d \geq 1$, $r \geq 3$, and $x \in \mathbb{T}_r$. Then $\tau_\beta^{\mathbb{T}_r}(x) \leq T_\beta^{(r)}(x)$.*

Proposition 8.1.5 suggests decomposing $\tau_\beta^{\mathbb{T}_r}(x)$ as

$$\tau_\beta^{\mathbb{T}_r}(x) = T_\beta^{(r)}(x) - (T_\beta^{(r)}(x) - \tau_\beta^{\mathbb{T}_r}(x)). \quad (8.1.25)$$

We estimate the two terms on the right-hand side separately as follows. Recall that $\chi(\beta)$ denotes the $\mathbb{Z}d$ susceptibility.

Proposition 8.1.6. *Let $d > 4$. There is a constant $c_1 > 0$ such that for all $r \geq 3$ and $\beta < \beta_c$,*

$$T_\beta^{(r)}(x) \leq \tau_\beta(x) + c_1 \frac{\chi(\beta)}{r^d} \quad (x \in \mathbb{T}_r). \quad (8.1.26)$$

Proposition 8.1.7. *Let $d > 4$ and $c_3 = \frac{4}{9}c_0^2$, where c_0 is the constant of (8.1.7). There is a constant $c_T > 0$ such that for all $r \geq 3$ and all $\beta \in [\beta_c - c_3r^{-2}, \beta_c)$,*

$$T_\beta^{(r)}(x) \geq \tau_\beta(x) + c_T \frac{\chi(\beta)}{r^d} \quad (x \in \mathbb{T}_r). \quad (8.1.27)$$

Proposition 8.1.8. *Let $d > 4$. Let c_T be the constant of Proposition 8.1.7. There are constants $c_4, c_5 > 0$ such that for all $r \geq 3$ and $\beta \leq \beta_c - c_4r^{-d/2}$,*

$$T_\beta^{(r)}(x) - \tau_\beta^{\mathbb{T}_r}(x) \leq \frac{1}{2}c_T \frac{\chi(\beta)}{r^d} \quad (x \in \mathbb{T}_r, \|x\|_\infty \geq c_5). \quad (8.1.28)$$

Proof of Theorem 8.1.1. The upper bound follows immediately from Propositions 8.1.5 and 8.1.6. Recall that $\beta_* = \beta_c - c_4r^{-d/2}$. For the lower bound, let $\beta \in [\beta_c - c_3r^{-2}, \beta_*]$ and $\|x\|_\infty \geq c_5$. Then by (8.1.25), and by Propositions 8.1.7 and 8.1.8,

$$\tau_\beta^{\mathbb{T}_r}(x) \geq \tau_\beta(x) + \left(1 - \frac{1}{2}\right) c_T \frac{\chi(\beta)}{r^d}. \quad (8.1.29)$$

This gives the desired result with $c_2 = \frac{1}{2}c_T$. $\square$

Proof of Corollary 8.1.2. The upper bound follows from (8.1.9), $\tau_\beta(x) \lesssim \|x\|^{-(d-2)}$ by (8.1.6), and the fact that $\chi(\beta_*) \asymp r^{-d/2}$ from (8.1.5).

For the lower bound, when $\|x\|_\infty \geq c_5$, we first observe that the lower bound (8.1.7) applies at $\beta = \beta_*$ provided that $r/2 \leq c_0(\beta_c - \beta_*)^{-1/2}$, i.e. $r^{(d-4)/4} \geq (2c_0)^{-1}c_4^{1/2}$. It then follows from (8.1.10) and $\chi(\beta) \geq \chi(\beta_*) \gtrsim r^{d/2}$ that

$$\tau_\beta^{\mathbb{T}_r}(x) \geq \tau_{\beta_*}(x) + c_2 \frac{\chi(\beta_*)}{r^d} \gtrsim \|x\|^{-(d-2)} + r^{-d/2}, \quad (8.1.30)$$

as desired. When $\|x\|_\infty \leq c_5$, we let $m_5 = \min_{\|x\|_\infty \leq c_5} \tau_{\beta_c/2}^{\Lambda_{2c_5}}(0, x) > 0$. By the second Griffiths inequality, when r is large we have

$$\tau_{\beta_*}^{\mathbb{T}_r}(x) \geq \tau_{\beta_*}^{\Lambda_{2c_5}}(0, x) \geq m_5 \geq \frac{m_5}{2} \frac{1}{\|x\|^{d-2}} + \frac{m_5}{2} \frac{1}{r^{d/2}}, \quad (8.1.31)$$

which completes the proof. $\square$

The proof of Theorem 8.1.4 uses the following proposition, which we prove in Appendix 8.B. The torus Ursell function $U_{4,\beta}^{\mathbb{T}_r}$ is defined by

$$U_{4,\beta}^{\mathbb{T}_r}(0, x, y, z) = \langle \sigma_0 \sigma_x \sigma_y \sigma_z \rangle_\beta^{\mathbb{T}_r} - \langle \sigma_0 \sigma_x \rangle_\beta^{\mathbb{T}_r} \langle \sigma_y \sigma_z \rangle_\beta^{\mathbb{T}_r} - \langle \sigma_0 \sigma_y \rangle_\beta^{\mathbb{T}_r} \langle \sigma_x \sigma_z \rangle_\beta^{\mathbb{T}_r} - \langle \sigma_0 \sigma_z \rangle_\beta^{\mathbb{T}_r} \langle \sigma_x \sigma_y \rangle_\beta^{\mathbb{T}_r}. \quad (8.1.32)$$

By the Lebowitz and Griffiths inequalities, $U_{4,\beta}^{\mathbb{T}_r}(0, x, y, z) \in [-2, 0]$. The torus *bubble diagram* is defined by

$$\mathbf{B}(\beta) = \sum_{x \in \mathbb{T}_r} \tau_\beta^{\mathbb{T}_r}(x)^2. \quad (8.1.33)$$

Proposition 8.1.9. *Let $d \geq 1$. For all $\beta > 0$ and for all $r \geq 3$,*

$$\sum_{x,y,z \in \mathbb{T}_r} |U_{4,\beta}^{\mathbb{T}_r}(0, x, y, z)| \geq \frac{1}{128d^2} \cdot \min \left\{ \beta(\chi^{\mathbb{T}_r})^2 \frac{\partial \chi^{\mathbb{T}_r}}{\partial \beta}, \frac{1}{\mathbf{B}} \left(\frac{\partial \chi^{\mathbb{T}_r}}{\partial \beta} \right)^2 \right\}. \quad (8.1.34)$$

Proof of Theorem 8.1.4. All quantities in this proof are torus quantities, so to simplify the notation we drop the label $\mathbb{T}_r$ everywhere. By a direct computation using $S_r = r^{-d} \sum_{x \in \mathbb{T}_r} \sigma_x$,

$$g(\beta) = -\frac{\langle S_r^4 \rangle_\beta - 3(\langle S_r^2 \rangle_\beta)^2}{(\langle S_r^2 \rangle_\beta)^2} = \frac{1}{\chi(\beta)^2 r^d} \sum_{x,y,z} |U_{4,\beta}(0, x, y, z)|. \quad (8.1.35)$$

We use Proposition 8.1.9 to bound from below the sum on the right-hand side. A lower bound on the derivative of the susceptibility, essentially due to [14] (see [95, (12.50)]), states that

$$\beta \frac{\partial \chi}{\partial \beta} \geq \frac{2d\chi^2}{1 + (2d)^2 \mathbf{B}} \left[1 - \frac{4d\mathbf{B}}{\chi} - \frac{\mathbf{B}}{\chi^2} \right]. \quad (8.1.36)$$

It follows from the upper bound of Corollary 8.1.2 that $\mathbf{B}(\beta_*) \lesssim 1$. Combined with $\chi(\beta_*) \asymp r^{d/2}$ from Corollary 8.1.3, we get

$$g(\beta_*) \gtrsim \frac{1}{r^d} \chi(\beta_*)^2 \asymp 1, \quad (8.1.37)$$

and the proof is complete. $\square$

8.2 Proof of Propositions 8.1.5–8.1.8

We now prove Propositions 8.1.5–8.1.8. The proofs of Propositions 8.1.5 and 8.1.8 depend on new diagrammatic estimates that we state as Theorem 8.2.1. We begin with the proofs of Propositions 8.1.6 and 8.1.7, which are easier.

8.2.1 Proof of Propositions 8.1.6 and 8.1.7

These two propositions provide upper and lower bounds on

$$T_\beta^{(r)}(x) = \sum_{u \in \mathbb{Z}^d} \tau_\beta(x + ru), \quad (8.2.1)$$

respectively. It is useful to observe that, since $\|x\|_\infty \leq \frac{r}{2}$ for all $x \in \mathbb{T}_r$, for all $0 \neq u \in \mathbb{Z}^d$ we have

$$\|x + ru\|_\infty \leq \|ru\|_\infty + \frac{r}{2} \leq \frac{3}{2}r\|u\|_\infty, \quad (8.2.2)$$

$$\|x + ru\|_\infty \geq \|ru\|_\infty - \frac{r}{2} \geq \frac{1}{2}r\|u\|_\infty. \quad (8.2.3)$$

Proposition 8.1.6 follows simply from (8.2.3) and the upper bound (8.1.6) on τ_β , and we defer its proof to Appendix 8.A where it is included in the elementary Lemma 8.A.2. We prove Proposition 8.1.7 here, using (8.2.2) and the lower bound (8.1.7) on τ_β .

Proof of Proposition 8.1.7. For the proof, we wish to apply the lower bound (8.1.7) to all $x + ru$ with $x \in \Lambda_r$ and $\|u\|_\infty \leq M$, with a well-chosen $M \geq 1$. Since (8.1.7) holds when $\|x + ru\|_\infty \leq c_0(\beta_c - \beta)^{-1/2}$, it suffices to have

$$M \geq 1, \quad \frac{r}{2} + rM \leq c_0(\beta_c - \beta)^{-1/2}. \quad (8.2.4)$$

We try $M = M(\beta, r) = \mu c_0(\beta_c - \beta)^{-1/2} r^{-1}$. The two inequalities of (8.2.4) then become

$$r^2(\beta_c - \beta) \leq \mu^2 c_0^2, \quad r^2(\beta_c - \beta) \leq 4(1 - \mu)^2 c_0^2. \quad (8.2.5)$$

The choice $\mu = 2/3$ makes the two right-hand sides equal, and in that case both inequalities assert $r^2(\beta_c - \beta) \leq \frac{4}{9}c_0^2$. Thus, if we choose $c_3 = \frac{4}{9}c_0^2$ and require $\beta \in [\beta_c - c_3 r^{-2}, \beta_c)$, then by using (8.1.7) and (8.2.2) in the definition of $T_\beta^{(r)}(x)$ we obtain

$$T_\beta^{(r)}(x) - \tau_\beta(x) \gtrsim \sum_{1 \leq \|u\|_\infty \leq M} \frac{1}{\|x + ru\|_\infty^{d-2}} \geq \sum_{1 \leq \|u\|_\infty \leq M} \frac{1}{(\frac{3}{2}r\|u\|_\infty)^{d-2}} \asymp \frac{M^2}{r^{d-2}}. \quad (8.2.6)$$

By the definition of M and the fact that $\chi(\beta) \asymp (\beta_c - \beta)^{-1}$ (by (8.1.5)), the right-hand side is bounded from below by a multiple of $r^{-d}\chi(\beta)$. This completes the proof. $\square$

8.2.2 Diagrammatic estimates

The proofs of Propositions 8.1.5 and 8.1.8 rely on the following theorem, whose proof is our main work. The proof of Theorem 8.2.1 is based on the random current representation and a new coupling of the Ising model on the torus and on $\mathbb{Z}^d$, and is deferred to Section 8.3. For the statement of Theorem 8.2.1, given $x, x' \in \mathbb{Z}^d$ we write $x \cong x'$ if $x - x' \in r\mathbb{Z}^d$, and we write πx for the unique point in Λ_r (which we identify with $\mathbb{T}_r$) such that $\pi x \cong x$. The bound (8.2.7) repeats the statement of Proposition 8.1.5.

Theorem 8.2.1. *Let $d \geq 1$. For any $x \in \mathbb{T}_r$,*

$$\tau_{\beta}^{\mathbb{T}_r}(x) \leq T_{\beta}^{(r)}(x), \quad (8.2.7)$$

$$T_{\beta}^{(r)}(x) - \tau_{\beta}^{\mathbb{T}_r}(x) \leq \sum_{x' \cong x} \sum_y \sum_{\substack{y' \cong y \\ y' \neq y}} \sum_z \tau_{\beta}(z) \tau_{\beta}(y - z) \tau_{\beta}(y' - z) \tau_{\beta}^{\mathbb{T}_r}(\pi y' - \pi z) \tau_{\beta}(x' - y), \quad (8.2.8)$$

where x', y, y', z range over $\mathbb{Z}^d$ with the indicated restrictions.

The following corollary, which gives an upper bound that simplifies the presentation of (8.2.8), will be important. We define the torus $\star$ convolution for functions $f, g : \mathbb{T}_r \rightarrow \mathbb{R}$ (periodic functions on $\mathbb{Z}^d$) by

$$(f \star g)(x) = \sum_{y \in \mathbb{T}_r} f(y)g(x - y) \quad (x \in \mathbb{T}_r). \quad (8.2.9)$$

We also use the $\mathbb{Z}^d$ convolution $(f * g)(x) = \sum_{y \in \mathbb{Z}^d} f(y)g(x - y)$ for absolutely summable functions on $\mathbb{Z}^d$, so it is important to mark the difference between $\star$ and $*$.

Corollary 8.2.2. *Let $d \geq 1$. For any $x \in \mathbb{T}_r$,*

$$T_{\beta}^{(r)}(x) - \tau_{\beta}^{\mathbb{T}_r}(x) \leq 2 \left(T_{\beta}^{(r)} \star (T_{\beta}^{(r)})^2 [T_{\beta}^{(r)} - (\tau_{\beta} \circ \pi)] \star T_{\beta}^{(r)} \right)(x). \quad (8.2.10)$$

Proof. We use (8.2.7) and the periodicity of $T_{\beta}^{(r)}$ to get $\tau_{\beta}^{\mathbb{T}_r}(\pi y' - \pi z) \leq T_{\beta}^{(r)}(y - z)$. We also rewrite the sums over x', y' using the definition of $T_{\beta}^{(r)}$. These allow us to bound the right-hand side of (8.2.8) by

$$\sum_y \sum_z \tau_{\beta}(z) \tau_{\beta}(y - z) (T_{\beta}^{(r)} - \tau_{\beta})(y - z) T_{\beta}^{(r)}(y - z) T_{\beta}^{(r)}(x - y). \quad (8.2.11)$$

Since every point in $\mathbb{Z}^d$ can be written uniquely in the form $\Lambda_r + r\mathbb{Z}^d$, we can write

$$x = \check{x}, \quad y = \check{y} + ru, \quad z = \check{z} + rv \quad (\check{x}, \check{y}, \check{z} \in \Lambda_r, u, v \in \mathbb{Z}^d), \quad (8.2.12)$$

so the two sums become four sums over $\check{y}, \check{z}, u, v$. Since $T_{\beta}^{(r)}$ is periodic, we have $T_{\beta}^{(r)}(y - z) = T_{\beta}^{(r)}(\check{y} - \check{z})$ and $T_{\beta}^{(r)}(x - y) = T_{\beta}^{(r)}(\check{x} - \check{y})$, independent of u . Hence, the sum over u gives the upper bound

$$\sum_u (\tau_{\beta} T_{\beta}^{(r)} - \tau_{\beta}^2)(\check{y} - \check{z} + r(u - v)) \leq T_{\beta}^{(r)}(\pi(\check{y} - \check{z}))^2 - \tau_{\beta}(\pi(\check{y} - \check{z}))^2, \quad (8.2.13)$$

where in the subtracted term we kept only one term in the sum as an upper bound. We further bound the above difference of squares by $2T_{\beta}^{(r)}[T_{\beta}^{(r)} - (\tau_{\beta} \circ \pi)]$. Finally, we complete the proof with the observation that $\sum_v \tau_{\beta}(\check{z} + rv) = T_{\beta}^{(r)}(\check{z})$. $\square$

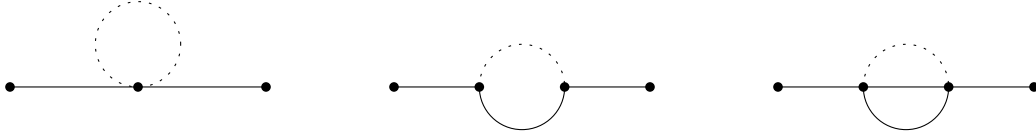


Figure 8.1: From left to right, diagrammatic representations of the upper bounds for the self-avoiding walk in (8.2.14), for percolation in (8.2.15), and for the Ising model in (8.2.16). The dotted line corresponds to $T - \tau \circ \pi$, which produces a factor χ/r^d for each model.

Remark 8.2.3. It is interesting to compare the diagrammatic upper bound (8.2.10) with diagrams in related models. The counterpart of $T_\beta^{(r)} - \tau_\beta^{\mathbb{T}_r}$ is denoted by $\psi - \psi^{\mathbb{T}}$ in [230, 154, 179]. For weakly or strictly self-avoiding walk, the bounds [230, (6.36)] and [179, (4.20)] can be written as

$$T \star [\delta \cdot (T - \tau \circ \pi)] \star T, \quad (8.2.14)$$

with δ the Kronecker delta. For percolation, the bounds of [154, (4.35), (4.42)] can be combined into the upper bound

$$T \star [T \cdot (T - \tau \circ \pi)] \star T. \quad (8.2.15)$$

For Ising, we found the bound

$$T \star [T^2 \cdot (T - \tau \circ \pi)] \star T. \quad (8.2.16)$$

The T^2 in the Ising bound, compared to the T in percolation, is responsible for lowering the critical dimension from 6 to 4. The three upper bounds are depicted in Figure 8.1.

Proof of Proposition 8.1.8. It follows from Corollary 8.2.2 and Proposition 8.1.6 that

$$T_\beta^{(r)}(x) - \tau_\beta^{\mathbb{T}_r}(x) \lesssim \frac{\chi(\beta)}{r^d} (T_\beta^{(r)} \star (T_\beta^{(r)})^2 \star T_\beta^{(r)})(x) \quad (x \in \mathbb{T}_r). \quad (8.2.17)$$

By Lemma 8.A.2, there is a constant $C > 0$ such that

$$(T_\beta^{(r)} \star (T_\beta^{(r)})^2 \star T_\beta^{(r)})(x) \leq (\tau_\beta \star \tau_\beta^2 \star \tau_\beta)(x) + C \frac{\chi(\beta)^2}{r^d} + C \left(\frac{\chi(\beta)^2}{r^d} \right)^2. \quad (8.2.18)$$

It follows from $\tau_\beta(x) \leq \tau_{\beta_c}(x) \lesssim \|x\|^{-(d-2)}$ and a standard convolution estimate [133, Proposition 1.7] that $(\tau_\beta \star \tau_\beta^2 \star \tau_\beta)(x) \lesssim \|x\|^{-(d-4)}$ when $d > 4$, so this term can be made as small as desired by taking $\|x\|_\infty \geq c_5$ with c_5 large enough, uniformly in $\beta \leq \beta_c$. Also, for $\beta \leq \beta_c - c_4 r^{-d/2}$, by (8.1.5) we have $r^{-d/2} \chi(\beta) \lesssim c_4^{-1}$, and we can make this as small as desired by taking c_4 large enough. With (8.2.18), this shows that the multiplier of $\chi(\beta)/r^d$ in (8.2.17) can be made as small as desired by taking $\|x\|_\infty \geq c_5$ and c_4, c_5 large. In particular, it can be made smaller than $\frac{1}{2}c_T$, which completes the proof. $\square$

8.3 Coupling and switching: Proof of Theorem 8.2.1

In this section, we introduce a new coupling between Ising models on $\mathbb{T}_r$ and on large finite subsets of $\mathbb{Z}^d$. The coupling uses the random current representation (see [76] for an introduction) and a percolation exploration process. To derive useful bounds, we develop a new switching lemma between currents on $\mathbb{T}_r$ and on large finite subsets of $\mathbb{Z}^d$, in Lemma 8.3.4.

8.3.1 Random currents and switching lemma

Definition 8.3.1. Let $\mathcal{G} = (V, E)$ be a finite graph. A *current* $\mathbf{n}$ on $\mathcal{G}$ is a function from E to $\mathbb{N}_0 = \{0, 1, \dots\}$. We write $\Omega_{\mathcal{G}}$ for the set of all currents on $\mathcal{G}$. The set of *sources* of $\mathbf{n}$, denoted by $\partial\mathbf{n}$, is defined as

$$\partial\mathbf{n} = \left\{ x \in V : \sum_{y: xy \in E} \mathbf{n}_{xy} \text{ is odd} \right\}. \quad (8.3.1)$$

It is often useful to interpret a current $\mathbf{n}$ as a multigraph on V with $\mathbf{n}_{xy}$ distinct edges connecting x and y for all $xy \in E$. A sourceless current can then be seen as the edge count of a multigraph consisting of a union of loops. Adding sources to $\mathbf{n}$ amounts to adding a collection of paths connecting the elements of $\partial\mathbf{n}$ pairwise. In particular, a current $\mathbf{n}$ with sources $\partial\mathbf{n} = \{u, v\}$ can be seen as the edge count of a multigraph consisting of a union of loops together with a path from u to v .

For $A \subset V$, we write $\sigma_A = \prod_{x \in A} \sigma_x$. It is well-known [4, 76] that the correlation functions of the Ising model (defined in (8.1.2)) are related to currents via

$$\langle \sigma_A \rangle_{\beta}^{\mathcal{G}} = \frac{\sum_{\partial\mathbf{n}=A} w_{\beta}(\mathbf{n})}{\sum_{\partial\mathbf{n}=\emptyset} w_{\beta}(\mathbf{n})} = \frac{Z_{\mathcal{G},\beta}^A}{Z_{\mathcal{G},\beta}^{\emptyset}}, \quad (8.3.2)$$

where

$$w_{\beta}(\mathbf{n}) = \prod_{xy \in E} \frac{\beta^{\mathbf{n}_{xy}}}{\mathbf{n}_{xy}!}, \quad Z_{\mathcal{G},\beta}^A = \sum_{\partial\mathbf{n}=A} w_{\beta}(\mathbf{n}). \quad (8.3.3)$$

For a set of currents $\mathcal{E} \subset \Omega_{\mathcal{G}}$, we also define

$$Z_{\mathcal{G},\beta}^A[\mathcal{E}] = \sum_{\partial\mathbf{n}=A} w_{\beta}(\mathbf{n}) \mathbb{1}\{\mathbf{n} \in \mathcal{E}\}. \quad (8.3.4)$$

A current $\mathbf{n}$ induces a bond percolation configuration on $\mathcal{G}$ defined by $(\mathbb{1}_{\mathbf{n}_e > 0})_{e \in E}$. The connectivity properties of this percolation configuration are known to play a crucial role in the analysis of the underlying Ising model. This motivates the following notation. For a current $\mathbf{n}$ on $\mathcal{G}$ and $x, y \in V$, we write $x \xleftrightarrow{\mathbf{n}} y$ if there exists a sequence of points $x_0 = x, x_1, \dots, x_k = y$ such that $x_i x_{i+1} \in E$ and $\mathbf{n}_{x_i x_{i+1}} > 0$ for all $0 \leq i < k$.

We will use the following minor extension of the switching lemma of [120, 4] (see also [76, Lemma 7]). Given two graphs $\mathcal{G}, \mathcal{H}$, two vertex sets $A \subset \mathcal{G}$, $B \subset \mathcal{H}$, and $\mathcal{E} \subset \Omega_{\mathcal{G}} \times \Omega_{\mathcal{H}}$, we write

$$Z_{\mathcal{G},\mathcal{H},\beta}^{A,B}[\mathcal{E}] = \sum_{\substack{\partial\mathbf{n}_1=A \\ \partial\mathbf{n}_2=B}} w_{\beta}(\mathbf{n}_1) w_{\beta}(\mathbf{n}_2) \mathbb{1}\{(\mathbf{n}_1, \mathbf{n}_2) \in \mathcal{E}\}. \quad (8.3.5)$$

We often drop the subscript β when it does not play a role. For two sets S_1, S_2 , we write $S_1 \Delta S_2 = (S_1 \setminus S_2) \cup (S_2 \setminus S_1)$ for their symmetric difference.

Lemma 8.3.2 (Switching lemma). *Let $\mathcal{G} = (V, E)$ and $\mathcal{H} = (V', E')$ be finite graphs with $\mathcal{H} \subset \mathcal{G}$. For a current $\mathbf{n}$ on $\mathcal{G}$, let $\mathbf{n}|_{\mathcal{H}}$ denote its restriction to edges of $\mathcal{H}$. Then, for any $A \subset V$, $B \subset V'$, $x, y \in V'$, and $\mathcal{E} \subset \Omega_{\mathcal{G}}$,*

$$Z_{\mathcal{G},\mathcal{H}}^{A,B}[\mathbf{n}_1 + \mathbf{n}_2 \in \mathcal{E}, x \xleftrightarrow{(\mathbf{n}_1 + \mathbf{n}_2)|_{\mathcal{H}}} y] = Z_{\mathcal{G},\mathcal{H}}^{A \Delta \{x,y\}, B \Delta \{x,y\}}[\mathbf{n}_1 + \mathbf{n}_2 \in \mathcal{E}, x \xleftrightarrow{(\mathbf{n}_1 + \mathbf{n}_2)|_{\mathcal{H}}} y]. \quad (8.3.6)$$

Proof. By definition, $w(\mathbf{n}_1)w(\mathbf{n}_2) = w(\mathbf{n}_1 + \mathbf{n}_2) \binom{\mathbf{n}_1 + \mathbf{n}_2}{\mathbf{n}_1}$. By the change of variables $\mathbf{m} = \mathbf{n}_1 + \mathbf{n}_2$ and $\mathbf{n} = \mathbf{n}_2$, the left-hand side of (8.3.6) can be written as

$$\sum_{\partial \mathbf{m} = A \Delta B} w(\mathbf{m}) \mathbb{1}\{\mathbf{m} \in \mathcal{E}\} \mathbb{1}\{x \xleftrightarrow{\mathbf{m}|_{\mathcal{H}}} y\} \sum_{\substack{\mathbf{n} \in \Omega_{\mathcal{H}}, \mathbf{n} \leq \mathbf{m} \\ \partial \mathbf{n} = B}} \binom{\mathbf{m}}{\mathbf{n}}. \quad (8.3.7)$$

We view $\mathbf{m}, \mathbf{n}$ as multigraphs $\mathcal{M}, \mathcal{N}$, and write $\partial \mathcal{N} \subset V'$ for the vertices with an odd number of incident edges in $\mathcal{N}$. From this viewpoint, the sum over $\mathbf{n}$ above exactly counts the number of subgraphs $\mathcal{N}$ of $\mathcal{M}$ with $\partial \mathcal{N} = B$.

On the event $x \xleftrightarrow{\mathbf{m}|_{\mathcal{H}}} y$, there is a path in $\mathcal{M}$ from x to y that uses only edges in E' . We denote such a path by $\mathcal{K}$. The involution $\mathcal{N} \mapsto \mathcal{N} \Delta \mathcal{K}$ then defines a bijection between subgraphs with source set B and source set $B \Delta \{x, y\}$. Thus,

$$\mathbb{1}\{x \xleftrightarrow{\mathbf{m}|_{\mathcal{H}}} y\} \sum_{\substack{\mathbf{n} \in \Omega_{\mathcal{H}}, \mathbf{n} \leq \mathbf{m} \\ \partial \mathbf{n} = B}} \binom{\mathbf{m}}{\mathbf{n}} = \mathbb{1}\{x \xleftrightarrow{\mathbf{m}|_{\mathcal{H}}} y\} \sum_{\substack{\mathbf{n} \in \Omega_{\mathcal{H}}, \mathbf{n} \leq \mathbf{m} \\ \partial \mathbf{n} = B \Delta \{x, y\}}} \binom{\mathbf{m}}{\mathbf{n}}. \quad (8.3.8)$$

Undoing the change of variables produces the right-hand side of (8.3.6). $\square$

8.3.2 A percolation exploration

We now consider bond percolation and introduce an exploration of the cluster of the origin that is particularly well-suited for the periodic setup to which we apply it. We follow the strategy initially introduced in [32] and revisited in [206, 142], with some modifications. We fix $r \geq 3$, and let $\mathcal{G} = (V, E)$ be a finite or infinite subgraph of $\mathbb{Z}^d$. Let $\omega = (\omega_e)_{e \in E}$ be a bond percolation configuration on $\mathcal{G}$. We will define an exploration $\mathcal{C}(0)$ of the cluster of 0 in ω that contains at most one representative for each vertex and edge of $\mathbb{T}_r$, under the equivalence that $x \cong x'$ if $x - x' \in r\mathbb{Z}^d$ (for $x, x' \in \mathcal{G}$). A simple example of the exploration is depicted in Figure 8.2.

We use the following colouring scheme for vertices and edges of $\mathcal{G}$:

- Vertices have three possible colourings: red (being or about to be explored), black (fully explored), or grey (has a coloured representative). We write $\mathbf{R}, \mathbf{B}, \mathbf{G}$ for the corresponding evolving sets.
- Edges have two possible colourings: white (explored and closed) or black (explored and open). We let $\mathfrak{W}, \mathfrak{B}$ be the corresponding evolving sets.

The construction of $\mathcal{C}(0)$ is as follows. We fix an ordering of the vertices in $\mathcal{G}$.

Step 0 Initially $\mathbf{R} = \{0\}$, $\mathbf{B} = \emptyset$, $\mathbf{G} = \{0' \mid 0' \cong 0, 0' \neq 0\}$, $\mathfrak{W} = \mathfrak{B} = \emptyset$.

Step 1 Let u be the earliest element of $\mathbf{R}$.

- Let v be the earliest vertex adjacent to u such that v is not coloured grey and uv is not coloured.
- If $\omega_{uv} = 0$, add the edge uv to $\mathfrak{W}$.

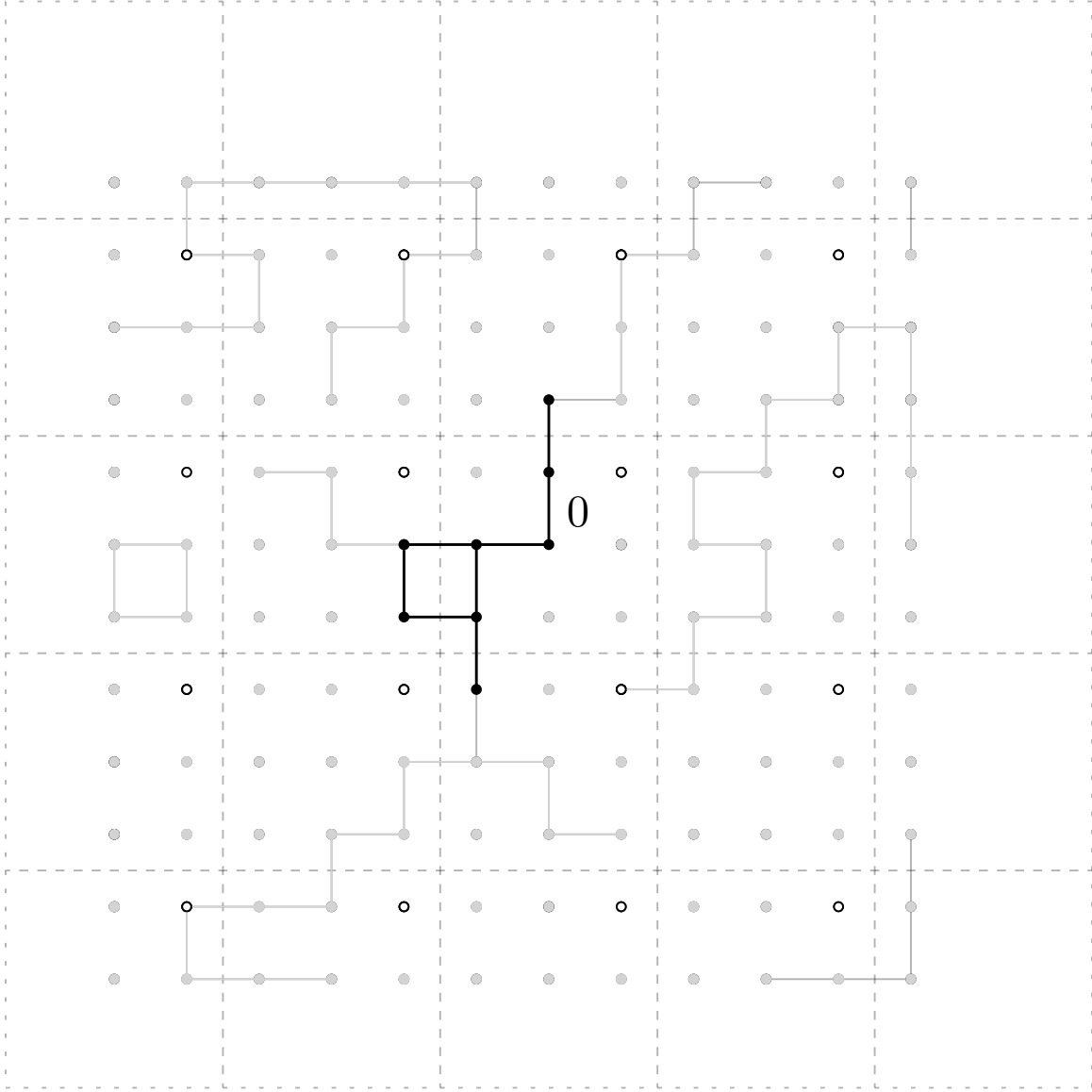


Figure 8.2: An illustration of the exploration algorithm with $r = 3$ and $\mathcal{G} = \mathbb{Z}^2$. Dotted boxes represent copies of Λ_3 . The bond percolation configuration is shown in black and grey; the black edges are explored. Vertices not explored by the algorithm are unfilled. The graph $\mathcal{C}(0)$ consists of the black vertices and edges.

- (iii) If $\omega_{uv} = 1$, add v to R (if it is not already in it) and all its representatives $v' \cong v$ ($v' \neq v$) to G , add uv to $\mathfrak{B}$.
- (iv) Restart from (i) until all the edges containing u are either coloured or connected to a grey vertex. Then add u to B .

Step 2 Restart from Step 1 until $R = \emptyset$.

At the end of the exploration, we obtain $B, G, \mathfrak{B}, \mathfrak{W}$. Note that all edges that have been explored belong to $\mathfrak{W} \cup \mathfrak{B}$. We define $\mathcal{C}(0)$ to be the graph $(B, \mathfrak{B})$.

We note the following properties of $\mathcal{C}(0)$:

- By construction, $\mathcal{C}(0)$ contains at most one representative of each vertex and edge, and $\mathcal{C}(0)$ is connected. Any path in $\mathcal{C}(0)$ has at most r^d vertices, so has length at most $r^d - 1$.
- Crucially, $\mathcal{C}(0) = (B, \mathfrak{B})$ is measurable in terms of the state of the edges in $\mathfrak{B} \cup \mathfrak{W}$, since $B = \{0\} \cup \{\text{vertices incident to } \mathfrak{B}\}$ by construction, and

$$\{\mathfrak{B} = B, \mathfrak{W} = W\} = \{\omega_e = 1 \text{ for } e \in B, \omega_e = 0 \text{ for } e \in W\}. \quad (8.3.9)$$

The $\subset$ inclusion is by definition of the algorithm. On the other hand, if (B, W) is a valid output of the algorithm and ω is an element of the right-hand side, then running the exploration on ω will produce $\mathfrak{B} = B, \mathfrak{W} = W$, so the $\supset$ inclusion follows.

8.3.3 A coupled cluster exploration

We now apply the exploration introduced in Section 8.3.2 to a well-chosen percolation configuration induced by random currents. This allows us to perform switching between a current on a large finite subgraph $\mathcal{G} \subset \mathbb{Z}^d$ and a current on $\mathbb{T}_r$.

Let $\mathbf{m}$ be a current on $\mathcal{G}$ and $\mathbf{n}$ be a current on $\mathbb{T}_r$. We write $\hat{\mathbf{n}}$ for the *lift* of $\mathbf{n}$ to $\mathcal{G}$, defined by

$$\hat{\mathbf{n}}(e) = \mathbf{n}(\pi e) \quad (e \in \mathcal{G}), \quad (8.3.10)$$

where $\pi : \mathbb{Z}^d \rightarrow \mathbb{T}_r$ is the canonical projection map. The current $\mathbf{m} + \hat{\mathbf{n}}$ induces a percolation configuration $(\mathbb{1}_{(\mathbf{m} + \hat{\mathbf{n}})_e > 0})_{e \in E}$ on $\mathcal{G}$. We run the exploration of Section 8.3.2 on this configuration and produce $(B, G, \mathfrak{B}, \mathfrak{W})$ and $\mathcal{C}(0) = (B, \mathfrak{B})$.

Lemma 8.3.3 (Properties of the exploration). *Let $d \geq 1$, $r \geq 3$, $x \in \mathbb{T}_r$, and fix finite $\mathcal{G} \supset \Lambda_{2r^d}$. Let $\mathbf{m} \in \Omega_{\mathcal{G}}$, and let $\mathbf{n} \in \Omega_{\mathbb{T}_r}$ with $\partial \mathbf{n} = \{0, x\}$. Then the exploration $(B, G, \mathfrak{B}, \mathfrak{W})$ of $(\mathbb{1}_{(\mathbf{m} + \hat{\mathbf{n}})_e > 0})_{e \in E}$ satisfies the following properties:*

- (i) $(\mathbf{m} + \hat{\mathbf{n}})_e = 0$ for all $e \in \mathfrak{W}$.
- (ii) $\hat{\mathbf{n}}_e = 0$ for every representative of $e \in \mathfrak{W}$.
- (iii) There exists exactly one representative of x , which we denote by x' , such that $x' \in \mathcal{C}(0)$.

Proof. The first two points follow directly from the algorithm of the exploration and the definition of $\hat{\mathbf{n}}$. The uniqueness of x' in (iii) has been noted already in Section 8.3.2.

For existence, since $\partial\mathbf{n} = \{0, x\}$, there is an open path in $\mathbf{n}$ from 0 to x , which we label as $(y_0, y_1, \dots, y_k)$ with $y_0 = 0$, $y_k = x$. We claim that, for any $j \in [0, k]$, there exists $y'_j \in \mathcal{G}$ such that $y'_j \cong y_j$ and $y'_j \in \mathcal{C}(0)$. Then, we can take $x' = y'_k$. We prove the claim by induction on j . The $j = 0$ case is trivial as we can take $y'_0 = 0$.

Assume the claim for some $j < k$. Observe that $y'_j \in \mathcal{C}(0) \subset \Lambda_{2r^d-1}$, because a path joining 0 to y'_j has length at most $r^d - 1$. Let $z = y'_j + (y_{j+1} - y_j) \in \Lambda_{2r^d} \subset \mathcal{G}$, so $z \cong y_{j+1}$. If $z \in \mathcal{C}(0)$, we take $y'_{j+1} = z$. Otherwise, z must be grey because $\hat{\mathbf{n}}_{y'_j, z} = \mathbf{n}_{y_j y_{j+1}} > 0$. This only happens when another representative z' of z is explored before z . In that case, we take $y'_{j+1} = z'$. This completes the inductive step and the proof. $\square$

We use the exploration to switch sources between currents on $\mathcal{G}$ and $\mathbb{T}_r$.

Lemma 8.3.4. *For any $C \subset \mathcal{G}$, $A \subset \mathbb{T}_r$, and $z \in \mathcal{G}$,*

$$Z_{\mathcal{G}, \mathbb{T}_r}^{C, A}[z \in \mathcal{C}(0)] = Z_{\mathcal{G}, \mathbb{T}_r}^{C \Delta \{0, z\}, A \Delta \{0, \pi z\}}[z \in \mathcal{C}(0)]. \quad (8.3.11)$$

In particular, for $x \in \mathbb{T}_r$, $A = \{0, x\}$, and $z = x'$ with $\pi z = x$,

$$Z_{\mathcal{G}, \mathbb{T}_r}^{\emptyset, 0x}[x' \in \mathcal{C}(0)] = Z_{\mathcal{G}, \mathbb{T}_r}^{0x', \emptyset}[x' \in \mathcal{C}(0)]. \quad (8.3.12)$$

Proof. By conditioning on $(\mathfrak{B}, \mathfrak{W})$, we have

$$Z_{\mathcal{G}, \mathbb{T}_r}^{C, A}[z \in \mathcal{C}(0)] = \sum_{B \ni z, W} Z_{\mathcal{G}, \mathbb{T}_r}^{C, A}[(\mathfrak{B}, \mathfrak{W}) = (B, W)], \quad (8.3.13)$$

where $B \ni z$ means that z is incident to some edge in B . Given an admissible pair (B, W) (i.e., realisable by the exploration and such that $B \ni z$), we make the decomposition

$$\mathbf{m} = \mathbf{m}_1 + \mathbf{m}_2, \quad \mathbf{m}_1(e) = \mathbf{m}(e) \mathbb{1}\{e \in B \cup W\}, \quad (8.3.14)$$

$$\mathbf{n} = \mathbf{n}_1 + \mathbf{n}_2, \quad \mathbf{n}_1(e) = \mathbf{n}(e) \mathbb{1}\{e \in \pi(B)\}. \quad (8.3.15)$$

Then $w(\mathbf{m}) = w(\mathbf{m}_1)w(\mathbf{m}_2)$ and $w(\mathbf{n}) = w(\mathbf{n}_1)w(\mathbf{n}_2)$. The current $\mathbf{n}_2$ is supported on $(\pi(B) \cup \pi(W))^c$, because $\mathbf{n} = 0$ on $\pi(W)$. Also, since no edges connect to grey vertices in the exploration, the projection map π restricts to an isomorphism between the graphs B and $\pi(B)$, whose inverse we denote by $\iota : \pi(B) \rightarrow B$. Let $\tilde{\mathbf{n}}_1$ denote the restriction of $\hat{\mathbf{n}}_1$ to B . Then

$$\iota^{-1}(\tilde{\mathbf{n}}_1) = \mathbf{n}_1, \quad w(\tilde{\mathbf{n}}_1) = w(\mathbf{n}_1), \quad \iota^{-1}(\partial\tilde{\mathbf{n}}_1) = \partial\mathbf{n}_1. \quad (8.3.16)$$

Let $\mathcal{E}_{(B, W)}$ denote the event $\{(\mathfrak{B}, \mathfrak{W}) = (B, W)\}$, which is measurable in $\mathbf{m}_1 + \tilde{\mathbf{n}}_1$ as indicated at (8.3.9). By conditioning on $\mathbf{m}_2, \mathbf{n}_2$,

$$Z_{\mathcal{G}, \mathbb{T}_r}^{C, A}[\mathcal{E}_{(B, W)}] = \sum_{\mathbf{m}_2, \mathbf{n}_2} w(\mathbf{m}_2)w(\mathbf{n}_2) \sum_{\substack{\partial\mathbf{m}_1 = \partial\mathbf{m}_2 \Delta C \\ \partial\mathbf{n}_1 = \partial\mathbf{n}_2 \Delta A}} w(\mathbf{m}_1)w(\tilde{\mathbf{n}}_1) \mathbb{1}_{\mathcal{E}_{(B, W)}} \mathbb{1}\{0 \xrightarrow{(\mathbf{m}_1 + \tilde{\mathbf{n}}_1)|_B} z\}, \quad (8.3.17)$$

where we have $0 \xleftarrow{(\mathbf{m}_1 + \tilde{\mathbf{n}}_1)|_B} z$ because $B \ni z$. The sum over currents $\mathbf{n}_1$ on $\pi(B)$ can be replaced by a sum over currents $\tilde{\mathbf{n}}_1$ on B , with the source constraint $\partial \tilde{\mathbf{n}}_1 = \iota(\partial \mathbf{n}_2 \Delta A)$. By Lemma 8.3.2 applied to the graphs $B \cup W$ and B ,

$$\begin{aligned} & \sum_{\substack{\partial \mathbf{m}_1 = \partial \mathbf{m}_2 \Delta C \\ \partial \tilde{\mathbf{n}}_1 = \iota(\partial \mathbf{n}_2 \Delta A)}} w(\mathbf{m}_1) w(\tilde{\mathbf{n}}_1) \mathbb{1}_{\mathcal{E}_{(B,W)}} \mathbb{1}_{\{0 \xleftarrow{(\mathbf{m}_1 + \tilde{\mathbf{n}}_1)|_B} z\}} \\ &= \sum_{\substack{\partial \mathbf{m}_1 = \partial \mathbf{m}_2 \Delta C \Delta \{0,z\} \\ \partial \tilde{\mathbf{n}}_1 = [\iota(\partial \mathbf{n}_2 \Delta A)] \Delta \{0,z\}}} w(\mathbf{m}_1) w(\tilde{\mathbf{n}}_1) \mathbb{1}_{\mathcal{E}_{(B,W)}} \mathbb{1}_{\{0 \xleftarrow{(\mathbf{m}_1 + \tilde{\mathbf{n}}_1)|_B} z\}}. \end{aligned} \quad (8.3.18)$$

Now, we use $\partial \mathbf{n}_1 = \iota^{-1}(\partial \tilde{\mathbf{n}}_1) = \partial \mathbf{n}_2 \Delta A \Delta \{0, \pi z\}$ to convert back to a sum over $\mathbf{n}_1$. The result is

$$Z_{\mathcal{G}, \mathbb{T}_r}^{C,A}[\mathcal{E}_{(B,W)}] = Z_{\mathcal{G}, \mathbb{T}_r}^{C \Delta \{0,z\}, A \Delta \{0, \pi z\}}[\mathcal{E}_{(B,W)}]. \quad (8.3.19)$$

Summation over all admissible pairs (B, W) then yields the desired result. $\square$

We can now prove the first bound of Theorem 8.2.1.

Proof of (8.2.7). Let $\mathcal{G} = \Lambda_r$ for some $R \geq 2r^d$. We introduce an auxiliary sourceless current $\mathbf{m}$ on Λ_r . Then (8.3.2) and Lemma 8.3.3(iii) give

$$\tau_{\beta}^{\mathbb{T}_r}(x) = \frac{Z_{\mathbb{T}_r}^{0x}}{Z_{\mathbb{T}_r}^{\emptyset}} = \frac{Z_{\Lambda_r, \mathbb{T}_r}^{\emptyset, 0x}}{Z_{\Lambda_r, \mathbb{T}_r}^{\emptyset, \emptyset}} = \frac{Z_{\Lambda_r, \mathbb{T}_r}^{\emptyset, 0x} [\sqcup_{x' \cong x} \{x' \in \mathcal{C}(0)\}]}{Z_{\Lambda_r, \mathbb{T}_r}^{\emptyset, \emptyset}} = \sum_{x' \in \Lambda_R: x' \cong x} \frac{Z_{\Lambda_r, \mathbb{T}_r}^{\emptyset, 0x} [x' \in \mathcal{C}(0)]}{Z_{\Lambda_r, \mathbb{T}_r}^{\emptyset, \emptyset}}. \quad (8.3.20)$$

Using (8.3.12) and $\{x' \in \mathcal{C}(0)\} \subset \{0 \xleftrightarrow{\mathbf{m}} x'\}$, we get

$$\tau_{\beta}^{\mathbb{T}_r}(x) = \sum_{x' \in \Lambda_R: x' \cong x} \frac{Z_{\Lambda_r, \mathbb{T}_r}^{0x', \emptyset} [x' \in \mathcal{C}(0)]}{Z_{\Lambda_r, \mathbb{T}_r}^{\emptyset, \emptyset}} \leq \sum_{x' \in \Lambda_R: x' \cong x} \tau_{\beta}^{\Lambda_r}(0, x'). \quad (8.3.21)$$

Then (8.2.7) follows from the fact that $\tau_{\beta}^{\Lambda_r}(0, x') \leq \tau_{\beta}(x')$ by the second Griffiths inequality. $\square$

8.3.4 Proof of (8.2.8)

Let $R \geq 2r^d$. We define

$$T_{\beta}^{(r,R)}(x) = \sum_{x' \in \Lambda_R: x' \cong x} \tau_{\beta}^{\Lambda_r}(0, x') \quad (x \in \mathbb{T}_r). \quad (8.3.22)$$

Then, by the first equality of (8.3.21),

$$T_{\beta}^{(r,R)}(x) - \tau_{\beta}^{\mathbb{T}_r}(x) = \sum_{x' \in \Lambda_R: x' \cong x} \frac{Z_{\Lambda_r, \mathbb{T}_r}^{0x', \emptyset} [x' \notin \mathcal{C}(0)]}{Z_{\Lambda_r, \mathbb{T}_r}^{\emptyset, \emptyset}}. \quad (8.3.23)$$

The inequality (8.2.8) is an immediate consequence of the following lemma, by taking the limit $R \rightarrow \infty$.

Lemma 8.3.5. *Let $d \geq 1$ and $R \geq 2r^d$. For any $x \in \mathbb{T}_r$, we have*

$$T_\beta^{(r,R)}(x) - \tau_\beta^{\mathbb{T}_r}(x) \leq \sum_{x' \cong x} \sum_y \sum_{\substack{y' \cong y \\ y' \neq y}} \sum_z \tau_\beta^{\Lambda_r}(0, z) \tau_\beta^{\Lambda_r}(z, y) \tau_\beta^{\Lambda_r}(z, y') \tau_\beta^{\mathbb{T}_r}(\pi y' - \pi z) \tau_\beta^{\Lambda_r}(y, x'), \quad (8.3.24)$$

where x', y, y', z range over Λ_r with the indicated restrictions.

The proof proceeds by bounding the terms on the right-hand side of (8.3.23). For each x' , since $\partial \mathbf{m} = \{0, x'\}$, there is an open path in $\mathbf{m}$ from 0 to x' . When $x' \notin \mathcal{C}(0)$, every such path must contain a grey site. (Note that x' might be uncoloured, because the sources do not guarantee that there is an $x'' \in \mathcal{C}(0)$ with $x'' \cong x$.) We let $y \in \Lambda_r$ denote the last grey site on one of the paths, so that $y \xleftrightarrow{\mathbf{m}} x'$ using only edges in $(\mathfrak{B} \cup \mathfrak{W})^c$. Also, since y is grey, there exists another representative $y' \cong y$, $y' \neq y$ such that $y' \in \mathcal{C}(0)$. This proves the inclusion

$$\{x' \notin \mathcal{C}(0)\} \subset \bigcup_{y \in \Lambda_r} \bigcup_{\substack{y' \cong y \\ y' \neq y}} \{y' \in \mathcal{C}(0)\} \cap \{y \xleftrightarrow{\mathbf{m}} x' \text{ in } (\mathfrak{B} \cup \mathfrak{W})^c\}. \quad (8.3.25)$$

In the following, we regard an edge subset (such as $(\mathfrak{B} \cup \mathfrak{W})^c$) as the graph whose edges are those in that subset and whose vertices are the vertices comprising those edges.

Lemma 8.3.6. *Let $\mathcal{A}(x', y, y') = \{y' \in \mathcal{C}(0)\} \cap \{y \xleftrightarrow{\mathbf{m}} x' \text{ in } (\mathfrak{B} \cup \mathfrak{W})^c\}$. Then,*

$$Z_{\Lambda_r, \mathbb{T}_r}^{0x', \emptyset}[\mathcal{A}(x', y, y')] \leq Z_{\Lambda_r, \mathbb{T}_r}^{0y, \emptyset}[y' \in \mathcal{C}(0)] \tau_\beta^{\Lambda_r}(y, x'). \quad (8.3.26)$$

Proof. For simplicity, we write $\mathcal{A} = \mathcal{A}(x', y, y')$. Since the last event in (8.3.25) is a connection outside of $\mathfrak{B} \cup \mathfrak{W}$, we can use it to switch sources with a third independent current without affecting $\mathfrak{B} \cup \mathfrak{W}$, as follows. Let ℓ be an independent current on Λ_r with $\partial \ell = \emptyset$. We again condition on the event $\mathcal{E}_{(B, W)} = \{(\mathfrak{B}, \mathfrak{W}) = (B, W)\}$. We decompose

$$\mathbf{m} = \mathbf{m}_1 + \mathbf{m}_2, \quad \mathbf{m}_1(e) = \mathbf{m}(e) \mathbb{1}\{e \in B \cup W\}, \quad (8.3.27)$$

and similarly for $\ell = \ell_1 + \ell_2$. By conditioning on $\mathbf{m}_1$ and ℓ_1 ,

$$Z_{\Lambda_r}^\emptyset Z_{\Lambda_r, \mathbb{T}_r}^{0x', \emptyset}[\mathcal{A}] = Z_{\Lambda_r}^\emptyset \sum_{B, W} Z_{\Lambda_r, \mathbb{T}_r}^{0x', \emptyset}[\mathcal{A} \cap \mathcal{E}_{(B, W)}] \quad (8.3.28)$$

$$= \sum_{B \ni y', W} Z_{\Lambda_r, \Lambda_r, \mathbb{T}_r}^{\emptyset, 0x', \emptyset}[\mathcal{E}_{(B, W)} \cap \{y \xleftrightarrow{\mathbf{m}} x' \text{ in } (B \cup W)^c\}] \quad (8.3.29)$$

$$= \sum_{B \ni y', W} \sum_{\substack{\ell_1, \mathbf{m}_1 \\ \partial \mathbf{m}_1 = \emptyset}} w(\ell_1) w(\mathbf{m}_1) w(\mathbf{n}) \mathbb{1}_{\mathcal{E}_{(B, W)}} \sum_{\substack{\partial \ell_2 = \partial \ell_1 \\ \partial \mathbf{m}_2 = \partial \mathbf{m}_1 \Delta \{0, x'\}}} w(\ell_2) w(\mathbf{m}_2) \mathbb{1}\{y \xleftrightarrow{\mathbf{m}_2} x'\}. \quad (8.3.30)$$

Since $\{y \xleftrightarrow{\mathbf{m}_2} x'\} \subset \{y \xleftrightarrow{\ell_2 + \mathbf{m}_2} x'\}$, Lemma 8.3.2 applied with $\mathcal{G} = \mathcal{H} = (B \cup W)^c$ gives

$$\sum_{\substack{\partial \ell_2 = \partial \ell_1 \\ \partial \mathbf{m}_2 = \partial \mathbf{m}_1 \Delta \{0, x'\}}} w(\ell_2) w(\mathbf{m}_2) \mathbb{1}\{y \xleftrightarrow{\mathbf{m}_2} x'\} \leq \sum_{\substack{\partial \ell_2 = \partial \ell_1 \Delta \{y, x'\} \\ \partial \mathbf{m}_2 = \partial \mathbf{m}_1 \Delta \{0, y\}}} w(\ell_2) w(\mathbf{m}_2) \mathbb{1}\{y \xleftrightarrow{\ell_2 + \mathbf{m}_2} x'\}. \quad (8.3.31)$$

We insert this into (8.3.28), and use $\mathbb{1}\{y \xrightarrow{\ell_2 + \mathbf{m}_2} x'\} \leq 1$, to get

$$Z_{\Lambda_r}^\emptyset Z_{\Lambda_r, \mathbb{T}_r}^{0x', \emptyset}[\mathcal{A}] \leq \sum_{B \ni y', W} Z_{\Lambda_r, \Lambda_r, \mathbb{T}_r}^{yx', 0y, \emptyset}[\mathcal{E}_{(B, W)}] = Z_{\Lambda_r}^{yx'} Z_{\Lambda_r, \mathbb{T}_r}^{0y, \emptyset}[y' \in \mathcal{C}(0)]. \quad (8.3.32)$$

The claim then follows from division by $Z_{\Lambda_r}^\emptyset$ and (8.3.2). $\square$

In the proof of Lemma 8.3.5, we use the notion of the backbone of a current [12, Section 4] (more on the backbone can be found in [11, 204]). A current $\mathbf{m}$ on a finite graph $\mathcal{G}$ with sources $\partial\mathbf{m} = \{u, v\}$ must have at least one path from u to v in the percolation configuration induced by $\mathbf{m}$. The backbone of $\mathbf{m}$ is an exploration of one of these paths. We fix some arbitrary ordering $\prec$ of the edges of $\mathcal{G}$.

Definition 8.3.7 (The backbone of a current). Let $\mathbf{m} \in \Omega_{\mathcal{G}}$ with $\partial\mathbf{m} = \{u, v\}$. The *backbone* of $\mathbf{m}$, denoted $\Gamma_{\mathbf{m}}$, is the unique oriented and edge self-avoiding path from u to v , supported on edges with $\mathbf{m}$ odd, that is minimal for $\prec$. The backbone $\Gamma_{\mathbf{m}} = \{x_i x_{i+1} : 0 \leq i < k\}$ is obtained via the following exploration process:

- (i) Let $x_0 = u$. The first edge $x_0 x_1$ of $\Gamma_{\mathbf{m}}$ is the earliest of all the edges e emerging from x_0 with $\mathbf{m}_e$ odd. Thus, all edges $e \ni x_0$ satisfying $e \prec x_0 x_1$ and $e \neq x_0 x_1$ are explored and have $\mathbf{m}_e$ even.
- (ii) Inductively, each edge $x_i x_{i+1}$, $i \geq 1$, is the earliest of all edges e emerging from x_i that have not been explored previously, and for which $\mathbf{m}_e$ is odd.
- (iii) The exploration stops when it reaches a vertex from which no more odd, non-explored edges are available. This happens at $x_k = v$.

Let $\bar{\Gamma}_{\mathbf{m}}$ denote the set of explored edges, i.e. $\Gamma_{\mathbf{m}}$ together with all explored even edges.

By definition, $\bar{\Gamma}_{\mathbf{m}}$ is determined by $\Gamma_{\mathbf{m}}$ and the ordering $\prec$, and similarly $\Gamma_{\mathbf{m}}$ can be determined from $\bar{\Gamma}_{\mathbf{m}}$ by selecting odd edges in $\bar{\Gamma}_{\mathbf{m}}$ according to $\prec$. Also, the current $\mathbf{m} \setminus \bar{\Gamma}_{\mathbf{m}}$, defined to be the restriction of $\mathbf{m}$ to the complement $\bar{\Gamma}_{\mathbf{m}}^c$, is sourceless.

Proof of Lemma 8.3.5. By (8.3.23), (8.3.25), a union bound, and Lemma 8.3.6,

$$T_{\beta}^{(r, R)}(x) - \tau_{\beta}^{\mathbb{T}_r}(x) \leq \sum_{x' \cong x} \sum_y \sum_{\substack{y' \cong y \\ y' \neq y}} \frac{Z_{\Lambda_r, \mathbb{T}_r}^{0y, \emptyset}[y' \in \mathcal{C}(0)]}{Z_{\Lambda_r, \mathbb{T}_r}^{\emptyset, \emptyset}} \tau_{\beta}^{\Lambda_r}(y, x'). \quad (8.3.33)$$

The connections used to bound the above right-hand side are depicted in Figure 8.3. We first apply Lemma 8.3.4 to switch on the event $y' \in \mathcal{C}(0)$. Using Lemma 8.3.4 with $z = y'$, $C = \{0, y\}$, and $A = \emptyset$, we get

$$Z_{\Lambda_r, \mathbb{T}_r}^{0y, \emptyset}[y' \in \mathcal{C}(0)] = Z_{\Lambda_r, \mathbb{T}_r}^{yy', 0\pi y'}[y' \in \mathcal{C}(0)]. \quad (8.3.34)$$

Since $\partial\mathbf{m} = \{y, y'\}$ now, we can form the backbone $\Gamma_{\mathbf{m}}$ of $\mathbf{m}$ from y' to y . Let $\bar{\Gamma}_{\mathbf{m}}$ be the set of edges explored during the formation of $\Gamma_{\mathbf{m}}$. Let $\mathcal{C}_{\mathbf{m} \setminus \bar{\Gamma}_{\mathbf{m}} + \hat{\mathbf{n}}}(0)$ denote the result of

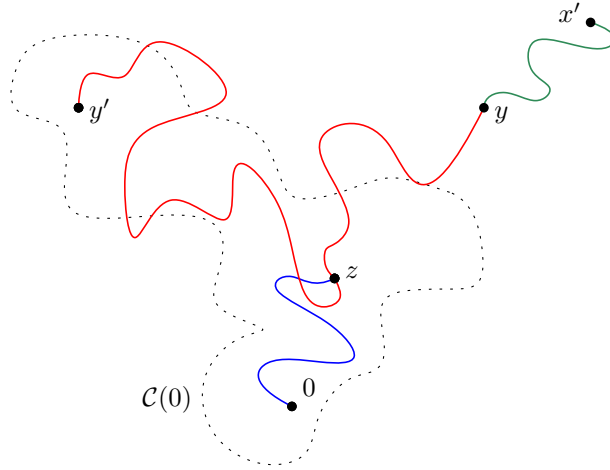


Figure 8.3: An illustration of the connections relevant to the right-hand side of (8.3.33). The region delimited by the dotted line corresponds to the exploration $\mathcal{C}(0)$. The connection from y to x' (in green) does not involve any edge of $(\mathfrak{B}, \mathfrak{W})$. The point z is defined by (8.3.35). Following the notation below (8.3.35): the red path corresponds to $\Gamma_{\mathbf{m}}$ after the switching performed in (8.3.34), the blue path corresponds to a connection in $\mathcal{C}_{\mathbf{m} \setminus \bar{\Gamma}_{\mathbf{m}} + \hat{\mathbf{n}}}(0)$. The additional torus two-point function appearing in the right-hand side of (8.3.24) is produced by the switching of (8.3.38).

the exploration process of Section 8.3.2 performed on the current $\mathbf{m} \setminus \bar{\Gamma}_{\mathbf{m}} + \hat{\mathbf{n}}$ (on the graph $\Lambda_r \setminus \bar{\Gamma}_{\mathbf{m}}$). We claim that

$$\{y' \in \mathcal{C}(0)\} \subset \bigcup_{z \in \Lambda_r} \{z \xleftrightarrow{\bar{\Gamma}_{\mathbf{m}}} y'\} \cap \{z \in \mathcal{C}_{\mathbf{m} \setminus \bar{\Gamma}_{\mathbf{m}} + \hat{\mathbf{n}}}(0)\}. \quad (8.3.35)$$

Indeed, if $\mathcal{C}_{\mathbf{m} \setminus \bar{\Gamma}_{\mathbf{m}} + \hat{\mathbf{n}}}(0) = \mathcal{C}(0)$, then (8.3.35) holds with $z = y'$. If $\mathcal{C}_{\mathbf{m} \setminus \bar{\Gamma}_{\mathbf{m}} + \hat{\mathbf{n}}}(0) \neq \mathcal{C}(0)$, then there is a first edge $zv \in \bar{\Gamma}_{\mathbf{m}}$ used in the construction of $\mathcal{C}(0)$ that makes the two exploration processes different. For this edge, we must have $z \in \mathcal{C}_{\mathbf{m} \setminus \bar{\Gamma}_{\mathbf{m}} + \hat{\mathbf{n}}}(0)$ and $\mathbf{m}_{zv} > 0$. Since one of z, v must be in $\Gamma_{\mathbf{m}}$, we see that $z \xleftrightarrow{\bar{\Gamma}_{\mathbf{m}}} y'$ also. This proves (8.3.35).

Conditional on $\Gamma_{\mathbf{m}} = \gamma$, we set $\bar{\gamma} = \bar{\Gamma}_{\mathbf{m}}$ and decompose $\mathbf{m}$ as

$$\mathbf{m} = \mathbf{k} + \ell, \quad \mathbf{k}(e) = \mathbf{m}(e) \mathbb{1}\{e \in \bar{\gamma}\}, \quad (8.3.36)$$

so $\mathbf{k}, \ell$ are currents on $\bar{\gamma}, \Lambda_r \setminus \bar{\gamma}$ respectively. Note that $w(\mathbf{m}) = w(\mathbf{k})w(\ell)$, $\partial \mathbf{k} = \{y, y'\}$, $\partial \ell = \emptyset$, and $\Gamma_{\mathbf{m}} = \Gamma_{\mathbf{k}}$. Using (8.3.35),

$$Z_{\Lambda_r, \mathbb{T}_r}^{yy', 0\pi y'}[y' \in \mathcal{C}(0)] \leq \sum_{z \in \Lambda_r} \sum_{\gamma} \sum_{\partial \mathbf{k} = \{y, y'\}} w(\mathbf{k}) \mathbb{1}\{\Gamma_{\mathbf{k}} = \gamma\} \mathbb{1}\{z \xleftrightarrow{\bar{\Gamma}_{\mathbf{k}}} y'\} Z_{\Lambda_r \setminus \bar{\gamma}, \mathbb{T}_r}^{\emptyset, 0\pi y'}[z \in \mathcal{C}_{\ell + \hat{\mathbf{n}}}(0)]. \quad (8.3.37)$$

By Lemma 8.3.4 with $\mathcal{G} = \Lambda_r \setminus \bar{\gamma}$,

$$\begin{aligned} Z_{\Lambda_r \setminus \bar{\gamma}, \mathbb{T}_r}^{\emptyset, 0\pi y'}[z \in \mathcal{C}_{\ell + \hat{\mathbf{n}}}(0)] &= Z_{\Lambda_r \setminus \bar{\gamma}, \mathbb{T}_r}^{0z, \pi z \pi y'}[z \in \mathcal{C}_{\ell + \hat{\mathbf{n}}}(0)] \\ &\leq Z_{\Lambda_r \setminus \bar{\gamma}}^{0z} Z_{\mathbb{T}_r}^{\pi z \pi y'} = Z_{\Lambda_r \setminus \bar{\gamma}}^{\emptyset} \tau_{\beta}^{\Lambda_r \setminus \bar{\gamma}}(0, z) Z_{\mathbb{T}_r}^{\pi z \pi y'}. \end{aligned} \quad (8.3.38)$$

Then we use $\tau_{\beta}^{\Lambda_r \setminus \bar{\gamma}} \leq \tau_{\beta}^{\Lambda_r}$ to obtain

$$Z_{\Lambda_r, \mathbb{T}_r}^{yy', 0\pi y'}[y' \in \mathcal{C}(0)] \leq \sum_{z \in \Lambda_r} Z_{\mathbb{T}_r}^{\pi z \pi y'} \sum_{\gamma} \sum_{\partial \mathbf{k} = \{y, y'\}} w(\mathbf{k}) \mathbf{1}\{\Gamma_{\mathbf{k}} = \gamma\} \mathbf{1}\{z \xleftarrow{\bar{\Gamma}_{\mathbf{k}}} y'\} Z_{\Lambda_r \setminus \bar{\gamma}}^{\emptyset} \tau_{\beta}^{\Lambda_r}(0, z) \quad (8.3.39)$$

$$= \sum_{z \in \Lambda_r} Z_{\mathbb{T}_r}^{\pi z \pi y'} \sum_{\gamma} Z_{\Lambda_r}^{yy'}[\Gamma_{\mathbf{m}} = \gamma, z \xleftarrow{\bar{\Gamma}_{\mathbf{m}}} y'] \tau_{\beta}^{\Lambda_r}(0, z) \quad (8.3.40)$$

$$= \sum_{z \in \Lambda_r} Z_{\mathbb{T}_r}^{\pi z \pi y'} Z_{\Lambda_r}^{yy'}[z \xleftarrow{\bar{\Gamma}_{\mathbf{m}}} y'] \tau_{\beta}^{\Lambda_r}(0, z). \quad (8.3.41)$$

For the middle factor on the right-hand side, we introduce an auxiliary sourceless current $\mathbf{m}'$ on Λ_r , and use Lemma 8.3.2 with $\mathcal{G} = \mathcal{H} = \Lambda_r$:

$$Z_{\Lambda_r}^{yy'}[z \xleftarrow{\bar{\Gamma}_{\mathbf{m}}} y'] \leq \frac{Z_{\Lambda_r, \Lambda_r}^{yy', \emptyset}[z \xleftarrow{\mathbf{m} + \mathbf{m}'} y']}{Z_{\Lambda_r}^{\emptyset}} = \frac{Z_{\Lambda_r, \Lambda_r}^{yz, zy'}}{Z_{\Lambda_r}^{\emptyset}} = Z_{\Lambda_r}^{yz} \tau_{\beta}^{\Lambda_r}(z, y'). \quad (8.3.42)$$

Inserting this yields

$$\frac{Z_{\Lambda_r, \mathbb{T}_r}^{yy', 0\pi y'}[y' \in \mathcal{C}(0)]}{Z_{\Lambda_r, \mathbb{T}_r}^{\emptyset, \emptyset}} \leq \sum_{z \in \Lambda_r} \frac{Z_{\mathbb{T}_r}^{\pi z \pi y'}}{Z_{\mathbb{T}_r}^{\emptyset}} \frac{Z_{\Lambda_r}^{yz}}{Z_{\Lambda_r}^{\emptyset}} \tau_{\beta}^{\Lambda_r}(z, y') \tau_{\beta}^{\Lambda_r}(0, z) \quad (8.3.43)$$

$$= \sum_{z \in \Lambda_r} \tau_{\beta}^{\mathbb{T}_r}(\pi y' - \pi z) \tau_{\beta}^{\Lambda_r}(z, y) \tau_{\beta}^{\Lambda_r}(z, y') \tau_{\beta}^{\Lambda_r}(0, z). \quad (8.3.44)$$

With (8.3.33) and (8.3.34), this concludes the proof. $\square$

8.A Torus convolution estimates

We present two elementary convolution lemmas. The first is a special case of [193, Lemma 3.4]. We include the simple proof for completeness, and as an opportunity to correct an omission in the proof of [193, Lemma 3.4] (repeated in [192, Lemma 4.2]) which however does not change the conclusion of those lemmas.

Lemma 8.A.1. *Let $d \geq 1$, $r \geq 1$, $a > 0$ and $\mu > 0$. There is a constant $C = C(d, a) > 0$ such that*

$$\sum_{u \in \mathbb{Z}^d: u \neq 0} \frac{1}{\|x + ru\|_{\infty}^{d-a}} e^{-\mu\|x+ru\|_{\infty}} \leq C e^{-\frac{1}{4}\mu r} \frac{1}{\mu^a r^d} \quad (x \in \Lambda_r). \quad (8.A.1)$$

Proof. For all nonzero $u \in \mathbb{Z}^d$ and $r \geq 1$, we have $\|x + ru\|_{\infty} \geq \frac{1}{2}\|ru\|_{\infty}$ by (8.2.3). Hence,

$$\begin{aligned} \sum_{u \in \mathbb{Z}^d: u \neq 0} \frac{1}{\|x + ru\|_{\infty}^{d-a}} e^{-\mu\|x+ru\|_{\infty}} &\leq 2^{d-a} \sum_{N=1}^{\infty} \sum_{u: \|u\|_{\infty}=N} \frac{1}{\|ru\|_{\infty}^{d-a}} e^{-\frac{1}{2}\mu\|ru\|_{\infty}} \\ &\lesssim r^{a-d} e^{-\frac{1}{4}\mu r} \sum_{N=1}^{\infty} N^{a-1} e^{-\frac{1}{4}\mu r N}. \end{aligned} \quad (8.A.2)$$

We set $f(x) = x^{a-1}e^{-bx}$ for $x > 0$, and $b = \frac{1}{4}\mu r > 0$, so the goal is to prove that $\sum_{N=1}^{\infty} f(N) \lesssim b^{-a}$. If $a \in (0, 1]$ then f is a decreasing function, so the sum is bounded by $\int_0^{\infty} f(x)dx = b^{-a}\Gamma(a)$. If $a > 1$ then $f(x)$ is increasing for $x < x_0 = \frac{a-1}{b}$ and is decreasing for $x > x_0$. We set $N_0 = \lfloor x_0 \rfloor$ and bound the sum by

$$\left(\int_1^{N_0} f(x)dx + f(N_0) \right) \mathbb{1}_{N_0 \geq 1} + f(N_0 + 1) + \int_{N_0+1}^{\infty} f(x)dx. \quad (8.A.3)$$

The sum of the two integrals is again bounded by $\int_0^{\infty} f(x)dx = b^{-a}\Gamma(a)$. For the other terms, if $N_0 = 0$ we use that $f(N_0 + 1) = f(1) = e^{-b} \leq c_a b^{-a}$ for all $b > 0$. If $N_0 \geq 1$, we use

$$f(N_0), f(N_0 + 1) \leq \max_{x \geq 0} f(x) = \frac{1}{b^{a-1}} \max_{t \geq 0} t^{a-1} e^{-t} \leq \frac{a-1}{b^a} c_a, \quad (8.A.4)$$

where the last inequality is due to $b \leq a-1$ since $x_0 \geq N_0 \geq 1$. This completes the proof. $\square$

The next lemma is a minor extension of [193, Lemma 3.6]. The definition of $T_{\beta}^{(r)}(x)$ is in (8.1.24), $\star$ denotes convolution on $\mathbb{T}_r$ and $*$ denotes convolution on $\mathbb{Z}^d$.

Lemma 8.A.2. *Let $d > 4$. There is a constant $C > 0$ such that for all $r \geq 3$, $\beta < \beta_c$, and $x \in \mathbb{T}_r$,*

$$T_{\beta}^{(r)}(x) \leq \tau_{\beta}(x) + C \frac{\chi(\beta)}{r^d}, \quad (8.A.5)$$

$$(T_{\beta}^{(r)} \star T_{\beta}^{(r)})(x) \leq (\tau_{\beta} * \tau_{\beta})(x) + C \frac{\chi(\beta)^2}{r^d}, \quad (8.A.6)$$

$$((T_{\beta}^{(r)})^2 \star T_{\beta}^{(r)} \star T_{\beta}^{(r)})(x) \leq (\tau_{\beta}^2 * \tau_{\beta} * \tau_{\beta})(x) + C \frac{\chi(\beta)^2}{r^d} + C \left(\frac{\chi(\beta)^2}{r^d} \right)^2. \quad (8.A.7)$$

Proof. We suppress the subscript β and superscript r in the notation, and write $m = (\beta_c - \beta)^{1/2}$. Then $m^{-2} \asymp \chi$ by (8.1.5). For (8.A.5), we separate the $u = 0$ term in the definition (8.1.24) of $T(x)$, and then use (8.1.6) and the $a = 2$ case of Lemma 8.A.1, to obtain

$$T(x) \leq \tau(x) + \sum_{u \neq 0} \frac{C_0}{\|x + ru\|_{\infty}^{d-2}} e^{-c_0 m \|x + ru\|_{\infty}} \leq \tau(x) + C \frac{\chi}{r^d}. \quad (8.A.8)$$

For (8.A.6), we first observe that

$$\begin{aligned} (T \star T)(x) &= \sum_{y \in \mathbb{T}_r} \sum_{u, v \in \mathbb{Z}^d} \tau(y + ru) \tau(x - y + rv) \\ &= \sum_{w \in \mathbb{Z}^d} \sum_{y \in \mathbb{T}_r} \sum_{v \in \mathbb{Z}^d} \tau(y - rv + rw) \tau(x - y + rv) = \sum_{w \in \mathbb{Z}^d} (\tau * \tau)(x - rw), \end{aligned} \quad (8.A.9)$$

where in the second equality we replaced u by $w - v$, and in the third we observe that $y - rv$ ranges over $\mathbb{Z}^d$ under the indicated summations. The $w = 0$ term is $(\tau * \tau)(x)$. For the remaining terms, we use the convolution estimate of [133, Proposition 1.7] and (8.1.6) to get $(\tau * \tau)(z) \lesssim \|z\|^{-(d-4)} \exp[-c_0 m \|z\|_{\infty}]$. We then apply Lemma 8.A.1 with $a = 4$ to see that

$$\sum_{w \neq 0} (\tau * \tau)(x - rw) \lesssim \frac{\chi^2}{r^d}, \quad (8.A.10)$$

which gives (8.A.6).

For (8.A.7), by applying the inequality $T \leq \tau \circ \pi + C\chi/r^d$ from (8.A.5), we get three terms by expanding the T^2 in $(T^2 \star T \star T)(x)$. The first term is

$$\begin{aligned} ((\tau \circ \pi)^2 \star T \star T)(x) &= \sum_{y,z \in \mathbb{T}_r} \sum_{v,w \in \mathbb{Z}^d} \tau^2(y) \tau(z + rw - y - rv) \tau(x - z - rw) \\ &= \sum_{y \in \mathbb{T}_r} \sum_{v \in \mathbb{Z}^d} \tau^2(y) (\tau \star \tau)(x - y - rv) \\ &\leq \sum_{u \in \mathbb{Z}^d} (\tau^2 \star \tau \star \tau)(x + ru). \end{aligned} \quad (8.A.11)$$

Since $d > 4$, by [133, Proposition 1.7], $\tau^2 \star \tau \star \tau$ obeys the same upper bound as the bound used on $\tau \star \tau$ in the proof of (8.A.6), so we again have a bound χ^2/r^d for the sum over nonzero u . The cross term is

$$2C \frac{\chi}{r^d} ((\tau \circ \pi) \star T \star T)(x) \lesssim \frac{\chi}{r^d} \|\tau\|_{L^1(\mathbb{T}_r)} \|T \star T\|_\infty \lesssim \frac{\chi}{r^d} \chi \left(1 + \frac{\chi^2}{r^d}\right), \quad (8.A.12)$$

by (8.A.6) together with the fact that $\|\tau \star \tau\|_\infty \lesssim 1$ when $d > 4$. The last term is

$$\left(C \frac{\chi}{r^d}\right)^2 (1 \star T \star T)(x) \lesssim \frac{\chi^2}{r^{2d}} \|1\|_\infty \|T\|_{L^1(\mathbb{T}_r)}^2 = \frac{\chi^2}{r^{2d}} \chi^2, \quad (8.A.13)$$

since $\|T\|_{L^1(\mathbb{T}_r)} = \chi$. This completes the proof. $\square$

8.B Proof of Proposition 8.1.9

We now prove Proposition 8.1.9. Although this differential inequality is essentially the same as [95, (12.118)], the justification in [95] involves an unpublished work of Fernández, and to our knowledge a proof has not been published elsewhere. We therefore present a proof here. We work on the torus and to simplify the notation we drop the labels β and $\mathbb{T}_r$. We apply the *dilution trick*, following the strategy of [12].

We begin with a reduction to the following proposition. Although we only need the nearest-neighbour case, we prove the result for a general Hamiltonian

$$H(\sigma) = - \sum_{\{x,y\} \subset \mathbb{T}_r} J_{x,y} \sigma_x \sigma_y, \quad (8.B.1)$$

where $J_{x,y} \geq 0$ is symmetric and translation invariant. We denote the associated measure as $\langle \cdot \rangle$. We write $|J| = \sum_{x \in \mathbb{T}_r} J_{0,x}$. For the nearest-neighbour model, $|J| = 2d$. The weights defined in (8.3.3) now become replaced by

$$w(\mathbf{n}) = \prod_{\{x,y\} \subset \mathbb{T}_r} \frac{(\beta J_{x,y})^{\mathbf{n}_{xy}}}{\mathbf{n}_{xy}!}. \quad (8.B.2)$$

Recall that the torus bubble diagram is $\mathbf{B}(\beta) = \sum_{x \in \mathbb{T}_r} \tau_\beta(x)^2$.

Proposition 8.B.1. *Let $d \geq 1$. For all $\beta > 0$, for all $r \geq 3$, and for all $p \in [0, 1]$,*

$$\sum_{x,y,z \in \mathbb{T}_r} |U_4(0, x, y, z)| \geq 2p\chi^2\beta \frac{\partial \chi}{\partial \beta} - 32p^2\beta^2 |J|^2 \chi^4 \mathbf{B}. \quad (8.B.3)$$

Proof of Proposition 8.1.9. The lower bound in (8.B.3) can be written as

$$ap - bp^2 \quad \text{with} \quad a = 2\chi^2\beta\frac{\partial\chi}{\partial\beta}, \quad b = 32\beta^2|J|^2\chi^4\mathbf{B}. \quad (8.B.4)$$

If $b \leq \frac{a}{2}$ then we take $p = 1$ and in this case we obtain

$$\sum_{x,y,z \in \mathbb{T}_r} |U_4(0, x, y, z)| \geq a - b \geq \frac{a}{2} = \chi^2\beta\frac{\partial\chi}{\partial\beta}. \quad (8.B.5)$$

If instead $b > a/2$, then $ap - bp^2$ is maximal at $p = \frac{a}{2b} \in (0, 1)$, with maximum $\frac{a^2}{4b}$. We use this value of p in (8.B.3), and obtain

$$\sum_{x,y,z \in \mathbb{T}_r} |U_4(0, x, y, z)| \geq \frac{a^2}{4b} = \frac{1}{32|J|^2\mathbf{B}} \left(\frac{\partial\chi}{\partial\beta} \right)^2. \quad (8.B.6)$$

Then (8.1.34) follows by choosing the worst alternative in (8.B.5)–(8.B.6). $\square$

It remains to prove Proposition 8.B.1. Given a vertex set $A \subset \mathbb{T}_r$, we define a probability measure $\mathbf{P}^A$ on the set Ω of currents on $\mathbb{T}_r$ by

$$\mathbf{P}^A[\mathbf{n}] = \mathbb{1}_{\partial\mathbf{n}=A} \frac{w(\mathbf{n})}{Z^A} \quad (\mathbf{n} \in \Omega), \quad (8.B.7)$$

where $Z^A = \sum_{\partial\mathbf{n}=A} w(\mathbf{n})$ is the normalisation constant. Given two vertex sets A_1, A_2 , we define

$$\mathbf{P}^{A_1, A_2} = \mathbf{P}^{A_1} \otimes \mathbf{P}^{A_2}, \quad (8.B.8)$$

and we write $\mathbf{E}^{A_1, A_2}$ for expectation with respect to $\mathbf{P}^{A_1, A_2}$.

We write $\mathcal{P}_2 = \mathcal{P}_2(\mathbb{T}_r)$ for the set of all unordered pairs of vertices in $\mathbb{T}_r$. Let $\mathbf{C}(u)$ denote the vertex cluster of $u \in \mathbb{T}_r$ in the percolation configuration constructed from a double current $\mathbf{n}_1 + \mathbf{n}_2$, and let $\mathbf{C}^b(u)$ denote the set of pairs $\{s, t\} \in \mathcal{P}_2$ such that either s or t belongs to $\mathbf{C}(0)$. Given $\theta \in [0, 1]$ and $E \subset \mathcal{P}_2$, we introduce the (random) interaction $(J_{u,v}^E(\theta))_{\{u,v\} \in \mathcal{P}_2}$ defined by

$$J_{u,v}^E(\theta) = \begin{cases} \theta J_{u,v} & (\{u, v\} \in \mathbf{C}^b(0) \cap E) \\ J_{u,v} & (\{u, v\} \notin \mathbf{C}^b(0) \cap E), \end{cases} \quad (8.B.9)$$

and we write $\langle \cdot \rangle_\theta$ for the Ising expectation with the random interaction $J^E(\theta)$. Given a subset $G \subset \mathcal{P}_2$, we regard G as a graph whose edges consist of the elements of G and whose vertices are the vertices contained in the elements of G , and we write $\langle \cdot \rangle^G$ for the Ising expectation on the graph G with interaction J . By definition, $\langle \cdot \rangle_1 = \langle \cdot \rangle$ and $\langle \cdot \rangle_0 = \langle \cdot \rangle^{(\mathbf{C}^b(0) \cap E)^c}$. According to our convention, the graph $(\mathbf{C}^b(0) \cap E)^c$ consists of the edges not in $\mathbf{C}^b(0) \cap E$, together with the vertices comprising those edges.

We start with a lower bound on U_4 expressed in terms of T_1, T_2 defined for $x, y, z \in \mathbb{T}_r$ and $E \subset \mathcal{P}_2$ by

$$T_1(x, y, z, E) = \beta \sum_{\{u,v\} \in E} J_{u,v} \langle \sigma_0 \sigma_x \rangle \mathbf{P}^{0x, \emptyset}[u \text{ or } v \in \mathbf{C}(0)] \langle \sigma_y \sigma_z; \sigma_u \sigma_v \rangle, \quad (8.B.10)$$

$$T_2(x, y, z, E) = \beta \sum_{\{u,v\} \in E} J_{u,v} \langle \sigma_0 \sigma_x \rangle \mathbf{E}^{0x, \emptyset} \left[\mathbb{1}[u \text{ or } v \in \mathbf{C}(0)] \int_0^1 \theta \partial_\theta \langle \sigma_y \sigma_z; \sigma_u \sigma_v \rangle_\theta d\theta \right]. \quad (8.B.11)$$

Lemma 8.B.2. For all $x, y, z \in \mathbb{T}_r$, and for any set $E \subset \mathcal{P}_2$,

$$\frac{1}{2}|U_4(0, x, y, z)| \geq T_1(x, y, z, E) - T_2(x, y, z, E). \quad (8.B.12)$$

Proof. By [4, Proposition 5.1], for all $x, y, z \in \mathbb{T}_r$,

$$\frac{1}{2}|U_4(0, x, y, z)| = \langle \sigma_0 \sigma_x \rangle \langle \sigma_y \sigma_z \rangle \mathbf{P}^{0x, yz}[\mathbf{C}(0) \cap \mathbf{C}(y) \neq \emptyset] \quad (8.B.13)$$

$$= \langle \sigma_0 \sigma_x \rangle \langle \sigma_y \sigma_z \rangle \left(1 - \mathbf{P}^{0x, yz}[\mathbf{C}(0) \cap \mathbf{C}(y) = \emptyset]\right). \quad (8.B.14)$$

We modify the definition (8.3.5) by defining, for $H \subset \mathcal{P}_2$ and for A, B subsets of vertices in the graph H ,

$$Z_H^{A, B}[\cdot] = \sum_{\substack{\partial \mathbf{n}_1 = A \\ \partial \mathbf{n}_2 = B}} w(\mathbf{n}_1) w(\mathbf{n}_2) \mathbb{1}\{(\mathbf{n}_1, \mathbf{n}_2) \in \cdot\}, \quad (8.B.15)$$

where the sums over $\mathbf{n}_1, \mathbf{n}_2$ are over currents on H and the dot represents any set of pairs of currents. When H is all of $\mathcal{P}_2$, we omit the subscript H . Then (8.B.13) can be written as

$$\frac{1}{2}|U_4(0, x, y, z)| = \langle \sigma_0 \sigma_x \rangle \langle \sigma_y \sigma_z \rangle - \frac{Z^{0x, yz}[\mathbf{C}(0) \cap \mathbf{C}(y) = \emptyset]}{Z^{\emptyset, \emptyset}}. \quad (8.B.16)$$

To prepare for an application of the switching lemma in the proof of Lemma 8.B.3, we wish to arrange that one of the two currents be sourceless (as they are in T_1 and T_2).

This can be achieved by the classical manoeuvre of *peeling off* the cluster of 0, by conditioning on $\mathbf{C}^b(0)$:

$$\begin{aligned} Z^{0x, yz}[\mathbf{C}(0) \cap \mathbf{C}(y) = \emptyset] &= \sum_{H \subset \mathcal{P}_2} Z_H^{0x, \emptyset}[\mathbf{C}^b(0) = H] Z_{H^c}^{\emptyset, yz} \\ &= \sum_{H \subset \mathcal{P}_2} Z^{0x, \emptyset}[\mathbf{C}^b(0) = H] \langle \sigma_y \sigma_z \rangle^{H^c} = Z^{0x, \emptyset} \mathbf{E}^{0x, \emptyset}[\langle \sigma_y \sigma_z \rangle^{\mathbf{C}^b(0)^c}]. \end{aligned} \quad (8.B.17)$$

For every $E \subset \mathcal{P}_2$, it follows from the second Griffiths inequality that

$$\langle \sigma_y \sigma_z \rangle^{\mathbf{C}^b(0)^c} \leq \langle \sigma_y \sigma_z \rangle^{(\mathbf{C}^b(0) \cap E)^c} = \langle \sigma_y \sigma_z \rangle_0. \quad (8.B.18)$$

Therefore,

$$Z^{0x, yz}[\mathbf{C}(0) \cap \mathbf{C}(y) = \emptyset] \leq Z^{0x, \emptyset} \mathbf{E}^{0x, \emptyset}[\langle \sigma_y \sigma_z \rangle_0] = Z^{\emptyset, \emptyset} \langle \sigma_0 \sigma_x \rangle \mathbf{E}^{0x, \emptyset}[\langle \sigma_y \sigma_z \rangle_0]. \quad (8.B.19)$$

We insert this into (8.B.16) and obtain

$$\frac{1}{2}|U_4(0, x, y, z)| \geq \langle \sigma_0 \sigma_x \rangle \mathbf{E}^{0x, \emptyset}[\langle \sigma_y \sigma_z \rangle_1 - \langle \sigma_y \sigma_z \rangle_0]. \quad (8.B.20)$$

By the Fundamental Theorem of Calculus,

$$\begin{aligned} \langle \sigma_y \sigma_z \rangle_1 - \langle \sigma_y \sigma_z \rangle_0 &= \int_0^1 \partial_{\theta_1} \langle \sigma_y \sigma_z \rangle_{\theta_1} d\theta_1 \\ &= \sum_{\{u, v\} \in E} \beta J_{u, v} \mathbb{1}[u \text{ or } v \in \mathbf{C}(0)] \int_0^1 \langle \sigma_y \sigma_z; \sigma_u \sigma_v \rangle_{\theta_1} d\theta_1. \end{aligned} \quad (8.B.21)$$

The desired inequality (8.B.12) then follows from

$$\langle \sigma_y \sigma_z; \sigma_u \sigma_v \rangle_{\theta_1} = \langle \sigma_y \sigma_z; \sigma_u \sigma_v \rangle - \int_{\theta_1}^1 \partial_\theta \langle \sigma_y \sigma_z; \sigma_u \sigma_v \rangle_\theta d\theta \quad (8.B.22)$$

and Fubini's theorem. $\square$

In combination with Lemma 8.B.2, the bounds on T_1 and T_2 obtained in the next two lemmas prove Proposition 8.B.1. These bounds use the *dilution trick*, which consists of picking $E = \omega$ with ω distributed according to an independent Bernoulli(p) bond percolation on $\mathcal{P}_2$, and averaging over E . We denote the law and expectation of ω by $\mathbb{P}_p$ and $\mathbb{E}_p$, respectively. The dilution method is comparable to the second moment method (see e.g. [9, Lemma 4.4]) but it is more *linear* in the sense that it allows for resummation. This facilitates working with averaged quantities.

Lemma 8.B.3. *For all $\beta > 0$ and for all $p \in [0, 1]$,*

$$\sum_{x,y,z \in \mathbb{T}_r} \mathbb{E}_p[T_1(x, y, z, \omega)] \geq p\chi^2 \beta \frac{\partial \chi}{\partial \beta}. \quad (8.B.23)$$

Proof. By definition,

$$T_1(x, y, z, E) = \beta \sum_{\{u,v\} \in E} J_{u,v} \langle \sigma_0 \sigma_x \rangle \mathbf{P}^{0x,\emptyset}[u \text{ or } v \in \mathbf{C}(0)] \langle \sigma_y \sigma_z; \sigma_u \sigma_v \rangle \quad (8.B.24)$$

$$= \frac{\beta}{2} \sum_{u,v \in \mathbb{T}_r} \mathbf{1}[\{u, v\} \in E] J_{u,v} \langle \sigma_0 \sigma_x \rangle \mathbf{P}^{0x,\emptyset}[u \text{ or } v \in \mathbf{C}(0)] \langle \sigma_y \sigma_z; \sigma_u \sigma_v \rangle. \quad (8.B.25)$$

We first apply the switching lemma (Lemma 8.3.2) to see that

$$\langle \sigma_0 \sigma_x \rangle \mathbf{P}^{0x,\emptyset}[u \text{ or } v \in \mathbf{C}(0)] \geq \langle \sigma_0 \sigma_x \rangle \mathbf{P}^{0x,\emptyset}[0 \xrightarrow{\mathbf{n}_1 + \mathbf{n}_2} u] = \langle \sigma_0 \sigma_u \rangle \langle \sigma_u \sigma_x \rangle. \quad (8.B.26)$$

This enables us to take the percolation expectation and obtain

$$\mathbb{E}_p[T_1(x, y, z, \omega)] \geq \frac{\beta p}{2} \sum_{u,v \in \mathbb{T}_r} J_{u,v} \langle \sigma_0 \sigma_u \rangle \langle \sigma_u \sigma_x \rangle \langle \sigma_y \sigma_z; \sigma_u \sigma_v \rangle. \quad (8.B.27)$$

Summation over the torus then gives

$$\sum_{x,y,z \in \mathbb{T}_r} \mathbb{E}_p[T_1(x, y, z, \omega)] \geq \frac{\beta p}{2} \sum_{y,z,u,v \in \mathbb{T}_r} J_{u,v} \langle \sigma_0 \sigma_u \rangle \chi \langle \sigma_y \sigma_z; \sigma_u \sigma_v \rangle. \quad (8.B.28)$$

By the change of variables $z = z' + y$, $u = u' + y$, and $v = v' + y$, the right-hand side becomes

$$\frac{\beta p}{2} \chi \sum_{y,z',u',v' \in \mathbb{T}_r} J_{u',v'} \langle \sigma_0 \sigma_{u'+y} \rangle \langle \sigma_0 \sigma_{z'}; \sigma_{u'} \sigma_{v'} \rangle = \beta p \chi^2 \frac{\partial \chi}{\partial \beta}, \quad (8.B.29)$$

and the proof is complete. $\square$

Lemma 8.B.4. *For all $\beta > 0$ and for all $p \in [0, 1]$,*

$$\sum_{x,y,z \in \mathbb{T}_r} \mathbb{E}_p[T_2(x, y, z, \omega)] \leq 16p^2 \beta^2 |J|^2 \chi^4 \mathbf{B}. \quad (8.B.30)$$

Proof. By definition,

$$T_2(x, y, z, E) = \beta \sum_{\{u, v\} \in E} J_{u, v} \langle \sigma_0 \sigma_x \rangle \mathbf{E}^{0x, \emptyset} \left[\mathbb{1}[u \text{ or } v \in \mathbf{C}(0)] \int_0^1 \theta \partial_\theta \langle \sigma_y \sigma_z; \sigma_u \sigma_v \rangle_\theta d\theta \right]. \quad (8.B.31)$$

An explicit computation gives

$$\partial_\theta \langle \sigma_y \sigma_z; \sigma_u \sigma_v \rangle_\theta = \beta \sum_{\{s, t\} \in E} J_{s, t} \mathbb{1}[s \text{ or } t \in \mathbf{C}(0)] \langle \sigma_y \sigma_z; \sigma_u \sigma_v; \sigma_s \sigma_t \rangle_\theta, \quad (8.B.32)$$

where

$$\langle \sigma_y \sigma_z; \sigma_u \sigma_v; \sigma_s \sigma_t \rangle_\theta = \langle \sigma_y \sigma_z; \sigma_u \sigma_v \sigma_s \sigma_t \rangle_\theta - \langle \sigma_u \sigma_v \rangle_\theta \langle \sigma_y \sigma_z; \sigma_s \sigma_t \rangle_\theta - \langle \sigma_s \sigma_t \rangle_\theta \langle \sigma_y \sigma_z; \sigma_u \sigma_v \rangle_\theta. \quad (8.B.33)$$

When the edges uv and st are the same edge, the first term on the right-hand side of (8.B.33) vanishes, as $\sigma_u \sigma_v \sigma_s \sigma_t = 1$. In this case, we have $\langle \sigma_y \sigma_z; \sigma_u \sigma_v; \sigma_u \sigma_v \rangle_\theta \leq 0$. Therefore,

$$T_2(x, y, z, E) \leq \frac{\beta^2}{4} \sum_{\substack{u, v, s, t \in \mathbb{T}_r \\ \{u, v\} \neq \{s, t\}}} \mathbb{1}[\{u, v\}, \{s, t\} \in E] J_{u, v} J_{s, t} \langle \sigma_0 \sigma_x \rangle \quad (8.B.34)$$

$$\times \mathbf{E}^{0x, \emptyset} \left[\mathbb{1}[u \text{ or } v \in \mathbf{C}(0)] \mathbb{1}[s \text{ or } t \in \mathbf{C}(0)] \int_0^1 \theta \langle \sigma_y \sigma_z; \sigma_u \sigma_v; \sigma_s \sigma_t \rangle_\theta d\theta \right]. \quad (8.B.35)$$

It is convenient to relabel $u = u_1, v = u_2, t = t_1, s = t_2$. For the integral over θ , we use a minor modification¹ of [12, Proposition 5.2], followed by the monotonicity of the two-point function in θ , to get

$$\langle \sigma_y \sigma_z; \sigma_{u_1} \sigma_{u_2}; \sigma_{t_1} \sigma_{t_2} \rangle_\theta \leq \sum_{i, j=1}^2 2 \langle \sigma_y \sigma_{u_i} \rangle \langle \sigma_{u_{\rho(i)}} \sigma_{t_j} \rangle \langle \sigma_{t_{\rho(j)}} \sigma_z \rangle + (y \Leftrightarrow z), \quad (8.B.36)$$

where $\rho(1) = 2, \rho(2) = 1$, and $(y \Leftrightarrow z)$ denotes the sum with the roles of y and z interchanged. This allows us to bound the integral using $\int_0^1 \theta d\theta = 1/2$. For the remaining expectation in (8.B.34), we use [9, Proposition A.3], which leads to

$$\langle \sigma_0 \sigma_x \rangle \mathbf{P}^{0x, \emptyset} [u_1 \text{ or } u_2 \in \mathbf{C}(0), t_1 \text{ or } t_2 \in \mathbf{C}(0)] \leq \sum_{i, j=1}^2 \langle \sigma_0 \sigma_{u_i} \rangle \langle \sigma_{u_i} \sigma_{t_j} \rangle \langle \sigma_{t_j} \sigma_x \rangle + (0 \Leftrightarrow x), \quad (8.B.37)$$

where $(0 \Leftrightarrow x)$ denotes the sum with the roles of 0 and x interchanged.

We insert the above two inequalities into (8.B.34), average over $E = \omega$, and then sum over $x, y, z \in \mathbb{T}_r$. The sums over x, y, z produce χ^3 . Then, by translation invariance, we can shift the origin to one of u_1, u_2, t_1, t_2 , and then sum over the former origin to get another factor of

¹In [12, Proposition 5.2], the Ising model is studied with an external magnetic field h . For the random current representation, this involves the introduction of a *ghost* vertex g with spin $\sigma_g = 1$, and [12, Proposition 5.2] gives a version of (8.B.36) in which the vertex z is equal to the ghost. The proof of [12, Proposition 5.2] adapts to our setting with minor modifications.

χ . The remaining sums are of three kinds and can all be written as convolutions. In terms of the torus convolution defined in (8.2.9), this produces

$$\begin{aligned} & \sum_{x,y,z \in \mathbb{T}_r} \mathbb{E}_p[T_2(x,y,z,\omega)] \\ & \leq \frac{\beta^2 p^2}{4} \left(\frac{1}{2}\right)(2)(4)\chi^4 \left(4\|J \star \tau^2 \star J\|_1 + 8\|J\|_1(\tau \star J \star \tau)(0) + 4(J \star \tau \star J \star \tau)(0)\right), \end{aligned} \quad (8.B.38)$$

where the factor 2 is from (8.B.36) and the first factor 4 is from $(0 \Leftrightarrow x)$ and $(y \Leftrightarrow z)$. We then apply Young's convolution inequality to see that

$$\|J \star \tau^2 \star J\|_1 \leq \|J\|_1^2 \|\tau^2\|_1, \quad (8.B.39)$$

$$\|J\|_1(\tau \star J \star \tau)(0) \leq \|J\|_1^2 \|\tau\|_2^2, \quad (8.B.40)$$

$$(J \star \tau \star J \star \tau)(0) \leq \|J\|_1^2 \|\tau\|_2^2. \quad (8.B.41)$$

These right-hand sides are all equal to $|J|^2 \mathbf{B}(\beta)$. Altogether, this gives an upper bound $16\beta^2 p^2 \chi^4 |J|^2 \mathbf{B}$ for the right-hand side of (8.B.38), which concludes the proof. $\square$

8.C Percolation

The conjectures for the Ising model in Section 8.1.3 have counterparts for Bernoulli bond (or site) percolation on $\mathbb{Z}^d$ in dimensions $d \geq 6$, which we discuss now.

Let W^* denote the Brownian excursion of length 1, let $\Psi(x) = \mathbb{E} \exp[x \int_0^1 W^*(t) dt]$ denote the moment generating function of the Brownian excursion area, let

$$F(x, s) = \frac{1}{6}x^3 - \frac{s}{2}x^2 + \frac{s^2}{2}x, \quad (8.C.1)$$

and define the conjectured *percolation profile*

$$f_{\text{perc}}(s) = \int_0^\infty x^2 d\sigma_s, \quad d\sigma_s = \frac{1}{\sqrt{2\pi}} x^{-5/2} \Psi(x^{3/2}) e^{-F(x,s)} dx. \quad (8.C.2)$$

On the torus, we write $\tau_p^{\mathbb{T}_r}(0, x) = \mathbb{P}_p^{\mathbb{T}_r}(0 \leftrightarrow x)$ for the percolation two-point function, and $\chi^{\mathbb{T}_r}(p) = \sum_{x \in \mathbb{T}_r} \tau_p^{\mathbb{T}_r}(0, x)$ for the susceptibility. On the infinite lattice $\mathbb{Z}^d$, we denote the two-point function by τ_p and the critical value by p_c .

Conjecture 4 (Universal profile for percolation). *For $d > 6$, with window scale $w_r = a_d r^{-d/3}$ for suitably chosen $a_d > 0$, and for $s \in \mathbb{R}$, as $r \rightarrow \infty$,*

$$\chi^{\mathbb{T}_r}(p_c + s w_r) \sim \text{const}_d f_{\text{perc}}(s) r^{d/3}, \quad (8.C.3)$$

$$\tau_{p_c + s w_r}^{\mathbb{T}_r}(0, x) - \tau_{p_c}(0, x) \sim \text{const}_d f_{\text{perc}}(s) r^{-2d/3}. \quad (8.C.4)$$

We also conjecture that (8.C.3) and (8.C.4) hold for percolation under free boundary conditions (i.e., only using bonds in Λ_r), after p_c is replaced by an effective critical value $p_{c,r} = p_c + v_r$ with $v_r \asymp r^{-2}$. If this is true, then since v_r is larger than the window w_r , the two FBC and PBC windows would not overlap.

Upper and lower bounds consistent with (8.C.3) and (8.C.4) are proved in [154] for spread-out percolation in dimensions $d > 6$, but asymptotic formulas have not been proved and the role of the profile has not been demonstrated. For $d = 6$, nothing close to Conjecture 4 has been proved, though we expect that the profile still applies after appropriate logarithmic corrections to w_r , v_r , and to the factors $r^{d/3}$ and $r^{-2d/3}$ in (8.C.3)–(8.C.4).

Logarithmic corrections for $d = 6$ are discussed in the physics literature, e.g., in [2, 213], on the basis of connections with the $q \rightarrow 1$ limit of the q -state Potts model and φ^3 field theory. Mathematically, the connection between percolation and φ^3 theory remains mysterious, and to our knowledge there are no rigorous results in this direction. Indeed, the very definition of the φ^3 model is problematic, since e^{φ^3} is not integrable. Nevertheless, there are indications of a φ^3 structure in the mathematical work on percolation: the percolation diagram in Figure 8.1 has vertices of degree 3, as do the tree-graph diagrams of [16], and the lace expansion diagrams of [134]. Also, the x^3 in the function F in $d\sigma_s$ is reminiscent of φ^3 .

The conjectured profile arises by analogy with the Erdős–Rényi random graph, where the properly rescaled cluster size (without taking expectation) is known to converge in distribution to a random variable described by the Brownian excursion [17]. The limiting random variable is also characterised by a point process [156]. The measure σ_s is the intensity of that point process and is taken from [156, Theorem 4.1]. The point process describes cluster sizes, in the sense that

$$n^{-2k/3} \sum_i |C_i|^k \Rightarrow \int_0^\infty x^k d\sigma_s \quad (k \geq 2) \quad (8.C.5)$$

in distribution [17]. The $k = 2$ case gives the properly rescaled susceptibility and identifies $f_{\text{perc}}(s)$ in (8.C.3). To our knowledge, the convergence of the rescaled susceptibility has not been proved, not even for the Erdős–Rényi random graph.

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Chapter 9

Triviality of critical Ising and φ^4 models of effective dimension at least four

We prove that any scaling limit of a critical reflection positive Ising or φ^4 model of effective dimension d_{eff} at least four is Gaussian. This extends the recent breakthrough work of Aizenman and Duminil-Copin (Ann. Maths., 2021)— which demonstrates the corresponding result in the setup of nearest-neighbour interactions in dimension four— to the case of long-range reflection positive interactions satisfying $d_{\text{eff}} = 4$. The proof relies on the random current representation which provides a geometric interpretation of the deviation of the models’ correlation functions from Wick’s law. When $d = 4$, long-range interactions are handled with the derivation of a criterion that relates the speed of decay of the interaction to two different mechanisms that entail Gaussianity: interactions with a sufficiently slow decay induce a faster decay at the level of the model’s two-point function, while sufficiently fast decaying interactions force a simpler geometry on the currents which allows us to extend nearest-neighbour arguments. When $1 \leq d \leq 3$ and $d_{\text{eff}} = 4$, the phenomenology is different as long-range effects play a prominent role.

This chapter is based on the preprint “*Triviality of the scaling limits of critical Ising and φ^4 models with effective dimension at least four*” [204].

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9.1 Introduction

9.1.1 Motivation

We are interested in ferromagnetic real-valued spin models on $\mathbb{Z}^d$ that arise in statistical mechanics. Mathematically, these models can be seen as probability measures on spin configurations $\tau : \mathbb{Z}^d \rightarrow \mathbb{R}$ formally given by

$$\langle F(\tau) \rangle_\beta = \frac{1}{Z} \int F(\tau) \exp \left(\beta \sum_{x,y \in \mathbb{Z}^d} J_{x,y} \tau_x \tau_y \right) \prod_{x \in \mathbb{Z}^d} d\rho(\tau_x), \quad (9.1.1)$$

where $\beta > 0$ is the inverse temperature, $J_{x,y} \geq 0$ are (possibly long-range) interactions, Z is a normalisation constant, and $d\rho$ is a single-site probability measure. Of particular interest to us are the Ising model which corresponds to choosing $d\rho$ to be the uniform measure on

$\{-1, +1\}$, and the φ^4 model which corresponds to confining the spins in a quartic potential given by

$$d\rho(\varphi) = \frac{1}{z_{g,a}} e^{-g\varphi^4 - a\varphi^2} d\varphi, \quad (9.1.2)$$

with $g > 0$, $a \in \mathbb{R}$ and $z_{g,a}$ a normalisation constant, and where $d\varphi$ is the Lebesgue measure on $\mathbb{R}$.

The study of these models plays a key role in two distinct, yet interacting, research areas: *constructive Euclidean field theory* and *statistical mechanics*.

Constructive Euclidean field theory aims at constructing random distributions on $\mathbb{R}^d$, with a particular focus on interacting, or non-*trivial*, field theories. This contrasts with the study of Gaussian fields, which have a trivial correlation function structure in the sense that it is entirely determined by their two-point function via Wick's law. A natural attempt to build non-Gaussian field theories is to try to define a measure on the set of functions $\mathbb{R}^d \rightarrow \mathbb{R}$ whose averages are given by

$$\langle F(\Phi) \rangle = \frac{1}{Z} \int F(\Phi) \exp(-H(\Phi)) \prod_{x \in \mathbb{R}^d} d\Phi_x, \quad (9.1.3)$$

with

$$H(\Phi) := \int_{\mathbb{R}^d} [A|\nabla\Phi(x)|^2 + B|\Phi(x)|^2 + P(\Phi(x))] dx, \quad (9.1.4)$$

where $A, B > 0$ and P is an even polynomial of degree 4 with a strictly positive leading coefficient. This choice corresponds to what would be the definition of the φ^4 field theory on $\mathbb{R}^d$. Due to the lack of a natural Lebesgue measure on infinite dimensional spaces, the above quantity is ill-defined. However, it is still possible to make sense of it using a pair of *ultraviolet* (short distance) and *infrared* (long distance) cutoffs. Highlights of this approach include the rigorous construction of the φ^4 measure, with infrared cutoff, in dimension two by Nelson [195], and in dimension three by Glimm and Jaffe [110]. These works were later extended to the infinite volume limit [188, 113]. A few years after these first results, Aizenman [4] and Fröhlich [102] showed that φ^4 is not a good candidate to construct interacting field theories when $d \geq 5$. In their works, they proved that any field obtained as a scaling limit of critical Ising or φ^4 models in dimension $d \geq 5$ is Gaussian. These papers, and other subsequent works [235, 14, 71, 107, 94, 139, 25], provided strong heuristics that the same result should hold in dimension $d = 4$. It was not until very recently that these heuristics were confirmed by the breakthrough work of Aizenman and Duminil-Copin [9].

Constructive Euclidean field theory is also closely related to *constructive quantum field theory* (CQFT). Indeed, the Osterwalder-Schrader Theorem [201, 202] provides a way to build quantum field theories in the sense proposed by Wightman [249] from Euclidean field theories. We refer to [113, 9, 6] for a more complete description of the CQFT point of view.

From the perspective of statistical mechanics, the Ising model and the φ^4 model are among the simplest examples which exhibit a phase transition¹ at a critical parameter $\beta_c \in (0, \infty)$. As proved in [10, 126], this phase transition is *continuous* for reflection positive interactions², and one of the main challenges of the field is to understand the nature of their scaling limits at criticality.

¹The conditions J must satisfy for a phase transition to occur are recalled below.

²This result holds for all reflection positive interactions if $d \geq 3$, and is restricted to some interactions when $d \in \{1, 2\}$.

The connection between the Ising model and the φ^4 model is predicted to be very rich: they are believed to belong to the same universality class. Renormalisation group heuristics (see [119, 160] or the recent book [29]) predict that at their respective critical points many of their properties (e.g. critical exponents) coincide exactly. Hints of these deep links were established by Griffiths and Simon in [224], where they show that the φ^4 model emerges as a certain near-critical scaling limit of a collection of mean-field Ising models. This permits one to rigourously transfer many useful properties of the Ising model, such as correlation inequalities, to the φ^4 model. In the other direction, the Ising model can be obtained as a limit of φ^4 using the following limit

$$\frac{\delta_{-1} + \delta_1}{2} = \lim_{g \rightarrow \infty} \frac{1}{z_{g, -2g}} e^{-g(\varphi^2 - 1)^2 + g} d\varphi. \quad (9.1.5)$$

The high dimension triviality results mentioned above are related to the simplicity of the critical exponents of these models, suggesting that for $d \geq 4$, they must take their *mean-field* values. Rigorous results in that direction have been obtained in [4, 14, 12, 13, 29, 192]. What appeared to be a negative result from the perspective of constructive Euclidean field theory is positive in the framework of statistical mechanics as it provides information at criticality for a wide class of non-integrable models.

The main step in the proof of Aizenman and Fröhlich in dimension $d \geq 5$ is the derivation of the so-called *tree diagram bound* through geometric representations, and the use of *reflection positivity* [104, 103] to argue that Ursell's four-point function's scaling limit always vanishes at criticality. Although initially presented in the case of nearest-neighbour interactions $J_{x,y} = \mathbb{1}_{|x-y|=1}$, these methods are robust and extend to more general (in particular long-range) reflection positive interactions. However, this is no longer true in dimension four, where only the case of nearest-neighbour interactions is treated [9].

The interest in the study of long-range interactions comes from the fact that the rate of decay of the interactions may change the effective dimension of the model by increasing it, meaning that one can recover high-dimensional features in some well-chosen one, two or three dimensional systems. An observation of this phenomenon was made for algebraically decaying long-range interactions of the form $1/r^{d+\alpha}$ by Fischer, Ma and Nickel [99] using renormalisation group heuristics. They noted that the parameter α had the effect of changing the value of the upper critical dimension³ into $d_c(\alpha) = \min(2\alpha, 4)$, suggesting that the effective dimension of the model should be given by $d_{\text{eff}}(\alpha) = d/(1 \wedge (\alpha/2))$ (see Figure 9.1). This was later studied by Aizenman and Fernández [13], and led to the observation that some Ising models in dimension $1 \leq d \leq 3$ present trivial scaling limits at criticality, which is not expected in the case of nearest-neighbour interactions⁴. Other rigorous results were obtained through lace expansion methods [145, 67, 68]. Conversely, if the interaction decays fast enough, the upper critical dimension of the model is unchanged (this corresponds to $\alpha \geq 2$ in the example above). The prediction is that only two situations may occur: either the interaction decays very fast and we expect to fall into the universality class of the nearest-neighbour models, or

³That is, the dimension above which the system simplifies drastically, adopting Gaussian features. This is also the dimension above which the *bubble diagram* converges, which is an indicator of mean-field behaviour as recalled below.

⁴Indeed, non-triviality of the nearest-neighbour Ising model has been proven for $d = 2$ in [4], while the case $d = 3$ remains open. Let us mention that the recent conformal bootstrap approach to the study of the critical 3d Ising model strongly supports this conjecture [215].

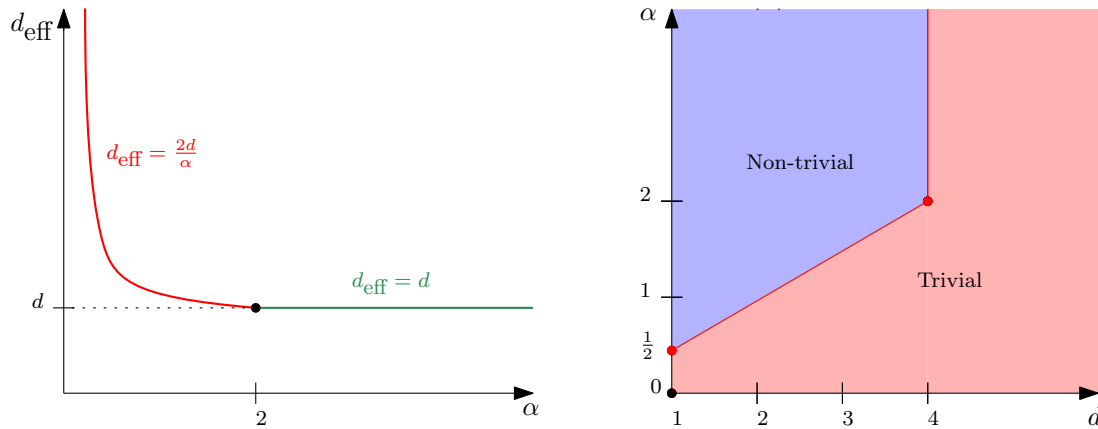


Figure 9.1: LEFT: The graph of $\alpha \mapsto d_{\text{eff}}(\alpha)$ for the interaction J given by $J_{x,y} = C|x-y|_1^{-d-\alpha}$ (for $d \geq 2$). The transition between the regime $d_{\text{eff}}(\alpha) > d$ and $d_{\text{eff}} = d$ occurs at $\alpha = 2$. At this point, we expect logarithmic corrections at the level of the decay of the critical two-point function. RIGHT: A summary of the expected behaviour of the critical scaling limits for the same interaction J . The red region (including segments and points), correspond to interactions which are expected to yield trivial scaling limits. The results of this chapter concern the study of the “marginal” cases that separate the two phases.

the decay is *exactly* fast enough for additional logarithmic corrections to appear. The latter scenario was shown to occur [68] in dimension $d \geq 4$, for (sufficiently spread out) interactions decaying such as $1/r^{d+2}$.

Finally, let us briefly mention that long-range interactions of the above type have been used to conduct rigorously the so-called “ $(d_c - \varepsilon)$ -expansions”—motivated by Wilson and Fisher through renormalisation group heuristics [252]—which give a precise understanding of the critical exponents of these models below the upper critical dimension, see [99, 238, 46, 227].

The goal of this chapter is threefold. First, we prove that reflection positive Ising or φ^4 models with effective dimension strictly above four are trivial. This revisits some of the results of [4, 13] together with the notion of effective dimension, and provides explicit examples of trivial models in dimensions one, two, and three. Second, we extend the results of [9] to near-critical and critical reflection positive Ising and φ^4 models in dimension $d = 4$ beyond the nearest-neighbour case. In particular, this case contains algebraically decaying interactions (as above) with $\alpha > 0$. The result was already known [13] for $\alpha \in (0, 2)$ but is new in the case $\alpha \geq 2$. Third, we prove triviality of the scaling limits of one, two, and three dimensional reflection positive Ising models with effective dimension four. This is the main novelty. Such examples of models can be obtained choosing $\alpha = d/2$ above for $1 \leq d \leq 3$. Our results apply to a wide class of single-site measures called the *Griffiths–Simon class* of measures (see Section 9.2), which in particular contains the examples mentioned below (9.1.1), and which can be recovered from weak limits of Ising-type single site measures.

As in [4, 9], we use the random current representation of the Ising model which enables, by means of the *switching lemma*, to express the correlation functions’ deviation from Wick’s law in terms of intersection probabilities of two independent random currents with distanced sources.

When $d = 4$, two situations may occur. First, the interaction’s decay may be “slow”,

in which case we observe a decay of the model's two-point function which is slightly better than the one obtained for nearest-neighbour interactions (see Corollary 9.3.10). We can then conclude using the *tree diagram bound* obtained in [4]. In particular, this first case contains reflection positive interactions of algebraic decay with $\alpha \in (0, 2]$. Second, the decay of the interaction may be “too fast”, in which case we observe no improvement at the level of the decay of the two-point function. As explained above, this case corresponds to the situation where we expect the model to behave like a nearest-neighbour one, this corresponds to choosing $\alpha > 2$ above. We follow the strategy of [9] and improve the tree diagram bound. The proof goes by arguing that in dimension four, just like random walks, if two independent random currents intersect at least once, they must re-intersect a large number of times. By the mean of a multi-scale analysis, the authors of [9] showed that intersections occur with large probability in a density of (well-chosen) scales. This essentially required three tools: regularity properties for the model's two-point function, a proof of the fact that intersections happen with (uniform) positive probability on all scales, and a mixing statement which allows one to argue that intersections at different scales are roughly independent events. However, in the case of long-range interactions, these steps fail. Indeed, the extension of the proof to the general setup requires an adaptation of the reflection positivity arguments to the case of arbitrary interactions which builds on a different view point on the spectral analysis of these models (see Section 9.3 and Appendix 9.A). This viewpoint was already introduced in [41, 203]. Then, long-range interactions may have the effect of making intersections less likely as it becomes possible to “jump” scales. Finally, long-range interactions may create more dependency between pieces of the current at different scales. We solve these problems by arguing that the currents do not jump above a $(4 - \varepsilon)$ -dimensional annulus with very high probability (see Section 9.6.2). As it turns out, this is enough to (essentially) recover the same geometric properties of currents as in the nearest-neighbour case.

When $1 \leq d \leq 3$ and $d_{\text{eff}} = 4$, the above improvements are not sufficient. The main reason is that in this precise regime, the decay of the interaction is too slow to exclude jumps above $(d - \varepsilon)$ -dimensional annuli (see Remark 9.6.10). This additional difficulty is treated by going one step further in the analysis of the currents (see Section 9.7.2). As a byproduct of our methods, we obtain a (quantitative) mixing statement that is valid for all models of effective dimension at least four (see Section 9.7.3).

To study the near-critical regime, it is important to introduce a typical length below which the model essentially behaves like a critical one. It is tempting to use the correlation length $\xi(\beta)$ defined for $\beta < \beta_c$, by

$$\xi(\beta) := - \left(\lim_{n \rightarrow \infty} \frac{\log \langle \tau_0 \tau_{n\mathbf{e}_1} \rangle_\beta}{n} \right)^{-1}. \quad (9.1.6)$$

However, this quantity is not relevant in the case of long-range interactions since one may have $\xi(\beta) = \infty$ for all $\beta < \beta_c$ (see [199, 20, 21]). Another contribution of this chapter is the introduction of the *sharp length* $L(\beta)$ (defined in Section 9.3.6), whose definition is inspired by [84].

We first prove an improved tree diagram bound for the Ising model, and then extend it to the φ^4 model (and more generally to every model in the Griffiths–Simon class) using its viewpoint as a generalised Ising model.

Let us mention that we also expect a *direct* analysis, meaning at the level of the φ^4 model and without any mention of the Ising model, to be possible with the use of the *random tangled*

current representation of φ^4 recently introduced in [126].

9.1.2 Definitions and statement of the results

We start by stating the results for the case of the Ising model.

9.1.2.1 Results for the Ising model

In what follows, Λ is a finite subset of $\mathbb{Z}^d$. Let $J = (J_{x,y})_{\{x,y\} \subset \mathbb{Z}^d}$ be an interaction (or a collection of coupling constants) and $h \in \mathbb{R}$. For $\sigma = (\sigma_x)_{x \in \Lambda} \in \{\pm 1\}^\Lambda$, introduce the *Hamiltonian*

$$H_{\Lambda,J,h}(\sigma) := - \sum_{\{x,y\} \subset \Lambda} J_{x,y} \sigma_x \sigma_y - h \sum_{x \in \Lambda} \sigma_x, \quad (9.1.7)$$

and define the associated finite volume Gibbs equilibrium measure $\langle \cdot \rangle_{\Lambda,J,h,\beta}$ at inverse temperature $\beta \geq 0$ to be the probability measure under which, for each $F : \{\pm 1\}^\Lambda \rightarrow \mathbb{R}$,

$$\langle F \rangle_{\Lambda,J,h,\beta} := \frac{1}{Z(\Lambda,J,h,\beta)} \sum_{\sigma \in \{\pm 1\}^\Lambda} F(\sigma) \exp(-\beta H_{\Lambda,J,h}(\sigma)), \quad (9.1.8)$$

where

$$Z(\Lambda,J,h,\beta) := \sum_{\sigma \in \{\pm 1\}^\Lambda} \exp(-\beta H_{\Lambda,J,h}(\sigma)), \quad (9.1.9)$$

is the *partition function* of the model. We make the following assumptions on the interaction J :

(A1) Ferromagnetic: For all $x, y \in \mathbb{Z}^d$, $J_{x,y} \geq 0$,

(A2) Locally finite: For any $x \in \mathbb{Z}^d$,

$$|J| := \sup_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} J_{x,y} < \infty, \quad (9.1.10)$$

(A3) Translation invariant: For all $x, y \in \mathbb{Z}^d$, $J_{x,y} = J_{0,y-x}$,

(A4) Irreducible: For all $x, y \in \mathbb{Z}^d$, there exist $x_1, \dots, x_k \in \mathbb{Z}^d$ such that

$$J_{x,x_1} J_{x_1,x_2} \dots J_{x_{k-1},x_k} J_{x_k,y} > 0, \quad (9.1.11)$$

(A5) Reflection positive: see Section 9.3.

We postpone the definition of reflection positivity to Section 9.3.1, but to fix the ideas the reader might keep in mind the following examples⁵ of interactions which satisfy **(A1)**–**(A5)**:

1. (nearest-neighbour interactions) $J_{x,y} = C \mathbb{1}_{|x-y|_1=1}$ for $C > 0$,
2. (exponential decay / Yukawa potentials) $J_{x,y} = C \exp(-\mu|x-y|_1)$ for $\mu, C > 0$,
3. (algebraic decay) $J_{x,y} = C|x-y|_1^{-d-\alpha}$ for $\alpha, C > 0$.

⁵Here, $|\cdot|_1$ refers to the ℓ^1 norm on $\mathbb{R}^d$.

Using Griffiths' inequalities [116], one can obtain the associated infinite volume Gibbs measure by taking weak limits of $\langle \cdot \rangle_{\Lambda, J, h, \beta}$ as $\Lambda \nearrow \mathbb{Z}^d$. We denote the limit by $\langle \cdot \rangle_{J, h, \beta}$. For convenience, in what follows, we omit the mention of the interaction in the notation of the Gibbs measures.

In dimensions $d > 1$, the model exhibits a phase transition for the vanishing of the *spontaneous magnetisation*. That is, if

$$m^*(\beta) := \lim_{h \rightarrow 0^+} \langle \sigma_0 \rangle_{\beta, h}, \quad (9.1.12)$$

then, $\beta_c := \inf\{\beta > 0, m^*(\beta) > 0\} \in (0, \infty)$. The above assumptions guarantee [97] that $\beta_c > 0$ (in fact $\beta_c \geq |J|^{-1}$), while Peierls' celebrated argument [208] yields the bound $\beta_c < \infty$. In dimension $d = 1$, the phase transition occurs [86] under the additional assumption that $J_{x,y} \asymp |x - y|^{-1-\alpha}$ with $\alpha \in (0, 1]$. We now assume that $h = 0$. Our results concern the nature of the scaling limits at⁶, or near the critical parameter β_c .

To determine the nature of the scaling limit, we look at the joint distribution of the *smearred observables*⁷ given for $\beta > 0$ and $L \geq 1$ by

$$T_{f,L,\beta}(\sigma) := \frac{1}{\sqrt{\Sigma_L(\beta)}} \sum_{x \in \mathbb{Z}^d} f\left(\frac{x}{L}\right) \sigma_x, \quad (9.1.14)$$

where f ranges over the set $\mathcal{C}_0(\mathbb{R}^d)$ of continuous, real valued, and compactly supported functions, and where

$$\Sigma_L(\beta) := \left\langle \left(\sum_{x \in \Lambda_L} \sigma_x \right)^2 \right\rangle_\beta = \sum_{x,y \in \Lambda_L} \langle \sigma_x \sigma_y \rangle_\beta, \text{ with } \Lambda_L := [-L, L]^d \cap \mathbb{Z}^d. \quad (9.1.15)$$

Definition 9.1.1. A discrete system as above is said to converge in distribution to a scaling limit if the collection of random variables $(T_{f,L,\beta}(\sigma))_{f \in \mathcal{C}_0(\mathbb{R}^d)}$ converges in distribution (in the sense of finite dimensional distributions) as L goes to infinity. Using Kolmogorov's extension theorem and the separability of $\mathcal{C}_0(\mathbb{R}^d)$, we can represent any scaling limit as a random field.

Our first result concerns the study of models of effective dimension $d_{\text{eff}} > 4$. We postpone the precise definition of effective dimension to Section 9.5 and illustrate this concept using the example of algebraically decaying reflection positive interactions mentioned above, i.e. $J_{x,y} = C|x - y|_1^{-d-\alpha}$ for $\alpha, C > 0$. In that case, we will see that $d_{\text{eff}} \geq \frac{d}{1 \wedge (\alpha/2)}$ so that the hypothesis $d_{\text{eff}} > 4$ corresponds to $d - 2(\alpha \wedge 2) > 0$.

Theorem 9.1.2. *Let $d \geq 1$. Let J be the interaction defined for $x \neq y \in \mathbb{Z}^d$ by $J_{x,y} = C_0|x - y|_1^{-d-\alpha}$ where $C_0, \alpha > 0$. We also assume that $d - 2(\alpha \wedge 2) > 0$. There exists $C =$*

⁶At a parameter $\beta < \beta_c$, finiteness of the susceptibility [8] implies that the scaling limit is the (Gaussian) white noise distribution [197].

⁷Note that for $f = \mathbf{1}_{[-1,1]^d}$,

$$\langle T_{f,L,\beta}(\sigma)^2 \rangle_\beta = 1, \quad (9.1.13)$$

and more generally for $f \neq 0$, one has $0 < c_f \leq \langle T_{f,L,\beta}(\sigma)^2 \rangle_\beta \leq C_f < \infty$, which means that the following quantity is bounded away from 0 and ∞ by constants that only depend on f . This indicates that this is the scaling that is the most likely to yield interesting limits.

$C(C_0, d), \gamma = \gamma(d) > 0$ such that for all $\beta \leq \beta_c$, $L \geq 1$, $f \in \mathcal{C}_0(\mathbb{R}^d)$ and $z \in \mathbb{R}$,

$$\left| \langle \exp(z T_{f,L,\beta}(\sigma)) \rangle_\beta - \exp\left(\frac{z^2}{2} \langle T_{f,L,\beta}(\sigma)^2 \rangle_\beta\right) \right| \leq \exp\left(\frac{z^2}{2} \langle T_{f,L,\beta}(\sigma)^2 \rangle_\beta\right) \frac{C(\beta^{-4} \vee \beta^{-2}) \|f\|_\infty^4 r_f^\gamma z^4}{L^{d-2(\alpha \wedge 2)}},$$

where $\|f\|_\infty = \sup_{x \in \mathbb{R}^d} |f(x)|$ and $r_f = \left(\max\{r \geq 0, \exists x \in \mathbb{R}^d, |x| = r, f(x) \neq 0\} \vee 1\right)$.

As a consequence, for $\beta \leq \beta_c$, every sub-sequential scaling limit (in the sense of Definition 9.1.1) of the model is Gaussian.

We now move the focus to the case $d = 4$. As explained in the introduction, and discussed in Section 9.4, the Gaussian behaviour of the model can be seen at the level of Ursell's four-point function [196, 4] defined for all $x, y, z, t \in \mathbb{Z}^d$ by

$$U_4^\beta(x, y, z, t) := \langle \sigma_x \sigma_y \sigma_z \sigma_t \rangle_\beta - \langle \sigma_x \sigma_y \rangle_\beta \langle \sigma_z \sigma_t \rangle_\beta - \langle \sigma_x \sigma_z \rangle_\beta \langle \sigma_y \sigma_t \rangle_\beta - \langle \sigma_x \sigma_t \rangle_\beta \langle \sigma_y \sigma_z \rangle_\beta. \quad (9.1.16)$$

In dimension $d > 4$, Aizenman [4] was able to conclude the triviality of the scaling limits using the tree diagram bound mentioned above,

$$|U_4^\beta(x, y, z, t)| \leq 2 \sum_{u \in \mathbb{Z}^d} \langle \sigma_x \sigma_u \rangle_\beta \langle \sigma_y \sigma_u \rangle_\beta \langle \sigma_z \sigma_u \rangle_\beta \langle \sigma_t \sigma_u \rangle_\beta, \quad (9.1.17)$$

together with the crucial input of reflection positivity, which implies (with the Messenger-Miracle-Solé inequalities [191]) the *infrared bound* [104, 103],

$$\langle \sigma_x \sigma_y \rangle_{\beta_c} \leq \frac{C}{|x - y|^{d-2}}, \quad (9.1.18)$$

where $|\cdot|$ denotes the infinite norm on $\mathbb{R}^d$. As noticed in [4], the relevant question is to see whether $|U_4^\beta(x, y, z, t)| / \langle \sigma_x \sigma_y \sigma_z \sigma_t \rangle_\beta$ vanishes or not, as the *mutual distance* $L(x, y, z, t) := \min_{u \neq v \in \{x, y, z, t\}} |u - v|$ between x, y, z and t goes to infinity but the distances between the pairs are all of the same order. The proof can be summed up by the following (incomplete) argument: assume that $\beta = \beta_c$ and that the two-point function is of comparable order for pairs of points at comparable distance; for a set of points x, y, z, t at mutual distance of order L , the sum of the right-hand side of (9.1.17) is of order $O(L^{8-3d})$, and we expect the four-point function $\langle \sigma_x \sigma_y \sigma_z \sigma_t \rangle_{\beta_c}$ to be of order of the product of two two-point functions, hence of order at least L^{4-2d} . As a result, we have

$$\frac{|U_4^{\beta_c}(x, y, z, t)|}{\langle \sigma_x \sigma_y \sigma_z \sigma_t \rangle_{\beta_c}} = O(L^{4-d}). \quad (9.1.19)$$

The above bound is clearly inconclusive in the case $d = 4$. However, in the case of nearest-neighbour interactions, (9.1.17) was improved by a logarithmic factor to obtain Gaussianity.

