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Full Network Nonlocality

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Networks have advanced the study of nonlocality beyond Bell's theorem. Here, we introduce the concept of full network nonlocality, which describes correlations that necessitate all links in a network to distribute nonlocal resources. Showcasing that this notion is stronger than standard network nonlocality, we prove that the most well-known network Bell test does not witness full network nonlocality. In contrast, we demonstrate that its generalization to star networks is capable of detecting full network nonlocality in quantum theory. More generally, we point out that established methods for analyzing local and theory-independent correlations in networks can be combined in order to systematically deduce sufficient conditions for full network nonlocality in any network and input-output scenario. We demonstrate the usefulness of these methods by constructing polynomial witnesses of full network nonlocality for the bilocal scenario. Then, we show that these inequalities can be violated via quantum elegant joint measurements.

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A network connects a number of parties using some configuration of independent sources. They are common in standard information technology (e.g., the internet) and their quantum counterparts have been developing rapidly in recent years [1–3]. In a quantum network, a source may distribute entanglement to a set of parties, who can then use entangled measurements to further propagate it along the network [4]. In the last decade, much research in quantum information has been dedicated to understanding correlations in networks (see Ref. [5] for a review).

Let a network be composed of m sources and n parties, each of whom selects a private input x_k and produces an output a_k (see, e.g., Fig. 1). The resulting correlations are said to admit a network local model if they can be understood by each source independently emitting a local variable λ_j:

$$p(\bar{a}|\bar{x}) = \int d\lambda_1 \mu_1(\lambda_1)\ldots \int d\lambda_m \mu_m(\lambda_m) \times p(a_1|x_1, \bar{\lambda}_1)\ldots p(a_n|x_n, \bar{\lambda}_n), \qquad (1)$$

where $\bar{a} = (a_1, \ldots, a_n)$, $\bar{x} = (x_1, \ldots, x_n)$, $\mu_1, \ldots, \mu_m$ are probability density functions, and $\bar{\lambda}_k$ is the set of local variables associated to the sources that connect to party k. This definition reduces to Bell's notion of local causality [6,7] when the network is trivial, i.e., when it only has a single source connecting all parties. If a correlation $p(\bar{a}|\bar{x})$ does not admit a model of the form (1), it is called network nonlocal (NN).

While it is generally believed that networks enable new forms of nonlocality as compared to standard Bell scenarios (see, e.g., [8–11]), it remains largely unclear how such phenomena arise and to what extent they are intrinsic to the network structure. This is partly due to properties inherent to the definition (1). For instance, if just two parties in a

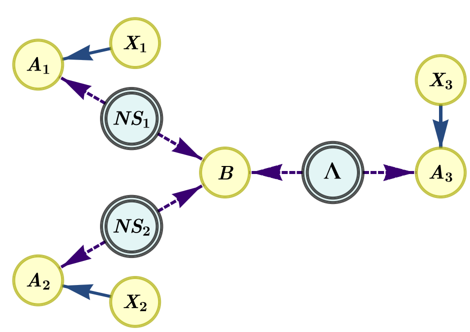

FIG. 1. Star network with three branches. Each branch connects a party (A_1, A_2, A_3) to the central party (B) via an independent source. All parties perform local measurements, for which the branch parties receive inputs (X_1, X_2, X_3). The outcome statistics is said to be fully network nonlocal if it cannot be explained in any model of the network where one of the three sources is represented by a classical random variable Λ and the other two by general no-signaling resources (NS).

large network violate the celebrated Clauser-Horne-Shimony-Holt (CHSH) Bell inequality [12], the generated distribution is NN regardless of the (perhaps quite trivial) correlations in the rest of the network. Such network nonlocality is arguably neither conceptually novel nor truly a network phenomenon. This motivates the need for conceptualizing stronger notions of nonlocality in networks.

Here, we introduce the concept of full network nonlocality. It is defined as follows.

