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Theory of the Scattering Operator II Multichannel Scattering*)

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Abstract. The mathematical theory of the scattering operator is developed for the general scattering systems involving an arbitrary number of channels. It includes as a special case the theory for 'simple scattering systems' given in an earlier paper. The scattering system is defined as a quantum mechanical system which satisfies certain asymptotic and completeness conditions given in Section 4. The existence of the S-operator as well as its unitary property is then a rigorous mathematical consequence of this property. A crucial step in these deductions is the orthogonality theorem for the left projections of the wave operators which is proved in Section 5. In the last Section 7, we discuss the various ways of introducing the 'in' and 'out'-operators and their relation to the S-operator.

1. Introduction

In a recent paper[1]) the author has developed the theory of scattering on a mathematically rigorous foundation for the so-called 'simple scattering systems'. This paper is an extension of this work to the case of multichannel scattering. The basic philosophy adopted in this paper is much the same as in the previous one. Only mathematically well defined concepts, symbols and operations are employed. All steps are mathematically rigorous. The space of the state vectors is the classical Hilbert space and, in view of recent discussions we may add, this implies a positive definite metric.

The main result of this paper is the precise formulation of the concept of a multichannel scattering system and the proof of the existence of the scattering operator as well as its unitary property in a certain subspace of the Hilbert space. As was already pointed out by EKSTEIN[2]),[3]) the operator defined as S in our earlier paper, does not have a generalization in the multichannel case. The suitable operator for this purpose is another one defined and discussed in Section 2.

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The results obtained are equally valid whether we are dealing with the relativistic or non-relativistic form of the theory. Furthermore, the formalism is developed in such a manner that the cases with bound states are also included. The number of channels need not be finite as it is the case for instance in nuclear reactions.

The formulation is sufficiently broad so as to include all known types of scattering and reaction processes. In particular, we feel confident that the formulation presented here will serve as a sufficiently precise mathematical framework within which it should be possible to accommodate the theories dealing with fundamental particles and their interactions.

2. The operators S and S' for simple scattering systems

A 'simple scattering system' was previously defined[1]) as a quantum mechanical system with the following properties:

For every $f \in \mathfrak{H}$ the strong limits

$$\lim_{t \to \mp\infty} V_t^* U_t f = f_\pm = \Omega_\pm f \tag{2.1}$$

exist and the range of the operators $\Omega_\pm$ is equal to the subspace N of the continuum states of V_t.

In these expressions $V_t = e^{-iHt}$ $(-\infty < t < +\infty)$ is the transformation group of the system and H is the total energy operator. The group $U_t = e^{-iH_0 t}$ represents the free motion of the system in the absence of the interaction. H_0 is the kinetic energy operator for the particles participating in the scattering process.

The conditions (2.1) have a direct interpretation in terms of the actual scattering process and they entail the existence of a unitary scattering operator

$$S = \Omega_-^* \, \Omega_+ . \tag{2.2}$$

The 'wave operators' $\Omega_\pm$ are isometries in all of $\mathfrak{H}$ and they satisfy for both signs

$$\left.\begin{aligned} \Omega^* \Omega &= I \\ \Omega \Omega^* &= E_N \end{aligned}\right\} \tag{2.3}$$

where E_N is the projection operator into the subspace of continuum states of H.

All these statements were proved in reference[1]) and are here briefly repeated for convenience.

The definition (2.2) of the scattering operator is the one which is usually implied in much of the current literature on this subject. It is possible to define a different scattering operator, which we shall temporarily denote by S':

$$S' = \Omega_+ \Omega_-^* . \tag{2.4}$$

This operator has properties very similar to S which we shall enumerate here briefly.

The interest in this operator arises from the fact that in a multichannel theory it has a proper generalization while the operator (2.2) cannot be so generalized.

We observe first that the operator product in (2.4) is well defined since the Ω and Ω^* are bounded operators and are therefore defined in the entire space $\mathfrak{H}$. We can further form without restriction the products

$$S' S'^* = \Omega_+ \Omega_-^* \Omega_- \Omega_+^* = \Omega_+ \Omega_+^* = E_N , \tag{2.5}$$

and similarly

$$S'^* S' = E_N . \tag{2.6}$$

Furthermore

$$E_N S' = \Omega_+ \Omega_+^* \Omega_+ \Omega_-^* = \Omega_+ \Omega_-^* = S' , \tag{2.7}$$

and similarly

$$S' E_N = S'. \tag{2.8}$$

The operator S' annihilates therefore all the elements in $M = N^\perp$ and in the invariant subspace N it is unitary. We shall say S' is 'quasi-unitary'.

The following relations between the two operators S and S' are direct consequences of the definitions (2.2), (2.4) and the relations (2.3).

$$\Omega_+^* S' \Omega_+ = S = \Omega_-^* S' \Omega_- , \tag{2.9}$$

$$\Omega_+^* S' \Omega_- = I , \tag{2.10}$$

$$\Omega_-^* S' \Omega_+ = S^2 . \tag{2.11}$$

The operator S' commutes with V_t but not with U_t. This follows from the intertwining property of Ω which was proved in reference[1]):

$$V_t \Omega = \Omega U_t . \tag{2.12}$$

For instance one obtains

$$S' V_t = \Omega_+ \Omega_-^* V_t = \Omega_+ U_t \Omega_-^* = V_t \Omega_+ \Omega_-^* = V_t S' .$$

Since U_t does not commute with S' the operator $S'(t) = U_t^* S U_t$ depends on t. It satisfies the following relations

$$S'(-\infty) = S'(+\infty) = S . \tag{2.13}$$

These limits are understood in the strong operator topology. This means for every $f \in \mathfrak{H}$ we have

$$\| (S'(t) - S) f \| \to 0 \qquad \text{for } t \to \pm\infty . \tag{2.14}$$

One may verify this by using the relations proved in reference[1]):

$$\begin{aligned} \Omega_+(+\infty) &= S = \Omega_-(-\infty) \\ \Omega_+(-\infty) &= I = \Omega_-(+\infty) , \end{aligned} \tag{2.15}$$

where $\Omega(t) = U_t^* \Omega U_t$. Thus for the limit $t \to \infty$ we write the decomposition

$$S'(t) - S = \Omega_+(t) (\Omega_-^*(t) - I) + (\Omega_+(t) - S) .$$

Using the boundedness of the $\Omega(t)$, we find

$$\| (S'(t) - S) f \| \le \| (\Omega_-^*(t) - I) f \| + \| (\Omega_+(t) - S) f \| . \tag{2.16}$$

Since both terms on the right hand side tend to zero with $t \to +\infty$ by (2.15) the left hand side does so too and one of the relations (2.13) is established. The other is verified similarly.

3. The channel energy

The most characteristic feature of the general scattering process is the occurrence of 'free' particles before and after the collision. Every set of free particles is characterized by a set of parameters which expresses the values of the rest masses, spin, charge and whatever other variables are needed for a full description of the free particles. Each possible set of values for these parameters determines a different 'channel' of the system.

