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Lukierski, Jerzy; Ruegg, Henri; Nowicki, Anatol

Quantum deformations of nonsemisimple algebras: The example of $D=4$ inhomogeneous rotations

Jerzy Lukierski[a] and Henri Ruegg
Département de Physique Théorique, 24, quai Ernest Ansermet, CH-1211 Genève 4, Switzerland

Anatol Nowicki
Institute of Physics, Pedagogical University, Plac Slowianski 6, 65-029 Zielona Gora, Poland

A general class of deformations of the complexified $D=4$ Poincaré algebra $O(3,1;C) \oplus T_4(C)$ is considered with a classical (undeformed) subalgebra $O(3;C) \oplus T_4(C)$ and deformed relations preserving the $O(3;C)$ tensor structure. We distinguish the class of quantum deformations—the complex noncocommutative Hopf algebras—which depend on one complex mass parameter κ. Further, we consider the real Hopf algebras, obtained by imposing the reality conditions. For any choice of real metric [$O(4)$, $O(3,1)$, or $O(2,2)$] the parameter κ becomes real. All (e.g., Minkowski as well as Euclidean) real quantum algebras with standard reality condition contain as nonlinearities the hyperbolic functions of the energy operator and can be interpreted as introducing an *imaginary time lattice*. The symmetries of the models with *real time lattice* are described by a real quantum algebra with nonstandard reality conditions and trigonometric nonlinearities.

I. INTRODUCTION

Recently the quantum deformations of the Poincaré algebra have been studied by several authors.[1–15] If we insist that the quantum deformation of a real Lie algebra is described by a real noncocommutative *-Hopf algebra it appears that in the literature the only quantum deformation of Poincaré algebra satisfying this stringent requirement is the one given firstly in Ref. 8. For the definitions and description of standard and nonstandard *-operations see Refs. 16 and 17. It should also be mentioned that we define the quantum deformations in a broader sense (in comparison with Ref. 18) because we do not assume the existence of a quasitriangular universal R-matrix. We would like to recall that some authors call quantum Poincaré algebra the quantum deformations of the 11-generator Weyl algebra.[3] This quantum algebra has been obtained by the particular contraction of the real quantum algebra $U_q(O(3,2))$, with q real, where $O(3,2)$ describes the anti-de-Sitter algebra, and it will be denoted further by $U_\kappa(\mathscr{P}_4)$, where κ describes a fundamental mass parameter. For the description of q-deformed anti-de-Sitter algebra see also Refs. 19 and 20. It takes the following form. The boost generators, denoted in our previous work[1,8,12] by L_i, are changed for N_i.

(a) Algebra sector:

$$[M_i, M_j] = i\epsilon_{ijk}M_k, \quad [P_\mu, P_\nu] = 0, \tag{1.1a}$$

$$[M_i, N_j] = i\epsilon_{ijk}N_k, \tag{1.1b}$$

$$[N_i, N_j] = -i\epsilon_{ijk}\left(M_k \cosh\frac{P_0}{\kappa} - \frac{1}{4\kappa^2} P_\kappa(\mathbf{PM})\right),$$ (1.1c)

$$[M_i, P_j] = i\epsilon_{ijk}P_k, \quad [M_i, P_0] = 0,$$ (1.1d)

$$[N_i, P_j] = i\kappa\delta_{ij} \sinh P_0/\kappa,$$ (1.1e)

$$[N_i, P_0] = iP_i.$$ (1.1f)

(b) Coalgebra sector:

$$\Delta(M_i) = M_i \otimes 1 + 1 \otimes M_i,$$

$$\Delta(N_i) = N_i \otimes e^{P_0/2\kappa} + e^{-P_0/2\kappa} \otimes N_i + \frac{1}{2\kappa} \epsilon_{ijk}(P_j \otimes M_k e^{P_0/2\kappa} + e^{-P_0/2\kappa}M_j \otimes P_k),$$

$$\Delta(P_i) = P_i \otimes e^{P_0/2\kappa} + e^{-P_0/2\kappa} \otimes P_i,$$ (1.2)

$$\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0.$$

(c) Antipodes:

$$S(M_i) = -M_i, \quad S(P_\mu) = -P_\mu, \quad S(N_i) = -N_i + \frac{3}{2}\frac{i}{\kappa} P_i.$$ (1.3)

