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Physics Letters B 293 (1992) 344–352
North-Holland

PHYSICS LETTERS B

New quantum Poincaré algebra and κ-deformed field theory

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Received 10 August 1992

We derive a new real quantum Poincaré algebra with standard real structure, obtained by contraction of $U_q(O(3, 2))$ (q real), which is a standard real Hopf algebra, depending on a dimension-full parameter κ instead of q. For our real quantum Poincaré algebra both Casimirs are given. The free scalar κ-deformed quantum field theory is considered. it appears that the κ-parameter introduced nonlocal $\tilde{q}$-time derivatives with $\ln \tilde{q} \sim 1/\kappa$.

1. Introduction

Recently the present authors considered quantum deformations of the $D=4$ Poincaré algebra [1–4] obtained by the contraction of a real form of quantum anti-de-Sitter algebra $U_q(O(3, 2))$. The method was based on finding the deformation of the Cartan–Weyl basis, with generators self-conjugate with respect to involutive homomorphisms (involutions) describing a real structure. In ref. [2] we considered all inner involutions of the Cartan–Weyl basis for $U_q(Sp(4; \mathbb{C}))$. We found in ref. [2] that only two (out of sixteen) provide examples of real forms of $U_q(Sp(4; \mathbb{C}))$ suitable for our contraction procedure to quantum the Poincaré algebra [#1]. Unfortunately these two real forms were described by nonstandard involutions (one ⊕ involution, used in ref. [1], and one ∗ involution listed in ref. [2] [#2]).

The standard + involution should be an antiautomorphism in the algebra sector and an automorphism in the coalgebra sector [5–7]

$$(a_1 \cdot a_2)^+ = a_2^+ a_1^+ , \quad (\Delta(a))^+ = \Delta(a^+) . \qquad (1.1)$$

In order to introduce the standard involution defining a real form of $U_q(Sp(4))$ suitable for our contraction procedure we are forced to consider involutions which take out from the Cartan–Weyl basis [#3]. It appears that one should consider the antipode-extended Cartan–Weyl basis $\{h_i, e_{\pm i}, e_{\pm A}, S(e_{\pm A})\}$ where $\{h_i, e_{\pm i}\}$ describe Cartan–Chevalley generators, and $e_{\pm A}$ describe the generators corresponding to nonsimple roots (for $Sp(4)$: $i=1, 2$; $A=3, 4$). In such a case the relations expressing physical real generators (in our case q-deformed $O(3, 2)$ generators) in terms of antipode-extended Cartan–Weyl basis contains additional freedom (in our case we express 10 physical generators in terms of 14 generators). In this paper we shall make a choice which after the contraction [#4] (we recall that q is real)

$$R \to \infty , \quad R \log q \to \kappa^{-1} \quad (0 < \kappa < \infty) \qquad (1.2)$$

provides a new quantum (κ-deformed) Poincaré al-

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[#1] We required in refs. [1–3] that the nonrelativistic $O(3)$ rotations as a quantum subalgebra of the quantum Poincaré algebra remains undeformed.
[#2] We use the notation described in ref. [2].
[#3] In fact, these inner involutions of $U_q(Sp(4; \mathbb{C}))$ were considered in ref. [2] (see ref. [2], formula (3.22)) but because they were not inner in the Cartan–Weyl basis they were not elaborated.
[#4] The contraction (1.2) was firstly introduced by the Firenze group [8,9].

0370-2693/92/$ 05.00

gebra, which is a Hopf algebra with standard real structure.

We would like to point out here that the contraction limit of $U_q(O(3,2))$ given in ref. [1] after the suitable nonlinear transformation of the κ-deformed boost generators (see ref. [10]) provides a simplified form of the real quantum Poincaré algebra. Interstingly enough, the contraction limits in ref. [1] (for $|q|=1$) and given in this paper (q real) which look quite different turn out after suitable nonlinear transformations, to be related simply by the replacement $\kappa \to i\kappa$. It should be stressed however that in this paper we avoid at least three difficulties related with nonstandard $\oplus$ involutions used in our earlier work on real Poincaré algebras (see refs. [1,2], also ref. [10]):

(α) The reality condition for the coproduct ($\Delta' = \tau \circ \Delta$)

$$(\Delta(a))^{\oplus} = \Delta'(a^{\oplus}) \tag{1.3}$$

implies that the real spectrum of the algebra becomes complex on tensor products (e.g. the total three-momenta of two independent subsystems becomes complex).

(β) If one wishes to define $\oplus$ involution as an adjoint operation in the representation space $|x\rangle$ endowed with scalar product $\langle x'|x\rangle$ it should be defined on the tensor product $|x_1, x_2\rangle = |x_1\rangle \oplus |x_2\rangle$ as follows:

$$\langle x_1', x_2' | x_1, x_2 \rangle = \langle x_1' | x_2 \rangle \langle x_2' | x_1 \rangle \ . \tag{1.4}$$

The scalar product (1.4) is not positive definite.

(γ) It is not known how to describe the dual objects to $\oplus$ real quantum Lie algebras which would describe standard real quantum groups.

