Projective Representation of the Poincaré Group in a Quaternionic Hilbert Space

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I. Introduction

A. Relativistic Quantum Mechanics

Theoretical physics in the first half of the twentieth century is dominated by two major developments: the discovery of the theory of relativity and the discovery of quantum mechanics. Both have led to profound modifications of basic concepts. Relativity in its special form proclaimed the invariance of physical laws with respect to Lorentz transformations and led to the inevitable consequence of the relativity of spatial and temporal relationships. Quantum mechanics, on the other hand, recognizes as basic the complementarity of certain measurable quantities for microsystems (uncertainty relations) and the concomitant indeterminism of physical measurements.

From the mathemetical point of view the central object in the special theory of relativity is a group, the Lorentz group, or more generally, the *Poincaré group*. For quantum mechanics the most important mathematical object is the *Hilbert space* and its linear operators. It is therefore not surprising that the most important mathematical problem in relativistic quantum mechanics is the representation theory of the Poincaré group in infinite-dimensional Hilbert space.

The representation theory of groups was first developed, at about the turn of the century, as a branch of algebra for the finite groups. The extension to compact Lie groups was a relatively easy generalization. However, these theories are too restrictive for the representation theory required by relativistic quantum mechanics. Two generalizations are needed in this case. First, not the vector representations but only representations up to a factor of modulus one are important in quantum mechanics. Such representations are called projective representations because they are encountered in projective geometry. Second, the Poincaré group is a noncompact group, and the faithful unitary representations of such groups are necessarily of infinite dimensions.

Until 1940 the unitary representation theory of noncompact groups in infinite-dimensional spaces was practically nonexistent. The first important results were obtained by Wigner (*1*) in 1939, and later by Bargmann (*2*). Wigner was able to adapt a method of Frobenius to the Poincaré group, and in this way he obtained a classification of all physically interesting irreducible representations of this group. Many questions of a mathematical nature remained unanswered by this work. A more complete and more general theory was given much later by Mackey, who generalized Frobenius's theorem to the case of noncompact groups of a certain class (*3*, *4*).

The study of projective representations led to the theory of the classes of equivalent factors developed especially by Bargmann (*5*). Thus the local and global theories of factors, together with the Mackey-Frobenius theory of the irreducible vector representations, constitute the main building blocks of the quantum-mechanical representation theory of the Poincaré group. They will be used in this article for a classification of elementary particles in quaternionic quantum mechanics.

B. General Quantum Mechanics

Quantum mechanics as it was discovered in connection with the problems in atomic physics has the peculiar feature that it is a theory that uses as its main tool a *complex* Hilbert space. The appearance of complex numbers in a basic physical theory can be of a rather trivial nature, such as, for instance, the representation in the complex plane of a periodic motion. In such a case the use of complex numbers is a matter of convenience, and it can be just as well avoided if we are willing to pay the price of more cumbersome formulas.

In quantum mechanics, however, the appearance of complex numbers seems to have a more fundamental significance, which has never been understood very well.

The question concerning the role of complex numbers was expressed early by Ehrenfest (*6*) and an answer was attempted by Pauli (*7*), for at least a special case. The question can be placed in a broader context if we examine more carefully just what properties of the complex Hilbert space are actually used in quantum mechanics. One way to do this is to reformulate quantum mechanics on an axiomatic basis as an algebraic structure, as was done by Birkoff and von Neumann (*8*) in 1936. In this formulation there is no need to introduce Hilbert space at all. The primary object is instead a lattice of the elementary propositions (yes–no experiment) pertaining to a given physical system. In conventional quantum mechanics this lattice is realized as the lattice of all the subspaces of a complex Hilbert space. In the abstract formulation of the proposition system the nature of the Hilbert space in a possible realization is left open. There is no obvious physical property that would force us to choose the complex numbers for the field of coefficients.

There is, however, one property of the field that one can motivate to some extent with physical considerations: the field should contain the reals as a subfield so that the representation of continuous quantities, such as the position of a particle, does not cause any difficulties. With this restriction the number of possible realizations of the abstract lattices is greatly reduced because, according to a celebrated theorem (*9*), there exist only three fields that contain the real numbers as a subfield, namely, the real numbers themselves, the complex numbers, and the quaternions. Thus it suffices to examine in detail quantum mechanics in real and in quaternionic Hilbert spaces.

Quantum mechanics in a real Hilbert space was studied by Stueckelberg in a number of papers (*10*, *11*). The result of these investigations is that the theory is in contradiction with the uncertainty relations unless we postulate the existence of a nontrivial operator $\mathcal{J}$ that commutes with all the observables. This operator $\mathcal{J}$ must in addition be antisymmetrical ($\mathcal{J}^\dagger = -\mathcal{J}$) and must satisfy $\mathcal{J}^2 = -I$. The latter property says that $\mathcal{J}$ is the square root of the negative identity operator. This implies that the theory is identical with conventional quantum mechanics in complex Hilbert spaces.

The situation is a little different for quaternion quantum mechanics. In fact, experience with real Hilbert spaces has shown that the question of the field is certainly connected with the question of superselection rules. The choice of the "wrong" field (for instance, the reals) can be compensated by restricting the number of operators that are admitted as observables. Such restrictions are called *superselection rules*. It was therefore natural to believe that the occurrence of superselection rules in nature might somehow find a natural explanation by a suitable choice of the number field. For this reason the study of quaternion quantum mechanics was undertaken by Finkelstein *et al.* in a number of publications (*12*, *13*). In spite of some interesting formal possibilities these attempts yielded no essentially new results that could be connected with the empirical facts of elementary particle physics, and the deeper significance of the complex numbers in quantum mechanics remains obscure.

In order to progress further it seemed natural to return to the lattice-theoretical approach of Birkhoff and von Neumann and to try to recover the number field from the structure properties of the lattice itself. It was especially emphasized by Finkelstein *et al.* that the abstract algebraic properties of the lattice are essentially nothing other than the formalization of the fundamental empirically given properties of the physical systems. On the other hand, from the experience we have had with the coordinate representations of projective geometries we expect that the nature of the field is essentially (that is, up to automorphisms) determined by this lattice structure.

This program of research was undertaken by Piron (*14*), who succeeded in formulating, in the precise mathematical language of lattice theory, a set of general quantum-mechanical axioms that embodied the basic empirical facts of quantum systems. He went beyond the work of Birkhoff and von Neumann by showing that for certain systems the axiom of modularity favored by these authors is in contradiction with the facts, and by supplying the correct axiom of *weak modularity*. He then stated and proved a representation theorem for the lattices encountered in Nature. For systems of finite dimensions this theorem is the well-known representation theorem of projective geometries; for infinite-dimensional systems it is a generalization of this theorem.

With Piron's result it became possible to affirm the representation of the lattice of a physical proposition system as subspaces in a Hilbert space with coefficients from a field. But still nothing was known about the physical properties that reflect the nature of the field.

C. Intervention of Group Theory

A new aspect was introduced with the study of the symmetry groups of proposition systems. It is known from examples that these symmetry groups have quite different structures for the different lattices. Since physical symmetries are often more easily recognized in Nature than, for instance, other detailed mechanical properties of the systems, this seemed a promising line of research to pursue.

The symmetries of a proposition system have two aspects. There is (as we shall show in detail in Section III, A, 3) a symmetry group of the proposition system that we shall call M. It consists of all automorphisms of the lattice. There is, in addition, the symmetry group G, which arises from the space-time frame of physical events. For relativistic quantum mechanics this group G is the Poincaré group. The study of elementary systems and their properties leads to the question of the isomorphisms (or homomorphisms) between G and the subgroups of M. In other words, we have here a representation problem of the Poincaré group.

The representation of groups as automorphisms of a lattice structure is a natural generalization of the representation of groups by unitary transformation of vectors in a complex Hilbert space. An intermediate stage in this generalization consists of the projective representations, which can be reduced to the vector representations via the theory of factors. There is virtually nothing known about representations of groups as automorphisms of lattices.

A special aspect of the problem could be revealed by studying the projective representations of the Poincaré group in quaternionic Hilbert space. This is essentially the same problem that Wigner had solved in 1939 for complex space, transferred and adapted to the situation in quaternionic space. The work of Mackey (*3*) and Bargmann (*5*) that intervened simplified the task considerably and made it possible to solve this problem with complete mathematical rigor. This was dome by Emch (*15*) in a thesis published in 1963. The result, which will be reported here, shows that the physical content of quaternionic Hilbert space is identical with that of complex space when it is combined with the principle of relativity. This result revealed, a little better than most previous attempts, why complex Hilbert space plays such an exceptional role in quantum mechanics. It is a good example of the efficacy of group-theoretical considerations in answering profound questions of fundamental physical theory.

II. The Lattice Structure of General Quantum Mechanics

A. The Proposition System

1. *The Elementary Propositions* (*Yes–No Experiments*)

All the information concerning the properties of a physical system is obtained by measurements. The results of such measurements depend on two things: the nature of the physical system and the state of that system. Although this distinction cannot always be carried through consistently in all cases, it is quite useful for most situations. Roughly speaking, the nature of the system is incorporated in all those measurable properties that are independent of the history of the system. We shall call then *intrinsic properties*. For instance, if the system consists of an elementary particle, the mass, charge, spin, and magnetic moment are some of the intrinsic properties. On the other hand, the position, energy, and orientation of the spin are some of the properties that depend on the state of the system.

The nature of the system can be characterized completely by specifying all the intrinsic properties of the system. In order to do this in the simplest and most systematic way, it is convenient to introduce a special class of experiments the yes–no experiments. These are experiments with equipment that can only respond with one of two alternatives. A typical example of such equipment is the *counter*, which is either triggered or remains silent. Every measurement of a measurable quantity can be broken up into a suitable set of yes–no experiments by the simple device of measuring only whether the quantity in question belongs to a given subset or not. For instance, the measurement of the position of a particle is accomplished if we know whether the values of its position coordinates belong to any given subset of the possible values of these coordinates.

We shall refer to the two alternatives of a yes–no experiment as (elementary) propositions for the system, which we denote in the following by $\mathscr{L}$. The determination of the nature and the state of a system is accomplished if we know the truth or falsehood of all propositions for the system.

2. *The Partial Ordering of Propositions*

One of the most important intrinsic properties of a physical system is expressed in a partial ordering of its proposition system. Certain pairs of propositions are not independent of each other. For instance, let proposition a locate a particle in a volume element V_a and proposition b locate the particle in volume element V_b. If $V_a \subseteq V_b$, then the two propositions clearly depend on each other because whenever a is true, b must be true too. Furthermore whenever b is false, a must be false too. We express this by the relation $a \subseteq b$ and recognize easily that it is a *partial ordering* of the proposition system that satisfies the following fundamental properties:

$$\begin{array}{llll} \text{(a)} & a \subseteq a & & \forall \quad a \in \mathscr{L}; \\ \text{(b)} & a \subseteq b & \text{and} & b \subseteq a \Leftrightarrow a = b; \\ \text{(c)} & a \subseteq b & \text{and} & b \subseteq c \Rightarrow a \subseteq c. \end{array} \qquad \text{(I)}$$

Property (b) may be considered as the definition of the equality of two propositions.

The fact that the ordering is only partial is very important. It gives rise to the existence of nontrivial symmetry groups for proposition systems.

3. *Intersection, Union, and Orthocomplement of Proposition*

In a partially ordered system it is natural to define the operations of intersection and union of its elements. If the system is a system of propositions, then we can give these operations a physical interpretation that enables us to verify in individual cases a system of axioms concerning them.

For these axioms the following have been found consistent with the empirically verifiable proposition systems.

Let I be an index set containing at least two elements and a_i $(i \in I)$ any subset of $\mathscr{L}$, $a_i \in \mathscr{L}$. Then there exists a proposition, denoted by $\bigcap_I a_i$, with the property

$$x \subseteq a_i \quad \forall \quad i \in I \quad \Leftrightarrow \quad x \subseteq \bigcap_I a_i \qquad (\forall \quad x \in \mathscr{L}). \tag{II}$$

It is called the *greatest lower bound*, or *intersection*, of the elements a_i. In the particular case that the index set I contains exactly two elements, we denote the intersection of two elements a and b by $a \cap b$.

In a similar way we define the *least upper bound*, or *union*, of an arbitrary subset of $\mathscr{L}$ by $\bigcup_I a_i$ with the property

$$a_i \subseteq x \quad \forall \quad i \in I \quad \Leftrightarrow \quad \bigcup_I a_i \subseteq x \qquad (\forall \quad x \in \mathscr{L}). \tag{III}$$

If the subset $\{a_i\}$ is identical with $\mathscr{L}$, we obtain two special elements of the set $\mathscr{L}$

$$\phi = \bigcap_{\mathscr{L}} a, \qquad I = \bigcup_{\mathscr{L}} a. \tag{1}$$

The element ϕ represents the *absurd* proposition, which is always false, while I is the *trivial* proposition, which is always true.

The next axiom (IV) asserts the existence of a unique *orthocomplement*: For every $a \in \mathscr{L}$ there exists another $a' \in \mathscr{L}$ such that

$$\begin{aligned} (a')' &= a, \\ a' \cap a &= \phi, \\ a \subseteq b &\Leftrightarrow b' \subseteq a'. \end{aligned} \tag{IV}$$

From the axioms stated so far follows immediately that for every subset $\{a_i\}$ $(i \in I)$ of $\mathscr{L}$ we have

$$\bigcup_I a_i = \left(\bigcap_I a_i'\right)'. \tag{2}$$

In particular, by taking for $\{a_i\}$ the set $\mathscr{L}$ itself, we obtain

$$\begin{aligned} \phi' &= I, \\ I' &= \phi. \end{aligned} \tag{3}$$

For every $x \in \mathscr{L}$ we verify also

$$x' \cup x = I. \tag{4}$$

If two elements a and b satisfy the symmetrical relation $a \subset b'$, we call them *disjoint* and we denote it by $a \perp b$.

The four axioms (I), (II), (III), and (IV) define an *orthocomplemented* and *complete lattice.*

4. *The States of a Physical System*

The properties that define the lattice structure of $\mathscr{L}$ contain the formalization of the intrinsic properties of a physical system. We shall now turn our attention to those properties that refer to the state of the system.

A state is the result of a set of physical manipulations that constitute the preparation of the system. The state can be determined by measuring the truth or falsehood of all the propositions of the system. In contradistinction to classical systems, however, not every proposition is necessarily true or false. The result of measurements on ensembles of identically prepared systems will yield the result that a given proposition may be true with a certain probability only. We are thus led to the following axiom.

A state is a function from $\mathscr{L}$ onto the interval $[0, 1]$ that satisfies

(i) $p(\phi) = 0, \qquad p(I) = 1.$

(ii) For every sequence a_i of pairwise disjoint elements we have

$$p\left(\bigcup_i a_i\right) = \sum_i p(a_i).$$

(iii) $p(a) = 1 = p(b) \Rightarrow p(a \cap b) = 1.$

If p_1 and p_2 are two different states, then $\lambda_1 p_1 + \lambda_2 p_2 \equiv p$ with $\lambda_1 + \lambda_2 = 1$, $\lambda_i > 0$ is also a state. Such a state p which can be constructed from two different states is called a *mixture*. A state that is not a mixture is said to be *pure*.