The two deformed Casimirs of $U_\kappa(\mathscr{P}_4)$ look as follows:

(1) mass square operator

$$C_1 = -\mathbf{P}^2 + 2\kappa^2\left(\cosh\frac{P_0}{\kappa} - 1\right) = -\mathbf{P}^2 + \left(2\kappa \sinh\frac{P_0}{2\kappa}\right)^2;$$ (1.4)

(2) relativistic spin square operator

$$C_2 = \left(\cosh\frac{P_0}{\kappa} - \frac{\mathbf{P}^2}{4\kappa^2}\right)(\mathbf{PM})^2 - \left(\kappa\mathbf{M} \sinh\frac{P_0}{\kappa} + \mathbf{P}\times\mathbf{N}\right)^2.$$ (1.5)

In this paper, using purely algebraic methods guided by the formulas (1.1)–(1.3), we would like to introduce a more general class of deformations of $D=4$ inhomogeneous rotation algebras. In the first part of this paper we extend the discussion presented recently in Ref. 12, where it was shown that the quantum algebras obtained in Refs. 1 and 8 are the unique solutions of an ansatz for the deformed Poincaré algebra depending on three arbitrary functions of P_0. {It should be mentioned that quite recently Bacry[15] independently considered the same class of deformations, with an additional function describing the freedom of nonlinear transformations of the energy operator $P_0 \rightarrow \tilde{P}_0(P_0)$. This function can be eliminated if we assume the conventional additivity law for the energies of two subsystems [i.e., $\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0$]. The results presented in Ref. 15 can be obtained from the formulas (1.1a)–(1.1f) by the substitution $\tilde{P}_0 = f(P_0)$, where $\tilde{P}_0$ satisfies the coproduct rule $\Delta(\tilde{P}_0) = f^{-1}(f(P_0) \otimes 1 + 1 \otimes f(P_0))$.} Here our most general ansatz will depend on five functions, depending on two variables $x = P_0/\kappa$ and $y = \mathbf{P}^2/\kappa'^2$. In such a way we obtain new deformations of the Poincaré algebra; however, we are not able to show that these deformations are quantum, i.e., can be supplemented with the Hopf algebra structure. In the second part of the

paper we consider the reality conditions defining the Minkowski $[O(3,1)]$, Euclidean $[O(4)$ and $O(2,2)]$ version of the quantum inhomogeneous $D=4$ rotation algebra and point out the difference between standard and nonstandard real Hopf algebras.

II. QUANTUM DEFORMATIONS OF THE COMPLEXIFIED POINCARÉ ALGEBRA

We shall consider the deformations of the Poincaré algebra $\mathscr{P}_4$ which

(α) do not modify the nonrelativistic classical algebra (1.1a) and (1.1d).

(β) The boost generators remain three vectors under rotations [i.e., (1.1b) is not changed].

(γ) The deformations of the $[N, N]$ and $[N, P]$ commutators remain manifestly $O(3)$-covariant, as well as invariant under the time parity transformations:

$$M_i \rightarrow M_i, \quad N_i \rightarrow -N_i, \quad P_i \rightarrow P_i, \quad P_0 \rightarrow -P_0. \tag{2.1}$$

(δ) The deformation functions depend on the four momenta only, i.e., rhs of (1.1c) after deformation remains linear in M.

The most general deformation consistent with the assumptions (α)–(δ) is the following:

$$[N_i, N_j] = -i\epsilon_{ijk}\left[M_k f_1\left(\frac{P_0}{\kappa}, \frac{\mathbf{P}^2}{\kappa'^2}\right) + \frac{P_k(\mathbf{PM})}{\kappa^2} f_2\left(\frac{P_0}{\kappa}, \frac{\mathbf{P}^2}{\kappa'^2}\right) + \epsilon_{klm}\frac{P_l M_m}{\kappa} f_5\left(\frac{P_0}{\kappa}, \frac{\mathbf{P}^2}{\kappa'^2}\right) \right],$$

$$\tag{2.2}$$

$$[N_i, P_j] = i\delta_{ij}\kappa f_3\left(\frac{P_0}{\kappa}, \frac{\mathbf{P}^2}{\kappa'^2}\right), \quad [N_i, P_0] = iP_i f_4\left(\frac{P_0}{\kappa}, \frac{\mathbf{P}^2}{\kappa'^2}\right),$$

where the masslike parameters κ, κ' have been introduced in order to obtain the functions $f_A(x,y)$ dimensionless.

