Differential Graded Manifolds and n-Categories

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Differential graded manifolds and n-categories

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Le Doyen

N.B. - La thèse doit porter la déclaration précédente et remplir les conditions énumérées dans les "Informations relatives aux thèses de doctorat à l'Université de Genève".

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Résumé

Dans cette thèse on étudie des extensions du troisième théorème de Lie pour les structures de Lie supérieures appelées *algébroïdes* L_∞.

Dans la première partie de la thèse on rappelle la théorie de Lie pour les algèbres de Lie (après Cartan, Sullivan et Duistermaat–Kolk), pour les algébroïdes de Lie (après Crainic–Fernandes) et pour les algèbres L_∞ (après Getzler et Henriques). Entretemps on introduit le langage de groupoïdes de Lie supérieurs qui généralisent les groupes de Lie. On introduit aussi la notion d'algébroïde L_∞. Jusqu'alors aucun résultat général concernant l'intégration de ces structures supérieures n'existait dans la littérature.

La deuxième partie est une article de Ševera et l'auteur intitulé "Integration of differential graded manifolds", dont l'objectif est de fournir un tel résultat général.

On suit de près la méthode d'intégration suggérée par Sullivan pour les algèbres de Lie de dimension finie. Il a montré que le groupe simplement connexe correspondant (sans sa structure lisse) s'obtient comme le groupe fondamental simplicial de la réalisation spatiale $K_\bullet^{\mathrm{big}}$ de l'algèbre différentielle graduée commutative (adgc) de Chevalley–Eilenberg de l'algèbre de Lie $\mathfrak{g}$. Il y a des analogues des adgc de Chevalley–Eilenberg pour les structures de Lie supérieures - on peut les voir comme les adgcs des fonctions sur certaines variétés positivement graduées, équipées par une différentielle Q (un champs de vecteurs de degré 1 et carré zéro). Leurs réalisations spatiales correspondantes ont des homotopies supérieures intéressantes ; en particulier, leurs types d'homotopie sont des groupoïdes L_∞ locaux.

Le premier résultat de l'article dit que $K_\bullet^{\mathrm{big}}$ est en fait une variété de Fréchet simpliciale, qui satisfait la version lisse de la condition de Kan (définie par A. Henriques). Cette intégration est "grande" : les variétés des n-simplexes ainsi obtenues sont de dimension infini (déjà dans le cas des algèbres de Lie). La construction de $K_\bullet^{\mathrm{big}}$ est fonctorielle.

Le deuxième résultat dit, que, en imposant une certaine condition de jauge (suivant l'idée de E. Getzler de son travail sur le théorie de Lie pour les algèbres L_∞ nilpotentes), on obtient une sous-variété simpliciale $K_\bullet^s$ de $K_\bullet^{\mathrm{big}}$, qui est un groupoïde L_∞ local. De plus, il existe une déformation par rétraction simpliciale de $K_\bullet^{\mathrm{big}}$ sur $K_\bullet^s$.

Finalement, on montre, que la construction $K_\bullet^s$ est fonctorielle à une homotopie près. Si on choisit un bon recouvrement de la base de l'algebroïde L_∞ et applique $K_\bullet^s$ sur le nerf du recouvrement, on obtient une intégration globale par un diagramme homotopiquement cohérent dans la catégorie de variétés simpliciales locales.

Contents

Introduction

The topic of this Thesis can be summed up in a single sentence as *extending Lie's third theorem to the case of L_∞-algebroids.* Here L_∞-algebroids can be thought of as Lie algebroids with higher homotopies. This has been a long-sought goal vividly yet minimalistically sketched in Ševera [**32**] via the fundamental ∞-groupoid based on ideas of Sullivan [**35**] from Rational Homotopy Theory. However the analytical details were not worked there. Up until now the problem of Lie integration for the special case of Lie algebroids was solved by Crainic, Fernandes [**7**] and the case of L_∞-algebras was solved by Getzler [**16**] and Henriques [**20**] (by independent methods). Apart from some partial cases (e.g. exact Courant algebroids which are Lie 2-algebroids with additional structure) there was no general result on integration of higher homotopy analogues of Lie algebroids. This problem was solved in Ševera, Širaň [**34**].

The first part of the Thesis is a survey of Lie integration techniques. The following Sections will gradually recall Lie theory for Lie structures, starting from the easiest case of Lie algebras building up all the way to the most general case of L_∞-algebroids. Along the way we offer reminders on the various higher geometrical and categorical structures involved. The survey should not be considered exhaustive; sacrifices were made to enhance readability. There are almost no proofs - the interested reader can find them in the references. The goal is to connect the methods and results of Ševera, Širaň [**34**] to the larger web of Lie theory which preceeded it and made it possible.

The second part of the Thesis is the article Ševera, Širaň [**34**]. The numbering of equations, figures, propositions and theorems is independent between the first and the second part of the Thesis.

Some remarks to notation follow. We will use abbreviations “gca” for graded commutative algebra and “dgca” for differential graded commutative algebra. Categories are denoted using $\mathcal{C}, \mathcal{D}$ or in bold for concrete categories as $\mathbf{Set}, \mathbf{Man}$ etc. Einstein summation convention is used (sum over repeated indices).

Part 1

A survey of Lie integration

1. Lie theory for Lie algebras

1.1. Modern approach. Let G be a (finite-dimensional) Lie group. Its tangent space at the unit element is naturally a (finite-dimensional) Lie algebra with the Lie bracket induced from the Lie algebra of left-invariant vector fields on G. This construction is functorial and we will write it correspondingly as

$$\mathrm{Lie} : \mathbf{LGrp} \to \mathbf{LAlg}. \tag{1}$$

A natural question is if given a Lie algebra $\mathfrak{g}$ there exists a Lie group G such that $\mathrm{Lie}(G) \cong \mathfrak{g}$, i.e. whether the functor (1) is essentially surjective. It is a classic result of Lie theory under the name *Lie's third theorem* that this is indeed the case, although the first full proof was given by Cartan (the ascription to Lie seems to be an influence of Serre's book [**31**]). Restricting to the category of connected, simply-connected Lie groups (1) becomes an equivalence of categories.

What Lie actually proved is a weaker result which can be expressed in modern language as existence of a local Lie group G integrating $\mathfrak{g}$. Today both results have standard proofs using high-powered techniques. For example, for local integration one can use the Baker-Campbell-Hausdorff formula

$$BCH(X,Y) := \log\left(\exp(X)\exp(Y)\right) \tag{2}$$

which can be famously written using only the Lie bracket. Since it converges in an open neighborhood $\mathfrak{g}_{\mathrm{BCH}} \subset \mathfrak{g}$ of the origin it makes $\mathfrak{g}_{\mathrm{BCH}}$ into a local Lie group. It is then possible to glue together such neighborhoods, see e.g. Serre's book [**31**] (he calls these open neighborhoods group chunks) to turn the method into global integration. The proof that the BCH formula methods work also globally is involved but the description is quite simple. Define

$$G_{BCH} := \left\{ \prod_{i=1}^{n} \exp(X_i), n \text{ finite}, X_i \in \mathfrak{g}_{\mathrm{BCH}} \right\},$$

where the products are formal. Then there is an equivalence relation $\sim$ on G_{BCH} such that

$$G := G_{BCH}/\sim$$

is the simply-connected Lie group integrating $\mathfrak{g}$. The equivalence relation is given by the BCH formula (2).

Another standard proof of the global version of Lie's third theorem involves Ado's Theorem, a deep result which states that all finite-dimensional Lie algebras have faithful finite-dimensional representations. Thus we can treat a finite-dimensional Lie algebra $\mathfrak{g}$ as a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ for some n. Since $\mathfrak{gl}_n(\mathbb{R})$ integrates to the Lie group $\mathrm{GL}_n(\mathbb{R})$ the integrating group of $\mathfrak{g}$ is generated by all exponentials of $\mathfrak{g}$ in $\mathfrak{gl}_n(\mathbb{R})$ and it is a Lie group by Frobenius theorem.

1.2. Cartan's method and the Maurer-Cartan equation. The above techniques work well for Lie algebras but they do not seem to have generalizations to higher Lie theory. It is instructory to recall Cartan's original method, which makes the Maurer-Cartan equation central. We follow the expositions of van Est [**36**] and Ebert [**14**].

A *transitive Maurer-Cartan* $\mathfrak{g}$*-structure* on M is a choice of a subspace $F \subset \Gamma(TM)$ such that $[F,F] \subset F$, $F \cong \mathfrak{g}$, the evaluation map $F \times M \to TM$ is an isomorphism of vector bundles and the group

$$G_F := \{ f : M \to M, f \text{ is a diffeomorphism}, \ f_* X = X \text{ for all } X \in F \}$$

acts transitively. The definition is motivated by rewriting of the properties of the subspace of left-invariant vector fields on a Lie group G. The key idea is that $G_F \cong M$ then inherits a smooth structure from M such that it becomes a Lie group.

The involution condition $[F,F] \subset F$ can be stated as a certain condition in the de Rham dgca $\Omega(M)$. A differential form is called F-constant iff it defines a constant function on vector fields in F. These F-constant forms form a graded subalgebra of $\Omega(M)$ which can be identified with $\wedge F^*$. This subalgebra is closed w.r.t. the de Rham differential on $\Omega(M)$ iff F is involutive. The de Rham differential reduces for $\alpha \in \wedge F^*$ to

$$d\alpha(X_0, \ldots, X_n) := \sum_{i<j} (-1)^{i+j} \alpha\left([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_n \right)$$

where $X_0, \dots, X_n \in F$. This is precisely the form of the differential on the *Chevalley-Eilenberg dgca of* $\mathfrak{g}$, denoted

$$(C(\mathfrak{g}), d_{CE}) := (\wedge \mathfrak{g}^*, d_{CE})\,.$$

Its cohomology is the Lie algebra cohomology. Thus one can rephrase the definition of a transitive Maurer-Cartan $\mathfrak{g}$-structure on M as a morphism of dg algebras

$$A_{\mathfrak{g},M} : (C(\mathfrak{g}), d_{CE}) \to (\Omega(M), d) \tag{3}$$

such that the evaluation map $\mathfrak{g}^* \times M \to T^*M$ is a vector bundle isomorphism. To integrate a Lie algebra $\mathfrak{g}$ it is thus neccesary to find a transitive Maurer-Cartan $\mathfrak{g}$-structure on some connected manifold M. Since (3) as a morphism of graded algebras is determined by its restriction to $\mathfrak{g}^* \to \Omega^1(M)$ it can be encoded as $A \in \Omega^1(M, \mathfrak{g})$. The condition on (3) to be a morphism of dgca is then the *Maurer-Cartan equation*

$$dA + \frac{1}{2}[A, A] = 0. \tag{4}$$

These observations have generalizations in higher Lie theory where an analogue of (4) appears.

It is clear that a Lie algebra with trivial center is integrable, since it is faithfully represented by the adjoint action.

Cartan developed a method to obtain a transitive Maurer-Cartan $\mathfrak{g}$-structure $A_{\mathfrak{g},M\times\mathbb{R}}$ on $M \times \mathbb{R}$ from a transitive Maurer-Cartan $\mathfrak{h}$-structure $A_{\mathfrak{h},M}$ on M with $H^2(M, \mathbb{R}) = 0$ where

$$0 \to \mathfrak{c} \hookrightarrow \mathfrak{g} \xrightarrow{p} \mathfrak{h} \to 0$$

is a short exact sequence of Lie algebras and $\mathfrak{c}$ is central in $\mathfrak{g}$ and one-dimensional. Suppose $A_{\mathfrak{h},M}$ is a transitive Maurer-Cartan $\mathfrak{h}$-structure on M. Pick a basis $\{e_1, \dots, e_n\}$ of $\mathfrak{g}$ with dual basis $\{e^1, \dots, e^n\}$ of $\mathfrak{g}^*$ such that $\{e^2, \dots, e^n\}$ are pullbacks via p of a basis $\{f^2, \dots, f^n\}$ of $\mathfrak{h}^*$ and the restriction of e^1 to $\mathfrak{c}$ is non-zero. Then

$$de^1 = -\frac{1}{2} c^1_{ij} e^i \wedge e^j$$

where $i, j = 2, \dots n$ as $\mathfrak{c}$ is central in $\mathfrak{g}$. Thus one can define a unique 2-cocycle $a \in \wedge^2 \mathfrak{h}^*$ by the equation

$$de^1 = p^* a.$$

Applying the transitive Maurer-Cartan $\mathfrak{h}$-structure on M we get $A_{\mathfrak{h},M}(a) \in \Omega^2(M)$. By the assumption $H^2(M, \mathbb{R}) = 0$ there exists $b \in \Omega^1(M)$ such that $a = db$. Let π be the projection

$$\pi : M \times \mathbb{R} \to M$$

and t the coordinate on $\mathbb{R}$. Cartan proved that the map

$$A_{\mathfrak{g},M\times\mathbb{R}} : \mathfrak{g}^* \to \Omega^1(M \times \mathbb{R}),$$
$$A_{\mathfrak{g},M\times\mathbb{R}}(e^i) := \begin{cases} \pi^* b + dt & i = 1, \\ \pi^*(A_{\mathfrak{g},M}(f^i)) & i > 1. \end{cases}$$

satisfies (4) and thus defines a transitive Maurer-Cartan $\mathfrak{g}$-structure on $M \times \mathbb{R}$.