A free particle is one which moves exactly the same as if other particles were not present. The Hamiltonian for the free motion of the particles in a channel (also called the channel Hamiltonian) is therefore merely the sum of the kinetic energy of all the particles in the channel. This Hamiltonian is in general not obtained from the total Hamiltonian by the mere omission of certain terms, as it was the case in the simple scattering systems.

We shall illustrate this on one of the simplest examples of multichannel scattering. Let us consider three fundamental particles labelled 1, 2 and 3.

Let the total Hamiltonian be of the form

$$H = \frac{p_1^2}{2 m_1} + \frac{p_2^2}{2 m_2} + \frac{p_3^2}{2 m_3} + V_{12} + V_{23} + V_{31}$$

where the potentials V_{rs} describe the three possible two-body interactions of these particles. Let us assume the interaction between 1 and 2 is such that there exists one or more bound states, represented by square integrable solutions of the Schrödinger equation

$$\left(\frac{p_1^2}{2\,m_1} + \frac{p_2^2}{2\,m_2} + V_{12}\right)\Psi = E\,\Psi$$

The state vector Ψ describes then a fragment which can enter a scattering process or which may be produced as a final state. The total mass of the fragment is

$$M = m_1 + m_2 - |E|\,,$$

and the total kinetic energy is given by

$$H_0 = \begin{cases} \frac{1}{2\,M}\,\boldsymbol{P}^2 & \text{non relativistic} \\ \sqrt{\boldsymbol{M}^2 + \boldsymbol{P}^2} & \text{relativistic} \end{cases}$$

where $\boldsymbol{P} = (\boldsymbol{p}_1 + \boldsymbol{p}_2)$ is the momentum of the centre of mass.

It is conceivable that the interaction between another pair, say 2 and 3, is sufficiently strong so as to produce also a bound state. This will give rise to a new fragment of the mass

$$M' = m_2 + m_3 - |E'|$$

where E' is the binding energy of particle 2 and 3. The corresponding expression for the kinetic energy of this fragment is then given by

$$H_0 = \begin{cases} \frac{1}{2\,M'}\,\boldsymbol{P}'^2 & \text{non relativistic} \\ \sqrt{M'^2 + \boldsymbol{P}'^2} & \text{relativistic} \end{cases}$$

where $\boldsymbol{P}' = (\boldsymbol{p}_2 + \boldsymbol{p}_3)$.

In this description every bound state which leads to a different binding energy is in a different channel. For a degenerate bound state we obtain a fragment with additional internal degrees of freedom.

It is clear from this example how one would obtain the most general channel Hamiltonian for the fragments composed of any number of particles. We shall not elaborate this approach however. Instead we shall formulate the properties of channel Hamiltonians which we believe to be essential for the mathematical formulation of multichannel scattering theory.

These properties are purposely fromulated sufficiently general so that they would presumably be satisfied for elementary particles as well as stable fragments. From the point of view of scattering theory the distinction between elementary particle and composite fragments is an entirely superficial one and this is as it should be. The physical picture of an elementary particle breaks down as soon as internal structure reveals itself in certain collision processes. On the other hand, a stable fragment may in every respect behave like a fundamental particle as long as the collisions are sufficiently slow to leave the fragment intact. In a good scattering theory therefore this distinction should not appear as an essential element.

In agreement with this requirement we shall describe the properties of the channel Hamiltonian which are derived from the description of channels in terms of fragments but which are presumed to have general validity for any kind of particle composite or elementary. Instead of working with the channel Hamiltonian H we shall immediately express the characteristic properties in terms of the 'channel operators'

$$U_t^{(\alpha)} \equiv e^{-iH_\alpha t} \, .$$

Channel operators $U_t^{(\alpha)}$ are a set of continuous unitary representations of the additive group of real numbers $(-\infty < t < +\infty)$ in the underlying Hilbert space of state vectors with the following three properties:

(1) All $U_t^{(\alpha)}$ commute with one another for all α and all t:

$$[U_t^{(\alpha)}, U_{t'}^{(\beta)}] = 0 \, , \text{ all } \alpha, \beta, t, t'. \tag{3.1}$$

(2) The family of spectral projections of the infinitesimal generators of the group are continuous. There are no discrete eigenvalues.

(3) The operators $U_t^{(\alpha)}$ are 'essentially different' for different indices. With this we mean

$$U_t^{(\alpha)} f = U_t^{(\beta)} f \text{ for all } t \text{ implies } f = 0 \text{ unless } \alpha = \beta \, . \tag{3.2}$$

4. The definition of a scattering system

The channel operator which we have defined in the preceding section are rarely known explicitly. Instead one usually knows only the total energy operator H or its corresponding unitary group

$$V_t = e^{-iHt} \, . \tag{4.1}$$

It is clear that all the information as to the possible occurrence of various particles as well as their physical properties should be contained in the structure of the group (4.1). The different particles form the different channels and are described by certain channel operators $U_t^{(\alpha)}$. We shall now formulate the condition which is needed in order that the group (4.1) describes particles associated with certain channel operators $U_t^{(\alpha)}$:

A unitary group $U_t^{(\alpha)}$ with the three properties described in the preceding section is a channel of the system defined by V_t if there exists at least one element $f \in \mathfrak{H}$, $f \neq 0$, for which the strong limits

$$\lim_{t \to \mp \infty} V_t^* U_t^{(\alpha)} f = f_{\pm}^{(\alpha)} \tag{4.2}$$

exist.

The condition (4.2) may be considered the defining property of a channel energy. It extracts from the total transformation operator the kind and energy of the stable fragments which can be associated with it. Since this condition is an asymptotic property the definition of particles which are obtained in this way includes already all the possible self interactions. In the language of field theory: The particles are 'dressed particles'.

On the other hand, we see that in general it is not possible of introducing one single free-particle Hamiltonian. Every channel, that is every kind of free particles, has its own energy operator associated with it.

As a first consequence of the defining property (4.2) we shall show that if there exists one single element f for which (4.2) is satisfied, then there exists an infinite-dimensional subspace of $\mathfrak{H}$ with the same property.

Let D_α be the set of elements f such that the limit (4.2) exists. We first observe that D_α is a closed linear manifold, that is a subspace of $\mathfrak{H}$. The linear property is obvious. In order to prove the closure property we consider an arbitrary sequence $f_n \in D_\alpha$ for which $f_n \to f$ with $n \to \infty$. We must show that the limit (4.2) exists for the element f.

Let us define $W_{t_1 t_2} \equiv V_{t_1}^* U_{t_1} - V_{t_2}^* U_{t_2}$, then

$$\| W_{t_1 t_2} f \| \leqslant \| W_{t_1 t_2} f_n \| + \| W_{t_1 t_2} (f - f_n) \| .$$

The second term may further be estimated by

$$\| W_{t_1 t_2} (f - f_n) \| \leqslant \| W_{t_1 t_2} \| \, \| f - f_n \| \leqslant 2 \, \| f - f_n \| .$$

The last step is a consequence of the triangle inequality

$$\| A + B \| \leqq \| A \| + \| B \|$$

for the bounds of bounded operators.

