Piecewise linear model for tree maps

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Abstract : We generalize to tree maps the theorems of Parry and Milnor-Thurston about the semi-conjugacy of a continuous piecewise monotone map f to a continuous piecewise linear map with constant slope, equal to the exponential of the entropy of f.

Introduction

Tree maps have recently attracted interest in the field of dynamical systems. In [de Carvalho and Hall, 2000] it was shown that they provide a good model for some large families of 2-dimensional systems. In [Baillif, 1999], kneading coordinates for tree maps were introduced, and used to prove the Milnor-Thurston formula relating the ζ-function and kneading determinant in this setting. Using a different approach, Alves and Sousa-Ramos [Alves and Sousa-Ramos, 2000] obtained a similar result. Other authors dealt with the topological entropy, pattern forcing and analogs of Sharkovskii's Theorem, among other aspects of the dynamics of tree endomorphisms (see, for example, [Alsedà et al., 1999], [Alsedà et al., 1997], [Ye, 2000] and references therein).

This recent abundance of results on the subject raises the following natural question: which parts of the "classical" theory of dynamical systems on the interval can be generalized to tree maps? Parry [Parry, 1966] and Milnor-Thurston [Milnor and Thurston, 1988] proved that a continuous, piecewise monotone map of the interval with positive topological entropy is semi-conjugated to a continuous piecewise affine map with constant slope and with the same topological entropy. We prove here the natural generalization of this theorem for tree maps. As was pointed out to us by the referee, a combinatorial proof using the methods from [Alsedà et al., 1993] should also be possible. We also obtain a result about the meromorphic continuation to the unit disk of the ζ-function of a tree endomorphism.

The proofs consist in adapting the lemmas of [Milnor and Thurston, 1988] to the coordinates given in [Baillif, 1999] and we proceed along the lines of the account given in de Melo and van Strien [de Melo and van Strien, 1993] (chapters 7 and 8).

1-Settings

The setting below is much the same as in [Baillif, 1999], except for some notational variations.

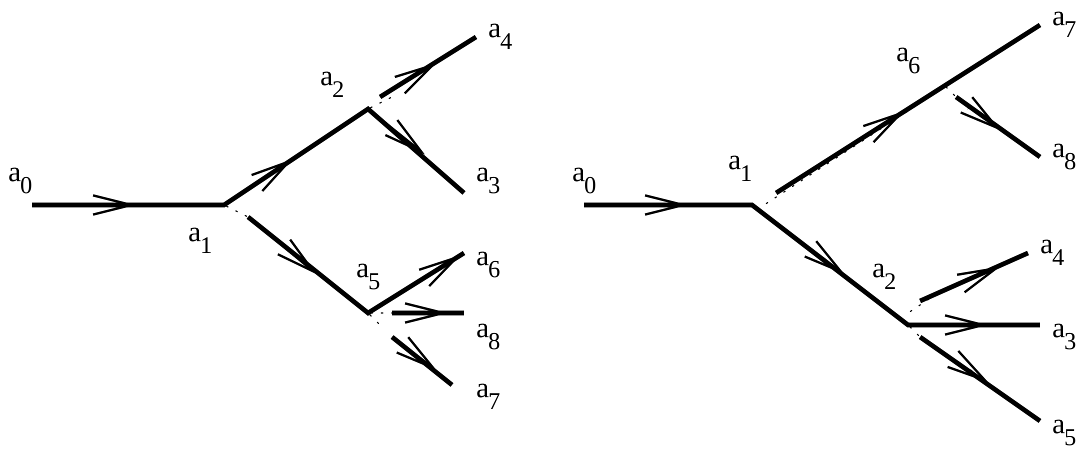

Figure 1: Two possible orders for the same oriented tree.