Thus the global version of Lie's Third Theorem follows from setting $M := H$ where H is the simply-connected Lie group integrating $\mathfrak{h} := \mathrm{ad}\ \mathfrak{g}$ and then proceeding by induction on the dimension of the center. The crucial and non-trivial part here is that

$$H^2(H, \mathbb{R}) = 0 \tag{5}$$

since H is a simply-connected Lie group (a proof of this can be found e.g. in Duistermaat, Kolk [**10**]). In fact the much more stronger $\pi_2(H) = 0$ is also true.

1.3. A non-integrable Banach-Lie algebra. The failure of the condition (5) for Banach-Lie groups is tied to non-integrability of Banach-Lie algebras. The following non-integrable Banach-Lie algebra was found by van Est, Korthagen [**37**]. Let

$$\mathfrak{h} := \left\{ \alpha : I \to \mathfrak{su}(2), \int_I \alpha(t) dt = 0 \right\}$$

be a Lie algebra with the pointwise Lie bracket. There is a skew-symmetric bilinear form on $\mathfrak{h}$

$$\langle \alpha, \beta \rangle := \int_I \mathrm{tr} \left(\int_{[0,t]} \alpha(s) ds \circ \beta(t) \right) dt. \tag{6}$$

The central extension of $\mathfrak{h}$ by $\mathbb{R}$

$$0 \to \mathbb{R} \to \mathfrak{g} \to \mathfrak{h} \to 0$$

w.r.t. the cocycle $\langle \cdot, \cdot \rangle$ (6) is a Banach-Lie algebra but there is no integrating Banach-Lie group.

1.4. A reminder on simplicial sets. Before we proceed to Sullivan's method [**35**] of integration we need to recall the language of simplicial sets (a generalization of the notion of a simplicial complex) as combinatorial models for the weak homotopy type of topological spaces. We will also require them in higher Lie integration, since the higher Lie groupoids are defined in terms of their nerves. Standard references to simplicial sets include May [**26**] and Goerss, Jardine [**17**].

Let us recall that the small simplex category Δ is the full subcategory of the category of small categories **Cat** given by free categories of finite linear directed graphs. It is also possible to think of Δ as the category with finite totally ordered sets as objects and order-preserving maps as morphisms. We will denote $[n]$ as the object of Δ given by the graph

$$[n] := \{0 \to 1 \to \ldots \to n\}.$$

Morphisms of Δ are generated by certain elementary maps, called the coface and codegeneracy maps denoted $\delta_i : [n-1] \to [n]$ and $\sigma_i : [n+1] \to [n]$, $0 \leq i \leq n$. These maps are defined as

$$\delta_i(k) := \begin{cases} k & k < i, \\ k+1 & k \geq i, \end{cases}$$
$$\sigma_i(k) := \begin{cases} k & k \leq i, \\ k-1 & k > i. \end{cases}$$

These maps satisfy the cosimplicial identities which are dual to (7) below.

Let $\mathcal{C}$ be a category. A simplicial object $K_\bullet$ in $\mathcal{C}$ is a presheaf on Δ with values in $\mathcal{C}$, i. e. a functor

$$K_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}.$$

In particular if $\mathcal{C}$ is the category of sets, $K_\bullet$ is called a simplicial set. Unpacking this definition we see that a simplicial set is a sequence of sets $\{K_n\}_{n\geq 0}$ where $K_n := K_\bullet([n])$ together with a collection of maps

$$d_i := K_\bullet(\delta_i) : K_n \to K_{n-1}$$

called the face maps and

$$s_j := K_\bullet(\sigma_j) : K_n \to K_{n+1}$$

called the degeneracy maps. These satisfy the simplicial identities

$$\begin{aligned} d_i d_j &= d_{j-1} d_i & i < j, \\ s_i s_j &= s_j s_{i-1} & i \leq j, \\ d_i s_j &= \begin{cases} s_{j-1} d_i & i < j, \\ \mathrm{id} & i = j, j+1, \\ s_j d_{i-1} & i > j+1. \end{cases} \end{aligned} \tag{7}$$

The simplicial n-simplex or the standard n-simplex $\Delta[n]$ is the simplicial set defined as the image of $[n]$ via the Yoneda embedding

$$\Delta[n] := \mathrm{Hom}_\Delta(\cdot, [n]).$$

For any simplicial set $K_\bullet$ we have by the Yoneda lemma

$$K_n = \mathrm{Hom}_{\mathbf{sSet}}(\Delta[n], K_\bullet).$$

The geometric realization of $\Delta[n]$ is the geometric n-simplex Δ^n defined as the topological subspace $\Delta^n \subset \mathbb{R}^{n+1}$

$$\Delta^n := \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1}, 0 \leq t_i \leq 1, \sum_{i=0}^n t_i = 1 \right\}.$$

Often we will consider Δ^n as a smooth manifold with corners.

The above construction extends to a functor

$$\begin{gathered} |\cdot| : \mathbf{sSet} \to \mathbf{Top} \\ |K_\bullet| := \left(\textstyle\bigsqcup_{n\geq 0} K_n \times \Delta^n \right) / \sim \end{gathered} \tag{8}$$

where $\sim$ is an equivalence relation $(x,p) \sim (y,q)$ iff

$$d_i x = y, \qquad \delta^i q = p,$$

or

$$s_j x = y, \qquad \sigma^j q = p.$$

Each K_n is given the discrete topology and $|K_\bullet|$ is given the quotient topology. The functor (8) is called the geometric realization. It is left adjoint to the singular complex functor

$$S_\bullet : \mathbf{Top} \to \mathbf{sSet}$$

which assigns to a topological space X its singular complex

$$S_\bullet(X) := \mathrm{Hom}_{\mathbf{Top}}(\Delta^\bullet, X).$$

1.5. Spatial realisation of a dgca. The geometric realization $|K_\bullet|$ is in general (unlike $\Delta^\bullet$) not a smooth manifold. However one can still work with forms defined on each simplex of $|K_\bullet|$ which are glued compatibly along the boundaries. There are several types of forms on $\Delta^\bullet$ which can be used, in particular: polynomial forms (for rational homotopy theory) and smooth forms (which is what we will use in Lie theory). We will denote $\Omega(\Delta^\bullet)$ the resulting simplicial dgca. There are two properties we require from $\Omega(\Delta^\bullet)$. The first is a sort of Poincaré lemma (vanishing of cohomology) and the second is the extension property (one can extend a form from the boundary of a simplex). These properties are satisfied for both choices of forms (for the extension property see Sullivan [**35**]). These properties are chosen so that there is an analogy of the de Rham theorem.

In this way Sullivan constructed a functor

$$\begin{gathered}\Omega : \mathbf{sSet} \to \mathbf{dgca}^{\mathrm{op}} \\ \Omega(K_\bullet) := \mathrm{Hom}_{\mathbf{sSet}}(K_\bullet, \Omega(\Delta^\bullet))\end{gathered} \tag{9}$$

which assigns to a simplicial set $K_\bullet$ a dgca $\Omega(K_\bullet)$ which can be thought of as its de Rham algebra. This functor is left adjoin to the *spatial realisation functor*

$$\begin{gathered}K_\bullet : \mathbf{dgca}^{\mathrm{op}} \to \mathbf{sSet} \\ K_\bullet(\mathcal{A}) := \mathrm{Hom}_{\mathbf{dgca}}(\mathcal{A}, \Omega(\Delta^\bullet)).\end{gathered} \tag{10}$$

1.6. Sullivan's method of integration. In the 1970s a new method of integration was suggested by Sullivan [**35**]. We need to stress that the result is the integrating simply-connected group without its smooth structure (this can be then remedied via a method of Duistermaat, Kolk [**10**], see below). The method makes use of the Maurer-Cartan equation (4), but does not solve it; instead the technique could be loosely described as integration without integration.

Let $\mathfrak{g}$ be a Lie algebra and $(C(\mathfrak{g}), d_{CE})$ its Chevalley-Eilenberg dgca. Its (smooth) spatial realisation is a simplicial set (in fact a simplicial manifold)

$$K_\bullet^{\mathrm{big}} := K_\bullet(\mathfrak{g}) = \mathrm{Hom}_{\mathbf{dgca}}(C(\mathfrak{g}), \Omega(\Delta^\bullet)). \tag{11}$$

The n-simplices of $K_\bullet^{\mathrm{big}}$ can be identified with elements $A \in \Omega^1(\Delta^n, \mathfrak{g})$ satisfying the Maurer-Cartan equation (4). In terms of the integrating simply-connected Lie group G of $\mathfrak{g}$ we can consider K_n^{big} as the space of flat connections on the trivial bundle $\Delta^n \times G \to \Delta^n$. Then by the identification of the space of flat connections with the space of horizontal sections we see that K_n^{big} is the space of maps $\Delta^n \to G$ modulo translation

$$K_n^{\mathrm{big}} := \mathrm{Hom}_{\mathbf{Man}}(\Delta^n, G)/G. \tag{12}$$

Let us describe the simplicial fundamental group of $K_\bullet^{\mathrm{big}}$ which is the integrating simply-connected group of $\mathfrak{g}$. Its elements are 1-simplices of $K_\bullet^{\mathrm{big}}$ which are (automatically flat) connections $A \in \Omega^1(\Delta^1, \mathfrak{g})$ modulo the equivalence relation $A \sim A'$ iff there exists a flat connection $A_{\ominus}$ on a bigon which restricts on the boundary semicircles to A resp. A' as can be seen on the left side of Figure 1.

Let us denote the equivalence class of A as g_A. Multiplication is given by $g_{A_1} g_{A_2} = g_{A_3}$ iff there exists a flat connection $A_\triangle$ which restricts on the boundary as illustrated on right side of Figure 1.

The reason we put the description big in $K_\bullet^{\mathrm{big}}$ is that K_n^{big} are infinite-dimensional manifolds. For any $g \in G$ there are infinitely many representing flat connections A on Δ^1. The same is true of the flat connections $A_\triangle$ filling the right side of Figure 1.

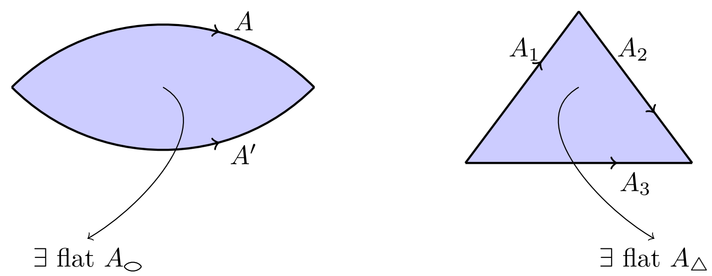

FIGURE 1. Left - two connections give the same group element. Right - group multiplication illustrated.

