On Infinite Groups and their Actions: From Group Actions on Graphs to Group Actions on Complexes

Schneeberger, Grégoire

This publication URL: https://archive-ouverte.unige.ch/unige:166747
Publication DOI: 10.13097/archive-ouverte/unige:166747

UNIVERSITÉ DE GENÈVE
Section de Mathématiques

FACULTÉ DES SCIENCES
Professeure Tatiana Smirnova-Nagnibeda

On Infinite Groups and Their Actions: From Group Actions On Graphs To Group Actions On Complexes

THÈSE

Présentée à la Faculté des Sciences de l'Université de Genève
pour obtenir le grade de Docteur ès Sciences, mention Mathématiques

par

Grégoire Jean-Marc SCHNEEBERGER
de
Bleienbach (BE)

Thèse N°5674

GENÈVE
Atelier d'impression ReproMail
2023

UNIVERSITÉ DE GENÈVE

FACULTÉ DES SCIENCES

DOCTORAT ÈS SCIENCES, MENTION MATHÉMATIQUES

Thèse de Monsieur Grégoire Jean-Marc SCHNEEBERGER

intitulée :

«On Infinite Groups and Their Actions: From Group Actions On Graphs To Group Actions On Complexes»

La Faculté des sciences, sur le préavis de Madame T. SMIRNOVA-NAGNIBEDA, professeure associée et directrice de thèse (Section de mathématiques), Monsieur A. KARLSSON, professeur associé (Section de mathématiques), Monsieur A. VALETTE, professeur (Institut de mathématiques, Université de Neuchâtel, Neuchâtel), autorise l'impression de la présente thèse, sans exprimer d'opinion sur les propositions qui y sont énoncées.

Genève, le 1er septembre 2022

Thèse - 5674 -

Le Doyen

N.B. - La thèse doit porter la déclaration précédente et remplir les conditions énumérées dans les "Informations relatives aux thèses de doctorat à l'Université de Genève".

There is a certain road in life most people walk on, because it is familiar, and they can jostle to get in front place. I prefer to take a different road that is less crowded, with many forks, where you get a wider view of life. I call it "the road less travelled". That is where I want to be.

Wayne Shorter

Abstract

Understanding how a group can act on a given type of space can be a valuable tool for proving properties of both the group and the space. This thesis focuses on three distinct topics involving actions of infinite groups on graphs, cube complexes and metric spaces.

In the first part, we answer a question by Grigorchuk asking whether it is possible to give an explicit and elementary description of a CAT(0) cube complex on which the groups he defined act and, if so, to describe these actions. Recall that the Grigorchuk groups are subgroups of finite types of the group of automorphisms of the rooted binary tree. This family of groups has been a prolific subject of study and has provided answers to many problems. The main ingredient allowing us to build such a complex is the existence of a Schreier graph with two ends which come from the action of these groups on the boundary of the binary tree. Using this graph, it is possible to construct a CAT(0) cube complex on which all Grigorchuk groups act without bounded orbit, or, equivalently, without a fixed point. Moreover, for a non-countable subfamily of Grigorchuk groups, we show that this action is faithful and proper. In this case, the CAT(0) cube complex is a model for the classifying space of the proper actions of the group.

In the following, we present the results of joint projects with P-H. Leemann where we are interested in the stability of certain properties of groups for the wreath product. We start by giving a proof of a result about the stability of the proprety FW for the wreath product which is very close to a theorem of Cherix-Martin-Valette and Neuhauser about the stability of the property (T). For this purpose, we use an obstruction to property FW described in terms of the number of ends in the Schreier graphs associated to the group.

Then, we study the stability of these properties for the wreath product in a more general framework. Recall that, for a countable group, property (T) is equivalent to the property FH defined as “every action on a Hilbert space has a fixed point”. By generalizing this example, we notice that there exists a whole family of algebraic properties associated to a group which are defined as “every action of this group on a certain type of metric space has a fixed point”. We can, for example, think about the property FA (any action on a tree has a fixed point) or the property FW (any action on a wall space has a fixed point). We show how the stability of a large family of such properties for the wreath product can be proved in a unified way.

Finally, we study the notion of expansion for objects of dimension greater than 1. We start by defining the notion of boundary expansion for CW complexes and then prove a link between this expansion and the spectrum of the Laplacian, in the same spirit as the Cheeger-Buser inequalities for graphs.

Résumé

Comprendre comment un groupe peut agir sur un type d'espace donné peut s'avérer être un outil précieux pour démontrer aussi bien des propriétés du groupe que de l'espace. Cette thèse s'intéresse à trois sujets distincts faisant intervenir des actions de groupes infinis sur des graphes, des complexes cubiques et des espaces métriques.

Dans la première partie, nous répondons à une question de Grigorchuk demandant s'il est possible de donner une description explicite et élémentaire d'un complexe cubique CAT(0) sur lequel les groupes qu'il a définis agissent et, le cas échéant, de décrire ces actions. Rappelons que les groupes de Grigorchuk sont des sous-groupes de types finis du groupe d'automorphismes de l'arbre binaire enraciné. Cette famille de groupes a été un sujet d'étude prolifique et à permis d'apporter des réponses à de nombreux problèmes. L'ingrédient principal nous permettant de construire un tel complexe est l'existence d'un Schreier à 2 bouts provenant de l'action de ces groupes sur le bord de l'arbre binaire. En utilisant ce graphe, il nous est possible de construire un complexe cubique CAT(0) sur lequel tous les groupes de Grigorchuk agissent sans orbite bornée, ou, de manière équivalente, sans point fixe. De plus, pour une sous-famille non dénombrable des groupes de Grigorchuk, nous montrons que cette action est fidèle et propre. Dans ce cas, le complexe cubique CAT(0) est un modèle pour l'espace classifiant des actions propres du groupe.

Par la suite, nous présentons les résultats de projets menés conjointement avec P-H. Leemann où nous nous intéressons à la stabilité de certaines propriétés des groupes pour le produit en couronne. Nous commençons par donner une preuve élémentaire d'un résultat concernant de la stabilité de la propriété FW pour ce produit. Ce résultat est très proche d'un théorème de Cherix-Martin-Valette et Neuhauser concernant de la stabilité de la propriété (T). Pour cela, nous utilisons une obstruction à la propriété FW décrite en termes de nombre de bouts dans les graphes de Schreier associés au groupe.

Nous étudions ensuite la stabilité de ces propriétés pour le produit en couronne dans un cadre plus général. Rappelons que, pour un groupe dénombrable, la propriété (T) est équivalente à la propriété FH définie comme « toute action de ce groupe sur un espace de Hilbert possède un point fixe ». En généralisant cet exemple, on remarque qu'il existe toute une famille de propriétés algébriques associées à un groupe qui sont définies comme « toute action de ce groupe sur un certain type d'espace métrique possède un point fixe ». Nous pouvons par exemple penser à la propriété FA (toute action sur un arbre possède un point fixe) ou à la propriété FW (toute action sur un espace à mur possède un point fixe). Nous montrons comment la stabilité d'une grande famille de propriétés de ce type pour le produit en couronne peut être démontrée d'une manière unifiée.

Finalement, nous étudions la notion d'expansion pour des objets de dimension plus grande que 1. Nous commençons par définir la notion d'expansion de bord pour des CW complexes puis nous montrons qu'il existe un lien entre cette expansion et le spectre des Laplaciens, dans le même esprit que les inégalités de Cheeger-Buser pour les graphes.

Remerciements

Tout d'abord, mes remerciements s'adressent à Tatiana Nagnibeda qui m'a suivi depuis le début de mes études et qui a su me laisser une grande liberté dans ma recherche tout en sachant me guider, me conseiller et insister quand il le fallait sur des directions que je rechignais parfois à suivre. Je lui suis aussi reconnaissant pour toutes les personnes que j'ai pu rencontrer grâce à elle et qui ont été déterminantes dans l'écriture de cette thèse.

Je remercie chaleureusement Alain Valette et Anders Karlsson de me faire l'honneur de participer à mon jury et pour les nombreuses discussions enrichissantes que nous avons pu avoir à propos de sujets divers et variés.

Un grand merci Pierre de la Harpe qui a toujours su écouter mes problèmes mathématiques, me suggérer des pistes pour les résoudre et me donner de précieux conseils tant sur le fond que sur la forme.

I am grateful to Rostislav Grigorchuk whose many questions, ideas and discussions have generated a fair amount of this work.

Merci à mon coauteur Paul-Henry Leemann qui a été un mentor aussi bien durant mes études que durant ces années de recherches.

Je remercie aussi particulièrement Dominik Francoeur pour les nombreuses fois où il m'a permis de résoudre en quelques heures des problèmes sur lesquels je travaillais depuis des semaines.

I am also grateful to all the researchers who took the time to discuss about my problems and give me some insights, in particular Alexander Lubotzky, who hosted me in Jerusalem for a few months.

Je tiens aussi à remercier tous les chercheurs de Genève et d'ailleurs avec qui j'ai eu des discussions, souvent à propos de mathématiques, mais pas uniquement : Pierre Bagnoud, Christophe Pittet, Anthony Genevois, Adrien Le Boudec, Nicolás Matte Bon, Anthony Conway, Jeremy Dubout, Sébastien Ott, Renaud Rivier, Elise Raphael, les doctorants de Neuchâtel, ainsi que toutes les personnes du département qui ont permis à ces dernières années d'être bien plus qu'un travail.

Un merci particulier à Thomas Tiercy qui m'a été d'une aide précieuse durant nos études.

Bien entendu, je n'oublie pas le personnel de la section qui rend la vie quotidienne agréable et stimulante, en particulier Joselle et Annick.

Je tiens à remercier toute ma famille et mes amis qui m'ont toujours soutenu, sans jamais vraiment savoir ce que je faisais et dans quel but.

Et surtout un grand merci à Clara et Mathias sans qui tout cela n'aurait pas été possible.

Contents

Chapter 1

Introduction

Groups are fundamental objects of mathematics, appearing naturally in almost all its branches. The study of these abstract structures has been developed in many different directions. Here, we will focus on the geometrical and combinatorial aspects of this theory.

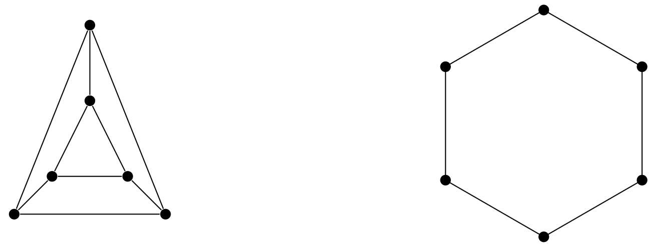
The way a group can act on certain types of spaces can help us understand this group. The simplest application of this idea is to consider actions of groups on sets. The action of a group G on a set X is a group homomorphism $\varphi : G \to \mathfrak{S}(X)$ between G and the group of permutations of X. Constructing and understanding such actions can be used to solve various problems, whether they are very concrete, such as counting the number of different necklaces that can be constructed with a given set of pearls, or more abstract, such as proving Fermat's little Theorem, Cauchy's Theorem or Sylow's Theorems.

Generally, when a group acts on a space that has more structure than a set, we will assume that this action preserves the structure of the space, in the sense that $\varphi : G \to \mathrm{Aut}(X)$ goes from G into the group of the automorphisms of X. For example, we will suppose that the group acts by isometries if X is a metric space, by invertible linear applications if X is a vector space, or by unitary operators if X is a Hilbert space.

This thesis presents the content of [64, 65, 92, 93] where we will mainly focus on the actions of infinite groups on graphs, CAT(0) cubic complexes, and metric spaces.

1.1 Cayley graphs and Schreier graphs

There is a natural way to associate a graph to a group G with a set of generators $S \subset G$. We define $\mathrm{Cay}(G, S)$, the (left) *Cayley graph* of G with respect to S, as follows. The set of vertices is the set of elements of the group and two vertices g and h are connected by an edge if and only if there is a generator $s \in S$ such that $g = sh$. Sometimes, we label the edges with the generator used to get from one vertex to the other. Even if the definition works in all generality, we will normally consider groups with a finite generating set S in order to obtain locally finite graphs. Usually, a group has more than one generating set and the choice of this one strongly affects the shape of the Cayley graph, see Figure 1.1. However, some properties of the Cayley graphs (such as the number of ends,



$\mathrm{Cay}\,(\mathfrak{S}_3, \{(1,2),(1,2,3),(1,3,2)\})$ $\mathrm{Cay}(\mathfrak{S}_3, \{(1,2),(2,3)\})$

Figure 1.1: Two examples of Cayley graphs of $\mathfrak{S}_3$

the growth rate, etc) are independent of the choice of the generating set. This can be proved formally using the notion of *quasi-isometries*, see [16, 27].

Cayley graphs allow us to use graph theory tools to study groups. For example, we can look at the number of ends of a Cayley graph. Recall that intuitively an *end* of a graph is a direction in which the graph extends to infinity. The number of ends of a Cayley graph does not depend on the generating set [12] and so we can define the number of ends of a finitely generated group. In the 1940s, Hopf and Freudenthal independently established the two following results about the number of ends and the structure of a group that has a linear Cayley graph.

Theorem. *The number of ends of a finitely generated group is* $0, 1, 2$ *or* ∞.

Theorem. *A finitely generated group has* 2 *ends if and only if it contains an infinite cyclic subgroup of finite index.*

Later, Stallings [95, 96] proved a similar result for the Cayley graphs with more than one end.

Theorem. *A finitely generated group G has at least 2 ends if and only if one of the following two points holds:*

- *The group G admits a decomposition $G = H *_C K$ as an amalgamated free product where C is a finite subgroup not equal to H or K.*
- *The group G is an HNN extension: $G = \langle H, t | t^{-1}C_1 t = C_2\rangle$ where C_1 and C_2 are 2 finite and isomorphic subgroups of H.*

Let G be a finitely generated group, $H \leq G$ an arbitrary subgroup and $S \subset G$ a generating set. We define $\mathrm{Sch}(G, H, S)$, the (left) *Schreier graph* of G with respect to S and H, as follows. The set of vertices is equal to the set of the left cosets $\{gH\}$ and two vertices g_1H and g_2H are connected by an edge if and only if there exists $s \in S$ such that $sg_1H = g_2H$. We can notice that if the subgroup H is normal in G, then the Schreier graph $\mathrm{Sch}(G, H, S)$ is isomorphic to the Cayley graph $\mathrm{Cay}(G/H, S)$. A legitimate question is to ask how the results proved for Cayley graphs generalize to Schreier graphs. If we study the possible number of ends of a Schreier graph, we can see that Hopf and Freudenthal's result does not generalize. Indeed, every $2d$-regular graph is the Schreier graph of a group, see [46, 63, 67]. It is therefore easy to construct Schreier graphs with an arbitrary number of ends. However, we will see in the following that the number of ends of the Schreier graph gives us essential information about the possible actions of the group on certain spaces.

1.2 Actions on graphs

One of the simplest objects of dimension one are graphs. Intuitively, a graph is a set of vertices with an adjacency relation describing which vertices are connected by an edge. This definition admits many variations: The edges can be oriented, labelled, multiple between 2 vertices or connect a vertex with itself. In the following, we will give precise definitions when necessary.

A homomorphism of graphs is an application between the sets of vertices of 2 graphs that preserves adjacency, in the sense that the images of 2 adjacent vertices are also adjacent. An action of a group on a graph will always be by graph automorphisms.

An interesting family of graphs is the one composed of graphs that do not contain a closed path. Such graphs are called *trees.* The trees have very good combinatorial properties, such as they are bipartite, median, planar, etc. A connected tree is simply connected and can be provided with a metric induced by the fact that there is exactly one path between 2 vertices.

The study of group actions on trees was initiated by Serre in his seminal book [94]. Let us consider an action $\varphi : G \to \mathrm{Aut}(T)$ of a group G on a tree T. A vertex v of the tree is a *fixed point* if it is fixed by all the automorphisms of the action, or, in other words, if $\varphi(g)(v) = v$ for any element g of G. We say that a group G acts without inversion if it does not send an edge on itself by exchanging its two extremal vertices. One of the questions that can be asked is: given a group G, is it possible to construct an action of this group on a tree without inversion that does not fix any point? We say that a group for which the answer is negative (all the actions have a fixed point) has the *property FA*. Serre has shown that the structure of such groups can be described.

Theorem. *A countable group G has the property FA if and only if the following 3 points are verified:*

1. *G is not a non trivial amalgamated product.*
2. *G has no quotient isomorphic to $\mathbf{Z}$.*
3. *G is of finite type.*

This theorem can be related to Stalling's theorem above to obtain the following reformulation.

Theorem. *Let G be a finitely generated group. Then, the following are equivalent:*

- *The group has at least two ends.*
- *The group does not have the property FA, i.e. there exists an action on a tree without fixed point and inversion.*

- *The group cannot be decomposed as a non trivial amalgamated product and it has no quotient isomorphic to* $\mathbf{Z}$*.*

In the same work, Serre gives an example of the application of actions on trees. He shows how to construct a tree associated to the group $\mathrm{SL}_2(\mathbf{Q}_p)$ admitting an action of this group and thanks to which he proves again, for example, a theorem of Ihara [59] showing that the group $\mathrm{SL}_2(\mathbf{Q}_p)$ can be decomposed into an amalgamated product of two factors $\mathrm{SL}_2(\mathbf{Z}_p)$.

1.3 Actions on CAT(0) cube complexes

When we consider actions on objects of dimension greater than 1, we can ask which ones generalize trees. The trees roughly can be seen as a gluing of intervals $[0,1]$ that does not contain a closed path. A way to generalize them is to glue cubes $[0,1]^n$ of different dimensions using isometries. Such objects are called *cube complexes*. The intrinsic euclidean metrics of the cubes $[0,1]^n$ induce a global metric on the complex. Recall that a metric verifies the condition CAT(0) if each triangle defined by geodesics is "thinner" than a comparable Euclidean triangle, see [2, 13] for detailed expositions of these notions. We may ask when the metric of a cubic complex verifies the CAT(0) condition. Generally, answering this question is a difficult problem, but fortunately, in the case of CAT(0) cube complexes, there exists a local characterization of this property introduced due to Gromov [44]. It is sufficient to prove that for each cube, the associated link does not contain the boundary of a triangle that is not filled. If the complex is, moreover, simply connected, then the metric verifies the CAT(0) condition. We notice that the trees are CAT (0) cube complexes of dimension 1.

The CAT(0) cube complexes have a rich combinatorial structure which facilitates their studies. The notion of hyperplane is, for example, very useful to understand how a group acts on a complex. A hyperplane is an equivalence class of edges. Two edges are equivalent if they are the opposite sides of a square. It can be defined equivalently as a codimension 1 subspace generated by the midpoints of equivalent edges. It has been shown that each hyperplane of a CAT(0) cube complex, cuts the complex into exactly 2 connected components, called *half-spaces*, see [90] for details. An action of a group on a CAT(0) cube complex naturally extends into an action on the hyperplanes and into an action on the half-spaces which can give us valuable information. Group actions on CAT(0) cube complexes have been a prolific subject of study and have led to significant advances in many fields such as low dimensional topology and geometric group theory, see for example [1, 53, 102].

Sageev [90] proved a theorem linking the actions on CAT(0) cube complexes and the number of ends of Schreier graphs in the same spirit as the results concerning the actions on the trees.

Theorem. *A finitely generated group G has a Schreier graph with at least 2 ends if and only if it acts on a CAT(0) cube complex without fixed point and transitively on hyperplanes.*

There is a famous family of groups for which the existence of Schreier graphs with at least 2 ends is known: the branched groups $\mathcal{G}_\omega$ introduced by Grigorchuk in [41]. These groups, which are known as the *Grigorchuk groups*, are indexed by sequences $\omega \in \{0,1,2\}^{\infty}$ and are defined as subgroups of the automorphism group of the binary tree. By Sageev's theorem, each of them acts without bounded orbit on a CAT(0) cube complex.

Grigorchuk asked us the following question:

Question. *Can we give an explicit and elementary description of a CAT (0) cube complex on which the groups $\mathcal{G}_\omega$ act without bounded orbit? Can we understand these actions more precisely?*

A positive answer is presented in Chapter 2 where we give an explicit construction of a CAT (0) cube complex on which all the groups $\mathcal{G}_\omega$ act without bounded orbit, see Theorem 2.4.3. Moreover, this action is proper and faithful for all the $\mathcal{G}_\omega$ which are indexed by a sequence without repetition, i.e. for the sequences ω such that $\omega_i \neq \omega_{i+1}$ for all i, see Theorem 2.4.15. In this case, we also prove that this CAT (0) cube complex is a model for the classifying spaces of proper actions.

1.4 Fixed point free actions

The actions of a group on CAT(0) cube complexes are related to the property (T). Indeed, if a countable group G acts without a fixed point on a CAT(0) cube complex, then it does not have the property (T), see [81]. Recall that for countable groups, the property (T) is equivalent to the property FH (any action on a Hilbert space has a fixed point) by results of Delorme and Guichardet [29, 47]. If we replace the Hilbert spaces in the definition of the property FH, we obtain other known properties, such as the property FA for actions on trees or the property FW for actions on wall spaces (or equivalently on CAT(0) cube complexes, see [82]).

When considering properties of groups, it is legitimate to ask whether they are stable under natural group operations. One such operation, of great use in geometric group theory, is the wreath product. Recall that the (restricted) wreath product of two groups G and H and a set X on which H acts is defined as

$$G \wr_X H = \left(\bigoplus_X G \right) \rtimes H.$$

We can ask the following question about the stability of the properties defined in the same spirit as the examples above:

Question. *Among the properties that can be characterized as "any action on a space of some subclass of metric spaces has a fixed point", which ones are stable for the wreath product?*

Answers to this question have been provided for some of these properties. For example, Neuhauser [77] and Cherix-Martin-Valette [21] handled the case of the property (T):

Theorem. *Let G, H be two finitely generated groups and X a set on which the group H acts. The wreath product $G \wr_X H$ has property (T) if and only if G and H have property (T) and if X is finite.*

In Chapter 3, we will give an elementary proof of a theorem about the stability of the property FW, see Theorem 3.1.1. The main ingredient of the proof is a result of Sageev [90] which proved that the number of ends in Schreier graphs of a group is an obstruction to the property FW.

More generally, the stability of many such properties for the wreath product can be proved in a unique way using the right framework. This will be the subject of Chapter 4.

1.5 Expansion for CW complexes

The last topic is also about cell complexes but seen from another point of view. Recall that the expansion constant (or Cheeger constant) of a finite graph X with a vertex set V is defined as

$$h(X) := \min \left\{ \frac{|\partial A|}{\min\{|A|, |A^c|\}} : \emptyset \subsetneq A \subsetneq V \right\}$$

where ∂A is the set of edges of X with one vertex in A and the other in A^c. This constant describes the difficulty of disconnecting the graph into two large subsets.

We notice right away that a finite connected graph always has an expansion constant strictly greater than 0. It is therefore interesting to consider families of graphs whose number of vertices grows and for which the expansion constants are uniformly bounded far from 0. However, there are also trivial examples, for example, the family $\{K_n\}_n$ of complete graphs on n vertices. We can notice that the number of edges of the graphs K_n increases extremely quickly and that therefore such graphs are certainly well connected but not in a very efficient way. This is why we will ask that the graphs have uniformly bounded degrees. This condition makes the construction of examples substantially more challenging. One can show that such families exist by a probabilistic argument, but constructing them explicitly is more difficult. One way to proceed is to consider Schreier graphs of groups with property (T), see [9, 68, 69].

The expansion of a graph is usually difficult to calculate precisely, especially if the graphs become large, but it is possible to estimate it by using the spectrum of the graph Laplacian, see [22] for a precise definition of this operator. This relation is given by the Cheeger-Buser inequality [15, 19].

Theorem. *Let X be a connected graph and λ the first non-trivial eigenvalue of the Laplacian, then*

$$\frac{\lambda}{2} \leq h(X) \leq \sqrt{2\lambda d}$$

where d is the maximal degree of a vertex.

One may wonder how this notion of expansion can be generalized for objects of dimension greater than one. Several approaches have been developed depending on the definition of the graph expansion considered. One can use the combinatorics, [48, 87, 88], the homology and the cohomology [74, 97] or the spectrum of Laplacian [61, 84, 85]. Contrary to the case of graphs, these notions are not equivalent in higher dimensions. The reader interested in all these different generalizations can refer to [70] and the references therein.

As in the one-dimensional case, it is possible to construct examples of families of expanding complexes, but the degrees are not bounded. The following question remains, to our knowledge, still open.

Question. *Can we explicitly construct a family of expanders of dimension greater than 1 that is of bounded degree?*

Inspired by the construction using property (T), we can ask whether algebraic properties of a group can be used to prove the expansion of higher dimensional complexes associated to the group, such as the presentation complexe or the Cayley complex. However, the theory of high dimensional expanders has been mostly developed for simplicial complexes and these associated complexes are, at best, polygonal complexes, at worst, CW complexes. This reflection leads us to define boundary expansion for CW complexes and to prove that there is a link between this expansion and the spectrum of Laplacians, in the same spirit as the Cheeger-Buser inequality, see Theorem 5.1.2. We present these results in Chapter 5.

Chapter **2**

Proper actions of the Grigorchuk groups on CAT(0) cube complexes

2.1 Introduction

CAT(0) cube complexes are nice examples of CAT(0) spaces that share many similarities with trees. Their combinatoric and geometric properties provide useful tools to study the large class of groups which admit non-trivial actions on CAT(0) cube complexes. Group acting on CAT(0) cube complexes were shown to play an important role in low dimensional topology [1, 52, 53, 102] and in geometric group theory [17, 18, 21, 80, 82, 89, 91].

In [40], Grigorchuk constructed an infinite finitely generated 2-group, now known as the *first Grigorchuk group*, and showed that this group has a lot of exotic properties. Most notably, it is of intermediate growth and hence amenable, but not elementary amenable. This example was generalized in [41] to an uncountable family of groups $\{\mathcal{G}_\omega\}_{\omega\in\{0,1,2\}^\infty}$, the *Grigorchuk groups*, that have since generated a lot of research. Grigorchuk groups are defined by their action by automorphisms on the infinite binary rooted tree. This action extends naturally to an action by homeomorphisms on the boundary of the tree. Schreier graphs of stabilizers of points in this action have linear structure and have proved an important tool in the study of these groups, see e.g. [5, 39, 42, 101]

In [90], Sageev showed that if a finitely generated group has a Schreier graph with more than one end, then one can construct a CAT(0) cube complex on which the group acts without bounded orbit. The way in which the complex and the action are constructed is explicitly described.

It is well known that Grigorchuk groups have Schreier graphs with 2 ends and hence it is tempting to apply Sageev's construction and to investigate the complex that it gives and the action of $\mathcal{G}_\omega$ on it for different sequences ω. Recall that the groups $\mathcal{G}_\omega$ are not quasi-isometric and may have different algebraic and geometric properties, for example, being torsion's groups or not. Following Sageev's method, we are able to construct a complex $\mathcal{X}_\omega$ on which the group $\mathcal{G}_\omega$ acts for each sequence ω. It turns out that these complexes are isometric for all ω. In this paper we present a different approach by constructing a unique CAT(0) cube complex $\mathcal{X}$ (isometric to the $\mathcal{X}_\omega$) on which all $\mathcal{G}_\omega$ act without bounded orbit, see Theorem 2.4.3.

