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Against a global conception of mathematical hinges*

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Epistemologists have developed a diverse group of theories, known as hinge epistemology, about our epistemic practices that resort to and expand on Wittgenstein's concept of 'hinges' in On Certainty. Within hinge epistemology there is a debate over the epistemic status of hinges. Some hold that hinges are non-epistemic (neither known, justified, nor warranted), while others contend that they are epistemic. Philosophers on both sides of the debate have often connected this discussion to Wittgenstein's later views on mathematics. Others have directly questioned whether there are mathematical hinges, and if so, these would be axioms. Here, we give a hinge epistemology account for mathematical practices based on their contextual dynamics. We argue that 1) there are indeed mathematical hinges (and they are not axioms necessarily), and 2) a given mathematical entity can be used contextually as an epistemic hinge, a non-epistemic hinge, or a non-hinge. We sustain our arguments exegetically and empirically.

Keywords: hinge epistemology; later Wittgenstein; mathematical hinges; philosophy of mathematical practice; *On Certainty*; context dependence.

I. Introduction

Over the last decades epistemologists have developed a diverse group of theories about our epistemic practices that resort to and expand on Wittgenstein's

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concept of 'hinges' in *On Certainty* (Wittgenstein 1969; hereinafter OC). This group of theories is known as *hinge epistemology*. Within hinge epistemology there is a debate over the epistemic status of hinges. Some hinge epistemologists hold that hinges are non-epistemic (neither known, justified, nor warranted), while others contend that they are epistemic. Philosophers on both sides of the debate have often connected this discussion to Wittgenstein's later views on mathematics (see McGinn 1989; Moyal-Sharrock 2004; Kusch 2016b; Coliva 2020; Martin 2022).[1] In this paper, we set out to address this debate and give a novel account of hinge epistemology for mathematical practices.

Section II provides a brief overview of hinge epistemology and the debate over the epistemic status of hinges. Section III connects these considerations to Wittgenstein's later philosophy of mathematics and previous literature on mathematical hinges. We defend that (i) there are indeed mathematical hinges (and these are not necessarily axioms) and (ii) a given proposition can be used as an epistemic or non-epistemic hinge, or non-hinge, in a context-dependent manner.

Section IV.1 develops a case study showcasing the considerations from Section III. This subsection focuses on mathematical practice at the community level, exploring the different and changing roles of mathematical propositions and the implications for the development of the practice.

Lastly, Section IV.2 explores mathematical practice at the level of the biography and work of the individual mathematician: mathematical propositions also play different roles at different stages of a mathematician's training and work.

II. Hinge epistemology

Hinge epistemology is an umbrella term for a diverse group of epistemological theories about justification and knowledge that expand on Wittgenstein's concept of 'hinge' in OC. This concept can be roughly defined as the fundamental presuppositions of one's worldview, which are exempt from doubt, and make it possible for us to perform other epistemic operations (such as discovering, justifying, verifying, evaluating beliefs as rational, and giving reasons

[1]There are two reasons for our focus on mathematical hinges. The first is pragmatic. Hinge epistemologists have been explicit about the application of their theories to mathematics, thereby making it an enticing case to explore the nature of mathematical hinges and discuss the research that has already been conducted. The second concerns the distinctive features of mathematics and mathematical practice. Wittgenstein's philosophy of mathematics illustrates pathways by which mathematical propositions acquire regulative roles in mathematical practices (see Section III.1). Of these, according to Wittgenstein's work, some are likely specific to mathematical practices, like 'petrification' or 'hardening' processes constituting rules of description, while others are shared with non-mathematical practices, like establishing definitions.

for belief and doubt) (OC: §341, §415).[2] Justification, knowledge, and doubt do not take place in vacuum, but rather depend on hinges. Just as the hinges must stay put for a door to move, so too must hinges be exempt from doubt for our epistemic practices to work. Examples of hinges include but are not limited to: 'There is an external world' or 'Sense of perception is reliable' (for more examples see OC: §93, §120, §146, §209; Moyal-Sharrock 2004; Pritchard 2011).

Philosophers have used a variety of notions, such as 'hinge propositions', 'hinge commitments' (Pritchard 2016), 'cornerstone propositions' (Wright 2004), or 'certainties' (Moyal-Sharrock 2004), to speak about Wittgenstein's views in OC and their application to epistemology. Throughout this paper, we will favour 'hinge', as it allows us to emphasize that hinges are the fundamental presuppositions of our worldviews and belief systems, without making any substantial claims about their epistemic status or the kind of attitudes individuals may hold towards their hinges. In other words, we can speak about hinges in a theory-neutral way, refraining from implicitly endorsing certain controversial claims in our preferred terminology. For instance, 'it's a matter of philosophical controversy whether the hinges are propositions' (Ranalli 2020: 4976; see Pritchard 2016 for similar pronouncements). Meanwhile, 'hinge commitments' could be understood as entailing a rejection of the idea that hinges are propositions or have epistemic properties, such as being true or false.

What hinges are meant to be remains highly controversial. Competing hinge theories have different views on how best to understand hinges, each with different implications for the analysis of our epistemic practices.[3] Although there are four main dimensions along which hinge theories differ (see Johnson 2022), we are going to focus on the division between theories in virtue of how they conceive the epistemic status of hinges.

On the one hand, there is a family of non-epistemic theories (Moyal-Sharrock 2004; Pritchard 2011; 2016; Coliva 2016), which share the core claim that hinges are neither known, justified, nor warranted.[4] For instance, Pritchard holds that while hinges are truth-apt statements, they are not believable. Pritchard's argument is thus. (P1) Beliefs are responsive to reasons. (P2) Our attitude to hinges is not responsive to reasons. Hinges are the basic presuppositions of our belief systems, which are exempt from rational evaluation. (C) This leads to a non-belief theory of hinges: our attitude to hinges is one of certainty, not belief.

[2]This description of hinges is used to articulate and clarify an initial way of understanding the concept by listing some of those features, which are most commonly associated with it in the literature. It is not used as a single overarching definition, which provides a definite and non-exhaustive description of the common features shared necessarily by all hinges. Thus, we do not exclude other ways of conceiving hinges, which may emphasize other relevant features.