In a given network and input/output scenario, $p(\bar{a}|\bar{x})$ is fully NN if and only if it cannot be modeled by allowing at least one source in the network to be of a local-variable nature, while all other sources may be general independent nonlocal resources.

We focus here on a literal definition due to the known difficulty of insightfully describing correlations produced by general nonlocal resources in networks [13,14], and we provide concrete mathematical formulations in specific scenarios throughout the work.

Put in other words, correlations $p(\bar{a}|\bar{x})$ that require all sources to be nonlocal resources are said to be fully NN. Such sources may emit entangled quantum states, but can also correspond to more general nonlocal resources [15]. Hence, full NN is defined independently of quantum theory. In the spirit of many works in the device-independent paradigm (see, e.g., [16,17]), this implies that the interest and relevance of quantum demonstrations of full NN do not hinge on the assumption that nature supports no stronger correlations than those of quantum theory. Full NN is clearly a notion strictly stronger than standard NN. Furthermore, full NN more faithfully adopts the perspective that nonlocality is a property relative to a network: correlations $p(\bar{a}|\bar{x})$ that are fully NN in a given network are not guaranteed to be if the network is expanded. This is not the case for standard NN, which is a property that is preserved when the network is enlarged.

This simple concept motivates several elementary questions. How, and to what extent, is full NN different from standard NN? Can we reexamine established network Bell experiments through the lens of full NN? How can we detect full NN, and is a general characterization possible? Does quantum theory allow for fully NN correlations? If yes, what are the experimental implications? In the following, we address these questions.

We begin by showing that full NN has major implications for the most well-known network Bell test. Consider the simplest network, known as the bilocal scenario, which features three parties. Alice and Bob are connected via a source and Bob and Charlie via another independent source (see Fig. 2). Alice and Charlie each have binary inputs, $x, z \in \{0,1\}$, and produce binary outcomes, $a, c \in \{0,1\}$, while Bob has a fixed measurement with four possible outcomes, $b \equiv (b_0, b_1) \in \{0,1\}^2$. In the bilocal scenario, models of the form (1) respect the network Bell inequality [18]

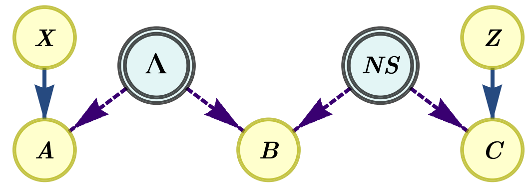

FIG. 2. Bilocal scenario in which Alice and Bob share a random variable (Λ) and Bob and Charlie share a general nonlocal resource (NS). Even though there is only one nonlocal resource in the network, correlations in this scenario [see Eq. (3)] can simulate every possible violation of the standard bilocal Bell inequality.

$$\mathcal{S}_2 := \sqrt{|I_0|} + \sqrt{|I_1|} \leq 1, \tag{2}$$

where $I_t := \frac{1}{4}\sum_{a,b,c,x,z}(-1)^{a+b_t+c+t(x+z)} p(a,b,c|x,z)$ for $t \in \{0,1\}$. Thus, a violation of Eq. (2) implies NN. The largest known quantum violation is $\mathcal{S}_2 = \sqrt{2}$ and it is obtained by each source emitting the singlet state $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$, Bob projecting onto the Bell basis and Alice and Charlie measuring suitable anticommuting observables [18]. Equation (2) has inspired many other network Bell inequalities (see, e.g., [19–24]), and has been (sometimes up to small modifications) the focus of several experiments [25–27].

We now show that it is possible to reproduce the quantum violation $\mathcal{S}_2 = \sqrt{2}$ using only a single nonlocal source. Consider that Bob and Charlie share a Popescu-Rohrlich (PR) box [28]. Let Alice and Bob share a binary local variable, $\lambda \in \{0,1\}$ with $p(\lambda) = \frac{1}{2}$, and have Alice determine her output as $a = x\lambda$. Bob uses λ as his input for the PR box. Thus $c \oplus b' = z\lambda$, where $b' \in \{0,1\}$ is the output of Bob's part of the PR box. Finally, Bob chooses his output to be $b_0 = b_1 = b'$. It is easily verified that, for this strategy, $I_0 = I_1 = \frac{1}{2}$ and consequently that $\mathcal{S}_2 = \sqrt{2}$.