The plan of our paper is the following: In section 2 we shall describe standard real quantum algebras $U_q(O(3,2))$ with q real, its antipode-extended basis and the corresponding 10 q-deformed $O(3,2)$ rotation generators. In section 3 we perform the contraction limit ( ) and obtain a new real quantum Poincaré algebra as standard $*$ Hopf algebra. In section 4 the nonlinear transformations simplifying the quantum Poincaré algebra are given and two κ-deformed Casimir operators for our real Poincaré algebra are given. In section 5 we outline the κ-deformed scalar (Klein–Gordon) free field theory and by taking the square root of the κ-deformed Klein–Gordon operator we propose a κ-deformed Dirac equation. From these models we see that the κ-deformation implies the replacement of the continuous time by a "$\tilde{q}$-lattice time", where $\ln \tilde{q} = i/2\kappa$. The comments and some open questions are presented in section 6.

It should be added that at present two versions of the q-deformed Poincaré algebra were proposed:

(i) The one discussed in this paper, with commuting four-momenta and Lorentz generators not forming a quantum subalgebra (see refs. [1–4]).

(ii) The one with the four-momenta forming a quadratic algebra and the Lorentz generators forming a quantum subalgebra. Such a structure was obtained from q-deformation of the $D=4$ conformal algebra [11,12] or from the realization of q-differential calculus on q-deformed Minkowski space [13].

The first approach has the advantage that the q-Poincaré algebra is a genuine Hopf algebra, with coproducts embedded in the tensor product of a q-Poincaré enveloping algebra. The second approach leads to a desirable property that q-Lorentz algebra is a Hopf subalgebra, but it is rather a quantum Weyl than a quantum Poincaré algebra (an eleventh scaling generator is needed for defining the coproducts for ten q-Poincaré generators).

2. Standard real form of $U_q(O(3, 2))$

Firstly we recall the basic formulae defining the Cartan–Weyl basis of $U_q(Sp(4;\mathbb{C}))$ [1,2]. Introducing the symmetrized Cartan matrix for the Lie algebra $C_2 \equiv Sp(4)$,

$$\alpha_{ij} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \tag{2.1}$$

the quantum Lie algebra $U_q(Sp(4;\mathbb{C}))$ is described by the following Cartan–Chevalley generators, corresponding to the simple roots of C_2 ($i=1,2$):

$$[e_i, e_{-j}] = \delta_{ij}[h_j]_q \ ,$$
$$[h_i, e_{\pm j}] = \pm \alpha_{ij} e_j \ ,$$
$$[h_i, h_j] = 0 \ , \tag{2.2}$$

restricted by the following q-Serre relations

$$[e_{\pm 1}, [e_{\pm 1}, [e_{\pm 1}, e_{\pm 2}]_{\tilde{q}^{\mp 1}}]_{\tilde{q}^{\mp 1}}]_{\tilde{q}^{\mp 1}} = 0 \ ,$$
$$[e_{\pm 2}, [e_{\pm 2}, e_{\pm 1}]_{\tilde{q}^{\mp 1}}]_{\tilde{q}^{\mp 1}} = 0 \ , \tag{2.3}$$

where $[e_\alpha, e_\beta]_{\tilde{q}} \equiv e_\alpha e_\beta - q^{-\langle\alpha,\beta\rangle} e_\beta e_\alpha$. The coproduct and antipodes are given by the formulae

$$\Delta(h_i) = h_i \otimes I + I \otimes h_i ,$$
$$\Delta(e_{\pm i}) = e_{\pm i} \otimes k_i + k_i^{-1} \otimes e_{\pm i} , \quad (2.4)$$

where $k_i = q^{h_i/2}$ and

$$S(h_i) = -h_i , \quad S(e_{\pm i}) = -q^{\pm d_i/2} e_{\pm i} , \quad (2.5)$$

where $d_i = (1, 2)$.

The Cartan–Weyl basis is defined as

$$e_3 = e_1 e_2 - q e_2 e_1 , \quad e_{-3} = e_{-2} e_{-1} - q^{-1} e_{-1} e_{-2} ,$$
$$e_4 = [e_1, e_3] , \quad e_{-4} = [e_{-3}, e_{-1}] . \quad (2.6a)$$

Introducing

$$\tilde{e}_3 = e_2 e_1 - q e_1 e_2 , \quad \tilde{e}_{-3} = e_{-1} e_{-2} - q^{-1} e_{-2} e_{-1} ,$$
$$\tilde{e}_4 = [\tilde{e}_3, e_1] , \quad \tilde{e}_{-4} = [e_{-1}, \tilde{e}_{-3}] , \quad (2.6b)$$

the formulae for the antipode take the form

$$S(e_{\pm 3}) = q^{\pm 3/2} \tilde{e}_{\pm 3} , \quad S(e_{\pm 4}) = -q^{\pm 2} \tilde{e}_{\pm 4} ,$$
$$S(\tilde{e}_{\pm 3}) = q^{\pm 3/2} e_{\pm 3} , \quad S(\tilde{e}_{\pm 4}) = -q^{\pm 2} e_{\pm 4} . \quad (2.7)$$

