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Feynman diagrams and the large charge expansion in $3-\varepsilon$ dimensions

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ABSTRACT

In arXiv:1909.01269 it was shown that the scaling dimension of the lightest charge n operator in the $U(1)$ model at the Wilson-Fisher fixed point in $D=4-\varepsilon$ can be computed semiclassically for arbitrary values of λn, where λ is the perturbatively small fixed point coupling. Here we generalize this result to the fixed point of the $U(1)$ model in $3-\varepsilon$ dimensions. The result interpolates continuously between diagrammatic calculations and the universal conformal superfluid regime for CFTs at large charge. In particular it reproduces the expectedly universal $\mathcal{O}(n^0)$ contribution to the scaling dimension of large charge operators in $3D$ CFTs.

1. Introduction

It is known that, even in a weakly coupled Quantum Field Theory (QFT), there exist situations where the ordinary Feynman diagram expansion fails. An example is given by amplitudes with a sufficiently large number of legs. This instance received some attention in the 90's, in the study of the production of a large number of massive bosons in high-energy scattering [1–4].[1] Multilegged amplitudes also occur in the correlators of operators carrying a large conserved internal charge, whose properties indeed defy perturbation theory for large enough charge.

Recently, it has been shown that, in conformal field theory (CFT), large charge operators can generally be associated, via the state-operator correspondence, to a superfluid phase of the theory on the cylinder [7–10]. The corresponding CFT data are then *universally* described, regardless of the details of the underlying CFT, by an *effective* field theory (EFT) for the hydrodynamic Goldstone modes [11,12] of the superfluid. The systematic derivative and field expansion of the resulting EFT coincide with an expansion in inverse powers of the charge.

While the effective superfluid description should equally well apply to strongly and weakly coupled theories, in the latter case it is also possible to work directly in the full theory, bypassing the EFT construction, or, in fact, deriving it. This was recently illustrated in [13], by focusing on the two-point function of the charge n operator ϕ^n in the $U(1)$ invariant Wilson-Fisher fixed point in $4-\varepsilon$ dimensions. For arbitrary n, the scaling dimension Δ_{ϕ^n} was computed semiclassically by expanding the path integral around a non-trivial trajectory. The result can be structured as a loop expansion in the coupling $\lambda \propto \varepsilon$ while treating λn as a fixed parameter, playing a role similar to that of the 't Hooft coupling in large N gauge theories.[2] The result encompasses the small charge regime ($\lambda n \ll 1$), where ordinary diagrammatic perturbation theory also applies, and the large charge regime ($\lambda n \gg 1$), described by a superfluid phase. Similar ideas were also shown to apply in the context of $\mathcal{N}=2$ superconformal theories [15], with the double expansion remarkably associated to a dual matrix model description.

In this paper we apply this methodology to compute the scaling dimension of ϕ^n in $(\bar{\phi}\phi)^3$ at its conformally invariant point in $3-\varepsilon$ dimensions. The result follows the same pattern observed in $(\bar{\phi}\phi)^2$ in $4-\varepsilon$ dimensions. Besides confirming the generality of the method [13], the main interest of $(\bar{\phi}\phi)^3$ in $D=3-\varepsilon$ lies in the possibility of non-trivially comparing to the universal predictions of the large charge EFT of 3D CFT [7]. Indeed the β function of $(\bar{\phi}\phi)^3$ arises only at 2-loops. At the 1-loop level the theory is therefore conformally invariant at $D=3$ for any value of λ. At this order, as λn is varied from small to large, our formulae non trivially

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[1] The claims made in a recent revival [5] seem controversial. See indeed [6] for a critical perspective.

[2] A similar double expansion exists at large N [14].

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interpolate between the prediction of standard Feynman diagram computations and those of the universal superfluid description of large charge states in 3D CFT. In particular for $t \equiv \lambda n/\sqrt{3}\pi \gg 1$ our result for the scaling dimension takes the form:

$$\Delta_{\phi^n} = t^{3/2}\left[c_{3/2} + c_{1/2}t^{-1} + \ldots\right] + t^0\left[d_0 + d_{-1}t^{-1} + \ldots\right], \quad (1)$$

with c's and d's having specific calculable values. This result nicely matches the universal predictions of the large charge EFT. Within the general EFT construction the c_k's are model dependent Wilson coefficients, but the d's are universally calculable effects associated to the 1-loop Casimir energy. Our result for the d's should thus match the general theory, and they do. In particular we find