The tree T

A tree is a connected graph containing no subset homeomorphic to the circle. The tree is finite if it contains only finitely many vertices (and therefore finitely many edges). Let T be a finite tree. It is possible to embed T in the plane, and the topology on T is the induced topology. We denote the set of its edges by E_T, and the set of its vertices by V_T. If $x \in V_T$, the valence of x, denoted by $val(x)$, is the number of connected components of $T \setminus \{x\}$. We allow vertices of valence 2. We partition V_T into two disjoint subsets: the *ends* of T, i.e., the collection of vertices of valence 1 of T, denoted by $Ends(T)$, and the set $V_T^{\geq 2}$, containing vertices of valence ≥ 2. From now on, we use the letter N to denote the number of edges of T, and $M^{\geq 2}$ will be $\# V_T^{\geq 2}$. Note that $N - M^{\geq 2} + 1 = \# Ends(T)$. If $x, y \in T$, we denote by $C(x, y)$ the smallest connected set in T containing x and y.

Let us choose an arbitrary point $a_0 \in Ends(T)$, which we call *the root* of the tree T. Let $E \in E_T$ be an edge, x, y being its endpoints, such that $x \in C(a_0, y)$ (therefore, $y \notin C(a_0, x)$). There is a homeomorphism $\phi : E \to [0, 1]$ so that $\phi(x) = 0$, $\phi(y) = 1$. We orient E so that ϕ preserves the orientation. Let $x, y \in T$. We call $C(x, y)$ a (closed) *oriented interval,* denoted by $[x, y]$ if there is an orientation preserving homeomorphism of $C(x, y)$ to $[0, 1]$. Note that our choice of orientation and the fact that T is a tree implies that the convex hull of any two points in T can be obtained as the union of at most 2 intervals. We also let $(x, y) = [x, y] \setminus \{x, y\}$.

To proceed, we will need to introduce a total order $\prec$ on T (see figure 1). Let us denote the vertices of T by a_0 (the root), $a_1, \cdots, a_N$ ($N = \# V_T - 1$).

We call *branch* of T any subset of T which is homeomorphic to an interval and which contains a_0 and some other end of T. The set of all branches of T is denoted by B_T. Let $B_0 \in B_T$. It is totally ordered from a_0 to some end a_j. The order $\prec$ on B_0 is given by the orientation, that is, $x \prec y$ if $x \in C(a_0, y)$. Moreover, we decide that if x, y are points in T with $x \in B_0$, and $y \notin B_0$, we have $x \prec y$. Let now $B' \in B_T \setminus \{B_0\}$ such that $B_0 \cap B'$ is maximal (for the inclusion) in $B_T \setminus \{B_0\}$ (there could be many such branches). Again, $B' \setminus B_0$ is totally ordered, and we say that $x \prec y$ $(x, y \in B' \setminus B_0)$ if x is in the convex hull in B' of a_0 and y. For $x \in B' \cup B_0 =: B_1$ and $y \notin B_1$, we declare that $x \prec y$. We proceed like this until we have exhausted the entire tree. We can then assume (up to renaming the vertices) that $a_0 \prec a_1 \prec \cdots \prec a_N$.

According to our orientation, we can approach $a_j \in V_T^{\geq 2}$ from the left in one way, and from the right in $val(a_j) - 1$ ways. We will denote this lefthand limit by a_j-, and the righthand limits by a_j+_k $(k = 1, \cdots, val(a_j) - 1)$. To be consistent with the order $\prec$, we decide that $a_j+_k \prec a_j+_\ell \Leftrightarrow k < \ell$.

The abundance of righthand limits forces us to use a notation with two indices for the edges. We denote by $I_{(k,\ell)}$ the edge 'containing' (in the obvious sense) c_k+_ℓ. Note that if $k < k'$, then for all $x \in I_{(k,\ell)}, y \in I_{(k',\ell')}$, we have $x \prec y$. Moreover, if $\ell < \ell'$, for all $x \in I_{(k,\ell)}, y \in I_{(k,\ell')}$, we have $x \prec y$.