Taking into consideration (2.1) and assuming that in the limit κ, $\kappa' \rightarrow \infty$ one obtains the classical Poincaré algebra, one obtains ($a = 1,2,4,5$)

$$f_a(x,y) = f_a(-x,y), \quad f_3(x,y) = -f_3(-x,y), \tag{2.3a}$$

and $(f' \equiv (\partial/\partial x)f; A = 1,2,3,4,5)$

$$f_A(0,0) = 1, \quad f_3'(0,0) = 1. \tag{2.3b}$$

The mass Casimir for the deformed Poincaré algebra (2.2) is described by the function $C_2(x,y)$ satisfying the following equation:

$$\frac{1}{\kappa}\frac{\partial C_2}{\partial x} f_4 + \frac{2\kappa^2}{\kappa'^2}\frac{\partial C_2}{\partial y} f_3 = 0. \tag{2.4a}$$

Subsequently, we can eliminate the function f_4 if we replace P_0 by $\tilde{P}_0(P_0/\kappa, \mathbf{P}^2/\kappa'^2)$ satisfying the following partial differential equation:

$$\frac{1}{\kappa}\frac{\partial \tilde{P}_0}{\partial x} f_4 + \frac{2\kappa^2}{\kappa'^2}\frac{\partial \tilde{P}_0}{\partial y} f_3 = 1. \tag{2.4b}$$

Substituting $P_0 \rightarrow \tilde{P}_0$ in the formulas (2.2) one obtains that f_4 becomes equal to 1. Further, one can proceed with the Jacobi identities. They are satisfied if

$$\tilde{f}_1 + \frac{4}{\kappa}f_3 f_2 + \frac{\mathbf{P}^2}{\kappa^2}\tilde{f}_2 = 0, \quad f_1 = \kappa \tilde{f}_3, \quad f_5 = 0, \tag{2.5a}$$

where

$$\tilde{f}_A = \hat{O}f_A = \frac{1}{\kappa}\left(\frac{\partial}{\partial x} + \frac{2\kappa^2}{\kappa'^2}f_3\frac{\partial}{\partial y}\right)f_A. \tag{2.5b}$$

It is also easy to see that

$$\hat{O}C_2(x,y) = 0. \tag{2.5c}$$

Introducing the change of variables $y \to C_2(x,y)$, i.e., $f_A(x,y) \equiv \hat{f}_A(x,C_2)$, one obtains from (2.5) that $\hat{O}\hat{f}_A = (1/\kappa)\hat{f}'_A$ $(\hat{f}'_A \equiv (\partial/\partial x)\hat{f}_A)$, i.e.,

$$\hat{f}'_1 + 4\hat{f}_3\hat{f}_2 + \frac{\mathbf{P}^2}{\kappa^2}\hat{f}'_2 = 0, \quad \hat{f}_1 = \hat{f}'_3. \tag{2.6}$$

Let us consider now the following general classes of deformations:
(i) $\hat{f}'_2 = 0$. In such a case one obtains

$$\hat{f}_2 = \hat{f}_2(C_2) \tag{2.7}$$

and one obtains further

$$\hat{f}'_1 = -4\hat{f}_2\hat{f}_3, \quad \hat{f}'_3 = \hat{f}_1. \tag{2.8}$$

Putting $4\hat{f}_2 = -\omega^2$ (ω complex) we obtain the following general solution of (2.8):

$$\hat{f}_3 = Ae^{\omega x} + Be^{-\omega x} \quad \hat{f}_1 = \omega(Ae^{\omega x} - Be^{-\omega x}), \tag{2.9}$$

where in general case A, B, and ω depend on C_2. Imposing the initial conditions (2.3) one obtains $A = -B = 1/2\omega$, i.e.,

$$\hat{f}_3 = (1/\omega)\sinh \omega x, \quad \hat{f}_1 = \cosh \omega x. \tag{2.10}$$