The case of long-range interactions is more subtle since we do not necessarily expect any improvement in the tree diagram bound in dimension 4. As it turns out, we do not need any such improvement when the decay of the interaction is sufficiently slow so that the decay of the model's two-point function is faster than (9.1.18).

To determine whether this decay is fast enough or not, it is (almost) enough to look at whether the following quantity is finite or not:

$$\mathbf{m}_2(J) := \sum_{x \in \mathbb{Z}^d} |x|^2 J_{0,x}. \quad (9.1.20)$$

When $\mathbf{m}_2(J) = \infty$, the decay of the interaction is slow enough to conclude using (9.1.17).

Theorem 9.1.3. *Let $d = 4$. Assume that J satisfies (A1)–(A5), and that $\mathbf{m}_2(J) = \infty$. Then, for all $\beta \leq \beta_c$, $f \in \mathcal{C}_0(\mathbb{R}^d)$ and $z \in \mathbb{R}$,*

$$\lim_{L \rightarrow \infty} \left| \langle \exp(z T_{f,L,\beta}(\sigma)) \rangle_\beta - \exp\left(\frac{z^2}{2} \langle T_{f,L,\beta}(\sigma)^2 \rangle_\beta\right) \right| = 0. \quad (9.1.21)$$

As a consequence, for $\beta \leq \beta_c$, every sub-sequential scaling limit of the model is Gaussian.

Remark 9.1.4. As we will see in Section 9.5, the rate of convergence to 0 can be expressed in terms of

$$\sum_{|x| \leq k} |x|^2 J_{0,x}. \quad (9.1.22)$$

For instance, in the case of J defined by $J_{x,y} = C|x - y|_1^{-d-2}$, one can check that $\mathbf{m}_2(J) = \infty$. The rate of convergence to 0 is then given by $C/\log L$.

We now discuss the case $\mathbf{m}_2(J) < \infty$. In fact, we will have to restrict our analysis to interactions J satisfying the following additional condition, which is slightly stronger:

(A6) There exist $\mathbf{C}, \varepsilon > 0$ such that for all $k \geq 1$,

$$\sum_{|x|=k} |x|^2 J_{0,x} \leq \frac{\mathbf{C}}{k^{1+\varepsilon}}. \quad (9.1.23)$$

As explained above, in this case we expect that the mechanism which leads to Gaussianity is the same as for the nearest-neighbour case. Hence, we first prove an improved tree diagram bound. The quantity $L(\beta)$ was briefly mentioned above and will be introduced in Section 9.3.6.

Theorem 9.1.5 (Improved tree diagram bound for $d = 4$). *Let $d = 4$. Assume that J satisfies (A1)–(A6). There exist $c, C > 0$ such that, for all $\beta \leq \beta_c$, for all $x, y, z, t \in \mathbb{Z}^4$ at mutual distance at least L of each other with $1 \leq L \leq L(\beta)$,*

$$|U_4^\beta(x, y, z, t)| \leq \frac{C}{B_L(\beta)^c} \sum_{u \in \mathbb{Z}^4} \langle \sigma_x \sigma_u \rangle_\beta \langle \sigma_y \sigma_u \rangle_\beta \langle \sigma_z \sigma_u \rangle_\beta \langle \sigma_t \sigma_u \rangle_\beta, \quad (9.1.24)$$

where $B_L(\beta)$ is the bubble diagram truncated at distance L defined by

$$B_L(\beta) := \sum_{x \in \Lambda_L} \langle \sigma_0 \sigma_x \rangle_\beta^2. \quad (9.1.25)$$

It is predicted, for an interaction J satisfying (A1)–(A6), that the bubble diagram diverges at criticality. This improves the $O(1)$ of (9.1.19) to a $O(B_L(\beta)^{-c})$.

Remark 9.1.6. As noticed in [4, 12], the *bubble condition*

$$B(\beta_c) < \infty, \quad (9.1.26)$$

implies that some of the model's critical exponents take their mean-field value. It is also possible to show that the bubble condition (together with some monotonicity properties of the two-point function), implies triviality of the scaling limits. We provide a proof of this fact in Appendix 9.D.

Remark 9.1.7. The reason why we restrict to interactions satisfying (A6) is technical and will become more transparent in Section 9.6.2. All the examples of reflection positive interactions we will encounter either satisfy (A6) or $\mathfrak{m}_2(J) = \infty$ (when $d = 4$). We strongly believe that the methods below are sufficient to conclude provided we know a sufficiently good estimate on quantities like (9.1.23).

One can check that the result still holds (see Remark 9.6.7) if one replaces (A6) by

(A6') There exist $\mathbf{C}, \varepsilon > 0$ such that for all $k \geq 1$,

$$\sum_{|x|=k} |x|^2 J_{0,x} \leq \frac{\mathbf{C}}{k(\log k)^{1+\varepsilon}}. \quad (9.1.27)$$

A natural question would be to ask what is the optimal condition J should satisfy when $\mathfrak{m}_2(J) < \infty$. For instance, is there any reflection positive interaction that does not satisfy (A6') but for which $\mathfrak{m}_2(J) < \infty$? The methods below should be optimal whenever the interaction J is such that the two-point function satisfies $\langle \sigma_0 \sigma_x \rangle_{\beta_c} \asymp |x|^{-2}$ as $|x| \rightarrow \infty$. As pointed out in [245, Section 7], the Green function associated to J has this decay provided

$$\sum_{|x| \geq k} J_{0,x} = O\left(\frac{1}{k^2 \log k}\right). \quad (9.1.28)$$

If we expect the two-point function to behave, at criticality, like the Green function of the random walk associated to J (as suggested by the infrared bound in Proposition 9.3.4), then the above appears to be the optimal condition. This leaves open the possibility of finding a reflection positive positive interaction which satisfies (A6') with $\varepsilon = 0$, but for which we still expect the triviality result to hold.

With the improved tree diagram bound, we can obtain a formulation of triviality similar to the one obtained in Theorem 9.1.3.

Corollary 9.1.8. *Let $d = 4$. Assume that J satisfies (A1)–(A6). There exist $C, c, \gamma > 0$ such that, for all $\beta \leq \beta_c$, $1 \leq L \leq L(\beta)$, $f \in \mathcal{C}_0(\mathbb{R}^d)$, and $z \in \mathbb{R}$,*

$$\left| \langle \exp(z T_{f,L,\beta}(\sigma)) \rangle_\beta - \exp\left(\frac{z^2}{2} \langle T_{f,L,\beta}(\sigma)^2 \rangle_\beta\right) \right| \leq \exp\left(\frac{z^2}{2} \langle T_{f,L,\beta}(\sigma)^2 \rangle_\beta\right) \frac{C \|f\|_\infty^4 r_f^\gamma z^4}{(\log L)^c}. \quad (9.1.29)$$

As a consequence, for $\beta = \beta_c$, every sub-sequential scaling limit of the model is Gaussian.

We now turn to the case of low dimensional models of effective dimension equal to four. As above, we illustrate the result by focusing on the case of algebraically decaying reflection positive interactions. The situation of interest corresponds to choosing $\alpha = d/2$ for $1 \leq d \leq 3$. More general versions of the following statements can be found in Section 9.7 (see Theorem 9.7.1 and Corollary 9.7.3).

For technical reasons (see Remark 9.7.2), the following statement is not as quantitative as the above ones, and for simplicity we restrict it to $\beta = \beta_c$.

Theorem 9.1.9 (Improved tree diagram bound for $1 \leq d \leq 3$). *Let $1 \leq d \leq 3$. Let J be the interaction defined for $x \neq y \in \mathbb{Z}^d$ by $J_{x,y} = C_0|x-y|_1^{-3d/2}$ (i.e. $\alpha = d/2$) where $C_0 > 0$. There exist $C > 0$ and a function $\psi : \mathbb{R} \rightarrow \mathbb{R}_{>0}$ which satisfies $\psi(t) \rightarrow \infty$ as $t \rightarrow \infty$, such that, for all $x, y, z, t \in \mathbb{Z}^d$ at mutual distance at least L of each other,*

$$|U_4^{\beta_c}(x, y, z, t)| \leq \frac{C}{\psi(B_L(\beta_c))} \sum_{u \in \mathbb{Z}^d} \langle \sigma_x \sigma_u \rangle_{\beta_c} \langle \sigma_y \sigma_u \rangle_{\beta_c} \langle \sigma_z \sigma_u \rangle_{\beta_c} \langle \sigma_t \sigma_u \rangle_{\beta_c}. \quad (9.1.30)$$

Remark 9.1.10. In fact, for $d = 1$, the result is much stronger and we recover the improvement of order $O(B_L(\beta)^{-c})$ obtained when $d = 4$. The precise statements will be given in Section 9.7.

We can still deduce a triviality statement from this improved tree diagram bound. It involves the so-called *renormalised coupling constant*. We begin with a definition. For $\sigma > 0$, we define the *correlation length of order σ* by: for $\beta < \beta_c$,

$$\xi_\sigma(\beta) := \left(\frac{\sum_{x \in \mathbb{Z}^d} |x|^\sigma \langle \sigma_0 \sigma_x \rangle_\beta}{\chi(\beta)} \right)^{1/\sigma}, \quad (9.1.31)$$

where $\chi(\beta) := \sum_{x \in \mathbb{Z}^d} \langle \sigma_0 \sigma_x \rangle_\beta$. As it turns out, the above quantity is well-defined when $J_{x,y} = C|x-y|^{-d-\alpha}$ as soon as $\sigma < \alpha$ (see for instance [199, 20, 21]). Also, by the results of [8], one has $\xi_\sigma(\beta) \rightarrow \infty$ as $\beta \rightarrow \beta_c$. For such a $\sigma > 0$, we introduce another convenient measure of the interaction called the renormalised coupling constant of order σ and defined for $\beta < \beta_c$ by:

$$g_\sigma(\beta) := -\frac{1}{\chi(\beta)^2 \xi_\sigma(\beta)^d} \sum_{x,y,z \in \mathbb{Z}^d} U_4^\beta(0, x, y, z). \quad (9.1.32)$$

The vanishing of the above quantity is known to imply triviality of the scaling limits of the model (see [196, Theorem 11] or [4, 235, 14]).

Corollary 9.1.11. *We keep the assumptions of Theorem 9.1.9. Then, for $\sigma \in (0, d/2)$,*

$$\lim_{\beta \nearrow \beta_c} g_\sigma(\beta) = 0. \quad (9.1.33)$$

As a consequence, for $\beta = \beta_c$, every sub-sequential scaling limit of the model is Gaussian.

9.1.2.2 Results for the φ^4 model

We now extend the above results to the φ^4 model. These results also extend to models in the Griffiths–Simon class of measures, whose definition is postponed to the next section. We refer to Section 9.8 for the general statement of triviality for these models.

We start with a proper definition of the φ^4 model. Let ρ be given by (9.1.2). As for the Ising model, the ferromagnetic φ^4 model on Λ is defined by the finite volume Gibbs equilibrium state: for $F : \mathbb{R}^\Lambda \rightarrow \mathbb{R}$,

$$\langle F(\varphi) \rangle_{\Lambda, \rho, \beta} = \frac{1}{Z(\Lambda, \rho, \beta)} \int F(\varphi) \exp(-\beta H_{\Lambda, J}(\varphi)) \prod_{x \in \Lambda} d\rho(\varphi_x), \quad (9.1.34)$$

where $Z(\Lambda, \rho, \beta)$ is the partition function and

$$H_{\Lambda, J}(\varphi) := - \sum_{\{x, y\} \subset \Lambda} J_{x, y} \varphi_x \varphi_y. \quad (9.1.35)$$

We call $\langle \cdot \rangle_{\rho, \beta}$ the model's infinite volume Gibbs measure. It is also possible to introduce a critical parameter $\beta_c(\rho)$, together with a sharp length $L(\rho, \beta)$, Ursell's four-point function $U_4^{\rho, \beta}$, and a renormalised coupling constant $g_\sigma(\rho, \beta)$. The extension of the results concerning models of effective dimension $d_{\text{eff}} > 4$ is quite straightforward and will be discussed in Section 9.5 (see Remark 9.5.7). We focus on the results for $1 \leq d \leq 4$ with $d_{\text{eff}} = 4$.

Theorem 9.1.12. *Let $d = 4$. Assume that J satisfies (A1)–(A5), and that $\mathfrak{m}_2(J) = \infty$. Then, for all $\beta \leq \beta_c(\rho)$, $f \in \mathcal{C}_0(\mathbb{R}^d)$ and $z \in \mathbb{R}$,*

$$\lim_{L \rightarrow \infty} \left| \langle \exp(z T_{f, L, \beta}(\varphi)) \rangle_{\rho, \beta} - \exp\left(\frac{z^2}{2} \langle T_{f, L, \beta}(\varphi)^2 \rangle_{\rho, \beta}\right) \right| = 0. \quad (9.1.36)$$

As a consequence, for $\beta = \beta_c$, every sub-sequential scaling limit of the model is Gaussian.

For the φ^4 model, the tree diagram bound takes a slightly different form.

Theorem 9.1.13. *Let $d = 4$. Assume that J satisfies (A1)–(A6). There exist $c, C > 0$ such that, for all $\beta \leq \beta_c(\rho)$, for all $x, y, z, t \in \mathbb{Z}^4$ at mutual distance at least L of each other with $1 \leq L \leq L(\rho, \beta)$,*

$$\begin{aligned} & |U_4^{\rho, \beta}(x, y, z, t)| \\ & \leq C \left(\frac{B_0(\rho, \beta)}{B_L(\rho, \beta)} \right)^c \sum_{u \in \mathbb{Z}^4} \sum_{u', u'' \in \mathbb{Z}^4} \langle \tau_x \tau_u \rangle_{\rho, \beta} \beta J_{u, u'} \langle \tau_{u'} \tau_y \rangle_{\rho, \beta} \langle \tau_z \tau_u \rangle_{\rho, \beta} \beta J_{u, u''} \langle \tau_{u''} \tau_t \rangle_{\rho, \beta}. \end{aligned}$$

Corollary 9.1.14. *Let $d = 4$. Assume that J satisfies (A1)–(A6). Consider a φ^4 model on $\mathbb{Z}^4$ with coupling constants J . There exist $C, c, \gamma > 0$ such that, for all $\beta \leq \beta_c(\rho)$, $1 \leq L \leq L(\rho, \beta)$, $f \in \mathcal{C}_0(\mathbb{R}^d)$, and $z \in \mathbb{R}$,*

$$\begin{aligned} & \left| \langle \exp(z T_{f, L, \beta}(\varphi)) \rangle_{\rho, \beta} - \exp\left(\frac{z^2}{2} \langle T_{f, L, \beta}(\varphi)^2 \rangle_{\rho, \beta}\right) \right| \\ & \leq \exp\left(\frac{z^2}{2} \langle T_{|f|, L, \beta}(\varphi)^2 \rangle_{\rho, \beta}\right) \frac{C \|f\|_\infty^4 r_f^\gamma z^4}{(\log L)^c}. \end{aligned}$$

As a consequence, for $\beta = \beta_c$, every sub-sequential scaling limit of the model is Gaussian.

We may also extend the results obtained for the Ising model for $1 \leq d \leq 3$ and $d_{\text{eff}} = 4$. The corresponding modifications of Theorem 9.1.9 and Corollary 9.1.11 will be stated in Section 9.8. Their main consequence for the φ^4 model is stated below.

Theorem 9.1.15. *Let $1 \leq d \leq 3$. Let J be the interaction defined for $x \neq y \in \mathbb{Z}^d$ by $J_{x,y} = C|x - y|_1^{-3d/2}$ (i.e. $\alpha = d/2$) where $C > 0$. Then, for $\sigma \in (0, d/2)$,*

$$\lim_{\beta \nearrow \beta_c(\rho)} g_\sigma(\rho, \beta) = 0. \quad (9.1.37)$$

As a consequence, for $\beta = \beta_c(\rho)$, every sub-sequential scaling limit of the model is Gaussian.

Organisation of the chapter

In Section 9.2, we define the Griffiths–Simon class of single-site measures to which our results apply. In Section 9.3, we recall the definition of reflection positivity and present the main properties it implies for the models under consideration (monotonicity of the two-point function, infrared bound, etc). The main result of this section is the derivation of the existence of regular scales in Propositions 9.3.28 and 9.3.29. In Section 9.4, we provide the basic knowledge on the random current representation of the Ising model and explain the heuristics it provides on Gaussianity of the scaling limits. We introduce the notion of effective dimension and prove a generalisation of Theorem 9.1.2 (see Theorem 9.5.5) to models with effective dimension $d_{\text{eff}} > 4$ in Section 9.5. Then, in Section 9.6, we prove Theorems 9.1.3 and 9.1.5, together with Corollary 9.1.8. The handling of long-range interactions is performed in Section 12.2. In Section 9.7, we prove more general versions of Theorem 9.1.9 and Corollary 9.1.11 (see Theorem 9.7.1 and Corollary 9.7.3). The main modifications in comparison to the $d = 4$ case are treated in Section 9.7.2. In Section 9.8, we extend the results to all the models introduced in Section 9.2.

In Appendix 9.A, we provide the proofs of the main spectral tools we will use for reflection positive models in the Griffiths–Simon class (see Theorem 9.3.11 and Proposition 9.3.16). In Appendix 9.B, we recall the definition of the backbone representation of the Ising model which will be useful in our proofs. In Appendix 9.C, we recall some useful bounds for the probability of connectivity events for the random current representation of Ising-type models in the Griffiths–Simon class. Finally, in Appendix 9.D, we provide an alternative proof of some of our results in the case where the bubble condition (9.1.26) is satisfied.

Notations We write a point $x \in \mathbb{R}^d$ as $x = (x_1, \dots, x_d)$ and denote by $\mathbf{e}_j$ the unit vector with $x_j = 1$. We will use the following notations for the standard norms on $\mathbb{R}^d$: for $x \in \mathbb{R}^d$, $|x|_1 := |x_1| + \dots + |x_d|$, $\|x\|_2^2 := x_1^2 + \dots + x_d^2$, and $|x| := \max_{1 \leq i \leq d} |x_i|$. Finally, for $k \geq 1$, denote $\Lambda_k := [-k, k]^d \cap \mathbb{Z}^d$.

If $(a_n)_{n \geq 0}, (b_n)_{n \geq 0} \in (\mathbb{R}_+^*)^{\mathbb{N}}$, we will write $a_n \gtrsim b_n$ (resp. $a_n \asymp b_n$) if there exists $C_1 = C_1(d)$ (resp. $C_1 = C_1(d), C_2 = C_2(d) > 0$) such that for all $n \geq 1$, $b_n \leq C_1 a_n$ (resp. $C_1 a_n \leq b_n \leq C_2 b_n$). We will also use Landau’s formalism and write $a_n = O(b_n)$ (resp. $a_n = o(b_n)$) if there exists $C = C(d) > 0$ such that for all $n \geq 1$, $a_n \leq C b_n$ (resp. $\lim_{n \rightarrow \infty} a_n/b_n = 0$).

9.2 The Griffiths–Simon class of measures

In this section, we define the proper class of single-site measures to which our results apply.

Definition 9.2.1 (The GS class of measures). A Borel measure ρ on $\mathbb{R}$ is said to belong to the Griffiths–Simon (GS) class of measures if it satisfies one of the following conditions:

- (i) there exist an integer $N \geq 1$, a renormalisation constant $Z > 0$, and sequences $(K_{i,j})_{1 \leq i,j \leq N} \in (\mathbb{R}^+)^{N^2}$ and $(Q_n)_{1 \leq n \leq N} \in \mathbb{R}^N$ such that for every $F : \mathbb{R} \rightarrow \mathbb{R}^+$ bounded and measurable,

$$\int_{\mathbb{R}} F(\tau) d\rho(\tau) = \frac{1}{Z} \sum_{\sigma \in \{\pm 1\}^N} F\left(\sum_{n=1}^N Q_n \sigma_n\right) \exp\left(\sum_{i,j=1}^N K_{i,j} \sigma_i \sigma_j\right), \quad (9.2.1)$$

- (ii) the measure ρ can be presented as a weak limit of probability measures of the above type, and it is of sub-Gaussian growth: for some $\alpha > 2$,

$$\int_{\mathbb{R}} e^{|\tau|^\alpha} d\rho(\tau) < \infty. \quad (9.2.2)$$

Measures that satisfy (i) are said to be of the “Ising type”.

The following result was proved in [118, 224] to extend the Lee–Yang theorem, together with Griffiths’ correlation inequalities, to the φ^4 model on $\mathbb{Z}^d$. We sketch its proof for sake of completeness.

Proposition 9.2.2 (The φ^4 measure belongs to the GS class, [224]). *Let $g > 0$ and $a \in \mathbb{R}$. The probability measure $\rho_{g,a}$ on $\mathbb{R}$ given by*

$$d\rho_{g,a}(\varphi) = \frac{1}{z_{g,a}} e^{-g\varphi^4 - a\varphi^2} d\varphi, \quad (9.2.3)$$

where $z_{g,a}$ is a renormalisation constant, belongs to the GS class.

Proof. Let $N \geq 1$. Let $\tilde{g} = (12g)^{-1/4}$ and $\tilde{a} = 2a\tilde{g}^2$. Define the coupling constants

$$c_N := \tilde{g}N^{-3/4}, \quad d_N := \frac{1}{N} \left(1 - \frac{\tilde{a}}{\sqrt{N}}\right). \quad (9.2.4)$$

Define the Ising Gibbs measure μ_N on the complete graph K_N , with Hamiltonian given by

$$\mathbf{H}_N(\sigma) := -d_N \sum_{\{i,j\} \subset K_N} \sigma_i \sigma_j. \quad (9.2.5)$$

Let ρ_N be the law of the random variable $\Phi_N := c_N \sum_{i=1}^N \sigma_i$. Then, ρ_N converges weakly to $\rho_{g,a}$. $\square$

9.3 Reflection positivity

In this section, we define reflection positivity and gather all the properties it implies in our setup (monotonicity, infrared bound, gradient estimates, etc). We refer to the review [36] or the original papers [104, 103], for more information on this notion.

The end-goal is to derive the existence of regular scales for general reflection positive interactions (see Propositions 9.3.28 and 9.3.29). Most of the results are classical and their

proofs in the case of nearest-neighbour ferromagnetic (n.n.f) models in the GS class were already derived in [9] using the spectral representation of the Ising model through the lens of transfer matrices. This approach is not optimal in the most general setup. Our viewpoint will be that of self-adjoint operators in infinite dimensional spaces, which allows us to import general results from [128].

The following statements apply to both Ising and φ^4 systems. To unify the notations, we refer to the spin or field variables by the symbol τ , with an a-priori spin distribution $d\rho(\tau)$ in the GS class which is supported on a set $\mathcal{S} \subset \mathbb{R}$. The expectation value functional with respect to the Gibbs measure, or functional integral, for a system in a domain Λ , is denoted $\langle \cdot \rangle_{\Lambda, \rho, \beta}$. We denote by $\langle \cdot \rangle_{\rho, \beta}$ the state's natural infinite volume limit. We also denote by $\beta_c(\rho)$ the critical inverse temperature, and $\xi(\rho, \beta)$ the correlation length. We sometimes omit ρ in the notations when it is clear from context.

We will use the following notation for the model's two-point function,

$$S_{\rho, \beta}(x) := \langle \tau_0 \tau_x \rangle_{\rho, \beta}. \quad (9.3.1)$$

Also, introduce *the finite-volume susceptibility*, for $L \geq 1$,

$$\chi_L(\rho, \beta) := \sum_{x \in \Lambda_L} S_{\rho, \beta}(x), \quad (9.3.2)$$

and define the *susceptibility* to be $\chi(\rho, \beta) := \lim_{L \rightarrow \infty} \chi_L(\rho, \beta)$. Finally, recall that $|J| = \sum_{x \in \mathbb{Z}^d} J_{0, x}$.

9.3.1 Definition of reflection positivity

Let $d \geq 1$. Consider the torus $\mathbb{T}_L := (\mathbb{Z}/L\mathbb{Z})^d$ with $L \geq 2$ an even integer. The torus is endowed with a natural reflection symmetry along hyperplanes $\mathcal{H}$ which are orthogonal to one of the lattice's directions. The hyperplane $\mathcal{H}$ either passes through sites of $\mathbb{T}_L$ or through mid-edges, and $\mathcal{H}$ divides the torus into two pieces $\mathbb{T}_L^+$ and $\mathbb{T}_L^-$. The two pieces are disjoint for mid-edges reflections and satisfy $\mathbb{T}_L^+ \cap \mathbb{T}_L^- = \mathcal{H}$ for site reflections. Denote by $\mathcal{A}^\pm$ the algebra of all real valued functions f that depend only on the spins in $\mathbb{T}_L^\pm$. Denote by Θ the reflection map associated with $\mathcal{H}$; it naturally acts on $\mathcal{A}^\pm$: for all $f \in \mathcal{A}^\pm$,

$$\Theta(f)(\tau) := f(\Theta(\tau)), \quad \forall \tau \in \mathcal{S}^{\mathbb{T}_L}. \quad (9.3.3)$$

If $J = (J_{x, y})_{x, y \in \mathbb{Z}^d} \in (\mathbb{R}^+)^{\mathbb{Z}^d \times \mathbb{Z}^d}$, we can view it as an interaction $J^{(L)}$ on $\mathbb{T}_L$ by setting

$$J_{x, y}^{(L)} := \sum_{z \in \mathbb{Z}^d} J_{x, y + Lz}. \quad (9.3.4)$$

Definition 9.3.1 (Reflection positivity). Let $J = (J_{x, y})_{x, y \in \mathbb{Z}^d} \in (\mathbb{R}^+)^{\mathbb{Z}^d \times \mathbb{Z}^d}$ be an interaction. The measure $\langle \cdot \rangle_{\mathbb{T}_L, \rho, \beta} = \langle \cdot \rangle_{\mathbb{T}_L, \rho, J^{(L)}, \beta}$ is called reflection positive (RP) with respect to Θ , if for all $f, g \in \mathcal{A}^+$,

$$\langle f \cdot \Theta(g) \rangle_{\mathbb{T}_L, \rho, \beta} = \langle \Theta(f) \cdot g \rangle_{\mathbb{T}_L, \rho, \beta}, \quad (9.3.5)$$

and,

$$\langle f \cdot \Theta(f) \rangle_{\mathbb{T}_L, \rho, \beta} \geq 0. \quad (9.3.6)$$

We say that J is reflection positive if for all $L \geq 2$ even, the associated measure $\langle \cdot \rangle_{\mathbb{T}_L, \rho, \beta}$ is reflection positive with respect to Θ for all such reflections Θ .

Before discussing the interest of studying such interactions let us briefly mention some examples (for more details see [104, 103, 13, 36]):

- (i) (nearest-neighbour interactions) $J_{x,y} = \mathbb{1}_{|x-y|_1=1}$,
- (ii) (exponential decay / Yukawa potentials) $J_{x,y} = C \exp(-\mu|x-y|_1)$ for $\mu, C > 0$,
- (iii) (power law decay) $J_{x,y} = C|x-y|_1^{-d-\alpha}$ for $\alpha, C > 0$,
- (iv) $J_{x,y} = 1/\prod_{i=1}^d(|x_i - y_i| + a_i^2)^{\tau_i}$, for $\tau_i \geq 0$ and $a_i \in \mathbb{R}$.

The last example above can be found in [103, 13] and is an example of a d -dimensional RP interaction constructed as the product of 1-dimensional RP interactions. Furthermore, models of the GS class whose couplings are linear combinations with positive coefficients of the couplings mentioned above are also reflection-positive. Note that all finite range reflection positive interaction satisfy $J_{0,x} = 0$ whenever $|x| > 1$ (see p.12 of [103] for more information).

We now turn to the main consequences of reflection positivity. In what follows, we consider reflection positive models with ρ in the GS class. We fix J satisfying (A1)–(A5).

9.3.2 Messenger-Miracle-Solé inequalities

The Messenger-Miracle-Solé inequality provides monotonicity properties for $S_{\rho,\beta}$ in the case of reflection positive interactions (see Figure 9.2 for an illustration of this result).

Proposition 9.3.2 (MMS inequalities, [140, 191, 219]). *Let Λ be a region endowed with reflection symmetry with respect to a plane $\mathcal{P}$. Let A, B be two sets of points on the same side of the reflection plane. If Θ is the reflection with respect to $\mathcal{P}$,*

$$\left\langle \prod_{x \in A} \tau_x \prod_{x \in B} \tau_x \right\rangle_{\Lambda, \rho, \beta} \geq \left\langle \prod_{x \in A} \tau_x \prod_{x \in \Theta(B)} \tau_x \right\rangle_{\Lambda, \rho, \beta}. \quad (9.3.7)$$

Moreover, in the infinite volume limit $\Lambda \nearrow \mathbb{Z}^d$ this result can be extended to reflections with respect to hyperplanes passing through sites or mid-edges; and to reflections with respect to diagonal hyperplanes or more precisely, reflections changing only two coordinates x_i and x_j which are sent to $x_i \pm L$ and $x_j \mp L$ respectively, for some $L \in \mathbb{Z}$.

As a result, we get the following monotonicity property for reflection positive models in the GS class.

Corollary 9.3.3 (Monotonicity of the two-point function). *Let $d \geq 1$ and $\beta > 0$. Then,*

(i) *for all $1 \leq j \leq d$, the sequence $(S_{\rho,\beta}(k\mathbf{e}_j))_{k \geq 0}$ is decreasing,*

(ii) *for $x \in \mathbb{Z}^d$,*

$$S_{\rho,\beta}(|x|, 0_\perp) \geq S_{\rho,\beta}(x) \geq S_{\rho,\beta}(|x|_1, 0_\perp), \quad (\text{MMS1})$$

where $0_\perp \in \mathbb{Z}^{d-1}$ is the null vector. In particular, for all $x, y \in \mathbb{Z}^d$ with $d|x| \leq |y|$,

$$S_{\rho,\beta}(x) \geq S_{\rho,\beta}(y). \quad (\text{MMS2})$$

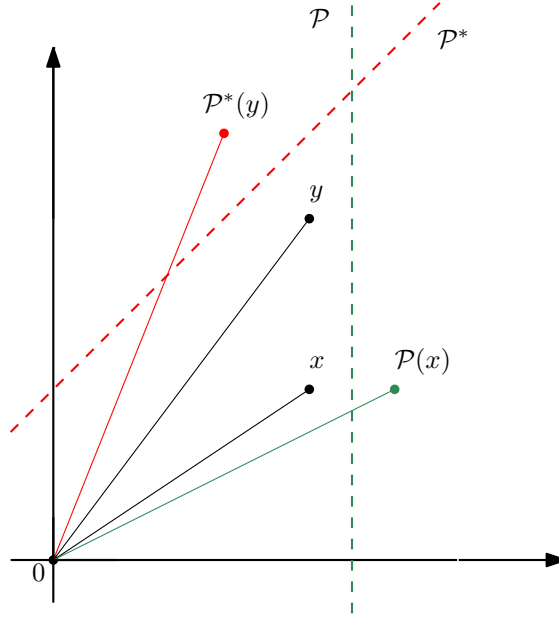


Figure 9.2: An illustration of Proposition 9.3.2. The two reflection planes are represented by the green and red dashed lines. The MMS inequalities state that $\langle \tau_0 \tau_x \rangle_{\rho, \beta} \geq \langle \tau_0 \tau_{\mathcal{P}(x)} \rangle_{\rho, \beta}$ and $\langle \tau_0 \tau_y \rangle_{\rho, \beta} \geq \langle \tau_0 \tau_{\mathcal{P}^*(y)} \rangle_{\rho, \beta}$.

9.3.3 The infrared bound

The second interesting property we can get on $S_{\rho, \beta}$ using reflection positivity is a quantitative estimate of its decay up to the critical point. This relies on the infrared bound which is recalled below.

We still work on the d -dimensional torus $\mathbb{T}_L$ (with L even). In view of the model's translation invariance, it is natural to introduce the Fourier transform of $S_{\rho, \beta}$,

$$\hat{S}_{\rho, \beta}^{(L)}(p) := \sum_{x \in \mathbb{T}_L} e^{ip \cdot x} \langle \tau_x \tau_y \rangle_{\mathbb{T}_L, \rho, \beta} \quad (9.3.8)$$

where p ranges over $\mathbb{T}_L^* := \left(\frac{2\pi}{L}\mathbb{Z}\right)^d \cap (-\pi, \pi]^d$. As it turns out, $\hat{S}_{\rho, \beta}^{(L)}(p)$ can be expressed in terms of the Fourier *spin-wave modes*, defined as

$$\hat{\tau}_\beta(p) := \frac{1}{\sqrt{(2L)^d}} \sum_{x \in \mathbb{T}_L} e^{ip \cdot x} \tau_x. \quad (9.3.9)$$

Indeed, one has for $p \in \mathbb{T}_L^*$,

$$\hat{S}_{\rho, \beta}^{(L)}(p) = \langle |\hat{\tau}_\beta(p)|^2 \rangle_{\mathbb{T}_L, \rho, \beta}, \quad (9.3.10)$$

so that in particular $\hat{S}_{\rho, \beta}^{(L)}(p) \geq 0$.

The following result was first proved and used in [104, 103].

Proposition 9.3.4 (Infrared bound). *For any $p \in \mathbb{T}_L^* \setminus \{0\}$,*

$$\widehat{S}_{\rho,\beta}^{(L)}(p) \leq \frac{1}{2\beta|J|(1 - \widehat{J}(p))}, \quad (9.3.11)$$

where $\widehat{J}(p) := \sum_{x \in \mathbb{Z}^d} e^{ip \cdot x} \frac{J_{0,x}}{|J|}$.

Remark 9.3.5. Note that

$$|J|(1 - \widehat{J}(p)) = 2 \sum_{x \in \mathbb{Z}^d} \sin^2 \left(\frac{p \cdot x}{2} \right) J_{0,x}. \quad (9.3.12)$$

Introduce for $\beta < \beta_c(\rho)$, and $p \in (-\pi, \pi]^d$,

$$\widehat{S}_{\rho,\beta}(p) := \sum_{x \in \mathbb{Z}^d} e^{ip \cdot x} S_{\rho,\beta}(x). \quad (9.3.13)$$

Note that this quantity is well defined since $\sum_{x \in \mathbb{Z}^d} S_{\rho,\beta}(x) < \infty$ for $\beta < \beta_c(\rho)$ as proved in [8]. The next result will use the celebrated Simon-Lieb inequality which we now recall.

Lemma 9.3.6 (Simon-Lieb inequality, [223, 178]). *Let $d \geq 1$. For every ferromagnetic model in the GS class on $\mathbb{Z}^d$ with translation invariant coupling constants, every $\beta > 0$, every finite subset Λ of $\mathbb{Z}^d$ containing 0, and every $x \notin \Lambda$,*

$$S_{\rho,\beta}(x) \leq \sum_{\substack{u \in \Lambda \\ v \notin \Lambda}} S_{\rho,\beta}(u) \beta J_{u,v} S_{\rho,\beta}(x - v). \quad (9.3.14)$$

The following result extends the infrared bound to $\widehat{S}_{\rho,\beta}$.

Proposition 9.3.7. *Let $\beta < \beta_c(\rho)$. Let $p \in (-\pi, \pi]^d$. Then,*

$$\widehat{S}_{\rho,\beta}(p) \leq \frac{1}{2\beta|J|(1 - \widehat{J}(p))}. \quad (9.3.15)$$

Proof. If $\beta < \beta_c(\rho)$, it is classical that there is only one infinite volume equilibrium state that we denote $\langle \cdot \rangle_{\rho,\beta}$. Moreover, for all $x \in \mathbb{Z}^d$,

$$\langle \tau_0 \tau_x \rangle_{\mathbb{T}_{L,\rho,\beta}} \xrightarrow{L \rightarrow \infty} \langle \tau_0 \tau_x \rangle_{\rho,\beta}. \quad (9.3.16)$$

Fix L even and take limits along sequences of the form $(L^k)_{k \geq 1}$ so that $\mathbb{T}_L^* \subset \mathbb{T}_{L^k}^*$ for all $k \geq 1$. For $p \in \mathbb{T}_L^*$, notice that by Fatou's lemma and (9.3.16),

$$\begin{aligned} \widehat{S}_{\rho,\beta}(p) + \chi(\rho, \beta) &= \sum_{x \in \mathbb{Z}^d} (1 + \cos(p \cdot x)) S_{\rho,\beta}(x) \\ &\leq \liminf \left(\widehat{S}_{\rho,\beta}^{(L^k)}(p) + \chi^{(L^k)}(\rho, \beta) \right) \\ &\leq \frac{1}{2\beta|J|(1 - \widehat{J}(p))} + \liminf \chi^{(L^k)}(\rho, \beta), \end{aligned}$$

where $\chi^{(L^k)}(\rho, \beta) := \widehat{S}_{\rho, \beta}^{(L^k)}(0)$ and $\chi(\rho, \beta) := \widehat{S}_{\rho, \beta}(0)$. It then suffices to show that $\chi^{(L^k)}(\rho, \beta)$ goes to $\chi(\rho, \beta)$ as k goes to infinity. Using the Simon–Lieb inequality (as in [84]), we get that for any $K \geq 0$,

$$\chi^{(L^k)}(\rho, \beta) \leq \widetilde{\varphi}_{\rho, \beta}(\Lambda_K) \chi^{(L^k)}(\rho, \beta) + \chi_K(\rho, \beta), \quad (9.3.17)$$

where $\widetilde{\varphi}_{\rho, \beta}(\Lambda_K) := \beta \sum_{\substack{x \in \Lambda_K \\ y \notin \Lambda_K}} J_{x, y} \langle \tau_0 \tau_x \rangle_{\rho, \beta}$.

Then, since $\chi(\rho, \beta) < \infty$, one has $\limsup_K \widetilde{\varphi}_{\rho, \beta}(\Lambda_K) = 0$. In particular, for every $\varepsilon > 0$,

$$\limsup \chi^{(L^k)}(\rho, \beta) \leq \frac{1}{1 - \varepsilon} \chi(\rho, \beta). \quad (9.3.18)$$

Which gives the result. $\square$

As a first consequence of the above result, using that $(1 - \widehat{J}(p)) \gtrsim |p|^2$ near 0, we see that if $d \geq 3$, there exists $C = C(d) > 0$ such that for $\beta < \beta_c(\rho)$,

$$\langle \tau_0^2 \rangle_{\rho, \beta} = \int_{(-\pi, \pi]^d} \widehat{S}_{\rho, \beta}(p) dp \leq \frac{C}{\beta |J|}. \quad (9.3.19)$$

Note that this bound extends to $\beta_c(\rho)$ by continuity. Since $\beta \mapsto \langle \tau_0^2 \rangle_{\rho, \beta}$ is non-decreasing⁸, we also get that for all $\beta \leq \beta_c(\rho)$,

$$\langle \tau_0^2 \rangle_{\rho, \beta} \leq \frac{C}{\beta_c(\rho) |J|}. \quad (9.3.20)$$

Proposition 9.3.4 together with the MMS inequalities also yield the following result.

Proposition 9.3.8 (Infrared bound). *Let $d \geq 3$. There exists $C = C(d) > 0$ such that for every $\beta \leq \beta_c(\rho)$, and every $x \in \mathbb{Z}^d \setminus \{0\}$,*

$$S_{\rho, \beta}(x) \leq \frac{C}{\beta |J| |x|^d} \int_{(-\pi|x|, \pi|x|]^d} \frac{e^{-\|p\|_2^2}}{1 - \widehat{J}(|p|/|x|)} dp. \quad (9.3.21)$$

In particular, for all $x \in \mathbb{Z}^d \setminus \{0\}$,

$$S_{\rho, \beta}(x) \leq S_{\rho, \beta_c(\rho)}(x) \leq \frac{C}{\beta_c(\rho) |J| |x|^{d-2}}. \quad (\text{IRB})$$

Proof. We prove the result for $\beta < \beta_c(\rho)$ and extend it to $\beta_c(\rho)$ with a continuity argument.

Using (MMS2), we get that for some $C_1 = C_1(d) > 0$,

$$S_{\rho, \beta}(x) \leq \frac{C_1}{|x|^d} \sum_{y \in \text{Ann}(|x|(2d)^{-1}, |x|^{d-1})} S_{\rho, \beta}(y) \leq \frac{C_1}{|x|^d} \chi_{|x|}(\rho, \beta). \quad (9.3.22)$$

We now observe that Proposition 9.3.7 provides a control on the finite volume susceptibility $\chi_L(\rho, \beta)$. Let

$$\widetilde{\chi}_L(\rho, \beta) := \sum_{x \in \mathbb{Z}^d} e^{-(\|x\|_2/L)^2} S_{\rho, \beta}(x). \quad (9.3.23)$$

⁸This is a classical consequence of Griffiths' inequalities.

There exists $C_2 = C_2(d) > 0$ such that, $\chi_L(\beta) \leq C_2 \tilde{\chi}_L(\rho, \beta)$. Using classical Fourier identities, we get $C_3 = C_3(d) > 0$ such that,

$$\chi_L(\rho, \beta) \leq C_2 \tilde{\chi}_L(\rho, \beta) \leq C_3 L^d \int_{(-\pi, \pi]^d} e^{-L^2 \|p\|_2^2} \hat{S}_{\rho, \beta}(p) dp. \quad (9.3.24)$$

With the change of variable $u = pL$, and Proposition 9.3.4,

$$\begin{aligned} \int_{(-\pi, \pi]^d} e^{-L^2 \|p\|_2^2} \hat{S}_{\rho, \beta}(p) dp &\leq \frac{1}{L^d} \int_{(-L\pi, L\pi]^d} e^{-\|u\|_2^2} \hat{S}_{\rho, \beta}(u/L) du \\ &\leq \frac{C_3}{\beta |J| L^d} \int_{(-L\pi, L\pi]^d} \frac{e^{-\|u\|_2^2}}{1 - \hat{J}(u/L)} du. \end{aligned}$$

The second part of the statement then follows from monotonicity in β of $S_\beta(x)$ together with the observation that $1 - \hat{J}(k) \gtrsim \|k\|_2^2$ as $k \rightarrow 0$. $\square$

The preceding result essentially gives that the decay of the two-point function is governed by the behaviour of $1 - \hat{J}(p)$ as p goes to 0. We have the following estimates for the examples of RP interactions given above,

- Nearest-neighbour interactions or Yukawa potentials: as $p \rightarrow 0$,

$$1 - \hat{J}(p) \asymp |p|^2. \quad (9.3.25)$$

- Power law decay interactions: as $p \rightarrow 0$,

$$1 - \hat{J}(p) \asymp \begin{cases} |p|^2 & \text{if } \alpha > 2, \\ |p|^2 \log \frac{1}{|p|} & \text{if } \alpha = 2, \\ |p|^\alpha & \text{if } \alpha \in (0, 2). \end{cases} \quad (9.3.26)$$

Together with Corollary 9.3.8, we get that for algebraic decay interactions,

$$\langle \tau_0 \tau_x \rangle_{\rho, \beta} \leq \frac{C}{\beta |J|} \begin{cases} |x|^{-(d-2)} & \text{if } \alpha > 2, \\ |x|^{-(d-2)} (\log |x|)^{-1} & \text{if } \alpha = 2, \\ |x|^{-(d-\alpha)} & \text{if } \alpha \in (0, 2). \end{cases} \quad (9.3.27)$$

Remark 9.3.9. The above bound is also valid for $d = 2$ in the case $\alpha \in (0, 2)$ and for $d = 1$ with $\alpha \in (0, 1)$, since in both cases $|p|^{-\alpha}$ is locally integrable.

Proposition 9.3.8 also yields the following improvement on the bound of the model's two-point function when the interaction J has a slow decay.

Corollary 9.3.10. *Let $d = 4$. Assume that $\mathfrak{m}_2(J) = \infty$. Then, as $|x| \rightarrow \infty$,*

$$\langle \tau_0 \tau_x \rangle_{\rho, \beta_c(\rho)} = o\left(\frac{1}{|x|^2}\right). \quad (9.3.28)$$

9.3.4 Spectral representation of reflection positive models and applications

In this subsection, we generalise the spectral representation used in [9]. The following lines are inspired by [41, 203]. We fix a reflection positive model on $\mathbb{Z}^d$ ($d \geq 1$) with an interaction J satisfying (A1)–(A5). The following result is classical for n.n.f models in the GS class (see [110, 235] or [9, Proposition A.6]). We present its proof in Appendix 9.A.

Theorem 9.3.11 (Spectral representation). *Let $d \geq 1$. For every $\beta \leq \beta_c(\rho)$ and every function $v : \mathbb{Z}^{d-1} \rightarrow \mathbb{C}$ in $\ell^2(\mathbb{Z}^{d-1})$, there exists a positive measure $\mu_{v,\beta}$ of finite mass*

$$\int_0^\infty d\mu_{v,\beta}(a) = \sum_{x_\perp, y_\perp \in \mathbb{Z}^{d-1}} v_{x_\perp} \overline{v_{y_\perp}} S_{\rho,\beta}((0, x_\perp - y_\perp)) \leq \|v\|_2^2 \langle \tau_0^2 \rangle_{\rho,\beta}, \quad (9.3.29)$$

such that for every $n \in \mathbb{Z}$,

$$\sum_{x_\perp, y_\perp \in \mathbb{Z}^{d-1}} v_{x_\perp} \overline{v_{y_\perp}} S_{\rho,\beta}((n, x_\perp - y_\perp)) = \int_0^\infty e^{-a|n|} d\mu_{v,\beta}(a). \quad (9.3.30)$$

The following result provides a very useful representation of the Fourier transform of $S_{\rho,\beta}$.

Corollary 9.3.12. *Let $\beta < \beta_c(\rho)$. Let $p = (p_1, p_\perp) \in (-\pi, \pi]^d$. There exists a measure $\mu_{p_\perp,\beta}$ such that*

$$\hat{S}_{\rho,\beta}(p) := \sum_{x \in \mathbb{Z}^d} e^{ip \cdot x} S_{\rho,\beta}(x) = \int_0^\infty \frac{e^a - e^{-a}}{\mathcal{E}_1(p_1) + (e^{a/2} - e^{-a/2})^2} d\mu_{p_\perp,\beta}(a), \quad (9.3.31)$$

where $\mathcal{E}_1(k) := 2(1 - \cos k) = 4 \sin^2(k/2)$.

Remark 9.3.13. The preceding result is still true under any permutation of the indices.

Proof. Consider for $L \geq 1$, $x_\perp \in \mathbb{Z}^{d-1}$ and $p_\perp \in (-\pi, \pi]^{d-1}$,

$$v_L(x_\perp) := \frac{e^{ip_\perp \cdot x_\perp}}{\sqrt{|\Lambda_L^{(d-1)}|}} \mathbf{1}_{x_\perp \in \Lambda_L^{(d-1)}}, \quad (9.3.32)$$

where $\Lambda_L^{(d-1)} := [-L, L]^{d-1} \cap \mathbb{Z}^{d-1}$. We apply Theorem 9.3.11 to the sequence $(v_L)_{L \geq 1}$ which yields that for $L \geq 1$, and $n \in \mathbb{Z}$

$$\frac{1}{|\Lambda_L^{(d-1)}|} \sum_{x_\perp, y_\perp \in \Lambda_L^{(d-1)}} e^{ip_\perp \cdot (x_\perp - y_\perp)} S_{\rho,\beta}((n, x_\perp - y_\perp)) = \int_0^\infty e^{-a|n|} d\mu_{v_L,\beta}(a). \quad (9.3.33)$$

Observe that one can rewrite the left-hand side above as

$$\sum_{z_\perp \in \mathbb{Z}^{d-1}} e^{ip_\perp \cdot z_\perp} S_{\rho,\beta}((n, z_\perp)) \frac{|\Lambda_L^{(d-1)} \cap (\Lambda_L^{(d-1)} - z_\perp)|}{|\Lambda_L^{(d-1)}|}. \quad (9.3.34)$$

Thus, (use [8] and the fact that $\beta < \beta_c(\rho)$ to justify that every sum converges absolutely)

$$\lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L^{(d-1)}|} \sum_{x_\perp, y_\perp \in \Lambda_L^{(d-1)}} e^{ip_\perp \cdot (x_\perp - y_\perp)} S_{\rho,\beta}((n, x_\perp - y_\perp)) = \sum_{z_\perp \in \mathbb{Z}^{d-1}} e^{ip_\perp \cdot z_\perp} S_{\rho,\beta}((n, z_\perp)). \quad (9.3.35)$$

The moments of $a \mapsto e^{-a}$ under $\mu_{v_L, \beta}$ converge as $L \rightarrow \infty$. Hence, the moment criterion for the convergence of positive measures over bounded intervals (here $[0, 1]$) allows us to conclude existence of the (weak) limit $\lim_{L \rightarrow \infty} \mu_{v_L, \beta} =: \mu_{p_\perp, \beta}$. We obtained that for any $n \in \mathbb{Z}$,

$$\sum_{z_\perp \in \mathbb{Z}^{d-1}} e^{ip_\perp \cdot z_\perp} S_{\rho, \beta}((n, z_\perp)) = \int_0^\infty e^{-a|n|} d\mu_{p_\perp, \beta}(a). \quad (9.3.36)$$

Taking the Fourier transform on each side of the above formula, and evaluating it at p_1 yields the desired formula. $\square$

Remark 9.3.14. We used the fact that for $p_1, a \in \mathbb{R}$,

$$\sum_{n \in \mathbb{Z}} e^{ip_1 n - a|n|} = \frac{e^a - e^{-a}}{\mathcal{E}_1(p_1) + (e^{a/2} - e^{-a/2})^2}. \quad (9.3.37)$$

Corollary 9.3.15. Let $\beta < \beta_c(\rho)$, $d \geq 1$. Then,

- (i) $\widehat{S}_{\rho, \beta}(p_1, \dots, p_d)$ is monotone decreasing in each $|p_j|$ over $[-\pi, \pi]$.
- (ii) $\mathcal{E}_1(p_1) \widehat{S}_{\rho, \beta}(p)$ and $|p_1|^2 \widehat{S}_{\rho, \beta}(p)$ are monotone increasing in $|p_1|$.

Proof. The result is a direct consequence of Corollary 9.3.12 and of the monotonicity of the following functions:

$$u \in [0, \pi] \mapsto \mathcal{E}_1(u), \quad u > 0 \mapsto \frac{u^2}{\mathcal{E}_1(u)}, \quad (9.3.38)$$

and for all $a \geq 0$,

$$u > 0 \mapsto \frac{\mathcal{E}_1(u)}{\mathcal{E}_1(u) + (e^{a/2} - e^{-a/2})^2}. \quad (9.3.39)$$

$\square$

The next result, obtained in [9] in the case of nearest-neighbour interactions, will be useful to derive Theorem 9.3.18. We derive it using the same methods used to obtain Theorem 9.3.11, and postpone the proof to Appendix 9.A.

Proposition 9.3.16. Let $d \geq 1$ and $\beta < \beta_c(\rho)$. Introduce for $p \in \mathbb{R}^d$,

$$\widehat{S}_{\rho, \beta}^{(\text{mod})}(p) := \widehat{S}_{\rho, \beta}(p) + \widehat{S}_{\rho, \beta}(p + \pi(1, 1, 0, \dots, 0)). \quad (9.3.40)$$

Then $\widehat{S}_{\rho, \beta}^{(\text{mod})}$ is monotone decreasing in $|p_1 - p_2|$ with $p_1 + p_2$ and $(p_3, \dots, p_d)$ constant.

Corollary 9.3.17. Let $d \geq 2$, $\beta < \beta_c(\rho)$. There exists $C = C(d) > 0$ such that for all $p \in [-\pi/2, \pi/2]^d$,

$$\widehat{S}_{\rho, \beta}(|p|, 0_\perp) \geq \widehat{S}_{\rho, \beta}(p) \geq \widehat{S}_{\rho, \beta}(|p|_1, 0_\perp) - \frac{C}{\beta}. \quad (9.3.41)$$

Proof. The first inequality is a direct consequence of the first item of Corollary 9.3.15. For the second inequality, notice that an iteration of Proposition 9.3.16 yields

$$\widehat{S}_{\rho, \beta}^{(\text{mod})}(p) \geq \widehat{S}_{\rho, \beta}^{(\text{mod})}(|p|_1, 0_\perp). \quad (9.3.42)$$

Now recall that by Corollary 9.3.12, $\widehat{S}_{\rho,\beta} \geq 0$, and that by the infrared bound, there exists $C > 0$ such that for $p \in [-\pi/2, \pi/2]^d$,

$$|\widehat{S}_{\rho,\beta}(p + \pi(1, 1, 0, \dots, 0))| \leq \frac{C}{\beta}. \quad (9.3.43)$$

□

The following result was first derived in [9]. The above work is sufficient to extend it to general reflection positive interactions.

Theorem 9.3.18 (Sliding-scale infrared bound, [9, Theorem 5.6]). *Let $d \geq 2$. There exists $C = C(d) > 0$ such that for all $\beta \leq \beta_c(\rho)$ and $1 \leq \ell \leq L$,*

$$\frac{\chi_L(\rho, \beta)}{L^2} \leq \frac{C}{\beta} \frac{\chi_\ell(\rho, \beta)}{\ell^2}. \quad (9.3.44)$$

Remark 9.3.19. In fact, using the results of Section 9.3.6, provided that the sharp length (defined below) satisfies $L(\rho, \beta) \geq 1$, we can replace the C/β by a C in the theorem above. This minor improvement will be used later to derive bounds with constants that are independent of β .

9.3.5 Gradient estimates

The following is a consequence of Theorem 9.3.11. It will play a crucial role in the proof of existence of regular scales that follows.

Proposition 9.3.20 (Gradient estimate, [9, Proposition 5.9]). *Let $d \geq 1$. There exists $C = C(d) > 0$ such that for every $\beta \leq \beta_c(\rho)$, every $x \in \mathbb{Z}^d$ and every $1 \leq i \leq d$,*

$$|S_{\rho,\beta}(x \pm \mathbf{e}_i) - S_{\rho,\beta}(x)| \leq \frac{F(|x|)}{|x|} S_{\rho,\beta}(x) \quad (9.3.45)$$

where $F(n) := C \frac{S_{\rho,\beta}(dn\mathbf{e}_1)}{S_{\rho,\beta}(n\mathbf{e}_1)} \log \left(\frac{2S_{\rho,\beta}(\frac{n}{2}\mathbf{e}_1)}{S_{\rho,\beta}(n\mathbf{e}_1)} \right)$.

The above estimate becomes particularly interesting whenever there exists $c_0 > 0$ such that,

$$S_{\rho,\beta}(2dn\mathbf{e}_1) \geq c_0 S_{\rho,\beta} \left(\frac{n}{2} \mathbf{e}_1 \right). \quad (9.3.46)$$

Indeed, in that case, one can find $C_0 = C_0(c_0, d) > 0$ such that for all $x \in \partial\Lambda_n$ and $1 \leq i \leq d$,

$$|S_{\rho,\beta}(x \pm \mathbf{e}_i) - S_{\rho,\beta}(x)| \leq \frac{C_0}{|x|} S_{\rho,\beta}(x). \quad (9.3.47)$$

9.3.6 The sharp length and a lower bound on the two-point function

Since we work with infinite range interactions, it is possible that $\xi(\rho, \beta) = \infty$ throughout the subcritical phase $\beta < \beta_c(\rho)$ (this is for instance the case for algebraically decaying RP interactions [199, 20]). This forces us to revisit the notion of “typical length” in these setups. As suggested by the work [84], the quantity defined below is a good candidate for the typical size of a box in which the model has a critical behaviour.

Definition 9.3.21 (Sharp length). Let $\beta > 0$. Let S be a finite subset of $\mathbb{Z}^d$ containing 0. Let

$$\varphi_{\rho,\beta}(S) := \beta \sum_{\substack{x \in S \\ y \notin S}} J_{x,y} \langle \tau_0 \tau_x \rangle_{S,\rho,\beta}. \quad (9.3.48)$$

Define⁹ the sharp length of parameter $\alpha \in (0, 1)$ by

$$L^{(\alpha)}(\rho, \beta) := \frac{1}{2d} \cdot \inf \left\{ k \geq 1, \exists S \subset \mathbb{Z}^d \text{ with } 0 \in S, \text{rad}(S) \leq 2k, \varphi_{\rho,\beta}(S) < \alpha \right\}, \quad (9.3.49)$$

where $\text{rad}(S) := \max\{|x - y|, x, y \in S\}$, and with the convention that $\inf \emptyset = \infty$. We will set $L(\rho, \beta) := L^{(1/2)}(\rho, \beta)$.

Remark 9.3.22. Using the work of [8], together with the strategy implemented in [84], we see that for any $\alpha \in (0, 1)$,

$$L^{(\alpha)}(\rho, \beta_c(\rho)) = \infty. \quad (9.3.50)$$

Indeed, using Simon-Lieb inequality as in (9.3.17), one can show that if S is a finite subset of $\mathbb{Z}^d$ containing 0 and satisfying $\varphi_{\rho,\beta_c(\rho)}(S) < 1$, then

$$\chi(\rho, \beta_c(\rho)) \leq \frac{|S| \langle \tau_0^2 \rangle_{\rho, \beta_c(\rho)}}{1 - \varphi_{\rho, \beta_c(\rho)}(S)}. \quad (9.3.51)$$

This is in contradiction with the infiniteness of the susceptibility at criticality. A similar argument gives that $L^{(\alpha)}(\rho, \beta)$ increases to infinity as β tends to $\beta_c(\rho)$.

Below $L(\rho, \beta)$, the two-point function can be lower bounded by an algebraically decaying function. We start by stating this result in the special case where $\mathfrak{m}_2(J) < \infty$.

Proposition 9.3.23 (Lower bound on the two-point function). *Let $d \geq 2$. Assume that J satisfies (A1)–(A5) and that $\mathfrak{m}_2(J) < \infty$. There exists $c = c(d, J) > 0$ such that for all $\beta \leq \beta_c(\rho)$, and for all $x \in \mathbb{Z}^d$ satisfying $1 \leq |x| \leq L(\rho, \beta)$,*

$$\langle \tau_0 \tau_x \rangle_{\rho, \beta} \geq \frac{c}{\beta |x|^{d-1}}. \quad (9.3.52)$$

Proof. Let $n \leq (2d)L(\rho, \beta)$. By definition of $L(\rho, \beta)$, one has $\varphi_{\rho,\beta}(\Lambda_n) \geq 1/2$. Using (IRB) and the assumption $\mathfrak{m}_2(J) < \infty$, one has for some $C_1 > 0$,

$$\beta \sum_{\substack{x \in \Lambda_{n/2} \\ y \notin \Lambda_n}} J_{x,y} \langle \tau_0 \tau_x \rangle_{\rho, \beta} \leq C_1 n^2 \sum_{|x| \geq n/2} J_{0,x} \leq 4C_1 \sum_{|x| \geq n/2} |x|^2 J_{0,x} \leq \frac{1}{4}, \quad (9.3.53)$$

provided that $n \geq N_0$ (where N_0 only depends on J). Hence, we now additionally assume that $(2d)L(\rho, \beta) \geq N_0$. We obtained that,

$$\beta \sum_{\substack{x \in \Lambda_n \setminus \Lambda_{n/2} \\ y \notin \Lambda_n}} J_{x,y} \langle \tau_0 \tau_x \rangle_{\Lambda_n, \rho, \beta} \geq \frac{1}{4}. \quad (9.3.54)$$

⁹The $(2d)^{-1}$ in the definition of $L^{(\alpha)}(\rho, \beta)$ is purely technical and will be useful in the proof of Proposition 9.3.23.

Then, using (MMS1), for some $C_2, C_3 > 0$,

$$\frac{1}{4} \leq C_2 \beta n^{d-1} \langle \tau_0 \tau_{(n/2)\mathbf{e}_1} \rangle_{\rho, \beta} \sum_{k=0}^n \sum_{|y| \geq k} J_{0,y} \leq C_3 \beta \mathbf{m}_2(J) n^{d-1} \langle \tau_0 \tau_{(n/2)\mathbf{e}_1} \rangle_{\rho, \beta}, \quad (9.3.55)$$

so that for some $c_1 > 0$,

$$\langle \tau_0 \tau_{(n/2)\mathbf{e}_1} \rangle_{\rho, \beta} \geq \frac{c_1}{\beta n^{d-1}}. \quad (9.3.56)$$

Hence, using this time (MMS2), there exists $c_2 > 0$ such that, if $(2d)^{-1}N_0 \leq k \leq L(\rho, \beta)$ and $x \in \partial\Lambda_k$,

$$\langle \tau_0 \tau_x \rangle_{\rho, \beta} \geq \langle \tau_0 \tau_{\frac{2d|x|}{2}\mathbf{e}_1} \rangle_{\rho, \beta} \geq \frac{c_2}{\beta |x|^{d-1}}. \quad (9.3.57)$$

We now handle the smaller values of $k = |x|$ by noticing that for $1 \leq k \leq (2d)^{-1}N_0 \wedge L(\rho, \beta)$, the hypothesis that $\varphi_{\rho, \beta}(\Lambda_k) \geq \frac{1}{2}$, together with (MMS1), yield

$$\langle \tau_0 \tau_{k\mathbf{e}_1} \rangle_{\rho, \beta} \geq \frac{c_3}{\beta}, \quad (9.3.58)$$

for some $c_3 = c_3(d, J)$. This concludes the proof. $\square$

As it turns out, it is also possible to extend this result with the following assumption: there exist $c_0, C_0, \alpha > 0$ such that,

$$\frac{c_0}{k^{1+\alpha}} \leq \sum_{|x|=k} J_{0,x} \leq \frac{C_0}{k^{1+\alpha}}, \quad \forall k \geq 1. \quad (9.3.59)$$

Using Proposition 9.3.8 we get that reflection positive interactions satisfying the above assumption also satisfy: there exists $C = C(d) > 0$ such that for all $\beta \leq \beta_c(\rho)$, for all $x \in \mathbb{Z}^d \setminus \{0\}$,

$$\langle \tau_0 \tau_x \rangle_{\rho, \beta} \leq \frac{C}{\beta_c(\rho) |x|^{d-\alpha/2} (\log |x|)^{\delta_{\alpha,2}}}. \quad (9.3.60)$$

The prototypical example of interactions satisfying (A1)–(A5) and (9.3.59) is given by algebraically decaying RP interactions.

Remark 9.3.24. Note that we may have models for which $\mathbf{m}_2(J) < \infty$ and yet Assumption 9.3.59 is not satisfied. The next proposition will be useful in the study of models with $d \in \{1, 2, 3\}$ and $d_{\text{eff}} = 4$, which do not satisfy $\mathbf{m}_2(J) < \infty$.

Proposition 9.3.25. *Let $d \geq 2$. Assume that J satisfies (A1)–(A5) and (9.3.59). There exists $c > 0$ which only depend on J and d such that, for all $\beta \leq \beta_c(\rho)$, for all $1 \leq |x| \leq cL(\rho, \beta)$,*

$$\langle \tau_0 \tau_x \rangle_{\rho, \beta} \geq \frac{c}{\beta |x|^{d-1}} \times \begin{cases} 1 & \text{if } \alpha > 1 \\ (\log |x|)^{-1} & \text{if } \alpha = 1 \\ |x|^{\alpha-1} & \text{if } \alpha \in (0, 1). \end{cases} \quad (9.3.61)$$

Moreover, the above result still holds for $d = 1$ and $\alpha \in (0, 1)$.

Remark 9.3.26. The lower bound matches (9.3.60) for $\alpha \in (0, 1)$.

Proof. We proceed like in the proof of Proposition 9.3.23. Notice that if $\varepsilon > 0$ is sufficiently small and $n \geq 1$,

$$\beta \sum_{\substack{x \in \Lambda_{\varepsilon n} \\ y \notin \Lambda_n}} J_{x,y} \langle \tau_0 \tau_x \rangle_{\rho, \beta} \leq C_1 \left(\sum_{x \in \Lambda_{\varepsilon n}} \langle \tau_0 \tau_x \rangle_{\rho, \beta} \right) \sum_{|u| \geq n/2} J_{0,u} \leq C_2 \varepsilon^{\alpha/2} < \frac{1}{4}, \quad (9.3.62)$$

where $C_1, C_2 > 0$, and where we used (9.3.59) together with (9.3.60) on the second inequality. For such a choice of ε , we have that for $1 \leq n \leq L(\rho, \beta)$,

$$\beta \sum_{\substack{x \in \Lambda_n \setminus \Lambda_{\varepsilon n} \\ y \notin \Lambda_n}} J_{x,y} \langle \tau_0 \tau_x \rangle_{\rho, \beta} \geq \frac{1}{4}. \quad (9.3.63)$$

Using (MMS1), we find that

$$\frac{1}{4\beta} \leq C_3 \langle \sigma_0 \sigma_{\varepsilon n \mathbf{e}_1} \rangle_{\beta} n^{d-1} \sum_{k=1}^n \sum_{|u| \geq k} J_{0,u} \leq C_4 \langle \sigma_0 \sigma_{\varepsilon n \mathbf{e}_1} \rangle_{\beta} n^{d-1} \sum_{k=1}^n \frac{1}{k^{\alpha}}, \quad (9.3.64)$$

from which we obtain the desired result. $\square$

9.3.7 Existence of regular scales

In this subsection, we introduce the notion of *regular* scales. These scales will be defined in such a way that, on them, the two-point function behaves “nicely” i.e. as if we knew that $\langle \tau_0 \tau_x \rangle_{\rho, \beta}$ decayed algebraically fast (for $1 \leq |x| \leq L(\rho, \beta)$).

Definition 9.3.27 (Regular scales). Fix $c, C > 0$. An annular region $\text{Ann}(n/2, 8n)$ is said to be (c, C) -regular if the following properties hold :

- (P1) For every $x, y \in \text{Ann}(n/2, 8n)$, $S_{\rho, \beta}(y) \leq C S_{\rho, \beta}(x)$,
- (P2) for every $x, y \in \text{Ann}(n/2, 8n)$, $|S_{\rho, \beta}(x) - S_{\rho, \beta}(y)| \leq \frac{C|x-y|}{|x|} S_{\rho, \beta}(x)$,
- (P3) $\chi_{2n}(\rho, \beta) \geq (1+c)\chi_n(\rho, \beta)$,
- (P4) for every $x \in \Lambda_n$ and $y \notin \Lambda_{Cn}$, $S_{\rho, \beta}(y) \leq \frac{1}{2} S_{\rho, \beta}(x)$.

A scale k is said to be *regular* if $n = 2^k$ is such that $\text{Ann}(n/2, 8n)$ is (c, C) -regular, a vertex $x \in \mathbb{Z}^d$ will be said to be *in a regular scale* if it belongs to an annulus $\text{Ann}(n, 2n)$ with $n = 2^k$ and k a regular scale.

We can now state the main result of this subsection.

Proposition 9.3.28 (Existence of regular scales). *Let $d \geq 3$. Let J satisfy (A1)–(A5) and $\mathbf{m}_2(J) < \infty$. Let $\gamma > 2$. There exist $c_0, c_1, C_0 > 0$ such that for every ρ in the GS class, every $\beta \leq \beta_c(\rho)$, and every $1 \leq n^\gamma \leq N \leq L(\rho, \beta)$, there are at least $c_1 \log_2 \left(\frac{N}{n} \right)$ (c_0, C_0) -regular scales between n and N .*

The proof of this result can be found in [9]. However, since it is a crucial tool for what follows, we decide to include it.

Proof. Using the lower bound of Proposition 9.3.23, together with (IRB), we get the existence of $c_1, c_2 > 0$ such that

$$\chi_N(\rho, \beta) \geq \frac{c_1}{\beta_c(\rho)} N \geq \frac{c_1}{\beta_c(\rho)} \left(\frac{N}{n}\right)^{\frac{\gamma-2}{\gamma-1}} n^2 \geq c_2 \left(\frac{N}{n}\right)^{\frac{\gamma-2}{\gamma-1}} \chi_n(\rho, \beta). \quad (9.3.65)$$

Using Theorem 9.3.18, we find $r, c_3 > 0$ and independent of n, N , such that there are at least $c_3 \log_2(N/n)$ scales $m = 2^k$ between n and N such that

$$\chi_{rm}(\rho, \beta) \geq \chi_{16dm}(\rho, \beta) + \chi_m(\rho, \beta). \quad (9.3.66)$$

We prove that such an m is a (c_0, C_0) -regular scale for a good choice of c_0, C_0 . Indeed, to show it satisfies (P1) it is enough¹⁰ to show that $S_{\rho, \beta}(\frac{1}{2}m\mathbf{e}_1) \leq C_4 S_{\rho, \beta}(16dm\mathbf{e}_1)$ for some constant $C_4 = C_4(d) > 0$. However, one has

$$\begin{aligned} |\text{Ann}(16dm, rm)| S_{\rho, \beta}(16dm\mathbf{e}_1) &\geq \chi_{rm}(\rho, \beta) - \chi_{16dm}(\rho, \beta) \\ &\geq \chi_m(\rho, \beta) \geq |\Lambda_{m/(2d)}| S_{\rho, \beta}(m\mathbf{e}_1/2) \end{aligned} \quad (9.3.67)$$

where in the first inequality we used (MMS1) to get that for all $x \in \text{Ann}(16dm, rm)$ one has $S_{\rho, \beta}(x) \leq S_{\rho, \beta}(16dm\mathbf{e}_1)$, in the second inequality we used (9.3.66), and in the third one we used (MMS1) again to argue that for all $x \in \Lambda_{m/(2d)}$ one has $S_{\rho, \beta}(x) \geq S_{\rho, \beta}(\frac{1}{2}m\mathbf{e}_1)$. This gives (P1). Note that (P2) follows from the remark below Proposition 9.3.20. Now, using again (9.3.67) and (MMS2), we get that for every $x \in \text{Ann}(m, 2m)$ one has

$$S_{\rho, \beta}(x) \geq S_{\rho, \beta}(16dm\mathbf{e}_1) \geq \frac{c_5}{m^d} \chi_m(\rho, \beta), \quad (9.3.68)$$

which implies (P3). Finally, we obtain¹¹ (P4) by observing that for every R , if $y \notin \Lambda_{dRm}$ and $x \in \Lambda_m$,

$$|\Lambda_{Rm}| S_{\rho, \beta}(y) \leq \chi_{Rm}(\rho, \beta) \leq C_6 R^2 \chi_m(\rho, \beta) \leq C_7 R^2 m^d S_{\rho, \beta}(x), \quad (9.3.69)$$

where we used (MMS1) in the first inequality, Theorem 9.3.18 in the second (more precisely the improvement discussed in Remark 9.3.19), and (9.3.68) in the last one. We obtain the result by choosing C_0 sufficiently large, and c_0 sufficiently small. $\square$

Using Proposition 9.3.25, we can extend the above result to interactions J satisfying (9.3.59).

Proposition 9.3.29. *Let $d \geq 1$. Let J satisfy (A1)–(A5) and (9.3.59) with $\alpha > 0$ if $d \geq 3$ and $\alpha \in (0, 1)$ if $d \in \{1, 2\}$. Let $\gamma > 2$. There exist $c, c_0, c_1, C_0 > 0$ such that for every ρ in the GS class, every $\beta \leq \beta_c(\rho)$, and every $1 \leq n^\gamma \leq N \leq cL(\rho, \beta)$, there are at least $c_1 \log_2 \left(\frac{N}{n}\right)$ (c_0, C_0) -regular scales between n and N .*

Proof. We only need to take care of the case $d \in \{1, 2\}$ and $\alpha \in (0, 1)$. As noticed above, in this case $S_{\rho, \beta}(x) \asymp |x|^{d-\alpha}$ below $L(\rho, \beta)$. The existence of regular scales in that case is then a direct consequence of the remark below Proposition 9.3.20. $\square$

¹⁰This comes from the fact that any $x \in \text{Ann}(m/2, 8m)$ satisfies

$$S_{\rho, \beta}(16dm\mathbf{e}_1) \leq S_{\rho, \beta}(x) \leq S_{\rho, \beta}\left(\frac{m\mathbf{e}_1}{2}\right).$$

¹¹This is the only place where the hypothesis $d \geq 3$ plays a role.

9.4 Random current representation

Let $d \geq 1$. Let J be an interaction on $\mathbb{Z}^d$ satisfying (A1)-(A5) and let Λ be a finite subset of $\mathbb{Z}^d$.

9.4.1 Definitions and the switching lemma

Definition 9.4.1. A *current* $\mathbf{n}$ on Λ is a function defined on the set $\mathcal{P}_2(\Lambda) := \{\{x, y\}, x, y \in \Lambda\}$ and taking its values in $\mathbb{N} = \{0, 1, \dots\}$. We denote by Ω_Λ the set of currents on Λ . The set of *sources* of $\mathbf{n}$, denoted by $\partial \mathbf{n}$, is defined as

$$\partial \mathbf{n} := \left\{ x \in \Lambda, \sum_{y \in \Lambda} \mathbf{n}_{x,y} \text{ is odd} \right\}. \quad (9.4.1)$$

We also set $w_\beta(\mathbf{n}) := \prod_{\{x,y\} \subset \Lambda} \frac{(\beta J_{x,y})^{\mathbf{n}_{x,y}}}{\mathbf{n}_{x,y}!}$.

There is a way to expand the correlation functions of the Ising model to relate them to currents. Indeed, if we use, for $\sigma \in \{\pm 1\}^\Lambda$, the expansion

$$\exp(\beta J_{x,y} \sigma_x \sigma_y) = \sum_{\mathbf{n}_{x,y} \geq 0} \frac{(\beta J_{x,y} \sigma_x \sigma_y)^{\mathbf{n}_{x,y}}}{\mathbf{n}_{x,y}!}, \quad (9.4.2)$$

we obtain that

$$Z(\Lambda, \beta) = 2^{|\Lambda|} \sum_{\partial \mathbf{n} = \emptyset} w_\beta(\mathbf{n}). \quad (9.4.3)$$

More generally, the correlation functions are given by: for $A \subset \Lambda$,

$$\langle \sigma_A \rangle_{\Lambda, \beta} = \frac{\sum_{\partial \mathbf{n} = A} w_\beta(\mathbf{n})}{\sum_{\partial \mathbf{n} = \emptyset} w_\beta(\mathbf{n})}, \quad (9.4.4)$$

where $\sigma_A := \prod_{x \in A} \sigma_x$.

A current configuration $\mathbf{n}$ with $\partial \mathbf{n} = \emptyset$ can be seen as the edge count of a multigraph obtained as a union of loops. Adding sources to a current configuration comes down to adding a collection of paths connecting pairwise the sources. For instance, a current configuration with sources $\partial \mathbf{n} = \{x, y\}$ can be seen as the edge count of a multigraph consisting of a family of loops together with a path from x to y .

Remark 9.4.2. It is possible to make the above remark rigorous by looking at the so-called *backbone representation* of the Ising model, which is a way to expand the correlation functions in terms of a weighted sum over a collection of source-pairing paths. This expansion will be useful later and we refer to Appendix 9.B for definitions and properties.

As we are about to see, connectivity properties of the multigraph induced by a current will play a crucial role in the analysis of the underlying Ising model, this motivates the following definition.

Definition 9.4.3. Let $\mathbf{n} \in \Omega_\Lambda$ and $x, y \in \Lambda$.

- (i) We say that x is connected to y in $\mathbf{n}$ and write $x \xleftrightarrow{\mathbf{n}} y$, if there exists a sequence of points $x_0 = x, x_1, \dots, x_m = y$ such that $\mathbf{n}_{x_i, x_{i+1}} > 0$ for $0 \leq i \leq m-1$.
- (ii) The cluster of x , denoted by $\mathbf{C}_{\mathbf{n}}(x)$, is the set of points connected to x in $\mathbf{n}$.

The main interest of the above expansion lies in the following result that allows one to switch the sources of two currents. This combinatorial result first appeared in [120] to prove the concavity of the magnetisation of an Ising model with positive external field, but the probabilistic picture attached to it was popularised in [4].

Lemma 9.4.4 (Switching lemma). *For any $A, B \subset \Lambda$ and any function F from the set of currents into $\mathbb{R}$,*

$$\begin{aligned} \sum_{\substack{\mathbf{n}_1 \in \Omega_\Lambda: \partial \mathbf{n}_1 = A \\ \mathbf{n}_2 \in \Omega_\Lambda: \partial \mathbf{n}_2 = B}} F(\mathbf{n}_1 + \mathbf{n}_2) w_\beta(\mathbf{n}_1) w_\beta(\mathbf{n}_2) \\ = \sum_{\substack{\mathbf{n}_1 \in \Omega_\Lambda: \partial \mathbf{n}_1 = A \Delta B \\ \mathbf{n}_2 \in \Omega_\Lambda: \partial \mathbf{n}_2 = \emptyset}} F(\mathbf{n}_1 + \mathbf{n}_2) w_\beta(\mathbf{n}_1) w_\beta(\mathbf{n}_2) \mathbf{1}_{(\mathbf{n}_1 + \mathbf{n}_2) \in \mathcal{F}_B}, \quad (\text{SL}) \end{aligned}$$

where $A \Delta B = (A \cup B) \setminus (A \cap B)$ is the symmetric difference of sets and $\mathcal{F}_B$ is given by

$$\mathcal{F}_B = \{\mathbf{n} \in \Omega_\Lambda, \exists \mathbf{m} \leq \mathbf{n}, \partial \mathbf{m} = B\}. \quad (9.4.5)$$

In fact, we will also need a slightly different version of the switching lemma, called the *switching principle*, whose proof can be found in [11, Lemma 2.1]. We use the representation of a current $\mathbf{n} \in \Omega_\Lambda$ into a multigraph $\mathcal{N}$ in which the vertex set is Λ and where there are exactly $\mathbf{n}_{x,y}$ edges between x and y . We will also use the notation $\partial \mathcal{N} = \partial \mathbf{n}$.

Lemma 9.4.5 (Switching principle). *For any multigraph $\mathcal{M}$ with vertex set Λ , any $A \subset \Lambda$, and any function f of a current,*

$$\sum_{\substack{\mathcal{N} \subset \mathcal{M} \\ \partial \mathcal{N} = A}} f(\mathcal{N}) = \mathbf{1}_{\exists \mathcal{K} \subset \mathcal{M}, \partial \mathcal{K} = A} \sum_{\substack{\mathcal{N} \subset \mathcal{M} \\ \partial \mathcal{N} = \emptyset}} f(\mathcal{N} \Delta \mathcal{K}). \quad (\text{SP})$$

The switching lemma provides probabilistic interpretations of several quantities of interest like differences or ratios of correlation functions. The natural probability measures are defined as follows. If $A \subset \Lambda$, define a probability measure $\mathbf{P}_{\Lambda, \beta}^A$ on Ω_Λ by: for $\mathbf{n} \in \Omega_\Lambda$,

$$\mathbf{P}_{\Lambda, \beta}^A[\mathbf{n}] := \mathbf{1}_{\partial \mathbf{n} = A} \frac{w_\beta(\mathbf{n})}{\sum_{\partial \mathbf{m} = A} w_\beta(\mathbf{m})}, \quad (9.4.6)$$

and for $A_1, \dots, A_k \subset \Lambda$, define

$$\mathbf{P}_{\Lambda, \beta}^{A_1, \dots, A_k} := \mathbf{P}_{\Lambda, \beta}^{A_1} \otimes \dots \otimes \mathbf{P}_{\Lambda, \beta}^{A_k}. \quad (9.4.7)$$

When $A = \{x, y\}$, we will write xy instead of $\{x, y\}$ in the above notation.

The first consequence of the switching lemma is the following expression of a ratio of correlation functions in terms of the probability of the occurrence of a certain connectivity event in a system of random currents. One has,

$$\frac{\langle \sigma_A \rangle_{\Lambda, \beta} \langle \sigma_B \rangle_{\Lambda, \beta}}{\langle \sigma_A \sigma_B \rangle_{\Lambda, \beta}} = \mathbf{P}_{\Lambda, \beta}^{A \Delta B, \emptyset}[\mathbf{n}_1 + \mathbf{n}_2 \in \mathcal{F}_B], \quad (9.4.8)$$

where $\mathcal{F}_B$ was defined in Lemma 9.4.4. In particular, for $0, x, u \in \Lambda$,

$$\frac{\langle \sigma_0 \sigma_u \rangle_{\Lambda, \beta} \langle \sigma_u \sigma_x \rangle_{\Lambda, \beta}}{\langle \sigma_0 \sigma_x \rangle_{\Lambda, \beta}} = \mathbf{P}_{\Lambda, \beta}^{0x, \emptyset} [0 \xleftrightarrow{\mathbf{n}_1 + \mathbf{n}_2} u]. \quad (9.4.9)$$

Later, it will also be interesting to control two-point connectivity probabilities. One can prove (see [9, Proposition A.3]) the following result: for every $x, u, v \in \Lambda$,

$$\begin{aligned} \mathbf{P}_{\Lambda, \beta}^{0x, \emptyset} [u, v \xleftrightarrow{\mathbf{n}_1 + \mathbf{n}_2} 0] &\leq \frac{\langle \sigma_0 \sigma_u \rangle_{\Lambda, \beta} \langle \sigma_u \sigma_v \rangle_{\Lambda, \beta} \langle \sigma_v \sigma_x \rangle_{\Lambda, \beta}}{\langle \sigma_0 \sigma_x \rangle_{\Lambda, \beta}} \\ &\quad + \frac{\langle \sigma_0 \sigma_v \rangle_{\Lambda, \beta} \langle \sigma_v \sigma_u \rangle_{\Lambda, \beta} \langle \sigma_u \sigma_x \rangle_{\Lambda, \beta}}{\langle \sigma_0 \sigma_x \rangle_{\Lambda, \beta}}. \end{aligned} \quad (9.4.10)$$

As proved in [10], the probability measure $\mathbf{P}_{\Lambda, \beta}^A$, for A a finite (even) subset of $\mathbb{Z}^d$, admits a weak limit as $\Lambda \nearrow \mathbb{Z}^d$ that we denote by $\mathbf{P}_\beta^A$. This yields infinite volume versions of the above results.

9.4.2 Ursell's four-point function

Newman [196] proved that the triviality of the scaling limits of the Ising model is equivalent to the vanishing of the scaling limit of Ursell's four-point function. This result was later quantified by Aizenman [4, Proposition 12.1] who obtained the following bound.

Proposition 9.4.6 (Deviation from Wick's law). *Let $d \geq 2$ and $n \geq 2$. For all $x_1, \dots, x_{2n} \in \mathbb{Z}^d$,*

$$\begin{aligned} &\left| \langle \sigma_{x_1} \dots \sigma_{x_{2n}} \rangle_\beta - \sum_{\substack{\pi \text{ pairing of } \{1, \dots, 2n\}}} \prod_{j=1}^n \langle \sigma_{x_{\pi(2j-1)}} \sigma_{x_{\pi(2j)}} \rangle_\beta \right| \\ &\leq \frac{3}{2} \sum_{1 \leq i < j < k < \ell \leq 2n} |U_4^\beta(x_i, x_j, x_k, x_\ell)| \sum_{\substack{\pi \text{ pairing of } \{1, \dots, 2n\} \setminus \{i, j, k, \ell\}}} \prod_{j=1}^{n-2} \langle \sigma_{x_{\pi(2j-1)}} \sigma_{x_{\pi(2j)}} \rangle_\beta. \end{aligned}$$

The switching lemma provides a probabilistic interpretation of Ursell's four-point function. This was a key step of the proof of triviality in [4, 9]. Although it was first stated in the case of nearest-neighbour interactions, the proof is valid on any graph and thus remains valid in the case of general interactions.

Proposition 9.4.7 (Representation of Ursell's four-point function). *For $x, y, z, t \in \mathbb{Z}^d$,*

$$U_4^\beta(x, y, z, t) = -2 \langle \sigma_x \sigma_y \rangle_\beta \langle \sigma_z \sigma_t \rangle_\beta \mathbf{P}_\beta^{xy, zt} [\mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_2}(x) \cap \mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_2}(z) \neq \emptyset]. \quad (9.4.11)$$

This identity might seem tricky to analyse due to the lack of independence between $\mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_2}(x)$ and $\mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_2}(z)$ but it is possible to show [4] that

$$\mathbf{P}_\beta^{xy, zt} [\mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_2}(x) \cap \mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_2}(z) \neq \emptyset] \leq \mathbf{P}_\beta^{xy, zt, \emptyset, \emptyset} [\mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_3}(x) \cap \mathbf{C}_{\mathbf{n}_2 + \mathbf{n}_4}(z) \neq \emptyset]. \quad (9.4.12)$$

In particular, if $\mathcal{I} := \mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_3}(x) \cap \mathbf{C}_{\mathbf{n}_2 + \mathbf{n}_4}(z)$, this leads to the following bound,

$$|U_4^\beta(x, y, z, t)| \leq 2 \langle \sigma_x \sigma_y \rangle_\beta \langle \sigma_z \sigma_t \rangle_\beta \mathbf{P}_\beta^{xy, zt, \emptyset, \emptyset} [|\mathcal{I}| > 0]. \quad (9.4.13)$$

The random current representation allows us to obtain an expression of U_4^β in terms of the probability of intersection of two independent random currents of prescribed sources. As explained in [4], the relevant question is then to see whether, in the limit $L(x, y, z, t) \rightarrow \infty$, the ratio $|U_4^\beta(x, y, z, t)| / \langle \sigma_x \sigma_y \sigma_z \sigma_t \rangle_\beta$ vanishes or not.

Heuristic for triviality. We work at $\beta \leq \beta_c$ and assume some regularity on the two-point function in the sense that it takes comparable values for pairs of points at comparable distances smaller than $L(\beta)$.

Let us first consider the case of the nearest-neighbour Ising model. If we expect the intersection properties of two independent random current clusters to behave essentially like the ones of two independent random walks in $\mathbb{Z}^d$ conditioned to start and end at x, y and z, t respectively, we expect the probability on the right-hand side of (9.4.13) to be very small in dimension $d > 4$. Following this analogy, Aizenman [4] argued the case $d > 4$ by using a first moment method on $|\mathcal{I}|$ which yields the so-called *tree diagram bound*,

$$|U_4^\beta(x, y, z, t)| \leq 2 \sum_{u \in \mathbb{Z}^d} \langle \sigma_x \sigma_u \rangle_\beta \langle \sigma_y \sigma_u \rangle_\beta \langle \sigma_z \sigma_u \rangle_\beta \langle \sigma_t \sigma_u \rangle_\beta. \quad (9.4.14)$$

As discussed in the introduction, (9.4.14) together with (IRB), imply

$$\frac{|U_4^\beta(x, y, z, t)|}{\langle \sigma_x \sigma_y \sigma_z \sigma_t \rangle_\beta} = O(L^{4-d}), \quad (9.4.15)$$

where $L \leq L(\beta)$ is the mutual distance between x, y, z and t .

In the case of the “marginal” dimension $d = 4$ the above bound yields no interesting result and we need to go one step further in the analysis of (9.4.13). Going back to the analogy with random walks, it is a well-known result that four is the critical dimension in terms of intersection for the simple random walk, meaning that two independent (simple) random walks with starting and ending points at mutual distance L , will intersect with probability $O(1/\log L)$ while the expected number of points in the intersection will typically be of order $\Omega(1)$ (see [172, Chapter 10]). This shows that when two independent (simple) random walks in dimension four intersect, they do so a logarithmic number of times. Transposing this idea in the realm of random currents suggests that the probability that two independent random currents with sources at mutual distance at least L intersect, but not so many times, should decay as $O(1/(\log L)^c)$ for some $c > 0$. This is indeed the result that Aizenman and Duminil-Copin obtained to improve by a logarithmic factor the tree diagram bound.

For general long-range interactions it is possible to extend these ideas. It is well known that long-range step distributions can virtually “increase” the effective dimension of a random walk to the point that some low-dimensional random walks start manifesting the above properties, only observed in dimension $d \geq 4$ for the simple random walk. This observation is made more explicit by the following computation (see Section 9.5).

$$\frac{|U_4^\beta(x, y, z, t)|}{\langle \sigma_x \sigma_y \sigma_z \sigma_t \rangle_\beta} = O(L^{4(d/d_{\text{eff}})-d}). \quad (9.4.16)$$

As a result, models with effective dimension strictly above four are easily shown to be trivial, this is the content of Section 9.5. The main contribution of this chapter is to treat the case $d_{\text{eff}} = 4$, and to show that we can still improve the tree diagram bound there.

9.5 Reflection positive Ising models satisfying $d_{\text{eff}} > 4$

In this section, we study models of effective dimension $d_{\text{eff}} > 4$ and prove a more general version of Theorem 9.1.2. As discussed in the introduction, choosing sufficiently slowly decaying interactions might have the effect of increasing the dimension of the model. As a result, we expect to find models in low dimensions which admit trivial scaling limits. These results had already been obtained in [4, 13], although not under this slightly stronger form. We begin with a definition.

Definition 9.5.1 (Effective dimension). Let $d \geq 1$. Assume that J satisfies (A1)–(A4). The effective dimension d_{eff} of the model is related to the critical exponent of the two-point function. Assume that there exists $\eta \geq 0$ such that, as $|x| \rightarrow \infty$,

$$\langle \sigma_0 \sigma_x \rangle_{\beta_c} \asymp \frac{1}{|x|^{d-2+\eta+o(1)}}. \quad (9.5.1)$$

The effective dimension d_{eff} is then given by

$$d_{\text{eff}} := \frac{d}{1 - (\eta \wedge 2)/2}. \quad (9.5.2)$$

Remark 9.5.2. The above formula can be justified using Fourier transform considerations, see [13]. We saw in (9.3.10) that the spin-wave mode squared averages $\langle |\hat{\tau}_\beta(p)|^2 \rangle_{\mathbb{T}_{L,\rho,\beta}}$ — or *thermal* averages— coincide with the two-point function’s Fourier transform. As it turns out, the relevant quantity to look at is often the density of these spin-wave modes— i.e. dp — expressed as a function of the *excitation level* which is measured by $\hat{S}_\beta$. For the Gaussian case, one has $\hat{S}(p) \asymp p^{-2}$ which leads to a density of levels $d\hat{S}^{-d/2}$. For the case of the Ising model, if we assume that $\hat{S}_{\beta_c}(p) \asymp p^{-(2-\eta)}$ for some critical exponent η , we end up with a density $d\hat{S}_{\beta_c}^{-d_{\text{eff}}/2}$ where $d_{\text{eff}} = d/(1 - \eta/2)$.

Note that for the above definition to make sense, we need the existence of the critical exponent η , which is expected to hold (but only known to exist in particular cases). However, we can still bound the effective dimension. Glimm and Jaffe [112] proved that for reflection positive interactions $\eta < 2$, which justifies that $d_{\text{eff}} < \infty$ in our setup. For reflection positive models, (IRB) yields that $\eta \geq 0$, so that

$$d_{\text{eff}} \geq d. \quad (9.5.3)$$

Remark 9.5.3 (Algebraically decaying RP interactions). In the case of reflection positive models with coupling constants of algebraic decay, i.e. $J_{x,y} = C|x - y|_1^{-d-\alpha}$ for $\alpha, C > 0$, Proposition 9.3.8 implies that $\eta \geq |2 - \alpha|_+$, which yields,

$$d_{\text{eff}} \geq \frac{d}{1 \wedge (\alpha/2)}. \quad (9.5.4)$$

As a consequence, if $d - 2(\alpha \wedge 2) > 0$, one has $d_{\text{eff}} > 4$.

More generally, with the definition given above, we see that $d_{\text{eff}} > 4$ if

$$d > 4(1 - \eta/2). \quad (9.5.5)$$

In what follows, we assume that $d_{\text{eff}} > 4$, that is, there exists $\mathbf{C} > 0$ and $\eta \in [0, 2)$ such that for all $x \in \mathbb{Z}^d \setminus \{0\}$,

$$\langle \sigma_0 \sigma_x \rangle_{\beta_c} \leq \frac{\mathbf{C}}{|x|^{d-2+\eta}}, \quad (9.5.6)$$

where $\eta \geq 0$ is such that $d + 2\eta > 4$.

Remark 9.5.4. Note that the above assumption is automatically satisfied when the interaction J satisfies (A1)–(A5) and (9.3.59) with $d - 2(\alpha \wedge 2) > 0$.

Theorem 9.5.5. *Let $d \geq 1$. Assume that J satisfies (A1)–(A5) and (9.5.6). There exist $C = C(\mathbf{C}, d), \gamma = \gamma(d) > 0$ such that for all $\beta \leq \beta_c$, $L \geq 1$, $f \in \mathcal{C}_0(\mathbb{R}^d)$ and $z \in \mathbb{R}$,*

$$\begin{aligned} & \left| \langle \exp(z T_{f,L,\beta}(\sigma)) \rangle_{\beta} - \exp\left(\frac{z^2}{2} \langle T_{f,L,\beta}(\sigma)^2 \rangle_{\beta}\right) \right| \\ & \leq \exp\left(\frac{z^2}{2} \langle T_{|f|,L,\beta}(\sigma)^2 \rangle_{\beta}\right) \frac{C(\beta^{-4} \vee \beta^{-2}) \|f\|_{\infty}^4 r_f^{\gamma} z^4}{L^{d+2\eta-4}}. \end{aligned}$$

Remark 9.5.6. If we assume that the critical exponent η exists, we see that $d + 2\eta - 4 = d[(4/d_{\text{eff}}) - 1]$, which is consistent with the decay of (9.4.16).

Proof. Let $f \in \mathcal{C}_0(\mathbb{R}^d)$, $\beta \leq \beta_c$ and $z \in \mathbb{C}$. Note that $r_f \geq 1$. Using Proposition 9.4.6 one gets for $n \geq 2$ (the inequality being trivial for $n = 0, 1$),

$$\begin{aligned} & \left| \langle T_{f,L,\beta}(\sigma)^{2n} \rangle_{\beta} - \frac{(2n)!}{2^n n!} \langle T_{f,L,\beta}(\sigma)^2 \rangle_{\beta}^n \right| \\ & \leq \frac{3}{2} (2n)^4 \langle T_{|f|,L,\beta}(\sigma)^{2n-4} \rangle_{\beta} \|f\|_{\infty}^4 S(\beta, L, f), \quad (9.5.7) \end{aligned}$$

where

$$S(\beta, L, f) := \sum_{x_1, x_2, x_3, x_4 \in \Lambda_{r_f L}} \frac{|U_4^{\beta}(x_1, x_2, x_3, x_4)|}{\Sigma_L(\beta)^2}. \quad (9.5.8)$$

Multiplying by $\frac{z^{2n}}{(2n)!}$ and summing (9.5.7) over n , one gets,

$$\begin{aligned} & \left| \langle \exp(z T_{f,L,\beta}(\sigma)) \rangle_{\beta} - \exp\left(\frac{z^2}{2} \langle T_{f,L,\beta}(\sigma)^2 \rangle_{\beta}\right) \right| \\ & \leq C_1 z^4 \exp\left(\frac{z^2}{2} \langle T_{|f|,L,\beta}(\sigma)^2 \rangle_{\beta}\right) \|f\|_{\infty}^4 S(\beta, L, f). \end{aligned}$$

Applying the tree diagram bound (9.4.14), we obtain

$$S(\beta, L, f) \leq 2 \sum_{\substack{x \in \mathbb{Z}^d \\ x_1, x_2, x_3, x_4 \in \Lambda_{r_f L}}} \frac{\langle \sigma_x \sigma_{x_1} \rangle_{\beta} \langle \sigma_x \sigma_{x_2} \rangle_{\beta} \langle \sigma_x \sigma_{x_3} \rangle_{\beta} \langle \sigma_x \sigma_{x_4} \rangle_{\beta}}{\Sigma_L(\beta)^2}. \quad (9.5.9)$$

Splitting the sum above,

$$S(\beta, L, f)/2 \leq \underbrace{\sum_{\substack{x \in \Lambda_{dr_f L} \\ x_1, x_2, x_3, x_4 \in \Lambda_{r_f L}}} (\dots)}_{(1)} + \underbrace{\sum_{\substack{x \notin \Lambda_{dr_f L} \\ x_1, x_2, x_3, x_4 \in \Lambda_{r_f L}}} (\dots)}_{(2)}. \quad (9.5.10)$$

• **Bound on (1)** The first term can be written

$$\sum_{\substack{x \in \Lambda_{dr_f L} \\ x_1, x_2, x_3, x_4 \in \Lambda_{r_f L}}} \frac{\langle \sigma_x \sigma_{x_1} \rangle_\beta \langle \sigma_x \sigma_{x_2} \rangle_\beta \langle \sigma_x \sigma_{x_3} \rangle_\beta \langle \sigma_x \sigma_{x_4} \rangle_\beta}{\Sigma_L(\beta)^2} = \sum_{x \in \Lambda_{dr_f L}} \frac{\left(\sum_{y \in \Lambda_{r_f L}} \langle \sigma_x \sigma_y \rangle_\beta \right)^4}{\Sigma_L(\beta)^2}. \quad (9.5.11)$$

Noticing that for $x \in \Lambda_{dr_f L}$, $\sum_{y \in \Lambda_{r_f L}} \langle \sigma_x \sigma_y \rangle_\beta \leq \chi_{2dr_f L}(\beta)$, using Theorem 9.3.18 to bound $\chi_{2dr_f L}(\beta)$ in terms of $\chi_L(\beta)$, and using that $\chi_L(\beta) \leq C_2 L^{-d} \Sigma_L(\beta)$, we get

$$(1) \leq C_3 \beta^{-4} r_f^{8+d} \frac{\chi_L(\beta)^2}{L^d}. \quad (9.5.12)$$

We can then use (9.5.6) to get the bound $\chi_L(\beta) \leq C_4 L^{2-\eta}$, so that

$$(1) \leq C_4 \beta^{-4} r_f^{8+d} L^{-(d+2\eta-4)}. \quad (9.5.13)$$

• **Bound on (2)** Combining (MMS2) and the sliding-scale infrared bound of Theorem 9.3.18, we get that for $i = 1, \dots, 4$,

$$\langle \sigma_x \sigma_{x_i} \rangle_\beta \leq \frac{C_5}{|x|^d} \chi_{|x|/d}(\beta) \leq \frac{C_6}{\beta L^2 |x|^{d-2}} \chi_L(\beta). \quad (9.5.14)$$

Bounding the terms indexed by x_1 and x_2 in the sum using (9.5.14) and the other two using (9.5.6), we get

$$\begin{aligned} (2) &\leq C_7 \beta^{-2} \sum_{x \notin \Lambda_{dr_f L}} \sum_{x_1, \dots, x_4 \in \Lambda_{r_f L}} \frac{\chi_L(\beta)^2}{\Sigma_L(\beta)^2 |x|^{2d-4} L^4} \frac{1}{|x - x_3|^{d-2+\eta}} \frac{1}{|x - x_4|^{d-2+\eta}} \\ &\leq C_8 \beta^{-2} r_f^{2d+4} L^{2d} \frac{\chi_L(\beta)^2}{\Sigma_L(\beta)^2} \sum_{x \notin \Lambda_{dr_f L}} \frac{1}{|x|^{2d-4+2\eta}} \\ &\leq C_9 \beta^{-2} r_f^{8+d-2\eta} L^{-(d+2\eta-4)}, \end{aligned}$$

where we used the inequality $|x - x_i| \geq (d-1)r_f L$ in the second line, and again $\chi_L(\beta) \leq C_2 L^{-d} \Sigma_L(\beta)$ in the third line. $\square$

Remark 9.5.7. It is possible—see [4] or Section 9.8—to obtain a version of the tree diagram bound for models in the GS class. In the general setup, it rewrites: for all $\beta > 0$, for all $x, y, z, t \in \mathbb{Z}^d$,

$$|U_4^{\rho, \beta}(x, y, z, t)| \leq 2 \sum_{u, u', u'' \in \mathbb{Z}^d} \langle \tau_x \tau_u \rangle_{\rho, \beta} \beta J_{u, u'} \langle \tau_{u'} \tau_y \rangle_{\rho, \beta} \langle \tau_z \tau_u \rangle_{\rho, \beta} \beta J_{u, u''} \langle \tau_{u''} \tau_t \rangle_{\rho, \beta}. \quad (9.5.15)$$

Using the results of Section 9.3, together with (9.5.15), we can easily extend the result of this section to the case of models in the GS class for which the two-point function $\langle \tau_0 \tau_x \rangle_{\rho, \beta_c(\rho)}$ satisfies condition (9.5.6).

9.6 Reflection positive Ising models in dimension $d = 4$

The goal of this section is to prove Theorems 9.1.3 and 9.1.5, and Corollary 9.1.8. The proof of the first result is a direct consequence of the methods developed in the preceding section.

Proof of Theorem 9.1.3. Let $d = 4$. Assume that the interaction J satisfies (A1)–(A5) and $\mathfrak{m}_2(J) = \infty$. Let $\beta \leq \beta_c$. Using Corollary 9.3.10 and adapting the proof of Section 9.5 to the case $\langle \sigma_0 \sigma_x \rangle_{\beta_c} = o(|x|^{2-d})$, we get (using the above notations) that $S(\beta, L, f) = o(1)$ as $L \rightarrow \infty$. The explicit rate of convergence to 0 is obtained by estimating (see Proposition 9.3.8)

$$\int_{(-\pi L, \pi L]^d} \frac{e^{-\|p\|_2^2}}{1 - \widehat{J}(|p|/L)} dp, \quad (9.6.1)$$

as $L \rightarrow \infty$. For instance, in the case of algebraically decaying RP interactions of the form $J_{x,y} = C|x - y|_1^{-d-2}$ ($\alpha = 2$), we get a decay of speed $O(1/\log L)$. $\square$

We now turn to the proof of Theorem 9.1.5. In the rest of the section, we fix $d = 4$ and an interaction J satisfying (A1)–(A6). Hence, there exist $\mathbf{C}, \varepsilon > 0$ such that for all $k \geq 1$,

$$\sum_{|x|=k} |x|^2 J_{0,x} \leq \frac{\mathbf{C}}{k^{1+\varepsilon}}. \quad (9.6.2)$$

We expect this case to be more subtle than the $\mathfrak{m}_2(J) = \infty$ case since we do not get any improvement on the effective dimension or on the decay of the two-point function. We will use the argument of [9] and adapt it to the long range setup. The proof follows essentially the same steps: we make up for the lack of precise knowledge on the behaviour of the two-point function at criticality by proving the existence of *regular scales* in which the two-point function behave nicely (see Proposition 9.3.28); then, we introduce a nice local intersection event which occurs with positive probability; and finally, we obtain a *mixing* statement (which is of independent interest) which allows us to argue that intersections at different scales are roughly independent events. As a result, we are able to prove that as soon as two independent currents intersect, they intersect a large number of time (see Proposition 9.6.2). One difficulty occurs in the process : since the model is long-range, the clusters of the sources might make big jumps and avoid scales which may drastically reduce the probability of the intersection event. Furthermore, the infinite range interactions may be problematic to obtain the mixing statement as they create more correlation between pieces of the current at different scales. The solution to get rid of that difficulty is to prove that these scale jumps occur with sufficiently small probability: this is the main technical point of this section, and the proof will be enabled by the assumption (9.6.2). As a consequence, we will be able to argue that the sources' clusters have a similar geometry as the one obtained in the nearest-neighbour case. This will be enough to adapt the multi-scale analysis of [9].

9.6.1 Proof of Theorem 9.1.5 conditionally on the clustering bound

We will need the following deterministic lemma which relates the number of points in a set $\mathcal{S} \subset \mathbb{Z}^d$ to the number of concentric annuli of the form $u + \text{Ann}(u_k, u_{k+1})$ with $u \in \mathcal{S}$ it

takes to cover $\mathcal{S}$. For any (possibly finite) increasing sequence $\mathcal{U} = (u_k)_{k \geq 0}$, any $u \in \mathbb{Z}^d$, and any $K \geq 0$, define,

$$\mathbf{M}_u(\mathcal{S}; \mathcal{U}, K) := |\{0 \leq k \leq K, \mathcal{S} \cap [u + \text{Ann}(u_k, u_{k+1})] \neq \emptyset\}|. \quad (9.6.3)$$

Lemma 9.6.1 (Covering Lemma, [9, Lemma 4.2]). *With the above notations, for any sequence $\mathcal{U} = (u_k)_{k \geq 1}$ with $u_1 \geq 1$ and $u_{k+1} \geq 2u_k$,*

$$|\mathcal{S}| \geq 2^{\min_{u \in \mathcal{S}} \mathbf{M}_u(\mathcal{S}; \mathcal{U}, K)/5}. \quad (9.6.4)$$

Fix D large enough. Define recursively a (possibly finite) sequence $\mathcal{L}$ of integers $\ell_k = \ell_k(\beta, D)$ by the formula: $\ell_0 = 0$ and

$$\ell_{k+1} = \inf\{\ell \geq \ell_k, B_\ell(\beta) \geq DB_{\ell_k}(\beta)\}. \quad (9.6.5)$$

Note that by (IRB), $B_L - B_\ell \leq C_0 \log(L/\ell)$. From this remark and the definition above one can deduce¹² that

$$D^k \leq B_{\ell_k}(\beta) \leq CD^k, \quad (9.6.6)$$

for every k and some large constant C independent of β, k , and D .

Theorem 9.1.5 will be a consequence of the following result. Recall that

$$\mathcal{I} = \mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_3}(x) \cap \mathbf{C}_{\mathbf{n}_2 + \mathbf{n}_4}(z). \quad (9.6.7)$$

Proposition 9.6.2 (Clustering bound). *For D large enough, there exists $\delta = \delta(D) > 0$ such that for all $\beta \leq \beta_c$, for all $K > 3$ with $\ell_{K+1} \leq L(\beta)$, and for all $u, x, y, z, t \in \mathbb{Z}^4$ with mutual distance between x, y, z, t larger than $2\ell_K$,*

$$\mathbf{P}_\beta^{ux, uz, uy, ut}[\mathbf{M}_u(\mathcal{I}; \mathcal{L}, K) < \delta K] \leq 2^{-\delta K}. \quad (9.6.8)$$

Let us see why this bound implies the improved tree diagram bound.

Proof of Theorem 9.1.5. Choose D large enough so that Proposition 9.6.2 holds. Fix x, y, z, t at mutual distance larger than $2\ell_K$. Using Lemma 9.6.1 together with the switching lemma (SL) we get,

$$\begin{aligned} \mathbf{P}_\beta^{xy, zt, \emptyset, \emptyset}[0 < |\mathcal{I}| < 2^{\delta K/5}] &\leq \sum_{u \in \mathbb{Z}^4} \mathbf{P}_\beta^{xy, zt, \emptyset, \emptyset}[u \in \mathcal{I}, \mathbf{M}_u(\mathcal{I}; \mathcal{L}, K) < \delta K] \\ &= \sum_{u \in \mathbb{Z}^4} \frac{\langle \sigma_x \sigma_u \rangle_\beta \langle \sigma_y \sigma_u \rangle_\beta \langle \sigma_z \sigma_u \rangle_\beta \langle \sigma_t \sigma_u \rangle_\beta}{\langle \sigma_x \sigma_y \rangle_\beta \langle \sigma_z \sigma_t \rangle_\beta} \mathbf{P}_\beta^{ux, uz, uy, ut}[\mathbf{M}_u(\mathcal{I}; \mathcal{L}, K) < \delta K], \end{aligned}$$

¹²Indeed, the lower bound is immediate and for the upper bound, write for $k \geq 1$,

$$\begin{aligned} B_{\ell_k-1}(\beta) &\leq DB_{\ell_{k-1}}(\beta) \leq DB_{\ell_{k-1}-1}(\beta) - CD \log\left(1 - \frac{1}{\ell_{k-1}}\right) \\ &\leq D^{k-1} B_{\ell_1-1}(\beta) + C \sum_{i=1}^{k-1} \frac{D^i}{\ell_{k-i}} \leq C_1 D^k \end{aligned}$$

for C_1 large enough (independent of D and k). Use (IRB) once again to write,

$$B_{\ell_k}(\beta) \leq B_{\ell_k-1} + C \log 2 \leq C_1 D^k + C \log 2 \leq C_2 D^k.$$

so that, with Proposition 9.6.2,

$$\mathbf{P}_\beta^{xy,zt,\emptyset,\emptyset}[0 < |\mathcal{I}| < 2^{\delta K/5}] \leq 2^{-\delta K} \sum_{u \in \mathbb{Z}^4} \frac{\langle \sigma_x \sigma_u \rangle_\beta \langle \sigma_y \sigma_u \rangle_\beta \langle \sigma_z \sigma_u \rangle_\beta \langle \sigma_t \sigma_u \rangle_\beta}{\langle \sigma_x \sigma_y \rangle_\beta \langle \sigma_z \sigma_t \rangle_\beta}. \quad (9.6.9)$$

Moreover,

$$\mathbf{P}_\beta^{xy,zt,\emptyset,\emptyset}[|\mathcal{I}| \geq 2^{\delta K/5}] \leq 2^{-\delta K/5} \sum_{u \in \mathbb{Z}^4} \frac{\langle \sigma_x \sigma_u \rangle_\beta \langle \sigma_y \sigma_u \rangle_\beta \langle \sigma_z \sigma_u \rangle_\beta \langle \sigma_t \sigma_u \rangle_\beta}{\langle \sigma_x \sigma_y \rangle_\beta \langle \sigma_z \sigma_t \rangle_\beta}, \quad (9.6.10)$$

which implies that

$$|U_4^\beta(x, y, z, t)| \leq \frac{2}{2^{\delta K/5}} \sum_{u \in \mathbb{Z}^4} \langle \sigma_x \sigma_u \rangle_\beta \langle \sigma_y \sigma_u \rangle_\beta \langle \sigma_z \sigma_u \rangle_\beta \langle \sigma_t \sigma_u \rangle_\beta.$$

Now, if $L := 2\ell_K$, observe that $\ell_{K+1} \geq L$ so that by (9.6.6), $B_L(\beta) \leq B_{\ell_{K+1}}(\beta) \leq CD^{K+1}$. Hence, we may find $c > 0$ sufficiently small (independent of L and β), such that $K \geq c \log B_L(\beta)$. This gives the result. $\square$

We now turn to the proof of Proposition 9.6.2. We start by showing that thanks to the hypothesis (A6), the interaction decays sufficiently fast so that the “jumps” made by the current are not so problematic. As a byproduct of these estimates, we are able to show that the clusters do not perform “back and forth” between different scales. This property was already a key step in the proof of [9].

9.6.2 Properties of the current

We will use a first moment method and argue that the expected number of long edges that have a non zero-weight under the current measure decays quickly in a certain sense. The following results rely on the existence of regular scales and thus require reflection positivity.

In what follows, we use the notion of backbone. The main properties of this object are gathered in Appendix 9.B.

First, we prove a bound on the probability that an edge is open in the percolation configuration deduced from the current measure. Note that this bound is in fact valid on any graph with any interactions.

Lemma 9.6.3 (Bound on open edge probability). *Let $d \geq 1$. Let $\beta > 0$. For $x, y, u, v \in \mathbb{Z}^d$, one has*

$$\begin{aligned} \mathbf{P}_\beta^{xy}[\mathbf{n}_{u,v} \geq 1] &\leq \mathbf{P}_\beta^{xy,\emptyset}[\mathbf{n}_{u,v} \geq 1] \\ &\leq \beta J_{u,v} \left(2\langle \sigma_u \sigma_v \rangle_\beta + \frac{\langle \sigma_x \sigma_u \rangle_\beta \langle \sigma_v \sigma_y \rangle_\beta}{\langle \sigma_x \sigma_y \rangle_\beta} + \frac{\langle \sigma_x \sigma_v \rangle_\beta \langle \sigma_u \sigma_y \rangle_\beta}{\langle \sigma_x \sigma_y \rangle_\beta} \right). \end{aligned}$$

Proof. The first inequality follows from a monotonicity argument. For the second inequality, we write,

$$\mathbf{P}_\beta^{xy,\emptyset}[\mathbf{n}_{u,v} \geq 1] \leq \mathbf{P}_\beta^{xy}[\mathbf{n}_{u,v} \geq 1] + \mathbf{P}_\beta^\emptyset[\mathbf{n}_{u,v} \geq 1]. \quad (9.6.11)$$

Then, we observe that $\mathbf{P}_\beta^\emptyset[\mathbf{n}_{u,v} \geq 1] \leq \beta J_{u,v} \langle \sigma_u \sigma_v \rangle_\beta$, which leads to

$$\begin{aligned} \mathbf{P}_\beta^{xy}[\mathbf{n}_{u,v} \geq 1] &\leq \beta J_{u,v} \frac{\langle \sigma_x \sigma_y \sigma_u \sigma_v \rangle_\beta}{\langle \sigma_x \sigma_y \rangle_\beta} \\ &\leq \beta J_{u,v} \left(\langle \sigma_u \sigma_v \rangle_\beta + \frac{\langle \sigma_x \sigma_u \rangle_\beta \langle \sigma_v \sigma_y \rangle_\beta}{\langle \sigma_x \sigma_y \rangle_\beta} + \frac{\langle \sigma_x \sigma_v \rangle_\beta \langle \sigma_u \sigma_y \rangle_\beta}{\langle \sigma_x \sigma_y \rangle_\beta} \right), \end{aligned}$$

where we used Lebowitz' inequality [173] to get $U_4^\beta(x, y, u, v) \leq 0$ (see also Proposition 9.4.7). $\square$

We now introduce the event we will be interested in. It is illustrated in Figure 9.3 below.

Definition 9.6.4 (Jump event). Let $1 \leq k \leq m$. We define $\text{Jump}(k, m)$ to be the event that there exist $u \in \Lambda_k$ and $v \notin \Lambda_m$ such that $\mathbf{n}_{u,v} \geq 1$.

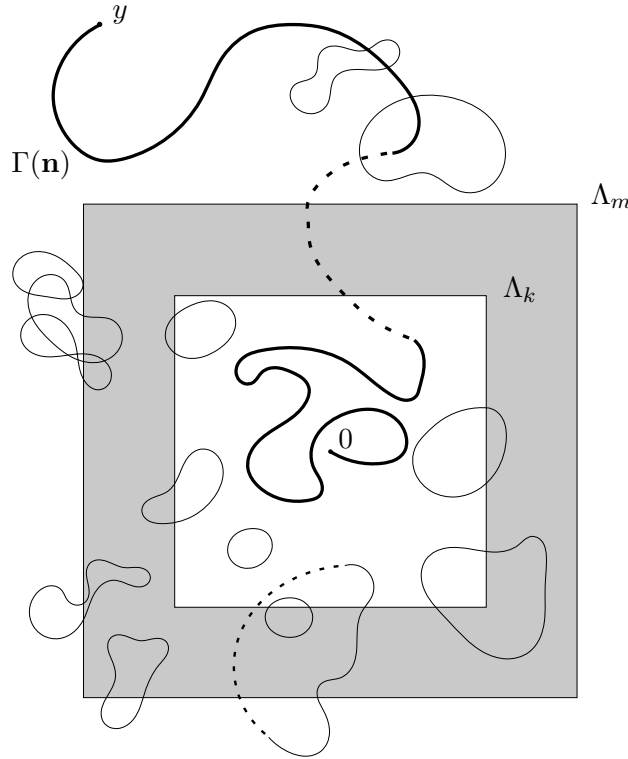


Figure 9.3: A realisation of the event $\text{Jump}(k, m)$ for a current $\mathbf{n}$ with source set $\partial \mathbf{n} = \{0, y\}$. The bold black path represents the backbone $\Gamma(\mathbf{n})$. The dashed curves represent long open edges that jump over the annulus $\text{Ann}(k, m)$.

We now prove that if we consider a current with two sources, and an annulus located between them, but “far away” from each of them, then with high probability the current does not “jump over it”. For convenience, we fix one of these sources to be the origin. Recall that $\varepsilon > 0$ is given by (9.6.2).

Lemma 9.6.5 (Jumping a scale is unlikely). *Let $d = 4$. Assume that J satisfies (A1)–(A6). Let $\nu \in (0, 1)$ be such that $\nu > \frac{d}{d+\varepsilon}$. There exist $C, \eta > 0$ such that for all $\beta \leq \beta_c$, for all $y \in \mathbb{Z}^d$ in a regular scale with $1 \leq |y| \leq L(\beta)$, and for all $k \geq 1$ such that $k^{1+4/\varepsilon} \leq |y|$,*

$$\mathbf{P}_\beta^{0y, \emptyset} [\text{Jump}(k, k + k^\nu)] \leq \frac{C}{k^\eta}. \quad (9.6.12)$$

Remark 9.6.6. If one takes $|y| \geq \ell_{K+1}$, this lemma ensures that the current visits the annuli $\text{Ann}(\ell_k, \ell_{k+1})$ for $1 \leq k \leq K$ with high probability. This property will be crucial in the proof of Proposition 9.6.2.

Proof. In what follows $C = C(d) > 0$ may change from line to line. It is sufficient to bound,

$$\sum_{u \in \Lambda_k, v \notin \Lambda_{k+k^\nu}} \mathbf{P}_\beta^{0y, \emptyset} [\mathbf{n}_{u,v} \geq 1]. \quad (9.6.13)$$

Using Lemma 9.6.3, we have (see Figure 9.4),

$$\begin{aligned} \sum_{u \in \Lambda_k, v \notin \Lambda_{k+k^\nu}} \mathbf{P}_\beta^{0y, \emptyset} [\mathbf{n}_{u,v} \geq 1] &\leq 2\beta \sum_{u \in \Lambda_k, v \notin \Lambda_{k+k^\nu}} J_{u,v} \left(\langle \sigma_u \sigma_v \rangle_\beta \right. \\ &\quad \left. + \frac{\langle \sigma_0 \sigma_u \rangle_\beta \langle \sigma_v \sigma_y \rangle_\beta}{\langle \sigma_0 \sigma_y \rangle_\beta} + \frac{\langle \sigma_0 \sigma_v \rangle_\beta \langle \sigma_u \sigma_y \rangle_\beta}{\langle \sigma_0 \sigma_y \rangle_\beta} \right) =: A_1 + A_2 + A_3. \end{aligned} \quad (9.6.14)$$

• **Bound on A_1 .** Write

$$\begin{aligned} \beta \sum_{u \in \Lambda_k, v \notin \Lambda_{k+k^\nu}} J_{u,v} \langle \sigma_u \sigma_v \rangle_\beta &\leq C \sum_{u \in \Lambda_k, v \notin \Lambda_{k+k^\nu}} \frac{|v - u|^2 J_{0,v-u}}{|v - u|^d} \\ &\leq C k^d \sum_{v \notin \Lambda_{k^\nu}} \frac{|v|^2 J_{0,v}}{|v|^d} \\ &\leq C k^{d(1-\nu)} \sum_{v \notin \Lambda_{k^\nu}} |v|^2 J_{0,v} \leq C k^{d(1-\nu)-\nu\varepsilon}, \end{aligned}$$

where we used (IRB) on the first line, translation invariance on the second line, and (9.6.2) on the last line.

• **Bound on A_2 .** For the second term, by the lower bound on the two-point function of Proposition 9.3.23 (together with the assumption that $1 \leq |y| \leq L(\beta)$) one has $\langle \sigma_0 \sigma_y \rangle_\beta^{-1} \leq C\beta|y|^{d-1}$. Using (IRB) and the translation invariance of J ,

$$\begin{aligned} \beta \sum_{u \in \Lambda_k, v \in \Lambda_{|y|/2}(y)} J_{u,v} \frac{\langle \sigma_0 \sigma_u \rangle_\beta \langle \sigma_v \sigma_y \rangle_\beta}{\langle \sigma_0 \sigma_y \rangle_\beta} &\leq C\beta^2 |y|^{d-1} \sum_{\substack{u \in \Lambda_k \\ v \in \Lambda_{|y|/2}(y)}} J_{0,v-u} \langle \sigma_0 \sigma_u \rangle_\beta \langle \sigma_v \sigma_y \rangle_\beta \\ &\leq Ck^2 |y|^{d-1} \sum_{v \in \Lambda_{2|y|/3}(y)} \frac{J_{0,v}}{(|v - y| + 1)^{d-2}} \\ &\leq Ck^2 |y|^{d-1} \sum_{p=1}^{2|y|/3} \frac{1}{p^{d-2}} \sum_{v \in \Lambda_{2|y|/3}(y)} J_{0,v} \mathbf{1}_{|y-v|=p}. \end{aligned}$$

Now, using¹³ (9.6.2),

$$\sum_{v \in \Lambda_{2|y|/3}(y)} J_{0,v} \mathbf{1}_{|y-v|=p} \leq Cp|y|^{-(3+\varepsilon)}. \quad (9.6.15)$$

Using the assumption that $|y| \geq k^{1+4/\varepsilon}$,

$$\sum_{u \in \Lambda_k, v \in \Lambda_{|y|/2}(y)} J_{u,v} \frac{\langle \sigma_0 \sigma_u \rangle_\beta \langle \sigma_v \sigma_y \rangle_\beta}{\langle \sigma_0 \sigma_y \rangle_\beta} \leq \frac{Ck^2 \log |y|}{|y|^\varepsilon} \leq \frac{Ck^2}{|y|^{\varepsilon/2}} \leq \frac{C}{k^{\varepsilon/2}}. \quad (9.6.16)$$

Finally, in the case $v \notin \Lambda_{|y|/2}(y) \cup \Lambda_{k+k^\nu}$, we may use **(P1)** (for $v \in \Lambda_{4|y|}(y) \setminus (\Lambda_{|y|/2}(y) \cup \Lambda_{k+k^\nu})$), as well as **(MMS2)** (for $v \notin \Lambda_{4|y|}(y)$), to show that $\langle \sigma_v \sigma_y \rangle_\beta \leq C_0 \langle \sigma_0 \sigma_y \rangle_\beta$, so that,

$$\begin{aligned} \beta \sum_{u \in \Lambda_k, v \notin \Lambda_{|y|/2}(y) \cup \Lambda_{k+k^\nu}} J_{u,v} \frac{\langle \sigma_0 \sigma_u \rangle_\beta \langle \sigma_v \sigma_y \rangle_\beta}{\langle \sigma_0 \sigma_y \rangle_\beta} &\leq C_0 \beta \sum_{u \in \Lambda_k, v \notin \Lambda_{|y|/2}(y) \cup \Lambda_{k+k^\nu}} J_{u,v} \langle \sigma_0 \sigma_u \rangle_\beta \\ &\leq Ck^{2(1-\nu)} \sum_{v \notin \Lambda_{k^\nu}} |v|^2 J_{0,v} \leq Ck^{2(1-\nu)-\nu\varepsilon}. \end{aligned}$$

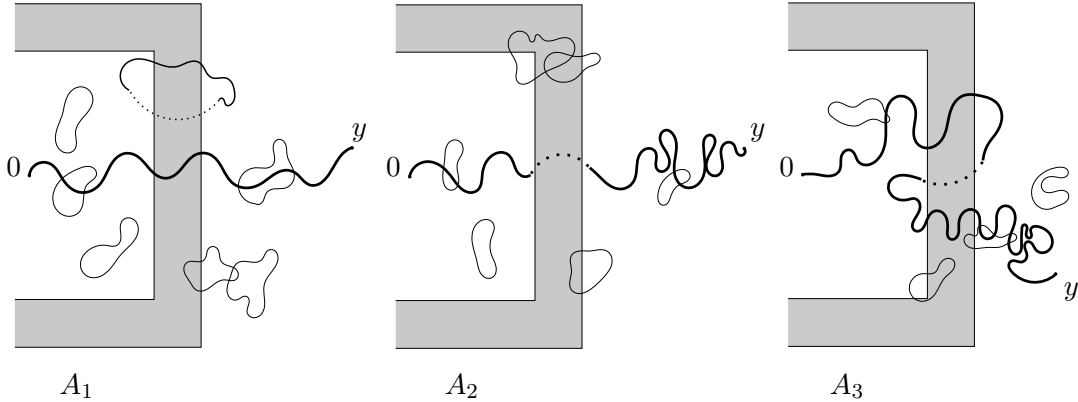


Figure 9.4: A graphical representation of the bound (9.6.14). We represented each potential contribution to A_1, A_2, A_3 (from left to right). The backbone is the bold path joining 0 and y . Long open edges are the dashed curves. The largest contribution should come from A_2 in which long edges are induced by the backbone.

¹³Note that this bound is very sub-optimal in some cases but is enough for our purpose. For instance, assuming more regularity on the interaction we could replace (9.6.2) by a bound of the form $J_{0,x} \leq C|x|^{-d-2-\varepsilon}$ for all $x \in \mathbb{Z}^d \setminus \{0\}$. In that case, we would have $J_{0,v} \lesssim |y|^{-d-2-\varepsilon}$ for $v \in \Lambda_{2|y|/3}(y)$ which leads to a bound of the first contribution of A_2 of order $k^2/|y|^{1+\varepsilon}$ for any $d \geq 4$.

• **Bound on A_3 .** Since $|y| \geq k^{1+4/\varepsilon}$, one has for $u \in \Lambda_k$, $\langle \sigma_u \sigma_y \rangle_\beta \leq C_0 \langle \sigma_0 \sigma_y \rangle_\beta$ by the property **(P1)** of regular scales. Using this remark, together with **(IRB)** and (9.6.2), yields,

$$\begin{aligned}
A_3 &\leq 2\beta C_0 \sum_{u \in \Lambda_k, v \notin \Lambda_{k+k^\nu}} J_{u,v} \langle \sigma_0 \sigma_v \rangle_\beta \\
&\leq C \sum_{u \in \Lambda_k, v \notin \Lambda_{k+k^\nu}} \frac{|v-u|^2 J_{0,v-u}}{|v-u|^2} \frac{1}{|v|^{d-2}} \\
&\leq \frac{C}{k^d} \sup_{\substack{u \in \Lambda_k \\ v \notin \Lambda_{k+k^\nu}}} \left(\frac{|v|}{|u-v|} \right)^2 \sum_{u \in \Lambda_k, v \notin \Lambda_{k+k^\nu}} |u-v|^2 J_{0,v-u} \\
&\leq C k^{2(1-\nu)} \sum_{v \notin \Lambda_{k^\nu}} |v|^2 J_{0,v} \leq C k^{2(1-\nu)-\nu\varepsilon}.
\end{aligned}$$

We then obtain the result taking η sufficiently small, and C large enough. $\square$

Remark 9.6.7. We could prove a similar result with the assumption **(A6)** replaced by **(A6')** as mentioned in the introduction. The only difference is that we now need to consider the event $\text{Jump}(k, k + k/(\log k)^{1-\nu})$ for $\nu \in (0, 1)$ sufficiently close to 1. Using the exact same method as above we obtain $\eta, C > 0$ such that, when $|y| \geq \exp(k^{1+4/\varepsilon})$,

$$\mathbf{P}_\beta^{0y, \emptyset}[\text{Jump}(k, k + k/(\log k)^{1-\nu})] \leq \frac{C}{(\log k)^\eta}. \quad (9.6.17)$$

Similar modifications can be applied below.

As a corollary of the above result, we can show that the probability that a backbone does a *zigzag* between two “distanced” scales is very small. This will be very useful later to argue that intersection events are (essentially) local.

Definition 9.6.8 (Zigzag event). For $1 \leq k \leq \ell \leq M$ and $u, v \in \mathbb{Z}^d$, let $\text{ZZ}(u, v; k, \ell, M)$ be the event that the backbone of $\mathbf{n}$ (with $\partial \mathbf{n} = \{u, v\}$) goes from u to a point in $\text{Ann}(\ell, M)$, then to a point in Λ_k , before finally hitting v . We let $\text{ZZ}(u, v; k, \ell, \infty)$ be the union of all $\text{ZZ}(u, v; k, \ell, M)$ for $M \geq \ell$.

Corollary 9.6.9 (No zigzag for the backbone). *Let $d = 4$. Assume that J satisfies **(A1)**–**(A6)**. Let $\nu \in (0, 1)$ be such that $\nu > \frac{d}{d+\varepsilon}$. There exist $C, \eta > 0$ such that for all $\beta \leq \beta_c$, for all $k, \ell \geq 1$ and $y \in \mathbb{Z}^d$ in a regular scale with $k^{8/(1-\nu)} \leq \ell$ and $\ell^{1+4/\varepsilon} \leq |y|$,*

$$\mathbf{P}_\beta^{0y}[\text{ZZ}(0, y; k, \ell, \infty)] \leq \frac{C}{\ell^\eta}. \quad (9.6.18)$$

Proof. Notice that,

$$\text{ZZ}(0, y; k, \ell, \infty) \subset \text{ZZ}(0, y; k, \ell, \ell + \ell^\nu) \cup \text{Jump}(\ell, \ell + \ell^\nu). \quad (9.6.19)$$

The chain rule for backbones (see Appendix 9.B), the assumption of regularity made on y , as well as **(IRB)**, yield

$$\begin{aligned}
\mathbf{P}_\beta^{0y}[\text{ZZ}(0, y; k, \ell, \ell + \ell^\nu)] &\leq \sum_{\substack{v \in \text{Ann}(\ell, \ell + \ell^\nu) \\ w \in \Lambda_k}} \frac{\langle \sigma_0 \sigma_v \rangle_\beta \langle \sigma_v \sigma_w \rangle_\beta \langle \sigma_w \sigma_y \rangle_\beta}{\langle \sigma_0 \sigma_y \rangle_\beta} \\
&\leq C \frac{k^4 \ell^{3+\nu}}{\ell^4} \leq \frac{C}{\ell^{(1-\nu)/2}}.
\end{aligned}$$

We conclude using Lemma 9.6.5. $\square$

Remark 9.6.10. Note that in the above result we heavily relied on the fact the current cannot jump over an annulus of dimension strictly smaller than four. We will see that Lemma 9.6.5 does not hold anymore in the case $d_{\text{eff}} = 4$ and $1 \leq d \leq 3$ which makes the study of the zigzag event more complicated.

This second technical result is a small modification of Lemma 9.6.5 but will be crucial in the proof of the mixing. It is a little easier since the sources are now located close to the origin: there is no “long backbone” which might help to jump scales. Note that this lemma does not rely on the existence of regular scales.