We have therefore in all

$$\| W_{t_1 t_2} f \| \leqslant \| W_{t_1 t_2} f_n \| + 2 \, \| f - f_n \| .$$

We choose first a fixed n such that, independently of t_1 and t_2 the last term is $\leq \epsilon/2$ for an arbitrary $\epsilon > 0$. We then determine a T, such that for $t_1 > T$ and $t_2 > T$ the first term is $\leq \epsilon/2$ too. This is possible because $f_n \in D_\alpha$. For such values of t_1 and t_2 we have then

$$\| W_{t_1 t_2} f \| \leqslant \epsilon \qquad \text{all } t_1 > T, \; t_2 > T .$$

The elements of the form $V_t^* U_t f$ satisfy therefore the Cauchy criterion and because of the completeness of $\mathfrak{H}$ $\lim_{t \to +\infty} V_t^* U_t f$ exists. The limit $t \to -\infty$ is established in the same manner. Thus D_α is closed, q.e.d.

We now show that the subspace D_α is invariant under the group $U_t^{(\alpha)}$. This means if $f \in D_\alpha$ then $U_\tau^{(\alpha)} f \in D_\alpha$. Indeed

$$\lim_{t \to \pm\infty} V_t^* U_t^{(\alpha)} U_\tau^{(\alpha)} f = \lim_{t \to \pm\infty} V_\tau V_t^* U_t^{(\alpha)} f = V_\tau f_\mp , \tag{4.3}$$

and D_α is seen to be invariant under $U_t^{(\alpha)}$.

We can now easily complete the proof that D_α is infinite dimensional. Assume to the contrary that D_α is finite dimensional. Then the reduction of $U_t^{(\alpha)}$ to the invariant subspace D_α furnishes a finite dimensional unitary representation of the group of real numbers. Such a representation has only discrete eigenvalues, this contradicts the basic property (2) of the previous section. Thus D_α is infinite dimensional, q. e. d.

It was shown previously[1]) that the mapping $f \to f_\pm^{(\alpha)}$ is a linear isometry. The ranges $R_\pm^{(\alpha)}$ of this correspondence are therefore closed linear manifolds, that is subspaces.

The set of subspaces $R_\pm^{(\alpha)}$ as α runs through all the channels of the system span a linear manifold which we denote by $\{R_\pm^{(\alpha)}\}$. Its closure shall be denoted by

$$R_\pm \equiv \overline{\{R_\pm^{(\alpha)}\}}. \tag{4.4}$$

We shall further write N for the subspace of continuum states of V_t, that is, the orthogonal complement of the subspace M of proper elements of V_t.

With these preliminaries out of the way we define the general scattering system as follows:

A quantum mechanical system, described by the unitary group V_t, is a scattering system if there exists a set of channels α, together with their unitary groups $U_t^{(\alpha)}$, such that

$$R_+ = N = R_- . \tag{4.5}$$

The physical meaning of the requirement (4.5) is simply this: Every continuum state must be a superposition of scattering states, that is, states which are in the ranges $R_\pm^{(\alpha)}$.

It is readily seen that this definition of the scattering system is a generalization of the definition[1]) for a 'simple scattering system' to which it reduces if there exists only one channel.

The number of channels may be finite or infinite. We shall see in the next section, however, that in case it is infinite it is necessarily countably inifnite.

This definition of a scattering system embodies the minimum requirements which must be placed on a scattering system. Yet there are some far reaching conclusions which can be drawn from it as we shall see in the following sections.

5. The wave operators

We shall now assume that V_t describes a general scattering system as defined in the preceding section. There exists then a set of commuting unitary groups $U_t^{(\alpha)}$ such that for all $f \in D_\alpha$

$$\lim_{t \to \mp\infty} V_t^* U_t^{(\alpha)} f = f_\pm^{(\alpha)} . \tag{5.1}$$

Let $R_\pm^{(\alpha)}$ be the set of elements $f_\pm^{(\alpha)}$. According to the previous paper[1]) the mapping of D_α onto $R_\pm^{(\alpha)}$ is a linear isometry, and $R_\pm^{(\alpha)}$ is a subspace (= closed linear manifold). This mapping defines therefore a bounded linear operator $\Omega_\pm^{(\alpha)}$ on D_α by the condition

$$\Omega_\pm^{(\alpha)} f = f_\pm^{(\alpha)} \quad f \in D_\alpha . \tag{5.2}$$

Such an operator can always be extended in a continuous manner to the whole space by, for instance, the following procedure: Let $f \in \mathfrak{H}$ and $f = g + h$, $g \in D_\alpha$, $h \in D_\alpha^\perp$. Define

$$\Omega_\pm^{(\alpha)} f = \Omega_\pm^{(\alpha)} g . \tag{5.3}$$

We shall denote this extended operator with the same symbol.

To every bounded operator $\Omega_\pm^{(\alpha)}$ can be associated uniquely[4]) an adjoint operator $\Omega_\pm^{(\alpha)*}$ by the condition

$$(\Omega_\pm^{(\alpha)*} f, g) = (f, \Omega_\pm^{(\alpha)} g) \tag{5.4}$$

for all $f, g \in \mathfrak{H}$. It follows from this definition that $\Omega_\pm^{(\alpha)*}$ vanishes on the orthogonal complement $R_\pm^{(\alpha)\perp}$ of the range of $\Omega_\pm^{(\alpha)}$.

In order to survey some of the general properties of the wave operators we shall introduce a few definitions and deduce from them some elementary consequences.

Definition 1:

A bounded linear operator Ω is a partial isometry if $E \equiv \Omega^* \Omega$ is a projection.

It follows that if Ω is a partial isometry then $F \equiv \Omega \Omega^*$ is also a projection.

In order to see this we note that

$$(\Omega(E-I))^* \Omega(E-I) = (E-I)\, \Omega^* \Omega\, (E-I) = (E-I)\, E(E-I) = 0\,.$$

Therefore for any $f \in \mathfrak{H}$

$$\| \Omega(E-I)\, f \|^2 = (f, (\Omega(E-I))^* \Omega(E-I)\, f) = 0\,,$$

that is $\Omega(E-I)\, f = 0$ for any $f \in \mathfrak{H}$,

or $$\Omega(E-I) = 0\,, \tag{5.5}$$

or also $$\Omega E = \Omega\,. \tag{5.6}$$

From this we obtain

$$F^2 = \Omega\Omega^*\, \Omega\Omega^* = \Omega E \Omega^* = \Omega\Omega^* = F$$

and $$F^* = F\,.$$

The last two relations express that F is a projection.