Thus, every known quantum violation of the inequality (2) can be simulated in a bilocal network with one local-variable source, if the other source is a general nonlocal resource. However, correlations arising in this scenario from more general probabilistic theories may still be fully NN. How can we characterize the most general nonfull NN correlations in the space of (I_0, I_1)? Reference [18] showed that standard no-signaling correlations (without requiring independent sources) satisfy $|I_0| + |I_1| \leq 1$. By varying $p(\lambda)$ and considering outputs flipping in our strategy, we can generate every pair (I_0, I_1) that satisfies this inequality. Thus, the inequality serves as a tight constraint on nonfull NN correlations.

However, although the projection of the correlations in the (I_0, I_1) plane is a convex set, the set of nonfull NN correlations is in general nonconvex. To see this, we can formalize the concept by defining nonfull NN correlations as those admitting the model in Fig. 2, namely,

$$p(a,b,c|x,z) = \int d\lambda q(\lambda) p(a|x,\lambda) p(b,c|\lambda,z), \quad (3)$$

where $q(\lambda)$ is a probability density, $p(a|x,\lambda)$ is a conditional probability distribution, and $p(b,c|\lambda,z)$ is a bipartite no-signaling distribution, i.e., it satisfies $\sum_b p(b,c|\lambda,z) = p(c|z)$ and $\sum_c p(b,c|\lambda,z) = p(b|\lambda)$. In general, to establish full NN, one needs also to consider the scenario when the sources are interchanged, since full NN requires that no simulation of the correlations is possible with a local-variable source anywhere in the network. Equation (3) is straightforwardly extended also to other networks [see, e.g., Eq. (S1) in the Supplemental Material [29] for a model for the network in Fig. 1]. Nonconvexity follows from the independence of the sources. A simple example is that $p_k(a,b,c) = \delta_{a,k}\delta_{b,k}\delta_{c,k}$ for $k \in \{0,1\}$ are both compatible with Eq. (3) but their uniform mixture is fully NN.

Naturally, however, the pivotal question is whether there exist full NN witnesses that can be violated in quantum theory. Here we consider two approaches: one based on entirely different quantum protocols in the bilocal scenario, and another that extends the ideas behind the inequality (2) to larger networks. We begin with the latter, considering star-shaped networks (see Fig. 1). Star networks are composed of n branch parties, which we consider to have binary inputs $(x_1,\ldots,x_n)$ and binary outputs $(a_1,\ldots,a_n)$. The branch parties are individually linked to a single party who performs a fixed measurement with an outcome represented by an n-bit string $b \equiv (b_1,\ldots,b_n) \in \{0,1\}^n$. Generalizing the inequality (2), every model of the form (1) respects the following network Bell inequality [31]:

$$\mathcal{S}_n \coloneqq \frac{1}{2^{n-2}} \sum_{t=1}^{2^{n-1}} |I_t|^{1/n} \leq 1, \quad (4)$$

where $I_t \coloneqq (1/2^n)\sum_{\bar{a},\bar{x},b}(-1)^{a_1+\cdots+a_n+\tilde{t}\cdot b+\hat{t}\cdot\bar{x}} p(\bar{a},b|\bar{x})$, the n-bit strings $\tilde{t}$ and $\hat{t}$ are $\tilde{t} = (1,t_1,t_2,\ldots)$ and $\hat{t} = (\bigoplus_{k=1}^{n-1} t_k, t_1, t_2, \ldots)$, and $(t_1,t_2,\ldots)$ is the string of $n-1$ bits representing t. The best known quantum protocol achieves $\mathcal{S}_n = \sqrt{2}$ by having each source emitting the singlet state $|\psi^-\rangle$, the central party performing a joint n-qubit entanglement swapping measurement, and letting the branch parties measure suitable anticommuting observables [31]. The inequality (or modifications of it) has been violated in experiments using complete entanglement swapping [32] and separable measurements [33].