1.7. Global integration by Duistermaat, Kolk. Duistermaat, Kolk [**10**] worked out a way how to obtain the integrating simply-connected Lie group (with its smooth structure) using the path method. Let $P(1, G)$ be the based path space of a Lie group G and $P(\mathfrak{g})$ the path space of its corresponding Lie algebra $\mathfrak{g}$. The map

$$\begin{gathered} D : P(1, G) \cap C^1 \to P(\mathfrak{g}) \\ D\gamma(t) := \gamma(t)^{-1} \left(\tfrac{d\gamma}{dt}(t) \right) \end{gathered} \tag{13}$$

is a homeomorphism and it satisfies

$$D(\gamma.\gamma')(t) = D\gamma(t) + (\mathrm{Ad}\ \gamma(t))(\gamma'(t)).$$

This can be used to define a product on $P(\mathfrak{g})$. Now the adjoint action is defined in terms of G, but it can be reformulated (for adjoint action of $\gamma(t)$) purely in terms of the Lie algebra $\mathfrak{g}$. In particular

$$A_{D\gamma}(t) := \mathrm{Ad}\ \gamma(t)$$

is a C^1 solution $A_{D\gamma}(t) : I \to GL(\mathfrak{g})$ of the following ODE

$$\begin{cases} \frac{d}{dt} A(t) = \mathrm{ad}\ D\gamma(t) \circ A(t) \\ A(0) = \mathrm{id} \end{cases} \tag{14}$$

Since D is a homeomorphism one can use this to define $\mathrm{Ad}_{\gamma(t)}$ using the corresponding path $D\gamma(t)$ without any reference to the Lie group G. It is straightforward to verify associativity of the product

$$\delta.\delta'(t) = \delta(t) + A_\delta(t)(\delta'(t)) \tag{15}$$

defined for any $\delta, \delta' \in P(\mathfrak{g})$. Furthermore the constant path at $0 \in \mathfrak{g}$ and the inverse

$$\gamma^{-1}(t) = -(A_\gamma(t))^{-1}(\gamma(t))$$

define a group structure on $P(\mathfrak{g})$. Unless $\mathfrak{g}$ is trivial $P(\mathfrak{g})$ is an infinite-dimensional Banach space. By linearity of ad and Equation 14 it follows that the map

$$\begin{gathered} A : P(\mathfrak{g}) \to C^1(I, GL(\mathfrak{g})) \\ A(\delta) := A_\delta(t) \end{gathered}$$

is an analytic map so that $P(\mathfrak{g})$ is a Banach-Lie group. Remarkably there is a closed normal Banach-Lie subgroup $P(\mathfrak{g})_0$ such that $P(\mathfrak{g})/P(\mathfrak{g})_0$ is the integrating 1-connected Lie group of $\mathfrak{g}$. This requires some care to show and we will be only sketchy. There are several ways to describe $P(\mathfrak{g})_0$. The obvious way is to consider the image of D for paths in G which all the paths are actually loops (based at 1) that is

$$P(\mathfrak{g})_0 := D(\Omega(1, G) \cap C^1).$$

Again it is necessary to write this intrinsically in terms of $\mathfrak{g}$. It can be shown that $\delta \in P(\mathfrak{g})_0$ iff there is a smooth path δ_s in $P(\mathfrak{g})$ from the constant path 0 to δ with the integral condition

$$\int_0^1 A_{\delta_s}(t)^{-1} \frac{\partial}{\partial s} \delta_s(t) dt = 0$$

for all $s \in I$. That $P(\mathfrak{g})_0$ is a normal subgroup is easy to verify. The difficult part is that it is a closed Banach-Lie subgroup of $P(\mathfrak{g})$ and this is a consequence of the non-trivial result that the second homotopy group of a 1-connected Lie group vanishes. To verify that the Lie algebra of $P(\mathfrak{g})/P(\mathfrak{g})_0$ is $\mathfrak{g}$ it is useful to write down the Lie bracket of the Lie algebra $P(\mathfrak{g})$ as

$$[X,Y](t) = \frac{d}{dt}[\mathrm{av}_t X, \mathrm{av}_t Y]$$

where $\mathrm{av}_t : P(\mathfrak{g}) \to \mathfrak{g}$, $t \in I$ is the averaging map

$$\mathrm{av}_t(X) := \int_0^t X(s)ds.$$

The Lie algebra of $P(\mathfrak{g})_0$ is then the kernel of av_1. Since av_1 is surjective (taking constant paths at the desired $X \in \mathfrak{g}$) this proves that $P(\mathfrak{g})/P(\mathfrak{g})_0$ is the sought Lie group.

2. Lie theory for Lie algebroids

2.1. A reminder on Lie groupoids. A group can be viewed as a category with a single object with the group elements being the morphisms of this category. A groupoid is a generalization of this notion - a small category with all morphisms being invertible. A Lie groupoid is a groupoid internal to the category of smooth manifolds.

Unpacking this definition a Lie groupoid $\mathcal{G} \rightrightarrows M$ is given by two smooth manifolds $\mathcal{G}$ (the possibly non-Hausdorff manifold of morphisms) and M (the manifold of objects) with surjective submersions $s, t : \mathcal{G} \to M$ (the source and target maps), a smooth map $u : M \to \mathcal{G}$ (the unit map), a smooth map $i : \mathcal{G} \to \mathcal{G}$ (the inversion map) and a smooth map $m : \mathcal{G}_s \times_t \mathcal{G} \to \mathcal{G}$ (the multiplication map). These have to satisfy the usual categorical axioms. Note that s, t are submersion so that the pullback $\mathcal{G}_s \times_t \mathcal{G}$ is a smooth manifold. Furthermore M is often treated as an embedded submanifold of $\mathcal{G}$ via the unit map. A Lie groupoid is said to be s-simply-connected (source simply-connected) iff the source fibers $s^{-1}(x)$ are simply-connected for all $x \in M$.

The isotropy group at $x \in M$ of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is the Lie group

$$\mathcal{G}_x := s^{-1}(x) \cap t^{-1}(x). \tag{16}$$

Lie groupoids were first studied by Ehresmann [**15**]. A good reference for the theory of Lie groupoids is Mackenzie [**25**].

2.2. A reminder on Lie algebroids. The corresponding infinitesimal notion to a Lie groupoid is a Lie algebroid which was introduced by Pradines [**28**]. The relationship between Lie groupoids and Lie algebroids mimicks the relationship between Lie groups and Lie algebras.

Recall that a Lie algebroid A is a vector bundle $A \to M$ together with a Lie bracket on $\Gamma(A)$ and a morphism of vector bundles $\rho : A \to TM$ called the anchor satisfying the Leibniz rule

$$[\alpha, f\beta] = f[\alpha, \beta] + \left(\mathcal{L}_{\rho(\alpha)} f\right) \beta \tag{17}$$

for any $\alpha, \beta \in \Gamma(A)$ and any $f \in C^\infty(M)$. The anchor ρ induces a Lie algebra morphism $\Gamma(A) \to \Gamma(TM)$ which is by abuse of notation also denoted ρ.

Any Lie groupoid can be linearized to a Lie algebroid following the idea from the Lie group case. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and denote by $T^s\mathcal{G}$ the source tangent bundle $T^s\mathcal{G} := \ker ds$. As usual we consider M embedded in in $\mathcal{G}$ using the unit map. The vector bundle $A := T^s\mathcal{G}|_M$ can be given the structure of a Lie algebroid as follows. There is an isomorphism between the vector space $\Gamma(A)$ and the vector space of right-invariant source-tangent vector fields on $\mathcal{G}$ since any $\alpha \in \Gamma(A)$ can be extended to a vector field $\alpha \in \Gamma(T^s\mathcal{G})$ by the formula

$$\alpha_f := R_f(\alpha_{t(f)})$$

(this is analogous to the Lie group case). This allows to define the Lie bracket on $\Gamma(A)$ using the corresponding Lie bracket of the Lie subalgebra of right-invariant vector fields in $\Gamma(T^s\mathcal{G})$. The anchor for A is given by $\rho := dt|_A$.

The isotropy Lie algebra at $x \in M$ of a Lie algebroid $A \to M$ is the Lie algebra $\mathfrak{g}_x := \ker \rho_x$. If $\mathcal{G} \rightrightarrows M$ is a s-simply-connected Lie groupoid with the Lie algebroid $A \to M$ then the Lie algebra of the isotropy Lie group at $x \in M$ given by (16) is isomorphic to $\mathfrak{g}_x$.

The Chevalley-Eilenberg dgca of a Lie algebroid $A \to M$ is the dgca

$$(C(A), d_{CE}) := (\Gamma(\wedge A^*), d_{CE})$$

where the tensor product and dualization is w.r.t. $C^\infty(M)$. The differential is given by a formula analogous to the Lie algebra case

$$\begin{aligned} d_{CE}\alpha(X_0, \ldots, X_n) := \sum_i (-1)^i \rho(X_i) \left(\alpha(X_0, \ldots, \hat{X}_i, \ldots, X_n)\right) \\ + \sum_{i<j} (-1)^{i+j+1} \alpha\left([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_n\right) \end{aligned}$$

for $X_0, \ldots, X_n \in \Gamma(A)$. Note that $(C(A), d_{CE})$ is an analogue of the de Rham dgca which is just the Chevalley-Eilenberg dgca of the standard Lie algebroid TM.

To define the category of Lie algebroids we need to define morphisms of Lie algebroids. Let $A_1 \to M_1$ and $A_2 \to M_2$ be two Lie algebroids then a Lie algebroid morphisms should be a vector bundle morphism

$$\begin{tikzcd} A_1 \arrow[r, "F"] \arrow[d] & A_2 \arrow[d] \\ M_1 \arrow[r, "f"] & M_2 \end{tikzcd} \tag{18}$$

compatible with the Lie brackets and anchors. The anchor compatibility is just the commutativity of the diagram

$$\begin{tikzcd} A_1 \arrow[r, "F"] \arrow[d, "\rho_1"'] & A_2 \arrow[d, "\rho_2"] \\ TM_1 \arrow[r, "Tf"] & TM_2 \end{tikzcd}$$

The compatibility for the Lie brackets is not difficult to write down, but we will instead describe it using two alternative approaches more suited for Lie theory. First, we can demand that the morphism (18) induces a dgca morphism of the Chevalley-Eilenberg dgcas. The second, equivalent, possibility is via connections on Lie algebroids. Let ∇ be a bilinear map

$$\nabla : \Gamma(A_2) \times \Gamma(A_2) \to \Gamma(A_2),$$
$$\nabla(\alpha, \beta) =: \nabla_\alpha \beta$$

which is $C^\infty(M_2)$-linear in the first argument and satisfies the Leibniz rule

$$\nabla_\alpha(g\beta) = g\nabla_\alpha\beta + \mathcal{L}_{\rho(\alpha)} g\beta$$

where $g \in C^\infty(M_2)$. This is a straightforward generalization of the notion of a connection to the setting of Lie algebroids. Now one can define via pullback a connection on f^*A_2 also denoted ∇. We can then calculate the curvature

$$R \in \Gamma\left(\wedge^2 A_1^* \otimes f^* A_2\right)$$
$$R_F(\alpha, \beta) := \nabla_\alpha(F(\beta)) - \nabla_\beta(F(\alpha)) - F([\alpha, \beta]) - T_\nabla(F(\alpha), F(\beta))$$

where T_∇ is the torsion. Then (18) is a Lie algebroid morphism iff $R_F = 0$ which can in turn be written as a Maurer-Cartan equation

$$d_\nabla(F) + \frac{1}{2}[F, F]_\nabla = 0.$$

In particular this reduces to the usual Maurer-Cartan equation if $A_1 := TM$ and $A_2 := \mathfrak{g}$. More details can be found in Crainic, Fernandes [**8**].

The special case of a Lie algebroid morphism $TI \to A$ is called an A-path. It is a pair (a, γ), where $a : I \to A$ is a path over $\gamma : I \to M$ such that

$$\rho(a(t)) = \frac{d\gamma}{dt}(t)$$

for all $t \in I$. The image of the anchor defines a (possibly singular) distribution

$$D_x := \mathrm{Im}\ \rho_x \subset T_x M$$

where $x \in M$. This in turn leads to a (possibly singular) foliation of M. The leaf of this foliation contating $x \in M$ is called the orbit of x and is denoted $\mathcal{O}_x$. The orbit of x can be also defined as the immersed submanifold consisting of all points $y \in M$ such that there exists an A-path covering a path $I \to M$ connecting x and y.

If $A \to M$ is a Lie algebroid and $\mathcal{O}_x$ is the orbit of $x \in M$, there is a Lie algebroid $\mathfrak{g}_{\mathcal{O}_x} \to \mathcal{O}_x$ given by

$$\mathfrak{g}_{\mathcal{O}_x} := \ker\ \rho|_{\mathcal{O}_x}$$

with the Lie bracket given fibrewise by the isotropy Lie algebras and trivial anchor. This is a particular type of a Lie algebroid called a bundle of Lie algebras.