[3]For a general exposition of existing views, see Pritchard (2011), Ranalli (2020), and Johnson (2022).

[4]Moyal-Sharrock and Pritchard hold the further claim that hinges are outside the scope of rational evaluation.

On the other hand, there is a family of epistemic theories (Wright 2004, 2014), which shares the core claim that hinges can be potentially justified, albeit in a non-paradigmatic way. For instance, Wright explains that numerous cognitive projects (i.e., a question-pair procedure to attain epistemic goods) presuppose hinges as an authenticity condition. We need to accept them as true for the procedure to be a legitimate way of finding an answer to our question, allowing us to form a wide variety of beliefs from certain grounds. Our justification for these hinges consists in having a default attitude of entitlement to trust or accept them, in the absence of evidence or anything which indicates that they are true. Entitlement may stem from the fact that doubting and not accepting them as true would stifle inquiry, preventing the attainment of epistemic goods, or have cognitively disastrous consequences, systematically leading to doubt many non-hinge propositions.

III. A contextual approach to mathematical hinges

Hinge epistemologists on all sides of the debate have connected this discussion to Wittgenstein's later views on mathematics. McGinn (1989) and Moyal-Sharrock (2004) appeal to Wittgenstein's work to advocate for the view that mathematical hinges are non-epistemic. Meanwhile, Kusch (2016b) argues for the opposed view, highlighting the epistemic nature of mathematical hinges. Lastly, Coliva (2020) expresses some reservations about the very possibility of mathematical hinges, arguing that if there are any (non-epistemic) hinges in mathematics these would have to be axioms (and even this concession would depart from Wittgenstein's philosophy of mathematics).

Hinge epistemologists have generally assumed that the status of mathematical hinges remains invariant. Namely, the status (i.e., epistemic or non-epistemic) of mathematical hinges is not merely specific to one example, but rather they are either strictly epistemic or non-epistemic in all contexts and it is the job of our hinge theory to explain this.[5] For example, Coliva (2020) holds that hinges are non-epistemic and display the following features: (i) they play a normative role, (ii) they cannot be epistemically justified, (iii) they cannot be doubted, and (iv) they are acquired through engagement with a community or practice of some kind. Accordingly, she highlights how a variety of mathematical signs and propositions (besides axioms) cannot be regarded as genuine hinges because they fail to meet said features.

In this section, we will argue that there is no reason to believe that the later Wittgenstein thought that (mathematical) hinges are strictly epistemic

[5] An exception is to be found in the work of Kusch (2016a,b, 2021), who offers a preliminary classification that distinguishes between eleven types of hinge commitments that fall into five distinct epistemic categories. Albeit it is important to highlight that Kusch still seemingly holds that some characteristics are shared by all hinges, such as the fact that they are all epistemic.

or non-epistemic looking at the Remarks on the Foundations of Mathematics (Wittgenstein 1978; hereinafter RFM), Lectures on the Foundations of Mathematics (Wittgenstein 1976; hereinafter LFM), and OC. Instead, we will defend mathematical hinges seem to be best described as complex phenomena, which may exhibit epistemic and non-epistemic statuses in different contexts and uses. To do so, we first recapitulate important features of the later Wittgenstein's philosophy of mathematics (Section III.1). Subsequently, we provide arguments against the assumption that the status of mathematical hinges remains invariant (Section III.2). Lastly, we outline a contextual approach of mathematical hinges (Section III.3).

III.1 Wittgenstein's later philosophy of mathematics

In this section, we recapitulate important features of the later Wittgenstein's philosophy of mathematics. This is important because the distinctive aspects of mathematics should be considered to clarify and further develop a hinge epistemology for mathematics.

First, mathematics is normative: it is exegetically uncontroversial that, according to the later Wittgenstein, mathematics has, in one way or another, normative power on empirical phenomena. For instance, it does not constitute descriptions of empirical phenomena, but 'hardened empirical regularities' that become rules of description (Fogelin 1995; Steiner 1996, 2009; Bangu 2012; Kusch 2016b; Coliva 2020; Pérez-Escobar 2022, 2023). This idea is still present in OC (e.g. OC: §657). Less uncontroversial are the following two views: 1) that mathematics, by virtue of being grammatical rules, admit only a non-epistemic interpretation (McGinn 1989; Moyal-Sharrock 2004), and 2) because rules of description are based on empirical regularities (Kusch 2016b), at least some mathematics is the petrification of empirical content, and thus admit epistemic justification. Kusch also argues that it makes sense to report mathematical certainties to others, which speaks for their epistemic character. In any case, Wittgenstein explicitly calls an example of mathematical proposition, $12 \times 12 = 144$, a hinge, two paragraphs before the claim above (OC: §657) that mathematical propositions are 'fossilised', suggesting a relationship between this 'hardening' and a hinge status of (at least some) mathematical propositions:

> The mathematical proposition has, as it were officially, been given the stamp of incontestability. I.e.: 'Dispute about other things; this is immovable — it is a hinge on which your dispute can turn. ' (OC: §655)

Secondly, meaning as use: another aspect of Wittgenstein's philosophy of mathematics, which he adapts from his philosophy of language, is that the meaning of mathematical propositions resides in their use and not in some essence intrinsic to their symbols (Maddy 1993; Rodych 1997; Pérez-Escobar and Sarikaya 2022). These uses can be roles within language or extra-linguistic uses. This idea is also still present in OC (e.g. §61).