Rescaling $\kappa \to \omega\kappa$ (or $x \to x/\omega$) one can write the commutators (2.2) in the form (1.1c), (1.1e) and (1.1f) with parameter $\kappa = \kappa(C_2)$ complex. It follows from our earlier work (see Ref. 8) that such a deformed $U_\kappa(\mathscr{P}_4)$ algebra can be endowed with the Hopf algebra structure given by the relations (1.2) and (1.3).
(ii) The functions $f_r(r=1,2,3)$ are linear in y:

$$f_r(x,y) = f_r^{(0)}(x) + yf_r^{(1)}(x). \tag{2.11}$$

One obtains from (2.11) the following five equations for six functions $f_r^{(0)}, f_r^{(1)}$:

$$f_1^{(0)} = (f_3^{(0)})' + \frac{2\kappa^2}{\kappa'^2}f_3^{(0)}f_3^{(1)}, \quad f_1^{(1)} = (f_3^{(1)})' + 2\frac{\kappa^2}{\kappa'^2}(f_3^{(1)})^2,$$

$$(f_1^{(1)})' + \frac{2\kappa^2}{\kappa'^2}f_3^{(1)}f_1^{(1)} + 6f_3^{(0)}f_2^{(1)} + 4f_3^{(1)}f_2^{(0)} + \frac{\kappa'^2}{\kappa^2}(f_2^{(0)})' = 0, \tag{2.12}$$

$$(f_2^{(1)})' + \frac{6\kappa^2}{\kappa'^2}f_3^{(1)}f_2^{(1)} = 0, \quad (f_1^{(0)})' + \frac{2\kappa^2}{\kappa'^2}f_3^{(0)}f_1^{(1)} + 4f_3^{(0)}f_2^{(0)} = 0.$$

Because the general solution of (2.12) is still very complicated, we shall further simplify this system in order to obtain only the parameter-dependent solutions and we assume additionally in (2.12) that $f_3^{(1)} \equiv 0$. In such a case the system (2.12) gives the following set of equations:

$$f_1^{(0)} = (f_3^{(0)})', \quad f_1^{(1)} = 0, \tag{2.13a}$$

$$(f_2^{(1)})' = 0 \to f_2^{(1)} = f = \text{const}, \tag{2.13b}$$

$$(f_2^{(0)})' + \frac{6\kappa^2}{\kappa'^2} f f_3^{(0)} = 0, \quad (f_1^{(0)})' = -4 f_3^{(0)} f_2^{(0)}. \tag{2.13c}$$

From (2.13c) one obtains that $(F = \kappa'^2/6\kappa^2 f)$

$$f_1^{(0)} = 2F(f_2^{(0)})^2 + c_1, \tag{2.14}$$

and from $f_1^{(0)} = f_3^{(0)'} = -F(f_2^{(0)})''$ one obtains $(\tilde{c}_1 = c_1/F)$

$$f_2^{(0)''} + 2(f_2^{(0)})^2 + \tilde{c}_1 = 0. \tag{2.15}$$

The whole deformation is determined by the solutions of (2.15). Putting $z(x) = -\frac{1}{3} f_2^{(0)}$ one obtains the equation

$$z''(x) = 6z^2(x) + \tilde{c}_1/3. \tag{2.16}$$

The formulas (2.13) give

$$f_3^{(0)} = 3Fz'(x), \quad f_1^{(0)} = 3Fz''(x). \tag{2.17}$$

We consider now separately

(α) $\tilde{c}_1 = 0$ In such a case Eq. (2.16) has the solution [see, e.g., Ref. 21, Chap. 7]

$$z(x) = C^2 \left[\frac{-k^2}{1+k^2} + \frac{1}{sn^2(C(x-x_1);k)} \right] = C^2 \left[\frac{-k^2}{1-k^2} + k^2 sn^2(C(x-x_0);k) \right], \tag{2.18}$$