2.3. Integration of Lie algebroids after Sullivan and Ševera. In Ševera [**32**], Sullivan's method of integration of Lie algebras was proposed to extend to the case of Lie algebroids (noting that the result might fail to be smooth). Let $A \to M$ be a Lie algebroid. Consider the spatial realisation of its Chevalley-Eilenberg dgca, the simplicial set (actually a Kan Fréchet manifold as will be shown later in higher Lie theory)

$$K_\bullet^{\mathrm{big}} := \mathrm{Hom}_{\mathbf{dgca}}\left((C(A), d_{CE}), (\Omega(\Delta^\bullet), d)\right).$$

Now instead of constructing the simplicial fundamental group we need to construct the simplicial fundamental groupoid as its integrating groupoid (again without its smooth structure). Let us phrase this in a more pedestrian terms. Suppose that the Lie algebroid $A \to M$ is associated with a Lie groupoid $\mathcal{G} \rightrightarrows M$. We have seen that A can be though of as the manifold of infinitesimal morphisms of $\mathcal{G}$. Thus to integrate A to $\mathcal{G}$ one needs to compose these infinitesimal morphisms along curves, which amounts to taking the set of all Lie algebroid morphisms $TI \to A$. This coincides with Sullivan's approach if $A = \mathfrak{g}$ for some finite dimensional Lie algebra $\mathfrak{g}$ as $\mathfrak{g}$-paths $TI \to \mathfrak{g}$ are $\mathfrak{g}$-connections on I. Taking the lead from this special case, for general A one introduces the equivalence relation $\gamma_1 \sim \gamma_2$ between two A-paths $a_1, a_2 : TI \to A$ iff there exists an A-homotopy between them (a morphism of Lie algebroids $a_{\circ} : T\circ \to A$, where $\circ$ is a bigon, which restricts on the semicircles to γ_1 and γ_2, here we are glossing over regularity issues). This can again be illustrated nicely by pictures, see left side of Figure 2.

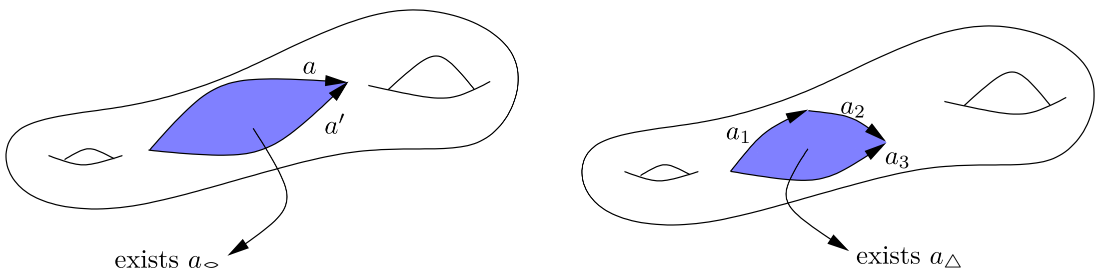

FIGURE 2. Left - two A-paths give the same groupoid element. Right - groupoid multiplication illustrated.

The groupoid multiplication is defined by concatenation of paths (again there are regularity issues, this is discussed below in the method of Crainic, Fernandes [**7**]) modulo the A-homotopy, i.e. the concatenation of two A-paths $a_1, a_2 : TI \to A$ will give the same groupoid element as the A-path $a_3 : TI \to A$ iff there exists a Lie algebroid morphism $a_{\triangle} : T\triangle \to A$ which restricts in on the boundary to the respective A-paths, see right side of Figure 2.

The integrating groupoid G might fail to be a smooth manifold. This is because the equivalence classes of $\sim$ might fail to be closed (this happens if the A-homotopies connecting close paths fail to be small). This is a phenomenon specific to Lie algebroids and also their higher homotopy analogues; the global version of Lie's third theorem is false for these Lie structures.

2.4. A non-integrable Lie algebroid. Pradines [**29**] incorrectly claimed that the global version of Lie's third theorem is valid for Lie algebroids. The first example of a non-integrable Lie algebroid is due to Almeida, Molino [**1**].

An easy example comes from geometric quantization. Let $\omega \in \Omega^2(M)$ be a closed 2-form and let $TM \oplus \mathbb{R}$ be the Lie algebroid with anchor the projection $TM \to \mathbb{R} \to TM$ and the Lie bracket defined by

$$[(X,f),(Y,g)] := ([X,Y], \mathcal{L}_X g - \mathcal{L}_Y f + \omega(X,Y)$$

for $X, Y \in \Gamma(TM)$ and $f, g \in C^\infty(M)$. If

$$\Big\{ \int_\alpha \omega, \alpha \in \pi_2(M) \Big\} \subset \mathbb{R}$$

is not a discrete subgroup, $TM \oplus \mathbb{R}$ is not integrable.

2.5. Crainic, Fernandes integration of Lie algebroids. While very intuitive, Ševera's approach to integration of Lie algebroids did not give precise conditions on integrability of Lie algebroids and it lacked a proof of smoothness of the integrating groupoid in the integrable case. Both of these problems were fully resolved by Crainic, Fernandes [**7**] following the Duistermaat, Kolk [**10**] method of integrating Lie algebras.

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and define the path space of $\mathcal{G}$ as

$$P(\mathcal{G}) := \{g : I \to \mathcal{G},\ g \text{ is } C^2, g(0) = \mathrm{id}_{s(g(0))}, s(g(t)) = s(g(0))\}$$

with the C^2-topology. Furthermore let $\pi : A \to M$ be the Lie algebroid corresponding to $\mathcal{G} \rightrightarrows M$ and let $P(A)$ be the space of A-paths with the C^1-topology. There is an analogue of the map (13)

$$\begin{aligned} D : P(\mathcal{G}) \cap C^2 &\to P(A) \\ (D\gamma)(t) &:= T_{\gamma(t)} R(\gamma(t))^{-1} \left(\tfrac{d\gamma}{dt}(t)\right). \end{aligned} \tag{19}$$

Here the higher regularity will be necessary to define a proper analogue (from the case of Lie groups) of the multiplication in $P(\mathcal{G})$ as the pointwise multiplication of $\mathcal{G}$-paths does not make sense for general Lie groupoids. However one replaces the pointwise multiplication from the case of Lie groups by a homotopically equivalent formula

$$g_1.g_2(t) = \begin{cases} g_2(2t) & 0 \le t \le \frac{1}{2}, \\ g_1(2t-1)g_2(1) & \frac{1}{2} < t \le 1, \end{cases}$$

which works also for $\mathcal{G}$-paths. The differentiation map (19) is a homeomorphism which enables us once again to transport the multiplication structure from the global object to the infinitesimal one. Thus a partial multiplication of two A-paths a_1, a_2 can be given by the concatenation

$$a_1.a_2(t) := \begin{cases} 2a_1(2t) & 0 \le t \le \frac{1}{2}, \\ 2a_2(2t-1) & \frac{1}{2} < t \le 1. \end{cases} \tag{20}$$

Here the multiplication is to be defined only if $\pi(a_1(1)) = \pi(a_2(0))$. This is essentially a replacement of the formula (15) which was obtained from the linearization of the (pointwise) multiplication of paths in the integrating Lie group.

There is a caveat, $a_1.a_2$ defined by (20) is only a piecewise smooth path. However this can be regulated by changing the composition formula with a use of a cutoff function in a way so that the multiplication is well defined on A-homotopy classes. Full details can be found in Crainic, Fernandes [**7**]. Passing to A-homotopy classes defines the Weinstein groupoid $\mathcal{G}(A)$ of a Lie algebroid A

$$\mathcal{G}(A) := P(A)/\sim. \tag{21}$$

The Weinstein groupoid $\mathcal{G}(A)$ is an s-simply connected topological groupoid and if A is integrable it is the unique s-simply connected Lie groupoid integrating A.

Crainic, Fernandes [**7**] described obstructions to integrability of a Lie algebroid A in terms of monodromy groups at each point of the body of A. Let us sketch their geometric origin using the exponential map for Lie algebroids. Given a connection ∇ on A there is a map $\exp_\nabla$ which is defined locally on a neighborhood $\mathcal{U}$ of the zero section of A

$$\exp_\nabla : \mathcal{U} \to \mathcal{G}(A)$$

such that if A is integrable then $\exp_\nabla$ is a local diffeomorphism which coincides with the exponential map defined using the pullback (along the t-maps) connections on each s-fiber. A simple construction of $\exp_\nabla$ for a (possibly non-integrable) Lie algebroid A is by the means of the geodesic equation

$$\begin{cases} \nabla_a a = 0 \\ a(0) = X, \end{cases} \tag{22}$$

which has a solution $a_X \in P(A)$ if $X \in A_x$ is sufficiently close to 0 (a solution exists always for a small time interval). The exponential map is then the A-homotopy class of the solution of (22), $\exp_\nabla(X) := [a_X]$. The *monodromy group of A at the point $x \in M$* is defined as

$$\mathcal{N}_x(A) := \{X \in Z(\mathfrak{g}_x), v \sim 0_x\}$$

where the relation $\sim$ is the A-homotopy of A-paths. The monodromy group of A at x is related to the second homotopy group of the orbit of x. In particular Crainic, Fernandes [**7**] constructed an exact sequence for a Lie algebroid A

$$\ldots \to \pi_2(\mathcal{O}_x) \xrightarrow{\partial} G_x \to \mathcal{G}(A)_x \to \pi_1(\mathcal{O}_x) \tag{23}$$

where G_x is the unique simply-connected Lie group integrating the isotropy Lie algebra $\mathfrak{g}_x$ and $\mathcal{G}(A)_x$ is the isotropy group at x of the Weinstein groupoid $\mathcal{G}(A)$ (this is not a Lie group in general). The exact sequence (23) is an associated homotopy exact sequence to the short exact sequence of Lie algebroids induced by the anchor

$$0 \to \mathfrak{g}_{\mathcal{O}_x} \to A|_{\mathcal{O}_x} \xrightarrow{\rho} T\mathcal{O}_x \to 0.$$

The intersection of Im ∂ from (23) with the connected component of the center of G_x is isomorphic to $\mathcal{N}_x(A)$.

A necessary condition for integrability of A is that for any $x \in M$ there is an open neighborhood U of $0 \in A_x$ in A such that there exist an open neighborhood V of x with the property

$$\mathcal{N}_y(A) \cap U = \{0\}$$

for all y in V. Otherwise $\exp_\nabla$ fails to be injective on any neighborhood of the origin and therefore A is not integrable. If the above condition holds one says that the monodromy groups $\mathcal{N}_x(A)$ are *uniformly locally discrete.* Crainic, Fernandes [**7**] further proved that this is also a sufficient condition for the integrability of A.

3. Lie theory for L_∞-algebras

3.1. A reminder on nerves. We first discuss the global objects corresponding to L_∞-algebras (which can be thought of as Lie algebras satisfying the Jacobi identity up to a system of coherent homotopies). As the name suggests the global objects should be called L_∞-groups. A standard way to define these (Getzler [**16**] and Henriques [**20**]) is via the nerve construction. The notion of a nerve of a category is an important link between category theory and homotopy theory. In particular together with the realization functor the nerve forms a pair of adjoint functors mimicking the singular homology theory. It furthermore provides us with a clue to higher category theory, specifically to models of $(\infty,1)$-categories and $(\infty,1)$-functors.

Let $\mathcal{C}$ be a locally small category (locally small means that all Homs are sets). To determine a realization of the standard n-simplices in $\mathcal{C}$ is to give a cosimplicial object $S : \Delta \to \mathcal{C}$. Then for any object X of the category $\mathcal{C}$ we have a simplicial set

$$\begin{gathered} N_\bullet(X) : \Delta^{\mathrm{op}} \to \mathbf{Set} \\ N_\bullet(X) := \mathrm{Hom}_{\mathcal{C}}(S([\bullet]), X) \end{gathered}$$

called the nerve of X. If S is a dense functor then the nerve functor is fully faithful.