We focus on $n=3$. In the Supplemental Material [29] we show that $|I_1|+|I_2|+|I_3|+|I_4| \leq 1$ is a tight constraint satisfied by all nonfull NN correlations. This implies the full NN witness:

$$\mathcal{S}_3 \leq 2^{1/3}, \quad (5)$$

meaning that a violation implies full NN. Importantly, it follows that quantum correlations reaching $\mathcal{S}_3 \in (1, 2^{1/3}]$ are certified to be NN but not fully NN, whereas quantum correlations reaching $\mathcal{S}_3 \in (2^{1/3}, \sqrt{2}]$ are both NN and fully NN. Because of the sizable gap between the bound on the right-hand side of Eq. (5) and the largest known quantum value of $\mathcal{S}_3$, a reasonable degree of noise can also be tolerated in the quantum realization. If each source emits a Werner state [34] $\rho_v = v\psi^- + [(1-v)/4]\mathbb{1}$, where $\psi^- = |\psi^-\rangle\langle\psi^-|$ and $v \in [0,1]$ is the visibility, a quantum demonstration of full NN is possible whenever $v > 2^{-1/6} \approx 89.1\%$.

Furthermore, star networks allow us to explore the relationship between standard NN and full NN in a scalable manner, i.e., by considering arbitrarily large values of n. It turns out that the network Bell inequalities (4) reveal differences between the two concepts in a rather extreme manner. To showcase this, imagine that in spite of having access to n sources, only one of them is of a nonlocal nature and the remaining $n-1$ sources correspond to local variables. This is sufficient to violate Eq. (4) for any n. Let each of the parties $k = 2,\ldots,n$ be connected to the central party via a local variable $\lambda_k \in \{0,1\}$ [with $p(\lambda_k) = \frac{1}{2}$] and locally assign their output $a_k = x_k\lambda_k$. The central party uses $\lambda_2 \oplus \ldots \oplus \lambda_n$ as a binary input for a PR box shared with the remaining branch party; thus yielding $a_1 \oplus b' = x_1(\lambda_2 \oplus \ldots \oplus \lambda_n)$. The output b' is then used to choose the final output of the central party as $b_1 = b'$ and $b_2 = \ldots = b_n = 0$. This gives $I_t = (1/2^{n-1})$ and $\mathcal{S}_n = 2^{1/n}$, giving a violation of Eq. (4) for every n. Notably, for $n=3$ it saturates the bound in Eq. (5). We note this may be surprising since only one nonlocal source is used, instead of two. However, based on some exploration of $\mathcal{S}_4$, we do not believe that correlations outperforming $\mathcal{S}_n = 2^{1/n}$ imply full NN when $n > 3$. Notably, extending the derivation in the Supplemental Material [29] to the case of $n=4$ gives the (perhaps not tight) full NN witness $\mathcal{S}_4 \leq \sqrt{2}$, which matches the best known quantum violation.

The fact that quantum full NN is possible through a violation of Eq. (5) in the three-branch star network motivates us to ask whether it can arise already in the simpler bilocal scenario. Because of our previous discussion, a positive answer would need an approach that is different from the one based on Eq. (2). Therefore, we consider the quantum correlations reported in Ref. [35]. These correlations arise in the bilocal scenario when both sources emit singlet states, Alice and Charlie each perform the three Pauli measurements, and Bob performs an elegant joint measurement. The latter is a projection of two qubits in a partially entangled basis with the defining property that the marginal states for each qubit forms a regular tetrahedron inside the Bloch sphere [35]. The tetrahedron radius corresponds with a parameter $\theta \in [0, \pi/2]$. Notably, $\theta = 0$ corresponds to the original elegant joint measurement introduced in Ref. [36] and $\theta = \pi/2$ corresponds to the Bell state measurement. In the Supplemental Material [29] we detail this family and the corresponding quantum correlations, which we denote $p_\theta(a,b,c|x,z)$.