where C, x_0, x_1 are arbitrary constants, $k^4 - k^2 + 1 = 0$, and $sn(u;k)$ is a Jacobi elliptic function, analytic at $u = 0$. In order to satisfy the conditions (2.3) we should choose the arbitrary constants in such a way that

$$z(0) = -\frac{1}{3}, \quad z'(0) = 0, \quad z''(0) = \frac{2}{3}. \tag{2.19}$$

(β) $\tilde{c}_1 \neq 0$ After simple normalization one can put in (2.16) $\tilde{c}_1 = -\frac{1}{2} g_2$, where $g_2 = 2(e_1^2 + e_2^2 + e_3^2)$, where e_i $(i = 1,2,3)$ determine the fundamental differential equation for the Weierstrass elliptic function $\mathscr{P}(x)$:[21,22]

$$(\mathscr{P}')^2 = 4(\mathscr{P} - e_1)(\mathscr{P} - e_2)(\mathscr{P} - e_3). \tag{2.20}$$

One can show that for the above choice of the constant $\tilde{c}_1$ one obtains

$$z(x) = \mathscr{P}(x),$$

$$\mathscr{P}(x) = \frac{1}{x^2} + \sum_{m,n=-\infty}^{+\infty}{}' \left\{ \frac{1}{(x - 2m\omega - 2n\omega')^2} - \frac{1}{(2m\omega + 2n\omega')^2} \right\}, \tag{2.21}$$

where Σ' means excluding m and n which are simultaneously zero. The quantities ω, ω' are two numbers with their ratio not real. The Weierstrass elliptic function $\mathscr{P}(x)$ implies the following expansion of the functions $f_r^{(0)}(x)$ $(r = 1,2,3)$:

$$f_1^{(0)} = 3F\mathscr{P}''(x) = 3F\left(\frac{6}{x^4} + \frac{g_2}{10} + O(x^2)\right),$$

$$f_2^{(0)} = -3\mathscr{P}(x) = -3\left(\frac{1}{x^2} + \frac{g_2 x^2}{20} + O(x^4)\right),$$ (2.22)

$$f_3^{(0)} = 3F\mathscr{P}'(x) = 3F\left(-\frac{2}{x^3} + \frac{g_2 x}{10} + O(x^3)\right),$$

where $g_3 = 4e_1 e_2 e_3$.

From the formulas (2.22) it follows that it is not possible to satisfy the conditions (2.3), i.e., we cannot treat (2.22) as the continuous deformation of the classical Poincaré algebra.

III. QUANTUM DEFORMATION OF $O(4-k,k) \ni T_4(k=0,1,2)$

Let us write the reality conditions for the complexified Poincaré algebra with four real metrics of $O(4;C)$:

$$O(3,1): M_i^* = M_i, \quad P_i^* = P_i, \quad N_i^* = N_i, \quad P_0^* = P_0,$$ (3.1a)

$$O(4): M_i^* = M_i, \quad P_i^* = P_i, \quad N_i^* = -N_i, \quad P_0^* = -P_0$$ (3.1b)

$$O(2,2): M_{1,2} = M_{1,2}^*, \quad M_3^* = -M_3, \quad N_{1,2} = N_{1,2}^*, \quad N_3^* = -N_3,$$

$$P_{1,2} = P_{1,2}^*, \quad P_3^* = -P_3, \quad P_0^* = P_0.$$ (3.1c)

We shall introduce the following two types of *-operations, defining the reality conditions.[16,17]

(i) The standard + involution, which is an antiautomorphism in the algebra sector and an automorphism in the coalgebra sector:

$$(a_1 \cdot a_2)^+ = a_2^+ a_1^+, \quad (a \otimes b)^+ = a^+ \otimes b^+.$$ (3.2a)

Further, the following consistency condition involving the antipode is satisfied:

$$S \circ + = + \circ S^{-1}.$$ (3.2b)

(ii) The nonstandard ⊕ involution, which is an antioautomorphism in both algebra and colgebra sector:

$$(a_1 \cdot a_2)^\oplus = a_2^\oplus a_1^\oplus, \quad (a \otimes b)^\oplus = b^\oplus \otimes a^\oplus.$$ (3.3a)

Further, we have

$$S \circ \oplus = \oplus \circ S.$$ (3.3b)

Now we shall describe six real forms of $U_\kappa(\mathscr{P}_4)$ with the Cartan–Weyl generators satisfying the reality conditions (3.1a)–(3.1c).