Since the category of all small categories $\mathbf{Cat}$ is a locally small category the cosimplicial object $i : \Delta \hookrightarrow \mathbf{Cat}$ specifies a functor

$$\begin{gathered} N_\bullet : \mathbf{Cat} \to \mathbf{sSet} \\ N_\bullet(\mathcal{C}) := \mathrm{Hom}_{\mathbf{Cat}}([\bullet], \mathcal{C}). \end{gathered} \tag{24}$$

Unpacking this definition we get the following description of $N_\bullet(\mathcal{C})$. The set of zero simplices $N_0(\mathcal{C})$ is the set of objects of $\mathcal{C}$. The set of n-simplices $N_n(\mathcal{C})$ for $n > 0$ is the set of all composable chains of morphisms of the category $\mathcal{C}$ of length n i.e. an n-simplex

$$\sigma = (f_0, \dots, f_{n-1}) \in N_n(\mathcal{C})$$

is a string

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} X_n.$$

The face maps

$$d_i : N_n(\mathcal{C}) \to N_{n-1}(\mathcal{C})$$

are defined as

$$\begin{gathered} d_0(\sigma) := X_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} X_n, \\ d_i(\sigma) := X_0 \xrightarrow{f_0} \dots \xrightarrow{f_i \circ f_{i-1}} X_{i+1} \xrightarrow{f_{i+1}} \dots \xrightarrow{f_{n-1}} X_n, \end{gathered}$$

where $0 < i < n$ and finally

$$d_n(\sigma) := X_0 \xrightarrow{f_0} \dots \xrightarrow{f_{n-1}} X_{n-2} \xrightarrow{f_{n-2}} X_{n-1}.$$

The degeneracy maps

$$s_i : N_n(\mathcal{C}) \to N_{n+1}(\mathcal{C})$$

are defined for $0 \le i \le n$ as

$$s_i(\sigma) := X_0 \xrightarrow{f_0} \dots \xrightarrow{f_{i-1}} X_i \xrightarrow{\mathrm{id}_{X_i}} X_i \xrightarrow{f_i} X_{i+1} \xrightarrow{f_{i+1}} \dots \xrightarrow{f_n} X_n.$$

In this case $N_\bullet$ is a fully faithful functor. It enables us to work with simplicial sets instead of categories (this approach is due to Grothendieck [**18**]). The nerve functor is not essentially surjective, simplicial sets isomorphic to a nerve of a category have to satisfy the Segal condition [**30**]. The nerve of a groupoid has similar characterization. Recall that the k-th horn of Δ^n is defined as the simplicial subset $\Delta[n,k] \subset \Delta[n]$ given by the union of all faces of $\Delta[n]$ except the k-th face. If $0 < k < n$ then $\Delta[n,k]$ is called an inner horn.

Proposition 3.1. *A simplicial set is isomorphic to a nerve of a category iff all inner horns have unique fillers, that is the natural maps induced by the inclusion* $\Delta[n,k] \subset \Delta[n]$

$$K_n \cong \mathit{Hom}(\Delta[n], K) \to \mathit{Hom}(\Delta[n,k], K) =: K_{n,k} \tag{25}$$

are bijections for $n > 1$ *and* $0 < k < n$. *It is isomorphic to a nerve of a groupoid iff (25) are bijections also for* $k = 0, n$.

Let us illustrate the geometric meaning of this Proposition by studying the horns $\wedge[2,\cdot]$. The condition (25) for filling the inner horn $\wedge[2,1]$ gives us composition as illustrated in Figure 3.

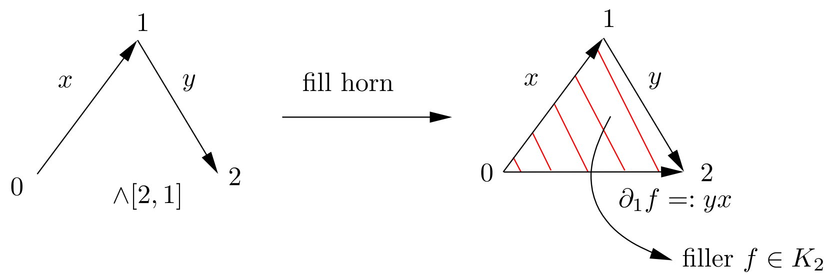

FIGURE 3. Composition in the nerve of a groupoid.

For the horns $\wedge[2,0]$ and $\wedge[2,2]$ the condition (25) gives inverses as illustrated in Figure 4. for $\wedge[2,0]$.

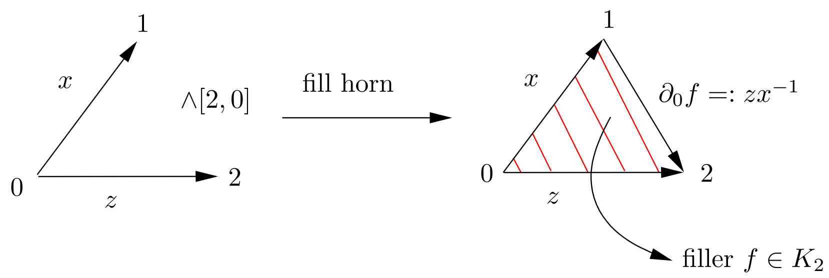

FIGURE 4. Inverse in the nerve of a groupoid.

Associativity is ensured by condition (25) on 3-horns. The higher horns follow automatically and do not give any additional information.

3.2. L_∞-groups. As we mentioned the nerve construction provides natural models of higher categories. Loosely speaking an ℓ-category should be a generalization of the notion of a category involving n-morphisms, which are morphisms between $(n-1)$-morphisms for n up to ℓ. The category of small categories itself is a 2-category with natural transformations as 2-morphisms. For Lie theory we will furthermore restrict to higher groupoids or ℓ-groupoids (where all n-morphisms are invertible). These were defined by Duskin [**12**] (by the name of hypergroupoids).

DEFINITION 3.1. *A* (weak) ℓ-groupoid *is a simplicial set $K_\bullet$ such that the maps (25) are surjective for $n \le \ell$ and $0 \le k \le n$ and bijections for $n > \ell$ and $0 \le k \le n$. A* (weak) ℓ-group *is a (weak) ℓ-groupoid such that $K_0 = \{*\}$.*

Thus in a (weak) ℓ-groupoid the composition of n-morphisms for $n < \ell$ is not uniquely defined, and the same is true for inverses. This is why the word "weak" is used in the definition. However the existence of fillers in higher dimensions controls the non-uniqueness, e.g. it gives associativity of composition of 1-morphisms up to a (non-unique) homotopy. This is illustrated on Figure 5.

By setting $\ell = \infty$ the definition of a (weak) ∞-groupoid coincides with the definition of a Kan complex.

Lie n-groups are smooth analogues of n-groups. We follow the definitions of Henriques [**20**]. A simplicial manifold $K_\bullet : \Delta^{\mathrm{op}} \to \mathbf{Man}$ is a Kan simplicial manifold iff the maps (25) are surjective submersions. Note that for this definition one needs to first prove that $K_{n,k}$ are smooth manifolds. This is done inductively (in n) in Henriques [**20**].

DEFINITION 3.2. *A Lie ℓ-group is a Kan simplicial manifold $K_\bullet$ with $K_0 = \{*\}$ such that the maps (25) are diffeomorphisms for all $n > \ell$ and all $0 \le k \le n$.*

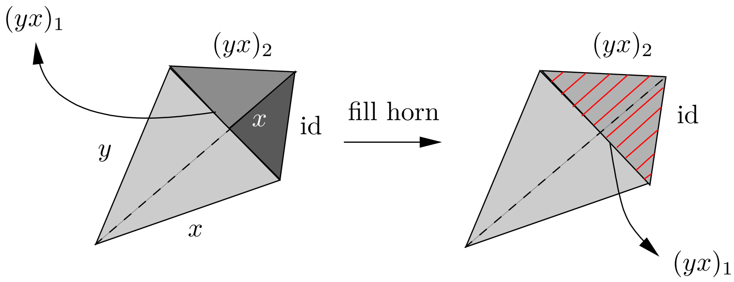

FIGURE 5. Associativity in an ℓ-groupoid. Here $(yx)_1$ and $(yx)_2$ denote two different compositions of x, y obtained as in Figure 3.

Thus Lie 1-groups are just (nerves) of Lie groups. Henriques's notion of a Lie 2-group is related to the Lie 2-group of Baez, Lauda [**2**] obtained via categorification. By setting $\ell = \infty$ we get that an L_∞-group is the same as a reduced Kan simplicial manifold (reduced means $K_0 = \{*\}$).

3.3. L_∞**-algebras.** We already mentioned that a L_∞-algebra is a notion similar to a Lie algebra except that the Jacobi identity holds only up to a series of coherent homotopies (given by higher brackets). For this reason L_∞-algebras are sometimes called strongly homotopy Lie algebras (or sh Lie algebras). Thus an L_∞-algebra is a graded vector space with a series of n-ary brackets satisfying some identites. There are various possible restrictions on the grading. Depending on the applications the definitions are sometimes restricted to only non-positively graded (this is common in Lie theory and we will follow this convention) or to only positively graded vector spaces (this is common in derived algebraic geometry). In some cases one needs both sides (as in the Batalin-Vilkovisky formalism, or more generally when working with derived stacks). Loosely speaking the non-negative part is tied to homotopy theory, while the non-positively graded part is tied to derived geometry.

From the point of view of Lie theory L_∞-algebras arise as the infinitesimal objects corresponding to L_∞-groups. The procedure will be recalled in the more general setting of L_∞-groupoids in the next Section following Ševera [**33**]. L_∞-algebras are ubiquitous to modern mathematics. From the homotopical viewpoint L_∞-algebras arise from dg Lie algebras via the homotopy transfer. L_∞-algebras also appear in closed string field theory (a second quantized version of closed string theory) of Zwiebach [**42**]. References for L_∞-algebras are Lada, Stasheff [**23**] and Lada, Markl [**24**].

Let us give a formal definition of an L_∞-algebra with the above grading restriction. We will write formulas for homogeneous elements of a non-positively graded vector space

$$\mathfrak{g} := \bigoplus_{i\leq 0} \mathfrak{g}_i$$

and then extend them linearly. If σ is a permutation of n elements $x_1, \ldots, x_n$ of $\mathfrak{g}$, the Koszul sign $\chi(\sigma, x_1, \ldots, x_n)$ is defined as

$$\chi(\sigma, x_1, \ldots, x_n) := (-1)^\sigma \epsilon(x_1, \ldots, x_n)$$

where $\epsilon(x_1, \ldots, x_n)$ is defined by

$$x_{\sigma(1)} \wedge \ldots \wedge x_{\sigma(n)} =: \epsilon(x_1, \ldots, x_n) x_1 \wedge \ldots \wedge x_n.$$

Finally $\pi^{\mathrm{un}}_{(i,n-i)}$ denotes the set of $(i, n-i)$-unshuffles.

DEFINITION 3.3. *An L_∞-algebra $\mathfrak{g}$ is a graded vector space $\mathfrak{g}$ together with n-ary bracket l_n for each $n \geq 1$*

$$l_n : \otimes^n \mathfrak{g} \to \mathfrak{g}$$

of degree $n-2$ such that each l_n is graded antisymmetric

$$l_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \chi(\sigma, x_1, \ldots, x_n) l_n(x_1, \ldots, x_n)$$

where χ is the Koszul sign and the generalized Jacobi identity holds

$$\sum_{i+j=n+1}\sum_{\sigma\in\pi^{un}_{(i,n-i)}}(-1)^{i(j-1)}\chi(\sigma,x_1,\dots,x_n)l_j\big(l_i(x_{\sigma(1)},\dots,x_{\sigma(i)}),x_{\sigma(i+1)},\dots,x_{\sigma(n)}\big)=0. \tag{26}$$

Note that $(-1)^{i(j-1)}=(-1)^{(n+1)i}$.

If l_n are trivial for $n>2$ we recover the notion of a dg Lie algebra with the differential l_1 and the Lie bracket l_2. An L_∞-algebra concentrated in degrees $>-\ell$ is called a *Lie ℓ-algebra*. Thus a Lie 1-algebra is just a Lie algebra. Lie ℓ-algebras are the infinitesimal counterparts to Lie ℓ-groups.

The generalized Jacobi identity (26) has a simple form if we treat the sequence of brackets l_n as a differential on certain graded coalgebra. This was described in Lada, Stasheff [**23**]. Consider the free graded cocommutative coalgebra on $\mathfrak{g}$ denoted $\vee\mathfrak{g}$. The underlying graded vector space of $\vee\mathfrak{g}$ is $S\mathfrak{g}$ and the coproduct is given by

$$\Delta(x_1\vee\ldots\vee x_n):=\sum_{1\le j\le n-1}\sum_{\sigma\in\pi^{\mathrm{un}}_{(i,n-i)}}\epsilon(\sigma)(x_{\sigma(1)}\vee\ldots\vee x_{\sigma(i)})\otimes(x_{\sigma(i+1)}\vee\ldots\vee x_{\sigma(n)}).$$

Since this coalgebra is (nilpotent) free, a differential d on $\vee\mathfrak{g}$ is given by its projections of the cogenerators $\mathfrak{g}$ of $\vee\mathfrak{g}$. Therefore we can write

$$d=\sum_{n\ge1}d_n$$

where d_n is the extension of l_n to $\vee\mathfrak{g}$.