Thirdly, context-dependence: this is the aspect of Wittgenstein's philosophy of mathematics that has been the least acknowledged in the literature on hinge epistemology. In fact, this aspect is closely connected to meaning as use: a proposition lacks intrinsic meaning, but its meaning resides on how it is used in a given situation. In the LFM and RFM, we find several extra-mathematical factors, which influence the use (and thus meaning) of mathematical propositions across situations, prominent ones being training and enculturation. It has recently been argued that such a context-dependence indeed affects the normative character of mathematics from a later Wittgensteinian perspective (Pérez-Escobar 2023): a given mathematical proposition may be used normatively in one context, but not in another. In fact, OC features several passages where this context-dependence is explicit. For instance, despite referring to propositions in general, the following remark features the 'hardening' typical of mathematical propositions (as claimed above) and context-dependence concerning normative power:

> It might be imagined that some propositions, of the form of empirical propositions, were hardened and functioned as channels for such empirical propositions as were not hardened but fluid; and that this relation altered with time, in that fluid propositions hardened, and hard ones became fluid. (OC: §96)

Thus, Wittgenstein indicates how 'hardened' propositions, despite being formulated similarly to empirical propositions, play a normative role in discourse, yet such a proposition may lack this use in other contexts and instead play the role of ordinary empirical propositions. Another passage indicates that this is also the case for logic:

> But if someone were to say 'So logic too is an empirical science' he would be wrong. Yet this is right: the same proposition may get treated at one time as something to test by experience, at another as a rule of testing. (OC: §98)

Overall, Wittgenstein is unusually explicit about the context-dependence on the normative power of propositions (whose use may shift from normative hinge to ordinary empirical proposition) and its relationship with 'hardening' (as in mathematics). Hereinafter, we employ these exegetical remarks to argue against the assumption that the status of mathematical hinges remains invariant and offer an alternative contextual approach. Also, in Sections IV.1 and IV.2, we will show how mathematical propositions change their meaning (i.e., are used differently in practice) contextually.

III.2 A problematic assumption about mathematical hinges

Wittgenstein's views on mathematics have been used to advocate for the interpretation that mathematical hinges are non-epistemic (McGinn 1989;

Moyal-Sharrock 2004), epistemic (Kusch 2016b), or to express reservations about the very possibility of mathematical hinges (Coliva 2020). One common assumption in these proposed interpretations is that the status of hinges must remain invariant in all contexts. However, there seems to be three good reasons to break with this assumption.

First, each hinge epistemologist appeals to different textual evidence in OC to provide an unequivocal set of claims, which stipulate the epistemic or non-epistemic status of all hinges. For instance, Moyal-Sharrock (2004: 72) holds that OC traces the arduous process by which Wittgenstein comes to see all hinges being indubitable, foundational, non-empirical, grammatical, ineffable, and enacted.[6] Unfortunately, if one goes through OC, there does not seem to be an unequivocal set of claims about the essential characteristics of hinges that constitutes the backbone of Wittgenstein's view (see Kusch 2016a; Coliva 2016, 2021 for a defence). Hence, by selecting specific claims, one may find textual evidence for a position or the other. For example, OC features claims that seem to advocate the epistemic incontestability of hinges (OC: §494, §655) while it also features claims that seem to suggest the reliance of hinges on the empirical domain (OC §321). Wittgenstein tells us that although he can enumerate various cases in which he can rightly say he cannot be making a mistake (i.e., cases in which one is certain of a hinge commitment, such as 'I have a body' or '$2 + 2 = 4$'), he cannot 'give any common characteristics' (OC: §674). Wittgenstein explicitly refrains from listing characteristics that are common to all hinges, settling for contextual investigations that clarify different aspects of hinges in particular contexts and uses. This applies to the divide that concerns us: hinges as epistemic or non-epistemic.

Secondly, every hinge epistemologist offers examples (within and outside of mathematics) to argue in favour of their preferred hinge theory and counterexamples against other theories (see Pritchard 2011 for an exposition of the strengths and weaknesses of existing hinge theories). The problem, however, lies in that, while each hinge theory can adequately explain specific instances of hinges in our epistemic practices, they are unable to offer a general and systematic explanation for all hinges. Take the use of hinge theories to explain deep disagreements. While epistemic hinge theories can provide intricate explanations of deep disagreements in science, they struggle to explain normative deep disagreements, such as moral ones, because the content of these disagreements does not include epistemic principles or epistemic properties—the opposite holds for non-epistemic hinge theories (Ranalli 2021: 993–4; see Ranalli 2020; Lavorerio 2021, for more examples of the limitations inherent to different hinge theories when explaining deep disagreements). Consequently,

[6] Moyal-Sharrock (2004: 100–1) does concede that there are some features which are not shared by all hinges. For instance, while some are universal, others are local hinges.

there is no set of epistemic practices or shared status that serve as conclusive evidence for one unique hinge theory to apply to the case of mathematics.

Thirdly, hinge epistemologists often agree on the claim that the same sign or proposition may serve as a hinge or a non-hinge in different contexts. For example, while people in the 1950s were certain about the fact that humans had not been on the moon, said hinge became a false proposition after 1969. However, we intend to go further: mathematical propositions not only can adopt hinge or empirical roles in different contexts but also can behave as different types of hinges (i.e., the epistemic/non-epistemic distinction). Kusch (2016b) contends against McGinn's (1989) and Moyal-Sharrock's (2004) views that mathematical hinges are non-epistemic by arguing for an important epistemic aspect of these hinges. Although Kusch suggests a primarily epistemic interpretation of Wittgenstein's mathematical hinges, his main goal is to challenge the view that Wittgenstein's mathematical hinges, by virtue of being grammatical rules, admit only a non-epistemic interpretation. This renders Kusch's position more moderate. In our view, compared to McGinn and Moyal-Sharrock, Kusch is on the right track given that he acknowledges the plurality of phenomena that Wittgenstein tries to characterize in OC. However, he does not capitalize enough on the context-dependency of propositions described above, that is, on how a given proposition plays different roles across contexts within practices. For instance, Kusch argues that the fact that it makes sense to report mathematical certainties to others speaks in favour of their epistemic dimension. However, this is not necessarily the case in all contexts: it may be a way to ensure that others apply a non-epistemic grammatical rule correctly. Furthermore, the later Wittgenstein also discusses mathematical definitions, which need not be justified in any way (i.e., not necessarily put into place as the 'hardening' of empirical regularities), which are still worth communicating. For instance, in LFM, p. 24, Wittgenstein claims that 'This is two: I I' is as good of a definition as Russell's work on $1 + 1 = 2$, a point that he made without appealing to a hardened regularity, but a similar proposition can be justified appealing to a hardened regularity in a different context (e.g. addition of discrete objects). In fact, epistemic and non-epistemic mathematical hinges play different normative roles, interact, and change their status contextually in mathematical practices, as Section IV will show.