A. Real quantum Poincaré algebra [$O(3,1)$ metric]

One obtains the following two real quantum Poincaré algebras:

(1) the standard real+Hopf algebra is obtained by putting in (1.1)–(1.3) $\kappa = \kappa^*$ (κ is real). Such an algebra has been obtained from the contraction of quantum anti-de-Sitter algebra $U_q(O(3,2))$ with q real.[8]

(2) The nonstandard real $\oplus$ Hopf algebra is obtained by putting in (1.1)–(1.3) $\kappa = -\kappa^*$ (κ is purely imaginary). Such an algebra has been obtained from the contraction of quantum anti-de-Sitter algebra $U_q(O(3,2))$ with $|q| = 1$.[1,6]

B. Real quantum Euclidean algebra [$O(4)$ metric]

Introducing the $D = 4$ Euclidean generators

$$\hat{M}_i = M_i, \quad \hat{P}_i = P_i, \quad \hat{N}_i = iN_i, \quad \hat{P}_0 = iP_0, \tag{3.4}$$

one can rewrite $U_\kappa(\mathscr{P}_4)$ as the κ-deformed $D = 4$ Euclidean inhomogeneous algebra (we write only the relations involving boost $\hat{N}_i$):

$$[\hat{M}_i, \hat{N}_j] = i\epsilon_{ijk}\hat{N}_k, \tag{3.5}$$

$$[\hat{N}_i, \hat{N}_j] = i\epsilon_{ijk}\left(\hat{M}_k \cos\frac{\hat{P}_0}{\kappa} - \frac{\hat{P}_k(\hat{\mathbf{P}}\hat{\mathbf{M}})}{4\kappa^2}\right),$$

$$[\hat{N}_i, \hat{P}_j] = i\delta_{ij}\kappa \sin(\hat{P}_0/\kappa), \quad [\hat{N}_i, \hat{P}_0] = -i\hat{P}_i, \tag{3.6}$$

where

$$\Delta(\hat{M}_i) = \hat{M}_i \otimes 1 + 1 \otimes \hat{M}_0, \quad \Delta(\hat{P}_0) = \hat{P}_0 \otimes 1 + 1 \otimes \hat{P}_0, \tag{3.7a}$$

$$\Delta(\hat{P}_i) = \hat{P}_i \otimes e^{-i\hat{P}_0/2\kappa} + e^{i\hat{P}_0/2\kappa} \otimes \hat{P}_i, \tag{3.7b}$$

$$\Delta(\hat{N}_i) = \hat{N}_i \otimes e^{-i\hat{P}_0/2\kappa} + e^{i\hat{P}_0/2\kappa} \otimes \hat{N}_i + (i/2\kappa)\epsilon_{ijk}(\hat{P}_j \otimes M_k e^{-i\hat{P}_0/2\kappa} + e^{i\hat{P}_0/2\kappa}\hat{M}_j \otimes \hat{P}_k) \tag{3.7c}$$

and

$$S(\hat{M}_i) = -\hat{M}_i, \quad S(\hat{P}_k) = -\hat{P}_k, \quad S(\hat{N}_i) = -\hat{N}_i - (3/2\kappa)\hat{P}_i. \tag{3.8}$$

The reality conditions (3.1b) imply that the generators occuring in the relations (3.5)–(3.7) are selfconjugate ($\hat{M}_i^* = \hat{M}_i, \hat{P}_i^* = \hat{P}_i, \hat{N}_i^* = \hat{N}_i, \hat{P}_0^* = \hat{P}_0$). One obtains the following two real quantum Euclidean algebras.

(1) The standard real+Hopf algebras is obtained by putting in (3.5)–(3.7) κ purely imaginary.
(2) The second, nonstandard real $\oplus$ Hopf algebra is obtained if we put in (3.5)–(3.7) κ real.