The states are thus a convex set of functionals over $\mathscr{L}$. The pure states are the boundary of this convex set.

The functional $\sigma_p(a) \equiv p(a) - p^2(a)$ measures the *dispersion* of a state. If $\sigma_p(a) \equiv 0 \ \forall \in \mathscr{L}$, we call the state *dispersion free*. A mixture always has dispersion, but a pure state is not necessarily dispersion free. For certain simple quantum-mechanical systems, such as a spin or an elementary particle, we can even show that there does not exist any dispersion-free state.

B. Distributivity, Modularity, and Atomicity

1. *Distributivity*

The lattice of propositions exists for any physical system, be it classical or quantal. The distinction between these two kinds of systems requires an additional structure property that is compatible with axioms (I)–(IV); a classical system has a proposition system that satisfies the axiom of *distributivity* as well as axioms (I)–(IV).

A lattice is distributive if for every triple $a, b, c \in \mathscr{L}$, the relations

$$\begin{aligned} a \cap (b \cup c) &= (a \cap b) \cup (a \cap c), \\ a \cup (b \cap c) &= (a \cup b) \cap (a \cup c) \end{aligned} \tag{D}$$

hold.

Such a lattice is called a Boolean lattice (or a Boolean algebra). If a lattice is not Boolean, it may at least contain Boolean sublattices. A sublattice $\mathscr{L}_0 \subset \mathscr{L}$ is a subset of $\mathscr{L}$ that satisfies all the axioms (I)–(IV). If $\mathscr{L}_i \subset \mathscr{L}$ is a famile of sublattices, then the set intersection $\mathscr{L}_0 \equiv \bigcap_i \mathscr{L}_i$ is also a sublattice.

Let $\gamma \subset \mathscr{L}$ be an arbitrary subset of $\mathscr{L}$. We may then consider the class of all sublattices $\mathscr{L}_i$ that contain γ. The intersection $\mathscr{L}_0 \equiv \bigcap_i \mathscr{L}_i$ will then also contain γ, and it is the smallest sublattice of $\mathscr{L}$ with this property. We call it the sublattice *generated* by γ and denote it by $\mathscr{L}(\gamma) \equiv \mathscr{L}_0$.

Of particular interest in the following are the subsets γ for which $\mathscr{L}(\gamma)$ is a Boolean sublattice of $\mathscr{L}$. We say then that the set γ is *classical*, or γ consists of pairwise *compatible elements*. If the set γ consists of exactly two elements $\gamma = \{a, b\}$, then these elements are compatible if and only if $\mathscr{L}(\{a, b\})$ is Boolean.

We have thus arrived at the important notion of *compatibility*, for which we introduce the special notation $a \leftrightarrow b$, which indicates that it is a symmetrical relation. It is clear from the preceding that in a classical system every pair of propositions is compatible. The converse is also true (*14*). The notion of compatibility defined here was first introduced in a slightly different way by Jordan (*16*) and it is discussed extensively in the mathematical literature (*17, 18*).

2. *Modularity and Weak Modularity*

It was clearly recognized by Birkhoff and von Neumann (*8*) that the distribution law is violated in Nature and that it has to be replaced by a weaker law. For this weakened structure property these authors proposed the so-called *modular law*.

It is an elementary exercise to show that in any lattice we have

$$x \cup (y \cap z) \subseteq (x \cup y) \cap z$$

for $x \subseteq z$.

If the lattice is such that for all $x \subseteq z$ the equality sign holds in this relation, then it is said to be modular. The modular law is thus expressed by

$$x \subseteq z \Rightarrow x \cup (y \cap z) = (x \cup y) \cap z. \tag{M}$$

It is clear that a Boolean lattice is always modular, but the converse is not true. Simple examples are found, for instance, in the work of Piron (*14*).

As was pointed out by Birkhoff (*19*), the modular lattices have many properties that make them quite attractive for the description of propositions in general quantum mechanics. It was established by Piron, however, that the notion of localizability, which is implied by that of an elementary particle, is incompatible with modularity (*14*, Proposition on p. 452). Piron also supplied the weaker axiom that is needed to describe the actually known physical systems. It is called the *weak modularity* axiom and can be stated in many equivalent forms. We choose the following, which lends itself most easily to a physical interpretation:

$$a \subseteq b \Rightarrow a \leftrightarrow b. \tag{P}$$

It is not difficult to verify that (M) implies (P) for a complete orthocomplemented lattice. The converse is not true. The most important example is the lattice of closed linear subspaces in an infinite-dimensional Hilbert space. This lattice satisfies (P) but is not modular (*14*).

Concluding this subsection, we state a theorem that is a rich source of alternative formulations of compatibility:

In a weakly modular lattice the following relations are equivalent [cf. (*14*, Theorem VII)].

(1) $a \leftrightarrow b$.
(2) $a \leftrightarrow b'$.
(3) $(a \cap b') \cup b \supseteq a$.
(4) $(a \cup b') \cap b \subseteq a$.
(5) Any three of the four elements a, b, a', b' satisfy a distributive law
$$x \cap (y \cup z) = (x \cap y) \cup (x \cap z).$$
(6) $(a \cap b) \cup (a \cap b') \cup (a' \cap b) \cup (a' \cap b') = I$.
(7) $(a \cup b) \cap (a \cup b') \cap (a' \cup b) \cap (a' \cup b') = \phi$.

3. *Atomicity*

The axion of atomicity consists of two parts. The first part expresses the existence of minimal propositions $P \in \mathscr{L}$ (called points) with the property

$$x \subset P \Rightarrow x = \phi. \tag{A.1}$$

The second part affirms the existence of minimal propositions over any other proposition.

For any point $P \in \mathscr{L}$

$$a \subseteq x \subseteq a \cup P \Rightarrow x = a \qquad \text{or} \qquad x = a \cup P. \tag{A.2}$$

We say then the proposition $a \cup P$ *covers* a.

The postulate of atomicity has the character of a technical axiom that is perhaps not indispensable for the description of actual physical systems, but that is mathematically useful. Recent experience, however, suggests that it may be possible to dispense with the axiom altogether (*20*). Work is now in progress to study the possibility in relation to weakly modular lattices.

For the rest of this chapter we shall designate as a *proposition system* a lattice that satisfies axioms (I)–(IV); (P); and (A.1) and (A.2).

C. Superposition Principle and Superselection Rules

1. *Reducible and Irreducible Lattices*

Consider two proposition systems $\mathscr{L}_1$ and $\mathscr{L}_2$. We can construct a third one, the elements of which are the pairs of elements (x_1, x_2) $x_1 \in \mathscr{L}_1$, $x_2 \in \mathscr{L}_2$ (the Cartesian product $\mathscr{L}_1 \times \mathscr{L}_2$). The partial ordering is defined by the rule

$$(x_1, x_2) \subseteq (y_1, y_2) \Leftrightarrow x_1 \subseteq y_1 \qquad \text{and} \qquad x_2 \subseteq y_2. \tag{5}$$

If we define further

$$(x_1, x_2)' = (x_1', x_2'), \tag{6}$$

$$(x_1, x_2) \cap (y_1, y_2) = (x_1 \cap y_1, x_2 \cap y_2), \quad \text{etc.}, \tag{7}$$

we obtain a new lattice, which we call the *direct union* of $\mathscr{L}_1$ and $\mathscr{L}_2$.

Any lattice that can thus be written as a direct union of two or more other lattices is called *reducible*. If this is not the case, it is called *irreducible* or *coherent*.

Every Boolean lattice is reducible except the trivial lattice consisting of only two elements ϕ and I. The occurrence of nontrivial irreducible lattices is thus an essential property of quantum systems.

Whenever a lattice is reducible, there exist nontrivial elements that are compatible with every other element in the lattice. The set of all such elements is called the *center* $\mathscr{C}$ of the lattice.

2. *The Superposition Principle*

An irreducible lattice $\mathscr{L}$ satisfies the *superposition principle*, which can be expressed as follows:

For every pair of distinct P, $Q \in \mathscr{L}$ there exists a third point $R \in \mathscr{L}$ such that

$$P \cup Q = P \cup R = R \cup Q. \tag{8}$$

Every proposition system can be decomposed in an essentially unique manner (except for the order of the irreducible parts) into a direct union of irreducible lattices (*14*). This theorem allows the reduction of the study of general proposition systems to that of irreducible ones. For an irreducible lattice the center is trivial (that is, it consist of only two elements ϕ and I). For any Boolean lattices it is identical with the entire lattice.

The lattices that actually occur in Nature are in general reducible. When this is the case, the superposition principle has only restricted validity. We say then that the system allows *superselection rules*, a notion introduced by P. Destouches-Février (*21*), and later again by Wick *et al.* (*22*)

III. The Group of Automorphisms in a Proposition System

A. Morphisms

1. *Definition of Morphisms*

Let $\mathscr{L}_1$, $\mathscr{L}_2$ be two proposition systems and m a bijective mapping with domain $\mathscr{L}_1$ and range $\mathscr{L}_2$ with the properties

$$\begin{aligned} &\text{(i)} \quad x \subseteq y \Leftrightarrow m(x) \subseteq m(y) \\ &\text{(ii)} \quad m(x') = m(x)' \end{aligned} \tag{9}$$

for every $x, y \in \mathscr{L}_1$.

Such a mapping is called a *morphism* of $\mathscr{L}_1$ onto $\mathscr{L}_2$. Every morphism admits an inverse m^{-1} with domain $\mathscr{L}_2$ and range $\mathscr{L}_1$ defined by

$$m^{-1}(m(x)) = x. \tag{10}$$

The inverse of a morphism is also a morphism.

2. *Various Invariance Properties*

The following properties are simple consequences of this definition. For the detailed proofs we refer to the work of Emch and Piron (*23*). If m is a morphism from $\mathscr{L}_1$ to $\mathscr{L}_2$ and $\{x_i\}$ any subset of $\mathscr{L}_1$, then

$$m\left(\bigcup_i x_i\right) = \bigcup_i m(x_i). \tag{11}$$

Similarly, we have

$$m\left(\bigcap_i x_i\right) = \bigcap_i m(x_i). \tag{12}$$

From this follows immediately

$$m(I_1) = I_2, \qquad m(\phi_1) = \phi_2, \tag{13}$$

and

$$x_1 \leftrightarrow x_2 \Leftrightarrow m(x_1) \leftrightarrow m(x_2). \tag{14}$$

If $P_1 \in \mathscr{L}_1$ is a point, then $m(P_1) = P_2 \in \mathscr{L}_2$ is also a point.

Furthermore, if x and y are contained in the same coherent component of $\mathscr{L}_1$, then $m(x)$ and $m(y)$ are also contained in the same coherent component of $\mathscr{L}_2$ [cf. Emch and Piron (*23*, Lemma 4)].

3. *Automorphisms*

If $\mathscr{L}_1 = \mathscr{L}_2 = \mathscr{L}$, then a morphism with domain and range $\mathscr{L}$ is called an *automorphism*. It is a permutation of the lattice that leaves the lattice structure invariant. An automorphism will also be called a *symmetry* of the lattice $\mathscr{L}$.

The set of all the automorphisms are a group, the symmetry group of the lattice. We shall denote it by M. The composition law of this group is defined by setting for the product of two automorphisms m_1 and m_2

$$m_1 m_2(x) = m_1(m_2(x)). \tag{15}$$

The identity element e of the group is represented by the trivial automorphism, which leaves every element of the lattice invariant: $e(x) = x$; and the inverse automorphism m^{-1} is the group inverse.

Every automorphism induces a transformation $p \to p^m$ of the states of a system through the formula

$$p^m(x) \equiv p(m^{-1}(x)) \qquad \forall \quad x \in \mathscr{L}. \tag{16}$$

It can easily be verified that if p is a state, then p^m, defined by Eq. (16), is a state too. If p is a pure state, then p^m is pure too.

B. The Symmetry Group of a Proposition System

1. *Topology in a Group of Automorphisms*

The group of automorphisms of a proposition system reflects many of the structure properties of the lattice. The study of these properties can therefore be reduced to some extent to the study of the group of its automorphisms. In this section we define the topology in the group of automorphisms so as to make this group a topological group.

A topology in an abstract space M is given by specifying a certain class of subsets, designated as the *open* sets of that space. In order to define them it is sufficient to give a *complete system* of *neighborhoods* of the set M. They form a *basis* in the sense that every open set can be represented as the union of such neighborhoods.

In the case of groups it suffices to designate only the neighborhoods of the identity element $e \in M$. Neighborhoods at other points $m_0 \neq e$ are then obtained by left or right translations of the neighborhoods at the identity. Thus if U is such a neighborhood, then the sets

$$m_0 U = \{m' | m' = m_0 m, m \in U\}$$

$$U m_0 = \{m' | m' = m_0 m, m \in U\}$$

are neighborhoods of the point m_0.

For the definition of the neighborhoods at $e \in M$ we look for a motivation in the physical interpretation of the lattice. The measurable quantities are the states, and proximity of two transformations of the lattice is therefore expressed most naturally in terms of the transformation of the states. Thus we define an ϵ neighborhood $N_\epsilon(e)$ of e as the set of automorphisms m such that

$$|p^m(x) - p(x)| < \epsilon \qquad \forall \quad x \in \mathscr{L} \text{ and all states } p. \tag{17}$$

It is easy to verify that this system of ϵ neighborhoods satisfies the five conditions of Theorem 10 of Pontrjagin (*9*, Section 17). Thus they define a topology in the group such that the group operations are continuous functions of its arguments. From now on we shall consider the group of automorphisms equipped with this topology so that it may be considered a topological group.

We may now consider various properties that depend on that topology. The following will be used frequently:

(a) *Closure:* A subset of M is *closed* if its complement in M is open.

(a) *Limit point:* A point m_0 is a *limit point* of a subset $A \subset M$ if every neighborhood of m_0 has at least one point $\neq m_0$ in common with A. A subset $A \subset M$ is closed if and only if it contains all its limit points.

(c) *Connectedness:* The space M is *connected* if there does not exist a subset A of M that is both open and closed. If such a set exists, then its complement $A' = M - A$ is also open and closed. It follows then that A is the union of two open sets and also the union of two closed sets.

(*d*) *Discreteness:* The space M is *discrete* if every subset of M is both open and closed.

(*e*) *Compactness:* The space M is *compact* if from every countable family of open sets in M we can select a finite subfamily that also covers M. In that case every infinite subset of M contains at least one limit point in M. If this is not the case, then the space is called noncompact.

If a space is not connected it may contain connected components. A closed subset M_0 of M is a connected component if it cannot be represented as the union of two nonintersecting closed sets.

The largest connected subset $M_0 \subset M$ that contains the element $e \in M$ is called the connected component of the unit element. It is easy to see that M_0 is an invariant subgroup of M.