Another recent view is that of Coliva (2020), who claims that, if there are mathematical hinges at all, these are the axioms. Coliva makes the point that most mathematical propositions, including the ones discussed by Wittgenstein in OC like $12 \times 12 = 144$, are not mathematical hinges, but have properties (like their normative role) that Wittgenstein uses to analogically illustrate properties of actual hinges. However, in Coliva's view, mathematical propositions like the above fail to meet two criteria to qualify as hinges: that 1) hinges cannot be epistemically justified and 2) that hinges are presupposed by further activities (in this case, by further mathematical activities, like counting and proving). These two criteria will not be our main worry in this work, for the following reasons. The first criterion is due to Coliva's adherence to the so-called 'framework' reading of OC, which stipulates a non-epistemic character of hinges. She also immunizes the framework reading against Kusch's account of epistemic hinges: mathematics has an epistemic dimension, but as she notes, there are no mathematical hinges in the first place if we adhere to Wittgenstein's view of mathematics. A main reason for this conclusion is that alleged mathematical hinges do not fulfil the second criterion above. Yet, it has been recently shown that simple mathematical propositions like $2 \times 2 = 4$ are presupposed by further mathematical activities and thus qualify as mathematical hinges, and that the notion of mathematical hinge is thus important to understand mathematical practice (Martin 2022).

Furthermore, Coliva raises a third worry about the characterization of mathematical hinges: she claims that, under Wittgenstein's characterization of mathematics, there is not a clear demarcation between mathematical propositions so that some would be hinges for others. This leads to the conclusion that either all mathematical propositions are hinges or none is, and she opts for the former option. We agree with Coliva that the idea that all mathematical propositions are hinges is strange: if some are hinges, others should hinge on them. This is a legitimate concern that previous literature (McGinn 1989; Moyal-Sharrock 2004; Kusch 2016b) did not address, and is at the heart of Coliva's rejection of mathematical hinges. However, given the short exegesis of Wittgentein's philosophy of mathematics above, we propose an alternative to Coliva's conclusion: it is not that mathematical propositions have or lack hinge status (a kind of essentialism that runs contrary to Wittgenstein's philosophy of language and mathematics), but that they are used as hinges or non-hinges contextually and dynamically, both at the level of community practices and at the level of the biographies of individual mathematicians (their training and work). Furthermore, these mathematical hinges can be epistemic or non-epistemic depending on the context.

III.3 Rethinking mathematical hinges

Wittgenstein's observations on the complexity of and lack of an essential epistemic or non-epistemic status, shared by all hinges together with their variety in different epistemic practices, give us good reasons to break with the assumption that hinges are either strictly epistemic or non-epistemic. Instead, they seem to be best described as complex phenomena, which may exhibit epistemic and non-epistemic statuses in different contexts and uses (see Section IV for examples). Mathematical hinges have both an epistemic aspect inasmuch as they capture 'petrified regularities' and therefore as justified (albeit non-paradigmatically) and can be true or false despite their normative power, and a non-epistemic, grammatical aspect that direct descriptions of empirical reality acting as framework rules without drawing epistemic justification whatsoever. The two aspects manifest in a context-dependent manner. Mathematical propositions, even axioms, can also be falsified when the context allows for it. This may seem too nuanced, but it is nuances like these which characterize Wittgenstein's philosophy.

Accordingly, rather than assuming that there must be one unique status for all mathematical hinges, the right thing might be to acknowledge their complexity and change course (cf. Wittgenstein 2009: §§65–83, §§92–131; Kuusela 2020). What emerges is a piecemeal hinge epistemology where we seek to clarify the status of hinges on a case-by-case basis in different contexts and uses, thereby gradually working towards a better grasp of their role in our epistemic practices without indicating a commonality about their epistemic status.

Henceforth (Section IV), we argue that mathematical practice provides evidence that a given mathematical hinge can be used epistemically and non-epistemically depending on contextual features like in which point of a research program we find ourselves. This means that our contextual view reflects actual mathematical practice. Specifically, we show how our use of a given proposition determines whether it is a:

a) Non-hinge (NH): a truth-apt proposition, which does not function as a hinge (e.g. it can be doubted and subjected to counter-evidence, it is not a background presupposition in our epistemic practices, it is justifiable, *etcetera*).
b) Non-epistemic hinge (NEH): a truth-apt proposition or non-factual rule, which functions as a hinge about which we have an attitude of certainty (it is neither known, justified, nor warranted).
c) Epistemic hinge (EH): a truth-apt proposition, which functions as a hinge about which we have an attitude of entitlement (it is non-paradigmatically justified and warranted).

We conclude this section with some words on the lack of definite properties of hinges and our acceptance of mathematical hinges. As we argue in Section III.2, OC describes many examples of hinges with very little in common besides having hinge-like properties. In fact, the notion of family resemblance may play a role in Wittgenstein's qualification of a proposition as a hinge: it may be possible that a common feature of two members of the group 'hinges' cannot be readily identified. This makes it reasonable to qualify mathematical propositions with hinge-like properties as mathematical hinges, especially given that OC does not identify mathematics as a special case the hinge-like propositions of which should be excluded from the family.[7]

[7]A recent response to Coliva (2020) has also noted the need for generality regarding the term 'hinge' and holds that mathematical hinges are 'those propositions that must be held fixed for an inquiry to remain the one that it is' (Martin 2022).

IV. Hinges and mathematical practice

In this section, we sketch the characteristics of mathematical practice that illustrate the above contextual dynamics of mathematical hinges, exemplifying how a mathematical proposition can be used as an NEH, an EH, or an NH. Section IV.1 does this at the community level and Section IV.2 at the level of the biographies of individual mathematicians.