The 14 generators h_i, $e_{\pm i}$ $(i=1, 2)$ and $e_{\pm A}$, $\tilde{e}_{\pm A}$ $(A=3, 4)$ form the antipode-extended Cartan–Weyl basis. One can introduce the following class of standard involutions, satisfying (1.1)

$$q \text{ real} , \quad h_i^+ = h_i \rightarrow k_i^+ = k_i ,$$
$$e_{\pm 1}^+ = \lambda \rho^{\pm 1} e_{\mp 1} , \quad e_{\pm 2}^+ = \epsilon \gamma^{\pm 1} e_{\mp 2} ,$$
$$e_{\pm 3}^+ = -\epsilon \lambda (\rho\gamma)^{\pm 1} q^{\pm} \tilde{e}_{\mp 3} , \quad \lambda^2 = \epsilon^2 = 1 ,$$
$$e_{\pm 4}^+ = \epsilon (\rho^2 \gamma)^{\pm 1} q^{\pm 1} \tilde{e}_{\mp 4} , \quad \rho, \gamma \text{ real} , \quad (2.8)$$

The involutions (2.8) (with $\rho = \gamma = 1$) can be related with the class of $\oplus$ involutions considered in refs. [1,2] (and in particular with the one considered in ref. [1]) if we introduce a complex linear morphism Q of the $U_q Sp(4; \mathbb{C})$ Hopf algebra, replacing q by q^{-1}, i.e. [#5]

$$Q: Qe_i = e_i , \quad Qh_i = h_i ,$$
$$Qe_{\pm 3} = -q^{\mp 1} \tilde{e}_{\pm 3} , \quad Qe_{\pm 4} = q^{\mp 1} \tilde{e}_{\pm 4} ,$$
$$Qq = q^{-1} . \quad (2.9)$$

Because Q is a $\oplus$ involution (see (1.3)) we have $+ = Q \circ \oplus$. In particular the $\oplus$ involution used in ref. [1] becomes after multiplying by Q the following standard $+$-involution:

$$e_1^+ = e_{-1} , \quad e_2^+ = -e_{-2} ,$$
$$e_3^+ = q\tilde{e}_{-3} , \quad e_4^+ = -q\tilde{e}_{-4} , \quad (2.10)$$

which is obtained from the relations (3.8) by putting $\rho = \gamma = 1$ and $\lambda = 1$, $\epsilon = -1$. Because for $q=1$ the morphism (2.9) describes an identity transformation, it follows that the real form defined by the involution (2.10) describes the deformation of O(3,2) Lie algebra, with the metric $g_{AB} = \text{diag}(-1, 1, 1, 1, -1)$.

It should be mentioned that the q-deformed generators M_{AB}, satisfying the condition $M_{AB} = M_{AB}^+$ can be introduced in many ways not necessarily leading to the Jacobi identities for the contracted quantum Poincaré algebra. We have performed the calculations for the following choice of the Cartan–Weyl basis for $U_q(O(3, 2))$ (q real):

$$M_3 = M_{12} = h_1 , \quad M_\pm = M_{23} \pm iM_{31} = \sqrt{2} e_{\pm 1} ,$$
$$L_3 = M_{34} = -\frac{1}{\sqrt{2}} (q^{-i/2} e_3 + q^{1+i/2} \tilde{e}_{-3}) ,$$
$$L_+ = M_{14} + iM_{24} = e_4 + e_{-2} ,$$
$$L_- = M_{14} - iM_{24} = -(e_2 + q\tilde{e}_{-4}) ,$$
$$RP_0 = M_{04} = h_3 ,$$
$$RP_3 = M_{03} = \frac{i}{\sqrt{2}} (q^{1+i/2} \tilde{e}_{-3} - q^{-i/2} e_3) ,$$
$$RP_+ = R(P_2 + iP_1) = M_{02} + iM_{01} = e_2 - q\tilde{e}_{-4} ,$$
$$RP_- = R(P_2 - iP_1) = M_{02} - iM_{01} = e_4 - e_{-2} . \quad (2.11)$$

Because in the definitions (2.11) enter the generators $\tilde{e}_{-3}$, $\tilde{e}_{-4}$, in order to calculate the quantum algebra of the generators (2.11) one should supplement the algebra satisfied by the generators h_i, $e_{\pm i}$ $(i=1, 2)$, $e_{\pm 3}$ and $e_{\pm 4}$ (q-deformed Cartan–Weyl basis of $U_q(Sp(4; \mathbb{C}))$ [1,2]) by the additional relations for the generators (2.6b). We have

[#5] The morphism (2.9) denoted by σ we found in ref. [14], p. 37.

$$[e_3, \tilde{e}_{-3}] = (q^{-1}-q)q^{h_1}e_{-2}e_2 + (q-q^{-1})q^{-h_2}e_{-1}e_1 + q^{h_1-h_2} - (h_1+h_2+1)\,,$$

$$[e_3, \tilde{e}_{-4}] = (q-q^{-1})(q-1)q^{-h_2}e_{-1}^2 e_1 + (1-q^2)q^{h_1}e_2\tilde{e}_{-3} + q^{-1}q^{-h_3}e_{-1}\,,$$

$$[e_4, \tilde{e}_{-3}] = (q-q^{-1})(q^{-1}-1)q^{-h_2}e_{-1}e_1^2 + (q^{-2}-1)\cdot q^{h_1}e_3e_{-2} - q^{-1}q^{-h_3}e_1\,,$$

$$\begin{aligned}[e_4, \tilde{e}_{-4}] &= (q-q^{-1})(q^{-1}-1)q^{-h_2}(qe_{-1}^2e_1^2 - e_{-1}e_1^2e_{-1}) \\ &+ (q^{-1}-q)q^{h_1}e_3e_{-3} + (1-q^2)q^{2h_1}e_2e_{-2} \\ &+ (q-q^{-1})\cdot q^{2h_1}e_{-2}e_2 + (q^{-1}-q)q^{h_1-h_2}e_{-1}e_1 \\ &+ q^{-1}q^{-h_3}[h_1] - q^{2h_1-h_2} + q^{h_1}(h_1+h_2+1)\,. \end{aligned} \tag{2.12}$$