Moreover, we can strengthen this result for an uncountable subfamily of groups $\mathcal{G}_\omega$, including the self-similar examples $\mathcal{G}_{(012)^\infty}$ (the first Grigorchuk group which is an infinite 2-group) and $\mathcal{G}_{(01)^\infty}$ (which has an element of infinite order), by proving that the actions of these groups on $\mathcal{X}$ are faithful and proper, see Theorem 2.4.15. Here, an action is called proper if the cubes' stabilizers are all finite.

All Grigorchuk groups $\mathcal{G}_\omega$ are amenable and therefore have the Haagerup property [8]. It is known that groups with the Haagerup property satisfy the Baum-Connes conjecture [56]. Recall that this conjecture links the K-homology of the classifying space of proper actions of a group G and the K-theory of its reduced C^*-algebra by conjecturing that the assembly map $\mu_i : RK_i^G(\underline{E}G) \to K_i(C_r^*(G))$ is an isomorphism for $i = 0, 1$, see [99]. Even if the conjecture is proved for a particular group, it is interesting to be able to explicit the isomorphisms μ_i. For this, we need a model for the classifying space of proper actions $\underline{E}G$. The Theorem 2.5.2 shows that the complex $\mathcal{X}$ gives such a model for all the groups $\mathcal{G}_\omega$ whose action on $\mathcal{X}$ is proper.

2.2 Definitions

CAT(0) cube complexes

A cube complex is a space obtained by gluing, via isometries, euclidean cubes with edges of length one. This space can be equipped with a metric induced by that of the cubes. A result of Gromov (see [13, Theorem II.5.20] for the finite dimension and [62, Theorem 40] for the general case), ensures that if the complex is simply connected and if the link of each cube does not contain the boundary of a triangle which is not filled[1], then the metric verifies the CAT(0) condition. Recall that the link associated to an n cube C in a cube complex is the simplicial complex defined as follows. The vertices are the $n+1$ cubes whose boundary contains C. A pair of vertices in the link are joined by an edge if the two corresponding $n+1$ cubes are in the boundary of a common $n+2$ cube. If the resulting 1-complex contains a triangle appears, we fill it if the 3 vertices that form the triangle are in the boundary of a common $n+3$ cube. If the boundary of a k-simplex appears, we fill it if the $k+1$ vertices that form the simplex are in the boundary of a $n+k+1$ cube.

A key feature of CAT(0) cube complexes is the existence of hyperplanes. Two edges are said equivalent if they are the opposite sides of a square. An equivalent class of edges is a *hyperplane*. It is also possible to define hyperplanes in a more geometrical way, as the codimension-1 subspace spanned by the midpoints of the edges. In the case of a CAT(0) cube complex, hyperplanes are also CAT(0) cube complexes which split the complex into exactly two connected components called *half-spaces*, see [90] for details.

Grigorchuk groups

Let $\mathcal{T}_2$ denote the infinite binary rooted tree. The vertices of this tree can be described as (finite) binary sequences, see Figure 2.1.

[1]Sometimes, this condition is formulated as: the link of each vertex does not contain the boundary of a simplex which is not filled. These two conditions are equivalent.

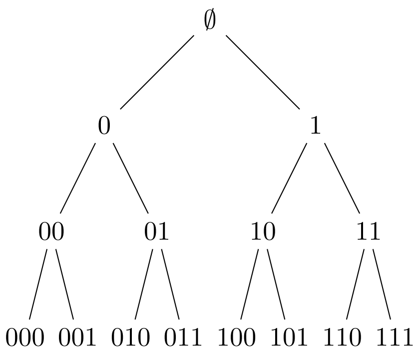


Figure 2.1: The first levels of $\mathcal{T}_2$

.

Definition 2.2.1. Let $\omega = \omega_1\omega_2\dots$ be an infinite sequence in $\{0,1,2\}^\infty$. Denote by σ the shift on the space $\{0,1,2\}^\infty$. The group $\mathcal{G}_\omega$ is the subgroup of the automorphisms group of $\mathcal{T}_2$ generated by the automorphisms $a, b_\omega, c_\omega, d_\omega$ where,

$$a(0x) = 1x \quad \text{and} \quad a(1x) = 0x$$

for all binary sequences x and

$$b_\omega(0x) = \begin{cases} 0a(x) & \text{if } \omega_1 \neq 2 \\ 0x & \text{if } \omega_1 = 2 \end{cases} \qquad b_\omega(1x) = 1b_{\sigma\omega}(x)$$

$$c_\omega(0x) = \begin{cases} 0a(x) & \text{if } \omega_1 \neq 1 \\ 0x & \text{if } \omega_1 = 1 \end{cases} \qquad c_\omega(1x) = 1c_{\sigma\omega}(x)$$

$$d_\omega(0x) = \begin{cases} 0a(x) & \text{if } \omega_1 \neq 0 \\ 0x & \text{if } \omega_1 = 0 \end{cases} \qquad d_\omega(1x) = 1d_{\sigma\omega}(x).$$

The actions of these generators can be represented graphically, see Figure 2.2.

Let v be a vertex of the n^{th} level of the tree, we can look at the action of an element $g \in \mathcal{G}_\omega$ on the subtree rooted at v. This subtree is isometric to $\mathcal{T}_2$ and then, we can define a new automorphism, which is an element of $\mathcal{G}_{\sigma^n\omega}$. This automorphism is called the *restriction of g on v* and is denoted by g_v.

An element g *stabilizes the n^{th} level* if $g(v) = v$ for every v on the n^{th} level. The subgroup of all such elements in $\mathcal{G}_\omega$ is $\mathrm{Stab}_\omega(n)$, the *stabilizer of the n^{th} level.* For an element g in $\mathrm{Stab}_\omega(1)$, we denote by (g_0, g_1) the two restrictions of g on the two subtrees rooted at the first level. If g does not stabilize the first level, we can still write it as $a(g_0, g_1)$. We refer the reader to the section VII of [27] and [43] for a detailed general exposition of these concepts.

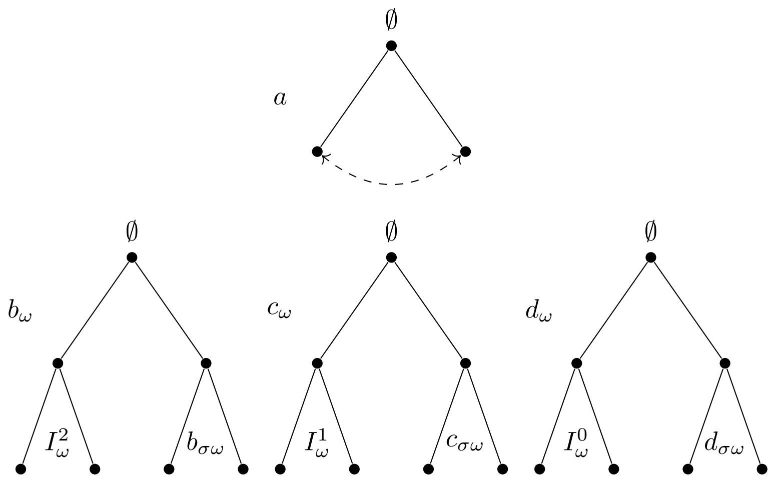


Figure 2.2: The actions of the generators, where $I^i_\omega = \begin{cases} 1 & \omega_1 = i \\ a & \omega_1 \neq i \end{cases}$.

2.3 Construction of the cube complex $\mathcal{X}$

The graph Γ

The action of $\mathcal{G}_\omega$ can be extended to $\partial\mathcal{T}_2$, the boundary of the tree. The points of the boundary $\partial\mathcal{T}_2$ of the tree can be seen as one-sided binary sequences. Consider the orbit of 0^∞ and define Γ_ω as the graph with $\mathcal{G}_\omega 0^\infty$ as vertices and $\{(g0^\infty, sg0^\infty) : g \in \mathcal{G}_\omega, s \in \{a, b_\omega, c_\omega, d_\omega\}\}$ as edges. This is the orbital graph of the action of $\mathcal{G}_\omega$ on 0^∞. These graphs are not isomorphic if we look at them as labelled[2] graphs, but the underlying unlabelled graphs are the same.

Proposition 2.3.1. *The vertices of the graphs Γ_ω are exactly the infinite sequences with a finite number of* 1*. Moreover all the Γ_ω are equal as unlabelled graphs.*

Proof. These facts are proved in [73] for the orbital graph of 1^∞, but the proofs can be translated for our case directly. □

We will denote by Γ the underlying unlabelled graph and by $V(\Gamma)$ its set of vertices.

The cube complex $\overline{\mathcal{X}}$

We begin by defining a huge cube complex $\overline{\mathcal{X}}$ with an action of $\mathcal{G}_\omega$ on it. This space is not CAT(0) and the actions do not preserve the connected components in general. However, each connected component is a CAT(0) cube complex.

The set $\overline{\mathcal{V}}$ of vertices of $\overline{\mathcal{X}}$, is the set of the 2-coloring of vertices of Γ, i.e.

$$\overline{\mathcal{V}} = \{v : V(\Gamma) \to \mathbf{Z}/2\mathbf{Z}\} = (\mathbf{Z}/2\mathbf{Z})^{V(\Gamma)}.$$

[2]By labelled graph we mean a graph with edges labelled by generators $\{a, b_\omega, c_\omega, d_\omega\}$

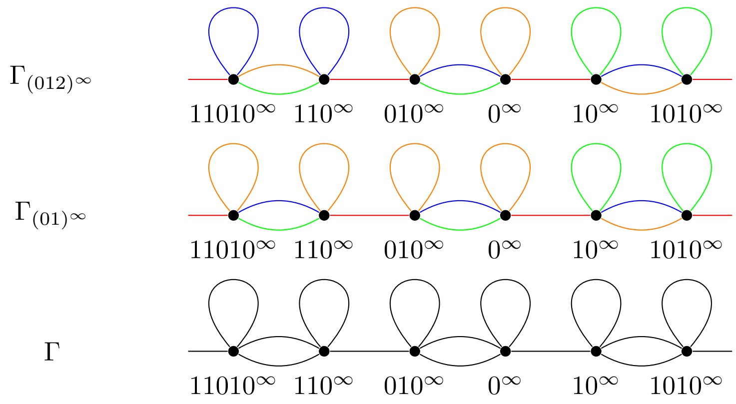


Figure 2.3: Some examples of graphs Γ_ω and the graph Γ. The red edges are labelled by a, the blue by b_ω, the green by c_ω and the orange by d_ω.

A vertex $v \in \overline{\mathcal{V}}$ can be seen as a subset of vertices of Γ whose image is equal to 1. This set is called the *support* of v.

We provide $\overline{\mathcal{V}}$ with a graph structure by adding an edge between two vertices v and w if and only if the symmetric difference of their support contains exactly one element, or in other words, if they are equal on all the vertices except one. Such an edge is labelled by the vertex of Γ on which the two functions differ. The neighborhood of a vertex v is all the vertices of the form $v + \delta_x$, for $x \in \Gamma$, where

$$\delta_x(y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$$

We can further add the higher dimensional cubes inductively. Whenever the boundary of a square appears, we fill it. After filling all the squares, we do the same for the cubes, then for the cubes of dimension 4, etc. In short, we add a n-cube at each appearance of its boundary. The resulting space is called $\overline{\mathcal{X}}$.

The combinatorial structure of the cubes of $\overline{\mathcal{X}}$ can be explicitly described. A subset $\{x_1, \ldots, x_n\}$ of $V(\Gamma)$ and a coloring v of $V(\Gamma)$ define a unique n cube in $\overline{\mathcal{X}}$ whose vertices are the colorings which differ from the coloring v only on vertices in $\{x_1, \ldots, x_n\}$. We will denote this cube by $\mathscr{C}(v, x_1, x_2, \ldots, x_n)$. Every cube of $\overline{\mathcal{X}}$ can be described in this way. The Figure 2.4 illustrates this characterization with an example.

As $V(\Gamma)$ is infinite, the set $\overline{\mathcal{V}}$, which is equal to $(\mathbf{Z}/2\mathbf{Z})^{V(\Gamma)}$, is uncountable. Moreover, $\overline{\mathcal{X}}$ is of infinite dimension, is not locally finite and is not connected. Indeed, there is no finite path between two vertices (= colorings) which differ on an infinite number of vertices of Γ. However, it turns out that each connected component of $\overline{\mathcal{X}}$ is a CAT(0) cube complex.

Proposition 2.3.2. *Each connected component of $\overline{\mathcal{X}}$ is a CAT(0) cube complex.*

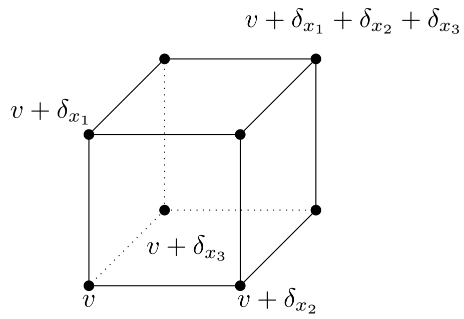


Figure 2.4: The 3-cube $\mathscr{C}(v, x_1, x_2, x_3)$.

Proof. Let Y be a connected component of $\overline{\mathcal{X}}$. To prove that this component is a CAT(0) complex, we will verify that it is simply connected and that the link of each cell does not contain the boundary of a triangle which is not filled.

Let γ be a closed path passing through the vertices $v_1, v_2, \dots, v_n$ of Y. By the description of a neighborhood, there exist $x_1, \dots, x_n$, some vertices of Γ, such that $v_i = v_{i-1} + \delta_{x_i}$ for $i = 2, \dots, n$ and $v_1 = v_n + \delta_{x_n}$. Using the description of cubes above, all these vertices are vertices of a cube of dimension at most n and thus γ is contractible.

Let C be an n-cube of Y and let C_1, C_2 and C_3 be $(n+1)$-cubes which form a triangle in the link of C. By definition of the link, C is in the boundary of the cubes C_1, C_2 and C_3. Using the characterization of the cubes above, if $C = \mathscr{C}(v, x_1, \dots, x_n)$, there exist x'_1, x'_2, x'_3, some different vertices of Γ, such that $C_i = \mathscr{C}(v, x_1, \dots, x_n, x'_i)$. If we consider $\mathscr{C}(v, x_1, \dots, x_n, x'_1, x'_2, x'_3)$, we can see that it is a $(n+3)$-cube of Y which contains C_1, C_2 and C_3 on its boundary. Then, the triangle in the link is filled. □

Using the action of a group $\mathcal{G}_\omega$ on the orbit of 0^∞, we define an action of $\mathcal{G}_\omega$ on $\overline{\mathcal{V}}$:

$$(gv)(x) = v(g^{-1}x)$$

for a vertex v of $\overline{\mathcal{V}}$, an element g of $\mathcal{G}_\omega$ and a vertex x of Γ. This action can be extended to an action on $\overline{\mathcal{X}}$ which preserves the structure of cube complex. Generally, it is not true that the connected components of $\overline{\mathcal{X}}$ are preserved by the action. For example, consider the vertex v of $\overline{\mathcal{X}}$ which colors in black the set L composed of the vertices of Γ which are left ends of edges labelled by a and only these ones. Then, av will color in black the complement of L in $V(\Gamma)$ and the distance between v and av will be infinite.

The CAT(0) cube complex $\mathcal{X}$.

We will now show that a particular connected component of $\overline{\mathcal{X}}$ is preserved under the action of $\mathcal{G}_\omega$. Let Γ_+ be the set of vertices of Γ including 0^∞ and all the vertices on the right side of it, see Figure 2.5. The vertices of Γ_+ can be described in an explicit way as follows,

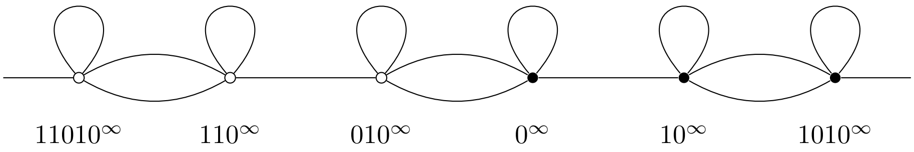


Figure 2.5: The black vertices are elements of Γ_+

Proposition 2.3.3. *The set Γ_+ consists of 0^∞ and all the vertices of the form $x = x_1x_2\dots x_n0^\infty$ with $x_n = 1$ and n odd.*

Proof. Let us begin by an observation. Given any element x of $\{0,1\}^\infty$, a generator s of $\mathcal{G}_\omega$ can only acts non-trivially in two ways:

- flip the first digit of x (if $s = a$)
- flip the digit after the first apparition of 0 (if $s \in \{b_\omega, c_\omega, d_\omega\}$)

It is straightforward to see that these two moves do not change the parity of the position of the last 1, except for 0^∞ and 10^∞. We constructed Γ in a way to have 10^∞ on the right of 0^∞, and then, the parity of the position of the last 1 is the same as for 10^∞ for all the other vertices on the right of 0^∞. $\square$

Definition 2.3.4. The cube complex $\mathcal{X}$ is the connected component of $\overline{\mathcal{X}}$ which contains the vertex v_0:

$$v_0(x) = \chi_{\Gamma_+}(x) = \begin{cases} 1 & x \in \Gamma_+ \\ 0 & x \notin \Gamma_+ \end{cases}.$$

We denote by $\mathcal{V}$ the set of vertices of $\mathcal{X}$. Let us show that the CAT(0) cube complex $\mathcal{X}$ is preserved by the action of every $\mathcal{G}_\omega$.

Proposition 2.3.5. *Let v be a vertex of $\mathcal{X}$ and g an element of $\mathcal{G}_\omega$. Then gv is a vertex of $\mathcal{X}$.*

Proof. By the construction of $\mathcal{V}$, it is sufficient to prove that $\Gamma_+ \Delta g\Omega$ is finite, where Ω is the support of v.

The first thing to show is that $\Gamma_+ \Delta g\Gamma_+$ is finite for every g. Let $Y = \Gamma_+ \setminus g\Gamma_+ = \{x \in \Gamma : x \in \Gamma_+ \text{ and } gx \notin \Gamma_+\}$. If n is the length of g with respect to the standard generating set, a vertex x in Y must be at distance at most n of Γ_+^c in the graph Γ. As this graph is locally finite, Y is necessarily finite. Similarly $g\Gamma_+ \setminus \Gamma_+$ is finite and so $\Gamma_+ \Delta g\Gamma_+$ is also finite.

We need the following property of symmetric differences. Let A, B, C be three sets. If $|A\Delta B|$ and $|B\Delta C|$ are finite, then $|A\Delta C|$ is finite.

The set $\Gamma_+ \Delta g\Gamma_+$ is finite by the first part of the proof. The action of $\mathcal{G}_\omega$ preserves the size of the subsets of Γ, then $|g\Gamma_+ \Delta g\Omega| = |g(\Gamma_+ \Delta \Omega)| = |\Gamma_+ \Delta \Omega|$ which is finite because v_0 and v are connected. Combining these equalities and using the property above, we prove that $|\Gamma \Delta g\Omega|$ is finite. $\square$

$$\begin{tikzcd}
v+\delta_x \arrow[r, dash] \arrow[d, dash, "e"'] & v+\delta_x+\delta_{x_1} \arrow[r, dash] \arrow[d, dash] & v+\delta_x+\delta_{x_1}+\delta_{x_2} \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] \arrow[d, dash] & w+\delta_x \arrow[d, dash, "f"] \\
v \arrow[r, dash] & v+\delta_{x_1} \arrow[r, dash] & v+\delta_{x_1}+\delta_{x_2} \arrow[r, dash] & \cdots \arrow[r, dash] & w
\end{tikzcd}$$

Figure 2.6: In bold, the sequence of equivalent edges between e and f

Corollary 2.3.6. *The action $\mathcal{G}_\omega \curvearrowright \overline{\mathcal{X}}$ can be restricted to an action $\mathcal{G}_\omega \curvearrowright \mathcal{X}$, for every $\omega \in \{0,1,2\}^\infty$.*

Remark 2.3.7. To find this invariant connected component, it is crucial to have a Schreier graph that has at least 2 ends. Indeed, this allows us to construct this function whose support cannot be moved too much by the action of the group. Such a set is sometimes called a *commensurated subset*, [24].

We next describe the hyperplanes of $\mathcal{X}$ and understand the induced action of the groups $\mathcal{G}_\omega$ on the set $\mathfrak{H}(\mathcal{X})$ of hyperplanes of $\mathcal{X}$.

Proposition 2.3.8. *There exists a bijection $\varphi : \Gamma \to \mathfrak{H}(\mathcal{X})$. More precisely, a hyperplane is composed of all the edges with the same label x, where x is a vertex of Γ.*

Proof. We will show that for each hyperplane $\mathfrak{h}$ there exists a vertex x in Γ such that the edges forming $\mathfrak{h}$ are exactly those labelled by x. It follows directly from the description of the squares, see Figure 2.4, that two opposite sides have the same label. Now let e and f be two edges with the same label x. We denote by v (resp. by w) the vertex of $\mathcal{X}$ which is the endpoint of e (resp. of f) such that $v(x) = 0$ (resp. $w(x) = 0$). The complex is connected, hence there exists a path from v to w and we can construct a sequence of adjacent squares, see Figure 2.6, and then v and w are on the same hyperplane. □

Proposition 2.3.9. *The action $\mathcal{G}_\omega \curvearrowright \mathcal{X}$ induces an action $\mathcal{G}_\omega \curvearrowright \mathfrak{H}(\mathcal{X})$ which is transitive, for every ω.*

Proof. Using the previous proposition, each hyperplane can be labelled by a vertex x of Γ and this is why we will denote them by $\mathfrak{h}_x$. A hyperplane $\mathfrak{h}_x$ splits $\mathcal{X}$ into 2 half-spaces $\mathfrak{h}_x^0$ and $\mathfrak{h}_x^1$:

$$\mathfrak{h}_x^0 = \{v \in \mathcal{V} : v(x) = 0\}$$
$$\mathfrak{h}_x^1 = \{v \in \mathcal{V} : v(x) = 1\}.$$

Let g be an element of $\mathcal{G}_\omega$ and v a vertex in $\mathfrak{h}_x^0$. As $(gv)(gx) = v(x) = 0$, then gv is in $\mathfrak{h}_{gx}^0$. Similarly, gv is in $\mathfrak{h}_{g'x}^1$ for each v in $\mathfrak{h}_x^1$. These images of half spaces lead us to define the induced action $\mathcal{G}_\omega \curvearrowright \mathfrak{H}(\mathcal{X})$ as

$$g\mathfrak{h}_x = \mathfrak{h}_{gx}.$$

The proof of transitivity follows directly from the definition of the action and the transitivity of $\mathcal{G}_\omega \curvearrowright \Gamma$. □

Remark 2.3.10. Recall that hyperplanes split CAT(0) cube complexes into two connected components. The action of $\mathcal{G}_\omega$ cannot swap these components delimited by a hyperplane. Indeed, for every g in $\mathcal{G}_\omega$ there exist vertices v and w in $\mathfrak{h}_x^0$ such that $v(gx) = 0$ and $w(gx) = 1$ and then $g\mathfrak{h}_x^0 \not\subset \mathfrak{h}_x^1$.

2.4 Properties of the action of $\mathcal{G}_\omega$ on $\mathcal{X}$

In this section we will prove our main results. Namely, we will first show that every action $\mathcal{G}_\omega \curvearrowright \mathcal{X}$ has no bounded orbit. Then, we will prove that this action is also proper and faithful for an uncountable subfamily of groups $\mathcal{G}_\omega$.

Bounded orbits and fixed points

The proofs of this section are inspired by [81].

Proposition 2.4.1. *Let ω be a sequence in $\{0,1,2\}^\infty$. Then, the action $\mathcal{G}_\omega \curvearrowright \mathcal{X}$ does not fix any vertex.*

Proof. Suppose by contradiction that there exists a vertex v of $\mathcal{X}$ which is fixed by the action of $\mathcal{G}_\omega$. We will show that then Γ_+ or Γ_+^c has to be finite which is obviously not the case.

Suppose that $v(0^\infty) = 0$. Then for every g in $\mathcal{G}_\omega$, we have

$$gv = v \Rightarrow v(g^{-1}0^\infty) = v(0^\infty) = 0.$$

As the action of $\mathcal{G}_\omega$ on the vertices of Γ is transitive, $v(x) = 0$ for every vertex x of Γ. The distance between v and v_0 is equal to the size of the symmetric difference between $\operatorname{supp} v$ and $\operatorname{supp} v_0$. As v is a vertex of $\mathcal{V}$, this distance must be finite and then $\operatorname{supp} v_0$, which is equal to Γ_+, must be finite.

In the same way, if $v(0^\infty) = 1$, then $v(x) = 1$ for every vertex of Γ and, therefore, Γ_+^c is finite. □

Proposition 2.4.2. *Let ω be a sequence in $\{0,1,2\}^\infty$. If the action $\mathcal{G}_\omega \curvearrowright \mathcal{X}$ has a bounded orbit, then $\mathcal{G}_\omega$ fixes a vertex of $\mathcal{X}$.*

Proof. We want to show that every bounded orbit has a center which will be a fixed point. The cube complex $\mathcal{X}$ is not locally finite and is therefore not a complete metric space. Then, the existence of such a center can not be proved using the usual general results and finding this point requires some work. To do this, we will embed $\mathcal{X}$ in a Hilbert space, where every bounded orbit has a center, and prove that this point comes from a vertex of $\mathcal{X}$.

We consider the Hilbert space of square summable functions on the vertices of Γ:

$$\ell^2(V(\Gamma)) = \{\varphi : \Gamma \to \mathbf{R} : \sum_{x \in V(\Gamma)} \varphi(x)^2 < \infty\}.$$

The set $\mathcal{V}$ of vertices of $\mathcal{X}$ is also a set of functions on the vertices of Γ and can be embedded isometrically in $\ell^2(V(\Gamma))$ via

$$\begin{aligned} p : \mathcal{V} &\to \ell^2(V(\Gamma)) \\ v &\mapsto p_v \end{aligned}$$

where v_0 is the vertex defined in Definition 2.3.4 and

$$p_v(x) = \begin{cases} 1 & \text{if } v_0(x) \neq v(x) \\ 0 & \text{if } v_0(x) = v(x) \end{cases}.$$

We define the action $\mathcal{G}_\omega \curvearrowright \ell^2(V(\Gamma))$ as

$$(g\varphi)(x) = \begin{cases} \varphi(g^{-1}x) & \text{if } v_0(g^{-1}x) = v_0(x) \\ 1 - \varphi(g^{-1}x) & \text{if } v_0(g^{-1}x) \neq v_0(x) \end{cases}$$

for every $g \in \mathcal{G}_\omega$, $\varphi \in \ell^2(V(\Gamma))$ and $x \in \Gamma$. It can be checked that this action is compatible with the action $\mathcal{G}_\omega \curvearrowright \mathcal{V}$, in the sense that $gp_v = p_{gv}$.