The map f

Let $f : T \to T$ be a continuous map. We say that f is piecewise strictly monotone if there is a finite subset $C_f \supset V_T^{\geq 2}$ of T such that for each connected component P of $T \setminus C_f$, $f|_P$ is a homeomorphism onto its image, and $f(P)$ is an oriented interval of T. The set C_f is called *a critical set* of f. Note that if T is homeomorphic to an interval, our definition coincides with the usual definition of a piecewise monotone map. If f is piecewise strictly monotone, note that f^n is also piecewise strictly monotone $\forall n$.
Let f be a continuous piecewise strictly monotone map of T. For convenience, we will suppose that $C_f = V_T^{\geq 2}$ (that is, we turn each critical point of f into a vertex of T, if necessary, and the tree contains no other vertex of valence 2). We denote the members of C_f by $c_1, \cdots, c_{M^{\geq 2}}$.
Let $C(f^n) := \cup_{i=0}^{n} f^{-i}(C_f)$ be the n-th critical set of f, and therefore a critical set for f^n.

Definition: *Let J be a connected subset of T (hence, a subtree of T) and f as above. The n-th lap number of f restricted to J, $\ell(f^n, J)$, is the number of connected components of $J \setminus C(f^n)$. For $\ell(f^n, T)$, which we denote simply by $\ell(f^n)$, an equivalent definition is $\ell(f^n) = \#C(f^n) + \#Ends(J) - 1$. The* *growth number s is given by* $s := \lim_{n\to\infty} (\ell(f^n, T))^{\frac{1}{n}}$.

Note: The limit $\lim_{n\to\infty}(\ell(f^n, T))^{\frac{1}{n}}$ exists since $\ell(f^n) \leq \ell(f^{n-m})^m$ for all n and $m \leq n$.

We also define the formal power series

$$L_f(J)(t) := \sum_{n\geq 0} \ell(f^n, J) \cdot t^n$$

and denote $L_f(T)(t)$ by $L_f(t)$.

The kneading coordinate

Recall that f is monotone on each edge of T. Set $\epsilon_{(k,\ell)}$ to be ± 1 according to whether f is orientation preserving or orientation reversing on $I_{(k,\ell)}$. Define the map $\epsilon : T \to \{0, \pm 1\}$ which takes constant value $\epsilon_{(k,\ell)}$ on the interior of $I_{(k,\ell)}$ and 0 on the vertices of T.
We now define the *address* $\alpha(x)$ of any point $x \in T$. It will be a vector in $\{0, \pm 1\}^{N-1}$ (where, again, $N = \#V_T - 1 = \#E_T$). Each entry of $\alpha(x)$ corresponds to an edge $I_{(k,\ell)}$ (except for the edge containing a_0 which will not be taken into account), and will then be denoted by $\alpha_{(k,\ell)}(x)$. We are aware that this notation can create some confusion since $\alpha(x)$ is a vector and not a matrix, but it is useful to be able to relate an edge (or some limit of some vertex) to its entry in the address. We therefore urge the reader to keep this in mind.
Let c_k be in $V_T^{\geq 2}$. Define $\alpha_{(k,\ell)}(x)$ to be -1 if $x \prec c_k$, $+1$ if there is a $y \neq c_k$, $y \in I_{(k,\ell)}$ with $y \prec x$, and 0 otherwise. Then put the entries of $\alpha(x)$ in the order corresponding to the order on the edges (or equivalently on the righthand limits of the vertices).

We may now define the *kneading coordinate* (see [Milnor and Thurston, 1988], [Baladi and Ruelle, 1994] and [Baillif, 1999]) of any point $x \in T$ as the following vector of power series in t:

$$\theta_f(x)(t) := \sum_{n\geq 1} t^n \cdot \Big(\prod_{i=0}^{n-1} \epsilon(f^i(x))\Big) \cdot \alpha(f^n(x))$$

By definition, $\theta_f(x)(t) \in \mathcal{H}(\mathbb{D})^{N-1}$, where $\mathbb{D}$ is the disc of radius 1 in $\mathbb{C}$, and $\mathcal{H}(\mathbb{D})$ is the space of holomorphic functions in $\mathbb{D}$.
To define the *kneading matrix* $K(t) \in \mathcal{H}(\mathbb{D})^{N-1\times N-1}$, we assign exactly $val(c_k) - 1$ rows of $K(t)$ to each $c_k \in V_T^{\geq 2} = C_f$: for each $1 \leq l < val(c_k) - 1$ set

$$K_{(k,\ell)}(t) := \frac{1}{2}\big(\theta_f(c_k +_\ell)(t) - \theta_f(c_k -)(t)\big).$$