The commutation relations for two tilded generators can be obtained from the formulae in ref. [1] by the action of antipode, e.g.

$$[\tilde{e}_3, \tilde{e}_4] = -q^{7/2}S([e_3, e_4]) = q^{-7/2}(1-q)S(e_4e_3) = (q-1)\tilde{e}_3\tilde{e}_4\,, \tag{2.13}$$

Calculation of coproducts for physical generators requires the knowledge of the following formulae:

$$\begin{aligned}\Delta(\tilde{e}_3) &= \tilde{e}_3 \otimes q^{h_3/2} + q^{-h_3/2} \otimes \tilde{e}_3 + (q^{-1}-q)q^{-h_1/2}e_2 \otimes e_1 q^{h_2/2}\,, \\ \Delta(\tilde{e}_{-3}) &= \tilde{e}_{-3} \otimes q^{h_3/2} + q^{-h_3/2} \otimes \tilde{e}_{-3} - (q^{-1}-q)q^{-h_2/2}e_{-1} \otimes e_{-2}q^{h_1/2}\,, \\ \Delta(\tilde{e}_4) &= \tilde{e}_4 \otimes q^{h_4/2} + q^{-h_4/2} \otimes \tilde{e}_4 + (q-q^{-1}) \\ &\cdot [(1-q^{-1})q^{-h_1}e_2 \otimes e_1^2 q^{h_2/2} - q^{-h_1/2}\tilde{e}_3 \otimes e_1 q^{h_3/2}]\,, \\ \Delta(\tilde{e}_{-4}) &= \tilde{e}_{-4} \otimes q^{h_4/2} + q^{-h_4/4} \otimes \tilde{e}_{-4} + (q-q^{-1}) \\ &\cdot [(q-1)q^{-h_2/2}e_{-1}^2 \otimes e_{-2}q^{h_1} + q^{-h_3/2}e_{-1} \otimes \tilde{e}_{-3}q^{h_1/2}]\,. \end{aligned} \tag{2.14}$$

3. The contraction to standard real quantum Poincaré algebra

In order to obtain our new q-Poincaré algebra we proceed further as follows:

(i) Using the formulae for the commutators and coproducts of the antipode-extended Cartan–Weyl basis and the definitions (2.11) we can write the q-deformation of the O(3, 2) Lie algebra as well as the coproduct relations for the q-deformed O(3, 2) generators.

(ii) We perform further the quantum de-Sitter contraction, obtained by the conventional rescaling of the O(3, 2) rotation generators

$$M_{\mu\nu} \text{ unchanged} \quad (M^+_{\mu\nu} = M_{\mu\nu})\,,$$

$$M_{0\mu} = RP_\mu \quad (P^+_\mu = P_\mu), \quad \mu, \nu = 1, ..., 4\,, \tag{3.1}$$

and the $R \to \infty$ limit described by (1.2).

As a result we obtain the following q-deformed Poincaré algebra:

(a) *Three-dimensional $O(3)$ rotations* ($M_\pm = M_1 + \mathrm{i}M_2 \equiv M_{23} \pm \mathrm{i}M_{31}$; $M_3 = M_{12}$).

(i) commutation relations:

$$[M_+, M_-] = 2M_3\,, \quad [M_3, M_\pm] = \pm M_\pm\,; \tag{3.2a}$$

(ii) coproducts:

$$\Delta M_i = M_i \otimes I + I \otimes M_i\,; \tag{3.2b}$$

(iii) antipode:

$$S(M_i) = -M_i\,. \tag{3.2c}$$

(b) *Boosts sector $O(3, 1)$* ($L_\pm = M_{14} \pm \mathrm{i}M_{24}$, $L_3 = M_{34}$).

(i) Commutation relations:

$$[L_+, L_-] = -2M_3 \cosh\frac{P_0}{\kappa} + \frac{1}{2\kappa^2}P_3^2 + \frac{1}{\kappa^2}M_3P_3^2 - \sinh\frac{P_0}{\kappa}\,,$$

$$[L_+, L_3] = \exp\left(-\frac{P_0}{\kappa}\right)M_+ + \frac{1}{2\kappa}(\mathrm{i}P_3L_+ + L_3P_-) - \frac{\mathrm{i}}{2\kappa^2}M_3P_3P_- + \frac{1}{4\kappa^2}(2-\mathrm{i})P_3P_-\,,$$

$$[L_-, L_3] = \exp\left(-\frac{P_0}{\kappa}\right)M_- + \frac{1}{2\kappa}(\mathrm{i}L_-P_3 - P_+L_3) - \frac{\mathrm{i}}{2\kappa^2}P_3P_+M_3 - \frac{1}{4\kappa^2}(2+\mathrm{i})P_3P_+\,, \tag{3.3a}$$

$$[M_3, L_3] = 0\,, \quad [M_3, L_\pm] = \pm L_\pm\,, \tag{3.3b}$$

$$[M_\pm, L_\pm] = \mp \frac{1}{2\kappa} M_\pm P_\mp ,$$

$$[M_+, L_-] = 2L_3 - \frac{1}{2\kappa} P_+ M_+ + \frac{i}{\kappa} M_3 P_3 + \frac{1}{\kappa} P_3 ,$$

$$[M_-, L_+] = -2L_3 + \frac{1}{2\kappa} M_- P_- + \frac{i}{\kappa} P_3 M_3 - \frac{1}{\kappa} P_3 ,$$

$$[M_+, L_3] = -L_+ + \frac{1}{2\kappa} M_3 P_- + \frac{i}{2\kappa} P_- ,$$

$$[M_-, L_3] = L_- - \frac{1}{2\kappa} P_+ M_3 + \frac{i}{2\kappa} P_+ ; \qquad (3.3\text{b cont'd})$$