A bounded set of a Hilbert space has a unique center, see [9, Lemma 2.2.7]. Suppose that there exists a bounded orbit in $\mathcal{X}$. The embedding p sends this orbit to a bounded orbit in $\ell^2(V(\Gamma))$ of which the center c is a fixed point. We claim that, as the action of $\mathcal{G}_\omega$ on the vertices Γ is transitive, this function c can only take values in $\{0, 1\}$. Indeed, for every pair x and y of vertices of Γ, there exists an element g of $\mathcal{G}_\omega$ such that $g^{-1}x = y$. If $v_0(x) = v_0(y)$, the equality $c(x) = gc(x)$ implies that $c(x) = c(y)$. If $v_0(x) \neq v_0(y)$, then

$$c(x) = gc(x) = 1 - c(g^{-1}x) = 1 - c(y).$$

In the same way, we can show that $c(y) = 1 - c(x)$ and then $c(x)$ and $c(y)$ take values only in $\{0, 1\}$.

By the definition of $\ell^2(V(\Gamma))$, all but finitely many values of c must be equal to 0. Then, using the definition of p, the point c is the image of the vertex of $\mathcal{X}$ which is equal to v_0 everywhere except on the vertices of Γ where c is equal to 1. □

The two previous propositions imply the following theorem.

Theorem 2.4.3. *Let ω be a sequence in $\{0, 1, 2\}^\infty$. Then, the action $\mathcal{G}_\omega \curvearrowright \mathcal{X}$ does not have a bounded orbit for every ω.*

Remark 2.4.4. We can ask if it is possible to construct an action of $\mathcal{G}_\omega$ without bounded orbit on a CAT(0) cube complex of finite dimension, or at least of locally finite dimension. A negative answer can be given for the groups $\mathcal{G}_\omega$ whose sequences ω contain an infinity of $0, 1$ and 2. Such groups are 2-groups, see [41], and Genevois proved that they can not act on smaller CAT(0) cube complexes without bounded orbit.

Theorem 2.4.5 ([34])**.** *Let G be a group acting on a CAT(0) cube complex X. Assume that there is a finite number of orbits of hyperplanes and that X does not contain an Hilbert cube*[3]*. If the action is purely periodic, i.e. every element g defines a periodic isometry of X, then G stabilises a finite dimensional cube.*

[3]A Hilbert cube is the product of countably infinitely many copies of intervals $[0, 1]^n$

Properness and Faithfulness

In this section, we will consider the subfamily of Grigorchuk groups $\mathcal{G}_\omega$ which are indexed by sequences ω in $\{0,1,2\}^\infty$ without repetition, i.e. $\omega_i \neq \omega_{i+1}$ for every i, and prove that their action on $\mathcal{X}$ is proper and faithful. This subfamily contains uncountably many groups.

The idea of the proof is to show that stabilizing a subset of vertices of Γ is a strong condition and only few elements of $\mathcal{G}_\omega$ succeed. We will decompose this proof into different steps. Firstly, we will compute the stabilizers of two particular subsets of $V(\Gamma)$. Secondly, we will show that the stabilizer of a subset of vertices of Γ which has a finite symmetric difference with the two particular subsets, is also finite. Finally, we will explain how these stabilizers of subsets can be related with the stabilizers of the cubes of $\mathcal{X}$.

Let us begin by defining precisely the notion of stabilizer of a subset of $V(\Gamma)$ and the 2 particular subsets that we will study.

Definition 2.4.6. Let $\Omega \subset V(\Gamma)$ be a subset of vertices of Γ and $\mathcal{G}_\omega$ a Grigorchuk group. The stabilizer of Ω for the action of $\mathcal{G}_\omega \curvearrowright \mathcal{X}$ is

$$\mathrm{Stab}_\omega(\Omega) = \{g \in \mathcal{G}_\omega : g\Omega = \Omega\}.$$

Definition 2.4.7. Two particular subsets of $V(\Gamma)$ that we will study are Γ_+, which we have already described above, and $\widetilde{\Gamma}_+ = \Gamma_+ \setminus \{0^\infty\}$.

The following lemma explains the behavior of the vertices of Γ between these two subsets when we add a digit at the beginning of their binary writing.

Lemma 2.4.8. *Let x be a vertex of Γ. Then*

- $x \in \Gamma_+ \Leftrightarrow 0x \in \widetilde{\Gamma}_+^c$.
- $x \in \Gamma_+^c \Leftrightarrow 0x \in \widetilde{\Gamma}_+$.
- $x \in \Gamma_+^c \Rightarrow 1x \in \widetilde{\Gamma}_+$.
- $x \in \widetilde{\Gamma}_+ \Leftrightarrow 0x \in \Gamma_+^c$.
- $x \in \widetilde{\Gamma}_+ \Leftrightarrow 1x \in \widetilde{\Gamma}_+^c$.
- $x \in \widetilde{\Gamma}_+^c \Leftrightarrow 0x \in \Gamma_+$.
- $x \in \widetilde{\Gamma}_+^c \Leftrightarrow 1x \in \widetilde{\Gamma}_+$.

Proof. We proved in Proposition 2.3.3 that Γ_+ can be fully described by looking at the parity of the position of the last digit 1. This allows us to give the following descriptions:

$$\begin{aligned}
\Gamma_+ &= \{0^\infty\} \cup \{x_1x_2\dots x_{2n+1}0^\infty\}\\
\Gamma_+^c &= \{x_1x_2\dots x_{2n}0^\infty\}\\
\widetilde{\Gamma}_+ &= \{x_1x_2\dots x_{2n+1}0^\infty\}\\
\widetilde{\Gamma}_+^c &= \{0^\infty\} \cup \{x_1x_2\dots x_{2n}0^\infty\}
\end{aligned}$$

where every last x_i is equal to 1. Every point of the lemma can be proved by direct computations using these descriptions. $\square$

We will study the restrictions of elements of $\mathrm{Stab}_\omega(\Gamma_+)$ and $\mathrm{Stab}_\omega(\widetilde{\Gamma}_+)$ on the first level of the tree.

Lemma 2.4.9. *Let ω be a sequence in $\{0,1,2\}^\infty$ and g an element of $\mathcal{G}_\omega$.*

1. *If g stabilizes the first level of the tree and $g \in \mathrm{Stab}_\omega(\Gamma_+)$, then $g_0, g_1 \in \mathrm{Stab}_{\sigma\omega}(\widetilde{\Gamma}_+)$.*
2. *If g stabilizes the first level of the tree and $g \in \mathrm{Stab}_\omega(\widetilde{\Gamma}_+)$, then $g_0 \in \mathrm{Stab}_{\sigma\omega}(\Gamma_+)$ and $g_1 \in \mathrm{Stab}_{\sigma\omega}(\Gamma_+) \cap \mathrm{Stab}_{\sigma\omega}(\widetilde{\Gamma}_+)$.*
3. *If g does not stabilize the first level of the tree and $g \in \mathrm{Stab}_\omega(\widetilde{\Gamma}_+)$, then $g_0, g_1 \in \mathrm{Stab}_{\sigma\omega}(\Gamma_+) \cap \mathrm{Stab}_{\sigma\omega}(\widetilde{\Gamma}_+)$.*

Proof. Let's start by noting that an element g is in $\mathrm{Stab}_\omega(\Gamma_+)$ (resp. $\mathrm{Stab}_\omega(\widetilde{\Gamma}_+)$) if and only if it is in $\mathrm{Stab}_\omega(\Gamma_+^c)$ (resp. $\mathrm{Stab}_\omega(\widetilde{\Gamma}_+^c)$). The proof is an application of Lemma 2.4.8.

1. Let g be an element of $\mathrm{Stab}_\omega(\Gamma_+)$ which stabilizes the first level of the tree. Then,
$$\begin{aligned} x \in \widetilde{\Gamma}_+ &\Rightarrow 0x \in \Gamma_+^c \\ &\Rightarrow g(0x) \in \Gamma_+^c \\ &\Rightarrow 0g_0(x) \in \Gamma_+^c \\ &\Rightarrow g_0(x) \in \widetilde{\Gamma}_+ \\ &\Rightarrow g_0 \in \mathrm{Stab}_{\sigma\omega}(\widetilde{\Gamma}_+), \end{aligned}$$
$$\begin{aligned} x \in \widetilde{\Gamma}_+^c &\Rightarrow 1x \in \Gamma_+ \\ &\Rightarrow g(1x) \in \Gamma_+ \\ &\Rightarrow 1g_1(x) \in \Gamma_+ \\ &\Rightarrow g_1(x) \in \widetilde{\Gamma}_+^c \\ &\Rightarrow g_1 \in \mathrm{Stab}_{\sigma\omega}(\widetilde{\Gamma}_+), \end{aligned}$$
2. Let g be an element of $\mathrm{Stab}_\omega(\widetilde{\Gamma}_+)$ which stabilizes the first level of the tree. Then,
$$\begin{aligned} x \in \Gamma_+^c &\Rightarrow 0x \in \widetilde{\Gamma}_+ \\ &\Rightarrow g(0x) \in \widetilde{\Gamma}_+ \\ &\Rightarrow 0g_0(x) \in \widetilde{\Gamma}_+ \\ &\Rightarrow g_0(x) \in \Gamma_+^c \\ &\Rightarrow g_0 \in \mathrm{Stab}_{\sigma\omega}(\Gamma_+), \end{aligned}$$
$$\begin{aligned} x \in \widetilde{\Gamma}_+ &\Rightarrow 1x \in \widetilde{\Gamma}_+^c \\ &\Rightarrow g(1x) \in \widetilde{\Gamma}_+^c \\ &\Rightarrow 1g_1(x) \in \widetilde{\Gamma}_+^c \\ &\Rightarrow g_1(x) \in \widetilde{\Gamma}_+ \\ &\Rightarrow g_1 \in \mathrm{Stab}_{\sigma\omega}(\widetilde{\Gamma}_+), \end{aligned}$$

$$\begin{aligned} x \in \Gamma_+^c &\Rightarrow 1x \in \widetilde{\Gamma}_+ \\ &\Rightarrow g(1x) \in \widetilde{\Gamma}_+ \\ &\Rightarrow 1g_1(x) \in \widetilde{\Gamma}_+ \\ &\Rightarrow g_1(x) \in \Gamma_+^c \\ &\Rightarrow g_1 \in \mathrm{Stab}_{\sigma\omega}(\Gamma_+). \end{aligned}$$

3. Let g be an element of $\mathrm{Stab}_\omega(\widetilde{\Gamma}_+)$ which does not stabilize the first level of the tree. Then,

$$\begin{aligned} x \in \Gamma_+ &\Rightarrow 0x \in \widetilde{\Gamma}_+^c \\ &\Rightarrow g(0x) \in \widetilde{\Gamma}_+^c \\ &\Rightarrow 1g_0(x) \in \widetilde{\Gamma}_+^c \\ &\Rightarrow g_0(x) \in \widetilde{\Gamma}_+ \subset \Gamma_+ \\ &\Rightarrow g_0 \in \mathrm{Stab}_{\sigma\omega}(\Gamma_+), \end{aligned}$$

$$\begin{aligned} x \in \widetilde{\Gamma}_+ \subset \Gamma_+ &\Rightarrow 0x \in \widetilde{\Gamma}_+^c \\ &\Rightarrow g(0x) \in \widetilde{\Gamma}_+^c \\ &\Rightarrow 1g_0(x) \in \widetilde{\Gamma}_+^c \\ &\Rightarrow g_0(x) \in \widetilde{\Gamma}_+ \\ &\Rightarrow g_0 \in \mathrm{Stab}_{\sigma\omega}(\widetilde{\Gamma}_+), \end{aligned}$$

$$\begin{aligned} x \in \Gamma_+^c &\Rightarrow 1x \in \widetilde{\Gamma}_+ \\ &\Rightarrow g(1x) \in \widetilde{\Gamma}_+ \\ &\Rightarrow 0g_1(x) \in \widetilde{\Gamma}_+ \\ &\Rightarrow g_1(x) \in \Gamma_+^c \\ &\Rightarrow g_1 \in \mathrm{Stab}_{\sigma\omega}(\Gamma_+), \end{aligned}$$

$$\begin{aligned} x \in \widetilde{\Gamma}_+^c &\Rightarrow 1x \in \widetilde{\Gamma}_+ \\ &\Rightarrow g(1x) \in \widetilde{\Gamma}_+ \\ &\Rightarrow 0g_1(x) \in \widetilde{\Gamma}_+ \\ &\Rightarrow g_1(x) \in \widetilde{\Gamma}_+^c \\ &\Rightarrow g_1 \in \mathrm{Stab}_{\sigma\omega}(\widetilde{\Gamma}_+). \end{aligned}$$

□

We recall the following classical lemma which gives an estimate of the size of restrictions of an element which fixes the first level of the tree.

Lemma 2.4.10. *Let ω be a sequence in $\{0,1,2\}^\infty$ and g an element of $\mathcal{G}_\omega$ which stabilizes the first level of the tree. Then,*

$$l(g_i) \leq \frac{l(g)+1}{2} \quad i = 1,2$$

where $l(g)$ is the length of g with respect to the corresponding generating set.

Proof. It is a straightforward generalization of [27, Lemma VIII.46]. □

We can now compute explicitly the stabilizers of Γ_+ and $\widetilde{\Gamma}_+$.

Proposition 2.4.11. *Let ω be a sequence without repetition. Then,*

$$\begin{aligned} \mathrm{Stab}_\omega(\Gamma_+) &= \langle a, u_{\omega,1} \rangle \cong D_8 \\ \mathrm{Stab}_\omega(\widetilde{\Gamma}_+) &= \langle b_\omega, c_\omega, d_\omega \rangle \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \end{aligned}$$

where $u_{\omega,1}$ is the generator in $\{b_\omega, c_\omega, d_\omega\}$ such that $u_{\omega,1}0^\infty = 0^\infty$, or explicitly,

$$u_{\omega,1} = \begin{cases} b_\omega & \omega_1 = 2 \\ c_\omega & \omega_1 = 1 \\ d_\omega & \omega_1 = 0 \end{cases}.$$

Proof. We will proceed by induction on the length of the elements of the group in parallel for the 2 parts of the proposition and all the sequences ω without repetition.

Let us begin by some notations. We denote by $u_{\omega,n}$ the generator in $\{b_\omega, c_\omega, d_\omega\}$ such that $u_{\omega,n}1^n0^\infty = 1^n0^\infty$, or explicitly,

$$u_{\omega,n} = \begin{cases} b_\omega & \omega_n = 2 \\ c_\omega & \omega_n = 1 \\ d_\omega & \omega_n = 0 \end{cases}.$$

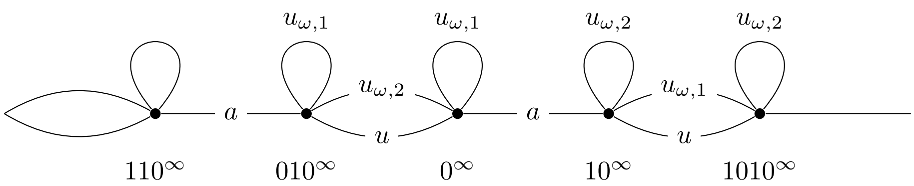
For the initial step, we verify the statements by hand for the elements of length smaller or equal to 2. It is clear that $\langle a, u_{\omega,1} \rangle \subset \mathrm{Stab}_\omega(\Gamma_+)$ and $\langle b_\omega, c_\omega, d_\omega \rangle \subset \mathrm{Stab}_\omega(\widetilde{\Gamma}_+)$. For all the other elements, an example of a vertex that comes out of the sets is given in Figure 2.7.

We now suppose that, for a fixed integer $n \geq 3$ and all the admissible sequences ω, all the elements of length at most n are in the desired subgroups. Recall that the stabilizer of the first level of the tree has an explicit generating set:

$$\mathrm{Stab}_\omega(1) = \langle b_\omega, c_\omega, d_\omega, ab_\omega a, ac_\omega a, ad_\omega a \rangle.$$

The restrictions of these generators are

$$\begin{aligned} b_\omega &= (I_\omega^2, b_{\sigma\omega}) \\ c_\omega &= (I_\omega^1, c_{\sigma\omega}) \\ d_\omega &= (I_\omega^0, d_{\sigma\omega}) \\ ab_\omega a &= (b_{\sigma\omega}, I_\omega^2) \\ ac_\omega a &= (c_{\sigma\omega}, I_\omega^1) \\ ad_\omega a &= (d_{\sigma\omega}, I_\omega^0). \end{aligned}$$



(a) The neighborhood of 0^∞ on the graph Γ_ω. The third generator which is not $u_{\omega,1} or u_{\omega,2}$ is denoted by u.

$u_{\omega,2}$	0^∞
w	0^∞
$au_{\omega,2}$	0^∞
aw	0^∞
$u_{\omega,2}a$	10^∞
wa	10^∞

(b) For every element of $\mathcal{G}_\omega$ of length smaller to 2 which is not in $\langle a, u_{\omega,1}\rangle$, we exhibit a vertex of Γ_+ whose image is in Γ_+^c.

a	10^∞
$au_{\omega,1}$	1010^∞
$au_{\omega,2}$	10^∞
aw	1010^∞
$u_{\omega,1}a$	10^∞
$u_{\omega,2}a$	10^∞
wa	10^∞

(c) For every element of $\mathcal{G}_\omega$ of length smaller to 2 of $\mathcal{G}_\omega$ which is not in $\langle u_{\omega,1}, u_{\omega,2}, u\rangle$, we exhibit a vertex of $\widetilde{\Gamma}_+$ whose image is in $\widetilde{\Gamma}_+^c$.

Figure 2.7: Initial step for the elements of small length.

Let g be an element of $\mathcal{G}_\omega$ of length $n+1$. We note that if σ is a sequence without repetition, $\sigma\omega$ is also. There are 3 cases to prove.

1. **If g is in $\mathrm{Stab}_\omega(\Gamma_+)$.** We notice that g is in $\mathrm{Stab}_\omega(\Gamma_+)$ if and only if ag also is. By supposing that g can be of length $n+2$, we can assume that g stabilizes the first level of the tree. By Lemmas 2.4.9 and 2.4.10, g_0 and g_1 are of length at most n and are in $\mathrm{Stab}_{\sigma\omega}(\widetilde{\Gamma}_+)$. Using induction's hypothesis, g_0 and g_1 are in $\langle b_{\sigma\omega}, c_{\sigma\omega}, d_{\sigma\omega}\rangle$. As the restriction g_0 does not contain the generator a in its writing, the element g can only contain the generator $u_{\omega,1}$ of the set $\{b_\omega, c_\omega, d_\omega\}$. In the same way, the restriction g_1 does not contain the generator a and then the element g can only contain the generator $au_{\omega,1}a$ of the set $\{ab_\omega a, ac_\omega a, ad_\omega a\}$. Therefore, g is in $\langle u_{\omega,1}, au_{\omega,1}a\rangle$, but we may have multiply g by a, then g is in $\langle a, u_{\omega,1}\rangle$.

2. **If g is in $\mathrm{Stab}_\omega(\widetilde{\Gamma}_+)$ and stabilizes the first level.** Combining Lemmas 2.4.9 and 2.4.10, the restriction g_0 is in $\langle a, u_{\sigma\omega,1}\rangle$ and the restriction g_1 is in $\langle u_{\sigma\omega,1}\rangle$. The form of g_0 allows the element g to contain only the generator $au_{\omega,2}a$ of the set $\{ab_{\sigma\omega}a, ac_{\sigma\omega}a, ad_{\sigma\omega}a\}$ because this is the only one which projects on $u_{\sigma\omega,1}$. The restriction g_1 does not contain the generator a, then the element g can only contain the element $au_{\omega,1}a$ of $\{ab_{\sigma\omega}a, ac_{\sigma\omega}a, ad_{\sigma\omega}a\}$. As $\omega_1 \neq \omega_2$, the element $au_{\omega,1}a$ is not equal to the element $au_{\omega,2}a$ and then g is in $\langle b_\omega, c_\omega, d_\omega\rangle$.

3. **If g is in $\mathrm{Stab}_\omega(\widetilde{\Gamma}_+)$ and does not stabilize the first level.** Once again by combining Lemmas 2.4.9 and 2.4.10, the restrictions g_0 and g_1

are in $\langle u_{\sigma\omega,1}\rangle$. The form of g_0 allows only the generators $\{u_{\omega,1}, au_{\omega,2}\}$ and g_1 only the generators $\{u_{\omega,2}, au_{\omega,1}\}$. As $\omega_1 \neq \omega_2$, these generators are different and then g is equal to 1.

For the explicit isomorphisms, it is left to the reader to generalize the case of $\mathcal{G}_{(012)^\infty}$ proved in [27, VIII.B.10] and [27, VIII.B.16]. □

The second step of the proof is to show that it is also difficult to stabilize subsets which differ only on a finite number of vertices with Γ_+.

Proposition 2.4.12. *Let ω be a sequence without repetition and Ω a subset of vertices of Γ such that $\Omega\Delta\Gamma_+$ is finite. Then $\mathrm{Stab}_\omega(\Omega)$ is finite.*

Proof. We denote by Λ the symmetric difference $\Omega\Delta\Gamma_+$. This is a finite set, therefore there exists an even integer n such that, for every x in Λ, there exists $x' \in \{0,1\}^n$ such that $x = x'0^\infty$.

Let x be an arbitrary prefix of length n such that $x0^\infty$ is an element of Ω and let y be an element of $\{0,1\}^\infty$. We claim that

$$xy \in \Omega \text{ if and only if } y \in \Gamma_+.$$

The claim is trivial if $y = 0^\infty$. Now, let us assume that y is not equal to 0^∞. As the length of x is even, the parities of the positions of the last 1 of y and of xy are equal. Then y is in Γ_+ if and only if xy is also in Γ_+. Moreover, the element xy is in Ω if and only if xy is in Γ_+. Indeed, all the elements of the symmetric difference Λ have a prefix of length shorter or equal than n which is not the case of xy as y is not equal to 0^∞. The claim is therefore true.

In the same way, let x be a prefix of length n such that $x0^\infty$ is not an element of Ω and let y be an element of $\{0,1\}^\infty$, we claim that

$$xy \in \Omega \text{ if and only if } y \in \widetilde{\Gamma}_+.$$

Let $K = \mathrm{Stab}_\omega(\Omega) \cap \mathrm{Stab}_\omega(n)$ be the subgroup of elements which stabilizes Ω as well as the n^{th} level of the tree. For every vertex x of the n^{th} level of the tree and every element g of K, the restriction g_x is an element of $\mathrm{Stab}_{\sigma^n\omega}(\Gamma_+) \cup \mathrm{Stab}_{\sigma^n\omega}(\widetilde{\Gamma}_+)$.

Indeed, if $x0^\infty$ is in Ω, the following equivalences are true for all $y \in \Gamma_+$:

$$\begin{aligned} y \in \Gamma_+ &\Leftrightarrow xy \in \Omega \\ &\Leftrightarrow g(xy) \in \Omega \\ &\Leftrightarrow xg_x(y) \in \Omega \\ &\Leftrightarrow g_x(y) \in \Gamma_+ \end{aligned}$$

and then g_x is in $\mathrm{Stab}_{\sigma^n\omega}(\Gamma_+)$. In the same way, if $x0^\infty$ is not in Ω:

$$\begin{aligned} y \in \widetilde{\Gamma}_+ &\Leftrightarrow xy \in \Omega \\ &\Leftrightarrow g(xy) \in \Omega \\ &\Leftrightarrow xg_x(y) \in \Omega \\ &\Leftrightarrow g_x(y) \in \widetilde{\Gamma}_+ \end{aligned}$$

and then g_x is in $\mathrm{Stab}_{\sigma^n\omega}(\widetilde{\Gamma}_+)$. Therefore, we can define the following embedding:

$$K \hookrightarrow \prod_{v\in\{0,1\}^n} \left(\mathrm{Stab}_{\sigma^n\omega}(\Gamma_+) \cup \mathrm{Stab}_{\sigma^n\omega}(\widetilde{\Gamma}_+)\right).$$

We proved in Proposition 2.4.11 that $\mathrm{Stab}_{\sigma^n\omega}(\Gamma_+)$ and $\mathrm{Stab}_{\sigma^n\omega}(\widetilde{\Gamma}_+)$ are finite and then K is also finite. As K is a subgroup of $\mathrm{Stab}_\omega(\Omega)$ of index at most 2^n, the stabilizer is finite. □

Corollary 2.4.13. *Let ω be a sequence without repetition, v a vertex of $\mathcal{X}$ with a support Λ and n an even integer such that the prefixes of the elements of $\Lambda\Delta\Gamma_+$ are at most of length n. Then,*

$$|\,\mathrm{Stab}_\omega(v)| \leq 8\cdot 4\cdot 4^n$$

Proof. It is a direct computation using the embedding above, the cardinals of D_8 and $\mathbf{Z}/2\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$ and $[\mathcal{G}_\omega, \mathrm{Stab}_\omega(n)] = 2^n$. □

We can now relate these stabilizers of subsets and the stabilizers of the cubes of $\mathcal{X}$.

Proposition 2.4.14. *Let ω be a sequence without repetition, then* $\mathrm{Stab}_\omega(C) = \{g\in\mathcal{G}_\omega : gC = C\}$ *is finite for every cube C of $\mathcal{X}$.*

Proof. We begin by the case of the vertices.

Let v be a vertex of $\mathcal{X}$. Let us denote by $\Omega\subset\Gamma$ the support of v. It is straightforward that

$$\mathrm{Stab}_\omega(v) = \mathrm{Stab}_\omega(\Omega).$$

The vertex v is in the connected cube $\mathcal{X}$, then the symmetric $\Omega\Delta\Gamma_+$ is finite and then $\mathrm{Stab}_\omega(\Omega)$ is finite by Proposition 2.4.12. Now, we will show that every n-cube has also finite stabilizer.

Let us suppose by contradiction that there exists an n-cube C with an infinite stabilizer. The elements of this stabilizer send vertices of C to vertices of C. We pick a vertex v on the boundary of the cube and we consider its orbit by the action of $\mathrm{Stab}_\omega(C)$. As this subgroup is infinite, there is at least one vertex w of C, which is reached infinitely many times. We define the set K as

$$K := \{g\in\mathrm{Stab}_\omega(C) : g.v = w\}.$$

It is infinite by the choose of w. For an element g of K, the subset Kg^{-1} is in $\mathrm{Stab}_\omega(w)$ and is therefore infinite, which is a contradiction with Proposition 2.4.12. □

We can now prove the main theorem of this section.

Theorem 2.4.15. *Let ω be an element of $\{0,1,2\}^\infty$ without repetition. Then the action of $\mathcal{G}_\omega$ on $\mathcal{X}$ is proper and faithful.*

Proof. The properness is a consequence of Proposition 2.4.14.

To prove that the action is faithful, we need to show that

$$\bigcap_{\substack{\Omega\subset V(\Gamma)\\ |\Omega\Delta V(\Gamma)|<\infty}} \mathrm{Stab}_\omega(\Omega) = \bigcap_{v\in\mathcal{V}} \mathrm{Stab}_\omega(v) = \{1\}.$$

By Proposition 2.4.11, the intersection $\mathrm{Stab}_\omega(\Gamma) \cap \mathrm{Stab}_\omega(\widetilde{\Gamma}_+) = \{1, d_\omega\}$. Moreover, d_ω does not stabilize $\Gamma_+ \setminus \{0^\infty, 1010^\infty\}$ because $d_\omega 101^\infty = 10^\infty$. Then, the intersection above is trivial. □

We can further refine the description of the stabilizers.