The entry of the matrix corresponding to $I_{(k,\ell)}$ (column) and to $c_{k'}+_{\ell'}$ (row) will then be denoted by $K_{(k',\ell'),(k,\ell)}(t)$. We will call the determinant of $K(t)$ the *kneading determinant* and denote it by $\Delta(t)$. By construction, $\Delta(t) \in \mathcal{H}(\mathbb{D})$.
Note that $K_{(k,\ell)}(0) = \frac{1}{2}(\alpha(c_k+_\ell) - \alpha(c_k-)) = (0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0)$, the number of 1's being exactly the number of vertices of T "between" c_k+_ℓ and c_k- according to $\prec$. Hence, $K(0)$ is upper-triangular with 1's on the diagonal so that $\Delta(0) = 1$ (see [Baillif, 1999] for more details).

2-Relating $\ell(f^n)$ to the kneading matrix

Let assume that s, the growth number of f, is strictly larger than 1. The following lemmas will prove that $L_f(t)$, which is analytic in $|t| < \frac{1}{s}$, can be extended meromorphically to $\mathbb{D}$. They are adaptations of the lemmas and theorems in [de Melo and van Strien, 1993] (pages 171-185).

Recall that $C_f = V_T^2 = \{c_1, \cdots, c_{M^{\geq 2}}\}$ is the critical set of f. Let $J \subset T$, and $c_k \in C_f$. We set

$$\begin{aligned} E^n_{(k,\ell)}(J) \;\; &:= \{x \in \mathrm{Int}(J) | f^n(x) = c_k \text{ and } f^i(x) \notin C_f \text{ for } i = 0, \cdots, n-1, \\ &\quad \text{and for all neighbourhoods } U \text{ of } x, \#(f^n(U) \cap I_{(k,\ell)}) > 1\} \end{aligned}$$

Note that if $c_k \in C_f$, then $c_k \in E^0_{(k,\ell)}$ $\forall \ell = 1, \cdots, val(c_k) - 1$.

Lemma 2.1 *Let $n \geq 1$, $J \subset T$ and $x \in E^n_{(k,\ell)}(J)$. Then:*

$$\frac{1}{2}(\theta_f(x+)(t) - \theta_f(x-))(t) = t^n \cdot K_{(k,\ell)}(t)$$

The proof is a straightforward computation. $\square$

In order to relate $\ell(f^n)$ to the kneading matrix, we introduce the numbers $\Gamma^n_{(k,\ell)}(J) := \#E^n_{(k,\ell)}(J)$ and the power series

$$\Gamma_{(k,\ell)}(J)(t) := \sum_{n \geq 0} \Gamma^n_{(k,\ell)}(J) \cdot t^n$$

Since $\Gamma^n_{(k,\ell)}(J) \leq \ell(f^n)$ for every n, $\Gamma_{(k,\ell)}(J)(t)$ is holomorphic in the disc of radius $1/s$.
For convenience, we will use the simplified notation $\sum_{(k,\ell)}$ instead of $\sum_{k=1}^{M^{\geq 2}} \sum_{\ell=1}^{val(c_k)-1}$ in what follows.

Lemma 2.2 *Let $J=[a,b]$ be an interval of T entirely contained in an edge. For all t in the disc of radius $1/s$, we have*

$$\frac{1}{2}\big(\theta_f(b-)(t)-\theta_f(a+)(t)\big)=\sum_{(k,\ell)}\Gamma_{(k,\ell)}(J)(t)\times K_{(k,\ell)}(t)$$