(ii) coproducts:

$$\Delta L_3 = L_3 \otimes \exp\left(\frac{P_0}{2\kappa}\right) + \exp\left(-\frac{P_0}{2\kappa}\right) \otimes L_3$$

$$+ \frac{1}{2\kappa} \exp\left(-\frac{P_0}{2\kappa}\right)(M_+ \otimes P_+ + M_- \otimes P_-) ,$$

$$\Delta L_\pm = L_\pm \otimes \exp\left(\frac{P_0}{2\kappa}\right) + \exp\left(-\frac{P_0}{2\kappa}\right) \otimes L_\pm$$

$$+ \frac{1}{2\kappa}\left[P_\mp \otimes M_3 \exp\left(\frac{P_0}{2\kappa}\right) - \exp\left(-\frac{P_0}{2\kappa}\right) M_3 \otimes P_\mp\right]$$

$$\mp \frac{i}{\kappa} \exp\left(-\frac{P_0}{2\kappa}\right) M_\pm \otimes P_3 ; \qquad (3.3\text{c})$$

(iii) antipode:

$$S(L_3) = -L_3 + \frac{i}{2\kappa} P_3 + \frac{1}{2\kappa}(M_+ P_+ + M_- P_-) ,$$

$$S(L_\pm) = -L_\pm \mp \frac{1}{\kappa} P_\mp \mp \frac{i}{\kappa} M_\pm P_3 . \qquad (3.3\text{d})$$

(c) *Translations sector* ($P_\pm = P_2 \pm iP_1, P_3, P_0$).
(i) Commutation relations ($\mu, \nu = 0, 1, 2, 3$):

$$[P_\mu, P_\nu] = 0 , \quad [M_i, P_j] = i\epsilon_{ijk} P_k ,$$

$$[M_i, P_0] = 0 , \qquad (3.4\text{a})$$

$$[L_3, P_0] = iP_3 ,$$

$$[L_3, P_3] = i\kappa \sinh \frac{P_0}{\kappa} - \frac{i}{2\kappa} P_+ P_- ,$$

$$[L_3, P_2] = \frac{i}{2\kappa} P_3 P_2 , \qquad (3.4\text{b})$$

$$[L_3, P_1] = \frac{i}{2\kappa} P_3 P_1 ,$$

$$[L_\pm, P_0] = iP_1 \mp P_2 ,$$

$$[L_\pm, P_2] = \mp \kappa \sinh \frac{P_0}{\kappa} \pm \frac{1}{2\kappa} P_3^2 ,$$

$$[L_\pm, P_3] = \mp \frac{1}{2\kappa} P_3 P_\mp ,$$

$$[L_\pm, P_1] = i\kappa \sinh \frac{P_0}{\kappa} + \frac{1}{2i\kappa} P_3^2 , \qquad (3.4\text{b cont'd})$$

(ii) coproducts:

$$\Delta P_0 = P_0 \otimes I + I \otimes P_0 ,$$

$$\Delta P_i = P_i \otimes \exp\left(\frac{P_0}{2\kappa}\right) + \exp\left(-\frac{P_0}{2\kappa}\right) \otimes P_i$$

$$(i = 1, 2, 3) , \qquad (3.4\text{c})$$

(iii) antipode:

$$S(P_\mu) = -P_\mu . \qquad (3.4\text{d})$$

4. Casimirs and elements of the representation theory

One can construct the quantum deformation of the quadratic Casimir, describing the quantum relativistic mass square operator. One gets

$$C_1 = P_1^2 + P_2^2 + P_3^2 + 2\kappa^2\left(1 - \cosh \frac{P_0}{\kappa}\right)$$

$$= \boldsymbol{p}^2 - \left(2\kappa \sinh \frac{P_0}{2\kappa}\right)^2 . \qquad (4.1)$$

It should be mentioned that recently the $D=4$ mass square Casimir was proposed in refs. [9,15] as the extension of the results obtained for $D=3$ Poincaré algebra. We can say therefore that the deformed square mass formula from refs. [9,15] found full theoretical justification in this paper.