The Chevalley-Eilenberg dgca of an L_∞-algebra is the gca $C(\mathfrak{g}):=S((\mathfrak{g}[1])^*)$ with the differential d_{CE} defined as follows. First take the duals of the maps l_n and assemble them to a degree 1 map $(\mathfrak{g}[1])^*\to C(\mathfrak{g})$. The differential d_{CE} is the unique extension of this map to a derivation on $C(\mathfrak{g})$.

3.4. Integration of L_∞-algebras. Lie integration of L_∞-algebras was done by Getzler [**16**] and Henriques [**20**] by different techniques.

Getzler [**16**], following Sullivan [**35**] starts from the spatial realisation of the Chevalley-Eilenberg dgca of a nilpotent L_∞-algebra

$$K^{\mathrm{big}}_\bullet:=\mathrm{Hom}_{\mathbf{dgca}}\big((C(\mathfrak{g}),d_{CE}),(\Omega(\Delta^\bullet),d)\big) \tag{27}$$

(he calls it the spectrum of $C(\mathfrak{g})$, since it generalizes the usual spectrum of a commutative algebra). He uses polynomial forms on $\Delta^\bullet$.

If $\mathfrak{g}$ is a nilpotent Lie algebra the simplicial set (27) is not the nerve of the integrating simply-connected group G, but it is simplicially homotopy equivalent. In particular, the algebraic path space $P(1,G)$ can be identified with K^{big}_1 via pullbacks of the (left invariant) Maurer-Cartan 1-form on G. $P(1,G)$ is foliated with the fibres of the evaluation map at the endpoint of a path and its leaf space is G. For nilpotent $\mathfrak{g}$ each leaf of the corresponding foliation of K^{big}_1 contains a unique constant 1-form and we have the inclusion $N_1G\hookrightarrow K^{\mathrm{big}}_1$. For higher simplices Getzler proposed using Dupont's [**11**] simplicial chain homotopy (from the proof of simplicial de Rham theorem)

$$s_\bullet:\Omega(\Delta^\bullet)\to\Omega(\Delta^\bullet)$$

retracting $\Omega(\Delta^\bullet)$ to define the simplicial set

$$K^s_\bullet:=\{A\in K^{\mathrm{big}}_\bullet, s_\bullet A=0\}. \tag{28}$$

He showed that $K^s_\bullet$ is a Kan complex and indeed $K^s_\bullet\cong N_\bullet G$.

The proofs work when one replaces the nilpotent Lie algebra $\mathfrak{g}$ with a nilpotent L_∞-algebra. In that case $K^s_\bullet$ is a (weak) ∞-group (in fact the integration procedure gives fillers for the horns).

We will discuss $s_\bullet$ in the next Section on integrating L_∞-algebroids, where we follow Getzler's ideas to construct a local L_∞-groupoid integrating an L_∞-algebroid.

Henriques [**20**] also starts from the spatial realisation (27) of $\mathfrak{g}$. Unlike Getzler, he does not restrict to nilpotent L_∞-algebras. He models forms on $\Delta^\bullet$ using C^r-forms with C^r exterior derivative to make use of Banach space techniques. His main result is that $K^{\mathrm{big}}_\bullet$ is a Kan simplicial Banach manifold. This is done by splitting $\mathfrak{g}$ into an acyclic and minimal part, followed by a construction of Postnikov towers with inverse limit being the minimal part. The pieces of the associated graded of this filtration are certain simpler L_∞-algebras, namely

(1) $\mathfrak{h} := \mathfrak{h}_{-n+1}$ with trivial brackets. Here $K_\bullet^{\mathrm{big}} = \Omega^n_{\mathrm{closed}}(\Delta^\bullet, \mathfrak{h}_{-n+1})$.
(2) $\mathfrak{h} := \mathfrak{h}_{-n} \oplus \mathfrak{h}_{-n+1}$ with the $l_1 : \mathfrak{h}_{-n} \to \mathfrak{h}_{-n+1}$ an isomorphism. Here $K_\bullet^{\mathrm{big}} = \Omega^n(\Delta^\bullet, \mathfrak{h}_{-n})$.
(3) $\mathfrak{h} := \mathfrak{h}_0$ a Lie algebra. Here $K_\bullet^{\mathrm{big}}$ is as in (12).

All of the above $K_\bullet^{\mathrm{big}}$ are Kan Banach manifolds. One then integrates $\mathfrak{g}$ step by step via the filtration, showing that the integrals form a Kan fibration.

4. Lie theory for L_∞-algebroids

4.1. A reminder on supergeometry. It turns out that all of the Lie structures discussed so far belong to the same class of geometric objects called *differential non-negatively graded manifolds*, or *NQ-manifolds*. Since the underlying geometric structures of these objects are supermanifolds we include a brief overview of the basics of supergeometry. A good reference is Varadarajan [**38**] and the lectures of Deligne, Freed, Morgan [**9**].

Serre and Grothendieck pioneered the locally ringed space approach to algebraic varieties and schemes (see e.g. Hartshorne [**19**]). In supergeometry we replace commutative rings by supercommutative rings, which are $\mathbb{Z}/2\mathbb{Z}$-graded rings such that the multiplication is graded commutative. To be more precise we will only need supercommutative algebras of a certain type since we are not interested in the more general theory of superschemes. A supercommutative ring R is called local iff it has a unique maximal homogeneous ideal $\mathfrak{m} \subset R$. This implies that the even part R_0 is a local ring with the unique maximal ideal $R_0 \cap \mathfrak{m}$. A pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of supercommutative rings on X such that at each point $x \in X$ the stalk

$$\mathcal{O}_{X,x} := \varinjlim_{x \in U} \mathcal{O}_X(U)$$

is a local supercommutative ring is called a superringed space. The sheaf $\mathcal{O}_X$ is often called the structure sheaf or the sheaf of functions of the superringed space $(X, \mathcal{O}_X)$. The definition of a supermanifold can be given by replicating the definition of a smooth manifold in terms of its sheaf of smooth functions. One first defines for $p, q \geq 0$ a superdomain $U^{p|q}$ of dimension $(p|q)$ to be a superringed space $(U, \mathcal{O}_U)$ where $U \subset \mathbb{R}^p$ is an open set and $\mathcal{O}_U$ is a sheaf of supercommutative algebras given for any $V \subset U$ open

$$\mathcal{O}_U(V) := C^\infty(V)[\theta^1, \ldots, \theta^q]$$

with the obvious restriction maps. Here θ^i have odd degree.

Definition 4.1. *A superringed space $(X, \mathcal{O}_X)$ is called a supermanifold of dimension $(p|q)$ iff it is locally isomorphic (in the category of superringed spaces) to a superdomain of dimension $(p|q)$. A morphism of supermanifolds*

$$f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

is a morphism of superringed spaces. The category of supermanifolds and morphisms of supermanifolds is denoted **supMan**.

In fact the underlying topological space X of a supermanifold $(X, \mathcal{O}_X)$ is automatically a smooth manifold and is called the body of $(X, \mathcal{O}_X)$. Taking global sections of the structure sheaves there is a natural bijection

$$\mathrm{Hom}_{\mathbf{supMan}}\big((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)\big) \cong \mathrm{Hom}_{\mathbf{supAlg}}\big(\mathcal{O}_Y(Y), \mathcal{O}_X(X)\big). \tag{29}$$

This is analogous to the smooth manifold case. Since $\mathcal{O}_X(X)$ should be considered as the superalgebra of global functions on $(X, \mathcal{O}_X)$ we will from now on write $Z := (X, \mathcal{O}_X)$ for a supermanifold and $C^\infty(Z) := \mathcal{O}_X(X)$. We will furthermore speak of local coordinates z^i etc. just like in the smooth manifold case. By this we mean that we take a neighborhood in the body such that $(U, \mathcal{O}_X|_U)$ is isomorphic to some superdomain so that

$$C^\infty(Z)|_U \cong C^\infty(U)[\theta^1, \ldots, \theta^q]$$

and the local coordinates are then $z^i = (x^a, \theta^b)$, where x^a are local coordinates in U. The body can be thus thought of as the ordinary smooth manifold part of a supermanifold (with ordinary even local coordinates x^a).

Any vector bundle $E \to M$ defines a supermanifold

$$\Pi E := (M, \mathcal{O}_{\wedge E^*}). \tag{30}$$

Here $\mathcal{O}_{\wedge E^*}$ is the sheaf of sections of the exterior bundle $\wedge E^*$. One can also understand ΠE as a vector bundle with shifted parity of the fibers (thus the local fiber coordinates are odd). The construction of Π extends from (30) to a functor

$$\Pi : \mathbf{VBund} \to \mathbf{supMan}.$$

By definition a supermanifold is locally isomorphic to ΠE for some vector bundle E. Batchelor [**3**] proved that this is true globally so that the functor Π is essentially surjective. However the identification of a supermanifold Z with ΠE for some vector bundle E is non-canonical. Futhermore **supMan** has a lot more morphisms since they do not need to preserve the fiber structure, so Π fails to be fully faithful. The categories **supMan** and **VBund** are not equivalent.

Many geometric constructions have direct analogues in the supergeometry case. In particular we will need the concept of a vector field, which can be defined as a derivation of the superalgebra $C^\infty(Z)$. Note that the space of vector fields is also $\mathbb{Z}/2\mathbb{Z}$-graded. Again we can work in local coordinates and write vector fields using $\partial_i := \partial_{z^i}$ defined in the obvious way.

4.2. ***NQ*-manifolds, L_∞-algebroids.** A *graded manifold* is a further generalization of a smooth manifold obtained by including coordinates of various degrees. There are several ways how to phrase a definition of a non-negatively graded manifold, also called as *N-manifold*.

Definition 4.2. *A* non-negatively graded manifold *(an N-*manifold*) is a supermanifold Z with an action of the semigroup $(\mathbb{R}, \times)$ such that $-1 \in \mathbb{R}$ acts as the parity operator P (changes the sign of the odd coordinates). Morphisms of N-manifolds are morphisms of the underlying supermanifolds compatible with the semigroup actions.*

This definition says that both the even and odd coordinates have additional non-negative degree with compatible parity. It is useful to write the additional grading in terms of the Euler vector field, expressed in local coordinates z^i as

$$E := w(z^i) z^i \partial_i. \tag{31}$$

Here $w(z^i)$ is the degree (eigenvalue, sometimes called the weight) of the local coordinate z^i. The highest degree of a local coordinate is called the *degree of Z*. Using the Euler vector field we can rephrase the condition on compatibility of the gradings as $P = (-1)^E$. The Euler vector field decomposes the cga of functions as

$$C^\infty(Z) = \bigoplus_{k \geq 0} C^\infty(Z)^k$$

where k is the degree.

Similarly to supermanifolds the most basic examples of N-manifolds arise from negatively graded vector bundles. Suppose $E \to M$ is a negatively graded vector bundle and consider $\mathcal{O}_{S(E^*)}$ the sheaf of sections of the bundle $S(E^*) \to M$. Working in a local trivializing neighborhood U one can choose a basis θ^I of sections such that $(U, \mathcal{O}_{S(E^*)}(U))$ is isomorphic to a superdomain $U^{p|q}$ with an additional grading. This shows that any negatively graded vector bundle defines an N-manifold and any N-manifold is locally isomorphic to a negatively graded vector bundle. The same remarks apply as in the case of supermanifolds - there is a generalization of Batchelor's theorem [**3**] that any N-manifold is (non-canonically) globally isomorphic to an N-manifold arising from a negatively graded vector bundle and the category of N-manifolds and the category of negatively graded vector bundles are not equivalent.

Vector fields on an N-manifold are defined analogously to vector fields on supermanifolds. They form a graded Lie algebra.

Definition 4.3. *An NQ-*manifold *(a* differential non-negatively graded manifold*) is an N-manifold equipped with a vector field Q of degree 1 such that $Q^2 = 0$. NQ-manifolds form a category **NQMan** with morphisms being morphisms of N-manifolds compatible with the differentials.*

An important example of an NQ-manifold is the shifted tangent bundle $T[1]M$ of a smooth manifold M. Here $C^\infty(T[1]M) = \Omega(M)$ and the differential is the de Rham differential. The construction extends to a functor

$$T[1] : \mathbf{Man} \to \mathbf{NQMan}$$

which is fully faithful.