Lemma 9.6.11. *Let $d = 4$. Assume that J satisfies (A1)–(A6). Let $\nu \in (0, 1)$ be such that $\nu > \frac{d}{d+\varepsilon}$. There exist $C, \eta > 0$ such that for all $\beta \leq \beta_c$, for all $n < m \leq M \leq k$ with $1 \leq M^{3/2} \leq k \leq L(\beta)$, for all $x \in \Lambda_n$ and all $u \in \text{Ann}(m, M)$,*

$$\mathbf{P}_\beta^{xu, \emptyset}[\text{Jump}(k, k + k^\nu)] \leq \frac{C}{k^\eta}. \quad (9.6.20)$$

Proof. We follow the same steps as in the proof of Lemma 9.6.5. Using Lemma 9.6.3,

$$\begin{aligned} \sum_{w \in \Lambda_k, v \notin \Lambda_{k+k^\nu}} \mathbf{P}_\beta^{xu, \emptyset}[\mathbf{n}_{w,v} \geq 1] \\ \leq 2\beta \sum_{w \in \Lambda_k, v \notin \Lambda_{k+k^\nu}} J_{w,v} \left(\langle \sigma_w \sigma_v \rangle_\beta + \frac{\langle \sigma_x \sigma_w \rangle_\beta \langle \sigma_v \sigma_u \rangle_\beta}{\langle \sigma_x \sigma_u \rangle_\beta} + \frac{\langle \sigma_x \sigma_v \rangle_\beta \langle \sigma_w \sigma_u \rangle_\beta}{\langle \sigma_x \sigma_u \rangle_\beta} \right). \end{aligned}$$

As for the bound of A_1 above,

$$\beta \sum_{w \in \Lambda_k, v \notin \Lambda_{k+k^\nu}} J_{w,v} \langle \sigma_w \sigma_v \rangle_\beta \leq C_1 k^{d(1-\nu)-\nu\varepsilon}. \quad (9.6.21)$$

Using the lower bound of Proposition 9.3.23 (which is allowed because $1 \leq |x - u| \leq L(\beta)$) together with (IRB) and (9.6.2), we get

$$\begin{aligned} \beta \sum_{w \in \Lambda_k, v \notin \Lambda_{k+k^\nu}} J_{w,v} \frac{\langle \sigma_x \sigma_v \rangle_\beta \langle \sigma_w \sigma_u \rangle_\beta}{\langle \sigma_x \sigma_u \rangle_\beta} &\leq \beta^2 C_2 M^{d-1} \sum_{w \in \Lambda_k, v \notin \Lambda_{k+k^\nu}} J_{w,v} \langle \sigma_x \sigma_v \rangle_\beta \langle \sigma_w \sigma_u \rangle_\beta \\ &\leq C_3 M^{d-1} k^2 k^{-(d-2)} \sum_{v \notin \Lambda_{k^\nu}} J_{0,v} \\ &\leq C_4 M^{d-1} k^{-\nu(2+\varepsilon)}. \end{aligned}$$

Finally, with the same reasoning, we also get

$$\sum_{w \in \Lambda_k, v \notin \Lambda_{k+k^\nu}} J_{w,v} \frac{\langle \sigma_x \sigma_w \rangle_\beta \langle \sigma_v \sigma_u \rangle_\beta}{\langle \sigma_x \sigma_u \rangle_\beta} \leq C_5 M^{d-1} k^{-\nu(2+\varepsilon)}. \quad (9.6.22)$$

The assumption made on ν and the inequality $M^3 \leq k^2$ yield the result. $\square$

Similarly, we can rule out the zigzag of the backbone in this setup.

Corollary 9.6.12. *Let $d = 4$. Assume that J satisfies (A1)–(A6). Let $\nu \in (0, 1)$ be such that $\nu > \frac{d}{d+\varepsilon}$. There exist $C, \eta > 0$ such that for all $\beta \leq \beta_c$, for all $n < m \leq M \leq k$ with $1 \leq M^{6/(1-\nu)} \leq k \leq L(\beta)$, for all $x \in \Lambda_n$ and all $u \in \text{Ann}(m, M)$,*

$$\mathbf{P}_\beta^{xu}[\text{ZZ}(x, u; M, k, \infty)] \leq \frac{C}{k^\eta}. \quad (9.6.23)$$

Proof. We repeat the argument used to prove Corollary 9.6.9, except that the proof is now easier since u already lies in Λ_M . Let $\nu \in (0, 1)$ be such that $\nu > \frac{d}{d+\varepsilon}$. One has,

$$\text{ZZ}(x, u; M, k, \infty) \subset \text{ZZ}(x, u; M, k, k + k^\nu) \cup \text{Jump}(k, k + k^\nu). \quad (9.6.24)$$

The chain rule for backbones, the lower bound of Proposition 9.3.23, and (IRB), yield

$$\begin{aligned} \mathbf{P}_\beta^{xu}[\text{ZZ}(x, u; M, k, k + k^\nu)] &\leq \sum_{v \in \text{Ann}(k, k + k^\nu)} \frac{\langle \sigma_x \sigma_v \rangle_\beta \langle \sigma_v \sigma_u \rangle_\beta}{\langle \sigma_x \sigma_u \rangle_\beta} \\ &\leq C \frac{M^{d-1} k^{3+\nu}}{k^4} \leq \frac{C}{k^{(1-\nu)/2}}. \end{aligned}$$

We conclude using Lemma 9.6.11. □

This final technical lemma will be useful to argue that for a current $\mathbf{n}$ with $\partial \mathbf{n} = \{x, y\}$, the restriction of $\mathbf{n}$ to $(\Gamma(\mathbf{n}))^c$ (that we denote by $\mathbf{n} \setminus \Gamma(\mathbf{n})$ below), where $\Gamma(\mathbf{n})$ is the backbone of $\mathbf{n}$ and $\overline{\Gamma(\mathbf{n})}$ is the set of edges revealed during the exploration of $\Gamma(\mathbf{n})$, essentially behaves like a sourceless current on a smaller graph. In that case, jump events should become even more unlikely since there is no backbone to create long connections anymore.

If $\mathbf{n}$ is a current, and E is a set of edges, we let $\mathbf{n}_E$ be the restriction of $\mathbf{n}$ to the edges in E .

Lemma 9.6.13. *Let $d = 4$. Assume that J satisfies (A1)–(A6). Let $\nu \in (0, 1)$ be such that $\nu > \frac{d}{d+\varepsilon}$. There exist $C, \eta > 0$ such that for all $\beta \leq \beta_c$, for all $k \geq 1$, for all $x, y \in \mathbb{Z}^d$,*

$$\mathbf{P}_\beta^{xy}[\mathbf{n} \setminus \overline{\Gamma(\mathbf{n})} \in \text{Jump}(k, k + k^\nu)] \leq \mathbf{P}_\beta^{xy, \emptyset}[(\mathbf{n}_1 + \mathbf{n}_2) \setminus \overline{\Gamma(\mathbf{n}_1)} \in \text{Jump}(k, k + k^\nu)] \leq \frac{C}{k^\eta}, \quad (9.6.25)$$

where by $(\mathbf{n}_1 + \mathbf{n}_2) \setminus \overline{\Gamma(\mathbf{n}_1)}$ we mean the restriction of $(\mathbf{n}_1 + \mathbf{n}_2)$ to the graph depleted of the edges belonging to $\overline{\Gamma(\mathbf{n}_1)}$.

Proof. The first inequality follows by monotonicity. Write $\mathcal{A} := \{(\mathbf{n}_1 + \mathbf{n}_2) \setminus \overline{\Gamma(\mathbf{n}_1)} \in \text{Jump}(k, k + k^\nu)\}$ and for a consistent (see Appendix 9.B for a definition) path $\gamma : x \rightarrow y$, $\mathcal{A}_\gamma := \{(\mathbf{n}_1 + \mathbf{n}_2)_{\bar{\gamma}^c} \in \text{Jump}(k, k + k^\nu)\}$. The idea is to condition on the backbone of $\mathbf{n}_1$. Going

to partition functions¹⁴, one has $\mathbf{P}_\beta^{xy, \emptyset}[\mathcal{A}] = Z_\beta^{xy, \emptyset}[\mathcal{A}] / Z_\beta^{xy, \emptyset}$ where,

$$\begin{aligned}
Z_\beta^{xy, \emptyset}[\mathcal{A}] &:= \sum_{\gamma: x \rightarrow y \text{ consistent}} \sum_{\substack{\partial \mathbf{n}_1 = \{x, y\} \\ \partial \mathbf{n}_2 = \emptyset}} w_\beta(\mathbf{n}_1) w_\beta(\mathbf{n}_2) \mathbb{1}_{\Gamma(\mathbf{n}_1) = \gamma} \mathbb{1}_{\mathcal{A}} \\
&= \sum_{\gamma} \sum_{\substack{\partial(\mathbf{n}_1)_{\bar{\gamma}} = \{x, y\} \\ \partial(\mathbf{n}_1)_{\bar{\gamma}^c} = \emptyset \\ \partial \mathbf{n}_2 = \emptyset}} w_\beta((\mathbf{n}_1)_{\bar{\gamma}}) w_\beta((\mathbf{n}_1)_{\bar{\gamma}^c}) w_\beta(\mathbf{n}_2) \mathbb{1}_{\Gamma((\mathbf{n}_1)_{\bar{\gamma}}) = \gamma} \mathbb{1}_{\mathcal{A}_\gamma} \\
&= \sum_{\gamma} Z_{\bar{\gamma}, \beta}^{xy}[\Gamma(\mathbf{n}_{\bar{\gamma}}) = \gamma] Z_{\bar{\gamma}^c, \mathbb{Z}^d, \beta}^{\emptyset, \emptyset}[\mathcal{A}_\gamma] \\
&= \sum_{\gamma} Z_\beta^{xy, \emptyset}[\Gamma(\mathbf{n}_1) = \gamma] \mathbf{P}_{\bar{\gamma}^c, \mathbb{Z}^d, \beta}^{\emptyset, \emptyset}[\mathcal{A}_\gamma].
\end{aligned}$$

Using Lemma 9.6.3 as well as Griffiths' inequality, for any γ as above,

$$\mathbf{P}_{\bar{\gamma}^c, \mathbb{Z}^d, \beta}^{\emptyset, \emptyset}[\mathcal{A}_\gamma] \leq \sum_{\substack{u \in \Lambda_k \\ v \notin \Lambda_{k+k\nu}}} 2\beta J_{u,v} \langle \sigma_u \sigma_v \rangle_\beta \leq Ck^{d(1-\nu)-\nu\varepsilon},$$

where the last inequality was obtained in the proof of Lemma 9.6.5. Hence,

$$Z_\beta^{xy, \emptyset}[\mathcal{A}] \leq Ck^{d(1-\nu)-\nu\varepsilon} Z_\beta^{xy, \emptyset}, \quad (9.6.26)$$

which yields the result since $\nu > \frac{d}{d+\varepsilon}$. $\square$

As before, having a control of the jump probability over an annulus of dimension < 4 (namely $3 + \nu$ for $\nu \in (0, 1)$) has consequences on the geometry of the current. In the context of Lemma 9.6.13, it prevents $(\mathbf{n}_1 + \mathbf{n}_2) \setminus \overline{\Gamma(\mathbf{n}_1)}$ to connect two (sufficiently) distanced scales. We begin with a definition.

Definition 9.6.14 (Crossing event). For $1 \leq k \leq \ell$ and $\mathbf{n}$ a current, we say that $\mathbf{n}$ realises the event $\text{Cross}(k, \ell)$ if $\mathbf{n}$ “crosses” $\text{Ann}(k, \ell)$, in the sense that there exists a cluster of $\mathbf{n}$ containing both a point in Λ_k and in Λ_ℓ^c .

Corollary 9.6.15. *Let $d = 4$. Assume that J satisfies (A1)–(A6). Let $\nu \in (0, 1)$ be such that $\nu > \frac{d}{d+\varepsilon}$. There exist $C, \eta > 0$ such that for all $\beta \leq \beta_c$, for all $k, \ell \geq 1$ with $k^{8/(1-\nu)} \leq \ell$, for all $x, u \in \mathbb{Z}^d$,*

$$\mathbf{P}_\beta^{xu}[\mathbf{n} \setminus \overline{\Gamma(\mathbf{n})} \in \text{Cross}(k, \ell)] \leq \mathbf{P}_\beta^{xu, \emptyset}[(\mathbf{n}_1 + \mathbf{n}_2) \setminus \overline{\Gamma(\mathbf{n}_1)} \in \text{Cross}(k, \ell)] \leq \frac{C}{\ell^\eta}. \quad (9.6.27)$$

Proof. The first inequality follows by monotonicity. Notice that,

$$\begin{aligned}
\{(\mathbf{n}_1 + \mathbf{n}_2) \setminus \overline{\Gamma(\mathbf{n}_1)} \in \text{Cross}(k, \ell)\} &\subset \bigcup_{v \in \Lambda_k, w \in \text{Ann}(\ell, \ell + \ell^\nu)} \{v \longleftrightarrow w \text{ in } (\mathbf{n}_1 + \mathbf{n}_2) \setminus \overline{\Gamma(\mathbf{n}_1)}\} \\
&\quad \cup \{(\mathbf{n}_1 + \mathbf{n}_2) \setminus \overline{\Gamma(\mathbf{n}_1)} \in \text{Jump}(\ell, \ell + \ell^\nu)\}.
\end{aligned}$$

¹⁴One would need to restrict to a finite subset Λ of $\mathbb{Z}^d$ first, and then take the limit $\Lambda \rightarrow \mathbb{Z}^d$. We omit this detail here.

The second event on the right-hand side above is handled using Lemma 9.6.13.

To handle the first event, we use the fact that the probability v and w are connected in $(\mathbf{n}_1 + \mathbf{n}_2) \setminus \overline{\Gamma(\mathbf{n}_1)}$ can be bounded by $\langle \sigma_v \sigma_w \rangle_\beta^2$. Indeed, this result follows from a generalisation of the switching lemma that can be found in [10, Lemma 2.2]. Proceeding as in the proof of Lemma 9.6.13, we get

$$\begin{aligned} \mathbf{P}_\beta^{xu, \emptyset} [u \longleftrightarrow v \text{ in } (\mathbf{n}_1 + \mathbf{n}_2) \setminus \overline{\Gamma(\mathbf{n}_1)}] \\ \leq \sum_{\gamma: x \rightarrow u \text{ consistent}} \mathbf{P}_\beta^{xu} [\Gamma(\mathbf{n}) = \gamma] \mathbf{P}_{\bar{\gamma}^c, \mathbb{Z}^d, \beta}^{\emptyset, \emptyset} [v \longleftrightarrow w \text{ in } (\mathbf{m}_1 + \mathbf{m}_2) \setminus \bar{\gamma}]. \end{aligned}$$

The above-mentioned generalisation of the switching lemma, together with Griffiths' inequality, yield

$$\mathbf{P}_{\bar{\gamma}^c, \mathbb{Z}^d, \beta}^{\emptyset, \emptyset} [v \longleftrightarrow w \text{ in } (\mathbf{m}_1 + \mathbf{m}_2) \setminus \bar{\gamma}] = \langle \sigma_v \sigma_w \rangle_{\bar{\gamma}^c, \beta} \langle \sigma_v \sigma_w \rangle_\beta \leq \langle \sigma_v \sigma_w \rangle_\beta^2. \quad (9.6.28)$$

As a result,

$$\begin{aligned} \mathbf{P}_\beta^{xu, \emptyset} \left[\bigcup_{v \in \Lambda_k, w \in \text{Ann}(\ell, \ell + \ell^\nu)} \left\{ v \longleftrightarrow w \text{ in } (\mathbf{n}_1 + \mathbf{n}_2) \setminus \overline{\Gamma(\mathbf{n}_1)} \right\} \right] &\leq \sum_{\substack{v \in \Lambda_k \\ w \in \text{Ann}(\ell, \ell + \ell^\nu)}} \langle \sigma_v \sigma_w \rangle_\beta^2 \\ &\leq C_1 \frac{k^4}{\ell^{1-\nu}} \leq \frac{C_1}{\ell^{(1-\nu)/2}}. \end{aligned}$$

This concludes the proof. □

9.6.3 Proof of the intersection property

Let $d = 4$. Assume that J satisfies (A1)–(A6). Let $\beta \leq \beta_c$. Recall the definition of $\mathcal{L} = \mathcal{L}(\beta, D)$ given at the beginning of the section: $\ell_0 = 0$ and

$$\ell_{k+1} = \inf \{ \ell \geq \ell_k, B_\ell(\beta) \geq DB_{\ell_k}(\beta) \}. \quad (9.6.29)$$

The existence of regular scales and the sliding-scale infrared bound have the following interesting consequence on how fast the bubble diagram grows from one scale to the other, it can be seen as an improvement over the bound $B_L(\beta) - B_\ell(\beta) \leq C_0 \log(L/\ell)$.

Lemma 9.6.16 (Scale to scale comparison of the bubble diagram, [9, Lemma 6.3]). *Let $d = 4$. There exists $C = C(d) > 0$ such that for every $\beta \leq \beta_c$, and for every $1 \leq \ell \leq L \leq L(\beta)$,*

$$B_L(\beta) \leq \left(1 + C \frac{\log_2(L/\ell)}{\log_2(\ell)} \right) B_\ell(\beta). \quad (9.6.30)$$

Proof. If $N \geq n$ and n is a regular scale, we may write,

$$\begin{aligned} B_{2N}(\beta) - B_N(\beta) &\leq C_1 N^{-4} \chi_{N/d}(\beta)^2 \leq C_2 \cdot n^{-4} \chi_n(\beta)^2 \leq C_3 \cdot n^{-4} (\chi_{2n}(\beta) - \chi_n(\beta))^2 \\ &\leq C_4 (B_{2n}(\beta) - B_n(\beta)), \end{aligned}$$

where we successively used (MMS2), the sliding-scale infrared bound, the property (P3) of regular scales, and the Cauchy–Schwarz inequality. There are $\log_2(L/\ell)$ scales between ℓ and L , and at least $c\log_2(\ell)$ regular scales between 1 and ℓ . Using the above computation,

$$\begin{aligned} B_L(\beta) - B_\ell(\beta) &= \sum_{N \text{ scale between } \ell \text{ and } L/2} B_{2N}(\beta) - B_N(\beta) \\ &\leq \frac{\log_2(L/\ell)}{c\log_2(\ell)} \sum_{\substack{n \text{ regular scale} \\ \text{between } 1 \text{ and } \ell}} B_{2n}(\beta) - B_n(\beta) \leq \frac{\log_2(L/\ell)}{c\log_2(\ell)} B_\ell(\beta). \end{aligned}$$

□

The above property has the following important consequence which ensures that the scales explode sufficiently fast. This will be used later to make sure there is “enough room” in the annuli $\text{Ann}(\ell_k, \ell_{k+1})$.

Remark 9.6.17 (Growth of $\mathcal{L}$). Using Lemma 9.6.16 we get that,

$$\log_2(\ell_k) \leq C \log_2(\ell_{k+1}/\ell_k) \frac{B_{\ell_k}(\beta)}{B_{\ell_{k+1}}(\beta) - B_{\ell_k}(\beta)} \leq \log_2(\ell_{k+1}/\ell_k) \frac{C}{D-1}, \quad (9.6.31)$$

so that,

$$\ell_{k+1} \geq \ell_k^{(D-1)/C}. \quad (9.6.32)$$

Recall that the event **Jump** was defined in Definition 9.6.18. In the remaining of this section, we fix $\nu \in (0, 1)$ with $\nu > \frac{d}{d+\varepsilon}$ such that Lemmas 9.6.5, 9.6.11, and 9.6.13 hold.

Definition 9.6.18 (Intersection event). Let $k \geq 1$ and $y \notin \Lambda_{\ell_{k+2}}$. A pair of currents $(\mathbf{n}, \mathbf{m})$ with $(\partial \mathbf{n}, \partial \mathbf{m}) = (\{0, y\}, \{0, y\})$ realises the event I_k if the following properties are satisfied:

- (i) The restrictions of $\mathbf{n}$ and $\mathbf{m}$ to edges with both endpoints in $\text{Ann}(\ell_k, \ell_{k+1} + \ell_{k+1}^\nu)$ contain a unique cluster “strongly crossing” $\text{Ann}(\ell_k, \ell_{k+1})$, in the sense that it contains a vertex in $\text{Ann}(\ell_k, \ell_k + \ell_k^\nu)$ and a vertex in $\text{Ann}(\ell_{k+1}, \ell_{k+1} + \ell_{k+1}^\nu)$.
- (ii) The two clusters described in (i) intersect.

Note that the event I_k is measurable in term of edges with both endpoints in the annulus $\text{Ann}(\ell_k, \ell_{k+1} + \ell_{k+1}^\nu)$.

The following lemma shows that intersections occur at every scale with a uniformly positive probability.

Lemma 9.6.19 (Intersection property). *Let $d = 4$. For D large enough, there exists $\kappa > 0$ such that for every $\beta \leq \beta_c$, every $k \geq 2$, and every $y \notin \Lambda_{\ell_{k+2}}$ in a regular scale with $1 \leq |y| \leq L(\beta)$,*

$$\mathbf{P}_\beta^{0y, 0y, \emptyset, \emptyset}[(\mathbf{n}_1 + \mathbf{n}_3, \mathbf{n}_2 + \mathbf{n}_4) \in I_k] \geq \kappa. \quad (9.6.33)$$

Proof. We restrict ourselves to the case of y in a regular scale to be able to use the properties (P1) and (P2). Introduce intermediate scales $\ell_k \leq n \leq m \leq M \leq N \leq \ell_{k+1}$ satisfying

$$\ell_k^{\frac{8}{1-\nu}+1} \geq n \geq \ell_k^{\frac{8}{1-\nu}}, \quad n^{\frac{8}{1-\nu}+1} \geq m \geq n^{\frac{8}{1-\nu}},$$

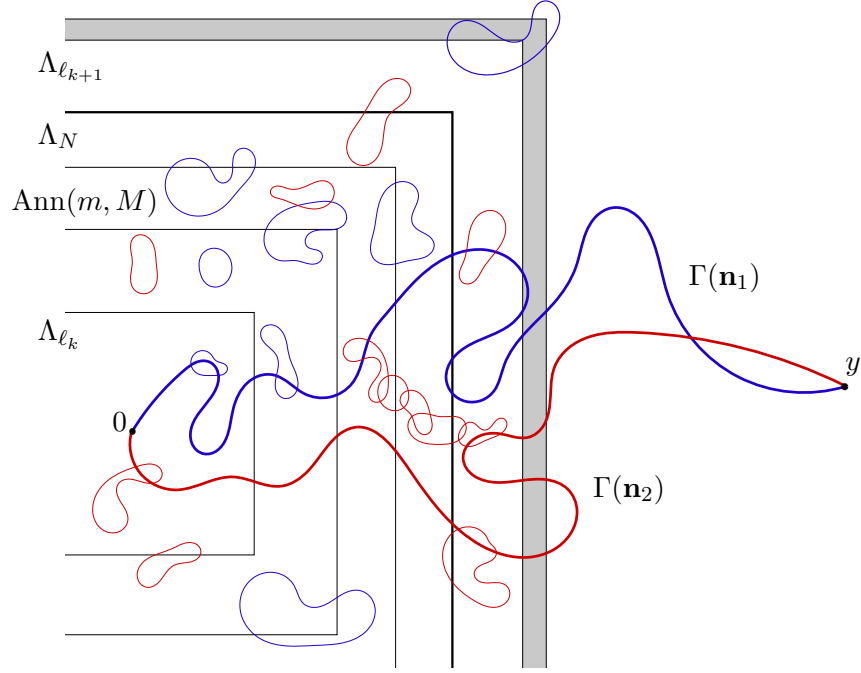


Figure 9.5: An example of a configuration for which $\mathcal{M} \neq \emptyset$ but I_k does not occur. $\mathbf{n}_1 + \mathbf{n}_3$ (resp. $\mathbf{n}_2 + \mathbf{n}_4$) is drawn in blue (resp. red), and the backbone of $\mathbf{n}_1$ (resp. $\mathbf{n}_2$) is the blue (resp. red) bold curve joining 0 and y . The outermost grey region represents the annulus $\text{Ann}(\ell_{k+1}, \ell_{k+1} + \ell_{k+1}^\nu)$. The clusters of the origin $\mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_3}(0)$ and $\mathbf{C}_{\mathbf{n}_2 + \mathbf{n}_4}(0)$ intersect in $\text{Ann}(m, M)$ thanks to a collection of red loops in $(\mathbf{n}_2 + \mathbf{n}_4) \setminus \overline{\Gamma(\mathbf{n}_2)}$ which crosses $\text{Ann}(M, N)$, which means that $\mathbf{n}_2 + \mathbf{n}_4$ realises $\mathcal{F}_3$. As a result, this intersection is not measurable in terms of edges with both endpoints in $\text{Ann}(\ell_k, \ell_{k+1} + \ell_{k+1}^\nu)$.

$$M^{\frac{8}{1-\nu}+1} \geq N \geq M^{\frac{8}{1-\nu}}, \quad N^{\frac{8}{1-\nu}+1} \geq \ell_{k+1} \geq N^{\frac{8}{1-\nu}},$$

which is possible provided D is large enough by Remark 9.6.17. Define

$$\mathcal{M} := \mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_3}(0) \cap \mathbf{C}_{\mathbf{n}_2 + \mathbf{n}_4}(0) \cap \text{Ann}(m, M). \quad (9.6.34)$$

Using the Cauchy–Schwarz inequality,

$$\mathbf{P}_\beta^{0y, 0y, \emptyset, \emptyset}[|\mathcal{M}| > 0] \geq \frac{\mathbf{E}_\beta^{0y, 0y, \emptyset, \emptyset}[|\mathcal{M}|]^2}{\mathbf{E}_\beta^{0y, 0y, \emptyset, \emptyset}[|\mathcal{M}|^2]}. \quad (9.6.35)$$

One has for some $c_1 > 0$,

$$\begin{aligned} \mathbf{E}_\beta^{0y, 0y, \emptyset, \emptyset}[|\mathcal{M}|] &= \sum_{u \in \text{Ann}(m, M)} \left(\frac{\langle \sigma_0 \sigma_u \rangle_\beta \langle \sigma_u \sigma_y \rangle_\beta}{\langle \sigma_0 \sigma_y \rangle_\beta} \right)^2 \\ &\geq c_1 (B_M(\beta) - B_{m-1}(\beta)), \end{aligned}$$

where we used the regularity to compare $\langle \sigma_u \sigma_y \rangle_\beta$ with $\langle \sigma_0 \sigma_y \rangle_\beta$. Using (9.4.10), for some $c_2 > 0$,

$$\begin{aligned} \mathbf{E}_\beta^{0y,0y,\emptyset,\emptyset}[|\mathcal{M}|^2] &\leq \sum_{u,v \in \text{Ann}(m,M)} \left(\frac{\langle \sigma_0 \sigma_u \rangle_\beta \langle \sigma_u \sigma_v \rangle_\beta \langle \sigma_v \sigma_y \rangle_\beta}{\langle \sigma_0 \sigma_y \rangle_\beta} + \frac{\langle \sigma_0 \sigma_v \rangle_\beta \langle \sigma_v \sigma_u \rangle_\beta \langle \sigma_u \sigma_y \rangle_\beta}{\langle \sigma_0 \sigma_y \rangle_\beta} \right)^2 \\ &\leq c_2 (B_M(\beta) - B_{m-1}(\beta)) B_{2M}(\beta), \end{aligned}$$

where we used once again the regularity assumption to compare $\langle \sigma_u \sigma_y \rangle_\beta$ and $\langle \sigma_v \sigma_y \rangle_\beta$ with $\langle \sigma_0 \sigma_y \rangle_\beta$. Using Lemma 9.6.16, we also get,

$$B_M(\beta) \geq \left(1 + C \frac{\log_2(\ell_{k+1}/M)}{\log_2(M)} \right)^{-1} B_{\ell_{k+1}}(\beta) \geq \frac{1}{1 + C_1} B_{\ell_{k+1}}(\beta), \quad (9.6.36)$$

where C is the constant of Lemma 9.6.16 and $C_1 = C_1(\nu) = C \left[\left(\frac{8}{1-\nu} + 1 \right)^2 - 1 \right] > 0$. Similarly,

$$B_{m-1}(\beta) \leq (1 + C_1) B_{\ell_k}(\beta) \leq \frac{1 + C_1}{D} B_{\ell_{k+1}}(\beta). \quad (9.6.37)$$

As a result, we obtain that for D large enough, there exists $c_3 > 0$ such that

$$\mathbf{P}_\beta^{0y,0y,\emptyset,\emptyset}[|\mathcal{M}| \neq 0] \geq c_3. \quad (9.6.38)$$

To conclude, we must prove uniqueness of the “crossing clusters” in $\mathbf{n}_1 + \mathbf{n}_3$ and $\mathbf{n}_2 + \mathbf{n}_4$. We reduce the argument to the one used in the nearest-neighbour case by not allowing the currents to make big jumps. More precisely, define the jump event $\mathbf{J}$ by

$$\mathbf{J} := \bigcup_{p \in \{\ell_k, \ell_{k+1}\}} \{ \mathbf{n}_1 + \mathbf{n}_3 \in \text{Jump}(p, p + p^\nu) \} \cup \{ \mathbf{n}_2 + \mathbf{n}_4 \in \text{Jump}(p, p + p^\nu) \}. \quad (9.6.39)$$

Using Lemma 9.6.5, we get that for some $\eta > 0$ and some constant $C_2 > 0$ (provided $\ell_{k+1}^{1+4/\varepsilon} \leq \ell_{k+2}$, which might require to increase D),

$$\mathbf{P}_\beta^{0y,0y,\emptyset,\emptyset}[\mathbf{J}] \leq \frac{C_2}{\ell_k^\eta}. \quad (9.6.40)$$

Note that on the complement of the above event, the cluster connecting 0 and y in $\mathbf{n}_1 + \mathbf{n}_3$ and $\mathbf{n}_2 + \mathbf{n}_4$ must go through $\text{Ann}(p, p + p^\nu)$ for $p \in \{\ell_k, \ell_{k+1}\}$. In particular, it satisfies the “strong crossing” constraint in the definition of I_k . Take D large enough so that there exists a constant $c_4 > 0$ such that,

$$\mathbf{P}_\beta^{0y,0y,\emptyset,\emptyset}[|\mathcal{M}| \neq 0, \mathbf{J}^c] \geq c_4. \quad (9.6.41)$$

If the event $\{|\mathcal{M}| \neq \emptyset\} \cap \mathbf{J}^c$ occurs but not I_k , then (see Figure 9.5) one of the following events must occur for the pair $(\mathbf{n}_1, \mathbf{n}_3)$ (or $(\mathbf{n}_2, \mathbf{n}_4)$):

- $\mathcal{F}_1 :=$ the backbone $\Gamma(\mathbf{n}_1)$ of $\mathbf{n}_1$ does a zigzag between scales ℓ_k and n , i.e. it belongs to $\text{ZZ}(0, y; \ell_k, n, \infty)$,
- $\mathcal{F}_2 := (\mathbf{n}_1 + \mathbf{n}_3) \setminus \overline{\Gamma(\mathbf{n}_1)}$ belongs to $\text{Cross}(n + n^\nu, m)$,

- $\mathcal{F}_3 :=$ the backbone $\Gamma(\mathbf{n}_1)$ of $\mathbf{n}_1$ does a zigzag between scales $N + N^\nu$ and ℓ_{k+1} , i.e. it belongs to $\text{ZZ}(0, y; N + N^\nu, \ell_{k+1}, \infty)$,
- $\mathcal{F}_4 := (\mathbf{n}_1 + \mathbf{n}_3) \setminus \overline{\Gamma(\mathbf{n}_1)}$ belongs to $\text{Cross}(M, N)$,
- $\mathcal{F}_5 := \text{Jump}(n, n + n^\nu) \cup \text{Jump}(N, N + N^\nu)$.

Note that the event $\mathcal{F}_5$ takes into account situations in which (for instance) $(\mathbf{n}_1 + \mathbf{n}_3) \setminus \overline{\Gamma(\mathbf{n}_1)}$ contains a cluster “almost realising” $\text{Cross}(M, N)$ and $\Gamma(\mathbf{n}_1)$ “almost” performs $\text{ZZ}(0, y; N + N^\nu, \ell_{k+1}, \infty)$, and these two pieces are connected by a long open edge of $\overline{\Gamma(\mathbf{n}_1)} \setminus \Gamma(\mathbf{n}_1)$ which jumps over $\text{Ann}(N, N + N^\nu)$.

Using Corollaries 9.6.9 and 9.6.15 we get the existence of $C, \eta > 0$ such that,

$$\mathbf{P}_\beta^{0y, \emptyset}[\mathcal{F}_1] \leq \frac{C}{n^\eta}, \quad \mathbf{P}_\beta^{0y, \emptyset}[\mathcal{F}_2] \leq \frac{C}{m^\eta}, \quad (9.6.42)$$

$$\mathbf{P}_\beta^{0y, \emptyset}[\mathcal{F}_3] \leq \frac{C}{\ell_{k+1}^\eta}, \quad \mathbf{P}_\beta^{0y, \emptyset}[\mathcal{F}_4] \leq \frac{C}{N^\eta}. \quad (9.6.43)$$

Moreover, as a consequence of Lemma 9.6.5, we also get that $\mathbf{P}_\beta^{0y, \emptyset}[\mathcal{F}_5] \leq C/n^\eta$.

As a result, taking D large enough, we get that the sum of the probabilities of the five events for the pairs $(\mathbf{n}_1, \mathbf{n}_3)$ and $(\mathbf{n}_2, \mathbf{n}_4)$ does not exceed $c_4/2$ so that, setting $\kappa := c_4/2$,

$$\mathbf{P}_\beta^{0y, 0y, \emptyset, \emptyset}[I_k] \geq \kappa. \quad (9.6.44)$$

□

9.6.4 Proof of the mixing

We now turn to the proof of the mixing property. Recall that J satisfies (A1)-(A6).

Theorem 9.6.20 (Mixing property). *Let $d = 4$ and $s \geq 1$. There exist $\gamma > 0$ and $C = C(s) > 0$, such that for every $1 \leq t \leq s$, every $\beta \leq \beta_c$, every $1 \leq n^\gamma \leq N \leq L(\beta)$, every $x_i \in \Lambda_n$ and $y_i \notin \Lambda_N$ ($i \leq t$), and every events E and F depending on the restriction of $(\mathbf{n}_1, \dots, \mathbf{n}_s)$ to edges with endpoints within Λ_n and outside Λ_N respectively,*

$$\left| \mathbf{P}_\beta^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}[E \cap F] - \mathbf{P}_\beta^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}[E] \mathbf{P}_\beta^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}[F] \right| \leq C \left(\log \frac{N}{n} \right)^{-1/2}. \quad (9.6.45)$$

Furthermore, for every $x'_1, \dots, x'_t \in \Lambda_n$ and $y'_1, \dots, y'_t \notin \Lambda_N$, we have that

$$\left| \mathbf{P}_\beta^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}[E] - \mathbf{P}_\beta^{x'_1 y'_1, \dots, x'_t y'_t, \emptyset, \dots, \emptyset}[E] \right| \leq C \left(\log \frac{N}{n} \right)^{-1/2}, \quad (9.6.46)$$

$$\left| \mathbf{P}_\beta^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}[F] - \mathbf{P}_\beta^{x'_1 y'_1, \dots, x'_t y'_t, \emptyset, \dots, \emptyset}[F] \right| \leq C \left(\log \frac{N}{n} \right)^{-1/2}. \quad (9.6.47)$$

Fix $\beta \leq \beta_c$. Fix two integers t, s satisfying $1 \leq t \leq s$. Introduce integers m, M such that $n \leq m \leq M \leq N$, $m/n = (N/n)^{\mu/2}$, and $N/M = (N/n)^{1-\mu}$ for μ small to be fixed. For $\mathbf{x} = (x_1, \dots, x_t)$ and $\mathbf{y} = (y_1, \dots, y_t)$, write:

$$\mathbf{P}_\beta^{\mathbf{xy}} := \mathbf{P}_\beta^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}, \quad \mathbf{P}_\beta^{\mathbf{xy}, \emptyset} := \mathbf{P}_\beta^{\mathbf{xy}} \otimes \mathbf{P}_\beta^{\emptyset, \dots, \emptyset}, \quad (9.6.48)$$

where $\mathbf{P}_\beta^{\emptyset, \dots, \emptyset}$ is the law of a sum of s independent sourceless currents that we denote by $(\mathbf{n}'_1, \dots, \mathbf{n}'_s)$.

If $p \geq 1$, define for $y \notin \Lambda_{2dp}$,

$$\mathbb{A}_y(p) := \left\{ u \in \text{Ann}(p, 2p) : \forall x \in \Lambda_{p/d}, \langle \sigma_x \sigma_y \rangle_\beta \leq \left(1 + C_0 \frac{|x - u|}{|y|} \right) \langle \sigma_u \sigma_y \rangle_\beta \right\}, \quad (9.6.49)$$

where C_0 is the constant in the definition of regular scales. Note that if y is in a regular scale, then $\mathbb{A}_y(p) = \text{Ann}(p, 2p)$.

Now, introduce the set $\mathcal{K}$ of regular scales k between m and $M/2$ with every element of $\mathcal{K}$ such that the 2^k ($k \in \mathcal{K}$) differ by a multiplicative constant at least C_0 (it will be useful later to apply **(P4)**). By the existence of regular scales of Proposition 9.3.28, we may assume that $|\mathcal{K}| \geq c_1 \log(N/n)$ for a sufficiently small $c_1 = c_1(\mu) > 0$. Introduce $\mathbf{U} := \prod_{i=1}^t \mathbf{U}_i$, where

$$\mathbf{U}_i := \frac{1}{|\mathcal{K}|} \sum_{k \in \mathcal{K}} \frac{1}{A_{x_i, y_i}(2^k)} \sum_{u \in \mathbb{A}_{y_i}(2^k)} \mathbf{1}[u \overset{\mathbf{n}_i + \mathbf{n}'_i}{\longleftrightarrow} x_i], \quad (9.6.50)$$

and,

$$a_{x,y}(u) := \frac{\langle \sigma_x \sigma_u \rangle_\beta \langle \sigma_u \sigma_y \rangle_\beta}{\langle \sigma_x \sigma_y \rangle_\beta}, \quad A_{x,y}(p) := \sum_{u \in \mathbb{A}_y(p)} a_{x,y}(u). \quad (9.6.51)$$

Using the switching lemma (**SL**), we see that $\mathbf{E}_\beta^{\mathbf{xy}, \emptyset}[\mathbf{U}] = 1$. We begin by importing a concentration inequality whose proof essentially relies on the definition of $\mathbb{A}_{y_i}(2^k)$ and the properties of regular scales¹⁵ (see [9, Proposition 6.6]).

Lemma 9.6.21 (Concentration of $\mathbf{U}$). *For all $\gamma > 2$, there exists $C = C(d, t, \gamma) > 0$ such that for all n sufficiently large satisfying $n^\gamma \leq N \leq L(\beta)$,*

$$\mathbf{E}_\beta^{\mathbf{xy}, \emptyset}[(\mathbf{U} - 1)^2] \leq \frac{C}{\log(N/n)}. \quad (9.6.52)$$

We now fix $\gamma > 2$ (it will be taken large enough later). Using the Cauchy–Schwarz inequality together with Lemma 9.6.21, we find $C_1 = C_1(d, t, \gamma) > 0$ such that

$$\begin{aligned} \left| \mathbf{P}_\beta^{\mathbf{xy}}[E \cap F] - \mathbf{E}_\beta^{\mathbf{xy}, \emptyset}[\mathbf{U} \mathbf{1}[(\mathbf{n}_1, \dots, \mathbf{n}_s) \in E \cap F]] \right| \\ \leq \sqrt{\mathbf{E}_\beta^{\mathbf{xy}, \emptyset}[(\mathbf{U} - 1)^2]} \leq \frac{C_1}{\sqrt{\log(N/n)}}. \end{aligned} \quad (9.6.53)$$

¹⁵This is the only place where we need the property **(P4)** of regular scales. It also heavily relies on **(P3)**.

At this stage of the proof, we need to analyse $\mathbf{E}_\beta^{\mathbf{xy}, \emptyset}[\mathbf{U} \mathbf{1}[(\mathbf{n}_1, \dots, \mathbf{n}_s) \in E \cap F]]$. By definition of $\mathbf{U}$, this term can be rewritten as a weighted sum of terms of the form

$$\mathbf{E}_\beta^{\mathbf{xy}, \emptyset} \left[\prod_{i=1}^t \mathbf{1}[u_i \xleftrightarrow{\mathbf{n}_i + \mathbf{n}'_i} y_i] \mathbf{1}[(\mathbf{n}_1, \dots, \mathbf{n}_s) \in E \cap F] \right] \quad (9.6.54)$$

where $u_i \in \mathbb{A}_{y_i}(2^{k_i})$ for some $k_i \in \mathcal{K}$. It would be very tempting to try to apply the switching lemma directly to turn the measure $\mathbf{P}_\beta^{\mathbf{xy}, \emptyset}$ into a measure $\mathbf{P}_\beta^{\mathbf{xu}, \mathbf{uy}}$ (up to a renormalisation weight). This would suggest that the occurrences of E and F are essentially due to $(\mathbf{n}_1, \dots, \mathbf{n}_s)$ and $(\mathbf{n}'_1, \dots, \mathbf{n}'_s)$ respectively under the measure $\mathbf{P}_\beta^{\mathbf{xu}, \mathbf{uy}}$. However, the occurrence of $E \cap F$ is only a function of $(\mathbf{n}_1, \dots, \mathbf{n}_s)$ which makes the use of the switching lemma impossible. Nevertheless, we can still use the switching principle. This motivates the introduction of the following event.

Definition 9.6.22. Let $\mathbf{u} = (u_1, \dots, u_t)$ with $u_i \in \text{Ann}(m, M)$ for every i . Introduce the event $\mathcal{G}(u_1, \dots, u_t) = \mathcal{G}(\mathbf{u})$ defined as follows: for every $i \leq s$, there exists $\mathbf{k}_i \leq \mathbf{n}_i + \mathbf{n}'_i$ such that $\mathbf{k}_i = 0$ on Λ_n , $\mathbf{k}_i = \mathbf{n}_i + \mathbf{n}'_i$ outside Λ_N , $\partial \mathbf{k}_i = \{u_i, y_i\}$ for $i \leq t$, and $\partial \mathbf{k}_i = \emptyset$ for $t < i \leq s$.

By the switching principle (**SP**), one has

$$\begin{aligned} \mathbf{P}_\beta^{\mathbf{xy}, \emptyset} \left[(\mathbf{n}_1, \dots, \mathbf{n}_s) \in E \cap F, u_i \xleftrightarrow{\mathbf{n}_i + \mathbf{n}'_i} y_i, \forall 1 \leq i \leq t, \mathcal{G}(\mathbf{u}) \right] \\ = \left(\prod_{i=1}^t a_{x_i, y_i}(u_i) \right) \mathbf{P}_\beta^{\mathbf{xu}, \mathbf{uy}} [(\mathbf{n}_1, \dots, \mathbf{n}_s) \in E, (\mathbf{n}'_1, \dots, \mathbf{n}'_s) \in F, \mathcal{G}(\mathbf{u})]. \end{aligned}$$

The trivial identity,

$$\mathbf{P}_\beta^{\mathbf{xu}, \mathbf{uy}} [(\mathbf{n}_1, \dots, \mathbf{n}_s) \in E, (\mathbf{n}'_1, \dots, \mathbf{n}'_s) \in F] = \mathbf{P}_\beta^{\mathbf{xu}} [(\mathbf{n}_1, \dots, \mathbf{n}_s) \in E] \mathbf{P}_\beta^{\mathbf{uy}} [(\mathbf{n}'_1, \dots, \mathbf{n}'_s) \in F] \quad (9.6.55)$$

motivates us to prove that under $\mathbf{P}_\beta^{\mathbf{xu}, \mathbf{uy}}$, the event $\mathcal{G}(\mathbf{u})$ occurs with high probability. This motivates the following result.

Lemma 9.6.23. Let $d = 4$. There exist $C, \epsilon > 0$, $\gamma = \gamma(\epsilon) > 0$ large enough and $\mu = \mu(\epsilon) > 0$ small enough such that for every $n^\gamma \leq N \leq L(\beta)$, and every $\mathbf{u}$ with $u_i \in \mathbb{A}_{y_i}(2^{k_i})$ with $k_i \in \mathcal{K}$ for $1 \leq i \leq t$,

$$\mathbf{P}_\beta^{\mathbf{xy}, \emptyset} \left[u_i \xleftrightarrow{\mathbf{n}_i + \mathbf{n}'_i} y_i, \forall 1 \leq i \leq t, \mathcal{G}(\mathbf{u}) \right] \left(\prod_{i=1}^t a_{x_i, y_i}(u_i) \right)^{-1} = \mathbf{P}_\beta^{\mathbf{xu}, \mathbf{uy}} [\mathcal{G}(\mathbf{u})^c] \leq C \left(\frac{n}{N} \right)^\epsilon. \quad (9.6.56)$$

Proof. Below, $C = C(d) > 0$ may change from line to line¹⁶. The equality follows from an application of the switching lemma (**SL**). If we write $\mathcal{G}(\mathbf{u}) = \cap_{1 \leq i \leq s} G_i$ (where the definition of G_i is implicit), then $H_i \cap F_i \subset G_i$ where,

$$H_i := \{\text{Ann}(M, N) \text{ is not crossed by a cluster in } \mathbf{n}_i\} = \{\mathbf{n}_i \notin \text{Cross}(M, N)\}, \quad (9.6.57)$$

¹⁶In fact, C will also depend on β_c (or more precisely on a lower bound on β_c).

and

$$F_i := \{\text{Ann}(n, m) \text{ is not crossed by a cluster in } \mathbf{n}'_i\} = \{\mathbf{n}'_i \notin \text{Cross}(n, m)\}. \quad (9.6.58)$$

Indeed, if $H_i \cap F_i$ occurs, we may define $\mathbf{k}_i$ as the sum of the restriction of $\mathbf{n}_i$ to the clusters intersecting Λ_N^c and the restriction of $\mathbf{n}'_i$ to the clusters intersecting Λ_m^c . In the following, we assume that $1 \leq i \leq t$. The argument for other values of i is easier and follows from the first case.

Introduce intermediate scales $n \leq r \leq m \leq M \leq R \leq N$ with r, R chosen below.

• **Bound on H_i .** Following the ideas developed in the proof of Lemma 9.6.19, we define,

$$J_i := \{\mathbf{n}_i \in \text{Jump}(R - R^\nu, R + R^\nu)\}. \quad (9.6.59)$$

Notice that¹⁷,

$$\begin{aligned} \mathbf{P}_\beta^{\mathbf{xu}, \mathbf{uy}}[H_i^c] &\leq \mathbf{P}_\beta^{\mathbf{xu}}[\Gamma(\mathbf{n}_i) \in \text{ZZ}(x_i, u_i; M, R - R^\nu, \infty)] \\ &\quad + \mathbf{P}_\beta^{\mathbf{xu}}[\mathbf{n}_i \setminus \overline{\Gamma(\mathbf{n}_i)} \in \text{Cross}(R + R^\nu, N)] + \mathbf{P}_\beta^{\mathbf{xu}}[J_i]. \end{aligned}$$

Assume $R = N^\iota$ where $\iota > \mu$ is chosen in such a way that $M^{6/(1-\nu)} \leq R$ and $R^{8/(1-\nu)} \leq N$ (note that this is possible if we choose μ sufficiently small and γ sufficiently large since $M \leq N^{\mu+1/\gamma}$). This choice allows us to use Corollaries 9.6.12 and 9.6.15 to get that for some $\eta > 0$,

$$\mathbf{P}_\beta^{\mathbf{xu}}[\mathbf{n}_i \in \text{ZZ}(x_i, u_i; M, R - R^\nu, \infty)] \leq \frac{C}{R^\eta}, \quad \mathbf{P}_\beta^{\mathbf{xu}}[\mathbf{n}_i \setminus \overline{\Gamma(\mathbf{n}_i)} \in \text{Cross}(R + R^\nu, N)] \leq \frac{C}{N^\eta}. \quad (9.6.60)$$

Moreover, at the cost of diminishing the value of ι (and hence also modifying μ and γ) to ensure that $R^{1+4/\varepsilon} \leq N$, we may use Lemma 9.6.5 to argue the existence of $\eta' > 0$ such that

$$\mathbf{P}_\beta^{\mathbf{xu}}[J_i] \leq \frac{C}{R^{\eta'}}. \quad (9.6.61)$$

Putting all the pieces together, we get for some $\epsilon > 0$,

$$\mathbf{P}_\beta^{\mathbf{xu}, \mathbf{uy}}[H_i^c] \leq C \left(\frac{n}{N} \right)^\epsilon. \quad (9.6.62)$$

• **Bound on F_i .** We proceed similarly for F_i although we now encounter an additional difficulty: we cannot rule out the possibility that $\mathbf{n}'_i$ will jump above the annulus $\text{Ann}(r - r^\nu, r + r^\nu)$, we can only rule it out in the complement of $\overline{\Gamma(\mathbf{n}'_i)}$. We set $r = (n^2 m)^{1/3}$. Recall that $m \geq N^{\mu/2}$. We claim that

$$\begin{aligned} \mathbf{P}_\beta^{\mathbf{xu}, \mathbf{uy}}[F_i^c] &\leq \mathbf{P}_\beta^{\mathbf{uy}}[\Gamma(\mathbf{n}'_i) \in \text{ZZ}(u_i, y_i; r + r^\nu, m, \infty)] \\ &\quad + \mathbf{P}_\beta^{\mathbf{uy}}[\mathbf{n}'_i \setminus \overline{\Gamma(\mathbf{n}'_i)} \in \text{Cross}(n, r - r^\nu)] + \mathbf{P}_\beta^{\mathbf{uy}}[K_i], \end{aligned} \quad (9.6.63)$$

¹⁷We need to create a “forbidden area” $\text{Ann}(R - R^\nu, R + R^\nu)$ between the two parts of the annulus $\text{Ann}(M, N)$ to rule out the situation in which $\overline{\Gamma(\mathbf{n}_i)} \setminus \Gamma(\mathbf{n}_i)$ connects an “almost successful” excursion of $\Gamma(\mathbf{n}_i)$ (in the sense that it almost crossed $\text{Ann}(M, R - R^\nu)$), with an “almost successful” excursion of $\mathbf{n}_i \setminus \overline{\Gamma(\mathbf{n}_i)}$. This is very similar to the argument used in the proof of Lemma 9.6.19.

where K_i is the event that there exists $a \in \Lambda_{r-r^\nu}$ and $b \notin \Lambda_{r+r^\nu}$ such that $(\mathbf{n}'_i)_{a,b} \geq 2$ and $\{a, b\} \in \overline{\Gamma(\mathbf{n}'_i)} \setminus \Gamma(\mathbf{n}'_i)$.

Indeed, if none of the events corresponding to the two first probabilities on the right-hand side of (9.6.63) occur, the only way we can find a cluster crossing $\text{Ann}(n, m)$ is if $\overline{\Gamma(\mathbf{n}'_i)} \setminus \Gamma(\mathbf{n}'_i)$ has a long (even) open edge which jumps above $\text{Ann}(r - r^\nu, r + r^\nu)$ (see Figure 9.6).

Using (IRB) to get $\langle \sigma_{u_i} \sigma_v \rangle_\beta \leq C m^{-2}$ and the assumption that $u_i \in \mathbb{A}_{y_i}(2^{k_i})$ to get that $\langle \sigma_v \sigma_{y_i} \rangle_\beta \leq C \langle \sigma_{u_i} \sigma_{y_i} \rangle_\beta$, we obtain

$$\begin{aligned} \mathbf{P}_\beta^{\text{uy}}[\Gamma(\mathbf{n}'_i) \in \text{ZZ}(u_i, y_i; r + r^\nu, m, \infty)] &\leq \sum_{v \in \Lambda_{r+r^\nu}} \frac{\langle \sigma_{u_i} \sigma_v \rangle_\beta \langle \sigma_v \sigma_{y_i} \rangle_\beta}{\langle \sigma_{u_i} \sigma_{y_i} \rangle_\beta} \\ &\leq C \frac{r^4}{m^2} = C \frac{n^{8/3}}{m^{2/3}} \leq C \frac{N^{\frac{8}{3\gamma}}}{N^{\frac{\mu}{3}}}. \end{aligned}$$

Moreover, using Corollary 9.6.15 (which requires that $n^{8/(1-\nu)} \leq r$ and hence decreases the values of μ and $1/\gamma$), there exist $\zeta > 0$ such that

$$\mathbf{P}_\beta^{\text{uy}}[\mathbf{n}'_i \setminus \overline{\Gamma(\mathbf{n}'_i)} \in \text{Cross}(n, r - r^\nu)] \leq \frac{C}{r^\zeta}. \quad (9.6.64)$$

We conclude the proof with the bound on K_i . If $\mathbf{n}'_i$ satisfies that $(\mathbf{n}'_i)_{a,b} \geq 2$ where $a \in \Lambda_{r-r^\nu}$ and $b \notin \Lambda_{r+r^\nu}$ with $\{a, b\} \in \overline{\Gamma(\mathbf{n}'_i)} \setminus \Gamma(\mathbf{n}'_i)$ being the earliest such edge. We consider the map $\mathbf{n}'_i \mapsto \mathbf{m}'_i$ where $(\mathbf{m}'_i)_{a,b} = (\mathbf{n}'_i)_{a,b} - 1$ and $\mathbf{m}'_i$ coincides with $\mathbf{n}'_i$ everywhere else. This maps $\mathbf{n}'_i$ to a current $\mathbf{m}'_i$ with sources $\{u_i, y_i\} \Delta \{a, b\}$ (b might coincide with u_i or y_i) such that the backbone $\Gamma(\mathbf{m}'_i)$ always connects u_i and b (by definition of the exploration). Hence, using the chain rule for backbones,

$$\begin{aligned} \mathbf{P}_\beta^{\text{uy}}[K_i] &\leq \sum_{\substack{a \in \Lambda_{r-r^\nu} \\ b \notin \Lambda_{r+r^\nu}}} \beta J_{a,b} \frac{\langle \sigma_{\{u_i, y_i\} \Delta \{a, b\}} \rangle_\beta}{\langle \sigma_{u_i} \sigma_{y_i} \rangle_\beta} \mathbf{P}_\beta^{\{u_i, y_i\} \Delta \{a, b\}}[\Gamma(\mathbf{m}'_i) \text{ connects } u_i \text{ and } b] \\ &\leq \sum_{\substack{a \in \Lambda_{r-r^\nu} \\ b \notin \Lambda_{r+r^\nu}}} \beta J_{a,b} \frac{\langle \sigma_{u_i} \sigma_b \rangle_\beta \langle \sigma_a \sigma_{y_i} \rangle_\beta}{\langle \sigma_{u_i} \sigma_{y_i} \rangle_\beta}. \end{aligned}$$

Since $u_i \in \mathbb{A}_{y_i}(2^{k_i})$, $\langle \sigma_a \sigma_{y_i} \rangle_\beta \leq C \langle \sigma_{u_i} \sigma_{y_i} \rangle_\beta$. Hence,

$$\beta \sum_{\substack{a \in \Lambda_{r-r^\nu} \\ b \notin \Lambda_{r+r^\nu}}} J_{a,b} \frac{\langle \sigma_{u_i} \sigma_b \rangle_\beta \langle \sigma_a \sigma_{y_i} \rangle_\beta}{\langle \sigma_{u_i} \sigma_{y_i} \rangle_\beta} \leq C \beta \sum_{\substack{a \in \Lambda_{r-r^\nu} \\ b \notin \Lambda_{r+r^\nu}}} J_{a,b} \langle \sigma_{u_i} \sigma_b \rangle_\beta. \quad (9.6.65)$$

We distinguish two-cases according to whether b is close to u_i or not. By (9.6.2) and (IRB),

$$\beta \sum_{\substack{a \in \Lambda_{r-r^\nu} \\ b \in \Lambda_{|u_i|/2}(u_i)}} J_{a,b} \langle \sigma_{u_i} \sigma_b \rangle_\beta \leq C r^4 \sum_{|x| \geq m/4} J_{0,x} \leq \frac{C r^4}{m^{2+\varepsilon}}. \quad (9.6.66)$$

Again, using (9.6.2) and (IRB),

$$\sum_{\substack{a \in \Lambda_{r-r^\nu} \\ b \notin \Lambda_{r+r^\nu} \cup \Lambda_{|u_i|/2}(u_i)}} J_{a,b} \langle \sigma_{u_i} \sigma_b \rangle_\beta \leq \frac{C r^4}{m^2} \sum_{|x| \geq r^\nu} J_{0,x} \leq \frac{C r^{4-\nu(2+\varepsilon)}}{m^2}. \quad (9.6.67)$$

By definition of r and m/n , we get the existence of $\zeta' > 0$ such that

$$\mathbf{P}_\beta^{\text{uy}}[K_i] \leq \frac{C}{m^{\zeta'}}. \quad (9.6.68)$$

As a result, if we choose μ and $1/\gamma$ sufficiently small, we get that for some $\epsilon' > 0$,

$$\mathbf{P}_\beta^{\text{xu,uy}}[F_i^c] \leq C \left(\frac{n}{N} \right)^{\epsilon'}. \quad (9.6.69)$$

□

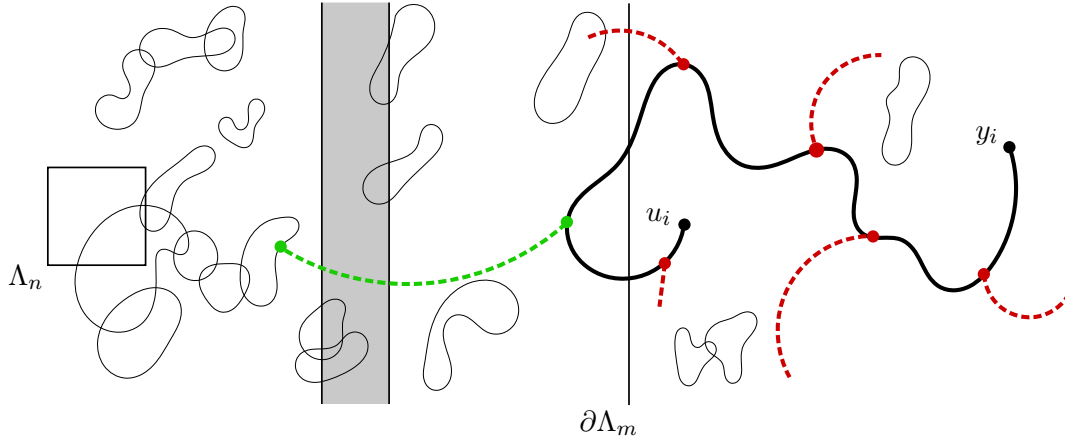


Figure 9.6: An illustration of the event K_i . We represented the current $\mathbf{n}'_i$ in black, with a bold line representing its backbone $\Gamma(\mathbf{n}'_i)$. The red/green dashed lines correspond to the open edges of $\overline{\Gamma(\mathbf{n}'_i)} \setminus \Gamma(\mathbf{n}'_i)$, they all carry a non-zero even weight. The grey region corresponds to the annulus $\text{Ann}(r - r^\nu, r + r^\nu)$. In this picture, $\Gamma(\mathbf{n}'_i)$ does not cross $\text{Ann}(r + r^\nu, m)$, $\mathbf{n}'_i \setminus \overline{\Gamma(\mathbf{n}'_i)}$ does not cross $\text{Ann}(n, r - r^\nu)$, yet the long green open edge of $\overline{\Gamma(\mathbf{n}'_i)} \setminus \Gamma(\mathbf{n}'_i)$ creates a cluster of $\mathbf{n}'_i$ crossing $\text{Ann}(n, m)$.

We now turn to the proof of Theorem 9.6.20.

Proof of Theorem 9.6.20. Introduce the coefficients $\delta(\mathbf{u}, \mathbf{x}, \mathbf{y})$ defined by,

$$\delta(\mathbf{u}, \mathbf{x}, \mathbf{y}) := \mathbb{1}[\exists(k_1, \dots, k_t) \in \mathcal{K}^t, \mathbf{u} \in \mathbb{A}_{y_1}(2^{k_1}) \times \dots \times \mathbb{A}_{y_t}(2^{k_t})] \prod_{i=1}^t \frac{a_{x_i, y_i}(u_i)}{|\mathcal{K}| A_{x_i, y_i}(2^{k_i})}. \quad (9.6.70)$$

Note that

$$\sum_{\substack{(k_1, \dots, k_t) \in \mathcal{K}^t \\ \mathbf{u} \in \mathbb{A}_{y_1}(2^{k_1}) \times \dots \times \mathbb{A}_{y_t}(2^{k_t})}} \delta(\mathbf{u}, \mathbf{x}, \mathbf{y}) = 1. \quad (9.6.71)$$

Equation (9.6.53) together with Lemma 9.6.23 yield,

$$\left| \mathbf{P}_\beta^{\mathbf{xy}}[E \cap F] - \sum_{\mathbf{u}} \delta(\mathbf{u}, \mathbf{x}, \mathbf{y}) \mathbf{P}_\beta^{\mathbf{xu}}[E] \mathbf{P}_\beta^{\mathbf{uy}}[F] \right| \leq \frac{C_1}{\sqrt{\log(N/n)}} + C_2 \left(\frac{n}{N} \right)^\epsilon \leq \frac{C_3}{\sqrt{\log(N/n)}}, \quad (9.6.72)$$

as long as $N \geq n^\gamma$ where $\gamma > 2$ is given by Lemma 9.6.23. We begin by proving (9.6.46) when y_i, y'_i are in regular scales (but not necessarily the same ones). Applying the above inequality once for $\mathbf{y}$ and once for $\mathbf{y}'$ with the event E and $F = \Omega_{\mathbb{Z}^d}$,

$$\left| \mathbf{P}_\beta^{\mathbf{xy}}[E] - \mathbf{P}_\beta^{\mathbf{xy}'}[E] \right| \leq \left| \sum_{\mathbf{u}} (\delta(\mathbf{u}, \mathbf{x}, \mathbf{y}) - \delta(\mathbf{u}, \mathbf{x}, \mathbf{y}')) \mathbf{P}_\beta^{\mathbf{xu}}[E] \right| + \frac{2C_3}{\sqrt{\log(N/n)}}. \quad (9.6.73)$$

Since all the y_i, y'_i are in regular scales, one has $\mathbb{A}_{y_i}(2^{k_i}) = \mathbb{A}_{y'_i}(2^{k_i}) = \text{Ann}(2^{k_i}, 2^{k_i+1})$. Moreover, using Property (P2) of regular scales,

$$|\delta(\mathbf{u}, \mathbf{x}, \mathbf{y}) - \delta(\mathbf{u}, \mathbf{x}, \mathbf{y}')| \leq C_4 \left(\frac{M}{N} \right) \delta(\mathbf{u}, \mathbf{x}, \mathbf{y}) \leq C_5 \left(\frac{n}{N} \right)^{1-\mu} \delta(\mathbf{u}, \mathbf{x}, \mathbf{y}), \quad (9.6.74)$$

where μ is also given by Lemma 9.6.23. Indeed, in that case $\delta(\mathbf{u}, \mathbf{x}, \mathbf{y})$ and $\delta(\mathbf{u}, \mathbf{x}, \mathbf{y}')$ are both close to

$$\prod_{i \leq t} \frac{\langle \sigma_{x_i} \sigma_{u_i} \rangle}{|\mathcal{K}| \sum_{v_i \in \text{Ann}(2^{k_i}, 2^{k_i+1})} \langle \sigma_{x_i} \sigma_{v_i} \rangle}. \quad (9.6.75)$$

This gives (9.6.46) in that case. Now, assume that $N \geq n^{2(\gamma/\mu)+1}$ (so that $m \geq n^\gamma$). Consider $\mathbf{z} = (z_1, \dots, z_t)$ with $z_i \in \text{Ann}(m, M)$ in a regular scale. Also, pick $\mathbf{y}$ on which we do not assume anything. We have,

$$\begin{aligned} \left| \mathbf{P}_\beta^{\mathbf{xy}}[E] - \mathbf{P}_\beta^{\mathbf{xz}}[E] \right| &= \left| \mathbf{P}_\beta^{\mathbf{xy}}[E] - \sum_{\mathbf{u}} \delta(\mathbf{u}, \mathbf{x}, \mathbf{y}) \mathbf{P}_\beta^{\mathbf{xu}}[E] \right| \\ &\leq \left| \mathbf{P}_\beta^{\mathbf{xy}}[E] - \sum_{\mathbf{u}} \delta(\mathbf{u}, \mathbf{x}, \mathbf{y}) \mathbf{P}_\beta^{\mathbf{xu}}[E] \right| + \frac{C_6}{\sqrt{\log(m/n)}} \\ &\leq \frac{C_7}{\sqrt{\log(N/n)}}, \end{aligned}$$

where in the second line we used (9.6.73) with $(\mathbf{y}, \mathbf{y}') = (\mathbf{y}, \mathbf{z})$ together with the fact that $m/n = (N/n)^{\mu/2} \geq n^\gamma$, and in the third line we used (9.6.72) with $F = \Omega_{\mathbb{Z}^d}$. This gives (9.6.46).

The same argument works for (9.6.47) for every $\mathbf{x}, \mathbf{x}', \mathbf{y}$, noticing that for every regular $\mathbf{u}$ for which $\delta(\mathbf{u}, \mathbf{x}, \mathbf{y}) \neq 0$, using once again (P2),

$$|\delta(\mathbf{u}, \mathbf{x}, \mathbf{y}) - \delta(\mathbf{u}, \mathbf{x}', \mathbf{y})| \leq C_8 \left(\frac{n}{m} \right) \delta(\mathbf{u}, \mathbf{x}, \mathbf{y}) \leq C_9 \left(\frac{n}{N} \right)^{\mu/2} \delta(\mathbf{u}, \mathbf{x}, \mathbf{y}). \quad (9.6.76)$$

To get (9.6.45) we repeat the same line of reasoning. We start by applying (9.6.72). Then, we replace each $\mathbf{P}_\beta^{\mathbf{xu}}[E]$ by $\mathbf{P}_\beta^{\mathbf{xy}}[E]$ using the above reasoning. Finally, replace $\mathbf{P}_\beta^{\mathbf{uy}}[F]$ by $\mathbf{P}_\beta^{\mathbf{xy}}[F]$ using the same methods. □

9.6.5 Proof of the clustering bound

With the intersection property and the mixing statement, we are now in a position to conclude. We will use the following (natural) monotonicity property of the currents measures.

Proposition 9.6.24 (Monotonicity in the number of sources, [9, Corollary A.2]). *For every $\beta > 0$, for every $x, y, z, t \in \mathbb{Z}^d$ and every set $S \subset \mathbb{Z}^d$, one has,*

$$\begin{aligned} \mathbf{P}_\beta^{0x,0z,0y,0t}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_3}(0) \cap \mathbf{C}_{\mathbf{n}_2+\mathbf{n}_4}(0) \cap S = \emptyset] \\ \leq \mathbf{P}_\beta^{0x,0z,\emptyset,\emptyset}[\mathbf{C}_{\mathbf{n}_1+\mathbf{n}_3}(0) \cap \mathbf{C}_{\mathbf{n}_2+\mathbf{n}_4}(0) \cap S = \emptyset]. \end{aligned}$$

Proof of Proposition 9.6.2. Fix $\gamma > 2$ sufficiently large so that Theorem 9.6.20 holds. Remember by Remark 9.6.17 that we may choose $D = D(\gamma)$ sufficiently large in the definition of $\mathcal{L}$ such that $\ell_{k+1} \geq \ell_k'$.

We may assume $u = 0$. Since x, y are at distance at least $2\ell_K$ of each other, one of them must be at distance at least ℓ_K of u . Without loss of generality we assume that this is the case of x and make the same assumption about z . Let $\delta > 0$ to be fixed below.

Let $\mathcal{S}_K^{(\delta)}$ denote the set of subsets of $\{2\delta K, \dots, K-3\}$ which contain only even integers. If $\{\mathbf{M}_u(\mathcal{I}; \mathcal{L}, K) < \delta K\}$ occurs, then there must be $S \in \mathcal{S}_K^{(\delta)}$ with $|S| \geq (1/2 - 2\delta)K$ such that $\mathfrak{B}_S$ occurs, where $\mathfrak{B}_S$ is the event that the clusters of 0 in $\mathbf{n}_1 + \mathbf{n}_3$ and $\mathbf{n}_2 + \mathbf{n}_4$ do not intersect in any of the annuli $\text{Ann}(\ell_i, \ell_{i+1})$ for $i \in S$. Using the monotonicity property recalled above,

$$\begin{aligned} \mathbf{P}_\beta^{0x,0z,0y,0t}[\mathbf{M}_u(\mathcal{I}; \mathcal{L}, K) < \delta K] &\leq \sum_{\substack{S \in \mathcal{S}_K^{(\delta)} \\ |S| \geq (1/2-2\delta)K}} \mathbf{P}_\beta^{0x,0z,0y,0t}[\mathfrak{B}_S] \\ &\leq \sum_{\substack{S \in \mathcal{S}_K^{(\delta)} \\ |S| \geq (1/2-2\delta)K}} \mathbf{P}_\beta^{0x,0z,\emptyset,\emptyset}[\mathfrak{B}_S]. \end{aligned} \quad (9.6.77)$$

Let $\mathbf{J}$ be the event defined by

$$\mathbf{J} := \bigcup_{k=2\delta K}^{K-3} \{\mathbf{n}_1 + \mathbf{n}_3 \in \text{Jump}(\ell_k, \ell_{k+k^\nu})\} \cup \{\mathbf{n}_2 + \mathbf{n}_4 \in \text{Jump}(\ell_k, \ell_{k+k^\nu})\}. \quad (9.6.78)$$

Using Lemma 9.6.5 and Remark 9.6.17, if D is large enough, there exist $C_0, C_1, \eta > 0$ such that

$$\mathbf{P}_\beta^{0x,0z,\emptyset,\emptyset}[\mathbf{J}] \leq \frac{C_0 K}{\ell_{\delta K}^\eta} \leq C_1 e^{-\eta 2^{\delta K}}. \quad (9.6.79)$$

Fix some $S \in \mathcal{S}_K^{(\delta)}$. Let $\mathfrak{A}_S$ be the event that none of the events I_k (defined in Definition 9.6.18) occur for $k \in S$.

We now make a crucial observation: if $k \in S$ and $\mathbf{J}^c$ occurs, the events I_k and $\mathfrak{B}_S$ are incompatible. Indeed, The occurrence of $\mathbf{J}^c \cap I_k$ ensures that the only cluster of $\mathbf{n}_1 + \mathbf{n}_3$ (resp. $\mathbf{n}_2 + \mathbf{n}_4$) crossing $\text{Ann}(\ell_k, \ell_{k+1})$ is the cluster of 0. Hence, $(\mathfrak{B}_S \cap \mathbf{J}^c) \subset \mathfrak{A}_S$. Using (9.6.79),

$$\begin{aligned} \mathbf{P}_\beta^{0x,0z,0y,0t}[\mathbf{M}_u(\mathcal{I}; \mathcal{L}, K) < \delta K] &\leq \sum_{\substack{S \in \mathcal{S}_K^{(\delta)} \\ |S| \geq (1/2-2\delta)K}} \mathbf{P}_\beta^{0x,0z,\emptyset,\emptyset}[\mathfrak{A}_S] + C_1 e^{-\eta 2^{\delta K}} \binom{(1/2-2\delta)K}{2\delta K}. \end{aligned} \quad (9.6.80)$$

Standard binomial estimates give that the second term on the right-hand side above is bounded by $C_2 2^{-\delta K}$. We are left with the study of $\mathbf{P}_\beta^{0x,0z,\emptyset,\emptyset}[\mathfrak{A}_S]$.

Since $\ell_K \leq L(\beta)$, we know by Proposition 9.3.28 that there exists $y \in \text{Ann}(\ell_{K-1}, \ell_K)$ in a regular scale. The event $\mathfrak{A}_S$ depends on edges with both endpoints in $\Lambda_{\ell_{K-2}}$. Using the mixing property together with Remark 9.6.17, we get that

$$\mathbf{P}_\beta^{0x,0z,\emptyset,\emptyset}[\mathfrak{A}_S] \leq \mathbf{P}_\beta^{0y,0y,\emptyset,\emptyset}[\mathfrak{A}_S] + \frac{C_3}{(\log \ell_{K-1})^c} \leq \mathbf{P}_\beta^{0y,0y,\emptyset,\emptyset}[\mathfrak{A}_S] + C_4 \left(\frac{D}{\log 2} \right)^{-c(K-1)}. \quad (9.6.81)$$

Let $s = \max S$. Using once again the mixing property between $n = \ell_{s-1}$ and $N = \ell_s$, we get,

$$\begin{aligned} \mathbf{P}_\beta^{0y,0y,\emptyset,\emptyset}[\mathfrak{A}_S] &\leq \mathbf{P}_\beta^{0y,0y,\emptyset,\emptyset}[I_s^c] \mathbf{P}_\beta^{0y,0y,\emptyset,\emptyset}[\mathfrak{A}_{S \setminus \{s\}}] + \frac{C_3}{(\log \ell_s / \ell_{s-1})^c} \\ &\leq (1 - \kappa) \mathbf{P}_\beta^{0y,0y,\emptyset,\emptyset}[\mathfrak{A}_{S \setminus \{s\}}] + C_5 (1 - \kappa)^{|S|-1}, \end{aligned}$$

where κ is given by Lemma 9.6.19 and where we chose D large enough. Iterating the above yields,

$$\mathbf{P}_\beta^{0y,0y,\emptyset,\emptyset}[\mathfrak{A}_S] \leq C_6 (1 - \kappa)^{|S|}. \quad (9.6.82)$$

Going back to (9.6.77), we get that

$$\mathbf{P}_\beta^{0x,0z,0y,0t}[\mathbf{M}_u(\mathcal{I}; \mathcal{L}, K) < \delta K] \leq C_7 \binom{(1/2 - 2\delta)K}{2\delta K} (1 - \kappa)^{(1/2 - 2\delta)K} + C_2 2^{-\delta K} \leq C_8 2^{-\delta K}, \quad (9.6.83)$$

for $\delta > 0$ sufficiently small. $\square$

9.6.6 Proof of Corollary 9.1.8

Now that we were able to obtain the improved tree diagram bound, the proof of Corollary 9.1.8 follows the strategy used in Section 9.5, although some additional technical difficulties appear due to the lack of knowledge on the growth of $B_L(\beta)$. We refer to [9] for the details of the proof.

Proof of Corollary 9.1.8. We import notations from the proof of Theorem 9.5.5. Applying the improved tree diagram bound we obtain

$$S(\beta, L, f) \leq C \sum_{\substack{x \in \mathbb{Z}^d \\ x_1, x_2, x_3, x_4 \in \Lambda_{r_f L}}} \frac{\langle \sigma_x \sigma_{x_1} \rangle_\beta \langle \sigma_x \sigma_{x_2} \rangle_\beta \langle \sigma_x \sigma_{x_3} \rangle_\beta \langle \sigma_x \sigma_{x_4} \rangle_\beta}{B_{L(x_1, x_2, x_3, x_4)}(\beta)^c \Sigma_L(\beta)^2}, \quad (9.6.84)$$

where $L(x_1, x_2, x_3, x_4)$ is the minimal distance between the x_i . We now fix $a \in (0, 1)$. The strategy consists in splitting the right-hand side of (9.6.84) according to the following four possibilities: $x \in \Lambda_{dr_f L}$ and $L(x_1, x_2, x_3, x_4) \leq L^a$, $x \notin \Lambda_{dr_f L}$ and $L(x_1, x_2, x_3, x_4) \leq L^a$, $x \in \Lambda_{dr_f L}$ and $L(x_1, x_2, x_3, x_4) > L^a$, $x \notin \Lambda_{dr_f L}$ and $L(x_1, x_2, x_3, x_4) > L^a$.

From there the conclusion builds on the same tools as the ones used in Section 9.5, and also on Lemma 9.6.16. $\square$

Remark 9.6.25. In [9], the proof of this result crucially relies on a *sharp* sliding-scale infrared bound which leads to a sharp bound in Lemma 9.6.16. This shows how remarkable this bound is since it yields a quantitative decay even though we do not even know that $B_L(\beta_c)$ explodes. This argument breaks down in the next section (for $d \in \{2, 3\}$) due to the lack of such a sharp result.

9.7 Reflection positive Ising models in dimension $1 \leq d \leq 3$ satisfying $d_{\text{eff}} = 4$

We now explain how to adapt the above strategy to the remaining “marginal case”: $d_{\text{eff}} = 4 > d$. As seen in Section 9.5 (and more precisely in Remark 9.5.3), for algebraically decaying RP interactions $J_{x,y} = C|x - y|_1^{-d-\alpha}$, this corresponds to choosing $d - 2(\alpha \wedge 2) = 0$.

We will assume that J satisfies (A1)–(A5) with the complementary assumption¹⁸ that there exist $\mathbf{c}, \mathbf{C} > 0$ such that, for all $x \in \mathbb{Z}^d \setminus \{0\}$,

$$\frac{\mathbf{c}}{|x|^{d+\alpha(d)}} \leq J_{0,x} \leq \frac{\mathbf{C}}{|x|^{d+\alpha(d)}}, \quad (9.7.1)$$

where $\alpha(d) = d/2$.

Our goal is to prove the following result, which is a slightly stronger version of Theorem 9.1.9.

Theorem 9.7.1 (Improved tree diagram bound). *Let $1 \leq d \leq 3$. Assume that J satisfies (A1)–(A5) together with (9.7.1). There exists $C > 0$ such that the following holds: for all $\beta \leq \beta_c$, there exists an increasing function $\phi_\beta : \mathbb{R} \rightarrow \mathbb{R}_>$ such that for all $x, y, z, t \in \mathbb{Z}^d$ at mutual distance at least L of each other with $1 \leq L \leq L(\beta)$,*

$$|U_4^\beta(x, y, z, t)| \leq \frac{C}{\phi_\beta(B_L(\beta))} \sum_{u \in \mathbb{Z}^d} \langle \sigma_x \sigma_u \rangle_\beta \langle \sigma_y \sigma_u \rangle_\beta \langle \sigma_z \sigma_u \rangle_\beta \langle \sigma_t \sigma_u \rangle_\beta. \quad (9.7.2)$$

If $B(\beta_c) = \infty$, one has $\phi_{\beta_c}(t) \rightarrow \infty$ as $t \rightarrow \infty$.

Remark 9.7.2. We could replace $\phi_\beta(B_L(\beta))$ by $B_L(\beta)^c$ (as in Theorem 9.1.8) provided we could prove the following (under the assumptions of Theorem 9.7.1): there exists $C > 0$ such that, if $1 \leq \ell \leq L \leq L(\beta)$,

$$\frac{\chi_N(\beta)}{N^{\gamma(d)}} \leq C \frac{\chi_n(\beta)}{n^{\gamma(d)}}, \quad (9.7.3)$$

where $\gamma(2) = 1$ and $\gamma(3) = 3/2$. This will become more transparent below.

When $d = 1$, such a sharp result is obtained thanks to the exact knowledge on the decay of the two-point function.

From this result, we will obtain

¹⁸This assumption is a little more restrictive than before since we now require some regularity property on the interaction. This restriction is essentially technical and we believe that the proof which follows should hold under more general assumptions.

Corollary 9.7.3. *We keep the assumptions of Theorem 9.7.1. Then, for $\sigma \in (0, d/2)$,*

$$\lim_{\beta \nearrow \beta_c} g_\sigma(\beta) = 0. \quad (9.7.4)$$

As a consequence, for $\beta = \beta_c$, every sub-sequential scaling limit of the model is Gaussian.

To prove the improved tree diagram bound, we will extend the strategy of Section 9.6 and prove a mixing statement together with the intersection property. The proofs heavily relies on a finer analysis of the geometry of the clusters compared to what was done in Section 9.6.2 (see Remark 9.6.10), that is in fact valid in a wider generality (i.e. also when $d_{\text{eff}} > 4$). Hence, we begin by proving the mixing statement in its most general form.

In Sections 9.7.1, 9.7.2, and 9.7.3 we consider an interaction J on $\mathbb{Z}^d$ ($d \geq 1$) satisfying (A1)–(A5) together with the following assumption: there exist $\mathbf{c}_1, \mathbf{C}_1 > 0$ such that, for all $x \in \mathbb{Z}^d \setminus \{0\}$,

$$\frac{\mathbf{c}_1}{|x|^{d+\alpha}} \leq J_{0,x} \leq \frac{\mathbf{C}_1}{|x|^{d+\alpha}}, \quad (\text{Assumption}_\alpha)$$

where $\alpha > 0$ will be specified below. By the results of Section 9.3, it has for consequence the existence of $\mathbf{C}_2 > 0$ such that: for all $\beta \leq \beta_c$, for all $x \in \mathbb{Z}^d \setminus \{0\}$,

$$\langle \sigma_0 \sigma_x \rangle_\beta \leq \frac{\mathbf{C}_2}{\beta_c |x|^{d-\alpha \wedge 2} (\log |x|)^{\delta_{2,\alpha}}}. \quad (\text{IRB}_\alpha)$$

Moreover, using Proposition 9.3.25, we find that there exists $\mathbf{C}_3 > 0$ such that: for all $\beta \leq \beta_c$, for all $x \in \mathbb{Z}^d \setminus \{0\}$ with $1 \leq |x| \leq L(\beta)$,

$$\langle \sigma_0 \sigma_x \rangle_\beta \geq \frac{\mathbf{C}_3}{\beta |x|^{d-1} f_\alpha(|x| + 1)}, \quad (\text{LB}_\alpha)$$

where if $t > 1$ $f_\alpha(t) := 1$ if $\alpha > 1$, $f_1(t) := \log t$, and $f_\alpha(t) := t^{1-\alpha}$ for $\alpha \in (0, 1)$.

9.7.1 Existence of weak regular scales

In the case $d \geq 3$, the existence of regular scales was proved in Section 9.3. For $d = 2$, the proof of (P4) failed. The reason behind this is purely technical: the sliding-scale infrared bound is not optimal in dimension 2 since we expect the growth of $\chi_n(\beta_c)$ to be smaller than n^2 . We can circumvent this technical difficulty by allowing ourselves a “weaker” property (P4') in the definition of a regular scale.

Definition 9.7.4 (Weak regular scales). Fix $c, C > 0$. An annular region $\text{Ann}(n/2, 8n)$ is said to be (c, C) -weak regular if it satisfies the properties (P1)–(P3) and

(P4') For every $x \in \Lambda_n$ and $y \notin \Lambda_{C(\log n)^2 n}$, $S_\beta(y) \leq \frac{1}{2} S_\beta(x)$.

A scale k is said to be *weak regular* if $n = 2^k$ is such that $\text{Ann}(n/2, 8n)$ is (c, C) -weak regular, a vertex $x \in \mathbb{Z}^d$ will be said to be in a weak regular scale if it belongs to an annulus $\text{Ann}(n, 2n)$ with $n = 2^k$ and k a weak regular scale.

Proposition 9.7.5 (Existence of weak regular scales). *Let $d \geq 1$. Assume that J satisfies (A1)–(A5) and (Assumption $_\alpha$) where $\alpha > 0$ if $d \geq 3$, $\alpha \in (0, 1]$ if $d = 2$, and $\alpha \in (0, 1)$ if $d = 1$. Let $\gamma > 2$. There exist $c_0, c_1, C_0 > 0$ such that for every $\beta \leq \beta_c$, and every $1 \leq n^\gamma \leq N \leq L(\beta)$, there are at least $c_1 \log_2 \left(\frac{N}{n} \right)$ (c_0, C_0) -weak regular scales between n and N .*

Proof. The cases $d \geq 3$, and $d \in \{1, 2\}$ with $\alpha \in (0, 1)$ were already settled in Propositions 9.3.28 and 9.3.29 since **(P4')** is weaker than **(P4)**. We only need to take care of the case $d = 2$ and $\alpha = 1$. Using **(LB $_{\alpha}$)** together with **(IRB $_{\alpha}$)**, we get that for $x \in \mathbb{Z}^d$ with $2 \leq |x| \leq L(\beta)$,

$$\frac{c_1}{\beta|x|(\log|x|)} \leq \langle \sigma_0 \sigma_x \rangle_{\beta} \leq \frac{C_1}{\beta|x|}. \quad (9.7.5)$$

Using (9.7.5) and the assumption¹⁹ $1 \leq n^{\gamma} \leq N$, we get the existence of $c_2, c_3, c_4 > 0$ such that,

$$\chi_N(\beta) \geq \frac{c_2}{\beta} \frac{N}{\log N} = \frac{c_2}{\beta} \left(\frac{N}{n}\right) \frac{n}{\log N} \geq c_3 \left(\frac{N}{n}\right) \frac{1}{\log N} \chi_n(\beta) \geq c_4 \left(\frac{N}{n}\right)^{1/2} \chi_n(\beta). \quad (9.7.6)$$

Using Theorem 9.3.18, we find $r, c_5 > 0$ and independent of n, N , such that there are at least $c_5 \log_2(N/n)$ scales $m = 2^k$ between n and N such that

$$\chi_{rm}(\beta) \geq \chi_{16dm}(\beta) + \chi_m(\beta). \quad (9.7.7)$$

We prove that such an m is a (c_0, C_0) -weak regular scale for a good choice of c_0, C_0 . The proof of **(P1)**–**(P3)** follows the same line as before. Now, using (9.7.5), we get that for $1 \leq \ell \leq L$ with $\ell \leq L(\beta)$,

$$\frac{\chi_L(\beta)}{L} \leq C_2 \frac{\log \ell}{\ell} \chi_{\ell}(\beta). \quad (9.7.8)$$

Let $R \geq 1$. Using the same strategy we used to obtain (9.3.69) and replacing the sliding-scale infrared bound by (9.7.8), we get for $y \notin \Lambda_{dR(\log m)^2 m}$ and $x \in \Lambda_m$,

$$|\Lambda_{R(\log m)^2 m}| S_{\beta}(y) \leq \chi_{R(\log m)^2 m}(\beta) \leq C_3 R (\log m)^3 \chi_m(\beta) \leq C_4 R (\log m)^3 m^2 S_{\beta}(x), \quad (9.7.9)$$

which implies that

$$S_{\beta}(y) \leq \frac{C_5}{R \log m} S_{\beta}(x) \leq \frac{1}{2} S_{\beta}(x), \quad (9.7.10)$$

if R is large enough. This concludes the proof. $\square$

9.7.2 Properties of the current

In this subsection, we let $d \geq 1$ and assume that J satisfies **(A1)**–**(A5)** together with **(Assumption $_{\alpha}$)** with $d - 2(\alpha \wedge 2) \geq 0$. As explained in Section 9.5, this choice corresponds to $d_{\text{eff}} \geq 4$.

We import the notations from Section 9.6.2. The main difficulty below will come from the fact that **Jump** $(k, k + k^{\nu})$ (for $\nu < 1$) now occurs with high probability (if $\alpha \in (0, 2]$). However, at the cost of considering thicker annuli we can keep a similar statement.

Many of the computations done here are very similar to what was done in Section 12.2 so we only present the main changes and omit the trivial modifications.

Lemma 9.7.6. *Let $d \geq 1$. Assume that J satisfies **(A1)**–**(A5)** and **(Assumption $_{\alpha}$)** with $\alpha > 0$. Let $\epsilon > 0$. There exist $C, \eta > 0$ such that for all $\beta \leq \beta_c$, $y \in \mathbb{Z}^d$ in a weak regular scale, with $1 \leq |y| \leq L(\beta)$, and for all $k \geq 1$ such that $k^4 \leq |y|$,*

$$\mathbf{P}_{\beta}^{0y, \emptyset}[\text{Jump}(k, k^{1+\epsilon})] \leq \frac{C}{k^{\eta}}. \quad (9.7.11)$$

¹⁹In fact $n \leq N/\log N$ is enough.

Proof. We repeat the strategy of proof of Lemma 9.6.5. Lemma 9.6.3 yields

$$\begin{aligned} \mathbf{P}_\beta^{0y, \emptyset}[\text{Jump}(k, k^{1+\epsilon})] &\leq 2\beta \sum_{u \in \Lambda_k, v \notin \Lambda_{k^{1+\epsilon}}} J_{u,v} \left(\langle \sigma_u \sigma_v \rangle_\beta \right. \\ &\quad \left. + \frac{\langle \sigma_0 \sigma_u \rangle_\beta \langle \sigma_v \sigma_y \rangle_\beta}{\langle \sigma_0 \sigma_y \rangle_\beta} + \frac{\langle \sigma_0 \sigma_v \rangle_\beta \langle \sigma_u \sigma_y \rangle_\beta}{\langle \sigma_0 \sigma_y \rangle_\beta} \right) =: A_1 + A_2 + A_3. \end{aligned}$$

Using (IRB $_\alpha$) and (Assumption $_\alpha$),

$$A_1 \leq C_1 \sum_{u \in \Lambda_k, v \notin \Lambda_{k^{1+\epsilon}}} \frac{1}{|u - v|^{d+\alpha+d-\alpha \wedge 2}} \leq C_2 \frac{k^d}{k^{(1+\epsilon)(d+\alpha-\alpha \wedge 2)}}. \quad (9.7.12)$$

Using (P1) of weak regular scales, together with (IRB $_\alpha$) and (Assumption $_\alpha$), we similarly obtain

$$A_2 \leq C_3 \frac{k^d}{k^{(1+\epsilon)(d+\alpha-\alpha \wedge 2)}}. \quad (9.7.13)$$

Finally, proceeding as in the proof of Lemma 9.6.5, using additionally (LB $_\alpha$),

$$\beta \sum_{u \in \Lambda_k, v \in \Lambda_{|y|/2}(y)} J_{u,v} \frac{\langle \sigma_0 \sigma_u \rangle_\beta \langle \sigma_v \sigma_y \rangle_\beta}{\langle \sigma_0 \sigma_y \rangle_\beta} \leq C_5 \frac{k^{\alpha \wedge 2} |y|^{d-1} f_\alpha(|y|) |y|^{\alpha \wedge 2}}{|y|^{d+\alpha}}, \quad (9.7.14)$$

where the right-hand side is bounded by $C_5 k^\alpha / |y|^\alpha$ for $\alpha \in (0, 1)$, by $C_5 k \log(|y|) / |y|$ for $\alpha = 1$, and by $C_5 k^{\alpha \wedge 2} / |y|$ for $\alpha > 1$. Hence, if $k^4 \leq |y|$, we always have that it is bounded by $C_4 / |y|^\delta$ for some $\delta = \delta(\alpha) > 0$.

Using again (P1) as in Lemma 9.6.5,

$$\beta \sum_{u \in \Lambda_k, v \notin \Lambda_{k^{1+\epsilon}} \cup \Lambda_{|y|/2}(y)} J_{u,v} \frac{\langle \sigma_0 \sigma_u \rangle_\beta \langle \sigma_v \sigma_y \rangle_\beta}{\langle \sigma_0 \sigma_y \rangle_\beta} \leq C_6 \frac{k^{\alpha \wedge 2}}{k^{(1+\epsilon)\alpha}}. \quad (9.7.15)$$

This concludes the proof. $\square$

Remark 9.7.7. The main contribution to the above probability comes from the term A_2 pictured in Figure 9.4.

Despite being equipped with a very weak version of Lemma 9.6.5, we can still obtain a version of Corollary 9.6.9 in our context.

Corollary 9.7.8 (No zigzag for the backbone). *Let $d \geq 1$. Assume that J satisfies (A1)–(A5) and (Assumption $_\alpha$) with $d - 2(\alpha \wedge 2) \geq 0$. Fix $\nu \in (0, 1)$ and $\epsilon > 0$. There exist $C, \eta > 0$ such that, for all $\beta \leq \beta_c$, for all $k, \ell \geq 1$ and $y \in \mathbb{Z}^d$ in a weak regular scale with $k^{\lfloor 2d/(1-\nu) \rfloor \vee \lfloor 2d/(\nu\alpha) \rfloor} \leq \ell$ and $\ell^4 \leq |y|$,*

$$\mathbf{P}_\beta^{0y}[\Gamma(\mathbf{n}_1) \in \text{ZZ}(0, y; k, \ell, \infty)] \leq \frac{C}{\ell^\eta}. \quad (9.7.16)$$

Proof. Notice that

$$\text{ZZ}(0, y; k, \ell, \infty) \subset \text{ZZ}(0, y; k, \ell, \ell^{1+\epsilon}) \cup \text{Jump}(\ell, \ell^{1+\epsilon}). \quad (9.7.17)$$

Using Lemma 9.7.6 we find $C_1, \eta > 0$ such that

$$\mathbf{P}_\beta^{0y}[\text{Jump}(\ell, \ell^{1+\epsilon})] \leq \frac{C_1}{\ell^\eta}. \quad (9.7.18)$$

If $\text{ZZ}(0, y; k, \ell, \ell^{1+\epsilon})$ occurs, there are two possibilities: either the backbone actually visits $\text{Ann}(\ell, \ell + \ell^\nu)$ before hitting Λ_k , an event we denote by $\mathbf{B}_1$; or it does not in which case there must be an open edge which jumps from Λ_ℓ to $\text{Ann}(\ell + \ell^\nu, \ell^{1+\epsilon})$, an event we denote by $\mathbf{B}_2$. By the chain rule for the backbone, we find that,

$$\mathbf{P}_\beta^{0y}[\mathbf{B}_1] \leq \sum_{\substack{u \in \text{Ann}(\ell, \ell + \ell^\nu) \\ v \in \Lambda_k}} \frac{\langle \sigma_0 \sigma_u \rangle_\beta \langle \sigma_u \sigma_v \rangle_\beta \langle \sigma_v \sigma_y \rangle_\beta}{\langle \sigma_0 \sigma_y \rangle_\beta} \leq C_2 \frac{k^d \ell^{d-1+\nu}}{\ell^{2d-2(\alpha \wedge 2)}} \leq \frac{C_2}{\ell^{(1-\nu)/2}}, \quad (9.7.19)$$

where we used **(IRB $_\alpha$)**, the property **(P2)** of weak regular scales to compare $\langle \sigma_v \sigma_y \rangle_\beta$ and $\langle \sigma_0 \sigma_y \rangle_\beta$, and the hypothesis $d - 2(\alpha \wedge 2) \geq 0$. Using (9.B.3), we see that for a one-step walk $\gamma : a \rightarrow b$, one has $\rho(\gamma) \leq \rho_{\{a,b\}}(\gamma) = \tanh(\beta J_{a,b})$. Combining this observation with the chain rule,

$$\mathbf{P}_\beta^{0y}[\mathbf{B}_2] \leq \sum_{\substack{a \in \Lambda_\ell \\ b \in \text{Ann}(\ell + \ell^\nu, \ell^{1+\epsilon}) \\ c \in \Lambda_k}} \frac{\langle \sigma_0 \sigma_a \rangle_\beta \tanh(\beta J_{a,b}) \langle \sigma_b \sigma_c \rangle_\beta \langle \sigma_c \sigma_y \rangle_\beta}{\langle \sigma_0 \sigma_y \rangle_\beta} \leq \frac{C_3 k^d \ell^{\alpha \wedge 2}}{\ell^{d-\alpha \wedge 2} \ell^{\nu \alpha}}, \quad (9.7.20)$$

where we used **(P2)** to compare $\langle \sigma_c \sigma_y \rangle_\beta$ and $\langle \sigma_0 \sigma_y \rangle_\beta$, **(IRB $_\alpha$)** to argue that

$$\sum_{c \in \Lambda_k} \langle \sigma_b \sigma_c \rangle_\beta \leq C_4 \frac{k^d}{\ell^{d-\alpha \wedge 2}}, \quad (9.7.21)$$

and **(IRB $_\alpha$)** once again with **(Assumption $_\alpha$)** to get

$$\sum_{\substack{a \in \Lambda_\ell \\ b \notin \Lambda_{\ell+\ell^\nu}}} \langle \sigma_0 \sigma_a \rangle_\beta \tanh(\beta J_{a,b}) \leq C_5 \frac{\ell^{\alpha \wedge 2}}{\ell^{\alpha \nu}}. \quad (9.7.22)$$

This concludes the proof. $\square$

We can also obtain the corresponding modification of Lemma 9.6.11.

Lemma 9.7.9. *Let $d \geq 1$. Assume that J satisfies **(A1)**–**(A5)** and **(Assumption $_\alpha$)** with $d - 2(\alpha \wedge 2) \geq 0$. Let $\epsilon > 0$. There exist $C, \eta > 0$ such that, for all $\beta \leq \beta_c$, for $n < m \leq M \leq k$ with $1 \leq M^{2/\epsilon} \leq k \leq L(\beta)$, for all $x \in \Lambda_n$, and all $u \in \text{Ann}(m, M)$,*

$$\mathbf{P}_\beta^{xu, \emptyset}[\text{Jump}(k, k^{1+\epsilon})] \leq \frac{C}{k^\eta}. \quad (9.7.23)$$

Proof. We repeat the proof of Lemma 9.6.11. Using Lemma 9.6.3,

$$\begin{aligned} \sum_{w \in \Lambda_k, v \notin \Lambda_{k^{1+\epsilon}}} \mathbf{P}_\beta^{xu, \emptyset}[\mathbf{n}_{w,v} \geq 1] \\ \leq 2\beta \sum_{w \in \Lambda_k, v \notin \Lambda_{k^{1+\epsilon}}} J_{w,v} \left(\langle \sigma_w \sigma_v \rangle_\beta + \frac{\langle \sigma_x \sigma_w \rangle_\beta \langle \sigma_v \sigma_u \rangle_\beta}{\langle \sigma_x \sigma_u \rangle_\beta} + \frac{\langle \sigma_x \sigma_v \rangle_\beta \langle \sigma_w \sigma_u \rangle_\beta}{\langle \sigma_x \sigma_u \rangle_\beta} \right). \end{aligned}$$

Using (Assumption $_\alpha$) and (IRB $_\alpha$),

$$\beta \sum_{w \in \Lambda_k, v \notin \Lambda_{k^{1+\epsilon}}} J_{w,v} \langle \sigma_w \sigma_v \rangle_\beta \leq C_1 k^d \sum_{p \geq k^{1+\epsilon}} \frac{p^{d-1}}{p^{d+\alpha+d-(\alpha \wedge 2)}} \leq \frac{C_2}{k^{d\epsilon}}. \quad (9.7.24)$$

Then, using (LB $_\alpha$) (which is licit since $1 \leq |x - u| \leq L(\beta)$) together with (Assumption $_\alpha$) and (IRB $_\alpha$), we get

$$\begin{aligned} \beta \sum_{w \in \Lambda_k, v \notin \Lambda_{k^{1+\epsilon}}} J_{w,v} \frac{\langle \sigma_x \sigma_v \rangle_\beta \langle \sigma_w \sigma_u \rangle_\beta}{\langle \sigma_x \sigma_u \rangle_\beta} &\leq \beta^2 C_3 M^{d-1} f_\alpha(M) \sum_{\substack{w \in \Lambda_k \\ v \notin \Lambda_{k^{1+\epsilon}}}} J_{w,v} \langle \sigma_x \sigma_v \rangle_\beta \langle \sigma_w \sigma_u \rangle_\beta \\ &\leq C_4 M^{d-1} f_\alpha(M) k^{\alpha \wedge 2} \sum_{v \notin \Lambda_{k^{1+\epsilon}}} |v|^{-(d-\alpha \wedge 2)} J_{0,v} \\ &\leq C_5 M^d k^{\alpha \wedge 2 - d(1+\epsilon)}. \end{aligned}$$

Finally, with the same reasoning we also get

$$\sum_{w \in \Lambda_k, v \notin \Lambda_{k^{1+\epsilon}}} J_{w,v} \frac{\langle \sigma_x \sigma_w \rangle_\beta \langle \sigma_v \sigma_u \rangle_\beta}{\langle \sigma_x \sigma_u \rangle_\beta} \leq C_6 M^d k^{\alpha \wedge 2 - d(1+\epsilon)}. \quad (9.7.25)$$

Now, clearly, if $M^{2/\epsilon} \leq k$, using that $d = 2(\alpha \wedge 2) \geq 0$, we can choose $\eta = d\epsilon/2$ and C a sufficiently large constant. $\square$

Corollary 9.7.10. *Let $d \geq 1$. Assume that J satisfies (A1)–(A5) and (Assumption $_\alpha$) with $d - 2(\alpha \wedge 2) \geq 0$. There exist $\nu \in (0, 1)$ and $C, \epsilon, \eta > 0$ such that for all $\beta \leq \beta_c$, for all $n < m \leq M \leq k$ with $1 \leq M^{[2(1+\epsilon)/\epsilon] \vee [2d/(1-\nu)]} \leq k \leq L(\beta)$, for all $x \in \Lambda_n$ and all $u \in \text{Ann}(m, M)$,*

$$\mathbf{P}_\beta^{xu}[\mathbf{ZZ}(x, u; M, k, \infty)] \leq \frac{C}{k^\eta}. \quad (9.7.26)$$

Proof. We repeat the strategy used to get Corollary 9.7.8. Let $\nu \in (0, 1)$ and $\epsilon > 0$ to be fixed below. Notice that,

$$\begin{aligned} \mathbf{ZZ}(x, u; M, k, \infty) &\subset \mathbf{ZZ}(x, u; M, k, k^{1+\epsilon}) \cap (\text{Jump}(k^{1/(1+\epsilon)}, k))^c \\ &\quad \cup \text{Jump}(k^{1/(1+\epsilon)}, k) \cup \text{Jump}(k, k^{1+\epsilon}). \end{aligned}$$

Using Lemma 9.7.9 together with the hypothesis on M and k , we find $C_{1,\eta} > 0$ such that

$$\mathbf{P}_\beta^{xu}[\text{Jump}(k^{1/(1+\epsilon)}, k) \cup \text{Jump}(k, k^{1+\epsilon})] \leq \frac{C_1}{k^\eta}. \quad (9.7.27)$$

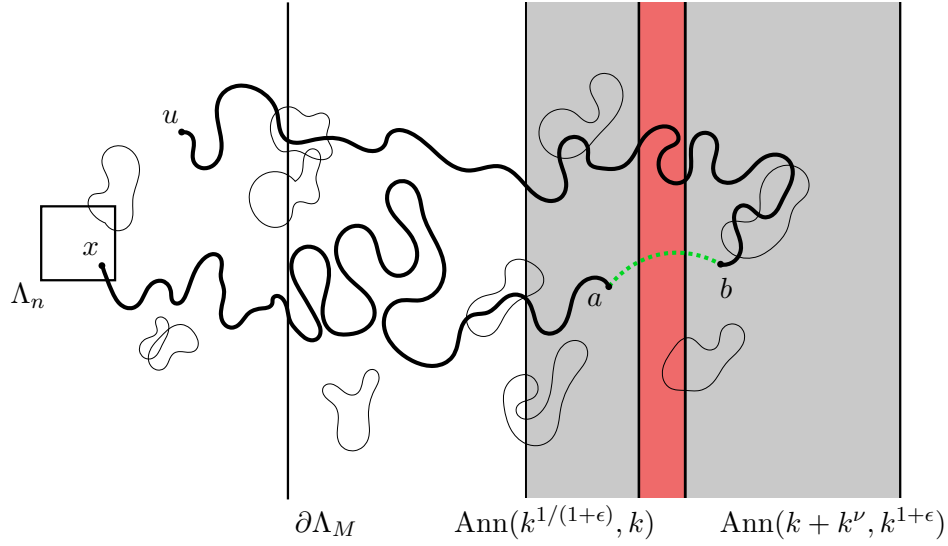


Figure 9.7: An illustration of the occurrence of the event B_2 defined in the proof of Corollary 9.7.10. The “exclusion zone” $\text{Ann}(k, k + k^\nu)$ is represented in red. The green dashed line illustrates the long open edge which jumps above it.

We handle $\text{ZZ}(x, u; M, k, k^{1+\epsilon}) \cap (\text{Jump}(k^{1/(1+\epsilon)}, k))^c$ as we did in the proof of Corollary 9.7.8 by splitting it into two events B_1 and B_2 according to whether or not the backbone reaches $\text{Ann}(k, k + k^\nu)$. Using the chain rule together with (LB_α) and (IRB_α) ,

$$\begin{aligned} \mathbf{P}_\beta^{xu}[B_1] &\leq \sum_{v \in \text{Ann}(k, k+k^\nu)} \frac{\langle \sigma_x \sigma_v \rangle_\beta \langle \sigma_v \sigma_u \rangle_\beta}{\langle \sigma_x \sigma_u \rangle_\beta} \\ &\leq \frac{C_2 M^{d-1} f_\alpha(M) k^{d-1+\nu}}{k^{2d-2(\alpha \wedge 2)}} \\ &\leq \frac{C_2 M^d}{k^{1-\nu}} \leq \frac{C_2}{k^{(1-\nu)/2}}, \end{aligned}$$

where we used the assumption that $d \geq 2(\alpha \wedge 2)$.

It remains to analyse B_2 . In that case, the backbone has to jump above $\text{Ann}(k, k + k^\nu)$: it goes from x to a point in $\text{Ann}(k^{1/(1+\epsilon)}, k)$ (recall that we excluded jumps above $\text{Ann}(k^{1/(1+\epsilon)}, k)$), then jumps in $\text{Ann}(k + k^\nu, k^{1+\epsilon})$, before finally hitting u (see Figure 9.7).

Using the chain rule for the backbone as we did in the proof of Corollary 9.7.8 together with $(\text{Assumption}_\alpha)$, (LB_α) , and (IRB_α) , we get

$$\begin{aligned} \mathbf{P}_\beta^{xu}[B_2] &\leq \sum_{\substack{a \in \text{Ann}(k^{1/(1+\epsilon)}, k) \\ b \in \text{Ann}(k+k^\nu, k^{1+\epsilon})}} \frac{\langle \sigma_x \sigma_a \rangle_\beta \tanh(\beta J_{a,b}) \langle \sigma_b \sigma_u \rangle_\beta}{\langle \sigma_x \sigma_u \rangle_\beta} \\ &\leq \frac{C_3 M^{d-1} f_\alpha(M) k^d k^{d(1+\epsilon)}}{k^{\frac{d-\alpha \wedge 2}{1+\epsilon}} k^{\nu(d+\alpha)} k^{d-\alpha \wedge 2}}. \end{aligned}$$

Now, recall that $M^d \leq k^{d\epsilon}$, so that we may find $\zeta = \zeta(\epsilon, \nu) > 0$ such that

$$\mathbf{P}_\beta^{xu}[\mathbf{B}_2] \leq \frac{C_3 k^{2d+3d\epsilon}}{k^{\frac{d-\alpha\wedge 2}{1+\epsilon}} k^{\nu(d+\alpha)} k^{d-\alpha\wedge 2}} \leq \frac{C_3}{k^\zeta}, \quad (9.7.28)$$

provided $\epsilon > 0$ is small enough, and ν is sufficiently close to 1. This concludes the proof. $\square$

Finally, as we did in Section 9.6.2, we conclude this subsection with some properties concerning the current $\mathbf{n} \setminus \overline{\Gamma(\mathbf{n})}$.

Lemma 9.7.11. *Let $d \geq 1$. Assume that J satisfies (A1)–(A5) and (Assumption $_\alpha$) with $\alpha > 0$. Let $\epsilon > 0$. There exist $C, \eta > 0$ such that for all $\beta \leq \beta_c$, for all $k \geq 1$, for all $x, y \in \mathbb{Z}^d$,*

$$\mathbf{P}_\beta^{xy}[\mathbf{n} \setminus \overline{\Gamma(\mathbf{n})} \in \text{Jump}(k, k^{1+\epsilon})] \leq \mathbf{P}_\beta^{xy, \emptyset}[(\mathbf{n}_1 + \mathbf{n}_2) \setminus \overline{\Gamma(\mathbf{n}_1)} \in \text{Jump}(k, k^{1+\epsilon})] \leq \frac{C}{k^\eta}. \quad (9.7.29)$$

Proof. Proceeding exactly as in the proof of Corollary 9.6.15 we find that

$$\mathbf{P}_\beta^{xy, \emptyset}[(\mathbf{n}_1 + \mathbf{n}_2) \setminus \overline{\Gamma(\mathbf{n}_1)} \in \text{Jump}(k, k^{1+\epsilon})] \leq 2 \sum_{u \in \Lambda_k, v \notin \Lambda_{k^{1+\epsilon}}} \beta J_{u,v} \langle \sigma_u \sigma_v \rangle_\beta. \quad (9.7.30)$$

This last sum is then smaller than $Ck^{-d\epsilon}$ as shown in (9.7.24). $\square$

Recall that the event Cross was defined in Definition 9.6.14.

Corollary 9.7.12. *Let $d \geq 1$. Assume that J satisfies (A1)–(A5) and (Assumption $_\alpha$) with $d - 2(\alpha \wedge 2) \geq 0$. There exist $\nu \in (0, 1)$ and $C, \epsilon, \eta > 0$ such that for all $\beta \leq \beta_c$, for all $k, \ell \geq 1$ with $k^{2d/(1-\nu)} \leq \ell$, for all $x, u \in \mathbb{Z}^d$,*

$$\mathbf{P}_\beta^{xu}[\mathbf{n} \setminus \overline{\Gamma(\mathbf{n})} \in \text{Cross}(k, \ell)] \leq \mathbf{P}_\beta^{xu, \emptyset}[(\mathbf{n}_1 + \mathbf{n}_2) \setminus \overline{\Gamma(\mathbf{n}_1)} \in \text{Cross}(k, \ell)] \leq \frac{C}{\ell^\eta}. \quad (9.7.31)$$

Proof. We use the ideas developed in the proofs of Lemma 9.6.13 and Corollary 9.7.10. Let $\nu \in (0, 1)$ and $\epsilon > 0$ to be fixed below. Start by writing,

$$\{(\mathbf{n}_1 + \mathbf{n}_2) \setminus \overline{\Gamma(\mathbf{n}_1)} \in \text{Cross}(k, \ell)\} = [(\mathbf{B}_1 \cup \mathbf{B}_2) \cap \mathbf{J}^c] \cup \mathbf{J}, \quad (9.7.32)$$

where $\mathbf{B}_1$ is the event that $(\mathbf{n}_1 + \mathbf{n}_2) \setminus \overline{\Gamma(\mathbf{n}_1)}$ crosses $\text{Ann}(k, \ell)$ by passing through $\text{Ann}(\ell, \ell + \ell^\nu)$, $\mathbf{B}_2$ is the complement of $\mathbf{B}_1$ in $\{(\mathbf{n}_1 + \mathbf{n}_2) \setminus \overline{\Gamma(\mathbf{n}_1)} \in \text{Cross}(k, \ell)\}$, and $\mathbf{J} = \{(\mathbf{n}_1 + \mathbf{n}_2) \setminus \overline{\Gamma(\mathbf{n}_1)} \in \text{Jump}(\ell^{1/(1+\epsilon)}, \ell)\} \cup \{(\mathbf{n}_1 + \mathbf{n}_2) \setminus \overline{\Gamma(\mathbf{n}_1)} \in \text{Jump}(\ell, \ell^{1+\epsilon})\}$.

Using Lemma 9.7.11, we get the existence of $C_1, \eta_1 > 0$ such that

$$\mathbf{P}_\beta^{xu, \emptyset}[\mathbf{J}] \leq \frac{C_1}{\ell^{\eta_1}}. \quad (9.7.33)$$

Using (IRB $_\alpha$) together with what was done in the proof of Corollary 9.6.15,

$$\mathbf{P}_\beta^{xu, \emptyset}[\mathbf{B}_1, \mathbf{J}^c] \leq \sum_{\substack{v \in \text{Ann}(\ell, \ell + \ell^\nu) \\ w \in \Lambda_k}} \langle \sigma_w \sigma_v \rangle_\beta^2 \leq \frac{C_2 \ell^{d-1+\nu} k^d}{\ell^{2d-2(\alpha \wedge 2)}} \leq \frac{C_2}{\ell^{(1-\nu)/2}}, \quad (9.7.34)$$

where we used that $d - 2(\alpha \wedge 2) \geq 0$. Finally, using a similar strategy as in the proof of Lemma 9.6.13,

$$\mathbf{P}_\beta^{xu, \emptyset}[\mathbf{B}_2, \mathbf{J}^c] \leq \sum_{\substack{\gamma: x \rightarrow u \\ \text{consistent}}} \sum_{\substack{a \in \Lambda_k \\ b \in \text{Ann}(\ell^{1/(1+\epsilon)}, \ell) \\ c \in \text{Ann}(\ell + \ell^\nu, \ell^{1+\epsilon})}} \mathbf{P}_\beta^{xu}[\Gamma(\mathbf{n}) = \gamma] \mathbf{P}_{\bar{\gamma}^c, \mathbb{Z}^d, \beta}^{\emptyset, \emptyset}[a \leftrightarrow b \text{ in } \bar{\gamma}^c, (\mathbf{n}_1 + \mathbf{n}_2)_{b,c} \geq 1]. \quad (9.7.35)$$

Using the generalisation of the switching lemma mentioned in the proof of Corollary 9.6.15 together with Griffith's inequality,

$$\begin{aligned} \mathbf{P}_{\bar{\gamma}^c, \mathbb{Z}^d, \beta}^{\emptyset, \emptyset}[a \leftrightarrow b \text{ in } \bar{\gamma}^c, (\mathbf{n}_1 + \mathbf{n}_2)_{b,c} \geq 1] &= \langle \sigma_a \sigma_b \rangle_{\bar{\gamma}^c, \beta} \langle \sigma_a \sigma_b \rangle_\beta \mathbf{P}_{\bar{\gamma}^c, \mathbb{Z}^d, \beta}^{ab, ab}[(\mathbf{n}_1 + \mathbf{n}_2)_{b,c} \geq 1] \\ &\leq \beta J_{b,c} \langle \sigma_a \sigma_c \rangle_{\bar{\gamma}^c, \beta} \langle \sigma_a \sigma_b \rangle_\beta + \beta J_{b,c} \langle \sigma_a \sigma_b \rangle_{\bar{\gamma}^c, \beta} \langle \sigma_a \sigma_c \rangle_\beta \\ &\leq 2 \langle \sigma_a \sigma_b \rangle_\beta \beta J_{b,c} \langle \sigma_c \sigma_a \rangle_\beta. \end{aligned}$$

We obtained,

$$\mathbf{P}_\beta^{xu, \emptyset}[\mathbf{B}_2, \mathbf{J}^c] \leq 2 \sum_{\substack{a \in \Lambda_k \\ b \in \text{Ann}(\ell^{1/(1+\epsilon)}, \ell) \\ c \in \text{Ann}(\ell + \ell^\nu, \ell^{1+\epsilon})}} \langle \sigma_a \sigma_b \rangle_\beta \beta J_{b,c} \langle \sigma_c \sigma_a \rangle_\beta. \quad (9.7.36)$$

Now, using (Assumption $_\alpha$) and (IRB $_\alpha$),

$$\mathbf{P}_\beta^{xu, \emptyset}[\mathbf{B}_2, \mathbf{J}^c] \leq \frac{C_3 k^d \ell^d \ell^{d(1+\epsilon)}}{\ell^{(d-\alpha \wedge 2)} \ell^{\frac{d-\alpha \wedge 2}{1+\epsilon}} \ell^{\nu(d+\alpha)}}. \quad (9.7.37)$$

We can now use that $k^d \leq \ell^{(1-\nu)}$ and choose $\epsilon > 0$ sufficiently small, and ν sufficiently close to 1 to conclude. $\square$

9.7.3 Mixing property for $d_{\text{eff}} \geq 4$

The goal of this subsection is to prove the following result.

Theorem 9.7.13 (Mixing property for $d_{\text{eff}} \geq 4$). *Let $d \geq 1$ and $s \geq 1$. Assume that J satisfies (A1)–(A5) and (Assumption $_\alpha$) with $d - 2(\alpha \wedge 2) \geq 0$. There exist $\gamma, C > 0$, such that for every $1 \leq t \leq s$, every $\beta \leq \beta_c$, every $1 \leq n^\gamma \leq N \leq L(\beta)$, every $x_i \in \Lambda_n$ and $y_i \notin \Lambda_N$ ($i \leq t$), and every events E and F depending on the restriction of $(\mathbf{n}_1, \dots, \mathbf{n}_s)$ to edges with endpoints within Λ_n and outside Λ_N respectively,*

$$\begin{aligned} \left| \mathbf{P}_\beta^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}[E \cap F] - \mathbf{P}_\beta^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}[E] \mathbf{P}_\beta^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}[F] \right| \\ \leq C \left(\frac{\log(N/n)}{\log \log(N/n)} \right)^{-1/2}. \end{aligned} \quad (9.7.38)$$

Furthermore, for every $x'_1, \dots, x'_t \in \Lambda_n$ and $y'_1, \dots, y'_t \notin \Lambda_N$, we have that

$$\left| \mathbf{P}_\beta^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}[E] - \mathbf{P}_\beta^{x'_1 y'_1, \dots, x'_t y'_t, \emptyset, \dots, \emptyset}[E] \right| \leq C \left(\frac{\log(N/n)}{\log \log(N/n)} \right)^{-1/2}, \quad (9.7.39)$$

$$\left| \mathbf{P}_\beta^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}[F] - \mathbf{P}_\beta^{x'_1 y'_1, \dots, x'_t y'_t, \emptyset, \dots, \emptyset}[F] \right| \leq C \left(\frac{\log(N/n)}{\log \log(N/n)} \right)^{-1/2}. \quad (9.7.40)$$

We follow the strategy employed above and import all the notations from Section 9.6.4. Fix $\beta \leq \beta_c$. Fix two integers t, s satisfying $1 \leq t \leq s$. Introduce integers m, M such that $n \leq m \leq M \leq N$, $m/n = (N/n)^{\mu/2}$, and $N/M = (N/n)^{1-\mu}$ for μ small to be fixed.

We fix $\nu \in (0, 1)$ and $\epsilon > 0$ such that Corollaries 9.7.10 and 9.7.12 hold.

Introduce the set $\mathcal{K}$ of (c_0, C_0) -weak regular scales k between m and $M/2$ with every 2^k for $k \in \mathcal{K}$ differing by a multiplicative factor at least $2C_0 \log(N/n)^2$. By Proposition 9.7.5, we may assume that $|\mathcal{K}| \geq c_1 \frac{\log(N/n)}{\log \log(N/n)}$ for a sufficiently small $c_1 = c_1(\mu) > 0$. Recall that $\mathbf{U}$ was defined in Section 9.6.4.

The property $(\mathbf{P4}')$ of weak regular scales allows us to prove,

Lemma 9.7.14 (Concentration of $\mathbf{U}$). *For all $\gamma > 2$, there exists $C_1 = C_1(d, t, \gamma) > 0$ such that for all n sufficiently large satisfying $n^\gamma \leq N \leq L(\beta)$,*

$$\mathbf{E}_\beta^{\mathbf{xy}, \emptyset}[(\mathbf{U} - 1)^2] \leq C_1 \frac{\log \log(N/n)}{\log(N/n)}. \quad (9.7.41)$$

The only place where the argument needs to be adapted is located in the proof of Lemma 9.6.23, which bounds the occurrence of $\mathcal{G}(\mathbf{u})^c$ (where $\mathcal{G}(\mathbf{u})$ was defined in Definition 9.6.22). The remainder of the subsection concerns the extension of this lemma to our setup.

Lemma 9.7.15. *We keep the assumptions of Theorem 9.7.13. There exist $C, \delta > 0$, $\gamma = \gamma(\delta) > 0$ large enough and $\mu = \mu(\delta) > 0$ small enough such that for every $n^\gamma \leq N \leq L(\beta)$, and every $\mathbf{u}$ with $u_i \in \mathbb{A}_{y_i}(2^{k_i})$ with $m \leq 2^{k_i} \leq M/2$ for every $1 \leq i \leq t$,*

$$\mathbf{P}_\beta^{\mathbf{xu}, \mathbf{uy}}[\mathcal{G}(\mathbf{u})^c] \leq C \left(\frac{n}{N} \right)^\delta. \quad (9.7.42)$$

Proof. Recall that we have fixed the values of ϵ and ν . We follow the notations used in the proof of Lemma 9.6.23.

Recall that $\mathcal{G}(\mathbf{u}) = \cap_{1 \leq i \leq s} G_i$, and $H_i \cap F_i \subset G_i$ where

$$H_i = \{\mathbf{n}_i \notin \text{Cross}(M, N)\}, \quad F_i = \{\mathbf{n}'_i \notin \text{Cross}(n, m)\}. \quad (9.7.43)$$

Introduce intermediate scales $n \leq r \leq m \leq M \leq R \leq N$ with r, R chosen below.

• **Bound on H_i .** Define,

$$J_i := \bigcup_{p \in \{R, N^{1/(1+\epsilon)}, N\}} \{\mathbf{n}_i \in \text{Jump}(p, p^{1+\epsilon})\}. \quad (9.7.44)$$

Notice that,

$$\begin{aligned} \mathbf{P}_\beta^{\mathbf{xu}, \mathbf{uy}}[H_i^c] &\leq \mathbf{P}_\beta^{\mathbf{xu}}[\Gamma(\mathbf{n}_i) \in \text{ZZ}(x_i, u_i; M, R, \infty)] \\ &\quad + \mathbf{P}_\beta^{\mathbf{xu}}[\mathbf{n}_i \setminus \overline{\Gamma(\mathbf{n}_i)} \in \text{Cross}(R^{1+\epsilon}, N)] + \mathbf{P}_\beta^{\mathbf{xu}}[J_i]. \end{aligned}$$

Assume $R = N^\iota$ where $\iota > \mu$ will be fixed below, and recall that $M \leq N^{\mu+1/\gamma}$. We might decrease μ and increase γ to ensure that $(2/\epsilon)(\mu + 1/\gamma) \leq \iota$. As a result, we may use Lemma 9.7.9 to obtain $C_1, \eta_1 > 0$ such that

$$\mathbf{P}_\beta^{\mathbf{xu}}[J_i] \leq \frac{C_1}{R^{\eta_1}}. \quad (9.7.45)$$

Moreover, thanks to Corollaries 9.7.10 and 9.7.12, if we additionally require that²⁰:

$$M^{[2(1+\epsilon)/\epsilon] \vee [2d/(1-\nu)]} \leq R, \quad R^{2d(1+\epsilon)/(1-\nu)} \leq N, \quad (9.7.46)$$

we find $C_2, \eta_2 > 0$ such that,

$$\mathbf{P}_\beta^{\text{xu}}[\Gamma(\mathbf{n}_i) \in \text{ZZ}(x_i, u_i; M, R, \infty)] \leq \frac{C_2}{R^{\eta_2}}, \quad \mathbf{P}_\beta^{\text{xu}}[\mathbf{n}_i \setminus \overline{\Gamma(\mathbf{n}_i)} \in \text{Cross}(R^{1+\epsilon}, N)] \leq \frac{C_2}{R^{\eta_2}}. \quad (9.7.47)$$

• **Bound on F_i .** We follow the exact same strategy as in the proof of Lemma 9.6.23. The modifications are similar to what was done for the bound on H_i . Again, we replace Corollary 9.6.15 by Corollary 9.7.12 and choose accordingly the values of μ and γ .

We set $r = m^\kappa$ with $2\kappa d(1+\epsilon) \leq \alpha \wedge 2$. Recall that $m \geq N^{\mu/2}$. We find that

$$\begin{aligned} \mathbf{P}_\beta^{\text{xu,uy}}[F_i^c] &\leq \mathbf{P}_\beta^{\text{uy}}[\Gamma(\mathbf{n}'_i) \in \text{ZZ}(u_i, y_i; r^{1+\epsilon}, m, \infty)] \\ &\quad + \mathbf{P}_\beta^{\text{uy}}[\mathbf{n}'_i \setminus \overline{\Gamma(\mathbf{n}'_i)} \in \text{Cross}(n, r)] + \mathbf{P}_\beta^{\text{uy}}[\widetilde{K}_i], \end{aligned}$$

where $\widetilde{K}_i$ is the event that there exists $a \in \Lambda_r$ and $b \notin \Lambda_{r^{1+\epsilon}}$ such that $(\mathbf{n}'_i)_{a,b} \geq 2$ and $\{a, b\} \in \overline{\Gamma(\mathbf{n}'_i)} \setminus \Gamma(\mathbf{n}'_i)$.

Using (IRB $_\alpha$) and the assumption that $u_i \in \mathbb{A}_{y_i}(2^{k_i})$ to get that $\langle \sigma_v \sigma_{y_i} \rangle_\beta \leq C_3 \langle \sigma_{u_i} \sigma_{y_i} \rangle_\beta$, we obtain

$$\begin{aligned} \mathbf{P}_\beta^{\text{uy}}[\Gamma(\mathbf{n}'_i) \in \text{ZZ}(u_i, y_i; r^{1+\epsilon}, m, \infty)] &\leq \sum_{v \in \Lambda_{r^{1+\epsilon}}} \frac{\langle \sigma_{u_i} \sigma_v \rangle_\beta \langle \sigma_v \sigma_{y_i} \rangle_\beta}{\langle \sigma_{u_i} \sigma_{y_i} \rangle_\beta} \\ &\leq C_4 \frac{r^{d(1+\epsilon)}}{m^{d-\alpha \wedge 2}} \leq \frac{C_4}{m^{(\alpha \wedge 2)/2}}, \end{aligned}$$

where we used that $d - \alpha \wedge 2 \geq \alpha \wedge 2$. Moreover, using Corollary 9.7.12 (which requires that $n^{2d/(1-\nu)} \leq r$ and hence decreases the values of μ and $1/\gamma$), there exist $\zeta > 0$ such that

$$\mathbf{P}_\beta^{\text{uy}}[\mathbf{n}'_i \setminus \overline{\Gamma(\mathbf{n}'_i)} \in \text{Cross}(n, r)] \leq \frac{C_5}{r^\zeta}. \quad (9.7.48)$$

We conclude the proof with the bound on $\widetilde{K}_i$. Proceeding as in the proof of Lemma 9.6.23,

$$\mathbf{P}_\beta^{\text{uy}}[\widetilde{K}_i] \leq C_6 \sum_{\substack{a \in \Lambda_r \\ b \notin \Lambda_{r^{1+\epsilon}}}} \beta J_{a,b} \langle \sigma_{u_i} \sigma_b \rangle_\beta. \quad (9.7.49)$$

Using (Assumption $_\alpha$) and (IRB $_\alpha$) we obtain $\zeta' > 0$ such that

$$\mathbf{P}_\beta^{\text{uy}}[K_i] \leq \frac{C_7}{m^{\zeta'}}. \quad (9.7.50)$$

This concludes the proof □

We are now in a position to conclude.

Proof of Theorem 9.7.13. The proof follows the exact same lines as the proof of Theorem 9.6.20 except that we replace Lemma 9.6.23 by Lemma 9.7.15. □

²⁰This might decrease μ and increase γ .

9.7.4 Proof of Theorem 9.7.1

The lack of precision of the sliding-scale infrared bound in comparison to the situation in $d = 4$ slightly weakens the result. The three cases of interest are a little different and thus are treated in different sections.

9.7.4.1 The case $d = 3$

We assume that $d = 3$ and that J satisfies (A1)–(A5) and (Assumption $_{\alpha}$) with $\alpha = 3/2$.

Let us first observe that the sliding-scale infrared bound of Theorem 9.3.18 is not sharp in this setup. Indeed, we expect the finite-volume susceptibility to grow like $n^{3/2}$ (below $L(\beta)$). (LB $_{\alpha}$) and (IRB $_{\alpha}$) can (almost) make up for this lack of precision as they yield the existence of $C > 0$ such that for $1 \leq n \leq N \leq L(\beta)$,

$$\frac{\chi_N(\beta)}{N^{3/2}} \leq C\sqrt{n} \frac{\chi_n(\beta)}{n^{3/2}}. \quad (9.7.51)$$

Recall that (IRB $_{\alpha}$) still gives $B_L(\beta) - B_{\ell}(\beta) \leq C_0 \log(L/\ell)$ in our setup. However, (9.7.51) not being sharp, we modify Lemma 9.6.16 accordingly.

Lemma 9.7.16. *There exists $C > 0$ such that for every $\beta \leq \beta_c$, and for every $1 \leq \ell \leq L \leq L(\beta)$,*

$$B_L(\beta) \leq \left(1 + C \frac{\log_2(L/\ell)}{\log_2(\ell)}\right) B_{\ell}(\beta). \quad (9.7.52)$$

Proof. We repeat the proof of Lemma 9.6.16 except that we replace the use of Theorem 9.3.18 by (9.7.51). $\square$

We define a (possibly finite) sequence $\mathcal{L}_3 = \mathcal{L}_3(\beta, D)$ by $\ell_0 = 0$ and

$$\ell_{k+1} = \inf \{ \ell \geq \ell_k, B_{\ell}(\beta) \geq D \cdot (\ell_k + 1) \cdot B_{\ell_k}(\beta) \}. \quad (9.7.53)$$

We also define a sequence $\mathcal{U}_3 = \mathcal{U}_3(\beta, D)$ by $u_k = \ell_{3k}$ for $k \geq 0$.

Remark 9.7.17. Note that the sequence $\mathcal{L}_3$ grows much faster than $\mathcal{L}$ introduced in Section 9.6. The reason why we need an additional sequence $\mathcal{U}_3$ is technical and will become transparent later.

We begin with a technical result on $\mathcal{L}_3$.

Lemma 9.7.18 (Growth of $\mathcal{L}_3$). *There exist $c, C > 0$ such that, for all $k \geq 1$,*

$$\prod_{i=0}^{k-1} [D \cdot (\ell_i + 1)] \leq B_{\ell_k}(\beta) \leq C \prod_{i=0}^{k-1} [D \cdot (\ell_i + 1)], \quad (9.7.54)$$

and, as long as $\ell_{k+1} \leq L(\beta)$,

$$\ell_{k+1} \geq \ell_k^{cD}. \quad (9.7.55)$$

Proof. We repeat the argument used to study $\mathcal{L}$ in Section 9.6. The lower bound is immediate and for the upper bound, using that $B_L(\beta) - B_\ell(\beta) \leq C_0 \log(L/\ell)$, for $k \geq 1$,

$$\begin{aligned} B_{\ell_{k-1}}(\beta) &\leq D(\ell_{k-1} + 1)B_{\ell_{k-1}}(\beta) \leq D(\ell_{k-1} + 1) \left(B_{\ell_{k-1}-1}(\beta) - C_0 \log \left(1 - \frac{1}{\ell_{k-1}} \right) \right) \\ &\leq \left(\prod_{i=1}^{k-1} [D \cdot (\ell_i + 1)] \right) B_{\ell_1-1}(\beta) + C_0 \sum_{i=1}^{k-1} \frac{\prod_{j=k-i}^{k-1} [D \cdot (\ell_j + 1)]}{\ell_{k-i}} \\ &\leq C \left(\prod_{i=0}^{k-1} [D \cdot (\ell_i + 1)] \right). \end{aligned}$$

for C large enough (independent of D and k). We conclude by noticing that

$$B_{\ell_k}(\beta) \leq B_{\ell_{k-1}}(\beta) + C_0 \log 2. \quad (9.7.56)$$

As for the growth of $\mathcal{L}_3$, we use Lemma 9.7.18 and proceed as in Remark 9.6.17 to get that

$$\log_2(\ell_k) \leq C \ell_k \log_2(\ell_{k+1}/\ell_k) \frac{B_{\ell_k}(\beta)}{B_{\ell_{k+1}}(\beta) - B_{\ell_k}(\beta)} \leq C \ell_k \log_2(\ell_{k+1}/\ell_k) \frac{1}{(D\ell_k - 1)}, \quad (9.7.57)$$

which yields, for some $c_1 > 0$, $\ell_{k+1} \geq \ell_k^{c_1 D}$. This concludes the proof. $\square$

Remark 9.7.19. The second part of the above statement is in some sense the most important one since it ensures that there is “room” between successive scales. This has been used before to apply the results of Section 12.2 and Theorem 9.6.20. It will also be useful in our case, and it explains the introduction of the additional multiplicative factor ℓ_k in the definition of $\mathcal{L}_3$. This choice backfires when we try to estimate $B_{\ell_k}(\beta)$.

Our goal now is to prove,

Proposition 9.7.20 (Clustering bound for $d = 3$ and $\alpha = 3/2$). *For D large enough, there exists $\delta = \delta(D) > 0$ such that for all $\beta \leq \beta_c$, for all $K > 3$ with $u_{K+1} \leq L(\beta)$, and for all $v, x, y, z, t \in \mathbb{Z}^3$ with mutual distance between x, y, z, t larger than $2u_K$,*

$$\mathbf{P}_\beta^{vx, vz, vy, vt}[\mathbf{M}_u(\mathcal{I}; \mathcal{U}_3, K) < \delta K] \leq 2^{-\delta K}. \quad (9.7.58)$$

We first see how this result implies Theorem 9.7.1.