IV.1 The development of mathematical notions at the community level

Despite common views of mathematics as unchanging, it has been argued that mathematical notions change over time, and so do mathematical practices at large (cf. Wagner 2022). We will see how the contextual dynamics of mathematical hinges is part of such changes.

Let us revisit the famous Lakatosean example, Euler's conjecture for polyhedra. The case is suited for this due to its simplicity, its philosophical significance, and the fact that it has been previously discussed from a Wittgensteinian perspective (Pérez-Escobar 2022; Zeng 2022, 2023). Here, we will emphasize the epistemic/non-epistemic distinction for hinges, but first we offer an overview of the discussion. Euler's conjecture for polyhedra consists in that, for any polyhedron, the number of vertices plus the number of faces minus the number of edges equals 2 ($V + F - E = 2$). We should note that the conjecture is a proposition that Euler did not feel the need to prove (the most distinctive and paradigmatic type of mathematical rational justification). This is even though conjectures, in general, are entities that await proof to become a theorem, and illustrates how a general picture of mathematics does not do justice to the richness of its practices. Instead, the conjecture was non-paradigmatically justified and warranted. Specifically, Euler's attitude to the conjecture seemed to be one of entitlement: he had a rational ground to accept it as true, despite not having any evidence or cognitive achievement (e.g. proof) that would be regarded as apt to constitute it as true or knowledge. The rational ground that motivated this entitlement is the repetition of an activity that yields consistent results: if we count the vertices, faces, and edges of the most salient polyhedra (tetrahedra, octahedra, dodecahedra …) and do the calculation, we will get 2. In *Proof and Refutations* (Lakatos 1976), we find the quote (attributed to Euler) that 'the truth of it has been established in so many cases, there can be no doubt that it holds good for any solid' (Lakatos 1976: 7). The repetition of this activity for different polyhedra makes the conjecture a 'petrified regularity' that, for Euler, is *true* (it is truth-apt, although its justification is non-paradigmatic mathematically speaking). This petrification or hardening, a key process in mathematics according to Wittgenstein (see

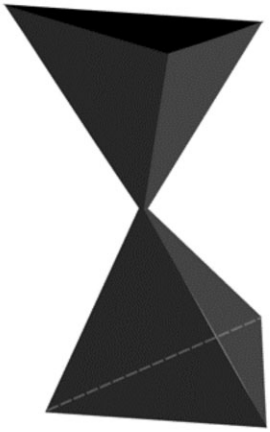

Figure 1. A counterexample. Taken from Pérez-Escobar (2023).

Section III), is at the base of why a mathematician may be entitled to it despite potential counterevidence (as seen next).

Overall, despite being neither an axiom nor formally proved at this stage, the conjecture is used as an EH: it is a truth-apt proposition that was non-paradigmatically justified insofar Euler (and historically, others too) felt entitled to and assumed it for all polyhedra. In turn, as we are about to see, this hinge plays a regulative role in mathematical practices about polyhedra. Later in the discussion, the counterexample is presented in Fig. 1.

Here, we have two joint tetrahedra. We may count one vertex where the two tetrahedra meet, and $V + F - E = 3$. Does this violate Euler's conjecture then? Lakatos considered that the mathematician could either discard the conjecture or do some 'monster-barring', claiming that this polyhedron is not a real polyhedron. The former option entails using the conjecture as an NH (the conjecture is an empirical proposition, which can be falsified), while the latter is a hinge use of the conjecture (inasmuch as the conjecture was used as a standard for what a polyhedron is). Yet there is a third, Wittgensteinian alternative presented in Pérez-Escobar (2022; see also Zeng 2022): the conjecture tells us how we should count, more specifically here how vertices should be counted, especially during informal stages of mathematical practice. For example, if we count separately for each tetrahedron, the regularity holds, since at the joint there are two vertices, one for each tetrahedron, and for each, $V + F - E = 2$; we could also not count any vertex at the joint, since 'it is a joint, not a vertex', and for the full figure, $V + F - E = 2$.

The second and third uses of the conjecture are normative, but they are distinct. Initially, the conjecture is an ordinary empirical proposition (an NH justified empirically that can be falsified). After its hardening, its justification and uses change. On one hand, it can be used an NEH: the conjecture can become (i.e., be used as) a definition; $V + F - E = 2$ does not apply to the figure that does not fulfil $V + F - E = 2$, and thus the proposition just refers to the set of figures for which $V + F - E = 2$ and forgoes empirical justification altogether. While it retains the formulation of its previous status as empirical proposition (see OC: §96), its role is different and regulates classification and monster-barring processes. On the other hand, it can be used as an EH: its justification is not foregone but re-regulated, as counting the elements of figures in a specific manner still yields the same quantitative regularity; however, this justification is non-paradigmatic and $V + F - E = 2$ has normative effects on counting practices (in other words, counting practices presuppose the conjecture; this was one of Coliva's major concerns about mathematical hinges). Thus, the different justifications (or lack thereof) of hinges are dynamic, alter their normative effects, and account for developments and changes in mathematical practices.

Therefore, we see how the conjecture admits different use cases: it can be used as an EH, an NEH, or as an NH. However, to fully appreciate this discussion under our view of hinges, we need to extend this analysis. In the discussion in *Proof and Refutations*, we find other key elements for our analysis: the different definitions of 'polyhedron' at stake. In fact, an implicit definition, Def 1, regulates what counts as a polyhedron for the student that presents the first counterexample. Def 1 is an NEH because it is never justified (paradigmatically or non-paradigmatically) and, in fact, it is never explicitly formulated (it is a set of implicit intuitions, maybe acquired by exposition to exemplars sharing some family resemblance, on what constitutes a polyhedron). Using Moyal-Sharrock's terminology, it is not said, but simply enacted: it shows itself in our practices. Def 1 had a normative role behind the scenes, and after being revealed and challenged (as we will see), it is characterized as 'a polyhedron is a solid whose surface consists of polygonal faces'.