2. *The Connected Component and Superselection Rules*

We shall here first examine the effect of the automorphisms in the connected component of a proposition system with superselection rules. We consider only the case of *discrete* superselection rules for which the lattice $\mathscr{L}$ is a direct union of a finite or countably infinite set of lattices $\mathscr{L}_i$.

We denote by $I_i = \{\phi_1, \phi_2, \ldots, I_i, \phi_{i+1}, \ldots\}$ the element that has the zero element at every position except at the ith position, where it has the unit element I_i of the lattice $\mathscr{L}_i$. The lattice $\mathscr{L}_i$ is then isomorphic with the segment $[\phi, I_i]$ that is the set of elements x such that $\phi \subseteq x \subseteq I_i$. So we may identify $\mathscr{L}_i$ with this segment. The elements I_i are all disjoint elements of the center of $\mathscr{L}$.

Let us now consider a state p_1 such that $p_1(I_i) = 1$. Because the I_i are disjoint, we have for any state $p(I_1) + p(I_i) = p(I_1 \cup I_i)$. It follows that $p_1(I_i) = 0$ for $i \neq 1$. Let us next choose a sufficiently small $\epsilon > 0$ and consider $m \in N_\epsilon(e)$ so that

$$|p_1(m^{-1}(I_1)) - p_1(I_1)| \lesssim \epsilon.$$

Since m^{-1} is an automorphism, $m^{-1}(I_1) = I_i$. But we have already shown that $p_1(I_i) = 0$ for $i \neq 1$. Hence $m^{-1}(I_i) = I_1$ (if $\epsilon < 1$, for instance) or $m(I_1) = I_1$ for all $m \in N_\epsilon(e)$.

We have thus established that for all $m \in N_\epsilon(e) = U$ we must have $m(I_1) = I_1$. It is now easy to extend this invariance of I_1 to the entire connected component $M_0 \subset M$ by using a theorem of Pontrjagin (*9*, Theorem 15) according to which every element in M can be written as a finite product of elements from U. With this we have proved the following

Theorem I. If a lattice $\mathscr{L}$ is a direct union of coherent lattices $\mathscr{L}_i$, then every morphism from the connected component that contains the unit element leaves every sublattice $\mathscr{L}_i$ invariant.

3. *Representations of Symmetry Groups*

The group of automorphisms M is in general much too large a group for the description of physical symmetries. The physical symmetry groups satisfy additional properties that are related to the physical content of the theory.

We shall say that the topological group G is a *symmetry group* of the system if there exists a homomorphism U of G into M. Such a homomorphism will be called a *projective representation* of the group G.

We remark here that by a projective representation we mean always a homomorphism as far as the group structure is concerned, and a homomorphism with respect to the topologies of G and M. A projective representation is thus always the continuous image of G in M.

We can now easily establish

Theorem 2. If the lattice $\mathscr{L}$ is a direct union of lattices $\mathscr{L}_i$ and if it admits a connected symmetry group G, then every $U_x \in M$ that is an image of $x \in G$ in the representation M leaves every component $\mathscr{L}_i$ invariant.

The proof follows from the remark that connectedness is invariant under a homeomorphism, hence all $U_x \in M_0$. We shall say the projective representation U in $\mathscr{L}$ (denoted by $(\mathscr{L}, U)$) of the group G is *irreducible* if

$$U_x a = a \qquad \forall \quad x \in G \Rightarrow a = \phi \qquad \text{or} \qquad a = I, \tag{18}$$

and we call such a pair $(\mathscr{L}, U)$ an *elementary system* with respect to the symmetry group G. We see immediately that every elementary system with respect to a symmetry group G is necessarily coherent. Indeed if it were not, it would have a nontrivial center and we have just seen that the elements of the center are all invariant under M_0. Since $U_x \in M_0$ for all $x \in G$, the conclusion follows that $\mathscr{L}$ is coherent.

The foregoing remarks contain the germ of a theory of elementary particles based on the phenomenology of physical systems. The idea is this: The phenomenology of a physical system is essentially contained in the lattice structure of the proposition system $\mathscr{L}$. This structure in turn determines the group M of its automorphisms, including all its subgroups. The irreducible representations of the group G in M are the possible elementary systems that are compatible with this lattice structure.

Unfortunately the representation theory of groups in lattices is a branch of mathematics that is not yet developed. Therefore the foregoing sketch of a program cannot yet be carried out. It is possible to pursue another road, however. Instead of working with abstract lattices, we can seek a representation of proposition systems and then study the automorphisms of such representations.

It is known that the closed linear subspaces (henceforth just called subspaces) of a Hilbert space have a lattice structure that satisfies all the axioms of a proposition system. These subspaces do furnish us, therefore, with a representation of a proposition system. This is, however, not the only representation possible. The task of finding all the representations of irreducible proposition systems was accomplished by Piron (*14*) and in the following subsection we shall give a brief outline of his and some related results.

C. Irreducible Proposition Systems as Subspaces of a Hilbert Space

1. *Proposition Systems and Projective Geometries*

There is a remarkable similarity between the proposition system of a quantum-mechanical system and the lattices that arise in the set of axioms of projective geometries. It is thus not surprising that representation theorems for proposition systems are modeled after those for projective geometries. In fact, the essence of the general representation theory of proposition systems is an embedding theorem that says that every proposition system can be embedded in a canonical way into a projective geometry. This theorem, then, establishes the link to the representation theory of the projective geometries and in this manner the representations of proposition systems can all be found.

The essential difference between projective geometries and proposition systems is that the former satisfy the modular law, whereas the latter, as we have seen, do not necessarily do so. If they do, they are, according to a theorem of Piron (*14*, Theorem V), direct unions of projective geometries of *finite dimensions*, where the dimension of a lattice is defined as the maximum of a chain $\phi \subset c \cdots \subset a \subset b \subset \cdots \subset I$ in the lattice.

In the case of infinite dimensions modularity is incompatible with the other axioms of a proposition system. This fact has been known for a long time, and for this reason von Neumann has expressed the conjecture that the *continuous geometries* discovered by him might give the mathematical frame of a generalized quantum mechanics. The continuous geometries do not contain any minimal elements ("point-less" geometries, as von Neumann called them) and thus they do not satisfy axiom (A.1).

Since there are proposition systems in Nature that are not modular (*14*, Proposition on p. 452), the strong constraint of modularity can be replaced by weak modularity and in that case it is possible to retain all the axioms of a proposition system, even for infinite systems, without contradiction. Projective geometries *are* modular, as we have seen. If they are infinite, the other axioms of a proposition system cannot hold for such projective geometries. The axiom that is violated for infinite projective geometries is axiom (IV), which affirms the existence of an orthocomplement. Infinite projective geometries are never orthocomplemented.

A standard example of an infinite projective geometry is the not necessarily closed linear manifolds of a Hilbert space. If union and intersection are defined as linear space and intersection, then the linear manifolds of such a space *are* modular (*18*, Theorem 9, p. 370; *24*). The complement is still defined. If a is not a closed linear manifold, we have $a \subset (a')'$. This violates axiom (IV).

For Boolean lattices representation theorems have been known for a long time: Every Boolean lattice may be realized as the lattice of subsets of some set (*25*, *26*).

2. *The Representation Theorem for Proposition Systems*

In subsection C, 1 we quoted the theorem that says that every reducible proposition system is the unique direct union of irreducible ones (*14*). The general representation problem of the lattices of proposition systems can thus be reduced to that of irreducible lattices. The following theorem is true for reducible or irreducible lattices (*14*, Theorem XVIII, p. 462).

Theorem 3 (Piron). If $\mathscr{L}$ is any lattice of propositions, then there exists always a projective geometry G_p and a canonical mapping α of $\mathscr{L}$ into G_p that satisfies the following properties.

(1) The restriction of α to the points of $\mathscr{L}$ is a one-to-one mapping onto the points of G_p.

(2) $a \subseteq b \Leftrightarrow \alpha(a) \subseteq \alpha(b)$.

(3) $\alpha\left(\bigcap_i a_i\right) = \bigcap_i \alpha(a_i)$.

(4) $\alpha(a \cup P) = \alpha(a) \cup \alpha(P) \qquad \forall \quad \text{points } P \in \mathscr{L}$.

It follows from these properties that if $\mathscr{L}$ is irreducible, then the canonically defined projective geometry G_p is irreducible, too.

This theorem establishes the bridge between the abstract proposition systems and the projective geometries. For the latter there exist well-known representation theorems that will yield similar theorems for the proposition systems. In order to formulate the fundamental representation theorem we need the following three concepts.

(a) A *chain* in $\mathscr{L}$ is a sequence of elements $\phi, \ldots, a, b, \ldots, I$ that satisfies $\phi \subset \cdots \subset a \subset b \subset \cdots \subset I$, where the inclusions are all proper. The number of elements in the chain is called its *length*.

(b) An *antiautomorphism* of a field $\mathfrak{F}$ is an involution $\alpha \to \alpha^*$ $(\alpha \in \mathfrak{F})$ with the property

$$\begin{aligned} (\alpha + \beta)^* &= \alpha^* + \beta^* \\ (\alpha\beta)^* &= \beta^* \alpha^* \\ (\alpha^*)^* &= \alpha \qquad \forall \quad \alpha, \beta \in \mathfrak{F}. \end{aligned} \tag{19}$$

An example of an antiautomorphism in the field of complex numbers is the complex conjugation. There are many others. But we can show that complex conjugation is the only one that is also continuous in the natural topology of these numbers. For the quaternions, on the other hand, every automorphism is continuous [cf. remark after Eq. (31)].

(c) A sesquilinear form over a vector space $\mathfrak{V}$ with coefficients from a field is a mapping f of $\mathfrak{V} \times \mathfrak{V}$ into $\mathfrak{F}$ such that

$$f(x+\alpha y, z) = f(x,z) + f(y,z)\alpha^*$$
$$f(x, y+\alpha z) = f(x,y) + \alpha f(y,z) \tag{20}$$
$$\forall \quad x,y \in \mathfrak{V} \quad \text{and} \quad \forall \quad \alpha \in \mathfrak{F}.$$

Such a form is called *Hermitian* if $f(x,y) = f^*(y,x)$, and it is *definite* if $f(x,x) = 0 \Rightarrow x = 0$. An example of such a form is the scalar product in a Hilbert space. The representation theorem of proposition systems can now be stated in the following form.

Theorem 4. Every irreducible proposition system that contains a chain of length at least equal to four can be realized by a linear vector space $\mathfrak{V}$ over a field $\mathfrak{F}$, an antiautomorphism of $\mathfrak{F}$, and a definite Hermitian sesquilinear form in $\mathfrak{V}$. Every proposition $a \in \mathscr{L}$ is represented by a subspace of vectors $x \in \mathfrak{V}$ that satisfy $f(x,y_i) = 0$ for some $y_i \in \mathfrak{V}$. If $a \in \mathscr{L}$ is represented by the subspace $M \subset \mathfrak{V}$ than a' is represented by the subspace $M^\perp \equiv N$ consisting of all $x \in \mathfrak{V}$ that satisfy $f(x,y) = 0$, $\forall\ y \in M$.

For the proof of this theorem we refer to Piron (*14*, Theorem XXI. [The proof of Theorem XXII in Piron (*14*) is incomplete. A corrected proof has been given by Amemiya and Araki (*29*).

We remark here that for irreducible proposition systems the field is essentially uniquely determined by the structure of the lattice. This is no longer the case for reducible lattices. This fact is at the origin of the connection between the field $\mathfrak{F}$ and the superselection rules mentioned in Section I, B.

If irreducibility is dropped, other representations are possible. We mention here particularly the representation of proposition systems by algebraic Hilbert spaces where the coefficients are no longer a field but only a matrix algebra. Such representations give an elegant and compact formulation of lattices with certain types of superselection rules (*27*, *28*).

D. Projective Representations of Symmetry Groups

1. *The Semilinear Transformations*

Let $\mathfrak{V}$ be a vector space over a field $\mathfrak{F}$ and let $\alpha \in \mathfrak{F}$. An *automorphism* $\alpha \to \alpha^s$ of the field $\mathfrak{F}$ is a permutation of the elements of $\mathfrak{F}$ that satisfies

$$(\alpha\beta)^s = \alpha^s\beta^s,$$
$$(\alpha+\beta)^s = \alpha^s + \beta^s. \tag{21}$$

A nonsingular *semilinear* transformation of $\mathfrak{B}$ is a one-to-one mapping S of $\mathfrak{B}$ onto itself that has the properties

$$\begin{aligned} S(u+v) &= Su + Sv \qquad \forall \quad u, v \in \mathfrak{B}, \\ S(\alpha u) &= \alpha^s Su, \\ Su = 0 &\Rightarrow u = 0. \end{aligned} \tag{22}$$

If $u = \sum_i u_i$ is a finite linear combination of vectors $u_i \in \mathfrak{B}$, then it follows from (22) that $Su = \sum Su_i$. Thus the lattice structure of the linear manifolds of $\mathfrak{B}$ is left invariant under a semilinear transformation. According to the so-called first fundamental theorem of projective geometry (*24*), the converse is true, too. That is, we have

Theorem 5. Every automorphism of the lattice of linear manifolds of a vector space $\mathfrak{B}$ over a field is induced by a nonsingular semilinear transformation S of the vectors in $\mathfrak{B}$.

2. *Automorphisms of Subspaces*

Let us now consider the vectór space $\mathfrak{B}$ associated with an irreducible proposition system $\mathscr{L}$. This space is endowed with the positive definite Hermitian form $f(x,y)$ of Theorem 4. We shall from now on write $f(x,y) = (x,y)$ and $f(x,x) = \|x\|^2$. The vector space $\mathfrak{B}$ then becomes a Hilbert space $\mathfrak{H}_{\mathfrak{F}}$ over the field $\mathfrak{F}$. The subspaces, images of the propositions in $\mathscr{L}$, are the closed linear manifolds in the norm topology of this space.

If S is a nonsingular bounded semilinear transformation, then there exists an inverse S^{-1} that is also such a transformation. Furthermore, is S_1 and S_2 are two such transformations, the $S_1 S_2$ is one, too. They are thus a group that is closely related to the group of automorphisms of the subspaces in $\mathfrak{H}_{\mathfrak{F}}$.

The precise nature of this relation is obtained if we consider the subgroup $M_0 \subset M_1$, which leaves all the subspaces of $\mathfrak{H}_{\mathfrak{F}}$ invariant. A transformation $T \in M_0$ is then of the form $Tx = \lambda x \ \forall \ x \in \mathfrak{H}_{\mathfrak{F}}$ for some fixed $\lambda \in \mathfrak{F}$. It is easily verified that M_0 is an invariant subgroup of M_1 and that the factor group M_1/M_0 is isomorphic to the group M of automorphisms.