Proposition 2.4.16. *Let ω be a sequence without repetition and H be a finite subgroup of $\mathcal{G}_\omega$. Then, there exists a vertex v of $\mathcal{X}$ such that $H < \mathrm{Stab}_\omega(v)$.*

Proof. We consider the subset $\Lambda \subset \Gamma$ defined as

$$\Lambda = \bigcup_{h \in H} h\Gamma_+.$$

In the proof of Proposition 2.3.5, we showed that the symmetric difference $\Gamma_+ \Delta g.\Gamma_+$ is always finite and so is $\Lambda \Delta \Gamma_+$. Therefore, the vertex $v = \chi_\Lambda$ is a vertex of $\mathcal{X}$ and , as $h\Lambda = \Lambda$, it is stabilized by H. □

Corollary 2.4.17. *Let ω be a sequence without repetition. The stabilizers of the vertices of $\mathcal{X}$ are arbitrarily large.*

Proof. The order of the elements of $\mathcal{G}_\omega$ is not bounded, then there exist finite arbitrarily large finite subgroups in $\mathcal{G}_\omega$. By the previous proposition, we can also find arbitrarily large stabilizers of vertices. □

2.5 Classifying space of proper actions

Definition 2.5.1. A classifying space of proper actions of a group G, denoted by $\underline{\mathrm{E}}\mathrm{G}$, is a topological space which admits a proper G-action and with the following property: if X is any space with a proper G-action, then there exists a G-equivariant map $f : X \to \underline{\mathrm{E}}\mathrm{G}$ and any two G-equivariant maps $X \to \underline{\mathrm{E}}\mathrm{G}$ admit a G-equivariant homotopie between them.

As proved in Theorem 2.4.15, the cube complex $\mathcal{X}$ is a topological space which admits a proper $\mathcal{G}_\omega$-action if ω does not contain repetition. We will show that it satisfies the property above.

Theorem 2.5.2. *Let $\omega \in \{0, 1, 2\}^\infty$ be a sequence without repetition. Then the CAT(0) cube complex $\mathcal{X}$ is a model for $\underline{\mathrm{E}}\mathrm{G}$, the classifying space of proper actions of $\mathcal{G}_\omega$.*

Proof. We use the reformulation of the universality definition proved in [7, Prop 1.8]. Applied to our situation, it means that in order to prove that CAT(0) cube complex is a model for $\underline{\mathrm{E}}\mathcal{G}_\omega$, it is enough to verify the following two points.

1. If H is a finite subgroup of $\mathcal{G}_\omega$, then there exists a point x in $\mathcal{X}$ which is fixed by H.

2. View $\mathcal{X} \times \mathcal{X}$ as a space endowed with the usual diagonal action $g(x_0, x_1) = (gx_0, gx_1)$. Denote by $p_0, p_1 : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ the two projections on each component, then there exists a $\mathcal{G}$-equivariant homotopie between p_0 and p_1.

The first point is a consequence of Proposition 2.4.16. For the second one, we define the following homotopy

$$h((x_0, x_1), t) = tx_0 + (1-t)x_1.$$

This is well-defined as $\mathcal{X}$ is convex. □

2.6 Further directions of research

A group action $G \curvearrowright X$ of a group G on a metric space X is called *metrically proper* if $d(x, g_n x) \to \infty$ for every infinite family $\{g_n\}$ of elements of $\mathcal{G}$ and every point x in X. This property is strictly stronger than the definition of proper action used above if the space X is not locally compact. For example, consider the Cayley graph of a free group generated by a countable infinite family of generators and the canonical associated action of the group on it. All the stabilizers are trivial hence the action is proper, but all the generators send the identity at distance 1 and so it is not metrically proper. We don't know if the actions $\mathcal{G}_\omega \curvearrowright \mathcal{X}$ are metrically proper.

Question 2.6.1. *Are there any sequences ω for which the action of $\mathcal{G}_\omega$ on $\mathcal{X}$ is metrically proper?*

Our intuition tells us that this is the case for the groups $\mathcal{G}_\omega$ where the action is proper. A positive answer will give an elegant proof of the Haagerup property for these groups by [21]. This property is known for all Grigorchuk groups using the fact that they have subexponential growth, whereas here we would have a more elementary proof.

It is not clear to us if the condition on the sequence ω appearing in Theorem 2.4.15 is purely technical or if it reflects a real difference in the behavior of actions. Indeed, we had never seen a condition of this type appear before in the study of Grigorchuk groups. It would be interesting to study the actions of the groups whose sequences have repetitions and to see if these are proper. However, it seems to us that the proof presented above is unlikely to be generalized to these cases.

Question 2.6.2. *Are the actions of the groups $\mathcal{G}_\omega$ proper if the sequence ω has a repetition?*

On Remark 2.4.4, we explain why the groups $\mathcal{G}_\omega$ which are 2-groups can not act without bounded orbit on a CAT(0) cube complex of locally finite dimension. However, we do not know any obstruction of the existence of such an action for groups with elements of infinite order.

Question 2.6.3. *Let $\mathcal{G}_\omega$ a group which contains an element of infinite order. Is it possible to construct an action of $\mathcal{G}_\omega$ on a CAT(0) cube complex of (locally) finite dimension without bounded orbit?*

There are several notions of boundaries of a CAT(0) cube complex, like the simplicial boundary [50, 51], the Roller boundary [89] and the Poisson-Furstenberg boundary [32, 33, 78]. An interesting question would be to study of the actions we have built on these boundaries and to understand what can be deduced from it.

Question 2.6.4. *Is it possible to understand the action of* $\mathcal{G}_\omega$ *on the boundaries of* $\mathcal{X}$*?*

Grigorchuk groups are examples of branch groups [6]. There are other classes of branch groups which share properties used in the construction of $\mathcal{X}$ and tools used in the proof (Schreier graph with more than one end, restriction of the action on the subtrees, reduction lemma, etc).

Question 2.6.5. *Is it possible to do the same construction for other finitely generated branch groups with a Schreier graph with at least 2 ends to obtain an action without bounded orbit, faithful, proper on a CAT(0) cube complex?*

Chapter 3

Property FW and wreath products of groups: a simple approach using Schreier graphs

3.1 Introduction

Property FW is a group property that is (for discrete groups) a weakening of the celebrated Kazdhan's property (T). It was introduced by Barnhill and Chatterji in [4]. It is a fixed point property for actions on wall spaces or, equivalently, on CAT(0) cube complexes. Therefore it stands between property FH (fixed points on Hilbert spaces, equivalent to property (T) for discrete groups) and property FA (fixed points on *arbres*[1]). It is known that all these properties are different, see [24] for examples of groups that distinguish them.

When working with group properties, it is natural to ask if they are stable under "natural" group operations. One such operation, of great use in geometric group theory, is the wreath product, that generalizes the direct product of two groups, see Definition 3.2.4.

In the context of properties defined by fixed points of actions, the first result concerning wreath products is due to Cherix, Martin and Valette and later refined by Neuhauser and concerns property (T).

Theorem 3.1.1 ([21,77]). *Let G, H be two discrete groups with G non-trivial and X a set on which H acts. The wreath product $G \wr_X H$ has property (T) if and only if G and H have property (T) and X is finite.*

In [24] Cornulier studied property FW using *cardinal definite functions* and while not explicitly stated in [24], the following result can be extracted from his work.

Theorem 3.1.2. *Let G, H be two finitely generated groups with G non-trivial and X a set on which H acts with finitely many orbits. The wreath product $G \wr_X H$ has property FW if and only if G and H have property FW and X is finite.*

[1] *Arbres* is the french word for trees.

The aim of this note is to give an explicit and elementary proof of this fact using a characterization of property FW via the number of ends of Schreier graphs, see Subsection 3.2 for the relevant definitions.

At this point, the curious reader might have two questions. First, is it possible to extend Theorem 3.1.2 beyond the realms of finitely generated groups and of actions with finitely many orbits? And secondly, is there a link between Theorems 3.1.1 and 3.1.2? In both cases, the answer is yes.

This is the subject of the more technical paper [65], which gives a unified proof of Theorems 3.1.1 and 3.1.2 as well as of similar results for the Bergman's property and more.

Organization of the paper The next section contains all the definitions as well as some examples, while Section 3.3 is devoted to the proof of Theorem 3.1.2 and some related results.

3.2 Definitions and examples

This section contains all the definitions, some standard but useful preliminary results as well as some examples.

Ends of Schreier graphs and property FW

In what follows, we will always assume that generating sets of groups are *symmetric*, that is we will look at $S \subset G$ such that $s \in S$ if and only if $s^{-1} \in S$. Our graphs will be undirected and we will sometimes identify a graph with its vertex set.

Let G be a group with symmetric generating set S, and let X be a non-empty set endowed with a left action $G \curvearrowright X$. The corresponding (left) *orbital graph* $\mathrm{Sch}_{\mathcal{O}}(G, X; S)$ is the graph with vertex set X and with an edge between x and y for every s in S such that $s.x = y$.

Definition 3.2.1. Let G be a group, H a subgroup of G and S a symmetric generating set. The *(left) Schreier graph* $\mathrm{Sch}(G, H; S)$ is the graph with vertices the left cosets $gH = \{gh \mid h \in H\}$ and for every $\{s, s^{-1}\} \subset S$ an edge between gH and sgH.

As the notation suggests, these two definitions are two faces of the same coin. More precisely, Schreier graphs of $G = \langle S\rangle$ are exactly the orbital graphs for transitive actions (equivalently: the connected components of orbital graphs) of $G = \langle S\rangle$. The correspondence is given by $H \mapsto G/H$ and $x_0 \in X \mapsto \mathrm{Stab}_G(x_0)$, where x_0 is an arbitrary element of X.

Schreier graphs are generalizations of the well-known *Cayley graphs*, with $\mathrm{Cay}(G; S) = \mathrm{Sch}(G, \{1\}; S)$, see Figures 3.1 and 3.2 for some examples.

If S and T are two generating sets of G, the graphs $\mathrm{Sch}(G, H; S)$ and $\mathrm{Sch}(G, H; T)$ does not need to be isomorphic. However, if both S and T are finite, then $\mathrm{Sch}(G, H; S)$ and $\mathrm{Sch}(G, H; T)$ are quasi-isometric, see [27, IV.B.21.iii] for a proof and Figure 3.1 for an example. The only fact we will use about quasi-isometries is that they preserve "large-scale properties" of the graph, as for example the number of ends. Observe that the requirement that both S and T are finite is crucial for the existence of a quasi-isometry between the corresponding Schreier graphs. Indeed, for every group $\mathrm{Cay}(G; G \setminus \{1\}) = \mathrm{Sch}(G, \{1\}; G \setminus \{1\})$ is always a complete graph on $|G|$ vertices, in particular $\mathrm{Cay}(\mathbf{Z}; \{1\})$ and $\mathrm{Cay}(\mathbf{Z}; \mathbf{Z} \setminus \{0\})$ are not quasi-isometric.

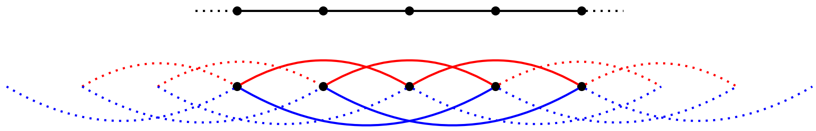


Figure 3.1: Fragments of two Cayley graphs of $\mathbf{Z}$ (2 ends), for the standard generating set $\{\pm 1\}$ and for the generating set $\{\pm 2, \pm 3\}$.

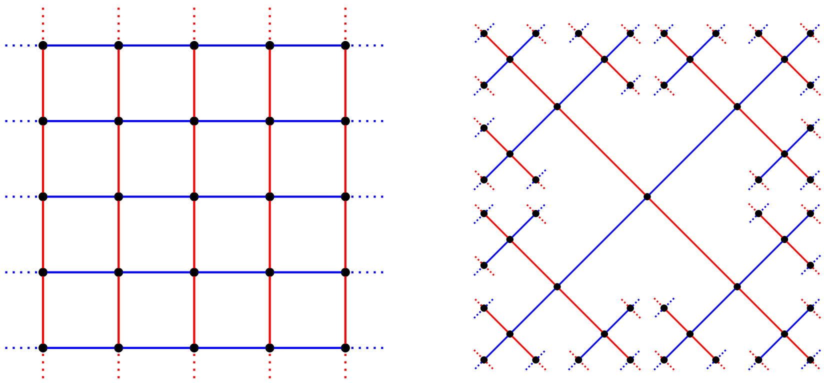


Figure 3.2: Fragments of the Cayley graphs of $\mathbf{Z}^2$ (1 end) on the left and of F_2 (infinitely many ends) on the right; with standard generating sets.

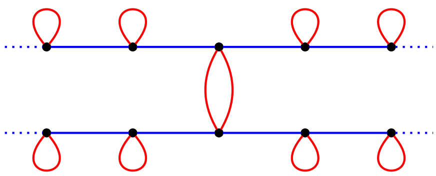


Figure 3.3: A fragment of a Schreier graph (with 4 ends) of the free group $F_2 = \langle x^{\pm 1}, y^{\pm 1} \rangle$ for the subgroup $H = \{x^2, y^n x y^{-n}, x y^n x y^{-n} x^{-1} \mid n \in \mathbf{Z} \setminus \{0\}\}$.

Let Γ be a graph and K a finite subset of vertices. The graph $\Gamma \setminus K$ is the subgraph of Γ obtained by deleting all vertices in K and all edges containing them. This graph is not necessarily connected.

Definition 3.2.2. Let Γ be a graph. The *number of ends* of Γ is the supremum, taken over all finite K, of the number of infinite connected components of $\Gamma \setminus K$.

There exist other characterizations of the number of ends in graphs, see [30] and the references therein, but Definition 3.2.2 is the one that best suits our purpose. A locally finite graph (i.e. such that every vertex has finite degree) is finite if and only if it has 0 end.

An important fact about the number of ends of a graph, is that it is an invariant of quasi-isometry, see [12]. In particular, if G is a finitely generated group it is possible to speak about the number of ends of the Schreier graph $\mathrm{Sch}(G, H; S)$ without specifying a particular finite generating set S. By a celebrated result of Hopf [58], the number of ends of a Cayley graph of a finitely generated group can only be 0, 1, 2 or infinite (in which case it is uncountable), see Figures 3.1 and 3.2 for some examples. On the other hand, Schreier graphs may have any number of ends in $\mathbf{N} \cup \{\infty\}$, see Figure 3.3 for an example of a graph with 4 ends. In fact, every regular graph of even degree is isomorphic to a Schreier graph, [46, 67].

We are now finally able to introduce property FW. Instead of giving the original definition in terms of actions on wall spaces, we will use an equivalent one for finitely generated groups, which essentially follows from [90], see [24] for a direct proof.

Definition 3.2.3. A finitely generated group G has *property FW* if all its Schreier graphs have at most one end.

It directly follows from the definition that all finite groups have property FW, but that $\mathbf{Z}$ does not have it. In fact, if G is a finitely generated group with a homomorphism onto $\mathbf{Z}$, then it does not have FW. Indeed, in this case $G \cong H \rtimes \mathbf{Z}$ for some H and the Schreier graph $\mathrm{Sch}(G, H; S)$ is isomorphic to a Cayley graph of $\mathbf{Z} \cong G/H$ and hence has 2 ends.

Property FW admits many distinct characterizations that allow to define it for groups that are non-necessarily finitely generated and even for topological groups. We refer the reader to [24] for a survey of these characterizations.

Wreath products

Let X be a set and G a group. We view $\bigoplus_X G$ as the set of functions from X to G with finite support:

$$\bigoplus_X G = \{\varphi\colon X \to G \mid \varphi(x) = 1 \text{ for all but finitely many } x\}.$$

This is naturally a group, where multiplication is taken pointwise.

If H is a group acting on X, then it naturally acts on $\bigoplus_X G$ by $(h.\varphi)(x) = \varphi(h^{-1}.x)$. This leads to the following standard definition.

Definition 3.2.4. Let G and H be groups and X be a set on which H acts. The *(retricted) wreath product* $G \wr_X H$ is the group $(\bigoplus_X G) \rtimes H$.

A prominent source of examples of wreath products is the ones of the form $G \wr_H H$, where H acts on itself by left multiplication. In particular, the group $(\mathbf{Z}/2\mathbf{Z}) \wr_{\mathbf{Z}} \mathbf{Z}$ is well-known under the name of the *lamplighter group*. The direct product $G \times H$ corresponds to wreath product over a singleton $G \wr_{\{*\}} H$.

Let S be a generating set of G and T a generating set of H. Let $\{x_i\}_{i \in I}$ be a choice of a representative in each H-orbit. Finally, let δ_x^s be the element of $\bigoplus_X G$ defined by $\delta_x^s(x) = s$ and $\delta_x^s(y) = 1_G$ if $y \neq x$ and let $\mathbf{1}$ be the constant function with value 1_G. It is a direct computation that

$$\{(\delta_{x_i}^s, 1_H) \mid s \in S, i \in I\} \cup \{(\mathbf{1}, t) \mid t \in T\}$$

is a generating set for $G \wr_X H$.

On the other hand, if $\{(\varphi_i, h_i) \mid i \in I\}$ is a generating set of $G \wr_X H$, then $\{h_i \mid i \in I\}$ is a generating set of H while $\{\varphi_i(x) \mid i \in I, x \in X\}$ is a generating set of G. Observe that since the φ_i take only finitely many values, if I is finite, so is $\{\varphi_i(x) \mid i \in I, x \in X\}$. We hence recover the following characterization of the finite generation of $G \wr_X H$.

Lemma 3.2.5. *The group $G \wr_X H$ is finitely generated if and only if both G and H are finitely generated and H acts on X with finitely many orbits.*

Proof. If $G \wr_X H$ is finitely generated, so is its abelianization $(G \wr_X H)^{\mathrm{ab}} \cong (\bigoplus_{X/H} G^{\mathrm{ab}}) \times H^{\mathrm{ab}}$, which implies that the orbit set X/H is finite. The other implications directly follow from the above discussion on generating sets. □

Using the above lemma, we could reformulate Theorem 3.1.2 in the following way: *Let G, H be two groups with G non-trivial and X a set on which H acts. Suppose that all three of G, H and $G \wr_X H$ are finitely generated. Then the wreath product $G \wr_X H$ has property FW if and only if G and H have property FW and X is finite.* While this formulation is formally equivalent to Theorem 3.1.2, it hints the fact that the finite generation hypothesis on G, H and $G \wr_X H$ are not necessary, but artefacts of using Schreier graphs in the proof. Indeed, the result remains true without these hypothesis, see [24, Propositions 5.B.3 & 5.B.4] and [65, Theorem A].

3.3 Proof of the main result

This section is devoted to the proof of Theorem 3.1.2. This proof is split into two parts: Lemma 3.3.2 and its Corollary 3.3.4, and Lemma 3.3.5.

We begin by an easy result on quotients.

Lemma 3.3.1. *Let G be a finitely generated group and H a quotient of G. If G has FW, then so does H.*

Proof. First, remark that if G is generated by a finite symmetric set S, the group H is generated by the projection of S that we will also denote by S. Moreover, any generating set of H can be obtained in such a way.

Let K be any subgroup of H and L its preimage in G. As $G/L \cong H/K$, forgetting about loops and multiedges, the Schreier graphs $\mathrm{Sch}(G, L; S)$ and $\mathrm{Sch}(H, K; S)$ are isomorphic. By assumption on G, the graph $\mathrm{Sch}(G, L; S)$ has at most one end. As adding loops or doubling edges do not change the number of ends, $\mathrm{Sch}(H, K; S)$ has also at most one end, and therefore H has property FW. □

We also have the following lemma on semi-direct products:

Lemma 3.3.2. *Let N and H be two finitely generated groups and $N \rtimes H$ a semi-direct product. Then*

1. *If $N \rtimes H$ has property FW, then so does H,*
2. *If both N and H have property FW, then $N \rtimes H$ also has property FW.*

Proof. The first part follows from Lemma 3.3.1.

For the second part, let S, respectively T, denote a finite generating set of N, respectively H. Then the group $G \coloneqq N \rtimes H$ is finitely generated by $U = (S \times \{1\}) \cup (\{1\} \times T)$.

Suppose that both N and H have property FW. We want to show that every Schreier graph of G has at most one end. If they are all finite, then there is nothing to prove (and G is finite). So let Γ be an infinite Schreier graph of G with respect to the generating set U. The groups N and H act on the vertices of Γ by restriction of the action of G. That is, $n.x = (n, 1).x$ and $h.x = (1, h).x$ for every vertex x of Γ. For each vertex x we define Γ_x^H (and respectively Γ_x^N) as the Schreier graph obtained from the action of H (respectively N) on the H-orbit (respectively N-orbit) of x. These are subgraphs of Γ. As N and H have property FW, the graphs Γ_x^H and Γ_x^N are either finite or one-ended; and we will prove that this implies that Γ has exactly one end.

Let K be a finite set of vertices of Γ. If x is in K and Γ_x^H is finite, add all vertices of Γ_x^H to K. By doing so for every x in K, we obtain a new finite set $K' \supset K$ of vertices of Γ. We will show that $\Gamma \setminus K'$ has only one infinite connected component. By definition of K', if x is not in K', then either Γ_x^H has one end or Γ_x^H does not contain vertices of K'.

Let x and y be two vertices, each of them lying in some infinite connected component of $\Gamma \setminus K'$. We will construct a path from x to y in $\Gamma \setminus K'$ as a concatenation of three smaller paths, see Figure 3.4, as follows. First, a path in $\Gamma_x^H \setminus K'$ from x to some z, then a path in $\Gamma_z^N \setminus K'$ from z to some $z' \in (\Gamma_z^N \cap \Gamma_y^H) \setminus K'$, and finally a path in $\Gamma_y^H \setminus K'$ from z' to y. In order to finish the proof, it remains to exhibit elements z and z' and the three desired paths.

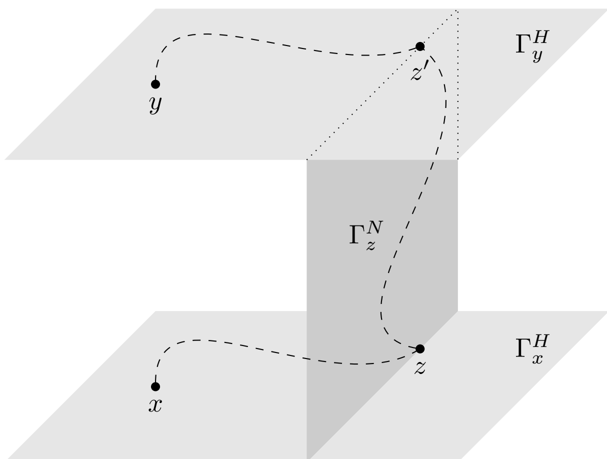


Figure 3.4: The path between x and y.

The action of G on Γ being transitive, there exists an element (n_0, h_0) of $N \rtimes H$ such that $(n_0, h_0).x = y$. Since K' is finite, the set $\Gamma_x^H \setminus K'$ is infinite. Moreover, there is infinitely many z in $\Gamma_x^H \setminus K'$ such that either Γ_z^N is one-ended or Γ_z^N does not intersect K'. For such a z there exists h such that $(1, h).x = z$. Now, the vertex $z' \coloneqq (hh_0^{-1}.n_0, h).x$ is both equal to $(hh_0^{-1}.n_0, 1)(1, h_0).x = (hh_0^{-1}.n_0, 1).z$ and to $(1, hh_0^{-1})(n_0, h_0).x = (1, hh_0^{-1}).y$.

That is, z' is in $\Gamma_z^N \cap \Gamma_y^H$. A direct computation shows us that the map $z \mapsto z'$ is injective: $z_1' = z_2'$ if and only if $z_1 = z_2$. Since K' is finite, there are only finitely many z' in K' and hence there are infinitely many $z \in \Gamma_x^H$ such that both z and z' are not in K' and either Γ_z^N is one-ended or Γ_z^N does not intersect K'.

In order to finish the proof, observe that the three graphs Γ_x^H, Γ_y^H and Γ_z^N are all either one-ended or do not intersect K'. Therefore, there is a path in $\Gamma_x^H \setminus K'$ from x to z as desired, and similarly for the paths from z to z' and z' to y. We have just proved that for any finite K the graph $\Gamma \setminus K$ has only one infinite connected component and therefore that Γ is one-ended. $\square$

We have the following result on direct products that can be obtained as a corollary of Lemma 3.3.2. It is also possible to give a short proof of it using Lemma 3.3.1 and a direct argument; details are left to the reader.

Corollary 3.3.3. *Let G and H be two finitely generated groups. Then $G \times H$ has property FW if and only if both G and H have property FW.*

By iterating Lemma 3.3.2, we obtain

Corollary 3.3.4. *Let G and H be two finitely generated groups and X a set on which H acts with finitely many orbits. Then,*

1. *If $G \wr_X H$ has property FW, then so does H,*
2. *If both G and H have property FW and X is finite, then $G \wr_X H$ has property FW.*

The following Lemma finishes the proof of Theorem 3.1.2.

Lemma 3.3.5. *Let G and H be two finitely generated groups with G non-trivial and such that H acts on some set X with finitely many orbits. If $G \wr_X H$ has property FW, then*

1. *G has property FW,*
2. *X is finite.*

Proof. Let us fix some finite generating sets S and T of G and H and let

$$U \coloneqq \{(\delta_x^s, 1_H) \mid s \in S\} \cup \{(\mathbf{1}, t) \mid t \in T\}$$

be the standard generating set of $G \wr_X H$.