Def 1 and the conjecture itself ($V + F - E = 2$) are initially used as hinges and they coexist harmonically. However, in *Proof and Refutations* (which considers the use case of the conjecture as an NEH, another implicit definition), it is the presentation of the first counterexample that breaks this harmony: it makes us choose between Def 1 and the conjecture. Either we accept that $V + F - E = 2$ for all polyhedra but the counterexample is not a real polyhedron (which leads to Def 1 becoming explicit for the first time and the formulation of a new definition Def 2 under which the counterexample is not a polyhedron anymore) or we simply discard the conjecture (in which case Def 1 may or may not become explicit). Thus, the conjecture and Def 1 are two competing NEHs in this scenario. Although with disagreement, the overall discussion takes the former path, but the point is that there is room to choose between these two alternatives (further evidenced by the fact that the students and the teacher disagree at different points on how to proceed). In this case we have competing

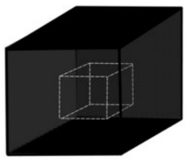

Figure 2. Conflicting hinges. Polyhedron taken from Pérez-Escobar (2023).

implicit definitions at this stage, as some students immediately disagree with the student who puts forward Def 1. In other words, we had a 'hidden' contradiction in the conglomerate of views of the different individuals.[8] Figure 2 illustrates this tension and competition featuring another problematic polyhedron.

We then have two conflicting hinges: the EH or NEH $V + F - E = 2$ and the NEH Def 1. Both are hinges inasmuch as they are not doubted and regulate mathematical practices on polyhedra. When the counterexample creates tension between the two, both temporarily lose their status as hinges in a resolution process where we briefly suspend our judgement, i.e., they are no longer cases in which we can rightfully say we are not making a mistake. In *Proof and Refutations*, the conjecture 'wins' in the first round, especially after Def 1 is made explicit and thereby loses part of its regulative power. Def 1 stops being a hinge altogether. This also fits Wittgenstein's view of hinges: two hinges coexisting in harmony may come into conflict and, for a period of time, both lose their properties as hinges as we briefly suspend our judgement. The resolution of the conflict involves that one of the two former hinges (the 'winner') regains its status of hinge (in terms of use: we resume its use as a hinge) while

[8]'Hidden' contradiction is also an important Wittgensteinian theme: we should not worry about them as long as they never appear. And even if they do appear, they can be put to good use in practice, thus posing no intrinsic danger. The point that Wittgenstein tries to make concerning hidden contradictions is that, according to him, they worry mathematicians excessively despite their practices working fine (for recent overviews, see Berg 2021 and Vanrie 2024). For Wittgenstein, contradictions do not exist a priori and possibly lurk hidden in mathematical formalisms. Instead, they may or may not be constituted in the particular developments of mathematical practices. In fact, as the analysis of the case study shows, some are 'created' by the changing dynamics of hinges; in this case the contradiction appears when both the conjecture and Def 1 are set as NEHs.

the other does not: the latter loses properties of hinges like being presupposed and regulating practices.

The discussion in *Proof and Refutations* implies a use of the conjecture as an NEH. However, as we saw, the conjecture also admits an EH use. In the latter case, the conjecture regulates not what counts as a polyhedron, but the practice of counting. If the conjecture takes this role, there is no conflict with Def 1: the alleged counterexample fulfilling Def 1 is not a counterexample because, when the conjecture regulates counting, the regularity holds. Thus, the resolution of the tension between two hinges may happen by Def 1 transitioning from NEH to NH, or by $V + F - E = 2$ transitioning from NEH to EH. Alternatively, the tension may never happen if the conjecture becomes an EH after petrification instead of an NEH, or if it never became a hinge and instead remained an NH (in the latter case, it would be falsified). Understanding these processes can shed further light on how mathematics develops despite its now more apparent open-endedness given its open texture (Tanswell 2018; Zayton 2022; Pérez-Escobar forthcoming).

Overall, this discussion reflects that propositions can be used in different ways depending on the context, or in other words, their properties 'shift'. Hinges shift from NEH, to EH to NH, and vice versa. Namely, they may shift from a truth-apt proposition or non-factual rule, which functions as a hinge about which we have an attitude of certainty (it is beyond justification and doubt), to a truth-apt proposition, which functions as a hinge about which we have an attitude of entitlement (it is non-paradigmatically justified and warranted), to a truth-apt proposition, which does not function as a hinge (it is paradigmatically justified and open to doubt), and vice versa. This also entails that an NEH can become epistemic via two pathways: by becoming an EH or by ceasing to be a hinge altogether (i.e., becoming an NH). Furthermore, as this case illustrates, the normative role of a proposition as an NEH may be different if used as an EH.

Overall, we have illustrated the epistemic and non-epistemic uses that one can make of mathematical propositions, drawing from and further building on Wittgensteinian considerations: 1) mathematical propositions may constitute petrified regularities that resist counterevidence, but also the petrified regularity may shatter if the counterevidence is contextually important; 2) mathematics are often used as rules of description, depending on the context; 3) there are both NEHs and EHs in mathematics; and 4) a given mathematical proposition can be used as more than just one type of hinge, or as an NH, on a contextual basis.

We see how mathematical practice admits choice in its development, specifically here as for whether some elements are used as EHs, NEHs, or NHs. However, there is no set rule on how we should proceed in every case about hinges; this is very contextual. We should factor in things like how central or valuable is the conjecture for our mathematics, whether discarding the mathematical proposition opens new interesting possibilities, whether using a proposition as a rule to increase coherence and consensus among practitioners, and similar. We think that Wittgenstein was aware of this indeterminacy looking at paragraphs like OC §139 or RFM VI §22.

Therefore, the context and the practice 'speaking for itself' fill the gaps of the rules. This indeterminacy may be bittersweet to the reader, who may hope for more determinate answers to questions like 'when should we use mathematical propositions epistemically (NE or EH) or non-epistemically (NEH)?'. We do not aim to provide categorical answers to this (neither did Wittgenstein): these questions cannot be answered categorically from outside practices themselves but find enacted resolutions in practice.