(The '$\times$' denotes the product between the scalar function $\Gamma_{(k,\ell)}(J)(t)$ and the (k,ℓ)-th row of the kneading matrix $K_{(k,\ell)}(t)$.)
Proof: Let $n\in\mathbb{N}$. Note that $C(f^n)\cap J=\cup_{i\leq n}\cup_{(k,\ell)}E^i_{(k,\ell)}(J)$. If $x\prec y$ are two consecutive points of $C(f^n)\cap J$, then $\forall z\in(x,y)$, $\theta_f(z)=\theta_f(x+)=\theta_f(y-)$ up to the order n. Hence, up to order n, Lemma 2.1 tells us that

$$\begin{aligned}\tfrac{1}{2}\big(\theta_f(b-)(t)-\theta_f(a+)(t)\big) &= \sum_{x\in C(f^n)\cap J}\big(\theta_f(x+)(t)-\theta_f(x-)(t)\big)\\ &=\sum_{i=0}^{n}\sum_{(k,\ell)}\sum_{x\in E^i_{(k,\ell)}(J)}t^iK_{(k,\ell)}(t)\\ &=\sum_{(k,\ell)}\Big(\sum_{i=0}^{n}\Gamma^i_{(k,\ell)}(J)t^i\cdot K_{(k,\ell)}(t)\Big)\end{aligned}$$

Since this is true for all n, the equality holds in the sense of power series, and thus the lemma follows. □

Let now J be a connected subset of T. We have $J=\cup_{I\in E_T}J\cap I=:\cup J_i$ (where the union is taken over a set of indices such that $J_i\neq\emptyset$). Since $\cup\mathrm{Int}(J_i)=\mathrm{Int}(J)\setminus V^2_T$, the difference between $\Gamma^n_{(k,\ell)}(J)$ and the sum of the $\Gamma^n_{(k,\ell)}(J_i)$ is 0 if $n>0$ or if $a_k\notin J$, and is 1 if $n=0$ and $a_k\in J$. Since J is connected, each J_i is connected and is therefore an interval $[a_i,b_i]$ entirely contained in an edge. Hence, we have the following corollary of Lemma 2.2 (this is specific to the tree case):

Corollary 2.3 *Let J be a connected subset of T. Then,*

$$\begin{aligned}\sum_i\frac{1}{2}\big(\theta_f(b_i-)(t)-\theta_f(a_i+)(t)\big) &= \sum_{(k,\ell)}\Gamma_{(k,\ell)}(J)(t)\times K_{(k,\ell)}(t)\\ &\quad+\sum_{c_k\in C_f\cap J}(val(c_k)-1)\end{aligned}$$

□

We can now state the next main result:

Corollary 2.4: *Let J be a connected subset of T. Then, for all k,ℓ, the holomorphic map $\Gamma_{(k,\ell)}(J)(t)$ has a meromorphic extension to $\mathbb{D}$, and its poles can only be at the zeroes of the kneading determinant $\Delta(t)$.*

PROOF: (See Corollary 1 in [de Melo and van Strien, 1993], page 182.) Since $\Delta(0)=1$ and $\Delta(t)$ is holomorphic in $\mathbb{D}$, there is (by Cramer's rule) a matrix $M(t)$, with entries that are meromorphic in $\mathbb{D}$, which is the inverse of $K(t)$. We use the same notation for rows and colums of $M(t)$ that we used for $K(t)$, i.e., we denote by $M_{(k,\ell),(k',\ell')}(t)$ the entry of $M(t)$ corresponding to the (k,ℓ)-th row and the (k',ℓ')-th column.
Looking at the coordinates, Corollary 2.3 gives us that

$$\begin{aligned}&-C(J)+\sum_i \frac{1}{2}\big(\theta_{(r,s)}(b_i-)(t)-\theta_{(r,s)}(a_i+)(t)\big)\\&=\sum_{(k,\ell)}\Gamma_{(k,\ell)}(J)(t)\cdot K_{(k,\ell),(r,s)}(t)\end{aligned}$$

where $C(J)$ is a constant, and $\theta_{(r,s)}$ is the (r,s)-th coordinate of θ_f.
Hence,