Proof of Theorem 9.7.1 for $d = 3$. We follow the proof of Section 9.6. The only change occurs in the connection between L and $B_L(\beta)$ when $L = 2u_K$. Using Lemma 9.7.18, we find that

$$B_L(\beta) \leq B_{\ell_{3K+1}}(\beta) \leq C \left(\prod_{i=0}^{3K} [D \cdot (\ell_i + 1)] \right) =: \Pi_{\beta, D}(3K), \quad (9.7.59)$$

so if $\Phi = \Phi_{\beta, D}$ is defined for $t \geq 1$ by:

$$\Phi(t) := \inf\{k \geq 0, \Pi_{\beta, D}(3k) \geq t\} \wedge (\lfloor N_0(\mathcal{L}_3)/3 \rfloor + 1), \quad (9.7.60)$$

where $N_0(\mathcal{L}_3)$ is the index of the last element of $\mathcal{L}_3$ (possibly equal to ∞), we find that $K \geq \Phi(B_L(\beta))$. This gives the result setting $\phi_\beta(t) := 2^{-\delta \Phi_{\beta, D}(t)/5}$. $\square$

As before, Proposition 9.7.20 will follow from a combination of Theorem 9.7.13 and of an intersection property. We begin by modifying the definition of the intersection event.

Below, we fix $\nu \in (0, 1)$ and $\epsilon > 0$ such that the results of Section 9.7.2 hold.

Definition 9.7.21 (Intersection event for $d = 3$ and $\alpha = 3/2$). Let $k \geq 1$ and $y \notin \Lambda_{u_{k+2}}$. A pair of currents $(\mathbf{n}, \mathbf{m})$ with $(\partial \mathbf{n}, \partial \mathbf{m}) = (\{0, y\}, \{0, y\})$ realises the event $\tilde{I}_k$ if the following properties are satisfied:

- (i) The restrictions of $\mathbf{n}$ and $\mathbf{m}$ to edges with both endpoints in $\text{Ann}(\ell_{3k}, \ell_{3k+3}^{1+\epsilon})$ contain a unique cluster “strongly crossing” $\text{Ann}(\ell_{3k}, \ell_{3k+3}^{1+\epsilon})$, in the sense that it contains a vertex in $\text{Ann}(\ell_{3k}, \ell_{3k}^{1+\epsilon})$ and a vertex in $\text{Ann}(\ell_{3k+3}, \ell_{3k+3}^{1+\epsilon})$.
- (ii) The two clusters described in (i) intersect.

Lemma 9.7.22 (Intersection property for $d = 3$ and $\alpha = 3/2$). *For D large enough, there exists $\kappa > 0$ such that for every $\beta \leq \beta_c$, every $k \geq 2$, and every $y \notin \Lambda_{u_{k+2}}$ in a weak regular scale with $1 \leq |y| \leq L(\beta)$,*

$$\mathbf{P}_\beta^{0y, 0y, \emptyset, \emptyset}[(\mathbf{n}_1 + \mathbf{n}_3, \mathbf{n}_2 + \mathbf{n}_4) \in \tilde{I}_k] \geq \kappa. \quad (9.7.61)$$

Proof. We repeat the two-step proof done in the preceding section. Introduce intermediate scales $u_k = \ell_{3k} \leq n \leq m \leq M \leq N \leq \ell_{3k+3} = u_{k+1}$ with $n = \sqrt{\ell_k \ell_{k+1}}$, $N = \sqrt{\ell_{k+2} \ell_{k+3}}$, $m = \ell_{3k+1}$, and $M = \ell_{3k+2}$. Keeping the same notations as in the proof of Lemma 9.6.19, one has for some $c_1 > 0$,

$$\mathbf{E}_\beta^{0y, 0y, \emptyset, \emptyset}[|\mathcal{M}|] \geq c_1(B_M(\beta) - B_{m-1}(\beta)), \quad (9.7.62)$$

and for some $c_2 > 0$

$$\mathbf{E}_\beta^{0y, 0y, \emptyset, \emptyset}[|\mathcal{M}|^2] \leq c_2(B_M(\beta) - B_{m-1}(\beta))B_{2M}(\beta). \quad (9.7.63)$$

Now, by definition of $\mathcal{L}_3$, one has $B_M(\beta) \geq D(\ell_{3k+1} + 1)B_m(\beta)$ so that $B_M(\beta) - B_{m-1}(\beta) \geq \frac{B_M(\beta)}{2}$ for D large enough. Moreover, by (IRB $_\alpha$), $B_{2M}(\beta) \leq B_M(\beta) + C \log 2$. As a result, we may find $c_3 > 0$ such that,

$$\mathbf{P}_\beta^{0y, 0y, \emptyset, \emptyset}[|\mathcal{M}| > 0] \geq c_3. \quad (9.7.64)$$

The conclusion of the proof follows the same lines as in Lemma 9.6.19: we replace Lemma 9.6.5 by Lemma 9.7.6, and Corollaries 9.6.9 and 9.6.15 by Corollaries 9.7.8 and 9.7.12. The proof is enabled by Lemma 9.7.18 which ensures that the different scales are sufficiently “distanced” when D is large enough. $\square$

We are now equipped to prove Proposition 9.7.20.

Proof of Proposition 9.7.20. Fix $\gamma > 2$ sufficiently large so that Theorem 9.7.13 holds. Remember by Lemma 9.7.18 that we may choose $D = D(\gamma)$ sufficiently large in the definition of $\mathcal{L}_3$ such that $\ell_{k+1} \geq \ell_k^\gamma$. We may assume $v = 0$. Since x, y are at distance at least $2u_K$ of each other, one of them must be at distance at least u_K of u . Without loss of generality we assume that this is the case of x and make the same assumption about z . Let $\delta > 0$ to

be fixed below. Recall that $\mathcal{S}_K^{(\delta)}$ denote the set of subsets of $\{2\delta K, \dots, K-3\}$ which contain only even integers.

As in the proof of Proposition 9.6.2,

$$\mathbf{P}_\beta^{0x,0z,0y,0t}[\mathbf{M}_u(\mathcal{I}; \mathcal{U}_3, K) < \delta K] \leq \sum_{\substack{S \in \mathcal{S}_K^{(\delta)} \\ |S| \geq (1/2 - 2\delta)K}} \mathbf{P}_\beta^{0x,0z,\emptyset,\emptyset}[\mathfrak{B}_S]. \quad (9.7.65)$$

Let J be the event defined by

$$J := \bigcup_{k=2\delta K}^{K-3} \{\mathbf{n}_1 + \mathbf{n}_3 \in \text{Jump}(u_k, u_k^{1+\epsilon})\} \cup \{\mathbf{n}_2 + \mathbf{n}_4 \in \text{Jump}(u_k, u_k^{1+\epsilon})\}. \quad (9.7.66)$$

Using Lemmas 9.7.6 and 9.7.18, if D is large enough, there exists $C_0, C_1, \eta > 0$ such that

$$\mathbf{P}_\beta^{0x,0z,\emptyset,\emptyset}[J] \leq \frac{C_0 K}{\ell_{3\delta K}^\eta} \leq C_1 e^{-\eta 2^{\delta K}}. \quad (9.7.67)$$

Fix some $S \in \mathcal{S}_K^{(\delta)}$. Let $\tilde{\mathfrak{A}}_S$ be the event that none of the events $\tilde{I}_k$ (defined in Definition 9.7.21) occur for $k \in S$. As above, if $k \in S$ and J^c occurs, the events $\tilde{I}_k$ and $\mathfrak{B}_S$ are incompatible. Using (9.7.67),

$$\mathbf{P}_\beta^{0x,0z,0y,0t}[\mathbf{M}_u(\mathcal{I}; \mathcal{U}_3, K) < \delta K] \leq \sum_{\substack{S \in \mathcal{S}_K^{(\delta)} \\ |S| \geq (1/2 - 2\delta)K}} \mathbf{P}_\beta^{0x,0z,\emptyset,\emptyset}[\tilde{\mathfrak{A}}_S] + C_1 e^{-\eta 2^{\delta K}} \binom{(1/2 - 2\delta)K}{2\delta K}. \quad (9.7.68)$$

From this point, the analysis follows the exact same lines as before and we refer to the proof of Proposition 9.6.2 for the rest of the argument. $\square$

9.7.4.2 The case $d = 2$

We now assume that $d = 2$ and that J satisfies (A1)–(A5) and (Assumption $_\alpha$) with $\alpha = 1$.

As before, the sliding-scale infrared bound of Theorem 9.3.18 is not sharp since we expect the finite-volume susceptibility to grow like n (below $L(\beta)$). Proposition 9.3.25 and (IRB $_\alpha$) yield the existence of $C > 0$ such that for $1 \leq n \leq N \leq L(\beta)$,

$$\frac{\chi_N(\beta)}{N} \leq C \log n \frac{\chi_n(\beta)}{n}. \quad (9.7.69)$$

Lemma 9.7.23. *There exists $C > 0$ such that for every $\beta \leq \beta_c$, and for every $1 \leq \ell \leq L \leq L(\beta)$,*

$$B_L(\beta) \leq \left(1 + C \frac{\log_2(L/\ell)}{\log_2(\ell)} \log \ell\right) B_\ell(\beta). \quad (9.7.70)$$

Proof. We repeat the proof of Lemma 9.7.16 using this time (9.7.69). $\square$

We define a (possibly finite) sequence $\mathcal{L}_2 = \mathcal{L}_2(\beta, D)$ by $\ell_0 = 1$ and

$$\ell_{k+1} = \inf \{\ell \geq \ell_k, B_\ell(\beta) \geq D \cdot (\log(\ell_k) + 1) \cdot B_{\ell_k}(\beta)\}. \quad (9.7.71)$$

We also define a sequence $\mathcal{U}_2 = \mathcal{U}_2(\beta, D)$ by $u_k = \ell_{3k}$ for $k \geq 0$. Adapting the proof of Lemma 9.7.18 to our setup, we obtain,

Lemma 9.7.24 (Growth of $\mathcal{L}_2$). *There exists $c, C_1, C_2 > 0$ such that, for all $k \geq 1$,*

$$\prod_{i=0}^{k-1} [D \cdot (\log(\ell_i) + 1)] \leq B_{\ell_k}(\beta) \leq C \prod_{i=0}^{k-1} [D \cdot (\log(\ell_i) + 1)], \quad (9.7.72)$$

and as long as $\ell_{k+1} \leq L(\beta)$,

$$\ell_{k+1} \geq \ell_k^{cD}. \quad (9.7.73)$$

The second part of Theorem 9.1.9 will follow from the following proposition.

Proposition 9.7.25 (Clustering bound for $d = 2$ and $\alpha = 1$). *For D large enough, there exists $\delta = \delta(D) > 0$ such that for all $\beta \leq \beta_c$, for all $K > 3$ with $u_{K+1} \leq L(\beta)$, and for all $v, x, y, z, t \in \mathbb{Z}^2$ with mutual distance between x, y, z, t larger than $2u_K$,*

$$\mathbf{P}_\beta^{vx, vz, vy, vt}[\mathbf{M}_u(\mathcal{I}; \mathcal{U}_2, K) < \delta K] \leq 2^{-\delta K}. \quad (9.7.74)$$

Proof. The proof follows the exact same lines as the proof of Proposition 9.7.20 (in particular, we keep the same intersection event $\tilde{I}_k$). $\square$

As above, this result easily implies 9.7.1 for $d = 2$.

Proof of Theorem 9.7.1 for $d = 2$. We follow the proof of Section 9.7.4.1. Using Lemma 9.7.24, we find that

$$B_L(\beta) \leq B_{\ell_{3K+1}}(\beta) \leq C \prod_{i=0}^{3K} [D \cdot (\log(\ell_i) + 1)] =: \Pi'_{\beta, D}(3K), \quad (9.7.75)$$

so that $K \geq \Phi'(B_L(\beta))$ where $\Phi' = \Phi'_{\beta, D}$ is defined for $t \geq 1$ by:

$$\Phi'(t) := \inf\{k \geq 0, \Pi'(3k) \geq t\} \wedge (\lfloor N_0(\mathcal{L}_2)/3 \rfloor + 1). \quad (9.7.76)$$

This concludes the proof. $\square$

9.7.4.3 The case $d = 1$

Finally, we treat the case $d = 1$. Assume that J satisfies (A1)–(A5) and (Assumption $_\alpha$) with $\alpha = 1/2$. The results of Section 9.3 give us the good rate of decay for S_β : there exist $c, C > 0$ such that for all $x \in \mathbb{Z}^d$ with $1 \leq |x| \leq L(\beta)$,

$$\frac{c}{|x|^{1/2}} \leq \langle \sigma_0 \sigma_x \rangle_\beta \leq \frac{C}{|x|^{1/2}}. \quad (9.7.77)$$

This observation greatly simplifies the proof²¹ and allows one to proceed like in Section 9.6.

As before, define a (possibly finite) sequence $\mathcal{L}_1 = \mathcal{L}_1(\beta, D)$ by $\ell_0 = 0$ and

$$\ell_{k+1} = \inf\{\ell \geq \ell_k, B_\ell(\beta) \geq DB_{\ell_k}(\beta)\}. \quad (9.7.78)$$

With the above work, it is easy to obtain,

²¹To avoid writing yet another proof of triviality we simply import the results of Section 9.6. However, note that the knowledge of the critical exponent η yields a shorter proof of the improved tree diagram bound, see [9, Section 4].

Proposition 9.7.26 (Clustering bound for $d = 1$ and $\alpha = 1/2$). *For D large enough, there exists $\delta = \delta(D) > 0$ such that for all $\beta \leq \beta_c$, for all $K > 3$ with $\ell_{K+1} \leq L(\beta)$, and for all $u, x, y, z, t \in \mathbb{Z}$ with mutual distance between x, y, z, t larger than $2\ell_K$,*

$$\mathbf{P}_\beta^{ux,uz,uy,ut}[\mathbf{M}_u(\mathcal{I}; \mathcal{L}, K) < \delta K] \leq 2^{-\delta K}. \quad (9.7.79)$$

From this result and (9.7.77), we can obtain Theorem 9.7.1 with $\phi_\beta(B_L(\beta)) = (\log L)^c$ for some constant $c > 0$.

9.7.5 Proof Corollary 9.7.3

Proof of Corollary 9.7.3. We use the same strategy as in Appendix 9.D. In particular, we will use Proposition 9.D.1. Recall that $\alpha = d/2$. Let $\sigma \in (0, d/2)$ so that $\xi_\sigma(\beta)$ is well defined for all $\beta < \beta_c$. We begin by noticing that there exists $C > 0$ such that, for $\beta < \beta_c$,

$$\chi(\beta) \leq C\xi_\sigma(\beta)^{d/2}. \quad (9.7.80)$$

Indeed, if $K > 0$, (IRB $_\alpha$) implies that for some $C_1 = C_1(d) > 0$,

$$\chi_{K\xi_\sigma(\beta)}(\beta) \leq C_1(K\xi_\sigma(\beta))^{d/2}. \quad (9.7.81)$$

Moreover, using Proposition 9.D.1,

$$\chi(\beta) - \chi_{K\xi_\sigma(\beta)}(\beta) \leq C_2 \frac{\chi(\beta)}{K^\sigma}. \quad (9.7.82)$$

Combining the two last inequalities, and choosing K large enough, we obtain (9.7.80).

Now, assume that we are given $1 \leq L \leq L(\beta)$. Write,

$$0 \leq g_\sigma(\beta) \leq A_1 + A_2, \quad (9.7.83)$$

where

$$A_1 := -\frac{1}{\chi(\beta)^2 \xi_\sigma(\beta)^d} \sum_{\substack{x,y,z \in \mathbb{Z}^d \\ L(0,x,y,z) \leq L}} U_4^\beta(0, x, y, z), \quad A_2 := -\frac{1}{\chi(\beta)^2 \xi_\sigma(\beta)^d} \sum_{\substack{x,y,z \in \mathbb{Z}^d \\ L(0,x,y,z) > L}} U_4^\beta(0, x, y, z). \quad (9.7.84)$$

Using the (standard) tree diagram bound (9.4.14), we get that

$$A_1 \leq C_3 L^d \frac{\chi(\beta)}{\xi_\sigma(\beta)^d} \leq C_4 L^d \xi_\sigma(\beta)^{-d/2}. \quad (9.7.85)$$

Moreover, using this time Theorem 9.7.1,

$$A_2 \leq \frac{C_5}{\phi_\beta(B_L(\beta))} \frac{\chi(\beta)^2}{\xi_\sigma(\beta)^d} \leq \frac{C_6}{\phi_\beta(B_L(\beta))}. \quad (9.7.86)$$

As a result, for any $L \geq 1$,

$$\limsup_{\beta \nearrow \beta_c} g_\sigma(\beta) \leq \frac{C_6}{\phi_{\beta_c}(B_L(\beta_c))}. \quad (9.7.87)$$

Hence, if $B(\beta_c) = \infty$, one obtains the result taking L to infinity. If $B(\beta_c) < \infty$, we may conclude using Theorem 9.D.2. $\square$

9.8 Extension of the results to models in the Griffiths–Simon class

In this section, we extend the results of Sections 9.6 and 9.7 to the case of single-site measures in the GS class. Let us mention that using Remark 9.5.7 we can adapt the proof of Theorem 9.1.3 to the case of measures in the GS class.

We focus on the results of Section 9.6 and briefly explains in Section 9.8.4 how similar considerations permit to extend the results of Section 9.7. Let J be an interaction satisfying (A1)–(A6), and ρ be a measure in the GS class.

Since the measure ρ might be of unbounded support, we will have to be careful in the derivation of the diagrammatic bounds. More precisely, to be able to take weak limits in ρ , we will need to write them in a spin-dimension balanced way. Let us give a concrete example. The tree diagram bound (9.4.14) has four spins on the left and four pairs of spins on the right. As such, it is not spin balanced. We can obtain a balanced version of this inequality by “site-splitting” each term where an Ising spin is repeated by using Lemma 9.3.6. The resulting bound is given in (9.5.15). Though they are more complicated, the diagrammatic bounds obtained via this procedure have the advantage of being spin-balanced. An alternative route²² would be to divide by $\langle \tau_0^2 \rangle_{\rho, \beta}$.

Below, we let $U_4^{\rho, \beta}$ be the corresponding Ursell’s four-point function for the field variable τ , and for $n \geq 1$,

$$B_n(\rho, \beta) := \sum_{x \in \Lambda_n} \langle \tau_0 \tau_x \rangle_{\rho, \beta}^2. \quad (9.8.1)$$

Also, we will use the definitions of $\beta_c(\rho)$ and $L(\rho, \beta)$ that were introduced in Section 9.3.

We will prove the following result, which in particular covers the case of the φ^4 lattice models by Proposition 9.2.2.

Theorem 9.8.1. *Let $d = 4$. Assume that J satisfies (A1)–(A6). Let $\kappa > 0$. There exist $c, C > 0$ such that, for all ρ in the GS class satisfying $\beta_c(\rho) \geq \kappa$, for all $\beta \leq \beta_c(\rho)$, for all $x, y, z, t \in \mathbb{Z}^4$ at mutual distance at least L with $1 \leq L \leq L(\rho, \beta)$,*

$$\begin{aligned} & |U_4^{\rho, \beta}(x, y, z, t)| \\ & \leq C \left(\frac{B_0(\rho, \beta)}{B_L(\rho, \beta)} \right)^c \sum_{u \in \mathbb{Z}^4} \sum_{u', u'' \in \mathbb{Z}^4} \langle \tau_x \tau_u \rangle_{\rho, \beta} \beta J_{u, u'} \langle \tau_{u'} \tau_y \rangle_{\rho, \beta} \langle \tau_z \tau_u \rangle_{\rho, \beta} \beta J_{u, u''} \langle \tau_{u''} \tau_t \rangle_{\rho, \beta}. \end{aligned} \quad (9.8.2)$$

As for the Ising model, we can deduce from Theorem 9.1.13 and Proposition 9.4.6 the following triviality statement for measures in the GS class.

Corollary 9.8.2. *Let $d = 4$. Assume that J satisfies (A1)–(A6). Let $\kappa > 0$. There exist $C, c, \gamma > 0$ such that, for any ρ in the GS class satisfying $\beta_c(\rho) \geq \kappa$, for all $\beta \leq \beta_c(\rho)$, $1 \leq L \leq L(\rho, \beta)$, $f \in \mathcal{C}_0(\mathbb{R}^d)$, and $z \in \mathbb{R}$,*

$$\begin{aligned} & \left| \langle \exp(z T_{f, L, \beta}(\tau)) \rangle_{\rho, \beta} - \exp \left(\frac{z^2}{2} \langle T_{f, L, \beta}(\tau)^2 \rangle_{\rho, \beta} \right) \right| \\ & \leq \exp \left(\frac{z^2}{2} \langle T_{|f|, L, \beta}(\tau)^2 \rangle_{\rho, \beta} \right) \frac{C \|f\|_\infty^4 r_f^\gamma z^4}{(\log L)^c}. \end{aligned}$$

²²The two methods are comparable through (9.3.20).

We will extend the results to the GS class using the following strategy.

Step 1 Fix a measure ρ_0 of the Ising-type in the GS class, i.e. a measure that falls into (i) of Definition 9.2.1. Prove that ρ_0 satisfies Theorem 9.1.13 with constants $c, C > 0$ which only depend on $\beta_c(\rho_0)$. To do so, we prove an analogous version of the intersection clustering bound that was derived in Proposition 9.6.2. We proceed as above by first proving that big jumps occur with small probability, and then by obtaining a version of Proposition 9.6.19, together with a mixing statement as in Theorem 9.6.20.

Step 2 Take any ρ in the GS class that is obtained as a weak limit of measures $(\rho_k)_{k \geq 1}$ of the type (i) in Definition 9.2.1. Prove that the statement available for each $k \geq 1$ passes to the limit $k \rightarrow \infty$. This requires a control of $(L(\rho_k, \beta))_{k \geq 1}$ and $(\beta_c(\rho_k))_{k \geq 1}$, together with “infinite volume” version of the GS approximation in the sense that: for all $\beta < \beta_c(\rho)$, for all $x, y \in \mathbb{Z}^d$,

$$\lim_{k \rightarrow \infty} \langle \tau_x \tau_y \rangle_{\rho_k, \beta} = \langle \tau_x \tau_y \rangle_{\rho, \beta}. \quad (9.8.3)$$

In Section 9.8.1, we prove Theorem 9.1.13 for measures of the Ising type in the GS class modulo an intermediate result (Proposition 9.8.3) that is similar to Proposition 9.6.2. In Section 9.8.2, we implement **Step 2** of the above strategy and extend the result to all measures in the GS class. In Section 9.8.3, we prove Proposition 9.8.3. Finally, in Section 9.8.4 we explain how this strategy can also be used to extend the results of Section 9.7.

9.8.1 Improved tree diagram bound for measures of the Ising type in the GS class

Fix ρ in the GS class of the Ising-type, and $\beta < \beta_c(\rho)$. The measure $\langle \cdot \rangle_{\rho, \beta}$ can be represented as an Ising measure on $\mathbb{Z}^d \times K_N$ that we still denote by $\langle \cdot \rangle_{\rho, \beta}$. In that case, we can identify τ_x with averages of the form

$$\sum_{i=1}^N Q_i \sigma_{(x, i)}, \quad (9.8.4)$$

where $Q_i \geq 0$ for $1 \leq i \leq N$. For $x \in \mathbb{Z}^d$, we will denote $\mathcal{B}_x := \{(x, i), 1 \leq i \leq N\}$. This point of view allows us to use the random current representation. We introduce a measure $\mathbb{P}_{\Lambda, \rho, \beta}^{xy}$ on $\Omega_{\Lambda \times K_N}$ which we define in the following two steps procedure:

- first, we sample two integers $1 \leq i, j \leq N$ with probability

$$\frac{Q_i Q_j \langle \sigma_{(x, i)} \sigma_{(y, j)} \rangle_{\rho, \beta}}{\langle \tau_x \tau_y \rangle_{\Lambda, \rho, \beta}}, \quad (9.8.5)$$

- then, sample a current according to the “usual” current measure $\mathbf{P}_{\rho, \beta}^{\{(x, i), (y, j)\}}$ introduced in Section 9.4.

It is also possible to define the infinite volume version of the above measure that we will denote $\mathbb{P}_{\rho, \beta}^{xy}$. Samples of $\mathbb{P}_{\rho, \beta}^{xy}$ are random currents with random sources in $\mathcal{B}_x$ and $\mathcal{B}_y$. The interest of this measure lies in the fact that it is better suited for the derivation of bounds

on connection probabilities in terms of the correlation functions of the field variable τ . These bounds can be directly imported from [9] and are recalled in Appendix 9.C.

Define a sequence $\mathcal{L}$ similarly as in (9.6.29), using this time $B_\ell(\rho, \beta)$. Call $\mathcal{I}_u$ the set of vertices in $v \in \mathbb{Z}^d$ such that $\mathcal{B}_v$ is connected in $\mathbf{n}_1 + \mathbf{n}_3$ and $\mathbf{n}_2 + \mathbf{n}_4$ to $\mathcal{B}_u$. With this definition, we now consider *coarse intersections* instead of proper intersections. This point of view is better for the analysis that follows.

For models in the GS class of the Ising type, the clustering bound takes the following form.

Proposition 9.8.3 (Clustering bound for models in the GS class). *Let $d = 4$. Assume that J satisfies (A1)-(A6). Let $\kappa > 0$. For D large enough, there exists $\delta = \delta(D, \kappa) > 0$ such that for every ρ in the GS class of the Ising-type with $\beta_c(\rho) \geq \kappa$, for every $\beta < \beta_c(\rho)$, every $K > 3$ with $\ell_{K+1} \leq L(\rho, \beta)$, every $x, y, z, t \in \mathbb{Z}^d$ with mutual distance between x, y, z, t larger than $2\ell_K$, every $u, u', u'' \in \mathbb{Z}^d$ with u', u'' J -neighbours²³ of u satisfying $|u - x|, |u - z| \geq \ell_K$,*

$$\mathbb{P}_{\rho, \beta}^{ux, uz, u'y, u''t}[\mathbf{M}_u(\mathcal{I}'_u; \mathcal{L}, K) < \delta K] \leq 2^{-\delta K}, \quad (9.8.6)$$

where $\mathcal{I}'_u$ is the set of vertices in $v \in \mathbb{Z}^d$ such that $\mathcal{B}_v$ is connected in²⁴ $\mathbf{n}_1 + \mathbf{n}_3 + \delta_{(\partial \mathbf{n}_1 \cap \mathcal{B}_u, \partial \mathbf{n}_3 \cap \mathcal{B}_{u'})}$ and $\mathbf{n}_2 + \mathbf{n}_4 + \delta_{(\partial \mathbf{n}_2 \cap \mathcal{B}_u, \partial \mathbf{n}_4 \cap \mathcal{B}_{u''})}$ to $\mathcal{B}_u$.

Remark 9.8.4. The reason why we have $\mathcal{I}'_u$ instead of $\mathcal{I}_u$ in the bound above is technical. This is a consequence of the form the switching lemma takes in that context, as seen in [9, Lemma A.7].

Remark 9.8.5. In fact, the same result holds with $L^{(\alpha)}(\rho, \beta)$, $\alpha \in (0, 1)$, instead of $L(\rho, \beta)$ (with a change of parameter²⁵ δ). This will be useful below.

We postpone the derivation of this bound to the next section and now explain how one can derive Theorem 9.1.13 for measures ρ of the Ising type.

Proof of Theorem 9.1.13 for a measure ρ of the Ising type in the GS class. As for the case of the Ising model, one can show (by summing (9.4.14) over points in $\mathcal{B}_x, \mathcal{B}_y, \mathcal{B}_z$ and $\mathcal{B}_t$) that

$$|U_4^{\rho, \beta}(x, y, z, t)| \leq 2 \langle \tau_x \tau_y \rangle_{\rho, \beta} \langle \tau_z \tau_t \rangle_{\rho, \beta} \mathbb{P}_{\rho, \beta}^{xy, zt, \emptyset, \emptyset}[\mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_3}(\partial \mathbf{n}_1) \cap \mathbf{C}_{\mathbf{n}_2 + \mathbf{n}_4}(\partial \mathbf{n}_2) \neq \emptyset], \quad (9.8.7)$$

where $\mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_3}(\partial \mathbf{n}_1)$ and $\mathbf{C}_{\mathbf{n}_2 + \mathbf{n}_4}(\partial \mathbf{n}_2)$ refer to the clusters in $\mathbf{n}_1 + \mathbf{n}_2$ and $\mathbf{n}_3 + \mathbf{n}_4$ of the (random) sources $\partial \mathbf{n}_1$ and $\partial \mathbf{n}_2$ respectively. As above, we may find $c_0 > 0$ such that if x, y, z, t are at mutual distance at least L , there exists $K = K(L)$ such that $K \geq c_0 \log(B_L(\rho, \beta)/B_0(\rho, \beta))$ and $2\ell_K \leq L$. The rest of the proof is conceptually identical to what was done before, except that now we look at coarse intersections. Let D be large enough so that Proposition 9.8.3 holds for some $\delta = \delta(D, \kappa) > 0$. Using Markov's inequality together with (9.C.1),

$$\begin{aligned} \langle \tau_x \tau_y \rangle_{\rho, \beta} \langle \tau_z \tau_t \rangle_{\rho, \beta} \mathbb{P}_{\rho, \beta}^{xy, zt, \emptyset, \emptyset}[|\mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_3}(\partial \mathbf{n}_1) \cap \mathbf{C}_{\mathbf{n}_2 + \mathbf{n}_4}(\partial \mathbf{n}_2)| \geq 2^{\delta K/5}] \\ \leq 2^{-\delta K/5} \sum_{u, u', u'' \in \mathbb{Z}^d} \langle \tau_x \tau_u \rangle_{\rho, \beta} \beta J_{u, u'} \langle \tau_{u'} \tau_y \rangle_{\rho, \beta} \langle \tau_z \tau_u \rangle_{\rho, \beta} \beta J_{u, u''} \langle \tau_{u''} \tau_t \rangle_{\rho, \beta}. \end{aligned}$$

²³In the sense that $J_{u, u'}, J_{u, u''} > 0$.

²⁴Here, for $u, v \in \mathbb{Z}^d$ such that $J_{u, v} > 0$, $\delta_{(u, v)}$ denotes the current identically equal to 0 except on the pair $\{u, v\}$ where it is equal to 1.

²⁵One way to see this is to observe that the constant c in Proposition 9.3.23 depends on α .

Now, using Lemma 9.6.1, write

$$\begin{aligned} & \langle \tau_x \tau_y \rangle_{\rho, \beta} \langle \tau_z \tau_t \rangle_{\rho, \beta} \mathbb{P}_{\rho, \beta}^{xy, zt, \emptyset, \emptyset} [0 < |\mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_3}(\partial \mathbf{n}_1) \cap \mathbf{C}_{\mathbf{n}_2 + \mathbf{n}_4}(\partial \mathbf{n}_2)| < 2^{\delta K/5}] \\ & \leq \sum_{u \in \mathbb{Z}^d} \mathbb{P}_{\rho, \beta}^{xy, zt, \emptyset, \emptyset} [\partial \mathbf{n}_1 \xrightarrow{\mathbf{n}_1 + \mathbf{n}_3} \mathcal{B}_u, \partial \mathbf{n}_2 \xrightarrow{\mathbf{n}_2 + \mathbf{n}_4} \mathcal{B}_u, \mathbf{M}_u(\mathcal{I}_u; \mathcal{L}, K) < \delta K]. \end{aligned}$$

Notice that by hypothesis u must satisfy $|u - x| \vee |u - y| \geq \ell_K$ and $|u - z| \vee |u - t| \geq \ell_K$. Hence, the upper bound above can be rewritten as the sum of four terms which represent each case. We assume without loss of generality that $|u - x| \geq \ell_K$ and $|u - z| \geq \ell_K$.

Using a proper formulation of the switching lemma in that context [9, Lemma A.7], we get

$$\begin{aligned} & \mathbb{P}_{\rho, \beta}^{xy, zt, \emptyset, \emptyset} [\partial \mathbf{n}_1 \xrightarrow{\mathbf{n}_1 + \mathbf{n}_3} \mathcal{B}_u, \partial \mathbf{n}_2 \xrightarrow{\mathbf{n}_2 + \mathbf{n}_4} \mathcal{B}_u, \mathbf{M}_u(\mathcal{I}_u; \mathcal{L}, K) < \delta K] \\ & \leq \sum_{u', u'' \neq u} \langle \tau_x \tau_u \rangle_{\rho, \beta} J_{u, u'} \langle \tau_{u'} \tau_y \rangle_{\rho, \beta} \langle \tau_z \tau_u \rangle_{\rho, \beta} J_{u, u''} \langle \tau_{u''} \tau_t \rangle_{\rho, \beta} \mathbb{P}_{\rho, \beta}^{ux, uz, u'y, u''t} [\mathbf{M}_u(\mathcal{I}'_u; \mathcal{L}, K) < \delta K]. \end{aligned}$$

We then conclude using Proposition 9.8.3. We obtained the existence of $c, C > 0$ such that: for all ρ in the GS class of the Ising type such that $\beta_c(\rho) \geq \kappa$, for all $\beta < \beta_c(\rho)$, for all $x, y, z, t \in \mathbb{Z}^d$ at mutual distance at least L with $L \leq L(\rho, \beta)$, (9.8.2) holds. We can then extend the result to $\beta_c(\rho)$ by a continuity argument²⁶ together with the observation that $L(\rho, \beta) \rightarrow \infty$ as $\beta \rightarrow \beta_c(\rho)$. □

Remark 9.8.6. Using Remark 9.8.5 we see that we also obtained the same result replacing $L(\rho, \beta)$ by $L^{(\alpha)}(\rho, \beta)$, for some $\alpha \in (0, 1)$. Note that this affects the constant c, C in Theorem 9.1.13.

9.8.2 Tree diagram bound for weak limits of Ising type measures

The goal of this section is to extend the improved tree diagram bound to the entire GS class of measures. The result will be a consequence of the following proposition.

Proposition 9.8.7. *Let ρ be a measure in the GS class. There exists a sequence of measures $(\rho_k)_{k \geq 1}$ of the Ising type in the GS class such that:*

- (1) $(\rho_k)_{k \geq 1}$ converges weakly to ρ ,
- (2) $\liminf \beta_c(\rho_k) \geq \beta_c(\rho)$,
- (3) for every $\beta < \beta_c(\rho)$, for every $x, y, z, t \in \mathbb{Z}^d$,

$$\lim_{k \rightarrow \infty} \langle \tau_x \tau_y \rangle_{\rho_k, \beta} = \langle \tau_x \tau_y \rangle_{\rho, \beta}, \quad \lim_{k \rightarrow \infty} \langle \tau_x \tau_y \tau_z \tau_t \rangle_{\rho_k, \beta} = \langle \tau_x \tau_y \tau_z \tau_t \rangle_{\rho, \beta}, \quad (9.8.8)$$

- (4) for every $\beta > 0$, $\liminf L^{(1/4)}(\rho_k, \beta) \geq L(\rho, \beta)$.

²⁶Here we use the left-continuity of the two-point and four-point correlation functions together with (IRB) which uniformly bounds the two-point function for $\beta \leq \beta_c(\rho)$.

Assuming this result, and knowing Theorem 9.1.13 for Ising type measures, we can easily extend Theorem 9.1.13 to all models in the GS class.

Proof of Theorem 9.1.13. Fix ρ in the GS class which is a weak limit of Ising type measure in the GS class of measures, and which satisfies $\beta_c(\rho) \geq \kappa$. Let also $(\rho_k)_{k \geq 1}$ be given by Proposition 9.8.7. By the property (2) of the same proposition, there exists $k_0 \geq 0$ such that for $k \geq k_0$, $\beta_c(\rho_k) \geq \kappa/2$. Since the tree diagram bound holds uniformly over Ising type measures ρ' satisfying $\beta_c(\rho') \geq \kappa/2$, and using Remark 9.8.6, there exist $C, c > 0$ such that for all $k \geq k_0$, for all $\beta \leq \beta_c(\rho_k)$, for all x, y, z, t at mutual distance at least L with $1 \leq L \leq L^{(1/4)}(\rho_k, \beta)$,

$$|U_4^{\rho_k, \beta}(x, y, z, t)| \leq C \left(\frac{B_0(\rho_k, \beta)}{B_L(\rho_k, \beta)} \right)^c \sum_{u, u', u'' \in \mathbb{Z}^d} \langle \tau_x \tau_u \rangle_{\rho_k, \beta} J_{u, u'} \langle \tau_{u'} \tau_y \rangle_{\rho_k, \beta} \langle \tau_z \tau_u \rangle_{\rho_k, \beta} J_{u, u''} \langle \tau_{u''} \tau_t \rangle_{\rho_k, \beta}. \quad (9.8.9)$$

Fix $\beta < \beta_c(\rho)$ and $1 \leq L \leq L(\rho, \beta)$. By properties (2) and (4) of Proposition 9.8.7, there exists $k_1 \geq k_0$ such that $\beta \leq \beta_c(\rho_k)$ and $L \leq L^{(1/4)}(\rho_k, \beta)$ for $k \geq k_1$. As a result, (9.8.9) holds with β and L for $k \geq k_1$. We now use (3) to pass the inequality to the limit. Using (IRB), we know that there exists $C = C(d) > 0$ such that for all $u, v \in \mathbb{Z}^d$, for $k \geq k_1$

$$\langle \tau_u \tau_v \rangle_{\rho_k, \beta} \leq \frac{C}{\beta_c(\rho) |J| |u - v|^{d-2}}. \quad (9.8.10)$$

This justifies passing to the limit in (9.8.9) and yields the result. We extend the result to $\beta = \beta_c(\rho)$ by a continuity argument as above. $\square$

We now prove Proposition 9.8.7. We split the statement into lemmas.

Lemma 9.8.8. *Assume that $(\rho_k)_{k \geq 1}$ converges weakly to ρ . Then,*

$$\liminf \beta_c(\rho_k) \geq \beta_c(\rho), \quad (9.8.11)$$

Proof. Assume $\beta > \liminf \beta_c(\rho_k)$. If $S \subset \mathbb{Z}^d$ is a finite set containing 0,

$$\lim_{k \rightarrow \infty} \varphi_{\rho_k, \beta}(S) = \varphi_{\rho, \beta}(S). \quad (9.8.12)$$

Since $\beta > \liminf \beta_c(\rho_k)$, one has that for k large enough $\beta \geq \beta_c(\rho_k)$ and hence²⁷ $\varphi_{\rho_k, \beta}(S) \geq 1$ (using the same argument as in Remark 9.3.22) so that

$$\varphi_{\rho, \beta}(S) \geq 1. \quad (9.8.13)$$

Since this holds for any finite set S containing 0, one has $\beta \geq \beta_c(\rho)$. $\square$

Lemma 9.8.9. *Assume that $(\rho_k)_{k \geq 1}$ converges weakly to ρ . Then, for all $\beta > 0$, for all $\alpha \in (0, 1/2)$*

$$L(\rho, \beta) \leq \liminf L^{(\alpha)}(\rho_k, \beta). \quad (9.8.14)$$

²⁷Otherwise the susceptibility would be finite at β .

Proof. If $\liminf L^{(\alpha)}(\rho_k, \beta) = \infty$ then the upper bound is trivial.

Otherwise, fix $n > (2d) \liminf L^{(\alpha)}(\rho_k, \beta)$. There exists $S \subset \mathbb{Z}^d$ finite and containing 0 with $\text{rad}(S) \leq 2n$ such that for all k sufficiently large,

$$\varphi_{\rho_k, \beta}(S) < \alpha. \quad (9.8.15)$$

Using 9.8.12, we get that $\varphi_{\rho, \beta}(S) \leq \alpha < 1/2$ so that $n \geq (2d)L(\rho, \beta)$. $\square$

Lemma 9.8.10. *Let $d = 4$. Assume that $(\rho_k)_{k \geq 1}$ converges weakly to ρ . Let $\beta < \beta_c(\rho)$. For every $x, y, z, t \in \mathbb{Z}^4$,*

$$\lim_{k \rightarrow \infty} \langle \tau_x \tau_y \rangle_{\rho_k, \beta} = \langle \tau_x \tau_y \rangle_{\rho, \beta}, \quad \lim_{k \rightarrow \infty} \langle \tau_x \tau_y \tau_z \tau_t \rangle_{\rho_k, \beta} = \langle \tau_x \tau_y \tau_z \tau_t \rangle_{\rho, \beta}. \quad (9.8.16)$$

Proof. We only prove the first part of the statement, the second part follows by a similar argument. Let $x, y \in \mathbb{Z}^d$. It is sufficient to show that

$$\lim_{n \rightarrow \infty} \sup_{k \geq 1} (\langle \tau_x \tau_y \rangle_{\rho_k, \beta} - \langle \tau_x \tau_y \rangle_{\Lambda_n, \rho_k, \beta}) = 0. \quad (9.8.17)$$

Fix n large enough such that $x, y \in \Lambda_n$. Recall that ρ_k is defined by averages on K_{N_k} for some $N_k \geq 1$. Using the switching lemma,

$$\begin{aligned} \langle \tau_x \tau_y \rangle_{\rho_k, \beta} - \langle \tau_x \tau_y \rangle_{\Lambda_n, \rho_k, \beta} &= \langle \tau_x \tau_y \rangle_{\rho_k, \beta} \mathbb{P}_{\mathbb{Z}^d, \Lambda_n, \rho_k, \beta}^{xy, \emptyset} \left[\partial \mathbf{n}_1 \cap \mathcal{B}_x \xleftrightarrow{(\mathbf{n}_1 + \mathbf{n}_2)|_{\Lambda_n \times K_{N_k}}} \partial \mathbf{n}_1 \cap \mathcal{B}_y \right] \\ &\leq \langle \tau_x \tau_y \rangle_{\rho, \beta} \mathbb{P}_{\rho_k, \beta}^{xy} \left[\partial \mathbf{n} \cap \mathcal{B}_x \xleftrightarrow{\mathbf{n}|_{\Lambda_n \times K_{N_k}}} \partial \mathbf{n} \cap \mathcal{B}_y \right]. \end{aligned}$$

Let $\ell := |x| + |y|$ and introduce the event $\text{ZZGS}_k(x, y; \ell, n, \infty)$ that the backbone of $\mathbf{n}$ goes from $\partial \mathbf{n} \cap \mathcal{B}_x$ to $\partial \mathbf{n} \cap \mathcal{B}_y$ by exiting $\Lambda_n \times K_{N_k}$. As explained below, it is possible to extend the proof of Corollary 9.6.12 to this setup to get that there exist $\eta, C_1 > 0$ such that for all n large enough, for all $k \geq 1$,

$$\mathbb{P}_{\rho_k, \beta}^{xy}[\text{ZZGS}_k(x, y; \ell, n, \infty)] \leq \frac{C_1}{n^\eta}. \quad (9.8.18)$$

The observation that $\left\{ \partial \mathbf{n} \cap \mathcal{B}_x \xleftrightarrow{\mathbf{n}|_{\Lambda_n \times K_{N_k}}} \partial \mathbf{n} \cap \mathcal{B}_y \right\} \subset \text{ZZGS}_k(x, y; \ell, n, \infty)$ gives (9.8.17). $\square$

Proof of Proposition 9.8.7. If ρ falls into (i) of Definition 9.2.1 the statement is trivial. Otherwise, fix any sequence $(\rho_k)_{k \geq 1}$ of Ising type measures in the GS class that converges weakly to ρ . Using the three above lemmas we verify that $(\rho_k)_{k \geq 1}$ satisfies all the desired properties. $\square$

9.8.3 Intersection clustering bound for models in the GS class

We now turn to the proof of Proposition 9.8.3. The proof follows the exact same lines as for the Ising case and is reduced to the adaptation of the results of Section 9.6.2, together with an extension of the intersection property of Lemma 9.6.19, and the mixing statement of Theorem 9.6.20. We fix $d = 4$ and an interaction J which satisfies (A1)–(A6).

We start by excluding the existence of “big jumps” in our context. As it turns out, the results of Section 9.6.2 directly follow from the following adaptation of Lemma 9.6.3.

Lemma 9.8.11. *Let $\beta > 0$. Let ρ be of the Ising type in the GS class. For $x, y, u, v \in \mathbb{Z}^d$,*

$$\begin{aligned} \mathbb{P}_{\rho, \beta}^{xy, \emptyset}[\exists i, j, \mathbf{n}_{(u, i), (v, j)} \geq 1] \\ \leq \beta J_{u, v} \left(2\langle \tau_u \tau_v \rangle_{\rho, \beta} + \frac{\langle \tau_x \tau_u \rangle_{\rho, \beta} \langle \tau_v \tau_y \rangle_{\rho, \beta}}{\langle \tau_x \tau_y \rangle_{\rho, \beta}} + \frac{\langle \tau_x \tau_v \rangle_{\rho, \beta} \langle \tau_u \tau_y \rangle_{\rho, \beta}}{\langle \tau_x \tau_y \rangle_{\rho, \beta}} \right). \end{aligned}$$

At this stage of the proof, the arguments essentially build on what was done in the Ising case together with a proper adaptation of the proofs, as already explained in [9]. Below we explain the main changes in the proofs and refer to [9] for more details. We first state the intersection property for Ising-type models in the GS class.

Lemma 9.8.12 (Intersection property for models in the GS class). *Let $\kappa > 0$. For $D = D(\kappa) > 0$ large enough, there exists $\delta = \delta(\kappa) > 0$ such that for every ρ of the Ising type in the GS class satisfying $\beta_c(\rho) \geq \kappa$, every $\beta \leq \beta_c(\rho)$, every $k \geq 2$, and every $y \notin \Lambda_{\ell_{k+2}}$ in a regular scale with $1 \leq |y| \leq L(\rho, \beta)$,*

$$\mathbb{P}_{\rho, \beta}^{0y, 0y, \emptyset, \emptyset}[(\mathbf{n}_1 + \mathbf{n}_3, \mathbf{n}_2 + \mathbf{n}_4) \in I_k(0)] \geq \delta, \quad (9.8.19)$$

where $I_k(0)$ is defined similarly to the intersection event of Definition 9.6.18, except that we now ask that the clusters of $\mathbf{n}_1 + \mathbf{n}_3$ and $\mathbf{n}_2 + \mathbf{n}_4$ coarse-intersect in the sense that there exists $v \in \text{Ann}(\ell_k, \ell_{k+1})$ such that $\mathcal{B}_v$ is connected to $\mathcal{B}_0$ in $\mathbf{n}_1 + \mathbf{n}_3$ and $\mathbf{n}_2 + \mathbf{n}_4$.

Proof. We keep the notations introduced in the proof of Lemma 9.6.19. Define,

$$\mathcal{M} := \sum_{v \in \text{Ann}(m, M)} \sum_{i, i'} Q_i^2 \mathbb{1}[\partial \mathbf{n}_1 \xrightarrow{\mathbf{n}_1 + \mathbf{n}_3} (v, i)] Q_{i'}^2 \mathbb{1}[\partial \mathbf{n}_2 \xrightarrow{\mathbf{n}_2 + \mathbf{n}_4} (v, i')]. \quad (9.8.20)$$

The extra $Q_i^2, Q_{i'}^2$ terms allow to rewrite moments of $\mathcal{M}$ in terms of correlation functions of the field variables $(\tau_z)_{z \in \mathbb{Z}^d}$. Using a similar computation as for the case of the Ising model, together with the results of Proposition 9.C.1, we get $c_1, C_1 > 0$ such that

$$\mathbb{E}_{\rho, \beta}^{0y, 0'y, \emptyset, \emptyset}[|\mathcal{M}|] \geq c_1(B_M(\rho, \beta) - B_{m-1}(\rho, \beta)), \quad (9.8.21)$$

$$\mathbb{E}_{\rho, \beta}^{0y, 0'y, \emptyset, \emptyset}[|\mathcal{M}|^2] \leq C_1 B_{\ell_{k+1}}(\rho, \beta)^2. \quad (9.8.22)$$

Similarly as above, we deduce, for some $c_2 > 0$,

$$\mathbb{P}_{\rho, \beta}^{0y, 0y, \emptyset, \emptyset}[\mathcal{M} \neq \emptyset] \geq c_2. \quad (9.8.23)$$

The second part of the proof consists in making the intersection event local. We proceed exactly as we did for the Ising case by first excluding the possibility of jumping any of the intermediate scales, and by then repeating the analysis that lead to the bounds on the events $\mathcal{F}_1, \dots, \mathcal{F}_5$. At this stage one needs to be careful in the use of the infrared bound and it is required to have bounds involving $\beta|J|$. This will ensure that the bound on the intersection probability we end up with does not depend on ρ . $\square$

Theorem 9.8.13 (Mixing property for models in the GS class). *Let $d = 4$. Let $\kappa > 0$ and $s \geq 1$. There exist $\gamma, C > 0$, such that for every ρ of the Ising-type in the GS class satisfying $\beta_c(\rho) \geq \kappa$, for every $1 \leq t \leq s$, every $\beta \leq \beta_c(\rho)$, every $n^\gamma \leq N \leq L(\rho, \beta)$, every $x_i \in \Lambda_n$ and*

$y_i \notin \Lambda_N$ ($i \leq t$), and every events E and F depending on the restriction of $(\mathbf{n}_1, \dots, \mathbf{n}_s)$ to edges with endpoints within Λ_n and outside Λ_N respectively,

$$\left| \mathbb{P}_{\rho, \beta}^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}[E \cap F] - \mathbb{P}_{\rho, \beta}^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}[E] \mathbb{P}_{\rho, \beta}^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}[F] \right| \leq C \left(\log \frac{N}{n} \right)^{-1/2}. \quad (9.8.24)$$

Furthermore, for every $x'_1, \dots, x'_t \in \Lambda_n$ and $y'_1, \dots, y'_t \notin \Lambda_N$, we have that

$$\left| \mathbb{P}_{\rho, \beta}^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}[E] - \mathbb{P}_{\rho, \beta}^{x'_1 y'_1, \dots, x'_t y'_t, \emptyset, \dots, \emptyset}[E] \right| \leq C \left(\log \frac{N}{n} \right)^{-1/2}, \quad (9.8.25)$$

$$\left| \mathbb{P}_{\rho, \beta}^{x_1 y_1, \dots, x_t y_t, \emptyset, \dots, \emptyset}[F] - \mathbb{P}_{\rho, \beta}^{x'_1 y'_1, \dots, x'_t y'_t, \emptyset, \dots, \emptyset}[F] \right| \leq C \left(\log \frac{N}{n} \right)^{-1/2}. \quad (9.8.26)$$

Proof. The main modification in the proof comes in the definition of $\mathbf{U}_i$:

$$\mathbf{U}_i := \frac{1}{|\mathcal{K}|} \sum_{k \in \mathcal{K}} \frac{1}{A_{x_i, y_i}(2^k)} \sum_{u \in \mathbb{A}_{y_i}(2^k)} \sum_{j=1}^N Q_j^2 \mathbb{1}[(u, j) \xleftrightarrow{\mathbf{n}_i + \mathbf{n}'_i} \partial \mathbf{n}_i], \quad (9.8.27)$$

where

$$a_{x, y}(u) := \frac{\langle \tau_x \tau_u \rangle_{\rho, \beta} \langle \tau_u \tau_y \rangle_{\rho, \beta}}{\langle \tau_x \tau_y \rangle_{\rho, \beta}}, \quad A_{x, y}(k) := \sum_{u \in \mathbb{A}_{y_i}(2^k)} a_{x, y}(u). \quad (9.8.28)$$

Note that, as above, the extra term Q_j^2 allows one to express the moments in terms of the field variables $(\tau_z)_{z \in \mathbb{Z}^d}$. Also, in the derivation of an analogue of Lemma 9.6.23, one will have to be careful to use infrared bounds involving $\beta|J|$. $\square$

We are now in a position to prove Proposition 9.8.3.

Proof of Proposition 9.8.3. The proof follows the exact same lines as for the Ising case, except that we need to slightly take care of the monotonicity property we want to use. We keep the notations introduced in the proof of Proposition 9.6.2. Let $\delta > 0$ to be fixed later. Let $S \in \mathcal{S}_K^{(\delta)}$. Let $\mathfrak{B}_S$ (resp. $\mathfrak{B}'_S$) be the event that the clusters of $\mathcal{B}_u$ in $\mathbf{n}_1 + \mathbf{n}_3$ and $\mathbf{n}_2 + \mathbf{n}_4$ (resp. $\mathbf{n}_1 + \mathbf{n}_3 + \delta(\partial \mathbf{n}_1 \cap \mathcal{B}_u, \partial \mathbf{n}_3 \cap \mathcal{B}_{u'})$ and $\mathbf{n}_2 + \mathbf{n}_4 + \delta(\partial \mathbf{n}_2 \cap \mathcal{B}_u, \partial \mathbf{n}_4 \cap \mathcal{B}_{u''})$), do not coarse intersect in any of the annuli $\text{Ann}(\ell_i, \ell_{i+1})$ for $i \in S$. Then, using an adaptation of the monotonicity argument of Proposition 9.6.24 to our context (see Proposition 9.C.2),

$$\begin{aligned} \mathbb{P}_{\rho, \beta}^{ux, uz, u'y, u''t}[\mathbf{M}_u(\mathcal{I}'_u; \mathcal{L}, K) < \delta K] &\leq \sum_{\substack{S \in \mathcal{S}_K^{(\delta)} \\ |S| \geq (1/2 - 2\delta)K}} \mathbb{P}_{\rho, \beta}^{ux, uz, u'y, u''t}[\mathfrak{B}'_S] \\ &\leq \sum_{\substack{S \in \mathcal{S}_K^{(\delta)} \\ |S| \geq (1/2 - 2\delta)K}} \mathbb{P}_{\rho, \beta}^{ux, uz, \emptyset, \emptyset}[\mathfrak{B}_S]. \end{aligned} \quad (9.8.29)$$

The rest of the proof is identical to what was done in Section 9.6. $\square$

9.8.4 Extension of the results of Section 9.7

We now briefly explain how to extend to results of Section 9.7 to models in the GS class. The strategy is very similar to what was done above so we only present the main modifications in the proof. We begin by discussing the modifications involved in the proofs of the results obtained in Sections 9.7.1 and 9.7.2

Let $d \geq 1$. We fix an interaction J on $\mathbb{Z}^d$ satisfying (A1)–(A5) and (Assumption $_{\alpha}$) with $d - 2(\alpha \wedge 2) \geq 0$. In that setup, we get that for any ρ in the GS class: if $\beta \leq \beta_c(\rho)$ and $x \in \mathbb{Z}^d \setminus \{0\}$,

$$\langle \tau_0 \tau_x \rangle_{\rho, \beta} \leq \frac{C}{\beta_c(\rho) |x|^{d-\alpha \wedge 2} (\log |x|)^{\delta_{2, \alpha}}}. \quad (9.8.30)$$

The first important observation is to notice that, although stated for the Ising model, the results of Section 9.7.1 extend *mutatis mutandis* to every single-site measure ρ in the GS class thanks to Proposition 9.3.25.

Similarly, we may extend the results of Section 9.7.2 to all measures ρ of the Ising type in the GS class by using Lemma 9.8.11 and (9.8.30). With these tools, it is possible to extend the results of Section 9.7 to measures of the Ising-type in the GS class by using the same strategy as in Section 9.8.3.

The extension to all measures in the GS class uses again the approximation step of Section 9.8.2. The only non-trivial modification concerns Proposition 9.8.7, and more precisely Lemma 9.8.10. We will prove the following result.

Lemma 9.8.14. *Let $1 \leq d \leq 3$. Assume that J satisfies (A1)–(A5) and (Assumption $_{\alpha}$) with $d - 2(\alpha \wedge 2) \geq 0$. Let ρ be a measure in the GS class, and let $(\rho_k)_{k \geq 1}$ be a sequence of measures of the Ising type which converges weakly to ρ . Let $\beta < \beta_c(\rho)$. For every $x, y, z, t \in \mathbb{Z}^d$,*

$$\lim_{k \rightarrow \infty} \langle \tau_x \tau_y \rangle_{\rho_k, \beta} = \langle \tau_x \tau_y \rangle_{\rho, \beta}, \quad \lim_{k \rightarrow \infty} \langle \tau_x \tau_y \tau_z \tau_t \rangle_{\rho_k, \beta} = \langle \tau_x \tau_y \tau_z \tau_t \rangle_{\rho, \beta}. \quad (9.8.31)$$

Proof. Again we only prove the first part of the statement. We follow the proof of Lemma 9.8.10. As before, if $\ell := |x| + |y|$, the key observation is that $\left\{ \partial \mathbf{n} \cap \mathcal{B}_x \overset{\mathbf{n}|\Lambda_n \times K_{N_k}}{\leftrightarrow} \partial \mathbf{n} \cap \mathcal{B}_y \right\}$ is included in the event $\text{ZZGS}_k(x, y; \ell, n, \infty)$. However, as explained above, we can extend the results of Section 9.7.2, and in particular Corollary 9.7.10, to obtain a bound the probability of the latter event. This is enough to conclude. $\square$

9.A Spectral representation of reflection positive Ising models

The aim of this appendix is to prove Theorem 9.3.11. We use the notations of Section 9.3. In what follows ρ is a measure in the GS class. We assume that J satisfies (A1)–(A5). We will make good use of the spectral theorem (see [128]) which will be applied to diagonalize the shift operator T given by,

$$T : x \in \mathbb{Z}^d \mapsto x + (1, 0, \dots, 0). \quad (9.A.1)$$

Before that, we introduce some notations and a proper Hilbert space.

Let $\beta > 0$. Let $\mathbf{e}_1 = (1, 0, \dots, 0)$. Let Σ be the hyperplane orthogonal to $\mathbf{e}_1$ passing through 0. Let Θ be the reflection through Σ . Notice that Σ cuts $\mathbb{Z}^d$ in two half-planes Λ_+

and Λ_- with $\Lambda_+ \cap \Lambda_- = \Sigma$. Let $\mathcal{A}_+$ be the algebra generated by local functions with support in Λ_+ . Reflection positivity with respect to Θ implies that for all $f \in \mathcal{A}_+$,

$$\langle \overline{\Theta(f)} f \rangle_{\rho, \beta} \geq 0. \quad (9.A.2)$$

We define a positive semi-definite bi-linear form on $\mathcal{A}_+$ by : for all $f, g \in \mathcal{A}_+$,

$$(f, g) := \langle \overline{\Theta(f)} g \rangle_{\rho, \beta}. \quad (9.A.3)$$

Quotienting $\mathcal{A}_+$ by the kernel of $(\cdot, \cdot)$ and completing the resulting space one obtains a Hilbert space $(\mathcal{H}, (\cdot, \cdot))$. We denote by $\|\cdot\|$ the norm on this Hilbert space and $\|\cdot\|^{\text{op}}$ the associated operator norm. The shift T in the $\mathbf{e}_1$ direction defines an operator on $\mathcal{H}$ whose properties are described in the next proposition whose proof can be found in [41, 203].

Proposition 9.A.1 (Properties of T). *The shift operator $T : \mathcal{H} \rightarrow \mathcal{H}$ has the following properties,*

- (i) T is self-adjoint,
- (ii) T is positive,
- (iii) T is bounded.

Proof. (i) Since $\langle \cdot \rangle_{\rho, \beta}$ is invariant under the action of T (by (A2)), for all $f, g \in \mathcal{H}$,

$$(Tf, g) = \langle \overline{\Theta(Tf)} g \rangle_{\rho, \beta} = \langle T^{-1}(\overline{\Theta(f)}) g \rangle_{\rho, \beta} = \langle \overline{\Theta(f)} Tg \rangle_{\rho, \beta} = (f, Tg), \quad (9.A.4)$$

so that T is self-adjoint.

(ii) For all $f \in \mathcal{H}$,

$$(f, Tf) = \langle \overline{\Theta(f)} Tf \rangle_{\rho, \beta} = \langle \overline{\Theta'(Tf)} Tf \rangle_{\rho, \beta} \geq 0, \quad (9.A.5)$$

where Θ' is the reflection through the hyperplane orthogonal to $\mathbf{e}_1$ passing through $\mathbf{e}_1$. We used reflection positivity to obtain the last inequality.

(iii) Iterating the Cauchy–Schwarz inequality as in [203], we get, for all $f \in \mathcal{H}$,

$$|(Tf, f)| \leq (f, f), \quad (9.A.6)$$

and thus, $\|T\|^{\text{op}} \leq 1$. □

In what follows we introduce many classical objects in the study of bounded self-adjoint operators in a Hilbert space. For all the definitions we refer to [128]. We are now in a position to apply the spectral theorem [128, Theorem 7.12].

Proposition 9.A.2. *There exists a unique projection valued measure μ^T such that*

$$T = \int_{\sigma(T)} \lambda d\mu^T(\lambda). \quad (9.A.7)$$

Remark 9.A.3. One has that $\sigma(T) \subset [0, 1]$.

We also state two propositions (which can be found in chapter 7 of [128]) that will allow us to make good use of the preceding proposition.

Proposition 9.A.4. *Let $f : \sigma(T) \rightarrow \mathbb{C}$ be a bounded measurable function. Then,*

$$f(T) = \int_{\sigma(T)} f(\lambda) d\mu^T(\lambda). \quad (9.A.8)$$

Proposition 9.A.5. *If $f : [0, 1] \rightarrow \mathbb{C}$ is a bounded measurable function, and $\psi \in \mathcal{H}$, there exists a (positive) real-valued measure μ_ψ such that*

$$\left(\psi, \left(\int_0^1 f(\lambda) d\mu^T(\lambda) \right) \psi \right) = \int_0^1 f d\mu_\psi. \quad (9.A.9)$$

Remark 9.A.6. The measure μ_ψ is given for $E \in \Omega$, by

$$\mu_\psi(E) = (\psi, \mu(E)\psi). \quad (9.A.10)$$

Recall that for $f, g \in \mathcal{H}$, the truncated correlation of f and g is given by

$$\langle f; g \rangle_{\rho, \beta} := \langle fg \rangle_{\rho, \beta} - \langle f \rangle_{\rho, \beta} \langle g \rangle_{\rho, \beta}. \quad (9.A.11)$$

Proposition 9.A.7 (Representation of truncated correlation functions). *For all $f \in \mathcal{H}$, and all $n \geq 0$, there exists $f_\perp \in \mathcal{H}$ such that,*

$$\langle \overline{\Theta(f)}; T^n f \rangle_{\rho, \beta} = (f_\perp, T^n f_\perp). \quad (9.A.12)$$

Proof. Let $\mathbf{1} \in \mathcal{H}$ be such that $T\mathbf{1} = \mathbf{1}$. By definition, for all $f \in \mathcal{H}$, $\langle f \rangle_{\rho, \beta} = (\mathbf{1}, f)$. Thus,

$$\langle \overline{\Theta(f)}; T^n f \rangle_{\rho, \beta} = (f, T^n f) - (\mathbf{1}, f)(\mathbf{1}, f). \quad (9.A.13)$$

Write $P_\perp$ the orthogonal projection on $\text{Vect}(\mathbf{1})^\perp$. Letting

$$f_\perp := P_\perp f = f - (\mathbf{1}, f)\mathbf{1}, \quad (9.A.14)$$

we find that

$$\langle \overline{\Theta(f)}; T^n f \rangle_{\rho, \beta} = (f_\perp, T^n f_\perp). \quad (9.A.15)$$

□

We are now in a position to prove the main result of this section.

Proof of Theorem 9.3.11. Let $\beta \leq \beta_c(\rho)$. Apply Proposition 9.A.5 to $f : x \in [0, 1] \mapsto x^n$ for $n \geq 0$, and $\psi = V_\perp$ where $V = \sum_{x_\perp \in \mathbb{Z}^{d-1}} v_{x_\perp} \tau_{(0, x_\perp)} \in \mathcal{H}$ to get

$$(V_\perp, T^n V_\perp) = \int_0^1 \lambda^n d\mu_{V_\perp}(\lambda). \quad (9.A.16)$$

Using Proposition 9.A.7, we obtain

$$\langle \overline{\Theta(V)} T^n V \rangle_{\rho, \beta} = \int_0^1 \lambda^n d\mu_{V_\perp}(\lambda). \quad (9.A.17)$$

Now, notice that $\langle \overline{\Theta(V)} T^n V \rangle_{\rho, \beta}$ is exactly the left-hand side of (9.3.30). Moreover, considering the push-forward of $\mu_{V_\perp}$ under the map $a \in [0, 1] \mapsto -\log a \in \mathbb{R}^+ \cup \{\infty\}$, that we denote $\mu_{v, \beta}$,

$$\int_0^1 \lambda^n d\mu_{V_\perp}(\lambda) = \int_0^\infty e^{-an} d\mu_{v, \beta}(a), \quad (9.A.18)$$

and the result follows for all $n \in \mathbb{Z}$ using that $\langle \cdot \rangle_{\rho, \beta}$ is invariant under T . □

Remark 9.A.8. Note that one may have

$$\mu_{v,\beta}(\{\infty\}) > 0, \quad (9.A.19)$$

which is exactly equivalent to the fact that $\xi(\rho, \beta) = \infty$.

We now present the proof of the monotonicity property of the two-point function's Fourier transform.

Proof of Proposition 9.3.16. First, notice that

$$\widehat{S}_{\rho,\beta}^{(\text{mod})}(p) = 2 \sum_{\substack{x \in \mathbb{Z}^d \\ x_1 + x_2 = 0[2]}} e^{ip \cdot x} S_{\rho,\beta}(x). \quad (9.A.20)$$

We follow the proof of Theorem 9.3.11 and keep the same notations. This time we introduce the operator $T' : x \mapsto x + (1, 1, 0, \dots)$. Let R be the reflection with respect to the hyperplane Σ' orthogonal to $\mathbf{e}_1 + \mathbf{e}_2$ passing through 0. Σ' cuts $\mathbb{Z}^d$ in two half-planes Λ'_+ and Λ'_- with $\Lambda'_+ \cap \Lambda'_- = \Sigma'$. Let $\mathcal{A}'_+$ be the algebra generated by local functions with support in Λ'_+ . Reflection positivity with respect to R implies that for all $f \in \mathcal{A}'_+$,

$$\langle \overline{R(f)} f \rangle_{\rho,\beta} \geq 0. \quad (9.A.21)$$

We define a positive semi-definite bilinear form on $\mathcal{A}'_+$ by : for all $f, g \in \mathcal{A}'_+$,

$$(f, g) := \langle \overline{R(f)} g \rangle_{\rho,\beta}. \quad (9.A.22)$$

Quotienting $\mathcal{A}'_+$ by the kernel of $(\cdot, \cdot)$ and completing the obtained space one obtains a Hilbert space $(\mathcal{H}', (\cdot, \cdot))$. Then, T' can be seen as an operator of $\mathcal{H}'$. Using the same arguments as in Proposition 9.A.1, we also have that T' is a self-adjoint, bounded and positive operator of $\mathcal{H}'$. Just as in Theorem 9.3.11, we obtain that for all $v : \mathbb{Z}^{d-1} \rightarrow \mathbb{C}$ in $\ell^2(\mathbb{Z}^{d-1})$, there exists a positive measure $\mu'_{v,\beta}$ such that, for all $n \in \mathbb{Z}$,

$$\sum_{(e, x_b), (e', y_b) \in \mathbb{Z}^{d-1}} v_{(e, x_b)} \overline{v_{(e', y_b)}} S_{\rho,\beta}(((e - e') + n, -(e - e') + n, x_b - y_b)) = \int_0^\infty e^{-a|n|} d\mu'_{v,\beta}(a). \quad (9.A.23)$$

Fix $p_b = (p_3, \dots, p_d)$. Let $q \in \mathbb{R}$ which will be fixed later. Considering the sequence of ℓ^2 functions given for $L \geq 1$ by

$$v_{(e, x_b)}^{(L)} = \frac{e^{iqe} e^{ip_b \cdot x_b}}{\sqrt{|\Lambda_L^{(d-1)}|}} \mathbf{1}_{(e, x_b) \in \Lambda_L^{(d-1)}}, \quad (9.A.24)$$

we get, just like in the proof of Corollary 9.3.12, that there exists a positive measure $\mu'_{q, p_b, \beta}$ such that for $r \in \mathbb{R}$,

$$\sum_{(n, e, z_b) \in \mathbb{Z}^d} e^{ir n + i q e + i p_b \cdot z_b} S_{\rho,\beta}(e + n, -e + n, z_b) = \int_0^\infty \frac{e^a - e^{-a}}{\mathcal{E}_1(r) + (e^{a/2} - e^{-a/2})^2} d\mu'_{q, p_b, \beta}(a). \quad (9.A.25)$$

Taking $r = p_1 + p_2$ and $q = p_1 - p_2$, we get that

$$\sum_{\substack{x \in \mathbb{Z}^d \\ x_1 + x_2 = 0[2]}} e^{ip \cdot x} S_{\rho, \beta}(x) = \int_0^\infty \frac{e^a - e^{-a}}{\mathcal{E}_1(p_1 + p_2) + (e^{a/2} - e^{-a/2})^2} d\mu'_{p_1 - p_2, p, \beta}(a). \quad (9.A.26)$$

Notice that in the formula above, one can use the symmetries of $\hat{S}_{\rho, \beta}^{(\text{mod})}(p)$ to change p_2 into $-p_2$. As a result, we obtain that

$$\hat{S}_{\rho, \beta}^{(\text{mod})}(p) = 2 \int_0^\infty \frac{e^a - e^{-a}}{\mathcal{E}_1(p_1 - p_2) + (e^{a/2} - e^{-a/2})^2} d\mu'_{p_1 + p_2, p, \beta}(a). \quad (9.A.27)$$

This yields the result using the monotonicity of $u \in [0, \pi] \mapsto \mathcal{E}_1(u)$, as in the proof of Corollary 9.3.15. $\square$

9.B The backbone representation of the Ising model

We recall the definition and some of the main properties of the backbone representation of the Ising model. This representation is closely related to the random current representation and was first introduced in [12, 8], and later used to capture fine properties of the Ising model [84, 11, 9]. We refer to these papers for more details about this object. We will use the notations of Section 9.4.

We fix a finite subset Λ of $\mathbb{Z}^d$ and fix any ordering of $(\{u, v\})_{u, v \in \Lambda}$ that we denote $\prec$.

Definition 9.B.1. Let $\mathbf{n} \in \Omega_\Lambda$. Assume that $\partial \mathbf{n} = \{x, y\}$. The backbone of $\mathbf{n}$, denoted $\Gamma(\mathbf{n})$, is the unique oriented and edge self-avoiding path from x to y supported on pairs $\{u, v\}$ with $\mathbf{n}_{u, v}$ odd which is minimal for $\prec$.

The backbone $\Gamma(\mathbf{n})$ can be obtained via the following exploration process:

- (1) Let $x_0 = x$. The first edge $\{x, x_1\}$ of $\Gamma(\mathbf{n})$ is the earliest one of all the edges emerging from x with $\mathbf{n}_{x, x_1}$ odd.
- (2) Each edge $\{x_i, x_{i+1}\}$ is the earliest of all edges emerging from x_i that have not been cancelled previously, and for which the flux number is odd.
- (3) The path stops when it reaches a site from which there are no more non-cancelled edges with odd flux number available. This always happen at a source of $\mathbf{n}$ (in that case y).

We let $\overline{\Gamma(\mathbf{n})}$ be the set of explored edges (this set is made of the $\{x_i, x_{i+1}\}$ together with all cancelled edges).

A path $\gamma : x \rightarrow y$ (viewed as a sequence of steps) is said to be consistent if no step of the sequence uses a bond cancelled by a previous step.

One can write

$$\langle \sigma_x \sigma_y \rangle_{\Lambda, \beta} = \sum_{\gamma: x \rightarrow y \text{ consistent}} \rho_\Lambda(\gamma), \quad (9.B.1)$$

where for a consistent path $\gamma : x \rightarrow y$,

$$\rho_\Lambda(\gamma) := \frac{\sum_{\partial \mathbf{n} = \partial \gamma} w_\beta(\mathbf{n}) \mathbb{1}[\Gamma(\mathbf{n}) = \gamma]}{\sum_{\partial \mathbf{n} = \emptyset} w_\beta(\mathbf{n})}. \quad (9.B.2)$$

The backbone representation has the following useful properties:

1. If γ is a consistent path and E is a subset of edges of Λ such that $\bar{\gamma} \cap E^c = \emptyset$, then

$$\rho_\Lambda(\gamma) \leq \rho_E(\gamma). \quad (9.B.3)$$

2. If a consistent path γ is the concatenation of γ_1 and γ_2 (which we denote by $\gamma = \gamma_1 \circ \gamma_2$),

$$\rho_\Lambda(\gamma) = \rho_\Lambda(\gamma_1) \rho_{\Lambda \setminus \bar{\gamma}_1}(\gamma_2). \quad (9.B.4)$$

This last property has the following consequence.

Proposition 9.B.2 (Chain rule for the backbone). *Let $x, y, u, v \in \Lambda$. Then,*

$$\mathbf{P}_{\Lambda, \beta}^{xy}[\Gamma(\mathbf{n}) \text{ passes through } u \text{ first and then through } v] \leq \frac{\langle \sigma_x \sigma_u \rangle_{\Lambda, \beta} \langle \sigma_u \sigma_v \rangle_{\Lambda, \beta} \langle \sigma_v \sigma_y \rangle_{\Lambda, \beta}}{\langle \sigma_x \sigma_y \rangle_{\Lambda, \beta}}. \quad (9.B.5)$$

Remark 9.B.3. The backbone expansion is very close to the Brydges-Fröhlich-Spencer random walk expansion introduced in [47], although it appears to be less canonical.

9.C Properties of currents for models of the Ising-type in the GS class

We recall a few classical bounds that can be found in [9, Appendix A.4]. We keep the notations introduced in Section 9.8. Fix a measure ρ of the Ising type in the GS class, and $\beta > 0$.

Proposition 9.C.1. *For every distinct $x, y, u, v \in \mathbb{Z}^d$,*

$$\mathbb{P}_{\rho, \beta}^{xy, \emptyset}[\partial \mathbf{n}_1 \xrightarrow{\mathbf{n}_1 + \mathbf{n}_2} \mathcal{B}_u] \leq \sum_{u' \in \mathbb{Z}^d} \frac{\langle \tau_x \tau_u \rangle_{\rho, \beta} (\beta J_{u, u'}) \langle \tau_{u'} \tau_y \rangle_{\rho, \beta}}{\langle \tau_x \tau_y \rangle_{\rho, \beta}}, \quad (9.C.1)$$

and

$$\mathbb{P}_{\rho, \beta}^{\emptyset, \emptyset}[\mathcal{B}_x \xrightarrow{\mathbf{n}_1 + \mathbf{n}_2} \mathcal{B}_y] \leq \sum_{x', y' \in \mathbb{Z}^d} \langle \tau_x \tau_y \rangle_{\rho, \beta} (\beta J_{y, y'}) \langle \tau_{y'} \tau_{x'} \rangle_{\rho, \beta} \beta J_{x', x}. \quad (9.C.2)$$

Moreover,

$$\begin{aligned} \mathbb{P}_{\rho, \beta}^{0x, \emptyset}[\partial \mathbf{n}_1 \xrightarrow{\mathbf{n}_1 + \mathbf{n}_2} \mathcal{B}_u, \mathcal{B}_v] &\leq \sum_{u', v' \in \mathbb{Z}^d} \frac{\langle \tau_x \tau_u \rangle_{\rho, \beta} (\beta J_{u, u'}) \langle \tau_{u'} \tau_v \rangle_{\rho, \beta} (\beta J_{v, v'}) \langle \tau_{v'} \tau_y \rangle_{\rho, \beta}}{\langle \tau_x \tau_y \rangle_{\rho, \beta}} \\ &\quad + \frac{\langle \tau_x \tau_v \rangle_{\rho, \beta} (\beta J_{v, v'}) \langle \tau_{v'} \tau_u \rangle_{\rho, \beta} (\beta J_{u, u'}) \langle \tau_{u'} \tau_y \rangle_{\rho, \beta}}{\langle \tau_x \tau_y \rangle_{\rho, \beta}} \end{aligned} \quad (9.C.3)$$

In the spirit of Proposition 9.6.24 we also have the following result.

Proposition 9.C.2 (Monotonicity in the number of sources for the GS class). *For every $x, y, z, t \in \mathbb{Z}^d$, every u, u', u'' with u', u'' J -neighbours of u , and every $S \subset \mathbb{Z}^d \times K_N$,*

$$\begin{aligned} \mathbb{P}_{\rho, \beta}^{ux, uz, u'y, u''t}[\mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_3 + \delta_{(\partial \mathbf{n}_1 \cap \mathcal{B}_u, \partial \mathbf{n}_3 \cap \mathcal{B}_{u'})}}(\partial \mathbf{n}_1) \cap \mathbf{C}_{\mathbf{n}_2 + \mathbf{n}_4 + \delta_{(\partial \mathbf{n}_2 \cap \mathcal{B}_u, \partial \mathbf{n}_4 \cap \mathcal{B}_{u''})}}(\partial \mathbf{n}_2) \cap S = \emptyset] \\ \leq \mathbb{P}_{\rho, \beta}^{ux, uz, \emptyset, \emptyset}[\mathbf{C}_{\mathbf{n}_1 + \mathbf{n}_3}(\partial \mathbf{n}_1) \cap \mathbf{C}_{\mathbf{n}_2 + \mathbf{n}_4}(\partial \mathbf{n}_2) \cap S = \emptyset]. \end{aligned}$$

9.D Triviality and finiteness of the Bubble diagram

In this appendix, we prove that models in the GS class for which the bubble diagram is finite at criticality behave trivially. This provides an alternative proof to the results of Section 9.5 but it also captures more cases (for instance we may apply it to the case of algebraically decaying RP interactions for $d = 4$ and $\alpha = 2$).

Below we fix a measure ρ in the GS class and, as in Section 9.3, we denote the spin-field by τ . The correlation length or order $\sigma > 0$, mentioned in the introduction is given by

$$\xi_\sigma(\rho, \beta) := \left(\frac{\sum_{x \in \mathbb{Z}^d} |x|^\sigma \langle \tau_0 \tau_x \rangle_{\rho, \beta}}{\chi(\rho, \beta)} \right)^{1/\sigma}. \quad (9.D.1)$$

We assume that we are given an interaction J on $\mathbb{Z}^d$ satisfying (A1)–(A5) and such that the above quantity can be defined for σ small enough throughout the critical phase.

The following result can be found in [235] and is a direct consequence of the Messenger-Miracle-Solé inequality.

Proposition 9.D.1. *Let $\beta < \beta_c(\rho)$. There exists a constant $C > 0$ such that for all $x \in \mathbb{Z}^d$,*

$$\langle \tau_0 \tau_x \rangle_{\rho, \beta} \leq C \frac{\chi(\rho, \beta) \xi_\sigma(\rho, \beta)^\sigma}{(1 + |x|)^{d+\sigma}}. \quad (9.D.2)$$

Proof. Using (MMS2),

$$|\{y \notin \Lambda_{|x|/(2d)}, S_{\rho, \beta}(y) \geq S_{\rho, \beta}(x)\}| \geq C_1(1 + |x|)^d, \quad (9.D.3)$$

for some $C_1 > 0$. As a consequence,

$$\chi(\rho, \beta) \xi_\sigma(\rho, \beta)^\sigma = \sum_{y \in \mathbb{Z}^d} |y|^\sigma S_{\rho, \beta}(y) \geq C_2(1 + |x|)^{d+\sigma} S_{\rho, \beta}(x), \quad (9.D.4)$$

which concludes the proof. $\square$

Recall that the *renormalised coupling constant* of order σ is defined by,

$$g_\sigma(\rho, \beta) := -\frac{1}{\chi(\rho, \beta)^2 \xi_\sigma(\rho, \beta)^d} \sum_{x, y, z \in \mathbb{Z}^d} U_4^{\rho, \beta}(0, x, y, z). \quad (9.D.5)$$

Theorem 9.D.2 (The bubble condition implies triviality). *Let $d \geq 2$. For a reflection positive model in $\mathbb{Z}^d$ with an interaction J satisfying the above conditions and such that*

$$B(\rho, \beta_c(\rho)) = \sum_{x \in \mathbb{Z}^d} \langle \tau_0 \tau_x \rangle_{\rho, \beta_c(\rho)}^2 < \infty, \quad (9.D.6)$$

one has,

$$\lim_{\beta \nearrow \beta_c(\rho)} g_\sigma(\rho, \beta) = 0. \quad (9.D.7)$$

Proof. Using the tree diagram bound (9.4.14), we get

$$0 \leq g_\sigma(\rho, \beta) \leq 2 \frac{\chi(\rho, \beta)^2}{\xi_\sigma(\rho, \beta)^d}. \quad (9.D.8)$$

Now, take $L \gg \varepsilon > 0$ to be fixed later. Write

$$\chi(\rho, \beta) = \underbrace{\chi_{\varepsilon\xi_\sigma(\rho, \beta)}(\rho, \beta)}_{(1)} + \underbrace{(\chi_{L\xi_\sigma(\rho, \beta)}(\rho, \beta) - \chi_{\varepsilon\xi_\sigma(\rho, \beta)}(\rho, \beta))}_{(2)} + \underbrace{(\chi(\rho, \beta) - \chi_{L\xi_\sigma(\rho, \beta)}(\rho, \beta))}_{(3)}.$$

Using the Cauchy–Schwarz inequality one gets for $C_1 > 0$,

$$(1) \leq C_1 \varepsilon^{d/2} \xi_\sigma(\rho, \beta)^{d/2} \sqrt{B(\rho, \beta_c(\rho))},$$

and for $C_2 = C_2(L, \varepsilon) > 0$,

$$(2) \leq C_2 \xi_\sigma(\rho, \beta)^{d/2} \sqrt{B_{L\xi_\sigma(\rho, \beta)}(\rho, \beta) - B_{\varepsilon\xi_\sigma(\rho, \beta)}(\rho, \beta)}.$$

Moreover, using Proposition 9.D.1, we get that for $C_3 > 0$,

$$(3) \leq C_3 \frac{\xi_\sigma(\rho, \beta)^{d/2}}{L^\sigma} \frac{\chi(\rho, \beta)}{\xi_\sigma(\rho, \beta)^{d/2}}.$$

Putting all the pieces together we get that,

$$\frac{\chi(\rho, \beta)}{\xi_\sigma(\rho, \beta)^{d/2}} \leq C_1 \sqrt{B(\rho, \beta_c)} \varepsilon^{d/2} + C_2 \sqrt{B_{L\xi_\sigma(\rho, \beta)}(\rho, \beta) - B_{\varepsilon\xi_\sigma(\rho, \beta)}(\rho, \beta)} + \frac{C_3}{L^\sigma} \frac{\chi(\rho, \beta)}{\xi_\sigma(\rho, \beta)^{d/2}}.$$

Fix $L > 0$ large enough so that $\frac{C_3}{L^\sigma} < 1$. Using the left-continuity of the two-point function, and the fact that $\xi_\sigma(\rho, \beta) \rightarrow \infty$ as $\beta \nearrow \beta_c(\rho)$ together with the monotone convergence theorem, we get that

$$B_{L\xi_\sigma(\rho, \beta)}(\rho, \beta), B_{\varepsilon\xi_\sigma(\rho, \beta)}(\rho, \beta) \xrightarrow{\beta \nearrow \beta_c(\rho)} B(\rho, \beta_c(\rho)),$$

so that for all $\varepsilon > 0$,

$$\limsup_{\beta \rightarrow \beta_c(\rho)} \frac{\chi(\rho, \beta)}{\xi_\sigma(\rho, \beta)^{d/2}} \leq \frac{C_1 \sqrt{B(\rho, \beta_c(\rho))}}{1 - \frac{C_3}{L^\sigma}} \varepsilon^{d/2},$$

which yields the result using (9.D.8). $\square$

Remark 9.D.3. The above result could be extended to more general models in the GS class: if J satisfies (A1)–(A4), and if we consider an interaction J for which both the bubble condition and the MMS inequalities hold (or more precisely (MMS2)), then the renormalised coupling constant vanishes at criticality. Using the proof of the MMS inequality of [11], we may then extend our result to finite-range interactions²⁸.

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²⁸Recall that these interactions are not reflection positive in general.

Chapter 10

Fractional Gaussian free field correlations in subcritical long-range Ising models

We study corrections to the scaling limit of subcritical long-range Ising models with (super)-summable interactions on $\mathbb{Z}^d$. For a wide class of models, the scaling limit is known to be white noise, as shown by Newman (1980). In the specific case of couplings $J_{x,y} = |x - y|^{-d-\alpha}$, where $\alpha > 0$ and $|\cdot|$ is the Euclidean norm, we find an emergence of fractional Gaussian free field correlations in appropriately renormalised and rescaled observables. The proof exploits the exact asymptotics of the two-point function, first established by Newman and Spohn (1998), together with the rotational symmetry of the interaction.

This chapter is based on the article “*Emergence of fractional Gaussian free field correlations in subcritical long-range Ising models*” [127] accepted for publication in *Annales de l’Institut Henri Poincaré B : Probabilités et Statistiques*. It is a joint work with Trishen S. Gunaratnam.

10.1 Introduction

In this chapter, we study the emergence of fractional Gaussian free field (FGFF) correlations in the scaling limit of long-range Ising models below the critical point.

Let $\mathbb{Z}^d$ be the hypercubic lattice of dimension $d \geq 2$. For $\alpha, \mathbf{C} > 0$, let $J = (J_{x,y})_{\{x,y\} \subset \mathbb{Z}^d}$ be a collection of ferromagnetic interactions defined by

$$J_{x,y} := \frac{\mathbf{C}}{|x - y|^{d+\alpha}}, \quad (10.1.1)$$

where $|\cdot|$ is the Euclidean norm¹ on $\mathbb{R}^d$.

The Ising model with interaction J is defined on finite subsets $\Lambda \subset \mathbb{Z}^d$ as follows. For $\sigma = (\sigma_x)_{x \in \Lambda} \in \{\pm 1\}^\Lambda$, define the Hamiltonian

$$H_{\Lambda,h}(\sigma) := - \sum_{\{x,y\} \subset \Lambda} J_{x,y} \sigma_x \sigma_y - h \sum_{x \in \Lambda} \sigma_x. \quad (10.1.2)$$

¹The choice of the Euclidean norm, which is spherically symmetric, is not just a technicality. See the discussion after Theorem 10.1.6 and also Remark 10.3.1

The finite volume Ising model $\langle \cdot \rangle_{\Lambda, \beta, h}$ at inverse temperature $\beta \geq 0$ and external field $h \in \mathbb{R}$ is the probability measure under which, for each $F : \{\pm 1\}^\Lambda \rightarrow \mathbb{R}$,

$$\langle F \rangle_{\Lambda, \beta, h} := \frac{1}{Z(\Lambda, \beta, h)} \sum_{\sigma \in \{\pm 1\}^\Lambda} F(\sigma) \exp(-\beta H_{\Lambda, h}(\sigma)), \quad (10.1.3)$$

where $Z(\Lambda, \beta, h) := \sum_{\sigma \in \{\pm 1\}^\Lambda} \exp(-\beta H_{\Lambda, h}(\sigma))$ is the partition function of the model.

Using Griffiths' inequalities [116, 117], one can obtain the associated infinite volume Ising model with free boundary conditions by taking weak limits of $\langle \cdot \rangle_{\Lambda, \beta, h}$ as $\Lambda \nearrow \mathbb{Z}^d$. Denote the limiting probability measure by $\langle \cdot \rangle_{\beta, h}$.

It is well known that the Ising model exhibits a phase transition for the vanishing of the *spontaneous magnetisation*. That is, if

$$m^*(\beta) := \lim_{h \rightarrow 0^+} \langle \sigma_0 \rangle_{\beta, h}, \quad (10.1.4)$$

then $\beta_c := \inf\{\beta > 0, m^*(\beta) > 0\} \in (0, \infty)$. It has been proved in [97] that $\beta_c > 0$, whilst the celebrated Peierls argument [208] yields the bound $\beta_c < \infty$. Throughout the rest of the chapter, we assume that $h = 0$.

We are interested in subcritical scaling limits, i.e. when $\beta < \beta_c$. As recalled in the proposition below, in this regime these models still exhibit polynomial decay of spin-spin correlations and thus may have a non-degenerate scaling limit.

Proposition 10.1.1 ([199, 20]). *Let $d \geq 2$, $\alpha, \mathbf{C} > 0$ in the interaction (10.1.1), and $\beta < \beta_c$. Then, there exists $\delta > 0$ such that for every $x \in \mathbb{Z}^d \setminus \{0\}$,*

$$\langle \sigma_0 \sigma_x \rangle_\beta = \frac{\beta \chi(\beta)^2}{2} \frac{\mathbf{C}}{|x|^{d+\alpha}} \left(1 + O(|x|^{-\delta})\right), \quad (10.1.5)$$

where $\chi(\beta) := \sum_{x \in \mathbb{Z}^d} \langle \sigma_0 \sigma_x \rangle_\beta$. As an immediate consequence, we obtain polynomial decay of correlations: there exists $C = C(\beta, J) > 0$ such that for all $x \in \mathbb{Z}^d \setminus \{0\}$,

$$\langle \sigma_0 \sigma_x \rangle_\beta \leq \frac{C}{|x|^{d+\alpha}}. \quad (10.1.6)$$

The relevant object to study the scaling limit is the set of smeared observables. Let $\mathcal{C}_c^\infty(\mathbb{R}^d)$ be the set of infinitely differentiable and compactly supported functions from $\mathbb{R}^d$ to $\mathbb{R}$.

Definition 10.1.2 (Smeared observable). Let $\beta > 0$ and $L \in \mathbb{N}^*$. For every $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, we define the smeared observable $T_{f, L, \beta} : \{\pm 1\}^{\mathbb{Z}^d} \rightarrow \mathbb{R}$ by,

$$T_{f, L, \beta}(\sigma) := \frac{2^{d/2}}{\sqrt{\Sigma_L(\beta)}} \sum_{x \in \mathbb{Z}^d} f\left(\frac{x}{L}\right) \sigma_x, \quad \sigma \in \{\pm 1\}^{\mathbb{Z}^d} \quad (10.1.7)$$

where

$$\Sigma_L(\beta) = \left\langle \left(\sum_{x \in \Lambda_L} \sigma_x \right)^2 \right\rangle_\beta = \sum_{x, y \in \Lambda_L} \langle \sigma_x \sigma_y \rangle_\beta, \quad \text{with } \Lambda_L := [-L, L]^d \cap \mathbb{Z}^d. \quad (10.1.8)$$

Remark 10.1.3. The natural scaling is guessed from considering the test function $f = \mathbf{1}_{[-1,1]^d}$. In this case, we find that

$$\langle T_{f,L,\beta}(\sigma)^2 \rangle_\beta = 1. \quad (10.1.9)$$

More generally, for positive functions f that are not identically zero, we have $0 < c_f \leq \langle T_{f,L,\beta}(\sigma)^2 \rangle_\beta \leq C_f < \infty$.

It is classical [4, 102] that when $\beta < \beta_c$, for each $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, any limit point in distribution of the random variables $T_{f,L,\beta}$ is necessarily a centred Gaussian random variable (see also Proposition 10.2.1). Our first result identifies the variance as $\int_{\mathbb{R}^d} f(\tilde{x})^2 d\tilde{x}$, hence proving full convergence. This implies that the set of averages, viewed as a stochastic process indexed by $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, converges in the sense of finite dimensional distributions to the standard white noise process on $\mathbb{R}^d$. This is somewhat counter-intuitive to the polynomial decay of spin-spin correlations since the white noise is a degenerate limit.

The same result was already obtained by Newman [197, 198] in a more general setup. Our approach is more specialised to the Ising model.

Definition 10.1.4. The white noise is a stochastic process $W_0 = \{W_0(f) : f \in \mathcal{C}_c^\infty(\mathbb{R}^d)\}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that: i) each $W_0(f)$ is a centred, Gaussian random variable; ii) the mapping $f \mapsto W_0(f)$ is linear; and, iii) the covariance is given by,

$$\mathbb{E}[W_0(f)W_0(g)] = \int_{\mathbb{R}^d} f(\tilde{x})g(\tilde{x})d\tilde{x}, \quad \forall f, g \in \mathcal{C}_c^\infty(\mathbb{R}^d). \quad (10.1.10)$$

Remark 10.1.5. The existence of white noise follows by the Kolmogorov extension theorem. Note that W_0 may be interpreted almost surely as a random element of $\mathcal{S}'(\mathbb{R}^d)$, the space of Schwartz distributions. Alternatively, we may canonically extend W_0 to a stochastic process indexed by $L^2(\mathbb{R}^d)$ — this is the Gaussian Hilbert space point of view. In doing so, we lose the interpretation as an almost sure element of a space of distributions, i.e. we lose the almost sure linearity property.

Theorem 10.1.6 (Convergence to the white noise). *Let $d \geq 2$ and $\alpha, \mathbf{C} > 0$ in the interaction (10.1.1), and $\beta < \beta_c$. Then, the stochastic process $\{T_{f,L,\beta}(\sigma) : f \in \mathcal{C}_c^\infty(\mathbb{R}^d)\}$ converges in the sense of finite dimensional distributions to the standard white noise. Moreover, we have the following quantitative rate of convergence: for every $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and $z \in \mathbb{R}$, there exists $C = C(\beta, d, J, z, f) > 0$, such that for $L \geq 1$,*

$$\left| \langle \exp(zT_{f,L,\beta}(\sigma)) \rangle_\beta - \exp\left(\frac{z^2}{2} \int_{\mathbb{R}^d} f(\tilde{x})^2 d\tilde{x}\right) \right| \leq CL^{-\frac{\alpha}{\alpha+1}}. \quad (10.1.11)$$

Remark 10.1.7. Theorem 10.1.6 (and in particular the polynomial rate of convergence) holds for translation-invariant Ising models on $\mathbb{Z}^d$ satisfying the following (super)-summable condition: for all $\beta < \beta_c$, there exists $\varepsilon, C > 0$ such that for all $x \in \mathbb{Z}^d \setminus \{0\}$,

$$\langle \sigma_0 \sigma_x \rangle_\beta \leq \frac{C}{|x|^{d+\varepsilon}}. \quad (10.1.12)$$

This class of models includes finite-range Ising models (see [8, 84]) as well as a wide class of ferromagnetic long-range interactions $(J_{x,y})_{x,y \in \mathbb{Z}^d}$. The former follows from the fact that the polynomial bound on spin-spin correlations in Proposition 10.1.1 is satisfied in greater generality (see [199, 20, 21]).

Remark 10.1.8. The proof below can be adapted to show convergence to white noise of any subcritical translation-invariant Ising models on $\mathbb{Z}^d$. Indeed, the key point in the proof is the finiteness of the susceptibility.

In the proof of Theorem 10.1.6, we explicitly compute the limiting covariance structure of smeared observables. Due to the sharp asymptotics of the spin-spin correlation functions, lower order contributions to this covariance structure can also be computed. The spherical symmetry of J allows us to find emergent fractional GFF correlations appearing for certain rescaled and renormalised product observables. Since Theorem 10.1.6 is a central limit theorem for the set of smeared observables, these higher order contributions can be viewed as corrections to the CLT.

Definition 10.1.9. Let $\alpha > 0$. The fractional GFF of Hurst parameter $H = H(\alpha) = -d/2 - \alpha/2$ is a stochastic process $W_\alpha = \{W_\alpha(f) : f \in \mathcal{C}_c^\infty(\mathbb{R}^d)\}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that: i) each $W_\alpha(f)$ is a centred, Gaussian random variable; ii) the mapping $f \mapsto W_\alpha(f)$ is linear; and, iii) the covariance is given by, $\forall f, g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,

$$\mathbb{E}[W_\alpha(f)W_\alpha(g)] = K_{\alpha,d}(f, g) \quad (10.1.13)$$

where, for $\alpha \in \mathbb{R}_{>0} \setminus 2\mathbb{N}^*$,

$$K_{\alpha,d}(f, g) := \int_{\mathbb{R}^d \times \mathbb{R}^d} C(\alpha, d) |\tilde{x} - \tilde{y}|^{-(d+\alpha)} \left(f(\tilde{x})g(\tilde{y}) - \sum_{j=0}^{\lfloor \alpha \rfloor_2/2} f(\tilde{x})H_j \Delta^j g(\tilde{x}) |\tilde{x} - \tilde{y}|^{2j} \right) d\tilde{x}d\tilde{y}, \quad (10.1.14)$$

and for $\alpha \in 2\mathbb{N}^*$,

$$K_{\alpha,d}(f, g) := (4\pi^2)^{-\alpha/2} \int_{\mathbb{R}^d \times \mathbb{R}^d} f(\tilde{x})(-\Delta)^{\alpha/2} g(\tilde{x}) d\tilde{x}. \quad (10.1.15)$$

Above, for $t > 0$, $[t]_2 := \max\{k \in 2\mathbb{N}, k < t\}$, Δ^j is the j -th power of the Laplacian, and

$$C(\alpha, d) = \frac{2^\alpha \pi^{-d/2} \Gamma(d/2 + \alpha/2)}{\Gamma(-\alpha/2)}, \quad \alpha \in \mathbb{R}_{>0} \setminus 2\mathbb{N}^*, \quad (10.1.16)$$

$$H_j = \frac{2\pi^{d/2} \Gamma(d/2)}{2^j j! d(d+2) \dots (d+2j-2)}.$$

Remark 10.1.10. (a) Fractional GFFs are surveyed in great generality and detail in the review paper [183]. When $\alpha > 0$, these random fields have singular short-range correlation structure. The case $\alpha = 1$ in $d = 3$ arises, for example, in the cosmology of the early universe [75]. Note that this range of parameters does not include “usual” GFF, which corresponds to $\alpha = -2$, or the fractional GFFs that usually arise from critical scaling limits of some long-range Ising models (see [13, 67]).

(b) There is a subtle but important distinction between the cases $\alpha \in \mathbb{R}_{>0} \setminus 2\mathbb{N}^*$ and $\alpha \in 2\mathbb{N}^*$. In the case $\alpha \in 2\mathbb{N}^*$, the short-distance correlations are singular, but retain some locality because powers of the Laplacian are local. On the other hand, in the case $\alpha \in \mathbb{R}_{>0} \setminus 2\mathbb{N}^*$, the short-distance correlations are both singular and non-local.

- (c) The fractional GFF can be viewed as a fractional Laplacian of the white noise W_0 . More precisely, the fractional GFF of Hurst parameter $H(\alpha)$ can be seen as $(-\Delta)^{\alpha/2}W_0$. Due to the non-integrable singularity at the origin, the kernel associated with the fractional Laplacian must in general be interpreted as a renormalised distribution. The details can be found in [171, Chapter 1].

We now introduce our renormalised smeared observables.

Definition 10.1.11. Let $d \geq 2$, $\alpha, \mathbf{C} > 0$ in the interaction (10.1.1), and $\beta < \beta_c$. Let $L \geq 1$. For $f, g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, and $\sigma \in \{\pm 1\}^{\mathbb{Z}^d}$, define

$$\mathcal{R}_{\alpha, \beta, L} [T_{f, L, \beta}(\sigma) T_{g, L, \beta}(\sigma)] := \frac{2^d}{\Sigma_L(\beta)} \sum_{x, y \in \mathbb{Z}^d} f(x/L) \left(g(y/L) - \text{Tay}_{[\alpha]_2} g(y/L; x/L) \right) \sigma_x \sigma_y, \quad (10.1.17)$$

where, for any $\tilde{x}, \tilde{y} \in \mathbb{R}^d$ and any $h \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,

$$\text{Tay}_{[\alpha]_2} h(\tilde{y}; \tilde{x}) := \sum_{j=0}^{[\alpha]_2} \frac{1}{j!} \nabla^j h(\tilde{x}) \underbrace{(\tilde{y} - \tilde{x}, \dots, \tilde{y} - \tilde{x})}_{j\text{-tuple}}, \quad (10.1.18)$$

and $\nabla^j h(\tilde{x})$ denotes the j -th Fréchet derivative of h at $\tilde{x}$.

The following is our main result.

Theorem 10.1.12 (Emergence of fractional GFF correlations). *Let $d \geq 2$, $\alpha, \mathbf{C} > 0$ in the interaction (10.1.1), and $\beta < \beta_c$. Then, for all $f, g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,*

$$\lim_{L \rightarrow \infty} L^\alpha \left\langle \mathcal{R}_{\alpha, \beta, L} [T_{f, L, \beta}(\sigma) T_{g, L, \beta}(\sigma)] \right\rangle_\beta = \frac{\beta \chi(\beta) \mathbf{C}}{2C(\alpha, d)} \mathbb{E}[W_\alpha(f) W_\alpha(g)], \quad \alpha \in \mathbb{R}_{\geq 0} \setminus 2\mathbb{N} \quad (10.1.19)$$

$$\lim_{L \rightarrow \infty} \frac{L^\alpha}{\log L} \left\langle \mathcal{R}_{\alpha, \beta, L} [T_{f, L, \beta}(\sigma) T_{g, L, \beta}(\sigma)] \right\rangle_\beta = \frac{\beta \chi(\beta) \mathbf{C} H_{\alpha/2}}{2(4\pi^2)^{-\alpha/2}} (-1)^{\alpha/2} \mathbb{E}[W_\alpha(f) W_\alpha(g)], \quad \alpha \in 2\mathbb{N}^*. \quad (10.1.20)$$

Remark 10.1.13. Quantitative error estimates are given in Propositions 10.3.2 and 10.3.3.

Theorem 10.1.12 appears to suggest that, for $\alpha \in (0, 2)$, there exists a copy of white noise W_0 and an independent FGFF W_α such that, for some explicit constant $C > 0$, for every $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, and for every $L \in \mathbb{N}$, there exists a random variable $F_{f, L}$ such that

$$T_{f, L, \beta} \stackrel{\text{law}}{=} W_0(f) + C^{-1/2} L^{-\alpha/2} W_\alpha(f) + L^{-\alpha/2} F_{f, L}, \quad (10.1.21)$$

and $F_{f, L}$ converges to 0 as $L \rightarrow 0$ in distribution. Similarly, for general $\alpha > 0$, Theorem 10.1.12 suggests that a similar expansion holds, albeit with additional terms corresponding to derivatives of white noise.

The following corollary establishes this rigorously for $\alpha \in (0, 2)$. It is a straightforward consequence of the quantitative triviality estimate of Proposition 10.2.1, the quantitative

error bounds in Propositions 10.3.2 and 10.3.3, and sufficiently good rate² of convergence of Riemann sums for smeared observables. The analogous result for general α can be established similarly, with the aforementioned additional terms/scalings incorporated. We omit this for clarity of exposition.

Corollary 10.1.14. *Let $d \geq 2$, $\alpha \in (0, 2)$, $\mathbf{C} > 0$, and $\beta < \beta_c$. For every $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, and every $z \in \mathbb{R}$, as $L \rightarrow \infty$,*

$$\langle \exp(zT_{f,L,\beta}(\sigma)) \rangle_\beta = \exp\left(\frac{z^2}{2} \mathbb{E}[W_0(f)^2]\right) \exp\left(\frac{z^2}{2} \frac{1}{L^\alpha} \frac{\beta \chi(\beta) \mathbf{C}}{2C(\alpha, d)} \mathbb{E}[W_\alpha(f)^2]\right) (1 + o(1)). \quad (10.1.22)$$

As a result, $T_{f,L,\beta}$ can be written as in (10.1.21), with $F_{f,L}$ converging to 0 in law.

Notations We write $|\cdot|_1$, $|\cdot|$, and $|\cdot|_\infty$ to denote the 1-norm, Euclidean norm, and ∞ -norm on $\mathbb{R}^d$, respectively. We also write these symbols for the corresponding norms on $\mathbb{Z}^d$.

In continuum, we use the following conventions. We write $B(\tilde{y}, r) = \{\tilde{x} \in \mathbb{R}^d : |\tilde{x} - \tilde{y}| \leq r\}$ to denote the ball of radius $r > 0$ and centred at $\tilde{y} \in \mathbb{R}^d$. We write $S(\tilde{y}, r) = \partial B(\tilde{y}, r)$ to denote the $(d-1)$ -sphere of radius r centred at $\tilde{y}$. On the lattice, for $x \in \mathbb{Z}^d$ and $L \in \mathbb{N}$, we write $\Lambda_L(x) := \{y \in \mathbb{Z}^d : |x - y|_\infty \leq L\}$ to denote the box of side length L centred at x .

If $(a_n)_{n \geq 0}, (b_n)_{n \geq 0} \in (\mathbb{R}_+^*)^\mathbb{N}$, we write $a_n = O(b_n)$ (resp. $a_n = o(b_n)$) if there exists $C = C(d, \beta, J) > 0$ such that for all $n \geq 1$, $a_n \leq Cb_n$ (resp. $\lim_{n \rightarrow \infty} a_n/b_n = 0$). Throughout the chapter, the constants $C, C_i > 0$ only depend on the parameters d, β, J , unless stated otherwise.

10.2 Convergence to white noise

In this section, we prove Theorem 10.1.6. We begin by showing the triviality of the smeared observables in the scaling limit. Then, we explicitly compute the covariance structure to fully characterise the limit as white noise.

10.2.1 Triviality of any limit point

The following proposition establishes that any limit point of a smeared observable is Gaussian in the subcritical phase, with quantitative bounds on the error in the Laplace transform.

Proposition 10.2.1. *Let $d \geq 2$, $\alpha, \mathbf{C} > 0$ in the interaction (10.1.1), and $\beta < \beta_c$. For all $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and $z \in \mathbb{R}$, there exists $C = C(d, \beta, J, f) > 0$ such that, for all $L \geq 1$,*

$$\left| \langle \exp(zT_{f,L,\beta}(\sigma)) \rangle_\beta - \exp\left(\frac{z^2}{2} \langle T_{f,L,\beta}(\sigma)^2 \rangle_\beta\right) \right| \leq \frac{C}{L^d}. \quad (10.2.1)$$

Proof. Recall that Ursell's four-point function U_4^β is defined for $x, y, z, t \in \mathbb{Z}^d$ by

$$U_4^\beta(x, y, z, t) := \langle \sigma_x \sigma_y \sigma_z \sigma_t \rangle_\beta - \langle \sigma_x \sigma_y \rangle_\beta \langle \sigma_z \sigma_t \rangle_\beta - \langle \sigma_x \sigma_z \rangle_\beta \langle \sigma_y \sigma_t \rangle_\beta - \langle \sigma_x \sigma_t \rangle_\beta \langle \sigma_y \sigma_z \rangle_\beta. \quad (10.2.2)$$

²For $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, the Riemann sum can be approximated up to an error that is $O(L^{-p})$ for any $p > 0$. This is a consequence of Poisson summation and the decay of the Fourier transform of f being faster than any polynomial.

Using [4, Proposition 12.1] as in [9, Section 6.3], we get that for every $L \geq 1$,

$$\left| \langle \exp(zT_{f,L,\beta}(\sigma)) \rangle_\beta - \exp\left(\frac{z^2}{2} \langle T_{f,L,\beta}(\sigma)^2 \rangle_\beta\right) \right| \leq C_1 z^4 \exp\left(\frac{z^2}{2} \langle T_{f,L,\beta}(\sigma)^2 \rangle_\beta\right) \|f\|_\infty^4 S(\beta, L, f), \quad (10.2.3)$$

where

$$S(\beta, L, f) = \sum_{x_1, x_2, x_3, x_4 \in \Lambda_{r_f L}} \frac{|U_4^\beta(x_1, x_2, x_3, x_4)|}{\Sigma_L(\beta)^2}, \quad (10.2.4)$$

$r_f = \max(r \geq 0, \exists x \in \mathbb{R}^d, |x| = r, f(x) \neq 0) \vee 1$, and $C_1 > 0$. The tree diagram bound [4, Proposition 5.3] asserts

$$|U_4^\beta(x_i, x_j, x_k, x_\ell)| \leq 2 \sum_{x \in \mathbb{Z}^d} \langle \sigma_x \sigma_{x_i} \rangle_\beta \langle \sigma_x \sigma_{x_j} \rangle_\beta \langle \sigma_x \sigma_{x_k} \rangle_\beta \langle \sigma_x \sigma_{x_\ell} \rangle_\beta. \quad (10.2.5)$$

Splitting the resulting sum,

$$S(\beta, L, f)/2 \leq \underbrace{\sum_{\substack{x \in \Lambda_{2r_f L} \\ x_1, x_2, x_3, x_4 \in \Lambda_{r_f L}}} (\dots)}_{(I)} + \underbrace{\sum_{\substack{x \notin \Lambda_{2r_f L} \\ x_1, x_2, x_3, x_4 \in \Lambda_{r_f L}}} (\dots)}_{(II)}. \quad (10.2.6)$$

Bound on (I). The first term can be written

$$\sum_{\substack{x \in \Lambda_{2r_f L} \\ x_1, x_2, x_3, x_4 \in \Lambda_{r_f L}}} \frac{\langle \sigma_x \sigma_{x_1} \rangle_\beta \langle \sigma_x \sigma_{x_2} \rangle_\beta \langle \sigma_x \sigma_{x_3} \rangle_\beta \langle \sigma_x \sigma_{x_4} \rangle_\beta}{\Sigma_L(\beta)^2} = \sum_{x \in \Lambda_{2r_f L}} \frac{\left(\sum_{y \in \Lambda_{r_f L}} \langle \sigma_x \sigma_y \rangle_\beta \right)^4}{\Sigma_L(\beta)^2}. \quad (10.2.7)$$

Noticing that for $x \in \Lambda_{2r_f L}$,

$$\sum_{y \in \Lambda_{r_f L}} \langle \sigma_x \sigma_y \rangle_\beta \leq \chi_{3r_f L}(\beta) \leq \chi(\beta), \quad (10.2.8)$$

one gets that for $C_2 > 0$,

$$(1) \leq \chi(\beta)^4 \frac{|\Lambda_{2r_f L}|}{\Sigma_L(\beta)^2} \leq \frac{C_2}{L^d}. \quad (10.2.9)$$

Bound on (II). By (10.1.6) there exists $C_3 > 0$ such that for all $u, v \in \mathbb{Z}^d$ such that $u \neq v$,

$$\langle \sigma_u \sigma_v \rangle_\beta \leq \frac{C_3}{|u - v|^{d+\alpha}}. \quad (10.2.10)$$

Thus, there exists $C_4, C_5, C_6 > 0$ such that

$$(2) \leq \frac{C_4}{\Sigma_L(\beta)^2} L^{4d} \sum_{k \geq r_f L} \frac{k^{d-1}}{k^{4d+4\alpha}} \leq C_5 \frac{L^{4d}}{L^{3d+4\alpha} \Sigma_L(\beta)^2} \leq \frac{C_6}{L^{d+4\alpha}}. \quad (10.2.11)$$

This concludes the proof. $\square$

10.2.2 Computation of the covariance kernel

We start with a useful lemma which estimates $\Sigma_L(\beta)$ as $L \rightarrow \infty$.

Lemma 10.2.2. *Let $d \geq 2$, $\alpha, \mathbf{C} > 0$ in the interaction (10.1.1), and $\beta < \beta_c$. Then, for all $L \geq 1$,*

$$\Sigma_L(\beta) = \chi(\beta) |\Lambda_L| \left(1 + O(L^{-\frac{\alpha}{\alpha+1}}) \right). \quad (10.2.12)$$

Proof. First note that, by translation-invariance,

$$\Sigma_L(\beta) = \sum_{x,y \in \Lambda_L} \langle \sigma_x \sigma_y \rangle_\beta \leq |\Lambda_L| \chi(\beta) \quad (10.2.13)$$

where we recall $\chi(\beta) = \sum_{x \in \mathbb{Z}^d} \langle \sigma_0 \sigma_x \rangle_\beta$ is the susceptibility.

On the other hand, denoting $L(\alpha) := L^{\frac{1}{\alpha+1}}$, we have

$$\sum_{x,y \in \Lambda_L} \langle \sigma_x \sigma_y \rangle_\beta \geq |\Lambda_{L-L(\alpha)}| \chi_{L(\alpha)}(\beta) \quad (10.2.14)$$

where $\chi_k(\beta) = \sum_{x \in \Lambda_k} \langle \sigma_0 \sigma_x \rangle_\beta$. By (10.1.6), there exists $C > 0$ such that

$$\chi_{L(\alpha)}(\beta) = \chi(\beta) \frac{\chi_{L(\alpha)}(\beta)}{\chi(\beta)} = \chi(\beta) \left(1 - \frac{\chi(\beta) - \chi_{L(\alpha)}(\beta)}{\chi(\beta)} \right) \geq \chi(\beta) \left(1 - CL^{-\frac{\alpha}{\alpha+1}} \right). \quad (10.2.15)$$

Therefore,

$$|\Lambda_{L-L(\alpha)}| \chi_{L(\alpha)}(\beta) \geq |\Lambda_L| \chi(\beta) \left(1 - CL^{-\frac{\alpha}{\alpha+1}} \right), \quad (10.2.16)$$

from which we conclude. $\square$

We now show that the covariance of the smeared observables is given by their L^2 -scalar product in the limit $L \rightarrow \infty$. The proof uses a summable polynomial upper bound on the spin-spin correlations and lattice symmetries of the model.

Proposition 10.2.3. *Let $d \geq 2$, $\alpha, \mathbf{C} > 0$ in the interaction (10.1.1), and $\beta < \beta_c$. Then, for all $f, g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and for all $L \geq 1$,*

$$\langle T_{f,L,\beta}(\sigma) T_{g,L,\beta}(\sigma) \rangle_\beta = \left(1 + O(L^{-\frac{\alpha}{\alpha+1}}) \right) \int_{\mathbb{R}^d} f(\tilde{x}) g(\tilde{x}) d\tilde{x}. \quad (10.2.17)$$

Proof. Using Lemma 10.2.2 above, $\Sigma_L(\beta) = |\Lambda_L| \chi(\beta) (1 + O(L^{-\frac{\alpha}{\alpha+1}})) = 2^d L^d \chi(\beta) (1 + O(L^{-\frac{\alpha}{\alpha+1}}))$ as $L \rightarrow \infty$.

Assume that $\text{supp}(f) \cup \text{supp}(g) \subset B(0, R_1)$, which is clearly contained in the unit ball of

radius R_1 in the $|\cdot|_\infty$ norm. Choose $R_2 > 2R_1$. Then,

$$\langle T_{f,L,\beta}(\sigma) T_{g,L,\beta}(\sigma) \rangle_\beta = \frac{2^d}{\Sigma_L(\beta)} \sum_{x,y \in \mathbb{Z}^d} f(x/L) g(y/L) \langle \sigma_x \sigma_y \rangle_\beta \quad (10.2.18)$$

$$= \frac{2^d}{\Sigma_L(\beta)} \sum_{x,y \in \Lambda_{LR_1}} f(x/L) g(y/L) \langle \sigma_x \sigma_y \rangle_\beta \quad (10.2.19)$$

$$= \frac{2^d}{\Sigma_L(\beta)} \sum_{x \in \Lambda_{LR_1}} f(x/L) \sum_{y \in \Lambda_{LR_2}(x)} g(y/L) \langle \sigma_x \sigma_y \rangle_\beta \quad (10.2.20)$$

$$= \frac{2^d}{\Sigma_L(\beta)} \sum_{x \in \Lambda_{LR_1}} f(x/L) \sum_{y \in \Lambda_{LR_2}(x)} \left(g(y/L) - g(x/L) \right) \langle \sigma_x \sigma_y \rangle_\beta \quad (10.2.21)$$

$$+ \frac{2^d}{\Sigma_L(\beta)} \sum_{x \in \Lambda_{LR_1}} f(x/L) g(x/L) \chi_{LR_2}(\beta). \quad (10.2.22)$$

Now, by Taylor's theorem,

$$g(y/L) - g(x/L) = \nabla g(x/L) \cdot \left(\frac{y-x}{L} \right) + O\left(\frac{|y-x|^2}{L^2} \right). \quad (10.2.23)$$

One has, by the invariance of the spin-spin correlations under lattice symmetries,

$$\sum_{y \in \Lambda_{LR_2}(x)} \nabla g(x/L) \cdot \left(\frac{y-x}{L} \right) \langle \sigma_x \sigma_y \rangle_\beta = 0. \quad (10.2.24)$$

By (10.1.6), we may choose $\varepsilon = \varepsilon(\alpha) = \frac{\alpha}{\alpha+1} \in (0, \alpha \wedge 1)$ and $C > 0$ such that

$$\langle \sigma_x \sigma_y \rangle_\beta \leq \frac{C}{|x-y|^{d+\varepsilon}}. \quad (10.2.25)$$

Then, there exists $C' > 0$ such that

$$\left| \frac{2^d}{\Sigma_L(\beta)} \sum_{x \in \Lambda_{LR_1}} f(x/L) \sum_{y \in \Lambda_{LR_2}(x)} O\left(\frac{|x-y|^2}{L^2} \right) \langle \sigma_x \sigma_y \rangle_\beta \right| \quad (10.2.26)$$

$$\leq \frac{2^d}{\Sigma_L(\beta)} \sum_{x \in \Lambda_{LR_1}} |f(x/L)| \sum_{y \in \Lambda_{LR_2}(x)} O\left(\frac{|x-y|^2}{L^2} \right) \frac{C}{|x-y|^{d+\varepsilon}} \quad (10.2.27)$$

$$\leq \frac{C'}{\Sigma_L(\beta)} \sum_{x \in \Lambda_{LR_1}} |f(x/L)| O(L^{-\varepsilon}) = O(L^{-\varepsilon}). \quad (10.2.28)$$

For the second term, by Riemann approximation and (10.1.5),

$$\frac{2^d}{\Sigma_L(\beta)} \sum_{x \in \Lambda_{LR_1}} f(x/L) g(x/L) \chi_{LR_2}(\beta) = (1 + O(L^{-\varepsilon})) \int_{\mathbb{R}^d} f(\tilde{x}) g(\tilde{x}) d\tilde{x}, \quad (10.2.29)$$

from which the proof follows. $\square$

10.2.3 Proof of white noise convergence

Proof of Theorem 10.1.6. The proof of (10.1.11) follows directly from Propositions 10.2.1 and 10.2.3. Note that, by the linearity of the smeared observables, we have that, for any $f_1, \dots, f_n \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, the vector $(T_{f_i, L, \beta}(\sigma))_{1 \leq i \leq n}$ converges in distribution to a Gaussian vector distributed as $\{W_0(f) : f \in \{f_1, \dots, f_n\}\}$. Thus, we have convergence of the set of smeared observables to white noise in finite dimensional distributions. $\square$

10.3 Emergence of fractional GFF correlations

In this section, we prove Theorem 10.1.12 and Corollary 10.1.14. We treat the cases $\alpha \in \mathbb{R}_{>0} \setminus 2\mathbb{N}^*$ and $\alpha \in 2\mathbb{N}^*$ separately. We start by showing that the expectation of the rescaled, renormalised product of smeared observables converges. This heavily exploits the exact asymptotic of the two-point function in Proposition 10.1.1. We combine this with Lemma 10.A.1 to show that the limiting expression agrees, up to a constant, with the covariance kernel of the fractional GFF.

Remark 10.3.1. The computation exploits *full* rotational symmetry of the interaction and this is the main reason why the Euclidean norm $|\cdot|$ is chosen in the interaction J . Otherwise, the choice of norm appears in the limit and does not agree with the fractional GFF kernel. In fact, it is not obvious that the limiting expression can be identified with a covariance kernel (it is not a priori symmetric or positive semi-definite).

10.3.1 Case $\alpha \in \mathbb{R}_{>0} \setminus 2\mathbb{N}^*$

Proposition 10.3.2. *Let $d \geq 2$, $\alpha \in \mathbb{R}_{>0} \setminus 2\mathbb{N}^*$ and $\mathbf{C} > 0$ in the interaction (10.1.1), and $\beta < \beta_c$. Then, there exists $\kappa > 0$ sufficiently small such that, for all $f, g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and for all $L \geq 1$,*

$$L^\alpha \left\langle \mathcal{R}_{\alpha, \beta, L} [T_{f, L, \beta}(\sigma) T_{g, L, \beta}(\sigma)] \right\rangle_\beta = \frac{\beta \chi(\beta) \mathbf{C}}{2} \tilde{K}_{\alpha, d}(f, g) (1 + O(L^{-\kappa})), \quad (10.3.1)$$

where

$$\tilde{K}_{\alpha, d}(f, g) = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(\tilde{x}) |\tilde{x} - \tilde{y}|^{-d-\alpha} (g(\tilde{y}) - \text{Tay}_{[\alpha]_2} g(\tilde{y}; \tilde{x})) d\tilde{x} d\tilde{y}. \quad (10.3.2)$$

Proof. By adding and subtracting the Taylor polynomial, we find

$$\langle T_{f, L, \beta}(\sigma) T_{g, L, \beta}(\sigma) \rangle_\beta = \frac{2^d}{\Sigma_L(\beta)} \sum_{x \in \mathbb{Z}^d} f(x/L) \sum_{y \in \mathbb{Z}^d \setminus \{x\}} (g(y/L) - \text{Tay}_{[\alpha]_2} g(y/L; x/L)) \langle \sigma_x \sigma_y \rangle_\beta \quad (10.3.3)$$

$$+ \frac{2^d}{\Sigma_L(\beta)} \sum_{x, y \in \mathbb{Z}^d} f(x/L) \text{Tay}_{[\alpha]_2} g(y/L; x/L) \langle \sigma_x \sigma_y \rangle_\beta. \quad (10.3.4)$$

Note that

$$\left\langle \mathcal{R}_{\alpha, \beta, L} [T_{f, L, \beta}(\sigma) T_{g, L, \beta}(\sigma)] \right\rangle_\beta \quad (10.3.5)$$

$$= \langle T_{f, L, \beta}(\sigma) T_{g, L, \beta}(\sigma) \rangle_\beta - \frac{2^d}{\Sigma_L(\beta)} \sum_{x, y \in \mathbb{Z}^d} f(x/L) \text{Tay}_{[\alpha]_2} g(y/L; x/L) \langle \sigma_x \sigma_y \rangle_\beta. \quad (10.3.6)$$

We analyse the remainder term. First, for $\varepsilon \in (0, 1)$,

$$T(L) := \frac{2^d}{\Sigma_L(\beta)} \sum_{x \in \mathbb{Z}^d} f(x/L) \sum_{y \in \mathbb{Z}^d \setminus \{x\}} \left(g(y/L) - \text{Taylor}_{[\alpha]_2} g(y/L; x/L) \right) \langle \sigma_x \sigma_y \rangle_\beta \quad (10.3.7)$$

$$= \frac{2^d}{\Sigma_L(\beta)} \sum_{x \in \mathbb{Z}^d} f(x/L) \sum_{y \in \Lambda_{L^\varepsilon}(x)} \left(g(y/L) - \text{Taylor}_{[\alpha]_2} g(y/L; x/L) \right) \langle \sigma_x \sigma_y \rangle_\beta \quad (10.3.8)$$

$$+ \frac{2^d}{\Sigma_L(\beta)} \sum_{x \in \mathbb{Z}^d} f(x/L) \sum_{y \in (\Lambda_{L^\varepsilon}(x))^c} \left(g(y/L) - \text{Taylor}_{[\alpha]_2} g(y/L; x/L) \right) \langle \sigma_x \sigma_y \rangle_\beta \quad (10.3.9)$$

$$=: \frac{2^d}{\Sigma_L(\beta)} \sum_{x \in \mathbb{Z}^d} f(x/L) T_1(x, L) + f(x/L) T_2(x, L). \quad (10.3.10)$$

Fix $x \in \mathbb{Z}^d$. Then, by Taylor's Theorem applied to $g(y/L)$ expanded around x to order $[\alpha]_2 + 2$, there exists $z(x) \in \mathbb{R}^d$ such that

$$T_1(x, L) = \sum_{y \in \Lambda_{L^\varepsilon}(x)} \left(\frac{1}{([\alpha]_2 + 1)!} \nabla^{[\alpha]_2 + 1} g(x/L)(y/L - x/L, \dots, y/L - x/L) \right. \quad (10.3.11)$$

$$\left. + \frac{1}{([\alpha]_2 + 2)!} \nabla^{[\alpha]_2 + 2} g(z(x)/L)(y/L - x/L, \dots, y/L - x/L) \right) \langle \sigma_x \sigma_y \rangle_\beta. \quad (10.3.12)$$

By symmetry of the two-point function,

$$\sum_{y \in \Lambda_{L^\varepsilon}(x)} \frac{1}{([\alpha]_2 + 1)!} \nabla^{[\alpha]_2 + 1} g(x/L)(y/L - x/L, \dots, y/L - x/L) \langle \sigma_x \sigma_y \rangle_\beta = 0. \quad (10.3.13)$$

Hence, by the upper bound on spin correlations (10.1.6), we have for some $C_1 > 0$,

$$\sup_{x \in \mathbb{Z}^d} |T_1(x, L)| \leq \frac{C_1 \|\nabla^{[\alpha]_2 + 2} g\|_{L^\infty}}{L^{[\alpha]_2 + 2}} \sum_{y \in \Lambda_{L^\varepsilon}(x)} \frac{1}{|x - y|^{d + \alpha - [\alpha]_2 - 2}} = O\left(\frac{L^{\varepsilon([\alpha]_2 + 2 - \alpha)}}{L^{[\alpha]_2 + 2}}\right) \quad (10.3.14)$$

Thus, by the above displays together with Proposition 10.1.1 and Lemma 10.2.2, choosing $\kappa > 0$ sufficiently small, we have that

$$L^\alpha \left\langle \mathcal{R}_{\alpha, \beta, L} [T_{f, L, \beta}(\sigma) T_{g, L, \beta}(\sigma)] \right\rangle_\beta \quad (10.3.15)$$

$$= \frac{\beta \chi(\beta)^2 \mathbf{C}}{2} \frac{2^d}{\Sigma_L(\beta) L^d} \sum_{x \in \mathbb{Z}^d} f(x/L) \sum_{y \in (\Lambda_{L^\varepsilon}(x))^c} \frac{g(y/L) - \text{Taylor}_{[\alpha]_2} g(y/L; x/L)}{|x/L - y/L|^{d + \alpha}} (1 + O(|x - y|^{-\delta})) \quad (10.3.16)$$

$$+ \frac{2^d}{\Sigma_L(\beta)} \sum_{x \in \mathbb{Z}^d} f(x/L) O\left(\frac{L^{\varepsilon([\alpha]_2 + 2 - \alpha)}}{L^{[\alpha]_2 + 2 - \alpha}}\right) \quad (10.3.17)$$

$$= \frac{\beta \chi(\beta) \mathbf{C}}{2} \tilde{K}_{\alpha, d}(f, g) (1 + O(L^{-\kappa})), \quad (10.3.18)$$

which concludes the proof. $\square$

10.3.2 Case $\alpha \in 2\mathbb{N}^*$

Proposition 10.3.3. *Let $d \geq 2$, $\alpha \in 2\mathbb{N}^*$ and $\mathbf{C} > 0$ in the interaction (10.1.1), and let $\beta < \beta_c$. Then, there exists $\kappa > 0$ sufficiently small such that, for all $f, g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and for all $L \geq 1$*

$$\frac{L^\alpha}{\log L} \left\langle \tilde{\mathcal{R}}_{\alpha, \beta, L} [T_{f, L, \beta}(\sigma) T_{g, L, \beta}(\sigma)] \right\rangle_\beta = (-1)^{\alpha/2} (4\pi^2)^{\alpha/2} H_{\alpha/2} \frac{\beta \chi(\beta) \mathbf{C}}{2} K_{\alpha, d}(f, g) \left(1 + O((\log L)^{-\kappa}) \right) \quad (10.3.19)$$

where $K_{\alpha, d}$ is defined in Definition 10.1.9.

Proof. We use the same notations as in the proof of Proposition 10.3.2, but with L^ε replaced by $\log L$ in the definitions of $T_1(x, L)$ and $T_2(x, L)$. By Taylor's Theorem (notice that in this case $\lfloor \alpha \rfloor_2 + 2 = \alpha$) and symmetry considerations, we have for some $C_1 > 0$,

$$\sup_{x \in \mathbb{Z}^d} |T_1(x, L)| \leq \frac{C_1 \|\nabla^\alpha g\|_{L^\infty}}{L^\alpha} \sum_{y \in \Lambda_{\log L}(x)} \frac{1}{|x - y|^d} = O\left(\frac{\log \log L}{L^\alpha}\right). \quad (10.3.20)$$

Hence, by Riemann approximation,

$$\frac{L^\alpha}{\log L} \frac{2^d}{\Sigma_L(\beta)} \sum_{x \in \mathbb{Z}^d} f(x/L) T_1(x, L) = O\left(\frac{\log \log L}{\log L}\right). \quad (10.3.21)$$

We now analyse

$$\frac{L^\alpha}{\log L} \frac{2^d}{\Sigma_L(\beta)} \sum_{x \in \mathbb{Z}^d} f(x/L) T_2(x, L). \quad (10.3.22)$$

Fix $x \in \mathbb{Z}^d$ and write

$$T_2(x, L) = T_{2,1}(x, L) + T_{2,2}(x, L), \quad (10.3.23)$$

where

$$T_{2,1}(x, L) := \sum_{y \in (\Lambda_L(x))^c} \left(g(y/L) - \text{Tay}_{\lfloor \alpha \rfloor_2} g(y/L; x/L) \right) \langle \sigma_x \sigma_y \rangle_\beta, \quad (10.3.24)$$

$$T_{2,2}(x, L) := \sum_{y \in (\Lambda_{\log L}(x))^c \cap (\Lambda_L(x))} \left(g(y/L) - \text{Tay}_{\lfloor \alpha \rfloor_2} g(y/L; x/L) \right) \langle \sigma_x \sigma_y \rangle_\beta. \quad (10.3.25)$$

By Taylor's Theorem, there exists $C_2 > 0$ such that for all $x, y \in \mathbb{Z}^d$,

$$\left| g(y/L) - \text{Tay}_{\lfloor \alpha \rfloor_2} g(y/L; x/L) \right| \leq C_2 \|\nabla^{\alpha-2} g\|_{L^\infty} |x - y|^{\alpha-2} L^{-\alpha+2}. \quad (10.3.26)$$

Hence, using (10.1.6) and translation invariance,

$$\sup_{x \in \mathbb{Z}^d} |T_{2,1}(x, L)| \leq C_3 L^{-\alpha+2} \sum_{y \in \Lambda_L^c} |y|^{\alpha-2} \langle \sigma_0 \sigma_y \rangle_\beta = O(L^{-\alpha}). \quad (10.3.27)$$

Moreover, by Taylor's Theorem and symmetry considerations,

$$T_{2,2}(x, L) = \sum_{y \in (\Lambda_{\log L}(x))^c \cap (\Lambda_L(x))} \left(\frac{1}{\alpha!} \nabla^\alpha g(x/L)(y/L - x/L, \dots, y/L - x/L) \right. \quad (10.3.28)$$

$$\left. + O\left(\|\nabla^{\alpha+2} g\|_{L^\infty} |x - y|^{\alpha+2} L^{-(\alpha+2)}\right) \right) \langle \sigma_x \sigma_y \rangle_\beta \quad (10.3.29)$$

$$= \sum_{y \in (\Lambda_{\log L}(x))^c \cap (\Lambda_L(x))} \frac{1}{\alpha!} \nabla^\alpha g(x/L)(y/L - x/L, \dots, y/L - x/L) \langle \sigma_x \sigma_y \rangle_\beta + O(L^{-\alpha}). \quad (10.3.30)$$

Hence,

$$\frac{L^\alpha}{\log L} T_{2,2}(x, L) = \sum_{y \in (\Lambda_{\log L}(x))^c \cap (\Lambda_L(x))} \frac{\beta \chi(\beta)^2 \mathbf{C}}{2\alpha!} \quad (10.3.31)$$

$$\times \frac{1}{\log L} \nabla^\alpha g(x/L)(y - x, \dots, y - x) \frac{1 + O(|x - y|^{-\delta})}{|x - y|^{d+\alpha}} + O((\log L)^{-1}) \quad (10.3.32)$$

$$= \frac{\beta \chi(\beta)^2 \mathbf{C}}{2} (1 + O((\log L)^{-\delta})) \quad (10.3.33)$$

$$\times \sum_{\substack{\gamma \in \mathbb{N}^d \\ |\gamma|_1 = \alpha}} \frac{1}{\gamma!} \partial^\gamma g(x/L) \frac{1}{\log L} \sum_{y \in (\Lambda_{\log L}(x))^c \cap (\Lambda_L(x))} \frac{(y - x)^\gamma}{|x - y|^{d+\alpha}} + O((\log L)^{-1}), \quad (10.3.34)$$

where in the second equality we use that, for $\tilde{z} \in \mathbb{R}^d$ and $j \geq 0$,

$$\frac{1}{j!} \nabla^j g(\tilde{x})(\tilde{z}, \dots, \tilde{z}) = \sum_{|\gamma|_1 = j} \frac{1}{\gamma!} \partial^\gamma g(\tilde{x}) \tilde{z}^\gamma, \quad (10.3.35)$$

and where the sum is over multi-indices $\gamma = (\gamma_1, \dots, \gamma_d) \in \mathbb{N}^d$, $\partial^\gamma = \partial_1^{\gamma_1} \dots \partial_d^{\gamma_d}$, $\tilde{z}^\gamma = \prod_{k=1}^d \tilde{z}_k^{\gamma_k}$, and $\gamma! := \gamma_1! \dots \gamma_d!$. Furthermore: i) note that by symmetry, we may restrict the above sum to $\gamma \in (2\mathbb{N})^d$; and, ii) up to an error of order $O(\log \log L / \log L)$, we can sum over $y \in (\Lambda_L(x)) \setminus \{x\}$ above.