IV.2 Hinges of the working mathematician at the individual level

Here, we analyse the different roles of hinges in mathematical practice at the level of individual mathematicians. In doing this, we show that the contextual dynamics of mathematical hinges are not only at the level of community practices or the evolution of language games during often relatively long timescales (a phenomenon that Martin 2022 notes in his account of mathematical hinges). The key insight is that mathematical activities are sometimes associated to mental representations, but not always. Say you are doing Euclidean geometry. You could have a representation in your mind of how you are using a ruler and a compass and drawing on a sheet of paper. You can always resort to your intuitions about what is possible in this context. Alternatively, you can proceed strictly axiomatically and perform very distinct operations. If you do the latter, you often need to use more than one axiom for even the most basic operation, like halving an angle.

We argue that phases with mental representations and more explicit intuitions are phases with EHs and those without are phases with NEHs. Here, it is of crucial importance that a mental representation typically involves a link to the physical world. As argued for instance by Freudenthal (1991) and by Kant and Sarikaya (2021), advanced mathematics is often abstracted from more concrete mathematics. So, when a topologist uses their intuition about their favourite topological space (let it be R or an interesting topological space with uncommon properties), a mental representation is involved. On the other hand, formal phases involve more recourse on axioms as in the idealized sense of Hilbert. In these phases, more detached from the physical world, the axioms only fix the relation between primitive terms without adding to the semantics of the terms. Think, for instance, about the duality results in projective geometry where lines and points in many theorems can be interchanged systematically.

In formal phases questions like 'but why?', 'does that really hold?', and the like are forbidden. In other words, demands for justification and the possibilities for doubting are prohibited and we are encouraged to adopt a non-epistemic attitude of certainty. This is common in advanced mathematics contexts. Then there are more intuition-led phases where we work with paradigmatic examples, intuitions, and heuristics (which nonetheless still play normative/regulative roles in mathematical practices). These stages are in accordance with the introspection and self-reflection of mathematicians, including the Fields medallist Terence Tao, who explained this in an interview and a blogpost.[9] He demarcated three phases: a pre-formal, a formal, and a post-formal phase.

The pre-formal phase is one of learning, the formal is one where rules are followed very strictly and rigorously, and the post-formal phase is one where intuitions are more allowed. In the words of Tao:

1. The pre-rigorous stage, in which mathematics is taught in an informal, intuitive manner, based on examples, fuzzy notions, and hand-waving (for instance, calculus is usually first introduced in terms of slopes, areas, rates of change, and so forth). The emphasis is more on computation than on theory. This stage generally lasts until the early undergraduate years.
2. The rigorous stage, in which one is taught that to do mathematics 'properly', one needs to work and think in a much more precise and formal manner (e.g. re-doing calculus by using epsilons and deltas everywhere). The emphasis is now primarily on theory, and one is expected to be able to comfortably manipulate abstract mathematical objects without focusing too much on what such objects actually 'mean', i.e., without concrete mental representations of them. This stage usually occupies the later undergraduate and early graduate years.
3. The post-rigorous stage, in which one has grown comfortable with all the rigorous foundations of one's chosen field of mathematics and is now ready to revisit and refine one's pre-rigorous intuition on the subject, but this time with the intuition solidly buttressed by rigorous theory. For instance, in this stage one would be able to quickly and accurately perform computations in vector calculus by using analogies with scalar calculus, or informal and semi-rigorous use of infinitesimals, big-O notation, and so forth, and be able to convert all such calculations into a rigorous argument whenever required. The emphasis is now on applications, intuition, and the 'big picture'. This stage usually occupies the late graduate years and beyond.

This can be presented schematically in Fig. 3 (adapted from Heuer and Sarikaya (2020), changing the focus from rigour to formality).

[9] See a blog post (Tao 2009) and the recordings of an interview of Tao conducted by the creators of YouTube channel Numberfile2 in Tao (2017). Heuer and Sarikaya (2020) analyse the interview and extend the account from problem-solving to theory building.

Figure 3. Tao's model.

It is of course problematic to rely on just one account for the characterization of mathematical practices.[10] However, this account closely mirrors how mathematics is taught, the experience of other important mathematicians besides Tao, the general trend to focus on more than proofs, and the growing debate on the relation between formal and informal proofs (as argued next).

When we first learn that the logical implications of $A \to B$ should hold regardless of the content of premises, we must just accept it to do classical logic. We can give examples where everything works fine (if it rains, the street gets wet; it doesn't rain, yet the streets may or may not get wet) but at some point—in classical logic—we need to exempt this from any type of doubt and justification, thus making it an NEH. This challenges students who are not yet used to working in such a formal phase. Later, in the post-formal stage, we return to an epistemic phase, where previous NEHs can be EHs or NHs. This stage allows for many more modes of argumentation. For instance, we are allowed to have structures or properties in mind which are of our interest and which we want to isolate. For this, we are allowed to choose axioms (instead of axioms being presented to us arbitrarily) in a precise way justified inasmuch as their properties allow us to single out abstract structures of our interest. The scheme of these phases does not involve a linear progression, as we will sketch later: instead, mathematicians often shift between them. A key point in our argument is that a Wittgensteinian account of mathematical hinges that fits actual mathematical practice needs to feature both EHs and NEHs contextually.

Now let us go through the steps of the model proposed by Tao. The path from phase 1 to phase 2 (as well as from phase 2 to phase 3) is in a sense related to the problem of alignment between mathematicians. If the pre-formal phase fails to establish similar mathematical rule-following in all members of a student community, there can be misunderstandings. Say you ask a student to evaluate the term e^0, or the empty product P_0, let $a_1, a_2, a_3, \ldots$ be a sequence of numbers, and let

$$P_m = \prod_{1}^{m} a_i$$

i.e., the product of the first factors a_m.