$$d_0 = -0.0937255(3) \quad (2)$$

in agreement with [8]. The prediction of d_{-1} is similarly matched, but the statement is less direct as it involves a correlation with the subleading corrections to the dispersion relations of the Goldstone; we discuss the precise expression in section 5. Previously, eqs. (1) and (2) were verified at large N for monopole operators [16]; the results of Monte-Carlo simulations for the $O(2)$ model at criticality are consistent with the expansion (1) [17], though their present precision is not sufficient to check the universal prediction for d_0. Our paper provides an alternative verification where the large charge regime is continuously connected, as λn is varied, to diagrammatic perturbation theory.

2. Lagrangian and conventions

We consider the following $U(1)$ symmetric theory in $D = 3 - \varepsilon$ dimensional euclidean space-time

$$\mathcal{L} = \partial\bar{\phi}\partial\phi + \frac{\lambda_0^2}{36}\left(\bar{\phi}\phi\right)^3 . \quad (3)$$

Within this convention for the Lagrangian, one can easily realize that λ_0 is the loop counting parameter by rescaling $\phi \to \phi/\sqrt{\lambda_0}$. The renormalized coupling and field are defined as

$$\phi = Z_\phi[\phi], \quad \lambda_0 = M^\varepsilon \lambda Z_\lambda , \quad (4)$$

where M denotes the sliding scale. The β-function is given by [18]

$$\frac{\partial\lambda}{\partial \log M} \equiv \beta(\lambda) = \lambda\left[-\varepsilon + \frac{7\lambda^2}{48\pi^2} + \mathcal{O}\left(\frac{\lambda^4}{(4\pi)^4}\right)\right]. \quad (5)$$

For $\varepsilon \ll 1$, this implies the existence of an IR-stable fixed point at

$$\frac{\lambda_*^2}{(4\pi)^2} = \frac{3}{7}\varepsilon + \mathcal{O}\left(\varepsilon^2\right). \quad (6)$$

Notice that the β-function (5) starts at two-loop order at $\varepsilon = 0$. Hence the model is conformally invariant up to $\mathcal{O}(\lambda)$ in exactly $D = 3$. This observation will be important for what follows. The field wave-function renormalization starts at four loops and we shall always neglect it in the following.

3. Anomalous dimension of large charge operators

In this paper we focus on the calculation of the scaling dimension of the $U(1)$ charge n operator ϕ^n, focusing on the regime $n \gg 1$. In complete analogy with the $(\phi\bar{\phi})^2$ case discussed in [13], the diagrammatic calculation for the anomalous dimension takes the form

$$\gamma_{\phi^n} = n\sum_{\ell=1}\lambda^\ell P_\ell(n), \quad (7)$$

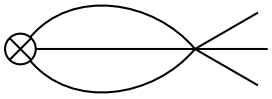

Fig. 1. Two-loop diagram contributing to the ϕ^n anomalous dimension. The crossed circle denotes the ϕ^n insertion.

where P_ℓ is a polynomial of degree ℓ for $\ell \le n$, and of degree n for $\ell > n$. Thus, the loop order ℓ contribution grows as $\lambda^\ell n^{\ell+1}$ for $\ell \le n$, implying that the diagrammatic expansion breaks down for sufficiently large λn. Re-organizing the series in (7), the scaling dimension can also be expanded as

$$\Delta_{\phi^n} = n\left(\frac{D}{2} - 1\right) + \gamma_{\phi^n} = \sum_{\kappa=-1}\lambda^\kappa \Delta_\kappa(\lambda n). \quad (8)$$

The main result of [13] is that it is possible to compute the functions $\Delta_\kappa(\lambda n)$ for arbitrary λn via a perturbative *semiclassical calculation* around a non-trivial saddle; the result can be organized as an expansion in $\lambda \ll 4\pi$ while treating λn as a fixed parameter, closely analogous to the 't Hooft coupling of large N theories. The goal of this paper is to compute the leading term and the first subleading correction in (8).

The scaling dimension (8) is a physical (scheme-independent) quantity only at the fixed-point (6). However, in light of the observation at the end of the previous section, working at order $\mathcal{O}(\lambda)$ we can take $\varepsilon \to 0$ without affecting the conformal invariance of the theory.[3] The leading order term $\Delta_{-1}(\lambda n)$ and the one-loop correction $\Delta_0(\lambda n)$ are hence scheme-independent for generic λ.