Among the semilinear transformations there are the semiunitary transformations. Such a transformation satisfies, in addition to (22), the relation

$$\|Ux\| = \|x\| \qquad \forall \quad x \in \mathfrak{H}_{\mathfrak{F}}. \tag{23}$$

Consider now any semilinear transformation $S \in M_1$ and define for any pair of elements $x, y \in \mathfrak{H}_{\mathfrak{F}}$ the Hermitian form $g(x,y) = (Sx, Sy)^{S^{-1}}$. Because S is also an automorphism of the subspaces of $\mathfrak{H}_{\mathfrak{F}}$ this form defines the same orthocomplementation in $\mathfrak{H}_{\mathfrak{F}}$ as the scalar product. According to a theorem of Baer (*30*) there exists then a number $\gamma \in \mathfrak{F}$ such that

$$g(x,y) = (x,y)\gamma \qquad \forall \quad x,y \in \mathfrak{H}_{\mathfrak{F}}. \tag{24}$$

Since g is Hermitian, γ is real and is in fact equal to $\gamma = g(x,x)/\|x\|^2$. If we define now $U = \gamma^{-1/2}S$, we find that U is semiunitary and is in the same equivalence class as S modulo M_0.

Thus we have shown: In every equivalence class modulo M_0 of semilinear transformations there exist semiunitary transformations. Two such transformations in the same class differ at most by a factor of modulus 1.

We shall now change the notation and designate henceforth as U the entire class of equivalent semiunitary transformations and as $u = U$ an element from this class.

We can then represent any automorphism $m \in M$ by one of these classes U_m and if E is a projection of $\mathfrak{H}_{\mathfrak{F}}$, $m(E)$ its image under the automorphism m, then we have the explicit formula

$$m(E) = u_m E u_m^{-1} \tag{25}$$

where $u_m \in U_m$ is any element from the class U_m.

3. *Wigner's Theorem*

Consider now a transformation in $\mathfrak{H}_{\mathfrak{C}}$ that maps unit rays into unit rays and conserves the magnitude of the scalar product for the unit vectors in the rays. Such a transformation preserves the order relation of subspaces and transforms orthogonal rays into orthogonal ones. It thus satisfies the two conditions of Eq. (9) for an automorphism. According to the preceding sections it is thus generated by a semiunitary transformation u. Since complex conjugation is the only continuous automorphism of the complex numbers, u is either unitary or antiunitary. Thus we have proved

Theorem 6 (Wigner). Every mapping of unit rays of a complex Hilbert space $\mathfrak{H}_{\mathfrak{C}}$ that preserves the magnitude of the scalar product between such rays can be induced by a unitary or antiunitary vector transformation of $\mathfrak{H}_{\mathfrak{C}}$.

We see from the proof we have given for this theorem that the hypotheses of Wigner's theorem are stronger than needed for the affirmation of the theorem. The only assumption we have used is that orthogonal rays are transformed into orthogonal ones. This generalization of the theorem was first given by Uhlhorn (*31*).

There exist many so-called elementary proofs of this theorem, beginning with the original (incomplete) proof of Wigner (*32*). Not all of these proofs were without error, as can be seen from the critical discussion by Uhlhorn (*31*); to complete the list given there, the elementary proofs that have appeared since (*33–35*) should be added. A more general theorem was proved by Emch and Piron (*23*).

4. *Unitary Projective Representations of Symmetry Groups*

Let G be a symmetry group of an irreducible physical system. There exists thus an isomorphism of G to a subgroup of M. Let $x \in G$ and $U_x \in M$ be the corresponding automorphism of the lattice $\mathscr{L}$ of subspaces. We say that we have a unitary projective representation of G if in every class U_x of semiunitary transformations there exists a unitary transformation.

Let U_x be such a representation and let $u_x \in U_x$ be a unitary transformation. It follows then that

$$u_x u_y = \omega(x, y)\, u_{xy}$$
$$\text{where } |\omega(x, y)| = 1, \qquad \omega(x, y) \in \mathfrak{F}. \tag{26}$$

The function $\omega(x, y)$ is called a *factor* of the unitary projective representation of the symmetry group G.

The theory of unitary projective representations can thus be divided into two parts. The first part is the theory of factors, which reduces the problem to the second part, the theory of unitary vector representations.

The theory of factors is quite different for the three different fields. For connected groups it can itself be subdivided into the theory of local factors and global theory. For complex Hilbert spaces and Lie groups the local theory and global theory of factors was developed by Bargmann (*36*). For quaternionic Hilbert spaces the theory of factors was given by Emch (*15*). It is interesting that the result for this case is much simpler than that for the complex case. We shall discuss it in Section IB, B, 1 and 2.

IV. Projective Representation of the Poincaré Group in Quaternionic Hilbert Space

A. Quaternionic Hilbert Space

1. *Quaternions*

The quaternions are an algebraic field endowed with a norm and a topology. As such they are a nontrivial but natural extension of the real numbers and the complex numbers. The central position occupied by the last two fields in all branches of mathematics and physics makes it desirable to understand the possible role of quaternions in fundamental physical theory, especially in quantum mechanics. This is all the more true since it can be shown that the complex numbers and the quaternions are the only possible algebraic fields endowed with a topology such that the algebraic operations are continuous in that topology and that they contain the real numbers as a subfield (*9*).

The quaternions contain three imaginary units, denoted by e_1, e_2, and e_3, which are assumed to satisfy the fundamental relations

$$\begin{aligned} e_i e_j &= e_k = -e_j e_i, \\ e_i^2 &= -1, \end{aligned} \tag{27}$$

where i, j, k are a cyclic permutation of 1, 2, 3.

A general quaternion q is then defined as a linear form

$$q = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3$$

with real coefficients a_r. We write sometimes $e_0 = 1$ and set $q = \sum_{r=0} a_r e_r$.

The sum and product of quaternions are defined by assuming the associative and distributive law with respect to both of these operations. Thus

$$q + q' = \sum_r (a_r + a_r') e_r \qquad \text{if} \quad q = \sum a_r e_r, q' = \sum a_r' e_r \tag{28}$$

and

$$qq' = \sum b_r e_r$$

with

$$\begin{aligned} b_0 &= a_0 a_0' - a_1 a_1' - a_2 a_2' - a_3 a_3' \\ b_1 &= a_0 a_1' + a_0 a_1' + a_2 a_3' - a_3 a_2' \\ b_2 &= a_0 a_2' + b_0 a_2' + a_3 a_1' - a_1 a_3' \\ b_3 &= a_0 a_3' + a_0 a_3' + a_1 a_2' - a_2 a_1'. \end{aligned} \tag{29}$$

We verify immediately that this product is *not* commutative: $qq' \neq q'q$.

The *norm* of quaternions is $|q| = [a_0^2 + a_1^2 + a_2^2 + a_3^2]^{1/2}$. It satisfies $|q + q'| \leqq |q| + |q'|$ and $|qq'| = |q|\,|q'|$ and it defines a topology by setting for the ϵ neighborhood of the element q_0 the quaternions q with $|q - q_0| < \epsilon$. With such a set of neighborhoods as a fundamental set, we have defined a topology for which the two operations of addition and multiplication are continuous operations (*q*).

The conjugation is defined by $q^{\mathfrak{Q}} = a_0 - a_1 e_1 - a_2 e_2 - a_3 e_3$. It follows that the norm is defined also by $|q|^2 = qq^{\mathfrak{Q}} = q^{\mathfrak{Q}} q$. Every quaternion $q \neq 0$ has an inverse given explicitly by

$$q^{-1} = (|q|)^{-1} q^{\mathfrak{Q}}. \tag{30}$$

The field of the quaternions thus defines two topological groups. The additive group is isomorphic to the group of vector addition in a four-dimensional real space. It is thus Abelian.

The multiplicative group is isomorphic to the covering group of $U(2, \mathfrak{C})$ the complex unitary group in two dimensions. It is thus not Abelian.

The quaternions ω of magnitude 1 are the invariant subgroup $SU(2, \mathfrak{C})$ of the multiplicative group. We denote these quaternions by Ω.

The center of the multiplicative group are the real quaternions $\mathfrak{R} \subset \mathfrak{Q}$. The center of Ω consists of the two elements ± 1. It is thus the cyclic group of order 2.

For every $\omega \in \Omega$ we can define an automorphism of the quaternions $\mathfrak{Q}$ by setting

$$q \to q^{\omega} = \omega q \omega^{-1}. \tag{31}$$

We prove in algebra that conversely every automorphism of the quaternions is of this form. The automorphisms are thus themselves a group that is isomorphic to the factor group $O^{+}(3) = SU(2, \mathfrak{Q})/Z_2$.

It is sometimes convenient to represent quaternions as pairs of complex numbers by setting

$$q = z_1 + e_2 z_2 = (z_1, z_2) \tag{32}$$

where

$$\begin{aligned} z_1 &= a_0 + a_3 e_3, \\ z_2 &= a_2 + a_1 e_3. \end{aligned} \tag{33}$$

We then identify e_3 with the imaginary unit i of the complex numbers. The multiplication law is then expressible by

$$\begin{gathered} q = (z_1, z_2), \qquad q' = (z_1', z_2'), \\ qq' = (z_1 z_1' - z_2^* z_2', z_2 z_1' + z_1^* z_2'). \end{gathered} \tag{34}$$

We shall call this representation of the quaternions by pairs of complex numbers the *symplectic decomposition.*

The symplectic decomposition furnishes us with a representation of the quaternions by 2×2 matrices in a complex space as follows. For any fixed quaternion $a \in \mathfrak{Q}$ with symplectic decomposition $a = (\alpha_1, \alpha_2)$ we set

$$q \to q' = aq. \tag{35}$$

We then interpret the quaternions q and q' as two component vectors with complex coefficients. Equation (35) is then equivalent with the linear transformation

$$q' = Aq,$$

where

$$A = \begin{pmatrix} \alpha_1 & -\alpha_2^* \\ \alpha_2 & \alpha_1^* \end{pmatrix}. \tag{36}$$

We shall refer to this as the *symplectic representation* of the quaternions. For the particular case that $a = e_r$ ($r = 1, 2, 3$) we obtain in the symplectic representation

$$e_r = -i\sigma_r \qquad (r = 1, 2, 3) \tag{37}$$

where σ_r are the three Pauli spin matrices. We should remark here that the symplectic decomposition can be made in a coordinate-free manner as follows: Let i be any fixed pure imaginary quaternion of magnitude 1 so that $i^{\mathfrak{Q}} = -i$, $i^2 = -1$. We write, for every quaternion $q = q_+ + q_-$, where $q_\pm = \frac{1}{2}(q \mp iqi)$, and define $q_+ = z_1$ and $q_- = iz_2$. The pair z_1 and z_2 can be considered as complex numbers with i as the imaginary unit. The correspondence $q \leftrightarrow (z_1, z_2)$ is unique in both directions and satisfies the rules (32). This is the symplectic decomposition with respect to i.

The symplectic decomposition will be very useful in the following because it can be extended to quaternionic Hilbert spaces, and it permits a certain reduction of quaternionic Hilbert spaces to pairs of complex spaces.

2. *Elementary Properties of Quaternionic Hilbert Space*

A quaternionic Hilbert space $\mathfrak{H}_{\mathfrak{Q}}$ is a linear vector space over the field of the quaternions. This means that in addition to the usual rule of vector addition there is also a left multiplication with scalars that associates with every $q \in \mathfrak{Q}$ and every $f \in \mathfrak{H}_{\mathfrak{Q}}$ an element $qf \in \mathfrak{H}_{\mathfrak{Q}}$.

This scalar multiplication shall satisfy the usual rules of distributivity and associativity, such as

$$\begin{aligned} q_1(q_2 f) &= (q_1 q_2) f, \\ q(f+g) &= qf + qg, \\ (q_1 + q_2) f &= q_1 f + g_2 f, \end{aligned} \tag{38}$$

for $q_r \in \mathfrak{Q}$ and $\forall f, g \in \mathfrak{H}_{\mathfrak{Q}}$.

Furthermore, we define a quaternion-valued scalar product $(f,g) \in \mathfrak{Q}$ by the axioms

$$\begin{aligned} &\text{(i)} \quad (qf,g) = (f,g)q^*. \\ &\text{(ii)} \quad (f+g,h) = (f,h) + (g,h). \\ &\text{(iii)} \quad (f,g) = (g,f)^*. \\ &\text{(iv)} \quad \|f\|^2 = (f,f) > 0; \\ &\qquad\quad \|f\|^2 = 0 \Rightarrow f = 0. \end{aligned} \tag{39}$$

Just as in the case of ordinary (complex) Hilbert space, we demonstrate then the inequalities of Cauchy and Minkowski:

$$\begin{aligned} &\text{(i)} \quad |(f,g)| \leqq \|f\|\,\|g\|, \\ &\text{(ii)} \quad \|f+g\| \leqq \|f\| + \|g\|. \end{aligned} \tag{40}$$

With the scalar product, strong and weak convergence can be defined in the usual manner.

3. *Linear and Semilinear Operators*

We define a semilinear operator t as a function tf, with a linear manifold as domain and values in $\mathfrak{H}_{\mathfrak{Q}}$, that satisfies the conditions

$$\begin{aligned} t(f+g) &= tf + tg, \\ t(qf) &= q^t(tf). \end{aligned} \tag{41}$$

Here q^t designates an automorphism of the quaternions independent of f. It follows that the range of a semilinear operator is also a linear manifold. We shall consider only nonsingular transformations such that $tf = 0 \Rightarrow f = 0$. The inverse t^{-1} then exists and it is also semilinear. The operator t is *linear* if $q^t = q \ \forall\, q \in \mathfrak{Q}$.

The Hermitian conjugate $t^\dagger$ of t is defined by the relation

$$(f, tg) = (t^\dagger f, g)^t.$$

We can verify that $t^\dagger$ is semilinear if t is, and if the automorphism associated with t is $q \to q^t$, then the automorphism associated with $t^\dagger$ is $q \to ((q^{\mathfrak{Q}})^s)^{\mathfrak{Q}}$ where $(q^s)^t = q$.

A semiunitary operator u is semilinear and in addition satisfies $\|uf\|^2 = \|f\|^2$ $\forall$ f $\in \mathfrak{H}_{\mathfrak{Q}}$. It is called unitary if it is also linear.

A simple example of a semilinear operator is a multiplication with a fixed quaternion $a \in \mathfrak{Q}$. Indeed, let

$$tf \equiv af \qquad a \in \mathfrak{Q}.$$

It follows that $t(qf) = a(qf) = (aq)f = aqa^{-1}af = q^a tf$. Thus we see that left multiplication with a fixed quaternion a induces a semilinear transformation that leaves every ray invariant.

4. *Ray Transformations*

Every semilinear transformation induces a ray transformation or, more generally, an automorphism of subspaces. A ray is defined as the set of vectors of the form qf with variable $q \in \mathfrak{Q}$ and fixed $f \in \mathfrak{H}_{\mathfrak{Q}}$. The image ray is given by the set of vectors ptf for all $p \in \mathfrak{Q}$.

We denote by F a ray that contains the vector f and by TF the mapping of the ray induced by a semilinear transformation t. We shall say that two semilinear transformations are *equivalent* if they induce the same ray transformation. This is clearly an equivalence relation. We can therefore identify the class $[t]$ of all equivalent transformations t with the ray transformation T.