Let us now consider Lie structures in the language of NQ-manifolds. We have seen that a Lie algebra can be thought of in terms of its Chevalley-Eilenberg dgca. Its underlying graded vector space is

$$\wedge \mathfrak{g}^* = S((\mathfrak{g}[1])^*).$$

This is the gca of functions on $\mathfrak{g}[1]$. Furthermore the Chevalley-Eilenberg differential can be written in coordinates ξ^a, $\deg \xi^a = 1$ dual to coordinates x^a of $\mathfrak{g}[1]$ as

$$Q := \frac{1}{2} f^a_{bc} \xi^b \xi^c \frac{\partial}{\partial \xi^a}. \tag{32}$$

Thus the Chevalley-Eilenberg dgca $(\wedge \mathfrak{g}^*, d_{CE})$ is the dgca of functions on a NQ-manifold corresponding to the negatively graded vector bundle (over a point) $\mathfrak{g}[1]$ with the differential Q given by (32). The condition $Q^2 = 0$ is equivalent to the Jacobi identity for the coefficients f^a_{bc}. Thus the correspondence works also the other way - a NQ-manifold of type $V[1]$ for some finite-dimensional vector space V defines a Lie algebra structure on V.

The case of L_∞-algebras is a natural generalisation of the above in one direction. If $\mathfrak{g}$ is a non-positively graded vector space consider $\mathfrak{g}[1]$ as a N-manifold. Then a differential Q on $\mathfrak{g}[1]$ is equivalent to a L_∞-algebra structure on $\mathfrak{g}$. Thus we can view NQ-manifolds with trivial base (but possibly non-trivial body) as L_∞-algebras. Again one can think of this correspondence as the identification of the Chevalley-Eilenberg dgca of the L_∞-algebra with the dgca of functions on the corresponding NQ-manifold.

Lie algebroids are also NQ-manifolds. Let $A \to M$ be a Lie algebroid. The underlying graded vector space of its Chevalley-Eilenberg dgca is

$$\Gamma(\wedge A^*) := \Gamma(S((A[1])^*)).$$

This suggests viewing the graded vector bundle $A[1]$ as a NQ-manifold with the differential Q being the Chevalley-Eilenberg differential which in local coordinates (x^i, ξ^a), $\deg x^i = 0$, $\deg \xi^a = 1$ is

$$Q := \frac{1}{2} f^a_{bc}(x) \xi^b \xi^c \frac{\partial}{\partial \xi^a} + a^i_a(x) \xi^a \frac{\partial}{\partial x^i}. \tag{33}$$

Here the NQ-manifolds come from (shifted) vector bundles, thus from non-trivial bases (and equivalently non-trivial bodies since we have only coordinates of degree 0 and 1). Again the condition that the differential (33) squares to zero is equivalent to the Lie algebroid axioms for the Lie bracket defined by f^a_{bc} and the anchor defined by a^i_a.

A general NQ-manifold can thus be seen as a simultaneous generalization of both Lie algebroids and L_∞-agebras and has thus a natural interpretation as an *L_∞-algebroid*. This name is not agreed on in the community. This is a reflection of the various grading conventions for L_∞-algebras. Again, in derived algebraic geometry one needs also negatively graded coordinates (a $\mathbb{Z}$-graded manifold). Since our interest lies in the homotopy side of the picture, we will adopt NQ-manifolds as L_∞-algebroids as was done by Ševera [**32**]. We can also define Lie ℓ-algebroids as NQ-manifolds of degree ℓ.

4.3. Local L_∞-groupoids and their jets. We saw in the previous Section that the higher homotopy analogues of Lie algebroids are L_∞-algebroids which are in turn NQ-manifolds. Even if a Lie algebroid is not integrable we can still think of it as the Lie algebroid corresponding to a certain local Lie groupoid. Similarily we expect L_∞-algebroids to be infinitesimal objects corresponding to *local L_∞-groupoids.*

A definition of these objects can be given in terms of a nerve of an n-groupoid together with certain smoothness properties. This is a straightforward generalization of a Lie ℓ-group. Again we follow the definitions of Getzler [**16**], Henriques [**20**]. The local version is from Zhu [**41**] and is obtained by eliminating surjectivity conditions (so that the ℓ-groupoid compositions and inverses are defined locally).

DEFINITION 4.4. *A simplicial manifold $K_\bullet$ is a* Kan simplicial manifold *iff the maps (25) $K_n \to K_{n,k}$ are surjective submersions for any $0 \leq k \leq n$. A* (weak) Lie ℓ-groupoid *is a finite-dimensional Kan simplicial manifold $K_\bullet$ such that the maps (25) $K_n \to K_{n,k}$ are diffeomorphisms for all $n \geq \ell$ and all $0 \leq k \leq n$. A* (weak) local Lie ℓ-groupoid *is a finite-dimensional simplicial manifold $K_\bullet$ such that the maps (25) are submersions for all $n \geq 1, 0 \leq k \leq n$ and open embeddings for all $n > \ell$ and all $0 \leq k \leq n$.*

Setting $\ell = \infty$ we obtain the definition of a (local) L_∞-groupoid.

The construction of an L_∞-algebroid associated to a local L_∞-groupoid is done via Ševera's [**33**] 1-jet functor. To describe this functor we need a construction of the de Rham complex due to Kontsevich [**21**].

The category **supMan** is monoidal with the obvious product. Thus it makes sense to speak of the internal hom for any pair (X, Y) of supermanifolds such that the presheaf

$$\underline{\mathrm{Hom}}_{\mathbf{supMan}}(X, Y) : \mathbf{supMan}^{\mathrm{op}} \to \mathbf{Set}$$

defined by

$$\underline{\mathrm{Hom}}_{\mathbf{supMan}}(X, Y)(W) := \mathrm{Hom}_{\mathbf{supMan}}(X \times W, Y) \tag{34}$$

is representable. As usual we will abuse notation and write $\underline{\mathrm{Hom}}_{\mathbf{supMan}}(X, Y)$ for both the presheaf (34) and the representing supermanifold.

If Z is a supermanifold then the internal hom $\underline{\mathrm{Hom}}_{\mathbf{supMan}}(\mathbb{R}^{0|1}, Z)$ is a representable functor in the category of supermanifolds and is isomorphic to the odd tangent bundle ΠTZ

$$\underline{\mathrm{Hom}}_{\mathbf{supMan}}(\mathbb{R}^{0|1}, Z)(X) = \mathrm{Hom}_{\mathbf{supMan}}(\mathbb{R}^{0|1} \times X, Z) \cong \mathrm{Hom}_{\mathbf{supMan}}(\Pi TZ, X).$$

The proof follows easily by working in local coordinates and checking the transformation rules on overlaps. A more geometric way of seeing the above isomorphism is to view the morphisms $\mathbb{R}^{0|1} \to Z$ as 1-jets of curves in Z (this is sometimes called an odd point of Z) since the data of the map consists of a point in Z and a tangent vector at that point. The right action of the super-semigroup $\underline{\mathrm{Hom}}_{\mathbf{supMan}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$ on $\underline{\mathrm{Hom}}_{\mathbf{supMan}}(\mathbb{R}^{0|1}, Z)$ turns the functions of ΠTZ into the de Rham complex $\Omega(Z)$. More precisely the infinitesimal generators of $\underline{\mathrm{Hom}}_{\mathbf{supMan}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$ are the de Rham differential and the Euler vector field (31).

Let **subm** be the category with objects surjective submersions $Y \to Z$ of supermanifolds with morphisms being commutative squares. Given a simplicial manifold $K_\bullet$ Ševera defined a presheaf on **subm**

$$\begin{gathered} 1 - \mathrm{Jet}_{K_\bullet} : \mathbf{subm}^{\mathrm{op}} \to \mathbf{sSet} \\ 1 - \mathrm{Jet}_{K_\bullet}(Y \to Z) := \mathrm{Hom}_{\mathbf{ssupMan}}(N_\bullet(Y \times_Z Y \rightrightarrows Y), K_\bullet) \end{gathered} \tag{35}$$

If $K_\bullet$ is a Kan simplicial manifold the functor $1 - \mathrm{Jet}_{K_\bullet}$ is representable. Since

$$\mathrm{Hom}_{\mathbf{subm}}(\mathbb{R}^{0|1} \times Z_1 \to Z_1, \mathbb{R}^{0|1} \times Z_2 \to Z_2) \cong \mathrm{Hom}_{\mathbf{supMan}}(Z_1, Z_2) \times \underline{\mathrm{Hom}}_{\mathbf{supMan}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})(Z_1),$$

restricting to the full subcategory of **subm** given by objects $\mathbb{R}^{0|1} \times N \to N$ the functor $1 - \mathrm{Jet}_{K_\bullet}$ gives a supermanifold Z with a right action of $\underline{\mathrm{Hom}}_{\mathbf{supMan}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$, which is an NQ-manifold or an L_∞-algebroid.

4.4. Integration of L_∞-algebroids. In this Section we present a brief overview of the article Ševera, Širaň [**34**] which is also reproduced in this Thesis.

Let Z be a NQ-manifold with the differential Q. We can non-canonically identify Z with a negatively graded vector bundle V with a differential Q on the gca

$$\mathcal{A}_V := \Gamma(S(V^*)). \tag{36}$$

Following Sullivan [**35**] the first major problem is to show that the spatial realisation simplicial set

$$K_\bullet^{\mathrm{big}} := \mathrm{Hom}_{\mathbf{dgca}}\big((\mathcal{A}_V, Q), (\Omega(\Delta^\bullet), d)\big) \tag{37}$$

is a Kan simplicial Fréchet manifold. The name $K_\bullet^{\mathrm{big}}$ is chosen to reflect the fact that this integral is too big compared to what we expect (as we saw it is already infinite dimensional for Lie algebras). This is in a certain sense a matter of taste, as from the homotopical point of view $K_\bullet^{\mathrm{big}}$ has the correct simplicial homotopy type, independent of the choice of identification with $(\mathcal{A}_V, Q)$. This global integration is more flexible in the sense that any NQ-manifold can be integrated in this way. However $K_\bullet^{\mathrm{big}}$ lacks the usual interpretation in terms of the nerve construction. Equivalently we can write (37) as

$$K_\bullet^{\mathrm{big}} = \mathrm{Hom}_{\mathbf{NQMan}}(T[1]\Delta^\bullet, Z)$$

which is due to a natural bijection analogous to that of (29). The integration procedure is essentially a construction of a fundamental ∞-groupoid. Analogous to the Lie algebroid case we consider NQ-manifold morphisms $T[1]I \to Z$ to be Z-paths, 2-morphisms are Z-discs connecting the paths, i.e. NQ-manifold morphisms $T[1]O \to Z$, see Figure 6 (for technical reasons it is more convenient to use simplices instead of balls). If Z is a Lie ℓ-algebroid one needs to construct the fundamental ℓ-groupoid.

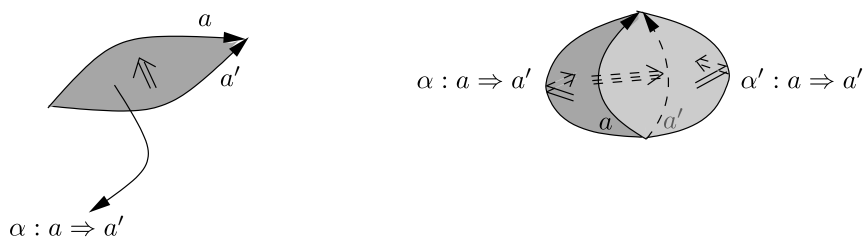

FIGURE 6. Construction of the fundamental ∞-groupoid. Left: 2-morphisms are homotopies $T[1]O \to Z$, right: 3-morphisms are homotopies between homotopies.

It is instructory to work first with the case of a trivial negatively graded bundle such as a restriction of V to a local trivializing neighborhood. Consider thus a trivial negatively graded vector bundle

$$W := W^{<0} \times W^0 \to W^0 \tag{38}$$

over a vector space W^0. We have

$$\mathcal{A}_W := C^\infty(W^0) \otimes S\left((W^{<0})^*\right).$$

Since $W^* \subset \mathcal{A}_W$ we can see a morphism of gca $A : \mathcal{A}_W \to \Omega(\Delta^\bullet)$ as an element of

$$\Omega(\Delta^\bullet, W)^0 := \bigoplus_{k\geq 0} \Omega^k(\Delta^\bullet) \otimes W^{-k}. \tag{39}$$

Now suppose a diffential Q is given turning $(\mathcal{A}_W, Q)$ into a dgca. Q is equivalent to $F_Q \in \mathcal{A}_W \otimes W$ defined by its action on $\xi \in W^*$ by

$$\langle F_Q, \xi\rangle := Q\xi.$$

One can also work in local coordinates - a choice of a homogeneous basis ξ^i of W^* allows us to specify Q by

$$F_Q^i := Q\xi^i \in \mathcal{A}_W.$$

The equation $Q^2 = 0$ becomes

$$F_Q^i \frac{\partial F_Q^j}{\partial \xi^i} = 0.$$

The condition on A to be a dgca morphism

$$A : (\mathcal{A}_W, Q) \to (\Omega(\Delta^\bullet), d)$$

is a *generalized Maurer-Cartan equation*

$$dA = F_Q(A) \tag{40}$$

where $A \in \Omega(\Delta^\bullet, W)^0$.