$$\begin{aligned}&\Big(-(N-1)C(J)+\sum_{(r,s)}\frac{1}{2}\sum_i\big(\theta_{(r,s)}(b_i-)(t)-\theta_{(r,s)}(a_i+)(t)\big)\Big)\cdot M_{(r,s),(u,v)}(t)\\&=\sum_{(r,s)}\sum_{(k,\ell)}\Gamma_{(k,\ell)}(J)(t)\cdot K_{(k,\ell),(r,s)}(t)\cdot M_{(r,s),(u,v)}(t)\\&=\sum_{(k,\ell)}\Gamma_{(k,\ell)}(J)(t)\sum_{(r,s)}(K_{(k,\ell),(r,s)}(t)\cdot M_{(r,s),(u,v)}(t))\\&=\sum_{(k,\ell)}\Gamma_{(k,\ell)}(J)(t)\delta_{(k,\ell),(u,v)}=\Gamma_{(u,v)}(J)(t)\end{aligned}$$

(just think of $(k,\ell),(r,s),(u,v)$ as being single indices) and $\Gamma_{(u,v)}(J)(t)$ is therefore meromorphic in $\mathbb{D}$. To conclude, it suffices to show that $\Delta(t)\cdot\Gamma_{(u,v)}(J)(t)$ is holomorphic in $\mathbb{D}$. The above lines tell us that $\Delta(t)\cdot\Gamma_{(u,v)}(J)(t)=h(t)\cdot M_{(r,s),(u,v)}(t)\cdot\Delta(t)$, where $h(t)$ is holomorphic in $\mathbb{D}$. Applying Cramer's rule again, $M_{(r,s),(u,v)}=\frac{d(t)}{\Delta(t)}$, where $d(t)$ is the determinant of some submatrix of $K(t)$. Since each entry of $K(t)$ is holomorphic in $\mathbb{D}$, $d(t)$ is holomorphic in $\mathbb{D}$, which implies the holomorphicity of $\Delta(t)\cdot\Gamma_{(u,v)}(J)(t)$. □

Corollary 2.5 *If J is a connected subset of T, then $L_f(J)(t)$ is meromorphic in $\mathbb{D}$, and its poles are contained in the set of the zeroes of $\Delta(t)$. Moreover, $L_f(T)(t)=L_f(t)$ has a pole at $t=1/s$.*

PROOF: Straightforward computations show that

$$L_f(J)(t)=\frac{\#Ends(J)-1}{1-t}+\frac{t}{1-t}\sum_{(k,\ell)}\Gamma_{(k,\ell)}(J)(t)$$

therefore $L_f(J)(t)$ is meromorphic in $\mathbb{D}$, and for $J=T$ has a pole at $t=1/s$ by definition of s. $\square$

3-The semi-conjugacy

All the work done so far aimed to show that $L_f(t)$ is meromorphic in $\mathbb{D}$ and has a pole at $1/s$. We will use this in order to define a probability measure Λ on T which will give the semi-conjugacy between f and a piecewise linear map. If J is a connected subset of T, we set

$$\Lambda(J) := \lim_{t\to 1/s} \frac{L_f(J)(t)}{L_f(t)}$$

The singularity at $1/s$ is removable in this quotient since $0 \leq L_f(J)(t) \leq L_f(t)$ for each $t \in (0,1/s)$. We also have trivially $0 \leq \Lambda(J) \leq 1$.

Lemma 3.1 *Assume that $s>1$.*
1. If $\#(J_1 \cap J_2) < \infty$ then $\Lambda(J_1 \cup J_2) = \Lambda(J_1) + \Lambda(J_2)$.
2. The number $\Lambda(J)$ depends continuously on the endpoints of J.
3. If f is monotone on J and $f(J) \cap V_T^{\geq 2} = \emptyset$, then $\Lambda(f(J)) = s\Lambda(J)$.