In order to investigate whether $p_\theta(a,b,c|x,z)$ is fully NN for some θ, we adopt a method based on inflation of networks [37]. Inflation is a technique by which the sources and measurement devices of a network are copied several times and arranged in different configurations. By studying the correlations that arise in these configurations, one can impose nontrivial constraints on the correlations that can be generated in the original network. Note that inflation is only a theoretical tool and does not mean that the actual network of interest is changed. By considering increasingly large inflations, one obtains increasingly strong conditions for the correlations of interest. Concretely, inflation-based methods have been developed to characterize correlations in networks where all sources are local [37], quantum [38], and general nonlocal resources [37,39]. By combining the ideas from the first and last, one obtains a computationally viable and generally applicable tool to analyze correlations in networks that are composed of both local and general nonlocal resources.

We consider the inflation of the bilocal network where the source between Alice and Bob is local and the source between Bob and Charlie is nonlocal. Our inflation is illustrated in Fig. S1 in [29]: following inflations for general nonlocal resources [37,39] we have duplicated Bob and Charlie and the source connecting them, and following local-variable inflations [37] we have copied the local variable originally shared between Alice and Bob. Via standard procedure [37,39], sufficient conditions for the incompatibility of p_θ with the hypothesized model in the bilocal network are obtained through a linear program (see the Supplemental Material [29] and the computational appendix [40]). In this manner, we find that, with the exception of the extreme points $\theta=0$ and $\theta=\pi/2$, the family of correlations p_θ is fully NN. We note that our inability to detect full NN for $\theta\in\{0,\pi/2\}$ is due to the correlations p_θ actually not being fully NN. In [29] we present explicit simulation models for both p_0 and $p_{\pi/2}$. That only the less symmetric distributions (in terms of θ) are fully NN is reminiscent of the fact that less entangled bipartite states are harder to simulate, e.g., with limited communication, with one PR box, exploiting the detection loophole or via the EPR2 decomposition [41,42].

By considering the duality theory of linear programming (see [29]), we can convert our proofs of full NN for p_θ into witnesses of full NN that apply not only to p_θ but to general distributions in the given network and input/output scenario. Although we have obtained many different witnesses in this way, we highlight one that is particularly elegant. A simultaneous violation of both the following inequalities implies full NN in the bilocal scenario where Alice and Charlie perform three dichotomic measurements and Bob performs a single measurement:

$$-\langle A_1B_2C_3\rangle-\langle A_2B_2\rangle+\langle C_3\rangle[\langle A_1B_2\rangle+\langle A_2B_2C_3\rangle+\langle C_3\rangle]\leq 1 \tag{6}$$

and

$$-\langle A_1B_2C_3\rangle+\langle B_2C_2\rangle+\langle A_1\rangle[\langle B_2C_3\rangle-\langle A_1B_2C_2\rangle+\langle A_1\rangle]\leq 1, \tag{7}$$

see the computational appendix [40] for their derivation. Generally, one may expect one network Bell inequality per arrangement of the local-variable source. A violation of Eq. (6) implies that the correlation does not admit a model of the type (3), and a simultaneous violation of Eq. (7) implies that the correlation neither can be generated if the source between Alice and Bob is nonlocal and the source between Bob and Charlie is local. The three-party expectation values are defined as $\langle A_xB_yC_z\rangle=\sum_{a,b,c}ab_y c\,p(a,b,c|x,z)$, where $a,c\in\{\pm1\}$ and b_y is the yth element in the bit string $(b_1,b_2,b_3)\in\{\pm1\}^3$ that satisfies $b_1b_2b_3=1$ (this is a handy way to represent Bob's outputs). The two-party and one-party expectation values are defined analogously. However, notice that the inequalities effectively only involve two outcomes on Bob.