Working at fixed n, at leading order in λ, the anomalous dimension of $\phi^n(x)$ is determined by the diagram in Fig. 1 and it is given by

$$\gamma_{\phi^n} = \frac{\lambda^2 n(n-1)(n-2)}{36(4\pi)^2} + \mathcal{O}\left(\frac{\lambda^4 n^5}{(4\pi)^4}\right). \quad (9)$$

Comparing with (8), one can readily extract the lowest order terms in the expansion of Δ_{-1} and Δ_0 at small λn. We will use this expression as a check of the more general result that we will derive in the next section.

4. Semiclassical computation

To compute the scaling dimension Δ_{ϕ^n} for arbitrary λn we proceed as in [13]. Here we review the logic and outline the main steps.

We first use a Weyl transformation to map the theory to the cylinder $\mathbb{R} \times S^{D-1}$. Parametrizing $\mathbb{R}^D$ by polar coordinates (r, Ω_{D-1}), where Ω_{D-1} collectively denotes the coordinates on S^{D-1}, and $\mathbb{R} \times S^{D-1}$ by (τ, Ω_{D-1}), the mapping is simply given by $r = Re^{\tau/R}$ with R the sphere radius [19,20]. The Lagrangian of the theory on the cylinder reads:

$$\mathcal{L}_{cyl} = \partial\bar{\phi}\partial\phi + m^2\bar{\phi}\phi + \frac{\lambda^2}{36}\left(\bar{\phi}\phi\right)^3 , \quad (10)$$

where $m^2 = \left(\frac{D-2}{2R}\right)^2 \overset{D=3}{=} \frac{1}{4R^2}$ arises from the $\mathcal{R}(g)\bar{\phi}\phi$ coupling to the Ricci scalar which is enforced by conformal invariance. Working at $\mathcal{O}(\lambda)$, we neglect the difference between bare and renormalized coupling, as that arises at $\mathcal{O}(\lambda^2)$.

For small λn, when diagrammatic perturbation theory holds, ϕ^n is the operator of lowest dimension with $U(1)$ charge n. Then for

[3] Dimensional regularization is still used in the intermediate steps.

generic λn, we *define* the operator ϕ^n to be the lowest dimension charge n operator. While this seems natural to us, the precise relation between such lowest dimension operator and its explicit functional expression in terms of field variables (ϕ^n, $\phi^{n-2}\partial^2(\phi)^2$, etc.) in the path integral, is not obvious in the regime $\lambda n \gg 4\pi$. It should however become clear from our discussion that the precise form of the lowest dimension operator is a separate issue. It does not affect our computation of its scaling dimension but it matters for the computation of the normalization of the correlator, and thus for the computation of higher point functions. We plan to explore this in future work.[4]

According to the above natural definition, Δ_{ϕ^n} is directly determined by studying the expectation value of the evolution operator e^{-HT} in an arbitrary state $|\psi_n\rangle$ with fixed charge n.[5] As long as there is an overlap between the state $|\psi_n\rangle$ and the lowest energy state (with charge n), in the limit $T \to \infty$ the expectation gets saturated by the latter

$$\langle\psi_n|e^{-HT}|\psi_n\rangle \underset{T\to\infty}{=} \tilde{\mathcal{N}} e^{-E_{\phi^n} T}\,, \quad E_{\phi^n} = \Delta_{\phi^n}/R\,. \tag{11}$$

To pick a specific state, we work in polar coordinates for the field:

$$\phi = \frac{\rho}{\sqrt{2}} e^{i\chi}\,, \quad \bar{\phi} = \frac{\rho}{\sqrt{2}} e^{-i\chi}\,. \tag{12}$$