For each $k \in \mathbb{N}^*$ and $\tilde{y} \in \mathbb{R}^d$, denote by

$$\text{SS}(\tilde{y}, k) = \{\tilde{x} \in \mathbb{R}^d : k - 1 \leq |\tilde{x} - \tilde{y}| < k\} \quad (10.3.36)$$

the spherical shell (closed at the interior boundary and open at the exterior boundary) of radius k and centred at $\tilde{y} \in \mathbb{R}^d$. Observe that $\bigsqcup_k \{\text{SS}(0, k) \cap \mathbb{Z}^d\} = \mathbb{Z}^d$. Thus, for any $\gamma \in (2\mathbb{N})^d$ such that $|\gamma|_1 = \alpha$, we have

$$\sum_{y \in (\Lambda_L(x)) \setminus \{x\}} \frac{\prod_{i=1}^d (y_i - x_i)^{\gamma_i}}{|x - y|^{d+\alpha}} = \sum_{r=1}^L \frac{1}{r} \frac{1}{r^{d-1}} \sum_{y \in \text{SS}(x, r) \cap (\Lambda_L(x))} \prod_{i=1}^d \left(\frac{y_i - x_i}{|x - y|} \right)^{\gamma_i} \quad (10.3.37)$$

$$= \log L \cdot \Omega_d \int_{\tilde{y} \in S(0,1)} \tilde{y}_1^{\gamma_1} \dots \tilde{y}_d^{\gamma_d} d\mu(\tilde{y}) (1 + O(L^{-1})), \quad (10.3.38)$$

where the last line is by Riemann approximation, μ is the normalised Haar measure on the sphere $S(0, 1)$, and Ω_d is the surface area of $S(0, 1)$, explicitly given by

$$\Omega_d := \frac{2\pi^{d/2}}{\Gamma(d/2)}. \quad (10.3.39)$$

We have thus obtained,

$$\frac{L^\alpha}{\log L} T_{2,2}(x, L) = \frac{\beta\chi(\beta)^2 \mathbf{C}}{2} \Omega_d \sum_{|\gamma|_1=\alpha} \frac{1}{\gamma!} \partial^\gamma g(x/L) \int_{\tilde{y} \in S(0,1)} \tilde{y}_1^{\gamma_1} \dots \tilde{y}_d^{\gamma_d} d\mu(\tilde{y}) \left(1 + O\left((\log L)^{-\delta}\right)\right). \quad (10.3.40)$$

Using symmetry considerations for the spherical integrals above as in the proof of Lemma 10.A.1, we have

$$\frac{L^\alpha}{\log L} T_{2,2}(x, L) = \frac{\beta\chi(\beta)^2 \mathbf{C}}{2} H_{\alpha/2} \Delta^{\alpha/2} g(\tilde{x}) \left(1 + O\left((\log L)^{-\delta}\right)\right). \quad (10.3.41)$$

Hence,

$$\frac{L^\alpha}{\log L} \left\langle \mathcal{R}_{\alpha,\beta,L} [T_{f,L,\beta}(\sigma) T_{g,L,\beta}(\sigma)] \right\rangle_\beta = \frac{\beta\chi(\beta) \mathbf{C}}{2} H_{\alpha/2} \int_{\mathbb{R}^d} f(\tilde{x}) \Delta^{\alpha/2} g(\tilde{x}) d\tilde{x} \left(1 + O\left((\log L)^{-\delta}\right)\right). \quad (10.3.42)$$

□

10.3.3 Proof of emergence of fractional GFF correlations

Proof of Theorem 10.1.12. The proof of Theorem 10.1.12 is a direct consequence of Propositions 10.3.2 and 10.3.3, together with Lemma 10.A.1. □

10.A Rotational symmetry and Taylor's theorem

Lemma 10.A.1. *Let $d \geq 2$ and $\alpha > 0$. Then, for any $f, g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |\tilde{x} - \tilde{y}|^{-(d+\alpha)} \left(f(\tilde{x})g(\tilde{y}) - f(\tilde{x})\text{Tay}_{[\alpha]_2} g(\tilde{y}; \tilde{x}) \right) d\tilde{x} d\tilde{y} \quad (10.A.1)$$

$$= \int_{\mathbb{R}^d \times \mathbb{R}^d} |\tilde{x} - \tilde{y}|^{-(d+\alpha)} \left(f(\tilde{x})g(\tilde{y}) - \sum_{j=0}^{[\alpha]_2/2} f(\tilde{x})H_j \Delta^j g(\tilde{x}) |\tilde{x} - \tilde{y}|^{2j} \right) d\tilde{x} d\tilde{y} \quad (10.A.2)$$

where the H_j 's are defined by (10.1.16).

Proof. For $\tilde{x} \in \mathbb{R}^d$,

$$\begin{aligned} & \int_{\mathbb{R}^d} |\tilde{x} - \tilde{y}|^{-(d+\alpha)} \left(g(\tilde{y}) - \text{Tay}_{[\alpha]_2} g(\tilde{y}; \tilde{x}) \right) d\tilde{y} \\ &= \int_{\mathbb{R}^d} |\tilde{x} - \tilde{y}|^{-(d+\alpha)} \left(g(\tilde{y}) - \sum_{j=0}^{[\alpha]_2/2} \frac{1}{(2j)!} \nabla^{2j} g(\tilde{x})(\tilde{y} - \tilde{x}, \dots, \tilde{y} - \tilde{x}) \right) d\tilde{y}. \end{aligned}$$

If $\varepsilon > 0$, notice that for $j \in \{0, \dots, \lfloor \alpha \rfloor/2\}$,

$$\begin{aligned} \frac{1}{(2j)!} \int_{B(x, \varepsilon)^c} |\tilde{x} - \tilde{y}|^{-(d+\alpha)} \nabla^{2j} g(\tilde{x})(\tilde{y} - \tilde{x}, \dots, \tilde{y} - \tilde{x}) d\tilde{y} \\ = \Omega_d \int_{\varepsilon}^{\infty} \frac{r^{d-1}}{r^{d+\alpha}} dr \sum_{|\gamma|_1=2j} \frac{1}{\gamma!} \partial_1^{\gamma_1} \dots \partial_d^{\gamma_d} g(\tilde{x}) \int_{S(0,1)} \tilde{z}_1^{\gamma_1} \dots \tilde{z}_d^{\gamma_d} d\mu(\tilde{z}). \end{aligned}$$

Note that in the sum above, by symmetry considerations we may restrict to multi-indices γ such that γ_k is even for all $1 \leq k \leq d$.

By a standard computation,

$$\int_{S(0,1)} \tilde{z}_1^{\gamma_1} \dots \tilde{z}_d^{\gamma_d} d\mu(\tilde{z}) = \frac{1}{d(d+2) \dots (d+2j-2)} \prod_{k=1}^d (\gamma_k - 1)!!, \quad (10.A.3)$$

where we recall that for $\ell \in \mathbb{N}$ odd, $\ell!! = \ell(\ell-2) \dots 1$. It remains to show that

$$\sum_{|\gamma|_1=2j, \gamma_k \text{ even}} \frac{1}{\gamma!} \partial_1^{\gamma_1} \dots \partial_d^{\gamma_d} g(\tilde{x}) \prod_{k=1}^d (\gamma_k - 1)!! = \frac{1}{2^j j!} \Delta^j g(\tilde{x}). \quad (10.A.4)$$

Indeed, notice that for $|\gamma|_1 = 2j$ with γ_k even for $1 \leq k \leq d$,

$$\prod_{k=1}^d (\gamma_k - 1)!! = \frac{1}{2^j} \prod_{k=1}^d \frac{\gamma_k!}{(\gamma_k/2)!}, \quad (10.A.5)$$

so that the left-hand side in (10.A.4) rewrites

$$\sum_{|\gamma|_1=2j, \gamma_k \text{ even}} \frac{1}{\gamma!} \partial_1^{\gamma_1} \dots \partial_d^{\gamma_d} g(\tilde{x}) \prod_{k=1}^d (\gamma_k - 1)!! = \frac{1}{2^j} \sum_{|\gamma|_1=2j, \gamma_k \text{ even}} \partial_1^{\gamma_1} \dots \partial_d^{\gamma_d} g(\tilde{x}) \prod_{k=1}^d \frac{1}{(\gamma_k/2)!} \quad (10.A.6)$$

Finally, since

$$\Delta^j g(\tilde{x}) = \sum_{i_1, \dots, i_j \in \{1, \dots, d\}} \partial_{i_1 i_1 \dots i_j i_j}^{2j} g(\tilde{x}), \quad (10.A.7)$$

we get

$$\begin{aligned} \sum_{|\gamma|_1=2j, \gamma_k \text{ even}} \partial_1^{\gamma_1} \dots \partial_d^{\gamma_d} g(\tilde{x}) \prod_{k=1}^d \frac{1}{(\gamma_k/2)!} &= \sum_{|\gamma|_1=2j, \gamma_k \text{ even}} \partial_1^{\gamma_1} \dots \partial_d^{\gamma_d} g(\tilde{x}) \binom{j}{\gamma_1/2, \dots, \gamma_d/2} \frac{1}{j!} \\ &= \frac{1}{j!} \sum_{|\gamma|_1=2j, \gamma_k \text{ even}} \sum_{\substack{i_1, \dots, i_j \in \{1, \dots, d\} \\ |\{1 \leq \ell \leq j, i_\ell = k\}| = \gamma_k/2}} \partial_{i_1 i_1 \dots i_j i_j}^{2j} g(\tilde{x}) \\ &= \frac{1}{j!} \Delta^j g(\tilde{x}). \end{aligned}$$

Using the definition of H_j (10.1.16),

$$\int_{B(x, \varepsilon)^c} |\tilde{x} - \tilde{y}|^{-(d+\alpha)} \nabla^{2j} g(\tilde{x})(\tilde{y} - \tilde{x}, \dots, \tilde{y} - \tilde{x}) d\tilde{y} = \int_{B(x, \varepsilon)^c} |\tilde{x} - \tilde{y}|^{-(d+\alpha)} H_j \Delta^j g(\tilde{x}) |\tilde{x} - \tilde{y}|^{2j} d\tilde{y}. \quad (10.A.8)$$

In order to conclude, we write

$$\int_{\mathbb{R}^d} |\tilde{x} - \tilde{y}|^{-(d+\alpha)} \left(g(\tilde{y}) - \text{Taylor}_{[\alpha]_2} g(\tilde{y}; \tilde{x}) \right) d\tilde{y} = \lim_{\epsilon \downarrow 0} \int_{B(x, \epsilon)^c} |\tilde{x} - \tilde{y}|^{-(d+\alpha)} \left(g(\tilde{y}) - \text{Taylor}_{[\alpha]_2} g(\tilde{y}; \tilde{x}) \right) d\tilde{y}, \quad (10.A.9)$$

use the considerations above, and take ϵ to 0. $\square$

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Part III

The φ^4 model

Chapter 11

Random tangled currents

We develop a novel geometric representation for the φ^4 model called the *random tangled current* representation. It is analogous to the random current representation of the Ising model. We then use it to prove that the set of automorphism invariant Gibbs measures of the φ^4 model on graphs of polynomial growth has at most two extremal measures at all values of β . We also give a sufficient condition to ensure that the set of all Gibbs measures is a singleton. As an application, we show that the spontaneous magnetisation of the nearest-neighbour φ^4 model on $\mathbb{Z}^d$ vanishes at criticality for $d \geq 3$.

This chapter is based on the preprint “*Random tangled currents for φ^4 : translation invariant Gibbs measures and continuity of the phase transition*” [126]. It is a joint work with Trishen S. Gunaratnam, Christoforos Panagiotis, and Franco Severo.

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11.1 Introduction

11.1.1 Motivation

The φ^4 model is a statistical mechanics model of ferromagnetism with real-valued spins attached to each vertex of a graph whose values are confined according to a quartic single-site potential. Mathematically, this is modelled by a probability measure on real-valued spin configurations on a graph $G = (V, E)$ formally given by

$$\langle F(\varphi) \rangle = \frac{1}{Z} \int F(\varphi) \exp \left(\beta \sum_{\{x,y\} \in E} \varphi_x \varphi_y - \sum_{x \in V} g \varphi_x^4 + a \varphi_x^2 \right) d\varphi \quad (11.1.1)$$

where $g > 0$ and $a \in \mathbb{R}$ are coupling constants, $\beta > 0$ is the inverse temperature, and Z is a normalisation constant. This model naturally interpolates between two famous models of statistical physics, namely the Gaussian free field and the Ising model. The Gaussian free field is defined by setting $g = 0$ and taking appropriate a . Formally, in the limit $a \rightarrow \infty$ when $g > 0$, the φ^4 model approximates a Gaussian free field of appropriate covariance. On the other hand, the Ising model is defined by integrating instead with the respect to the Bernoulli measure on $\{\pm 1\}$. In this case, the limit $g \rightarrow \infty$ for $a < 0$ formally corresponds to the Ising model. This can be seen easily if one takes a joint scaling limit when $a = -2g$:

$$\frac{1}{2}(\delta_{-1} + \delta_1) = \lim_{g \rightarrow \infty} \frac{1}{Z_g} e^{-g(\varphi^2 - 1)^2} d\varphi. \quad (11.1.2)$$

The model has its origins in Euclidean quantum field theory, where its analogue on $\mathbb{R}^d$ is amongst the simplest candidates for a non-trivial (i.e. non-Gaussian) measure. These measures in dimensions $d \geq 2$ are a priori ill-defined due to the notorious problem of ultraviolet divergences [113]. Two major breakthroughs in the constructive field theory programme of the 1960s-1970s were the solutions to these problems in dimensions 2 and 3 due to Nelson [195], and Glimm and Jaffe [110]. In dimensions $d \geq 4$ a non-Gaussian measure cannot be obtained due to the triviality results that we mention below.

It was later realised that the φ^4 model on the lattice $\mathbb{Z}^d$ has importance in statistical physics. Its viewpoint as a generalised Ising model was first rigourously explored in the landmark paper by Guerra, Rosen, and Simon [125]. The approximation to Ising was further exploited by Glimm, Jaffe, and Spencer [114], where they adapt the celebrated Peierls' argument [208] to prove that the phase transition occurs at a finite non-zero temperature for φ^4 in $d \geq 2$. In particular, there is a critical inverse temperature β_c , depending on the fixed values of g, a , such that: for $\beta < \beta_c$, the two point correlation tends to 0; and for $\beta > \beta_c$, the two point correlation does not tend to 0 and there is long-range order.

The relationship between φ^4 and the Ising model is predicted to be very deep: they are a fundamental example of two models that are believed to belong to the same universality class. Indeed, it is predicted from renormalisation group heuristics (see [119, 160] or the recent book [29]) that at their respective critical points many properties of these two models, such as critical exponents, exactly coincide. A manifestation of this was discovered by Griffiths and Simon in [224], where they show that the φ^4 model arises as a certain near-critical scaling limit of mean-field Ising models. Indeed, this allows one to transfer many special properties of the Ising model, such as correlation inequalities, to φ^4 . This makes it one of the simplest non-integrable models to study, even in spite of the unbounded spins.

More direct results towards universality were obtained in $d \geq 5$ by Aizenman [4] and Fröhlich [102], where triviality of their scaling limits, both in the infinite volume and continuum, were established. The case of $d = 4$ was settled in the beautiful paper by Aizenman and Duminil-Copin [9], where stochastic geometric representations for the model, as in this chapter, play a central role. Establishing rigourously universality results in dimensions 2 and 3 remains a difficult challenge. We refer to the recent review [6] concerning φ^4 for further discussion.

In this chapter, we are interested in understanding the structure of Gibbs measures for φ^4 , and in particular identifying the behaviour at the critical point. To do this, we build on ideas that have led to recent breakthroughs towards these questions in the case of the Ising model. In particular, the continuity of the magnetisation at the critical point in $d \geq 3$ was established by Aizenman, Duminil-Copin, and Sidoravicius [10] using the so-called random current representation, first introduced by Aizenman in [4]. These techniques were further exploited by Raoufi in [211] to show that the set of translation invariant Gibbs measures on any amenable graph at any temperature is given by convex combinations of two natural extremal measures. This was previously known for temperatures not equal to the critical point on $\mathbb{Z}^d$, see the results of Bodineau [37]. First, we rigourously define the quantities of interest and state our main result on the structure of Gibbs measures. Then, we state its consequences for continuity of the phase transition (as well as a precise definition of what this means).

11.1.2 Definitions and structure theorem

Let $G = (V, E)$ be a connected, infinite, and locally finite graph. Let $J = (J_{x,y})_{x,y \in V}$ be a collection of ferromagnetic interactions, i.e. such that $J_{x,y} \geq 0$ for all $x, y \in V$. For $g > 0$ and $a \in \mathbb{R}$, let $\rho_{g,a}$ be the single-site measure on $\mathbb{R}$ defined by

$$d\rho_{g,a}(t) = \frac{1}{z_{g,a}} e^{-gt^4 - at^2} dt \quad (11.1.3)$$

where $z_{g,a} = \int_{\mathbb{R}} e^{-gt^4 - at^2} dt$. Let $\Lambda \subset V$ be finite. Let $h = (h_x)_{x \in V}, \eta = (\eta_x)_{x \in V} \in \mathbb{R}^V$ be the external magnetic field and boundary condition, respectively. We define the Hamiltonian

$$H_{\Lambda,h,J}^{\eta}(\varphi) := - \sum_{x,y \in \Lambda} J_{x,y} \varphi_x \varphi_y - \sum_{\substack{x \in \Lambda \\ y \in V \setminus \Lambda}} J_{x,y} \varphi_x \eta_y - \sum_{x \in \Lambda} h_x \varphi_x, \quad \forall \varphi \in \mathbb{R}^{\Lambda} \quad (11.1.4)$$

whenever $\sum_{y \in V \setminus \Lambda} J_{x,y} |\eta_y| < \infty \quad \forall x \in \Lambda$.

The φ^4 model on Λ with coupling constants (J, g, a) , boundary condition η , inverse temperature $\beta > 0$, and external magnetic field h , is the measure $\langle \cdot \rangle_{\Lambda, \beta, h, J, g, a}^{\eta}$ defined via the expectation values

$$\langle f \rangle_{\Lambda, \beta, h, J, g, a}^{\eta} := \frac{1}{Z_{\Lambda, \beta, h, J, g, a}^{\eta}} \int_{\varphi \in \mathbb{R}^{\Lambda}} f(\varphi) e^{-\beta H_{\Lambda, h, J}^{\eta}(\varphi)} d\rho_{\Lambda, g, a}(\varphi) \quad (11.1.5)$$

where $d\rho_{\Lambda, g, a}(\varphi) = \prod_{x \in \Lambda} d\rho_{g, a}(\varphi_x)$, $f : \mathbb{R}^{\Lambda} \rightarrow \mathbb{R}$ is a bounded measurable function and $Z_{\Lambda, \beta, h, J, g, a}^{\eta}$ is the partition function, i.e. the normalisation constant such that $\langle 1 \rangle_{\Lambda, \beta, h, J, g, a}^{\eta} = 1$. When clear from context we drop (J, g, a) from the notation.

In this chapter, we impose a restriction on the growth of the graph. We fix $o \in G$ a distinguished origin and let B_k denote the ball of radius k centred at o with respect to the graph distance d_G . Let G be a *vertex-transitive graph* of polynomial growth, i.e. a graph for which there exists $d \geq 1$ (called the dimension of G) and $c, C > 0$ such that, for all $k \geq 1$,

$$ck^d \leq |B_k| \leq Ck^d. \quad (11.1.6)$$

As such, G is amenable:

$$\inf_{A \subset V, |A| < \infty} \frac{|\partial A|}{|A|} = 0 \quad (11.1.7)$$

where $\partial A = \{x \in V \setminus A \text{ such that there exists } y \in A \text{ with } \{x, y\} \in E\}$ and $|\cdot|$ denotes cardinality. We fix Γ a vertex-transitive subgroup of the automorphism group of G , denoted $\text{Aut}(G)$.

A set of interactions $J = (J_{x,y})_{x,y \in V}$ is *admissible* with respect to G and Γ if:

- C1 (Ferromagnetic)** $J_{x,y} \geq 0$ for all $x, y \in V$;
- C2 (Γ -invariance)** $J_{\gamma(x), \gamma(y)} = J_{x,y}$ for all $\gamma \in \Gamma$;
- C3 (Irreducibility)** for every $x, y \in V$, there exists a sequence $x_0 = x, \dots, x_k = y$ such that $J_{x_0, x_1} \dots J_{x_{k-1}, x_k} > 0$;
- C4 (Integrability)** $\exists C > 0$ and $\varepsilon > 0$ such that $J_{x,y} \leq Cd_G(x, y)^{-(d+\varepsilon)}$ for all $x \neq y \in V$.

We remark that **(C4)** indeed implies that $\sum_{x \in V} J_{o,x} < \infty$, see the calculation in Lemma 11.C.7.

We say that a Borel measure ν on $\mathbb{R}^V$ is *tempered* if:

- for every $\Lambda \subset V$ finite, $\nu|_\Lambda$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^\Lambda$;
- ν is supported on the event

$$R = \left\{ (\varphi_x)_{x \in V} : \exists C \in (0, \infty), \sum_{x \in B_k} \varphi_x^2 \leq C|B_k| \text{ for all } k \in \mathbb{N} \right\} \quad (11.1.8)$$

i.e. $\nu(R) = 1$.

Definition 11.1.1 (Gibbs measures). We say that a Borel probability measure ν on $\mathbb{R}^V$ is a Gibbs measure at (β, h) if it is tempered and for every finite $\Lambda \subset V$ and $f \in \mathbb{R}^\Lambda$ bounded and measurable, the DLR equation

$$\nu(f) = \int_{\eta \in \mathbb{R}^V} \langle f \rangle_{\Lambda, \beta, h}^\eta d\nu(\eta) \quad (11.1.9)$$

holds. In particular, we assume that ν is almost surely supported on $\eta \in \mathbb{R}^V$ such that $\langle \cdot \rangle_{\Lambda, \beta, h}^\eta$ is well-defined.

We denote by $\mathcal{G}(\beta, h)$ the set of Gibbs measures at (β, h) . We say that a measure $\nu \in \mathcal{G}(\beta, h)$ is Γ -invariant if for all $\gamma \in \Gamma$, the push-forward measure $\gamma_*\nu$ is equal to ν . We denote by $\mathcal{G}_\Gamma(\beta, h) \subset \mathcal{G}(\beta, h)$ the set of Γ -invariant Gibbs measures. The set $\mathcal{G}_\Gamma(\beta, h)$ has a structure of a Choquet simplex, i.e. admits a *unique* extremal ergodic decomposition (see [212, Theorem 5.8]).

Our main result is a structure theorem for $\mathcal{G}_\Gamma(\beta, 0) = \mathcal{G}_\Gamma(\beta)$. It states that there are at most two extremal measures $\langle \cdot \rangle_\beta^+$ and $\langle \cdot \rangle_\beta^-$ which we define as follows. First, for any $h \in \mathbb{R}$, let $\langle \cdot \rangle_{\beta, h}^0 = \lim_{\Lambda \uparrow V} \langle \cdot \rangle_{\Lambda, \beta, h}^0$, which exists by tightness (see Proposition 11.C.4) and a standard monotonicity argument using Griffiths' inequality (see Proposition 11.3.17). Define $\langle \cdot \rangle_\beta^+$ and $\langle \cdot \rangle_\beta^-$ via the following monotonic weak limits (again by Griffiths' inequality, see Remark 11.3.19):

$$\langle \cdot \rangle_\beta^+ = \lim_{h \downarrow 0} \langle \cdot \rangle_{\beta, h}^0, \quad \langle \cdot \rangle_\beta^- = \lim_{h \uparrow 0} \langle \cdot \rangle_{\beta, h}^0. \quad (11.1.10)$$

The measures $\langle \cdot \rangle_\beta^+$ and $\langle \cdot \rangle_\beta^-$ are ergodic (in fact mixing) and extremal – see Proposition 11.2.8.

Theorem 11.1.2. *Let G be a connected, vertex-transitive graph of polynomial growth and $\Gamma \subset \text{Aut}(G)$ a vertex-transitive subgroup. Let J be an admissible set of coupling constants with respect to G and Γ , and fix $g > 0$ and $a \in \mathbb{R}$. Then, for any $\beta > 0$,*

$$\mathcal{G}_\Gamma(\beta) = \left\{ \lambda \langle \cdot \rangle_\beta^+ + (1 - \lambda) \langle \cdot \rangle_\beta^-, \lambda \in [0, 1] \right\}. \quad (11.1.11)$$

Moreover, if

$$\liminf_{d_G(o, x) \rightarrow \infty} \langle \varphi_o \varphi_x \rangle_\beta^0 = 0 \quad (11.1.12)$$

then there exists only one Gibbs measure, i.e.

$$|\mathcal{G}(\beta)| = |\mathcal{G}_\Gamma(\beta)| = 1. \quad (11.1.13)$$

To the best of our knowledge, Theorem 11.1.2 is the first result that characterises the set of translation invariant Gibbs measures for φ^4 at any β . It has natural consequences for the continuity of the phase transition, i.e. condition (11.1.12) can be verified at criticality in certain cases; these are explored in the next subsection. We believe that certain ideas in the proof may be useful to study the universality between φ^4 and Ising, and perhaps of use in studying $P(\varphi)$ models (i.e. when the quartic polynomial in the single-site measure is replaced by an arbitrary even polynomial P of positive leading coefficient). We also remark that this seems to be the first instance where the φ^4 model is studied on general graphs beyond $\mathbb{Z}^d$ and the techniques developed may be useful to study universality with respect to the underlying choice of graph, i.e. microscopic structure. The restriction to (amenable) graphs of polynomial growth is for technical reasons; our proof of the construction of general Gibbs measures works under certain geometric conditions, but we cannot give an example of such an amenable graph which exhibits superpolynomial growth. We refer to Appendix 11.C for further discussion. However, the main techniques - in particular all the stochastic geometric tools developed - that we employ should extend to the case of a general amenable graph.

The starting point to prove Theorem 11.1.2 is a characterisation of the structure of Γ -invariant Gibbs measures for φ^4 - see Proposition 11.2.15. This is similar in spirit to the Lebowitz characterisation for the Ising model [174], but there are significant complications that arise due to the presence of unbounded spins. The importance of this characterisation is that it reduces the problem to showing that the plus and free measures, i.e. $\langle \cdot \rangle_\beta^+$ and $\langle \cdot \rangle_{\beta, h=0}^0$, coincide on two-point correlations. Thus, we need to control differences of two-point correlations. For the case of the Ising model [10, 211] this is enabled by the celebrated random current representation, and in particular by the switching lemma [4, Lemma 3.2] which allows one to give a probabilistic interpretation of these quantities.

It is well-known that the φ^4 model on a *finite* graph Λ is approximated by an Ising model on $\Lambda \times K_N$ with special coupling constants, where K_N is the complete graph on N vertices. This is called the Griffiths–Simon approximation [224]. One natural question, in light of the above, is why the naive approach of proving the structure theorem for Gibbs measures directly via the Griffiths–Simon approximation of φ^4 fails. This approach would go by proving a corresponding statement for the Ising model on $\Lambda \times K_N$ and then taking the limit $\Lambda \uparrow G$ and $N \rightarrow \infty$ appropriately. Since the proof in the case of the Ising model heavily relies on ergodicity, it only gives information about the model in infinite volume without any quantitative control in finite volume. Therefore, this approach would require a proof of the convergence of the Griffiths–Simon model on $G \times K_N$ to the φ^4 model on G . However, to the best of our knowledge, such a convergence result in the infinite volume is not known. Indeed, the notion of single-site measure, which is crucial to the argument of [224], disappears in the infinite volume limit.

The core of the chapter is the development of a new geometric representation of spin correlations for φ^4 , based on the random current representation for the Ising model, that we call a *random tangled current representation*. A current $\mathbf{n}$ is a positive integer valued function on the J -edges of the graph, meaning the pairs $\{x, y\} \subset V$ such that $J_{x,y} > 0$. A tangled current is a pair $(\mathbf{n}, \mathfrak{t})$ obtained as follows: first, the current $\mathbf{n}$ is lifted to its natural multigraph $\mathcal{N}$, where each J -edge $\{x, y\}$ is replaced by $\mathbf{n}_{x,y}$ edges. Then, each vertex x is replaced by an ordered block of $\Delta \mathbf{n}(x)$ vertices, where $\Delta \mathbf{n}(x)$ is the degree of x in $\mathcal{N}$ and each of the $\Delta \mathbf{n}(x)$ edges is incident to a distinct vertex in the block. Finally, a choice of

an even partition at each vertex, $\mathfrak{t}$, naturally tangles the currents at each block. The key technical tool is the switching principle of Theorem 12.4.2, which, in particular, allows us to express ratios and differences of correlation functions in terms of connectivity properties of random tangled currents. With this switching principle, the proof of Theorem 11.1.2 broadly follows ideas developed in [10] and [211]. However, there are severe obstacles that require a fine knowledge of the behaviour of the single-site tangles, for example that any partition occurs with positive probability. There are also additional essential difficulties caused by the presence of unbounded spin.

The construction of the random tangled currents and the proof of the switching principle starts from the Griffiths–Simon approximation. Indeed, since the model in the prelimit of $N \rightarrow \infty$ is an Ising model, we may use the standard switching lemma. Capturing the stable quantities as $N \rightarrow \infty$ requires isolating the geometric structures that persist in the scaling limit, which then leads us to random tangled currents. We stress, however, that random tangled currents and the switching principle are stated on the level of φ^4 and make no reference to the prelimit; it is only their construction that does. It is an interesting question to prove such a switching principle directly for φ^4 , without any reference to the Griffiths–Simon approximation.

11.1.3 Continuity of the phase transition for φ^4 on $\mathbb{Z}^d$

We now discuss an important consequence of Theorem 11.1.2 to the (continuity of) phase transition for φ^4 . We restrict ourselves to the hypercubic lattice $G = \mathbb{Z}^d$. We fix $g > 0$ and $a \in \mathbb{R}$. We also fix a collection of interactions J satisfying (C1–C4).

We begin by defining the phase transition rigourously. There are several natural ways to do this, which are all a posteriori equivalent. We consider here perhaps the most classical definition in terms of spontaneous magnetisation. Other definitions involve existence of long range order (as mentioned above), uniqueness of Gibbs measure, finiteness of susceptibility, etc. We refer the reader to Appendix 11.B for these definitions and results relating them. The spontaneous magnetisation at inverse temperature β is defined as

$$m^*(\beta) := \langle \varphi_0 \rangle_\beta^+. \quad (11.1.14)$$

We can now define

$$\beta_c := \inf\{\beta > 0, m^*(\beta) > 0\}. \quad (11.1.15)$$

For $d \geq 2$ it has been proved [114] (see also [113]) that $\beta_c(g, a) \in (0, \infty)$ for all $g > 0$ and $a \in \mathbb{R}$.

In view of the definition above, it is a natural question to ask whether $m^*(\beta_c) = 0$ (i.e. the spontaneous magnetisation vanishes at criticality). If so, the model is said to undergo a *continuous* phase transition. This is a fundamental assumption on which arguments of universality (e.g. via renormalisation group heuristics) are built on. It turns out that continuity of the phase transition for lattice φ^4 models boils down to proving that $|\mathcal{G}(\beta_c)| = 1$, see Proposition 11.B.3. Theorem 11.1.2 gives a sufficient condition (11.1.12) for that to be satisfied.

Very few tools are available to obtain condition (11.1.12) at criticality. For the Ising model in dimension 2, this can be verified by either using exact integrability established in the famous work of Onsager [200] or planar duality of the interaction as realised by Werner [247]. However, we stress these techniques only work for the nearest-neighbour model, and

have not been extended to finite range interactions in general, or the case of φ^4 . In higher dimensions, one can use the lace expansion for either the Ising model [216] or φ^4 [217]. In the intermediate dimensions, such as the physically relevant dimension 3 for nearest-neighbour interactions, the infrared/Gaussian domination bound of Fröhlich, Simon, and Spencer [104] (see also [103]) for reflection positive models (see [36, Definition 5.1]) is the main tool we have at our disposal. As already observed in [10], the infrared bound is a good candidate to ensure continuity.

We now assume that the interaction J is admissible and reflection positive – see Section 11.7 for the definition. The canonical examples of reflection positive interactions are

- (Nearest-neighbor interactions) $J_{x,y} = \delta_{|x-y|_1=1}$;
- (Exponential decay / Yukawa potentials) $J_{x,y} = C \exp(-\mu|x-y|_1)$ for $\mu, C > 0$;
- (Power law decay) $J_{x,y} = C|x-y|_1^{-d-\alpha}$ for $\alpha, C > 0$,

where $|\cdot|_1$ refers to the ℓ^1 -norm on $\mathbb{R}^d$. For each J , we consider the associated random walk $(X_k)_{k \geq 0}$ with step distribution given by

$$\mathbb{P}[X_{k+1} = y \mid X_k = x] = \frac{J_{x,y}}{|J|} \quad (11.1.16)$$

where $|J| := \sum_{x \in \mathbb{Z}^d} J_{0,x}$. The infrared bound relates the two-point function of a reflection positive model to the Green's function of the associated random walk X . In the case that the random walk is transient, one concludes that the condition (11.1.12) from Theorem 11.1.2 holds. We then obtain the following result— see Section 11.7 for the details.

Theorem 11.1.3 (Continuity of the phase transition). *Let $d \geq 1$. Let J be admissible, reflection positive, and such that the associated random walk on $\mathbb{Z}^d$ is transient. Then,*

$$m^*(\beta_c) = 0. \quad (11.1.17)$$

In particular, $m^*(\beta_c) = 0$ for the nearest-neighbour φ^4 model on $\mathbb{Z}^d$ when $d \geq 3$.

Remark 11.1.4. Theorem 11.1.3 gives that the magnetisation is continuous at β_c when J is any admissible and reflection positive interaction in dimension at least 3. In dimension 2, it yields the same conclusion when J has power law decay of the form $J_{x,y} = C|x-y|_1^{-d-\alpha}$ with $\alpha \in (0, 2)$. Finally, in dimension 1, provided that the model undergoes a non-trivial phase transition¹, we get that the magnetisation is continuous at β_c when J has power law decay with $\alpha \in (0, 1)$.

11.1.4 Open problems

We now give what, in our opinion, are some beautiful open problems that naturally lead on from Theorems 11.1.2 and 11.1.3. In what follows, we fix $G = \mathbb{Z}^2$ and only consider the nearest-neighbour case.

The most striking omission from Theorem 11.1.3 is the behaviour at β_c for the nearest-neighbour model in $d = 2$. Indeed, the infrared bound is not useful in this context and the

¹We expect this to be true whenever $J_{x,y} \geq c|x-y|^{-1-\alpha}$ for some $\alpha \in (0, 1)$, as in the case of the one-dimensional Ising model [86].

analogous result for the Ising model relies on exact integrability [200] or planar duality [247], which only works in the nearest-neighbour case; we stress that the case of finite range Ising is open. It is widely believed that the phase transition for φ^4 , just as for Ising, is continuous in dimension 2.

Open problem 4 (Continuity in $d = 2$). *Let $d = 2$ and J be the nearest-neighbour interaction. Prove that*

$$m^*(\beta_c) = 0. \quad (11.1.18)$$

The only rigorous evidence that we are aware of universality with respect to changes in the Hamiltonian² for the Ising model in $d = 2$ are the celebrated works of Pinson and Spencer [209] and Giuliani, Greenblatt, and Mastropietro [109], which treat the scaling limit of energy-energy correlations for critical finite range Ising models (with a perturbative condition on the range) via renormalisation group techniques; and the work of Aizenman, Duminil-Copin, Tassion, and Warzel [11], which proves asymptotic Pfaffian relations for boundary point correlations for any critical finite range model (with no perturbative condition) via stochastic geometric techniques. Both cases are coherent with universality, i.e. with the corresponding results established for the critical nearest-neighbour model; see [220, 123] for the energy-energy and boundary point correlations, respectively. We defer to the articles cited above for precise statements in the case of Ising.

Open problem 5 (Pfaffian relations in $d = 2$). *For the nearest-neighbour φ^4 model on $\mathbb{Z}^2$, prove that the boundary spin correlations satisfy the Pfaffian relations asymptotically at criticality.*

Our final problem concerns the set of *all* Gibbs measures in $d = 2$. For the Ising model, it is known [3, 146] that in $d = 2$, there are no non-translation invariant Gibbs measures. This is in stark contrast to the existence of (non-translation invariant) Dobrushin measures in $d \geq 3$ as established in [74]. For φ^4 , the existence of Dobrushin measures is known in $d \geq 3$, see [222]. The following problem is thus natural.

Open problem 6 (Aizenman–Higuchi result in $d = 2$). *Let $d = 2$ and J be the nearest-neighbour interaction. Prove that, for all $\beta > \beta_c$,*

$$\mathcal{G}(\beta) = \mathcal{G}_T(\beta) \quad (11.1.19)$$

where $\mathcal{G}_T(\beta)$ is the set of translation invariant Gibbs measures.

11.2 Gibbs measures and correlation functions

In this section, we prove important properties about Gibbs measures. First, we construct the plus and minus measures, and show that they are mixing and extremal. Then, we provide a Lebowitz-type characterisation on the structure of Γ -invariant Gibbs measures that reduces the proof of Theorem 11.1.2 to checking that even correlations under the plus and free measures coincide. We fix $\beta > 0$.

²This is in contrast to universality results with respect to changing the underlying *planar* graph for nearest-neighbour models, i.e. beyond the square lattice, for which much progress has been made. We refer to the review by Chelkak [63] for further details.

11.2.1 Regularity of Gibbs measures

It is crucial to our techniques that $\langle \cdot \rangle_\beta^+$ and $\langle \cdot \rangle_\beta^-$ are realised as infinite volume limits of measures with appropriate boundary conditions at $h = 0$. This is not a priori clear from their definitions via limits as $h \downarrow 0$ and $h \uparrow 0$ of $\langle \cdot \rangle_{\beta,h}^0$, respectively. Indeed, it is not even clear that they are even Gibbs measures at $h = 0$. As such, we show that any Gibbs measure satisfies a regularity condition that puts a constraint on the growth of the configurations that it is supported on. This allows us to define natural analogues of plus and minus boundary condition measures obtained from finite volume measures that are mixing and extremal, which we shall do in the next subsection.

Definition 11.2.1. Let ν be a measure on $\mathbb{R}^V$. For $\gamma, \delta > 0$, we say that ν is (γ, δ) -regular if, for every $\Lambda \subset V$ finite, the density of the projection measure satisfies

$$\frac{d\nu}{d\varphi_\Lambda}(\varphi_\Lambda) \leq e^{-\sum_{x \in \Lambda} (\gamma \varphi_x^4 - \delta)} \quad (11.2.1)$$

where $d\varphi_\Lambda$ is the Lebesgue measure on $\mathbb{R}^\Lambda$.

The following proposition states that (tempered) Gibbs measures are regular. The proof is technical and relies on estimates developed in [175], adapted to our more general setting on graphs of polynomial growth. We therefore defer the proof to Appendix 11.C, c.f. Proposition 11.C.6.

Proposition 11.2.2. *There exist constants $\gamma, \delta > 0$ depending on G, g, a, J, β, h such that ν is (γ, δ) -regular for every $\nu \in \mathcal{G}(\beta)$. Moreover, the constants γ and δ are continuous in g, a, J, β, h .*

Regular measures concentrate on weakly growing configurations.

Lemma 11.2.3. *Let ν be (γ, δ) -regular. For every $M = M(\gamma, \delta) > 0$ sufficiently large, we have*

$$\nu[\Xi(M)] = 1 \quad (11.2.2)$$

where

$$\Xi(M) := \{\varphi \in \mathbb{R}^V : \exists \Lambda_\varphi \subset V \text{ finite, } \varphi_x^4 \leq M \log d_G(o, x) \ \forall x \in V \setminus \Lambda_\varphi\}. \quad (11.2.3)$$

Proof. This is a direct consequence of Definition 11.2.1, a union bound, and Borel-Cantelli. $\square$

In light of Lemma 11.2.3, there is a natural notion of “maximal boundary condition”.

Definition 11.2.4. For $M > 0$, we define the configurations $\mathfrak{p} = \mathfrak{p}(M)$ and $\mathfrak{m} = \mathfrak{m}(M)$ such that

$$\mathfrak{p}_x = \sqrt[4]{M \log(d_G(o, x) \vee 1)}, \quad \forall x \in V \quad (11.2.4)$$

and $\mathfrak{m} = -\mathfrak{p}$. We refer to $\langle \cdot \rangle_{\Lambda, \beta}^{\mathfrak{p}}$ and $\langle \cdot \rangle_{\Lambda, \beta}^{\mathfrak{m}}$ as the plus and minus (finite volume) measures.

Remark 11.2.5. Proposition 11.2.2 and Lemma 11.2.3 imply that any Gibbs measure is a limit of finite volume (regular) φ^4 measures $\langle \cdot \rangle_{\Lambda, \beta}^\eta$ with (possibly random) boundary conditions η satisfying $|\eta_x| \leq \mathfrak{p}_x(M) \ \forall x \in V \setminus \Lambda$, provided M is taken sufficiently large. This is established

in Appendix 11.C. One of the key results to proving Proposition 11.2.2 is that finite volume measures with boundary condition satisfying $|\eta| \leq \mathfrak{p}$ satisfy an analogous estimate to (11.2.1) with (γ, δ) uniform over subsets sufficiently far away from the boundary, see Corollary 11.C.13 for a precise statement. Conversely, any sequence of finite volume measures with boundary conditions satisfying $|\eta| \leq \mathfrak{p}$ is tight and any accumulation point is a Gibbs measure, see Propositions 11.C.4 and 11.C.5.

11.2.2 Extremality and mixing of the plus and minus measures

Recall that $\langle \cdot \rangle_\beta^+$ and $\langle \cdot \rangle_\beta^-$ were defined as the respective monotone limits $\lim_{h \downarrow 0} \langle \cdot \rangle_{\beta, h}^0$ and $\lim_{h \uparrow 0} \langle \cdot \rangle_{\beta, h}^0$. We now show that they can be obtained by taking infinite volume limits of measures with maximal boundary conditions at $(\beta, h = 0)$, thus giving a proof that they are in $\mathcal{G}(\beta)$.

Proposition 11.2.6. *For M sufficiently large, one has that $\langle \cdot \rangle_{\Lambda, \beta}^{\mathfrak{p}} \rightarrow \langle \cdot \rangle_\beta^+$ and $\langle \cdot \rangle_{\Lambda, \beta}^{\mathfrak{m}} \rightarrow \langle \cdot \rangle_\beta^-$ as $\Lambda \uparrow V$. Moreover, the measures are maximal and minimal, respectively, in the sense that, for any $\langle \cdot \rangle \in \mathcal{G}(\beta)$ and any increasing and integrable function f ,*

$$\langle f \rangle_\beta^- \leq \langle f \rangle \leq \langle f \rangle_\beta^+. \quad (11.2.5)$$

Proof. We only prove the convergence results; the monotonicity (11.2.5) is a direct consequence of this together with Remark 11.2.5, and the monotonicity on boundary conditions in finite volume (see Proposition 11.A.4). Without loss of generality, we only consider the case of $\langle \cdot \rangle_\beta^+$. First, since the measures $\langle \cdot \rangle_{\Lambda, \beta}^{\mathfrak{p}}$ are uniformly regular (see Remark 11.2.5), it is straightforward to prove that they form a tight sequence of measures. Now, let $\Lambda_n \uparrow V$ be any sequence such that $\langle \cdot \rangle_{\Lambda_n, \beta}^{\mathfrak{p}}$ converges to a probability measure, which we denote by $\langle \cdot \rangle_\beta^{\mathfrak{p}}$. Note that this is a Gibbs measure. We prove that $\langle \cdot \rangle_\beta^{\mathfrak{p}} = \langle \cdot \rangle_\beta^+$, which in particular implies that the full sequence $\langle \cdot \rangle_{\Lambda, \beta}^{\mathfrak{p}}$ converges.

By monotonicity³ in h , for any f local, increasing and bounded, and $h > 0$,

$$\langle f \rangle_\beta^{\mathfrak{p}} = \lim_{n \rightarrow \infty} \langle f \rangle_{\Lambda_n, \beta}^{\mathfrak{p}} \leq \lim_{n \rightarrow \infty} \langle f \rangle_{\Lambda_n, \beta, h}^{\mathfrak{p}} = \nu_{\beta, h}[f] \quad (11.2.6)$$

where $\nu_{\beta, h} := \langle \cdot \rangle_{\beta, h}$ is the unique measure in $\mathcal{G}(\beta, h)$ for $h \neq 0$: this is a consequence of the Lee–Yang theorem for φ^4 [224] and [175, Theorem 5.6]. Since by definition $\nu_{\beta, h} \rightarrow \langle \cdot \rangle_\beta^+$ as $h \rightarrow 0$, we conclude that

$$\langle f \rangle_\beta^{\mathfrak{p}} \leq \langle f \rangle_\beta^+. \quad (11.2.7)$$

For the reverse inequality, let f be local, increasing, and bounded. For $h > 0$ and $\Lambda \subset V$ finite, let

$$\varepsilon(\Lambda, \beta, h) := \|f\|_\infty \nu_{\beta, h}[\exists x \notin \Lambda : |\eta_x| > \mathfrak{p}_x(M)]. \quad (11.2.8)$$

Provided M is large enough, by continuity of the regularity coefficients and concentration on regular boundary conditions, $\sup_{h \in [0, 1]} \varepsilon(\Lambda, h) = o(1)$ as $\Lambda \uparrow V$. Thus, by the DLR equations

³This follows by differentiating in h and using the FKG inequality of Proposition 11.A.1 with the observation that the identity map $\varphi_x \mapsto \varphi_x$ is increasing.

and monotonicity in boundary conditions (see Proposition 11.A.4), we have for n sufficiently large

$$\langle f \rangle_\beta^+ := \lim_{h \rightarrow 0^+} \nu_{\beta,h}[f] = \lim_{h \rightarrow 0^+} \int \langle f \rangle_{\Lambda_n}^\eta d\nu_{\beta,h}(\eta) \quad (11.2.9)$$

$$\leq \lim_{h \rightarrow 0^+} \left(\varepsilon(\Lambda_n, \beta, h) + \int \langle f \rangle_{\Lambda_n, \beta, h}^p d\nu_{\beta,h}(\eta) \right) \quad (11.2.10)$$

$$\leq \sup_{h \in [0,1]} \varepsilon(\Lambda_n, \beta, h) + \lim_{h \rightarrow 0^+} \langle f \rangle_{\Lambda_n, \beta, h}^p \xrightarrow{n \rightarrow \infty} \langle f \rangle_\beta^p. \quad (11.2.11)$$

Therefore, $\langle \cdot \rangle_\beta^+$ and $\langle \cdot \rangle_\beta^p$ agree on all local bounded increasing functions, hence $\langle \cdot \rangle_\beta^p = \langle \cdot \rangle_\beta^+$, as we wanted to prove. $\square$

In what follows, we fix M sufficiently large so that Lemma 11.2.3 and Proposition 11.2.6 apply.

Corollary 11.2.7 (Extremality). *The measures $\langle \cdot \rangle_\beta^+$ and $\langle \cdot \rangle_\beta^-$ are extremal in the following sense. Let $\# \in \{+, -\}$. If there exists $\nu_1, \nu_2 \in \mathcal{G}(\beta)$ and $\lambda \in (0, 1)$ such that*

$$\langle \cdot \rangle_\beta^\# = \lambda \nu_1 + (1 - \lambda) \nu_2 \quad (11.2.12)$$

then necessarily $\nu_1 = \nu_2 = \langle \cdot \rangle_\beta^\#$.

Proof. This is a direct consequence of (11.2.5). $\square$

We now prove that the measures $\langle \cdot \rangle_\beta^+$ and $\langle \cdot \rangle_\beta^-$ are mixing in the following sense.

Proposition 11.2.8 (Mixing). *Let $\# \in \{+, -\}$. Then, for all local bounded functions f and g ,*

$$\lim_{d_G(o, x) \rightarrow \infty} \langle f \cdot \gamma_x g \rangle_\beta^\# = \langle f \rangle_\beta^\# \langle g \rangle_\beta^\# \quad (11.2.13)$$

for all $\gamma_x \in \Gamma$ such that $\gamma_x o = x$.

Proof. Without loss of generality, we prove the assertion for $\langle \cdot \rangle_\beta^+$. First, it is easy to see that $\langle \cdot \rangle_\beta^+ \in \mathcal{G}_\Gamma(\beta)$.

To show that $\langle \cdot \rangle_\beta^+$ is mixing, let f and g be local bounded increasing functions. Note that by the FKG inequality (see Proposition 11.A.1) and Γ -invariance,

$$\langle f \rangle_\beta^+ \langle g \rangle_\beta^+ \leq \langle f \cdot \gamma_x g \rangle_\beta^+ \quad (11.2.14)$$

for any $x \in V$ and any $\gamma_x \in \Gamma$ such that $\gamma_x o = x$.

For the reverse inequality, consider some $x \in V$, and let $\gamma_x \in \Gamma$ such that $\gamma_x o = x$. Let $\varepsilon > 0$. Note that, by regularity, we can find $\Lambda \subset V$ finite and containing the support of f and g such that

$$\varepsilon/2 > \|f\| \|g\| \left(\langle \mathbf{1}\{\exists y \notin \Lambda, |\eta_y| > \mathfrak{p}_y\} \rangle_\beta^+ + \langle \mathbf{1}\{\exists y \notin \gamma_x \Lambda, |\eta_y| > \mathfrak{p}_{\gamma_x y}\} \rangle_\beta^+ \right) \quad (11.2.15)$$

$$= 2\|f\| \|g\| \langle \mathbf{1}\{\exists y \notin \Lambda, |\eta_y| > \mathfrak{p}_y\} \rangle_\beta^+. \quad (11.2.16)$$

Therefore, uniformly in $x \in V$,

$$\langle f \cdot \gamma_x g \rangle_\beta^+ \leq \left\langle \langle f \cdot \gamma_x g \rangle_{\Lambda(x), \beta}^\eta \mathbb{1}\{|\eta| \leq \min\{\mathfrak{p}, \gamma_x \mathfrak{p}\} \text{ in } \Lambda(x)^c\} \right\rangle_\beta^+ + \varepsilon/2 \quad (11.2.17)$$

where $\Lambda(x) = \Lambda \cup \gamma_x \Lambda$ and the outer integration is performed with respect to η . Note that, again by regularity, we can find $C = C(\varepsilon, \Lambda)$ sufficiently large such that, uniformly in x ,

$$\langle f \cdot \gamma_x g \rangle_\beta^+ \leq \left\langle \langle f \cdot \gamma_x g \mathbb{1}_A \rangle_{\Lambda(x), \beta}^\eta \mathbb{1}\{|\eta| \leq \min\{\mathfrak{p}, \gamma_x \mathfrak{p}\} \text{ in } \Lambda(x)^c\} \right\rangle_\beta^+ + \varepsilon, \quad (11.2.18)$$

where $A = \{|\varphi_y| \leq C \forall y \in \Lambda(x)\}$.

By the integrability assumption on J , we know that for any $|\eta| \leq \mathfrak{p}$

$$\sum_{u \in \Lambda} \sum_{v \in V} \beta J_{u,v} |\eta_v| < \infty. \quad (11.2.19)$$

Hence, there exists $n = n(\varepsilon, \Lambda, C, \beta)$ sufficiently large such that for all η satisfying $|\eta| \leq \min\{\mathfrak{p}, \gamma_x \mathfrak{p}\}$,

$$\sum_{u \in \Lambda(x)} \sum_{v \in V, d_G(u,v) > n} \beta J_{u,v} |\eta_v| < \frac{\varepsilon}{2C} \quad (11.2.20)$$

uniformly in x .

Let J^n be the family of interactions defined by $J_{u,v}^n = J_{u,v} \mathbb{1}_{d_G(u,v) \leq n}$, for all $u, v \in V$. Using the estimate (11.2.20) and that $\sup_{z \in \Lambda(x)} |\varphi_z| \leq C$ on A , we have

$$\sup_x \left| \beta H_{\Lambda(x), J^n}^\eta(\varphi) - \beta H_{\Lambda(x), J}^\eta(\varphi) \right| \leq \varepsilon/2. \quad (11.2.21)$$

Hence,

$$\langle f \cdot \gamma_x g \rangle_\beta^+ \leq e^\varepsilon \left\langle \langle f \cdot \gamma_x g \mathbb{1}_A \rangle_{\Lambda(x), \beta, J^n}^\eta \mathbb{1}\{|\eta| \leq \min\{\mathfrak{p}, \gamma_x \mathfrak{p}\} \text{ in } \Lambda(x)^c\} \right\rangle_\beta^+ + \varepsilon \quad (11.2.22)$$

uniformly in x . We now choose x such that $d_G(o, x)$ is sufficiently large so that $\bigcup_{u \in \Lambda} B_n(u)$ and $\bigcup_{u \in \gamma_x \Lambda} B_n(u)$ are disjoint, where $B_n(u)$ is the ball of radius n centered at $u \in V$. Then, by independence, monotonicity in J and the boundary conditions, and regularity, we have that

$$\langle f \cdot \gamma_x g \mathbb{1}_A \rangle_{\Lambda(x), \beta, J^n}^\eta = \langle f \rangle_{\Lambda, \beta, J^n}^\eta \langle \gamma_x g \rangle_{\gamma_x \Lambda, \beta, J^n}^\eta \leq \langle f \rangle_{\Lambda, \beta, J}^\mathfrak{p} \langle \gamma_x g \rangle_{\gamma_x \Lambda, \beta, J}^{\gamma_x \mathfrak{p}}. \quad (11.2.23)$$

Thus,

$$\langle f \cdot \gamma_x g \rangle_\beta^+ \leq e^\varepsilon \langle f \rangle_{\Lambda, \beta, J}^\mathfrak{p} \langle \gamma_x g \rangle_{\gamma_x \Lambda, \beta, J}^{\gamma_x \mathfrak{p}} \langle \mathbb{1}\{|\eta| \leq \min\{\mathfrak{p}, \gamma_x \mathfrak{p}\} \text{ in } \Lambda(x)^c\} \rangle_\beta^+ + \varepsilon. \quad (11.2.24)$$

Hence, sending $\Lambda \uparrow V$, using Γ -invariance, and then $\varepsilon \rightarrow 0$, we get

$$\limsup_{d_G(o, x) \rightarrow \infty} \langle f \cdot \gamma_x g \rangle_\beta^+ \leq \langle f \rangle_\beta^+ \langle g \rangle_\beta^+. \quad (11.2.25)$$

Combining this with (11.2.14), we prove mixing for any local bounded increasing functions. Since increasing functions generate the σ -algebra, the result follows from approximation. $\square$

11.2.3 Ergodicity of the free measure under even functions

Recall that the free measure, denoted $\langle \cdot \rangle_\beta^0$, is the weak limit of $\langle \cdot \rangle_{\Lambda, \beta}^0$ as $\Lambda \uparrow V$. The free measure is a Gibbs measure. Indeed, by Remark 11.2.5, we have tightness and that the (unique) limit point is tempered and satisfies the DLR equations.

Proposition 11.2.9. *The measure $\langle \cdot \rangle_\beta^0$ is mixing on even functions in the following sense. For all local even functions f and g such that f, g are square integrable,*

$$\lim_{d_G(o, x) \rightarrow \infty} \langle f \cdot \gamma_x g \rangle_\beta^0 = \langle f \rangle_\beta^0 \langle g \rangle_\beta^0 \quad (11.2.26)$$

for all $\gamma_x \in \Gamma$ such that $\gamma_x o = x$.

Proof. Let $A : V \rightarrow \mathbb{N}$ be of finite support and such that $\sum_x A_x$ is even. Define $\varphi_A := \prod_{x \in V} \varphi_x^{A_x}$. Let $(\tilde{J}_{x,y})_{x,y \in V}$ be a set of interactions supported on finitely many pairs in V such that

$$-J_{x,y} \leq \tilde{J}_{x,y} \leq 0. \quad (11.2.27)$$

We define

$$H_{\tilde{J}}(\varphi) = - \sum_{\{x,y\} \in \mathcal{P}_2(V)} \tilde{J}_{x,y} \varphi_x \varphi_y \quad (11.2.28)$$

where $\mathcal{P}_2(V) = \{\{x, y\} : x, y \in V\}$. Note that, by taking infinite volume limits, the measure $\langle \varphi_A \rangle_{\beta, J + \gamma_x^{-1} \tilde{J}}^0$ is well-defined and satisfies

$$\langle \varphi_A \rangle_{\beta, J + \gamma_x^{-1} \tilde{J}}^0 = \frac{\langle \varphi_A e^{-\beta H_{\tilde{J}}(\gamma_x \varphi)} \rangle_{\beta, J}^0}{\langle e^{-\beta H_{\tilde{J}}(\gamma_x \varphi)} \rangle_{\beta, J}^0}. \quad (11.2.29)$$

Now for $x \in V$ such that $n := d_G(o, x)$ is sufficiently large, by monotonicity in the volume and Griffiths' inequality, we have that

$$\langle \varphi_A \rangle_{B_{n/2}, \beta, J}^0 = \langle \varphi_A \rangle_{B_{n/2}, \beta, J + \gamma_x^{-1} \tilde{J}}^0 \leq \langle \varphi_A \rangle_{\beta, J + \gamma_x^{-1} \tilde{J}}^0 \leq \langle \varphi_A \rangle_{\beta, J}^0. \quad (11.2.30)$$

Hence, by taking $d_G(o, x) \rightarrow \infty$ and using convergence of correlations in the infinite volume limit, we have shown that

$$\lim_{d_G(o, x) \rightarrow \infty} \langle \varphi_A e^{-\beta H_{\tilde{J}}(\gamma_x \varphi)} \rangle_{\beta, J}^0 = \langle \varphi_A \rangle_{\beta, J}^0 \langle e^{-\beta H_{\tilde{J}}(\varphi)} \rangle_{\beta, J}^0. \quad (11.2.31)$$

This establish Proposition 11.2.9 for $f(\varphi) = \varphi_A$ and $g(\varphi) = e^{-\beta H_{\tilde{J}}(\varphi)}$. By standard approximation arguments via expanding the exponential, resumming, and using Stone–Weierstrass, the mixing extends to all even functions. $\square$

Remark 11.2.10. We cannot use the same strategy for the proofs of mixing for $\langle \cdot \rangle_\beta^+$ and $\langle \cdot \rangle_\beta^0$, unlike for the case of the Ising model. This comes from the fact that we do not have clear monotonicity in the volume of the quantity $\langle f \rangle_{\Lambda, \beta}^{\mathbf{p}}$ (for f local, increasing and bounded). This is a consequence of the unboundedness of the spins. As a result, we must use regularity to “create” a maximal boundary condition $\mathbf{p}$, which comes with an error term, and temperedness to bound the spins inside a large box. As a byproduct of the methods developed above, we see that for $\Lambda_1 \subset \Lambda_2$ two finite subsets of V , and for f local (with support in Λ_1), increasing, bounded,

$$\langle f \rangle_{\Lambda_2, \beta}^{\mathbf{p}} \leq \langle f \rangle_{\Lambda_1, \beta}^{\mathbf{p}} + \varepsilon(\Lambda_1), \quad (11.2.32)$$

where $\varepsilon(\Lambda_1) \rightarrow 0$ as $\Lambda_1 \uparrow V$.

11.2.4 A characterisation of Γ -invariant Gibbs measures via correlations

The main result of this subsection is a characterisation for the structure of the Γ -invariant Gibbs measures, see Proposition 11.2.15. The analogous result for the Ising model was obtained by Lebowitz [174]. In our setting, there are significant complications that come from the unboundedness of the spins.

A central tool in the proof is the following correlation inequality. Recall that if $A : V \rightarrow \mathbb{N}$ is finitely supported, we denote

$$\varphi_A = \prod_{x \in V} \varphi_x^{A_x}. \quad (11.2.33)$$

Proposition 11.2.11 (Ginibre inequality). *Let $A, B : V \rightarrow \mathbb{N}$ be finitely supported. Then, if Λ contains the support of A and B , for any $\eta, \eta' \in \Xi(M)$ such that $|\eta_x| \leq \eta'_x$ for all $x \notin \Lambda$,*

$$\langle \varphi_A \varphi_B \rangle_{\Lambda, \beta}^{\eta'} - \langle \varphi_A \varphi_B \rangle_{\Lambda, \beta}^{\eta} \geq \left| \langle \varphi_A \rangle_{\Lambda, \beta}^{\eta'} \langle \varphi_B \rangle_{\Lambda, \beta}^{\eta} - \langle \varphi_A \rangle_{\Lambda, \beta}^{\eta} \langle \varphi_B \rangle_{\Lambda, \beta}^{\eta'} \right|. \quad (11.2.34)$$

Moreover, if $\langle \cdot \rangle_{\beta} \in \mathcal{G}(\beta)$,

$$\langle \varphi_A \varphi_B \rangle_{\beta}^+ - \langle \varphi_A \varphi_B \rangle_{\beta} \geq \left| \langle \varphi_A \rangle_{\beta}^+ \langle \varphi_B \rangle_{\beta} - \langle \varphi_A \rangle_{\beta} \langle \varphi_B \rangle_{\beta}^+ \right|. \quad (11.2.35)$$

Proof. See Appendix 11.A.2. □

We make heavy use of the fact that the sign field of the φ^4 measure can be viewed as an Ising model with random coupling constants coming from the absolute value field. Given $x \in \mathbb{R}$, we define the sign of x as follows:

$$\text{sgn}(x) := \begin{cases} 1, & \text{for } x \geq 0 \\ -1, & \text{for } x < 0. \end{cases} \quad (11.2.36)$$

We have the following coupling result.

Lemma 11.2.12. *Let $\Lambda \subset V$ be a finite set and let $\eta \in \mathbb{R}^V$ such that $|\eta| \leq \mathbf{p}$ (outside Λ). There exists a coupling $(\mathbb{P}_{\Lambda, \beta}, \varphi^{\mathbf{p}}, \varphi^{\eta})$, $\varphi^{\mathbf{p}} \sim \langle \cdot \rangle_{\Lambda, \beta}^{\mathbf{p}}$, $\varphi^{\eta} \sim \langle \cdot \rangle_{\Lambda, \beta}^{\eta}$, such that $\mathbb{P}_{\Lambda, \beta}$ -almost surely,*

$$|\varphi_x^{\mathbf{p}}| \geq |\varphi_x^{\eta}| \quad \text{for every } x \in \Lambda, \quad (11.2.37)$$

$$\mathbb{E}_{\Lambda, \beta} \left[\text{sgn}(\varphi_x^{\mathbf{p}}) \text{sgn}(\varphi_y^{\mathbf{p}}) \mid |\varphi^{\mathbf{p}}|, |\varphi^{\eta}| \right] \geq 0 \quad \text{for every } x, y \in \Lambda, \quad (11.2.38)$$

and

$$\mathbb{E}_{\Lambda, \beta} \left[\text{sgn}(\varphi_x^{\mathbf{p}}) \text{sgn}(\varphi_y^{\mathbf{p}}) - \text{sgn}(\varphi_x^{\eta}) \text{sgn}(\varphi_y^{\eta}) \mid |\varphi^{\mathbf{p}}|, |\varphi^{\eta}| \right] \geq 0 \quad \text{for every } x, y \in \Lambda. \quad (11.2.39)$$

This proof of this result requires the following lemma.

Lemma 11.2.13 (Stochastic domination of absolute value fields). *Let $\Lambda \subset V$ be a finite set, let $\beta > 0$, and let $\eta \in \mathbb{R}^V$ such that $|\eta| \leq \mathbf{p}$ (outside Λ). Then $\langle |\cdot| \rangle_{\Lambda, \beta}^{\mathbf{p}}$ stochastically dominates $\langle |\cdot| \rangle_{\Lambda, \beta}^{\eta}$, in the sense that for any local increasing and bounded function f ,*

$$\langle f(|\varphi|) \rangle_{\Lambda, \beta}^{\mathbf{p}} \geq \langle f(|\varphi|) \rangle_{\Lambda, \beta}^{\eta}. \quad (11.2.40)$$

Proof. Let

$$J^\eta(|\varphi|)_{x,y} = \begin{cases} J_{x,y}|\varphi_x\varphi_y|, & \text{for } x, y \in \Lambda \\ J_{x,y}|\varphi_x\eta_y|, & \text{for } x \in \Lambda, y \in V \setminus \Lambda. \end{cases} \quad (11.2.41)$$

It follows from a direct calculation that

$$d\langle \cdot \rangle_{\Lambda,\beta}^\eta(|\varphi|) = \frac{Z_{\Lambda,J^\eta(|\varphi|),\beta}^{\text{Ising,sgn}(\eta)}}{Z_{\Lambda,\beta}^\eta} \prod_{x \in \Lambda} d\rho(|\varphi_x|), \quad (11.2.42)$$

where, for any $\tau \in \{\pm 1\}^V$,

$$Z_{\Lambda,J^\eta(|\varphi|),\beta}^{\text{Ising},\tau} := \sum_{\sigma \in \{\pm 1\}^\Lambda} e^{-\beta H_{\Lambda,J^\eta(|\varphi|)}^{\text{Ising},\tau}(\sigma)} \quad (11.2.43)$$

and for $\sigma \in \{\pm 1\}^\Lambda$

$$H_{\Lambda,J^\eta(|\varphi|)}^{\text{Ising},\tau}(\sigma) := - \sum_{x,y \in \Lambda} J^\eta(|\varphi|)_{x,y} \sigma_x \sigma_y - \sum_{\substack{x \in \Lambda \\ y \in V \setminus \Lambda}} J^\eta(|\varphi|)_{x,y} \sigma_x \tau_y. \quad (11.2.44)$$

For simplicity, we write $+$ for the configuration $\tau_x = 1, \forall x \in V$.

Thus, for any increasing function $F : [0, \infty)^\Lambda \mapsto \mathbb{R}$,

$$\langle F(|\varphi|) \rangle_{\Lambda,\beta}^{\mathfrak{p}} = \frac{\langle F(|\varphi|) G(|\varphi|) \rangle_{\Lambda,\beta}^\eta}{\langle G(|\varphi|) \rangle_{\Lambda,\beta}^\eta}, \quad (11.2.45)$$

where

$$G(|\varphi|) = \frac{Z_{\Lambda,J^{\mathfrak{p}}(|\varphi|),\beta}^{\text{Ising},+}}{Z_{\Lambda,J^\eta(|\varphi|),\beta}^{\text{Ising,sgn}(\eta)}}. \quad (11.2.46)$$

Note that $G(|\varphi|)$ is an increasing function of $|\varphi|$, which follows from taking the logarithmic derivative and using the Ginibre inequality for the Ising model (see [174]). We can now use Proposition 11.A.2 to conclude that

$$\langle F(|\varphi|) \rangle_{\Lambda,\beta}^{\mathfrak{p}} \geq \langle F(|\varphi|) \rangle_{\Lambda,\beta}^\eta, \quad (11.2.47)$$

as desired. $\square$

Proof of Lemma 11.2.12. Using Lemma 11.2.13 and Strassen's theorem [237], we obtain a coupling $(\mathbb{P}_{\Lambda,\beta}, \psi^{\mathfrak{p}}, \psi^\eta)$, where $\psi^{\mathfrak{p}} \sim \langle |\cdot| \rangle_\Lambda^{\mathfrak{p}}$ and $\psi^\eta \sim \langle |\cdot| \rangle_\Lambda^\eta$, such that $\mathbb{P}_{\Lambda,\beta}$ -almost surely,

$$\psi_x^{\mathfrak{p}} \geq \psi_x^\eta \quad \text{for every } x \in \Lambda. \quad (11.2.48)$$

Recall the notation introduced in the proof of Lemma 11.2.13. Enlarging our probability space, we can assume that in the same probability space, there is a family $I = \{\sigma^{\mathfrak{p}}(\mathbf{J}) \sim \langle \cdot \rangle_{\Lambda,\mathbf{J},\beta}^{\text{Ising},+}, \sigma^\eta(\mathbf{J}) \sim \langle \cdot \rangle_{\Lambda,\mathbf{J},\beta}^{\text{Ising,sgn}(\eta)} \mid \mathbf{J} : \mathcal{P}_2(\Lambda) \mapsto [0, \infty)\}$ of independent Ising models that are also independent from $\psi^{\mathfrak{p}}$ and ψ^η . Now let $\varphi^{\mathfrak{p}} := \psi^{\mathfrak{p}} \cdot \sigma^{\mathfrak{p}}(J^{\mathfrak{p}}(\psi^{\mathfrak{p}}))$ and $\varphi^\eta := \psi^\eta \cdot \sigma^\eta(J^\eta(\psi^\eta))$. It follows that $\varphi^{\mathfrak{p}} \sim \langle \cdot \rangle_\Lambda^{\mathfrak{p}}$, $\varphi^\eta \sim \langle \cdot \rangle_\Lambda^\eta$, and

$$|\varphi_x^{\mathfrak{p}}| \geq |\varphi_x^\eta| \quad \text{for every } x \in \Lambda. \quad (11.2.49)$$

Moreover,

$$\mathbb{E}_{\Lambda,\beta} \left[\text{sgn}(\varphi_x^{\mathfrak{p}}) \text{sgn}(\varphi_y^{\mathfrak{p}}) - \text{sgn}(\varphi_x^{\eta}) \text{sgn}(\varphi_y^{\eta}) \mid |\varphi^{\mathfrak{p}}|, |\varphi^{\eta}| \right] \quad (11.2.50)$$

$$= \langle \sigma_x \sigma_y \rangle_{\Lambda, J^{\mathfrak{p}}(\psi^{\mathfrak{p}}), \beta}^{\text{Ising}, +} - \langle \sigma_x \sigma_y \rangle_{\Lambda, J^{\eta}(\psi^{\eta}), \beta}^{\text{Ising}, \text{sgn}(\eta)} \geq 0, \quad (11.2.51)$$

where the equality follows from independence and the last inequality from the Ginibre inequality for the Ising model. Similarly, by Griffiths' inequality for the Ising model,

$$\mathbb{E}_{\Lambda,\beta} \left[\text{sgn}(\varphi_x^{\mathfrak{p}}) \text{sgn}(\varphi_y^{\mathfrak{p}}) \mid |\varphi^{\mathfrak{p}}|, |\varphi^{\eta}| \right] = \langle \sigma_x \sigma_y \rangle_{\Lambda, J^{\mathfrak{p}}(\psi^{\mathfrak{p}}), \beta}^{\text{Ising}, +} \geq 0. \quad (11.2.52)$$

□

The following Lemma is the crucial place in this section where Γ -invariance is required.

Lemma 11.2.14. *Let $\beta > 0$. Let $\langle \cdot \rangle_{\beta} \in \mathcal{G}_{\Gamma}(\beta)$. Then, for any $x, y \in V$ such that $J_{x,y} > 0$,*

$$\langle \varphi_x \varphi_y \rangle_{\beta}^{+} \geq \langle \varphi_x \varphi_y \rangle_{\beta} \geq \langle \varphi_x \varphi_y \rangle_{\beta}^0. \quad (11.2.53)$$

Moreover, for any $x \in V$,

$$\langle \varphi_x^2 \rangle_{\beta}^{+} \geq \langle \varphi_x^2 \rangle_{\beta} \geq \langle \varphi_x^2 \rangle_{\beta}^0. \quad (11.2.54)$$

Proof. We start by proving (11.2.53). The first inequality is a direct consequence of Proposition 11.2.11. The second inequality in (11.2.53) is more involved and requires Γ -invariance. We show that a change in temperature is more costly than a change of boundary condition. More precisely, define the function $t \mapsto (\tilde{J}_{u,v}(t))_{u,v \in V}$ as $\tilde{J}_{u,v}(t) = tJ_{u,v}$ if there is $\gamma \in \Gamma$ such that $u = \gamma x$ and $v = \gamma y$, and $\tilde{J}_{u,v}(t) = J_{u,v}$ otherwise. For $t \in (0, 1)$, define $\alpha(t) := \langle \varphi_x \varphi_y \rangle_{\beta, \tilde{J}(t)}^{+}$. In addition, define $b = \langle \varphi_x \varphi_y \rangle_{\beta}$. Assume that, for any $t \in (0, 1)$,

$$\alpha(t) \leq b. \quad (11.2.55)$$

We now argue that the second inequality in (11.2.53) is satisfied. Indeed, as a consequence of the Ginibre inequality of Proposition 11.2.11, we have that $\alpha(t) \geq \langle \varphi_x \varphi_y \rangle_{\beta, \tilde{J}(t)}^0$. The desired inequality then follows from the left-continuity⁴ at $t = 1$ of $t \mapsto \langle \varphi_x \varphi_y \rangle_{\beta, \tilde{J}(t)}^0$. It remains to prove (11.2.55).

Fix $t \in (0, 1)$ and let $\alpha = \alpha(t)$. Let S_n be the set of pairs $u, v \in B_n$ such that there exists $\gamma \in \Gamma$ with $\gamma x = u$ and $\gamma y = v$. Note that there exists $c_1 > 0$ such that $|S_n| \geq c_1 |B_n|$. We define

$$f(n) := \sum_{\{u,v\} \in S_n} \varphi_u \varphi_v. \quad (11.2.56)$$

Note that by Γ -invariance,

$$\langle f(n) \rangle_{\beta} = b |S_n|, \quad \langle f(n) \rangle_{\beta, \tilde{J}(t)} = \alpha |S_n|. \quad (11.2.57)$$

Let $\varepsilon \in (0, \alpha)$. Define the event

$$C_n = \{f(n) \leq (\alpha - 2\varepsilon_0) |S_n|\}. \quad (11.2.58)$$

⁴This follows from the fact that this function can be realised as a increasing limit of increasing functions.

The key result we need to establish is the following large deviation estimate: there exists $c > 0$ such that, for n sufficiently large

$$\langle \langle \mathbb{1}_{C_n} \rangle_{B_n, \beta, J}^\eta \mathbb{1}_{K_n} \rangle_\beta \leq e^{-c|B_n|} \quad (11.2.59)$$

where $K_n = \{|\eta| \leq \mathfrak{p} \text{ in } B_n^c\}$. We assume this for now to establish (11.2.55).

On the one hand, by the Cauchy–Schwarz inequality and (11.2.59), we have that

$$\langle \langle f(n) \mathbb{1}_{C_n} \rangle_{B_n, \beta, J}^\eta \mathbb{1}_{K_n} \rangle_\beta = o(1). \quad (11.2.60)$$

On the other hand, by a union bound and regularity,

$$\langle \mathbb{1}_{\{\exists x \notin B_n, |\eta_x| > \mathfrak{p}_x\}} \rangle_\beta^+ \leq \sum_{x \notin B_n} \langle \mathbb{1}_{\{|\eta_x| > \mathfrak{p}_x\}} \rangle_\beta^+ \leq \sum_{x \notin B_n} \frac{1}{d_G(o, x)^M} \lesssim \frac{1}{n^{M-d}}. \quad (11.2.61)$$

Thus, by the Cauchy–Schwarz inequality,

$$\left| \langle \langle f(n) \rangle_{B_n, \beta, J}^\eta \mathbb{1}_{K_n^c} \rangle_\beta \right| \leq \sqrt{\langle f(n)^2 \rangle_\beta} \sqrt{\langle \mathbb{1}_{\{\exists x \notin B_n, |\eta_x| > \mathfrak{p}_x\}} \rangle_\beta} = o(|S_n|) \quad (11.2.62)$$

provided M is chosen large enough.

Hence, by the above estimates and the definition of C_n ,

$$b|S_n| = \langle f(n) \rangle_\beta = \langle \langle f(n) \rangle_{B_n, \beta, J}^\eta \mathbb{1}_{K_n} \rangle_\beta + \langle \langle f(n) \rangle_{B_n, \beta, J}^\eta \mathbb{1}_{K_n^c} \rangle_\beta \quad (11.2.63)$$

$$\geq (\alpha - 2\varepsilon_0)|S_n| \langle \langle \mathbb{1}_{C_n^c} \rangle_{B_n, \beta, J}^\eta \mathbb{1}_{K_n} \rangle_\beta - o(|S_n|) \quad (11.2.64)$$

$$= (\alpha - 2\varepsilon_0)|S_n| \left(\langle \mathbb{1}_{K_n} \rangle_\beta - \langle \langle \mathbb{1}_{C_n} \rangle_{B_n, \beta, J}^\eta \mathbb{1}_{K_n} \rangle_\beta \right) - o(|S_n|). \quad (11.2.65)$$

As a result, by applying (11.2.59),

$$b|S_n| \geq (\alpha - 2\varepsilon_0)|S_n| - o(|S_n|). \quad (11.2.66)$$

Sending n to infinity we obtain $b \geq \alpha - 2\varepsilon_0$, which implies that $b \geq \alpha$, as desired.

Finally, we turn to the proof of the large deviation estimate (11.2.59). By the DLR equations we have

$$\langle f(n) \rangle_{B_n, \beta, \tilde{J}(t)}^{\mathfrak{p}} \geq \langle f(n) \rangle_{\beta, \tilde{J}(t)}^+ - \varepsilon(n) = \alpha|S_n| - \varepsilon(n) \quad (11.2.67)$$

where, arguing as above,

$$\varepsilon(n) := \langle |f(n)| \mathbb{1}_{\{\exists x \notin B_n, |\eta_x| > \mathfrak{p}_x\}} \rangle_{\beta, \tilde{J}(t)}^+ = o(|S_n|). \quad (11.2.68)$$

Define

$$A_n = \{f(n) \geq (\alpha - \varepsilon_0)|S_n|\} \quad (11.2.69)$$

For every n large enough, the above considerations imply

$$\langle f(n) \mathbb{1}_{A_n} \rangle_{B_n, \beta, \tilde{J}(t)}^{\mathfrak{p}} = \langle f(n) \rangle_{B_n, \beta, \tilde{J}(t)}^{\mathfrak{p}} - \langle f(n) \mathbb{1}_{A_n^c} \rangle_{B_n, \beta, \tilde{J}(t)}^{\mathfrak{p}} \geq \varepsilon_0|S_n| - o(|S_n|) \geq \varepsilon_0|S_n|/2. \quad (11.2.70)$$

Observe, furthermore, that

$$\langle f(n) \mathbb{1}_{A_n} \rangle_{B_n, \beta, \tilde{J}(t)}^{\mathfrak{p}} = \frac{\langle \mathbb{1}_{A_n} e^{\beta(t-1)f(n)} f(n) \rangle_{\Lambda, \beta, J}^{\mathfrak{p}}}{\langle e^{\beta(t-1)f(n)} \rangle_{B_n, \beta, J}^{\mathfrak{p}}} \leq \frac{\langle \mathbb{1}_{A_n} e^{\beta(t-1)f(n)} |f(n)| \rangle_{\Lambda, \beta, J}^{\mathfrak{p}}}{\langle \mathbb{1}_{C_n} e^{\beta(t-1)f(n)} \rangle_{B_n, \beta, J}^{\mathfrak{p}}} \quad (11.2.71)$$

$$\leq e^{\beta \varepsilon_0(t-1)|S_n|} \frac{\langle |f(n)| \rangle_{B_n, \beta, J}^{\mathfrak{p}}}{\langle \mathbb{1}_{C_n} \rangle_{B_n, \beta, J}^{\mathfrak{p}}}. \quad (11.2.72)$$

Since $\langle \varphi_x \varphi_y \rangle_{B_n, \beta, J}^{\mathfrak{p}} \rightarrow \langle \varphi_x \varphi_y \rangle_{\beta, J}^+$, we have $\langle |f(n)| \rangle_{B_n, \beta, J}^{\mathfrak{p}} \leq |S_n| \langle \varphi_x \varphi_y \rangle_{B_n, \beta, J}^{\mathfrak{p}} \leq C_1 |S_n|$ for some constant $C_1 > 0$ independent of n . Re-arranging, this yields a bound on $\langle \mathbb{1}_{C_n} \rangle_{B_n, \beta, J}^{\mathfrak{p}}$. We want to replace the boundary $\mathfrak{p}$ by an arbitrary boundary condition in K_n .

Let $\eta \in K_n$. Define

$$g_n(\eta) := \sum_{\substack{x \in B_n \\ y \notin B_n}} J_{x,y} \varphi_x(\mathfrak{p}_y - \eta_y). \quad (11.2.73)$$

Using the above inequalities, we have that

$$\frac{\langle \mathbb{1}_{C_n} e^{\beta g_n(\eta)} \rangle_{B_n, \beta, J}^{\eta}}{\langle e^{\beta g_n(\eta)} \rangle_{B_n, \beta, J}^{\eta}} = \langle \mathbb{1}_{C_n} \rangle_{B_n, \beta, J}^{\mathfrak{p}} \leq \frac{2C_1}{\varepsilon_0} e^{-\frac{\beta \varepsilon_0}{c_1}(1-t)|B_n|}. \quad (11.2.74)$$

Note that by the Cauchy–Schwarz inequality

$$\left(\langle \mathbb{1}_{C_n} \rangle_{B_n, \beta, J}^{\eta} \right)^2 \leq \langle \mathbb{1}_{C_n} e^{\beta g_n(\eta)} \rangle_{B_n, \beta, J}^{\eta} \langle e^{-\beta g_n(\eta)} \rangle_{B_n, \beta, J}^{\eta}, \quad (11.2.75)$$

so that,

$$\langle \mathbb{1}_{C_n} \rangle_{B_n, \beta, J}^{\eta} \leq \sqrt{\frac{2C_1}{\varepsilon_0}} e^{-\frac{\beta \varepsilon_0}{2c_1}(1-t)|B_n|} \langle e^{\beta |g_n(\eta)|} \rangle_{B_n, \beta, J}^{\eta}. \quad (11.2.76)$$

We seek an estimate on $\langle e^{\beta |g_n(\eta)|} \rangle_{B_n, \beta, J}^{\eta}$ which is uniform over $\eta \in K$. To this end, observe that, by the definition of $\mathfrak{p}$, and by the condition **(C4)** satisfied by J ,

$$|g_n(\eta)| \leq 2 \sum_{\substack{x \in B_n \\ y \notin B_n}} J_{x,y} |\varphi_x| \mathfrak{p}_y \quad (11.2.77)$$

$$\leq 2C \sum_{x \in B_n} |\varphi_x| \sum_{y \notin B_n} \frac{\sqrt[4]{M \log d_G(o, y)}}{d_G(x, y)^{d+\varepsilon}}. \quad (11.2.78)$$

Let $\delta > 0$ be such that $|B_n \setminus B_{n-1}| = O(n^{-\delta} |B_n|)$ for every $n \geq 1$ (such a δ exists by [241]—see Remark 11.C.2 in the Appendix). Note that for every $x \in B_n$ and $y \notin B_n$, using that $d_G(x, y) \geq 1$ and $d_G(o, x) \leq n$, we have

$$d_G(o, y) \leq d_G(o, x) + d_G(x, y) \leq (n+1) d_G(x, y). \quad (11.2.79)$$

Moreover, for every $x \in B_n$ and $y \notin B_n$, if $d_G(o, x) \leq n - n^{\delta/2}$, then since $d_G(o, y) \geq n$ we have $d_G(x, y) \geq n^{\delta/2}$. Therefore, with the same condition on $d_G(o, x)$, we have

$$\frac{\sqrt[4]{M \log d_G(o, y)}}{d_G(x, y)^{d+\varepsilon}} \leq \frac{\sqrt[4]{M \log(n+1)} + \sqrt[4]{M \log d_G(x, y)}}{d_G(x, y)^{d+\varepsilon}} \leq \frac{1}{d_G(x, y)^{d+\varepsilon/2}} \quad (11.2.80)$$

provided that n is large enough.

By Lemma 11.C.9,

$$\sum_{y \in V \setminus \{x\}} \frac{1}{d_G(x, y)^{d+\varepsilon/2}} < \infty. \quad (11.2.81)$$

Thus, combining the above estimates and summing over y , we find

$$|g_n(\eta)| \leq C' \sum_{x \in B_n} |\varphi_x| \left(\sqrt[4]{M \log n} \mathbf{1}_{x \in B_n \setminus B_{c_n}} + o(1) \right) \quad (11.2.82)$$

for some constant $C' > 0$, where $c_n = \lfloor n - n^{\delta/2} \rfloor$ and $o(1)$ does not depend $\eta \in K_n$. Plugging this inequality into (11.2.76) and integrating over K_n yields

$$\langle \langle \mathbf{1}_{C_n} \rangle_{B_n, \beta, J}^\eta \mathbf{1}_{K_n} \rangle_\beta \leq \sqrt{\frac{2C_1}{\varepsilon_0}} e^{-\frac{\beta \varepsilon_0}{2c_1} (1-t)|B_n|} \left\langle e^{\beta C' \sqrt[4]{M \log n} \sum_{x \in B_n \setminus B_{c_n}} |\varphi_x| + o(1) \sum_{x \in B_n} |\varphi_x|} \right\rangle_\beta. \quad (11.2.83)$$

By regularity of $\langle \cdot \rangle_\beta$,

$$\left\langle e^{\beta C' \sqrt[4]{M \log n} \sum_{x \in B_n \setminus B_{c_n}} |\varphi_x| + o(1) \sum_{x \in B_n} |\varphi_x|} \right\rangle_\beta = e^{o(|B_n|)} \quad (11.2.84)$$

thus establishing (11.2.59) to complete the proof of (11.2.53).

To show the remaining estimate (11.2.54) of Lemma 11.2.14, one can argue similarly by varying the parameter a of the single-site measure. $\square$

We now turn to the main result of this section.

Proposition 11.2.15 (Lebowitz-type characterisation). *Let $\beta > 0$. The following are equivalent.*

1. *For any Γ -invariant Gibbs measure $\langle \cdot \rangle_\beta \in \mathcal{G}_\Gamma(\beta)$, there exists $\lambda \in [0, 1]$ such that $\langle \cdot \rangle_\beta = \lambda \langle \cdot \rangle_\beta^+ + (1 - \lambda) \langle \cdot \rangle_\beta^-$.*
2. *For any Γ -invariant Gibbs measure $\langle \cdot \rangle_\beta \in \mathcal{G}_\Gamma(\beta)$, and for any bounded and measurable even function f , we have $\langle f \rangle_\beta^+ = \langle f \rangle_\beta$.*
3. *For any Γ -invariant Gibbs measure $\langle \cdot \rangle_\beta \in \mathcal{G}_\Gamma(\beta)$, and for any $x, y \in V$ such that $J_{x,y} > 0$, any vertices $z_1, z_2, \dots, z_k \in V$, $k \geq 0$, and any positive integers $a_1, a_2, \dots, a_k$, we have*

$$\left\langle \varphi_x \varphi_y \prod_{i=1}^k \varphi_{z_i}^{2a_i} \right\rangle_\beta^+ = \left\langle \varphi_x \varphi_y \prod_{i=1}^k \varphi_{z_i}^{2a_i} \right\rangle_\beta. \quad (11.2.85)$$

4. *For any $x, y \in V$ such that $J_{x,y} > 0$,*

$$\langle \varphi_x \varphi_y \rangle_\beta^+ = \langle \varphi_x \varphi_y \rangle_\beta^0 \quad \text{and} \quad \langle \varphi_x^2 \rangle_\beta^+ = \langle \varphi_x^2 \rangle_\beta^0. \quad (11.2.86)$$

Proof. Since $\langle \cdot \rangle_\beta^+$ and $\langle \cdot \rangle_\beta^-$ coincide on even functions, it is straightforward that $1 \Rightarrow 4$.

Let us prove that $4 \Rightarrow 3$. To this end, by Lemma 11.2.14 we have that for any $\langle \cdot \rangle_\beta \in \mathcal{G}_\Gamma(\beta)$,

$$\langle \varphi_x \varphi_y \rangle_\beta^+ = \langle \varphi_x \varphi_y \rangle_\beta \quad \text{and} \quad \langle \varphi_w^2 \rangle_\beta^+ = \langle \varphi_w^2 \rangle_\beta \quad (11.2.87)$$

for every $w \in \{x, y, z_1, z_2, \dots, z_k\}$. Consider some $\varepsilon \in (0, 1)$, and a finite set Λ that contains $x, y, z_1, \dots, z_k$. Let

$$A_\Lambda := \{\eta \mid |\eta_z| \leq \mathfrak{p} \ \forall z \in \Lambda^c, \langle \varphi_x \varphi_y \rangle_{\Lambda, \beta}^{\mathfrak{p}} \leq \langle \varphi_x \varphi_y \rangle_{\Lambda, \beta}^\eta + \varepsilon \quad (11.2.88)$$

$$\text{and } \langle \varphi_w^2 \rangle_{\Lambda, \beta}^{\mathfrak{p}} \leq \langle \varphi_w^2 \rangle_{\Lambda, \beta}^\eta + \varepsilon \ \forall w = x, y, z_1, \dots, z_k\}. \quad (11.2.89)$$

Note that $\langle \mathbf{1}_{A_\Lambda} \rangle_\beta \rightarrow 1$ as $\Lambda \uparrow V$. Since the inequality

$$\left\langle \varphi_x \varphi_y \prod_{i=1}^k \varphi_{z_i}^{2a_i} \right\rangle_{\Lambda, \beta}^{\mathfrak{p}} \geq \left\langle \varphi_x \varphi_y \prod_{i=1}^k \varphi_{z_i}^{2a_i} \right\rangle_{\Lambda, \beta}^\eta \quad (11.2.90)$$

holds for every η such that $|\eta| \leq \mathfrak{p}$ outside Λ by the Ginibre inequality, it suffices to obtain an upper bound for

$$\left\langle \varphi_x \varphi_y \prod_{i=1}^k \varphi_{z_i}^{2a_i} \right\rangle_{\Lambda, \beta}^{\mathfrak{p}} - \left\langle \varphi_x \varphi_y \prod_{i=1}^k \varphi_{z_i}^{2a_i} \right\rangle_{\Lambda, \beta}^\eta, \quad (11.2.91)$$

which converges to 0 uniformly in $\eta \in A_\Lambda$ as $\Lambda \rightarrow V$ and $\varepsilon \rightarrow 0$.

Consider the coupling $(\mathbb{P}_{\Lambda, \beta}, \varphi^{\mathfrak{p}}, \varphi^\eta)$ of Lemma 11.2.12 and let

$$\varphi^{\mathfrak{p}} = \varphi_x^{\mathfrak{p}} \varphi_y^{\mathfrak{p}} \prod_{i=1}^k (\varphi_{z_i}^{\mathfrak{p}})^{2a_i} \quad \text{and} \quad \varphi^\eta = \varphi_x^\eta \varphi_y^\eta \prod_{i=1}^k (\varphi_{z_i}^\eta)^{2a_i}. \quad (11.2.92)$$

Then,

$$\left\langle \varphi_x \varphi_y \prod_{i=1}^k \varphi_{z_i}^{2a_i} \right\rangle_{\Lambda, \beta}^{\mathfrak{p}} - \left\langle \varphi_x \varphi_y \prod_{i=1}^k \varphi_{z_i}^{2a_i} \right\rangle_{\Lambda, \beta}^\eta = \mathbb{E}_{\Lambda, \beta} [\varphi^{\mathfrak{p}} - \varphi^\eta] \quad (11.2.93)$$

$$= \mathbb{E}_{\Lambda, \beta} [(\varphi^{\mathfrak{p}} - \varphi^\eta) \mathbf{1}_B] + \mathbb{E}_{\Lambda, \beta} [(\varphi^{\mathfrak{p}} - \varphi^\eta) \mathbf{1}_{B^c}] \quad (11.2.94)$$

where B is the event $\{(\varphi_w^{\mathfrak{p}})^2 \leq (\varphi_w^\eta)^2 + \sqrt{\varepsilon} \text{ for every } w \in \{x, y, z_1, \dots, z_k\}\}$. Letting

$$f^{\mathfrak{p}}(x, y) := \mathbb{E}_{\Lambda, \beta} [\text{sgn}(\varphi_x^{\mathfrak{p}}) \text{sgn}(\varphi_y^{\mathfrak{p}}) \mid |\varphi^{\mathfrak{p}}|, |\varphi^\eta|] \quad (11.2.95)$$

and

$$f^\eta(x, y) := \mathbb{E}_{\Lambda, \beta} [\text{sgn}(\varphi_x^\eta) \text{sgn}(\varphi_y^\eta) \mid |\varphi^{\mathfrak{p}}|, |\varphi^\eta|], \quad (11.2.96)$$

we have

$$\begin{aligned} \mathbb{E}_{\Lambda, \beta} [(\varphi^{\mathfrak{p}} - \varphi^\eta) \mathbf{1}_B] &\leq \mathbb{E}_{\Lambda, \beta} [|\varphi^\eta| (f^{\mathfrak{p}}(x, y) - f^\eta(x, y))] + O(\varepsilon^{1/4}) \\ &\leq \sqrt{\mathbb{E}_{\Lambda, \beta} [(\varphi^\eta)^2]} \sqrt{\mathbb{E}_{\Lambda, \beta} [(f^{\mathfrak{p}}(x, y) - f^\eta(x, y))^2]} + O(\varepsilon^{1/4}) \\ &\leq \sqrt{\mathbb{E}_{\Lambda, \beta} [(\varphi^\eta)^2]} \sqrt{2\mathbb{E}_{\Lambda, \beta} [f^{\mathfrak{p}}(x, y) - f^\eta(x, y)]} + O(\varepsilon^{1/4}), \end{aligned}$$

where the constant in the $O(\varepsilon^{1/4})$ term is uniform in Λ . In the first line, we used the inequalities

$$|\varphi_w^{\mathfrak{p}}| \leq \sqrt{(\varphi_w^\eta)^2 + \sqrt{\varepsilon}} \leq |\varphi_w^\eta| + \varepsilon^{1/4} \quad (11.2.97)$$

for $w \in \{x, y, z_1, \dots, z_k\}$, and the fact that $f^p(y, z) \geq f^\eta(y, z)$ to remove $\mathbf{1}_B$. In the second line, we used the Cauchy–Schwarz inequality, and in the third line we used that $0 \leq f^p(y, z) - f^\eta(y, z) \leq 2$.

Now, define S to be the event $\{|\varphi_x^\eta \varphi_y^\eta| \geq \sqrt{\varepsilon}\}$, and note that

$$\begin{aligned} \varepsilon &\geq \langle \varphi_x \varphi_y \rangle_{\Lambda, \beta}^p - \langle \varphi_x \varphi_y \rangle_{\Lambda, \beta}^\eta \\ &\geq \mathbb{E}_{\Lambda, \beta} \left[|\varphi_x^\eta \varphi_y^\eta| (f^p(x, y) - f^\eta(x, y)) \right] \\ &\geq \mathbb{E}_{\Lambda, \beta} \left[|\varphi_x^\eta \varphi_y^\eta| (f^p(x, y) - f^\eta(x, y)) \mathbf{1}_S \right] \\ &\geq \sqrt{\varepsilon} \mathbb{E}_{\Lambda, \beta} [(f^p(x, y) - f^\eta(x, y)) \mathbf{1}_S]. \end{aligned}$$

In the second line, we used that $|\varphi_x^p \varphi_y^p| \geq |\varphi_x^\eta \varphi_y^\eta|$ and that $f^p(x, y) \geq 0$, and in the third line, we used that $f^p(x, y) \geq f^\eta(x, y)$. It follows that

$$\mathbb{E}_{\Lambda, \beta} [(f^p(x, y) - f^\eta(x, y)) \mathbf{1}_S] \leq \sqrt{\varepsilon}. \quad (11.2.98)$$

On the other hand,

$$\mathbb{E}_{\Lambda, \beta} [(f^p(x, y) - f^\eta(x, y)) \mathbf{1}_{S^c}] \leq 2\mathbb{P}_{\Lambda, \beta}[S^c], \quad (11.2.99)$$

and the right-hand side converges to 0 uniformly in Λ and η as ε goes to 0. Since $\mathbb{E}_{\Lambda, \beta} [(\varphi^\eta)^2]$ remains bounded uniformly in η by regularity of $\langle \cdot \rangle_{\Lambda, \beta}^\eta$, we can first send Λ to V and then ε to 0 to obtain that $\mathbb{E}_{\Lambda, \beta} [(\varphi^p - \varphi^\eta) \mathbf{1}_B]$ converges to 0 uniformly in η .

To handle $\mathbb{E}_{\Lambda, \beta} [(\varphi^p - \varphi^\eta) \mathbf{1}_{B^c}]$, note that

$$\mathbb{E}_{\Lambda, \beta} [(\varphi^p - \varphi^\eta) \mathbf{1}_{B^c}] \leq \sqrt{\mathbb{E}_{\Lambda, \beta} [(\varphi^p - \varphi^\eta)^2]} \sqrt{\mathbb{P}_{\Lambda, \beta}[B^c]} \quad (11.2.100)$$

by the Cauchy–Schwarz inequality. The first term in the right-hand side is bounded uniformly in Λ . For the second term, using a union bound we get,

$$\begin{aligned} \mathbb{P}_{\Lambda, \beta}[B^c] &\leq \sum_{w \in \{x, y, z_1, \dots, z_k\}} \mathbb{P}_{\Lambda, \beta} [(\varphi_w^p)^2 > (\varphi_w^\eta)^2 + \sqrt{\varepsilon}] \\ &\leq (k+2)\sqrt{\varepsilon} \end{aligned}$$

where we used Markov's inequality on the second line, together with the inequalities $\langle \varphi_w^2 \rangle_{\Lambda, \beta}^p \leq \langle \varphi_w^2 \rangle_{\Lambda, \beta}^\eta + \varepsilon$ and $|\varphi_w^p| \geq |\varphi_w^\eta|$. Taking first the limit as $\Lambda \uparrow V$ and then $\varepsilon \rightarrow 0$, we can now conclude that $\mathbb{E}_{\Lambda, \beta} [(\varphi^p - \varphi^\eta) \mathbf{1}_{B^c}]$ converges to 0, which implies that

$$\left\langle \varphi_x \varphi_y \prod_{i=1}^k \varphi_{z_i}^{2a_i} \right\rangle_\beta^+ = \left\langle \varphi_x \varphi_y \prod_{i=1}^k \varphi_{z_i}^{2a_i} \right\rangle_\beta, \quad (11.2.101)$$

as desired.

We now prove $3 \Rightarrow 2$. Notice that it is sufficient⁵ to prove that for any $A : V \rightarrow \mathbb{N}$ finitely supported with $\sum_{x \in V} A_x$ even, one has

$$\langle \varphi_A \rangle_\beta^+ = \langle \varphi_A \rangle_\beta. \quad (11.2.102)$$

⁵Indeed, if this holds, going to Laplace transforms, one finds that any finite (even) dimensional vector $(\varphi_{x_1}, \dots, \varphi_{x_{2k}})$ has the same law under the two measures which in turns yields that the measures $\langle \cdot \rangle_\beta^+$ and $\langle \cdot \rangle_\beta$ coincide on the set of even measurable functions.

First, we prove that for any $x, y \in V$, any $z_1, \dots, z_k$ with $k \geq 0$, one has,

$$\left\langle \varphi_x \varphi_y \prod_{i=1}^k \varphi_{z_i}^{2a_i} \right\rangle_{\beta}^+ = \left\langle \varphi_x \varphi_y \prod_{i=1}^k \varphi_{z_i}^{2a_i} \right\rangle_{\beta} \quad (11.2.103)$$

Let us start with the case where x and y are such that there exists $z \in V$ such that $J_{x,y} J_{y,z} > 0$. Apply Ginibre's inequality to $A = \mathbb{1}_x + \mathbb{1}_y + \sum_{i=1}^k 2a_i \mathbb{1}_{z_i}$ and $B = \mathbb{1}_z + \mathbb{1}_y$, and use the hypothesis to get

$$\begin{aligned} 0 &= \left\langle \varphi_x \varphi_z \varphi_y^2 \prod_{i=1}^k \varphi_{z_i}^{2a_i} \right\rangle_{\beta}^+ - \left\langle \varphi_x \varphi_z \varphi_y^2 \prod_{i=1}^k \varphi_{z_i}^{2a_i} \right\rangle_{\beta} \\ &\geq \left| \left\langle \varphi_x \varphi_y \prod_{i=1}^k \varphi_{z_i}^{2a_i} \right\rangle_{\beta}^+ \langle \varphi_z \varphi_y \rangle_{\beta} - \left\langle \varphi_x \varphi_y \prod_{i=1}^k \varphi_{z_i}^{2a_i} \right\rangle_{\beta} \langle \varphi_z \varphi_y \rangle_{\beta}^+ \right|, \end{aligned}$$

so that $\left\langle \varphi_x \varphi_y \prod_{i=1}^k \varphi_{z_i}^{2a_i} \right\rangle_{\beta}^+ = \left\langle \varphi_x \varphi_y \prod_{i=1}^k \varphi_{z_i}^{2a_i} \right\rangle_{\beta}$. To get the result in the general case, we use the irreducibility **(C3)** of J and proceed inductively on the number p such that there exists $(x_i)_{0 \leq i \leq p}$ with $x_0 = x$, $x_p = y$, and $J_{x_0, x_1} \dots J_{x_{p-1}, x_p} > 0$. Notice that (11.2.103) also holds with $x = y$.

To conclude, it is sufficient to prove that for any $x_1, \dots, x_{2p}, z_1, \dots, z_k \in V$ with $p \geq 1$ and $k \geq 0$, and any $a_1, \dots, a_k \in \mathbb{N}$, one has,

$$\left\langle \varphi_{x_1} \dots \varphi_{x_{2p}} \prod_{i=1}^k \varphi_{z_i}^{2a_i} \right\rangle_{\beta}^+ = \left\langle \varphi_{x_1} \dots \varphi_{x_{2p}} \prod_{i=1}^k \varphi_{z_i}^{2a_i} \right\rangle_{\beta}. \quad (11.2.104)$$

Once again, let us proceed inductively on p . The case $p = 1$ was proved above. Assume the result holds for any collection of $2p - 2$ points with $p \geq 1$ and consider $x_1, \dots, x_{2p} \in V$ with $z_1, \dots, z_k \in V$ and $k \geq 0$. Apply Ginibre's inequality to $A = \sum_{i=1}^{2p} \mathbb{1}_{x_i} + \sum_{i=1}^k 2a_i \mathbb{1}_{z_i}$ and $B = \mathbb{1}_{x_{2p-1}} + \mathbb{1}_{x_{2p}}$ to get the result.

We now prove $2 \Rightarrow 1$. Let $\langle \cdot \rangle_{\beta} \in \mathcal{G}_{\Gamma}(\beta)$. We can assume that $\langle \cdot \rangle_{\beta}$ is extremal. Define the measure $\langle \cdot \rangle_{\beta}^{\text{sym}}$ as follows: for any bounded measurable function $X : \mathbb{R}^V \rightarrow \mathbb{R}$,

$$\langle X(\varphi) \rangle_{\beta}^{\text{sym}} = \frac{1}{2} \langle X(\varphi) \rangle_{\beta} + \frac{1}{2} \langle X(-\varphi) \rangle_{\beta}. \quad (11.2.105)$$

Then, notice that for any odd function X , one has $\langle X(\varphi) \rangle_{\beta}^{\text{sym}} = \langle X(\varphi) \rangle_{\beta}^0 = 0$. If X is even, then $\langle X(\varphi) \rangle_{\beta}^{\text{sym}} = \langle X(\varphi) \rangle_{\beta}^0$ by the hypothesis, hence $\langle \cdot \rangle_{\beta}^{\text{sym}} = \langle \cdot \rangle_{\beta}^0$. Now, the hypothesis applied to $\langle \cdot \rangle_{\beta}^0$ yields the relation $\langle \cdot \rangle_{\beta}^0 = \frac{1}{2} \langle \cdot \rangle_{\beta}^+ + \frac{1}{2} \langle \cdot \rangle_{\beta}^-$. The uniqueness of the decomposition into extremal states then allows us to conclude that $\langle \cdot \rangle_{\beta} \in \{ \langle \cdot \rangle_{\beta}^+, \langle \cdot \rangle_{\beta}^- \}$. $\square$

11.3 Switching principle for random tangled currents

In this section, we develop a random tangled current representation for φ^4 which is analogous to the classical random current representation for Ising models. In comparison to the latter representation, in our setup, an additional complexity appears through the notion of tanglings which we define below. When clear from context, we sometimes write $\langle \cdot \rangle_{\Lambda, \beta, h} = \langle \cdot \rangle_{\Lambda, \beta, h}^0$ to denote the free measure at (β, h) .

11.3.1 Current expansion

Let $G = (V, E)$ be a countably infinite, locally finite graph. Let Λ be a finite subset of V . To include boundary conditions, we introduce a ghost vertex $\mathfrak{g}$ with the condition $\varphi_{\mathfrak{g}} \equiv 1$ and extend the model on the augmented set $\Lambda_{\mathfrak{g}} = \Lambda \cup \{\mathfrak{g}\}$. Let $\Omega_{\Lambda_{\mathfrak{g}}} := \mathbb{N}^{\mathcal{P}_2(\Lambda_{\mathfrak{g}})}$ be the space of currents on $\Lambda_{\mathfrak{g}}$, where $\mathbb{N} = \{0, 1, 2, \dots\}$. For $\mathbf{n} \in \Omega_{\Lambda_{\mathfrak{g}}}$ and $\{x, y\} \subset \Lambda_{\mathfrak{g}}$, we use the following notations to denote the value of $\mathbf{n}$ on the pair $\{x, y\}$: $\mathbf{n}(\{x, y\}) = \mathbf{n}_{x,y} = \mathbf{n}(x, y)$.

Given $\mathbf{n} \in \Omega_{\Lambda_{\mathfrak{g}}}$, let

$$\partial \mathbf{n} := \{x \in \Lambda : \Delta \mathbf{n}(x) \text{ is odd}\} \quad (11.3.1)$$

be the set of sources of $\mathbf{n}$, where

$$\Delta \mathbf{n}(x) := \sum_{y \in \Lambda_{\mathfrak{g}}} \mathbf{n}_{x,y} \quad (11.3.2)$$

is the $\mathbf{n}$ -degree of x . Additionally, let $|\mathbf{n}|$ denote the L^1 norm of $\mathbf{n}$, i.e.

$$|\mathbf{n}| = \sum_{\{x,y\} \in \mathcal{P}_2(\Lambda_{\mathfrak{g}})} \mathbf{n}_{x,y}. \quad (11.3.3)$$

Let

$$\mathcal{M}(\Lambda_{\mathfrak{g}}) := \left\{ A \in \mathbb{N}^{\Lambda_{\mathfrak{g}}}, A_{\mathfrak{g}} \leq 1, \sum_{x \in \Lambda_{\mathfrak{g}}} A_x \text{ is even} \right\} \quad (11.3.4)$$

be the set of admissible moments on $\Lambda_{\mathfrak{g}}$. Given $A, B \in \mathcal{M}_{\mathfrak{g}}$, define $A + B \in \mathcal{M}(\Lambda_{\mathfrak{g}})$ by

$$(A + B)_x := A_x + B_x \quad (11.3.5)$$

for all $x \in \Lambda$, and

$$(A + B)_{\mathfrak{g}} := A_{\mathfrak{g}} + B_{\mathfrak{g}} \pmod{2}. \quad (11.3.6)$$

Given $A \in \mathcal{M}(\Lambda_{\mathfrak{g}})$, denote by $\partial A := \{x \in \Lambda, A_x \text{ is odd}\}$ its set of sources (note $\mathfrak{g}$ is not included by convention). The above definitions also apply naturally to $\mathcal{M}(\Lambda)$, the set of admissible moments on Λ .

In the following proposition, we obtain an expression for the spin correlations in terms of random currents, and as a corollary we obtain the first Griffiths' inequality. We recall that for $A \in \mathcal{M}(\Lambda_{\mathfrak{g}})$, $\varphi_A = \prod_{x \in \Lambda} \varphi_x^{A_x}$.

Proposition 11.3.1 (Current expansion for φ^A). *Let $\beta > 0$. For any $A \in \mathcal{M}(\Lambda_{\mathfrak{g}})$ and $h = (h_x)_{x \in \Lambda}$ external field,*

$$\langle \varphi_A \rangle_{\Lambda, \beta, h} = \frac{\sum_{\partial \mathbf{n} = \partial A} w_{A, \beta, h}(\mathbf{n})}{\sum_{\partial \mathbf{n} = \emptyset} w_{0, \beta, h}(\mathbf{n})}, \quad (11.3.7)$$

where

$$w_{A, \beta, h}(\mathbf{n}) := \prod_{\{x,y\} \subset \Lambda} \frac{(\beta J_{x,y})^{\mathbf{n}_{x,y}}}{\mathbf{n}_{x,y}!} \prod_{x \in \Lambda} \frac{(\beta h_x)^{\mathbf{n}_{x,\mathfrak{g}}}}{\mathbf{n}_{x,\mathfrak{g}}!} \left\langle \varphi^{\Delta \mathbf{n}(x) + A_x} \right\rangle_0 \quad (11.3.8)$$

and we recall that $\langle \cdot \rangle_0$ is integration with respect to the single-site measure $\rho_{g,a}$.

In particular, Griffiths' first inequality holds: if $h_x \geq 0$ for every $x \in \Lambda$, then

$$\langle \varphi_A \rangle_{\Lambda, \beta, h} \geq 0. \quad (11.3.9)$$

Proof. Expanding the partition function of the φ^4 model in the same fashion as for the Ising model, we obtain the current representation of the φ^4 model. More precisely,

$$\begin{aligned} Z_{\Lambda, \beta, h} &= \int_{\mathbb{R}^\Lambda} \exp \left(\beta \sum_{\{x, y\} \subset \Lambda} J_{x, y} \varphi_x \varphi_y + \beta \sum_{x \in \Lambda} h_x \varphi_x \right) \prod_{x \in \Lambda} \rho_{g, a}(\mathrm{d}\varphi_x) \\ &= \int_{\mathbb{R}^\Lambda} \sum_{\mathbf{n} \in \Omega_{\Lambda_g}} \prod_{\{x, y\} \subset \Lambda} \frac{(\beta J_{x, y})^{\mathbf{n}_{x, y}}}{\mathbf{n}_{x, y}!} \prod_{x \in \Lambda} \frac{(\beta h_x)^{\mathbf{n}_{x, g}}}{\mathbf{n}_{x, g}!} \prod_{x \in \Lambda} \varphi_x^{\Delta \mathbf{n}(x)} \prod_{x \in \Lambda} \rho_{g, a}(\mathrm{d}\varphi_x) \\ &= \sum_{\mathbf{n} \in \Omega_{\Lambda_g}} \prod_{\{x, y\} \subset \Lambda} \frac{(\beta J_{x, y})^{\mathbf{n}_{x, y}}}{\mathbf{n}_{x, y}!} \prod_{x \in \Lambda} \frac{(\beta h_x)^{\mathbf{n}_{x, g}}}{\mathbf{n}_{x, g}!} \prod_{x \in \Lambda} \int_{\mathbb{R}} \varphi_x^{\Delta \mathbf{n}(x)} \rho_{g, a}(\mathrm{d}\varphi). \end{aligned}$$

Now, note that for $z \in \partial \mathbf{n}$,

$$\int_{\mathbb{R}} \varphi^{\Delta \mathbf{n}(x)} \rho_{g, a}(\mathrm{d}\varphi) = 0 \quad (11.3.10)$$

since $\mathrm{d}\rho_{g, a}$ is an even measure. Then,

$$Z_{\Lambda, \beta, h} = z_{g, a}^{|\Lambda|} \sum_{\substack{\mathbf{n} \in \Omega_{\Lambda_g} \\ \partial \mathbf{n} = \emptyset}} \prod_{\{x, y\} \subset \Lambda} \frac{(\beta J_{x, y})^{\mathbf{n}_{x, y}}}{\mathbf{n}_{x, y}!} \prod_{x \in \Lambda} \frac{(\beta h_x)^{\mathbf{n}_{x, g}}}{\mathbf{n}_{x, g}!} \langle \varphi^{\Delta \mathbf{n}(x)} \rangle_0. \quad (11.3.11)$$

By the same argument as above, we see that for $A \in \mathcal{M}_g$,

$$\int_{\mathbb{R}^\Lambda} \varphi_A \exp \left(\beta \sum_{\{x, y\} \subset \Lambda} J_{x, y} \varphi_x \varphi_y + \beta \sum_{x \in \Lambda} h_x \varphi_x \right) \prod_{x \in \Lambda} \rho_{g, a}(\mathrm{d}\varphi_x) = z_{g, a}^{|\Lambda|} \sum_{\substack{\mathbf{n} \in \Omega_{\Lambda_g} \\ \partial \mathbf{n} = \partial A}} w_{A, \beta, h}(\mathbf{n}). \quad (11.3.12)$$

□

Remark 11.3.2. Note that we can also obtain a current expansion for $\langle \varphi_A \rangle_{\Lambda, \beta, h}^\eta$ by viewing the boundary condition as an external field $h'_x := \sum_{y \notin \Lambda} J_{x, y} \eta_y$ and then writing $\langle \varphi_A \rangle_{\Lambda, \beta, h}^\eta = \langle \varphi_A \rangle_{\Lambda, \beta, h+h'}^0$.

Remark 11.3.3. Due to presence of the moments $\langle \varphi^{\Delta \mathbf{n}(x)} \rangle_0$, all known proofs of the classical switching lemma for the Ising model do not apply to the above current weights. In order to recover a switching principle, we need the notion of tangled currents, which is introduced in the next section.

Remark 11.3.4. We may drop the h or β dependency on $w_{A, \beta, h}$ when clear from context.

11.3.2 Tangled currents

In this section, we introduce the notion of tanglings for single and double currents. Let us start with the former case.

We fix $A \in \mathcal{M}(\Lambda_g)$, and let $\mathbf{n} \in \Omega_{\Lambda_g}$ be such that $\partial \mathbf{n} = \partial A$. For each $z \in \Lambda$, we define the block $\mathcal{B}_z(\mathbf{n}, A)$ as follows : for each $y \in \Lambda_g$, it contains $\mathbf{n}_{z, y}$ points labelled $(zy(k))_{1 \leq k \leq \mathbf{n}_{z, y}}$, and A_z points labelled $(za(k))_{1 \leq k \leq A_z}$. Note that $\mathcal{B}_z(\mathbf{n}, A)$ has cardinality $\Delta \mathbf{n}(z) + A_z$, which is an even number. We also write $\mathcal{B}_g(\mathbf{n}, A) = \{g\}$, and when $A_g = 1$ we may also write

$\mathfrak{g}a(1) = \mathfrak{g}$. For each $z \in \Lambda$, let $\mathcal{T}_{\mathbf{n},A}(z)$ be the set of *even partitions* of the block $\mathcal{B}_z(\mathbf{n}, A)$, i.e. for every $P = \{P_1, P_2, \dots, P_k\} \in \mathcal{T}_{\mathbf{n},A}(z)$, each $|P_i|$ is even. Define

$$\mathcal{T}_{\mathbf{n},A} = \bigotimes_{z \in \Lambda} \mathcal{T}_{\mathbf{n},A}(z). \quad (11.3.13)$$

An element $\mathbf{t} \in \mathcal{T}_{\mathbf{n},A}$ is called a *tangling*. We sometimes refer to an element $\mathbf{t}_z$ of $\mathcal{T}_{\mathbf{n},A}(z)$ as a tangling (of $\mathcal{B}_z(\mathbf{n}, A)$), this will be clear from the context. We call the pair $(\mathbf{n}, \mathbf{t})$ a *tangled current*.

Let $\mathcal{H}(\mathbf{n}, A)$ be the graph consisting of vertex set

$$\bigcup_{z \in \Lambda_{\mathfrak{g}}} \mathcal{B}_z(\mathbf{n}, A) \quad (11.3.14)$$

and edge set

$$\left(\bigcup_{\{x,y\} \subset \Lambda} \{\{xy(k), yx(k)\}, 1 \leq k \leq \mathbf{n}_{x,y}\} \right) \cup \left(\bigcup_{x \in \Lambda} \{\{x\mathfrak{g}(k), \mathfrak{g}\}, 1 \leq k \leq \mathbf{n}_{x,\mathfrak{g}}\} \right). \quad (11.3.15)$$

Given any tangling $\mathbf{t} \in \mathcal{T}_{\mathbf{n},A}$, the graph $\mathcal{H}(\mathbf{n}, A)$ naturally projects to a multigraph $\mathcal{H}(\mathbf{n}, \mathbf{t}, A)$ where for each block $\mathcal{B}_z(\mathbf{n}, A)$, vertices in the same partition class given by $\mathbf{t}_z$ are identified as one. This gives a canonical notion of connectivity in a tangled current $(\mathbf{n}, \mathbf{t})$.

Remark 11.3.5. In the case where $A = \emptyset$, we write $\mathcal{H}(\mathbf{n}, \mathbf{t})$ instead of $\mathcal{H}(\mathbf{n}, \mathbf{t}, \emptyset)$.

Remark 11.3.6. We can also extend the definitions for $\mathbf{n} \in \Omega_{\Lambda}$ and $A \in \mathcal{M}(\Lambda)$ by simply noticing that $\mathbf{n}$ can be seen as an element of $\Omega_{\Lambda_{\mathfrak{g}}}$ and A can be seen as an element of $\mathcal{M}(\Lambda_{\mathfrak{g}})$. For a tangling $\mathbf{t} \in \mathcal{T}(\mathbf{n}, A)$, the graph $\mathcal{H}(\mathbf{n}, \mathbf{t}, A)$ is obtained via the same procedure as above, removing $\mathcal{B}_{\mathfrak{g}}(\mathbf{n}, A)$.

In the perspective of constructing a switching principle for tangled currents, it is natural to extend the above definitions to the case where $\mathbf{n}$ is replaced by the sum of two currents.

Let $\Lambda' \subset \Lambda$ be two finite subsets of V . Fix $A \in \mathcal{M}(\Lambda_{\mathfrak{g}})$ and $B \in \mathcal{M}(\Lambda'_{\mathfrak{g}})$. Let $\mathbf{n}_1 \in \Omega_{\Lambda_{\mathfrak{g}}}$ and $\mathbf{n}_2 \in \Omega_{\Lambda'_{\mathfrak{g}}}$ satisfy $\partial \mathbf{n}_1 = \partial A$ and $\partial \mathbf{n}_2 = \partial B$. We trivially view $\mathbf{n}_2$ as an element of $\Omega_{\Lambda_{\mathfrak{g}}}$ by setting $\mathbf{n}_2(x, y) = 0$ for all pairs $\{x, y\}$ with $x \in \Lambda$ or $y \in \Lambda$. Additionally, we view B as an element of $\mathcal{M}(\Lambda_{\mathfrak{g}})$. Let $\mathbf{n} := \mathbf{n}_1 + \mathbf{n}_2 \in \Omega_{\Lambda_{\mathfrak{g}}}$. To define a notion of tangled currents for $\mathbf{n}$, we use the same procedure as above. We define $\mathcal{B}_z(\mathbf{n}, A, B)$ in the following way: for every $z \in \Lambda_{\mathfrak{g}}$, i.e. for each $y \in \Lambda$, it contains $\mathbf{n}_{z,y}$ points labelled $(zy(k))_{1 \leq k \leq \mathbf{n}_{z,y}}$, A_z points labelled $(za(k))_{1 \leq k \leq A_z}$, and B_z points labelled $(zb(k))_{1 \leq k \leq B_z}$. There is a natural injection of $\mathcal{B}_z(\mathbf{n}_1, A)$ and $\mathcal{B}_z(\mathbf{n}_2, B)$ in $\mathcal{B}_z(\mathbf{n}, A, B)$. We write $\mathcal{T}_{\mathbf{n},A,B}(z)$ for the set of even partitions of the set $\mathcal{B}_z(\mathbf{n}, A, B)$ whose restriction to $\mathcal{B}_z(\mathbf{n}_1, A)$ and $\mathcal{B}_z(\mathbf{n}_2, B)$ is also an even partition, i.e. for every $P = \{P_1, P_2, \dots, P_k\} \in \mathcal{T}_{\mathbf{n},A,B}(z)$, both $|P_i \cap \mathcal{B}_z(\mathbf{n}_1, A)|$ and $|P_i \cap \mathcal{B}_z(\mathbf{n}_2, B)|$ are even numbers. The elements of $\mathcal{T}_{\mathbf{n},A,B}(z)$ are called *admissible* partitions. As above, we define

$$\mathcal{T}_{\mathbf{n},A,B} = \bigotimes_{z \in \Lambda} \mathcal{T}_{\mathbf{n},A,B}(z). \quad (11.3.16)$$

For any tangling in $\mathcal{T}_{\mathbf{n},A,B}$, we obtain a multigraph $\mathcal{H}(\mathbf{n}, \mathbf{t}, A, B)$ which provides a natural notion of connectivity on the pair $(\mathbf{n}, \mathbf{t}) = (\mathbf{n}_1 + \mathbf{n}_2, \mathbf{t})$.

Remark 11.3.7. As in Remark 11.3.6, we can obtain a notion of tangled currents for pairs of currents $(\mathbf{n}_1, \mathbf{n}_2)$ such that $\mathbf{n}_1 \in \Omega_{\Lambda_{\mathfrak{g}}}$ and $\mathbf{n}_2 \in \Omega_{\Lambda}$, or $\mathbf{n}_1 \in \Omega_{\Lambda}$ and $\mathbf{n}_2 \in \Omega_{\Lambda'}$, or $\mathbf{n}_1 \in \Omega_{\Lambda}$ and $\mathbf{n}_2 \in \Omega_{\Lambda'}$.

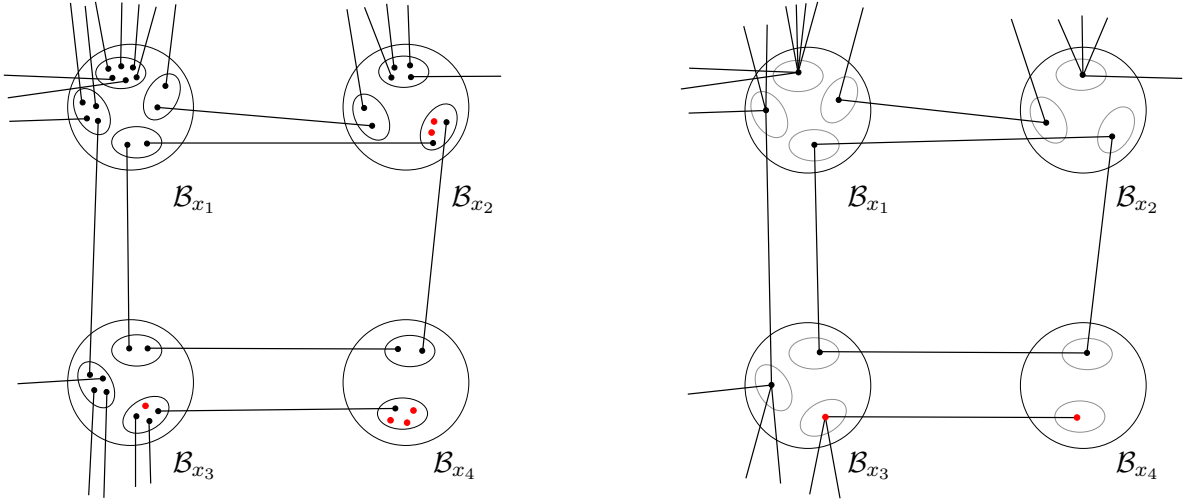


Figure 11.1: An example of a tangled current $(\mathbf{n}, \mathbf{t})$. On the left picture, the circles represent the blocks $\mathcal{B}(\mathbf{n}, A)$; the small ellipses inside represent the elements of the partitions $\mathbf{t}$. Notice that x_3, x_4 are sources of $\mathbf{n}$, and the red vertices correspond to the vertices associated with A . In this case, $A_{x_1} = 0$, $A_{x_2} = 2$, $A_{x_3} = 1$ and $A_{x_4} = 3$. On the right picture, the multigraph $\mathcal{H}(\mathbf{n}, \mathbf{t}, A)$ is obtained by identifying vertices in the same ellipses. The red vertices on the right picture correspond to sources of the multigraph.

11.3.3 A switching principle for tangled currents

Let $\Lambda' \subset \Lambda$ be finite subsets of $V \cup \{\mathbf{g}\}$. We define $\Omega_{\Lambda}^{\mathcal{T}}$ to be the set of tangled currents on Λ . Fix $\beta > 0$ and $h = (h_x)_{x \in V} \in (\mathbb{R}^+)^V$. Before stating the main result of the section, we need a definition.

Definition 11.3.8. Let $B \in \mathcal{M}(\Lambda')$. Define $\mathcal{F}_B^{\Lambda'}$ to be the subset of $\Omega_{\Lambda}^{\mathcal{T}}$ consisting of all tangled currents $(\mathbf{n}, \mathbf{t})$ for which every connected component of $(\mathbf{n}, \mathbf{t})_{\Lambda'}$ intersects $\{zb(k) : z \in \Lambda', 1 \leq k \leq B_z\}$ an even number of times. In words, $\mathcal{F}_B^{\Lambda'}$ consists of all tangled currents that pair the elements of B in its associated multigraph. The set $\mathcal{F}_B^{\Lambda'}$ is measurable with respect to the natural σ -algebra on $\Omega_{\Lambda}^{\mathcal{T}}$.

Remark 11.3.9. The above definition is to be compared with the pairing event which naturally appears in the switching lemma of the Ising model.

Remark 11.3.10. Note that for the above definition to make sense, we need to have a natural injection of B into the (ordered) degree function of $\mathbf{n}$.

Theorem 11.3.11. Let $A \in \mathcal{M}(\Lambda)$, $B \in \mathcal{M}(\Lambda')$. For every $\mathbf{n}_1 \in \Omega_{\Lambda}$ and $\mathbf{n}_2 \in \Omega_{\Lambda'}$ satisfying $\partial \mathbf{n}_1 = \partial A$ and $\partial \mathbf{n}_2 = \partial B$ and for every $z \in \Lambda$, there are probability measures $\rho_{z, A, B}^{\mathbf{n}_1, \mathbf{n}_2}$ on $\mathcal{T}_{\mathbf{n}, A, B}(z)$ such that the following holds. For any measurable and bounded $F : \Omega_{\Lambda}^{\mathcal{T}} \rightarrow \mathbb{R}$,

$$\sum_{\substack{\partial \mathbf{n}_1 = \partial A \\ \partial \mathbf{n}_2 = \partial B}} w_{A, \beta}(\mathbf{n}_1) w_{B, \beta}(\mathbf{n}_2) \rho_{A, B}^{\mathbf{n}_1, \mathbf{n}_2} [F(\mathbf{n}_1 + \mathbf{n}_2, \mathbf{t})] \quad (11.3.17)$$

$$= \sum_{\substack{\partial \mathbf{n}_1 = \partial(A+B) \\ \partial \mathbf{n}_2 = \emptyset}} w_{A+B, \beta}(\mathbf{n}_1) w_{\emptyset, \beta}(\mathbf{n}_2) \rho_{A+B, \emptyset}^{\mathbf{n}_1, \mathbf{n}_2} [F(\mathbf{n}_1 + \mathbf{n}_2, \mathbf{t}) \mathbf{1}_{\mathcal{F}_B^{\Lambda'}}], \quad (11.3.18)$$

where

$$\rho_{A,B}^{\mathbf{n}_1, \mathbf{n}_2} := \bigotimes_{z \in \Lambda} \rho_{z,A,B}^{\mathbf{n}_1, \mathbf{n}_2}, \text{ and } \rho_{A+B, \emptyset}^{\mathbf{n}_1, \mathbf{n}_2} := \bigotimes_{z \in \Lambda} \rho_{z,A+B, \emptyset}^{\mathbf{n}_1, \mathbf{n}_2}, \quad (11.3.19)$$

are measures on $\mathcal{T}_{\mathbf{n}, A, B}$ and $\mathcal{T}_{\mathbf{n}, A+B, \emptyset}$ respectively, and $\mathcal{F}_B^{\Lambda'}$ is the event defined above.

Proof. See Section 11.3.5. □

Remark 11.3.12. Note that, for any z , the measures $\rho_{z,A,B}^{\mathbf{n}_1, \mathbf{n}_2}$ depend on g , a , and $(\Delta \mathbf{n}_1(z), \Delta \mathbf{n}_2(z))$, but they do not depend on Λ , β , h or the rest of $\mathbf{n}_1, \mathbf{n}_2$.

We can also state the switching lemma in a probabilistic way. We start by introducing the random tangled current measures of interest. We define $\rho_{z,A}^{\mathbf{n}} := \rho_{z,A, \emptyset}^{\mathbf{n}, 0}$ and

$$\rho_A^{\mathbf{n}} := \bigotimes_{z \in \Lambda} \rho_{z,A}^{\mathbf{n}}. \quad (11.3.20)$$

We view $\rho_A^{\mathbf{n}}$ as a measure on single currents.