Definition 2:

If Ω is a partial isometry then
$E = \Omega^*\Omega$ is called the right-projection and
$F = \Omega\Omega^*$ the left-projection of Ω.
It follows obviously that conversely E is the left-projection and F the right-projection of Ω^*.
Left- and right-projections have a maximal property as follows:
The projection E is the smallest projection with the property

$$\Omega E = \Omega$$

and F is the smallest projection with the property

$$F\, \Omega = \Omega\,.$$

We recall the partial ordering of projections: The projection E is smaller than E_1, if for every $f \in \mathfrak{H}$

$$\| E f \| \leqslant \| E_1 f \|\,.$$

This will be written as $E \leq E_1$. An equivalent fromulation of this relation is $E_1 E = E E_1 = E$.

In order to verify the maximal property of the projection E let us assume that E_1 is another projection such that

$$\Omega\, E_1 = \Omega\,.$$

We have then

$$\Omega^*\Omega E_1 = \Omega^*\Omega = E$$

or

$$EE_1 = E$$

Since both E and E_1 are projections we obtain also by taking the adjoint

$$E_1 E = E$$

which means $E \leq E_1$. The second half of the assertion is proved in a similar manner.

The subspaces which are the ranges of the projections E and F are respectively the ranges of Ω^* and of Ω. For instance if f is in the range of F then $Ff = f = \Omega\Omega^* f$, hence it is in the range of Ω. Conversely if f is in the range of Ω there exists a g such that $f = \Omega g = F\Omega g$. Therefore f is also in the range of F. The two ranges are identical. In a similar way one verifies the second half of the statement.

We shall now apply these concepts to the operators $\Omega_\pm^{(\alpha)}$. From the earlier paper[1]) and the extension (5.3) it follows that they are partial isometries. We define

$$E_\alpha = \Omega_\pm^{(\alpha)*}\, \Omega_\pm^{(\alpha)} \tag{5.7}$$

$$F_\pm^{(\alpha)} = \Omega_\pm^{(\alpha)} \Omega_\pm^{(\alpha)*}\,. \tag{5.8}$$

For the projections E_α the distinction between the two cases $\pm$ is unnecessary since the two projections are identical.

We shall now prove the main result of this section:

Theorem:

The projections $F_\pm^{(\alpha)}$ are orthogonal for different channels, in the sense

$$F_+^{(\alpha)} F_+^{(\beta)} = F_-^{(\alpha)} F_-^{(\beta)} = 0 \quad \text{for} \quad \alpha \neq \beta\,. \tag{5.9}$$

Proof:

We shall prove the case with the minus sign and omit the sign index.

A fully equivalent statement of the theorem is the following: Let $f^{(\alpha)}$ and $g^{(\beta)}$ be any two elements of $\mathfrak{H}$ with the property

$$F^{(\alpha)} f^{(\alpha)} = f^{(\alpha)},\; F^{(\beta)} g^{(\beta)} = g^{(\beta)} \tag{5.10}$$

then

$$(f^{(\alpha)}, g^{(\beta)}) = 0 \quad \text{for} \quad \alpha \neq \beta\,. \tag{5.11}$$

We shall prove the theorem in this form.

According to the preceding remarks the elements $f^{(\alpha)}$ and $g^{(\beta)}$ are in the ranges of the operators $\Omega^{(\alpha)}$ and $\Omega^{(\beta)}$ respectively. This means there exists element $f \in D_\alpha$ and $g \in D_\beta$ such that

$$f_t^{(\alpha)} \equiv V_t^* U_t^{(\alpha)} f \rightarrow f^{(\alpha)} \quad \text{for} \quad t \rightarrow \infty \tag{5.12}$$

$$g_t^{(\beta)} \equiv V_t^* U_t^{(\beta)} f \rightarrow g^{(\beta)} \quad \text{for} \quad t \rightarrow \infty . \tag{5.13}$$

We shall first show that for any pair of elements f and g the operators

$$W_t \equiv U_t^{(\alpha)*} U_t^{(\beta)} \tag{5.14}$$

which are also a unitary group, converge weakly to zero in the limit $t \rightarrow \infty$. This means

$$(f, W_t g) \rightarrow 0 , \quad \text{for } t \rightarrow \infty \; ; \; f \in D_\alpha , \; g \in D_\beta . \tag{5.15}$$

We shall first show that the limit exists and then that it is zero.

That it exists may be seen from

$$(f, W_t g) = (f_t^{(\alpha)}, g_t^{(\beta)}) . \tag{5.16}$$

Both $f_t^{(\alpha)}$ and $g_t^{(\beta)}$ strongly converge to $f^{(\alpha)}$ and $g^{(\beta)}$ respectively and their scalar product converges to $(f^{(\alpha)}, g^{(\beta)})$:

$$| (f_t^{(\alpha)}, g_t^{(\beta)}) - (f^{(\alpha)}, g^{(\beta)}) |$$

$$\leqslant | ((f_t^{(\alpha)} - f^{(\alpha)}) , g_t^{(\beta)})| + | (f^{(\alpha)}, (g_t^{(\beta)} - g^{(\beta)})) |$$

$$\leqslant \| f_t^{(\alpha)} - f^{(\alpha)} \| \; \| g^{(\beta)} \| + \| f^{(\alpha)} \| \; \| g_t^{(\beta)} - g^{(\beta)} \| .$$

We have made use of Schwartz's inequality and $\| g_t^{(\beta)} \| = \| g^{(\beta)} \|$. Since each of the t-dependent factors goes to zero (Equation (5.12) and (5.13)) the assertion that the limit $(f, W_t g)$ exists for $t \rightarrow \infty$ follows. Moreover, it is shown that this limit is equal to $(f^{(\alpha)}, g^{(\beta)})$.

We shall next show that this limit is zero. This is easily established by using a corollary of the mean ergodic theorem of v. NEUMANN[5]). According to this

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (f, W_t g) = (f, Pg) \tag{5.17}$$

where P is the projection operator into the subspace of proper elements of W_t with eigenvalue $+ 1$. On the other hand, since the integrand has the limit $(f^{(\alpha)}, g^{(\beta)})$ for $t \rightarrow \infty$ we also have

$$(f, Pg) = (f^{(\alpha)}, g^{(\beta)}) . \tag{5.18}$$

It remains to show that P is the zero operator.

Suppose f is in the range of P then by definition of P

$$W_t f = f \quad \text{for all } t$$

or

$$U_t^{(\alpha)} f = U_t^{(\beta)} f \quad (\alpha \neq \beta). \tag{5.19}$$

According to property (3) of the channel operators (Section 3) this imples $f = 0$. Consequently $P = 0$ and therefore by (5.18) $(f^{(\alpha)}, g^{(\beta)}) = 0$. Since $f^{(\alpha)}$ and $g^{(\beta)}$ were arbitrary elements in the ranges $F^{(\alpha)}$ and $F^{(\beta)}$ we have

$$F^{(\alpha)} F^{(\beta)} = 0 \quad \text{for} \quad \alpha \neq \beta. \text{ q.e.d.}$$

We shall now mention a few useful consequences of this theorem.