Following [8], we then consider the following path integral:

$$\begin{aligned}\langle\psi_n|e^{-HT}|\psi_n\rangle = \mathcal{Z}^{-1} \int \mathcal{D}\chi_i \mathcal{D}\chi_f \psi_n(\chi_i)\psi_n^*(\chi_f) \\ \times \int_{\rho=f,\ \chi=\chi_i}^{\rho=f,\ \chi=\chi_f} \mathcal{D}\rho\mathcal{D}\chi e^{-S}\,,\end{aligned} \tag{13}$$

where the insertions of the wave-functional

$$\psi_n(\chi) = \exp\left(\frac{in}{R^{D-1}\Omega_{D-1}} \int d\Omega_{D-1}\chi\right) \tag{14}$$

ensure that the initial and final states have charge n, while the boundary value f for ρ is arbitrary and will be fixed later by convenience. The factor $\mathcal{Z}$ ensures that the vacuum to vacuum amplitude is normalized to unity:

$$\mathcal{Z} = \int \mathcal{D}\rho\mathcal{D}\chi e^{-S}\,. \tag{15}$$

The structure of the expansion (8) follows from performing the path-integral in a saddle-point approximation. This is easily seen rewriting eq. (13) as

$$\langle\psi_n|e^{-HT}|\psi_n\rangle = \mathcal{Z}^{-1} \int_{\rho=f}^{\rho=f} \mathcal{D}\rho\mathcal{D}\chi e^{-S_{eff}}\,, \tag{16}$$

where the action on the right hand side is given by

$$\begin{aligned}S_{eff} = \int_{-T/2}^{T/2} d\tau \int d\Omega_{D-1} \Big[\frac{1}{2}(\partial\rho)^2 + \frac{1}{2}\rho^2(\partial\chi)^2 \\ + \frac{m^2}{2}\rho^2 + \frac{\lambda^2}{2(12)^2}\rho^6 + i\frac{n}{R^{D-1}\Omega_{D-1}}\dot{\chi}\Big]\,.\end{aligned} \tag{17}$$

Rescaling then the field as $\rho \to \rho/\lambda^{1/2}$ and collecting up front the overall λ^{-1}, one immediately recognizes eq. (8) as the result of performing the path-integral (16) as a systematic loop expansion around a saddle-point (see [13] for details).

Properly tuning the initial and final value $\rho_i = \rho_f = f$ in eq. (16), the stationary configuration for the action (17) is given by a superfluid configuration:

$$\rho = f\,, \qquad \chi = -i\mu\tau + \text{const.}\,, \tag{18}$$

where μ is interpreted as the chemical potential of the system and μ and f satisfy

$$\mu^2 - m^2 = \frac{\lambda^2}{48} f^4, \qquad \mu f^2 R^{D-1}\Omega_{D-1} = n\,. \tag{19}$$

Given the constraint $f^2 \geq 0$, the eqs. (19) admit a unique solution. In particular, in $D = 3$ and for $n > 0$, μ reads:

$$R\mu = \frac{\sqrt{1+\sqrt{1+\frac{\lambda^2 n^2}{12\pi^2}}}}{2\sqrt{2}}\,. \tag{20}$$

For $\lambda n < 0$ the chemical potential is given by minus the expression in (20) and is hence discontinuous at $\lambda n = 0$.[6] In the following we always assume $\lambda n > 0$.

Plugging the solution into the classical action we extract the leading order contribution to the scaling dimension:

$$S_{eff}/T = \frac{n}{3}\left(2\mu + \frac{m^2}{\mu}\right) \overset{D=3}{=} \frac{1}{R}\frac{\Delta_{-1}(\lambda n)}{\lambda}\,. \tag{21}$$

Explicitly, the result reads

$$\Delta_{-1}(\lambda n) = \lambda n\, F_{-1}\left(\frac{\lambda^2 n^2}{12\pi^2}\right)\,, \tag{22}$$

where

$$F_{-1}(x) = \frac{1+\sqrt{1+x}+x/3}{\sqrt{2}\,(1+\sqrt{1+x})^{3/2}}\,. \tag{23}$$

To compute the one-loop correction we expand the fields around the saddle point configuration:

$$\rho(x) = f + r(x)\,, \quad \chi(x) = -i\mu\tau + \frac{1}{f\sqrt{2}}\pi(x)\,. \tag{24}$$

The action (17) at quadratic order in the fluctuations reads

$$\begin{aligned}S^{(2)} = \int_{-T/2}^{T/2} d\tau \int d\Omega_{D-1} \Big[\frac{1}{2}(\partial r)^2 + \frac{1}{2}(\partial\pi)^2 \\ - 2i\mu\, r\partial_\tau\pi + 2(\mu^2 - m^2)r^2\Big]\,.\end{aligned} \tag{25}$$