Definition 11.3.13. Let $\Lambda' \subset \Lambda$ be finite subsets of $V \cup \{\mathbf{g}\}$. Let $A \in \mathcal{M}(\Lambda)$ and $B \in \mathcal{M}(\Lambda')$. We define a measure $\mathbf{P}_{\Lambda, \beta}^A$ on the set of tangled currents on Λ with source set A : for any $(\mathbf{n}, \mathbf{t}) \in \Omega_{\Lambda, \mathcal{T}}$,

$$\mathbf{P}_{\Lambda, \beta}^A[(\mathbf{n}, \mathbf{t})] = \mathbb{1}_{\partial \mathbf{n} = \partial A} \frac{w_{A, \beta}(\mathbf{n}) \rho_A^{\mathbf{n}}[(\mathbf{n}, \mathbf{t})]}{\sum_{\partial \mathbf{m} = \partial A} w_{A, \beta}(\mathbf{m})}. \quad (11.3.21)$$

We also define a measure $\mathbf{P}_{\Lambda, \Lambda', \beta}^{A, B}$ on the set of tangled currents on Λ with source set $A + B$: for $(\mathbf{n}, \mathbf{t}) \in \Omega_{\Lambda, \mathcal{T}}$,

$$\mathbf{P}_{\Lambda, \Lambda', \beta}^{A, B}[(\mathbf{n}, \mathbf{t})] = \mathbb{1}_{\partial \mathbf{n} = \partial(A+B)} \frac{\sum_{\substack{\partial \mathbf{n}_1 = \partial A \\ \partial \mathbf{n}_2 = \partial B}} w_{A, \beta}(\mathbf{n}_1) w_{B, \beta}(\mathbf{n}_2) \mathbb{1}_{\mathbf{n}_1 + \mathbf{n}_2 = \mathbf{n}} \rho_{A, B}^{\mathbf{n}_1, \mathbf{n}_2}[(\mathbf{n}, \mathbf{t})]}{\sum_{\substack{\partial \mathbf{n}_1 = \partial A \\ \partial \mathbf{n}_2 = \partial B}} w_{A, \beta}(\mathbf{n}_1) w_{B, \beta}(\mathbf{n}_2)}. \quad (11.3.22)$$

We write $\mathbf{E}_{\Lambda, \beta}^A$ (resp. $\mathbf{E}_{\Lambda, \Lambda', \beta}^{A, B}$) the expectation with respect to the measure $\mathbf{P}_{\Lambda, \beta}^A$ (resp. $\mathbf{P}_{\Lambda, \Lambda', \beta}^{A, B}$).

Using these definitions, we can reformulate the switching lemma in a more probabilistic way.

Corollary 11.3.14 (Probabilistic version of the switching lemma). *Let $A \in \mathcal{M}(\Lambda)$, $B \in \mathcal{M}(\Lambda')$. For any measurable and bounded $F : \Omega_{\Lambda, \mathcal{T}} \rightarrow \mathbb{R}$,*

$$\frac{\langle \varphi_A \rangle_{\Lambda, \beta, h} \langle \varphi_B \rangle_{\Lambda', \beta, h}}{\langle \varphi_{A+B} \rangle_{\Lambda, \beta, h}} \mathbf{E}_{\Lambda, \Lambda', \beta}^{A, B}[F(\mathbf{n}, \mathbf{t})] = \mathbf{E}_{\Lambda, \Lambda', \beta}^{A+B, \emptyset}[F(\mathbf{n}, \mathbf{t}) \mathbb{1}_{\mathcal{F}_B^{\Lambda'}}]. \quad (11.3.23)$$

In what follows, β is often fixed and to lighten the notations we omit it in the notation of the tangling measures.

11.3.4 Application: correlation inequalities

We fix $\beta > 0$ and drop it from the notation.

Proposition 11.3.15. *Let Λ be a finite subset of V . Let $h = (h_x)_{x \in \Lambda} \in (\mathbb{R}_+)^{\Lambda}$. For $A, B \in \mathcal{M}(\Lambda_{\mathfrak{g}})$,*

$$\frac{\langle \varphi_A \rangle_{\Lambda, h} \langle \varphi_B \rangle_{\Lambda, h}}{\langle \varphi_{A+B} \rangle_{\Lambda, h}} = \mathbf{P}_{\Lambda_{\mathfrak{g}}, \Lambda_{\mathfrak{g}}}^{A+B, \emptyset}[(\mathbf{n}_1 + \mathbf{n}_2, \mathbf{t}) \in \mathcal{F}_B] \quad (11.3.24)$$

Proof. This is a direct consequence of Corollary 11.3.14 applied to $F = 1$. $\square$

Proposition 11.3.16 (Second Griffiths' inequality). *Let Λ be a finite subset of V , and $h = (h_x)_{x \in \Lambda} \in (\mathbb{R}_+)^{\Lambda}$. Let $A, B \in \mathcal{M}(\Lambda_{\mathfrak{g}})$.*

$$\langle \varphi_A \varphi_B \rangle_{\Lambda, h} \geq \langle \varphi_A \rangle_{\Lambda, h} \langle \varphi_B \rangle_{\Lambda, h}. \quad (11.3.25)$$

Proof. This is a direct consequence of Proposition 11.3.15 $\square$

Proposition 11.3.17 (Monotonicity in volume). *Let $\Lambda' \subset \Lambda$ be finite subsets of V , and $h = (h_x)_{x \in \Lambda} \in (\mathbb{R}_+)^{\Lambda}$. Let $B \in \mathcal{M}(\Lambda_{\mathfrak{g}})$. Then,*

$$\langle \varphi_B \rangle_{\Lambda, h} \geq \langle \varphi_B \rangle_{\Lambda', h}. \quad (11.3.26)$$

Proof. Apply Corollary 11.3.14 to $A = 0$ and $F = 1$ to obtain that

$$\langle \varphi_B \rangle_{\Lambda', h} = \langle \varphi_B \rangle_{\Lambda, h} \mathbf{P}_{\Lambda_{\mathfrak{g}}, \Lambda'_{\mathfrak{g}}}^{B, \emptyset}[\mathcal{F}_B^{\Lambda'_{\mathfrak{g}}}] \quad (11.3.27)$$

$\square$

Proposition 11.3.18. *Let Λ be a finite subset of V , $h = (h_x)_{x \in \Lambda} \in (\mathbb{R}_+)^{\Lambda}$, and $A \in \mathcal{M}(\Lambda_{\mathfrak{g}})$. Then,*

- (i) $\beta \mapsto \langle \varphi_A \rangle_{\Lambda, \beta, h, g, a}$ is non-decreasing on $\mathbb{R}_+$,
- (ii) $g \mapsto \langle \varphi_A \rangle_{\Lambda, \beta, h, g, a}$ is non-increasing on $\mathbb{R}$,
- (iii) $a \mapsto \langle \varphi_A \rangle_{\Lambda, \beta, h, g, a}$ is non-increasing on $\mathbb{R}$.

Proof. Notice that

$$\frac{d}{d\beta} \langle \varphi_A \rangle_{\Lambda, \beta, h, g, a} = \sum_{\{x, y\} \subset \Lambda} J_{x, y} \langle \varphi_A; \varphi_x \varphi_y \rangle_{\Lambda, \beta, h}, \quad (11.3.28)$$

$$\frac{d}{dg} \langle \varphi_A \rangle_{\Lambda, \beta, h, g, a} = - \sum_{x \in \Lambda} \langle \varphi_A; \varphi_x^4 \rangle_{\Lambda, \beta, h, g, a}, \quad (11.3.29)$$

and

$$\frac{d}{da} \langle \varphi_A \rangle_{\Lambda, \beta, h, g, a} = - \sum_{x \in \Lambda} \langle \varphi_A; \varphi_x^2 \rangle_{\Lambda, \beta, h, g, a}, \quad (11.3.30)$$

where we used the notation $\langle \varphi_A; \varphi_B \rangle_{\Lambda, \beta, h, g, a} := \langle \varphi_A \varphi_B \rangle_{\Lambda, \beta, h} - \langle \varphi_A \rangle_{\Lambda, \beta, h} \langle \varphi_B \rangle_{\Lambda, \beta, h}$. $\square$

Remark 11.3.19. Note that there is also monotonicity in h and J analogous to the statements in Proposition 11.3.18. They follow by similar considerations.

11.3.5 Proof of switching principle

In this section, we use the Griffiths–Simon approximation [224] to construct a switching principle for a random tangled current representation of φ^4 on a finite subset Λ of V . The prelimit is an Ising model on the product of Λ with the complete graph on N vertices. We use the switching principle of the random current representation of the Ising model to obtain the switching principle for φ^4 by taking limits, as appropriate. Along the way, we identify the underlying geometric structures that persist in the scaling limit.

11.3.5.1 Griffiths–Simon approximation

Let $N \in \mathbb{N}$. Define $\Lambda_N := \Lambda \times \{1, \dots, N\}$. For $x \in \Lambda$, define

$$B_{x,N} := \{(x, i)\}_{1 \leq i \leq N} \quad (11.3.31)$$

and set $B_{\mathfrak{g},N} := \{\mathfrak{g}\}$.

Let $\mathcal{G}_N$ be the graph with vertex set $V(\mathcal{G}_N) := \Lambda_N \cup \{\mathfrak{g}\}$ and edge set

$$E(\mathcal{G}_N) := E^{\text{int}}(\mathcal{G}_N) \cup E^{\text{ext}}(\mathcal{G}_N) \quad (11.3.32)$$

where

$$E^{\text{int}}(\mathcal{G}_N) := \bigcup_{x \in \Lambda} \mathcal{P}_2(B_{x,N}) \quad (11.3.33)$$

$$E^{\text{ext}}(\mathcal{G}_N) := \left\{ \{u, v\} \text{ such that there exists } \{x, y\} \in E(\Lambda_{\mathfrak{g}}) \text{ with } u \in B_{x,N} \text{ and } v \in B_{y,N} \right\} \quad (11.3.34)$$

and, above, $\mathcal{P}_2(B_{x,N})$ denotes the set of pairs in $B_{x,N}$.

Remark 11.3.20. An edge of the form $\{(x, j), (x, k)\} \in E^{\text{int}}$ is called *internal*, whereas an edge of the form $\{(x, j), (y, k)\} \in E^{\text{ext}}$ is called *external*.

We now introduce the Griffiths–Simon model. Let $\tilde{g} = (12/g)^{1/4}$ and $\tilde{a} = a\tilde{g}^2$. Define the coupling constants

$$c_N := \tilde{g}N^{-3/4}, \quad d_N := \frac{1}{N} \left(1 - \frac{\tilde{a}}{\sqrt{N}} \right). \quad (11.3.35)$$

For $\beta > 0$, $h \in (\mathbb{R}_+)^{\Lambda}$ and σ in $\{\pm 1\}^{\Lambda_N}$, define the Hamiltonian

$$\mathbf{H}_{\Lambda, \beta, N}^h(\sigma) := -\beta c_N^2 \sum_{\substack{\{x, y\} \subset \Lambda \\ j, k \in \{1, \dots, N\}}} J_{x, y} \sigma_{(x, j)} \sigma_{(y, k)} \quad (11.3.36)$$

$$- \beta c_N \sum_{(x, j) \in \Lambda_N} h_x \sigma_{(x, j)} - d_N \sum_{\substack{x \in \Lambda \\ j, k \in \{1, \dots, N\}}} \sigma_{(x, j)} \sigma_{(x, k)}. \quad (11.3.37)$$

Definition 11.3.21. Let $\beta > 0$. Let $\mu_{\Lambda, N, \beta}^h$ be the probability measure on $\{\pm 1\}^{\Lambda_N}$ defined for $\sigma \in \{\pm 1\}^{\Lambda_N}$ by

$$\mu_{\Lambda, N, \beta}^h(\sigma) = \frac{\exp(-\mathbf{H}_{\Lambda, \beta, N}^h(\sigma))}{\mathbf{Z}(\Lambda, N, \beta, h)} \quad (11.3.38)$$

where $\mathbf{Z}(\Lambda, N, \beta, h)$ is the partition function. In the specific case where Λ is reduced to a point x , we denote this measure $\mu_{x, N, \beta}^h$. Note that this measure does not depend on x .

Observe that $\mu_{\Lambda, N, \beta}^h$ is an Ising model on Λ_N . The coupling constants are such that the model has a scaling limit as $N \rightarrow \infty$, which is the φ^4 model on Λ with parameters (g, a) at (β, h) . The natural observable to state the convergence is the block-spin moment, defined for $x \in \Lambda$ by

$$\varphi_{x, N} := c_N \sum_{j=1}^N \sigma_{(x, j)}. \quad (11.3.39)$$

The following result was first derived in [224] to obtain, amongst other things, the Lee–Yang theorem for the φ^4 model on $\mathbb{Z}^d$. It was later reproved for a larger class of models in [85].

Proposition 11.3.22. *Let $\beta > 0$ and $h \in (\mathbb{R}_+)^{\Lambda}$. Then, for any $p \in \mathbb{N}^{\Lambda}$,*

$$\lim_{N \rightarrow \infty} \mu_{\Lambda, N, \beta}^h \left[\prod_{x \in \Lambda} \varphi_{x, N}^{p_x} \right] = \left\langle \prod_{x \in \Lambda} \varphi_x^{p_x} \right\rangle_{\Lambda, \beta, h}. \quad (11.3.40)$$

Remark 11.3.23. Proposition 11.3.22 can be seen as a generalisation of the central limit theorem to a class of dependent random variables. The different rescaling factor $c_N \sim N^{-3/4}$ from the usual $N^{-1/2}$ scaling for independent random variables is intrinsically linked to the non-Gaussian limiting random variable. A more sophisticated viewpoint is that the coupling constants are chosen such that the $N \rightarrow \infty$ limit corresponds to a near-critical scaling limit of the Curie–Weiss model. Thus, the convergence of moments of the block-spin magnetisations to the φ^4 moments can be seen as a statement about the model lying in a φ^4 universality class. Indeed, generalisations of these results leading to densities with respect to φ^{2k} measures, i.e. distinct universality classes, can be found in [91].

We strengthen the above convergence result for the specific case of a single block-spin. More precisely, we show that in this instance we can neglect diagonal terms.

Lemma 11.3.24. *For any $p \in \mathbb{N}$ and $x \in \Lambda$,*

$$(c_N N)^p \mu_{x, N, \beta}^0 \left[\prod_{j=1}^p \sigma_{(x, j)} \right] \xrightarrow{N \rightarrow \infty} \langle \varphi^p \rangle_0. \quad (11.3.41)$$

Proof. For $p = 1$, there is nothing to show because both the sequence and $\langle \varphi \rangle_0$ are equal to 0, so let us assume that $p \geq 2$. Let $N \geq p$. Expanding $\varphi_{x, N}^p$ and taking cases according to whether some index repeats or not we obtain

$$\mu_{x, N, \beta}^0 [\varphi_{x, N}^p] = c_N^p \sum_{\substack{(i_1, \dots, i_p) \in \{1, \dots, N\}^p \\ \forall k \neq \ell, i_k \neq i_\ell}} \mu_{x, N, \beta}^0 \left[\prod_{j=1}^p \sigma_{(x, i_j)} \right] + c_N^p \sum_{\substack{(i_1, \dots, i_p) \in \{1, \dots, N\}^p \\ \exists k < \ell, i_k = i_\ell}} \mu_{x, N, \beta}^0 \left[\prod_{j=1}^p \sigma_{(x, i_j)} \right]. \quad (11.3.42)$$

Note that

$$c_N^p \sum_{\substack{(i_1, \dots, i_p) \in \{1, \dots, N\}^p \\ \forall k \neq \ell, i_k \neq i_\ell}} \mu_{x, N, \beta}^0 \left[\prod_{j=1}^p \sigma_{(x, i_j)} \right] = c_N^p p! \binom{N}{p} \mu_{x, N, \beta}^0 \left[\prod_{j=1}^p \sigma_{(x, j)} \right] \quad (11.3.43)$$

$$= (1 + o(1)) (c_N N)^p \mu_{x, N, \beta}^0 \left[\prod_{j=1}^p \sigma_{(x, j)} \right] \quad (11.3.44)$$

by the symmetries of the complete graph, and

$$0 \leq c_N^p \sum_{\substack{(i_1, \dots, i_p) \in \{1, \dots, N\}^p \\ \exists k < \ell, i_k = i_\ell}} \mu_{x, N, \beta}^0 \left[\prod_{j=1}^p \sigma_{(x, i_j)} \right] \leq c_N^2 \binom{p}{2} N \mu_{x, N, \beta}^0 [\varphi_{x, N}^{p-2}] \xrightarrow{N \rightarrow \infty} 0 \quad (11.3.45)$$

where we have used that each summand is non-negative by Griffiths' inequality. The desired assertion then follows from Proposition 11.3.22. $\square$

11.3.5.2 A conditional proof

We now introduce the random current representation for $\mu_{\Lambda, N, \beta}^h$. For ease of notation, we adopt the following conventions. First, we write $\Omega_{\Lambda_{\mathfrak{g}}}^N := \Omega_{\mathfrak{G}_N}$ to denote the space of currents on $\mathcal{G}_N$. Second, for $\tilde{\mathbf{n}} \in \Omega_{\Lambda_{\mathfrak{g}}}^N$, $x, y \in \Lambda$ and $j, k \in \{1, \dots, N\}$, we set

$$\tilde{\mathbf{n}}(x, j, y, k) := \tilde{\mathbf{n}}((x, j), (y, k)), \quad \tilde{\mathbf{n}}(x, j, \mathfrak{g}) := \mathbf{n}((x, j), \mathfrak{g}). \quad (11.3.46)$$

The random current expansion for $\mu_{\Lambda, N, \beta}^h$, takes the following form: for any $S \subset \Lambda_N$,

$$\mu_{\Lambda, N, \beta}^h[\sigma_S] = \frac{\sum_{\partial \tilde{\mathbf{n}}=S} w_{N, \beta}(\tilde{\mathbf{n}})}{\sum_{\partial \tilde{\mathbf{n}}=\emptyset} w_{N, \beta}(\tilde{\mathbf{n}})}, \quad (11.3.47)$$

where

$$w_{N, \beta}(\tilde{\mathbf{n}}) := \prod_{\substack{\{x, y\} \subset \Lambda \\ j, k \in \{1, \dots, N\}}} \frac{(\beta c_N^2 J_{x, y})^{\tilde{\mathbf{n}}(x, j, y, k)}}{\tilde{\mathbf{n}}(x, j, y, k)!} \prod_{\substack{x \in \Lambda \\ j, k \in \{1, \dots, N\}}} \frac{(d_N)^{\tilde{\mathbf{n}}(x, j, x, k)}}{\tilde{\mathbf{n}}(x, j, x, k)!} \quad (11.3.48)$$

$$\times \prod_{(x, j) \in \Lambda_N} \frac{(\beta c_N h_x)^{\tilde{\mathbf{n}}(x, j, \mathfrak{g})}}{\tilde{\mathbf{n}}(x, j, \mathfrak{g})!}. \quad (11.3.49)$$

Let $\Theta_N : \Omega_{\Lambda_{\mathfrak{g}}}^N \rightarrow \Omega_{\Lambda_{\mathfrak{g}}}$ be the projection map defined, for all $\tilde{\mathbf{n}} \in \Omega_{\Lambda_{\mathfrak{g}}}^N$ and $\{x, y\} \subset \Lambda_{\mathfrak{g}}$, by

$$\Theta_N(\tilde{\mathbf{n}})_{\{x, y\}} := \sum_{j, k \in \{1, \dots, N\}} \tilde{\mathbf{n}}(x, j, y, k). \quad (11.3.50)$$

We then have that $\partial \Theta_N(\tilde{\mathbf{n}}) = \partial A$ where A is defined for $x \in \Lambda$ by $A_x = |\partial \tilde{\mathbf{n}} \cap B_{x, N}|$.

Conversely, we can lift any $\mathbf{n} \in \Omega_{\Lambda_{\mathfrak{g}}}$ to a set of currents in $\Omega_{\Lambda_{\mathfrak{g}}}^N$ once we have fixed an injection of admissible moments $A \in \mathcal{M}(\Lambda_{\mathfrak{g}})$ to subsets of Λ_N . For convenience, we fix the natural injection: for $A \in \mathcal{M}(\Lambda_{\mathfrak{g}})$, define

$$\tilde{A} := \bigcup_{x \in \Lambda} \{(x, j), 1 \leq j \leq A_x\}. \quad (11.3.51)$$

Then, the lift of $\mathbf{n} \in \Omega_{\Lambda_{\mathfrak{g}}}$ with source set ∂A is the set of $\tilde{\mathbf{n}} \in \Omega_{\Lambda_{\mathfrak{g}}}^N$ such that $\Theta_N(\tilde{\mathbf{n}}) = \mathbf{n}$ and $\partial \tilde{\mathbf{n}} = \tilde{A}$.

Motivated by this observation and Lemma 11.3.24, we define the renormalised weight for $\mathbf{n} \in \Omega_{\Lambda_{\mathfrak{g}}}$ by

$$\bar{w}_{N,A,\beta}(\mathbf{n}) := \frac{(c_N N)^{|A|}}{\mathbf{Z}_N^{|A|}} \sum_{\substack{\tilde{\mathbf{n}} \in \Omega_{\Lambda_{\mathfrak{g}}}^N \\ \Theta_N(\tilde{\mathbf{n}}) = \mathbf{n} \\ \partial \tilde{\mathbf{n}} = \bar{A}}} w_{N,\beta}(\tilde{\mathbf{n}}) \quad (11.3.52)$$

where $|A| = \sum_{x \in \Lambda} A_x$ and

$$\mathbf{Z}_N := \sum_{\substack{\tilde{\mathbf{n}} \in \Omega_{B_{o,N}} \\ \partial \tilde{\mathbf{n}} = \emptyset}} w_{N,\beta}(\tilde{\mathbf{n}}). \quad (11.3.53)$$

We emphasise that the renormalisation factor $c_N N \sim N^{1/4}$ is directly linked to Proposition 11.3.22.

An important step in the proof of the switching principle is that the renormalised weights converge to those of the φ^4 current expansion. To prove this, we need to understand the geometric structure of the currents that persist in the limit. They consist of currents which belong to the set W_N defined below.

Definition 11.3.25. Let $W_N(\Lambda_{\mathfrak{g}}) = W_N := W_{1,N} \cap W_{2,N}$, where

$$W_{1,N} := \left\{ \tilde{\mathbf{n}} \in \Omega_{\Lambda_{\mathfrak{g}}}^N : \forall (x, j) \in \partial \tilde{\mathbf{n}} \setminus \{\mathfrak{g}\}, \forall e \in E^{\text{ext}} \text{ with } (x, j) \in e, \tilde{\mathbf{n}}_e = 0 \right\}, \quad (11.3.54)$$

$$W_{2,N} := \left\{ \tilde{\mathbf{n}} \in \Omega_{\Lambda_{\mathfrak{g}}}^N : \forall (x, j) \in \Lambda_N, \sum_{e \in E^{\text{ext}}: (x,j) \in e} \tilde{\mathbf{n}}_e \leq 1 \right\}. \quad (11.3.55)$$

In words, the currents in $W_{1,N}$ consists of those whose sources are separated from exterior block-spins by other internal vertices; whereas $W_{2,N}$ consists of currents whose weight on external edges does not exceed 1. We have the following result.

Proposition 11.3.26. For all $A \in \mathcal{M}(\Lambda_{\mathfrak{g}})$ and $\mathbf{n} \in \Omega_{\Lambda_{\mathfrak{g}}}$,

$$\frac{(c_N N)^{|A|}}{\mathbf{Z}_N^{|A|}} \sum_{\substack{\tilde{\mathbf{n}} \in W_N \\ \Theta_N(\tilde{\mathbf{n}}) = \mathbf{n} \\ \partial \tilde{\mathbf{n}} = \bar{A}}} w_{N,\beta}(\tilde{\mathbf{n}}) \xrightarrow{N \rightarrow \infty} w_{A,\beta}(\mathbf{n}), \quad (11.3.56)$$

and,

$$\frac{(c_N N)^{|A|}}{\mathbf{Z}_N^{|A|}} \sum_{\substack{\tilde{\mathbf{n}} \in W_N^c \\ \Theta_N(\tilde{\mathbf{n}}) = \mathbf{n} \\ \partial \tilde{\mathbf{n}} = \bar{A}}} w_{N,\beta}(\tilde{\mathbf{n}}) \xrightarrow{N \rightarrow \infty} 0. \quad (11.3.57)$$

Proof. See section 11.3.5.3. □

Corollary 11.3.27. For all $A \in \mathcal{M}(\Lambda_{\mathfrak{g}})$ and $\mathbf{n} \in \Omega_{\Lambda_{\mathfrak{g}}}$,

$$\bar{w}_{N,A,\beta}(\mathbf{n}) \xrightarrow{n \rightarrow \infty} w_{A,\beta}(\mathbf{n}). \quad (11.3.58)$$

Guided by this intuition and Proposition 11.3.26, we isolate how these geometric structures intertwine in the statement of the usual switching lemma for $\mu_{\Lambda, \beta, N}^h$, which takes the following form.

In what follows we fix $\Lambda' \subset \Lambda$ two finite subsets of $V \cup \{\mathfrak{g}\}$. This is slightly different from the convention we had above in which Λ was a subset of V .

Lemma 11.3.28. *Let $S \subset \Lambda_N$, $T \subset \Lambda'_N$, and let $F : \Omega_{\Lambda_g}^N \rightarrow \mathbb{R}$ be a bounded measurable function. Then,*

$$\sum_{\substack{\tilde{\mathbf{n}}_1 \in \Omega_{\Lambda}^N, \tilde{\mathbf{n}}_2 \in \Omega_{\Lambda'}^N \\ \partial \tilde{\mathbf{n}}_1 = S, \partial \tilde{\mathbf{n}}_2 = T}} w_N(\tilde{\mathbf{n}}_1) w_N(\tilde{\mathbf{n}}_2) F(\tilde{\mathbf{n}}_1 + \tilde{\mathbf{n}}_2) \quad (11.3.59)$$

$$= \sum_{\substack{\tilde{\mathbf{n}}_1 \in \Omega_{\Lambda}^N, \tilde{\mathbf{n}}_2 \in \Omega_{\Lambda'}^N \\ \partial \tilde{\mathbf{n}}_1 = S \Delta T, \partial \tilde{\mathbf{n}}_2 = \emptyset}} w_N(\tilde{\mathbf{n}}_1) w_N(\tilde{\mathbf{n}}_2) F(\tilde{\mathbf{n}}_1 + \tilde{\mathbf{n}}_2) \mathbb{1}[\mathcal{F}_T^{\Lambda'_N}] \quad (11.3.60)$$

where $\mathcal{F}_T^{\Lambda'_N}$ is the event that every cluster of the multigraph induced by $(\tilde{\mathbf{n}}_1 + \tilde{\mathbf{n}}_2)_{\Lambda'_N}$ intersects T an even number of times.

If we proceed naively in expressing the lefthand side and the righthand side in terms of projected currents and renormalised weights, we encounter a homogeneity issue. Indeed, let $\mathbf{n}_1 \in \Omega_{\Lambda}$, and $\mathbf{n}_2 \in \Omega_{\Lambda'}$ with source sets $\partial A, \partial B$ associated to $A \in \mathcal{M}(\Lambda)$, and $B \in \mathcal{M}(\Lambda')$ respectively. Suppose we fix injections such that $\tilde{A}$ and $\tilde{B}$ may intersect. Then, the lefthand side of the switching lemma has a scaling factor $(c_N N)^{|A|+|B|}$, whereas the righthand side has a scaling factor $(c_N N)^{|A \Delta B|}$, which is strictly smaller.

To resolve this issue, we require a consistent convention for injecting the sources of fixed admissible moments, A and B , in such a way that their lifts $\tilde{A}$ and $\tilde{B}$ satisfy $\tilde{A} \cap \tilde{B} = \emptyset$. The natural such injection of the ordered pair (A, B) is given by

$$\tilde{A} := \bigcup_{x \in \Lambda} \{(x, j) : 1 \leq j \leq A_x\} \quad (11.3.61)$$

$$\tilde{B} := \bigcup_{x \in \Lambda} \{(x, j) : A_x + 1 \leq j \leq A_x + B_x\}. \quad (11.3.62)$$

Consider the lefthand side of (11.3.59) for $S = \tilde{A}$ and $T = \tilde{B}$. By conditioning on the underlying projected current in Ω_{Λ} or $\Omega_{\Lambda'}$, and then splitting the sum over W_N and W_N^c , we obtain

$$(\text{LHS}) := \sum_{\substack{\tilde{\mathbf{n}}_1 \in \Omega_{\Lambda}^N, \tilde{\mathbf{n}}_2 \in \Omega_{\Lambda'}^N \\ \partial \tilde{\mathbf{n}}_1 = \tilde{A}, \partial \tilde{\mathbf{n}}_2 = \tilde{B}}} w_N(\tilde{\mathbf{n}}_1) w_N(\tilde{\mathbf{n}}_2) F(\tilde{\mathbf{n}}_1 + \tilde{\mathbf{n}}_2) \quad (11.3.63)$$

$$= \sum_{\substack{\mathbf{n}_1 \in \Omega_{\Lambda}, \partial \mathbf{n}_1 = \partial A \\ \mathbf{n}_2 \in \Omega_{\Lambda'}, \partial \mathbf{n}_2 = \partial B}} \sum_{\substack{\Theta_N(\tilde{\mathbf{n}}_1) = \mathbf{n}_1, \Theta_N(\tilde{\mathbf{n}}_2) = \mathbf{n}_2 \\ \partial \tilde{\mathbf{n}}_1 = \tilde{A}, \partial \tilde{\mathbf{n}}_2 = \tilde{B}}} w_N(\tilde{\mathbf{n}}_1) w_N(\tilde{\mathbf{n}}_2) F(\tilde{\mathbf{n}}_1 + \tilde{\mathbf{n}}_2) \quad (11.3.64)$$

$$= \sum_{\substack{\mathbf{n}_1 \in \Omega_{\Lambda}, \partial \mathbf{n}_1 = \partial A \\ \mathbf{n}_2 \in \Omega_{\Lambda'}, \partial \mathbf{n}_2 = \partial B}} (\Sigma_{1,N}(\mathbf{n}_1, \mathbf{n}_2) + \Sigma_{2,N}(\mathbf{n}_1, \mathbf{n}_2)) \quad (11.3.65)$$

where

$$\Sigma_{1,N}(\mathbf{n}_1, \mathbf{n}_2) := \sum_{\substack{\tilde{\mathbf{n}}_1 \in W_N(\Lambda), \tilde{\mathbf{n}}_2 \in W_N(\Lambda') \\ \Theta_N(\tilde{\mathbf{n}}_1) = \mathbf{n}_1, \Theta_N(\tilde{\mathbf{n}}_2) = \mathbf{n}_2 \\ \partial \tilde{\mathbf{n}}_1 = \tilde{A}, \partial \tilde{\mathbf{n}}_2 = \tilde{B}}} w_N(\tilde{\mathbf{n}}_1) w_N(\tilde{\mathbf{n}}_2) F(\tilde{\mathbf{n}}_1 + \tilde{\mathbf{n}}_2) \quad (11.3.66)$$

$$\Sigma_{2,N}(\mathbf{n}_1, \mathbf{n}_2) := \sum_{\substack{\tilde{\mathbf{n}}_1 \notin W_N(\Lambda) \text{ or } \tilde{\mathbf{n}}_2 \notin W_N(\Lambda') \\ \Theta_N(\tilde{\mathbf{n}}_1) = \mathbf{n}_1, \Theta_N(\tilde{\mathbf{n}}_2) = \mathbf{n}_2 \\ \partial \tilde{\mathbf{n}}_1 = \tilde{A}, \partial \tilde{\mathbf{n}}_2 = \tilde{B}}} w_N(\tilde{\mathbf{n}}_1) w_N(\tilde{\mathbf{n}}_2) F(\tilde{\mathbf{n}}_1 + \tilde{\mathbf{n}}_2). \quad (11.3.67)$$

As a consequence of Proposition 11.3.26, we find that (with the proper scaling) $\Sigma_{2,N}$ converges to 0 as $N \rightarrow \infty$.

Lemma 11.3.29. *Let $F : \Omega_\Lambda^N \rightarrow \mathbb{R}$ be bounded. Then, for any $A \in \mathcal{M}(\Lambda)$, $B \in \mathcal{M}(\Lambda')$ and $(\mathbf{n}_1, \mathbf{n}_2) \in \Omega_\Lambda \times \Omega_{\Lambda'}$ such that $(\partial \mathbf{n}_1, \partial \mathbf{n}_2) = (\partial A, \partial B)$,*

$$\lim_{N \rightarrow \infty} \frac{(C_N N)^{|A|+|B|}}{\mathbf{Z}_N^{|\Lambda|+|\Lambda'|}} \Sigma_{2,N}(\mathbf{n}_1, \mathbf{n}_2) = 0 \quad (11.3.68)$$

Proof. Since F is bounded,

$$\left| \frac{(C_N N)^{|A|+|B|}}{\mathbf{Z}_N^{2|\Lambda|+|\Lambda'|}} \Sigma_{2,N}(\mathbf{n}_1, \mathbf{n}_2) \right| \leq \|F\|_\infty ((1) + (2)) \quad (11.3.69)$$

where

$$(1) = \frac{(C_N N)^{|A|}}{\mathbf{Z}_N^{|\Lambda|}} \sum_{\substack{\tilde{\mathbf{n}}_1 \in W_N^c(\Lambda) \\ \Theta_N(\tilde{\mathbf{n}}_1) = \mathbf{n}_1 \\ \partial \tilde{\mathbf{n}}_1 = \tilde{A}}} w_N(\tilde{\mathbf{n}}_1) \frac{(C_N N)^{|B|}}{\mathbf{Z}_N^{|\Lambda'|}} \sum_{\substack{\tilde{\mathbf{n}}_2 \in \Omega_{\Lambda'}^N \\ \Theta_N(\tilde{\mathbf{n}}_2) = \mathbf{n}_2 \\ \partial \tilde{\mathbf{n}}_2 = \tilde{B}}} w_N(\tilde{\mathbf{n}}_2) \quad (11.3.70)$$

$$(2) = \frac{(C_N N)^{|A|}}{\mathbf{Z}_N^{|\Lambda|}} \sum_{\substack{\tilde{\mathbf{n}}_1 \in \Omega_\Lambda^N \\ \Theta_N(\tilde{\mathbf{n}}_1) = \mathbf{n}_1 \\ \partial \tilde{\mathbf{n}}_1 = \tilde{A}}} w_N(\tilde{\mathbf{n}}_1) \frac{(C_N N)^{|B|}}{\mathbf{Z}_N^{|\Lambda'|}} \sum_{\substack{\tilde{\mathbf{n}}_2 \in W_N^c(\Lambda') \\ \Theta_N(\tilde{\mathbf{n}}_2) = \mathbf{n}_2 \\ \partial \tilde{\mathbf{n}}_2 = \tilde{B}}} w_N(\tilde{\mathbf{n}}_2). \quad (11.3.71)$$

The lemma then follows from rewriting the weights in terms of renormalised weights and applying Proposition 11.3.26. $\square$

The above lemma implies that it is sufficient to restrict our attention to pairs of currents that belong to $W_N(\Lambda) \times W_N(\Lambda')$ in the switching lemma of (11.3.59). In order to understand the geometry of the sum of two such currents, we further decompose into internal and external components.

Definition 11.3.30. Let

$$W_N^{\text{ext}}(\Lambda) = \{\tilde{\mathbf{n}} \in W_{2,N} : \tilde{\mathbf{n}}(x, i, x, j) = 0 \text{ for all } x \in \Lambda\}, \quad (11.3.72)$$

$$W_N^{\text{int}}(\Lambda) = \{\tilde{\mathbf{n}} \in \Omega_\Lambda^N : \tilde{\mathbf{n}}(x, i, y, j) = 0 \text{ if } x \neq y\} \quad (11.3.73)$$

Note that any $\tilde{\mathbf{n}} \in W_N(\Lambda)$ admits the decomposition $\tilde{\mathbf{n}} = \tilde{\mathbf{n}}^{\text{ext}} + \tilde{\mathbf{n}}^{\text{int}}$ with $\tilde{\mathbf{n}}^{\text{ext}} \in W_N^{\text{ext}}(\Lambda)$ and $\tilde{\mathbf{n}}^{\text{int}} \in W_N^{\text{int}}(\Lambda)$ such that $\partial\tilde{\mathbf{n}}^{\text{ext}} \cap \partial\tilde{\mathbf{n}} = \emptyset$ and $\partial\tilde{\mathbf{n}}^{\text{int}} = \partial\tilde{\mathbf{n}}^{\text{ext}} \cup \partial\tilde{\mathbf{n}}$. Furthermore, we can write $\tilde{\mathbf{n}}^{\text{int}} = \sum_{x \in \Lambda} \tilde{\mathbf{n}}^{\text{int}}(x)$, where $\tilde{\mathbf{n}}^{\text{int}}(x) \in W_{x,N}^{\text{int}}(\Lambda)$ for

$$W_{x,N}^{\text{int}}(\Lambda) = \{\tilde{\mathbf{n}} \in W_N^{\text{int}}(\Lambda) : \tilde{\mathbf{n}}(y, i, y, j) = 0 \text{ for all } y \neq x\}. \quad (11.3.74)$$

These definitions extend to Λ' and since a current on $\Omega_{\Lambda'}$ can naturally be extended to a current on Ω_{Λ} , one has that $W_{x,N}^{\text{int}}(\Lambda') = W_{x,N}^{\text{int}}(\Lambda)$ for all $x \in \Lambda'$.

In order to identify the scaling limit of $\Sigma_{1,N}$, we have to encode how the connectivity properties inside the block-spins intertwine when summing over the internal components. We formalise this as follows.

Definition 11.3.31. Let $N \in \mathbb{N}$. For $S_x, T_x \in \mathbb{N}$, denote by $\tilde{S}_x := \{(x, i) : 1 \leq i \leq S_x\} \subset B_{x,N}$ and $\tilde{T}_x := \{(x, i) : S_x + 1 \leq i \leq S_x + T_x\}$. Let $\pi_{x,N}^{\tilde{S}_x}$ be the measure on $\Omega_{B_{x,N}}$ defined by

$$\pi_{x,N}^{\tilde{S}_x}(\tilde{\mathbf{n}}^x) := (c_N N)^{S_x} \frac{w_N(\tilde{\mathbf{n}}^x) \mathbf{1}_{\partial\tilde{\mathbf{n}}^x = \tilde{S}_x}}{\sum_{\tilde{\mathbf{n}}^x \in W_{x,N}^{\text{int}}, \partial\tilde{\mathbf{n}}^x = \emptyset} w_N(\tilde{\mathbf{n}}^x)}. \quad (11.3.75)$$

Furthermore, denote by $\pi_{x,N}^{\tilde{S}_x, \tilde{T}_x}$ the pushforward of the product measure $\pi_{x,N}^{\tilde{S}_x} \otimes \pi_{x,N}^{\tilde{T}_x}$ on $\Omega_{B_{x,N}} \times \Omega_{B_{x,N}}$ by the mapping $(\mathbf{n}^x, \mathbf{m}^x) \mapsto \mathbf{n}^x + \mathbf{m}^x$.

Remark 11.3.32. Note that

$$\pi_{x,N}^{\tilde{S}_x}(\Omega_{B_{x,N}}) = (c_N N)^{S_x} \mu_{x,N,\beta}^0 \left[\prod_{i=1}^{S_x} \sigma_{(x,i)} \right], \quad (11.3.76)$$

and

$$\pi_{x,N}^{\tilde{S}_x, \tilde{T}_x}(\Omega_{B_{x,N}}) = (c_N N)^{S_x + T_x} \mu_{x,N,\beta}^0 \left[\prod_{i=1}^{S_x} \sigma_{(x,i)} \right] \mu_{x,N,\beta}^0 \left[\prod_{i=S_x+1}^{S_x+T_x} \sigma_{(x,i)} \right]. \quad (11.3.77)$$

Let $A_x \in \mathbb{N}$. Consider the following equivalence relation on $\{\tilde{\mathbf{n}}^x \in \Omega_{B_{x,N}} : \partial\tilde{\mathbf{n}}^x = \tilde{A}_x\}$. We say that $\tilde{\mathbf{n}}^x$ and $\tilde{\mathbf{m}}^x$ are equivalent, and write $\tilde{\mathbf{n}}^x \simeq \tilde{\mathbf{m}}^x$, if for all $1 \leq i, j \leq A_x$,

$$\{(x, i) \xleftrightarrow{\tilde{\mathbf{n}}^x} (x, j)\} \Leftrightarrow \{(x, i) \xleftrightarrow{\tilde{\mathbf{m}}^x} (x, j)\}. \quad (11.3.78)$$

Following the notations of Definition 11.3.31, denote by $\tilde{\rho}_{x,N}^{\tilde{S}_x}$ and $\tilde{\rho}_{x,N}^{\tilde{S}_x, \tilde{T}_x}$ the induced measures corresponding to $\pi_{x,N}^{\tilde{S}_x}$ and $\pi_{x,N}^{\tilde{S}_x, \tilde{T}_x}$, respectively, on the quotient σ -algebra. Note that these measures are finite due to the finiteness of the underlying space. Let $\rho_{x,N}^{\tilde{S}_x}$ and $\rho_{x,N}^{\tilde{S}_x, \tilde{T}_x}$ denote the corresponding probability measures, which can be identified as probability distribution on even partitions of K_{S_x} and $K_{S_x+T_x}$, respectively.

Lemma 11.3.33. *Let $S_x, T_x \in \mathbb{N}$. The two sequences of probability measures $(\rho_{x,N}^{\tilde{S}_x})_{N \in \mathbb{N}}$ and $(\rho_{x,N}^{\tilde{S}_x, \tilde{T}_x})$ are tight. In both cases, the convergence along subsequences is pointwise on events. We denote their respective limits by $\rho_x^{\tilde{S}_x}$ and $\rho_x^{\tilde{S}_x, \tilde{T}_x}$.*

Proof. Since the measures considered here are defined on a finite space, it suffices to prove that the measure of the entire space converges in each case. But this is a direct consequence of Remark 11.3.32 and Lemma 11.3.24. $\square$

Remark 11.3.34. For every $N \geq 1$, we fix a canonical embedding of K_N into the countably infinite complete graph K_∞ , and we equip the latter with the metric (here we fix an enumeration of the edges of K_∞)

$$d(\mathbf{n}_1, \mathbf{n}_2) := \sum_{k \geq 0} \frac{1}{2^k} \frac{|\mathbf{n}_1(e_k) - \mathbf{n}_2(e_k)|}{1 + |\mathbf{n}_1(e_k) - \mathbf{n}_2(e_k)|}. \quad (11.3.79)$$

Then the measures $(\pi_{x,N}^{\tilde{S}_x})_{N \in \mathbb{N}}$ admit by compactness a subsequential limit $\pi_x^{\tilde{S}_x}$. The measure $\pi_x^{\tilde{S}_x}$ can be seen as a lift of the measure $\rho_x^{\tilde{S}_x}$, where an even partition of K_{S_x} is mapped to all current configurations on K_∞ that realise this connectivity property.

Definition 11.3.35. If P, Q are two partitions of a set $\mathcal{S}$, we say that P is coarser than Q , and write $P \succ Q$, if any element of P can be written as a union of elements of Q .

The following proposition is a direct consequence of the constructions above.

Proposition 11.3.36 (The measure $\rho_x^{\tilde{S}_x, \tilde{T}_x}$ stochastically dominates $\rho_x^{\tilde{S}_x} \sqcup \rho_x^{\tilde{T}_x}$). *Let $S_x, T_x \in 2\mathbb{N}$. There exists a coupling of $\rho_x^{\tilde{S}_x, \tilde{T}_x}$, $\rho_x^{\tilde{S}_x}$ and $\rho_x^{\tilde{T}_x}$ such that if $(X, Y, Z) \sim (\rho_x^{\tilde{S}_x, \tilde{T}_x}, \rho_x^{\tilde{S}_x}, \rho_x^{\tilde{T}_x})$ and if $Y \sqcup Z$ is the partition whose partition classes are the partition classes of Y and Z ,*

$$X \succ Y \sqcup Z, \quad a.s. \quad (11.3.80)$$

We write,

$$\rho_x^{\tilde{S}_x, \tilde{T}_x} \succ \rho_x^{\tilde{S}_x} \sqcup \rho_x^{\tilde{T}_x}. \quad (11.3.81)$$

Definition 11.3.37. Let $A \in \mathcal{M}(\Lambda)$, $B \in \mathcal{M}(\Lambda')$ and let $(\mathbf{n}_1, \mathbf{n}_2) \in \Omega_\Lambda \times \Omega_{\Lambda'}$ with $(\partial \mathbf{n}_1, \partial \mathbf{n}_2) = (\partial A, \partial B)$. Let $F : \Omega_\Lambda \rightarrow \mathbb{R}$ be bounded and measurable with respect to the quotient σ -algebra induced by the equivalence relation $\simeq$. We define

$$\tilde{\rho}_{A,B,N}^{\mathbf{n}_1, \mathbf{n}_2}[F] := \frac{1}{|W_N^{\text{ext}}(\mathbf{n}_1, \mathbf{n}_2; A, B)|} \left(\sum_{(\tilde{\mathbf{n}}_1^{\text{ext}}, \tilde{\mathbf{n}}_2^{\text{ext}}) \in W_N^{\text{ext}}(\mathbf{n}_1, \mathbf{n}_2; A, B)} \bigotimes_{z \in \Lambda} \tilde{\rho}_{z,N}^{\partial \tilde{\mathbf{n}}_1^{\text{ext}}(z) \sqcup \tilde{A}_z, \partial \tilde{\mathbf{n}}_2^{\text{ext}}(z) \sqcup \tilde{B}_z} \right) [F] \quad (11.3.82)$$

where

$$W_N^{\text{ext}}(\mathbf{n}_1, \mathbf{n}_2; A, B) := \left\{ (\tilde{\mathbf{n}}_1^{\text{ext}}, \tilde{\mathbf{n}}_2^{\text{ext}}) \in W^{\text{ext}}(\Lambda) \times W^{\text{ext}}(\Lambda') : \begin{array}{l} \Theta_N(\tilde{\mathbf{n}}_1^{\text{ext}}) = \mathbf{n}_1, \Theta_N(\tilde{\mathbf{n}}_2^{\text{ext}}) = \mathbf{n}_2 \\ \partial \tilde{\mathbf{n}}_1^{\text{ext}} \cap \tilde{A} = \emptyset, \partial \tilde{\mathbf{n}}_2^{\text{ext}} \cap \tilde{B} = \emptyset \end{array} \right\} \quad (11.3.83)$$

and we use the convention that $\tilde{A}_z, \tilde{B}_z = \emptyset$ if $A_z, B_z = 0$. We denote by $\rho_{A,B,N}^{\mathbf{n}_1, \mathbf{n}_2}$ the associated probability measures.

Lemma 11.3.38. *Let $A \in \mathcal{M}(\Lambda)$, $B \in \mathcal{M}(\Lambda')$ and let $(\mathbf{n}_1, \mathbf{n}_2) \in \Omega_\Lambda \times \Omega_{\Lambda'}$ with $(\partial \mathbf{n}_1, \partial \mathbf{n}_2) = (\partial A, \partial B)$. Then, the sequence of probability measures $(\rho_{A,B,N}^{\mathbf{n}_1, \mathbf{n}_2})_{N \in \mathbb{N}}$ is tight.*

Proof. As in Lemma 11.3.33, this result is a consequence of the finiteness of the underlying probability space together with the convergence

$$\tilde{\rho}_{A,B,N}^{\mathbf{n}_1, \mathbf{n}_2}[1] \xrightarrow{N \rightarrow \infty} \prod_{z \in \Lambda} \left\langle \varphi^{\Delta \mathbf{n}_1(z) + A_z} \right\rangle_0 \left\langle \varphi^{\Delta \mathbf{n}_2(z) + B_z} \right\rangle_0. \quad (11.3.84)$$

□

Corollary 11.3.39. *There exists an increasing sequence $(N_k)_{k \geq 1}$ such that for any $A \in \mathcal{M}(\Lambda)$, $B \in \mathcal{M}(\Lambda')$, and for any $(\mathbf{n}_1, \mathbf{n}_2) \in \Omega_\Lambda \times \Omega_{\Lambda'}$ satisfying $\partial \mathbf{n}_1 = \partial A$ and $\partial \mathbf{n}_2 = \partial B$, the sequence of measures $(\rho_{A,B,N_k}^{\mathbf{n}_1, \mathbf{n}_2})_{k \geq 1}$ weakly converges to a measure $\rho_{A,B}^{\mathbf{n}_1, \mathbf{n}_2}$.*

The following estimate on the cardinality of $W_N^{\text{ext}}(\mathbf{n}_1, \mathbf{n}_2; A, B)$ will be useful throughout this section.

Lemma 11.3.40. *Let $A \in \mathcal{M}(\Lambda)$, $B \in \mathcal{M}(\Lambda')$ and let $(\mathbf{n}_1, \mathbf{n}_2) \in \Omega_\Lambda \times \Omega_{\Lambda'}$ with $(\partial \mathbf{n}_1, \partial \mathbf{n}_2) = (\partial A, \partial B)$. Then we have*

$$|W_N^{\text{ext}}(\mathbf{n}_1, \mathbf{n}_2; A, B)| = (1 + o(1)) \prod_{\{x,y\} \subset \Lambda} \frac{N^{2\mathbf{n}_1(x,y) + 2\mathbf{n}_2(x,y)}}{\mathbf{n}_1(x,y)! \mathbf{n}_2(x,y)!} \prod_{x \in \Lambda} \frac{N^{\mathbf{n}_1(x,\mathbf{g}) + \mathbf{n}_2(x,\mathbf{g})}}{\mathbf{n}_1(x,\mathbf{g})! \mathbf{n}_2(x,\mathbf{g})!}. \quad (11.3.85)$$

Proof. It suffices to estimate the cardinality of $W_N^{\text{ext}}(\mathbf{n}; A) := W_N^{\text{ext}}(\mathbf{n}, 0; A, 0)$. To this end, we introduce the following procedure. We first enumerate the elements of $\mathcal{P}_2(\Lambda)$. We will count inductively the number of lifts $\tilde{\mathbf{n}}$ of $\mathbf{n}$ which satisfy $\tilde{\mathbf{n}} \in W_N^{\text{ext}}$ and $\partial \tilde{\mathbf{n}} \cap \tilde{A} = \emptyset$.

Starting from the first element $\{x, y\}$ according to this ordering and taking into account that we cannot use any point in $\tilde{A}$, we see that we have

$$\binom{N - A_x}{\mathbf{n}(x, y)} \binom{N - A_y}{\mathbf{n}(x, y)} \mathbf{n}(x, y)! \quad (11.3.86)$$

choices for the piece of current between the blocks $B_{x,N}$ and $B_{y,N}$ when $\{x, y\} \subset \Lambda$, and

$$\binom{N - A_x}{\mathbf{n}(x, \mathbf{g})} \quad (11.3.87)$$

choices if $x \in \Lambda$ and $y = \mathbf{g}$. We then proceed to the second element $\{z, w\}$. In that case, the only additional condition we need to take into account is that we have attributed $\mathbf{n}(x, y)$ vertices of $B_{x,N}$ and $B_{y,N}$ in the preceding step that we cannot re-use (because the lifts we construct belong to $W_{2,N}$). If $z, w \in \Lambda$, then we have

$$\binom{N - A_z - \mathbf{1}_{z \in \{x,y\}} \mathbf{n}(x, y)}{\mathbf{n}(z, w)} \binom{N - A_w - \mathbf{1}_{w \in \{x,y\}} \mathbf{n}(x, y)}{\mathbf{n}(z, w)} \mathbf{n}(z, w)! \quad (11.3.88)$$

choices for the piece of lift between x and z . We can iterate this procedure to compute the number of suitable lifts of $\mathbf{n}$.

It follows from the above procedure that

$$\prod_{\{x,y\} \subset \Lambda} \binom{N - \Delta \mathbf{n}(x) - A_x}{\mathbf{n}(x, y)} \binom{N - \Delta \mathbf{n}(y) - A_y}{\mathbf{n}(x, y)} \mathbf{n}(x, y)! \prod_{x \in \Lambda} \binom{N - \Delta \mathbf{n}(x) - A_x}{\mathbf{n}(x, \mathbf{g})} \leq$$

$$|W_N^{\text{ext}}(\mathbf{n}; A)| \leq \prod_{\{x,y\} \in \Lambda} \binom{N}{\mathbf{n}(x, y)}^2 \mathbf{n}(x, y)! \prod_{x \in \Lambda} \binom{N}{\mathbf{n}(x, \mathbf{g})},$$

hence

$$|W_N^{\text{ext}}(\mathbf{n}; A)| = (1 + o(1)) \prod_{\{x,y\} \subset \Lambda} \frac{N^{2\mathbf{n}(x,y)}}{\mathbf{n}(x,y)!} \prod_{x \in \Lambda} \frac{N^{\mathbf{n}(x,\mathfrak{g})}}{\mathbf{n}(x,\mathfrak{g})!}, \quad (11.3.89)$$

as desired. $\square$

In what follows, we fix a sequence $(N_k)_{k \geq 1}$ as in Corollary 11.3.39.

Lemma 11.3.41. *Let $A \in \mathcal{M}(\Lambda)$, $B \in \mathcal{M}(\Lambda')$ and let $(\mathbf{n}_1, \mathbf{n}_2) \in \Omega_\Lambda \times \Omega_{\Lambda'}$ with $(\partial \mathbf{n}_1, \partial \mathbf{n}_2) = (\partial A, \partial B)$. Then, along the subsequence $(N_k)_{k \geq 1}$,*

$$\lim_{k \rightarrow \infty} \frac{(c_{N_k} N_k)^{|A|+|B|}}{\mathbf{Z}_{N_k}^{|A|+|\Lambda'|}} \Sigma_{1,N_k}(\mathbf{n}_1, \mathbf{n}_2) = w_{A,\beta}(\mathbf{n}_1) w_{B,\beta}(\mathbf{n}_2) \rho_{A,B}^{\mathbf{n}_1, \mathbf{n}_2}[F]. \quad (11.3.90)$$

Proof. By definition

$$\frac{(c_N N)^{|A|+|B|}}{\mathbf{Z}_N^{|A|+|\Lambda'|}} \Sigma_{1,N}(\mathbf{n}_1, \mathbf{n}_2) = \frac{(c_N N)^{|A|+|B|}}{\mathbf{Z}_N^{|A|+|\Lambda'|}} \sum_{\substack{\tilde{\mathbf{n}}_1 \in W_N(\Lambda), \tilde{\mathbf{n}}_2 \in W_N(\Lambda') \\ \Theta_N(\tilde{\mathbf{n}}_1) = \mathbf{n}_1, \Theta_N(\tilde{\mathbf{n}}_2) = \mathbf{n}_2 \\ \partial \tilde{\mathbf{n}}_1 = \tilde{A}, \partial \tilde{\mathbf{n}}_2 = \tilde{B}}} w_N(\tilde{\mathbf{n}}_1) w_N(\tilde{\mathbf{n}}_2) F(\tilde{\mathbf{n}}_1 + \tilde{\mathbf{n}}_2). \quad (11.3.91)$$

For $i = 1, 2$, write $\tilde{\mathbf{n}}_i = \tilde{\mathbf{n}}_i^{\text{ext}} + \tilde{\mathbf{n}}_i^{\text{int}}$ with $\tilde{\mathbf{n}}_i^{\text{ext}} \in W_N^{\text{ext}}$ satisfying $\partial \tilde{\mathbf{n}}_i^{\text{ext}} \cap \partial \tilde{\mathbf{n}}_i = \emptyset$, and $\tilde{\mathbf{n}}_i^{\text{int}} \in W_N^{\text{int}}$. Then, if $\mathbf{n}_1^{\text{ext}}, \tilde{\mathbf{n}}_2^{\text{ext}} \in W_N^{\text{ext}}$ are such that $\Theta_N(\tilde{\mathbf{n}}_1^{\text{ext}}) = \mathbf{n}_1$, $\Theta_N(\tilde{\mathbf{n}}_2^{\text{ext}}) = \mathbf{n}_2$, one has

$$w_N(\tilde{\mathbf{n}}_1^{\text{ext}}) w_N(\tilde{\mathbf{n}}_2^{\text{ext}}) = \prod_{\{x,y\} \subset \Lambda} (\beta c_N^2 J_{x,y})^{\mathbf{n}_1(x,y) + \mathbf{n}_2(x,y)} \prod_{x \in \Lambda} (\beta c_N h_x)^{\mathbf{n}_1(x,\mathfrak{g}) + \mathbf{n}_2(x,\mathfrak{g})}. \quad (11.3.92)$$

As a consequence,

$$\begin{aligned} & \frac{(c_N N)^{|A|+|B|}}{\mathbf{Z}_N^{|A|+|\Lambda'|}} \Sigma_{1,N}(\mathbf{n}_1, \mathbf{n}_2) \\ &= \frac{|W_N^{\text{ext}}(\mathbf{n}_1, \mathbf{n}_2; A, B)|}{(c_N N)^{|\mathbf{n}_1|+|\mathbf{n}_2|}} \prod_{\{x,y\} \subset \Lambda} (\beta c_N^2 J_{x,y})^{\mathbf{n}_1(x,y) + \mathbf{n}_2(x,y)} \cdot \prod_{x \in \Lambda} (\beta c_N h_x)^{\mathbf{n}_1(x,\mathfrak{g}) + \mathbf{n}_2(x,\mathfrak{g})} \tilde{\rho}_{A,B,N}^{\mathbf{n}_1, \mathbf{n}_2}[F]. \end{aligned} \quad (11.3.93)$$

Recall now Lemma 11.3.40 and that

$$\tilde{\rho}_{A,B}^{\mathbf{n}_1, \mathbf{n}_2}[1] = \prod_{x \in \Lambda} \left\langle \varphi_x^{\Delta \mathbf{n}_1(x) + A_x} \right\rangle_0 \left\langle \varphi_x^{\Delta \mathbf{n}_2(x) + B_x} \right\rangle_0. \quad (11.3.94)$$

The desired convergence along the subsequence $(N_k)_{k \geq 1}$ follows readily. $\square$

Using the same method as above, we also obtain the following.

Lemma 11.3.42. *Let $A \in \mathcal{M}(\Lambda)$, $B \in \mathcal{M}(\Lambda')$ and let $(\mathbf{n}_1, \mathbf{n}_2) \in \Omega_\Lambda \times \Omega_{\Lambda'}$ with $(\partial \mathbf{n}_1, \partial \mathbf{n}_2) = (\partial A, \partial B)$. Then, along the subsequence $(N_k)_{k \geq 1}$,*

$$\begin{aligned} & \frac{(c_N N)^{|A|+|B|}}{\mathbf{Z}_N^{|A|+|\Lambda'|}} \sum_{\substack{\tilde{\mathbf{n}}_1 \in W_N(\Lambda), \tilde{\mathbf{n}}_2 \in W_N(\Lambda') \\ \Theta_N(\tilde{\mathbf{n}}_1) = \mathbf{n}_1, \Theta_N(\tilde{\mathbf{n}}_2) = \mathbf{n}_2 \\ \partial \tilde{\mathbf{n}}_1 = \tilde{A} \sqcup \tilde{B}, \partial \tilde{\mathbf{n}}_2 = \emptyset}} w_N(\tilde{\mathbf{n}}_1) w_N(\tilde{\mathbf{n}}_2) F(\tilde{\mathbf{n}}_1 + \tilde{\mathbf{n}}_2) \mathbb{1}_{\mathcal{F}_B^{\Lambda'_N}} \\ & \longrightarrow w_{A+B,\beta}(\mathbf{n}_1) w_{\emptyset,\beta}(\mathbf{n}_2) \rho_{A+B,\emptyset}^{\mathbf{n}_1, \mathbf{n}_2} \left[F(\mathbf{n}_1 + \mathbf{n}_2, \mathbf{t}) \mathbb{1}_{\mathcal{F}_B^{\Lambda'_N}} \right], \end{aligned} \quad (11.3.95)$$

where $\mathcal{F}_B^{\Lambda'_N}$ is the event defined in Theorem 11.3.11.

We are now equipped to complete the conditional proof of Theorem 11.3.11

Proof of Theorem 11.3.11. We first apply (11.3.59) for $F \cdot \mathbb{1}_C$, where

$$C := \left\{ \sum_{i=1}^N (\Delta \tilde{\mathbf{n}}_1^{\text{ext}}(x, i) + \Delta \tilde{\mathbf{n}}_2^{\text{ext}}(x, i)) \leq M, \forall x \in \Lambda_g \right\} \quad (11.3.96)$$

for some $M \geq 1$. Note that $F\mathbb{1}_C$ is bounded. Using the two preceding lemmas and taking the limit along an appropriate sequence $(N_k)_{k \geq 1}$ gives

$$\sum_{\substack{\mathbf{n}_1 \in \Omega_\Lambda, \partial \mathbf{n}_1 = \partial A \\ \mathbf{n}_2 \in \Omega_{\Lambda'}, \partial \mathbf{n}_2 = \partial B}} \mathbb{1}_{(\mathbf{n}_1, \mathbf{n}_2) \in C} w_{A, \beta}(\mathbf{n}_1) w_{B, \beta}(\mathbf{n}_2) \rho_{A, B}^{\mathbf{n}_1, \mathbf{n}_2} [F(\mathbf{n}_1 + \mathbf{n}_2, \mathbf{t})] \quad (11.3.97)$$

$$= \sum_{\substack{\mathbf{n}_1 \in \Omega_\Lambda, \partial \mathbf{n}_1 = \partial(A+B) \\ \mathbf{n}_2 \in \Omega_{\Lambda'}, \partial \mathbf{n}_2 = \emptyset}} \mathbb{1}_{(\mathbf{n}_1, \mathbf{n}_2) \in C} w_{A+B, \beta}(\mathbf{n}_1) w_{\emptyset, \beta}(\mathbf{n}_2) \rho_{A+B, \emptyset}^{\mathbf{n}_1, \mathbf{n}_2} [F(\mathbf{n}_1 + \mathbf{n}_2, \mathbf{t}) \mathbb{1}_{\mathcal{F}_B^{\Lambda'}}]. \quad (11.3.98)$$

Sending M to infinity and using the dominated convergence theorem we obtain

$$\sum_{\substack{\mathbf{n}_1 \in \Omega_\Lambda, \partial \mathbf{n}_1 = \partial A \\ \mathbf{n}_2 \in \Omega_{\Lambda'}, \partial \mathbf{n}_2 = \partial B}} w_{A, \beta}(\mathbf{n}_1) w_{B, \beta}(\mathbf{n}_2) \rho_{A, B}^{\mathbf{n}_1, \mathbf{n}_2} [F(\mathbf{n}_1 + \mathbf{n}_2, \mathbf{t})] \quad (11.3.99)$$

$$= \sum_{\substack{\mathbf{n}_1 \in \Omega_\Lambda, \partial \mathbf{n}_1 = \partial(A+B) \\ \mathbf{n}_2 \in \Omega_{\Lambda'}, \partial \mathbf{n}_2 = \emptyset}} w_{A+B, \beta}(\mathbf{n}_1) w_{\emptyset, \beta}(\mathbf{n}_2) \rho_{A+B, \emptyset}^{\mathbf{n}_1, \mathbf{n}_2} [F(\mathbf{n}_1 + \mathbf{n}_2, \mathbf{t}) \mathbb{1}_{\mathcal{F}_B^{\Lambda'}}]. \quad (11.3.100)$$

It remains to prove that for $A \in \mathcal{M}(\Lambda)$, $B \in \mathcal{M}(\Lambda')$ and $(\mathbf{n}_1, \mathbf{n}_2) \in \Omega_\Lambda \times \Omega_{\Lambda'}$ with $(\partial \mathbf{n}_1, \partial \mathbf{n}_2) = (\partial A, \partial B)$, the measure $\rho_{A, B}^{\mathbf{n}_1, \mathbf{n}_2}$ is a product measure. It suffices to show that $\tilde{\rho}_{A, B}^{\mathbf{n}_1, \mathbf{n}_2}$ is a product measure, and for that, we can assume that F is bounded. We write

$$\tilde{\rho}_{A, B, N}^{\mathbf{n}_1, \mathbf{n}_2} [F] = C_1 + C_2 \quad (11.3.101)$$

where

$$C_1 := \frac{1}{|W_N^{\text{ext}}(\mathbf{n}_1, \mathbf{n}_2; A, B)|} \sum_{\substack{(\tilde{\mathbf{n}}_1^{\text{ext}}, \tilde{\mathbf{n}}_2^{\text{ext}}) \in W_N^{\text{ext}} \\ \Theta_N(\tilde{\mathbf{n}}_1^{\text{ext}}) = \mathbf{n}_1, \Theta_N(\tilde{\mathbf{n}}_2^{\text{ext}}) = \mathbf{n}_2 \\ \partial \tilde{\mathbf{n}}_1^{\text{ext}} \cap \tilde{A} = \emptyset, \partial \tilde{\mathbf{n}}_2^{\text{ext}} \cap \tilde{B} = \emptyset \\ (\partial \tilde{\mathbf{n}}_1^{\text{ext}} \sqcup \tilde{A}) \cap (\partial \tilde{\mathbf{n}}_2^{\text{ext}} \sqcup \tilde{B}) = \emptyset}} \left(\bigotimes_{z \in \Lambda} \tilde{\rho}_{z, N}^{\partial \tilde{\mathbf{n}}_1^{\text{ext}}(z) \sqcup \tilde{A}_z, \partial \tilde{\mathbf{n}}_2^{\text{ext}}(x) \sqcup \tilde{B}_z} \right) [F], \quad (11.3.102)$$

and

$$C_2 := \frac{1}{|W_N^{\text{ext}}(\mathbf{n}_1, \mathbf{n}_2; A, B)|} \sum_{\substack{(\tilde{\mathbf{n}}_1^{\text{ext}}, \tilde{\mathbf{n}}_2^{\text{ext}}) \in W_N^{\text{ext}} \\ \Theta_N(\tilde{\mathbf{n}}_1^{\text{ext}}) = \mathbf{n}_1, \Theta_N(\tilde{\mathbf{n}}_2^{\text{ext}}) = \mathbf{n}_2 \\ \partial \tilde{\mathbf{n}}_1^{\text{ext}} \cap \tilde{A} = \emptyset, \partial \tilde{\mathbf{n}}_2^{\text{ext}} \cap \tilde{B} = \emptyset \\ (\partial \tilde{\mathbf{n}}_1^{\text{ext}} \sqcup \tilde{A}) \cap (\partial \tilde{\mathbf{n}}_2^{\text{ext}} \sqcup \tilde{B}) \neq \emptyset}} \left(\bigotimes_{z \in \Lambda} \tilde{\rho}_{z, N}^{\partial \tilde{\mathbf{n}}_1^{\text{ext}}(z) \sqcup \tilde{A}_z, \partial \tilde{\mathbf{n}}_2^{\text{ext}}(x) \sqcup \tilde{B}_z} \right) [F]. \quad (11.3.103)$$

Notice that because of the symmetries of the model, all the product measures appearing in C_1 are equal. Thus, for a fixed pair of currents $(\tilde{\mathbf{n}}_1^{\text{ext}}, \tilde{\mathbf{n}}_2^{\text{ext}})$ satisfying the conditions of the sum of the numerator in C_1 , one finds,

$$C_1 = \bigotimes_{z \in \Lambda} \left(\rho_{z,N}^{\partial \tilde{\mathbf{n}}_1^{\text{ext}}(z) \sqcup \tilde{A}_z, \partial \tilde{\mathbf{n}}_2^{\text{ext}}(x) \sqcup \tilde{B}_z} \right) [F] \frac{1}{|W_N^{\text{ext}}(\mathbf{n}_1, \mathbf{n}_2; A, B)|} \sum_{\substack{(\tilde{\mathbf{n}}_1^{\text{ext}}, \tilde{\mathbf{n}}_2^{\text{ext}}) \in W^{\text{ext}}_N \\ \Theta_N(\tilde{\mathbf{n}}_1^{\text{ext}}) = \mathbf{n}_1, \Theta_N(\tilde{\mathbf{n}}_2^{\text{ext}}) = \mathbf{n}_2 \\ \partial \tilde{\mathbf{n}}_1^{\text{ext}} \cap \tilde{A} = \emptyset, \partial \tilde{\mathbf{n}}_2^{\text{ext}} \cap \tilde{B} = \emptyset \\ (\partial \tilde{\mathbf{n}}_1^{\text{ext}} \sqcup \tilde{A}) \cap (\partial \tilde{\mathbf{n}}_2^{\text{ext}} \sqcup \tilde{B}) = \emptyset}} 1. \quad (11.3.104)$$

Arguing as in the proof of Lemma 11.3.40 we obtain

$$\sum_{\substack{(\tilde{\mathbf{n}}_1^{\text{ext}}, \tilde{\mathbf{n}}_2^{\text{ext}}) \in W^{\text{ext}}_N \\ \Theta_N(\tilde{\mathbf{n}}_1^{\text{ext}}) = \mathbf{n}_1, \Theta_N(\tilde{\mathbf{n}}_2^{\text{ext}}) = \mathbf{n}_2 \\ \partial \tilde{\mathbf{n}}_1^{\text{ext}} \cap \tilde{A} = \emptyset, \partial \tilde{\mathbf{n}}_2^{\text{ext}} \cap \tilde{B} = \emptyset \\ (\partial \tilde{\mathbf{n}}_1^{\text{ext}} \sqcup \tilde{A}) \cap (\partial \tilde{\mathbf{n}}_2^{\text{ext}} \sqcup \tilde{B}) = \emptyset}} 1 \geq \prod_{i=1}^2 \prod_{\{x,y\} \subset \Lambda} \binom{N_{x,y}}{\mathbf{n}_i(x,y)}^2 \mathbf{n}_i(x,y)! \prod_{x \in \Lambda} \binom{N_{x,\mathfrak{g}}}{\mathbf{n}_i(x,\mathfrak{g})},$$

where $N_{x,y} = N - \Delta \mathbf{n}(x) - \Delta \mathbf{n}(y) - A_x - A_y - B_x - B_y$ for $\mathbf{n} = \mathbf{n}_1 + \mathbf{n}_2$, hence

$$\frac{1}{|W_N^{\text{ext}}(\mathbf{n}_1, \mathbf{n}_2; A, B)|} \sum_{\substack{(\tilde{\mathbf{n}}_1^{\text{ext}}, \tilde{\mathbf{n}}_2^{\text{ext}}) \in W^{\text{ext}}_N \\ \Theta_N(\tilde{\mathbf{n}}_1^{\text{ext}}) = \mathbf{n}_1, \Theta_N(\tilde{\mathbf{n}}_2^{\text{ext}}) = \mathbf{n}_2 \\ \partial \tilde{\mathbf{n}}_1^{\text{ext}} \cap \tilde{A} = \emptyset, \partial \tilde{\mathbf{n}}_2^{\text{ext}} \cap \tilde{B} = \emptyset \\ (\partial \tilde{\mathbf{n}}_1^{\text{ext}} \sqcup \tilde{A}) \cap (\partial \tilde{\mathbf{n}}_2^{\text{ext}} \sqcup \tilde{B}) = \emptyset}} 1 \xrightarrow{N \rightarrow \infty} 1, \quad (11.3.105)$$

which implies that along $(N_k)_{k \geq 1}$ the term C_1 converges to a product measure. To handle C_2 , notice that

$$|C_2| \leq D_N \|F\|_\infty \frac{1}{|W_N^{\text{ext}}(\mathbf{n}_1, \mathbf{n}_2; A, B)|} \sum_{\substack{(\tilde{\mathbf{n}}_1^{\text{ext}}, \tilde{\mathbf{n}}_2^{\text{ext}}) \in W^{\text{ext}}_N \\ \Theta_N(\tilde{\mathbf{n}}_1^{\text{ext}}) = \mathbf{n}_1, \Theta_N(\tilde{\mathbf{n}}_2^{\text{ext}}) = \mathbf{n}_2 \\ \partial \tilde{\mathbf{n}}_1^{\text{ext}} \cap \tilde{A} = \emptyset, \partial \tilde{\mathbf{n}}_2^{\text{ext}} \cap \tilde{B} = \emptyset \\ (\partial \tilde{\mathbf{n}}_1^{\text{ext}} \sqcup \tilde{A}) \cap (\partial \tilde{\mathbf{n}}_2^{\text{ext}} \sqcup \tilde{B}) \neq \emptyset}} 1, \quad (11.3.106)$$

where

$$D_N := \prod_{z \in \Lambda} \max_{\substack{k \leq \Delta \mathbf{n}_1(z) + A_z \\ p \leq \Delta \mathbf{n}_2(z) + B_z}} \langle \varphi_z^k \rangle_0 \langle \varphi_z^p \rangle_0 + o(1). \quad (11.3.107)$$

It follows from (11.3.105) that

$$\frac{1}{|W_N^{\text{ext}}(\mathbf{n}_1, \mathbf{n}_2; A, B)|} \sum_{\substack{(\tilde{\mathbf{n}}_1^{\text{ext}}, \tilde{\mathbf{n}}_2^{\text{ext}}) \in W^{\text{ext}}_N \\ \Theta_N(\tilde{\mathbf{n}}_1^{\text{ext}}) = \mathbf{n}_1, \Theta_N(\tilde{\mathbf{n}}_2^{\text{ext}}) = \mathbf{n}_2 \\ \partial \tilde{\mathbf{n}}_1^{\text{ext}} \cap \tilde{A} = \emptyset, \partial \tilde{\mathbf{n}}_2^{\text{ext}} \cap \tilde{B} = \emptyset \\ (\partial \tilde{\mathbf{n}}_1^{\text{ext}} \sqcup \tilde{A}) \cap (\partial \tilde{\mathbf{n}}_2^{\text{ext}} \sqcup \tilde{B}) \neq \emptyset}} 1 = o(1), \quad (11.3.108)$$

hence

$$C_2 \xrightarrow{N \rightarrow \infty} 0. \quad (11.3.109)$$

□

11.3.5.3 Proof of Proposition 11.3.26

We now turn to the proof of Proposition 11.3.26. As explained above, we will argue that the main contribution in the weights $\bar{w}_{N,A,\beta}(\mathbf{n})$ comes from currents $\tilde{\mathbf{n}}$ which belong to the set W_N introduced in Definition 11.3.25. We start by showing that this contribution admits an explicit limit, which is this first part of Proposition 11.3.26.

Lemma 11.3.43. *For all $A \in \mathcal{M}(\Lambda_{\mathfrak{g}})$ and $\mathbf{n} \in \Omega_{\Lambda_{\mathfrak{g}}}$,*

$$\frac{(c_N N)^{|A|}}{\mathbf{Z}_N^{|A|}} \sum_{\substack{\tilde{\mathbf{n}} \in W_N \\ \Theta_N(\tilde{\mathbf{n}}) = \mathbf{n} \\ \partial \tilde{\mathbf{n}} = \tilde{A}}} w_{N,\beta}(\tilde{\mathbf{n}}) \xrightarrow{N \rightarrow \infty} w_{A,\beta}(\mathbf{n}). \quad (11.3.110)$$

Proof. Fix $A \in \mathcal{M}(\Lambda_{\mathfrak{g}})$ and $\mathbf{n} \in \Omega_{\Lambda_{\mathfrak{g}}}$. Notice that for every $\tilde{\mathbf{n}} \in W_N$, we have a unique decomposition $\tilde{\mathbf{n}} = \tilde{\mathbf{n}}_1 + \tilde{\mathbf{n}}_2$ with $\tilde{\mathbf{n}}_1 \in W_N^{\text{int}}$ and $\tilde{\mathbf{n}}_2 \in W_N^{\text{ext}}$ (these sets were introduced in Definition 11.3.30). This decomposition yields the following factorization

$$w_{N,\beta}(\tilde{\mathbf{n}}) = w_{N,\beta}(\tilde{\mathbf{n}}_1) w_{N,\beta}(\tilde{\mathbf{n}}_2). \quad (11.3.111)$$

Now write,

$$\begin{aligned} \sum_{\substack{\tilde{\mathbf{n}} \in W_N \\ \Theta_N(\tilde{\mathbf{n}}) = \mathbf{n} \\ \partial \tilde{\mathbf{n}} = \tilde{A}}} w_{N,\beta}(\tilde{\mathbf{n}}) &= \sum_{\substack{\tilde{\mathbf{n}} \in \Omega_{\Lambda_{\mathfrak{g}}}^N \\ \Theta_N(\tilde{\mathbf{n}}) = \mathbf{n} \\ \partial \tilde{\mathbf{n}} = \tilde{A}}} w_{N,\beta}(\tilde{\mathbf{n}}_1) w_{N,\beta}(\tilde{\mathbf{n}}_2) \mathbf{1}_{\tilde{\mathbf{n}}_1 + \tilde{\mathbf{n}}_2 \in W_N} \\ &= \sum_{\substack{\tilde{\mathbf{n}}_2 \in W_N^{\text{ext}} \\ \Theta_N(\tilde{\mathbf{n}}_2) = \mathbf{n} \\ \partial \tilde{\mathbf{n}}_2 \cap \tilde{A} = \emptyset}} w_{N,\beta}(\tilde{\mathbf{n}}_2) \sum_{\substack{\tilde{\mathbf{n}}_1 \in W_N^{\text{int}} \\ \partial \tilde{\mathbf{n}}_1 = \tilde{A} \sqcup \partial \tilde{\mathbf{n}}_2}} w_{N,\beta}(\tilde{\mathbf{n}}_1). \end{aligned}$$

Notice that for $\tilde{\mathbf{n}}_2 \in W_N^{\text{ext}}$ satisfying $\Theta_N(\tilde{\mathbf{n}}_2) = \mathbf{n}$, and $\partial \tilde{\mathbf{n}}_2 \cap \tilde{A} = \emptyset$,

$$\frac{1}{\mathbf{Z}_N^{|A|}} \sum_{\substack{\tilde{\mathbf{n}}_1 \in W_N^{\text{int}} \\ \partial \tilde{\mathbf{n}}_1 = \tilde{A} \sqcup \partial \tilde{\mathbf{n}}_2}} w_{N,\beta}(\tilde{\mathbf{n}}_1) = \prod_{x \in \Lambda} \mu_{x,N}^0 \left[\prod_{j=1}^{A_x + \Delta \mathbf{n}(x)} \sigma_{(x,j)} \right]. \quad (11.3.112)$$

Using this we get,

$$\begin{aligned} \frac{(c_N N)^{|A|}}{\mathbf{Z}_N^{|A|}} \sum_{\substack{\tilde{\mathbf{n}} \in W_N \\ \Theta_N(\tilde{\mathbf{n}}) = \mathbf{n} \\ \partial \tilde{\mathbf{n}} = \tilde{A}}} w_{N,\beta}(\tilde{\mathbf{n}}) &= (c_N N)^{|A|} \prod_{x \in \Lambda} \mu_{x,N}^0 \left[\prod_{j=1}^{A_x + \Delta \mathbf{n}(x)} \sigma_{(x,j)} \right] \sum_{\substack{\tilde{\mathbf{n}}_2 \in W_N^{\text{ext}} \\ \Theta_N(\tilde{\mathbf{n}}_2) = \mathbf{n} \\ \partial \tilde{\mathbf{n}}_2 \cap \tilde{A} = \emptyset}} w_{N,\beta}(\tilde{\mathbf{n}}_2). \quad (11.3.113) \end{aligned}$$

Now, recall that $W_N^{\text{ext}} \subset W_{2,N}$ so that for $\tilde{\mathbf{n}}_2 \in W_N^{\text{ext}}$ with $\Theta_N(\tilde{\mathbf{n}}_2) = \mathbf{n}$, one has

$$w_{N,\beta}(\tilde{\mathbf{n}}_2) = \prod_{\{x,y\} \subset \Lambda} (\beta c_N^2 J_{x,y})^{\mathbf{n}_{x,y}} \prod_{x \in \Lambda} (\beta c_N h_x)^{\mathbf{n}_{x,g}}. \quad (11.3.114)$$

This implies for the left hand side of (11.3.113) that

$$\begin{aligned} (\text{LHS}) &= (c_N N)^{|\Lambda|} \prod_{x \in \Lambda} \mu_{x,N}^0 \left[\prod_{j=1}^{A_x + \Delta \mathbf{n}(x)} \sigma_{(x,j)} \right] \prod_{\{x,y\} \subset \Lambda} (\beta c_N^2 J_{x,y})^{\mathbf{n}_{x,y}} \\ &\quad \times \prod_{x \in \Lambda} (\beta c_N h_x)^{\mathbf{n}_{x,g}} |\{\tilde{\mathbf{n}}_2 \in W_N^{\text{ext}} : \Theta_N(\tilde{\mathbf{n}}_2) = \mathbf{n}, \partial \tilde{\mathbf{n}}_2 \cap \tilde{A} = \emptyset\}|. \end{aligned}$$

By (11.3.89),

$$|\{\tilde{\mathbf{n}}_2 \in W_N^{\text{ext}} : \Theta_N(\tilde{\mathbf{n}}_2) = \mathbf{n}, \partial \tilde{\mathbf{n}}_2 \cap \tilde{A} = \emptyset\}| = (1 + o(1)) \prod_{\{x,y\} \subset \Lambda} \frac{N^{2\mathbf{n}_{x,y}}}{\mathbf{n}_{x,y}!} \prod_{x \in \Lambda} \frac{N^{\mathbf{n}_{x,g}}}{\mathbf{n}_{x,g}!}, \quad (11.3.115)$$

hence

$$\begin{aligned} (\text{LHS}) &= (1 + o(1)) \prod_{\{x,y\} \subset \Lambda} \frac{(\beta J_{x,y})^{\mathbf{n}_{x,y}}}{\mathbf{n}_{x,y}!} \prod_{x \in \Lambda} \frac{(\beta h_x)^{\mathbf{n}_{x,g}}}{\mathbf{n}_{x,g}!} \\ &\quad \times \prod_{x \in \Lambda} (c_N N)^{A_x + \Delta \mathbf{n}(x)} \mu_{x,N}^0 \left[\prod_{j=1}^{A_x + \Delta \mathbf{n}(x)} \sigma_{(x,j)} \right]. \end{aligned}$$

Using Lemma 12.2.1, we obtain that

$$\frac{(c_N N)^{|\Lambda|}}{\mathbf{Z}_N^{|\Lambda|}} \sum_{\substack{\tilde{\mathbf{n}} \in W_N \\ \Theta_N(\tilde{\mathbf{n}}) = \mathbf{n} \\ \partial \tilde{\mathbf{n}} = \tilde{A}}} w_{N,\beta}(\tilde{\mathbf{n}}) \xrightarrow{N \rightarrow \infty} \prod_{\{x,y\} \subset \Lambda} \frac{(\beta J_{x,y})^{\mathbf{n}_{x,y}}}{\mathbf{n}_{x,y}!} \prod_{x \in \Lambda} \frac{(\beta h_x)^{\mathbf{n}_{x,g}}}{\mathbf{n}_{x,g}!} \prod_{x \in \Lambda} \langle \varphi^{A_x + \Delta \mathbf{n}(x)} \rangle_0$$

which is the desired result. $\square$

We now prove the second part of Proposition 11.3.26 by induction on $|\mathbf{n}|$, namely (11.3.57). We only write the proof in the special case $h = 0$ in which the ghost $\mathbf{g}$ plays no role. It is easy to extend the result to the general case $h \geq 0$ (see Remark 11.3.44 below).

Proof of second half of Proposition 11.3.26. Recall that $h = 0$. We proceed by induction on $|\mathbf{n}|$. For $k \geq 0$, introduce

$\mathbf{H}_k$: For every $\mathbf{n} \in \Omega_{\Lambda_g}$ with $|\mathbf{n}| = k$, and every $A \in \mathcal{M}(\Lambda)$,

$$\bar{w}_{N,A,\beta}(\mathbf{n}) \xrightarrow{N \rightarrow \infty} w_{A,\beta}(\mathbf{n}). \quad (11.3.116)$$

First, if $|\mathbf{n}| = 0$, then, proceeding as in the proof of Lemma 11.3.43, we get

$$\bar{w}_{N,A,\beta}(\mathbf{n}) = \prod_{x \in \Lambda} (c_N N)^{A_x + \Delta \mathbf{n}(x)} \mu_{x,N}^0 \left[\prod_{j=1}^{A_x + \Delta \mathbf{n}(x)} \sigma_{(x,j)} \right], \quad (11.3.117)$$

and the right-hand side in (11.3.117) converges to $w_{A,\beta}(\mathbf{n})$ by Lemma 11.3.24.

Let $k \geq 0$ and assume that $\mathbf{H}_i$ are true for all $0 \leq i \leq k$. Let $\mathbf{n} \in \Omega_\Lambda$ such that $|\mathbf{n}| = k+1$. By Lemma 11.3.43, it is sufficient to prove that $\alpha_1(N)$ and $\alpha_2(N)$ vanish where

$$\alpha_i(N) := \frac{(c_N N)^{|\Lambda|}}{\mathbf{Z}_N^{|\Lambda|}} \sum_{\substack{\tilde{\mathbf{n}} \in \Omega_\Lambda^N \setminus W_{i,N} \\ \Theta_N(\tilde{\mathbf{n}}) = \mathbf{n} \\ \partial \tilde{\mathbf{n}} = \tilde{A}}} w_{N,\beta}(\tilde{\mathbf{n}}). \quad (11.3.118)$$

We first prove that $\alpha_1(N) \rightarrow 0$. Notice that

$$\Omega_\Lambda^N \setminus W_{1,N} = \bigcup_{\substack{u,v \in \Lambda \\ u \neq v}} \bigcup_{1 \leq i \leq A_u} \bigcup_{1 \leq j \leq N} V_{i,j}(u,v) \quad (11.3.119)$$

where

$$V_{i,j}(u,v) := \{\tilde{\mathbf{n}} \in \Omega_\Lambda^N, \tilde{\mathbf{n}}(u,i,v,j) \geq 1\}. \quad (11.3.120)$$

Then, by subtracting 1 to the weight of the edge $\{(u,i),(v,j)\}$ for every $\tilde{\mathbf{n}} \in V_{i,j}(u,v)$, we get

$$\begin{aligned} \alpha_1(N) &\leq \sum_{u \neq v} \sum_{i=1}^{A_u} \sum_{j=1}^N \frac{(c_N N)^{|\Lambda|}}{\mathbf{Z}_N^{|\Lambda|}} \sum_{\substack{\tilde{\mathbf{n}} \in V_{i,j}(u,v) \\ \Theta_N(\tilde{\mathbf{n}}) = \mathbf{n} \\ \partial \tilde{\mathbf{n}} = \tilde{A}}} w_{N,\beta}(\tilde{\mathbf{n}}) \\ &\leq \sum_{u \neq v} \sum_{i=1}^{A_u} \sum_{j=1}^N \beta J_{u,v} c_N^2 \frac{(c_N N)^{|\Lambda|}}{\mathbf{Z}_N^{|\Lambda|}} \sum_{\substack{\tilde{\mathbf{m}} \in \Omega_\Lambda^N \\ \Theta_N(\tilde{\mathbf{m}}) = \mathbf{n}^- \\ \partial \tilde{\mathbf{m}} = \tilde{A}_{i,j}}} w_{N,\beta}(\tilde{\mathbf{m}}) \end{aligned}$$

where $\mathbf{n}^-$ is the current satisfying $\mathbf{n}_e^- = \mathbf{n}_e$ for all edges $e \neq uv$, and $\mathbf{n}_{uv}^- = \mathbf{n}_{uv} - 1$; and $\tilde{A}_{i,j} = \tilde{A} \Delta \{(u,i),(v,j)\}$. Notice that $\tilde{A}_{i,j} = \tilde{A} \setminus \{(u,i),(v,j)\}$ if $1 \leq j \leq A_v$, and $\tilde{A}_{i,j} = (\tilde{A} \setminus \{(u,i)\}) \cup \{(v,j)\}$ otherwise. With this in mind, we associate to these new sets of sources their corresponding elements in $\mathcal{M}$: define A' from A by only changing its value on u and v where $A'_u = A_u - 1$ and $A'_v = A_v - 1$; also define A'' from A by only changing its value on u and v where $A''_u = A_u - 1$ and $A''_v = A_v + 1$. Then, using the induction hypothesis ($|\mathbf{n}^-| = |\mathbf{n}| - 1$),

$$\begin{aligned} \alpha_1(N) &\leq \sum_{\substack{u \neq v \\ 1 \leq i \leq A_u}} \beta J_{u,v} c_N^2 \left(\sum_{j=1}^{A_v} (c_N N)^2 \bar{w}_{N,A',\beta}(\mathbf{n}^-) + \sum_{j=A_v+1}^N \bar{w}_{N,A'',\beta}(\mathbf{n}^-) \right) \\ &\leq \sum_{\substack{u \neq v \\ 1 \leq i \leq A_u}} \beta J_{u,v} c_N^2 \left(\sum_{j=1}^{A_v} (c_N N)^2 O(1) + \sum_{j=A_v+1}^N O(1) \right) \\ &= O(1/\sqrt{N}) \xrightarrow{N \rightarrow \infty} 0. \end{aligned}$$

We used the fact that $(c_N N)^2 c_N^2 \rightarrow 0$ and $c_N^2 N \rightarrow 0$. Moreover, to bound the second sum in the right-hand side of the first inequality, we used the symmetries of the model to write that

for all $A_v + 1 \leq j \leq N$,

$$\frac{(c_N N)^{|A|}}{\mathbf{Z}_N^{|A|}} \sum_{\substack{\tilde{\mathbf{m}} \in \Omega_{\Lambda}^N \\ \Theta_N(\tilde{\mathbf{m}}) = \mathbf{n}^- \\ \partial \tilde{\mathbf{m}} = \tilde{A}_{i,j}}} w_{N,\beta}(\tilde{\mathbf{m}}) = \bar{w}_{N,A'',\beta}(\mathbf{n}^-). \quad (11.3.121)$$

We now prove that $\alpha_2(N) \rightarrow 0$. The idea is essentially the same: we seek a non-zero weight between to different blocks that we diminish to apply the induction hypothesis. Notice that if $\tilde{\mathbf{n}} \in \Omega_{\Lambda_g}^N \setminus W_{2,N}$ then there exist $u \in \Lambda$ and $i \in \{1, \dots, N\}$, such that

$$\sum_{\substack{(u,i) \in e \\ e \notin B_{u,N}}} \tilde{\mathbf{n}}_e \geq 2. \quad (11.3.122)$$

There are two cases: either for some (v, j) with $u \neq v$, one has $\tilde{\mathbf{n}}(u, i, v, j) \geq 2$ and in such case we can subtract 2 to the weight of this edge (which does not change the set of sources) to gain a $(c_N^2)^2$ which is in competition with a factor $O(N^2)$, and hence vanishes; or we can find two distinct vertices (v, j) and (w, k) with $u \neq v, w$ such that $\tilde{\mathbf{n}}(u, i, v, j) = 1$ and $\tilde{\mathbf{n}}(u, i, w, k) = 1$ so that the deletion of these weights adds two new sources, (v, j) and (w, k) , and gives a factor $(c_N^2)^2$ which is in competition with the factor $(c_N N)^{-2} O(N^3) = O(N^{5/2})$, and once again vanishes. More formally, define $X_{1,N}$ (resp. $X_{2,N}$) to be the subset of currents of $\Omega_{\Lambda_g}^N \setminus W_{2,N}$ that falls into the first case (resp. second case). Then,

$$\begin{aligned} \frac{(c_N N)^{|A|}}{\mathbf{Z}_N^{|A|}} \sum_{\substack{\tilde{\mathbf{n}} \in X_{1,N} \\ \Theta_N(\tilde{\mathbf{n}}) = \mathbf{n} \\ \partial \tilde{\mathbf{n}} = \tilde{A}}} w_{N,\beta}(\tilde{\mathbf{n}}) &\leq \sum_{u \neq v} \sum_{1 \leq i \leq N} \sum_{1 \leq j \leq N} \frac{(c_N N)^{|A|}}{\mathbf{Z}_N^{|A|}} \sum_{\substack{\tilde{\mathbf{n}}(u,i,v,j) \geq 2 \\ \Theta_N(\tilde{\mathbf{n}}) = \mathbf{n} \\ \partial \tilde{\mathbf{n}} = \tilde{A}}} w_{N,\beta}(\tilde{\mathbf{n}}) \\ &\leq \sum_{u \neq v} \sum_{1 \leq i, j \leq N} (\beta J_{u,v} c_N^2)^2 \bar{w}_{N,A,\beta}(\mathbf{n}^{2-}) \\ &= O(N^2 c_N^4) \xrightarrow{N \rightarrow \infty} 0, \end{aligned}$$

where $\mathbf{n}^{2-}$ is the current satisfying $\mathbf{n}_e^{2-} = \mathbf{n}_e$ for all edges $e \neq uv$ and $\mathbf{n}_{u,v}^{2-} = \mathbf{n}_{u,v} - 2$. We have used the induction hypothesis on the last line (since $|\mathbf{n}^{2-}| = |\mathbf{n}| - 2$).

Now, for the second case, bear in mind that if we can find, for $\tilde{\mathbf{n}} \in X_{2,N}$, (u, i) , (v, j) and (w, k) satisfying the properties described above, with one of them belonging to $\partial \tilde{\mathbf{n}}$ then $\tilde{\mathbf{n}}$ belongs to $\Omega_{\Lambda}^N \setminus W_{1,N}$ and the contribution of such currents goes to zero since $\alpha_1(N) \rightarrow 0$. Thus,

$$\begin{aligned} \frac{(c_N N)^{|A|}}{\mathbf{Z}_N^{|A|}} \sum_{\substack{\tilde{\mathbf{n}} \in X_{2,N} \\ \Theta_N(\tilde{\mathbf{n}}) = \mathbf{n} \\ \partial \tilde{\mathbf{n}} = \tilde{A}}} w_{N,\beta}(\tilde{\mathbf{n}}) &\leq \alpha_1(N) \\ &+ \sum_{\substack{u,v,w \in \Lambda \\ v \neq u \\ w \neq u}} \sum_{i=A_u+1}^N \sum_{j=A_v+1}^N \sum_{k=A_w+1}^N \frac{(c_N N)^{|A|}}{\mathbf{Z}_N^{|A|}} \sum_{\substack{\tilde{\mathbf{n}}(u,i,v,j)=1 \\ \tilde{\mathbf{n}}(u,i,w,k)=1 \\ \Theta_N(\tilde{\mathbf{n}}) = \mathbf{n} \\ \partial \tilde{\mathbf{n}} = \tilde{A}}} w_{N,\beta}(\tilde{\mathbf{n}}). \end{aligned}$$

Notice that for $i \geq A_u + 1$, $j \geq A_v + 1$ and $k \geq A_w + 1$,

$$\frac{(c_N N)^{|\Lambda|}}{\mathbf{Z}_N^{|\Lambda|}} \sum_{\substack{\tilde{\mathbf{n}}(u,i,v,j)=1 \\ \tilde{\mathbf{n}}(u,i,w,k)=1 \\ \Theta_N(\tilde{\mathbf{n}})=\mathbf{n} \\ \partial \tilde{\mathbf{n}}=\tilde{A}}} w_{N,\beta}(\tilde{\mathbf{n}}) = \beta^2 J_{u,v} J_{u,w} c_N^4 (c_N N)^{-2} \bar{w}_{N,B,\beta}(\mathbf{n}^{\cdot,-}) \quad (11.3.123)$$

where $B \in \mathcal{M}$ is defined from A by only changing its values on v and w where $B_v = A_v + 1$ and $B_w = A_w + 1$, and $\mathbf{n}^{\cdot,-}$ is obtained from $\mathbf{n}$ by reducing by 1 the weights of the edges uv and uw . We can once again apply the induction hypothesis to get

$$\frac{(c_N N)^{|\Lambda|}}{\mathbf{Z}_N^{|\Lambda|}} \sum_{\substack{\tilde{\mathbf{n}} \in X_{2,N} \\ \Theta_N(\tilde{\mathbf{n}})=\mathbf{n} \\ \partial \tilde{\mathbf{n}}=\tilde{A}}} w_{N,\beta}(\tilde{\mathbf{n}}) \leq O(c_N^2 N) \xrightarrow{N \rightarrow \infty} 0. \quad (11.3.124)$$

We finally obtained $\alpha_2(N) \rightarrow 0$ which yields $\mathbf{H}_{k+1}$. $\square$

Remark 11.3.44. In the case where we add an external field, the preceding proof may be adapted using the following ideas. For the vanishing of $\alpha_1(N)$ above, adding the external field means considering the case where $v = \mathbf{g}$ which adds a $O(c_N(c_N N))$ term (because in this case removing 1 to the weight of the edge reduces by 1 the number of sources in Λ_N). For $\alpha_2(N)$, in the first case, consider the situation where v is equal to $\mathbf{g}$ which turns a $O(N^2 c_N^4)$ into a $O(N c_N^2)$; and in the second case, consider the situation where v or w is equal to $\mathbf{g}$ which turns a $O((c_N^2)^2 N^{5/2})$ into a $O((c_N^3)(c_N N)^{-1} N^2) = O(c_N^2 N)$ (because in this case we add exactly one source in Λ_N).

11.4 Positive probability of admissible partitions

The goal of this section is to prove that any element of $\mathcal{T}_{\mathbf{n}_1+\mathbf{n}_2,A,B}(z)$ has positive probability under $\rho_{z,A,B}^{\mathbf{n}_1,\mathbf{n}_2}$. In particular, this shows that the distribution of tanglings in each block is not deterministic. This is of crucial importance for the proof of Theorem 11.1.2.

In Chapter 12, we provide a much stronger result by computing exactly the probability of occurrence of any tangling under $\rho_{z,A,B}^{\mathbf{n}_1,\mathbf{n}_2}$.

Let Λ be a finite subset of $V \cup \{\mathbf{g}\}$. Let $A, B \in \mathcal{M}(\Lambda)$, and consider a pair of currents $(\mathbf{n}_1, \mathbf{n}_2)$ on Λ satisfying $(\partial \mathbf{n}_1, \partial \mathbf{n}_2) = (\partial A, \partial B)$. We write $\mathbf{n} = \mathbf{n}_1 + \mathbf{n}_2$. Recall that up to relabelling the elements of $\mathcal{B}_z(\mathbf{n}, A, B)$, the measure $\rho_{z,A,B}^{\mathbf{n}_1,\mathbf{n}_2}$ only depends on $\Delta \mathbf{n}_1(z), \Delta \mathbf{n}_2(z), A_z$, and B_z . The following proposition is the main result of this section.

Proposition 11.4.1. *For every integer $M \geq 1$, there exists a constant $\kappa = \kappa(M) > 0$ such that for every $A, B, \mathbf{n}_1, \mathbf{n}_2$ as above, and $z \in \Lambda$ satisfying $\max(A_z, B_z, \Delta \mathbf{n}(z)) \leq M$, we have*

$$\rho_{z,A,B}^{\mathbf{n}_1,\mathbf{n}_2}(P) \geq \kappa, \quad (11.4.1)$$

for all $P \in \mathcal{T}_{\mathbf{n},A,B}(z)$. In particular,

$$\rho_{z,A}^{\mathbf{n}_1}(P) \geq \kappa, \quad (11.4.2)$$

for all $P \in \mathcal{T}_{\mathbf{n}_1,A}(z)$.

Remark 11.4.2. The occurrence of the partition consisting of one element, i.e. the partition for which all the points of the block are tangled together, plays the role of an insertion tolerance property inside the block $\mathcal{B}_z$ of $\mathcal{H}(\mathbf{n}_1 + \mathbf{n}_2, \mathbf{t}, A, B)$. We call this event $\text{ECT}(z)$. Proposition 11.4.1 implies, in particular, that $\text{ECT}(z)$ occurs with positive probability.

Let us recall that $\rho_{z,A,B}^{\mathbf{n}_1, \mathbf{n}_2}$ is constructed as the weak subsequential limit of the probability measures $\rho_{z,N}^{\tilde{S}, \tilde{T}}$ where $S = \Delta \mathbf{n}_1(z) + A_z$ and $T = \Delta \mathbf{n}_2(z) + B_z$ (recall that $\tilde{S} \cap \tilde{T} = \emptyset$ by our choice of projection, as explained in Definition 11.3.31). This convergence is in fact strong, i.e. pointwise on events, since the underlying space is finite dimensional. The measures in the prelimit are themselves defined as the pushforward of the block-spin double current measures $\mathbb{P}_{K_N}^{\tilde{S}, \tilde{T}}$ under the partitions on sources induced by the clusters of $\tilde{\mathbf{n}}_1 + \tilde{\mathbf{n}}_2 \sim \mathbb{P}_{K_N}^{\tilde{S}, \tilde{T}}$.

A fundamental step to proving Proposition 11.4.1 is understanding the large N asymptotics of the size of the cluster of a source. We then use this to prove two lemmas that are used in the proof of Proposition 11.4.1. The first one asserts that pairings occur with positive probability, where a *pairing* of $\tilde{S}$ is a partition of it into sets of cardinality 2, and the second one can be seen as merging the clusters of the sources at finite cost. In the following and later on this in section, we make the dependency on a explicit as it is of great importance.

11.4.1 Large N asymptotics of the size of the cluster

Let $S \in \mathbb{N}$ be finite and recall that, for each $N \geq S$, $\tilde{S}$ is its natural injection into K_N . Given a vertex $x \in \tilde{S}$, we write $\mathcal{C}_x = \mathcal{C}_x(\tilde{\mathbf{n}})$ for the connected component of x in $\tilde{\mathbf{n}}$.

Proposition 11.4.3. *Let $S = 2k$, $k > 0$, and let $a_0 \in \mathbb{R}$. Then there exists a constant $C = C(k, a_0) > 0$ such that for every $x \in \tilde{S}$ and every $N \geq 2k$, we have*

$$\sup_{a \geq a_0} \mathbb{E}_{K_N, a}^{\tilde{S}, \emptyset} \left[\frac{|\mathcal{C}_x|}{\sqrt{N}} \right] \leq C. \quad (11.4.3)$$

Proof. Note that for every $N \geq 2k$, we trivially have

$$\sup_{a \in I} \mathbb{E}_{K_N, a}^{\tilde{S}, \emptyset} \left[\frac{|\mathcal{C}_x|}{\sqrt{N}} \right] \leq \sqrt{N}, \quad (11.4.4)$$

thus it suffices to handle the case of large N . Let $u \notin \tilde{S}$. Then, for every $a \geq a_0$, we have

$$\mathbb{E}_{K_N, a}^{\tilde{S}, \emptyset} \left[\frac{|\mathcal{C}_x|}{\sqrt{N}} \right] = \frac{1}{\sqrt{N}} \left((N - 2k) \mathbb{P}_{K_N, a}^{\tilde{S}, \emptyset} [x \longleftrightarrow u] + O(1) \right) \quad (11.4.5)$$

$$= \frac{N - 2k}{\sqrt{N}} \frac{\langle \sigma_{\tilde{S} \Delta \{x, u\}} \rangle_{K_N, a}}{\langle \sigma_{\tilde{S}} \rangle_{K_N, a}} \langle \sigma_x \sigma_u \rangle_{K_N, a} + O(1/\sqrt{N}) \quad (11.4.6)$$

$$\leq \frac{N - 2k}{\sqrt{N}} \langle \sigma_x \sigma_u \rangle_{K_N, a_0} + O(1/\sqrt{N}) \quad (11.4.7)$$

$$\xrightarrow{N \rightarrow \infty} \sqrt{g/12} \langle \varphi^2 \rangle_{0, a_0}. \quad (11.4.8)$$

In the first line, we use the symmetry of the model. In the second line, we use the random current representation and the usual switching lemma for the Ising model. In the third line, we use that $\langle \sigma_{\tilde{S} \Delta \{x, u\}} \rangle_{K_N, a} = \langle \sigma_{\tilde{S}} \rangle_{K_N, a}$ by symmetry (recall that $x \in \tilde{S}$ but $u \notin \tilde{S}$), and the

monotonicity of $a \mapsto \langle \sigma_x \sigma_u \rangle_{K_N, a}$, which follows from differentiating and applying Griffiths' inequalities. Finally, in the last line we use Lemma 11.3.24. The desired assertion follows readily. $\square$

Remark 11.4.4. From the proof above, one can show $\mathbb{E}_{K_N, a}^{\tilde{S}, \emptyset} \left[\frac{|\mathcal{C}_x|}{(c_N N)^2} \right] \rightarrow \langle \varphi^2 \rangle_{0, a}$ as $N \rightarrow \infty$. It would be of great interest to understand the scaling limit of the size of the rescaled cluster.

We now make an observation⁶ that is subtle and means that arguments involving exploration of clusters naturally change the underlying model.

Remark 11.4.5. Let $u \in \mathbb{R}$. The scaling limit of the Ising model on $K_{N-r_N(u)}$ with interaction $d_N(a) := \frac{1}{N} \left(1 - \frac{a\sqrt{12/g}}{\sqrt{N}} \right)$, where $r_N(u) = \lceil u\sqrt{N} \rceil$, coincides with the φ^4 single-site measure with parameter $a(u) = a + u\sqrt{g/12}$ in place of a . Indeed, writing $\tilde{N} := N - r_N(u)$, we have

$$d_N(a) = \frac{1}{\tilde{N}} \left(1 - \frac{(a + u\sqrt{g/12})\sqrt{12/g}}{\sqrt{\tilde{N}}} + O\left(\frac{1}{\tilde{N}}\right) \right). \quad (11.4.9)$$

11.4.2 Pairings occur with positive probability

Let $\mathcal{P}$ denote the (random) even partition of the source set $\tilde{S}$ induced by the cluster of $\tilde{\mathbf{n}}_1 + \tilde{\mathbf{n}}_2 \sim \mathbb{P}_{K_N}^{\tilde{S}, \emptyset}$. Recall that a is a parameter of the single site measure.

Lemma 11.4.6 (Pairings happen with positive probability). *Let $S = 2k$, $k > 0$, and let $I = [\ell, L]$ be a compact interval. Then, there exists a constant $c = c(I, k) > 0$ such that for every pairing P of $\tilde{S}$ and for every $N \geq 2k$,*

$$\inf_{a \in I} \mathbb{P}_{K_N, a}^{\tilde{S}, \emptyset} [\mathcal{P} = P] \geq c. \quad (11.4.10)$$

Proof. We prove the assertion by induction on k . The case $k = 1$ is trivial. Let us assume that the statement holds for some $k \geq 1$. We will prove it for $k + 1$.

Note that for every $N \geq 2k + 2$, there exists $\alpha_N > 0$ such that

$$\inf_{a \in I} \mathbb{P}_{K_N, a}^{\tilde{S}, \emptyset} [\mathcal{P} = P] \geq \alpha_N, \quad (11.4.11)$$

hence we can assume that N is sufficiently large. To simplify notation, we identify $\tilde{S}$ with $\{1, 2, \dots, 2k + 2\}$. Consider a parameter $a \in I$, and let us write $P_0 = \{\{1, 2\}, \dots, \{2k + 1, 2k + 2\}\}$. Then, conditioning on $\mathcal{C}_x$, and letting $x \in \{1, 2\}$,

$$\mathbb{P}_{K_N, a}^{\tilde{S}, \emptyset} [\mathcal{P} = P_0] = \frac{1}{\langle \sigma_{\tilde{S}} \rangle_{K_N, a}} \sum_{S \subset (K_N \setminus \tilde{S}) \cup \{1, 2\}} \frac{Z_{K_N, a}^{\tilde{S}, \emptyset} [\mathcal{C}_x = S, \mathcal{P} = P_0]}{Z_{K_N, a}^{\emptyset, \emptyset}}. \quad (11.4.12)$$

Notice that for $P'_0 = \{\{3, 4\}, \dots, \{2k + 1, 2k + 2\}\}$,

$$\begin{aligned} Z_{K_N, a}^{\tilde{S}, \emptyset} [\mathcal{C}_x = S, \mathcal{P} = P_0] &= Z_{S, a}^{\{1, 2\}, \emptyset} [\mathcal{C}_x = S] Z_{K_N \setminus S, a}^{\tilde{S} \setminus \{1, 2\}, \emptyset} [\mathcal{P} = P'_0] \\ &= Z_{K_N, a}^{\{1, 2\}, \emptyset} [\mathcal{C}_x = S] \frac{Z_{K_N \setminus S, a}^{\tilde{S} \setminus \{1, 2\}, \emptyset} [\mathcal{P} = P'_0]}{Z_{K_N \setminus S, a}^{\emptyset, \emptyset}}. \end{aligned}$$

⁶This observation is not new, see [4, Lemma 11.2].

Using the above equalities, we obtain

$$\mathbb{P}_{K_N,a}^{\tilde{S},\emptyset}[\mathcal{P} = P_0] = \sum_{\mathcal{S} \subset (K_N \setminus \tilde{S}) \cup \{1,2\}} Q_{\mathcal{S},N}(a) \mathbb{P}_{K_N,a}^{\{1,2\},\emptyset}[\mathcal{C}_x = \mathcal{S}] \mathbb{P}_{K_N \setminus \mathcal{S},a}^{\tilde{S} \setminus \{1,2\},\emptyset}[\mathcal{P} = P'_0], \quad (11.4.13)$$

where

$$Q_{\mathcal{S},N}(a) = \frac{\langle \sigma_1 \sigma_2 \rangle_{K_N,a} \langle \sigma_{\tilde{S} \setminus \{1,2\}} \rangle_{K_N \setminus \mathcal{S},a}}{\langle \sigma_{\tilde{S}} \rangle_{K_N,a}}. \quad (11.4.14)$$

Let us now consider a constant $C_0 > 0$ such that

$$\mathbb{P}_{K_N,a}^{\{1,2\},\emptyset}[A] \geq \frac{1}{2} \quad \text{for every } a \in I, \quad (11.4.15)$$

where $A = A(N) = \{|\mathcal{C}_x| < C_0 \sqrt{N}, (\mathcal{C}_x \setminus \{1,2\}) \cap \tilde{S} = \emptyset\}$. The existence of such a constant follows from Proposition 11.4.3 and the symmetry of the model, provided that N is large enough. Using Griffiths inequality yields that for every $\mathcal{S}$ such that $|\mathcal{S}| < C_0 \sqrt{N}$ and every $a \in I$,

$$Q_{\mathcal{S},N}(a) \geq Q_N := \frac{\langle \sigma_1 \sigma_2 \rangle_{K_N,L} \langle \sigma_{\tilde{S} \setminus \{1,2\}} \rangle_{K_N - \lceil C_0 \sqrt{N} \rceil, L}}{\langle \sigma_{\tilde{S}} \rangle_{K_N, \ell}}. \quad (11.4.16)$$

Since every summand in (11.4.13) is non-negative, using the above inequalities we obtain

$$\mathbb{P}_{K_N,a}^{\tilde{S},\emptyset}[\mathcal{P} = P_0] \geq \frac{1}{2} Q_N \inf_{b \in I} \inf_{|\mathcal{S}| < C_0 \sqrt{N}} \mathbb{P}_{K_N \setminus \mathcal{S},b}^{\tilde{S} \setminus \{1,2\},\emptyset}[\mathcal{P} = P'_0] \quad (11.4.17)$$

for every $a \in I$.

Using Remark 11.4.5 and Lemma 11.3.24 we also obtain that

$$\lim_{N \rightarrow \infty} Q_N = \frac{\langle \varphi^2 \rangle_{0,L} \langle \varphi^{2k} \rangle_{0,L+C_0 \sqrt{g/12}}}{\langle \varphi^{2k+2} \rangle_{0,\ell}}. \quad (11.4.18)$$

Finally, it follows from our induction hypothesis and Remark 11.4.5 that

$$\inf_{b \in I} \inf_{|\mathcal{S}| < C_0 \sqrt{N}} \mathbb{P}_{K_N \setminus \mathcal{S},b}^{\tilde{S} \setminus \{1,2\},\emptyset}[\mathcal{P} = P'_0] \geq c(k, I'), \quad (11.4.19)$$

where $I' = [\ell, L + C_0 \sqrt{g/12}]$. This concludes the proof. $\square$

11.4.3 Cluster merging at finite cost

Given a partition $P = \{P_1, P_2, \dots, P_n\}$ of $\tilde{S}$, we write $P^{i,j}$ for the partition of $\tilde{S}$ with P_i and P_j merged together, i.e. $P^{i,j} = (P \setminus \{P_i, P_j\}) \cup \{P_i \cup P_j\}$.

Lemma 11.4.7 (Cluster merging at finite cost). *Let $S = 2k$, $k > 0$. For every $\varepsilon \in (0, 1)$, there exists $\delta = \delta(\varepsilon) > 0$ such that the following holds for every N large enough. Assume that for a partition P of $\tilde{S}$ we have*

$$\mathbb{P}_{K_N}^{\tilde{S},\emptyset}[\mathcal{P} = P] \geq \varepsilon. \quad (11.4.20)$$

Then for every i, j we have

$$\mathbb{P}_{K_N}^{\tilde{S},\emptyset}[\mathcal{P} = P^{i,j}] \geq \delta \mathbb{P}_{K_N}^{\tilde{S},\emptyset}[\mathcal{P} = P]. \quad (11.4.21)$$

Proof. Consider a partition $P = \{P_1, P_2, \dots, P_n\}$ satisfying our assumption (11.4.20). Fix $i, j \in \{1, 2, \dots, n\}$ distinct and let $r \in \{i, j\}$. We start by defining $\Pi(P_r)$ to be the (random) set of (vertex-)self-avoiding walks satisfying the following properties:

- each self-avoiding walk in $\Pi(P_r)$ traverses only edges with non-zero value in $\tilde{\mathbf{n}} := \tilde{\mathbf{n}}_1 + \tilde{\mathbf{n}}_2$;
- and, each self-avoiding walk in $\Pi(P_r)$ starts and ends in P_r .

We write $\pi(P_r)$ to denote the set of vertices visited by the self-avoiding walks in $\Pi(P_r)$. It is possible that some vertices of $\mathcal{C}_{P_r} := \bigcup_{x \in P_r} \mathcal{C}_x$ are not contained in $\pi(P_r)$, which can happen when $\tilde{\mathbf{n}}$ contains a loop starting and ending at some vertex in $\pi(P_r)$.

We first show that with high probability $\pi(P_r)$ has size $\Omega(\sqrt{N})$. Indeed, let $m > 0$ and $x, y \in P_r$, with $x \neq y$. For a self-avoiding walk γ in K_N which starts at x , ends at y , and has length γ , we write $\{x \xrightarrow{\gamma} y\}$ for the event that $\tilde{\mathbf{n}}_1(e) > 0$ for all edges e of γ . When $\tilde{\mathbf{n}}_1 \in \{x \xrightarrow{\gamma} y\}$, the current

$$\tilde{\mathbf{m}}_1(e) := \begin{cases} \tilde{\mathbf{n}}_1(e) - 1, & \text{for } e \text{ edge of } \gamma, \\ \tilde{\mathbf{n}}_1(e), & \text{otherwise,} \end{cases} \quad (11.4.22)$$

has sources $\tilde{S} \setminus \{x, y\}$. Thus, using the inequality $\frac{1}{\tilde{\mathbf{n}}_1(e)} \leq \frac{1}{\tilde{\mathbf{m}}_1(e)}$ and arguing as in the proof of Proposition 11.4.3, we obtain (for the single current measure)

$$\mathbb{P}_{K_N}^{\tilde{S}}[x \xrightarrow{\gamma} y] \leq d_N^m \frac{Z_{K_N}^{\tilde{S} \setminus \{x, y\}}}{Z_{K_N}^{\tilde{S}}} = d_N^m \frac{\langle \sigma_{\tilde{S} \setminus \{x, y\}} \rangle_{K_N}}{\langle \sigma_{\tilde{S}} \rangle_{K_N}} \leq C_1 d_N^m \sqrt{N} \quad (11.4.23)$$

where d_N is defined in (11.3.35), m is the length of γ , and $C_1 = C_1(k) > 0$ is a constant.

Note that there are at most N^{m-1} self-avoiding walks of length m connecting x and y in K_N . Moreover, when $|\pi(P_r)| \leq m$, each $x \in P_r$ is connected to some $y \in P_r$ by a self-avoiding walk of length at most m . Since adding a second current cannot decrease $|\pi(P_r)|$, for every $1 > \eta > 0$

$$\mathbb{P}_{K_N}^{\tilde{S}, \emptyset} \left[\{\mathcal{P} = P\} \cap \left(\{|\pi(P_i)| \leq \eta\sqrt{N}\} \cup \{|\pi(P_j)| \leq \eta\sqrt{N}\} \right) \right] \leq C_2 d_N \eta N \leq C_3 \eta \quad (11.4.24)$$

where $C_2 = C_2(k) > 0, C_3 = C_3(k) > 0$ are constants. Thus, because of our assumption (11.4.20) there exists a constant $c > 0$ such that

$$\mathbb{P}_{K_N}^{\tilde{S}, \emptyset} [\{\mathcal{P} = P\} \cap \{|\pi(P_i)|, |\pi(P_j)| \geq c\sqrt{N}\}] \geq \frac{1}{2} \mathbb{P}_{K_N}^{\tilde{S}, \emptyset} [\mathcal{P} = P] \quad (11.4.25)$$

for every $N \geq 2k$.

Let $(\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2) \in \{\mathcal{P} = P\} \cap \{|\pi(P_i)|, |\pi(P_j)| \geq c\sqrt{N}\}$. For every $x \in \pi(P_i) \setminus \tilde{S}$ and $y \in \pi(P_j) \setminus \tilde{S}$, define $(\tilde{\mathbf{n}}_1^{x,y}, \tilde{\mathbf{n}}_2^{x,y})$ as follows. First, we identify x and y in the natural way, labelling the point obtained from the gluing by x . Also, we preserve the labelling of all other vertices. The gluing procedure gives rise to a pair of currents $(\tilde{\mathbf{n}}_1^{x,y}, \tilde{\mathbf{n}}_2^{x,y})$ which has the property that $\mathcal{P} = P^{i,j}$. The caveat is that they are defined on $N - 1$ vertices. To fix this, we add a new vertex, labelled y , and extend $\tilde{\mathbf{n}}_1^{x,y}$ and $\tilde{\mathbf{n}}_2^{x,y}$ in the natural way such that y is of degree 0 in their sum.

We define f as the multi-valued map which associates to each $(\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2)$ the set of all possible $(\tilde{\mathbf{n}}_1^{x,y}, \tilde{\mathbf{n}}_2^{x,y})$. Note that, since $|\pi(P_i)|, |\pi(P_j)| \geq c\sqrt{N}$, we have

$$|f(\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2)| \geq (c\sqrt{N} - |\tilde{S}|)^2 \geq c_1 N, \quad (11.4.26)$$

for some constant $c_1 > 0$, provided that N is large enough. Moreover, since no new edges were added, $\mathbb{P}_{K_N}^{\tilde{S}, \emptyset}[(\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2)] = \mathbb{P}_{K_N}^{\tilde{S}, \emptyset}[(\tilde{\mathbf{n}}_1^{x,y}, \tilde{\mathbf{n}}_2^{x,y})]$. Thus, for every $(\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2) \in \{\mathcal{P} = P\} \cap \{|\pi(P_i)|, |\pi(P_j)| \geq c\sqrt{N}\}$,

$$\mathbb{P}_{K_N}^{\tilde{S}, \emptyset}[(\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2)] = \sum_{(\tilde{\mathbf{n}}, \tilde{\mathbf{m}}) \in f(\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2)} \frac{\mathbb{P}_{K_N}^{\tilde{S}, \emptyset}[(\tilde{\mathbf{n}}, \tilde{\mathbf{m}})]}{|f(\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2)|} \leq \sum_{(\tilde{\mathbf{n}}, \tilde{\mathbf{m}}) \in f(\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2)} \frac{\mathbb{P}_{K_N}^{\tilde{S}, \emptyset}[(\tilde{\mathbf{n}}, \tilde{\mathbf{m}})]}{c_1 N} \quad (11.4.27)$$

which, by Lemma 11.4.25, yields

$$\frac{1}{2} \mathbb{P}_{K_N}^{\tilde{S}, \emptyset}[\mathcal{P} = P] \leq \sum_{\substack{(\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2) \in \{\mathcal{P} = P\} \\ |\pi(P_i)|, |\pi(P_j)| \geq c\sqrt{N}}} \sum_{(\tilde{\mathbf{n}}, \tilde{\mathbf{m}}) \in f(\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2)} \frac{\mathbb{P}_{K_N}^{\tilde{S}, \emptyset}[(\tilde{\mathbf{n}}, \tilde{\mathbf{m}})]}{c_1 N} \quad (11.4.28)$$

$$\leq \sum_{(\tilde{\mathbf{n}}, \tilde{\mathbf{m}}) \in \{\mathcal{P} = P^{i,j}\}} \frac{|f^{-1}(\tilde{\mathbf{n}}, \tilde{\mathbf{m}})|}{c_1 N} \mathbb{P}_{K_N}^{\tilde{S}, \emptyset}[(\tilde{\mathbf{n}}, \tilde{\mathbf{m}})]. \quad (11.4.29)$$

We now estimate $|f^{-1}(\tilde{\mathbf{n}}, \tilde{\mathbf{m}})|$. Let $(\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2) \in f^{-1}(\tilde{\mathbf{n}}, \tilde{\mathbf{m}})$. Then, there exists $x \in \pi(P_i), y \in \pi(P_j)$ such that $\tilde{\mathbf{n}} = \tilde{\mathbf{n}}_1^{x,y}$ and $\tilde{\mathbf{m}} = \tilde{\mathbf{n}}_2^{x,y}$. Consider the number of self-avoiding walks between vertices of P_i in $(\tilde{\mathbf{n}}, \tilde{\mathbf{m}})$ and the number of self-avoiding walks between vertices of P_j in $(\tilde{\mathbf{n}}, \tilde{\mathbf{m}})$. Notice that *both* numbers strictly decrease when we remove x from $\mathcal{C}_{P_i \cup P_j}$ and that no other point has this property. Since we do not relabel x in the construction of $(\tilde{\mathbf{n}}_1^{x,y}, \tilde{\mathbf{n}}_2^{x,y})$, the topological consideration above implies that this point is the same for any $(\tilde{\mathbf{n}}_3, \tilde{\mathbf{n}}_4) \in f^{-1}(\tilde{\mathbf{n}}, \tilde{\mathbf{m}})$. See Figure 11.2. Thus, $\mathcal{C}_{P_i}(\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2) = \mathcal{C}_{P_i}(\tilde{\mathbf{n}}_3, \tilde{\mathbf{n}}_4)$.

Since the labels of all other vertices are preserved after gluing except for y , this means that any $(\tilde{\mathbf{n}}_3, \tilde{\mathbf{n}}_4) \in f^{-1}(\tilde{\mathbf{n}}, \tilde{\mathbf{m}})$ can be mapped bijectively to $(\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2)$ by swapping the label of y with a vertex of degree zero in $(\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2)$. Thus, $|f^{-1}(\tilde{\mathbf{n}}, \tilde{\mathbf{m}})| \leq N$. Putting this into (11.4.28) gives the desired result. $\square$

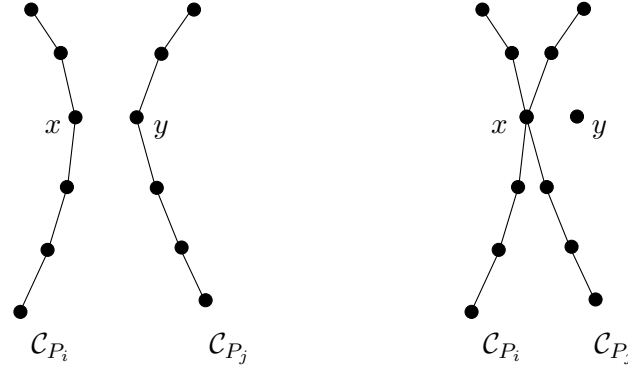
11.4.4 Proof of Proposition 11.4.1

We prove the estimate for the case of a double current; the case of a single current follows similarly. Let P be an admissible partition of $\tilde{S} \sqcup \tilde{T}$. Note that,

$$\mathbb{P}_{K_N}^{\tilde{S}, \tilde{T}}[P] = \frac{\langle \sigma_{\tilde{S} \sqcup \tilde{T}} \rangle_{K_N}}{\langle \sigma_{\tilde{S}} \rangle_{K_N} \langle \sigma_{\tilde{T}} \rangle_{K_N}} \mathbb{P}_{K_N}^{\tilde{S} \sqcup \tilde{T}, \emptyset}[P, \mathcal{F}_{\tilde{T}}] \quad (11.4.30)$$

$$= (1 + o(1)) \frac{\langle \varphi^{S+T} \rangle_0}{\langle \varphi^S \rangle_0 \langle \varphi^T \rangle_0} \mathbb{P}_{K_N}^{\tilde{S} \sqcup \tilde{T}, \emptyset}[P] \quad (11.4.31)$$

since any configuration $\tilde{\mathbf{n}}_1 + \tilde{\mathbf{n}}_2 \sim \mathbb{P}_{K_N}^{\tilde{S}, \tilde{T}}$ that realises P immediately realises $\mathcal{F}_{\tilde{T}}$ because of the admissibility constraint on P . This computation gives Lemmas 11.4.6 and 11.4.7 for the

Figure 11.2: An example illustrating the merging of $\mathcal{C}_{P_i}$ and $\mathcal{C}_{P_j}$.

measure $\mathbb{P}_{K_N}^{\tilde{S}, \tilde{T}}$. As a consequence, there exists a constant $c > 0$ such that, if N is large enough and if P is an admissible pairing of $\tilde{S} \sqcup \tilde{T}$, $\mathbb{P}_{K_N}^{\tilde{S}, \tilde{T}}[P = P] \geq c$. Note that by successively merging the partition classes of a pairing, we can obtain any admissible partition. Using Lemma 11.4.7, we obtain that $\mathbb{P}_{K_N}^{\tilde{S}, \tilde{T}}[P = P] \geq c'$ uniformly in N (sufficiently large) for a constant $c' > 0$. Letting N go to infinity, we obtain the desired result. $\square$

11.5 Infinite volume measures

11.5.1 Infinite volume distributions on currents

Denote by $(\Omega_V, \mathcal{A}_V)$ the space of currents on V equipped with the σ -algebra generated by cylinder events (i.e. events depending on finitely many edges). For $\Lambda \subset V$ finite, denote by $(\Omega_\Lambda, \mathcal{A}_\Lambda)$ the space of currents on Λ equipped with its Borel σ -algebra. There are natural injections $\Omega_\Lambda \hookrightarrow \Omega_V$ and $\mathcal{A}_\Lambda \hookrightarrow \mathcal{A}_V$. Thus, we canonically identify measures on $(\Omega_\Lambda, \mathcal{A}_\Lambda)$ as measures on $(\Omega_V, \mathcal{A}_V)$. Similarly, denote by $(\Omega_{\Lambda_g}, \mathcal{A}_{\Lambda_g})$ the space of currents on Λ_g equipped with its Borel σ -algebra. We can also canonically identify measures on $(\Omega_\Lambda, \mathcal{A}_\Lambda)$ as measures on $(\Omega_V, \mathcal{A}_V)$. We consider $\beta > 0, h = 0$ fixed in this section and drop from it from notation when convenient.

Definition 11.5.1. For $\Lambda \subset V$ finite, let $P_\Lambda^\emptyset$ and $P_{\Lambda_g}^\emptyset$ be the probability measures on Ω_Λ and Ω_{Λ_g} , respectively, defined by

$$P_\Lambda^\emptyset[\mathbf{n}] = \frac{w_{\emptyset, \beta, 0}(\mathbf{n}) \mathbb{1}_{\partial \mathbf{n} = \emptyset}}{\sum_{\mathbf{m} \in \Omega_\Lambda} w_{\emptyset, \beta, 0}(\mathbf{m}) \mathbb{1}_{\partial \mathbf{m} = \emptyset}} \quad (11.5.1)$$

$$P_{\Lambda_g}^\emptyset[\mathbf{n}] = \frac{w_{\emptyset, \beta, h_\Lambda(\mathbf{p})}(\mathbf{n}) \mathbb{1}_{\partial \mathbf{n} = \emptyset}}{\sum_{\mathbf{m} \in \Omega_{\Lambda_g}} w_{\emptyset, \beta, h_\Lambda(\mathbf{p})}(\mathbf{m}) \mathbb{1}_{\partial \mathbf{m} = \emptyset}} \quad (11.5.2)$$

where $h_\Lambda(\mathbf{p})_x = \sum_{y \notin \Lambda} J_{x,y} \mathbf{p}_x$ for $x \in \Lambda$. We denote the expectations with respect to $P_\Lambda^\emptyset$ and $P_{\Lambda_g}^\emptyset$ by $E_\Lambda^\emptyset$ and $E_{\Lambda_g}^\emptyset$, respectively. Later, when clear from context, we will drop $h_\Lambda(\mathbf{p})$ from the notation.

Proposition 11.5.2. The probability measures $P_\Lambda^\emptyset$ (resp. $P_{\Lambda_g}^\emptyset$) converge weakly as $\Lambda \uparrow V$ to a probability measure P^0 (resp. P^+) on Ω_V . The convergence is in fact strong for cylinder

events, i.e. for any $A \in \mathcal{A}_V$ depending on finitely many edges,

$$\lim_{\Lambda \uparrow V} \mathbf{P}_\Lambda^\emptyset[A] = \mathbf{P}^0[A] \quad \text{and} \quad \lim_{\Lambda \uparrow V} \mathbf{P}_{\Lambda_g}^\emptyset[A] = \mathbf{P}^+[A]. \quad (11.5.3)$$

Furthermore, $\mathbf{P}^0$ and $\mathbf{P}^+$ are invariant and mixing (hence, ergodic) under automorphisms in Γ^7 .

In order to prove the above proposition, we need the following lemma.

Lemma 11.5.3. *Let $\Lambda \subset V$ finite. Then for every $x \in \Lambda$ we have*

$$\mathbf{E}_{\Lambda_g, \Lambda}^{\emptyset, \emptyset}[\Delta \mathbf{n}_1(x) + \Delta \mathbf{n}_2(x)] = 4g \left(\langle \varphi_x^4 \rangle_\Lambda^0 + \langle \varphi_x^4 \rangle_\Lambda^p \right) + 2a \left(\langle \varphi_x^2 \rangle_\Lambda^0 + \langle \varphi_x^2 \rangle_\Lambda^p \right) - 2. \quad (11.5.4)$$

Proof. For $k \geq 0$, let $u_{2k} := \langle \varphi^{2k} \rangle_0$. An integration by parts yields that

$$(2k+1)u_{2k} = 2a \cdot u_{2k+2} + 4g \cdot u_{2k+4}. \quad (11.5.5)$$

Note that

$$\begin{aligned} \mathbf{E}_{\Lambda_g, \Lambda}^{\emptyset, \emptyset}[\Delta \mathbf{n}_1(x) + \Delta \mathbf{n}_2(x)] &= \frac{\sum_{\partial \mathbf{n}_1 = \emptyset} (\Delta \mathbf{n}_1(x) + 1) w_\emptyset(\mathbf{n}_1)}{\sum_{\partial \mathbf{n}_1 = \emptyset} w_\emptyset(\mathbf{n}_1)} \\ &\quad + \frac{\sum_{\partial \mathbf{n}_2 = \emptyset} (\Delta \mathbf{n}_2(x) + 1) w_\emptyset(\mathbf{n}_2)}{\sum_{\partial \mathbf{n}_2 = \emptyset} w_\emptyset(\mathbf{n}_2)} - 2, \end{aligned}$$

and for $i \in \{1, 2\}$,

$$\begin{aligned} (\Delta \mathbf{n}_i(x) + 1) w_\emptyset(\mathbf{n}_i) &= w(\mathbf{n}_i) \prod_{y \neq x} u_{\Delta \mathbf{n}_i(y)} \cdot (\Delta \mathbf{n}_i(x) + 1) u_{\Delta \mathbf{n}_i(x)} \\ &= w(\mathbf{n}_i) \prod_{y \neq x} u_{\Delta \mathbf{n}_i(y)} (2a \cdot u_{\Delta \mathbf{n}_i(x)+2} + 4g \cdot u_{\Delta \mathbf{n}_i(x)+4}) \\ &= 4g \cdot w_{\{x, x, x, x\}, a}(\mathbf{n}_i) + 2a \cdot w_{\{x, x\}, a}(\mathbf{n}_i). \end{aligned}$$

The desired result follows. $\square$

Without loss of generality, we only prove Proposition 11.5.2 for $\mathbf{P}^0$. We first construct the limit on special cylinder events (which generate the σ -algebra) where both the bond parities and degrees of the current are locally specified. Fix $F \subset \mathcal{P}_2(V)$ finite. Let $V(F)$ be the set of $x \in V$ which belong to an edge in F . For $d_F \in (2\mathbb{N})^{V(F)}$, we consider the following event

$$\mathcal{C}_{F, d_F} := \left\{ \mathbf{n} \in \Omega_V \mid \begin{array}{l} \mathbf{n}_e \text{ is even for all } e \in F, \text{ and} \\ \Delta \mathbf{n}_F(x) = d_F(x) \text{ for all } x \in V(F) \end{array} \right\}, \quad (11.5.6)$$

where $\mathbf{n}_F$ is the restriction of $\mathbf{n}$ to pairs in F .