We now have the following important property:

Theorem 7. Every equivalence class T of semilinear transformations in a Hilbert space $\mathfrak{H}_{\mathfrak{Q}}$ contains at least one linear transformation t_0.

Proof. Let $q \to q^t$ be the automorphism induced in $\mathfrak{Q}$ by the semilinear transformation t. Since every such automorphism is inner, there exists an $\omega \in \Omega$ (quaternions of norm 1) such that

$$q^t = \omega q \omega^{-1}. \tag{42}$$

Define $t_0 = \omega^{-1}t$. It is equivalent to t and we find $t_0(qf) = \omega^{-1}t(qf) = \omega^{-1}\omega q \omega^{-1} tf = qt_0 f$. Thus t_0 is linear. This proves Theorem 7.

If t_0' is another linear transformation in the same class than $t_0^{-1}t_0'$ is a linear transformation that leaves every ray invariant. Such a transformation is of the form

$$f \to \lambda f \qquad \text{with} \quad \lambda \in \mathfrak{R} \quad \text{and} \quad \lambda \neq 0.$$

In the particular case that t is also unitary we must have $\lambda^2 = 1$ or $\lambda = \pm 1$. If we combine this result with the result of subsection D, 2, we obtain the following

Corollary. Every equivalence class T of semilinear transformations of a quaternionic Hilbert space contains exactly two unitary transformations. They differ only by a sign.

B. Projective Representations of Symmetry Groups in Quaternionic Hilbert Space

1. *Local Lifting of Factors*

We consider now a topological group G and a projective representation that associates with every $x \in G$ a ray transformation U_x. According to the preceding subsection, every such transformation can be represented by two unitary operators $u_x \in U_x$ that differ only by a sign. If we choose in an arbitrary manner in each class U_x one of the two representatives u_x then we obtain a projective representation of the symmetry group G by unitary operators that satisfies

$$u_x u_y = \omega(x, y) u_{xy} \tag{43}$$

where $\omega(x,y) = \pm 1$. From the foregoing it is clear that every ray representation of a topological group in a quaternionic Hilbert space can be brought into this form. If we choose $u_e = I$, then the factors $\omega(x,y)$ also satisfy

$$\omega(e, y) = \omega(x, e) = 1 \tag{44}$$

for all x, y in G.

It is natural to ask at this point whether it is possible to choose in a suitable neighborhood of the identity $e \in G$ the representatives u_x in such a way that the factors $\omega(x,y) = 1$. This is indeed the case. The relevant theorem is due to Bargmann (*36*), and it states that for every representation of a topological group in a complex Hilbert space there exists a suitable neighborhood $N(e)$ of the identity so that $\omega(x,y)$ is a *continuous* function of its two arguments. This theorem is also valid in quaternionic Hilbert spaces. The proof for this case was given by Emch (*15*).

The application of this result to the representation $x \to U_x$ leads to

Theorem 8 (Emch). Every ray representation $x \to U_x$ of a topological group G in a quaternionic Hilbert space can be induced by a strongly continuous unitary representation $x \to u_x \in U_x$ in a suitable neighborhood of the identity.

It is worth pointing out here that this theorem is false for complex Hilbert spaces. The deeper reason for this fundamental difference of the two spaces has been analyzed by Emch (*15*) and is due to the fact that $SU(2, \mathbb{C}) = \Omega$ of the quaternions of magnitude 1 is semisimple, whereas the corresponding group of phase transformations in a complex space is not (it is in fact Abelian).

Theorem 8 leads to a considerable simplification of the theory of projective representations of groups. It suffices to study the locally unitary vector representations.

2. *Global Lifting of Factors*

We must next examine the question whether it is possible to extend the vector representation $x \to u_x$ to the entire group G. For simply connected groups the answer is easy. We have in fact

Theorem 9. Every ray representation $x \to U_x$ of a simply connected topological group in a quaternionic Hilbert space can be induced by a unitary vector representation $x \to u_x$.

Proof. According to Theorem 8 there exists a neighborhood $N(e)$ of the identity and a local lifting of the factors such that $u_x u_y = u_{xy}\ \forall\ x, y \in N(e)$. According to Theorem 15 of Pontrjagin (*9*) every element $x \in G$ admits a representation $x = \prod_{i=1}^{n} x_i$, $x_i \in N(e)$ and $n < \infty$. Since the correspondence $x \to u_x$ is a vector representation for all $x \in N(e)$, this theorem permits us to conclude that it remains true for all $x \in G$. This proves Theorem 9. [For the details of this part of the proof we refer to Bargmann (*36*).]

The case of multiply connected groups can be reduced to the case of simply connected groups via the theory of the universal covering group. In the application that constitutes the main topic of this article we need only the result for doubly connected groups, which we shall state with

Theorem 10. Every ray representation of a doubly connected topological group G in a quaternionic Hilbert space can be induced by a unitary vector representation $x \to u_x$ of its simply connected covering group G. There are two and only two distinct cases possible. Either $x \to u_x$ is also a vector representation of G or it is a double-valued vector representation that satisfies only

$$u_x u_y = \pm u_{xy}.$$

The proof of this theorem is exactly the same as in the case of complex spaces. We can therefore omit it here (*15*).

3. *Schur's Lemma and Its Corollary*

The lemma of Schur plays a fundamental role in the representation theory of groups. For the quaternionic case we shall need its generalization, which can be stated as follows.

Lemma (Schur). Let $\mathfrak{H}_{\mathfrak{Q}}^{(r)}$ $(r = 1, 2)$ be two quaternionic Hilbert spaces, G a topological group, and $u_x^{(r)}$ irreducible unitary representations of G in $\mathfrak{H}_{\mathfrak{Q}}^{(r)}$. Furthermore, let t be a bounded colinear mapping of $\mathfrak{H}_{\mathfrak{Q}}^{(1)}$ into $\mathfrak{H}_{\mathfrak{Q}}^{(2)}$ such that

$$t u_x^{(1)} = u_x^{(2)} t \qquad \forall \quad x \in G;$$

then t either admits an inverse or it is zero.

The proof of this lemma requires only small adaptations to be valid for the case of quaternionic spaces as well, and we shall omit it here [for details cf. Emch (*15*)].

Although Schur's lemma is identical in the quaternionic and complex cases, the situation is quite different for its corollary. We state it in the form of

Theorem 11. Let u_x be an irreducible representation of the group G in a quaternionic Hilbert space $\mathfrak{H}_{\mathfrak{Q}}$ and t a bounded linear operator in $\mathfrak{H}_{\mathfrak{Q}}$ such that $tu_x = u_x t \; \forall \; x \in G$; then t is of the form $t = rI + s\mathcal{J}$ where r, s are real, I is the identity in $\mathfrak{H}_{\mathfrak{Q}}$ and $\mathcal{J}$ is a unitary and anti-Hermitian operator in $\mathfrak{H}_{\mathfrak{Q}}$.

If we compare this theorem with the corollary of Schur's lemma in $\mathfrak{H}_{\mathfrak{Q}}$, we note that the essential difference is the appearance of a linear operator $\mathcal{J}$ that is unitary and anti-Hermitian. Such a $\mathcal{J}$ satisfies $\mathcal{J}^\dagger = -\mathcal{J}$ and $\mathcal{J}^\dagger = \mathcal{J}^\dagger \mathcal{J} = -\mathcal{J}^2 = I$.

In a complex space such an operator is always of the form $\mathcal{J} = \pm iI$ where $i = (-1)^{1/2}$ and it is seen that in this case the corollary reduces to the corollary for complex spaces.

Before giving a formal proof of the theorem, let us verify it for the case of a one-dimensional space. The vectors in this space are the quaternions q. Linear operators are multiplication from the right with another quaternion. The unitary operators are multiplication from the right with a quaternion of magnitude 1. Thus we may write $uq = q\omega$, $q \in \mathfrak{Q}$, $\omega \in \Omega$.

A linear operator t that commutes with u must have the form

$$tq = qa \qquad a \in \mathfrak{Q} \quad \text{and} \quad \omega a = a\omega.$$

Let us write for $\omega = \omega_0 + \omega \cdot e$. We define $e_\omega = |\omega|^{-1} \omega \cdot e$ so that $\omega = \omega_0 + |\omega| \, e_\omega$. We find then easily that a must have the form $a = r + se_\omega$ with r and s real. Thus t is of the form

$$tq = rq + sqe_\omega$$

and we have verified the theorem for this case if we show that $\mathcal{J}q = qe_\omega$ is unitary and anti-Hermitian. This is indeed the case, since $e_\omega^\dagger = -e_\omega$ and $e_\omega^2 = -1$, so that $\mathcal{J}^\dagger q = -qe_\omega$ and $\mathcal{J}^2 = -I$.

Let us now prove Theorem 11. Assume first that t is Hermitian, so that $t^\dagger = t$. In that case not only t but also every function of t commutes with u. In particular, the spectral projections associated with t do the same. Since u is irreducible, all these spectral projections are either 0 or I. From this follows that t is a multiplum of $I: t = r \cdot I$ with r real. This proves the theorem for Hermitian t.

Let us now examine the case of anti-Hermitian $t: t^\dagger = -t$. It follows then that $t^\dagger t$ is a positive operator, since $(f, t^\dagger tf) = 0 \Rightarrow (tf, tf) = 0$ or $tf = 0$. By

Schur's lemma this is only possible if $f = 0$. Hence $t^\dagger t$ is positive and Hermitian. According to a well-known theorem *(37)* there exists then a unique positive square root $(t^\dagger t)^{1/2} = |t|$ that is Hermitian, possesses an inverse, and commutes with t. We define then

$$\mathscr{J} = t|t|^{-1} = |t|^{-1} t$$

so that

$$\mathscr{J}^\dagger = -\mathscr{J} \quad \text{and} \quad \mathscr{J}^2 = -I.$$

We verify then that t also commutes with u, so according to the preceding paragraph it is of the form $s \cdot I$. Thus we have proved that $t = s \cdot \mathscr{J}$ with $\mathscr{J}^2 = -I$, $\mathscr{J}^\dagger = -\mathscr{J}$ if $t^\dagger = -t$.

The general case, where t is neither Hermitian nor anti-Hermitian, is now easily reduced to the preceding two special cases. We write $t = t_1 + t_2$, where $t_1 = \frac{1}{2}(t + t^\dagger)$, $t_2 = \frac{1}{2}(t - t^\dagger)$, so that $t_1^\dagger = t_1$, $t_2^\dagger = -t_2$. Moreover, both t_1 and t_2 commute separably with u. Thus $t_1 = r \cdot I$, $t_2 = s \cdot \mathscr{J}$, and $t = r \cdot I + s \cdot \mathscr{J}$. This proves the theorem.

It should have become obvious by now that the operator $\mathscr{J}$ is related to the symplectic decomposition of the complex numbers. Indeed the $\mathscr{J}$ plays the role of an imaginary unit in the quaternionic Hilbert space. This will be discussed in detail in the following subsection.

4. *The Symplectic Decomposition of* $\mathfrak{H}_{\mathfrak{Q}}$

We recall that the symplectic decomposition for quaternions (cf. Section IV, A, 1) was obtained by distinguishing one of the quaternionic units and decomposing the quaternions into two distinct classes, those that commute with this unit and those that anticommute. This process can be extended to Hilbert spaces.

Let $\mathscr{J}$ be a linear operator in $\mathfrak{H}_{\mathfrak{Q}}$ such that

$$\mathscr{J}\mathscr{J}^\dagger = \mathscr{J}^\dagger \mathscr{J} = I, \qquad \mathscr{J}^\dagger = -\mathscr{J}. \tag{45}$$

We observe first that every vector $f \in \mathfrak{H}_{\mathfrak{Q}}$ is an eigenvector of $\mathscr{J}$ and every pure imaginary quaternion of magnitude 1 is an eigenvalue.

To see this let $f \in \mathfrak{H}_{\mathfrak{Q}}$ be an arbitrary vector and define $g = \mathscr{J}f$. We decompose f with respect to the ray $F = \{f\}$, which is a one-dimensional subspace of $\mathfrak{H}_{\mathfrak{Q}}$:

$$\mathscr{J}f = g = qf + h, \qquad \text{where} \quad (h, f) = 0. \tag{46}$$

It follows from this and the properties (45) that

$$(f, \mathscr{J}h) = (f, \mathscr{J}(\mathscr{J}f - qf)) = -(f, f) - q(f, (qf + h)) = -(1 + q^2)(f, f). \tag{47}$$

On the other hand

$$(f, \mathscr{J} h) = -(\mathscr{J} f, h) = -(h,h) = -(1 - |q|^2)(f, f). \tag{48}$$

From this we obtain

$$q^2 = -|q|^2 \quad \text{and} \quad |q|^2 \leqq 1. \tag{49}$$

Using $\mathscr{J}^2 = -I$, we obtain further

$$0 = \mathscr{J}^2 f + f = (1 + q^2) f + \mathscr{J} h = (1 - |q|^2) f + \mathscr{J} h,$$

so that

$$(h, h) = (\mathscr{J} h, \mathscr{J} h) = (1 - |q|^2)^2 (f, f). \tag{50}$$

Comparing Eq. (50) with Eq. (48) we find

$$|q|^2 = 1, \qquad q^2 = -1, \qquad h = 0. \tag{51}$$

Thus we have proved: every vector $f \in \mathfrak{H}_{\mathfrak{Q}}$ is an eigenvector of $\mathscr{J}$ and the eigenvalue is a pure imaginary quaternion of magnitude 1.

Consider now any $f \in \mathfrak{H}_{\mathfrak{Q}}$ and assume $\mathscr{J} f = if$ where i is pure imaginary and $i^2 = -1$. Let $\omega \in \Omega$ and evaluate

$$\mathscr{J} \omega f = \omega \mathscr{J} f = \omega i f = \omega i \omega^{-1} \omega f.$$

Thus we see: If f is an eigenvector of $\mathscr{J}$ with eigenvalue i, then ωf is an eigenvector of $\mathscr{J}$ with eigenvalue $\omega i \omega^{-1}$. If ω runs through Ω, we obtain with $\omega i \omega^{-1}$ every imaginary quaternion of norm 1. Thus we have proved

Theorem 12. Every vector f in a ray F is an eigenvector of the operator $\mathscr{J}$. The eigenvalues are pure imaginary quaternions of magnitude 1. As f runs through the ray, the eigenvalues run through all such quaternions.