This action describes two modes, with dispersion relations given by

$$\omega_\pm^2(\ell) = J_\ell^2 + 2(2\mu^2 - m^2) \pm 2\sqrt{J_\ell^2\mu^2 + (2\mu^2 - m^2)^2}\,, \tag{26}$$

where $J_\ell^2 = \ell(\ell + D - 2)/R^2$ is the eigenvalue of the Laplacian on the sphere. The first mode has a gap $\omega_+(0) = 2\sqrt{2\mu^2 - m^2} \propto \sqrt{\lambda n}$

[4] In [13] the analyticity of Δ_{ϕ^n} in λn as it directly emerges from the computation was taken as indication that there is no level crossing as λn is varied. However, unlike argued in [13], we now realize that does not imply that the field expression for the lowest dimension charge n operator remains ϕ^n for all values of λn.

[5] Alternatively, one could include explicitly the operator insertions in the action as sources; this was done in [21] in the limit $\lambda^2 n^2 \ll (4\pi)^2$.

[6] This discontinuity is required as the scaling dimension of ϕ^n and the conjugated operator, $\bar{\phi}^n$, must be the same.

for $\lambda n \gg 1$. The dispersion relation $\omega_-(\ell)$ describes instead a gapless mode, the Goldstone boson for the spontaneously broken $U(1)$ symmetry. These modes, except for the zero mode of the Goldstone which relates different charge sectors, provide a basis for the Fock space describing charge n operators with higher scaling dimension. In particular, the descendants, obtained by acting q times with the P_i generators of the conformal algebra, correspond to states involving a number q of massless spin one quanta, each increasing the energy by $\omega_-(1) = 1/R$. Other modes describe different primaries; non-trivially, the expressions (26) include the leading λn corrections to the free theory values, effectively resumming the effect of infinitely many loop diagrams in standard diagrammatic calculations.

In the large λn regime we can integrate out the gapped mode and describe the lightest states at charge n through the superfluid effective theory for the gapless mode [8]. In this limit the dispersion relation of the Goldstone boson can be expanded in inverse powers of λn and reads

$$R\omega_-(\ell) = \left[\frac{1}{\sqrt{2}} - \frac{\sqrt{3}\pi}{\sqrt{2}\lambda n} + \mathcal{O}\left(\frac{1}{(\lambda n)^2}\right)\right] J_\ell + \left[\frac{\sqrt{3}\pi}{2\sqrt{2}} + \mathcal{O}\left(\frac{1}{\lambda n}\right)\right]\frac{J_\ell^3}{\lambda n} + \mathcal{O}\left(\frac{J_\ell^5}{(\lambda n)^2}\right). \quad (27)$$

From this expression we see that the Goldstone sound speed approaches the value $c_s = 1/\sqrt{2}$ as $\lambda n \to \infty$, as dictated by conformal invariance in the superfluid phase.

The one-loop correction Δ_0 is determined by the fluctuation determinant around the leading trajectory (18). Explicitly, we find[7]

$$\Delta_0(\lambda n) = \frac{1}{2}\sum_{\ell=0}^{\infty} n_\ell \left[\omega_+(\ell) + \omega_-(\ell) - 2\omega_0(\ell)\right], \quad (28)$$

where $\omega_0^2(\ell) = J_\ell^2 + m^2 = \left(\ell + \frac{D-2}{2}\right)^2/R^2$ is the free theory dispersion relation and $n_\ell = \frac{(2\ell+D-2)\Gamma(\ell+D-2)}{\Gamma(\ell+1)\Gamma(D-1)}$ is the multiplicity of the Laplacian on the $(D-1)$-dimensional sphere. The analytic continuation to negative D of the sum (28) is convergent; the final result is finite in the limit $D \to 3$, consistently with the coupling not being renormalized at one-loop. Eventually, Δ_0 can be written in terms of an infinite convergent sum as in [13]

$$\Delta_0(\lambda n) = \frac{1}{4} - 3(R\mu)^2 + \frac{\sqrt{8R^2\mu^2 - 1}}{2} + \frac{1}{2}\sum_{\ell=1}^{\infty}\sigma(\ell), \quad (29)$$

where $\sigma(\ell)$ is obtained from the summand in (28) by subtracting the divergent piece:

$$\sigma(\ell) = (1+2\ell)R\left[\omega_+(\ell) + \omega_-(\ell)\right] - 4\ell(\ell+1) - \left(6R^2\mu^2 - \frac{1}{2}\right). \quad (30)$$

In (29) all quantities are evaluated in $D = 3$, hence μ is given by eq. (20) and $m = \frac{1}{2R}$.