Lemma 11.5.4. *Let $F \subset \mathcal{P}_2(V)$ finite and $d_F \in V(F)^{2\mathbb{N}}$. The probabilities $\mathbf{P}_\Lambda^\emptyset[\mathcal{C}_{F, d_F}]$ converge as $\Lambda \uparrow V$. We define $\mathbf{P}^0[\mathcal{C}_{F, d_F}]$ to be this limit.*

⁷Recall we fix a vertex-transitive subgroup $\Gamma \subset \text{Aut}(G)$.

Proof. Let $\Lambda \supset V(F)$. Define $\mathring{V}(F) \subset V(F)$ be the set of $x \in V(F)$ such that if $J_{x,y} > 0$ for $y \in V$, then $y \in V(F)$, i.e. vertices whose J -support are contained in $V(F)$, which might be empty if J has infinite range.

For $\mathbf{n} \in \Omega_\Lambda$, define

$$w(\mathbf{n}) := \prod_{\{x,y\} \subset \Lambda} \frac{(\beta J_{x,y})^{\mathbf{n}_{x,y}}}{(\mathbf{n}_{x,y})!} \quad \text{and} \quad Z_\Lambda^A := \sum_{\partial \mathbf{n} = \partial A} w_{A,\beta}(\mathbf{n}). \quad (11.5.7)$$

We write $\mathbf{n}_{\Lambda \setminus F}$ to denote the restriction of $\mathbf{n}$ to edges in the complement of F and denote $Z_{\Lambda \setminus F}^A$ for the corresponding partition function.

Observe that we can write,

$$w_{\emptyset,\beta}(\mathbf{n}) \mathbb{1}[\mathbf{n} \in \mathcal{C}_{F,d_F}] = w(\mathbf{n}_F) \mathbb{1}[\mathbf{n}_F \in \mathcal{C}_{F,d_F}] \prod_{x \in \mathring{V}(F)} \langle \varphi^{d_F(x)} \rangle_0 \times w(\mathbf{n}_{\Lambda \setminus F}) \quad (11.5.8)$$

$$\times \prod_{x \in \Lambda \setminus \mathring{V}(F)} \langle \varphi^{\Delta \mathbf{n}_{\Lambda \setminus F}(x) + d_F(x) \mathbb{1}[x \in V(F)]} \rangle_0. \quad (11.5.9)$$

Therefore,

$$\mathbf{P}_\Lambda^\emptyset[\mathcal{C}_{F,d_F}] = \frac{\sum_{\mathbf{n} \in \Omega_\Lambda} w_{\emptyset,\beta}(\mathbf{n}) \mathbb{1}[\mathbf{n} \in \mathcal{C}_{F,d_F}] \mathbb{1}[\partial \mathbf{n} = \emptyset]}{Z_\Lambda^\emptyset} = \frac{Z_{\Lambda \setminus F}^{d_F}}{Z_\Lambda^\emptyset} f(F, d_F), \quad (11.5.10)$$

where

$$f(F, d_F) := \prod_{x \in \mathring{V}(F)} \langle \varphi^{d_F(x)} \rangle_0 \sum_{\partial \mathbf{n}_F = \emptyset} w(\mathbf{n}_F) \mathbb{1}[\mathbf{n}_F \in \mathcal{C}_{F,d_F}]. \quad (11.5.11)$$

As a result, we get

$$\mathbf{P}_\Lambda^\emptyset[\mathcal{C}_{F,d_F}] = \left\langle e^{-K_F} \prod_{x \in V(F) \setminus \mathring{V}(F)} \varphi_x^{d_F(x)} \right\rangle_{\Lambda,\beta}^0 f(F, d_F), \quad (11.5.12)$$

where

$$K_F = \beta \sum_{\{x,y\} \in F} J_{x,y} \varphi_x \varphi_y. \quad (11.5.13)$$

Note that by Young's inequality, for any $(\varphi_x)_{x \in V(F)} \in \mathbb{R}^{V(F)}$,

$$e^{-K_F} \prod_{x \in V(F) \setminus \mathring{V}(F)} \varphi_x^{d_F(x)} \leq e^{\beta \left(\sup_{\{x,y\} \in F} J_{x,y} \right) \sum_{x \in V(F)} \varphi_x^2} \prod_{x \in V(F) \setminus \mathring{V}(F)} \varphi_x^{d_F(x)}. \quad (11.5.14)$$

The expectation of the righthand side of (11.5.14) is finite for $\langle \cdot \rangle_\beta^0$ (i.e. the infinite volume measure). Hence, the the lefthand side of (11.5.14) is integrable with respect to $\langle \cdot \rangle_\beta^0$. Since the lefthand side is a local function, the convergence as $\Lambda \uparrow V$ follows by the DLR equations.

The observations above yield that $\lim_{\Lambda \uparrow V} \mathbf{P}_\Lambda^\emptyset(\mathcal{C}_{F,d_F})$ exists and is equal to

$$\left\langle e^{-K_F} \prod_{x \in V(F) \setminus \mathring{V}(F)} \varphi_x^{d_F(x)} \right\rangle_\beta^0 f(F, d_F). \quad (11.5.15)$$

□

Proof of Proposition 11.5.2. We first prove the existence of P^0 . Let $A \in \mathcal{A}_V$ be a cylinder event. Consider a sequence of finite sets of vertices Λ converging to V . We show that $(P_\Lambda^\emptyset[A])_\Lambda$ is Cauchy. The limit does not depend on the sequence since, by Lemma 11.5.4, all limit points have the same values on events $\mathcal{C}_{F,d_F}$, for any $F \subset \mathcal{P}_2(V)$ finite and $d_F \in V(F)^{2\mathbb{N}}$.

Let $F \subset \mathcal{P}_2(V)$ finite and such that $A \in \mathcal{A}_{V(F)}$. Let $M \in \mathbb{N}$. Note that $\mathcal{C}_F := \{\mathcal{C}_{F',d_{F'}} : F' \subset F, d_{F'} \in V(F')^{2\mathbb{N}}\}$ is stable under intersection. Thus, by inclusion-exclusion, there exists $N = N(M) > 0$ such that

$$P_\Lambda^\emptyset \left[\mathcal{A} \cap \left\{ \sup_{x \in V(F)} \Delta \mathbf{n}_F(x) \leq M \right\} \right] = \sum_{n=1}^N p_{n,\Lambda}, \quad (11.5.16)$$

where $p_{n,\Lambda}$ are proportional to probabilities involving finite intersections and unions of events in $\mathcal{C}_F$. By Lemma 11.5.4, $p_{n,\Lambda} \rightarrow p_n$ as $\Lambda \uparrow V$.

On the other hand, note that

$$\sup_{\Lambda: V(F) \subset \Lambda \subset V} P_\Lambda^\emptyset \left[A \cap \left\{ \sup_{x \in V(F)} \Delta \mathbf{n}_F(x) \leq M \right\}^c \right] \quad (11.5.17)$$

$$\leq \sup_{\Lambda: V(F) \subset \Lambda \subset V} \sum_{x \in V(F)} P_\Lambda^\emptyset [\Delta \mathbf{n}_E(x) > M] \quad (11.5.18)$$

$$\leq \frac{1}{M} \sum_{x \in V(F)} \sup_{\Lambda: V(F) \subset \Lambda \subset V} E_\Lambda^\emptyset [\Delta \mathbf{n}(x)] \quad (11.5.19)$$

$$\leq \frac{C(F, \beta)}{M} \xrightarrow{M \rightarrow \infty} 0 \quad (11.5.20)$$

where the first inequality is by trivial inclusion and a union bound; the second inequality is by Markov's inequality; and the third inequality is by the moment bounds of $\Delta \mathbf{n}$ in Lemma 11.5.3 together with the convergence of $\langle \varphi_x^{2k} \rangle_\Lambda^0$ as $\Lambda \uparrow V$.

Hence, $(P_\Lambda^\emptyset[A])_\Lambda$ is Cauchy. Define $P^0[A]$ to be this limit. Since cylinder events generate $\mathcal{A}_V$, this extends to a unique probability measure on Ω_V .

Now we establish the invariance under Γ . By (11.5.15) and the fact that $\langle \cdot \rangle_\beta^0$ is invariant under Γ , we have that for all $x \in V$, $F \subset \mathcal{P}_2(V)$ finite, and $d_F \in V(F)^{2\mathbb{N}}$,

$$P^0[\gamma_x \mathcal{C}_{F,d_F}] = P^0[\mathcal{C}_{F,d_F}]. \quad (11.5.21)$$

Γ -invariance then follows by a truncation argument as above.

In order to establish mixing, notice that (again by a truncation argument) it suffices to show that for two finite sets $F, F' \subset E$,

$$\lim_{d_G(o,x) \rightarrow \infty} P^0[\mathcal{C}_{F,d_F} \cap \gamma_x \mathcal{C}_{F',d_{F'}}] = P^0[\mathcal{C}_{F,d_F}] P^0[\mathcal{C}_{F',d_{F'}}], \quad (11.5.22)$$

where $\gamma_x \in \Gamma$ is such that $\gamma_x \cdot o = x$.

Taking x such that $d_G(o, x)$ is large enough, we notice that

$$\mathcal{C}_{F,d_F} \cap \gamma_x \mathcal{C}_{F',d_{F'}} = \mathcal{C}_{F \cup (\gamma_x F'), d_F \cup d_{\gamma_x F'}}. \quad (11.5.23)$$

Thus,

$$\lim_{d_G(o,x) \rightarrow \infty} f(F \cup (\gamma_x F'), d_F \cup d_{\gamma_x F'}) = f(F, d_F) f(F', d_{F'}). \quad (11.5.24)$$

Moreover, by Proposition 11.2.9, the restriction of $\langle \cdot \rangle_\beta^0$ to even functions is ergodic under Γ . Hence.

$$\begin{aligned} \lim_{d_G(o,x) \rightarrow \infty} \left\langle \prod_{y \in V(F \cup (\gamma_x F')) \setminus \dot{V}(F \cup (\gamma_x F'))} \varphi_y^{(\Delta_F \cup \Delta_{\gamma_x F'})(x)} e^{-K_{F \cup (\gamma_x F')}} \right\rangle_\beta^0 \\ = \left\langle \prod_{y \in V(F) \setminus \dot{V}(F)} \varphi_y^{\Delta_F(x)} e^{-K_F} \right\rangle_\beta^0 \left\langle \prod_{y \in V(F') \setminus \dot{V}(F')} \varphi_y^{\Delta_{F'}(x)} e^{-K_{F'}} \right\rangle_\beta^0. \end{aligned} \quad (11.5.25)$$

Combining (11.5.24) and (11.5.25) yields (11.5.22). $\square$

Remark 11.5.5. Note that connections to the ghost vertex $\mathbf{g}$ disappear in the weak limit. Indeed, for any sourceless current $\mathbf{n}$ with $\mathbf{n}_{o,\mathbf{g}} > 0$, consider the configuration $\mathbf{n}'_{o,\mathbf{g}} = \mathbf{n}_{o,\mathbf{g}} - 1$ and $\mathbf{n}'_{x,y} = \mathbf{n}_{x,y}$ otherwise and notice that $w_{\emptyset,\beta,h_\Lambda(\mathbf{g})}(\mathbf{n}) \leq \beta h(\mathbf{g})_o w_{\{o\},\beta,h_\Lambda(\mathbf{g})}(\mathbf{n}')$. Summing over all such $\mathbf{n}$ we obtain

$$\mathbf{P}_{\Lambda_{\mathbf{g}}}^\emptyset[\mathbf{n}_{o,\mathbf{g}} > 0] \leq \beta h_\Lambda(\mathbf{g})_o \langle \varphi_o \rangle_\Lambda^p \rightarrow 0 \quad (11.5.26)$$

where the convergence is ensured by the integrability assumption on J .

In addition, no “long” edges remain in the weak limit. More precisely, fix $\Delta \subset V$ and take any n large enough such that $\Lambda := B_n \supset \Delta$. We claim that

$$\sum_{y \in \Lambda \setminus \Delta} \mathbf{P}_{\Lambda_{\mathbf{g}}}^\emptyset[\mathbf{n}_{o,y} > 0] \leq C(\Delta) \quad (11.5.27)$$

where $C(\Delta)$ is uniform over Λ and such that $C(\Delta) \rightarrow 0$ as $\Delta \uparrow V$. Indeed, proceeding as above, we have

$$\sum_{y \in \Lambda \setminus \Delta} \mathbf{P}_{\Lambda_{\mathbf{g}}}^\emptyset[\mathbf{n}_{o,y} > 0] \leq \sum_{y \in \Lambda \setminus \Delta} \beta J_{o,y} \langle \varphi_o \varphi_y \rangle_\Lambda^p \leq \beta \sqrt{\langle \varphi_o^2 \rangle_\Lambda^p} \sum_{y \in \Lambda \setminus \Delta} J_{o,y} \sqrt{\langle \varphi_y^2 \rangle_\Lambda^p} \quad (11.5.28)$$

where we used Cauchy–Schwarz in the last inequality. By Corollary 11.C.13 and Remark 11.C.14 there exists $C > 0$ such that $\langle \varphi_y^2 \rangle_\Lambda^p \leq \sqrt{C \max(\log(d_G(o,y)), 1)}$ uniformly in Λ . Hence, there exists $c_1 > 0$ such that, uniformly in Λ and Δ ,

$$\sum_{y \in \Lambda \setminus \Delta} \mathbf{P}_{\Lambda_{\mathbf{g}}}^\emptyset[\mathbf{n}_{o,y} > 0] \leq c_1 \sum_{y \in V \setminus \Delta} J_{o,y} \sqrt{C \max(\log(d_G(o,y)), 1)} =: C(\Delta) \quad (11.5.29)$$

After a summation by parts, the fact that $C(\Delta) \rightarrow 0$ is ensured by the integrability assumption on J , c.f. the calculation in Lemma 11.C.7.

The above considerations imply that certain global properties, such as the degree of a vertex in the random current remaining even, is preserved in the infinite volume limit. This can be seen as a strengthening of weak convergence of these measures to a type of local limit.

11.5.2 Infinite volume distributions on tangled currents

For $\Lambda \subset V$ finite, equip $\Omega_{\Lambda_{\mathbf{g}}}^\mathcal{T}$ with its natural Borel σ -algebra $\mathcal{A}_{\Lambda_{\mathbf{g}}}^\mathcal{T}$. Equip $\Omega_V^\mathcal{T}$ with the σ -algebra generated by local (tangled) events, denoted by $\mathcal{A}_V^\mathcal{T}$. We canonically identify measures on $\Omega_{\Lambda_{\mathbf{g}}}^\mathcal{T}$ with their pushforward under the natural injection $\mathcal{A}_{\Lambda_{\mathbf{g}}}^\mathcal{T} \hookrightarrow \mathcal{A}_V^\mathcal{T}$, by viewing the ghost as the contraction of Λ^c to a single vertex.

Definition 11.5.6. For $\Lambda \subset V$ finite, let $\mathbf{P}_{\Lambda_g, \Lambda}^{\emptyset, \emptyset}$, $\mathbf{P}_{\Lambda_g}^{\emptyset}$, and $\mathbf{P}_{\Lambda}^{\emptyset}$ be the probability measures on $\Omega_{\Lambda_g}^{\mathcal{T}}$ defined, for $A \in \mathcal{A}_{\Lambda_g}^{\mathcal{T}}$, by

$$\mathbf{P}_{\Lambda_g, \Lambda}^{\emptyset, \emptyset}[A] := \frac{\sum_{\mathbf{n}_1 \in \Omega_{\Lambda_g}, \mathbf{n}_2 \in \Omega_{\Lambda}} \mathbb{1}\{\partial \mathbf{n}_1 = \emptyset, \partial \mathbf{n}_2 = \emptyset\} w_{\emptyset, \beta, h_{\Lambda}(\mathbf{p})}(\mathbf{n}_1) w_{\emptyset, \beta}(\mathbf{n}_2) \rho_{\emptyset, \emptyset}^{\mathbf{n}_1, \mathbf{n}_2}[A]}{\sum_{\mathbf{n}_1 \in \Omega_{\Lambda_g}, \mathbf{n}_2 \in \Omega_{\Lambda}} \mathbb{1}\{\partial \mathbf{n}_1 = \emptyset, \partial \mathbf{n}_2 = \emptyset\} w_{\emptyset, \beta, h_{\Lambda}(\mathbf{p})}(\mathbf{n}_1) w_{\emptyset, \beta}(\mathbf{n}_2)} \quad (11.5.30)$$

$$\mathbf{P}_{\Lambda_g}^{\emptyset}[A] := \frac{\sum_{\mathbf{n}_1 \in \Omega_{\Lambda_g}} \mathbb{1}\{\partial \mathbf{n}_1 = \emptyset\} w_{\emptyset, \beta, h_{\Lambda}(\mathbf{p})}(\mathbf{n}_1) \rho_{\emptyset}^{\mathbf{n}_1}[A]}{\sum_{\mathbf{n}_1 \in \Omega_{\Lambda_g}} \mathbb{1}\{\partial \mathbf{n}_1 = \emptyset\} w_{\emptyset, \beta, h_{\Lambda}(\mathbf{p})}(\mathbf{n}_1)} \quad (11.5.31)$$

$$\mathbf{P}_{\Lambda}^{\emptyset}[A] := \frac{\sum_{\mathbf{n}_1 \in \Omega_{\Lambda}} \mathbb{1}\{\partial \mathbf{n}_1 = \emptyset\} w_{\emptyset, \beta, 0}(\mathbf{n}_1) \rho_{\emptyset}^{\mathbf{n}_1}[A]}{\sum_{\mathbf{n}_1 \in \Omega_{\Lambda}} \mathbb{1}\{\partial \mathbf{n}_1 = \emptyset\} w_{\emptyset, \beta, 0}(\mathbf{n}_1)}. \quad (11.5.32)$$

Proposition 11.5.7. The probability measures $\mathbf{P}_{\Lambda_g, \Lambda}^{\emptyset, \emptyset}$, $\mathbf{P}_{\Lambda_g}^{\emptyset}$, and $\mathbf{P}_{\Lambda}^{\emptyset}$ converge weakly as $\Lambda \uparrow V$, respectively, to probability measures $\mathbf{P}^{+, 0}$, $\mathbf{P}^+$, $\mathbf{P}^0$ on $\Omega_V^{\mathcal{T}}$. The convergence is in fact strong for cylinder events, i.e. for any $A \in \mathcal{A}_V^{\mathcal{T}}$ depending on finitely many edges and tanglings,

$$\lim_{\Lambda \uparrow V} \mathbf{P}_{\Lambda_g, \Lambda}^{\emptyset, \emptyset}[A] = \mathbf{P}^{+, 0}[A], \quad \lim_{\Lambda \uparrow V} \mathbf{P}_{\Lambda_g}^{\emptyset}[A] = \mathbf{P}^+[A], \quad \lim_{\Lambda \uparrow V} \mathbf{P}_{\Lambda}^{\emptyset}[A] = \mathbf{P}^0[A]. \quad (11.5.33)$$

Furthermore, $\mathbf{P}^{+, 0}$, $\mathbf{P}^+$, and $\mathbf{P}^0$ are invariant and mixing (hence, ergodic) under automorphisms in Γ .

Proof. Without loss of generality, we prove the result for $\mathbf{P}^{+, 0}$. First observe that $\mathbf{P}_{\Lambda_g, \Lambda}^{\emptyset, \emptyset}$ is obtained via the following coupling: first, sample $\mathbf{n} = \mathbf{n}_1 + \mathbf{n}_2$ according to the measure $\mathbf{P}_{\Lambda_g}^{\emptyset} \otimes \mathbf{P}_{\Lambda}^{\emptyset}$. Then, independently on each block $\mathcal{B}_x(\mathbf{n}, \emptyset, \emptyset)$ for $x \in \Lambda$, sample a tangling $\mathbf{t}_x$ according to the measure $\rho_{x, \emptyset, \emptyset}^{\mathbf{n}_1, \mathbf{n}_2}$ constructed in Theorem 11.3.11. Thus, to define $\mathbf{P}^{+, 0}$, we extend the coupling to infinite volume in the obvious way by using Proposition 11.5.2.

The Γ -invariance and mixing is a direct consequence of Proposition 11.5.2 and the construction via the coupling. $\square$

Remark 11.5.8. Observe that for any $A \in \mathcal{A}_V$, we have

$$\mathbf{P}^{+, 0}[A] = \mathbf{P}^+ \otimes \mathbf{P}^0[A], \quad \mathbf{P}^+[A] = \mathbf{P}^+[A], \quad \mathbf{P}^0[A] = \mathbf{P}^0[A] \quad (11.5.34)$$

where we have identified $\mathcal{A}_V \subset \mathcal{A}_V^{\mathcal{T}}$.

11.5.3 Uniqueness of the infinite cluster

We prove that samples under the measures $\mathbf{P}^{+, 0}$, $\mathbf{P}^+$, and $\mathbf{P}^0$, viewed as the graph $\mathcal{H}(\mathbf{n}, \mathbf{t})$ defined in Section 11.3.2, almost surely have at most one infinite cluster. We start by proving an insertion tolerance property for local events that are in $\mathcal{A}_V$, i.e. do not depend on the tanglings. This, together with Proposition 11.4.1, implies a (weak) insertion tolerance property for local events in $\mathcal{A}_V^{\mathcal{T}}$.

Definition 11.5.9. For $\Lambda \subset V$ finite, define $\varphi_{\Lambda} : \Omega_V \times \Omega_V \rightarrow \Omega_V \times \Omega_V$ as the map $(\mathbf{n}_1, \mathbf{n}_2) \mapsto (\mathbf{n}'_1, \mathbf{n}'_2)$, where, for $x, y \in V$,

$$(\mathbf{n}'_1, \mathbf{n}'_2)\{x, y\} = \begin{cases} (2, 0), & \text{if } (\mathbf{n}_1, \mathbf{n}_2)\{x, y\} = (0, 0), x, y \in \Lambda \text{ and } J_{x, y} > 0, \\ (\mathbf{n}_1, \mathbf{n}_2)\{x, y\}, & \text{otherwise.} \end{cases} \quad (11.5.35)$$

In particular, $\mathbf{n}'_2 = \mathbf{n}_2$.

Proposition 11.5.10. *For $\Lambda \subset V$ finite, there exists $c = c(\Lambda, \beta, J) > 0$ such that, for all $A \in \mathcal{A}_V$,*

$$\mathbf{P}^+ \otimes \mathbf{P}^\emptyset[\varphi_\Lambda(A)] \geq c \mathbf{P}^+ \otimes \mathbf{P}^\emptyset[A]. \quad (11.5.36)$$

Proof. It is sufficient to consider A depending only on finitely many edges. Recall that B_n is the ball of radius n centred on the fixed origin $o \in V$. Let N be such that $A \in \mathcal{A}_{B_N}$. We show the estimate (11.5.36) holds for the measures $\mathbf{P}_{B_n \sqcup \{\mathfrak{g}\}}^\emptyset \otimes \mathbf{P}_{B_n}^\emptyset$ with constant uniform in $n > N$. The result then follows by Proposition 11.5.2.

Note that for every $(\mathbf{n}'_1, \mathbf{n}'_2)$ we have $|\varphi_\Lambda^{-1}(\mathbf{n}'_1, \mathbf{n}'_2)| \leq 2^{|\mathcal{P}_2(\Lambda)|}$, hence

$$\mathbf{P}_{B_n \sqcup \{\mathfrak{g}\}}^\emptyset \otimes \mathbf{P}_{B_n}^\emptyset[\varphi_\Lambda(A)] \geq \quad (11.5.37)$$

$$\frac{2^{-|\mathcal{P}_2(\Lambda)|}}{Z_{B_n \sqcup \{\mathfrak{g}\}, B_n}^{\emptyset, \emptyset}} \sum_{\substack{\mathbf{n}_1 \in \Omega_{B_n \sqcup \{\mathfrak{g}\}}, \mathbf{n}_2 \in \Omega_{B_n} \\ \partial \mathbf{n}_1 = \emptyset, \partial \mathbf{n}_2 = \emptyset}} w_{\emptyset, \beta}(\mathbf{n}'_1) w_{\emptyset, \beta}(\mathbf{n}'_2) \mathbf{1}\{\mathbf{n} \in A\} \quad (11.5.38)$$

where we recall $\varphi_\Lambda(\mathbf{n}_1, \mathbf{n}_2) = (\mathbf{n}'_1, \mathbf{n}'_2)$, and $\mathbf{n} = \mathbf{n}_1 + \mathbf{n}_2$. By definition,

$$w_{\emptyset, \beta}(\mathbf{n}'_1) w_{\emptyset, \beta}(\mathbf{n}'_2) = w_{\emptyset, \beta}(\mathbf{n}_1) w_{\emptyset, \beta}(\mathbf{n}_2) \prod_{x \in \Lambda} \frac{\langle \varphi^{\Delta \mathbf{n}'_1(x)} \rangle_0}{\langle \varphi^{\Delta \mathbf{n}_1(x)} \rangle_0} \prod_{\{x, y\} \subset \Lambda} \frac{(J_{x, y})^2}{2}, \quad (11.5.39)$$

where the final product is over all $x, y \in \Lambda$ such that $J_{x, y} > 0$ and $(\mathbf{n}_1, \mathbf{n}_2)\{x, y\} = (0, 0)$. By Proposition 11.A.5, there exists $c_1 > 0$ such that, for all $k \geq 0$, $\frac{\langle \varphi_x^{2k+2} \rangle_0}{\langle \varphi_x^{2k} \rangle_0} \geq c_1$. Since $\Delta \mathbf{n}_1 \leq \Delta \mathbf{n}'_1(x) \leq \Delta \mathbf{n}_1(x) + 2|\Lambda|$, we conclude that

$$\prod_{x \in \Lambda} \frac{\langle \varphi^{\Delta \mathbf{n}'_1(x)} \rangle_0}{\langle \varphi^{\Delta \mathbf{n}_1(x)} \rangle_0} \geq \prod_{x \in \Lambda} \min(1, c_1^{|\Lambda|}) = \min(1, c_1^{|\Lambda|^2}) > 0. \quad (11.5.40)$$

Inserting (11.5.39) and (11.5.40) into (11.5.37), we get the existence of $c = c(\Lambda, \beta, J)$ such that, for all $n > N$,

$$\mathbf{P}_{B_n \sqcup \{\mathfrak{g}\}}^\emptyset \otimes \mathbf{P}_{B_n}^\emptyset[\varphi_\Lambda(A)] \geq c \mathbf{P}_{B_n \sqcup \{\mathfrak{g}\}}^\emptyset \otimes \mathbf{P}_{B_n}^\emptyset[A]. \quad (11.5.41)$$

Taking $n \rightarrow \infty$ concludes the proof. $\square$

Definition 11.5.11. For $\Lambda \subset V$ finite, define $\varphi_\Lambda^\mathcal{T} : \Omega_V^\mathcal{T} \rightarrow \Omega_V^\mathcal{T}$ as the map $(\mathbf{n}, \mathbf{t}) \mapsto (\mathbf{n}', \mathbf{t}')$, given by $\mathbf{n}' = \varphi_\Lambda(\mathbf{n})$ and

$$\mathbf{t}'_x = \begin{cases} \{\mathcal{B}_x(\mathbf{n}')\}, & \text{if } x \in \Lambda, \\ \mathbf{t}_x, & \text{otherwise,} \end{cases} \quad (11.5.42)$$

i.e. in $(\mathbf{n}', \mathbf{t}')$, the event $\text{ECT}(x)$ happens for all $x \in \Lambda$.

Proposition 11.5.12. *If $A \in \mathcal{A}_V$ is such that $\mathbf{P}^{+,0}[A] > 0$, then $\mathbf{P}^{+,0}[\varphi_\Lambda^\mathcal{T}(A)] > 0$ for every $\Lambda \subset V$ finite.*

Proof. Recall that, conditionally on $\mathbf{n}$, the tanglings $\mathbf{t} = (\mathbf{t}_x)_{x \in V}$ are independent and the distribution of $\mathbf{t}_x$ only depends on $(\Delta \mathbf{n}_1(x), \Delta \mathbf{n}_2(x))$. Therefore, the result readily follows from Propositions 11.4.1 and 11.5.10. $\square$

Let $\mathfrak{N}$ denote the random variable that counts the number of infinite clusters in $\mathcal{H}(\mathbf{n}, \mathbf{t})$. Define for $\ell \in \mathbb{N} \cup \{\infty\}$,

$$\mathcal{E}_\ell := \{\mathfrak{N} = \ell\}. \quad (11.5.43)$$

Note that by ergodicity one has that for all ℓ , $\mathbf{P}^{+,0}[\mathcal{E}_\ell] \in \{0, 1\}$. The following result establishes that almost surely there exists at most one infinite cluster.

Proposition 11.5.13 (Uniqueness of infinite cluster). *We have*

$$\mathbf{P}^{+,0}[\mathcal{E}_0] + \mathbf{P}^{+,0}[\mathcal{E}_1] = 1. \quad (11.5.44)$$

The same holds for $\mathbf{P}^+$ and $\mathbf{P}^0$.

Proof. We only prove the result for $\mathbf{P}^{+,0}$, as the proof for $\mathbf{P}^+$ and $\mathbf{P}^0$ is analogous.

We start by constructing a convenient sequence of finite domains $(\Lambda_N)_N$ as follows. For each $x \in V$, consider a sequence of vertices $\pi_{o,x} = (u_0 = o, u_1, \dots, u_k = x)$ with $J_{u_i, u_{i+1}} > 0$ for all $0 \leq i \leq k-1$, which exists by the irreducibility assumption. We then consider the domains $\Lambda_N := \bigcup_{x \in B_N} \pi_x$.

We first establish that $\mathbf{P}^{+,0}[\mathcal{E}_\ell] = 0$ for $\ell \in \mathbb{N} \setminus \{0, 1\}$. Let $\ell \in \mathbb{N}$ with $\ell \geq 2$ and assume by contradiction that $\mathbf{P}^{+,0}[\mathcal{E}_\ell] = 1$. Let $\mathcal{I}_N = \mathcal{I}_N(\ell) \subset \mathcal{E}_\ell$ be the event that the ℓ infinite clusters intersect the blocks of the vertices in B_N . Observe that $\mathbf{P}^{+,0}[\mathcal{I}_N] \xrightarrow{N \rightarrow \infty} 1$, so in particular there exists N such that $\mathbf{P}^{+,0}[\mathcal{I}_N] > 0$. By Proposition 11.5.12, we have

$$\mathbf{P}^{+,0}[\varphi_{\Lambda_N}^{\mathcal{I}_N}(\mathcal{I}_N)] > 0. \quad (11.5.45)$$

However, one can easily see that $\varphi_{\Lambda_N}^{\mathcal{I}_N}(\mathcal{I}_N) \subset \mathcal{E}_1$, and therefore $\mathbf{P}^{+,0}[\mathcal{E}_1] > 0$, which is a contradiction with $\mathbf{P}^{+,0}[\mathcal{E}_\ell] = 1$, as desired.

Now we establish $\mathbf{P}^{+,0}[\mathcal{E}_\infty] = 0$. Assume for a contradiction that $\mathbf{P}^{+,0}[\mathcal{E}_\infty] = 1$. Let $\mathcal{J}_N$ be the event that at least three distinct infinite clusters intersect the blocks of the vertices in B_N . Note that $\mathbf{P}^{+,0}[\mathcal{J}_N] \xrightarrow{N \rightarrow \infty} 1$, hence there is some N , fixed throughout, such that

$$\mathbf{P}^{+,0}[\mathcal{J}_N] > 0. \quad (11.5.46)$$

Given $\Lambda \subset V$ and a tangled current $(\mathbf{n}, \mathbf{t})$, we denote by $\mathcal{L}(\Lambda, \mathbf{n}, \mathbf{t})$ the lift of Λ in $\mathcal{H}(\mathbf{n}, \mathbf{t})$, i.e. the subgraph of $\mathcal{H}(\mathbf{n}, \mathbf{t})$ spanned by $\bigcup_{x \in \Lambda} \mathcal{B}_x(\mathbf{n})$. We introduce the following notion of coarse trifurcation. Let $\mathbf{CT}_o$ denote the set of $(\mathbf{n}, \mathbf{t})$ such that:

- $\mathbf{n}(x, y) > 0$ for all edges $xy \in \Lambda_N$.
- $\text{ECT}(x)$ happens for all vertices $x \in \Lambda_N$.
- The cluster of the origin in $\mathcal{H}(\mathbf{n}, \mathbf{t})$ breaks into at least three distinct infinite clusters by removing $\mathcal{L}(\Lambda_N, \mathbf{n}, \mathbf{t})$.

One can easily verify that $\varphi_{\Lambda_N}(\mathcal{J}_N) \subset \mathbf{CT}_o$, hence by Proposition 11.5.12 we have

$$\mathbf{P}^{+,0}[\mathbf{CT}_o] > 0. \quad (11.5.47)$$

Furthermore, note that by Γ -invariance, we have $\mathbf{P}^{+,0}[\mathbf{CT}_0] = \mathbf{P}^{+,0}[\mathbf{CT}_x]$ for all $x \in V$, where $\mathbf{CT}_x = \gamma_x \mathbf{CT}_0$, and $\gamma_x \in \Gamma$ is an automorphism such that $\gamma_x o = x$. A vertex $x \in V$ is called a coarse trifurcation if $\mathbf{CT}_x$ occurs.

For each $x \in V$ denote $\Lambda_N(x) = \gamma_x \Lambda_N$. Let $n \in \mathbb{N}$ and define $S_n \subset B_n$ to be a set of (sparse) vertices such that $\Lambda_N(x) \cap \Lambda_N(y) = \emptyset$ for all distinct $x, y \in S_n$, and S_n is maximal with respect to this property. Then

$$|S_n| \geq \frac{|B_n|}{|B_K|}, \quad (11.5.48)$$

where $K > N$ is large enough so that $\Lambda_N \subset B_K$. Let $T_n = T(S_n)$ denote the set of coarse trifurcations in S_n , and L_n denote the set of edges in $\mathcal{H}(\mathbf{n}, \mathbf{t})$ between $\mathcal{L}(B_n, \mathbf{n}, \mathbf{t})$ and $\mathcal{L}(V \setminus B_n, \mathbf{n}, \mathbf{t})$. There is a natural way of constructing a forest where every leaf corresponds to an element of L_n , while every element of T_n corresponds to a vertex of degree at least three.

To see this, define a new (random) graph H as follows. For $t \in T_n$, we identify all vertices of $\mathcal{L}(\Lambda_N(t), \mathbf{n}, \mathbf{t})$ into a single vertex $v(t)$. The vertex set of H is then given by the union of the vertices in $\mathcal{H}(\mathbf{n}, \mathbf{t}) \setminus \bigcup_{t \in T_n} \mathcal{L}(\Lambda_N(t))$ together with the vertices $\{v(t) : t \in T_n\}$. The edge set of H is the natural edge set induced from $\mathcal{H}(\mathbf{n}, \mathbf{t})$ under the equivalence relation defining $v(t)$. Note that for all $t \in T_n$, $v(t)$ is a trifurcation in H , i.e. the cluster of $v(t)$ in H splits into at least 3 infinite clusters when we remove $v(t)$. We now restrict H to the graph H' which is induced by the edges with at least one endpoint in $\Pi(B_n)$, where Π is the natural map $G \mapsto H$. Take a spanning forest of H' and successively delete vertices of degree 1 from the spanning forest whose pre-image lies $\mathcal{L}(B_n, \mathbf{n}, \mathbf{t})$; note that this does not affect the vertices $v(t)$, which still have degree at least 3. Moreover, the only leaves of the resulting graph correspond to elements of L_n .

Therefore, we have

$$|T_n| \leq |L_n|. \quad (11.5.49)$$

On one hand, by (11.5.47) and Γ -invariance,

$$\mathbf{E}^{+,0}[|T_n|] = |S_n| \mathbf{P}^{+,0}[\text{CT}_0] \geq c(N)|B_n|, \quad (11.5.50)$$

for some constant $c(N) > 0$. On the other hand, for every $m > 0$ we have, by Γ -invariance,

$$\mathbf{E}^{+,0}[|L_n|] = \sum_{\substack{x \in B_n \\ y \in V \setminus B_n}} \mathbf{E}^{+,0}[\mathbf{n}_{x,y}] \quad (11.5.51)$$

$$\leq \sum_{x \in \partial_{\text{in}}^m B_n} \mathbf{E}^{+,0}[\Delta \mathbf{n}(x)] + \sum_{x \in B_n \setminus \partial_{\text{in}}^m B_n} \mathbf{E}^{+,0}[\Delta^m \mathbf{n}(x)] \quad (11.5.52)$$

$$\leq |\partial_{\text{in}}^m B_n| \mathbf{E}^{+,0}[\Delta \mathbf{n}(o)] + |B_n| \mathbf{E}^{+,0}[\Delta^m \mathbf{n}(o)] \quad (11.5.53)$$

where $\partial_{\text{in}}^m B_n := \{x \in B_n \mid d_G(x, V \setminus B_n) \leq m\}$ and $\Delta^m \mathbf{n}(x) := \sum_{y \in V: d_G(x,y) > m} \mathbf{n}_{x,y}$. Notice that $|\partial_{\text{in}}^m B_n| \leq |B_m| |\partial B_n|$. Moreover, recall that $c_0 := \mathbf{E}^{+,0}[\Delta \mathbf{n}(o)] < \infty$ by Lemma 11.5.3. In particular, $c_m := \mathbf{E}^{+,0}[\Delta^m \mathbf{n}(o)] \rightarrow 0$ as $m \rightarrow \infty$. Altogether, we have

$$\mathbf{E}^{+,0}[|L_n|] \leq c_0 |B_m| |\partial B_n| + c_m |B_n| \quad (11.5.54)$$

Combining (11.5.49), (11.5.50) and (11.5.54), we obtain

$$c(N) \leq c_0 |B_m| \frac{|\partial B_n|}{|B_n|} + c_m. \quad (11.5.55)$$

Letting $n \rightarrow \infty$ and then $m \rightarrow \infty$, we find that $c(N) \leq 0$, thus obtaining the desired contradiction. $\square$

Remark 11.5.14. As a byproduct of the above proof we also obtain that the infinite cluster (if it exists) in $(\mathbf{n}, \mathbf{t})$ has at most two ends almost surely. Indeed, if the infinite cluster has at least three ends, one can proceed as above to prove that coarse trifurcations happen with positive probability and then obtain a contradiction.

11.6 Γ -invariant Gibbs measures

In this section, we prove Theorem 11.1.2. Fix $\beta > 0$, which we may sometimes omit from the notation without risk of confusion. By Theorem 11.2.15, we first need to prove that the condition 4 therein holds. A straightforward argument then yields that under (11.1.12), $|\mathcal{G}(\beta)| = 1$ to complete the proof. We therefore make the following assumption for the rest of this section, and in the end, we derive a contradiction.

Assumption 1. Assume that there exist $x, y \in V$ such that $J_{x,y} > 0$ and either of the two following things happens

$$\langle \varphi_x \varphi_y \rangle_\beta^+ \neq \langle \varphi_x \varphi_y \rangle_\beta^0 \quad \text{or} \quad \langle \varphi_x^2 \rangle_\beta^+ \neq \langle \varphi_x^2 \rangle_\beta^0. \quad (\text{H})$$

Under this assumption, we show that the following event happens with positive probability.

Definition 11.6.1 (Bridges). We say that a pair $x, y \in V$ is a bridge for a tangled current $(\mathbf{n}, \mathbf{t})$ if there exist two vertices $\bar{x}$ and $\bar{y}$ in the blocks of x and y in the multigraph $\mathcal{H}(\mathbf{n}, \mathbf{t})$ such that there is exactly one edge e between $\bar{x}$ and $\bar{y}$, and furthermore the cluster of $\bar{x}, \bar{y}$ splits into two disjoint infinite clusters after removing e . We denote the event that $\{x, y\}$ is a bridge by $\mathcal{A}_{x,y}$.

In what follows, we write $A \longleftrightarrow B$ to denote the event that A is connected to B in $\mathcal{H}(\mathbf{n}, \mathbf{t})$. To simplify the notation, for $x, y \in V$, we write $x \longleftrightarrow y$ instead of $\mathcal{B}_x \longleftrightarrow \mathcal{B}_y$. Moreover, we identify a moment $A \in \mathcal{M}(V)$ with the corresponding multiset it defines, where each $x \in V$ appears as many times as the value of A_x , e.g. we identify the moment A defined as $A_x = A_y = 2$ and $A_z = 0$ otherwise, with $xyxy$.

Proposition 11.6.2. Assume (H). Then,

$$\mathbf{P}_\beta^{+,0}[\mathcal{A}_{x,y}] > 0 \text{ and } \mathbf{P}_\beta^+[\mathcal{A}_{x,y}] > 0. \quad (11.6.1)$$

Proof. Let $\mathcal{A}_{x,y}^f$ denote the event that there exist two vertices $\bar{x}$ and $\bar{y}$ in the blocks of x and y in the multigraph $\mathcal{H}(\mathbf{n}, \mathbf{t})$ such that there is exactly one edge e between $\bar{x}$ and $\bar{y}$, and furthermore the cluster of $\bar{x}, \bar{y}$ splits into two disjoint clusters after removing e that are both connected to $\mathbf{g}$. By Proposition 11.5.7 and Remark 11.5.5, it suffices to show that there exists $c > 0$ such that $\mathbf{P}_{B_N \sqcup \{\mathbf{g}\}, B_N}^{\emptyset, \emptyset}[\mathcal{A}_{x,y}^f] > c$ and $\mathbf{P}_{B_N \sqcup \{\mathbf{g}\}}^\emptyset[\mathcal{A}_{x,y}^f] > c$ for every N large enough.

We treat the two cases in (H) separately. First assume that $\langle \varphi_x \varphi_y \rangle_\beta^+ \neq \langle \varphi_x \varphi_y \rangle_\beta^0$. Let $c := \langle \varphi_x \varphi_y \rangle_\beta^+ - \langle \varphi_x \varphi_y \rangle_\beta^0 > 0$ and take N large enough so that $\langle \varphi_x \varphi_y \rangle_{B_N}^{\mathbf{p}} - \langle \varphi_x \varphi_y \rangle_{B_N}^0 > c/2$. By the switching principle of Theorem 12.4.2, we have

$$c/2 < \langle \varphi_x \varphi_y \rangle_{B_N}^{\mathbf{p}} - \langle \varphi_x \varphi_y \rangle_{B_N}^0 = \langle \varphi_x \varphi_y \rangle_{B_N}^{\mathbf{p}} \mathbf{P}_{B_N \sqcup \{\mathbf{g}\}, B_N}^{xy, \emptyset}[\tilde{x} \not\longleftrightarrow \tilde{y} \text{ in } B_N], \quad (11.6.2)$$

where $\tilde{x} = xa(1), \tilde{y} = ya(1)$ are the extra vertices added in $\mathcal{B}_x, \mathcal{B}_y$ due to the sources. Notice that due to the sources constraint, $\tilde{x}$ is connected to $\tilde{y}$ in $B_N \sqcup \{\mathbf{g}\}$, in particular on the event

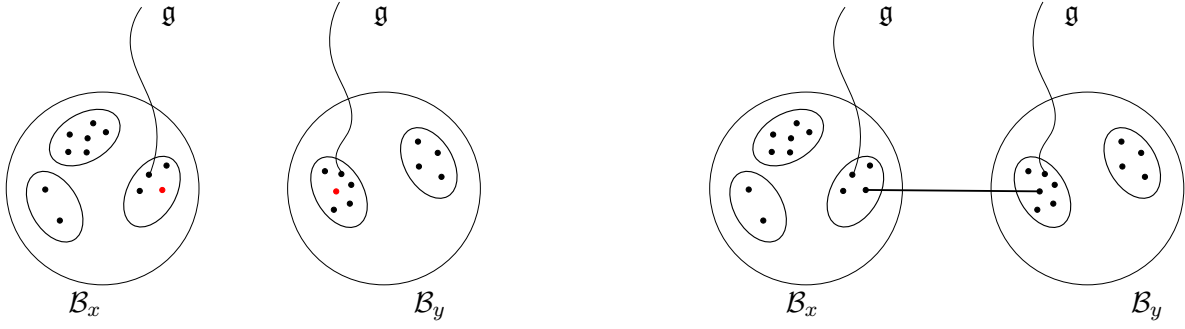


Figure 11.3: The mapping in the first case. On the left, an example of $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{t})$; on the right, its image $(\mathbf{n}'_1, \mathbf{n}_2, \mathbf{t}')$ by the map. The red vertices represent $\tilde{x}, \tilde{y}$. One can see that $\{\tilde{x}, \tilde{y}\}$ is a bridge for the associated multigraph $\mathcal{H}(\mathbf{n}'_1 + \mathbf{n}_2, \mathbf{t}')$.

in (11.6.2) both $\tilde{x}$ and $\tilde{y}$ are connected to $\mathbf{g}$, and therefore $\{x, y\}$ would become a bridge by adding an edge between $\tilde{x}$ and $\tilde{y}$ in $\mathcal{H}$. We now implement this modification rigourously. First, take M large enough (depending on c only) such that

$$\mathbf{P}_{B_N \sqcup \{\mathbf{g}\}, B_N}^{xy, \emptyset}[\tilde{x} \not\leftrightarrow \tilde{y} \text{ in } B_N, \Delta \mathbf{n}(x) \leq M, \Delta \mathbf{n}(y) \leq M] > c/4. \quad (11.6.3)$$

To each $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{t})$ in the event above, we consider the modified configuration $(\mathbf{n}'_1, \mathbf{n}_2, \mathbf{t}')$ given constructed as follows. First, set $\mathbf{n}'_1(x, y) = \mathbf{n}_1(x, y) + 1$ and $\mathbf{n}'_1(e) = \mathbf{n}_1(e) \forall e \neq \{x, y\}$ (notice that $\partial \mathbf{n}'_1 = \emptyset$). In order to define $\mathbf{t}'$, first set $\mathbf{t}'_z = \mathbf{t}_z \forall z \notin \{x, y\}$, which makes sense as $\mathcal{B}_z(\mathbf{n}_1, \mathbf{n}_2, xy, \emptyset) = \mathcal{B}_z(\mathbf{n}'_1, \mathbf{n}_2, \emptyset, \emptyset) \forall z \notin \{x, y\}$. By seeing the previously extra vertices $\tilde{x}, \tilde{y}$ now as the endpoints of the added edge between x and y in $\mathbf{n}'_1$, we construct a natural bijection $\psi_z : \mathcal{B}_z(\mathbf{n}_1, \mathbf{n}_2, xy, \emptyset) \rightarrow \mathcal{B}_z(\mathbf{n}'_1, \mathbf{n}_2, \emptyset, \emptyset)$ for $z \in \{x, y\}$. We can then set $\mathbf{t}'_x = \psi_x(\mathbf{t}_x)$ and $\mathbf{t}'_y = \psi_y(\mathbf{t}_y)$. One can see that $\{x, y\}$ is a bridge in the configuration $(\mathbf{n}'_1, \mathbf{n}_2, \mathbf{t}')$, with $\bar{x} = \tilde{x}$ and $\bar{y} = \tilde{y}$ – see Figure 11.3. On the one hand, notice that the map $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{t}) \mapsto (\mathbf{n}'_1, \mathbf{n}_2, \mathbf{t}')$ is at most $(M+1)^2$ -to-1 due to the degree constraints. On the other hand, by (12.4.12) and Proposition 11.4.1, the ratio between the probabilities of $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{t})$ and $(\mathbf{n}'_1, \mathbf{n}_2, \mathbf{t}')$ under $\mathbf{P}_{B_N \sqcup \{\mathbf{g}\}, B_N}^{xy, \emptyset}$ and $\mathbf{P}_{B_N \sqcup \{\mathbf{g}\}, B_N}^{\emptyset, \emptyset}$, respectively, is bounded above by a constant $C = C(M) \in (0, \infty)$. All in all, we conclude that for N large enough,

$$\mathbf{P}_{B_N \sqcup \{\mathbf{g}\}, B_N}^{\emptyset, \emptyset}[\mathcal{A}_{x,y}^f] > \frac{c}{4C(M+1)^2}, \quad (11.6.4)$$

as we wanted to prove. For the single tangled current, simply notice that by the stochastic domination (11.3.81) we have $\mathbf{P}_{B_N \sqcup \{\mathbf{g}\}}^{xy}[\tilde{x} \not\leftrightarrow \tilde{y} \text{ in } B_N] \geq \mathbf{P}_{B_N \sqcup \{\mathbf{g}\}, B_N}^{xy, \emptyset}[\tilde{x} \not\leftrightarrow \tilde{y} \text{ in } B_N] > c/2$, and the above proof adapts readily.

We now assume that $\langle \varphi_x^2 \rangle_\beta^+ \neq \langle \varphi_x^2 \rangle_\beta^0$. We proceed similarly to the first case, except that our mapping construction is slightly different. First, as before, we can find constants $c, M \in (0, \infty)$ such that for all N sufficiently large we have

$$\mathbf{P}_{B_N \sqcup \{\mathbf{g}\}, B_N}^{xx, \emptyset}[\tilde{x}_1 \not\leftrightarrow \tilde{x}_2 \text{ in } B_N, \Delta \mathbf{n}(x) \leq M, \Delta \mathbf{n}(y) \leq M] > c,$$

where $\tilde{x}_1 = xa(1), \tilde{x}_2 = xa(2)$ are the extra vertices added in $\mathcal{B}_x$ due to the sources. To each $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{t})$ in the event above we consider the modified configuration $(\mathbf{n}'_1, \mathbf{n}_2, \mathbf{t}')$ constructed

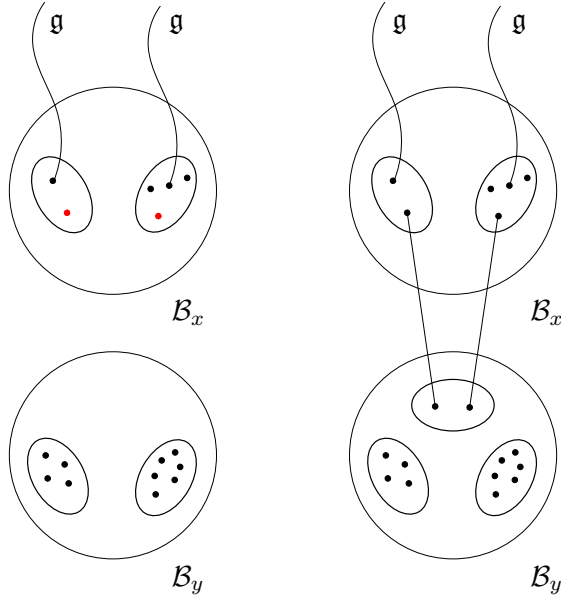


Figure 11.4: The mapping in the second case. On the left, an example of $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{t})$; on the right, its image $(\mathbf{n}'_1, \mathbf{n}_2, \mathbf{t}')$ by the map. The red vertices represent $\tilde{x}_1, \tilde{x}_2$. One can see that both edges corresponding to $\tilde{x}_1$ and $\tilde{x}_2$ are bridges for $\mathcal{H}(\mathbf{n}'_1 + \mathbf{n}_2, \mathbf{t}')$.

as follows. First, let y be such that $J_{x,y} > 0$. Now set $\mathbf{n}'_1(x, y) = \mathbf{n}_1(x, y) + 2$ and $\mathbf{n}'_1(e) = \mathbf{n}_1(e) \forall e \neq \{x, y\}$ (notice that $\partial \mathbf{n}'_1 = \emptyset$). Notice that $\mathcal{B}_z(\mathbf{n}'_1, \mathbf{n}_2, \emptyset, \emptyset) = \mathcal{B}_z(\mathbf{n}_1, \mathbf{n}_2, xx, \emptyset) \forall z \notin \{x, y\}$, and we can set $\mathbf{t}'_z = \mathbf{t}_z \forall z \notin \{x, y\}$. As before, we can see the previously extra vertices $\tilde{x}_1, \tilde{x}_2$ in $\mathcal{B}_x(\mathbf{n}_1, \mathbf{n}_2, xx, \emptyset)$ as the endpoints in $\mathcal{B}_x(\mathbf{n}'_1, \mathbf{n}_2, \emptyset, \emptyset)$ corresponding to the two added edges between x and y in $\mathbf{n}'_1$. This leads to a natural bijection $\psi_x : \mathcal{B}_x(\mathbf{n}_1, \mathbf{n}_2, xx, \emptyset) \rightarrow \mathcal{B}_x(\mathbf{n}'_1, \mathbf{n}_2, \emptyset, \emptyset)$, and we set $\mathbf{t}'_x = \psi_x(\mathbf{t}_x)$. Notice that $\mathcal{B}_y(\mathbf{n}'_1, \mathbf{n}_2, \emptyset, \emptyset) = \mathcal{B}_y(\mathbf{n}_1, \mathbf{n}_2, xx, \emptyset) \cup \{\tilde{x}_1, \tilde{x}_2\}$, where $\tilde{y}_1$ and $\tilde{y}_2$ are the endpoints in the block of y induced by the two added edges (say with the other endpoints being $\tilde{x}_1$ and $\tilde{x}_2$, respectively). Finally, we set $\mathbf{t}'_y = \mathbf{t}_y \cup \{\{\tilde{y}_1, \tilde{y}_2\}\}$. One can then see that $\{x, y\}$ is a bridge in the configuration $(\mathbf{n}'_1, \mathbf{n}_2, \mathbf{t}')$ (with, say, $\bar{x} = \tilde{x}_1$ and $\bar{y} = \tilde{y}_1$) – see Figure 11.4. As before, the map $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{t}) \mapsto (\mathbf{n}'_1, \mathbf{n}_2, \mathbf{t}')$ is at most $(M+2)^2 M^2$ -to-1 and the ratio between the probabilities of $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{t})$ and $(\mathbf{n}'_1, \mathbf{n}_2, \mathbf{t}')$ under $\mathbf{P}_{B_N \sqcup \{\mathbf{g}\}, B_N}^{xx, \emptyset}$ and $\mathbf{P}_{B_N \sqcup \{\mathbf{g}\}, B_N}^{\emptyset, \emptyset}$, respectively, is bounded above by a constant $C = C(M) \in (0, \infty)$. All in all, we conclude that for N large enough

$$\mathbf{P}_{B_N \sqcup \{\mathbf{g}\}, B_N}^{\emptyset, \emptyset}[\mathcal{A}_{x,y}^f] > \frac{c}{C(M+2)^2 M^2}. \quad (11.6.5)$$

as we wanted to prove. Once again, the proof for the single tangled current follows the same lines as above. $\square$

Definition 11.6.3 (Flows). Given two vertices $\tilde{x}, \tilde{y}$ in a multigraph $\mathcal{H}$, we define the flow $f(\tilde{x}, \tilde{y}, \mathcal{H})$ between $\tilde{x}$ and $\tilde{y}$ in $\mathcal{H}$ as the maximal number of edge disjoint paths between $\tilde{x}$ and $\tilde{y}$ in $\mathcal{H}$. We also define $f(\tilde{x}, \infty, \mathcal{H})$ as the maximal number of edge disjoint infinite path in $\mathcal{H}$ starting from $\tilde{x}$. For a tangled current $(\mathbf{n}, \mathbf{t})$ on V and $x, y \in V$, we denote by $f(x, y, \mathbf{n}, \mathbf{t})$ the

maximum of $f(\tilde{x}, \tilde{y}, \mathcal{H}(\mathbf{n}, \mathbf{t}))$ over all $\tilde{x} \in \mathcal{B}_x$ and $\tilde{y} \in \mathcal{B}_y$, and by $f(x, \infty, \mathbf{n}, \mathbf{t})$ the maximum of $f(\tilde{x}, \infty, \mathcal{H}(\mathbf{n}, \mathbf{t}))$ over all $\tilde{x} \in \mathcal{B}_x$.

Lemma 11.6.4. *Assume (H). For $\mathbf{P} = \mathbf{P}_\beta^{+,0}$ and $\mathbf{P}_\beta^+$, one has*

$$\lim_{d_G(o,y) \rightarrow \infty} \mathbf{P}[f(x, y, \mathbf{n}, \mathbf{t}) \geq 2] = 0, \quad (11.6.6)$$

$$\mathbf{P}[f(x, \infty, \mathbf{n}, \mathbf{t}) > 2] = 0. \quad (11.6.7)$$

Proof. Note that by ergodicity and Proposition 11.6.2, almost surely there are infinitely many bridges. Also, there is a unique infinite cluster, which has at most two ends almost surely by Proposition 11.5.13 and Remark 11.5.14. For any configuration $(\mathbf{n}, \mathbf{t})$ satisfying the properties above, it is proved in [211, Lemma 3] that $f(x, \infty, \mathbf{n}, \mathbf{t}) \leq 2$ and $\lim_{d_G(o,y) \rightarrow \infty} \mathbb{1}\{f(x, y, \mathbf{n}, \mathbf{t}) \geq 2\} = 0$. Then, (11.6.6) follows by dominated convergence and (11.6.7) follows directly. $\square$

Lemma 11.6.5. *Let $x \in V$ and $\beta > 0$. If $\mathbf{P}_\beta^{+,0}[x \longleftrightarrow \infty] > 0$, then $\inf_{y \in V} \langle \varphi_x \varphi_y \rangle_\beta^0 > 0$.*

Proof. Since β is fixed, we shall omit it from the notation. By the uniqueness of the infinite cluster, for every $y \in V$,

$$\mathbf{P}^{+,0}[x \longleftrightarrow y] = \mathbf{P}^{+,0}[x \longleftrightarrow \infty, y \longleftrightarrow \infty] \quad (11.6.8)$$

$$+ \mathbf{P}^{+,0}[x \longleftrightarrow y, x \not\longleftrightarrow \infty]. \quad (11.6.9)$$

The first term converges to $\left(\mathbf{P}^{+,0}[x \longleftrightarrow \infty]\right)^2$ by the mixing of Proposition 11.5.7, while the second term converges to 0. Thus

$$\lim_{d_G(o,y) \rightarrow \infty} \mathbf{P}^{+,0}[x \longleftrightarrow y] = \left(\mathbf{P}^{+,0}[x \longleftrightarrow \infty]\right)^2, \quad (11.6.10)$$

hence $\mathbf{P}^{+,0}[x \longleftrightarrow y]$ remains bounded away from 0 uniformly in x, y . We wish to show that there exists a constant $C > 0$ independent of x, y such that

$$\mathbf{P}^{+,0}[x \longleftrightarrow y] \leq C \langle \varphi_x \varphi_y \rangle^0. \quad (11.6.11)$$

To this end, we shall work at finite volume. For every $m > 0$ large enough so that $x, y \in B_m$,

$$\begin{aligned} \mathbf{P}_{B_N \sqcup \{\mathfrak{g}\}, B_N}^{\emptyset, \emptyset}[x \longleftrightarrow y \text{ in } B_m] &\leq \mathbf{P}_{B_N \sqcup \{\mathfrak{g}\}, B_N}^{\emptyset, \emptyset}[x \longleftrightarrow y] \leq \mathbf{P}_{B_N \sqcup \{\mathfrak{g}\}, B_N}^{\emptyset, \emptyset}[x \longleftrightarrow y \text{ in } B_m] \\ &\quad + \mathbf{P}_{B_N \sqcup \{\mathfrak{g}\}, B_N}^{\emptyset, \emptyset}[x, y \longleftrightarrow B_m^c, x \not\longleftrightarrow y \text{ in } B_m], \end{aligned} \quad (11.6.12)$$

hence sending N and m to infinity, and using the uniqueness of the infinite cluster, we deduce that

$$\lim_{N \rightarrow \infty} \mathbf{P}_{B_N \sqcup \{\mathfrak{g}\}, B_N}^{\emptyset, \emptyset}[x \longleftrightarrow y] = \mathbf{P}^{+,0}[x \longleftrightarrow y], \quad (11.6.13)$$

which implies that $\mathbf{P}_{B_N \sqcup \{\mathfrak{g}\}, B_N}^{\emptyset, \emptyset}[x \longleftrightarrow y]$ remains bounded away from 0 for every large enough N . We can now find constants $M, C_1, C_2 > 0$ independent of x, y and $N_0 = N_0(x, y) > 0$ such that for every $N \geq N_0$,

$$\begin{aligned} \mathbf{P}_{B_N \sqcup \{\mathfrak{g}\}, B_N}^{\emptyset, \emptyset}[x \longleftrightarrow y] &\leq C_1 \mathbf{P}_{B_N \sqcup \{\mathfrak{g}\}, B_N}^{\emptyset, \emptyset}[x \longleftrightarrow y, \text{ECT}(x), \text{ECT}(y), \Delta \mathbf{n}(x) \leq M, \Delta \mathbf{n}(y) \leq M] \\ &\leq C_2 \langle \varphi_x^2 \varphi_y^2 \rangle_{B_N}^0 \mathbf{P}_{B_N \sqcup \{\mathfrak{g}\}, B_N}^{A, \emptyset}[x \longleftrightarrow y, \text{ECT}(x), \text{ECT}(y), \Delta \mathbf{n}(x) \leq M, \Delta \mathbf{n}(y) \leq M] \\ &\leq C_2 \langle \varphi_x \varphi_y \rangle_{B_N}^{\mathbf{p}} \langle \varphi_x \varphi_y \rangle_{B_N}^0, \end{aligned}$$

where $A = xxyy$. Sending N to infinity and using that

$$\langle \varphi_x \varphi_y \rangle^+ \leq \langle \varphi_x^2 \rangle^+ < \infty \quad (11.6.14)$$

by the Cauchy–Schwarz inequality and Γ -invariance, we obtain that (11.6.11) holds. It follows that $\inf_{y \in V} \langle \varphi_x \varphi_y \rangle^0 > 0$, as desired. $\square$

11.6.1 Proof of Theorem 11.1.2

Proof of (11.1.11) in Theorem 11.1.2. Assuming (H), we have $\mathbf{P}^{+,0}[\mathcal{A}_{x,y}] > 0$ by Proposition 11.6.2. In particular $\mathbf{P}^{+,0}[x \longleftrightarrow \infty] > 0$, which implies

$$c_1 := \inf_{y \in V} \langle \varphi_x \varphi_y \rangle_\beta^0 > 0 \quad (11.6.15)$$

by Lemma 11.6.5. We now consider two cases.

First case. $\mathbf{P}^0[x \longleftrightarrow \infty] > 0$.

Equivalently, we have $\mathbf{P}^0[f(x, \infty, \mathbf{n}, \mathbf{t}) \geq 1] > 0$. We also know by Proposition 11.6.2 that $\mathbf{P}^+[\mathcal{A}_{x,y}] > 0$, so in particular $\mathbf{P}^+[f(x, \infty, \mathbf{n}, \mathbf{t}) \geq 2] > 0$. Let $(\mathbf{n}_1, \mathbf{t}_1) \sim \mathbf{P}^+$ and $(\mathbf{n}_2, \mathbf{t}_2) \sim \mathbf{P}^0$ be independent and $(\mathbf{n}, \mathbf{t}) \sim \mathbf{P}^{+,0}$, then by (11.3.81) we have $\mathbf{t} \succ \mathbf{t}_1 \sqcup \mathbf{t}_2$. Using this stochastic domination and inserting the event $\text{ECT}(x)$, we obtain

$$\mathbf{P}^{+,0}[f(x, \infty, \mathbf{n}_1, \mathbf{t}) \geq 2, f(x, \infty, \mathbf{n}_2, \mathbf{t}) \geq 1, \text{ECT}(x)] > 0. \quad (11.6.16)$$

Since $\{f(x, \infty, \mathbf{n}_1, \mathbf{t}_1) \geq 2, f(x, \infty, \mathbf{n}_2, \mathbf{t}_2) \geq 1, \text{ECT}(x)\} \subset \{f(x, \infty, \mathbf{n}, \mathbf{t}) \geq 3\}$, we obtain a contradiction with Lemma 11.6.4 – see (11.6.7).

Second case. $\mathbf{P}^0[x \longleftrightarrow \infty] = 0$.

We prove the following lemmas.

Lemma 11.6.6. *Assume $\mathbf{P}^0[x \longleftrightarrow \infty] = 0$. Then for every $x, y \in V$ there exists an integer $m = m(x, y)$ such that*

$$\mathbf{P}_{B_n}^{xy}[x \longleftrightarrow y \text{ in } B_m] > 1/2 \quad (11.6.17)$$

for all n sufficiently large.

Lemma 11.6.7. *Assume (H). Then, there exists $c_2 > 0$ such that, for all $x \in V$ there exists infinitely many $y \in V$ such that, there exists an integer $m = m(x, y)$ such that*

$$\langle \varphi_x \varphi_y \rangle_{B_n}^{\mathbf{p}} \mathbf{P}_{B_n \sqcup \{\mathbf{g}\}}^{xy}[x \longleftrightarrow y \text{ in } B_m] > c_2 \quad (11.6.18)$$

for all n sufficiently large.

Before proving Lemmas 11.6.6 and 11.6.7, we conclude the proof of the main result. Let x, y, m, n be such that both (11.6.17) and (11.6.18) hold. By taking the product between (11.6.17), (11.6.18) and $\langle \varphi_x \varphi_y \rangle_{B_n}^0$ (which we can assume to be larger than $c_1/2$ by (11.6.15)), using (11.3.81), and bounding $\Delta \mathbf{n}(x)$ and $\Delta \mathbf{n}(y)$ to insert $\text{ECT}(x)$, $\text{ECT}(y)$, we obtain

$$\langle \varphi_x \varphi_y \rangle_{B_n}^{\mathbf{p}} \langle \varphi_x \varphi_y \rangle_{B_n}^0 \mathbf{P}_{B_n \sqcup \{\mathbf{g}\}, B_n}^{xy, xy}[f(x, y, \mathbf{n}, \mathbf{t}) \geq 2 \text{ in } B_m] > c_3 \quad (11.6.19)$$

for some uniform constant $c_3 > 0$. Applying the switching principle of Theorem 11.3.11 we obtain

$$\langle \varphi_x^2 \varphi_y^2 \rangle_{B_n}^{\mathbf{p}} \mathbf{P}_{B_n \sqcup \{\mathbf{g}\}, B_n}^{xxyy, \emptyset}[f(x, y, \mathbf{n}, \mathbf{t}) \geq 2 \text{ in } B_m] > c_3. \quad (11.6.20)$$

We can now remove the sources $A = xxyy$ at a finite cost by bounding $\Delta \mathbf{n}(x)$ and $\Delta \mathbf{n}(y)$, and inserting $\text{ECT}(x)$, $\text{ECT}(y)$, to obtain

$$\mathbf{P}_{B_n \sqcup \{\mathfrak{g}\}, B_n}^{\emptyset, \emptyset} [f(x, y, \mathbf{n}, \mathbf{t}) \geq 2 \text{ in } B_m] > c_4 \quad (11.6.21)$$

for some uniform constant $c_4 > 0$. Taking $n \rightarrow \infty$,

$$\mathbf{P}^{+, 0} [f(x, y, \mathbf{n}, \mathbf{t}) \geq 2] \geq c_4. \quad (11.6.22)$$

Taking $d_G(x, y) \rightarrow \infty$ contradicts (11.6.6) from Lemma 11.6.4, thus concluding the proof of condition 4 in Proposition 11.2.15 and, hence, establishing (11.1.11) in Theorem 11.1.2. $\square$

Proof of Lemma 11.6.6. Assume for a contradiction that (11.6.17) does not hold. Then there exist $x, y \in V$ such that for every $m \geq 0$,

$$\mathbf{P}_{B_n}^{xy} [x \longleftrightarrow \partial B_m] \geq 1/2 \quad (11.6.23)$$

for all n sufficiently large, since there is a path from x to y in $(\mathbf{n}, \mathbf{t})$ due to the source constraints. We now wish to compare $\mathbf{P}_{B_n}^{\emptyset} [x \longleftrightarrow \partial B_m]$ to $\mathbf{P}_{B_n}^{xy} [x \longleftrightarrow \partial B_m]$ to deduce that the former remains bounded away from 0, thus obtaining a contradiction.

To this end, fix a sequence of vertices $u_0 = x, u_1, \dots, u_k = y$ such that $J_{u_i, u_{i+1}} > 0$ for all $i = 0, 1, \dots, k-1$, and let $M > 0$ be large enough (depending on x, y only) so that

$$\mathbf{P}_{B_n}^{xy} [x \longleftrightarrow \partial B_m, \Delta \mathbf{n}(u_i) \leq M, \forall i = 0, 1, \dots, k] \geq 1/4 \quad (11.6.24)$$

for every $m \geq 0$ and every n sufficiently large. Now given a current $\mathbf{n}$ with sources x, y , let $\mathbf{n}'$ be the sourceless current which is defined as

$$\mathbf{n}'(s, t) = \begin{cases} \mathbf{n}(s, t) + 1, & \text{for } \{s, t\} \in \{\{u_0 u_1\}, \dots, \{u_{k-1} u_k\}\}, \\ \mathbf{n}(s, t), & \text{otherwise.} \end{cases} \quad (11.6.25)$$

By (11.3.8), Proposition 11.4.1 and the fact that $\Delta \mathbf{n}(u_i) \leq M$ for all $i = 0, 1, \dots, k$, we obtain

$$w_{xy, \beta}(\mathbf{n}) \leq C_5 w_{\emptyset, \beta}(\mathbf{n}') \quad \text{and} \quad \rho_{xy}^{\mathbf{n}} [x \longleftrightarrow \partial B_m] \leq C_5 \rho_{\emptyset}^{\mathbf{n}'} [x \xleftrightarrow{(\mathbf{n}', \mathbf{t})} \partial B_m] \quad (11.6.26)$$

for a constant $C_5 > 0$ depending on x, y and M only. Since the map $\mathbf{n} \mapsto \mathbf{n}'$ is injective, it follows that

$$\langle \varphi_x \varphi_y \rangle_{B_n}^0 \mathbf{P}_{B_n}^{xy} [x \longleftrightarrow \partial B_m, \Delta \mathbf{n}(u_i) \leq M \forall i = 0, 1, \dots, k] \leq C_5^2 \mathbf{P}_{B_n}^{\emptyset} [x \longleftrightarrow \partial B_m]. \quad (11.6.27)$$

By the irreducibility property, one can easily prove that $\langle \varphi_x \varphi_y \rangle_{B_n}^0 \geq c_6$ for n large enough, where $c_6 > 0$ depends on x, y only. Recalling (11.6.24), we conclude that $\mathbf{P}_{B_n}^{\emptyset} [x \longleftrightarrow \partial B_m] \geq (c_6/4)C_5^2$ for every m and every n large enough. Taking $n \rightarrow \infty$ and then $m \rightarrow \infty$, we obtain $\mathbf{P}^0 [x \longleftrightarrow \infty] > 0$, which concludes the proof by contradiction. $\square$

Proof of Lemma 11.6.7. Let x, x' be such that $c_7 := \mathbf{P}^+[\mathcal{A}_{x, x'}] > 0$. Then by mixing

$$\lim_{y \rightarrow \infty} \mathbf{P}^+[\mathcal{A}_{x, x'}, \mathcal{A}_{y, y'}] = c_7^2, \quad (11.6.28)$$

where $y' = \gamma \cdot x'$ for an automorphism $\gamma \in \Gamma$ such that $\gamma \cdot x = y$. Thus for every y sufficiently far away from x , $\mathbf{P}^+[\mathcal{A}_{x,x'}, \mathcal{A}_{y,y'}] > c_7^2/2$. We now wish to modify our current so that it has sources x, x', y, y' . Assume without loss of generality that they are all distinct.

Given $(\mathbf{n}, \mathbf{t}) \in \mathcal{A}_{x,x'} \cap \mathcal{A}_{y,y'}$, let $(\mathbf{n}', \mathbf{t}')$ be the tangled current with sources x, x', y, y' defined as

$$\mathbf{n}'(e) = \begin{cases} \mathbf{n}(e) - 1, & \text{for } e \in \{\{x, x'\}, \{y, y'\}\}, \\ \mathbf{n}(e), & \text{otherwise,} \end{cases} \quad (11.6.29)$$

where $\mathbf{t}'$ coincides with $\mathbf{t}$ up to redefining $\bar{x}, \bar{x}', \bar{y}, \bar{y}'$ to be the extra vertices in the blocks added due to the sources, similarly to the proof of Proposition 11.6.2. Note that up to possibly exchanging x and x' , and y and y' , $(\mathbf{n}', \mathbf{t}') \in \mathcal{B}_m(x, x', y, y')$ for some $m > 0$, where

$$\mathcal{B}_m(x, x', y, y') := \{(\mathbf{n}, \mathbf{t}) : \bar{x} \longleftrightarrow \bar{y} \text{ in } B_m, \bar{x} \not\longleftrightarrow \bar{x}', \bar{y} \not\longleftrightarrow \bar{y}'\}, \quad (11.6.30)$$

and now $\bar{x}, \bar{x}', \bar{y}, \bar{y}'$ denote the extra vertices in the blocks. Thus a union bound implies that

$$\mathbf{P}^+[(\mathbf{n}', \mathbf{t}') \in \mathcal{B}_m(x, x', y, y')] > c_7^2/8, \quad (11.6.31)$$

which in turn implies that

$$\mathbf{P}_{B_n \sqcup \{\mathfrak{g}\}}^\emptyset[(\mathbf{n}', \mathbf{t}') \in \mathcal{B}_m(x, x', y, y')] > c_7^2/8 \quad (11.6.32)$$

for all n sufficiently large. After bounding the degrees, it follows that there exists a constant $c_8 > 0$ independent of x, x', y, y' such that for every n large enough

$$\langle \varphi_x \varphi_{x'} \varphi_y \varphi_{y'} \rangle_{B_n}^{\mathbf{p}} \mathbf{P}_{B_n \sqcup \{\mathfrak{g}\}}^{xx'yy'}[\mathcal{B}_m(x, x', y, y')] \quad (11.6.33)$$

$$\geq c_8 \mathbf{P}_{B_n \sqcup \{\mathfrak{g}\}}^\emptyset[(\mathbf{n}', \mathbf{t}') \in \mathcal{B}_m(x, x', y, y')] \quad (11.6.34)$$

hence for every n sufficiently large,

$$\sum_{\substack{\mathbf{n} \in \Omega_{B_n \sqcup \{\mathfrak{g}\}} \\ \partial \mathbf{n} = \{x, x', y, y'\}}} w_{xx'yy', \beta}(\mathbf{n}) \rho_{xx'yy'}^{\mathbf{n}}[\mathcal{B}_m(x, x', y, y')] \quad (11.6.35)$$

$$\geq c_8 \sum_{\substack{\mathbf{n} \in \Omega_{B_n \sqcup \{\mathfrak{g}\}} \\ \partial \mathbf{n} = \emptyset}} w_{\emptyset, \beta}(\mathbf{n}) \rho_{\emptyset}^{\mathbf{n}}[(\mathbf{n}', \mathbf{t}') \in \mathcal{B}_m(x, x', y, y')]. \quad (11.6.36)$$

Our goal now is to prove that

$$\begin{aligned} \sum_{\substack{\mathbf{n} \in \Omega_{B_n \sqcup \{\mathfrak{g}\}} \\ \partial \mathbf{n} = \{x, x', y, y'\}}} w_{xx'yy', \beta}(\mathbf{n}) \rho_{xx'yy'}^{\mathbf{n}}[\mathcal{B}_m(x, x', y, y')] &\leq \\ \langle \varphi_{x'} \varphi_{y'} \rangle_{B_n}^{\mathbf{p}} \sum_{\substack{\mathbf{n} \in \Omega_{B_n \sqcup \{\mathfrak{g}\}} \\ \partial \mathbf{n} = \{x, y\}}} w_{xy, \beta}(\mathbf{n}) \rho_{xx'yy'}^{\mathbf{n}}[\mathcal{B}_m(x, x', y, y')]. \end{aligned} \quad (11.6.37)$$

Indeed, with this inequality and (11.6.33),

$$\begin{aligned} \frac{c_7^2}{8} c_8 &\leq \langle \varphi_x \varphi_{x'} \varphi_y \varphi_{y'} \rangle_{B_n}^{\mathbf{p}} \mathbf{P}_{B_n \sqcup \{\mathfrak{g}\}}^{xx'yy'}[\mathcal{B}_m(x, x', y, y')] \\ &\leq \langle \varphi_{x'} \varphi_{y'} \rangle_{B_n}^{\mathbf{p}} \langle \varphi_x \varphi_y \rangle_{B_n}^{\mathbf{p}} \mathbf{P}_{B_n \sqcup \{\mathfrak{g}\}}^{xy}[x \longleftrightarrow y \text{ in } B_m] \\ &\leq \sqrt{\langle \varphi_{x'}^2 \rangle_{B_n}^{\mathbf{p}} \langle \varphi_{y'}^2 \rangle_{B_n}^{\mathbf{p}} \langle \varphi_x \varphi_y \rangle_{B_n}^{\mathbf{p}} \mathbf{P}_{B_n \sqcup \{\mathfrak{g}\}}^{xy}[x \longleftrightarrow y \text{ in } B_m]}. \end{aligned}$$

This is sufficient to conclude since for $n \geq 1$ sufficiently large one has $\langle \varphi_{x'}^2 \rangle_{B_n}^p, \langle \varphi_{y'}^2 \rangle_{B_n}^p \leq 2\langle \varphi_x^2 \rangle^+ < \infty$.

We now prove (11.6.37). To do this, we analyse the prelimit, i.e. go back to the Griffiths–Simon approximation. Defining $h(\mathbf{p})_z := \sum_{y \notin B_n} J_{z,y} \mathbf{p}_y$ for all $z \in B_n$, we have that

$$\mu_{B_n, N, \beta}^{h(\mathbf{p})} [\varphi_{x', N} \varphi_{y', N}] \xrightarrow{N \rightarrow \infty} \langle \varphi_{x'} \varphi_{y'} \rangle_{B_n}^p \quad (11.6.38)$$

where $\mu_{B_n, N, \beta}^{h(\mathbf{p})}$ has been defined in Definition 11.3.21, and where we have used Proposition 11.3.22. Note that by the symmetries of the block-spins for any $i, j \in \{1, \dots, N\}$,

$$\mu_{B_n, N, \beta}^{h(\mathbf{p})} [\sigma_{\tilde{x}'} \sigma_{\tilde{y}'}] = \mu_{B_n, N, \beta}^{h(\mathbf{p})} [\sigma_{(x', i)} \sigma_{(y', j)}] \quad (11.6.39)$$

where $\tilde{x}' = (x', 1)$ and $\tilde{y}' = (y', 1)$, so that the above convergence can be rephrased as

$$(c_N N)^2 \mu_{B_n, N, \beta}^{h(\mathbf{p})} [\sigma_{\tilde{x}'} \sigma_{\tilde{y}'}] \xrightarrow{N \rightarrow \infty} \langle \varphi_{x'} \varphi_{y'} \rangle_{B_n}^p. \quad (11.6.40)$$

Recall from Section 11.3.5.1 that the Ising measure $\mu_{B_n, N, \beta}^{h(\mathbf{p})}$ on $B_n \times K_N$ is defined by the Hamiltonian

$$\mathbf{H}_{B_n, \beta, N}^{h(\mathbf{p})}(\sigma) = -\beta c_N^2 \sum_{\substack{\{u, v\} \subset B_n \\ j, k \in \{1, \dots, N\}}} J_{u, v} \sigma_{(u, j)} \sigma_{(v, k)} \quad (11.6.41)$$

$$- \beta c_N \sum_{(u, j) \in B_n \times K_N} h(\mathbf{p})_u \sigma_{(u, j)} - d_N \sum_{\substack{u \in B_n \\ j, k \in \{1, \dots, N\}}} \sigma_{(u, j)} \sigma_{(u, k)} \quad (11.6.42)$$

$$= - \sum_{\substack{u, v \in B_n \\ j, k \in \{1, \dots, N\}}} J_N((u, i), (v, j)) \sigma_{(u, i)} \sigma_{(v, j)} - \beta c_N \sum_{(u, j) \in B_n \times K_N} h(\mathbf{p})_u \sigma_{(u, j)}, \quad (11.6.43)$$

where J_N is defined implicitly. Hence, we may as well write $\mu_{B_n, N, \beta, J_N}^{h(\mathbf{p})} = \mu_{B_n, N, \beta}^{h(\mathbf{p})}$. This notation allows us later to consider interactions other than J_N .

By Proposition 11.3.26, that the left-hand side of (11.6.37) can be obtained as the limit of

$$\frac{(c_N N)^4}{\mathbf{Z}_N^{|B_n|}} \sum_{\substack{\tilde{\mathbf{n}} \in \Omega_{B_n \cup \{\mathbf{g}\}}^N \\ \partial \tilde{\mathbf{n}} = \{\tilde{x}, \tilde{x}', \tilde{y}, \tilde{y}'\}}} w_{N, \beta}(\tilde{\mathbf{n}}) \mathbf{1}_{\mathcal{B}_m^N(\tilde{x}, \tilde{x}', \tilde{y}, \tilde{y}')} \quad (11.6.44)$$

as N goes to infinity, where $\tilde{z} = (z, 1)$ for $z \in \{x, x', y, y'\}$ and

$$\mathcal{B}_m^N(\tilde{x}, \tilde{x}', \tilde{y}, \tilde{y}') := \{\tilde{\mathbf{n}} \in \Omega_{B_n \cup \{\mathbf{g}\}}^N : \tilde{x} \xleftrightarrow{\tilde{\mathbf{n}}} \tilde{y} \text{ in } B_m \times K_N, \tilde{x} \not\xleftrightarrow{\tilde{\mathbf{n}}} \tilde{x}', \tilde{y} \not\xleftrightarrow{\tilde{\mathbf{n}}} \tilde{y}'\}. \quad (11.6.45)$$

Now, denote by $\mathcal{C}_{\tilde{x}}$ the cluster of $\tilde{x}$. We say that a subset of $B_n \times K_N \cup \{\mathbf{g}\}$ is admissible if it satisfies $\{\mathcal{C}_{\tilde{x}} = C\} \subset \mathcal{B}_m^N(\tilde{x}, \tilde{x}', \tilde{y}, \tilde{y}')$. If C is admissible, then

$$\sum_{\substack{\tilde{\mathbf{n}} \in \Omega_{B_n \cup \{\mathbf{g}\}}^N \\ \partial \tilde{\mathbf{n}} = \{\tilde{x}, \tilde{x}', \tilde{y}, \tilde{y}'\}}} w_{N, \beta, h(\mathbf{p})}(\tilde{\mathbf{n}}) \mathbf{1}_{\mathcal{C}_{\tilde{x}} = C} = \mu_{B_n, N, \beta, J_N^C}^{h(\mathbf{p})} [\sigma_{\tilde{x}'} \sigma_{\tilde{y}'}] \sum_{\substack{\tilde{\mathbf{n}} \in \Omega_{B_n \cup \{\mathbf{g}\}}^N \\ \partial \tilde{\mathbf{n}} = \{\tilde{x}, \tilde{y}\}}} w_{N, \beta, h(\mathbf{p})}(\tilde{\mathbf{n}}) \mathbf{1}_{\mathcal{C}_{\tilde{x}} = C} \quad (11.6.46)$$

where J_N^C is defined by

$$J_N^C((u, i), (v, j)) = \begin{cases} J_N((u, i), (v, j)) & \text{if } (u, i), (v, j) \notin C \\ 0 & \text{otherwise.} \end{cases} \quad (11.6.47)$$

and $h^C(\mathfrak{p}) = 0$ if $\mathfrak{g} \in C$ and $h^C(\mathfrak{p}) = h(\mathfrak{p})$ otherwise. Since $J_N^C \leq J_N$ and $h^C(\mathfrak{p}) \leq h(\mathfrak{p})$, by Griffiths' inequality,

$$\mu_{B_n, N, \beta, J_N^C}^{h^C(\mathfrak{p})}[\sigma_{\tilde{x}'} \sigma_{\tilde{y}'}] \leq \mu_{B_n, N, \beta, J_N}^{h(\mathfrak{p})}[\sigma_{\tilde{x}'} \sigma_{\tilde{y}'}] \quad (11.6.48)$$

so that

$$\sum_{\substack{\tilde{\mathbf{n}} \in \Omega_{B_n \sqcup \{\mathfrak{g}\}}^N \\ \partial \tilde{\mathbf{n}} = \{\tilde{x}, \tilde{x}', \tilde{y}, \tilde{y}'\}}} w_{N, \beta}(\tilde{\mathbf{n}}) \mathbf{1}_{C_{\tilde{x}}=C} \leq \mu_{B_n, N, \beta, J_N}^{h(\mathfrak{p})}[\sigma_{\tilde{x}'} \sigma_{\tilde{y}'}] \sum_{\substack{\tilde{\mathbf{n}} \in \Omega_{B_n \sqcup \{\mathfrak{g}\}}^N \\ \partial \tilde{\mathbf{n}} = \{\tilde{x}, \tilde{y}\}}} w_{N, \beta}(\tilde{\mathbf{n}}) \mathbf{1}_{C_{\tilde{x}}=C}. \quad (11.6.49)$$

Summing over C admissible yields

$$\begin{aligned} \sum_{\substack{\tilde{\mathbf{n}} \in \Omega_{B_n \sqcup \{\mathfrak{g}\}}^N \\ \partial \tilde{\mathbf{n}} = \{\tilde{x}, \tilde{x}', \tilde{y}, \tilde{y}'\}}} w_{N, \beta}(\tilde{\mathbf{n}}) \mathbf{1}_{\mathcal{B}_m^N(\tilde{x}, \tilde{x}', \tilde{y}, \tilde{y}')} &\leq \mu_{B_n, N, \beta, J_N}^{h(\mathfrak{p})}[\sigma_{\tilde{x}'} \sigma_{\tilde{y}'}] \\ &\times \sum_{\substack{\tilde{\mathbf{n}} \in \Omega_{B_n \sqcup \{\mathfrak{g}\}}^N \\ \partial \tilde{\mathbf{n}} = \{\tilde{x}, \tilde{y}\}}} w_{N, \beta}(\tilde{\mathbf{n}}) \mathbf{1}_{\mathcal{B}_m^N(\tilde{x}, \tilde{x}', \tilde{y}, \tilde{y}')} \end{aligned} \quad (11.6.50)$$

Multiplying by $(c_N N)^4 / |\mathbf{Z}_N|^{B_n}$ on each side of (11.6.50) and letting N go to infinity yields (11.6.37) by Proposition 11.3.26 and (11.6.40). $\square$

We now finish the proof of Theorem 11.1.2. It remains to prove that (11.1.12) is sufficient to establish that $|\mathcal{G}(\beta)| = 1$. By Proposition 11.B.3 it suffices to show that $m^*(\beta) = 0$. From the preceding arguments, we know that $\langle \cdot \rangle^0$ and $\langle \cdot \rangle^+$ coincide on even functions, so condition (11.1.12) reads

$$\liminf_{d_G(o, x) \rightarrow \infty} \langle \varphi_o \varphi_x \rangle_\beta^0 = \liminf_{d_G(o, x) \rightarrow \infty} \langle \varphi_o \varphi_x \rangle_\beta^+ = 0. \quad (11.6.51)$$

However, by Griffiths' inequality and Γ -invariance, for all $x \in V$,

$$\langle \varphi_o \varphi_x \rangle_\beta^+ \geq m^*(\beta)^2 \quad (11.6.52)$$

which implies that $m^*(\beta) = 0$. $\square$

11.7 Continuity of the phase transition

In this section we prove Theorem 11.1.3. Consider the torus $\mathbb{T}_L^d := (\mathbb{Z}/L\mathbb{Z})^d$ with L even. The torus has a natural reflection symmetry along hyperplanes $\mathcal{P}$ which are orthogonal to one of the lattice directions. Any such hyperplane $\mathcal{P}$ either passes through sites of $\mathbb{T}_L$ or through mid-edges, and $\mathcal{P}$ divides the torus into two pieces $\mathbb{T}_L^+$ and $\mathbb{T}_L^-$. The two pieces are disjoint for mid-edges reflections, whereas for site reflections they satisfy $\mathbb{T}_L^+ \cap \mathbb{T}_L^- = \mathcal{P}$. Denote by $\mathcal{A}^\pm$ the algebra of all real valued functions f that depend only on the spins in $\mathbb{T}_L^\pm$. Denote

by Θ the reflection map associated with $\mathcal{P}$. This can be associated to a reflection operator $\Theta : \mathcal{A}^\pm \rightarrow \mathcal{A}^\mp$ which acts on $f \in \mathcal{A}^\pm$ by

$$\Theta(f)(\varphi) = f(\varphi \circ \Theta), \quad \forall \varphi \in \mathbb{R}^{\mathbb{T}_L^d}. \quad (11.7.1)$$

We consider the φ^4 model on the d -dimensional torus $\mathbb{T}_L^d$ with interactions J^L and inverse temperature β defined in the natural way, and denote it by $\langle \cdot \rangle_{\mathbb{T}_L^d, \beta, J^L}$.

Definition 11.7.1 (Reflection positivity). Let $L \in \mathbb{N}$ be even. Fix a hyperplane $\mathcal{P}$ on $\mathbb{T}_L^d$ as above and let Θ denote the associated reflection operator. We say that the interaction $J^L = (J_{x,y}^L)_{x,y \in \mathbb{T}_L^d}$ is reflection positive with respect to Θ , if for all $f, g \in \mathcal{A}^+$, and for all $\beta > 0$

$$\langle f\Theta(g) \rangle_{\mathbb{T}_L, \beta, J^L} = \langle \Theta(f)g \rangle_{\mathbb{T}_L, \beta, J^L} \quad (11.7.2)$$

and,

$$\langle f\Theta(f) \rangle_{\mathbb{T}_L, \beta, J^L} \geq 0. \quad (11.7.3)$$

We say that an interaction $J = (J_{x,y})_{x,y \in \mathbb{Z}^d}$ is reflection positive if, for every L even, the associated interaction on $\mathbb{T}_L^d$ given by $J_{x,y}^L := \sum_{z \in \mathbb{Z}^d} J_{x,y+Lz}$ is reflection positive with respect to any hyperplane as described above.

Fix a reflection positive interaction J on $\mathbb{Z}^d$. We suppress J from the notation when clear from context. For each such J , recall the associated random walk $(X_k)_{k \geq 0}$ (started at 0) with step distribution given by

$$\mathbb{P}[X_{n+1} = y \mid X_n = x] = \frac{J_{x,y}}{|J|} \quad (11.7.4)$$

where $|J| := \sum_{x \in \mathbb{Z}^d} J_{0,x}$. It is known that $(X_k)_{k \geq 0}$ is transient if and only if

$$\int_{[-\pi, \pi]^d} \frac{1}{1 - \mathbb{E}[e^{ip \cdot X_1}]} \frac{dp}{(2\pi)^d} < \infty. \quad (11.7.5)$$

A proof of this statement can be found in [172, Proposition 4.2.3]; it crucially uses the reflection symmetry and aperiodicity of the random walk.

We now state a version of the infrared bound that is convenient in our setting.

Proposition 11.7.2. *Let J be reflection positive and such that the associated random walk is transient, i.e. (11.7.5) is verified. For any $(v_x)_{x \in \mathbb{Z}^d} \in \mathcal{C}^{\mathbb{Z}^d}$ of finite support, and any $\beta < \beta_c(J)$,*

$$\sum_{x,y \in \mathbb{Z}^d} v_x \overline{v_y} \langle \varphi_x \varphi_y \rangle_\beta^0 \leq \frac{1}{2\beta|J|} \sum_{x,y \in \mathbb{Z}^d} v_x \overline{v_y} G(x,y) \quad (11.7.6)$$

where G is the Green's function of $(X_k)_{k \in \mathbb{N}}$ given by

$$G(x,y) := \lim_{L \rightarrow \infty} \frac{1}{L^d} \sum_{p \in \mathbb{T}_L^*} \frac{e^{ip \cdot (x-y)}}{1 - \mathbb{E}[e^{ip \cdot X_1}]} = \int_{[-\pi, \pi]^d} \frac{e^{ip \cdot (x-y)}}{1 - \mathbb{E}[e^{ip \cdot X_1}]} \frac{dp}{(2\pi)^d} < \infty. \quad (11.7.7)$$

Proof. The convergence and finiteness of $G(x,y)$ is a consequence of (11.7.5). The infrared bound (11.7.6) is a direct consequence of [36, Lemma 4.5] together with the fact that the spontaneous magnetisation vanishes for $\beta < \beta_c$. We stress a subtle point in the reasoning above: one can show, by using similar regularity estimates as in Corollary 11.C.13 and arguing as in the proof of [175, Theorem 4.5], that the φ^4 measures on $\mathbb{T}_L^d$ converge weakly as $L \rightarrow \infty$ to the unique (since $\beta < \beta_c$, see Proposition 11.B.3) Gibbs measure on $\mathbb{Z}^d$. $\square$

By left-continuity of $\beta \mapsto \langle \varphi_x \varphi_y \rangle_\beta^0$ for $x, y \in \mathbb{Z}^d$, we can also extend (11.7.6) to $\beta = \beta_c$. Using equation (11.7.6) with v defined by $v_x = |B_n|^{-2} \mathbf{1}_{B_n}(x)$ yields

$$\frac{1}{|B_n|^2} \sum_{x, y \in B_n} \langle \varphi_x \varphi_y \rangle_{\beta_c}^0 \leq \frac{1}{2\beta_c |J|} \frac{1}{|B_n|^2} \sum_{x, y \in B_n} G(x, y). \quad (11.7.8)$$

The Riemann-Lebesgue Lemma gives that $G(x, y) \xrightarrow{|x-y|_1 \rightarrow \infty} 0$. In particular,

$$\frac{1}{|B_n|^2} \sum_{x, y \in B_n} G(x, y) \xrightarrow{n \rightarrow \infty} 0. \quad (11.7.9)$$

In particular, by positivity of correlations we have that the lefthand side of (11.7.8) converges to 0, which in turn implies that (11.1.12) holds at $\beta = \beta_c$. By Theorem 11.1.2, one concludes that $|\mathcal{G}_\Gamma(\beta_c)| = 1$, and therefore $m^*(\beta_c) = 0$ by Proposition 11.B.3. $\square$

11.A Correlation inequalities

In this section, all the boundary conditions considered are implicitly assumed to satisfy the summability condition defined in the introduction, i.e. $\sum_{y \notin \Lambda} J_{x,y} \eta_y < \infty \quad \forall x \in \Lambda$.

11.A.1 The FKG inequality and absolute value FKG

Proposition 11.A.1 (FKG). *Let $\Lambda \subset V$ finite, $\eta \in \mathbb{R}^V$ and $h \in \mathbb{R}^\Lambda$. Then, for any increasing and bounded functions $F, G : \mathbb{R}^\Lambda \rightarrow \mathbb{R}$,*

$$\langle F(\varphi) G(\varphi) \rangle_{\Lambda, \beta, h}^\eta \geq \langle F(\varphi) \rangle_{\Lambda, \beta, h}^\eta \langle G(\varphi) \rangle_{\Lambda, \beta, h}^\eta. \quad (11.A.1)$$

Proof. See [113, Theorem 4.4.1]. $\square$

Proposition 11.A.2 (Absolute value FKG). *Let $\Lambda \subset V$ finite, $\eta \in \mathbb{R}^V$ and $h \in \mathbb{R}^\Lambda$. Then, for any non-negative increasing functions $F, G : (\mathbb{R}_+)^{\Lambda} \rightarrow \mathbb{R}$,*

$$\langle F(|\varphi|) G(|\varphi|) \rangle_{\Lambda, \beta, h}^\eta \geq \langle F(|\varphi|) \rangle_{\Lambda, \beta, h}^\eta \langle G(|\varphi|) \rangle_{\Lambda, \beta, h}^\eta. \quad (11.A.2)$$

Proof. We rewrite the Hamiltonian as

$$H_\Lambda^\eta(\varphi) := \sum_{x, y \in \Lambda} V_{x,y}(\varphi_x - \varphi_y) - \sum_{x, y \in \Lambda} J_{x,y} \frac{\varphi_x^2 + \varphi_y^2}{2} - \sum_{\substack{x \in \Lambda \\ y \in V \setminus \Lambda}} J_{x,y} \varphi_x \eta_y - \sum_{x \in \Lambda} h_x \varphi_x, \quad (11.A.3)$$

where $V_{x,y}(z) = \frac{1}{2} J_{x,y} z^2$. Note that $V_{x,y}$ satisfies the following properties:

1. $V_{x,y}$ is convex,
2. $V_{x,y}$ is symmetric, i.e. $V_{x,y}(-z) = V_{x,y}(z)$,
3. the second derivative of $V_{x,y}$ is a non-increasing function.

The desired assertion follows from [170, Corollary 6.4]. $\square$

11.A.2 Proof of the Ginibre inequality

Recall that if $A : V \rightarrow \mathbb{N}$ is finitely supported, we write

$$\varphi_A = \prod_{x \in V} \varphi_x^{A_x}. \quad (11.A.4)$$

Let $A, B : V \rightarrow \mathbb{N}$ be finitely supported. Let Λ be a finite subset of V that contains the support of A and B , and let $|\eta| \leq \eta'$ outside Λ . We want to prove that

$$\langle \varphi_A \varphi_B \rangle_{\Lambda, \beta}^{\eta'} - \langle \varphi_A \varphi_B \rangle_{\Lambda, \beta}^{\eta} \geq \left| \langle \varphi_A \rangle_{\Lambda, \beta}^{\eta'} \langle \varphi_B \rangle_{\Lambda, \beta}^{\eta} - \langle \varphi_A \rangle_{\Lambda, \beta}^{\eta} \langle \varphi_B \rangle_{\Lambda, \beta}^{\eta'} \right| \quad (11.A.5)$$

which is equivalent to

$$\begin{aligned} - \left(\langle \varphi_A \varphi_B \rangle_{\Lambda, \beta}^{\eta'} - \langle \varphi_A \varphi_B \rangle_{\Lambda, \beta}^{\eta} \right) &\leq \langle \varphi_A \rangle_{\Lambda, \beta}^{\eta'} \langle \varphi_B \rangle_{\Lambda, \beta}^{\eta} - \langle \varphi_A \rangle_{\Lambda, \beta}^{\eta} \langle \varphi_B \rangle_{\Lambda, \beta}^{\eta'} \\ &\leq \langle \varphi_A \varphi_B \rangle_{\Lambda, \beta}^{\eta'} - \langle \varphi_A \varphi_B \rangle_{\Lambda, \beta}^{\eta}. \end{aligned}$$

Let $h_x := \sum_{y \notin \Lambda} J_{x,y} \eta_y$ and $h'_x := \sum_{y \notin \Lambda} J_{x,y} \eta'_y$. Then, using Ginibre inequality at the level of the Griffiths–Simon graph $\Lambda_N = \Lambda \times K_N$ (we kept the notations of Section 11.3.5.1), and using linearity,

$$\begin{aligned} - \left(\langle \varphi_{A,N} \varphi_{B,N} \rangle_{\Lambda_N, \beta, h'}^0 - \langle \varphi_{A,N} \varphi_{B,N} \rangle_{\Lambda_N, \beta, h}^0 \right) \\ \leq \langle \varphi_{A,N} \rangle_{\Lambda_N, \beta, h'}^0 \langle \varphi_{B,N} \rangle_{\Lambda_N, \beta, h}^0 - \langle \varphi_{A,N} \rangle_{\Lambda_N, \beta, h}^0 \langle \varphi_{B,N} \rangle_{\Lambda_N, \beta, h'}^0 \\ \leq \langle \varphi_{A,N} \varphi_{B,N} \rangle_{\Lambda_N, \beta, h'}^0 - \langle \varphi_{A,N} \varphi_{B,N} \rangle_{\Lambda_N, \beta, h}^0 \end{aligned}$$

where

$$\varphi_{A,N} := \prod_{x \in \Lambda} \varphi_{x,N}^{A_x}, \quad \varphi_{B,N} := \prod_{x \in \Lambda} \varphi_{x,N}^{B_x}. \quad (11.A.6)$$

Now, taking the limit as N goes to infinity and using Proposition 11.3.22, we get the desired result. Note that we used the interpretation of boundary conditions as non-homogeneous external fields: $\langle \cdot \rangle_{\Lambda, \beta, h'}^0 = \langle \cdot \rangle_{\Lambda, \beta}^{\eta'}$ and $\langle \cdot \rangle_{\Lambda, \beta, h}^0 = \langle \cdot \rangle_{\Lambda, \beta}^{\eta}$.

Moreover, if $\langle \cdot \rangle_{\beta} \in \mathcal{G}(\beta)$,

$$\langle \varphi_A \varphi_B \rangle_{\beta}^+ - \langle \varphi_A \varphi_B \rangle_{\beta} \geq \left| \langle \varphi_A \rangle_{\beta}^+ \langle \varphi_B \rangle_{\beta} - \langle \varphi_A \rangle_{\beta} \langle \varphi_B \rangle_{\beta}^+ \right|. \quad (11.A.7)$$

Indeed, if we fix $\eta' = \mathfrak{p}$ above and integrate against $\langle \cdot \rangle_{\beta}$ among η satisfying $|\eta| \leq \mathfrak{p}$ outside Λ ,

$$\begin{aligned} \langle \varphi_A \varphi_B \rangle_{\Lambda, \beta}^{\mathfrak{p}} \langle \mathbb{1}_{\forall x \notin \Lambda, |\eta_x| \leq \mathfrak{p}_x} \rangle_{\beta} - \left\langle \langle \varphi_A \varphi_B \rangle_{\Lambda, \beta}^{\eta} \mathbb{1}_{\forall x \notin \Lambda, |\eta_x| \leq \mathfrak{p}_x} \right\rangle_{\beta} \\ \geq \left| \langle \varphi_A \rangle_{\Lambda, \beta}^{\mathfrak{p}} \left\langle \langle \varphi_B \rangle_{\Lambda, \beta}^{\eta} \mathbb{1}_{\forall x \notin \Lambda, |\eta_x| \leq \mathfrak{p}_x} \right\rangle_{\beta} - \left\langle \langle \varphi_A \rangle_{\Lambda, \beta}^{\eta} \mathbb{1}_{\forall x \notin \Lambda, |\eta_x| \leq \mathfrak{p}_x} \right\rangle_{\beta} \langle \varphi_B \rangle_{\Lambda, \beta}^{\mathfrak{p}} \right|. \end{aligned}$$

Using the results of Section 11.2 and taking $\Lambda \uparrow V$ yields (11.A.7). $\square$

11.A.3 Monotonicity in boundary conditions

We have the following finite volume monotonicity properties for correlations and increasing functions.

Proposition 11.A.3. *Let $\Lambda \subset V$ finite. Let $A : \Lambda \rightarrow \mathbb{N}$ be a degree function. For $|\eta| \leq \eta'$,*

$$\langle \varphi_A \rangle_{\Lambda, \beta}^{\eta} \leq \langle \varphi_A \rangle_{\Lambda, \beta}^{\eta'} \quad (11.A.8)$$

Proof. This is a consequence of the Ginibre inequality. $\square$

Proposition 11.A.4. *Let $\eta \leq \eta'$. Then, for any f increasing and integrable,*

$$\langle f \rangle_{\Lambda, \beta}^{\eta} \leq \langle f \rangle_{\Lambda, \beta}^{\eta'} \quad (11.A.9)$$

Proof. This is a consequence of the FKG inequality $\square$

11.A.4 Single-site moment bounds

Proposition 11.A.5. *There exists $C > 0$ such that for all $k \geq 0$,*

$$\frac{\langle \varphi^{2k} \rangle_0}{\langle \varphi^{2k+2} \rangle_0} \leq C. \quad (11.A.10)$$

Proof. Note that

$$\begin{aligned} \frac{\langle \varphi^{2k} \rangle_0}{\langle \varphi^{2k+2} \rangle_0} &\leq \frac{\int_{-1}^1 s^{2k} \rho(ds)}{\int_{-1}^1 s^{2k+2} \rho(ds)} + 1 \\ &\leq \frac{M}{m} \frac{2k+3}{2k+1} + 1, \end{aligned}$$

where

$$M := \max_{s \in [-1, 1]} e^{-gs^4 - as^2} \quad \text{and} \quad m := \min_{s \in [-1, 1]} e^{-gs^4 - as^2}. \quad (11.A.11)$$

The desired assertion follows. $\square$

11.B Uniqueness of the phase transition

Definition 11.B.1. Define,

$$\beta_c^{\text{MAG}} := \inf \left\{ \beta > 0 : \langle \varphi_o \rangle_{\beta}^+ > 0 \right\} \quad (11.B.1)$$

and

$$\beta_c^{\text{LRO}} := \inf \left\{ \beta > 0 : \inf_{x \in V} \langle \varphi_o \varphi_x \rangle_{\beta}^0 > 0 \right\} \quad (11.B.2)$$

and

$$\beta_c^{\text{SHARP}} := \inf \left\{ \beta > 0, \sum_{x \in V} \langle \varphi_o \varphi_x \rangle_{\beta}^0 = \infty \right\}. \quad (11.B.3)$$

Proposition 11.B.2. *Let G be a countably infinite and vertex-transitive graph of polynomial growth. Then,*

$$\beta_c^{\text{MAG}} = \beta_c^{\text{LRO}} = \beta_c^{\text{SHARP}}. \quad (11.B.4)$$

Proof. The assertion that $\beta_c^{\text{MAG}} = \beta_c^{\text{SHARP}}$ was established in [8]. We now prove that $\beta_c^{\text{MAG}} = \beta_c^{\text{LRO}}$. Let $\beta > \beta_c^{\text{LRO}}$ and write $c := \inf_{x \in V} \langle \varphi_o \varphi_x \rangle_\beta^0 > 0$. Then, by monotonicity in the boundary conditions and the mixing of $\langle \cdot \rangle_\beta^+$ we have

$$c \leq \langle \varphi_o \varphi_x \rangle_\beta^0 \leq \langle \varphi_o \varphi_x \rangle_\beta^+ \xrightarrow{d_G(o,x) \rightarrow \infty} \left(\langle \varphi_o \rangle_\beta^+ \right)^2. \quad (11.B.5)$$

This implies that $\langle \varphi_o \rangle_\beta^+ > 0$.

Let now $\beta < \beta_c^{\text{LRO}}$. Consider some $\beta < \beta' < \beta_c^{\text{LRO}}$. It follows from (11.2.55) that

$$\langle \varphi_o \varphi_x \rangle_\beta^+ \leq \langle \varphi_o \varphi_x \rangle_{\beta'}^0. \quad (11.B.6)$$

Letting $d_G(o, x) \rightarrow \infty$ and using the mixing of $\langle \cdot \rangle_\beta^+$ as above we obtain that $\langle \varphi_o \rangle_\beta^+ = 0$. The desired assertion follows. $\square$

Proposition 11.B.3. *Let G be as in the proposition above. Let $\beta > 0$. The following are equivalent,*

- (i) $|\mathcal{G}(\beta)| = 1$,
- (ii) $\langle \cdot \rangle_\beta^+ = \langle \cdot \rangle_\beta^-$,
- (iii) $\langle \varphi_o \rangle_\beta^+ = \langle \varphi_o \rangle_\beta^- = 0$.

Proof. The implication (i) $\implies$ (ii) is trivial. For the implication (ii) $\implies$ (iii), note that if $\varphi \sim \langle \cdot \rangle_\beta^+$, then $-\varphi \sim \langle \cdot \rangle_\beta^-$. Hence, $\langle \varphi_o \rangle_\beta^+ = -\langle \varphi_o \rangle_\beta^-$, which implies that $\langle \varphi_o \rangle_\beta^+ = \langle \varphi_o \rangle_\beta^- = 0$.

Finally, to show that (iii) $\implies$ (i), let $\langle \cdot \rangle_\beta \in \mathcal{G}(\beta)$. Using Proposition 11.2.6 and Strassen's theorem [237] we obtain a coupling $(\mathbb{P}, \varphi^+, \varphi)$, $\varphi^+ \sim \langle \cdot \rangle_\beta^+$, $\varphi \sim \langle \cdot \rangle_\beta$, such that $\mathbb{P}$ -almost surely, $\varphi_x^+ \geq \varphi_x$ for every $x \in V$. Again by Proposition 11.2.6 we have $\langle \varphi_x \rangle_\beta = \langle \varphi_x \rangle_\beta^+ = 0$ for every $x \in V$. It follows that $\mathbb{P}$ -almost surely, $\varphi_x^+ = \varphi_x$ for every $x \in V$, hence $\langle \cdot \rangle_\beta = \langle \cdot \rangle_\beta^+$, as desired. $\square$

11.C Probabilistic estimates for Gibbs measures

In this section we develop the probabilistic estimates for Gibbs measures associated to the φ^4 model on graphs of polynomial growth. The results in this section follow from adaptations of the foundational papers on the subject [175] and [212], which establish the theory for more general measures on $\mathbb{Z}^d$. We optimise our estimates to the case of φ^4 , which has stronger concentration properties than the class considered in [175].

11.C.1 Setup and main results

We start by recalling some notation and definitions. We work on a graph $G = (V, E)$ which is connected, vertex-transitive graph and has polynomial growth, defined as follows. For a distinguished origin $o \in V$, we write B_k to denote the ball of radius $k \in \mathbb{N}$ centred at o with respect to the graph distance $d_G(\cdot, \cdot)$. We assume that there exists $d \geq 1$ (called the dimension of G) and $c_G, C_G > 0$ such that

$$c_G k^d \leq |B_k| \leq C_G k^d, \quad \forall k \geq 1. \quad (11.C.1)$$

Remark 11.C.1. Usually a graph is said to have (at most) polynomial growth if there are constants $d \geq 1$ and $C_G > 0$ such that the upper bound $|B_k| \leq C_G k^d$ is satisfied for every $k \geq 1$. That one can always find c_G, C_G and d such that (11.C.1) holds follows from Gromov's theorem [124].

Remark 11.C.2. It was proved in [241] (see also [44] for a relevant result) that for every vertex-transitive graph of polynomial growth there are constants $C > 0$ and $\delta > 0$ such that

$$\frac{|B_n \setminus B_{n-1}|}{|B_n|} \leq C n^{-\delta} \quad \text{for every } n \geq 1. \quad (11.C.2)$$

In fact, this was proved under the weaker assumption that the graph satisfies a (uniform) doubling property, i.e. without the assumption of vertex-transitivity, which is easy to check in our case. To the best of our knowledge, it is not known whether one can take $\delta = 1$ for all vertex-transitive graphs of polynomial growth.

Remark 11.C.3. The complications in generalising the results of [212, 175] to the context of vertex-transitive graphs of polynomial growth arise from the fact that it is not known whether one can take $\delta = 1$ in the remark above. The results of the aforementioned articles apply to a much more general class of unbounded spin systems, which includes in particular the case $g = 0$ for some a chosen appropriately, and generalising our results to the whole class of unbounded spin systems considered there without a good understanding of the underlying graph structure seems rather complicated. The fact that we have a quartic term in our single site measure, i.e. that $g > 0$, plays an important role in our analysis.

We fix a set of interactions $J = (J_{x,y})_{x,y \in V}$ such that $J_{x,y} \geq 0$ for all $x, y \in V$ and

$$J_{x,y} \leq \Psi(d_G(x, y)), \quad \text{for all } x \neq y \in V \quad (11.C.3)$$

where $\Psi : \mathbb{N} \rightarrow \mathbb{R}_+$ is defined as $\Psi(k) = C_\Psi k^{-d-\varepsilon}$ for some constants $C_\Psi > 0$ and $\varepsilon > 0$. In addition, we fix (β, h, g, a) . Recall that for $\eta \in \mathbb{R}^V$, the φ^4 model with these parameters on $\Lambda \subset V$ is the measure $\langle \cdot \rangle_{\Lambda, \beta, h}^\eta$ defined by (11.1.5). In this section, we only keep the dependence on Λ explicit and it is sometimes convenient to write ν_Λ^η to denote the corresponding measure.

For a Borel probability measure λ on $\mathbb{R}^V$, we write ν_Λ^λ to denote the corresponding measure with random boundary conditions sampled from λ . Let

$$\Xi(M) = \{\varphi \in \mathbb{R}^V : \exists \Lambda_\varphi \subset V \text{ finite, } \varphi_x^4 \leq M \log d_G(o, x) \quad \forall x \in V \setminus \Lambda_\varphi\}. \quad (11.C.4)$$

We call λ a law on weakly growing boundary conditions if λ is a Borel probability measure on $\mathbb{R}^V$ supported on $\Xi(M)$ for some $M > 0$. Finally, for $\Delta \subset \Lambda$, we write $\nu_\Delta^{(\Lambda),\lambda}$ to denote the restriction of ν_Λ^λ to $\mathbb{R}^\Delta$, i.e. the measure defined by

$$\nu_\Delta^{(\Lambda),\lambda}(\varphi_\Delta) = \frac{1}{Z_\Lambda^\lambda} \int_{\mathbb{R}^{\Lambda \setminus \Delta}} e^{-U_\Lambda^\lambda(\varphi_\Delta \sqcup \varphi_{\Lambda \setminus \Delta})} d\varphi_{\Lambda \setminus \Delta} \quad (11.C.5)$$

where $\varphi_\Delta \sqcup \varphi_{\Lambda \setminus \Delta}$ denotes the concatenation of two configurations and

$$U_\Lambda^\lambda(\varphi) = \beta H_\Lambda^\lambda(\varphi) + \sum_{x \in \Lambda} (g\varphi_x^4 + a\varphi_x^2) \quad (11.C.6)$$

is the potential.

We now recall the definition of regular measures. Let ν be a measure on $\mathbb{R}^V$. For $\gamma, \delta > 0$, we say ν is (γ, δ) -regular if, for every $\Lambda \subset V$ finite, the density of the projection measure satisfies

$$\frac{d\nu}{d\varphi_\Lambda}(\varphi_\Lambda) \leq e^{-\sum_{x \in \Lambda} (\gamma\varphi_x^4 - \delta)} \quad (11.C.7)$$

where $d\varphi_\Lambda$ is the Lebesgue measure on $\mathbb{R}^\Lambda$. Note that, for any $\Lambda \subset V$ finite and λ a law on weakly growing boundary conditions, ν_Λ^λ is (γ, δ) -regular with constants depending only on (β, h, g, a) . Moreover, the constants are continuous in these dependencies and uniform over all subsets of Λ . This is a direct consequence of repeatedly applying Hölder's inequalities on $|H_\Lambda(\varphi)|$.

Finally, we restate the definition of a Gibbs measure. Recall that a Borel probability measure ν on $\mathbb{R}^V$ is *tempered* if:

- for every $\Lambda \subset V$ finite, $\nu|_\Lambda$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^\Lambda$;
- ν is supported on the event

$$R = \left\{ (\varphi_x)_{x \in V} : \exists C \in (0, \infty), \sum_{x \in B_k} \varphi_x^2 \leq C|B_k| \text{ for all } k \in \mathbb{N} \right\} \quad (11.C.8)$$

i.e. $\nu(R) = 1$.

Furthermore, recall that a Borel probability measure ν on $\mathbb{R}^V$ is a Gibbs measure at (β, h) if it is tempered and for every finite $\Lambda \subset V$ and $f \in \mathbb{R}^\Lambda$ bounded and measurable, the DLR equation

$$\nu(f) = \int_{\eta \in \mathbb{R}^V} \langle f \rangle_{\Lambda, \beta, h}^\eta d\nu(\eta) \quad (11.C.9)$$

holds. We call $\mathcal{G}(\beta, h)$ the set of Gibbs measures.

The main results of this section are the following.

Proposition 11.C.4. *Let $(\nu_{\Lambda_n})_{n \geq 0}$ be a sequence of finite volume measures with (random) weakly growing boundary conditions. Then, the sequence is tight and any limit point is a tempered probability measure.*

Proposition 11.C.5. *Let $(\nu_{\Lambda_n})_{n \geq 0}$ be a sequence of finite volume measure with (random) weakly growing boundary conditions such that ν_{Λ_n} converges weakly to a tempered probability measure ν . Then, ν is a Gibbs measure, i.e. satisfies the DLR equations.*

Proposition 11.C.6. *For any (β, h) , there exist $\gamma = \gamma(\beta, h), \delta = \delta(\beta, h) > 0$ such that every Gibbs measure $\nu \in \mathcal{G}(\beta, h)$ is (γ, δ) -regular. Moreover, the constants (γ, δ) depend continuously on (β, h) .*

The proofs of Propositions 11.C.4–11.C.6 are based on the proofs of Theorems 4.3, 4.5, and 4.4 in [175], respectively⁸, adapted from the case of $\mathbb{Z}^d$ to our setting. The extensions of these proofs require us to develop some technical results, which broadly speaking allow us to distinguish as well as quantify bulk and boundary contributions from the potential. The key geometric estimate is contained in Proposition 11.C.11. The key probabilistic estimate is contained in Proposition 11.C.12, which in particular implies uniform regularity estimates for finite volume measures, see Corollary 11.C.13. Both key estimates require multiscale control over the decay and oscillations of the kernel Ψ – which in turn controls the decay of the interactions J – on scales determined by the geometry of the graph; this naturally imposes conditions jointly on the allowed graphs and interactions. For graphs of polynomial growth, these conditions are explicit; some partial results are obtained for general amenable graphs that satisfy certain regularity conditions, see Section 11.C.4. Since, given Proposition 11.C.12 and Corollary 11.C.13, the proofs of Propositions 11.C.4–11.C.6 are identical to [175], so we omit them and refer to the exact references given above.

11.C.2 Technical results on bulk-boundary contributions

We first verify that Ψ is indeed summable on G .

Lemma 11.C.7. *We have that*

$$\|\Psi\| := \sup_{x \in V} \sum_{y \in V} \Psi(d_G(x, y)) = \sum_{y \in V} \Psi(d_G(o, y)) < \infty. \quad (11.C.10)$$

Proof. By summation by parts and Γ -invariance, we have that

$$\sum_{y \in V} \Psi(d_G(x, y)) = \sum_{k=0}^{\infty} (\Psi(k) - \Psi(k+1)) |B_k|. \quad (11.C.11)$$

Note that $\Psi(k) - \Psi(k+1) = O(k^{-(d+1+\varepsilon)})$, from which the result follows. $\square$

We are interested in quantifying the oscillatory behaviour of Ψ at large scales. In our analysis we require the scales to have a nonlinear dependence depending on the dimension. Let $\alpha = \frac{d}{\varepsilon} + 1$ and define $F_n := B_{t_n}$, where $t_n = \lfloor n^\alpha \rfloor$. We write $V_n = |F_n|$. We often refer to n as the scale even though, strictly speaking, the scale is t_n .

Definition 11.C.8. For $q, k \in \mathbb{N}$, define

$$\Psi_{k,q} = \sup_{x \in F_q, y \in V \setminus F_{q+k}} \Psi(d_G(x, y)). \quad (11.C.12)$$

⁸Note that the proof of [175, Theorem 4.5] is given in the appendix of [175], where it is mislabelled as proof of Theorem 4.4.

The coefficient $\Psi_{k,q}$ tracks the oscillation of Ψ from scale q to scale $q+k$. The next lemma states that the decay condition on Ψ is enough to ensure that asymptotically the oscillations are small with respect to the volume.

Lemma 11.C.9. *We have that*

$$\lim_{q \rightarrow \infty} \sum_{k=1}^{\infty} (\Psi_{k,q} - \Psi_{k+1,q}) V_{q+k+1} = 0. \quad (11.C.13)$$

Proof. By summation by parts

$$\sum_{k=1}^{\infty} (\Psi_{k,q} - \Psi_{k+1,q}) V_{q+k+1} \leq \Psi_{1,q} V_{q+2} + C_G \sum_{k=2}^{\infty} \Psi_{k,q} (t_{q+k+1}^d - t_{q+k}^d) \quad (11.C.14)$$

where C_G is as in (11.C.1). Note that $\Psi_{k,q} = C_\Psi (t_{q+k} + 1 - t_q)^{-d-\varepsilon}$. On one hand, by telescoping and a simple calculation

$$t_{q+k} + 1 - t_q \geq \sum_{i=1}^k t_{q+i} - t_{q+i-1} \geq c_1 \sum_{i=\lfloor k/2 \rfloor}^k (q+i)^{\alpha-1} \geq c_2 k (q+k)^{\alpha-1} \quad (11.C.15)$$

for some constants $c_1, c_2 > 0$, hence $\Psi_{k,q} \leq C_\Psi (c_2 k)^{-d-\varepsilon} (q+k)^{-(\alpha-1)(d+\varepsilon)}$. On the other hand, $t_{q+k+1}^d - t_{q+k}^d = O((q+k)^{\alpha d-1})$. Since $(\alpha-1)(d+\varepsilon) > \alpha d - 1$, the desired assertion follows. $\square$

We now introduce a sequence $(\psi_j)_{j \in \mathbb{N}}$ that plays the role of C in (11.C.8) from the definition of tempered measures. It is chosen to increase from scale to scale sufficiently slowly so that the following holds.

Lemma 11.C.10. *There exists an increasing sequence $(\psi_j)_{j \in \mathbb{N}}$ converging to infinity such that $\psi_1 \geq 1$,*

$$\lim_{j \rightarrow \infty} \frac{\psi_{j+1}}{\psi_j} = 1 \quad (11.C.16)$$

and

$$\lim_{q \rightarrow \infty} \sum_{k=1}^{\infty} (\Psi_{k,q} - \Psi_{k+1,q}) \psi_{q+k+2} V_{q+k+2} = 0. \quad (11.C.17)$$

Proof. Let $(k_n)_{n \in \mathbb{N}}$ be a strictly increasing sequence of positive integers such that

$$\sum_{k=m_{n,q}}^{\infty} (\Psi_{k,q} - \Psi_{k+1,q}) V_{q+k+2} \leq 2^{-n} \quad \text{for every } q \geq 1 \quad (11.C.18)$$

where $m_{n,q} := \max\{1, k_n - q - 2\}$. Such a sequence exists because of (11.C.13) together with the fact that $V_{q+k+2} = O(V_{q+k+1})$.

Define

$$\psi_j := \begin{cases} 1, & \text{for } j < k_1 \\ n, & \text{for } k_n \leq j < k_{n+1}, n \geq 1. \end{cases} \quad (11.C.19)$$

Then, $\psi_1 \geq 1$, $\psi_{j+1}/\psi_j \rightarrow 1$, and for every $q \geq k_1 - 1$,

$$\sum_{k=1}^{\infty} (\Psi_{k,q} - \Psi_{k+1,q}) \psi_{q+k+2} V_{q+k+2} = \sum_{n=n(q)}^{\infty} \sum_{k=m_{n,q}}^{m_{n+1,q}-1} (\Psi_{k,q} - \Psi_{k+1,q}) \psi_{q+k+2} V_{q+k+2} \quad (11.C.20)$$

$$\leq \sum_{n=n(q)}^{\infty} n 2^{-n} \rightarrow 0 \quad (11.C.21)$$

as $q \rightarrow \infty$, where $n(q) := \max\{n \geq 1 \mid m_{n,q} = 1\}$. $\square$

We now state a technical result which plays an important role in the proof of the key probabilistic estimate in the next section. Loosely speaking, it states that, at the last scale q at which the bulk contribution of $\sum_{x \in F_q} \varphi_x^2$ dominates the volume, the boundary behaviour on the subsequent scale can be controlled by the bulk behaviour.

Proposition 11.C.11. *There exists a constant $C_1 > 0$ such that the following holds. Let $\varphi \in \mathbb{R}^V$. Suppose there exists $q = q(\varphi) \geq 1$ such that q is the largest integer for which*

$$\sum_{x \in F_q} \varphi_x^2 \geq \psi_q V_q. \quad (11.C.22)$$

Then,

$$\frac{1}{2} \sum_{x \in F_{q+1}} \sum_{y \in V \setminus F_{q+1}} \Psi(d_G(x, y)) (\varphi_x^2 + \varphi_y^2) \leq C_1 \sum_{x \in F_{q+1}} \varphi_x^2. \quad (11.C.23)$$

Proof. First observe that, for $k \geq 0$, we have

$$\begin{aligned} \sum_{k=1}^{\infty} \Psi_{k,q+1} \sum_{x \in F_{q+k+2} \setminus F_{q+k+1}} \varphi_x^2 &= \sum_{k=1}^{\infty} (\Psi_{k,q+1} - \Psi_{k+1,q+1}) \sum_{x \in F_{q+k+2} \setminus F_{q+2}} \varphi_x^2 \\ &\leq \sum_{k=1}^{\infty} (\Psi_{k,q+1} - \Psi_{k+1,q+1}) \psi_{q+k+2} V_{q+k+2}. \end{aligned} \quad (11.C.24)$$

We now handle the double sum by taking cases according to the position of x and y . Let us write

$$D := \sum_{x \in F_{q+1}} \sum_{y \in V \setminus F_{q+1}} \Psi(d_G(x, y)) (\varphi_x^2 + \varphi_y^2) \quad (11.C.25)$$

$$= \sum_{y \in F_{q+2} \setminus F_{q+1}} \varphi_y^2 \sum_{x \in F_{q+1}} \Psi(d_G(x, y)) + \sum_{y \in V \setminus F_{q+2}} \varphi_y^2 \sum_{x \in F_{q+1}} \Psi(d_G(x, y)) \quad (11.C.26)$$

$$+ \sum_{x \in F_{q+1}} \varphi_x^2 \sum_{y \in V \setminus F_{q+1}} \Psi(d_G(x, y)) \quad (11.C.27)$$

$$=: D_1 + D_2 + D_3. \quad (11.C.28)$$

Observe that, by Lemma 11.C.7,

$$D_1 + D_3 \leq \|\Psi\| \sum_{x \in F_{q+2}} \varphi_x^2 \leq \|\Psi\| \psi_{q+2} V_{q+2}. \quad (11.C.29)$$

Since $V_{q+2} = O(V_q)$ and $\psi_{q+2}/\psi_q \rightarrow 1$ as $q \rightarrow \infty$, it then follows that

$$D_1 + D_3 \leq C_3 \psi_q V_q \leq C_3 \sum_{x \in F_{q+1}} \varphi_x^2 \quad (11.C.30)$$

for some $C_3 > 0$ large enough.

To handle D_2 , we bound $\Psi(d_G(x, y))$ by $\Psi_{k, q+1}$ for $x \in F_{q+1}$ and $y \in F_{q+k+2} \setminus F_{q+k+1}$ to obtain that

$$D_2 \leq V_{q+1} \sum_{k=1}^{\infty} \Psi_{k, q} \sum_{y \in F_{q+k+2} \setminus F_{q+k+1}} \varphi_y^2. \quad (11.C.31)$$

Using (11.C.24) we obtain that

$$D_2 \leq V_{q+1} \sum_{k=1}^{\infty} (\Psi_{k, q+1} - \Psi_{k+1, q+1}) \psi_{q+k+2} V_{q+k+2}. \quad (11.C.32)$$

By (11.C.17) and the fact that $V_{q+1} = O(V_q)$ we have

$$D_2 \leq C_4 \psi_q V_q \leq C_4 \sum_{x \in F_{q+1}} \varphi_x^2 \quad (11.C.33)$$

for some constant $C_4 > 0$. The desired assertion follows readily. $\square$

11.C.3 Key probabilistic regularity estimates

We now prove the key probabilistic estimates which justify the extension of the proofs of Propositions 11.C.4, 11.C.5 and 11.C.6 for the case of $\mathbb{Z}^d$ to our setting. We state it for deterministic boundary conditions; the proof is exactly the same in the case of random tempered boundary conditions, although with heavier notation.

Proposition 11.C.12. *Let $q_0 > 0$ and let $\Lambda = B_n$ for $n > t_{q_0}$ so that $F_{q_0} \subset \Lambda$. There exists $c_1 = c_1(q_0) > 0$ such that, for all $\Delta \subset \Lambda$ with $F_{q_0} \subset \Delta \subset \{x \in \Lambda : d_G(x, \partial\Lambda) > n^{\delta/2}\}$ where δ is as in Remark 11.C.2, and for any $x \in \Delta$, the following decomposition holds. For all $\eta \in \mathbb{R}^V$ such that $|\eta_z| \leq \mathfrak{p}_z$ for all $z \in \Lambda^c$,*

$$\nu_{\Delta}^{(\Lambda), \eta}[\varphi_{\Delta}] = \nu_{\Delta}^{1, \eta}[\varphi_{\Delta}] + \nu_{\Delta}^{2, \eta}[\varphi_{\Delta}], \quad \forall \varphi_{\Delta} \in \mathbb{R}^{\Delta} \quad (11.C.34)$$

and

$$\nu_{\Delta}^{1, \eta}[\varphi_{\Delta}] \leq c_1^{|\Delta|} e^{-\frac{g}{2} \varphi_x^4} \nu_{\Delta \setminus \{x\}}^{(\Lambda)}[\varphi_{\Delta \setminus \{x\}}] \quad (11.C.35)$$

$$\nu_{\Delta}^{2, \eta}[\varphi_{\Delta}] \leq \sum_{q \geq q_0} e^{-\sum_{z \in F_{q+1}(x) \cap \Delta} \frac{g}{2} \varphi_z^4 + c_1 V_{q+1}} \nu_{\Delta \setminus F_{q+1}(x)}^{(\Lambda)}[\varphi_{\Delta \setminus F_{q+1}(x)}]. \quad (11.C.36)$$

We stress that c_1 uniform over all such Δ and η , and moreover that it is continuous in the parameters of the model.

Proof. Let $q_0 \in \mathbb{N}$ be a sufficiently large constant to be a posteriori. Fix $x \in \Delta$. Write

$$A_{\Delta, q_0}(x) = \{\varphi_{\Delta} \in \mathbb{R}^{\Delta} : \sum_{y \in F_q(x)} \varphi_y^2 \leq \psi_q V_q \text{ for all } q \geq q_0\} \quad (11.C.37)$$

where $F_q(x) = \gamma_x F_q$ where γ_x is an automorphism such that $\gamma_x o = x$. Define

$$\nu_\Delta^{1,\eta}[\cdot] := \nu_\Delta^{(\Lambda),\eta}[\cdot \mathbb{1}\{A_{\Delta,q_0}(x)\}], \quad \nu_\Delta^{2,\eta} := \nu_\Delta^{(\Lambda)} - \nu_\Delta^{1,\eta}. \quad (11.C.38)$$

On the event $A_{\Delta,q_0}(x)$, we are able to control the bulk behaviour (centred at x) of φ_y^2 in spatial average; on the complement we have control via the oscillation and decay bounds.