C. Real quantum inhomogenous $O(2,2)$ algebra

The $O(2,2)$ generators are introduced as follows:

$$\tilde{M}_r = iM_r, \quad \tilde{M}_3 = M_3, \quad \tilde{P}_r = P_r, \quad \tilde{P}_3 = iP_3, \quad \tilde{N}_r = N_r, \quad \tilde{N}_3 = iN_3, \quad \tilde{P}_0 = P_0. \tag{3.9}$$

Substitution of the generators (3.8) in (1.1)–(1.3) does not change the form of the nolinearities, but due to the replacement of Minkowski metric by $O(2,2)$ metric there appear some changes of signs. Again one obtains two real quantum inhomogeneous $O(2,2)$ algebras.

(1) The standard $+$-Hopf algebra is obtained if the deformation parameter κ is real.
(2) The nonstandard $\oplus$-Hopf algebra is obtained if κ is purely imaginary. One can therefore conclude the following.

(i) The standard real Hopf algebras for all three $O(4-k,k)$ metrics ($k = 0,1,2$) are characterized by the hyperbolic nonlinearities. If one introduces the coordinate space in a standard way,

i.e., by classical Fourier transform, these hyperbolic nonlinearities in P_0 imply the replacement of the usual time derivatives by the finite difference time derivatives (see also Refs. 8 and 14)

$$\partial_t f(t\ldots) \Rightarrow D_t^\kappa f(t,\ldots) = \frac{f(t+\Delta) - f(t-\Delta)}{2\Delta},$$ (3.10)

where Δ is proportional to κ^{-1} and it is *purely imaginary*.

(ii) The nonstandard real Hopf algebras for all three metrics are characterized by the trigonometric nonlinearities. In such a case the quantum deformation leads again to the replacement (3.10), but with Δ *real*. In such a case the quantum derivative (3.10) can be identified with the derivatives on the lattice (see, e.g., Refs. 23 and 24) in time direction, i.e., one can treat the quantum κ-deformation as introducing the time lattice with the lattice length proportional to $1/\kappa$. We see that the real lattice interpretation is related only with a *nonstandard* real Hopf algebra with star operation satisfying the relations (3.3a) and (3.3b). These nonstandard real Hopf algebras [obtained in the Minkowski case by putting in (1.1)–(1.3) the parameter κ purely imaginary] have several features which are not encouraging from the mathematical as well as the physical point of view. We shall mention two of them.

(1) The reality condition (3.3a) for the coproduct implies that the real spectrum of the quantum Poincaré generators becomes complex on the tensor product. Let us consider for example two states $|p_i^{(r)}\rangle$ $(r=1,2)$, which are the eigenvalues of the three-momentum operator. For purely imaginary $\kappa = i\kappa'$ $[\kappa' = (\kappa')^*]$ one obtains from (1.2) that $(|p_i^{(1)}, p_i^{(2)}\rangle \equiv |p_i^{(1)}\rangle \otimes |p_i^{(2)}\rangle)$

$$P_i^{(1+2)} |p_\mu^{(1)}, p_\nu^{(2)}\rangle = \left[p_i^{(1)} \exp\left(-i \frac{p_0^{(2)}}{\kappa'} \right) + p_i^{(2)} \exp\left(i \frac{p_0^{(1)}}{\kappa'} \right) \right] |p_i^{(1)}, p_i^{(2)}\rangle$$

$$= \left[\left(p_i^{(1)} \cos\frac{p_0^{(2)}}{\kappa'} + p_i^{(2)} \cos\frac{p_0^{(1)}}{\kappa'} \right) - i\left(p_i^{(1)} \sin\frac{p_0^{(2)}}{\kappa'} - p_i^{(2)} \sin\frac{p_0^{(1)}}{\kappa'} \right) \right] |p_i^{(1)}, p_i^{(2)}\rangle,$$ (3.11)

i.e., the total three momentum of two independent subsystems with real three momenta becomes complex.