5. Analysis of the result

Eqs. (22) and (29) provide the first two terms of the expansion (8) for the scaling dimension of the operator ϕ^n, Δ_{ϕ^n}. The result holds for arbitrary values of λn. Here we explicitly show that Δ_{ϕ^n} matches the diagrammatic result (9) and the large charge prediction (1) in the two extreme regimes of, respectively, small and large λn.

Let us consider first the small λn regime. From eq. (20) it follows that the chemical potential, and consequently all the functions Δ_κ, can be expanded in powers of $\lambda^2 n^2$. Explicitly neglecting terms of order $\mathcal{O}\left(\frac{\lambda^6 n^7}{(4\pi)^6}\right)$, we get:

$$\Delta_{\phi^n} = \frac{n}{2} + \frac{\lambda^2}{(4\pi)^2}\left[\frac{n^3 - 3n^2}{36} + \mathcal{O}(n)\right] - \frac{\lambda^4}{(4\pi)^4}\left[\frac{n^5}{144} - \frac{n^4(64 - 9\pi^2)}{1152} + \mathcal{O}\left(n^3\right)\right] + \dots. \quad (31)$$

In this regime we can compare eq. (31) with the diagrammatic result (9), finding perfect agreement.

Let us now discuss the large λn regime. The classical result (22) is easily seen to admit an expansion in inverse powers of λn with the expected form. The one-loop contribution (29) can be evaluated numerically for large $\mu \sim (\lambda n)^{1/2}$ and then fitted[8] to the functional form (1). When doing this we also verified that the coefficients of terms which might modify the form of the expansion, such as a term linear in λn, are compatible with zero within the numerical uncertainty. The final result reads

$$\Delta_{\phi^n} = t^{3/2}\left[c_{3/2} + c_{1/2}t^{-1} + c_{-1/2}t^{-2} + \dots\right] + \left[d_0 + d_{-1}t^{-1} + \dots\right], \quad (32)$$

where we defined $t = \frac{\lambda n}{\sqrt{3}\pi}$ and the coefficients read

$$\begin{aligned} c_{3/2} &= \frac{\sqrt{3}\pi}{6\lambda} - 0.0653 + \mathcal{O}\left(\frac{\lambda}{\sqrt{3}\pi}\right), \\ c_{1/2} &= \frac{\sqrt{3}\pi}{2\lambda} + 0.2088 + \mathcal{O}\left(\frac{\lambda}{\sqrt{3}\pi}\right), \\ c_{-1/2} &= -\frac{\sqrt{3}\pi}{4\lambda} - 0.2627 + \mathcal{O}\left(\frac{\lambda}{\sqrt{3}\pi}\right), \\ d_0 &= -0.0937255(3), \\ d_{-1} &= 0.096(1) + \mathcal{O}\left(\frac{\lambda}{\sqrt{3}\pi}\right). \end{aligned} \quad (33)$$

The parentheses show the numerical error on the last digit, when the latter is not negligible at the reported precision.

To interpret this result notice that, as already mentioned above eq. (27), in the large λn regime we can integrate out the gapped mode. We are then left with an *effective theory* for the Goldstone boson on the cylinder. The form of the latter is determined by $U(1)$ and Weyl invariance and, in $D = 3$, reads:

$$\mathcal{L}/\sqrt{g} = -\frac{1}{\lambda}\left\{\alpha_1|\partial\chi|^3 + \alpha_2\mathcal{R}_{\mu\nu}\frac{\partial^\mu\chi\,\partial^\nu\chi}{|\partial\chi|} - \alpha_3|\partial\chi|\left[\mathcal{R} + 2\frac{\left(\partial_\mu|\partial\chi|\right)^2}{|\partial\chi|^2}\right] + \dots\right\}. \quad (34)$$

The field is expanded around the classical value $\chi = -i\mu\tau$ and the factor $1/\lambda$ in front ensures that the Wilson coefficients α_i scale as λ^0. In the EFT the derivative expansion coincides with an

[7] In (28) we neglect the integration over the zero mode associated to the $U(1)$ symmetry, whose result is independent of T and hence does not contribute to E_{ϕ^n} in eq. (11).