We first establish (11.C.35). For $\Lambda = \Lambda_1 \sqcup \Lambda_2$, for all $\varphi'_\Lambda, \varphi_\Lambda \in \mathbb{R}^\Lambda$, define

$$W_\Lambda(\varphi'_\Lambda, \varphi_\Lambda) := U_{\Lambda_1}^\eta(\varphi'_\Lambda) + U_{\Lambda_2}^\eta(\varphi_\Lambda) - U_\Lambda^\eta(\varphi'_\Lambda \sqcup \varphi_\Lambda). \quad (11.C.39)$$

In particular,

$$W_\Lambda(\varphi_{\Lambda_1}, \varphi_{\Lambda_2}) = \left\{ \sum_{x,y \in \Lambda_1} + \sum_{x,y \in \Lambda_2} - \sum_{x,y \in \Lambda} \right\} J_{x,y} \varphi_x \varphi_y = - \sum_{x \in \Lambda_1, y \in \Lambda_2} J_{x,y} \varphi_x \varphi_y. \quad (11.C.40)$$

In particular, observe that for our fixed $x \in \Delta$,

$$W_\Lambda(\varphi_x, \varphi_{\Lambda \setminus \{x\}}) = W_\Lambda(\varphi'_x, \varphi_{\Lambda \setminus \{x\}}) - \sum_{y \in \Lambda \setminus \{x\}} J_{x,y} (\varphi_x - \varphi'_x) \varphi_y. \quad (11.C.41)$$

Thus, by Young's inequality,

$$\left| \sum_{y \in \Lambda \setminus \{x\}} J_{x,y} (\varphi_x - \varphi'_x) \varphi_y \right| \leq \frac{\sum_{y \in \Lambda \setminus \{x\}} J_{x,y}}{2} (\varphi_x^2 + (\varphi'_x)^2) + T_1 \quad (11.C.42)$$

where

$$T_1 = \sum_{y \in \Lambda \setminus \{x\}} J_{x,y} \varphi_y^2 \leq \sum_{y \in F_{q_0}(x)} J_{x,y} \varphi_y^2 + \sum_{y \in \Lambda \setminus F_{q_0}(x)} J_{x,y} \varphi_y^2 := T_2 + T_3. \quad (11.C.43)$$

By the definition of $A_{\Delta,q_0}(x)$,

$$T_2 \leq |J|_\infty \psi_{q_0} V_{q_0}. \quad (11.C.44)$$

In order to handle T_3 , we observe that

$$\sum_{y \in F_q(x)} \varphi_y^2 \leq \psi_q V_q, \quad \forall q \geq q_0. \quad (11.C.45)$$

Moreover,

$$\sum_{y \in V \setminus F_{q_0}(x)} \Psi(d_G(x, y)) \varphi_y^2 \leq \sum_{k \geq 1} \Psi_{k,q_0-1} \sum_{y \in F_{q_0+k}(x) \setminus F_{q_0-1+k}(x)} \varphi_y^2 \quad (11.C.46)$$

$$= \sum_{k \geq 1} (\Psi_{k,q_0-1} - \Psi_{k+1,q_0-1}) \sum_{y \in F_{q_0+k}(x) \setminus F_{q_0}(x)} \varphi_y^2 \quad (11.C.47)$$

$$\leq \sum_{k \geq 1} (\Psi_{k,q_0-1} - \Psi_{k+1,q_0-1}) \psi_{q_0+k} V_{q_0+k} \quad (11.C.48)$$

where in the first inequality we have bounded $\Psi(d_G(x, y))$ by Ψ_{k,q_0-1} for $y \in F_{q_0+k}(x) \setminus F_{q_0-1+k}(x)$ (we use that $x \in F_{q_0-1}(x)$), then in the second inequality we use a summation by

parts. Combining the above estimates together and using Lemma 11.C.10, we have established that

$$T_3 \leq \sum_{k \geq 1} (\Psi_{k,q_0-1} - \Psi_{k+1,q_0-1}) \psi_{q_0+k} V_{q_0+k} \leq C_1. \quad (11.C.49)$$

All together we have established

$$|W_\Lambda(\varphi_x, \varphi_{\Lambda \setminus \{x\}}) - W_\Lambda(\varphi'_x, \varphi_{\Lambda \setminus \{x\}})| \leq \frac{\|\Psi\|}{2} (\varphi_x^2 + (\varphi'_x)^2) + |J|_\infty \psi_{q_0} V_{q_0} + C_1. \quad (11.C.50)$$

We now bound $\nu_\Delta^{1,\eta}[\varphi_\Delta]$. First observe that, for any $\varphi'_x \in [-A, A]$, where $A > 0$,

$$\nu_\Delta^{1,\eta}[\varphi_\Delta] = Z_\Lambda^{-1} \int e^{-U_{\{x\}}^\eta(\varphi_x) - U_{\Lambda \setminus \{x\}}^\eta(\varphi_{\Lambda \setminus \{x\}}) + W(\varphi_x, \varphi_{\Lambda \setminus \{x\}})} d\varphi_{\Lambda \setminus \Delta} \quad (11.C.51)$$

$$= e^{-U_{\{x\}}^\eta(\varphi_x)} Z_\Lambda^{-1} \int e^{-U_{\Lambda \setminus \{x\}}^\eta(\varphi_{\Lambda \setminus \{x\}}) + W(\varphi'_x, \varphi_{\Lambda \setminus \{x\}}) - W(\varphi'_x, \varphi_{\Lambda \setminus \{x\}}) + W(\varphi_x, \varphi_{\Lambda \setminus \{x\}})} d\varphi_{\Lambda \setminus \Delta} \quad (11.C.52)$$

$$\lesssim e^{-U_{\{x\}}^\eta(\varphi_x)} e^{\frac{\sum_{y \in V} J_{0,y}}{2} (\varphi_x^2 + (\varphi'_x)^2)} \int e^{-U_{\Lambda \setminus \{x\}}^\eta(\varphi_{\Lambda \setminus \{x\}}) + W(\varphi'_x, \varphi_{\Lambda \setminus \{x\}})} d\varphi_{\Lambda \setminus \Delta} \quad (11.C.53)$$

$$\lesssim e^{-\frac{g}{2} \varphi_x^4} \int e^{-U_\Lambda^\eta(\varphi_{\Lambda \setminus \{x\}} \sqcup \varphi'_x) + U_{\{x\}}^\eta(\varphi'_x)} d\varphi_{\Lambda \setminus \Delta} \quad (11.C.54)$$

$$\lesssim e^{-\frac{g}{2} \varphi_x^4} \int e^{-U_\Lambda^\eta(\varphi_{\Lambda \setminus \{x\}} \sqcup \varphi'_x)} d\varphi_{\Lambda \setminus \Delta}. \quad (11.C.55)$$

Above, the first two lines are straightforward manipulations; the third line follows from the preceding bounds; the fourth line follows from Young's inequality inside the exponentials, the fact that $|\varphi'_x| \leq A$, and estimates similar to (11.2.82) to handle the boundary terms in $U_{\{x\}}^\eta(\varphi_x)$; and the final line is by using that $|\varphi'_x| \leq A$. Thus, taking the integral mean,

$$\nu_\Delta^{1,\eta}[\varphi_\Delta] \leq c_1^{|\Delta|} e^{-\frac{g}{2} \varphi_x^4} \int_{-A}^A \int e^{-U_\Lambda^\eta(\varphi_{\Lambda \setminus \{x\}} \sqcup \varphi'_x)} d\varphi_{\Lambda \setminus \Delta} d\varphi'_x \leq c_1^{|\Delta|} e^{-\frac{g}{2} \varphi_x^4} \int e^{-U_\Lambda^\eta(\varphi_\Lambda)} d\varphi_{\Lambda \setminus \Delta \cup \{x\}}. \quad (11.C.56)$$

This last bound establishes (11.C.35).

We now turn to (11.C.36). Write

$$B_{\Delta,q}(x) = \{\varphi_\Delta \in \mathbb{R}^\Delta : \sum_{y \in F_q(x)} \varphi_y^2 > \psi_q V_q, \sum_{y \in F_k(x)} \varphi_y^2 \leq \psi_k V_k, \forall k > q\}. \quad (11.C.57)$$

Then, by a union bound and re-ordering terms

$$\nu_{\Delta}^{2,\eta}[\varphi_{\Delta}] \leq \sum_{q \geq q_0} \frac{\mathbb{1}_{\{\varphi_{\Delta} \in B_{\Delta,q}(x)\}}}{Z_{\Lambda}} \int e^{-U_{\Lambda}^{\eta}(\varphi_{\Lambda})} d\varphi_{\Lambda \setminus \Delta} \quad (11.C.58)$$

$$= \sum_{q \geq q_0} \frac{\mathbb{1}_{\{\varphi_{\Delta} \in B_{\Delta,q}(x)\}}}{Z_{\Lambda}} \int e^{-U_{F_{q+1}(x) \cap \Delta}^{\eta}(\varphi_{F_{q+1}(x) \cap \Delta}) - U_{\Lambda \setminus (F_{q+1}(x) \cap \Delta)}^{\eta}(\varphi_{\Lambda \setminus (F_{q+1}(x) \cap \Delta)})} \quad (11.C.59)$$

$$\times e^{W_{\Lambda}(\varphi_{F_{q+1}(x) \cap \Delta}, \varphi_{\Lambda \setminus (F_{q+1}(x) \cap \Delta)})} d\varphi_{\Lambda \setminus \Delta} \quad (11.C.60)$$

$$\leq \sum_{q \geq q_0} \frac{\mathbb{1}_{\{\varphi_{\Delta} \in B_{\Delta,q}\}}}{Z_{\Lambda}} e^{-\sum_{y \in F_{q+1}(x) \cap \Delta} \frac{3g}{4} \varphi_y^4 + C V_{q+1}} \quad (11.C.61)$$

$$\times \int e^{-U_{\Lambda \setminus (F_{q+1}(x) \cap \Delta)}^{\eta}(\varphi_{\Lambda \setminus (F_{q+1}(x) \cap \Delta)}) + W_{\Lambda}(\varphi_{F_{q+1}(x) \cap \Delta}, \varphi_{\Lambda \setminus (F_{q+1}(x) \cap \Delta)})} d\varphi_{\Lambda \setminus \Delta} \quad (11.C.62)$$

where the third line is uses similar reasoning as above to handle $U_{F_{q+1}(x) \cap \Delta}^{\eta}(\varphi_{F_{q+1}(x) \cap \Delta})$.

Let $\varphi'_{F_{q+1}(x) \cap \Delta} \in \mathbb{R}^{F_{q+1}(x) \cap \Delta}$ be such that $|\varphi'_z| \leq A$ for all $z \in F_{q+1}(x) \cap \Delta$. Observe that we can write

$$W_{\Lambda}(\varphi_{F_{q+1}(x) \cap \Delta}, \varphi_{\Lambda \setminus (F_{q+1}(x) \cap \Delta)}) = W(\varphi'_{F_{q+1}(x) \cap \Delta}, \varphi_{\Lambda \setminus (F_{q+1}(x) \cap \Delta)}) + T_4 \quad (11.C.63)$$

where

$$|T_4| \leq \sum_{\substack{z \in F_{q+1}(x) \cap \Delta \\ y \in \Lambda \setminus (F_{q+1}(x) \cap \Delta)}} \Psi(d_G(z, y)) \left(\frac{1}{2} \varphi_z^2 + \frac{1}{2} (\varphi'_z)^2 + \varphi_y^2 \right) \quad (11.C.64)$$

On the event $B_{\Delta,q}(x)$, we can apply Proposition 11.C.11 and Γ -invariance to obtain

$$(11.C.64) \leq C_2 \sum_{z \in F_{q+1} \cap \Delta} \varphi_z^2 + C_3 \sum_{z \in F_{q+1}(x) \cap \Delta} (\varphi'_z)^2. \quad (11.C.65)$$

Thus, by Young's inequality, using the boundedness of φ'_x , we obtain

$$\nu_{\Delta}^{2,\eta}[\varphi_{\Delta}] \leq \sum_{q \geq q_0} \frac{1}{Z_{\Lambda}} e^{-\sum_{y \in F_{q+1}(x) \cap \Delta} \frac{g}{2} \varphi_y^4 + C_4 V_{q+1}} \quad (11.C.66)$$

$$\times \int e^{-U_{\Lambda \setminus (F_{q+1}(x) \cap \Delta)}^{\eta}(\varphi_{\Lambda \setminus (F_{q+1}(x) \cap \Delta)}) + W_{\Lambda}(\varphi'_{F_{q+1}(x) \cap \Delta}, \varphi_{\Lambda \setminus (F_{q+1}(x) \cap \Delta)})} d\varphi_{\Lambda \setminus \Delta} \quad (11.C.67)$$

$$= \sum_{q \geq q_0} \frac{1}{Z_{\Lambda}} e^{-\sum_{y \in F_{q+1}(x) \cap \Delta} \frac{g}{2} \varphi_y^4 + C_4 V_{q+1}} \quad (11.C.68)$$

$$\times \int e^{-U_{\Lambda \setminus (F_{q+1}(x) \cap \Delta)}^{\eta}(\varphi_{\Lambda \setminus (F_{q+1}(x) \cap \Delta)}) \sqcup \varphi'_{F_{q+1}(x) \cap \Delta} + U_{F_{q+1}(x) \cap \Delta}^{\eta}(\varphi'_{F_{q+1}(x) \cap \Delta})} d\varphi_{\Lambda \setminus \Delta}. \quad (11.C.69)$$

The desired estimate follows by using the boundedness of $|\varphi'_{F_{q+1}(x) \cap \Delta}|$ and taking the integral mean with respect to $\varphi'_{F_{q+1} \cap \Lambda}$, similarly as for the estimate on $\nu_{\Delta}^{1,\eta}$. $\square$

Corollary 11.C.13. *There exists $c_1 > 0$ such that the following holds. Let $n \geq 1$ and let $\Lambda \subset V$ be finite such that $\Lambda \supset B_n$. Let $\Delta \subset \Lambda$ such that $\Delta \subset B_m$, where $m = n - n^{\frac{\delta}{2}}$ and δ is as in Remark 11.C.2. Then, for all $\eta \in \mathbb{R}^V$ such that $|\eta_z| \leq \mathfrak{p}_z$ for all $z \in \Lambda^c$,*

$$\nu_{\Delta}^{(\Lambda), \eta}[\varphi_{\Delta}] \leq e^{-\sum_{x \in \Delta} \left(\frac{g}{2} \varphi_x^4 - c_1 \right)}. \quad (11.C.70)$$

Proof. The result is based on an induction argument that, given Proposition 11.C.12, is identical to the proof of [212, Theorem 2.2]. We omit the details. $\square$

Remark 11.C.14. The result of Corollary 11.C.13 can be extended up to the boundary with the cost of setting $c_1 = O(\log d_G(o, \partial\Lambda))$. In particular, by using an exponential Chebyshev argument, one can bound moments; for instance

$$\sup_{\Delta \subset \Lambda} \sup_{x \in \Delta} \nu_{\Delta}^{(\Lambda), \eta}[\varphi_x^2] \leq \sqrt{C \log d_G(o, \partial\Lambda)} \quad (11.C.71)$$

for some $C > 0$.

11.C.4 Generalisation to tempered amenable graphs

The results of this section can be generalised to (supposedly) a larger class of amenable graphs that we now introduce. Recall that a connected, locally finite, and vertex-transitive graph $G = (V, E)$ is called amenable if

$$\inf_{A \subset V, |A| < \infty} \frac{|\partial A|}{|A|} = 0. \quad (11.C.72)$$

Equivalently, G is amenable if and only if it admits a Følner sequence $(F_n)_{n \in \mathbb{N}} \subset V$, i.e. a sequence $(F_n)_{n \in \mathbb{N}}$ such that $\frac{|\partial F_n|}{|F_n|} \rightarrow 0$ as $n \rightarrow \infty$. We call $(F_n)_{n \in \mathbb{N}}$ a Følner exhaustion if it is a Følner sequence and, in addition, $F_n \uparrow V$ as $n \rightarrow \infty$.

We say that a connected, locally finite, vertex-transitive graph G is a *tempered amenable graph* if it admits a function $f : \mathbb{N} \rightarrow (0, \infty)$ such that $f(x+y) \leq f(x)f(y)$ and a Følner exhaustion $(F_n)_{n \in \mathbb{N}}$ that satisfy

$$\sum_{x \in V \setminus \{o\}} \frac{1}{f(d_G(o, x))} < \infty \quad (11.C.73)$$

$$\limsup_{n \rightarrow \infty} \frac{|F_{n+1}|}{|F_n|} < \infty \quad (11.C.74)$$

$$F_n \cup \partial F_n \subset F_{n+1} \quad (11.C.75)$$

$$\lim_{n \rightarrow \infty} \frac{(|F_{n+1}| - |F_n|) \max_{x \in F_{n+1} \setminus F_n} \{ \sqrt[4]{\log f(d_G(o, x))} \}}{|F_{n+1}|} = 0. \quad (11.C.76)$$

We restrict to kernels $\Psi : \mathbb{N} \rightarrow (0, \infty)$ which are decreasing and satisfy

$$\lim_{q \rightarrow \infty} \sum_{k=1}^{\infty} (\Psi_{k,q} - \Psi_{k+1,q}) |F_{q+k+1}| = 0. \quad (11.C.77)$$

Moreover, we define

$$\Xi(M) = \left\{ \varphi : \exists V_\varphi \subset V, |\varphi_x| \leq \sqrt[4]{M \log f(d_G(o, x))} \forall x \in V_\varphi^c \right\}. \quad (11.C.78)$$

Then, one can check that the proofs of Propositions 11.C.4-11.C.6 extend to this setting.

Remark 11.C.15. Any connected, vertex-transitive graph of polynomial growth is a tempered amenable graph. It would be interesting to find amenable graphs of intermediate or exponential growth that satisfy our assumptions; we do not know of any such examples as of yet.

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Chapter 12

Random currents on the complete graph

We study the percolation configuration arising from the random current representation of the near-critical Ising model on the complete graph. We compute the scaling limit of the cluster size distribution for an arbitrary set of sources in the single and the double current measures. As a byproduct, we compute the tangling probabilities introduced in Chapter 11. This provides a new perspective on the switching lemma for the φ^4 model introduced in the same paper: in the Gaussian limit we recover Wick’s law, while in the Ising limit we recover the corresponding tool for the Ising model.

This chapter is based on the preprint “*Scaling limit of the cluster size distribution for the random current measure on the complete graph*” [169]. It is a joint work with Dmitrii Krachun and Christoforos Panagiotis.

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12.1 Introduction

For $n \geq 1$, consider the complete graph $K_n = (V_n, \mathcal{P}_2(V_n))$ on n vertices, where $\mathcal{P}_2(V_n)$ is the set of pairs of elements of V_n . Let $\lambda \in \mathbb{R}$, and define the inverse temperature

$$d_n(\lambda) := \frac{1}{n} \left(1 - \frac{\lambda}{\sqrt{n}} \right). \quad (12.1.1)$$

For any $\sigma \in \{\pm 1\}^{V_n}$, define the Hamiltonian of the Ising model on the complete graph K_n by

$$\mathcal{H}_{n,\lambda}(\sigma) := - \sum_{\{i,j\} \in \mathcal{P}_2(V_n)} d_n(\lambda) \sigma_i \sigma_j. \quad (12.1.2)$$

The associated Ising measure is defined via the expectation values

$$\langle F(\sigma) \rangle_{n,\lambda} := \frac{1}{\mathcal{Z}_n(\lambda)} \sum_{\sigma \in \{\pm 1\}^{V_n}} F(\sigma) \exp(-\mathcal{H}_{n,\lambda}(\sigma)) \quad (12.1.3)$$

for $F : \{\pm 1\}^{V_n} \rightarrow \mathbb{R}^+$, where $\mathcal{Z}_n(\lambda)$ is the partition function of the model. This model, which is also known as the *near-critical Curie–Weiss* model, has been introduced independently by many people, including [240, 153], and since then, it has been extensively studied. See [101, Chapter 2] and references therein.

We are interested in a particular expansion of the partition function of the model called the *random current expansion* which provides an interesting geometric representation of the correlation functions of the model. This expansion was initially introduced in [120] to prove concavity of the magnetisation with a positive external field, but the probabilistic picture attached to it was developed in [4]. We refer to [76] for more information about this expansion.

A *current* on K_n is a function $\mathbf{n} : \mathcal{P}_2(V_n) \rightarrow \mathbb{N} := \{0, 1, \dots\}$. For $\{i, j\} \in \mathcal{P}_2(V_n)$ and $\mathbf{n} \in \Omega_n$, we will write $\mathbf{n}_{i,j} = \mathbf{n}(i, j)$ for the value of $\mathbf{n}$ on the edge $\{i, j\}$. We denote by Ω_n the set of currents on K_n . The set of *sources* of $\mathbf{n}$, denoted by $\partial \mathbf{n}$, is defined as

$$\partial \mathbf{n} := \left\{ x \in V_n : \sum_{y \in V_n \setminus \{x\}} \mathbf{n}_{x,y} \text{ is odd} \right\}. \quad (12.1.4)$$

It is an easy exercise to check that $|\partial \mathbf{n}|$ is always even. For a current $\mathbf{n}$ on K_n , introduce the weight

$$w(\mathbf{n}) = w_{n,\lambda}(\mathbf{n}) := \prod_{\{i,j\} \in \mathcal{P}_2(V_n)} \frac{d_n(\lambda)^{\mathbf{n}_{i,j}}}{\mathbf{n}_{i,j}!}. \quad (12.1.5)$$

For S an even subset of V_n (i.e. of even cardinality), let $Z_n^S(\lambda) := \sum_{\partial \mathbf{n} = S} w(\mathbf{n})$. When $S = \emptyset$, we let $Z_n^\emptyset(\lambda) = Z_n(\lambda)$. It is classical [120, 4] that

$$Z_n(\lambda) = 2^n Z_n(\lambda), \quad (12.1.6)$$

and furthermore, if $\sigma_S := \prod_{j \in S} \sigma_j$,

$$\langle \sigma_S \rangle_{n,\lambda} = \frac{Z_n^S(\lambda)}{Z_n^\emptyset(\lambda)}. \quad (12.1.7)$$

To each current $\mathbf{n}$, we can associate a percolation configuration by looking at the trace of the edges that carry a non-zero weight, i.e. $(\mathbf{1}[\mathbf{n}_{i,j} > 0])_{\{i,j\} \in \mathcal{P}_2(V_n)}$. As is standard in percolation theory, our interest lies in the connectivity properties of this percolation configuration. Let us import some standard terminology. If $x, y \in V_n$, we say that x is *connected* to y in $\mathbf{n}$ and write $x \xrightarrow{\mathbf{n}} y$, if there exists a sequence of points $x_0 = x, x_1, \dots, x_m = y$ such that $\mathbf{n}_{x_i, x_{i+1}} > 0$ for $0 \leq i \leq m-1$. For $x \in V_n$, we let $\mathcal{C}_x(\mathbf{n})$ be the set of vertices of K_n which are connected to x in $\mathbf{n}$. Note that for any $x \in V_n$, $\mathcal{C}_x(\mathbf{n})$ contains an even number of sources.

We naturally define a probability measure $\mathbb{P}_{n,\lambda}^S$ on the set of currents with source set S by letting

$$\mathbb{P}_{n,\lambda}^S[\mathbf{n}] := \frac{1}{Z_n^S(\lambda)} w(\mathbf{n}) \mathbb{1}[\partial \mathbf{n} = S]. \quad (12.1.8)$$

The main interest in the current expansion of the Ising model arises from the existence of a combinatorial tool called the *switching lemma*, which was initially introduced in [120] and used in many subsequent works to capture fine properties of the Ising model [4, 14, 12, 8, 84, 11, 9, 204], see [76] for more information. This tool relates ratios and differences of correlation functions to a system of two independent currents. This motivates us to also consider, for $S_1, S_2 \subset V_n$, the measure

$$\mathbb{P}_{n,\lambda}^{S_1, S_2} := \mathbb{P}_{n,\lambda}^{S_1} \otimes \mathbb{P}_{n,\lambda}^{S_2}. \quad (12.1.9)$$

Applying the switching lemma [76, Lemma 4.3], we find that for any event $\mathcal{A}$ which depends on $\mathbf{n}_1 + \mathbf{n}_2$,

$$\mathbb{P}_{n,\lambda}^{S_1, S_2}[\mathcal{A}] = \frac{\langle \sigma_{S_1 \Delta S_2} \rangle_{n,\lambda}}{\langle \sigma_{S_1} \rangle_{n,\lambda} \langle \sigma_{S_2} \rangle_{n,\lambda}} \mathbb{P}_{n,\lambda}^{S_1 \Delta S_2, \emptyset}[\mathcal{A}, \mathcal{F}_{S_2}], \quad (12.1.10)$$

where $\mathcal{F}_{S_2}$ is the event that there exists a sub-current $\mathbf{m} \leq \mathbf{n}_1 + \mathbf{n}_2$ satisfying $\partial \mathbf{m} = S_2$. Note that if $S_2 = \{i, j\}$, $\mathcal{F}_{S_2}$ coincides with the event that $i \xleftrightarrow{\mathbf{n}} j$. This observation and Lemma 12.2.1 (which provides asymptotics as $n \rightarrow \infty$ for the ratio of correlation functions appearing in the right-hand side of (12.1.10)) reduce the analysis to the case $S_2 = \emptyset$.

Let S be an even subset of V_n . Let $\mathcal{P}_{\text{even}}(S)$ be the set of even partitions of S , i.e. the set made of partitions of S consisting only of even subsets of S . If $\mathbf{n}$ is a current on K_n with sources S , we denote by $\mathfrak{P}(\mathbf{n})$ the partition of S in the associated percolation configuration. Note that for a current $\mathbf{n}$ one always has that $\mathfrak{P}(\mathbf{n}) \in \mathcal{P}_{\text{even}}(\partial \mathbf{n})$.

In [126, Proposition 4.3], the authors proved¹ that the clusters of the sources have size of order $\sqrt{n}$. Our main result strengthens this observation. For simplicity we focus on the clusters of the sources. Beyond these clusters, there could be several other clusters of size of order $\sqrt{n}$. Their study reduces to the case below as explained in Remark 12.3.5.

Theorem 12.1.1. *Let $\lambda \in \mathbb{R}$ and $k \geq 1$. Set $S = \{1, \dots, 2k\}$. Let $\mathbf{n}_1$ (resp. $\mathbf{n}_2$) be distributed according to $\mathbb{P}_{n,\lambda}^S$ (resp. $\mathbb{P}_{n,\lambda}^\emptyset$). There exist two random variables $X_{k,\lambda}^s, X_{k,\lambda}^d$ on $(\mathbb{R}_{\geq 0})^{2k} \times \mathcal{P}_{\text{even}}(S)$ such that*

$$\left(\frac{|\mathcal{C}_1(\mathbf{n}_1)|}{\sqrt{n}}, \dots, \frac{|\mathcal{C}_{2k}(\mathbf{n}_1)|}{\sqrt{n}}, \mathfrak{P}(\mathbf{n}_1) \right) \xrightarrow{n \rightarrow \infty} X_{k,\lambda}^s, \quad (12.1.11)$$

$$\left(\frac{|\mathcal{C}_1(\mathbf{n}_1 + \mathbf{n}_2)|}{\sqrt{n}}, \dots, \frac{|\mathcal{C}_{2k}(\mathbf{n}_1 + \mathbf{n}_2)|}{\sqrt{n}}, \mathfrak{P}(\mathbf{n}_1 + \mathbf{n}_2) \right) \xrightarrow{n \rightarrow \infty} X_{k,\lambda}^d, \quad (12.1.12)$$

where both convergences are in distribution. Moreover, each of the first $2k$ coordinates of $X_{k,\lambda}^s$ (resp. $X_{k,\lambda}^d$) is strictly positive almost surely.

Remark 12.1.2. In Section 12.3, we will obtain exact formulas for the density of these random variables (see Proposition 12.3.2).

¹This observation was already made in [4].

To prove Theorem 12.1.1, we will first analyse the structure of the clusters of the sources in the limit $n \rightarrow \infty$, see Section 12.2. As it turns out, the clusters have a simple combinatorial description in terms of a family of multigraphs we call *backbone multigraphs* which in particular carry the information of how the sources are connected together, see Definition 12.3.4. Conditionally on this backbone multigraph, the scaling limit of the cluster size distribution can be computed exactly, see (12.3.18) and (12.3.21).

In comparison with our model, for the well-studied (near-)critical Erdős–Rényi model², the size of the largest cluster is of order $n^{2/3}$. The asymptotic joint distribution of components sizes after normalising by $n^{-2/3}$ was obtained by Aldous [17], which lead to the continuum limit of Erdős–Rényi components [1], as well as to results in the spirit of Theorem 12.1.1 for several other models, see e.g. [38, 18, 194]. See also [147] and references therein.

As a corollary of Theorem 12.1.1 and Proposition 12.2.6, where we obtain the scaling limit of the total number of vertices of positive degree, we get the following result. Recall that a random variable X on $\mathbb{R}^d$ is said to have Gaussian tails if there exists $c > 0$ such that for all $t \geq 1$,

$$\mathbb{P}[|X| \geq t] \leq \frac{1}{c} e^{-ct^2}, \quad (12.1.13)$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$.

Corollary 12.1.3 (Scaling limit of the cluster size distribution). *Let $\lambda \in \mathbb{R}$ and $k \geq 1$. Set $S = \{1, \dots, 2k\}$. Let $\mathbf{n}_1$ (resp. $\mathbf{n}_2$) be distributed according to $\mathbb{P}_{n,\lambda}^S$ (resp. $\mathbb{P}_{n,\lambda}^\emptyset$). Let $Y_{k,\lambda}^s$ (resp. $Y_{k,\lambda}^d$) be the projection on the first $2k$ coordinates of $X_{k,\lambda}^s$ (resp. $X_{k,\lambda}^d$). Then, $Y_{k,\lambda}^s, Y_{k,\lambda}^d$ have Gaussian tails and,*

$$\left(\frac{|\mathcal{C}_1(\mathbf{n}_1)|}{\sqrt{n}}, \dots, \frac{|\mathcal{C}_{2k}(\mathbf{n}_1)|}{\sqrt{n}} \right) \xrightarrow{n \rightarrow \infty} Y_{k,\lambda}^s, \quad (12.1.14)$$

$$\left(\frac{|\mathcal{C}_1(\mathbf{n}_1 + \mathbf{n}_2)|}{\sqrt{n}}, \dots, \frac{|\mathcal{C}_{2k}(\mathbf{n}_1 + \mathbf{n}_2)|}{\sqrt{n}} \right) \xrightarrow{n \rightarrow \infty} Y_{k,\lambda}^d, \quad (12.1.15)$$

where both convergences are in distribution.

One striking property of the near-critical Ising model introduced above is that its scaling limit is (under appropriate scaling) the φ^4 measure, see [224] or Appendix 12.A. This observation plays a central role in [126] where a switching lemma for the discrete φ^4 model (see Section 12.4 for a definition) is obtained by approximating the latter model with a well-chosen sequence of Ising models of the above type. As it turns out, the limiting objects are related to the distribution of $\mathfrak{P}(\mathbf{n})$. In the latter paper, even partitions of S are referred to as *tanglings*, see Section 12.4.

In the proof of Theorem 12.1.1, we will also obtain that the projections of $X_{k,\lambda}^s$ and $X_{k,\lambda}^d$ on $\mathcal{P}_{\text{even}}(S)$ give positive probability to any even partition. This yields the following result which in particular improves on [126, Lemma 4.7], where the full convergence of the tangling probabilities is missing.

²The critical threshold for the Erdős–Rényi model on K_n is $\frac{1}{n}$. The near-critical regime is given by a parameter of the form $\frac{1}{n} \left(1 - \frac{\lambda}{n^{1/3}} \right)$ with $\lambda \in \mathbb{R}$.

Corollary 12.1.4 (Convergence of the tanglings probabilities). *Let $\lambda \in \mathbb{R}$ and $k \geq 1$. Set $S = \{1, \dots, 2k\}$. Fix $\mathbf{P} \in \mathcal{P}_{\text{even}}(S)$. Let $\mathbf{n}_1$ (resp. $\mathbf{n}_2$) be distributed according to $\mathbb{P}_{n,\lambda}^S$ (resp. $\mathbb{P}_{n,\lambda}^\emptyset$). Let $P_{k,\lambda}^s$ (resp. $P_{k,\lambda}^d$) be the projection on $\mathcal{P}_{\text{even}}(S)$ of $X_{k,\lambda}^s$ (resp. $X_{k,\lambda}^d$). Then,*

$$\mathfrak{P}(\mathbf{n}_1) \xrightarrow[n \rightarrow \infty]{} P_{k,\lambda}^s, \quad (12.1.16)$$

$$\mathfrak{P}(\mathbf{n}_1 + \mathbf{n}_2) \xrightarrow[n \rightarrow \infty]{} P_{k,\lambda}^d, \quad (12.1.17)$$

where both convergences are in distribution. Moreover, the support of $P_{k,\lambda}^s$ and $P_{k,\lambda}^d$ is equal to $\mathcal{P}_{\text{even}}(S)$.

As mentioned above, this last corollary has interesting consequences regarding the study of the discrete φ^4 model. In particular, it will be used to provide a new perspective on the switching lemma developed in [126], by showing that this combinatorial tool interpolates between the corresponding tool for Ising (as λ tends to $-\infty$) and Wick's law (as λ tends to $+\infty$). See Section 12.4 for more details.

Notations We will use Landau's formalism: if $(u_n)_{n \geq 0}, (v_n)_{n \geq 0} \in (\mathbb{R}_+^*)^{\mathbb{N}}$, write $u_n = O(v_n)$ (resp. $u_n = o(v_n)$) if there exists $C > 0$ such that for all $n \geq 1$, $u_n \leq C v_n$ (resp. $\lim_{n \rightarrow \infty} u_n/v_n = 0$).

12.2 The random current model on the complete graph.

We begin by importing a result from [126]. For $g > 0$ and $a \in \mathbb{R}$, let $\rho_{g,a}$ be the single-site measure on $\mathbb{R}$ defined by

$$d\rho_{g,a}(t) = \frac{1}{z_{g,a}} e^{-gt^4 - at^2} dt, \quad (12.2.1)$$

where $z_{g,a} = \int_{\mathbb{R}} e^{-gt^4 - at^2} dt$. If $g > 0$ and $\lambda \in \mathbb{R}$, let $\tilde{g} = (12g)^{1/4}$ and choose $a = a(\lambda)$ such that $\lambda = 2a\tilde{g}^{-2}$. We now fix $g > 0$ for the rest of the chapter. Let

$$c_n := \tilde{g}^{-1} n^{-3/4}, \quad (12.2.2)$$

so that $(c_n n)^2 = \tilde{g}^{-2} \sqrt{n}$. The following result is a consequence of [224] which states that the scaling limit of the Ising model under consideration is the φ^4 single-site measure. We reprove this result for sake of completeness.

Lemma 12.2.1 ([126, Lemma 3.24]). *Let $g > 0$ and $\lambda \in \mathbb{R}$. For any $p \in \mathbb{N}$,*

$$(c_n n)^p \left\langle \prod_{j=1}^p \sigma_j \right\rangle_{n,\lambda} \xrightarrow[n \rightarrow \infty]{} \langle \varphi^p \rangle_0, \quad (12.2.3)$$

where $\langle \cdot \rangle_0 = \langle \cdot \rangle_{0,g,a(\lambda)}$ is the average with respect to the (normalised) measure $\rho_{g,a(\lambda)}$.

Proof. For $p = 1$, there is nothing to show because both $\langle \sigma_1 \rangle_{n,\lambda}$ and $\langle \varphi \rangle_0$ are equal to 0, so let us assume that $p \geq 2$.

Applying Proposition 12.A.1 for $\lambda = 2a\tilde{g}^{-2}$ and doing the change of variables $s = \tilde{g}^2 u$ we get that for every $k \geq 0$

$$c_n^k \left\langle \left(\sum_{i=1}^n \sigma_i \right)^k \right\rangle_{n,\lambda} \xrightarrow{n \rightarrow \infty} \langle \varphi^k \rangle_0.$$

Let $n \geq p$. Expanding $(\sum_{i=1}^n \sigma_i)^p$ and taking cases according to whether some index repeats or not we obtain

$$\left\langle \left(\sum_{i=1}^n \sigma_i \right)^p \right\rangle_{n,\lambda} = \sum_{\substack{(i_1, \dots, i_p) \in V_n^p \\ \forall k \neq \ell, i_k \neq i_\ell}} \left\langle \prod_{j=1}^p \sigma_{i_j} \right\rangle_{n,\lambda} + \sum_{\substack{(i_1, \dots, i_p) \in V_n^p \\ \exists k < \ell, i_k = i_\ell}} \left\langle \prod_{j=1}^p \sigma_{i_j} \right\rangle_{n,\lambda}. \quad (12.2.4)$$

Note that

$$\sum_{\substack{(i_1, \dots, i_p) \in V_n^p \\ \forall k \neq \ell, i_k \neq i_\ell}} \left\langle \prod_{j=1}^p \sigma_{i_j} \right\rangle_{n,\lambda} = p! \binom{n}{p} \left\langle \prod_{j=1}^p \sigma_j \right\rangle_{n,\lambda} = (1 + o(1)) n^p \left\langle \prod_{j=1}^p \sigma_j \right\rangle_{n,\lambda} \quad (12.2.5)$$

by the symmetries of the complete graph. Furthermore,

$$0 \leq \sum_{\substack{(i_1, \dots, i_p) \in V_n^p \\ \exists k < \ell, i_k = i_\ell}} \left\langle \prod_{j=1}^p \sigma_{i_j} \right\rangle_{n,\lambda} \leq \binom{p}{2} n \left\langle \left(\sum_{i=1}^n \sigma_i \right)^{p-2} \right\rangle_{n,\lambda} \xrightarrow{n \rightarrow \infty} 0 \quad (12.2.6)$$

by choosing 2 indices k, ℓ for which $i_k = i_\ell$, and the element of V_n that they are equal. Here we have also used that each summand is non-negative by Griffiths' inequality. $\square$

We now turn to some useful estimates on the geometry of the clusters under the measures $\mathbb{P}_{n,\lambda}^S$ and $\mathbb{P}_{n,\lambda}^{S,\emptyset}$. Below, we let $k \geq 1$ and $S = \{1, \dots, 2k\}$. The constant C may depend on λ . We let $\mathbb{P}$ denote either $\mathbb{P}_{n,\lambda}^S$ or $\mathbb{P}_{n,\lambda}^{S,\emptyset}$, and we write $\mathbb{E}$ for the expectation with respect to $\mathbb{P}$.

Lemma 12.2.2 (Multiplicity of the edges). *There exists $C > 0$ such that for all n large enough:*

- (i) $\mathbb{E} \left[\sum_{1 \leq i < j \leq n} \mathbf{1}[\mathbf{n}_{i,j} \geq 1] \right] \leq C\sqrt{n},$
- (ii) $\mathbb{E} \left[\sum_{1 \leq i < j \leq n} \mathbf{1}[\mathbf{n}_{i,j} \geq 2] \right] \leq C,$
- (iii) $\mathbb{E} \left[\sum_{1 \leq i < j \leq n} \mathbf{1}[\mathbf{n}_{i,j} \geq 3] \right] \leq \frac{C}{n^{3/2}}.$

Remark 12.2.3. The above result implies that with high probability: there are at most $O(\sqrt{n})$ open edges, there is a bounded number of edges of weight 2, and there are no edges of weight at least 3.

Proof. We first show the result for $\mathbb{P} = \mathbb{P}_{n,\lambda}^S$. We begin by proving (i). Consider $1 \leq i < j \leq n$. Let $\mathbf{n}$ be a current such that $\mathbf{n}_{i,j} \geq 1$, and define the current $\mathbf{m}$ that satisfies $\mathbf{m}_{i,j} = \mathbf{n}_{i,j} - 1$ and otherwise coincides with $\mathbf{n}$. Then, $\mathbf{m}$ has sources $S\Delta\{i, j\}$ and $w(\mathbf{n}) \leq d_n(\lambda)w(\mathbf{m})$. Thus,

$$\mathbb{P}_{n,\lambda}^S[\mathbf{n}_{i,j} \geq 1] \leq d_n(\lambda) \frac{Z_n^{S\Delta\{i,j\}}(\lambda)}{Z_n^S(\lambda)} = (1 + o(1)) d_n(\lambda) (c_n n)^{|S| - |S\Delta\{i,j\}|} \frac{\langle \varphi^{|S\Delta\{i,j\}|} \rangle_0}{\langle \varphi^{|S|} \rangle_0}. \quad (12.2.7)$$

Summing over all $i < j$, and splitting the sum according to the three possibilities $|S| - |S\Delta\{i, j\}| \in \{-2, 0, 2\}$, we see by Lemma 12.2.1 that

$$\mathbb{E}_{n,\lambda}^S \left[\sum_{1 \leq i < j \leq n} \mathbb{1}[\mathbf{n}_{i,j} \geq 1] \right] \leq O \left(n^2 d_n(\lambda) \frac{1}{\sqrt{n}} \right) + O(nd_n(\lambda)) + O(d_n(\lambda)\sqrt{n}) = O(\sqrt{n}). \quad (12.2.8)$$

We now turn to (ii). As above, let $\mathbf{n}$ be a current such that $\mathbf{n}_{i,j} \geq 2$, and define the current $\mathbf{m}$ that satisfies $\mathbf{m}_{i,j} = \mathbf{n}_{i,j} - 2$ and otherwise coincides with $\mathbf{n}$. Then, $\mathbf{m}$ has sources S and $w(\mathbf{n}) \leq d_n(\lambda)^2 w(\mathbf{m})$. Thus $\mathbb{P}_{n,\lambda}^S[\mathbf{n}_{i,j} \geq 2] \leq d_n(\lambda)^2$, which implies that

$$\mathbb{E}_{n,\lambda}^S \left[\sum_{1 \leq i < j \leq n} \mathbb{1}[\mathbf{n}_{i,j} \geq 2] \right] = O(1). \quad (12.2.9)$$

Finally, we prove (iii). We have

$$\mathbb{P}_{n,\lambda}^S[\mathbf{n}_{i,j} \geq 3] \leq d_n(\lambda)^3 \frac{Z_n^{S\Delta\{i,j\}}(\lambda)}{Z_n^S(\lambda)} = (1 + o(1))d_n(\lambda)^3 (c_n n)^{|S| - |S\Delta\{i,j\}|} \frac{\langle \varphi^{|S\Delta\{i,j\}|} \rangle_0}{\langle \varphi^{|S|} \rangle_0}. \quad (12.2.10)$$

Hence, proceeding as above,

$$\mathbb{E}_{n,\lambda}^S \left[\sum_{1 \leq i < j \leq n} \mathbb{1}[\mathbf{n}_{i,j} \geq 3] \right] \leq O(n^{-3/2}). \quad (12.2.11)$$

The proof for $\mathbb{E} = \mathbb{P}_{n,\lambda}^{S,\emptyset}$ is very similar. For (i), it follows from the proof above together with the observation that

$$\mathbb{P}_{n,\lambda}^{S,\emptyset}[\mathbf{n}_{i,j} \geq 1] \leq \mathbb{P}_{n,\lambda}^S[\mathbf{n}_{i,j} \geq 1] + \mathbb{P}_{n,\lambda}^{\emptyset}[\mathbf{n}_{i,j} \geq 1]. \quad (12.2.12)$$

For (ii), we observe that,

$$\mathbb{P}_{n,\lambda}^{S,\emptyset}[\mathbf{n}_{i,j} \geq 2] \leq \mathbb{P}_{n,\lambda}^S[\mathbf{n}_{i,j} \geq 2] + \mathbb{P}_{n,\lambda}^{\emptyset}[\mathbf{n}_{i,j} \geq 2] + \mathbb{P}_{n,\lambda}^{S,\emptyset}[\mathbf{n}_1(i, j) \geq 1, \mathbf{n}_2(i, j) \geq 1]. \quad (12.2.13)$$

We handle the sum over $1 \leq i < j \leq n$ of the first two probabilities using the same method as above. For the last one, notice that

$$\begin{aligned} \mathbb{P}_{n,\lambda}^{S,\emptyset}[\mathbf{n}_1(i, j) \geq 1, \mathbf{n}_2(i, j) \geq 1] &\leq d_n(\lambda)^2 \frac{Z_n^{S\Delta\{i,j\},\{i,j\}}(\lambda)}{Z_n^{S,\emptyset}(\lambda)} \\ &= (1 + o(1))d_n(\lambda)^2 (c_n n)^{|S| - |S\Delta\{i,j\}| - 2} \frac{\langle \varphi^{|S\Delta\{i,j\}|} \rangle_0 \langle \varphi^2 \rangle_0}{\langle \varphi^{|S|} \rangle_0}. \end{aligned}$$

Finally, we proceed similarly for (iii) by noticing that

$$\begin{aligned} \mathbb{P}_{n,\lambda}^{S,\emptyset}[\mathbf{n}_{i,j} \geq 3] &\leq \mathbb{P}_{n,\lambda}^S[\mathbf{n}_{i,j} \geq 3] + \mathbb{P}_{n,\lambda}^{\emptyset}[\mathbf{n}_{i,j} \geq 3] + \mathbb{P}_{n,\lambda}^{S,\emptyset}[\mathbf{n}_1(i, j) \geq 2, \mathbf{n}_2(i, j) \geq 1] \\ &\quad + \mathbb{P}_{n,\lambda}^{S,\emptyset}[\mathbf{n}_1(i, j) \geq 1, \mathbf{n}_2(i, j) \geq 2]. \end{aligned}$$

□

For $x \in V_n$ and a current $\mathbf{n}$, we let $\Delta_{\mathbf{n}}(x) := \sum_{y \neq x} \mathbf{n}_{x,y}$ be the degree of x in the multigraph associated with $\mathbf{n}$.

Lemma 12.2.4 (The degree is either 1, 2 or 4). *There exists $C > 0$ such that for all n large enough:*

- (i) $\mathbb{E} \left[\sum_{1 \leq i < j \leq n} \mathbf{1}[\mathbf{n}_{i,j} \geq 2] \mathbf{1}[\Delta_{\mathbf{n}}(i) \geq 3 \text{ or } \Delta_{\mathbf{n}}(j) \geq 3] \right] \leq \frac{C}{n^{1/2}},$
- (ii) *for all $i \in S$, $\mathbb{P}[\Delta_{\mathbf{n}}(i) \geq 2] \leq \frac{C}{n^{1/2}},$*
- (iii) $\mathbb{E} \left[\sum_{x \in V_n} \mathbf{1}[\Delta_{\mathbf{n}}(x) \geq 4] \right] \leq C,$
- (iv) $\mathbb{E} \left[\sum_{x \in V_n} \mathbf{1}[\Delta_{\mathbf{n}}(x) \geq 5] \right] \leq \frac{C}{n^{1/2}}.$

Remark 12.2.5. The first property above proves that double edges (i.e. of multiplicity at least two) are isolated with high probability (w.h.p). The second one shows that the sources have degree one w.h.p., the third one states the tightness of the total number of vertices of degree 4, and finally, the fourth inequality above rules out w.h.p the existence of vertices of degree at least 5.

Proof. We prove the result for $\mathbb{P} = \mathbb{P}_{n,\lambda}^S$. The proof for $\mathbb{P}_{n,\lambda}^{S,\emptyset}$ uses similar arguments as in the proof of Lemma 12.2.2.

For (i) we already proved in Lemma 12.2.2 that there exists $C_1 > 0$ such that

$$\mathbb{E}_{n,\lambda}^S \left[\sum_{1 \leq i < j \leq n} \mathbf{1}[\mathbf{n}_{i,j} \geq 3] \right] \leq \frac{C_1}{n^{3/2}}. \quad (12.2.14)$$

Hence, it remains to handle $\mathbb{E}_{n,\lambda}^S \left[\sum_{1 \leq i < j < k \leq n} \mathbf{1}[\mathbf{n}_{i,j} = 2, \mathbf{n}_{j,k} \geq 1] \right]$. We have

$$\mathbb{P}_{n,\lambda}^S [\mathbf{n}_{i,j} = 2, \mathbf{n}_{j,k} \geq 1] \leq d_n(\lambda)^3 \frac{Z_n^{S\Delta\{j,k\}}(\lambda)}{Z_n^S(\lambda)}, \quad (12.2.15)$$

hence, taking into account whether j, k belong to S or not (as in (12.2.8)),

$$\mathbb{E}_{n,\lambda}^S \left[\sum_{1 \leq i < j < k \leq n} \mathbf{1}[\mathbf{n}_{i,j} = 2, \mathbf{n}_{j,k} \geq 1] \right] \leq O(n^{-1/2}). \quad (12.2.16)$$

For (ii), we have already showed that

$$\mathbb{P}_{n,\lambda}^S [\mathbf{n}_{i,j} \geq 2] \leq d_n(\lambda)^2, \quad (12.2.17)$$

hence, for a fixed $i \in S$,

$$\mathbb{E}_{n,\lambda}^S \left[\sum_{j \neq i} \mathbf{1}[\mathbf{n}_{i,j} \geq 2] \right] = O(1/n). \quad (12.2.18)$$

It remains to estimate $\mathbb{E}_{n,\lambda}^S \left[\sum_{1 \leq j < k \leq n} \mathbf{1}[\mathbf{n}_{i,j} \geq 1, \mathbf{n}_{i,k} \geq 1] \right]$. Note that

$$\mathbb{P}_{n,\lambda}^S [\mathbf{n}_{i,j} \geq 1, \mathbf{n}_{i,k} \geq 1] \leq d_n(\lambda)^2 \frac{Z_n^{S\Delta\{j,k\}}(\lambda)}{Z_n^S(\lambda)}, \quad (12.2.19)$$

which yields,

$$\mathbb{E}_{n,\lambda}^S \left[\sum_{j < k} \mathbf{1}[\mathbf{n}_{i,j} \geq 1, \mathbf{n}_{i,k} \geq 1] \right] \leq O(n^{-1/2}). \quad (12.2.20)$$

For (iii), we only need to estimate $\mathbb{E}_{n,\lambda}^S \left[\sum_{1 \leq i < j < k < \ell < m \leq n} \mathbf{1}[\mathbf{n}_{i,x} \geq 1, \forall x \in \{j, k, \ell, m\}] \right]$. Indeed, if i has an outgoing double edge, then with high probability it has degree 2 by (i). To estimate the latter expectation, note that

$$\mathbb{P}_{n,\lambda}^S [\mathbf{n}_{i,x} \geq 1, \forall x \in \{j, k, \ell, m\}] \leq d_n(\lambda)^4 \frac{Z_n^{S \Delta \{j,k,\ell,m\}}(\lambda)}{Z_n^S(\lambda)}, \quad (12.2.21)$$

hence, letting $r = |\{j, k, \ell, m\} \cap S^c|$ we see that the main contribution comes from when $r = 4$, and we get using Lemma 12.2.1 again

$$\mathbb{E}_{n,\lambda}^S \left[\sum_{1 \leq i < j < k < \ell < m \leq n} \mathbf{1}[\mathbf{n}_{i,x} \geq 1, \forall x \in \{j, k, \ell, m\}] \right] \leq (1 + o(1)) \tilde{g}^4 \frac{\langle \varphi^{|S|+4} \rangle_0}{\langle \varphi^{|S|} \rangle_0}. \quad (12.2.22)$$

Similarly, for (iv), we have

$$\mathbb{P}_{n,\lambda}^S [\mathbf{n}_{i,x} \geq 1, \forall x \in \{j, k, \ell, m, y\}] \leq d_n(\lambda)^5 \frac{Z_n^{S \Delta \{i,j,k,\ell,m,y\}}(\lambda)}{Z_n^S(\lambda)} \quad (12.2.23)$$

hence

$$\mathbb{E}_{n,\lambda}^S \left[\sum_{1 \leq i < j < k < \ell < m < y \leq n} \mathbf{1}[\mathbf{n}_{i,x} \geq 1, \forall x \in \{j, k, \ell, m, y\}] \right] = O \left(d_n(\lambda)^5 \sum_{r=0}^6 n^r (c_n n)^{6-2r} \right) \quad (12.2.24)$$

$$= O(n^{-1/2}). \quad (12.2.25)$$

□

The two above results have the following interesting consequence for the number of vertices with non-zero degree. For a current $\mathbf{n}$, let $\mathcal{N}(\mathbf{n})$ be the number of vertices $i \in V_n$ such that $\Delta_{\mathbf{n}}(i) > 0$. Recall that for $g > 0$, $\tilde{g} = (12g)^{1/4}$.

Proposition 12.2.6. *Let $\lambda \in \mathbb{R}$, $g > 0$, and $k \geq 0$. Set $S = \{1, \dots, 2k\}$ if $k \geq 1$ and $S = \emptyset$ otherwise. Let $\mathbf{n}$ be distributed according to $\mathbb{P}_{n,\lambda}^S$. Then,*

$$\frac{\mathcal{N}(\mathbf{n})}{(c_n n)^2} \xrightarrow{n \rightarrow \infty} \frac{\tilde{g}^4}{2} Z_k^2, \quad (12.2.26)$$

where the convergence holds in distribution, and Z_k is the random variable with density³

$$\frac{\varphi^{2k}}{\langle \varphi^{2k} \rangle_0} d\rho_{g,a(\lambda)}. \quad (12.2.27)$$

³Recall that $a = a(\lambda)$ was defined in the beginning of Section 12.2.

Proof. By Lemma 12.2.4, “most vertices” of positive degree have degree 2. As a result, with high probability we have that $\mathcal{E}(\mathbf{n}) := \sum_{1 \leq i < j \leq n} \mathbb{1}[\mathbf{n}_{i,j} \geq 1] = (1 + o(1))\mathcal{N}(\mathbf{n})$. Hence, it suffices to prove the convergence (in distribution) of $\mathcal{E}(\mathbf{n})/(c_n n)^2$.

We will first show convergence of all moments. Let $r > 0$. Recall that by Lemma 12.2.2, most open edges are simple (i.e. carry weight 1) with high probability. As a result, in order to estimate $\mathbb{E}_{n,\lambda}^S \left[\frac{\mathcal{E}^r}{(c_n n)^{2r}} \right]$, it is sufficient to consider the probabilities

$$\mathbb{P}_{n,\lambda}^S [\mathbf{n}_{i_s, j_s} = 1, \forall s \in \{1, 2, \dots, r\}] \quad (12.2.28)$$

for $i_1, j_1, i_2, j_2, \dots, i_r, j_r \in V_n$ not necessarily distinct. Letting ℓ be the number of distinct edges $\{i_s, j_s\}$ and $m = |S\Delta\{i_1, j_1, \dots, i_r, j_r\}|$ we have

$$\mathbb{P}_{n,\lambda}^S [\mathbf{n}_{i_s, j_s} = 1, \forall s \in \{1, 2, \dots, r\}] = d_n(\lambda)^\ell \frac{Z_n^{S\Delta\{i_1, j_1, \dots, i_r, j_r\}}(\lambda) [\mathbf{n}_{i_s, j_s} = 0, \forall s \in \{1, 2, \dots, r\}]}{Z_n^S(\lambda)} \quad (12.2.29)$$

$$= (1 + o(1)) d_n(\lambda)^\ell \frac{Z_n^{S\Delta\{i_1, j_1, \dots, i_r, j_r\}}(\lambda)}{Z_n^S(\lambda)} \quad (12.2.30)$$

$$= (1 + o(1)) d_n(\lambda)^\ell (c_n n)^{2k-m} \frac{\langle \varphi^m \rangle_0}{\langle \varphi^{2k} \rangle_0}, \quad (12.2.31)$$

where we used that,

$$\begin{aligned} \frac{Z_n^{S\Delta\{i_1, j_1, \dots, i_r, j_r\}}(\lambda) [\mathbf{n}_{i_s, j_s} = 0, \forall s \in \{1, 2, \dots, r\}]}{Z_n^{S\Delta\{i_1, j_1, \dots, i_r, j_r\}}(\lambda)} &= \mathbb{P}_{n,\lambda}^S [\mathbf{n}_{i_s, j_s} = 0, \forall s \in \{1, 2, \dots, r\}] \\ &= 1 - o(1). \end{aligned}$$

Summing these probabilities we see that the main contribution comes from the case where $i_1, j_1, i_2, j_2, \dots, i_r, j_r \in V_n \setminus S$ are all distinct. Thus,

$$\mathbb{E}_{n,\lambda}^S \left[\frac{\mathcal{E}^r}{(c_n n)^{2r}} \right] = (1 + o(1)) \frac{n^{2r}}{2^r} d_n(\lambda)^r (c_n n)^{-4r} \frac{\langle \varphi^{2k+2r} \rangle_0}{\langle \varphi^{2k} \rangle_0} \quad (12.2.32)$$

$$= (1 + o(1)) \frac{\tilde{g}^{4r}}{2^r} \frac{\langle \varphi^{2k+2r} \rangle_0}{\langle \varphi^{2k} \rangle_0} \quad (12.2.33)$$

$$= (1 + o(1)) \mathbb{E} \left[\frac{\tilde{g}^{4r}}{2^r} Z_k^{2r} \right], \quad (12.2.34)$$

where Z_k is a random variable with density $\frac{\varphi^{2k}}{\langle \varphi^{2k} \rangle_0} d\rho_{g,a}$.

The convergence of all moments implies that the sequence $(\mathcal{N}(\mathbf{n})/(c_n n)^2)_{n \geq 1}$ is tight, and any limiting random variable has r -th moment given by $\frac{\tilde{g}^{4r}}{2^r} \frac{\langle \varphi^{2k+2r} \rangle_0}{\langle \varphi^{2k} \rangle_0}$. Since $\frac{\tilde{g}^4}{2^r} Z_k^2$ has finite exponential moments, it follows that there exists a unique distribution with r -th moment given by $\frac{\tilde{g}^{4r}}{2^r} \frac{\langle \varphi^{2k+2r} \rangle_0}{\langle \varphi^{2k} \rangle_0}$. Thus $(\mathcal{N}(\mathbf{n})/(c_n n)^2)_{n \geq 1}$ converges to $\frac{\tilde{g}^4}{2^r} Z_k^2$, as desired. $\square$

12.3 Scaling limit of the cluster size distribution

In this section, we will use the results of Section 12.2 and Appendix 12.A to prove Theorem 12.1.1.

We begin with a definition.

Definition 12.3.1. Let $n \geq 1$. Let S be an even subset of V_n and $\mathbf{P} = \{P_1, \dots, P_r\} \in \mathcal{P}_{\text{even}}(S)$. Let $\mathbf{a}, \mathbf{b} \in (\mathbb{R}_{\geq 0})^r$ with $a_i \leq b_i$ for each $1 \leq i \leq r$. Define the subset $\mathcal{I}_n(\mathbf{a}, \mathbf{b}, \mathbf{P})$ of Ω_n as

$$\mathcal{I}_n(\mathbf{a}, \mathbf{b}, \mathbf{P}) := \{\mathbf{n} \in \Omega_n, \mathfrak{P}(\mathbf{n}) = \mathbf{P}, a_i \sqrt{n} \leq |\mathcal{C}_{P_i}(\mathbf{n})| \leq b_i \sqrt{n} \forall 1 \leq i \leq r\}, \quad (12.3.1)$$

where for each $1 \leq i \leq r$, $\mathcal{C}_{P_i}(\mathbf{n})$ is the cluster of $\mathbf{n}$ containing the elements of P_i .

As we will see, it is sufficient to study the above events to obtain Theorem 12.1.1. In fact, the latter will be a consequence of the following result.

Proposition 12.3.2. Let $\lambda \in \mathbb{R}$ and $k \geq 1$. Set $S := \{1, \dots, 2k\}$ and consider $\mathbf{P} = \{P_1, \dots, P_r\} \in \mathcal{P}_{\text{even}}(S)$. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}_{\geq 0}^r$ with $a_i \leq b_i$ for each $1 \leq i \leq r$. Then, the following limits exist

$$\lim_{n \rightarrow \infty} \mathbb{P}_{n,\lambda}^S[\mathcal{I}_n(\mathbf{a}, \mathbf{b}, \mathbf{P})], \quad \lim_{n \rightarrow \infty} \mathbb{P}_{n,\lambda}^{S,\emptyset}[\mathcal{I}_n(\mathbf{a}, \mathbf{b}, \mathbf{P})]. \quad (12.3.2)$$

Moreover, both limits are continuous as functions of $\mathbf{a}$ and $\mathbf{b}$.

We now show how to obtain our main result using Proposition 12.3.2.

Proof of Theorem 12.1.1. We give the proof for the case of $\mathbb{P}_{n,\lambda}^S$, with the proof for the double current measure being similar. The main observation we need to make is that with high probability the clusters of the sources always remain of order at most $(c_n n)^2 \asymp \sqrt{n}$. This fact follows immediately from the results of Section 12.2. Indeed, using Lemma 12.2.2 (i) and Markov's inequality, we get the existence of $C > 0$ such that, for any $A > 0$,

$$\limsup_{n \rightarrow \infty} \mathbb{P}_{n,\lambda}^S [\cup_{1 \leq i \leq 2k} \{|\mathcal{C}_i(\mathbf{n})| \geq A\sqrt{n}\}] \leq \frac{2kC}{A}. \quad (12.3.3)$$

In particular, this proves that the sequence of random variables $\left(\frac{|\mathcal{C}_1(\mathbf{n})|}{\sqrt{n}}, \dots, \frac{|\mathcal{C}_{2k}(\mathbf{n})|}{\sqrt{n}}, \mathfrak{P}(\mathbf{n})\right)_{n \geq 1}$ is tight on $(\mathbb{R}_{\geq 0})^{2k} \times \mathcal{P}_{\text{even}}(S)$ (equipped with the standard product σ -algebra). Proposition 12.3.2 then fully characterises the limit which yields the desired convergence. The fact that the first $2k$ coordinates of the limiting random variables $X_{k,\lambda}^s$ and $X_{k,\lambda}^d$ are strictly positive almost surely is a consequence of the second part of Proposition 12.3.2, since the event $\mathcal{I}_n(\mathbf{a}, \mathbf{b}, \mathbf{P})$ is empty if one of the b_i 's is 0. $\square$

It remains to prove Proposition 12.3.2. Below, we fix $g > 0$, $\lambda \in \mathbb{R}$ and $k \geq 1$. We recall that $S = \{1, \dots, 2k\}$. We work on the complete graph K_n with inverse temperature $d_n(\lambda)$ and study the measures $\mathbb{P}_{n,\lambda}^S$ and $\mathbb{P}_{n,\lambda}^{S,\emptyset}$.

Let $\mathbf{P} = \{P_1, \dots, P_r\} \in \mathcal{P}_{\text{even}}(S)$. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}_{\geq 0}^r$ with $a_i \leq b_i$ for each $1 \leq i \leq r$. We begin by analysing the typical form of a current $\mathbf{n}$ which belongs to $\mathcal{I}_n(\mathbf{a}, \mathbf{b}, \mathbf{P})$. The results of Section 12.2 motivate the introduction of the following event.

Definition 12.3.3 (Good event). For $A > 0$, let $\mathcal{G}_A$ be the event that the current $\mathbf{n}$ satisfies the following properties: for every $1 \leq i \leq 2k$ the cluster $\mathcal{C}_i(\mathbf{n})$ has only simple edges (i.e. edges e with $\mathbf{n}_e = 1$) and $\Delta_{\mathbf{n}}(i) = 1$, there are no vertices of degree 5 or larger, and the number of vertices of degree 4 is at most A .

Recall that the elements of S^c always have even degree. This means that if $\mathbf{n} \in \mathcal{G}_A$, vertices in S^c have degree 0, 2 or 4 in $\mathbf{n}$. Lemmas 12.2.2 and 12.2.4 imply that there exists $C > 0$ such that for all $n \geq 1$,

$$\mathbb{P}_{n,\lambda}^S[\mathcal{G}_A] \geq 1 - \frac{C}{A} - \frac{C}{\sqrt{n}}, \quad \mathbb{P}_{n,\lambda}^{S,\emptyset}[\mathcal{G}_A] \geq 1 - \frac{C}{A} - \frac{C}{\sqrt{n}}. \quad (12.3.4)$$

As a result, we can restrict our attention to currents $\mathbf{n}$ satisfying $\mathbf{n} \in \mathcal{I}_n(\mathbf{a}, \mathbf{b}, \mathbf{P}) \cap \mathcal{G}_A$. Let $\mathbf{n}$ be such a current. Denote by $B(\mathbf{n})$ the set of vertices of degree 4 in $\cup_{i=1}^r \mathcal{C}_{P_i}(\mathbf{n})$. Then, each cluster $\mathcal{C}_{P_i}(\mathbf{n})$ consists of a union of self-avoiding walks whose internal (meaning not the endpoints) vertices have degree 2 in $\mathbf{n}$, together with a collection of cycles of length at least 3 which contain exactly one element of $B(\mathbf{n}) \cap \mathcal{C}_{P_i}(\mathbf{n})$. Note that these self-avoiding walks begin and end at either an element of $B(\mathbf{n})$ or at a source. See Figure 12.1.

This observation motivates the following definition.

Definition 12.3.4 (Backbone multigraph). Let $\mathbf{n} \in \mathcal{G}_A$ satisfy $\mathfrak{P}(\mathbf{n}) = \mathbf{P}$. Let $G = (V, E)$ be a multigraph (i.e. we allow multiple edges with the same endpoints and loops attached to a vertex) such that $V = \{1, 2, \dots, v\}$ for some $2k \leq v \leq n$, all vertices in $\{1, 2, \dots, 2k\}$ have degree 1, and all vertices in $\{2k+1, \dots, v\}$ have degree 4, where we use the standard convention that loops contribute twice to the degree of a vertex. We say that G is a *backbone multigraph* of $\mathbf{n}$ if there exists an automorphism π of K_n which fixes each element of S such that: $\pi(V)$ consists of the union of the source set S with the set of vertices of degree 4 lying in one of the clusters $\mathcal{C}_{P_i}(\mathbf{n})$; the set $\pi(E)$ is obtained by contracting each self-avoiding walk or cycle in the above description of $\cup_{i=1}^r \mathcal{C}_{P_i}(\mathbf{n})$.

We call G_i the connected component of G which contains P_i , and we write $\mathcal{B}(G)$ for the event that G is a backbone multigraph of $\mathbf{n}$. Moreover, we define $\Gamma(k, A, \mathbf{P})$ as the set of isomorphism classes of backbone multigraphs for the currents $\mathbf{n} \in \mathcal{G}_A$ such that $\mathfrak{P}(\mathbf{n}) = \mathbf{P}$, where we identify two backbone multigraphs G and G' if there is an isomorphism from G to G' which fixes each element of S .

We are now equipped to prove Proposition 12.3.2. We will use the above observations to show that, under the measures of interest, the probabilities of events of the type $\mathcal{I}_n(\mathbf{a}, \mathbf{b}, \mathbf{P}) \cap \mathcal{G}_A \cap \mathcal{B}(G)$ for a fixed *admissible* (i.e. compatible with the event $\mathcal{I}_n(\mathbf{a}, \mathbf{b}, \mathbf{P}) \cap \mathcal{G}_A$) backbone multigraph G converge as $n \rightarrow \infty$. Proposition 12.3.2 will then follow from summing the above quantities among the (finite) set of admissible backbone multigraphs.

Proof of Proposition 12.3.2. We start with the simpler case of a single random current. As we will see, the case of double random currents can be handled similarly, with only a few modifications. We work on the complete graph K_n and consider the random current measure $\mathbb{P}_{n,\lambda}^S$ with inverse temperature $d_n(\lambda)$. Recall that $S := \{1, \dots, 2k\}$ and that we fixed $\mathbf{P} = \{P_1, \dots, P_r\} \in \mathcal{P}_{\text{even}}(S)$ and $\mathbf{a} \leq \mathbf{b} \in (\mathbb{R}_{\geq 0})^r$.

With the above definitions in hands we have,

$$\mathbb{P}_{n,\lambda}^S[\mathcal{I}_n(\mathbf{a}, \mathbf{b}, \mathbf{P}), \mathcal{G}_A] = \sum_{[G] \in \Gamma(k, A, \mathbf{P})} \mathbb{P}_{n,\lambda}^S[\mathcal{I}_n(\mathbf{a}, \mathbf{b}, \mathbf{P}), \mathcal{G}_A, \mathcal{B}(G)], \quad (12.3.5)$$

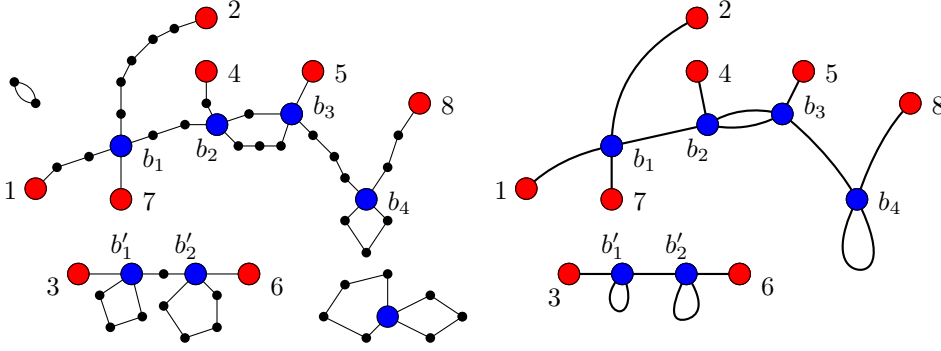


Figure 12.1: LEFT: the multigraph associated with a typical sample $\mathbf{n}$ of $\mathbb{P}_{n,\lambda}^S$ with $S = \{1, \dots, 8\}$. RIGHT: the backbone multigraph associated to $\mathbf{n}$. The sources are represented in red, the set $B(\mathbf{n})$ of vertices of degree 4 is represented in blue, and the vertices of degree 2 in black. The edges of multiplicity 2 do not appear in the backbone graph since they do not contribute to the cluster of the sources with high probability.

where G is an arbitrary representative of the corresponding isomorphism class.

It remains to estimate the probability $\mathbb{P}_{n,\lambda}^S[\mathcal{I}_n(\mathbf{a}, \mathbf{b}, \mathbf{P}), \mathcal{G}_A, \mathcal{B}(G)]$. To this end, let us write m_i for the number of vertices of degree 2 in $\mathcal{C}_{P_i}(\mathbf{n})$, and m for the number of vertices of degree 2 in $\cup_{i=1}^r \mathcal{C}_{P_i}(\mathbf{n})$. Note that $m = \sum_{i=1}^r m_i$. We also let ℓ_i be the number of edges of G_i , so that $\ell := \sum_{i=1}^r \ell_i$ is the number of edges in G . Define, $\mathcal{H}_n(\mathbf{a}, \mathbf{b}, A, G, (m_i)_{1 \leq i \leq r})$ to be the set of currents $\mathbf{n}$ which satisfy: $\mathbf{n} \in \mathcal{I}_n(\mathbf{a}, \mathbf{b}, \mathbf{P}) \cap \mathcal{G}_A \cap \mathcal{B}(G)$, and $\mathcal{C}_{P_i}(\mathbf{n})$ contains m_i vertices of degree 2.

Let $\mathcal{C}_S(\mathbf{n}) = \cup_{i=1}^{2k} \mathcal{C}_i(\mathbf{n})$. A current $\mathbf{n} \in \mathcal{H}_n(\mathbf{a}, \mathbf{b}, A, G, (m_i)_{1 \leq i \leq r})$ can be written as:

$$\mathbf{n} = \mathbf{m}_1(\mathbf{n}) + \mathbf{m}_2(\mathbf{n}), \quad (12.3.6)$$

where $\mathbf{m}_1 = \mathbf{m}_1(\mathbf{n})$ has source set S and is supported on $\mathcal{P}_2(\mathcal{C}_S(\mathbf{n}))$, and $\mathbf{m}_2 = \mathbf{m}_2(\mathbf{n})$ is sourceless and is supported on $\mathcal{P}_2(\mathcal{C}_S(\mathbf{n})^c)$. Moreover, one has that $w(\mathbf{n}) = w(\mathbf{m}_1)w(\mathbf{m}_2)$ with $w(\mathbf{m}_1) = d_n(\lambda)^{m+\ell}$. Indeed, by definition $\mathbf{m}_1(\mathbf{n})$ only contains edges of multiplicity one. Hence, its weight is exactly $d_n(\lambda)^{|\{e, \mathbf{m}_1(e) > 0\}|}$, and $|\{e, \mathbf{m}_1(e) > 0\}|$ is nothing but the number of open edges in $\mathcal{C}_S(\mathbf{n})$. By the handshake lemma applied to G we get that $2\ell = 2k + 4(v - 2k)$. Now, applying the handshake lemma to $\mathcal{C}_S(\mathbf{n})$ we get that its number of edges is equal to $k + m + 2(v - 2k) = m + \ell$.⁴ Hence, for each possible value $\mathbf{m}$ that $\mathbf{m}_1(\mathbf{n})$ may take,

$$\sum_{\mathbf{n} \in \mathcal{H}_n(\mathbf{a}, \mathbf{b}, A, G, (m_i)_{1 \leq i \leq r}), \mathbf{m}_1(\mathbf{n}) = \mathbf{m}} w(\mathbf{n}) = d_n(\lambda)^{m+\ell} \times \sum_{\mathbf{m}_2 \in \Omega_{n-v-m}, \partial \mathbf{m}_2 = \emptyset} w_{n,\lambda}(\mathbf{m}_2). \quad (12.3.7)$$

Let $x = m/\sqrt{n}$. Notice that,

$$d_n(\lambda) = \frac{1}{n} \left(1 - \frac{\lambda}{\sqrt{n}} \right) = \frac{1}{n - v - m} \left(1 - (1 + o(1)) \frac{\lambda + x}{\sqrt{n - v - m}} \right), \quad (12.3.8)$$

where the $o(1)$ term goes to 0 uniformly over x in a bounded interval. Thus, one obtains that

$$\sum_{\mathbf{m}_2 \in \Omega_{n-v-m}, \partial \mathbf{m}_2 = \emptyset} w_{n,\lambda}(\mathbf{m}_2) = Z_{n-v-m}^\emptyset[(\lambda + x)(1 + o(1))]. \quad (12.3.9)$$

⁴ An alternative way to see this is to observe that every self-avoiding walk and every cycle that corresponds to an edge or a loop of G contains one more edge than vertices of degree 2.

Since the right-hand side of (12.3.7) does not depend on the value of $\mathbf{m}_1(\mathbf{n})$, it remains to count the number of values $\mathbf{m}_1(\mathbf{n})$ may take for $\mathbf{n} \in \mathcal{H}_n(\mathbf{a}, \mathbf{b}, A, G, (m_i)_{1 \leq i \leq r})$. We claim that the number of possibilities for $\mathbf{m}_1(\mathbf{n})$ is equal to

$$\begin{aligned} & \frac{1+o(1)}{2^L |\mathfrak{A}_G|} \sum_{m_1=a_1\sqrt{n}}^{b_1\sqrt{n}} \cdots \sum_{m_r=a_r\sqrt{n}}^{b_r\sqrt{n}} \binom{n-2k}{v-2k} (v-2k)! \binom{n-v}{m} \\ & \quad \times \binom{m}{m_1, \dots, m_r} \prod_{i=1}^r m_i! \frac{m_i^{\ell_i-1}}{(\ell_i-1)!} = \\ & \frac{1+o(1)}{2^L |\mathfrak{A}_G|} \sum_{m_1=a_1\sqrt{n}}^{b_1\sqrt{n}} \cdots \sum_{m_r=a_r\sqrt{n}}^{b_r\sqrt{n}} \binom{n-2k}{v-2k} (v-2k)! \binom{n-v}{m} m! \prod_{i=1}^r \frac{m_i^{\ell_i-1}}{(\ell_i-1)!}, \quad (12.3.10) \end{aligned}$$

where L is the number of loops of G , and $\mathfrak{A}_G$ is the set of automorphisms of G which fix each element of S . To see this, note that we have $\binom{n-2k}{v-2k}$ ways to choose the vertices of degree 4 in $\mathcal{C}_S(\mathbf{n})$, and $\frac{(v-2k)!}{|\mathfrak{A}_G|}$ ways to associate each of these vertices to a vertex of G in a unique way, see Figure 12.2.

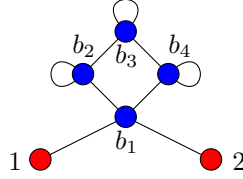


Figure 12.2: A backbone multigraph G with two automorphisms in $\mathfrak{A}_G$.

Moreover, there are $\binom{n-v}{m}$ ways to choose the m vertices of degree 2. Next, we split these m vertices into ordered sets of size $m_1, m_2, \dots, m_r$, corresponding to the vertices of degree 2 in each $\mathcal{C}_{P_i}(\mathbf{n})$, for which there are $\binom{m}{m_1, \dots, m_r}$ possibilities. Then, we split each of these m_i vertices into ℓ_i sets corresponding to the ℓ_i edges of G_i , and order them within the sets. These ordered sets may be empty except for those corresponding to the cycles of $\mathcal{C}_S(\mathbf{n})$ that are attached to a vertex of degree 4. Thus, the number of possibilities to split them into ℓ_i ordered sets is sandwiched between⁵ $m_i! \binom{m_i+\ell_i-1}{\ell_i-1} - L m_i! \binom{m_i+\ell_i-2}{\ell_i-2} = (1+o(1)) m_i! \frac{m_i^{\ell_i-1}}{(\ell_i-1)!}$ and $m_i! \binom{m_i+\ell_i-1}{\ell_i-1} = (1+o(1)) m_i! \frac{m_i^{\ell_i-1}}{(\ell_i-1)!}$. Finally, the term 2^{-L} accounts for the cyclic ordering of the vertices in the cycles corresponding to the loops of G .

Gathering (12.3.7), (12.3.9), and (12.3.10) yields that $\mathbb{P}_{n,\lambda}^S[\mathcal{I}_n(\mathbf{a}, \mathbf{b}, \mathbf{P}), \mathcal{G}_A, \mathcal{B}(G)]$ is equal to

$$\begin{aligned} & \frac{1+o(1)}{Z_n^S(\lambda) 2^L |\mathfrak{A}_G|} \sum_{m_1=a_1\sqrt{n}}^{b_1\sqrt{n}} \cdots \sum_{m_r=a_r\sqrt{n}}^{b_r\sqrt{n}} \binom{n-2k}{v-2k} (v-2k)! \\ & \quad \times \binom{n-v}{m} m! \prod_{i=1}^r \frac{m_i^{\ell_i-1}}{(\ell_i-1)!} d_n(\lambda)^{m+\ell} Z_{n-v-m}^\emptyset[(\lambda+x)(1+o(1))]. \quad (12.3.11) \end{aligned}$$

⁵Here the term $m_i! \binom{m_i+\ell_i-2}{\ell_i-2}$ upper bounds the number of possibilities in which a particular set corresponding to one of the loops is empty, and we recall that L is the number of loops.

We now proceed with the asymptotic analysis of each of these terms. Note that for $x_i = m_i/\sqrt{n}$ we have

$$\binom{n-2k}{v-2k} (v-2k)! = (1+o(1)) n^{v-2k}, \quad d_n(\lambda)^{m+\ell} = (1+o(1)) n^{-m-\ell} e^{-\lambda x}, \quad (12.3.12)$$

$$m_i^{\ell_i-1} = x_i^{\ell_i-1} n^{(\ell_i-1)/2}. \quad (12.3.13)$$

Then, by the asymptotic estimate of the partition function obtained in Proposition 12.A.2,

$$Z_{n-v-m}^\emptyset[(\lambda+x)(1+o(1))] = (1+o(1)) \frac{n^{1/4}}{\sqrt{2\pi e}} \int_{\mathbb{R}} e^{-\frac{1}{12}s^4 - \frac{\lambda+x}{2}s^2} ds, \quad (12.3.14)$$

where the $o(1)$ term on the right-hand side is uniform in x thanks to the monotonicity of $Z_k^\emptyset(u)$ in u . Moreover, we have

$$\log \left\{ \frac{(n-v)!}{(n-v-m)! n^m} \right\} = \sum_{i=v}^{v+m-1} \log \left(\frac{n-i}{n} \right) = \sum_{i=v}^{v+m-1} \left(-\frac{i}{n} + O\left(\frac{i^2}{n^2}\right) \right) = -\frac{m^2}{2n} + o(1),$$

hence,

$$\binom{n-v}{m} m! = (1+o(1)) e^{-x^2/2} n^m. \quad (12.3.15)$$

Multiplying all these together, and recalling that $2\ell = 4(v-2k)+2k$, we see that the exponent of n is equal to

$$v-2k + (\ell-r)/2 - \ell + 1/4 = v-2k - \ell/2 - (r/2 - 1/4) = -k/2 - (r/2 - 1/4). \quad (12.3.16)$$

Thus, using Riemann approximation, we obtain that (12.3.11) is equal to

$$\begin{aligned} & (1+o(1)) \frac{n^{-k/2+1/4}}{\sqrt{2\pi e} 2^L |\mathfrak{A}_G|} \frac{1}{n^{r/2}} \sum_{m_1=a_1\sqrt{n}}^{b_1\sqrt{n}} \cdots \sum_{m_r=a_r\sqrt{n}}^{b_r\sqrt{n}} e^{-x^2/2-\lambda x} \prod_{i=1}^r \frac{x_i^{\ell_i-1}}{(\ell_i-1)!} \int_{\mathbb{R}} e^{-\frac{1}{12}s^4 - \frac{\lambda+x}{2}s^2} ds \\ &= (1+o(1)) \frac{n^{-k/2+1/4}}{\sqrt{2\pi e} 2^L |\mathfrak{A}_G|} \int_{a_1}^{b_1} \cdots \int_{a_r}^{b_r} \int_{\mathbb{R}} \prod_{i=1}^r \frac{x_i^{\ell_i-1}}{(\ell_i-1)!} e^{-\frac{1}{2}x^2 - \lambda x - \frac{1}{12}s^4 - \frac{\lambda+x}{2}s^2} \prod_{i=1}^r dx_i ds. \end{aligned} \quad (12.3.17)$$

By Proposition 12.A.2, $Z_n^S(\lambda) = (1+o(1)) K_S(\lambda) n^{-k/2+1/4}$ for some constant $K_S(\lambda) > 0$, hence $\mathbb{P}_{n,\lambda}^S[\mathcal{I}_n(\mathbf{a}, \mathbf{b}, \mathbf{P}), \mathcal{G}_A, \mathcal{B}(G)]$ is asymptotically equal to

$$\frac{1+o(1)}{K_S(\lambda) \sqrt{2\pi e} 2^L |\mathfrak{A}_G|} \int_{a_1}^{b_1} \cdots \int_{a_r}^{b_r} \int_{\mathbb{R}} \prod_{i=1}^r \frac{x_i^{\ell_i-1}}{(\ell_i-1)!} e^{-\frac{1}{2}x^2 - \lambda x - \frac{1}{12}s^4 - \frac{\lambda+x}{2}s^2} \prod_{i=1}^r dx_i ds. \quad (12.3.18)$$

This completes the proof in the case of a single random current, and now we proceed with the proof in the case of the measure $\mathbb{P}_{n,\lambda}^{S,\emptyset}$. Introduce the event $\mathcal{F}_n$ that $\Delta_{\mathbf{n}}(i) = 0$ for every $i \in S$, and recall that Lemma 12.2.4 (ii) implies that $\mathbb{P}_{n,\lambda}^\emptyset[\mathcal{F}_n] = 1 - O(1/\sqrt{n})$. Thus, it suffices to estimate the probability $\mathbb{P}_{n,\lambda}^{S,\emptyset}[\mathbf{n}_1 + \mathbf{n}_2 \in \mathcal{I}_n(\mathbf{a}, \mathbf{b}, \mathbf{P}), \mathbf{n}_1 + \mathbf{n}_2 \in \mathcal{G}_A, \mathbf{n}_2 \in \mathcal{F}_n, \mathcal{B}(G)]$. Using the same notations as above, we see that the weight of $\mathbf{m}_1(\mathbf{n}_1 + \mathbf{n}_2)$ is again equal to

$d_n(\lambda)^{m+\ell}$, and the main differences in the estimates come from the number of possibilities for $\mathbf{m}_1(\mathbf{n}_1 + \mathbf{n}_2)$.

Given a realisation of the clusters $\mathcal{C}_{P_i}(\mathbf{n}_1 + \mathbf{n}_2)$ there are several ways to assign each open edge of these clusters to either $\mathbf{n}_1$ or $\mathbf{n}_2$. Observe that if $u \in \mathcal{C}_{P_i}(\mathbf{n}_1 + \mathbf{n}_2)$ is a vertex of degree 2 with neighbours z and w , then either $\mathbf{n}_1(u, z) = \mathbf{n}_1(u, w) = 1$ or $\mathbf{n}_2(u, z) = \mathbf{n}_2(u, w) = 1$. As a consequence, $\mathcal{C}_{P_i}(\mathbf{n}_1 + \mathbf{n}_2)$ consists of self-avoiding walks and cycles which are open either in $\mathbf{n}_1$ or in $\mathbf{n}_2$. This implies that the number of ways to assign each open edge of $\mathcal{C}_S(\mathbf{n}_1 + \mathbf{n}_2)$ to either $\mathbf{n}_1$ or $\mathbf{n}_2$ is equal to number of ways to colour the edges G red and blue in such a way that the edges incident to S are all coloured red, and the graph consisting of red edges has even degree for every $u \in V \setminus S$, where we recall that loops contribute twice. We denote this number C_G . We can now conclude that the number of possibilities for $\mathbf{m}_1(\mathbf{n})$ is equal to

$$\frac{(1+o(1))C_G}{2^L|\mathfrak{A}_G|} \sum_{m_1=a_1\sqrt{n}}^{b_1\sqrt{n}} \cdots \sum_{m_r=a_r\sqrt{n}}^{b_r\sqrt{n}} \binom{n-2k}{v-2k} (v-2k)! \binom{n-v}{m} m! \prod_{i=1}^r \frac{m_i^{\ell_i-1}}{(\ell_i-1)!}. \quad (12.3.19)$$

Using once again Proposition 12.A.2,

$$Z_{n-v-m}^{\emptyset, \emptyset}[(\lambda+x)(1+o(1))] = (1+o(1)) \frac{n^{1/2}}{2\pi e} \left(\int_{\mathbb{R}} e^{-\frac{1}{12}s^4 - \frac{\lambda+x}{2}s^2} ds \right)^2. \quad (12.3.20)$$

Arguing as above, we find that $\mathbb{P}_{n,\lambda}^{S,\emptyset}[\mathbf{n}_1 + \mathbf{n}_2 \in \mathcal{I}_n(\mathbf{a}, \mathbf{b}, \mathbf{P}), \mathbf{n}_1 + \mathbf{n}_2 \in \mathcal{G}_A, \mathbf{n}_2 \in \mathcal{F}_n, \mathcal{B}(G)]$ is equal to

$$\frac{(1+o(1))C_G 2^{-L}}{K'_S(\lambda)\sqrt{2\pi e}|\mathfrak{A}_G|} \int_{a_1}^{b_1} \cdots \int_{a_r}^{b_r} \left(\int_{\mathbb{R}} e^{-\frac{1}{12}s^4 - \frac{\lambda+x}{2}s^2} ds \right)^2 \prod_{i=1}^r \frac{x_i^{\ell_i-1}}{(\ell_i-1)!} e^{-\frac{1}{2}x^2 - \lambda x} \prod_{i=1}^r dx_i \quad (12.3.21)$$

for some constant $K'_S(\lambda) > 0$. □

We now turn to the proofs of Corollaries 12.1.3 and 12.1.4.

Proof of Corollary 12.1.3. The convergence in distribution is a direct consequence of Theorem 12.1.1. The existence of Gaussian tails for the limiting measures follows from Proposition 12.2.6. Indeed, looking at the case of a single current (the double current case is similar), one has

$$\left| \left(\frac{|\mathcal{C}_1(\mathbf{n}_1)|}{\sqrt{n}}, \dots, \frac{|\mathcal{C}_{2k}(\mathbf{n}_1)|}{\sqrt{n}} \right) \right|_{\infty} \leq \frac{\mathcal{N}(\mathbf{n}_1)}{\sqrt{n}}, \quad (12.3.22)$$

where $\mathcal{N}(\mathbf{n}_1)$ was defined in Proposition 12.2.6 and where $|\cdot|_{\infty}$ is the ℓ^{∞} norm on $\mathbb{R}^d$. Now, by Proposition 12.2.6, the sequence of random variables $(\frac{\mathcal{N}(\mathbf{n}_1)}{\sqrt{n}})_{n \geq 1}$ converges in law to a multiple of Z_k^2 , where Z_k has quartic tails, and hence has Gaussian tails. □

Proof of Corollary 12.1.4. Again, Theorem 12.1.1 gives convergence in law of $(\mathfrak{P}(\mathbf{n}))_{n \geq 1}$. The second part of the statement follows from the explicit formula for the limit obtained in the proof of Proposition 12.3.2. □

Remark 12.3.5 (Study of the clusters of points which are not sources). The above analysis does not exclude the possibility of existence of clusters of size of order $\sqrt{n}$ which contain no sources. It is possible to study these clusters via the following observation. Notice that

removing an edge of weight 1 creates at most two new sources. More precisely, if $i, j \in V_n$, we can asymptotically relate the measure $\mathbb{P}_{n,\lambda}^S[\cdot \mid \mathbf{n}_{i,j} = 1]$ to the measure $\mathbb{P}_{n,\lambda}^{S\Delta\{i,j\}}$. This allows one to study the scaling limit of the cluster size distribution of i (or j) under the measure $\mathbb{P}_{n,\lambda}^S[\cdot \mid \mathbf{n}_{i,j} = 1]$.

12.4 Consequences for the φ^4 model

The goal of this section is to give a new perspective on the switching lemma for the φ^4 model obtained in [126], and recalled below. We begin with some useful definitions.

Let $\Lambda = (V, E)$ be a finite graph. Let $g > 0$ and $a \in \mathbb{R}$. The ferromagnetic φ^4 model on Λ with parameters (g, a) at inverse temperature $\beta > 0$ is defined by the finite volume Gibbs equilibrium state: for $F : \mathbb{R}^V \rightarrow \mathbb{R}$,

$$\langle F(\varphi) \rangle_{\Lambda, g, a, \beta} = \frac{1}{Z_{\Lambda, g, a, \beta}} \int F(\varphi) \exp(-\beta H_\Lambda(\varphi)) \prod_{x \in V} d\rho_{g, a}(\varphi_x), \quad (12.4.1)$$

where $Z_{\Lambda, g, a, \beta}$ is the partition function and

$$H_\Lambda(\varphi) := - \sum_{\{x, y\} \in E} \varphi_x \varphi_y. \quad (12.4.2)$$

Choosing $a > 0$, it is clear that the limit of $\rho_{g, a}$ as g goes to 0 is the Gaussian law ρ_a given by

$$d\rho_a(t) = \frac{1}{z_a} e^{-at^2} dt. \quad (12.4.3)$$

We will denote by $\langle \cdot \rangle_{0, a}$ the average with respect to ρ_a . Hence, if we additionally require that $a > 4d\beta$, the corresponding φ^4 measure on Λ converges (as g goes to 0) to the rescaled massive discrete Gaussian free field (GFF) of mass $m = a - 4d\beta$, defined via the expectation values

$$\langle F \rangle_{\Lambda, a, \beta}^{\text{GFF}} := \frac{1}{Z_{\Lambda, a, \beta}^{\text{GFF}}} \int_{\varphi \in \mathbb{R}^V} F(\varphi) \exp(-\beta H_\Lambda(\varphi)) \prod_{x \in V} d\rho_a(\varphi_x), \quad (12.4.4)$$

where $Z_{\Lambda, a, \beta}^{\text{GFF}}$ is the partition function.

Moreover, choosing $a = -2g$ and taking the limit as $g \rightarrow \infty$,

$$\frac{\delta_{-1} + \delta_1}{2} = \lim_{g \rightarrow \infty} \frac{1}{z_{g, -2g}} e^{-g(\varphi^2 - 1)^2 + g} d\varphi, \quad (12.4.5)$$

the corresponding φ^4 measure on Λ converges to the Ising model on Λ defined via the expectation values

$$\langle F \rangle_{\Lambda, \beta}^{\text{Ising}} := \frac{1}{Z_{\Lambda, \beta}^{\text{Ising}}} \sum_{\sigma \in \{\pm 1\}^V} F(\sigma) \exp(-\beta H_\Lambda(\sigma)). \quad (12.4.6)$$

The above observations suggest that the φ^4 model interpolates between the discrete GFF and the Ising model. Interestingly, Theorem 12.1.1 allows us to extend this observation at the level of the switching lemma for φ^4 : under the appropriate limits mentioned above, one recovers Wick's law (on the GFF side), or the switching lemma (on the Ising model side). More precisely, the proof will reduce to showing that the random variable $P_{k, \lambda}^d$ converges

in distribution to either: the uniform measure over pairings in the GFF limit, or the Dirac measure $\delta_{\{1,\dots,2k\}}$ in the Ising limit.

This gives an interpretation⁶ of the switching lemma for φ^4 as a type of Wick's law for the non-Gaussian φ^4 field.

12.4.1 Switching lemma for φ^4 and full description of the tangling probabilities

We recall the switching lemma for φ^4 in its simplest case and import some terminology from [126].

Let $\Lambda = (V, E)$ be a finite graph. Write $\Omega_\Lambda := \mathbb{N}^E$ for the space of currents on Λ . Given $\mathbf{n} \in \Omega_\Lambda$, let $\Delta_{\mathbf{n}}(x) := \sum_{y \in V} \mathbf{n}_{x,y}$ be the $\mathbf{n}$ -degree of x . We define

$$\mathcal{M}(\Lambda) := \left\{ A \in \mathbb{N}^V, \sum_{x \in V} A_x \text{ is even} \right\} \quad (12.4.7)$$

to be the set of admissible moments on Λ , and write $\partial A := \{x \in V : A_x \text{ is odd}\}$. We will write $A = \emptyset$ if $A_x = 0$ for all $x \in \Lambda$. Given $A, B \in \mathcal{M}(\Lambda)$, define $A + B \in \mathcal{M}(\Lambda)$ by

$$(A + B)_x := A_x + B_x \quad (12.4.8)$$

for all $x \in V$.

Fix $A, B \in \mathcal{M}(\Lambda)$. Let $\mathbf{n}_1, \mathbf{n}_2 \in \Omega_\Lambda$ satisfy $\partial \mathbf{n}_1 = \partial A$ and $\partial \mathbf{n}_2 = \partial B$. For each $x \in V$, we define the block $\mathcal{B}_x(\mathbf{n}_1, A)$ as follows: for each $y \in V$, it contains $\mathbf{n}_{x,y}$ points labelled $(xy(k))_{1 \leq k \leq \mathbf{n}_1(x,y)}$, and A_x points labelled $(ya(k))_{1 \leq k \leq A_x}$. We also define $\mathcal{B}_x(\mathbf{n}_2, B)$ similarly. To define a notion of tangled currents for $\mathbf{n} = \mathbf{n}_1 + \mathbf{n}_2$, we define $\mathcal{B}_x(\mathbf{n}, A, B)$ in the following way: for every $y \in \Lambda$, it contains $\mathbf{n}_{x,y}$ points labelled $(xy(k))_{1 \leq k \leq \mathbf{n}_{x,y}}$, A_x points labelled $(xa(k))_{1 \leq k \leq A_x}$, and B_x points labelled $(xb(k))_{1 \leq k \leq B_x}$. There is a natural injection of $\mathcal{B}_x(\mathbf{n}_1, A)$ and $\mathcal{B}_x(\mathbf{n}_2, B)$ in $\mathcal{B}_x(\mathbf{n}, A, B)$. We write $\mathcal{T}_{\mathbf{n}, A, B}(x)$ for the set of even partitions of the set $\mathcal{B}_x(\mathbf{n}, A, B)$ whose restriction to $\mathcal{B}_x(\mathbf{n}_1, A)$ and $\mathcal{B}_x(\mathbf{n}_2, B)$ is also an even partition, i.e. for every $P = \{P_1, P_2, \dots, P_r\} \in \mathcal{T}_{\mathbf{n}, A, B}(x)$, both $|P_i \cap \mathcal{B}_x(\mathbf{n}_1, A)|$ and $|P_i \cap \mathcal{B}_x(\mathbf{n}_2, B)|$ are even numbers. The elements of $\mathcal{T}_{\mathbf{n}, A, B}(x)$ are called *admissible* partitions. We also define

$$\mathcal{T}_{\mathbf{n}, A, B} = \bigotimes_{x \in V} \mathcal{T}_{\mathbf{n}, A, B}(x) \quad (12.4.9)$$

to be the set of admissible *tanglings*. For any admissible A and B and any current $\mathbf{n} = \mathbf{n}_1 + \mathbf{n}_2$ with $\partial \mathbf{n}_1 = \partial A$ and $\partial \mathbf{n}_2 = \partial B$, we let $\mathcal{H}(\mathbf{n}, A, B)$ be the graph with vertex set

$$\bigcup_{z \in V} \mathcal{B}_z(\mathbf{n}, A, B), \quad (12.4.10)$$

and edge set

$$\bigcup_{\{x,y\} \in E} \{\{xy(k), yx(k)\}, 1 \leq k \leq \mathbf{n}_{x,y}\}. \quad (12.4.11)$$

Given any tangling $\mathbf{t} \in \mathcal{T}_{\mathbf{n}, A, B}$, the graph $\mathcal{H}(\mathbf{n}, A, B)$ naturally projects to a multigraph $\mathcal{H}(\mathbf{n}, \mathbf{t}, A, B)$ where for each block $\mathcal{B}_x(\mathbf{n}, A, B)$, vertices in the same partition class given by $\mathbf{t}_x$ are identified as one. See Figure 12.3.

⁶This property can already be seen at the level of the formulas obtained for the tangling probabilities — see (12.3.18) and (12.3.21), which are expressed as a sum over backbone multigraphs, similarly to Gaussian correlations being expressed as a sum over pairings.

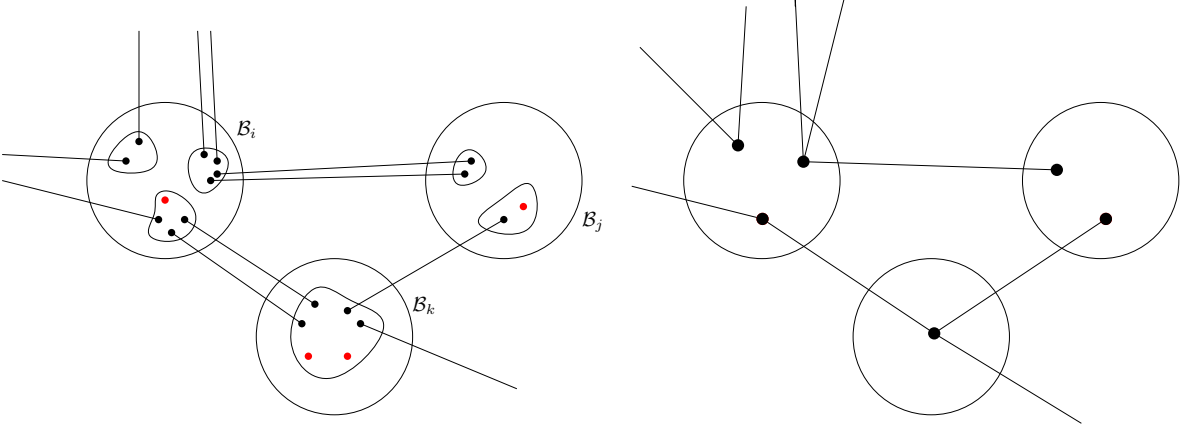


Figure 12.3: An example of a tangled current $(\mathbf{n}, \mathfrak{t})$ with a moment function A satisfying $A_i = 1, A_j = 1$, and $A_k = 2$. On the left picture, the circles represent the blocks $\mathcal{B}(\mathbf{n}, A)$; the small ellipses inside represent the elements of the partitions $\mathfrak{t}$. Notice that i, j are sources of $\mathbf{n}$, and the red vertices correspond to the vertices associated with A . On the right picture, the multigraph $\mathcal{H}(\mathbf{n}, \mathfrak{t}, A)$ is obtained by identifying vertices in the same ellipses.

We define $\Omega_\Lambda^\mathcal{T}$ to be the set of tangled currents on Λ . Fix $\beta > 0$. Before recalling the switching lemma for φ^4 , we need a definition.

Definition 12.4.1. Let $B \in \mathcal{M}(\Lambda)$. Define $\mathcal{F}_B^\Lambda$ to be the subset of $\Omega_\Lambda^\mathcal{T}$ consisting of all tangled currents $(\mathbf{n}, \mathfrak{t})$ for which every connected component intersects $\{xb(k) : x \in \Lambda, 1 \leq k \leq B_x\}$ an even number of times. In words, $\mathcal{F}_B^\Lambda$ consists of all tangled currents that pair the elements of B in its associated multigraph. The set $\mathcal{F}_B^\Lambda$ is measurable with respect to the natural σ -algebra on $\Omega_\Lambda^\mathcal{T}$.

We let

$$w_{A,\beta}(\mathbf{n}) := \prod_{\{x,y\} \in E} \frac{\beta^{\mathbf{n}_{x,y}}}{\mathbf{n}_{x,y}!} \prod_{x \in \Lambda} \langle \varphi^{\Delta_{\mathbf{n}}(x) + A_x} \rangle_{0,g,a}. \quad (12.4.12)$$

Theorem 12.4.2 (Switching lemma for φ^4 , [126, Theorem 3.11]). *Let $A, B \in \mathcal{M}(\Lambda)$. For every $\mathbf{n}_1, \mathbf{n}_2 \in \Omega_\Lambda$ satisfying $\partial \mathbf{n}_1 = \partial A$ and $\partial \mathbf{n}_2 = \partial B$ and for every $z \in \Lambda$, there are probability measures $\rho_{z,A,B}^{\mathbf{n}_1, \mathbf{n}_2} = \rho_{z,A,B,g,a}^{\mathbf{n}_1, \mathbf{n}_2}$ on $\mathcal{T}_{\mathbf{n},A,B}(z)$ such that the following holds. For any measurable and bounded $F : \Omega_\Lambda^\mathcal{T} \rightarrow \mathbb{R}$,*

$$\sum_{\substack{\partial \mathbf{n}_1 = \partial A \\ \partial \mathbf{n}_2 = \partial B}} w_{A,\beta}(\mathbf{n}_1) w_{B,\beta}(\mathbf{n}_2) \rho_{A,B}^{\mathbf{n}_1, \mathbf{n}_2} [F(\mathbf{n}_1 + \mathbf{n}_2, \mathfrak{t})] \quad (12.4.13)$$

$$= \sum_{\substack{\partial \mathbf{n}_1 = \partial(A+B) \\ \partial \mathbf{n}_2 = \emptyset}} w_{A+B,\beta}(\mathbf{n}_1) w_{\emptyset,\beta}(\mathbf{n}_2) \rho_{A+B,\emptyset}^{\mathbf{n}_1, \mathbf{n}_2} [F(\mathbf{n}_1 + \mathbf{n}_2, \mathfrak{t}) \mathbf{1}_{\mathcal{F}_B^\Lambda}], \quad (12.4.14)$$

where

$$\rho_{A,B}^{\mathbf{n}_1, \mathbf{n}_2} := \bigotimes_{z \in \Lambda} \rho_{z,A,B}^{\mathbf{n}_1, \mathbf{n}_2}, \text{ and } \rho_{A+B,\emptyset}^{\mathbf{n}_1, \mathbf{n}_2} := \bigotimes_{z \in \Lambda} \rho_{z,A+B,\emptyset}^{\mathbf{n}_1, \mathbf{n}_2}, \quad (12.4.15)$$

are measures on $\mathcal{T}_{\mathbf{n},A,B}$ and $\mathcal{T}_{\mathbf{n},A+B,\emptyset}$ respectively, and $\mathcal{F}_B^\Lambda$ is the event defined above.

In [126], the probability measures $\rho_{z,A,B}^{\mathbf{n}_1, \mathbf{n}_2}$ were obtained by taking sub-sequential limits of certain measures defined below. Our main result implies full convergence of these measures. We will need the following definitions.

If S_1, S_2 are two even and disjoint subsets of V_n , we let $\tilde{\rho}_{n,\lambda}^{S_1, S_2}$ be the measure $\mathbb{P}_{n,\lambda}^{S_1, S_2}$ quotiented by the relation:

$$\mathbf{n} \sim \mathbf{m} \iff \mathfrak{P}(\mathbf{n}) = \mathfrak{P}(\mathbf{m}). \quad (12.4.16)$$

In words, $\tilde{\rho}_{n,\lambda}^{S_1, S_2}$ is the law of $\mathfrak{P}(\mathbf{n})$ for $\mathbf{n}$ sampled according to $\mathbb{P}_{n,\lambda}^{S_1, S_2}$. Note that the measure $\tilde{\rho}_{n,\lambda}^{S_1, S_2}$ is supported on the set $\mathcal{P}_{\text{even}}(S_1, S_2)$ which consists of even partitions $\mathbf{P} = \{P_1, \dots, P_r\}$ of $S_1 \sqcup S_2$ which are compatible⁷ with both S_1 and S_2 in the sense that: for all $1 \leq i \leq r$, $|P_i \cap S_1|$ and $|P_i \cap S_2|$ are even.

For $x \in V$, the measure $\rho_{x,A,B}^{\mathbf{n}_1, \mathbf{n}_2}$ is defined in [126] as a subsequential limit of the sequence of measures $(\tilde{\rho}_{n,\lambda}^{S_1, S_2})_{n \geq 1}$ where $|S_1| = A_x + \Delta_{\mathbf{n}_1}(x)$ and $|S_2| = B_x + \Delta_{\mathbf{n}_2}(x)$ (and S_1, S_2 are disjoint).

As mentioned in (12.1.9), the switching lemma (for the Ising model), gives a connection between $\tilde{\rho}_{n,\lambda}^{S_1, S_2}$ and $\tilde{\rho}_{n,\lambda}^{S_1 \sqcup S_2, \emptyset}$. Indeed, notice that the event $\mathcal{F}_{S_2}$ defined below (12.1.9) satisfies $\{\mathbf{n}, \mathfrak{P}(\mathbf{n}) = \mathbf{P}\} \subset \mathcal{F}_{S_2}$ for any $\mathbf{P} \in \mathcal{P}_{\text{even}}(S_1, S_2)$. As a result, for any such $\mathbf{P}$,

$$\tilde{\rho}_{n,\lambda}^{S_1, S_2}[\mathbf{P}] = \frac{\langle \sigma_{S_1 \sqcup S_2} \rangle_{n,\lambda}}{\langle \sigma_{S_1} \rangle_{n,\lambda} \langle \sigma_{S_2} \rangle_{n,\lambda}} \tilde{\rho}_{n,\lambda}^{S_1 \sqcup S_2, \emptyset}[\mathbf{P}]. \quad (12.4.17)$$

Proposition 12.4.3. *Let $\lambda \in \mathbb{R}$. For any S_1, S_2 even and disjoint subsets of V_n . The sequence of measures $(\tilde{\rho}_{n,\lambda}^{S_1, S_2})_{n \geq 1}$ converges in the sense of distribution.*

Proof. Since the sequence of measures $(\tilde{\rho}_{n,\lambda}^{S_1, S_2})_{n \geq 1}$ lives on the finite space $\mathcal{P}_{\text{even}}(S_1, S_2)$ we just need to prove the convergence for $\mathbf{P} \in \mathcal{P}_{\text{even}}(S_1, S_2)$ of $(\tilde{\rho}_{n,\lambda}^{S_1, S_2}[\mathbf{P}])_{n \geq 1}$. Using Corollary 12.1.4, Lemma 12.2.1, and (12.4.17), we find that

$$\lim_{n \rightarrow \infty} \tilde{\rho}_{n,\lambda}^{S_1, S_2}[\mathbf{P}] = \frac{\langle \varphi^{|S_1|+|S_2|} \rangle_0}{\langle \varphi^{|S_1|} \rangle_0 \langle \varphi^{|S_2|} \rangle_0} \mathbb{P}[P_{k,\lambda}^{\mathbf{d}} = \mathbf{P}], \quad (12.4.18)$$

where $\langle \cdot \rangle_0 = \langle \cdot \rangle_{0,g,a(\lambda)}$ was defined in Lemma 12.2.1, and $k = (|S_1| + |S_2|)/2$. $\square$

As mentioned in [126], we can also give a probabilistic interpretation of Theorem 12.4.2. We begin by defining the natural probability measures on the set of tangled currents.

Definition 12.4.4. Let $A, B \in \mathcal{M}(\Lambda)$. We define a measure $\mathbf{P}_{\Lambda,g,a,\beta}^{A,B}$ on the set of tangled currents on Λ with source set $A + B$ as follows: for $(\mathbf{n}, \mathbf{t}) \in \Omega_{\Lambda, \mathcal{T}}$,

$$\mathbf{P}_{\Lambda,g,a,\beta}^{A,B}[(\mathbf{n}, \mathbf{t})] = \mathbf{1}_{\partial \mathbf{n} = \partial(A+B)} \frac{\sum_{\substack{\partial \mathbf{n}_1 = \partial A \\ \partial \mathbf{n}_2 = \partial B}} w_{A,\beta}(\mathbf{n}_1) w_{B,\beta}(\mathbf{n}_2) \mathbf{1}_{\mathbf{n}_1 + \mathbf{n}_2 = \mathbf{n}} \rho_{A,B,g,a}^{\mathbf{n}_1, \mathbf{n}_2}[(\mathbf{n}, \mathbf{t})]}{\sum_{\substack{\partial \mathbf{n}_1 = \partial A \\ \partial \mathbf{n}_2 = \partial B}} w_{A,\beta}(\mathbf{n}_1) w_{B,\beta}(\mathbf{n}_2)}. \quad (12.4.19)$$

We write $\mathbf{E}_{\Lambda,\beta}^A$ for the expectation with respect to the measure $\mathbf{P}_{\Lambda,\beta}^A$.

⁷Note also that $\mathcal{P}_{\text{even}}(S_1, \emptyset) = \mathcal{P}_{\text{even}}(S_1)$.

Corollary 12.4.5 (Probabilistic version of the switching lemma). *Let $A, B \in \mathcal{M}(\Lambda)$. For any measurable and bounded $F : \Omega_{\Lambda, \mathcal{T}} \rightarrow \mathbb{R}$,*

$$\frac{\langle \varphi_A \rangle_{\Lambda, g, a, \beta} \langle \varphi_B \rangle_{\Lambda, g, a, \beta}}{\langle \varphi_{A+B} \rangle_{\Lambda, g, a, \beta}} \mathbf{E}_{\Lambda, \beta}^{A, B} [F(\mathbf{n}, \mathbf{t})] = \mathbf{E}_{\Lambda, g, a, \beta}^{A+B, \emptyset} [F(\mathbf{n}, \mathbf{t}) \mathbf{1}_{\mathcal{F}_B^\Lambda}], \quad (12.4.20)$$

where for $S \in \mathcal{M}(\Lambda)$, $\langle \varphi_S \rangle_{\Lambda, g, a, \beta} := \langle \prod_{x \in \Lambda} \varphi_x^{S_x} \rangle_{\Lambda, g, a, \beta}$.

12.4.2 An alternative proof of Wick's law

Let $U_{x, A, B}^{\mathbf{n}_1, \mathbf{n}_2}$ be the uniform measure on pairings in $\mathcal{T}_{\mathbf{n}, A, B}(x)$.

Proposition 12.4.6. *Let $a > 0$ and $x \in V$. As g tends to 0, $\rho_{x, A, B, g, a}^{\mathbf{n}_1, \mathbf{n}_2}$ converges in distribution to $U_{x, A, B}^{\mathbf{n}_1, \mathbf{n}_2}$.*

Proof. Fix S_1, S_2 even and disjoint subsets of V_n of size $A_x + \Delta_{\mathbf{n}_1}(x)$ and $B_x + \Delta_{\mathbf{n}_2}(x)$, respectively. Let $\mu_{x, A, B}^{\mathbf{n}_1, \mathbf{n}_2}$ be any limiting measure as g tends to 0. Due to the invariance of $\mathbb{P}_{n, \lambda}^{S_1, S_2}$ with respect to permuting the elements of S_1 and the elements of S_2 , $\mu_{x, A, B}^{\mathbf{n}_1, \mathbf{n}_2}$ also has this property, and the desired result will follow once we prove that $\mu_{x, A, B}^{\mathbf{n}_1, \mathbf{n}_2}$ is supported on (admissible) pairings.

To this end, note that if there are no vertices of degree 4 (or more) under $\mathbb{P}_{n, \lambda}^{S_1, S_2}$, then $\mathfrak{P}(\mathbf{n})$ consists of pairs, see the discussion above Definition 12.3.4. Let us estimate the probability of having a vertex of degree at least 4. By (12.4.17), it is sufficient to study this question for the measure $\mathbb{P}_{n, \lambda}^{S_1 \sqcup S_2, \emptyset}$. Let $S := S_1 \sqcup S_2$.

By Lemma 12.2.4 (ii), all sources have degree 1 with high probability, and it remains to consider the degree of the vertices in $V_n \setminus S$. Let $i \in V_n \setminus S$. If $\Delta_{\mathbf{n}}(i) \geq 4$ then one of the following occurs: (i) $\Delta_{\mathbf{n}_1}(i) \geq 4$ or $\Delta_{\mathbf{n}_2}(i) \geq 4$, (ii) $\Delta_{\mathbf{n}_1}(i) \geq 2$ and $\Delta_{\mathbf{n}_2}(i) \geq 2$. The contribution coming from case (i) is handled by (12.2.22). The contribution coming from (ii) is handled by noticing that for $j, k \notin S$,

$$\begin{aligned} \mathbb{P}_{n, \lambda}^{S, \emptyset} [\mathbf{n}_1(i, j) \geq 1, \mathbf{n}_2(i, k) \geq 1] &\leq (1 + o(1)) d_n(\lambda)^2 (c_n n)^{-4} \frac{\langle \varphi^{|S|+2} \rangle_0 \langle \varphi^2 \rangle_0}{\langle \varphi^{|S|} \rangle_0} \\ &= (1 + o(1)) \tilde{g}^4 n^{-3} \frac{\langle \varphi^{|S|+2} \rangle_0 \langle \varphi^2 \rangle_0}{\langle \varphi^{|S|} \rangle_0}, \end{aligned}$$

which implies that

$$\mathbb{E}_{n, \lambda}^{S, \emptyset} \left[\sum_{i \notin S} \mathbf{1} [\Delta_{\mathbf{n}_1}(i) \geq 2, \Delta_{\mathbf{n}_2}(i) \geq 2] \right] \leq (1 + o(1)) \tilde{g}^4 \frac{\langle \varphi^{|S|+2} \rangle_0 \langle \varphi^2 \rangle_0}{\langle \varphi^{|S|} \rangle_0}. \quad (12.4.21)$$

Hence, one can find $C > 0$ (which does not depend on g) such that,

$$\limsup_{n \rightarrow \infty} \mathbb{E}_{n, \lambda}^{S, \emptyset} [|\{i \in V_n, \Delta_{\mathbf{n}}(i) \geq 4\}|] \leq C \tilde{g}^4 \left(\frac{\langle \varphi^{|S|+2} \rangle_{0, g, a} \langle \varphi^2 \rangle_{0, g, a}}{\langle \varphi^{|S|} \rangle_{0, g, a}} + \frac{\langle \varphi^{|S|+4} \rangle_{0, g, a}}{\langle \varphi^{|S|} \rangle_{0, g, a}} + \langle \varphi^4 \rangle_{0, g, a} \right). \quad (12.4.22)$$

If $p \geq 0$, as g tends to 0, $\langle \varphi^{2p} \rangle_{0, g, a}$ tends to $\langle \varphi^{2p} \rangle_{0, a}$. As a consequence, using Markov's inequality,

$$\lim_{g \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}_{n, \lambda}^{S, \emptyset} [\exists i \in V_n, \Delta_{\mathbf{n}}(i) \geq 4] = 0. \quad (12.4.23)$$

The desired result follows. $\square$

Combining Theorem 12.4.2 with the above result yields a new proof of Wick's law for the GFF.

Corollary 12.4.7 (Wick's law). *Let $\Lambda = (V, E)$ be a finite graph. Consider the Gaussian free field at parameter $a > 0$ and inverse temperature β on Λ . Then for any $p \geq 1$, for any $(i_k)_{1 \leq k \leq 2p} \in V^{2p}$, one has,*

$$\mathbb{E}_{\Lambda, a, \beta}^{\text{GFF}} \left[\prod_{k=1}^{2p} \varphi_{i_k} \right] = \sum_{\pi} \prod_{k=1}^p \mathbb{E}_{\Lambda, a, \beta}^{\text{GFF}} \left[\varphi_{i_{\pi(2k-1)}} \varphi_{i_{\pi(2k)}} \right], \quad (12.4.24)$$

where the sum is over pairings π of $\{1, \dots, 2p\}$.

Proof. For $x \in \Lambda$, set $A_x := |\{k, i_k = x\}|$. This defines an $A \in \mathcal{M}(\Lambda)$. Fix $u \in \Lambda$ such that $A_u > 0$. If $v \in \Lambda$ satisfies $A_v > 0$, define $C = C_{u,v} := \mathbb{1}_u + \mathbb{1}_v$ and $B = B_{u,v} := A - C_{u,v}$.

By Corollary 12.4.5 applied to $F \equiv 1$: for all $1 \leq j \leq A_v$ (except if $u = v$ in which case⁸ we require $j \neq 1$),

$$\frac{\langle \varphi_B \rangle_{\Lambda, a, \beta}^{\text{GFF}} \langle \varphi_C \rangle_{\Lambda, a, \beta}^{\text{GFF}}}{\langle \varphi_A \rangle_{\Lambda, a, \beta}^{\text{GFF}}} = \mathbf{P}_{\Lambda, a, \beta}^{A, \emptyset}[\mathcal{F}_C^\Lambda] = \mathbf{P}_{\Lambda, a, \beta}^{A, \emptyset}[ua(1) \xleftrightarrow{\mathcal{H}(\mathbf{n}, \mathbf{t}, A, \emptyset)} va(j)], \quad (12.4.25)$$

where $\mathbf{P}_{\Lambda, a, \beta}^{A, \emptyset} := \lim_{g \rightarrow 0} \mathbf{P}_{\Lambda, g, a, \beta}^{A, \emptyset}$. In particular, by Proposition 12.4.6 this measure is defined similarly as in (12.4.19) replacing the measures $\rho_{x, A, \emptyset, g, a}^{\mathbf{n}_1 + \mathbf{n}_2}$ by $U_{x, A, \emptyset}^{\mathbf{n}_1, \mathbf{n}_2}$. Now, since $U_{x, A, \emptyset}^{\mathbf{n}_1, \mathbf{n}_2}$ is supported on pairings, almost surely under $\mathcal{F}_B^\Lambda$, all vertices in $\mathcal{H}(\mathbf{n}, \mathbf{t}, A, \emptyset)$ have degree 2 except for the sources that have degree 1. This implies that $ua(k)$ is connected to only one element of $\mathcal{S}_u := \{va(t), 1 \leq t \leq A_v, v \in \Lambda \setminus \{u\}\} \cup \{ua(t), 2 \leq t \leq A_u\}$ in $\mathcal{H}(\mathbf{n}, \mathbf{t}, A, \emptyset)$. Thus,

$$\sum_{s \in \mathcal{S}_u} \mathbf{P}_{\Lambda, a, \beta}^{A, \emptyset}[ua(1) \xleftrightarrow{\mathcal{H}(\mathbf{n}, \mathbf{t}, A, \emptyset)} s] = 1. \quad (12.4.26)$$

As a result, summing (12.4.25) over $va(j) \in \mathcal{S}_u$ and using (12.4.26) yields

$$\langle \varphi_A \rangle_{\Lambda, a, \beta}^{\text{GFF}} = \sum_{v \in \Lambda} \sum_{j=1+\delta_u(v)}^{A_v} \langle \varphi_{B_{u,v}} \rangle_{\Lambda, a, \beta}^{\text{GFF}} \langle \varphi_{C_{u,v}} \rangle_{\Lambda, a, \beta}^{\text{GFF}}, \quad (12.4.27)$$

which implies the result by induction. $\square$

12.4.3 Recovering the switching lemma for the Ising model

Below, we call ECT (for “Everybody Connected Together”) the event that $\mathfrak{P}(\mathbf{n}) = \{\partial \mathbf{n}\}$.

Proposition 12.4.8. *Let $a = -2g$. Then, as g tends to ∞ , $\rho_{x, A, B}^{\mathbf{n}_1, \mathbf{n}_2}$ converges in distribution to the Dirac measure δ_{ECT} .*

Proof. First, observe that if $p \geq 0$, as g tends to ∞ , $\langle \varphi^{2p} \rangle_{0, g, -2g}$ tends to 1. Fix S_1, S_2 even and disjoint subsets of V_n of size $A_x + \Delta_{\mathbf{n}_1}(x)$ and $B_x + \Delta_{\mathbf{n}_2}(x)$, respectively. Using (12.4.17) and the observation above, we see that for any $\mathbf{P} \in \mathcal{P}_{\text{even}}(S_1, S_2)$,

$$\lim_{g \rightarrow \infty} \rho_{x, A, B, g, -2g}^{\mathbf{n}_1, \mathbf{n}_2}[\mathbf{P}] = \lim_{g \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{P}_{n, \lambda(g)}^{S, \emptyset}[\mathfrak{P}(\mathbf{n}_1 + \mathbf{n}_2) = \mathbf{P}], \quad (12.4.28)$$

⁸For this case to occur, one must have $A_u \geq 2$.

where $S = S_1 \sqcup S_2$ and $\lambda(g)$ is such that $a(\lambda(g)) = -2g$. We can always assume that $S = \{1, \dots, |S|\}$. Clearly, if $P \neq \{S\}$, there exists $1 \leq i < j \leq |S|$ such that i is not connected to j in $\mathbf{n}_1 + \mathbf{n}_2$. As a result, for such a P ,

$$\mathbb{P}_{n, \lambda(g)}^{S, \emptyset}[\mathfrak{P}(\mathbf{n}_1 + \mathbf{n}_2) = P] \leq \sum_{1 \leq i < j \leq |S|} \left(1 - \mathbb{P}_{n, \lambda(g)}^{S, \emptyset}[i \xleftrightarrow{\mathbf{n}_1 + \mathbf{n}_2} j]\right). \quad (12.4.29)$$

Using the switching lemma for the Ising model (12.1.10) and Lemma 12.2.1, we get that

$$\mathbb{P}_{n, \lambda(g)}^{S, \emptyset}[i \xleftrightarrow{\mathbf{n}_1 + \mathbf{n}_2} j] = (1 + o(1)) \frac{\langle \varphi^{|S|-2} \rangle_{0, g, -2g} \langle \varphi^2 \rangle_{0, g, -2g}}{\langle \varphi^{|S|} \rangle_{0, g, -2g}}. \quad (12.4.30)$$

This readily implies that for $P \neq \{S\}$,

$$\lim_{g \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{P}_{n, \lambda(g)}^{S, \emptyset}[\mathfrak{P}(\mathbf{n}_1 + \mathbf{n}_2) = P] = 0 \quad (12.4.31)$$

and the result follows. $\square$

The above result immediately implies that one can recover the switching lemma for the Ising model when taking the Ising limit in the identity of Theorem 12.4.2. Indeed, the measures $\rho_{x, A, B}^{\mathbf{n}_1, \mathbf{n}_2}$ converge to δ_{ECT} which has as a consequence that the vertex set of the multigraph $\mathcal{H}(\mathbf{n}, \mathbf{t}, A, B)$ coincides with V , hence we can identify $\mathcal{H}(\mathbf{n}, \mathbf{t}, A, B)$ with the current $\mathbf{n}$.

12.A Explicit Griffiths–Simon approximation for φ^4

In this appendix, we obtain asymptotic estimates for the partition functions of the Ising and random current models on the complete graph. Along the way, we reprove the main result of [224].

Proposition 12.A.1. *Let $\lambda \in \mathbb{R}$. Let $(\sigma_1, \dots, \sigma_n)$ be distributed as spins of the Ising model on the complete graph K_n with inverse temperature $d_n(\lambda)$. Then, one has*

$$\frac{\sigma_1 + \dots + \sigma_n}{n^{3/4}} \xrightarrow[n \rightarrow \infty]{} \frac{1}{z(\lambda)} e^{-\frac{1}{12}s^4 - \frac{\lambda}{2}s^2} \mathrm{d}s,$$

where the convergence is in distribution and $z(\lambda) := \int_{\mathbb{R}} e^{-\frac{1}{12}s^4 - \frac{\lambda}{2}s^2} \mathrm{d}s$ is a renormalising factor. Moreover, for the partition function of the Ising model on K_n we have

$$\mathcal{Z}_n(\lambda) = (1 + o(1)) \frac{2^n n^{1/4}}{\sqrt{2\pi e}} \int_{\mathbb{R}} e^{-\frac{1}{12}s^4 - \frac{\lambda}{2}s^2} \mathrm{d}s.$$

Proof. We assume that n is even, the other case is similar. We denote by $\mathbf{P}_{n, \lambda}$ the measure of the Ising model on K_n with inverse temperature $d_n(\lambda)$. Note that

$$2 \sum_{1 \leq i < j \leq n} \sigma_i \sigma_j = \left(\sum_{i=1}^n \sigma_i \right)^2 - n,$$

so that, for any $m \in [0, n/2]$ we have,

$$\mathbf{P}_{n,\lambda} \left[\sum_{i=1}^n \sigma_i = 2m \right] = \frac{1}{\mathcal{Z}_n(\lambda)} \binom{n}{n/2+m} \exp \left(d_n(\lambda) \frac{4m^2 - n}{2} \right) = \frac{\exp(-d_n(\lambda) \frac{n}{2})}{\mathcal{Z}_n(\lambda)} f_n(m),$$

where $f_n(m) := \binom{n}{n/2+m} \exp(2d_n(\lambda)m^2)$. We have

$$\log \frac{f_n(m)}{f_n(m-1)} = \log \frac{n/2 - m + 1}{n/2 + m} + (1 - \lambda/\sqrt{n}) \frac{4m-2}{n} \quad (12.A.1)$$

$$= \log \frac{1 - 2m/n}{1 + 2m/n} + (1 - \lambda/\sqrt{n}) \frac{4m}{n} + O\left(\frac{1}{n}\right). \quad (12.A.2)$$

Expanding in powers of m/n and using that $\log(1-2t) - \log(1+2t) = -4t - 16t^3/3 + O(t^5)$, we observe that the first order term cancels and we get

$$\log \frac{f_n(m)}{f_n(m-1)} = -\frac{16m^3}{3n^3} - \lambda \frac{4m}{n\sqrt{n}} + O\left(\frac{m^5}{n^5} + \frac{1}{n}\right).$$

By telescoping, we obtain that

$$\log \frac{f_n(m)}{f_n(0)} = -\frac{4m^4}{3n^3} - \lambda \frac{2m^2}{n\sqrt{n}} + O\left(\frac{m^6}{n^5} + \frac{m}{n}\right). \quad (12.A.3)$$

By symmetry, the above holds also for negative m up to an error term $O\left(\frac{m^6}{n^5} + \frac{|m|}{n}\right)$. Letting $g(x) = \exp\left(-\frac{4}{3}x^4 - 2\lambda x^2\right)$, for any $a, b \in \mathbb{R}$ with $a < b$ we have

$$\mathbf{P}_{n,\lambda} \left[an^{3/4} \leq \sum_{i=1}^n \sigma_i \leq bn^{3/4} \right] = (1 + o(1)) \frac{\exp(-d_n(\lambda) \frac{n}{2}) f_n(0)}{\mathcal{Z}_n(\lambda)} \sum_{\frac{a}{2}n^{3/4} \leq m \leq \frac{b}{2}n^{3/4}} g\left(\frac{m}{n^{3/4}}\right) \quad (12.A.4)$$

$$= (1 + o(1)) \frac{e^{-1/2} f_n(0)}{\mathcal{Z}_n(\lambda)} n^{3/4} \int_{\frac{a}{2}}^{\frac{b}{2}} g(\lambda) d\lambda \quad (12.A.5)$$

$$= (1 + o(1)) \frac{e^{-1/2} f_n(0)}{2\mathcal{Z}_n(\lambda)} n^{3/4} \int_a^b e^{-\frac{1}{12}s^4 - \frac{\lambda}{2}s^2} ds. \quad (12.A.6)$$

It remains to estimate $\mathcal{Z}_n(\lambda)$, and for that we need to handle the case where m is much larger than $n^{3/4}$. Since all terms in the Taylor expansion of $\log(1-2t) - \log(1+2t)$ are non-positive it follows that

$$\log \frac{f_n(m)}{f_n(m-1)} \leq -\frac{16m^3}{3n^3} - \lambda \frac{4m}{n^{3/2}} + O\left(\frac{1}{n}\right).$$

Thus, there exists $C > 0$ such that $f_n(m) \leq C f_n(0) g\left(\frac{m}{n^{3/4}}\right)$, from which we can conclude by arguing as above that

$$\mathcal{Z}_n(\lambda) = (1 + o(1)) \frac{e^{-1/2} f_n(0)}{2} n^{3/4} \int_{\mathbb{R}} e^{-\frac{1}{12}s^4 - \frac{\lambda}{2}s^2} ds. \quad (12.A.7)$$

Sending n to infinity we get the desired convergence in distribution.

Finally, we have $f_n(0) = (1 + o(1)) 2^n / \sqrt{\pi n/2}$ by Stirling's approximation, hence

$$\mathcal{Z}_n(\lambda) = (1 + o(1)) \frac{2^n n^{1/4}}{\sqrt{2\pi e}} \int_{\mathbb{R}} e^{-\frac{1}{12}s^4 - \frac{\lambda}{2}s^2} ds,$$

as desired. $\square$

Recall that for $g > 0$ and $\lambda \in \mathbb{R}$, the measure $\langle \cdot \rangle_0$ is the average with respect to $\rho_{g,a(\lambda)}$ with $a(\lambda) = \frac{\lambda \tilde{g}^2}{2}$.

Proposition 12.A.2 (Estimate of the partition function of the current). *Let $\lambda \in \mathbb{R}$, $g > 0$, and $k \geq 1$. Set $S = \{1, \dots, 2k\}$. Then,*

$$Z_n^S(\lambda) = (1 + o(1)) \tilde{g}^{2k} \frac{\langle \varphi^{2k} \rangle_0}{\sqrt{2\pi e}} n^{1/4-k/2} \int_{\mathbb{R}} e^{-\frac{1}{12}s^4 - \frac{\lambda}{2}s^2} ds.$$

Moreover, for every $n \geq 1$, $Z_n^S(\lambda)$ is decreasing as a function of $\lambda \in (-\infty, \sqrt{n}]$.

Proof. Recall that by Lemma 12.2.1

$$\frac{Z_n^S(\lambda)}{Z_n(\lambda)} = \langle \sigma_S \rangle_{n,\lambda} = (1 + o(1)) (c_n n)^{-2k} \langle \varphi^{2k} \rangle_0, \quad (12.A.8)$$

and as mentioned in (12.1.6), $Z_n(\lambda) = 2^{-n} \mathcal{Z}_n(\lambda)$. Thus, by Proposition 12.A.1, and since $(c_n n)^2 = \tilde{g}^{-2} \sqrt{n}$,

$$Z_n^S(\lambda) = (1 + o(1)) (c_n n)^{-2k} \frac{\langle \varphi^{2k} \rangle_0}{\sqrt{2\pi e}} n^{1/4} \int_{\mathbb{R}} e^{-\frac{1}{12}s^4 - \frac{\lambda}{2}s^2} ds \quad (12.A.9)$$

$$= (1 + o(1)) \tilde{g}^{2k} \frac{\langle \varphi^{2k} \rangle_0}{\sqrt{2\pi e}} n^{1/4-k/2} \int_{\mathbb{R}} e^{-\frac{1}{12}s^4 - \frac{\lambda}{2}s^2} ds. \quad (12.A.10)$$

The second part of the statement follows from the monotonicity of $d_n(\lambda)$ as a function of λ . $\square$

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