First we observe that the number of channels is either finite or countably infinite. Indeed, since $\mathfrak{H}$ is separable and the projections $F^{(\alpha)}$ orthogonal, their number is at most countably infinite and it is equal to the number of channels.

Furthermore since by the definition of the scattering system the closed linear manifold spanned by the ranges $R_{\pm}^{(\alpha)}$ is the whole of the space of continuum states N, we have

$$E_N = \sum_{\alpha} F_{+}^{(\alpha)} = \sum_{\beta} F_{-}^{(\beta)}, \tag{5.20}$$

where E_N is the projection operator with range N.

As a further consequence of the orthogonality relations (5.9) we have

$$\Omega_{+}^{(\alpha)*} \Omega_{+}^{(\beta)} = E_\alpha \delta_{\alpha\beta} = \Omega_{-}^{(\alpha)*} \Omega_{-}^{(\beta)}. \tag{5.21}$$

For instance the first of these equations is obtained from

$$\Omega_{+}^{(\alpha)*} \Omega_{+}^{(\beta)} = \Omega_{+}^{(\alpha)*} F_{+}^{(\alpha)} F_{+}^{(\beta)} \Omega_{+}^{(\beta)} = \Omega_{+}^{(\alpha)*} F_{+}^{(\alpha)} \Omega_{+}^{(\beta)} \delta_{\alpha\beta} = \Omega_{+}^{(\alpha)*} \Omega_{+}^{(\beta)} \delta_{\alpha\beta} = E_\alpha \delta_{\alpha\beta}.$$

Finally we note that the operators $\Omega^{(\alpha)}$ have the intertwining property

$$V_t \Omega^{(\alpha)} = \Omega^{(\alpha)} U_t^{(\alpha)} \tag{5.22}$$

from which follows, among other things, that the projections $F_{\pm}^{(\alpha)}$ commute with V_t

$$[F_{\pm}^{(\alpha)}, V_t] = 0 \quad \text{all } t, \text{ all } \alpha. \tag{5.23}$$

There is no orthogonality theorem for the domains D_α, or their projections E_α. In fact it is quite possible that one of the domains D_α may be the entire space $\mathfrak{H}$. In any case the different E_α are in general not orthogonal.

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6. The scattering operator

The operator to be defined is the generalization of the operator S' of Section 2. In order to avoid a too cumbersome notation, we shall omit the prime.

We define a sequence of operators S_n by

$$S_n = \sum_{\alpha=1}^{n} \Omega_{+}^{(\alpha)} \Omega_{-}^{(\alpha)*} \tag{6.1}$$

and investigate the limiting properties of these operators as n increases indefinitely. When the number of channels is finite then the sequence S_n is finite and stops for some number n_0. In this case there is no limit problem. We define

$$S = S_{n_0}. \tag{6.2}$$

In case the number of channels is infinite S_n converges strongly with $n \to \infty$ to a limit S on all of $\mathfrak{H}$.

To show this consider an arbitrary element $f \in \mathfrak{H}$ and the sequence

$$f_n = S_n f. \tag{6.3}$$

We must show f_n converges to some element g. Because $\mathfrak{H}$ is complete it suffices to show that f_n is a Cauchy sequence.
Assuming $n \geq m$, we have

$$\| f_n - f_m \|^2 = \| (S_n - S_m) f \|^2 = \left(f, \sum_{\alpha,\beta=m}^{n} \Omega_{-}^{(\alpha)} \Omega_{+}^{(\alpha)*} \Omega_{+}^{(\beta)} \Omega_{-}^{(\beta)*} f \right).$$

The general term under the summation sign can be simplified by using Equation (5.7) and (5.8).

$$\Omega_{-}^{(\alpha)} \Omega_{+}^{(\alpha)*} \Omega_{+}^{(\beta)} \Omega_{-}^{(\beta)*} = \delta_{\alpha\beta} \Omega_{-}^{(\alpha)} E_\alpha \Omega_{-}^{(\beta)*} = \delta_{\alpha\beta} \Omega_{-}^{(\alpha)} \Omega_{-}^{(\alpha)*} = \delta_{\alpha\beta} F_{-}^{(\alpha)}. \tag{6.4}$$

Hence

$$\| f_n - f_m \|^2 = \left(f, \sum_{\alpha=m}^{n} F_{-}^{(\alpha)} f \right). \tag{6.5}$$

According to Equation (5.9) the projections are all mutually orthogonal. Hence

$$G_n \equiv \sum_{\alpha=1}^{n} F_{-}^{(\alpha)}$$

is a sequence of non-decreasing projections. Such a sequence converges strongly to a projection. The sequence of elements

$$g_n \equiv G_n f$$

is therefore a Cauchy sequence and since

$$\| f_n - f_m \|^2 = \| g_n - g_m \|^2$$

f_n is also a Cauchy sequence. The strong limit $n \to \infty$ exists

$$g = \lim_{n \to \infty} f_n$$

and the correspondence

$$f \to g = Sf$$

defines the linear operator

$$S = \sum_{\alpha=1}^{\infty} \Omega_{+}^{(\alpha)} \Omega_{-}^{(\alpha)*} . \tag{6.6}$$

This is the scattering operator.

Next we prove that S is quasi-unitary:

$$S^*S = SS^* = E_N . \tag{6.7}$$

For instance, using the identity (6.4), we find with (5.20)

$$S^*S = \sum_{\alpha=1}^{\infty} F_{-}^{(\alpha)} = E_N \tag{6.8}$$

$$SS^* = \sum_{\alpha=1}^{\infty} F_{+}^{(\alpha)} = E_N . \tag{6.9}$$

Since V_t commutes with all terms in the sum (6.6) (cf. Equation (5.22)) we have also

$$[V_t, S] = 0 . \tag{6.10}$$

The following relations are also useful

$$\Omega_{-}^{(\beta)*} S \Omega_{-}^{(\alpha)} = \Omega_{+}^{(\beta)*} S \Omega_{+}^{(\alpha)} = \Omega_{-}^{(\beta)*} \Omega_{+}^{(\alpha)} . \tag{6.11}$$

The physical interpretation of the scattering operator is obtained from the matrix elements. Let the system be at $t \to -\infty$ in a channel α. With this we mean that it approaches in the norm for $t \to -\infty$ the state $U_t^{(\alpha)} f_\alpha$, where $f_\alpha \in D_\alpha$ and $\| f_\alpha \| = 1$. Because of the basic property of scattering systems, this also approaches in the norm $V_t f_{+}^{(\alpha)}$ as $t \to -\infty$. The probability for finding the system at time t in the state $U_t^{(\alpha)} f_\beta$ ($f_\beta \in D_\beta$) of channel β is then

$$P_{\beta\alpha}(t) = | (U_t^{(\beta)} f , \; V_t f_{+}^{(\alpha)}) |^2 .$$

In the limit $t \to \infty$ we have

$$(U_t^{(\beta)} f_\beta, \; V_t f_+^{(\alpha)}) = (V_t^* U_t^{(\beta)} f_\beta, \; f_+^{(\alpha)}) \to (f_-^{(\beta)}, f_+^{(\alpha)}) .$$

We can write the last expression in the equivalent forms (cf. Equation (6.11))

$$\begin{aligned}(f_-^{(\beta)}, f_+^{(\alpha)}) &= (f_-^{(\beta)}, S f_-^{(\alpha)}) = (f_+^{(\beta)}, S f_+^{(\alpha)}) \\ &= (f_\beta, \, \Omega_-^{(\beta)*} S \Omega_-^{(\alpha)} f_\alpha) \\ &= (f_\beta, \, \Omega_+^{(\beta)*} S \Omega_+^{(\alpha)} f_\alpha) . \end{aligned} \qquad (6.12)$$

These expressions lead to the usual formulae for scattering cross-sections and decay time when properly specialized.