Let us now select an arbitrary but fixed pure imaginary quaternion i of magnitude 1. In every ray F we select the ensemble of vectors f such that $\mathscr{J} f = if$. The totality of such vectors from all rays defines a subset of $\mathfrak{H}_{\mathfrak{Q}}$ that we denote by $\mathfrak{H}_{\mathfrak{C}}^{(i)}$; thus

$$\mathfrak{H}_{\mathfrak{C}}^{(i)} = \{ f \in \mathfrak{H}_{\mathfrak{Q}} | \mathscr{J} f = if \}.$$

We verify without effort that $\mathfrak{H}_{\mathfrak{C}}^{(i)}$ is a complex Hilbert space when the complex numbers $\mathfrak{C}$ are defined by $z = x + iy$ (x, y real). Thus for instance if we have $f, g \in \mathfrak{H}_{\mathfrak{C}}^{(i)}$, then $\mathscr{J}(f+g) = \mathscr{J} f + \mathscr{J} g = if + ig$; $i(f+g)$. If $z \in \mathfrak{C}$, then $\mathscr{J}(zf) = z \mathscr{J} f = zif = i(zf)$. Furthermore, if $f, g \in \mathfrak{H}_{\mathfrak{C}}^{(i)}$, then $i(f,g) = (-if, g) = (-\mathscr{J} f, g) = (f, \mathscr{J} g) = (f, ig) = (f,g)i$. Hence $(f,g) \in \mathfrak{C}$.

Finally, if $f_n \in \mathfrak{H}_{\mathfrak{C}}^{(i)}$ is a sequence such that $\|f_n - f_m\| \to 0$ for $n, m \to \infty$, then there exists a limit $f \in \mathfrak{H}_{\mathfrak{Q}}$ such that $f_n \to f$. For this element f we find $\|\mathscr{J}f - \mathscr{J}f_n\| = \|f - f_n\|$, so that $\mathscr{J}f$ is also the limit of $\mathscr{J}f = if_n$. This means $\mathscr{J}f = if$ and $f \in \mathfrak{H}_{\mathfrak{C}}^{(i)}$. With this we have verified that $\mathfrak{H}_{\mathfrak{C}}^{(i)}$ is indeed a complex Hilbert space.

We remark also that the space $\mathfrak{H}_{\mathfrak{C}}^{(i)}$ is total in $\mathfrak{H}_{\mathfrak{C}}$ in the sense that every $f \in \mathfrak{H}_{\mathfrak{C}}$ can be written as linear combinations of vectors $f_+ \in \mathfrak{H}_{\mathfrak{C}}^{(i)}$ with coefficients from $\mathfrak{Q}$. Indeed, let $f \in \mathfrak{H}_{\mathfrak{Q}}$. We define $f_{\pm} = \frac{1}{2}(I \mp i\mathscr{J})f$ and then choose an arbitrary imaginary quaternion j that anticommutes with i. By setting $f_+' = -jf_-$ we find

$$\mathscr{J}f_+ = if_+,$$
$$\mathscr{J}f_+' = if_+', \tag{52}$$

and

$$f = f_+' + jf_+'.$$

Thus every vector $f \in \mathfrak{H}_{\mathfrak{Q}}$ admits a decomposition into pairs of vectors f_+, $f_+' \in \mathfrak{H}_{\mathfrak{C}}^{(i)}$ such that f is a linear combination of such a pair with coefficients from $\mathfrak{Q}$. This is the symplectic decomposition of the quaternionic Hilbert space.

We summarize the results of this subsection with

Theorem 13. Every unitary anti-Hermitian operator $\mathscr{J}$ in a quaternionic Hilbert space defines for each imaginary quaternion i of magnitude 1 a family $\mathfrak{H}_{\mathfrak{C}}^{(i)}$ of vectors f all of which satisfy $\mathscr{J}f = if$. They are a complex Hilbert space that is total in $\mathfrak{H}_{\mathfrak{Q}}$.

5. *Restriction and Extension of Representations*

As before, let $\mathscr{J}$ denote a unitary anti-Hermitian operator in $\mathfrak{H}_{\mathfrak{Q}}$, $\mathfrak{H}_{\mathfrak{C}}^{(i)}$ the complex Hilbert space associated with an imaginary quaternion, and t a bounded linear operator that commutes with $\mathscr{J}$. If $f \in \mathfrak{H}_{\mathfrak{C}}^{(i)}$, then $\mathscr{J}tf = t\mathscr{J}f = t(if) = itf$. Thus $tf \in \mathfrak{H}_{\mathfrak{C}}^{(i)}$. We may therefore define the *restriction* $t_{(i)}$ of the operator as the operator with domain $\mathfrak{H}_{\mathfrak{C}}^{(i)}$. For all $f \in \mathfrak{H}_{\mathfrak{C}}^{(i)}$ it is defined by $t^{(i)}f = tf$.

Conversely, if $t_{(i)}$ is any bounded linear operator in $\mathfrak{H}_{\mathfrak{C}}^{(i)}$, we define its extension t to $\mathfrak{H}_{\mathfrak{Q}}$ by the conditions

(i) t is linear

(ii) $tf = t^{(i)}f \quad \forall \quad f \in \mathfrak{H}_{\mathfrak{C}}^{(i)}$. (53)

Let us show that this extension is always possible and that is is unique. This can be seen directly from the symplectic decomposition (52). Thus we *define* tf by

$$tf = t^{(i)} f_+ + j t^{(i)} f'_+. \tag{54}$$

Let t' be any other extension. Because it is linear we have for any f

$$t'f = t^{(i)} f_+ + j t^{(i)} f'_+ = tf. \tag{55}$$

This proves that the extension is unique.

Let us now consider the Hermitian conjugate of t. It is defined by the relation $(f, tg) = (t^\dagger f, g)\ \forall\, f, g \in \mathfrak{H}_{\mathfrak{Q}}$. Since $\mathscr{J}$ commutes with t we have also for $t^{(i)}$ $(f, t^{(i)} g) = (t^{(i)\dagger} f, g)\ \forall\, f, g \in \mathfrak{H}_{\mathfrak{C}}^{(i)}$. If in the first of these two relations we restrict f, g to $\mathfrak{H}_{\mathfrak{C}}^{(i)}$, we evidently obtain $(t^{\dagger(i)} f, g) = (f, t^{(i)} g)$, from which we conclude that $t^{\dagger(i)} = t^{(i)\dagger}$.

The following assertions are immediate consequences of this.

(a) If t is Hermitian, then $t^{(i)}$ is Hermitian, too.
(b) If t is a projection, then $t^{(i)}$ is a projection, too.
(c) If $t = u$ is unitary, then $u^{(i)}$ is unitary, too.
(d) If $t_1 t_2$ commute, then $t_1^{(i)} t_2^{(i)}$ commute.
(e) If t is an irreducible system all commuting with $\mathscr{J}$, then $t^{(i)}$ is an irreducible system, too.
(f) If t_n is a sequence of t_n all of which commute with $\mathscr{J}$ and tending weakly, strongly, or uniformly to a limit t, then t commutes with $\mathscr{J}$ too, and $t_n^{(i)}$ tends weakly, strongly, or uniformly to $t^{(i)}$.

We retain the part that is relevant for the group representations in

Theorem 14. If $x \to u_x$ is a unitary representation of the topological group G in a quaternionic Hilbert space $\mathfrak{H}_{\mathfrak{Q}}$ that commutes with a unitary and anti-Hermitian linear operator $\mathscr{J}$, then for each pure imaginary quaternion the restriction $u_x^{(i)}$ is defined and

(a) the $u_x^{(i)}$ are a unitary representation of G in $\mathfrak{H}_{\mathfrak{C}}^{(i)}$;
(b) if u_x is irreducible in $\mathfrak{H}_{\mathfrak{Q}}$, then $u_x^{(i)}$ is irreducible in $\mathfrak{H}_{\mathfrak{C}}^{(i)}$.

This theorem gives us complete information as to the properties of the restriction of a representation that commutes with a unitary anti-Hermitian operator. It is natural to ask the question about the converse problem: If we extend a representation from $\mathfrak{H}_{\mathfrak{C}}^{(i)}$ with the unique process described at the beginning of this subsection, what happens to a representation? The answer is contained in

Theorem 15. If $x \to u_x^{(i)}$ is a representation of a topological group G in a complex Hilbert space $\mathfrak{H}_{\mathfrak{C}}^{(i)}$ and it is of class $+1$ or 0 in the sense of Frobenius and Schur, then x_x is an irreducible representation of G in $\mathfrak{H}_{\mathfrak{Q}}$. On the other hand, if $u_x^{(i)}$ is of class -1, then u^x is reducible.

The classification of Frobenius and Schur that is needed here is defined in the following way.

Let $\mathfrak{H}_x$ be a complex Hilbert space, and K a conjugation of $\mathfrak{H}_{\mathfrak{C}}$, that is, an antiunitary involutive mapping of $\mathfrak{H}_{\mathfrak{C}}$ onto itself. If $x \to u_x$ is an irreducible representation of a group G, we can define a conjugate representation $\tilde{u}_x = Ku_xK$. Then Frobenius and Schur have observed that exactly three cases may occur.

(a) $\tilde{u}$ is equivalent to u. There exists then a unitary operator C such that $u_x = C^{-1}u_xC$. If $CKCK = I$, then the representation is of class $+1$.

(b) $\tilde{u}$ is equivalent to u and $CKCK = I$. The representation is then of class -1.

(c) $\tilde{u}$ is not equivalent to u. It is then said to be of class 0.

The proof of Theorem 15 is given by Finkelstein *et al.* (*12*) and Emch (*15*). [The second part (concerning the class -1) is, however, proved only for compact groups by Finkelstein *et al.* (*12*).]

6. *Representation of Abelian Groups*

It is well known that the only irreducible vector representations of an Abelian group in a complex Hilbert space are one-dimensional. Let us now establish this same theorem for the quaternionic vector representation. Assume $x \to u_x$ to be such a representation. It follows, then, from the corollary of Schur's lemma, that $u_x = r(x)I + s(x)\mathscr{J}(x)$ where the $\mathscr{J}(x)$ are unitary and anti-Hermitian operators that all commute with one another. The operators $\mathscr{J}(x)\mathscr{J}(y)$ are thus Hermitian and they all commute with each other and with all the u_x. Thus all the $\mathscr{J}(x)$ are multiples of one another. We can thus write $u_x = r(x)I + s(x)\mathscr{J}$. According to Section IV, A, 4, every vector is an eigenvector of $\mathscr{J}$. Thus u_x leaves every ray invariant, and since the u_x are irreducible, the representation $x \to u_x$ is one-dimensional.

Let us now examine the properties of these irreducible representations of G. Every vector f in a one-dimensional quaternionic Hilbert space may be represented by a quaternion $q \in \mathfrak{Q}$. The operator I is then multiplication with 1 and the linear operator $\mathscr{J}$ is multiplication from the right with an arbitrary pure imaginary quaternion i, so that

$$u_xq = q(r(x) + s(x)\,i). \tag{56}$$

The unitarity of u_x implies $r(x)^2 + s(x)^2 = 1$. We may thus write

$$u_xq = qe^{i\theta(x)} \tag{57}$$

with

$$tg\theta(x) = \frac{s(x)}{r(x)} \qquad (0 \leqq \theta(x) < 2\pi).$$

The representation property $u_x u_y = u_{xy}$ leads then to the relation

$$\theta(x) + \theta(y) = \theta(xy) \qquad (\text{mod } 2\pi). \tag{58}$$

The correspondence $u_x \to \theta(x)$ is thus a continuous homomorphism of the group G onto the additive group of real numbers modulo 2π, called the *circle group*. The image $\theta(x)$ of such a homomorphism is called a *character* of the group G.

The characters of an Abelian group G are themselves a group, the character group X, and there exists a natural procedure to define a topology in X such that this group becomes a topological group. The group operations in X are defined by setting for any two characters $\theta_1(x)$ and $\theta_2(x)$

$$(\theta_1 \theta_2)(x) = \theta_1(x) + \theta_2(x) \qquad (\text{mod } 2\pi). \tag{59}$$

Just as in the complex case so we can here, too, characterize the inequivalent irreducible representations of the Abelian group G by their characters. In order to see this, let us assume that $u_x^{(1)}$ and $u_x^{(2)}$ are two equivalent irreducible representations. There exists then a unitary (hence linear) operator u such that $u_x^{(1)} = u u_x^{(2)} u^{-1}$. Recalling that unitary operators in a one-dimensional quaternionic Hilbert space are multiplication from the *right* with a quaternion $\omega \in \Omega$, we see that

$$u_x^{(1)} q = q \exp(i_1 \theta_1(x)) = u u_x^{(2)} u^{-1} q = q \omega^{-1} \exp(i_2 \theta_2(x))\, \omega.$$

Thus

$$\omega^{-1} \exp(i_2 \theta_2(x))\, \omega = \exp(i_1 \theta_1(x)), \tag{60}$$

which implies

$$\omega^{-1} i_2 \omega = i_1 \qquad \text{and} \qquad \theta_1(x) = \theta_2(x) \qquad (\text{mod } 2\pi). \tag{61}$$

The second part of Eq. (61) says that the two characters are equal. Conversely, if the two characters are equal, then we can always choose an $\omega \in \Omega$ such that for any two pure imaginary quaternions i_1 and i_2 we have $i_1 = \omega^{-1} i_2 \omega$. This ω interpreted as a right multiplication in $\mathfrak{Q}$ furnishes us with the unitary operator u that establishes the equivalence between the two representations.

This result enables us to reduce the problem of finding all irreducible representations of an Abelian group G to that of finding all the characters of G.

The group that interests us in the following is the group of translations in the four-dimensional Minkowski space. For this case all the characters are known. They are of the form

$$\theta(x) = p \cdot x \qquad (\mathrm{mod}\, 2\pi) \tag{62}$$

where p is a fixed four-component vector in Minkowski space, x is the four-vector of the translation x, and $p \cdot x$ is the scalar product in the Minkowski metric of these two four-vectors.

Let us now proceed to the discussion of reducible representations. In the complex case the structure of the reducible representations of a locally compact Abelian group can be characterized by a projection-valued measure on the group of characters θ. This is the theorem of Stone-Neumark-Ambrose-Godement [in the following referred to as the SNAG theorem; (*38–41*)], which may be stated as follows.

Every unitary representation of a locally compact connected Abelian topological group G defines a unique projection valued measure dE on the character group X such that

$$u_x = \int_X e^{i\theta(x)} \mathrm{d}\mathrm{E}. \tag{63}$$

This result can be described as a kind of generalization of the spectral resolution of unitary operators.

This theorem can be transferred to the quaternionic case. The only problem is to construct the analog of the imaginary unit i that appears in the expression (63). It is clear that this analog must be replaced by a unitary anti-Hermitian operator $\mathcal{J}$ that commutes with all u_x. The construction of such an operator is always possible (*15*, Lemma 4.2, p. 766).