[8] We computed Δ_0 numerically for $R\mu = 10, 11, \dots 210$ to perform the fit; the final results are obtained using six fitting parameters in the expansion (1).

expansion in inverse powers of λn; the loop counting parameter is $\lambda/(\lambda n)^{3/2}$ instead. It follows that the scaling dimension of the lightest charged operator takes the form (32), where the first line corresponds to short distance (classical plus quantum) contribution from both the radial and Goldstone mode, while the second line corresponds the one-loop Casimir energy of the Goldstone mode. This second contribution is thus a genuinely long distance one. Matching the explicit calculation in the full theory with the result of the effective theory we can determine the Wilson coefficients α_1 and α_3 to next to leading order in λ through the relations:

$$\lambda c_{3/2} = \frac{\pi}{3^{3/4}\sqrt{\alpha_1}}, \qquad \lambda c_{1/2} = \frac{4\pi\alpha_3}{3^{1/4}\sqrt{\alpha_1}}. \tag{35}$$

From these we extract $\alpha_1 = 4/\sqrt{3} + 0.3326\,\lambda + \mathcal{O}(\lambda^2)$ and $\alpha_3 = \sqrt{3}/4 + 0.0644\,\lambda + \mathcal{O}(\lambda^2)$. Notice that the coefficient α_2 does not contribute to the scaling dimension at order $(\lambda n)^{1/2}$ since $\mathcal{R}_{00} = 0$.

To discuss the value of the coefficients d's in eq. (33), notice first that from the Lagrangian (34) one derives the dispersion relation of the Goldstone boson as:

$$R\omega_-(\ell) = \left[\frac{1}{\sqrt{2}} - \frac{4\pi(\alpha_3+2\alpha_2)}{\sqrt{2}\,\lambda n} + \mathcal{O}\left(\frac{1}{(\lambda n)^2}\right)\right] J_\ell + \left[\sqrt{2}\pi\alpha_3 + \mathcal{O}\left(\frac{1}{\lambda n}\right)\right]\frac{J_\ell^3}{\lambda n} + \dots . \tag{36}$$

Comparing this equation to eqs. (27), (33) and (35), at leading order we find $\alpha_2 = 0$, and we can also check the consistency of the result for α_3: $\alpha_3 = \sqrt{3}/4$.[9] Moreover with (36) we can compute the one-loop Casimir energy of the Goldstone mode and determine the second line of (32) in terms of the EFT Wilson coefficients:

$$d_0 = -0.0937255\,, \tag{37}$$

$$d_{-1} = \alpha_2 \times 0.4329 + \alpha_3 \times 0.2236\,. \tag{38}$$

As remarked in [7], d_0 is a theory independent number. The result of the explicit computation in the full model (33) agrees with its value (37) almost to seven digits accuracy. Using the previously extracted values for the α_i, the EFT prediction in eq. (38) gives $d_{-1} = 0.0968$, again in agreement with the explicit result in eq. (33) within its numerical accuracy.

6. Conclusions

In conclusion, in the tricritical $U(1)$ CFT in $3-\varepsilon$ dimensions we computed the scaling dimension of the operator ϕ^n at the next-to-leading order in the coupling λ, but for arbitrary values of λn. Our results nicely interpolate between the small λn regime, when it is given by (31), in agreement with diagrammatic calculations, and the large λn regime, where it reads as in (32) and it agrees with the expectation for the universal conformal superfluid phase of CFTs at large charge. The remarkable agreement between the form of the quantum corrections in eqs. (37) and (38) and the explicit result (32) provides a nontrivial check of the validity of our methodology.

By further developing these ideas, in the future it would be interesting to study the transition from diagrammatic perturbation theory to semiclassics in other observables studied by the large charge expansion in CFT. Possible examples include three-point functions of charged operators [8] or the scaling dimension of charged operators with large spin [22,23]. Perhaps, these ideas might be applied as well in the study of inhomogeneous phases, which are conjectured to describe operators in mixed symmetric representations of the $O(n)$ models [24,25].

[9] That $\alpha_2 = 0$ at the tree level in the effective lagrangian simply follows from the fact that, in the microscopic lagrangian, χ only appears through $(\partial\chi)^2$.

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