7. Asymptotic properties of observables

In this section we shall investigate the question how the asymptotic behaviour of observables is related to the S-operator in a general scattering system. This question is of interest because the asymptotic operators have often been introduced as auxiliary quantities in the definition of the scattering operator[6],[7]). In fact in more recent works the tendency has been to formulate the scattering theory in such a manner that no explicit reference is made to the free Hamiltonian describing 'bare' particles. Instead one attempts to replace it by the asymptotic properties of a sufficiently complete set of observables (or field operators). The relation between these observables in the past and the future is then used for the definition of the scattering operator[3]),[8]).

We shall examine here the asymptotic properties of observables from the point of view of the rigorous scattering theory.

Let A be a self-adjoint, bounded operator in $\mathfrak{H}$, representing an observable. We shall assume that A is independent of time in the Schrödinger picture in which state vectors change in time according to

$$f_t = V_t f . \qquad (7.1)$$

In the Heisenberg picture the operator A varies in time according to

$$A_t = V_t^* A \, V_t \qquad (7.2)$$

while the state vectors are constant.

We shall first discuss the case of a 'simple scattering system', that is, a system with one channel only[1]). The transformation operator for this channel shall be denoted, as before, by $U_t = e^{-iH_0 t}$.

The asymptotic condition as usually formulated is to assume that with every operator A there are associated two operators A_{in} and A_{out} which in some sense, to be made precise, represent the operator A in the distant past and the remote future.

The A_{in} and A_{out} are considered the observables of free particle motion. The definition of the operators A_{in} and A_{out} is not unique even for the case of simple scattering systems. The ambiguity is related to the ambiguity in the definition of the scattering operator discussed in section 2 of this paper.

One sense in which the asymptotic condition can be made precise is to require the weak convergence of the operators $U_t A_t U_t^*$ to the constant operators A_{in} and A_{out} for the respective limits $t \to -\infty$, and $t \to +\infty$.

In other words, for any two elements $f, g \in \mathfrak{H}$ we shall have

$$(f, U_t A_t U_t^* g) \to (f, A_{\text{in}} g) \quad \text{for } t \to -\infty,$$

and

$$(f, U_t A_t U_t^* g) \to (f, A_{\text{out}} g) \quad \text{for } t \to +\infty. \tag{7.3}$$

We shall now examine whether this condition is in fact satisfied for simple scattering systems and whether it can be used for the definition of the S-operator.

From the identity

$$U_t V_t^* U_t^* - \Omega_-^* A \Omega_- = U_t V_t^* A (V_t U_t^* - \Omega_-) + (U_t V_t^* - \Omega_-^*) A \Omega_-$$

we find

$$| (f, (U_t A_t U_t^* - \Omega_- A \Omega_-) g) |$$
$$\leqslant | (f, U_t V_t^* A (V_t U_t^* - \Omega_-) g) | + | f, (U_t V_t^* - \Omega_-^*) A \Omega_- g) | .$$

For the first term we have by Schwartz' inequality

$$| (f, U_t V_t^* A (V_t U_t^* - \Omega_-) g) |$$
$$\leqslant \| f \| \ \| A \| \ \ \| (V_t U_t^* - \Omega_-) g \|$$

This term tends to zero with $t \to -\infty$ for all $f, g \in \mathfrak{H}$ as a consequence of the basic property of a simple scattering system.

For the second term we have

$$| (f, (U_t V_t^* - \Omega_-^*) A \Omega_- g) |$$
$$\leqslant \| f \| \ \| A \| \ \ \| (V_t U_t^* - \Omega_-) g \|$$

and this is again convergent to zero for $t \to -\infty$. We have thus verified the weak convergence of

$$U_t A_t U_t^* \text{ to } A_{\text{in}} \equiv \Omega_-^* A \Omega_- \text{ for } t \to -\infty .$$

We remark that we could have asserted strong convergence for operators A which leave the subspace N of continuum states invariant. Indeed we have

$$\| (U_t A_t U_t^* - A_{\text{in}}) f \| \leqslant \| U_t V_t^* A (\Omega_- - V_t U_t^*) f \| + \| (\Omega_-^* - U_t V_t^*) A \Omega_- f \| .$$

The first term converges to zero without restriction for $t \to -\infty$. For the second term on the other hand we can only assert convergence if $A\, \Omega_- f \in N$. Since $\Omega_- f \in N$ this is assured if N is invariant under A.

If $N = \mathfrak{H}$, that is if there are no bound states, this is no restriction and strong convergence can always be asserted in that case.

In a similar way one can discuss the operators A_{out} and the convergence of the corresponding expressions in the limit $t \to +\infty$. The relation between A_{in} and A_{out} is then obtained from the two defining equations

$$\begin{aligned} A_{\text{in}} &= \Omega_-^* A\, \Omega_- \\ A_{\text{out}} &= \Omega_+^* A\, \Omega_+ . \end{aligned} \tag{7.4}$$

If we multiply the second Equation (7.4) from the left with Ω_+ and from the right with Ω_+^*, we obtain first

$$\Omega_+ A_{\text{out}}\, \Omega_+^* = F A F \tag{7.5}$$

where F is the left projection of Ω_+ with range N. Multiplying now (7.5) from the left with Ω_-^* and from the right with Ω_- we obtain

$$\Omega_-^*\, \Omega_+ A_{\text{out}}\, \Omega_+^*\, \Omega_- = A_{\text{in}} . \tag{7.6}$$

Here we have used the fact that the left projection of $\Omega_\pm$ are the same for simple scattering systems. This leads to the definition of the S-operator according to

$$A_{\text{out}} = S^* A_{\text{in}} S \tag{7.7}$$

with $S = \Omega_-^*\, \Omega_+$, in agreement with Equation (9) reference[8]). The S-operator obtained in this manner is the S-operator which does not allow a generalization to the multichannel case. The definition of S through Equations (7.3) and (7.5) is therefore only suitable for the 'simple scattering systems'.