We exemplify the use of the inequalities (6) and (7) by detecting full NN in the quantum correlations based on elegant joint measurements. To also take noise into the analysis, we consider that the quantum protocol is performed with each source emitting a Werner state ρ_v. For simplicity, we let both Werner states have the same visibility. A direct calculation [29] gives that both the left-hand sides in Eqs. (6) and (7) are equal to $\frac{1}{2}v(v+v\sin\theta+\cos\theta)$. The critical visibility per source required for a violation of Eqs. (6) and (7) is, thus,

$$v_{\text{crit}}=\frac{4}{\cos\theta+\sqrt{8+8\sin\theta+\cos^2\theta}}. \tag{8}$$

We have $v_{\text{crit}}<1$ for every $\theta\in(0,\pi/2)$. Moreover, the best tolerance to noise is obtained at $\theta=\arccos(\sqrt{5}/3)\approx 0.7297$ for which it becomes $v_{\text{crit}}=(2/\sqrt{5})\approx 0.8944$. Conversely, if we are given sources of visibility v, the best choice of measurement is given by $\theta=\arctan(v)$. We have also conducted a numerical search for a simultaneous quantum violation of the inequalities (6) and (7) and found no improvement on the presented protocol. Even though the critical visibility here is slightly larger than that required for violating (5), it is important to note that this scenario is considerably simpler: it requires fewer copies, fewer parties, and a simpler joint measurement. Also, it highlights the relevance of more general entangled measurements in network nonlocality. Finally, note that the best known visibility for standard NN in the bilocal scenario is $v=(1/\sqrt{2})$ [18].

We now discuss some practical and conceptual aspects of full NN, as well as future challenges. An interesting future task is to experimentally demonstrate full NN. To this end, one may consider the violation of one of our above discussed full NN witnesses. In fact, the elegant joint measurement has been implemented in both optics and superconducting circuits [32,43]. Nevertheless, our witnesses are tailored to appealing theoretical properties, rather than experimental friendliness. Therefore, we present in the Supplemental Material [29] a quantum protocol for full NN in the bilocal scenario that can be implemented in photonic systems using only linear optics and no auxiliary photons. Since our protocol requires only pairs of entangled qubits, a partial Bell state measurement and a visibility per source of $v \gtrsim 0.92$, we believe a proof-of-principle violation is well within reach of present technology. We note that previous entanglement-swapping-based demonstrations of NN do not constitute demonstrations of full NN; either due to the fundamental reasons already discussed or due to insufficient magnitudes of violations (see, e.g., [32]). We mention also that the traditional entanglement swapping protocol [4] combined with an event-ready violation of the CHSH inequality [44] (which significantly predate the modern topic of Bell nonlocality in networks) may be interpreted as a proof of both standard NN and full NN in the bilocal scenario. However, alike most other works on this topic, our interest is to consider qualitatively different quantum protocols. Nonetheless, it remains an interesting open problem to pinpoint a natural notion of nonlocality in networks that discards event-ready protocols.

An interesting extension of our work is to consider full nonlocality in quantum networks. This amounts to assuming that all sources in a network are described by quantum theory and attempt to identify correlations that necessitate entanglement in all sources. Although this notion is arguably less fundamental, it may be a relevant consideration for quantum information technologies. In analogy with our approach here, we expect that such correlations can be characterized by combining the inflation techniques reported in Refs. [37,38].

Full network nonlocality is rightfully viewed as a first step in a quest to identify notions of network nonlocality that are more "genuine" than the standard definition (1). It is not unreasonable to suspect that many conceptually different forms of network nonlocality are yet to be identified, with features desirable in different situations. For instance, full NN is not stable under composition, so distributions that are not fully NN can become so by grouping parties. In this aspect, full NN is not different from genuinely multipartite nonlocality [7] or genuinely multipartite entanglement [45,46]. A particularly interesting next step would be to consider correlations that not only necessitate all sources to be nonlocal, but also assert the role of entanglement swapping on the level of the correlations.

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