If $G \wr_X H$ has property FW, then X is finite. Since H acts on X with finitely many orbits, it is enough to show that each orbit is finite. So let X' be an orbit and x_0 be an arbitrary vertex of X'. The group G acting on itself by left multiplications, we have the so-called *imprimitive action* of the wreath-product $G \wr_X H$ on $Y \coloneqq G \times X'$:

$$(\varphi, h).(g, x) \coloneqq (\varphi(h.x)g, h.x).$$

Since both $G \curvearrowright G$ and $H \curvearrowright X'$ are transitive, the action $G \wr_X H \curvearrowright Y$ is also transitive. Therefore, the orbital Schreier graph of $G \wr_X H \curvearrowright Y$ is isomorphic to the Schreier graph $\Gamma \coloneqq \mathrm{Sch}\left(G \wr_X H, \mathrm{Stab}((1_G, x_0)), U\right)$. We decompose this graph into leaves of the form $Y_g = \{g\} \times X'$. There are two types of edges in Γ, which are coming from the two sets of generators, see Figure 3.5. The first ones, of the form $(\mathbf{1}, t)$, give us on each leaf a copy of the orbital Schreier graph of $H \curvearrowright X'$. Indeed,

$$(\mathbf{1}, t).(g, x) = (g, t.x).$$

The second ones, of the form $(\delta^s_{x_0}, 1)$, give us loops everywhere except on vertices of the form (g, x_0). By direct computation, we see that the vertices (g, x_0) and (sg, x_0) connect the leaves Y_g and Y_{sg},

$$(\delta^s_{x_0}, 1).(g, x) = \begin{cases} (g, x) & \text{if } x \neq x_0, \\ (sg, x) & \text{if } x = x_0. \end{cases}$$

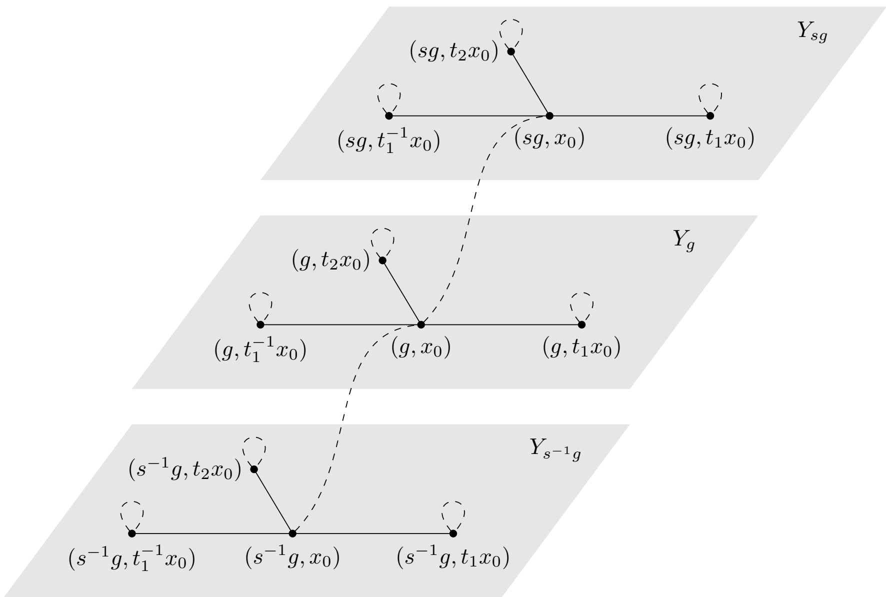


Figure 3.5: The leaf structure of the orbital Schreier graph of $G \wr_X H \curvearrowright Y$. Plain edges correspond to generators of the form $(\mathbf{1}, t)$ while dotted edges correspond to generators of the form $(\delta^s_{x_0}, 1)$.

If we remove a vertex (g, x_0) we disconnect the leaf Y_g from the rest of Γ. Since Γ has at most one end and there is $|G| \geq 2$ leaves, we deduce that all leaves Y_g are finite, and hence that X' itself is finite.

The group G has property FW. Let K be any subgroup of G. We will show that $\mathrm{Sch}(G, K, S)$ has at most one end. Let x_0 be any point of X and X' be its orbit under the action of H. We have the imprimitive action of $G \wr_X H$ on $G/K \times X$, which restricts to an action on $G/K \times X'$:

$$(\varphi, h).(gK, x) = (\varphi(h.x)gK, h.x).$$

As above, the action is transitive and the orbital Schreier graph of this action is isomorphic to a Schreier graph Γ of $G \wr_X H$. We decompose this graph into leaves in the same way. Now observe that $\mathrm{Sch}(G, K, S)$ is isomorphic to the subgraph Δ of Γ consisting of vertices $\{(g, x_0) \mid g \in G\}$ and edges $\{(\delta^s_{x_0}, 1) \mid s \in S\}$. Due to the leaves structure of Γ, the number of ends of Δ is bounded above by the number of ends of Γ, and hence is at most one. We conclude that $\mathrm{Sch}(G, K, S)$ too has at most one end. $\square$

CHAPTER **4**

Wreath products of groups acting with bounded orbits

4.1 Introduction

When working with group properties, it is natural to ask if they are stable under “natural” group operations. One such operation, of great use in geometric group theory, is the wreath product, see Section 4.2 for all the relevant definitions.

An **S**-space is a metric space with an “additional structure” and we will say that a group G has *property B**S*** if every action by isometries which preserves the structure on an **S**-space has bounded orbits. Formally, this means that **S** is a subcategory of the category of metric spaces, and that the actions are by **S**-automorphisms. We note that having one bounded orbit implies that all the orbits are bounded.

In the context of properties defined by actions with bounded orbits, the first result concerning wreath products, due to Cherix, Martin and Valette and later refined by Neuhauser, concerns Kazhdan’s property (T).

Theorem 4.1.1 ([21, 77]). *Let G and H be two discrete groups with G non-trivial and let X be a set on which H acts. The wreath product $G \wr_X H$ has property (T) if and only if G and H have property (T) and X is finite.*

For countable groups (and more generally for σ-compact locally-compact topological groups), property (T) is equivalent, by the Delorme-Guichardet’s Theorem, to property FH (every action on an affine real Hilbert space has bounded orbits), see [9, Thm. 2.12.4]. Hence, Theorem 4.1.1 can also be viewed, for countable groups, as a result on property FH.

The corresponding result for property FA (every action on a tree has bounded orbits) and property F**R** (every action on a real tree has bounded orbits), is a little more convoluted and was obtained a few years later by Cornulier and Kar.

Theorem 4.1.2 ([26]). *Let G and H be two groups with G non-trivial and let X be a set on which H acts with finitely many orbits and without fixed points. Then $G \wr_X H$ has property FA (respectively property F**R**) if and only if H has property FA (respectively property F**R**), G has no quotient isomorphic to $\mathbf{Z}$ and can not be written as a countable increasing union of proper subgroups.*

Observe that our statement of Theorem 4.1.2 differs of the original statement of [26]. Indeed, where we ask G to have uncountable cofinality and no quotient isomorphic to $\mathbf{Z}$, the authors of [26] ask G to have uncountable cofinality and finite abelianization. However, these two sets of conditions are equivalent. One implication is trivial, as finite Abelian groups do not project onto $\mathbf{Z}$. For the other implication, suppose that G has uncountable cofinality but infinite abelianization $G/[G,G]$. The group $G/[G,G]$ being an infinite Abelian group, it has a countably infinite quotient A — a classical fact that Y. de Cornulier kindly reminded us, see [55][§16.11.c] for a proof. The quotient A has uncountable cofinality, see Lemma 4.3.2, and is therefore an infinite finitely generated Abelian group, which hence projects onto $\mathbf{Z}$.

Finally, we have an analogous of Theorem 4.1.1 for property FW (every action on a CAT(0) cube complexe has bounded orbits):

Theorem 4.1.3 ([24,64])**.** *Let G and H be two groups with G non-trivial and let X be a set on which H acts. Suppose that all three of G, H and $G \wr_X H$ are finitely generated. Then the wreath product $G \wr_X H$ has property FW if and only if G and H have property FW and X is finite.*

It is straightforward to prove that the wreath product $G \wr_X H$ is finitely generated if and only if both G and H are finitely generated and the number of orbits of the action of H on X is finite.

Theorem 4.1.3 was first proved, using *cardinal definite functions*, for arbitrary groups by Cornulier [24, Propositions 5.B.3 and 5.B.4], but without the implication "if $G \wr_X H$ has property FW, then G has property FW". The authors then gave an elementary proof of it via *Schreier graphs* for the specific case of finitely generated groups [64]. Y. Stalder has let us know (private communication) that, using *space with walls* instead of Schreier graphs, the arguments of [64] can be adapted to replace the finite generation hypothesis of Theorem 4.1.3 by the condition that all three of G, H and X are at most countable. Finally, A. Genevois published in [35] a proof of Theorem 4.1.3 for wreath products of the form $G \wr_H H$, based on his *diadem product of spaces.*

The above results on properties FH, FW and FA were obtained with distinct methods even if the final results have a common flavor. In the same time, all three properties FH, FW and FA can be characterized by the fact that any isometric action on a suitable metric space (respectively affine real Hilbert space, connected median graph and tree) has bounded orbits, see Definition 4.2.4. But more group properties can be characterized in terms of actions with bounded orbits. This is, for example, the case of the Bergman's property (actions on metric spaces), the property $\mathrm{FB_r}$ (actions on reflexive real Banach spaces) or of uncountable cofinality (actions on ultrametric spaces).

By adopting the point of view of actions with bounded orbits, we obtain a unified proof of the following result; see also Theorem 4.3.1 for the general (and more technical) statement.

Theorem. *Let $B\boldsymbol{S}$ be any one of the following properties: Bergman's property, property FB_r, property FH or property FW. Let G and H be two groups with G non-trivial and let X be a set on which H acts. Then the wreath product $G \wr_X H$ has property $B\boldsymbol{S}$ if and only if G and H have property $B\boldsymbol{S}$ and X is finite.*

With a little twist, we also obtain a similar result for groups with uncountable cofinality:

Proposition. *Let G and H be two groups with G non-trivial and let X be a set on which H acts. Then the wreath product $G \wr_X H$ has uncountable cofinality if and only if G and H have uncountable cofinality and H acts on X with finitely many orbits.*

A crucial ingredient of our proofs is that the spaces under consideration admit a natural notion of Cartesian product. In particular, some of our results do not work for trees and property FA, nor do they for real trees and the corresponding property F$\mathbf{R}$. Nevertheless, we are still able to show that if $G \wr_X H$ has property FA, then H acts on X with finitely many orbits. Combining this with Theorem 4.1.2 we obtain

Theorem. *Let G and H be two groups with G non-trivial and X a set on which H acts. Suppose that H acts on X without fixed points. Then $G \wr_X H$ has property FA (respectively has property F$\mathbf{R}$) if and only if H has property FA (respectively has property F$\mathbf{R}$), H acts on X with finitely many orbits, G has no quotient isomorphic to $\mathbf{Z}$ and can not be written as a countable increasing union of proper subgroups.*

4.2 Definitions and examples

This section contains all the definitions, as well as some useful preliminary facts and some examples.

Wreath products

Let X be a set and G a group. We view $\bigoplus_X G$ as the set of functions from X to G with finite support:

$$\bigoplus_X G = \{\varphi\colon X \to G \mid \varphi(x) = 1 \text{ for all but finitely many } x\}.$$

This is naturally a group, where multiplication is taken componentwise.

If H is a group acting on X, then it naturally acts on $\bigoplus_X G$ by $(h.\varphi)(x) = \varphi(h^{-1}.x)$. This leads to the following standard definition

Definition 4.2.1. Let G and H be groups and X be a set on which H acts. The *(restricted* [1]*) wreath product* $G \wr_X H$ is the group $(\bigoplus_X G) \rtimes H$.

A prominent particular case of wreath products is of the form $G \wr_H H$, where H acts on itself by left multiplication. They are sometimes called *standard wreath products* or simply *wreath products*, while general $G \wr_X H$ are sometimes called *permutational wreath products*. Best known example of wreath product is the so called *lamplighter group* $(\mathbf{Z}/2\mathbf{Z}) \wr_{\mathbf{Z}} \mathbf{Z}$. Other (trivial) examples of wreath products are direct products $G \times H$ which correspond to wreath products over a singleton $G \wr_{\{*\}} H$.

[1] There exists an unrestricted version of this product where the direct sum is replaced by a direct product.

Classical actions with bounded orbits

In this subsection we discuss some classical group properties, which are defined by actions with bounded orbits on various metric spaces.

Median graphs For u and v two vertices of a connected[2] graph $\mathcal{G}$, we define the total interval $[u, v]$ as the set of vertices that lie on some shortest path between u and v. A connected graph $\mathcal{G}$ is *median* if for any three vertices u, v, w, the intersection $[u, v] \cap [v, w] \cap [u, w]$ consists of a unique vertex, denoted $m(u, v, w)$. A graph is *median* if each of its connected components is median. For more on median graphs and spaces see [3, 17, 60]. If X and Y are both (connected) median graphs, then their Cartesian product is also a (connected) median graph. The class of median graphs was introduced by Nebeský in 1971 [76] and Gerasimov [36,37], Roller [89] and Chepoï [20] realized independently that this class coincides with the class of 1-skeleta of CAT(0) cube complexes. Trees are the simplest examples of connected median graphs, while the ensuing classical example show that any power set can be endowed with a median graph structure.

Example 4.2.2. Let X be a set and let $\mathcal{P}(X) = 2^X$ be the set of all subsets of X. Define a graph structure on $\mathcal{P}(X)$ by putting an edge between E and F if and only if $\#(E\Delta F) = 1$, where Δ is the symmetric difference. Therefore, the distance between two subsets E and F is $\#(E\Delta F)$ and the connected component of E is the set of all subsets F with $E\Delta F$ finite. For E and F in the same connected component, $[E, F]$ consists of all subsets of X that both contain $E \cap F$ and are contained in $E \cup F$. In particular, $\mathcal{P}(X)$ is a median graph, with $m(D, E, F)$ being the set of all elements belonging to at least two of D, E and F. In other words, $m(D, E, F) = (D \cap E) \cup (D \cap F) \cup (E \cap F)$.

These graphs are useful due to the following fact. Any action of a group G on a set X naturally extends to an action of G on $\mathcal{P}(X)$ by graph homomorphisms: $g.Y \coloneqq \{g.y \mid y \in Y\}$ for $Y \subset X$. Note that the action of G on $\mathcal{P}(X)$ may exchange the connected components. In fact, the connected component of $E \in \mathcal{P}(X)$ is stabilized by G if and only if E is *commensurated* by G, that is if for every $g \in G$ the set $E\Delta g.E$ is finite.

Uncountable cofinality Recall that a metric space (X, d) is *ultrametric* if d satisfies the strong triangular inequality: $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for any x, y and z in X. A group G has *uncountable cofinality* if every action on ultrametric spaces has bounded orbits. The following characterization of groups of countable cofinality can be extracted from [23] and we include a proof only for the sake of completeness. It implies in particular that a countable group has uncountable cofinality if and only if it is finitely generated.

Lemma 4.2.3. *Let G be a group. Then the following are equivalent:*

1. *G can be written as a countable increasing union of proper subgroups,*
2. *G does not have uncountable cofinality, i.e. there exists an ultrametric space X on which G acts with an unbounded orbit,*

[2] We will always assume that our connected graphs are non-empty. This is coherent with the definition that a connected graph is a graph with exactly one connected component.

3. *There exists a G-invariant (for the action by left multiplication) ultrametric d on G such that $G \curvearrowright G$ has an unbounded orbit.*

Proof. It is clear that the third item implies the second.

Let (X,d) be an ultrametric space on which G acts with an unbounded orbit $G.x_0$. For any $n \in \mathbf{N}$ let H_n be the subset of G defined by

$$H_n \coloneqq \{g \in G \mid d(x_0, g.x_0) \leq n\}.$$

Then G is the union of the (countably many) H_n, which are subgroups of G. Indeed, H_n is trivially closed under taking the inverse, and is also closed under taking products as we have $d(x_0, gh.x_0) \leq \max\{d(x_0, g.x_0), d(g.x_0, gh.x_0)\} = \max\{d(x_0, g.x_0), d(x_0, h.x_0)\}$. As $G.x_0$ is unbounded, they are proper subgroups. Since they are proper subgroups and $H_n \leq H_{n+1}$, we can extract an increasing subsequence $(H_{r_n})_n$ that still satisfies $G = \bigcup_n H_{r_n}$.

Finally, suppose that $G = \bigcup_{n \in \mathbf{N}} H_n$, where the H_n form an increasing sequence of proper subgroups. It is always possible to suppose that $H_0 = \{1\}$. Define d on G by $d(g,h) \coloneqq \min\{n \mid g^{-1}h \in H_n\}$. One easily verifies that d is a G-invariant ultrametric. Moreover, the orbit of 1 contains all of G and is hence unbounded. □

Some classical group properties We now discuss the bounded orbits properties for actions on various classes of metric spaces.

Definition 4.2.4. Let G be a group. It is said to have

- *Bergman's property* if any action by isometries on a metric space has bounded orbits,
- *Property FB$_r$* if any action by affine isometries on a reflexive real Banach space has bounded orbits,
- *Property FH* if any action by affine isometries on a real Hilbert space has bounded orbits,
- *Property FW* if any action by graph isomorphisms on a connected median graph has bounded orbits,
- *Property F**R*** if any action by isometries on a real tree has bounded orbits,
- *Property FA* if any action by graph isomorphisms on a tree has bounded orbits,
- *Uncountable cofinality* if any action by isometries on an ultrametric space has bounded orbits.

In the above, we insisted on the fact that actions are supposed to preserve the structure of the metric space under consideration, or in other words to be by automorphisms of the considered category. This is sometimes automatic, as for example any isometry of a real Banach or Hilbert space is affine by the Mazur-Ulam theorem. However, for cube complexes, for example, this is not the case; a 2-regular tree has automorphism group $\mathbf{Z} \rtimes (\mathbf{Z}/2\mathbf{Z})$, while its isometry group is $\mathbf{R} \rtimes (\mathbf{Z}/2\mathbf{Z})$. In other words, an isometry of a cube complex is not necessarily a cube complex isomorphism. See also Example 4.2.8.

In the following, we will often do a slight abuse of notation and simply speak of a *group action* on a space X, without always specifying by which kinds of maps the group acts, which should always be clear from the context.

The names $\mathrm{FB_r}$, FH, FW, F$\mathbf{R}$ and FA come from the fact that these properties admit a description in terms of (and were fist studied in the context of) the existence of a fixed point for actions on reflexive Banach spaces, on Hilbert spaces, on spaces with walls (or equivalently on CAT(0) cube complexes, see [18, 82]), on real trees and on trees (*Arbres* in french). However this is equivalent with the bounded orbit property, see Proposition 4.2.16 and the discussion below it. Observe that space with walls admit a natural pseudo-metric on them, which is not necessarily a metric.

The Bergman's property can also be characterized via length function, see for example the beginning of [98].

For a survey on property $\mathrm{FB_r}$, see [83] and the references therein.

For countable groups (and more generally for σ-compact locally compact groups), property FH is equivalent, by the Delorme-Guichardet theorem, to the celebrated Kazhdan's property (T), but this is not true in general. Indeed by [10] symmetric groups over infinite sets are uncountable discrete group with Bergman's property which, as we will see just below, implies property FH. Such groups cannot have property (T) as, for discrete groups, it implies finite generation.

A classical result of the Bass-Serre theory of groups acting on trees [94], is that a group G has property FA if and only if it satisfies the following three conditions: G has uncountable cofinality, G has no quotient isomorphic to $\mathbf{Z}$ and G is not a non-trivial amalgam. In view of this characterization, Theorem 4.1.2 says that property FA almost behaves well under wreath products.

Proposition 4.2.5. *There are the following implications between the properties of Definition 4.2.4:*

$$\begin{tikzcd}
\text{Bergman's property} \arrow[r, Rightarrow] & \mathrm{FB_r} \arrow[r, Rightarrow] & \mathrm{FH} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathrm{FW} \arrow[d, Rightarrow] & \\
 & & \mathrm{F}\mathbf{R} \arrow[r, Rightarrow] & \mathrm{FA} \arrow[r, Rightarrow] & \text{uncountable cofinality}
\end{tikzcd}$$

Moreover, except maybe for the implication [Bergman's property $\implies$ $\mathrm{FB_r}$], *all implications are strict.*

Proof. The implications [Bergman's property $\implies$ $\mathrm{FB_r}$ $\implies$ FH] and [FW $\implies$ FA $\impliedby$ F$\mathbf{R}$] trivially follow from the fact that Hilbert spaces are reflexive Banach spaces, which are themselves metric spaces and that trees are both real trees and connected median graphs. The implication [FH $\implies$ FW] follows from the fact that a group G has property FW if and only if any affine action on a real Hilbert space which preserves integral points has bounded orbits [24, Proposition 7.I.3]. The implication [FH $\implies$ F$\mathbf{R}$] follows from the fact that real trees are median metric spaces, and that such spaces can be embedded into L^1-spaces (see for instance [100, Theorem V.2.4]). Finally, the implication [FA $\implies$ uncountable cofinality] is due to Serre [94]: if G is an increasing union of subgroups G_i, then $\bigsqcup G/G_i$ admits a tree structure by joining any $gG_i \in G/G_i$ to $gG_{i+1} \in G/G_{i+1}$. The action of G by multiplication on $\bigsqcup G/G_i$ is by graph isomorphisms and with unbounded orbits.

We now present some examples demonstrating the strictness of the implications. Countable groups with property $\mathrm{FB_r}$ are finite by [14], while infinite finitely generated groups with property (T), e.g. $\mathrm{SL}_3(\mathbf{Z})$, have property FH. The group $\mathrm{SL}_2(\mathbf{Z}[\sqrt{2}])$ has property FW but not FH, see [25]. If G is a non-trivial finite group and H is an infinite group with property FA (respectively property F$\mathbf{R}$), then $G \wr_H H$ has property FA (respectively property F$\mathbf{R}$) by Theorem 4.1, but does not have property FW by Theorem 4.1. The group $\mathbf{Z}$ has uncountable cofinality, while it acts by translations and with unbounded orbits on the infinite 2-regular tree. Finally, Minasyan constructed examples of groups with FA but without F$\mathbf{R}$ in [75]. $\square$

The reader familiar with triangles groups $\Delta(l,m,n) = \langle a,b,c \,|\, a^2 = b^2 = c^2 = (ab)^l = (bc)^m = (ca)^n = 1\rangle$ with $l,m,n \in \{1,2,\ldots\} \cup \{\infty\}$ will be happy to observe that they provide explicit examples of groups with property FA but not property FW. Indeed, if l, m and n are all three integers, then $\Delta(l,m,n)$ has property FA by Serre [94, Section 6.5, Corollaire 2]. And if $\kappa(l,m,n) := \frac{1}{l} + \frac{1}{m} + \frac{1}{n} \leq 1$, then $\Delta(l,m,n)$ is the infinite symmetric group of a tilling of the Euclidean plane (if $\kappa(l,m,n) = 1$) or of the hyperbolic plane (if $\kappa(l,m,n) < 1$) and hence acts on a space with walls without fixed point, which implies that it does not have property FW.

In view of Proposition 4.2.5, two questions remain open: is the implication [Bergman's property $\implies$ $\mathrm{FB_r}$] strict, and does property FW implies property F$\mathbf{R}$?

It is possible to consider relative versions of the properties appearing in Definition 4.2.4. Let $\mathbf{S}$ be any classes of metric spaces considered in Definition 4.2.4 and let B$\mathbf{S}$ be the corresponding group property. If G is a group and H a subgroup of G, we say that the pair (G,H) has *relative property B*$\boldsymbol{S}$ if for every G-action on an $\mathbf{S}$-space, the H orbits are bounded. A group G has property B$\mathbf{S}$ if and only if for every subgroup H the pair (G,H) has relative property B$\mathbf{S}$, and if and only if for every overgroup L the pair (L,G) has relative property B$\mathbf{S}$.

Groups acting with bounded orbits on S-spaces

It is possible to define other properties in the spirit of Definition 4.2.4 for any "subclass of metric spaces", or more precisely for any subcategory of pseudo-metric spaces. A reader not familiar with category theory and interested only in one specific subclass of metric spaces may forget all these general considerations and only verify that the arguments of Section 4.3 apply for their favourite subclass of metric spaces.

Definition 4.2.6. A *pseudo-metric space* is a set X with a map $d\colon X \times X \to \mathbf{R}_{\geq 0}$, called a *pseudo-distance*, such that

1. $d(x,x) = 0$ for all $x \in X$,
2. $d(x,y) = d(y,x)$ for all $x,y \in X$,
3. $d(x,z) \leq d(x,y) + d(y,z)$.

If moreover $d(x, y) \neq 0$ for $x \neq y$, the map d is a *distance* and (X, d) is a *metric space*. On the other hand, an *ultra-pseudo-metric space* is a pseudo-metric space (X, d) such that d satisfies the strong triangular inequality: $d(x, z) \leq \max\{d(x, y), d(y, z)\}$. A *morphism* (or *short map*) between two pseudo-metric spaces (X_1, d_1) and (X_2, d_2) is a distance non-increasing map $f\colon X_1 \to X_2$, that is $d_2(f(x), f(y)) \leq d_1(x, y)$ for any x and y in X_1. If f is bijective and distance preserving, then it is an *isomorphism* (or *isometry*). Pseudo-metric spaces with short maps form a category **PMet**, of which the category of metric spaces (with short maps) **Met** is a full subcategory.

If (X, d) is a pseudo-metric space, we have a natural notion of the *diameter* of a subset $Y \subset X$ with values in $[0, \infty]$, defined by $\mathrm{diam}(Y) := \sup\{d(x, y) \mid x, y \in Y\}$, and we say that Y is *bounded* if it has finite diameter.

Remind that a *subcategory of **PMet***, is a category **S** whose objects are pseudo-metric spaces, and whose morphisms are short maps. The subcategory **S** is *full* if given two **S**-objects X and Y, any short map from X to Y is a **S**-morphism. A *G-action on an **S**-space* X is simply a group homomorphism $\alpha\colon G \to \mathrm{Aut}_{\mathbf{S}}(X)$.

In practice, a lot of examples of (full) subcategories of **PMet** are already subcategories of **Met**. Obvious examples of full subcategories of **Met** include metric spaces and ultrametric spaces (with short maps). Affine real Hilbert and Banach spaces and more generally normed vector spaces (with affine maps) are also full subcategories of **Met** if we restrict ourselves to morphisms that do not increase the norm (that is such that $\|f(x)\| \leq \|x\|$). In particular, for us isomorphisms of Hilbert and Banach spaces will always be affine isometries.

For connected graphs (and hence for its full subcategories of connected median graphs and of trees), one looks at the category **Graph** where objects are connected simple graphs $G = (V, E)$ and where a morphism $f\colon (V, E) \to (V', E')$ is a function between the vertex sets such that if (x, y) is an edge then either $f(x) = f(y)$ or $(f(x), f(y))$ is an edge. There are two natural ways to see **Graph** as a subcategory of **Met**. The first one consists to look at the vertex set V endowed with the shortest-path metric: $d(v, w)$ is the minimal number of edges on a path between v and w. The second one, consists at looking at the so called geometric realization of (V, E), where each edge is seen as an isometric copy of the segment $[0, 1]$. Similarly to what happens for cube complexes (see the discussion after Definition 4.2.4), the geometric realization of a graph gives an embedding **Graph** $\hookrightarrow$ **Met** which is not full. Nevertheless, for our purpose, the particular choice of one of the above two embeddings **Graph** $\hookrightarrow$ **Met** will make no difference.

We can now formally define the group property B**S** as:

Definition 4.2.7. Let **S** be a subcategory of **PMet**. A group G has *property* B**S** if every G-action by **S**-automorphisms on an **S**-space has all its orbits bounded. A pair (G, H) of a group and a subgroup has *relative property* B**S** if for every G-action by **S**-automorphisms on an **S**-space, the H-orbits are bounded.

Observe that a group G has property B**S** if and only if any G-action on an **S**-space has at least one bounded orbit.

All the properties of Definition 4.2.4 are of the form B**S**. Another example of property of the form B**S** can be found in [28, Definition 6.22]: a group has property ($\mathrm{FHyp}_{\mathbf{C}}$) if any action on a real or complex hyperbolic space of finite dimension has bounded orbits. This property is implied by property FH, but does not imply property FA [28, Corollary 6.23 and Example 6.24]. One can also want to look at the category of all Banach spaces (the corresponding property BB hence stands between the Bergman's property and property FB_{r}), or the category of L^p-spaces for p fixed [11] (if $p \notin \{1, \infty\}$, then $\mathrm{B}L^p$ is implied by FB_{r}).