A different definition of A_{in} and A_{out} which is used by EKSTEIN[3]), allows a generalization to the multichannel case. We shall again use the same notation as before even though the operators to be defined now differ from the ones discussed so far.

We verify first that for $f, g \in N$

$$(f, A_t g) \to (f, \Omega_- A(t)\, \Omega_-^* g) \quad \text{for} \quad t \to +\infty . \tag{7.8}$$

Where

$$A_t = V_t^* A\, V_t$$

$$A(t) = U_t^* A\, U_t .$$

Indeed

$$| (f, A_t g) - (f, \Omega_- A(t)\, \Omega_-^* g) |$$

$$\leqslant | (f, V_t^* A (V_t - U_t\, \Omega_-^*)\, g) | + | (f, (V_t^* - \Omega_- U_t^*)\, A\, U_t\, \Omega_-^* g) | .$$

For the first term we have

$$| (f, V_t^* A (V_t - U_t\, \Omega_-^*)\, g) | \leqslant \| f \| \;\; \| A \| \;\; \| (I - V_t^* U_t \Omega_-^*)\, g \| .$$

Since g $\in N$

$$\lim_{t \to +\infty} V_t^* U_t \Omega_-^* g = \Omega_- \Omega_-^* g = F g = g .$$

Therefore the right hand side vanishes with $t \to +\infty$. For the second term we obtain

$$| (f, (V_t^* - \Omega_- U_t^*)\, A\, U_t\, \Omega_t^* g) |$$

$$\leqslant \| (I - V_t^* U_t \Omega_-^*)\, f \| \;\; \| A \| \;\; \| g \|$$

and this vanishes for $t \to \infty$ because $f \in N$. Thus relation (7.8) is established. A similar reasoning for $t \to -\infty$ leads to a corresponding relation. We can then define two time dependent observables $A_{in}(t)$ and $A_{out}(t)$ according to

$$\left.\begin{aligned} A_{\text{out}}(t) &\equiv \Omega_- A(t)\; \Omega_-^* \\ A_{\text{in}}\;(t) &= \Omega_+ A(t)\; \Omega_+^* \end{aligned}\right\} \tag{7.9}$$

As was pointed out by EKSTEIN[3]) these are not 'free' operators since their time dependence is only apparently governed by the free transformation operator U_t as may be seen from

$$\Omega_- A(t)\; \Omega_-^* = \Omega_- U_t^* A\; U_t \Omega_-^* = V_t^*\, \Omega_- A\; \Omega_-^* V_t .$$

By eliminating from (7.9) the operators $A(t)$, we obtain

$$A_{\text{out}}(t) = \Omega_- \Omega_+^*\, A_{\text{in}}(t)\; \Omega_+ \Omega_-^* \tag{7.10}$$

or

$$A_{\text{out}}(t) = S^* A_{\text{in}}(t)\, S\, , \tag{7.11}$$

with

$$S = \Omega_+ \Omega_-^* . \tag{7.12}$$

The similarity of (7.11) with (7.7) is superficial and misleading (as is the notation) since all the quantities involved in (7.11) are quite different from the ones occurring in (7.7). We shall not refer to (7.7) any more in the rest of the paper.

The generalization of these considerations to the multichannel case are now fairly straightforward. We need to show for instance the weak convergence for $f, g \in N$

$$(f, A_t g) \to (f, A_{\mathrm{out}}(t)\, g) \quad \text{for} \quad t \to +\infty\,, \tag{7.13}$$

with

$$A_{\mathrm{out}}(t) = \sum_\alpha \Omega_-^{(\alpha)} U_t^{(\alpha)*} A \sum_\beta U_t^{(\beta)} \Omega_-^{(\beta)*}\,. \tag{7.14}$$

To show this we use formula (5.20), and obtain for instance

$$g = E_N g = \sum_\alpha F_-^{(\alpha)} g\,, \tag{7.15}$$

and a similar decomposition for f. We shall write $f^{(\beta)} = F_-^{(\beta)} f$ and $g^{(\beta)} = F_-^{(\beta)} g$, so that

$$(f, A_t g) = \sum_{\alpha, \beta} (f^{(\alpha)}, A_t g^{(\beta)})\,, \tag{7.16}$$

and show that the general term under the summation sign converges to $(f^{(\alpha)}, A_{out}(t)\, g^{(\beta)})$ for $t \to +\infty$. This last expression may also be written as

$$(f^{(\alpha)}, \Omega_-^{(\alpha)} U_t^{(\alpha)*} A\; U_t^{(\beta)} \Omega_-^{(\beta)}) \quad \text{since } F_-^{(\alpha')} \Omega_-^{(\alpha)} = \delta_{\alpha'\alpha} \Omega_-^{(\alpha)}\,.$$

The stated convergence is now proved by writing

$$\begin{aligned} &| (f^{(\alpha)}, (A_t - A_{\mathrm{out}}(t))\, g^{(\beta)}) | \\ &\leqslant | (f^{(\alpha)}, V_t^* A\; (V_t - U_t^{(\beta)} \Omega_-^{(\beta)})\, g^{(\beta)}) | \\ &+ | (f^{(\alpha)}, (V_t^* - \Omega_-^{(\alpha)} U_+^{(\alpha)*})\; A\; U_t^{(\beta)} \Omega_-^{(\beta)*}\, g^{(\beta)}) |\,. \end{aligned}$$

The vanishing of the first factor in the limit $t \to +\infty$ follows now from

$$\| (V_t - U_t^{(\beta)} \Omega_-^{(\beta)})\, g^{(\beta)} \| \to 0 \quad \text{for} \quad t \to +\infty\,.$$

Similarly one concludes the vanishing of the second factor.

We summarize the result for convenience:

$$(f, A_t\, g) \to (f, A_{\substack{\mathrm{out}\\ \mathrm{in}}}(t)\, g) \quad \text{for} \quad t \to \pm\infty \quad \text{and} \quad f, g \in N\,. \tag{7.17}$$

$$A_{\substack{\mathrm{out}\\ \mathrm{in}}}(t) = \sum_\alpha \Omega_\mp^{(\alpha)} U_t^{(\alpha)*} A \sum_\beta U_t^{(\beta)} \Omega_\mp^{(\beta)*}\,. \tag{7.18}$$

The relation between $A_{out}(t)$ and $A_{in}(t)$ is exactly as in the single channel case

$$A_{\text{out}}(t) = S^* A_{\text{in}}(t)\, S\,, \tag{7.19}$$

with S now defined as

$$S = \sum_{\alpha} \Omega_+^{(\alpha)}\, \Omega_-^{(\alpha)*}\,. \tag{7.20}$$

To prove this we eliminate A from the two relations (7.18). For instance we obtain by multiplying from the left with $\Omega_+^{(\alpha)*}$ and from the right with $\Omega_+^{(\beta)}$ using (5.21)

$$\Omega_+^{(\alpha)*} A_{\text{in}}(t)\, \Omega_+^{(\beta)} = E_\alpha\, U_t^{(\alpha)*} A\, U_t^{(\beta)}\, E_\beta\,.$$

This relation we multiply next from the left with $\Omega_-^{(\alpha)}$ and from the right with $\Omega_-^{(\beta)*}$ and sum over the indices α and β. Using the fact that E_α is right projection of Ω_α we find

$$S^* A_{\text{in}}(t)\, S = A_{\text{out}}(t)$$

with S given by (7.20).