In order to establish the SNAG theorem for the quaternionic representations, we proceed as follows. We are given a representation $x \to u_x$ in $\mathfrak{H}_{\mathfrak{Q}}$. We choose a unitary anti-Hermitian $\mathcal{J}$ that commutes with all u_x and select an arbitrary pure imaginary quaternion i. According to Theorem 13 this defines a complex Hilbert space $\mathfrak{H}_{\mathfrak{C}}^{(i)}$ that is invariant under all u_x. The restriction of u_x to $\mathfrak{H}_{\mathfrak{C}}^{(i)}$ is denoted by $u_x^{(i)}$. It satisfies the hypotheses of the SNAG theorem. Hence there exists a unique projection-valued measure $dE^{(i)}$ on the character group X so that for this $u_x^{(i)}$ we have a formula

$$u_x^{(i)} = \int_X e^{i\theta(x)} \, dE^{(i)}. \tag{64}$$

The unique extension procedure described in subsection 5 defines projections dE in $\mathfrak{H}_{\mathfrak{Q}}$ and an operator $\mathcal{J}$ such that

$$u_x = \int_X e^{\mathcal{J}\theta(x)}\, dE. \tag{65}$$

Thus we have established

Theorem 16. Let $x \to u_x$ be a representation of a locally compact connected Abelian group G in a quaternionic Hilbert space $\mathfrak{H}_{\mathfrak{Q}}$. Then there exists a unitary anti-Hermitian operator $\mathcal{J}$ and a projection-valued measure dE on the character group X of G such that u_x can be represented by formula (65).

We remark here that the uniqueness of the measure cannot be affirmed as in the complex case because the operator $\mathcal{J}$ need not be unique. There is a trivial ambiguity for $\mathcal{J}$ because, on the subspace $M = \{f \,|\, u f_x = f \quad \forall\, x \in G\}$ that reduces u_x, $\mathcal{J}$ is completely arbitrary. This situation already exists in the complex case, but in neither case has it any consequences for the definition of the spectral measure.

In the quaternionic case there is a further ambiguity for $\mathcal{J}$, even for the part of $\mathcal{J}$ that belongs to the space $M^{\perp}$.

For the case of the Poincaré group it is relatively easy to formulate physically motivated conditions on the representation that imply uniqueness of the operator $\mathcal{J}$ in that case. This will be done in the subsection C, 2.

C. Representation Theory of the Poincaré Group

1. *The Poincaré Group*

The Poincaré group G is defined as the group of real linear transformations in four variables that leave the metric of Minkowski space invariant. We shall choose for this metric the tensor $g_{00} = +1$, $g_{ii} = -1$ for $i = 1, 2, 3$ and $g_{\mu\nu} = 0$ for $\mu \neq \nu$.

The translations T are an Abelian invariant subgroup. The homogeneous transformations constitute another subgroup L, called the Lorentz group. This subgroup consists of four disconnected components that contain, respectively, the identity e, space inversion σ, time reversal $\mathcal{T}$, and combined inversion $\vartheta = \sigma\mathcal{T}$. The connected component of the Poincaré group will be denoted by G_e and that of the Lorentz group by L_e.

The composition law can be expressed in terms of the translation vector $a \in T$ and an arbitrary Lorentz transformation by

$$(a, \Lambda)(a', \Lambda') = (a'', \Lambda'') \tag{67}$$

where

$$a'' = a + \Lambda a', \qquad \Lambda'' = \Lambda\Lambda'. \tag{68}$$

The subgroup T consists of the elements of the form (a, I) while L is represented by the elements of the form $(0, \Lambda)$. The connected component L_c is doubly connected. Its simply connected covering group is the group $S(2,\mathfrak{C})$.

2. *Physical Heuristics*

It is now time to consider some of the physical aspects of the representation problem of the Poincaré group. If we compare the representation theory of groups in complex and quaternionic Hilbert spaces, then we observe that up to a certain point the two theories run more or less parallel without, however, being exactly identical. The point where the two theories begin to differ in a deeper way is met when we introduce the unitary anti-Hermitian operator $\mathscr{J}$. In a complex Hilbert space such an operator is always the direct sum of $\pm i$ times the identity operator.

In a quaternionic space such an operator has a much richer structure because there exist an infinity of different square roots of -1. Consequently we expect that the representation of groups in a quaternionic space will depart from the complex case in an essential way if we admit for the operators $\mathscr{J}$ the most general possibilities.

Instead of studying the most general possibilities for the operator $\mathscr{J}$, we want to examine the problem from a physical point of view and see whether we can find in the physical interpretation a motive for restricting the possibilities for the operator $\mathscr{J}$. The operator $\mathscr{J}$ is met when the Abelian subgroups of the Poincaré group are studied. Such groups are, for instance, the one-parameter subgroups. If $s \to u_s$ is the representation of such a one-parameter subgroup, then we can always define in a unique manner [cf. Finkelstein *et al.* (*13*)] an anti-Hermitian operator A by setting

$$A = s - \lim_{s \to 0} (1/s)(u_s - I).$$

This limit always exists on a dense linear manifold of vectors that is the domain for this operator A.

In complex quantum mechanics the self-adjoint operator $P = -iA$ is always an observable. Thus the reconstruction of an observable from the generator of an infinitesimal one-parameter symmetry transformation is a unique process in complex quantum mechanics. In quaternion quantum mechanics any operator $\mathscr{J}$ can be used for defining a self-adjoint operator P by setting, for instance, $P(\mathscr{J}) = -\mathscr{J}A$. However, only A is determined uniquely by the group, but A is not an observable, because it is not self-adjoint. The $P(\mathscr{J})$ *is* self-adjoint, but is is not unique. Since only self-adjoint operators can represent observables, we cannot associate observables in this manner with the infinitesimal generators of symmetry transformations without restricting the operator $\mathscr{J}$ in some way.

The simplest way to restrict the operator $\mathscr{J}$ is to require that it commute with all the transformations of the Poincaré group. Let us examine whether this condition can by physically motivated.

The infinitesimal generators A of the translation group behave under Lorentz transformations like a four-vector. We can make a good case that the self-adjoint momentum operators $P = -\mathscr{J} A$ associated with these operators should have the same property. This means physically that the measured values of these operators transform like a four-vector under Lorentz transformations. This is only possible if the operator $\mathscr{J}$ commutes with all the u_x of the given representation.

A further restriction is obtained by requiring the energy operator P_0 to have a positive definite spectrum. It is interesting to note that in quaternion quantum mechanics this can always be accomplished by a suitable choice of $\mathscr{J}$ *(13)*.

We formulate therefore the following two postulates:

Postulate 1. The observables P associated with the translations in Minkowski space (momentum operators) transform under Lorentz transformations like a four-vector.

Postulate 2. The energy P_0 has a positive definite spectrum.

It is seen that these postulates are quite reasonable from the point of view of physics. We want to point out, however, that there are possible representations of the Poincaré group that do not satisfy these requirements. In view of recent developments in fundamental particle physics there might even be some interest in these representations, for instance, for a relativistic theory of the recently discussed hypothetical units called *quarks*. That the infinitesimal generators for the translations do not give rise to unique observables is not such a compelling objection to quarks, which do not seem to be observable in the usual sense of the word. In fact, they reveal their presence (if present they are) only through a structure of partial symmetries for strongly interacting particles.

In the rest of this chapter we shall not dwell, however, on these speculative aspects of the unknown quaternionic representations of the Poincaré group. We now proceed to the classification of the irreducible representations that satisfy Postulates 1 and 2; we shall call these the *physical representations*.

3. *The Physical Representations of the Connected Component*

We denote by G_e the connected component of the Poincaré group and we shall determine all the physical ray representations of this group in a quaternionic Hilbert space. The theory of the preceding section (notably Section IV, B, 5 and 6) permits us to reduce this problem to the complex case, where it is already solved. The steps in this reduction can be outlined as follows.

(a) We assume that we have an irreducible ray representation $x \to U_x$ of the connected Poincaré group G_e into the ray transformations of the quaternionic Hilbert space $\mathfrak{H}_{\mathfrak{Q}}$. Theorem 8 tells us that this representation can be induced by a unitary vector representation $x \to u_x$ by a suitable choice of the factors. Because the group G_e is doubly connected, $x \to u_x$ is a unitary representation of the simply connected covering group $\tilde{G}_e$. The representation of G_e is thus either unitary (if the kernel of the homomorphism $\tilde{G}_e \to G_e$ is represented by the unit operators), or unitary but double valued (if it is represented by $\pm I$; Theorem 10).

(b) According to Theorem 16 there exists a unitary anti-Hermitian operator $\mathscr{J}$ and a projection-valued measure on the characters of the translation group $T \subset G_e$ such that

$$u_x = \int_X e^{J\theta(x)}\, dE \tag{69}$$

where $x \in T$, $\theta(x)$ is the character, and the integral is extended over the entire character group X. Every character $\theta(x)$ has the form (62) where x is the translation vector and p is an arbitrary fixed vector in Minkowski space.

(c) According to Postulate 1 the operator $\mathscr{J}$ commutes with all operators u_x with $x \in G_e$. According to Theorems 18 and 14 we can, for each arbitrary but fixed pure imaginary quaternion i, define a complex Hilbert space $\mathfrak{H}_{\mathfrak{C}}^{(i)}$ and a restriction $u_x^{(i)}$ of the representation u_x to this space. According to Section IV, B, 5, this restriction is an irreducible single- or double-valued unitary vector representation of G_e in the complex space $\mathfrak{H}_{\mathfrak{C}}^{(i)}$.

(d) Conversely, if $x \to u_x^{(i)}$ is an irreducible (possibly double-valued) representation of G_e and if it is not of type -1 then it can, according to Theorem 15, be extended in a unique way to a unitary representation in $\mathfrak{H}_{\mathfrak{Q}}$.

The problem of finding all the physically meaningful representations of the Poincaré group in a quaternionic space $\mathfrak{H}_{\mathfrak{Q}}$ is thus reduced to that of finding these representations in the complex space. This problem is solved and all these representations are known.

We shall summarize the method and results for the complex case in the following subsections.

4. *Induced Representations* (*Discrete Case*)

Herein we review briefly the theory of induced representations in a complex

Hilbert space for the case of finite groups. Although finite groups are not our primary concern, they serve as a useful example for the discussion of the purely *algebraic* aspects of the theory. The application of these results to the infinite Poincaré group is then possible by supplementing this algebraic part by some *measure*-theoretical and *topological* considerations.

The notion of *induced* representations is a generalization of that of the *regular* representations. Characteristic for both is that the group plays a double role: First, it is the group to be represented and second it is also a Hilbert space. For finite groups this space is finite-dimensional; in fact, its dimension is equal to the order of the group.

This space is defined as the set of all functions $f(x)$ from the group G to the complex numbers $\mathfrak{C}$. If we define the norm of such functions by

$$\|f\|^2 = \sum_{x \in G} |f(x)|^2, \tag{70}$$

we evidently obtain a Hilbert space $\mathfrak{H}(R)$.

The regular representation is then obtained by defining for any $s \in G$ the unitary operator R_s:

$$(R_s f)(x) = f(xs). \tag{71}$$

If s_1 and s_2 are two elements from G, we have evidently

$$(R_{s_1}(R_{s_2} f))(x) = (R_{s_2} f)(xs_1) = f(xs_1 s_2).$$

Therefore we may set

$$R_{s_1} R_{s_2} = R_{s_1 s_2}. \tag{72}$$

The correspondence $s \to R_s$ is a unitary representation of the abstract group in the Hilbert space $\mathfrak{H}(R)$. This is the *regular* representation of the group G.

We shall now generalize this notion in successive steps until we arrive at the induced representation in sufficient generality for use in connection with the Poincaré group.

Let $H \subset G$ be a subgroup of G. We can then decompose G into its right cosets by the formula

$$G = H + Hx_1 + Hx_2 + \cdots \qquad x_1 \notin H, x_2 \notin H, x_2 \notin Hx, \text{etc.} \tag{73}$$

We denote the set of right cosets of G by G_H. Two elements x and y in the same coset are said to be equivalent modulo H, and we write

$$x \equiv y(H). \tag{74}$$

We consider now functions $f(x)$ defined on right cosets G_H. If the number of right cosets G_H is j, then these functions define a j-dimensional vector space, which can be made into a Hilbert space by defining the norm

$$\|f\|^2 = \sum_{x \in G_H} |f(x)|^2 \tag{75}$$

where the summation is (as indicated) extended only over the cosets. Since $f(x)$ is assumed to be constant in a coset, it suffices to select from each coset one x and carry out the sum (75).

The induced representation is now obtained by setting for all $s \in G$

$$(U_s f)(x) = f(xs). \tag{76}$$

We remark here that this definition is meaningful; that is, the right-hand side is again a function on the cosets because as x runs through one coset the image xs runs through another.

We again easily verify that this is indeed a representation in a j-dimensional space and that it is unitary.

For the special case where H consists only of the unit element, we obtain the regular representation. Thus U_s is seen to be a generalization of the regular representation. For the other extreme case where $H = G$, we obtain the trivial unit representation for which every $s \in G$ is represented by 1.

In the next step we consider functions $f(x)$ that are not necessarily constant in the cosets. Let, for instance, $\chi(s)$ be a one-dimensional representation of G so that

$$\begin{aligned} \chi(s_1)\,\chi(s_2) &= \chi(s_1 s_2) \\ |\chi(s)| &= 1 \end{aligned} \tag{77}$$

and define

$$f(\xi x) = \chi(\xi) f(x) \qquad \forall \quad \xi \in H, x \in G. \tag{78}$$

Such functions still define a j-dimensional vector space since the values of $f(x)$ are determined in each coset by its value for one particular element in the coset. With the norm defined again by

$$\|f\|^2 = \sum_{x \in G_H} |f(x)|^2 \tag{79}$$

we obtain a Hilbert space.

An induced representation is now obtained by again setting

$$(U_s f)(x) = f(xs). \tag{80}$$

The next and final generalization is obtained by starting with any unitary representation $h \to L_h$ of H in a representation space $\mathfrak{H}_0$ of dimension n_0. We then define functions $f(x)$ on G with values in $\mathfrak{H}_0$. That is, for each $x \in G$ we associate a vector $f(x) \in \mathfrak{H}_0$. This vector-valued function is assumed to satisfy

$$f(hx) = L_h f(x) \qquad \forall \quad h \in H, x \in G. \tag{81}$$

With the norm defined by

$$\|f\|^2 = \sum_{x \in G_H} \|f(x)\|^2 \tag{82}$$

we obtain a $(j \cdot n_0)$-dimensional vector space and an induced unitary representation

$$(U_s f)(x) = f(xs) \qquad \forall \quad s \in G. \tag{83}$$

This is the induced representation denoted by U^L.

5. *Induced Representations* (*Continuous Case*)

In this subsection we describe the generalization of this method for constructing representations of G to the case of topological groups. In the following we shall apply this only to Lie groups, but many of the definitions and theorems are applicable for locally compact topological groups.

Let us then assume that G is such a group, and $H \subset G$ is a subgroup. The first difference from the finite case already becomes evident: In the finite case we could admit any subgroup; in the case of topological groups, however, we must add the condition that H is *closed* in G ($\bar{H} = H$). We shall see that for the applications we have in mind this is always the case.