PROOF: See [de Melo and van Strien, 1993], page 184, Lemma 8.5. The difference between a tree and an interval plays no role in this proof. $\square$

Let us now construct the semi-conjugacy. For this, we will build a new tree T' which has at most as many edges as T (see [de Melo and van Strien, 1993] for an example where T' has strictly fewer edges than T in the case where T and T' are intervals). We recall that B_T is the set of the branches of T, say $B_T = \{B_1, \cdots, B_l\}$. For $i = 1, \cdots, l$, we put $B_i' := [0, \Lambda(B_i)]$. We then define $\lambda_i : B_i \to B_i'$ to be $x \mapsto \Lambda([a_0, x])$ (since B_i is a branch, $C(a_0, x)$ is an interval for all $x \in B_i$). By Lemma 3.1, λ_i is continuous, (not necessarily strictly) monotone and surjective.
Let now $T' := \sqcup B_i' / \sim$, where $\sim$ is the following equivalence relation:

$$x' \sim y' \leftrightarrow \begin{cases} x' = y' \\ \text{or} \\ \exists x \in T, \exists i,j \text{ s.t. } x' = \lambda_i(x), y' = \lambda_j(x) \end{cases}$$

and we define $\lambda : T \to T'$ to be the induced map. (We just glued together the images under different λ_i's.) It is also possible to consider T as the quotient of $\sqcup_i B_i$ by the obvious equivalence relation. We denote by π, π' the projections of $\sqcup B_i, \sqcup B_i'$ on T, T' (see Figure 2).

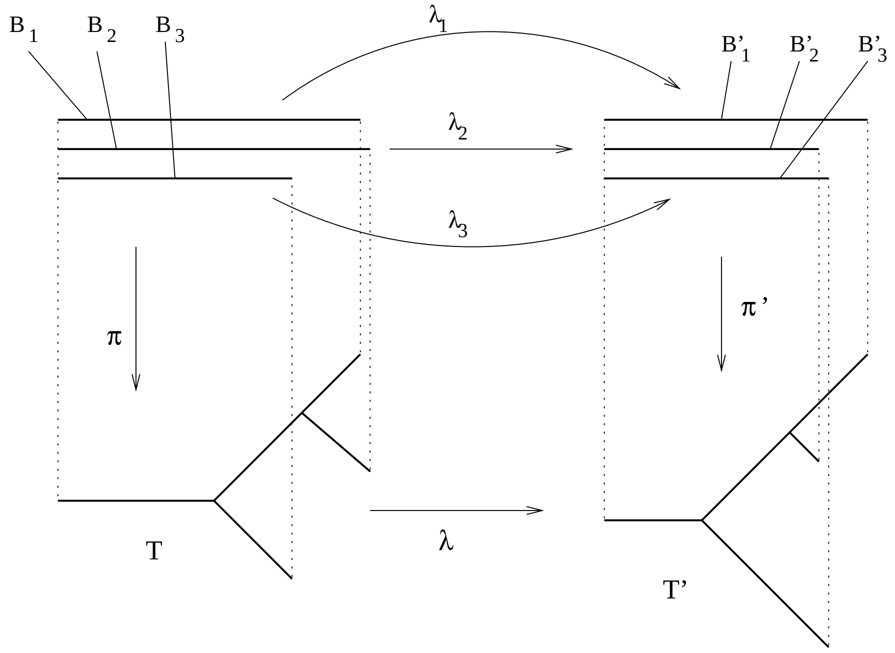

Figure 2: The conjugacy.

Claim 1: If $\lambda(x) = \lambda(y)$, then $\lambda(f(x)) = \lambda(f(y))$

Proof of Claim 1: Suppose first that x and y belong to the same branch B. We can subdivide $[x, y]$ in $J_1, \cdots J_s$ such that $f|_{J_i}$ is monotone and $f(J_i)$ is contained into an edge, and such that $\#(J_i \cap J_j) \leq 1$. By Lemma 3.1, and since $\Lambda([c_0, x]) = \Lambda([c_0, y])$, we have that $0 = \Lambda([x, y]) = \sum_i \Lambda(J_i) = \frac{1}{s}\sum_i \Lambda(f(J_i))$. Hence, $\Lambda(f(J_i)) = 0$, and since $f(J_i)$ is contained in a branch B_j, then $\lambda = \lambda_j$ collapses $f(J_i)$ to a point of T'.
By continuity, $f([x, y]) \subset \cup_i(f(J_i))$ is connected, and then λ squashes $f([x, y])$ into a point. Hence $\lambda(f(x)) = \lambda(f(y))$.
Now, suppose that x and y belong to different branches. Then, there is a unique vertex $a \in V_T^{\geq 2}$ such that $C(x, y) = [a, x] \cup [a, y]$. Since $\lambda(x) = \lambda(y)$, we have that $\lambda(a) = \lambda(x) = \lambda(y)$ (otherwise $\lambda(x)$ and $\lambda(y)$ would belong to different edges of T' by definition of T'). Hence, $\lambda(f(a)) = \lambda(f(x))$ and $\lambda(f(a)) = \lambda(f(y))$, and Claim 1 is proven.