Another interesting example of a property of the form B**S** is the fact to have no quotient isomorphic to **Z**, see Example 4.2.8. The main interest for us of this example is that property FA is the conjonction of three properties, two of them (uncountable cofinality and having no quotient isomorphic to **Z**) still being of the form B**S**.

Example 4.2.8. Let Z be the 2-regular tree, or in other words the Cayley graph of **Z** for the standard generating set. Then $\mathrm{Aut}_{\mathbf{Graph}}(Z) = \mathbf{Z} \rtimes (\mathbf{Z}/2\mathbf{Z})$ is the infinite dihedral group and its subgroup of orientation preserving isometries is isomorphic to **Z**. Let **S** be the category with one object Z and with morphisms the orientation preserving isometries. Hence, we obtain that a group G has no quotient isomorphic to **Z** if and only if every G-action on **S**-space has bounded orbits. Let us denote by B**Z** this property.

Since Z is a tree, property FA implies property B**Z**. This implication is strict as demonstrated by **Q**. In fact, the counterexample **Q** shows that B**Z** does not imply uncountable cofinality, while **Z** demonstrate that uncountable cofinality does not implies B**Z**.

An example of an uninteresting property B**S** is given by taking **S** to be the category of metric spaces of finite diameter (together with short maps), in which case any group has B**S**. On the opposite, if **S** is the category of extended pseudo-metric spaces (d takes values in $\mathbf{R}_+ \cup \{\infty\}$), only the trivial group has B**S**. Indeed, one can put the extended metric $d(x, y) = \infty$ if $x \neq y$ on G and then the action by left multiplication of G on (G, d) is transitive and with an unbounded orbit as soon as G is non trivial.

The category **PMet** has the advantage (over **Met**) of behaving more nicely with respect to categorical constructions and quotients. However, we have

Lemma 4.2.9. *A group G has Bergman's property (respectively uncountable cofinality) if and only if any isometric G-action on a pseudo-metric (respectively ultra-pseudo-metric) space has bounded orbits.*

Proof. One direction is trivial. For the other direction, let (X, d) be a pseudo-metric space on which G acts by isometries. Let $X' := X/\sim$ be the quotient of X for the relation $x \sim y$ if $d(x, y) = 0$ and let d' be the quotient of d. Then (X', d') is a metric space, the action of G passes to the quotient and $G.x$ is d-bounded if and only if $G.[x]$ is d'-bounded. Finally, if d satisfies the strong triangular inequality, then so does d'. $\square$

On the other hand, the following result is perhaps more surprising.

Lemma 4.2.10. *A group G has Bergman's property if and only if any G-action by graph automorphisms on a connected graph has bounded orbits.*

Proof. The left-to-right implication is clear.

For the other direction, let X be a metric space. Let $G(X)$ denotes the graph obtained from a vertex-set X by applying the following process: for any two $x, y \in X$ add a path of length $\lfloor d(x,y) \rfloor + 1$ between x and y. $G(X)$ is connected and the obvious inclusion $\iota\colon X \to G(X)$ is a quasi-isometric embedding. Moreover, the construction is canonical, so every group action on X extends to a group action on $G(X)$, making ι equivariant. So if a group satisfying the bounded orbit property on connected graphs acts on a metric space X, then its induced action on $G(X)$ has bounded orbits, which implies that its orbits in X are bounded. □

Observe that an alternative proof of the above Lemma can be easily deduced from the following characterization of Bergman's property due to Cornulier [23]: a group G has Bergman's property if and only if it has uncountable cofinality and for every generating set T of G, the Calyey graph $\mathrm{Cay}(G;T)$ is bounded. Details are left to the reader.

While we will be able to obtain some results for a general subcategory of **PMet**, we will sometimes need to restrict ourselves to subcategories satisfying good properties. In particular, we will use three axioms: one on the existence of non-trivial G-action, one on the existence of finite Cartesian powers and one on infinite Cartesian powers. Our Cartesian powers will need to be in some sense compatible with the bornology, but the conditions for finite and infinite powers will not be the same. A summary of which axioms are satisfied by the above mentioned subcategories of **PMet** can be found in page 54.

Definition 4.2.11 (Axiom (A1))**.** A subcategory **S** of **PMet** has *non-trivial group actions* if for every non-trivial group G there exists an **S**-space X and an action $G \curvearrowright X$ by **S**-automorphisms moving at least one point.

Examples of categories **S** with non-trivial group actions include (ultra-) metric spaces and metric spaces of finite diameter (with the action by multiplication of G on itself, endowed with the discrete metric), (reflexive) Banach spaces and Hilbert spaces (with $X = l^2(G)$ and the permutation action of G), L^p spaces (G acting by permutation on $\ell^p(G)$) and finally connected median graphs and (real) trees (X has one central vertex to which we attach an edge for every $g \in G$ and the action of G is by left multiplication). On the opposite side, both Z from Example 4.2.8 and real and complex finite dimensional hyperbolic spaces do not have non-trivial group actions. Indeed, a group acts non-trivially on Z if and only if it projects onto $\mathbf{Z}$. For hyperbolic spaces, a group G acts on a hyperbolic space of dimension n if and only if it projects onto a subgroup of $\mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$. In particular, if the action is non-trivial, then G projects onto a non-trivial subgroup of $\mathrm{GL}_n(\mathbf{C})$, whose all finitely generated subgroups are residually finite. We conclude that a finitely generated infinite simple group G does not admit a non-trivial action on a real or complexe hyperbolic space of finite dimension.

Before introducing the axioms about Cartesian powers, let us recall the definition and some properties of the product distances d_p.

Definition 4.2.12. For a real $p \geq 1$ and a collection of non-empty metric spaces $(X_i, d_i)_{i\in I}$, we have the maps

$$\begin{aligned} d_p \colon \prod_{i\in I} X_i \times \prod_{i\in I} X_i &\to \mathbf{R} \\ (f,g) &\mapsto \Big(\sum_{i\in I} d_i(f(i),g(i))^p\Big)^{\frac{1}{p}} \end{aligned}$$

and

$$\begin{aligned} d_\infty \colon \prod_{i\in I} X_i \times \prod_{i\in I} X_i &\to \mathbf{R} \\ (f,g) &\mapsto \sup_{i\in I} d_i(f(i),g(i)), \end{aligned}$$

with the convention that a sum with uncountably non-zero summands is infinite.

If I is finite, then d_p is a distance on $\prod_{i\in I} X_i$. Moreover, it is compatible with the bornology in the sense that if $E \subseteq X$ is unbounded, then the diagonal $\mathrm{diag}(E) \subset X^n$ is also unbounded. The following definition generalizes this comportement to other distances.

Definition 4.2.13 (Axiom (A2))**.** A subcategory $\mathbf{S}$ of $\mathbf{PMet}$ satisfies *axiom (A2)* if for any $\mathbf{S}$-space X and any integer n, there exists an $\mathbf{S}$-object, called a *n^{th} Cartesian power of* X and written X^n, such that:

1. As a set, X^n is the n^{th} Cartesian power of X,
2. The canonical image of $\mathrm{Aut}_{\mathbf{S}}(X)^n \rtimes \mathrm{Sym}(n)$ in $\mathrm{Bij}(X^n)$ lies in $\mathrm{Aut}_{\mathbf{S}}(X^n)$,
3. If $E \subset X$ is unbounded, then the diagonal $\mathrm{diag}(E) \subset X^n$ is unbounded.

A good heuristic is that your favourite subcategory of **PMet** would satisfies axiom (A2) in the above sense if and only if it already has a classical operation which is called *Cartesian product*. An $\mathbf{S}$-object satisfying the first two properties of Definition 4.2.13 will be called a finite Cartesian power.

For metric spaces (of finite diameter), the categorical product (corresponding to the metric $d_\infty = \max\{d_X, d_Y\}$) works fine, but any product metric of the form $d_p = (d_X^p + d_Y^p)^{\frac{1}{p}}$ for $p \in [1,\infty]$ works as well. For ultra-metric spaces, the categorical product d_∞ works fine. For L^p spaces (and hence for Hilbert spaces) we take the usual Cartesian product (which is also the categorial product), which corresponds to the metric d_p. For (reflexive) Banach spaces, any product metric of the form d_p works. For connected graphs, the usual Cartesian product, which corresponds to $d_1 = d_X + d_Y$ works well, but one can also take the strong product (which is the categorial product in **Graph**), that is the distance d_∞. For connected median graphs, only the usual Cartesian product with d_1 works.[3] On the other hand, (real) trees, Z from Example 4.2.8 and hyperbolic spaces do not have finite Cartesian powers and hence cannot satisfy axiom (A2).

[3]The category of median graphs does not have categorial products.

As the above examples illustrate, there can be multiple non-isomorphic spaces playing the role of X^n. As our results will not depend on a particular choice of a Cartesian power, we will sometimes make a slight abuse of language and speak of *the* Cartesian power X^n.

The situation for infinite products is more complicated. Indeed, if I is infinite then the map d_p is not necessary a distance on $\prod_{i\in I} X_i$ as it can take infinite values. The solution consists of looking at the subset of $\prod_{i\in I} X_i$ on which d_p takes finite values. Formally we first need to chose a base-point x_i in X_i for each $i \in I$, which gives us an element $f_0 \in \prod_{i\in I} X_i$ defined by $f_0(i) = x_i$. We can then define

$$\bigoplus_{i\in I} X_i := \{f \in \prod_{i\in I} X_i \mid f(i) = f_0(i) \text{ for all but finitely many } i\}$$

and

$$\bigoplus_{i\in I}^p X_i := \Big\{f \in \prod_{i\in I} X_i \,\Big|\, \sum_i d_i\big(f(i), f_0(i)\big)^p < \infty\Big\}.$$

A priori, the above definitions depend on the choice of the x_i. However, since our results will not depend on a particular choice of base-points, we will omit to specify it. Moreover, if $\mathrm{Aut}_{\mathbf{S}}(X)$ acts transitively on X, then $\bigoplus_{i\in I}^p X$ will not depend on the choice of $x \in X$. If all the X_i are equal to $\mathbf{R}$ with $x_i = 0$, then $\bigoplus_{i\in I}^p X_i$ is the classical Banach space $\ell^p(I)$ while $\bigoplus_{i\in I} X_i$ is the (non-complete) sequence space $c_{00}(I)$.

It is straightforward to verify that $\bigoplus_{i\in I} X_i \subseteq \bigoplus_{i\in I}^p X_i \subseteq \prod_{i\in I} X_i$ and that the first inclusion is an equality if the X_i are uniformly discrete with an uniform lower bound on their packing radius. Moreover the restriction of the map d_p to $\bigoplus_{i\in I}^p X_i$ is a distance. While $(\bigoplus_{i\in I} X_i, d_p)$ is a metric space, it is in general not complete even when the X_i are complete, which is why we needed to define $\bigoplus_{i\in I}^p X_i$; this will be important for Banach and Hilbert spaces.

One common feature of the product distances d_p for $p \neq \infty$ is that, in some rough way, they are able to detect the number of coordinates on which f differs from f_0. Our last axiom will generalize this comportment.

Definition 4.2.14 (Axiom (A3))**.** A subcategory $\mathbf{S}$ of **PMet** satisfies *axiom (A3)* if for any $\mathbf{S}$-space X, any element x_0 of X and any infinite set I, there exists an $\mathbf{S}$-object, called the *I^{th} Cartesian power of* X and written $\bigoplus_I^{\mathbf{S}} X$, such that:

1. As sets we have the inclusions $\bigoplus_I X \subseteq \bigoplus_I^{\mathbf{S}} X \subseteq \prod_I X$,
2. The canonical image of $\mathrm{Aut}_{\mathbf{S}}(X) \wr_I \mathrm{Sym}(I)$ in $\mathrm{Bij}(\prod_I X)$ lies in $\mathrm{Aut}_{\mathbf{S}}(\bigoplus_I^{\mathbf{S}} X)$,
3. For any y in X with $d(y, x_0) > 0$, the following set has infinite diameter

$$\Big\{f \in \bigoplus_I X \,\Big|\, f(i) = y \text{ for finitely many } i \text{ and otherwise } f(i) = x_0\Big\}.$$

An object satisfying the first two items of Definition 4.2.14 will be called an infinite Cartesian power.

In practice, the above definition is often easy to verify. Indeed, in most cases when $\mathbf{S}$ has finite Cartesian powers it is for some product metric of the form d_p.

Then the metric space $(\bigoplus_I^p X, d_p)$ will usually be an infinite Cartesian power in **S** and, if $p \neq \infty$, it will satisfies (A3). In particular, (A3) holds in the following categories: metric spaces, (reflexive) Banach spaces (with d_p for $1 < p < \infty$), Hilbert spaces and L^p spaces if $p \neq \infty$, connected (median) graphs. On the other hand, (real) trees, hyperbolic spaces and Z from Example 4.2.8 do not have a sensible notion of infinite Cartesian powers. Finally, while ultra-metric spaces, L^∞ spaces and spaces of finite diameter have infinite Cartesian powers (for d_∞), axiom (A3) does not hold as the diameter of the set appearing in Definition 4.2.14 is $d(y, x_0)$.

Finally, we introduce one last definition

Definition 4.2.15. A subcategory **S** of **PMet** has *bornological Cartesian powers* if it satisfies both axiom (A2) and (A3).

Variations and generalizations

This subsection is devoted to variations and generalizations of property B**S**. It is intended as a note for the interested reader, and can be skipped without any harm.

Groups acting with fixed point on S-spaces Some of the properties that are of interest for us have been historically defined via the existence of a fixed point for some action. More generally, we say that a group G has *property F**S*** if any G-action on an **S**-space has a fixed point.

Since our actions are by isometries, property F**S** implies property B**S**. The other implication holds as soon as we have a suitable notion of the center of a (non-empty) bounded subset X. For a large class of metric spaces, this is provided by the following result of Bruhat and Tits:

Proposition 4.2.16 ([28, Chapter 3.b])**.** *Let (X, d) be a complete metric space such that the following two conditions are satisfied:*

1. *For all x and y in X, there exists a unique $m \in X$ (the middle of $[x, y]$) such that $d(x, m) = d(y, m) = \frac{1}{2} d(x, y)$;*

2. *For all x, y and z in X, if m is the middle of $[y, z]$ we have the median's inequality $2d(x, m)^2 + \frac{1}{2} d(y, z)^2 \leq d(x, y)^2 + d(x, z)^2$.*

Then if G is a group acting by isometries on X with a bounded orbit, it has a fixed point.

Examples of complete metric spaces satisfying Proposition 4.2.16 include among others: Hilbert spaces, Bruhat-Tits Buildings, Hadamard spaces (i.e. complete CAT(0) spaces and in particular CAT(0) cube complexes which are either finite dimensional or locally finite), trees and **R**-trees; with the caveat that for trees and **R**-trees, the fixed point is either a vertex or the middle of an edge. See [28, Chapter 3.b] and the references therein for more on this subject. On the other hand, [9, Lemma 2.2.7] gives a simple proof of the existence of a center for bounded subsets of Hilbert spaces, and more generally of reflexive Banach spaces, but this also directly follows from the Ryll-Nardzewski fixed-point theorem. Finally, properties F**S** and B**S** are equivalent for the category of separable uniformly convex Banach spaces by the existence of the Chebyshev center of a (nonempty) bounded set.

For action on metric spaces or on connected median graphs, F**S** is strictly stronger than B**S**. Indeed, this trivially follows from the action by rotation of $\mathbf{Z}/4\mathbf{Z}$ on the square graph. However, by [37, 89] if a group G acts on a connected median graph with a bounded orbit, then it has a finite orbit. For actions on an ultra-metric spaces F**S** is strictly stronger than B**S** since the finite group $\mathbf{Z}/2\mathbf{Z}$ acts without fixed point on the Cantor space $X \subset [0,1]$ by $x \mapsto 1-x$. Property F**S** is also strictly stronger than B**S** for **S** the category of all Banach spaces. Indeed, by [79] any infinite discrete group admits an action without fixed point on some Banach space and hence does not have property F**S**, while there exists infinite groups with the Bergmann's property which implies property B**S**.

Actions with uniformly bounded orbits One might wonder what happens if in Definition 4.2.4 we replace the requirement of having bounded orbits by having uniformly bounded orbits. It turns out that this is rather uninteresting, as a group G is trivial if and only if any G-action on a metric space (respectively on an Hilbert space, on a connected median graph, on a tree or on an ultrametric space) has uniformly bounded orbits. Indeed, if G is nontrivial, then, for the action of G on the Hilbert space $\ell^2(G)$ the orbit of $n \cdot \delta_g$ has diameter $n\sqrt{2}$. For a tree (and hence also for a connected median graph), one may look at the tree T obtained by taking a root r on which we glue an infinite ray for each element of G. Then G naturally acts on T by permuting the rays. The orbits for this action are the $\mathcal{L}_n = \{v \mid d(v,r) = n\}$ which have diameter $2n$. Finally, it is possible to put an ultradistance on the vertices of T by $d_\infty(x,y) \coloneqq \max\{d(x,r), d(y,r)\}$ if $x \neq y$. Then the orbits are still the $\mathcal{L}_n$, but this time with diameter n.

Topological groups One can wonder what happens for topological groups. While, the wreath product of topological groups is not in general a topological group, this is the case if G is discrete and X is a discrete set endowed with a continuous H-action. In this particular context, Theorem 4.3.1, as well as its proof, remains true. The details are left to the interested reader.

Categorical generalizations In the above, we defined property B**S** for **S** a subcategory of **PMet**. It is possible to generalize this definition to more general categories. We are not aware of any example of the existence of a group property arising in this general context that is not equivalent to a property B**S** in the sense of Definition 4.2.7, but still mention it fur the curious reader.

On one hand, we can replace **PMet** with a more general category. For example, one can look at the category **M** of sets X endowed with a map $d\colon X \times X \to \mathbf{R}_{\geq 0}$ satisfying the triangle inequality. That is, d is a pseudo-distance, except that is is not necessary symmetric and $d(x,x)$ may be greater than 0. All the statements and the proofs remain true for **S** a subcategory of **M**.

On the other hand, we can define property B**S** for any category **S** over **PMet**, that is for any category **S** endowed with a faithful functor $F\colon \mathbf{S} \to \mathbf{PMet}$. Such a couple $(\mathbf{S}, F\colon \mathbf{S} \to \mathbf{PMet})$ is sometimes called a *structure over*

PMet, and F is said to be *forgetful*. In this context, we need to be careful to define Cartesian powers (Definitions 4.2.13 and 4.2.14) using F, but apart for that all the statements and all the proofs remain unchanged. An example of such an $\mathbf{S}$ that cannot be expressed as a subcategory of **PMet** is the category of edge-labeled graphs, where the morphisms are graph morphisms that induce a permutation on the set of labels. However, in this case the property B$\mathbf{S}$ is equivalent to the Bergman's property.

One can also combine the above two examples and look at couples $(\mathbf{S}, F\colon \mathbf{S} \to \mathbf{M})$, with F faithful.

Finally, in view of Definitions 4.2.7, 4.2.13 and 4.2.14, the reader might ask why we are working in **PMet** or **M** instead of **Born**, the category of bornological spaces together with bounded maps. The reason behind this is the forthcoming Lemma 4.3.3 and its corollaries, which fail for general bornological spaces. In fact, all the statements and the proofs remain true for a general $(\mathbf{S}, F\colon \mathbf{S} \to \mathbf{Born})$ as soon as $\mathbf{S}$ satisfies Lemma 4.3.3. Here is an example of such an $\mathbf{S}$ which does not appear as a category over $\mathbf{M}$. Take κ to be an infinite cardinal and let $\mathbf{S}_\kappa$ be the subcategory of **Born** where a subset E of a $\mathbf{S}_\kappa$-space is bounded if and only if $|E| < \kappa$. A group G has property B$\mathbf{S}_\kappa$ if and only if $|G| < \kappa$.

4.3 Proofs of the main results

Throughout this section, $\mathbf{S}$ will denote a subcategory of **PMet** and B$\mathbf{S}$ the group property *every action by* $\boldsymbol{S}$*-automorphisms on an* $\boldsymbol{S}$*-space has bounded orbits*. In Subsection 4.2, we defined 3 axioms that $\mathbf{S}$ might satisfy. Axiom (A1) simply states that a non-trivial group acts non-trivially on some $\mathbf{S}$-space. Axioms (A2) and (A3) guarantee the existence of finite and infinite Cartesian powers, which should be compatible in some sense with the bornology. Finally, $\mathbf{S}$ has bornological Cartesian powers if it satisfy both axioms (A2) and (A3). In Table 4.1 we present a short reminder on whenever these axioms are satisfied for some subcategories of **PMet** that were mentioned in Sections 4.1 and 4.2.

The main result of this section is the following theorem that implies Theorem 4.1.

Theorem 4.3.1. *Suppose that* $\boldsymbol{S}$ *has non-trivial group actions and bornological Cartesian powers. Let* G *and* H *be two groups with* G *non-trivial and let* X *be a set on which* H *acts. Then the wreath product* $G \wr_X H$ *has property B*$\boldsymbol{S}$ *if and only if* G *and* H *have property B*$\boldsymbol{S}$ *and* X *is finite.*

Theorem 4.3.1 is a direct consequence of the forthcoming Corollary 4.3.6 and Lemmas 4.3.10 and 4.3.12. The conclusion of Theorem 4.3.1 remains true if the hypothesis on $\mathbf{S}$ are replaced by "$\mathbf{S}$ satisfies (A2) and property B$\mathbf{S}$ implies property FW", see the discussion after Lemma 4.3.10 for more details.

We now state two elementary but useful results.

Lemma 4.3.2. *Let* G *be a group and* H *be a quotient. If* G *has property B*$\boldsymbol{S}$*, then so has* H.

Proof. If H acts on some $\mathbf{S}$-space X with an unbounded orbit, then the surjection $G \twoheadrightarrow H$ gives us a G-action on X, with the same orbits as the H-action. □

Category **S**	Corresponding group property	satisfies axiom		
		(A1)	(A2)	(A3)
Metric spaces	Bergmann's property	✓	✓	✓
Banach spaces	BB	✓	✓	✓
Reflexive Banach spaces	$\mathrm{FB_r}$	✓	✓	✓
L^p spaces (p fixed)	$\mathrm{B}L^p$	✓	✓	*
Hilbert spaces	FH	✓	✓	✓
R and **C** hyperbolic spaces	$\mathrm{FHyp}_{\mathbf{C}}$	✗	✗	✗
Median graphs	FW	✓	✓	✓
Real trees	$\mathrm{F}\mathbf{R}$	✓	✗	✗
Trees	FA	✓	✗	✗
Ultrametric spaces	uncountable cofinality	✓	✓	✗
Z= 2-regular tree with Isom^+	$\mathrm{B}\mathbf{Z}$	✗	✗	✗
Spaces of finite diameter	hold for all groups	✓	✓	✗

*: if and only if $p \neq \infty$.

Table 4.1: Axioms for category **S**. (A1) = non-trivial group actions (Definition 4.2.11), (A2) = Definition 4.2.13, (A3) = Definition 4.2.14, bornological Cartesian powers = (A2)+(A3).

Lemma 4.3.3. *Let G be a group and A an B be two subgroups such that $G = AB$. If both (G, A) and (G, B) have relative property $\mathrm{B}\boldsymbol{S}$, then G has property $\mathrm{B}\boldsymbol{S}$.*

Proof. Let X be an **S**-space on which G acts and let x be an element of X. Let D_1 be the diameter of $A.x$ and D_2 the diameter of $B.x$. By assumption, they both are finite. Since A acts by isometries, all the $a.Bx$ have diameter D_2. Let y be an element of $G.x$. There exists $a \in A$ such that y belongs to $a.Bx$. Since 1 belongs to B, y is at distance at most D_2 of $a.x$ and hence at distance at most $D_1 + D_2$ of x. Therefore, the diameter of $G.x$ is finite. □

By combining Lemmas 4.3.2 and 4.3.3, we obtain the following three corollaries on direct, semi-direct and wreath products.

Corollary 4.3.4. *Let G and H be two groups. Then $G \times H$ has property $\mathrm{B}\boldsymbol{S}$ if and only if both G and H have property $\mathrm{B}\boldsymbol{S}$.*

Corollary 4.3.5. *Let $N \rtimes H$ be a semidirect product. Then*

1. *If $N \rtimes H$ has property $\mathrm{B}\boldsymbol{S}$, then so has H,*

2. *If both N and H have property $\mathrm{B}\boldsymbol{S}$, then $N \rtimes H$ also has property $\mathrm{B}\boldsymbol{S}$.*

Corollary 4.3.6. *Let G and H be two groups and X a set on which H acts. Then,*

1. *If $G \wr_X H$ has property $\mathrm{B}\boldsymbol{S}$, then so has H,*

2. *If both G and H have property $\mathrm{B}\boldsymbol{S}$ and X is finite, then $G \wr_X H$ has property $\mathrm{B}\boldsymbol{S}$.*

When **S** has a suitable notion of quotients (by a group of isometries), it is possible to obtain a strong version of Lemmas 4.3.2 and 4.3.3. Here is the corresponding result for Bergman's property and uncountable cofinality.

Proposition 4.3.7. *Let B**S** be either Bergman's property or the property of having uncountable cofinality. Let $1 \to N \to G \to H \to 1$ be a group extension. Then G has property B**S** if and only if H has property B**S** and the pair (G, N) has relative B**S** property.*

Proof. One direction is simply Lemma 4.3.2 and the definition of relative property B**S**.

On the other hand, let (X, d) be a pseudo-metric space on which G acts by isometries and let x be an element of X. Let $\{g_i \mid i \in I\}$ be a transversal for N, that is $H \cong \{g_i N \mid i \in I\}$ with the quotient multiplication. By assumption, $N.x$ is bounded of diameter D_1 and for any $i \in I$ the subset $g_i N.x$ of X has also diameter D_1. Since N is a subgroup of isometries of X, the map $d' \colon X/N \times X/N \to \mathbf{R}$ defined by $d'([x],[y]) := \inf\{d(x', y') \mid x' \in N.x, y' \in N.y\}$ is the quotient pseudo-distance on X/N. Indeed, while the map d' might not satisfies the triangle inequality for a generic quotient $X/\sim$, this is the case if the quotient is by a subgroup of isometries; details are left to the reader. Moreover, if d satisfies the strong triangle inequality, then so does d'. The quotient action of $H \cong G/N$ on X/N is by isometries and the diameter of $H.xN$ is bounded, say by D_2. In particular, for any i and j in I, the distance between the subsets $g_i N.x$ and $g_j N.x$ of X is bounded by D_2. Since this distance is an infimum, there exist actual elements of $g_i N.x$ and $g_j N.x$ at distance less than $D_2 + 1$. Altogether, we obtain that any y in $G.x$ is at distance at most $D_1 + D_2 + 1$ of x. Hence, the orbit $G.x$ is bounded. $\square$

Since the triangle graph, which is not median, is a quotient of the 2-regular infinite tree by a subgroup of isometries, the proof of Proposition 4.3.7 does not carry over for properties FW and FA. Similarly, the quotient of $\mathbf{R}$ by the action of $\mathbf{Z}/2\mathbf{Z}$ given by $x \mapsto -x$ is not a Banach space and hence the proof of Proposition 4.3.7 does not apply to properties FH and $\mathrm{FB_r}$. However, the statement of Proposition 4.3.7 (stability under extensions) remains true for properties FH, $\mathrm{FB_r}$, FW and FA. For properties FH and $\mathrm{FB_r}$, this follows from the fixed-point definition and the fact that a non-empty closed subset of an Hilbert space (respectively of a reflexive Banach space) is an Hilbert space (respectively a reflexive Banach space) itself. For property FW and FA, see [24, Proposition 5.B.3] and [94].