(2) In the nonstandard real quantum Poincaré algebra, trigonometric nonlinearity implies the mass-shell condition

$$-\mathbf{P}^2 + (2\kappa \sin P_0/2\kappa)^2 = M_0^2.$$ (3.12)

For $M_0 = 0$ the κ-deformed formula for the light velocity looks as follows:

$$v_i = \frac{\partial}{\partial p_i}\left(\arcsin\frac{|p|}{2\omega} \right) = \frac{1}{\sqrt{1 - \mathbf{p}^2/4\kappa^2}}\frac{p_i}{|p_i|}.$$ (3.13)

(Such a modification of relativistic velocity was discussed long time ago by Caldirola.[25]) We see that the velocity for the particles with vanishing κ-mass M_0 $(M_0 = 0)$ achieves at $|\mathbf{p}| = 2\kappa$ an infinite value, i.e., the Einstein principle of maximal finite velocity of any signals is strongly violated. To the contrary, if we consider the standard real quantum Poincaré algebra, with the κ-deformed mass-shell condition given by (1.4), the formula for the velocity of κ-massless quanta looks as follows[10,11]

$$v_i = \frac{\partial}{\partial p_i}\left(\arcsin\frac{|p|}{2\omega} \right) = \frac{1}{\sqrt{1 + \mathbf{p}^2/4\kappa^2}}\frac{p_i}{|p_i|},$$ (3.14)

i.e., one obtains a more acceptable modification of Einstein's value $c = 1$.

It should be stressed however that

(i) The application of $D=2$ inhomogeneous quantum algebras with trigonometric nonlinearities to the phonon excitations[26] as well as Heisenberg spin models[27] (one-dimensional real spin lattices) were successful and provided an algebraic scheme consistent with the Bethe ansatz for solving $D=2$ integrable models. Further, it has been shown[26,27] that the coproduct is a correct operation describing algebraically in $D=2$ (one space, one time dimension) the multiexcited states (e.g., two-magnon states).

(ii) For q being a root of unity the difficulties with introducing tensor product of representations for nonstandard real Hopf algebras can be avoided if one introduces suitable projection operators on the spaces of physical interest. In such a case one arrives at the notion of quasiassociative quasi Hopf algebras.[20,28,29]

(iii) In this paper we consider the application of quantum groups to the deformation of four-dimensional symmetries. Unfortunately the physical interpretation of "nonsymmetric" coproduct rules for the four-dimensional κ-Poincaré generators is yet not clarified and should be further studied.

IV. FINAL REMARKS

We presented in this paper a new type of quantum deformation which can be generalized to any semidirect product $\hat{g} \oplus \hat{t}$ where $\hat{t} \equiv (t_r)$ are the Abelian generators and $\hat{g}$ describes the semisimple Lie algebra. The case of analogous deformation for $\hat{g}$ given by the algebra of rank one [SU(2) or SU(1,1)] was given in Ref. 30. In such a simple case one can find an explicit transformation of the generators which eliminate the nonlinearities from the algebra, i.e., the whole quantum deformation appears only in the noncocommutative coproduct sector. In our case the concrete form of the analogous transformation reducing the quantum algebra (1.1) to the classical Poincaré algebra is under consideration.

We would like to point out that there is an analogy between the Drinfeld–Jimbo deformation scheme of simple Lie algebras $\hat{g}$ (Refs. 18 and 31) and our scheme. In the Drinfeld–Jimbo method the nonpolynomial deformations are introduced as the functions on the elements of the maximal Abelian subalgebra of Cartan generators; for nonsemisimple algebra $\hat{g} \oplus \hat{t}$ an analogous role is played by the Abelian subalgebra $\hat{t}$. In principle, for nonsemisimple algebra one can form "mixed" Abelian subalgebras taking suitable parts of the Cartan subalgebra of $\hat{g}$ as well as of the subalgebra $\hat{t}$. It would be interesting to find an example of quantum deformation with nonpolynomial functions of the generators belonging to such a "mixed" Abelian subalgebra sector.

Finally let us mention that our deformation implies the deformation in one direction of the four-dimensional space, which was chosen to be the time axis. It is very easy to see that the considerations in the present paper can be extended to the description of the deformation into one spacelike direction. {For Euclidean case [see, e.g., (3.5)–(3.7)] such a deformation is obtained by reindexing the generators.} It is interesting, however, to consider the multidimensional deformations (e.g., cubic[13] or rectangular lattices) and see whether the real Hopf algebra structure (standard or nonstandard) remains valid.

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