In order to introduce the κ-deformation of second Casimir of Poincaré algebra described by the Pauli–Lubanski fourvector square, following ref. [10] we introduce a nonlinear transformation of the boost generators

$$\tilde{L}_+ = L_+ + \frac{\mathrm{i}}{2\kappa} M_+ P_3 - \frac{1}{2\kappa} P_- ,$$

$$\tilde{L}_- = L_- - \frac{\mathrm{i}}{2\kappa} P_3 M_- - \frac{1}{2\kappa} P_+ ,$$

$$\tilde{L}_3 = L_3 - \frac{1}{4\kappa} (M_+ P_+ + P_- M_-) + \frac{1}{2\kappa} P_3 , \qquad (4.2)$$

simplifying the κ-Poincaré algebra substantially. The new boosts satisfy the following relations:

$$[M_i, \tilde{L}_j] = \mathrm{i}\epsilon_{ijk} \tilde{L}_k ,$$

$$[P_0, \tilde{L}_k] = -\mathrm{i}P_k ,$$

$$[P_k, \tilde{L}_j] = -\mathrm{i}\kappa\delta_{kj} \sinh\frac{P_0}{\kappa} ,$$

$$[\tilde{L}_i, \tilde{L}_j] = -\mathrm{i}\epsilon_{ijk}\left(M_k \cosh\frac{P_0}{\kappa} - \frac{1}{4\kappa^2} P_k (\boldsymbol{P}\cdot\boldsymbol{M})\right) . \qquad (4.3)$$

It is interesting to observe that the algebra (4.3) differs from the one obtained in ref. [10] only by the replacement $\kappa \to \mathrm{i}\kappa$. The same holds for the coproduct formulae:

$$\Delta(\tilde{L}_i) = \tilde{L}_i \otimes \exp\left(\frac{P_0}{2\kappa}\right) + \exp\left(-\frac{P_0}{2\kappa}\right) \otimes \tilde{L}_i + \frac{1}{2\kappa}\epsilon_{ijk}\left[P_j \otimes M_k \exp\left(\frac{P_0}{2\kappa}\right) + \exp\left(-\frac{P_0}{2\kappa}\right) M_j \otimes P_k \right] . \qquad (4.4)$$

The coproduct (4.4) which satisfies the relation (1.1) permits to define the tensor product representations in Hilbert space. For completeness we give also the antipodes:

$$S(\tilde{L}_i) = -\tilde{L}_i + \frac{3\mathrm{i}}{2\kappa} P_i . \qquad (4.5)$$

The second Casimir can be obtained by introducing the κ-deformed Pauli–Lubanski fourvector:

$$W_0 = \boldsymbol{P}\cdot\boldsymbol{M} ,$$

$$W_k = \kappa M_k \sinh\frac{P_0}{\kappa} + \epsilon_{kij} P_i \tilde{L}_j , \qquad (4.6)$$

Writing the commutators of W_μ with themselves and $\tilde{L}_i$ one obtains the relations, presented in ref. [10], modified only by the replacement $\kappa \to \mathrm{i}\kappa$. The formula for the second Casimir takes the form

$$C_2\left(\cosh\frac{P_0}{\kappa} - \frac{\boldsymbol{P}^2}{4\kappa^2}\right) W_0^2 - \boldsymbol{W}^2 . \qquad (4.7)$$

The formulae (4.3) simplify substantially the κ-deformed Poincaré algebra and can be used as a starting point for the description of the realizations of the κ-deformed Poincaré algebra. Using the spinless realization of κ-Poincaré algebra, for which $\boldsymbol{P}\cdot\boldsymbol{M} = 0$ [#6]

$$P_\mu = \frac{1}{\mathrm{i}} \frac{\partial}{\partial x^\mu} . \qquad (4.8a)$$

$$M_i = \frac{1}{\mathrm{i}} \epsilon_{ijk} x_j P_k , \quad \tilde{L}_i = \frac{1}{\mathrm{i}} (x_0 P_i - x_i \tilde{P}_0) , \qquad (4.8b)$$

where

$$\tilde{P}_0(P_0) = \kappa \sinh\frac{P_0}{\kappa} ; \qquad (4.8c)$$

we see that we obtain a new type of realization containing the derivatives of arbitrarily high orders. The κ-deformed boosts L_i act explicitly on the scalar field $\phi(\boldsymbol{x}, t)$ as follows [#7]:

$$\hat{L}_i \phi(\boldsymbol{x}, t) = \frac{1}{\mathrm{i}} x_0 \frac{\partial}{\partial x_i} \phi(\boldsymbol{x}, t) - \kappa \frac{x_i}{\mathrm{i}} \left[\phi\left(\boldsymbol{x}, t + \frac{\mathrm{i}}{\kappa}\right) - \phi\left(\boldsymbol{x}, t - \frac{\mathrm{i}}{\kappa}\right)\right] . \qquad (4.9)$$

In order to obtain the finite-dimensional (nonunitary) matrix realizations of the algebra (4.3) one can expand the function $\phi(\boldsymbol{x}, t)$ in power series. In particular considering the five-dimensional representation, described by a linear function $\phi_1(x) = V + V_\mu x^\mu$ one obtains the classical result. The κ-dependence starts to be seen if we consider the tensors described by the third order polynomial $\phi_3(x) = V + V_\mu x^\mu + V_{\mu\nu} x^\mu x^\nu + V_{\mu\nu\rho} x^\mu x^\nu x^\rho$.

It should be added that in the discussion of the κ-deformed Poincaré invariance the following nonlinear Fourier transform is useful:

[#6] The realization (4.8) was firstly obtained independently by P. Zaugg and the Lodz group [10]. We were informed [16] that the realization (4.8) was extended by the Lodz group to arbitrary spin.