By Claim 1, we may now define

$$F : T' \to T'$$
$$F(x') = \lambda(f(\lambda^{-1}(x'))$$

Thus, F is continuous, and $F(\lambda(x)) = \lambda(f(x))$.

Claim 2: F is piecewise linear with slope $\pm s$.

Proof of Claim 2: We show that F is piecewise linear as a map from the branch that contains $\lambda(x)$ to the branch that contains $\lambda(f(x))$ for all $x \in T$. Let $J = [a, b] \subset T$ be such that $f|_J$ is monotone and preserves the orientation. Let us say that $J \subset B_i$ and $f(J) \subset B_j$. For all $x \in J$ we have (seeing B_j as $[0, \Lambda(B_j)]$):

$$\begin{aligned}
F(\lambda(x)) &= \lambda(f(x)) \\
&= \lambda_j(f(x)) \\
&= \Lambda([c_0, f(x)]) \\
&= \Lambda([c_0, f(a)]) + \Lambda([f(a), f(x)]) \\
&= \Lambda([c_0, f(a)]) + \Lambda(f[a, x]) \\
&= \Lambda([c_0, f(a)]) + s \cdot \Lambda([a, x]) \\
&= \underbrace{\Lambda([c_0, f(a)]) - s \cdot \Lambda([c_0, a])}_{K} + s \cdot \underbrace{\Lambda([c_0, x])}_{\lambda_i(x)} \\
&= K + s \cdot \lambda(x)
\end{aligned}$$

The other case (f reverses the orientation on J) being handled similarly, claim 2 is therefore proved.

4-Results about the entropy

The following two results do not deserve a proof here, since the ones provided in [de Melo and van Strien, 1993] could be very easily adapted for tree maps. The first is due to Misiurewicz and Szlenk, but the proof given in [de Melo and van Strien, 1993] is due to L.S. Young. It uses symbolic dynamics, which can be introduced in our settings using the partition given by the critical set C_f (Lemma 7.3 and Theorem 7.2, pages 168-170 in [de Melo and van Strien, 1993]). The second uses the linearity of f and the fact that $[0, 1]$ is of length 1 (Corollary, page 171 in [de Melo and van Strien, 1993]).

Lemma 4.1: *Let $f : T \to T$ be a continuous piecewise monotone tree map and s its growth number. Then, the topological entropy of f is equal to $log(s)$.*

Lemma 4.2: *Let $f : T \to T$ be a continuous piecewise linear map with slope $\pm\sigma$, and T be of total length 1. Then, the topological entropy of f is equal to $max(0, log(\sigma))$.*

Combining the definition of the reduced zeta function $\zeta_f^R(t)$ and the results given in [Baillif, 1999] with Corollary 2.5, Section 3 and Lemmas 4.1 and 4.2, we find :

Theorem 4.3: *Let $f : T \to T$ be a continuous, piecewise strictly monotone tree map with positive entropy $\sigma = log(s) > 0$. Then, there is a semi-conjugacy λ between f and a continuous piecewise affine tree map $F : T' \to T'$ with slope $\pm s$. The semi-conjugacy λ is the composition of a homeomorphism with a map which collapses at most countably many subintervals of T. Moreover, the reduced zeta function $\zeta_f^R(t)$ is meromorphic in the unit disc and its first pole occurs at $t = 1/s$.*

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