We now state a result on infinite direct sums.

Lemma 4.3.8. *Let G and $(G_x)_{x\in X}$ be non-trivial groups and let H be a group acting on X. Then*

1. *$\bigoplus_{x\in X} G_x$ has uncountable cofinality if and only if all the G_x have uncountable cofinality and X is finite,*
2. *If $G \wr_X H$ has uncountable cofinality, then H acts on X with finitely many orbits.*

It is of course possible to prove Lemma 4.3.8 using the characterization of uncountable cofinality in terms of subgroups, in which case the proof is a short exercise left to the reader. However, we find enlightening to prove it using the characterization in terms of actions on ultrametric spaces.

Proof of Lemma 4.3.8. One direction of the first assertion is simply Corollary 4.3.4 and holds for any property of the form B**S**. For the other direction, for any **S**, if $\bigoplus_{x\in X} G_x$ has property B**S** then all its quotients, and hence all the G_x, have property B**S**. Hence, we have to prove that an infinite direct sum of non-trivial groups does not have uncountable cofinality. If X is infinite, there exists a countable subset $Y \subset X$. Let $Z \coloneqq X \setminus Y$, thus we have $X = Y \sqcup Z$. We can decompose the direct sum as $\bigoplus_{x\in X} G_x = (\bigoplus_{x\in Y} G_x) \times (\bigoplus_{x\in Z} G_x)$ and then, by Corollary 4.3.4, if $\bigoplus_{x\in Y} G_x$ does not have uncountable cofinality, then neither does $\bigoplus_{x\in X} G_x$. So let us fix an enumeration of Y and let $K \coloneqq \bigoplus_{i\geq 1} G_i$ and for each i, choose $g_i \neq 1$ in G_i. Let $d_{\max}(f,g) \coloneqq \max\{i \mid f(i) \neq g(i)\}$. This is a K-invariant ultra-metric for the action by left multiplication of G on itself. Then for every integer n, the orbit $K.\{1,1,\dots\}$ contains $\{g_1,\dots,g_n,1,\dots\}$ which is at distance n of $\{1,1,\dots\}$ for $d_{\max}$ if the g_i are not equal to 1.

The second assertion is a simple variation on the first. Indeed, we have

$$G \wr_X H \cong (\bigoplus_{Y\in X/H} L_Y) \rtimes H \qquad \text{with} \qquad L_Y \cong \bigoplus_{y\in Y} G_y,$$

where X/H is the set of H-orbits. The important fact for us is that H fixes the decomposition into L_Y factors: for all Y we have $H.L_Y = L_Y$. Up to regrouping some of the L_Y together we hence have $G \wr_X H \cong \left(\bigoplus_{i\geq 1} L_i\right) \rtimes H$ with $H.L_i = L_i$ for all i. Now, we have an ultradistance $d_{\max}$ on $L \coloneqq \bigoplus_{i\geq 1} L_i$ as above and we can put the discrete distance d on H. Then $d'_{\max} \coloneqq \max\{d_{\max}, d\}$ is an ultradistance on $\left(\bigoplus_{i\geq 1} L_i\right) \rtimes H$, which is $\left(\bigoplus_{i\geq 1} L_i\right) \rtimes H$-invariant (for the action by left multiplication). From a practical point of view, we have $d'_\infty\big((\varphi,h),(\varphi',h')\big) \coloneqq \max\{i \mid \varphi(i) \neq \varphi'(i)\}$ if $\varphi \neq \varphi'$ and $d'_{\max}\big((\varphi,h),(\varphi,h') = 1$ if $h \neq h'$. Since the action of L on itself has an unbounded orbit for $d_{\max}$, the action of $\left(\bigoplus_{i\geq 1} L_i\right) \rtimes H$ on itself has an unbounded orbit for $d'_{\max}$. ☐

We directly obtain

Corollary 4.3.9. *Suppose that B**S** implies having uncountable cofinality. Let G and $(G_x)_{x\in X}$ be non-trivial groups and let H be a group acting on X.*

1. *$\bigoplus_{x\in X} G_x$ has property B**S** if and only if all the G_x have property B**S** and X is finite,*

2. *If $G \wr_X H$ has property B**S**, then H acts on X with finitely many orbits.*

While the statement (and the proof) of Corollary 4.3.9.1 is expressed in terms of uncountable cofinality, it is also possible to state it and prove it for a subcategory **S** of **PMet** without a priori knowing if B**S** is stronger than having uncountable cofinality. The main idea is to find a "natural" **S**-space on which $G = \bigoplus_{i\geq 1} G_i$ acts. For example, for (reflexive) Banach, Hilbert and L^p spaces, one can take $\bigoplus_{i\geq 1} \ell^p(G_i)$. For connected median graphs, one takes the connected component of $\{1_{G_1}, 1_{G_2}, \dots\}$ in $\mathcal{P}(\bigsqcup_{i\geq 1} G_i)$. For (real) trees, it is possible to put a forest structure on $\mathcal{P}(\bigsqcup_{i\geq 1} G_i)$ in the following way. For $E \in \mathcal{P}(\bigsqcup_{i\geq 1} G_i)$, and for each i such that $E \cap G_j$ is empty for all $j \leq i$, add an edge from E to $E \cup \{g\}$ for each $g \in G_i$. The graph obtained this way is a G-invariant subforest of the median graph on $\mathcal{P}(\bigsqcup_{i\geq 1} G_i)$. For Corollary 4.3.9.2 we also need that the corresponding structure is invariant by the action of H, which is the case of the above examples, except for the tree structure.

It is also possible to give a proof of Corollary 4.3.9.1 using axioms similar to (A1)-(A3). More precisely, we need a variation of (A3) for countable Cartesian products (for morphisms we only ask that $\bigoplus_{i\in\mathbf{N}} \mathrm{Aut}_{\mathbf{S}}(X_i) \subseteq \mathrm{Aut}_{\mathbf{S}}(\bigoplus_{i\in\mathbf{N}} X_i))$ and a strong version of (A1) saying that there exists an universal bound M such that any non-trivial group G acts on some $\mathbf{S}$-space moving a point at distance at least M. The axiomatization of Corollary 4.3.9.2 is a little more complex. However, since in the following we will use Corollary 4.3.9 only for (real) trees, which do not have Cartesian powers, we will not elaborate on the details and let the proof to the interested reader. Instead, we will give an axiomatic proof of the following variation of Corollary 4.3.9.2.

Lemma 4.3.10. *Suppose that* $\boldsymbol{S}$ *has non-trivial group actions and satisfies axiom (A3). Let* G *and* H *be two groups with* G *non-trivial and let* X *be a set on which* H *acts. If* $G \wr_X H$ *has B*$\boldsymbol{S}$*, then* X *is finite.*

Proof. We will prove that if X is infinite, then $G \wr_X H$ does not have property B$\mathbf{S}$. Suppose that X is infinite. By non-trivial groups actions, there exists an $\mathbf{S}$-space Y on which G acts non-trivially by moving some element y_0 to another element $z_0 \neq y_0$. Let $\bigoplus_X Y$ be the corresponding Cartesian power and f_0 be the constant function $f_0(x) = y_0$. By assumption, the natural action of $G \wr_X H$ on $\bigoplus_X Y$ is by $\mathbf{S}$-automorphisms. Since X is infinite, it contains a countable subset $I = \{i_1, i_2, \ldots\}$. For every integer n, the function

$$f_n(x) := \begin{cases} z_0 & \text{if } x = i_m, m \leq n \\ y_0 & \text{otherwise} \end{cases}$$

is in the $G \wr_X H$-orbit of f_0. By axiome (A3) this orbit is unbounded and $G \wr_X H$ does not have property B$\mathbf{S}$. □

As a direct corollary, we obtain that if property B$\mathbf{S}$ implies property B$\mathbf{S}$' for some $\mathbf{S}$' with non-trivial group actions and (A3) (example: B$\mathbf{S}$'=FW), then the conclusion of Lemma 4.3.10 holds even if $\mathbf{S}$ might not satisfy its premises. Conversely, it follows from Theorem 4.1.2 and Proposition 4.1 that Lemma 4.3.10 do not holds for property F$\mathbf{R}$, property FA or having uncountable cofinality.

We now turn our attention to properties that behave well under finite Cartesian products in the sense of axiom (A2). We first describe the comportment of property B$\mathbf{S}$ under finite index subgroups.

Lemma 4.3.11. *Let* G *be a group and let* H *be a finite index subgroup.*

1. *If* H *has property B*$\boldsymbol{S}$*, then so has* G*,*
2. *If* $\boldsymbol{S}$ *satisfies (A2) and* G *has property B*$\boldsymbol{S}$*, then* H *has property B*$\boldsymbol{S}$*.*

Proof. Suppose that G does not have B$\mathbf{S}$ and let X be an $\mathbf{S}$-space on which G acts with an unbounded orbit O. Then H acts on X and O is a union of at most $[G : H]$ orbits. This directly implies that H has an unbounded orbit and therefore does not have B$\mathbf{S}$.

On the other hand, suppose that $H \leq G$ is a finite index subgroup of G without property B**S**. Let $\alpha\colon H \curvearrowright X$ be an action of H on an **S**-space (X, d_X) such that there is an unbounded orbit O. Similarly to the classical theory of representations of finite groups, we have the induced action $\mathrm{Ind}_H^G(\alpha)\colon G \curvearrowright X^{G/H}$ on the set $X^{G/H}$. Since H has finite index, $X^{G/H}$ is an **S**-space and the action is by **S**-automorphisms. On the other hand, the subgroup $H \leq G$ acts diagonally on $X^{G/H}$, which implies that diag(O) is contained in a G-orbit. Since diag(O) is unbounded, G does not have property B**S**.

For readers that are not familiar with representations of finite groups, here is the above argument in more details. Let $(f_i)_{i=1}^n$ be a transversal for G/H. The natural action of G on G/H gives rise to an action of G on $\{1, \ldots, n\}$. Hence, for any g in G and i in $\{1, \ldots, n\}$ there exists a unique $h_{g,i}$ in H such that $gf_i = f_{g.i}h_{g,i}$. That is, $h_{g,i} = f_{g.i}^{-1}gf_i$. We then define $g.(x_1, \ldots, x_n) \coloneqq (h_{g,g^{-1}.1}.x_{g^{-1}.1}, \ldots, h_{g,g^{-1}.n}.x_{g^{-1}.n})$. This is indeed an action by **S**-automorphisms on $X^{G/H}$ by Condition 2 of Definition 4.2.13. Moreover, every element $h \in H$ acts diagonally by $h.(x_1, \ldots, x_n) = (h.x_1, \ldots, h.x_n)$. In particular, this G-action has an unbounded orbit. □

We now prove one last lemma that will be necessary fo the proof of Theorem 4.1.

Lemma 4.3.12. *Suppose that **S** satisfies (A2). If X is finite and $G \wr_X H$ has property B**S**, then G has property B**S**.*

Proof. Suppose that G does not have B**S** and let (Y, d_Y) be an **S**-space on which G acts with an unbounded orbit $G.y$. Then (Y^X, d) is an **S**-space and we have the *primitive action* of the wreath product $G \wr_X H$ on Y^X:

$$\big((\varphi, h).\psi\big)(x) = \varphi(h^{-1}.x).\psi(h^{-1}.x).$$

By Condition 2 of Definition 4.2.13, this action is by **S**-automorphisms. The orbit $G.y$ embeds diagonally and hence diag$(G.y)$ is an unbounded subset of some $G \wr_X H$-orbit, which implies that $G \wr_X H$ does not have property B**S**. □

It is also possible to derive Lemma 4.3.12 directly from Lemma 4.3.11.2, with a more algebraic proof. Indeed, using the notation and hypothesis of Lemma 4.3.12, let H' be the kernel of the action of H on X and $\pi\colon G \wr_X H \to H$ be the canonical projection. Then $\pi^{-1}(H') \cong \bigoplus_X G \oplus H'$ is a finite index subgroup of $G \wr_X H$ and hence has property B**S**. Since G is a quotient of $\bigoplus_X G \oplus H'$ we conclude that it also has property B**S**.

We now proceed to prove Proposition 4.1. As for Lemma 4.3.8, it is also possible to prove it using the characterization of uncountable cofinality in terms of subgroups, in which case it is an easy exercise, but we will only give a proof using the characterization in terms of actions on ultrametric spaces.

Proof of Proposition 4.1. By Corollary 4.3.6 and Lemma 4.3.8 we already know that if $G \wr_X H$ has uncountable cofinality, then H has uncountable cofinality and it acts on X with finitely many orbits. We will now prove that if $G \wr_X H$ has uncountable cofinality so does G. Let us suppose that G has countable cofinality. By Lemma 4.2.3, there exists an ultrametric d on G such that the action of G on itself by left multiplication has an unbounded orbit. But then we have the primitive action of the wreath product $G \wr_X H$ on $G^X \cong \prod_X G$, which preserves $\bigoplus_X G$. It is easy to check that the map $d_\infty\colon \bigoplus_X G \times \bigoplus_X G \to \mathbf{R}$ defined by $d_\infty(\psi_1, \psi_2) := \max\{d\bigl(\psi_1(x), \psi_2(x)\bigr) \mid x \in X\}$ is a $G \wr_X H$-invariant ultrametric. Finally, let $g_0 \in G$ be an element of unbounded G-orbit for d and let x_0 be any element of X. For g in G and x in X, we define the following analog of Kronecker's delta functions

$$\delta_x^g(y) := \begin{cases} g & \text{if } y = x \\ 1 & \text{if } y \neq x. \end{cases}$$

Then we have $(\delta_{x_0}^g, 1).\delta_{x_0}^{g_0} = \delta_{x_0}^{gg_0}$ and hence $d_\infty(\delta_{x_0}^{g_0}, \delta_{x_0}^{gg_0}) = d(g_0, gg_0)$ is unbounded.

Suppose now that both G and H have uncountable cofinality and that H acts on X with finitely many orbits. We want to prove that $G \wr_X H$ has uncountable cofinality.

Let (Y, d) be an ultrametric space on which $G \wr_X H$ acts. Then H and all the G_x act on Y with bounded orbits. Let $H.x_1, \ldots, H.x_n$ be the H-orbits on X and let y be any element of Y. Then $H.y$ has finite diameter D_0 while $G_{x_i}.y$ has finite diameter D_i. For any $x \in X$, there exists $1 \leq i \leq n$ and $h \in H$ such that $x = h.x_i$. We have

$$\begin{aligned} d\bigl((\delta_{x_i}^g, h^{-1}).y, y\bigr) &\leq \max\{d\bigl((\delta_{x_i}^g, h^{-1}).y, (\delta_{x_i}^g, 1).y\bigr), d\bigl((\delta_{x_i}^g, 1).y, y\bigr)\} \\ &= \max\{d\bigl((1, h^{-1}).y, y\bigr), d\bigl((\delta_{x_i}^g, 1).y, y\bigr)\} \\ &\leq \max\{D_0, D_i\}, \end{aligned}$$

which implies that the diameter of $G_{x_i}h^{-1}.y$ is bounded by $\max\{D_0, D_i\}$. But $G_{x_i}h^{-1}.y$ has the same diameter as $hG_{x_i}h^{-1}.y = G_{h.x_i}.y = G_x.y$.

On the other hand, the diameter of $\bigoplus_X G.y$ is bounded by the supremum of the diameters of the $G_x.y$, and hence bounded by $\max\{D_0, D_1, \ldots, D_n\}$. Finally, for (φ, h) in $G \wr_X H$ we have

$$\begin{aligned} d\bigl(y, (\varphi, h).y\bigr) &\leq \max\{d\bigl(y, (\varphi, 1).y\bigr), d\bigl((\varphi, 1).y, (\varphi, h).y\bigr)\} \\ &= \max\{d\bigl(y, (\varphi, 1).y\bigr), d\bigl(y, (1, h).y\bigr)\} \\ &\leq \max\{\max\{D_0, D_1, \ldots, D_n\}, D_0\}. \end{aligned}$$

That is, the diameter of $G \wr_X H.y$ is itself bounded by $\max\{D_0, D_1, \ldots, D_n\}$, which finishes the proof. □

While the fact that trees do not have Cartesian powers is an obstacle to our methods, we still have a weak version of Theorem 4.3.1 for properties FA and F$\mathbf{R}$. Before stating it, remind that we already know, by Proposition 4.1, the behavior of uncountable cofinality under wreath products. On the other hand, we have the following result:

Lemma 4.3.13. *The group $G \wr_X H$ has no quotient isomorphic to $\mathbf{Z}$ if and only if both G and H have no quotient isomorphic to $\mathbf{Z}$.*

Proof. The desired result follows from $(G \wr_X H)^{\mathrm{ab}} \cong \bigoplus_{X/H}(G^{\mathrm{ab}}) \times H^{\mathrm{ab}}$ and the claim that a direct sum $\bigoplus_{y \in Y} K_y$ has a quotient isomorphic to $\mathbf{Z}$ if and only if at least one of the factor has a quotient isomorphic to $\mathbf{Z}$. Indeed, one direction of the claim is trivial. For the other direction, remind that K does not project onto $\mathbf{Z}$ if and only if any action of K by orientation preserving isomorphisms on Z, the 2-regular tree, has bounded orbits. But the only possibility for such an action to have bounded orbits is to be trivial. If none of the K_y projects onto $\mathbf{Z}$, all their actions on Z are trivial and so is any action of $\bigoplus_{y\in Y} K_y$, which can therefore not project onto $\mathbf{Z}$. $\square$

By Corollary 4.3.6, Proposition 4.1 and Lemma 4.3.13, we directly obtain the following partial version of Theorem 4.1.2.

Proposition 4.3.14. *Let G and H be two groups with G non-trivial and X a set on which H acts. Then*

1. *If $G \wr_X H$ has property FA (respectively property F$\boldsymbol{R}$), then H has property FA (respectively property F$\boldsymbol{R}$), H acts on X with finitely many orbits, G has no quotient isomorphic to $\mathbf{Z}$ and G has uncountable cofinality,*

2. *If both G and H have no quotient isomorphic to $\mathbf{Z}$, have uncountable cofinality and H acts on X with finitely many orbits, then $G \wr_X H$ has no quotient isomorphic to $\mathbf{Z}$ and has uncountable cofinality,*

3. *If both G and H have property FA (respectively property F$\boldsymbol{R}$) and X is finite, then $G \wr_X H$ has property FA (respectively property F$\boldsymbol{R}$).*

Moreover, by using Corollary 4.3.9 we can get ride of the *finitely many orbits* hypothesis in Theorem 4.1.2 in order to obtain Theorem 4.1.

Chapter **5**

A Cheeger-Buser-type inequality on CW complexes

5.1 Introduction

The expander graphs have been a prolific field of research in the last four decades (see for example [69] for an excellent survey). For a graph X with a vertex set V the classical expansion constant (or *Cheeger constant*) is defined by

$$h(X) := \min \left\{ \frac{|\partial A|}{\min\{|A|, |A^c|\}} : \emptyset \subsetneq A \subsetneq V \right\}$$

where ∂A is the set of edges with one vertex in A and the other in A^c. A central result of this field is the Cheeger-Buser inequality, which describes the relation between the expansion and the spectrum of the Laplacian.

Theorem 5.1.1 (Cheeger-Buser inequality)**.** *Let X be a connected graph and λ the first non-trivial eigenvalue of the Laplacian, then*

$$\frac{\lambda}{2} \leq h(X) \leq \sqrt{2\lambda d}$$

where d is the maximal degree of a vertex.

For more details see [57, Theorem 2.4].

In recent years, theories for expansion of higher dimensional simplicial complexes have emerged. The combinatorial definitions allow generalizations of the Cheeger-Buser inequality, see [38, 48, 88]. Other results, like Expander Mixing Lemma or generalization of Alon-Boppana theorem, can be proved using this formalism, see [86]. We can also use homology and cohomology with coefficients in $\mathbf{Z}/2\mathbf{Z}$ to define boundary expansion, see [97], and coboundary expansion, see [31, 45, 49, 66, 74, 97]. A Cheeger-Buser-type inequality is proved for boundary expansion in [97]. The coboundary expansion has the advantage that it coincides with the standard Cheeger constant in the one dimensional case, but the Cheeger-Buser inequality does not stay true in higher dimensions, see [49, 97] for counterexamples. Nevertheless there exist some indications that suggest a connection between these two notions, particularly for the Cheeger's part (the upper bound) which holds for Riemannian manifolds [19].

Recall that one of the first explicit constructions of expander graphs used Cayley graphs of finite quotients of a group with Kazhdan's Property (T), see [69, Chapter 3] and [9, Chapter 6]. There exist higher dimensional objects which can be associated to groups in the same spirit, as, for example, Cayley complexes [71, Chapter 3] or presentation complexes [54]. For example, the group $\mathrm{SL}_3(\mathbf{Z})$ has Property (T) and any family of its finite quotients will give an expanding family of Cayley graphs. One can wonder whether the corresponding Cayley complexes are also expanding. More generally, it would be interesting to establish which properties of the group may imply high-dimensional expansion in its finite quotients. One technical issue that has to be addressed while proceeding with this program is that high dimensional expansion has been mainly defined and studied for simplicial complexes, while the higher-dimensional objects naturally associated to groups are typically CW-complexes. Working out the formalism of high dimensional expansion for CW complexes is the aim of the present note.

We will begin by recalling some classical definitions about CW complexes, groups of cochains with coefficients in an abelian group G, Laplacians and their spectra. Then, considering cochains with coefficients in $\mathbf{Z}/2\mathbf{Z}$, we will introduce $h_n(X)$, the n^{th} boundary expansion constant for every integer n and we will prove a Cheeger-Buser-type inequality in the same spirit as the original result when n equal to the dimension of the complex. This Theorem generalizes [97].

Theorem 5.1.2. *Let X be a regular CW complex of dimension d and λ_d the smallest non trivial eigenvalue of the d^{th} lower Laplacian, then*

1. *If X is orientable*
$$\lambda_d(X) \leq h_d(X).$$
2. *If the maximal degree of a $(d-1)$-cell is 2, then*
$$h_d(X) \leq \sqrt{2m\lambda_d}$$
where $m = \max\left\{\sum_\mu \left|[e^d_\lambda : e^{d-1}_\mu]\right| : e^d_\lambda \in X^d\right\}$.

5.2 Definitions

CW Complexes

We will begin by fixing some definitions and notations about CW complexes that we will use in the following. All the details can be found in [72].

A *CW complex* X is a topological space obtained inductively by gluing euclidean balls, called *cells*, via continuous maps called *attaching maps*. In what follows, all complexes will be regular, which means that the attaching maps are homeomorphisms on their images. The *n-skeleton*, denoted by X^n, is the set of all the cells of dimension n which are called the *n-cells*. The *dimension* of X is the maximal dimension of a cell.

Cohomology with coefficients

We will now define the cohomology groups with coefficients associated to a CW complex. All the details of this classic construction can be found in [72].

The group of *n-chains* $C_n(X)$ is the free abelian group generated by the n-cells:

$$C_n(X) = \bigoplus_{e^n_\lambda \in X^n} \mathbf{Z}.$$

By using classical homology tools, that we will not detail here, these groups can be provided with a structure of chain complex using boundary operators $\partial_n : C_n(X) \to C_{n-1}(X)$. In each infinite cyclic summand in the direct sum above, one can choose between the cyclic generator and its inverse: b^n_λ or $\bar{b}^n_\lambda$. This choice defines the orientation of the n-cell e^n_λ. The set $\{b^n_\lambda\}_\lambda$ forms a basis of $C_n(X)$. The boundary operator is completely determined by the values on basis elements:

$$\partial_n(b^n_\lambda) = \sum_\mu \left[b^n_\lambda : b^{n-1}_\mu\right] b^{n-1}_\mu$$

where the coefficients $[b^n_\lambda : b^{n-1}_\mu]$ are integers called the *incidence numbers* of the cells e^n_λ and e^{n-1}_μ with respect to the chosen orientations. This incidence number can be thought intuitively as the number of times an $n-1$ cell appears in the boundary of an n cell, the sign depending of the consistency of the chosen orientations. The *degree* of an oriented cell b^n_λ is the sum $\sum_\mu \left|[b^{n+1}_\mu : b^n_\lambda]\right|$.

As the gluing functions are homeomorphisms, the incidence numbers take values in $\{-1, 0, 1\}$. Two oriented cells $b^n_{\lambda_1}$ and $b^n_{\lambda_2}$ that have a common $(n-1)$-cell b^{n-1}_μ in their boundary are either *dissimilarly oriented* if $[b^n_{\lambda_1} : b^{n-1}_\mu] = [b^n_{\lambda_2} : b^{n-1}_\mu]$ or *similarly oriented* if $[b^n_{\lambda_1} : b^{n-1}_\mu] \neq [b^n_{\lambda_2} : b^{n-1}_\mu]$. If there exists an orientation on a d-dimensional CW complex such that all the d-cells are similarly oriented, X is said to be *orientable*.

The coboundary operator is defined using these incidence numbers :

$$\begin{aligned} \delta_n : C_n(X) &\to C_{n+1}(X) \\ b^n_\lambda &\mapsto \sum_\mu [b^{n+1}_\mu : b^n_\lambda] b^{n+1}_\mu. \end{aligned}$$

We will often omit the indices of the operators.

Definition 5.2.1. Let G be an abelian group and X be a CW complex. The n-cochains group with coefficients in G is the group of homomorphisms between the n-chains and G,

$$C^n(X : G) \coloneqq \mathrm{Hom}(C_n(X), G).$$

It follows from the definition that $f(\bar{b}^n_\lambda) = -f(b^n_\lambda)$ for all $f \in C^n(X : G)$ and $n \geq 1$. We define operators, also denoted by ∂_n and δ_n, between the cochains groups:

$$\partial_n f(e^{n-1}_\mu) = f(\delta_n(e^{n-1}_\mu)) = \sum_\lambda [e^n_\lambda : e^{n-1}_\mu] f(e^n_\lambda)$$
$$\delta_n f(e^{n+1}_\mu) = f(\partial_n(e^{n+1}_\mu)) = \sum_\lambda [e^{n+1}_\mu : e^n_\lambda] f(e^n_\lambda)$$

It is sometimes more convenient to add (-1)-chains consisting only of the empty set and operators

$$\partial_0(\sum_\lambda g_\lambda e^0_\lambda) = \sum_\lambda g_\lambda \text{ and } \delta_{-1}(g) = \sum_\lambda g e^0_\lambda.$$

The following subgroups will be used in the following:

$$B_n \coloneqq \operatorname{Im} \partial_{n+1} \qquad Z_n \coloneqq \operatorname{Ker} \partial_n$$

Laplacians and eigenvalues

Consider the case $G = \mathbf{R}$. The cochains $C^n(X : \mathbf{R})$ can be turned into real Hilbert spaces using

$$\langle f, g \rangle = \sum_{\lambda} f(e^n_\lambda) g(e^n_\lambda).$$

In this case, ∂ and δ are adjoint operators. Combining them, we define the n^{th} *lower Laplacian*,

$$\Delta_n^- = \delta_{n-1}\partial_n.$$

It can be noted that the elements of B_n are in the kernel of Δ_n^-. Indeed, if f is in B_n, there exists g in $C^{n+1}(X : \mathbf{R})$ such that $f = \partial_{n+1} g$ and then $\Delta_n^- f = \delta_{n-1}\partial_n f = \delta_{n-1}\partial_n\partial_{n+1} g = 0$. This part will be called the *trivial* part of the spectrum and we will be interested in the smallest eigenvalue on the other parts.