The question needs to be examined whether this procedure can be used for the definition of the scattering operator. The answer is yes if the asymptotic operators $A_{out}(t)$ are known for a sufficiently large set of operators in A so that they generate in N an irreducible operator ring. The S-operator is then uniquely determined by (7.19) in the subspace N up to a numerical factor of magnitude 1.

It is also possible to introduce time-independent operators A_{in} and A_{out} by defining

$$A_{\substack{\text{out}\\ \text{in}}}(t) = V_t^* A_{\substack{\text{out}\\ \text{in}}} V_t\,. \tag{7.21}$$

By using the intertwining property (5.22) one obtains

$$A_{\substack{\text{out}\\ \text{in}}} = \sum_{\alpha} \Omega_\mp^{(\alpha)} A \sum_{\beta} \Omega_\mp^{(\beta)*}, \tag{7.22}$$

and

$$A_{\text{out}} = S^* A_{\text{in}} S\,. \tag{7.23}$$

Appendix

In reference[1]) we have stated the following lemma: let $V_t = e^{-iHt}$ and $U_t = e^{-iH_0 t}$ be the unitary transformation groups associated with a 'simple scattering system'. Let Ω denote either Ω_+ or Ω_- and let Ω_1 be any bounded intertwining operator such that

$$V_t\, \Omega_1 = \Omega_1\, U_t\,. \tag{1}$$

Denote by R the range of Ω (which is common to Ω_+ and Ω_-) and by R_1 the range of Ω_1 then

$$R_1 \subseteq R . \tag{2}$$

The proof given in reference[1]) was incorrect as indicated in the footnote. We shall here give a correct proof.

We remark first of all that it is sufficient to prove the theorem for intertwining operators which are isometries. This simplification is possible because if T is any bounded intertwining operator, then there exists another one W which is an isometry and which has exactly the same range as T.

This is a simple consequence of the well-known theorem on the polar decomposition of bounded linear operators[9]). According to this theorem one can associate with every bounded linear operator T a unique linear isometry W with the same left and right projections as T. It is related to T by the formulas

$$T = W R \tag{3}$$

and

$$R = (T^* T)^{\frac{1}{2}} . \tag{4}$$

The operator W is defined as follows: on all elements g of the form $g = Rf$, we have $Wg = Tf$. On the linear manifold $R(\mathfrak{H})$ of such elements W is homogeneous, additive and isometric. Such a transformation can be uniquely extended by continuity to the closed linear manifold $\overline{R(\mathfrak{H})}$, preserving these properties. On the orthogonal complement of this subspace W is zero by definition. W is then a partial isometry. Its left and right projections are the same as the left and right projections of T.

To apply this theorem for our case we assume that T is an intertwining operator such that

$$V_t T = T U_t . \tag{5}$$

We aim to show that then also

$$V_t W = W U_t \tag{6}$$

where W is the partial isometry just defined. In order to see this we evaluate the left-hand side of (6) as follows. Let the arbitrary f be decomposed $f = g + h$ with $g \in R(\mathfrak{H})$ and $h \in \overline{R(\mathfrak{H})}^{\perp}$. We have then $Wf = Wg$. Since g is of the form Ru with $u \in \mathfrak{H}$, or a limit point of such elements we have

$Wf = Tu$ or the corresponding limiting relation.

Consequently

$$V_t W f = V_t T u = T U_t u \tag{7}$$

On the other hand we have

$$U_t f = U_t g + U_t h\,,\ U_t g \in \overline{R(\mathfrak{H})} \text{ and } U_t h \in R(\mathfrak{H})\perp$$

because R commutes with U_t.
Therefore

$$W U_t f = W U_t g = W U_t R u = W R\, U_t u = T\, U_t u\,. \tag{8}$$

Combining (7) and (8) we find for all f

$$V_t W f = W\, U_t f\,. \tag{9}$$

This is the content of Equation (6). It is thus sufficient to prove the theorem for partial isometries.

Let now Ω, be a partial isometry with the intertwining property

$$V_t\, \Omega_1 = \Omega_1\, U_t\,. \tag{1}$$

The relation (1) can be extended to a certain class of bounded functions of V_t and U_t by a standard procedure*). In particular if F_λ and E_λ are the spectral revolutions associated with V_t and U_t such that

$$U_t = \int_{-\infty}^{+\infty} e^{-t\lambda i}\, dE_\lambda \tag{10}$$

$$V_t = \int_{-\infty}^{+\infty} e^{-i\lambda t}\, dF_\lambda\,. \tag{11}$$

Then we have also

$$F_\lambda \Omega_1 = \Omega_1 E_\lambda \tag{12}$$

from which follows

$$F_\lambda F_1 = \Omega_1\, E_\lambda\, \Omega_1^* = F_1\, F_\lambda \tag{13}$$

where

$$F_1 = \Omega_1\, \Omega_1^* \tag{14}$$

is the left-projection of Ω_1.
Consider now an element f in the range of Ω_1 such that

$$f = F_1 f\,. \tag{15}$$

We decompose it according to

$$f = g + h$$

with $g \in R \equiv N$ and $h \in N^\perp = M$.
We wish to show that $h = 0$ for arbitrary choice of $f \in R_1$.

*) See for instance ref.[4]), p. 341.

Let λ_i be the eigenvalues of the operators V_t and

$$P_i \equiv F_{\lambda_i + 0} - F_{\lambda_i - 0}$$

the corresponding projection operators. The total projection P corresponding to the subspace M of bound states is then the sum

$$P = \sum_i P_i$$

Since $h \in M$ we have $Ph = h$ and since $h = Pf$ and $[P, F_1] = 0$, we also have

$$F_1 h = F_1 P f = P F_1 f = P f = h \tag{16}$$

Thus h is also in the range of Ω_1.

We express now the norm of h in the following manner

$$\| h \|^2 = (h, P h) = (h, \sum P_i h) = (h, \sum P_i F_1 h)$$

$$= (h, \Omega_1 \sum_i (E_{\lambda_i + 0} - E_{\lambda_i - 0}) \Omega_1^* h) \text{ , or finally}$$

$$\| h \|^2 = (\Omega_t^* h , \sum_i (E_{\lambda_i + 0} - E_{\lambda_i - 0}) \Omega_1^* h)$$

The right-hand side is zero, because U_t has a spectral revolution with no discontinuities. Therefore $h = 0$ for all f, or $f \in R$, or finally $R_1 \subseteq R$, q.e.d.

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