We can now define, in complete analogy to the discrete case, the right cosets, but we cannot expect them to form a finite or even a discrete set. Thus instead of a formula such as (73), which would not be correct for the continuous case, we define the space G_H of the right cosets simply as the equivalence classes of the elements $x \in G$ modulo the subgroup H. Two elements x_1, $x_2 \in G$ are said to be equivalent modulo H if there exists an element $y \in H$ such that $x_1 = yx_2$. We shall denote by ξ the class of equivalent elements Hx that contains the element $x \in G$. The correspondence $\xi = \pi(x)$ is called the canonical mapping of G onto the equivalence classes or right cosets G_H.

The cosets G_H inherit a natural topology from the topological space G: A subset $\Delta \subset G_H$ is open if and only if $\pi^{-1}(\Delta)$ is open in the topology of G. Here $\pi^{-1}(\Delta)$ denotes the set

$$\pi^{-1}(\Delta) = \{x | x \in G, \pi(x) \in \Delta\}. \tag{84}$$

With this topology the mapping $\pi(x)$ from G to G_H is continuous.

We also need a measure on the cosets, since we need to replace the sum of the discrete case by an integral. The ideal type of measure would be invariant under right translations, but such a measure is not always possible. Fortunately the weaker requirement of "quasi-invariance" will be sufficient to construct the induced representation.

A measure μ on G_H is said to be quasi-invariant if the translated measure $\mu([\Delta]x) \equiv \mu_x(\Delta)$ has the same null sets as the measure $\mu(\Delta)$. We can prove that such a measure always exists (*43*) on the groups that interest us. The translated measure μ_x is then absolutely continuous with respect to the original one and we can define the Radon-Nikodym derivative (*44*)

$$\frac{d\mu_x}{d\mu}(\xi) = \rho_x(\xi). \tag{85}$$

The function $\rho_x(\xi)$ is positive and essentially bounded and satisfies in addition the identity

$$\rho_{xy}(\xi) = \rho_x(\xi)\,\rho_y(\xi). \tag{86}$$

We now consider the set of all functions from the topological measure space G_H, the vectors $f(\xi)$ of a fixed Hilbert space $\mathfrak{H}_0$, and a unitary representation L of H, which satisfy the following conditions:

(a) $(f(x), g)$ is a Borel function in x for all $g \in \mathfrak{H}_0$.

(b) For all $x \in G$ and all $h \in H$ we have

$$f(hx) = L_h f(x). \tag{87}$$

(c) $\int_{G_H} \|f(x)\|^2 d\mu(\xi) < \infty.$

The integration in this last expression makes sense because $\|f(x)\|^2$ is, on account of the unitarity of the representation L, only a function of the cosets.

We can define more generally a scalar product

$$(f, g) = \int_{G_H} (f(x), g(x))\, d\mu(\xi) \tag{88}$$

so that the set of functions $f = \{f(x)\}$ becomes a Hilbert space. We now define the *induced representation* U_x^L in this Hilbert space by setting.

$$(U_x^L f)(y) = f(yx)\,[\rho_x(y)]^{1/2}. \tag{89}$$

It can be verified without difficulty that this is a unitary representation in the (generally) infinite-dimensional Hilbert space $\mathfrak{H}$. In the special case where the subgroup H is the identity element this construction still works. For the measure we can choose the right-invariant Haar measure and the representation that we obtain is again called the regular representation.

The usefulness of the induced representation is that with it we can construct the representation of groups from those of certain subgroups. In order to use this method effectively it is necessary to know something more about the properties of the induced representations when H and L are given. It would be particularly useful to know when the representation U^L is irreducible. A great deal of research has been devoted to this problem with only partial results (*42*).

The Poincaré group belongs to a certain class of groups for which there exists a complete and satisfactory theory of the induced representation. The groups of this class are the so-called *semidirect products*, which we shall discuss in the following subsection.

6. *Semidirect Products*

We consider now a special class of groups, the semidirect products. Let $G_1 \subset G$ be an invariant Abelian subgroup of G and G_2 another subgroup of G such that $G_1 \cap G_2 = e$ and such that every element $z \in G$ can be written as a product

$$z = xy \qquad \text{with} \quad x \in G_1, y \in G_2.$$

Because $G_1 \cap G_2 = e$, this product representation is unique.

The semidirect product can also be written as pairs of elements $(x,y) = z$ with the composition law

$$(x_1, y_1)(x_2, y_2) = (x_1 y_1 x_2 y_1^{-1}, y_1 y_2). \tag{90}$$

The transformations $x \rightarrow y[x] = yxy^{-1}$ constitute an automorphism of G_1. The semidirect product can thus also be considered as composed of the pairs of which one element is an element from G_1 and the other is an automorphism of G_1.

Examples of semidirect products are many and important. They may occur in discrete or continuous groups. We shall here mention three.

Example 1. Probably the simplest example of a semidirect product, which may serve to illustrate many of the concepts and theorems, is the group S_3, the permutation group of three objects. It is of order 6. The Abelian invariant subgroup G_1 consists of the two cyclic permutations plus the identity and it is of order 3. The group G_2 of automorphisms of G_1 consists of exactly two elements, the identity and the interchange of the two cyclic permutations. Thus the semidirect product $G_1 \wedge G_2$ consists of six elements. Examination of the group table shows that it is isomorphic to the permutation group of three elements.

Example 2. The group of Euclidean motions in the plane consists of translations and rotations around a fixed point in the plane. The translations are an Abelian subgroup G_1 and the rotations G_2 induce an automorphism in G_1. Thus this group, too, is a semidirect product, since every Euclidean motion may be represented as a rotation followed by a translation.

Example 3. The connected component of the Poincaré group G_2 contains as an invariant Abelian subgroup G_1 the translations in Minkowski space. The Lorentz transformations induce an automorphism in G_1 and every element of G can be represented as a product of an element from G_1 with an element from G_2. The composition law (68) is already in the form that shows that G_2 is indeed a semidirect product.

In the following we shall be concerned primarily with the last example. Consequently we shall adopt from now notation conforming to that introduced earlier (in Section IV, C, 1), according to which the invariant Abelian subgroup G_1 will be the translation group T. An element $a \in T$ is represented by the four-vector a. The group G_2 is to be identified with the group L of the Lorentz transformation Λ.

It is also convenient to use the following notation for the characters and the character group of T. Each character will be represented by a function on the group T of modulus one (instead of the exponents mod 2π, as we did in Section IV, B, 6). A general such function will be denoted by $\hat{a}$ and its dependence on a will be written $\langle a, \hat{a} \rangle$. For the translation group T these characters are represented by four-vectors $\hat{a}$ and the foregoing functions take the form (cf. Section IV, B, 6)

$$\langle a, \hat{a} \rangle = \exp(ia \cdot \hat{a}). \tag{91}$$

This notation is convenient since it emphasizes the complete symmetry between the group T and its character group $\hat{T}$ that permits us to identify the characters of the characters with the elements of T by writing $\hat{\hat{a}} = a$.

The automorphisms of the group T induced by Lorentz transformations Λ may be written $a \rightarrow \Lambda[a] = \Lambda a$. They induce a dual automorphism in the character group $\hat{a} \rightarrow [\hat{a}]\Lambda$ defined by

$$\langle a, [\hat{a}]\Lambda \rangle = \langle \Lambda[a], \hat{a} \rangle. \tag{92}$$

We shall now introduce two concepts that are convenient at this point. Two characters $\hat{a}_1$ and $\hat{a}_2$ that can be transformed into each other by a Lorentz transformation are said to be *equivalent*, and we define the *orbit* of characters by the

Definition. An *orbit* O in the character group is a class of equivalent characters.

We can also introduce the family of all Lorentz transformations that leave a given character $\hat{a}$ invariant. This family is a group attached to the character $\hat{a}$. It is called the little group.

Definition. The *little group* $G_{\hat{a}}$ of the character $\hat{a}$ is the set of all transformations $\Lambda \in L$ that leave $\hat{a}$ invariant: In a formula

$$G_{\hat{a}} = \{\Lambda | \Lambda \in L, [\hat{a}]\Lambda = \hat{a}\}. \tag{93}$$

For the Lorentz group the orbits consist of the family of four-vectors $\hat{a}$ that satisfy a relation $\hat{a}\cdot\hat{a} = m^2 = \text{const}$.

The little group is naturally a subgroup of the Lorentz group L. By combining it with the translation group T we can make it into a subgroup H of the Poincaré group. Thus to every character $\hat{a}$ we associate a subgroup $H = T \wedge G_{\hat{a}}$. This group H is thus the semidirect product of T with the little group $G_{\hat{a}}$. Every irreducible representation L of a group H has the form $L = \hat{a}M$ where $\hat{a}$ is a character and M is an irreducible representation of the little group $G_{\hat{a}}$.

We have now all the concepts needed for the formulation of the fundamental theorem of Mackey and Frobenius

Theorem 17 (Mackey-Frobenius). Let $L = \hat{a}M$ be an irreducible representation of the subgroup $H = T \wedge G_{\hat{a}}$ where $\hat{a}$ is an arbitrary character for T; then the induced representation U^L of G is irreducible. Moreover, every irreducible representation of G can be obtained in this form and two irreducible representations $L = \hat{a}M$ and $L' = \hat{a}' M'$ of G are equivalent if and only if $\hat{a}$ and $\hat{a}'$ are in the same orbit and M is unitarily equivalent to M'.

The proof of this theorem is given by Mackey (*3, 4*); it is rather long and cannot be reproduced here. Its usefulness for us is that it permits a further reduction of the representation problem of the Poincaré group. In fact the representations of this group can now be completely classified by following these six steps (*45*):

(a) Determine all the characters of T.
(b) Find the orbits O in the charactergroup T.
(c) Select a character $\hat{a}$ in each orbit.
(d) Determine the little group $G_{\hat{a}}$.

(e) Determine all irreducible representations M of the little group $G_{\hat{a}}$ and construct with them the irreducible representations $L = \hat{a}M$ of the subgroup $H = T \wedge G_{\hat{a}}$.

(f) Construct the induced representations U^L.

The most difficult part in this program is usually step (e), as will be seen by examining it for the case of the Poincaré group. For the latter (or rather, its covering group) we have the following situation:

(a) The characters $\hat{a}$ are all of the form

$$\langle a, \hat{a} \rangle = \exp(ia \cdot \hat{a}).$$

(b) Each orbit is determined by the value of the invariant product $m^2 = \hat{a} \cdot \hat{a}$. Here Postulate 1 will restrict the values to $m^2 \geqq 0$, and $\hat{a}^0 \geqq 0$. The orbits are thus hyperboloids in the forward light-cone of the Minkowski space.

(c) For $m^2 \neq 0$ we can select the special character $\hat{a} = (1,0,0,0)$ on the unit hyperboloid. For $m^2 = 0$ there are two possibilities: (i) The character $\hat{a} = (1,0,0,1)$; and (ii) the singular case $\hat{a} = (0,0,0,0)$.

(d) The little group associated with the character $\hat{a} = (1,0,0,0)$ is the group $SU(2, \mathfrak{C})$, that for $\hat{a} = (1,0,0,1)$ is isomorph to the group E_2 of Euclidean motions of a plane, that for $a = (0,0,0,0)$ is the covering group of the Lorentz group, that is, $SL(2, \mathfrak{C})$.

(e) The irreducible representations of $SU(2, \mathfrak{C})$ are the well-known finite-dimensional representations of dimension $2s + 1$ with $s = \frac{1}{2}, 1, \frac{3}{2}, \ldots$, etc. The irreducible representations of E_2 are of two kinds; only one kind seems to be of physical interest, and it corresponds to a finite and discrete value of the spin. The case (c), (ii) does not represent a particle since the momentum and energy are identically zero in this case.

(f) For each of the irreducible representations listed under (e) we construct the induced representation according to formula (89). In all of these cases the measure can be so constructed that $\rho_x(y) = 1$. With this step the problem is solved.

V. Conclusion

We recapitulate the essential steps that led us to the physical representations of the Poincaré group in a quaternionic Hilbert space.

We started with the systems of elementary propositions and we have given some reasons why such a system is always an orthocomplemented, complete, weakly modular, and atomic lattice. Such lattices are thus the basic structures of any physical theory that is concerned with measurable physical quantities. The distributive lattices are characteristic for classical mechanics. In such a lattice every proposition is compatible with every other. At the other extreme we have quantum mechanics, where we find that propositions may form coherent components.

Every coherent component can be represented as the lattice of subspaces in a Hilbert space with coefficients from a field. If this field contains the reals as a subfield (as it must if we have continuous measurable quantities), then there are only three possibilities, since mathematics tells us that there are only three such fields possible: the reals, the complex numbers, and the quaternions.

A lattice has a natural symmetry group, the group of automorphisms. Translated into the language of Hilbert spaces these automorphisms become the ray transformations. A symmetry group of a physical system appears thus as a homomorphism of this group into the group of ray transformations. This is called a ray—or projective—representation of the symmetry group. Relativistic invariance is thus introduced by considering the projective representations of the Poincaré group.

There is a connection between the projective representations and the vector representations that may be rather involved in the complex case. In quaternionic Hilbert spaces $\mathfrak{H}_{\mathfrak{Q}}$ this connection is extremely simple, since we can show that every projective representation of every group can be induced by a unitary representation. This happens to be also true in the complex case for the Poincaré group. But only in the quaternionic space is it true for every group.

The next step is the construction of the *unitary* representation of the Poincaré group in quaternionic space $\mathfrak{H}_{\mathfrak{Q}}$. Here we postulated for physical reasons that the momentum operators must behave under Lorentz transformations like a four-vector. This implies the existence of a unitary and anti-Hermitian linear operator $\mathscr{J}$ that must commute with all the unitary operators of the representation. The existence of the operator $\mathscr{J}$ permits, for every pure imaginary quaternion i, the extraction from $\mathfrak{H}_{\mathfrak{Q}}$ of a complex Hilbert space $\mathfrak{H}_{\mathfrak{C}}^{(i)}$.

The study of these Hilbert spaces shows that there exist simple relations between the unitary representations in $\mathfrak{H}_{\mathfrak{C}}^{(i)}$ and in $\mathfrak{H}_{\mathfrak{Q}}$. These relations were described under the heading of *contractions* and *expansions* of representations.

The final result is that the physical representations in $\mathfrak{H}_{\mathfrak{Q}}$ can always be obtained as expansions of complex representations.

There remains thus the construction of all the complex representations of the Poincaré group. This can be accomplished with the help of the theorem of Mackey-Frobenius for semidirect products. The Poincaré group is such a group and the theorem is directly applicable, giving very quickly all the results obtained by Wigner in 1939.

The theorem of Mackey-Frobenius can also be used for the construction of the nonphysical representations, since the validity of the theorem is independent of the nature of the field. For quaternions, however, the irreducible representations for the subgroup $H = T \wedge G_{\dot{a}}$ [step (*e*)] can in general no longer be constructed in the same way, since the one-dimensional characters need not commute with the representations of the little group $G_{\dot{a}}$. Thus we know nothing about the possible representations that do not satisfy the physically motivated Postulates 1 and 2. Since the latter give essentially the same result as in the complex case, we may look for unexpected possibilities only in the as yet unexplored "unphysical" representations.

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