Definition 5.2.2. The smallest non trivial eigenvalue of Δ_n^-, denoted by λ_n, is defined as

$$\lambda_n = \min \operatorname{Spec} \Delta_n^-|_{B_n^\perp}.$$

It can be computed using Rayleigh's quotients:

$$\lambda_n \coloneqq \min \left\{ \frac{\|\partial_n f\|^2}{\|f\|^2} : f \in B_n^\perp, f \neq 0 \right\} = \min \left\{ \frac{\|\partial_n f\|^2}{\|f + B_n|^2} : f \notin B_n \right\}$$

where $\|f + B_n\| = \min\{\|f + g\| : g \in B_n\}$.

Boundary expansion

Let us consider $G = \mathbf{Z}/2\mathbf{Z}$ to define the notion of boundary expansion for CW complexes. The cochains groups can be endowed with the Hamming's norm:

$$\|\alpha\| = |\operatorname{supp} \alpha|$$

for $\alpha \in C^n(X : \mathbf{Z}/2\mathbf{Z})$. We can define the following notion of expansion for CW complexes.

Definition 5.2.3. Let X be a CW complex. The n^{th} boundary expansion constant of X is:

$$h_n(X) \coloneqq \min \left\{ \frac{\|\partial\alpha\|}{\|\alpha + B_n\|} : \alpha \in C^n(X : \mathbf{Z}/2\mathbf{Z}) \setminus B_n \right\}.$$

where $\|\alpha + B_n\| = \min\{\|\alpha + \beta\| : \beta \in B_n\}$

5.3 Proof of the Theorem

Proof of 1). Let α be an element of $C^d(X : \mathbf{Z}/2\mathbf{Z})$ which realizes the minimum in h_d. We can find a cochain f in $C^d(X : \mathbf{R})$, which assigns 1 to every d-cells in $\operatorname{supp}\alpha$ and 0 to all the others. Since X is orientable, $\partial_d\alpha$ is equivalent to $\partial_d f$. Then,

$$\begin{aligned} h_d &= \frac{\|\partial_d\alpha\|}{\|\alpha\|} \\ &= \frac{\|\partial_d f\|_2^2}{\|f\|_2^2} \\ &\geq \min\left\{\frac{\|\partial_d g\|^2}{\|g\|^2} : g \in C^d(X : \mathbf{R}), g \notin B_d = 0\right\} \\ &= \lambda_d. \end{aligned}$$

□

Proof of 2). Let f be a real cochain which is an eigenvector of Δ_d^- of eigenvalue λ_d. We chose an orientation on the d-cells such that all the values of f are positive. We do not assume that they are similarly oriented. We put an order on $X^d = \{e_1^d, e_2^d, \ldots, e_N^d\}$ such that

$$0 \leq f(e_1^d) \leq f(e_2^d) \leq \ldots \leq f(e_N^d).$$

The boundary of X is the $(d-1)$-cells with degree 1,

$$\partial X := \{e_\lambda^{d-1} \in X^{d-1} : \deg e_\lambda^{d-1} = 1\}$$

For each e_λ^{d-1} in ∂X, we add another $(d-1)$-cell ,$e_{\lambda'}^{d-1}$, via the attaching map $\varphi_{\lambda'} = \varphi_\lambda$. We can add a d-cell on X, whose attaching map goes homeomorphically into $e_\lambda^{d-1} \cup e_{\lambda'}^{d-1}$. We denote by X_∂^d the set of these new d-cells and put an order $X_\partial^d = \{e_0^d, e_{-1}^d, \ldots, e_{1-M}^d\}$ on it. The function f is defined as $f = 0$ on the cells of X_∂^d. When two d-cells have a common $(d-1)$-cell in their boundary, we say they are *low adjacent* and write $e_\lambda^d \sim e_\lambda^d$. It is possible that there are more than one $(d-1)$-cells in the intersection of the boundary of two d-cells. We say that we count the cells that realize $e_\lambda^d \sim e_\lambda^d$ with *multiplicity* in this case, i.e. the pair $\{e_j^d, e_k^d\}$ appears a number of time equal of the number of common $(d-1)$-cells in their boundary. We define

$$C_i := \left\{\{e_j^d, e_k^d\} : 1 - M \leq j \leq i < k \leq N \text{ and } e_j^d \sim e_k^d\right\}$$

counted with multiplicity. Consider the quantity

$$H[f] := \min_{0 \leq i \leq N-1} \frac{|C_i|}{N - i}.$$

We can show that $H[f] \geq h_d$. Indeed, let $\mathbf{i}$ be the i which realizes the minimum of $H[f]$ and $\alpha \in C(X : \mathbf{Z}/2\mathbf{Z})$ defined as follows,

$$\alpha(e_k^d) = \begin{cases} 1 & \mathbf{i} < k \\ 0 & \mathbf{i} \geq k \end{cases}.$$

So the $(d-1)$-cells that are in the support of $\partial_d \alpha$ are in the boundary of one e_k^d with $k > i$ and another with $k \leq i$. Then, we have

$$H[f] = \frac{|C_{\mathbf{i}}|}{N - \mathbf{i}} = \frac{\|\partial_d \alpha\|}{\|\alpha\|} \geq h_d.$$

We can now prove our inequality. All the sums on d-cells are on $X^d \cup X_\partial^d$ and are taken with multiplicity.

$$\begin{aligned}
\lambda_d &= \frac{\|\partial_d f\|_2^2}{\|f\|_2^2} \\
&= \frac{\sum_\mu \partial_d f(e_\mu^{d-1})^2}{\sum_\lambda f(e_\lambda^d)^2} \\
&= \frac{\sum_{e_i^d \sim e_j^d} \left(f(e_i^d) \pm f(e_j^d)\right)^2}{\sum_\lambda f(e_\lambda^d)^2} \cdot \frac{\sum_{e_i^d \sim e_j^d} \left(f(e_i^d) \mp f(e_j^d)\right)^2}{\sum_{e_i^d \sim e_j^d} \left(f(e_i^d) \mp f(e_j^d)\right)^2} && (5.1) \\
&\geq \frac{\left(\sum_{e_i^d \sim e_j^d} |f(e_i^d)^2 - f(e_j^d)^2|\right)^2}{\left(\sum_\lambda f(e_\lambda^d)^2\right) \cdot \left(\sum_{e_i^d \sim e_j^d} (f(e_i^d) \mp f(e_j^d))^2\right)} && (5.2) \\
&\geq \frac{\left(\sum_{e_i^d \sim e_j^d} |f(e_i^d)^2 - f(e_j^d)^2|\right)^2}{\left(\sum_\lambda f(e_\lambda^d)^2\right) \cdot 2\left(\sum_{e_i^d \sim e_j^d} f(e_i^d)^2 + f(e_j^d)^2\right)} \\
&\geq \frac{\left(\sum_{e_i^d \sim e_j^d} |f(e_i^d)^2 - f(e_j^d)^2|\right)^2}{\left(\sum_\lambda f(e_\lambda^d)^2\right) \cdot \left(2m \sum_\lambda f(e_\lambda^d)^2\right)} \\
&= \frac{\left(\sum_{i=0}^{N-1} (f(e_{i+1}^d)^2 - f(e_i^d)^2)|C_i|\right)^2}{2m \left(\sum_\lambda f(e_\lambda^d)^2\right)^2} && (5.3) \\
&\geq \frac{\left(\sum_{i=0}^{N-1} (f(e_{i+1}^d)^2 - f(e_i^d)^2) H[f](N-i)\right)^2}{2m \left(\sum_\lambda f(e_\lambda^d)^2\right)^2} \\
&= \frac{H[f]^2}{2m} \cdot \frac{\left(\sum_\lambda f(e_\lambda^d)^2\right)^2}{\left(\sum_\lambda f(e_\lambda^d)^2\right)^2} \\
&\geq \frac{h_d^2(X)}{2m}.
\end{aligned}$$

The equality (5.1) is a consequence of $f|_{X_\partial^d} = 0$ and $\partial_d f = 0$ for a $(d-1)$-cell of degree 0 and (5.2) follows from Cauchy-Schwartz. For (5.3), we want to show that

$$\sum_{e_i^d \sim e_j^d} |f(e_i^d)^2 - f(e_j^d)^2| = \sum_{i=0}^{N-1} (f(e_{i+1}^d)^2 - f(e_i^d)^2)|C_i|.$$

This can be seen by counting the number of times each $f(e_i^d)^2$ appears on each side. On the left, $f(e_i^d)^2$ appears

$$\begin{aligned}
\Sigma_i &:= \left|\left\{\{e_i^d, e_j^d\} : j < i \text{ and } e_i^d \sim e_j^d \text{ with multiplicity}\right\}\right| \\
&\quad - \left|\left\{\{e_i^d, e_k^d\} : i < k \text{ and } e_i^d \sim e_k^d \text{ with multiplicity}\right\}\right|.
\end{aligned}$$

On the other side, each $f(e_i^d)^2$ appears $|C_{i-1}| - |C_i|$ times. Remark that for $j < k$ such that $\{e_j^d, e_k^d\}$ is in C_{i-1}, $\{e_j^d, e_k^d\}$ is also in C_i if $k \neq i$. Similarly, $\{e_j^d, e_k^d\}$ in C_i is also in C_{i-1} if $j \neq i$. Then,

$$|C_{i-1}| - |C_i| = \Sigma_i$$

$\square$

Bibliography

[1] I. Agol. The virtual Haken conjecture. *Doc. Math.*, 18:1045–1087, 2013.

[2] W. Ballmann. *Lectures on spaces of nonpositive curvature*, volume 25 of *DMV Seminar*. Birkhäuser Verlag, Basel, 1995.

[3] H.-J. Bandelt and V. Chepoi. Metric graph theory and geometry: a survey. In *Surveys on discrete and computational geometry*, volume 453 of *Contemp. Math.*, pages 49–86. Amer. Math. Soc., Providence, RI, 2008.

[4] A. Barnhill and I. Chatterji. Property (T) versus Property FW. In *Guido's book of conjectures*, volume 40 of *Monographies de L'Enseignement Mathématique [Monographs of L'Enseignement Mathématique]*, chapter 9. L'Enseignement Mathématique, Geneva, 2008.

[5] L. Bartholdi and R. I. Grigorchuk. On the spectrum of Hecke type operators related to some fractal groups. *Tr. Mat. Inst. Steklova*, 231(Din. Sist., Avtom. i Beskon. Gruppy):5–45, 2000.

[6] L. Bartholdi, R. I. Grigorchuk, and Z. Šunić. Branch groups. In *Handbook of algebra, Vol. 3*, volume 3 of *Handb. Algebr.*, pages 989–1112. Elsevier/North-Holland, Amsterdam, 2003.

[7] P. Baum, A. Connes, and N. Higson. Classifying space for proper actions and K-theory of group C^*-algebras. In *C^*-algebras: 1943–1993 (San Antonio, TX, 1993)*, volume 167 of *Contemp. Math.*, pages 240–291. Amer. Math. Soc., Providence, RI, 1994.

[8] B. Bekka, P.-A. Cherix, and A. Valette. Proper affine isometric actions of amenable groups. In *Novikov conjectures, index theorems and rigidity, Vol. 2 (Oberwolfach, 1993)*, volume 227 of *London Math. Soc. Lecture Note Ser.*, pages 1–4. Cambridge Univ. Press, Cambridge, 1995.

[9] B. Bekka, P. de la Harpe, and A. Valette. *Kazhdan's property (T)*, volume 11 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2008.

[10] G. M. Bergman. Generating infinite symmetric groups. *Bull. London Math. Soc.*, 38(3):429–440, 2006.

[11] M. Bourdon. Un théorème de point fixe sur les espaces L^p. *Publ. Mat.*, 56(2):375–392, 2012.

[12] S. G. Brick. Quasi-isometries and ends of groups. *J. Pure Appl. Algebra*, 86(1):23–33, 1993.

[13] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[14] N. Brown and E. Guentner. Uniform embeddings of bounded geometry spaces into reflexive Banach space. *Proc. Amer. Math. Soc.*, 133(7):2045–2050, 2005.

[15] P. Buser. A note on the isoperimetric constant. *Ann. Sci. École Norm. Sup. (4)*, 15(2):213–230, 1982.

[16] J. W. Cannon. Geometric group theory. In *Handbook of geometric topology*, pages 261–305. North-Holland, Amsterdam, 2002.

[17] I. Chatterji, C. Druţu, and F. Haglund. Kazhdan and Haagerup properties from the median viewpoint. *Adv. Math.*, 225(2):882–921, 2010.

[18] I. Chatterji and G. Niblo. From wall spaces to CAT(0) cube complexes. *Internat. J. Algebra Comput.*, 15(5-6):875–885, 2005.

[19] J. Cheeger. A lower bound for the smallest eigenvalue of the Laplacian. In R. C. Gunning, editor, *Problems in analysis (Papers dedicated to Salomon Bochner, 1969)*, pages 195–199. Princeton University Press, Princeton, N.J., 1970.

[20] V. Chepoi. Graphs of some CAT(0) complexes. *Adv. in Appl. Math.*, 24(2):125–179, 2000.

[21] P.-A. Cherix, F. Martin, and A. Valette. Spaces with measured walls, the Haagerup property and property (T). *Ergodic Theory Dynam. Systems*, 24(6):1895–1908, 2004.

[22] F. R. K. Chung. *Spectral graph theory*, volume 92 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1997.

[23] Y. Cornulier. Strongly bounded groups and infinite powers of finite groups. *Comm. Algebra*, 34(7):2337–2345, 2006.

[24] Y. Cornulier. Group actions with commensurated subsets, wallings and cubings. *arXiv e-prints*, 2013.

[25] Y. Cornulier. Irreducible lattices, invariant means, and commensurating actions. *Math. Z.*, 279(1-2):1–26, 2015.

[26] Y. Cornulier and A. Kar. On Property (FA) for wreath products. *Journal of Group Theory*, 14(1):165–174, jan 2011.

[27] P. de la Harpe. *Topics in geometric group theory.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000.

[28] P. de la Harpe and A. Valette. *La propriété* (T) *de Kazhdan pour les groupes localement compacts (avec un appendice de Marc Burger).* Astérisque, No. 175. Société Mathématique de France, Paris, 1989.

[29] P. Delorme. 1-cohomologie des représentations unitaires des groupes de Lie semi-simples et résolubles. Produits tensoriels continus de représentations. *Bull. Soc. Math. France*, 105(3):281–336, 1977.

[30] R. Diestel and D. Kühn. Graph-theoretical versus topological ends of graphs. *J. Combin. Theory Ser. B*, 87(1):197–206, 2003.

[31] D. Dotterrer. *The (co)isoperimetric problem in (random) polyhedra.* PhD thesis, University of Toronto, 2013.

[32] T. Fernós. The Furstenberg-Poisson boundary and CAT(0) cube complexes. *Ergodic Theory Dynam. Systems*, 38(6):2180–2223, 2018.

[33] T. Fernós, J. Lécureux, and F. Mathéus. Random walks and boundaries of CAT(0) cubical complexes. *Comment. Math. Helv.*, 93(2):291–333, 2018.

[34] A. Genevois. *Algebraic properties of groups acting on median graphs.* In preparation, 2022.

[35] A. Genevois. Lamplighter groups, median spaces, and Hilbertian geometry. *Proceedings of the Edinburgh Mathematical Society*, 65(2):500–529, 2022.

[36] V. Gerasimov. Semi-splittings of groups and actions on cubings. In *Algebra, geometry, analysis and mathematical physics (Russian) (Novosibirsk, 1996)*, pages 91–109,190. Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Mat., Novosibirsk, 1997.

[37] V. Gerasimov. Fixed-point-free actions on cubings [translation of *Algebra, geometry, analysis and mathematical physics (Russian) (Novosibirsk, 1996)*, 91–109, 190, Izdat. Ross. Akad. Nauk Sibirsk. Otdel. Inst. Mat., Novosibirsk, 1997; MR. *Siberian Adv. Math.*, 8(3):36–58, 1998.

[38] K. Golubev and O. Parzanchevski. Spectrum and combinatorics of two-dimensional Ramanujan complexes. *Israel J. Math.*, 230(2):583–612, 2019.

[39] R. Grigorchuk, D. Lenz, T. Nagnibeda, and D. Sell. Subshifts with leading sequences, uniformity of cocycles and spectra of Schreier graphs. *Adv. Math.*, 407:Paper No. 108550, 34, 2022.

[40] R. I. Grigorchuk. On Burnside's problem on periodic groups. *Funktsional. Anal. i Prilozhen.*, 14(1):53–54, 1980.

[41] R. I. Grigorchuk. Degrees of growth of finitely generated groups and the theory of invariant means. *Izv. Akad. Nauk SSSR Ser. Mat.*, 48(5):939–985, 1984.

[42] R. I. Grigorchuk, D. Lenz, and T. Nagnibeda. Spectra of Schreier graphs of Grigorchuk's group and Schroedinger operators with aperiodic order. *Math. Ann.*, 370(3-4):1607–1637, 2018.

[43] R. I. Grigorchuk and I. Pak. Groups of Intermediate Growth: an Introduction for Beginners. *arXiv e-prints*, 2006.

[44] M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.

[45] M. Gromov. Singularities, expanders and topology of maps. Part 2: From combinatorics to topology via algebraic isoperimetry. *Geom. Funct. Anal.*, 20(2):416–526, 2010.

[46] J. L. Gross. Every connected regular graph of even degree is a Schreier coset graph. *J. Combinatorial Theory Ser. B*, 22(3):227–232, 1977.

[47] A. Guichardet. Sur la cohomologie des groupes topologiques. II. *Bull. Sci. Math. (2)*, 96:305–332, 1972.

[48] A. Gundert and M. Szedlák. Higher dimensional discrete Cheeger inequalities. *J. Comput. Geom.*, 6(2):54–71, 2015.

[49] A. Gundert and U. Wagner. On Laplacians of random complexes. In *Computational geometry (SCG'12)*, pages 151–160. ACM, New York, 2012.

[50] M. F. Hagen. The simplicial boundary of a CAT(0) cube complex. *Algebr. Geom. Topol.*, 13(3):1299–1367, 2013.

[51] M. F. Hagen. Corrigendum to the article The simplicial boundary of a CAT(0) cube complex. *Algebr. Geom. Topol.*, 18(2):1251–1256, 2018.

[52] F. Haglund. Finite index subgroups of graph products. *Geom. Dedicata*, 135:167–209, 2008.

[53] F. Haglund and D. T. Wise. A combination theorem for special cube complexes. *Ann. of Math. (2)*, 176(3):1427–1482, 2012.

[54] A. Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[55] E. Hewitt and K. A. Ross. *Abstract harmonic analysis. Vol. I*, volume 115 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin-New York, second edition, 1979.

[56] N. Higson and G. Kasparov. Operator K-theory for groups which act properly and isometrically on Hilbert space. *Electron. Res. Announc. Amer. Math. Soc.*, 3:131–142, 1997.

[57] S. Hoory, N. Linial, and A. Wigderson. Expander graphs and their applications. *Bulletin of the American Mathematical Society*, 43(4):439–561, 2006.

[58] H. Hopf. Enden offener Räume und unendliche diskontinuierliche Gruppen. *Comment. Math. Helv.*, 16:81–100, 1944.

[59] Y. Ihara. On discrete subgroups of the two by two projective linear group over p-adic fields. *Journal of the Mathematical Society of Japan*, 18(3):219–235, 1966.

[60] S. Klavžar and H. M. Mulder. Median graphs: characterizations, location theory and related structures. *J. Combin. Math. Combin. Comput.*, 30:103–127, 1999.

[61] A. Knowles and R. Rosenthal. Eigenvalue confinement and spectral gap for random simplicial complexes. *Random Structures Algorithms*, 51(3):506–537, 2017.

[62] I. J. Leary. A metric Kan-Thurston theorem. *J. Topol.*, 6(1):251–284, 2013.

[63] P.-H. Leemann. Up to a double cover, every regular connected graph is isomorphic to a Schreier graph. *Bulletin of the Belgian Mathematical Society - Simon Stevin*, 28(4), 2022.

[64] P.-H. Leemann and G. Schneeberger. Property FW and wreath products of groups: a simple approach using Schreier graphs. *Expositiones Mathematicae*, 40(4):1261—-1270, 2022.

[65] P.-H. Leemann and G. Schneeberger. Wreath products of groups acting with bounded orbits. *L'Enseignement Mathématique*, accepted, 2022.

[66] N. Linial and R. Meshulam. Homological connectivity of random 2-complexes. *Combinatorica*, 26(4):475–487, 2006.

[67] A. Lubotzky. Cayley graphs: eigenvalues, expanders and random walks. In *Surveys in combinatorics, 1995 (Stirling)*, volume 218 of *London Math. Soc. Lecture Note Ser.*, pages 155–189. Cambridge Univ. Press, Cambridge, 1995.

[68] A. Lubotzky. *Discrete groups, expanding graphs and invariant measures.* Modern Birkhäuser Classics. Birkhäuser Verlag, Basel, 2010.

[69] A. Lubotzky. Expander graphs in pure and applied mathematics. *Bull. Amer. Math. Soc. (N.S.)*, 49(1):113–162, 2012.

[70] A. Lubotzky. High dimensional expanders. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. I. Plenary lectures*, pages 705–730. World Sci. Publ., Hackensack, NJ, 2018.

[71] R. C. Lyndon and P. E. Schupp. *Combinatorial group theory.* Classics in Mathematics. Springer-Verlag, Berlin, 2010.

[72] W. S. Massey. *Singular homology theory*, volume 70 of *Graduate Texts in Mathematics.* Springer-Verlag, New York-Berlin, 1980.

[73] N. Matte Bon. Topological full groups of minimal subshifts with subgroups of intermediate growth. *J. Mod. Dyn.*, 9:67–80, 2015.

[74] R. Meshulam and N. Wallach. Homological connectivity of random k-dimensional complexes. *Random Structures Algorithms*, 34(3):408–417, 2009.

[75] A. Minasyan. New examples of groups acting on real trees. *J. Topol.*, 9(1):192–214, 2016.

[76] L. Nebeský. Median graphs. *Comment. Math. Univ. Carolinae*, 12:317–325, 1971.

[77] M. Neuhauser. Relative property (T) and related properties of wreath products. *Mathematische Zeitschrift*, 251(1):167–177, 2005.

[78] A. Nevo and M. Sageev. The Poisson boundary of CAT(0) cube complex groups. *Groups Geom. Dyn.*, 7(3):653–695, 2013.

[79] L. Nguyen Van Thé and V. G. Pestov. Fixed point-free isometric actions of topological groups on Banach spaces. *Bull. Belg. Math. Soc. Simon Stevin*, 17(1):29–51, 2010.

[80] G. Niblo and L. Reeves. Groups acting on CAT(0) cube complexes. *Geom. Topol.*, 1:approx. 7 pp. 1997.

[81] G. A. Niblo and M. A. Roller. Groups acting on cubes and Kazhdan's property (T). *Proc. Amer. Math. Soc.*, 126(3):693–699, 1998.

[82] B. Nica. Cubulating spaces with walls. *Algebr. Geom. Topol.*, 4:297–309, 2004.

[83] P. W. Nowak. Group actions on Banach spaces. In *Handbook of group actions. Vol. II*, volume 32 of *Adv. Lect. Math. (ALM)*, pages 121–149. Int. Press, Somerville, MA, 2015.

[84] I. Oppenheim. Local spectral expansion approach to high dimensional expanders Part I: Descent of spectral gaps. *Discrete Comput. Geom.*, 59(2):293–330, 2018.

[85] I. Oppenheim. Local spectral expansion approach to high dimensional expanders Part II: Mixing and geometrical overlapping. *Discrete Comput. Geom.*, 64(3):1023–1066, 2020.

[86] O. Parzanchevski. *High Dimensional Expanders*. PhD thesis, Hebrew University of Jerusalem, 2013.

[87] O. Parzanchevski. Mixing in high-dimensional expanders. *Combin. Probab. Comput.*, 26(5):746–761, 2017.

[88] O. Parzanchevski, R. Rosenthal, and R. J. Tessler. Isoperimetric inequalities in simplicial complexes. *Combinatorica*, 36(2):195–227, 2016.

[89] M. A. Roller. *Poc Sets, Median Algebras and Group Actions*. PhD thesis, Universität Regensburg, 1998.

[90] M. Sageev. Ends of group pairs and non-positively curved cube complexes. *Proc. London Math. Soc. (3)*, 71(3):585–617, 1995.

[91] M. Sageev and D. T. Wise. The Tits alternative for CAT(0) cubical complexes. *Bull. London Math. Soc.*, 37(5):706–710, 2005.

[92] G. Schneeberger. Proper actions of Grigorchuk groups on CAT(0) cube complexes. *arXiv e-prints*, 2022.

[93] G. Schneeberger. A Cheeger-Buser-Type inequality on CW complexes. *International Journal of Group Theory*, 12(3):197 – 204, 2023.

[94] J.-P. Serre. *Arbres, amalgames,* SL_2. Société Mathématique de France, Paris, 1977.

[95] J. R. Stallings. On torsion-free groups with infinitely many ends. *Ann. of Math. (2)*, 88:312–334, 1968.

[96] J. R. Stallings. *Group theory and three-dimensional manifolds.* Yale University Press, New Haven, Conn.-London, yale mathe edition, 1971.

[97] J. Steenbergen, C. Klivans, and S. Mukherjee. A Cheeger-type inequality on simplicial complexes. *Adv. in Appl. Math.*, 56:56–77, 2014.

[98] R. Tessera and A. Valette. Locally compact groups with every isometric action bounded or proper. *J. Topol. Anal.*, 12(2):267–292, 2020.

[99] A. Valette. *Introduction to the Baum-Connes conjecture.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2002.

[100] E. R. Verheul. *Multimedians in metric and normed spaces*, volume 91 of *CWI Tract*. Stichting Mathematisch Centrum, Centrum voor Wiskunde en Informatica, Amsterdam, 1993.

[101] Y. Vorobets. Notes on the Schreier graphs of the Grigorchuk group. In *Dynamical systems and group actions*, volume 567 of *Contemp. Math.*, pages 221–248. Amer. Math. Soc., Providence, RI, 2012.

[102] D. T. Wise. *From riches to raags: 3-manifolds, right-angled Artin groups, and cubical geometry*, volume 117 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2012.