[#7] For the quantum Poincaré algebra given in ref. [1] the shift of the time arguments is real.

$$F_g(\boldsymbol{x},\tau)=\frac{1}{(2\pi)^4}\int d^3p\int dp_0\,\tilde{F}(\boldsymbol{p},p_0)$$
$$\times\exp\{i[\boldsymbol{p}\cdot\boldsymbol{x}+g(p_0)\tau]\}\,. \tag{4.10}$$

Putting $g(p_0)=\kappa\sinh p_0/\kappa$ one obtains from (4.8b) the classical Minkowski rotations in the $(\boldsymbol{x},\tau)$ plane; putting $g(p_0)=2\kappa\sinh p_0/2\kappa$ one can write the κ-deformed Klein–Gordon and Dirac equations in a classical form, as will be seen in the next section.

5. κ-deformed free field theory

Using the realization (4.8a) one gets the following κ-deformed Klein–Gordon equation for free spinless (scalar) filed ($\Delta=\partial_i\partial_i$):

$$\left[\Delta-2\kappa^2\left(1-\cos\frac{\partial_t}{\kappa}\right)\right]\phi(\boldsymbol{x},t)$$
$$=\left[\Delta-\left(2\kappa\sin\frac{\partial_t}{2\kappa}\right)^2\right]\phi(\boldsymbol{x},t)$$
$$=m^2\phi(\boldsymbol{x},t)\,, \tag{5.1}$$

which can be obtained from the usual KG equation by the replacement of the standard time derivative by the $\tilde{q}$-deformed one (see e.g. refs. [17,18]) with the value $\ln\tilde{q}=i/2\kappa$. The lagrangian providing eq. (5.1) has the form

$$\mathscr{L}=\tfrac{1}{2}\int d^4x\,\phi(x)\left[\Delta-\left(2\kappa\sin\frac{\partial_t}{2\kappa}\right)^2-m^2\right]\phi(x)\,. \tag{5.2}$$

The action (5.2) resembles the ones discussed by Pais and Uhlenbeck [19] in the early days of renormalization theory. Using the general quantization scheme (see e.g. ref. [20]) one gets the following formulae for the κ-extension of the scalar field Green functions:

(a) *κ-deformed Pauli–Jordan commutator function.*

$$[\phi(x),\phi(x')]=i\Delta_\kappa(x-x';m^2)\,, \tag{5.3}$$

where ($x\equiv(\boldsymbol{x},t)$),

$$\Delta_\kappa(x;m^2)=\frac{-i}{(2\pi)^3}\int d^4p\,\epsilon(p_0)$$
$$\times\delta\left(\boldsymbol{p}^2+m^2-\left(2\kappa\sinh\frac{p_0}{2\kappa}\right)^2\right)$$
$$\times\exp[i\boldsymbol{p}\cdot\boldsymbol{x}-\omega_\kappa(\boldsymbol{p})t]$$
$$=\frac{-1}{(2\pi)^3}\int\frac{d^3p}{f_\kappa(\boldsymbol{p})}\exp(i\boldsymbol{p}\cdot\boldsymbol{x})\kappa\sin[\omega_\kappa(\boldsymbol{p})t]\,, \tag{5.4}$$

and ($\omega=\sqrt{\boldsymbol{p}^2+m^2}$), $f_\kappa=\omega\sqrt{1+\frac{\omega^2}{4\kappa^2}}$

$$p_0=\pm 2\kappa\,\mathrm{arcsinh}\frac{\omega}{2\kappa}=\pm\omega_\kappa \tag{5.4'}$$

solves the κ-mass-shell condition

$$\boldsymbol{p}^2+m^2=\left(2\kappa\sinh\frac{p_0}{2\kappa}\right)^2\,. \tag{5.4''}$$

The commutator function (5.4) satisfies the homogeneous κ–Klein–Gordon equation

$$\left[\Delta-\left(2\kappa\sinh\frac{p_0}{2\kappa}\right)^2-m^2\right]\Delta_\kappa(x;m^2)=0\,,$$
$$p_0=-i\partial_t\,, \tag{5.5}$$

and ET relations

$$\Delta_\kappa(\boldsymbol{x},0;m^2)=0\,,\quad \partial_t\Delta(\boldsymbol{x},t;m^2)|_{t=0}$$
$$=\frac{-1}{(2\pi)^3}\int\frac{d^3p}{f(\boldsymbol{p})}\,\omega_\kappa\exp(i\boldsymbol{p}\cdot\boldsymbol{x})\,. \tag{5.6}$$

(b) *κ-deformed Feynman propagator.* We assume that

$$\Delta_\kappa^{(F)}(x;m^2)=\frac{1}{(2\pi)^4}$$
$$\times\int d^4p\frac{\exp(ipx)}{\boldsymbol{p}^2+m^2-(2\kappa\sinh P_0/2\kappa)^2-i\epsilon}\,. \tag{5.7}$$

Because the light-cone invariant is defined by the propagation of massless particles, $s^2\sim[\Delta^F(x;0)]^{-1}$, one can employ a similar definition of the κ-deformed light cone.

It is easy to see that the p_0-integration in (4.7) is damped exponentially. Let us observe further that the static Yukawa potential described by $\int dt\,\Delta_\kappa^F(\boldsymbol{x},t;m^2)$ is not modified.