What is P_0? Some students might be tempted to say $P_0 = 0$ and others to say $P_0 = 1$. Others may object more generally before responding and demand

[10]See Rittberg (2019), Rittberg and Van Kerkhove (2019), or Kant, Pérez-Escobar, and Sarikaya (2021).

m being at least 1. Typically, the mathematical community agrees to define $P_0 = 1$; this is the neutral element of multiplication and in typical practice it prevents annoying case distinctions. This can now be established as a non-factual rule of multiplication. Note that the establishment of such an NEH is a community phenomenon, as argued in the previous subsection. In this section, however, we discuss the perspective of an individual who is exposed to the current body of mathematics as it is. For them it is not allowed to discuss the appropriateness of these definitions when learning mathematics. Once a definition is agreed by a community or accepted by an individual, it is not up for discussion anymore. It becomes a fundamental presupposition of our practices on which large parts of our knowledge building depend. In Wittgensteinian terms: 'if I want the door to turn, the hinges must stay put' (OC: §343); these assumptions constitute the 'bedrock' (OC: §498) of our research and actions (OC: §§87–88) and our doubt and enquiry (OC: §151).

Once we become used to accepting definitions (and hopefully have a good teacher who renders it unproblematic to just go along with definitions as preparation for the third phase) and we learn to abstract knowledge from textbooks, we take definitions for what they are—just definitions. We analyse examples, test why they fall under the definition, and we even learn and embrace seemingly paradoxical results with great joy. Consider for instance the Banach–Tarski paradox: it is the theorem that you can take the sphere and partition it in a finite number of subsets in such a way that you can arrange them building up two spheres each of the exact size as the first one (i.e., doubling the volume). This sounds ridiculous because it goes against our pre-formal intuitions, but a mathematics student accepts it as meaning that not all subsets have a translation invariant size, i.e., some subsets get bigger while moving them in Euclidean space. Consider the many paradoxes of infinity as well, where a proper subset is of the same size as the whole set (say for instance the natural numbers and the even numbers), which we accept to do set theory.

Those NEHs are just to be accepted at 'rigorous', formal stages: one can do all kinds of performative acts to make this process easier but in the end they are not epistemically questionable or justifiable; they are things about which we are certain we are not making a mistake.

Another example is our attitude towards many paradoxes in mathematics. Let us consider measure theory. If we want to assign every subset of the Euclidean space a size, it is reasonable to assume that this measure does not change when we move this subset around the Euclidean space, or more technically, it is reasonable to define it as a property of any measure to be translation invariant. Note that the term 'reasonable' above is to be read as epistemic (it fits daily life intuitions of mathematicians and non-mathematicians) concerning the attitude of the community of mathematicians who make this a part of the theory of measures. On the other hand, we learn that by the axiom of choice we can divide the unit sphere into proper subsets, so that we can rearrange those subsets (by translations, i.e., shifts and rotations) in a way that we get two unit spheres. This so-called Banach–Tarski paradox is no strict paradox, but it challenges our most basic intuitions of size. We will physically not encounter this phenomenon, but on the level of formal axioms/definitions, i.e., on the level of our NEHs (i.e., in phase 2 of Tao's model, at the level of an individual working mathematician), the situation is different. Once we want to work with such measures of size, like the Lebesgue measure, this apparent paradox becomes the theorem that there are subsets of the Euclidean space, which are non-Lebesgue measurable. Similarly, we learn that changes on a Lebesgue zero set do not influence the integral of a function, or that possible events in a probability space (things that can happen) have probability 0. We must accept those statements while doing mathematics; anything else would be a mere change of subject or a reduction of scope (e.g. we do not reject Euclidean geometry when it does not work; instead, we resort to non-Euclidean geometry).

NEHs and EHs do not always need to be well defined, as we saw in the case of the initial implicit definition of polyhedron in Section IV.1. Take for instance the observation that '() ()' is no palindrome but '()) (' is one. This goes against a fuzzy intuition (instead of an internalized formal notion) about the symmetry of palindromes (like 'OTTO' and 'MOM') that regulates our practice while implicit in the background and using it as a shortcut.

As Tao explains, when we master mathematics (i.e., when we approach phase 3), we build up experience, learn 'shortcuts' after multiple experiences with similar cases, and so on. We get used to the objects (speaking metaphysically loosely) that we have so often encountered before in our mathematical doings that we develop intuitions about those things. This is a phase of great productivity as we have many tools at hand and we are allowed to use many modes of argumentation. Here, hinges constitute intuitions, methods, and tools that empower us to do mathematics and guide our mathematical practice. In this phase, there are many epistemic hinges: in many situations, we no longer must just accept, for instance, axioms, but have developed expert intuitions and (non-paradigmatic) justifications for the ones we end up using. We should also note here that, in this process, the intuitions can stop being hinges altogether: they can be challenged, which is exactly the point of Tao's model. He motivates the shift back to phase 2 precisely to check and test the intuitions that guided the work once problems and challenges arise. In this third phase, given the development of personal intuitions by the mathematician, the added complexity, and the increased number of modes of argumentation allowed, productivity increases but things can get out of control, leading to inconsistencies that are eventually not tolerated. This forces us from time to time to go back to stage 2 to reevaluate the EHs of phase 3 and insert new NEHs. This back and forth in the mathematical practice at the level of the individual mathematician provides further empirical support to our reading of the later Wittgenstein's philosophy of mathematics and hinge epistemology.

Finally, it is worth noting that the fact that different mathematicians find themselves in different phases where mathematical propositions may play different roles and support different intuitions may account for why, at the community level, there may be disagreements and hinge-based conflicts like the ones in Section IV.1. Future work will engage with this possibility in depth.

V. Conclusion

This work has shown that there are good reasons to adhere to a pluralist conception of mathematical hinges, both from an exegetical and practice-faithfulness standpoints. Not only are there mathematical hinges, but they can change their epistemic status contextually and even cease to be a hinge altogether, and eventually become a hinge again. This represents a contribution to hinge epistemology and the understanding of actual mathematical practices. Further work should assess whether other purported properties of hinges, for instance being truth-apt/non-truth-apt, doxastic/non-doxastic, and rational/non-rational, vary contextually in a similar manner. Furthermore, given that this work has focused on mathematical hinges, future work should assess the merits of this contextual approach for non-mathematical hinges.

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Conflict of interest statement

None declared.

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