We would like to add the following:

(i) Using the know techniques for higher order lagrangian theories [20,21] one can calculate from the lagrangian (5.2) the hamiltonian, energy–momentum tensors, Poincaré generators etc.,

(ii) The κ-deformed Dirac equation can be defined by a square root of κ-deformed KG operator

$$\left[\gamma^i \partial_i + 2\kappa\gamma^4 \sin\left(\frac{\partial_t}{2\kappa}\right) - m\right]\psi = 0 , \qquad (5.8)$$

where (γ^i, γ^4) are usual $D=4$ Minkowski γ_μ-matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\eta_{\mu\nu}(\eta_{\mu\nu} = \mathrm{diag}(1, 1, 1, -1))$.

We see from (5.8) that the κ-deformation of the Dirac operator is given by the replacement of the time derivative by a $\tilde{q}$-deformed time derivative, with κ-dependent deformation parameter $\ln \tilde{q} = \mathrm{i}/2\kappa$.

(iii) An interesting question is to reconcile the nontrivial coproducts with the form of the vertex operators describing interaction. We expect that the appearance of the κ-parameter will smooth the divergences in local QFT. In particular it will be interesting to κ-regularize the nonrenormalizable QFT, e.g. quantum gravity.

6. Discussion

The aim of this paper is to present a new quantum Poincaré algebra, with standard reality condition having properties given by (1.1). In such a way the problems, related with the choice of ⊕ involution in refs. [1–4] are avoided. In particular the composition law of two three-momenta leads to the following real value of the total fourmomentum (see also (3.4))

$$P_i^{(1+2)} = P_i^{(1)} \exp\left(\frac{P_0^{(2)}}{2\kappa}\right) + P_i^{(2)} \exp\left(-\frac{P_0^{(1)}}{2\kappa}\right)$$
$$= P_i^{(1)} + P_i^{(2)} + \frac{1}{2\kappa}(P_i^{(1)}P_0^{(2)} - P_i^{(2)}P_0^{(1)})$$
$$+ \mathrm{O}\left(\frac{1}{\kappa^2}\right), \qquad (6.1)$$

with the classical addition formula valid for $P_0^{(i)} \ll \kappa$. The existence of coproduct implies the existence of the tensor product of representations, i.e. one can pass from irreducible representations in Hilbert space (quantum mechanics) to the reducible representations in Fock space (free quantum fields).

In particular in order to define the multiparticle states correctly, one has to do it consistently with nontrivial coproduct rules. This program, as well as studying the vertex functions and the S-matrix in κ-deformed QFT is under consideration.

Physically the κ-deformation means introducing the discrete "$\tilde{q}$-lattice" time, with preserving almost all classical properties of three-dimensional euclidean space. In such a way the κ-deformation introduces a separation between the space and time degrees of freedom, with linear algebra replaced by a nonlinear (nonpolynomial) one. In the formalism there is a new dimension-full deformation parameter $\kappa(\ln \tilde{q} \sim 1/\kappa)$ which permits to exponentiate the time derivatives carrying the dimension. In particular looking at the Dirac equation (5.8) we see that it can be written as follows:

$$\tilde{\mathrm{D}}^t_\kappa \psi(\boldsymbol{x}, t) = H_{\mathrm{cl}}^{\mathrm{DIR}} \psi(\boldsymbol{x}, t) \qquad (6.2)$$

where $(i=1, 2, 3)$,

$$H_{\mathrm{cl}}^{\mathrm{DIR}} = \gamma^4(\gamma_i \partial_i - m) ,$$

$$\tilde{\mathrm{D}}^t_\kappa f(t) = \frac{1}{2\Delta t}[f(t+\Delta t) - f(t-\Delta t)]\,|_{\Delta t = \mathrm{i}/2\kappa}, \qquad (6.3)$$

which differs slightly from ref. [17]. We see therefore that contrary to most regularizations in QFT it is the time variable for which the continuous limits are replaced by finite time difference expressions in κ-deformed theory.

We would also like to add that

(i) one can develop the calculus of differential forms, and introduce the κ-deformation of Cartan–Maurer equations corresponding to our "nonlinearization procedure" of Poincaré algebra for finite κ.

(ii) For both real Poincaré algebras – from ref. [1] as well as from the present paper – the contraction of the universal R-matrix for $\mathrm{U}_q(\mathrm{O}(3, 2))$ (see e.g. ref. [22]) leads to divergencies. It appears however that there exists a whole class of contractions of $\mathrm{U}_q(\mathrm{O}(3, 2))$ for which the universal R-matrix can be found [23,24]. These contraction limits describe twisted classical Lie algebras.

(iii) In this paper we propose a contraction scheme leading to the κ-deformation of the time variable with finite difference derivatives in time. It appears that if we use the contractions of $\mathrm{U}_q(\mathrm{O}(3, 2))$ leaving the

O(2, 1) subalgebra invariant [2,24], the contraction limit describes the geometry with κ-deformation of one of the space directions. It would be interesting to describe the contractions of the new quantum algebras (generalized Sklyanin algebras describing the multiparameter deformation of O(3, 2)) permitting to obtain independent quantum deformations in space and time directions.

Acknowledgement

Two of the authors (J.L. and A.N.) would like to thank Professor Pierre Minnaert and Université de Bordeaux I, for the hospitality during the initial stage of this work and the financial support. The first author (J.L.) would also like to thank Université de Genève for the hospitality and financial support when the paper was being completed. Useful discussions with M. Dubois-Violette, C. Gomez, S. Woronowicz and V.N. Tolstoy are acknowledged.

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