We thus need to show that solutions of (40) form a Kan simplicial Fréchet manifold. The Fréchet manifold structure is obtained via an integral transform which maps solutions of (40) to solutions of another generalized Maurer-Cartan equation

$$dB = 0 \tag{41}$$

with $B \in \Omega(\Delta^\bullet, W)^0$. Thus the transformation kills the differential Q. Since the solutions of (41) form a simplicial Fréchet space one can transport the Fréchet manifold structure to $K_\bullet^{\mathrm{big}}$ provided the transformation has certain properties.

The integral transform taking solutions of (40) to solutions of (41) uses the de Rham homotopy $h : \Omega(\Delta^\bullet) \to \Omega(\Delta^\bullet)$ operator (of degree -1) associated to the deformation retraction of $\Delta^\bullet \subset \mathbb{R}^\bullet$ to the vertex at the origin. The integral transform is then defined as

$$\kappa(A) := A - h(F_Q(A)).$$

We call this the *Kuranishi map* to acknowledge similarity with a certain mapping appearing in Kuranishi [**22**]. It has the property that $A \in \Omega(\Delta^\bullet, W)$ satisfies (40) iff $B := \kappa(A)$ satisfies (41).

Imposing certain regularity constraints (it is necessary to pass to Banach manifolds to use stable manifold techniques) κ is a smooth open embedding of Banach manifolds.

$K_\bullet^{\text{big}}$ is a Fréchet Kan simplicial manifold. This can be seen as follows. Let $A_{\text{horn}} \in \Omega(\Delta[n,k],W)^{MC}$. The Kuranishi map applied on the faces of $\Delta[n,k] \subset \mathbb{R}^n$ takes A_{horn} to $B_{\text{horn}} \in \Omega(\Delta[n,k],W)^{0,cl}$. Since $\Omega(\Delta[n,k],W)^{0,cl}$ is a simplicial vector space, it is a Kan complex; we can thus fill the horn B_{horn} to $B \in \Omega(\Delta^n,W)^{0,cl}$. However this B may fail to be in the image of κ; to remedy this one actually considers κ constructed on a suitable open neighborhood N of $\Delta[n,k]$ in Δ^n so that $\kappa^{-1}(B)$ exists. The situation is illustrated on Figure 7.

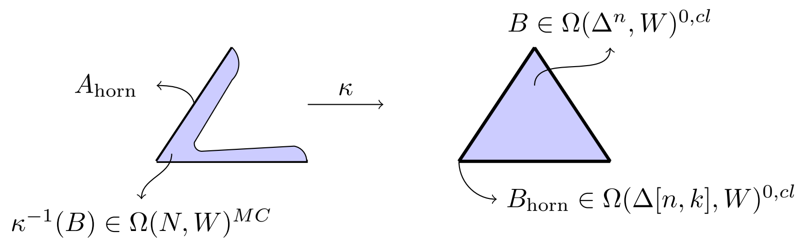

FIGURE 7. Filling horns in $K_\bullet^{\text{big}}$.

Let ϕ be a smooth bump function $\phi \in C^\infty(\Delta^n)$, $\phi|_{\Delta[n,k]} = 1$ and $\phi|_{\Delta^n - N} = 0$ and let $f : \Delta^n \to N$ be given by $f(x) = \phi(x)x$. Then $A := f^*\kappa^{-1}(B)$ is a filler of A_{horn}.

Thus the map $K_n^{\text{big}} \to K_{n,k}^{\text{big}}$ is surjective. To show that it is a submersion, notice first that the linear map $\Omega(\Delta^n,W)^{0,cl} \to \Omega(\Delta[n,k],W)^{0,cl}$ is a submersion. Indeed, it is a surjective linear map, admitting a (continuous) right inverse. The fact that $K_n^{\text{big}} \to K_{n,k}^{\text{big}}$ is a submersion now follows by use of the Kuranishi map: we have the commutative diagram

$$\begin{array}{ccc} K_n^{\text{big}} & \xrightarrow{\kappa} & \Omega(\Delta^n,W)^{0,cl} \\ \downarrow & & \downarrow \\ K_{n,k}^{\text{big}} & \xrightarrow{\kappa} & \Omega(\Delta[n,k],W)^{0,cl} \end{array}$$

where the horizontal arrows are open embeddings.

Furthermore there is a way to extend this integration method from the above trivial negatively graded bundle case (38) to arbitrary $(\mathcal{A}_V, Q)$, that is, to arbitrary NQ-manifolds. The construction of $K_\bullet^{\text{big}}$ is functorial and is referred to as the *big integration*.

4.5. Gauge integration to local Lie ℓ-groupoids. There is a local integration result if one puts constraints on $K_\bullet^{\text{big}}$, such that the simplicial homotopy type of $K_\bullet^{\text{big}}$ does not change.

To express the result we will need the following definition.

DEFINITION 4.5. *A local morphism of simplicial manifolds $X_\bullet \to Y_\bullet$ is a smooth simplicial map $U_\bullet \to Y_\bullet$, where $U_\bullet \subset X_\bullet$ is an open simplicial submanifold containing all fully degenerate simplices. We identify local morphisms if they coincide on some $U_\bullet$.*

Following Getzler's [**16**] construction (28), we restrict solutions of (40) by imposing a certain *gauge condition* to obtain a finite-dimensional Kan manifold $K_\bullet^s$. This should be considered the correct analogy to usual Lie theory as $K_\bullet^s$ has the interpretation of (the nerve of) a local L_∞-groupoid integrating the L_∞-algebroid given by the NQ-manifold defined by $(\mathcal{A}_W, Q)$. Here the letter s in $K_\bullet^s$ stands for small (compared to $K_\bullet^{\text{big}}$) as well as for the choice of a *gauge* $s_\bullet$. A gauge is a simplicial chain homotopy retracting the de Rham complex $\Omega(\Delta^\bullet)$ to the subcomplex of elementary Whitney forms [**40**] $E(\Delta^\bullet)$ with the additional condition that $s_\bullet^2 = 0$. The space of elementary Whitney k-forms on Δ^n is given as the linear span of the differential forms

$$\omega_{i_0,\dots,i_k} := k! \sum_{j=0}^{k} (-1)^j t_{i_j} dt_{i_0} \dots \widehat{dt_{i_j}} \dots dt_{i_k}$$

where we used the standard coordinates $(t_0, \ldots, t_n, \sum_k t_k = 1)$ on $\Delta^n \subset \mathbb{R}^n$. These forms are widely used in numerical mathematics.

The definition of a gauge is motivated from Dupont's [**11**] proof of the simplicial de Rham theorem where he constructed an explicit gauge in terms of Whitney's elementary forms and de Rham homotopy operators. Dupont's gauge is

$$s_n := \sum_{k=0}^{n-1} \sum_{i_0 < \ldots < i_k} \omega_{i_0 \ldots i_k} h_{i_k} \ldots h_{i_0}$$

where $n \geq 0$ and h_i is the de Rham homotopy operator associated to the retraction of $\Delta^\bullet$ to the i-th vertex.

$K^s_\bullet$ is now defined in an analogous way to (28) as

$$K^s_\bullet := \{A \in K^{\mathrm{big}}_\bullet, A \text{ small}, s_\bullet A = 0\}.$$

There is a smallness condition on the forms in the sense that they are in some neighborhood of the constant dg morphisms. $K^s_\bullet$ is a local Lie ℓ-groupoid with ℓ being the degree of the NQ-manifold.

Different choices of a gauge yield non-canonically isomorphic local L_∞-groupoids. Furthermore there is a local simplicial deformation retraction of $K^{\mathrm{big}}_\bullet$ to $K^s_\bullet$ on any local coordinate neighborhood. Thus the two integrals are locally equivalent as local L_∞-groupoids. We call the technique of constructing $K^s_\bullet$ the *gauge integration*.

4.6. Gauge integration to quasi-functors. The gauge integration is not globalizable (since it uses local coordinates) and this seems to be related to higher analogues of the obstruction theory for Lie algebroids. However, there is a global integration result if one loosens the requirement for the integrating object to be a local L_∞-groupoid.

We noted above that the big integration $K^{\mathrm{big}}_\bullet$ is functorial. This is not true for the gauge integration $K^s_\bullet$, since this construction uses local coordinates. However, since there is a local simplicial deformation retraction of $K^{\mathrm{big}}_\bullet$ to $K^s_\bullet$ the functoriality of the big integration gives functoriality up to homotopy of $K^s_\bullet$. More precisely $K^s_\bullet$ is a *quasi-functor* between appropriately chosen simplicially enriched categories. These notions will be explained below.

Let us recall that a simplicially enriched category (sometimes confusingly called a simplicial category) is a small category enriched over **sSet**. Functors of simplicially enriched categories are defined in the obvious manner. The corresponding category is denoted **sSetCat**.

There is a natural generalization of the nerve functor for simplicially enriched categories. Given a choice of a cosimplicial object in a simplicially enriched category

$$S : \Delta \to \mathbf{sSetCat}$$

the *homotopy coherent nerve* (*simplicial nerve*) is a functor

$$N_\bullet : \mathbf{sSetCat} \to \mathbf{sSet},$$
$$N_\bullet(\mathcal{C}) := \mathrm{Hom}_{\mathbf{sSetCat}}(S[\bullet], \mathcal{C}).$$

This is a generalization of the construction (24). We will use the same notation $N_\bullet$ for the nerve and the homotopy coherent nerve functor (this will be clear from context).

There is a natural choice of S as follows. For $[n] \in \Delta$ $S[n]$ is a simplicially enriched category with objects $\{0, \ldots, n\}$ and the simplicial sets

$$\mathrm{Hom}_{S[n]}(i, j) := N_\bullet(P_{ij})$$

where P_{ij} is the poset of paths in the category $[n]$ from i to j with the partial order given by the natural inclusion. This simplicial set is isomorphic to the cube I^{j-i-1}. Finally the composition in $S[n]$ is given by concatenation of the paths.

This definition was given by Cordier [**5**] building on Vogt [**39**]. It was shown by Cordier, Porter [**6**] that if $\mathcal{C}$ is a locally Kan simplicially enriched category then $N_\bullet(\mathcal{C})$ is a quasi-category (introduced by Boardman, Vogt [**4**]). Here a quasi-category is just a weak Kan complex. This means that the natural maps

$$N_n(\mathcal{C}) \to N_{n,k}(\mathcal{C})$$

are surjective for $n > 1$ and $0 < k < n$. This corresponds to (non-unique) filling of inner horns. Thus quasi-categories are similar to ∞-groupoids except that ℓ-morphisms are not required to be (weakly) invertible.

A homotopy coherent diagram in $\mathcal{D}$ of a shape of a small category $\mathcal{C}$ can be thought of as a diagram of shape $\mathcal{C}$ in $\mathcal{D}$ which is not commutative - instead homotopies are assigned to the faces of the diagram in a coherent manner. Let us illustrate this on the case $\mathcal{C} = [3]$ and $\mathcal{D} = \mathbf{sSet}$ using Figure (8).

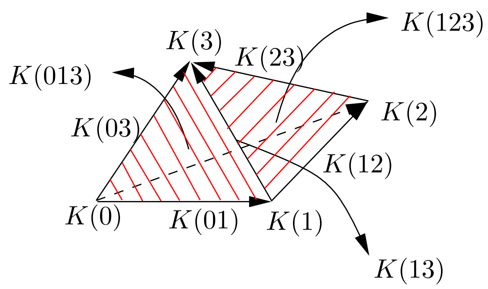

FIGURE 8. Homotopy coherent diagram of shape [3] in **sSet**.

Here $K(0), \dots, K(3)$ are simplicial sets and $K(01), \dots, K(23)$ are simplicial maps. These form homotopy commutative diagram with $K(012), \dots, K(123)$ as simplicial homotopies and $K(0123)$ as a simplicial homotopy of simplicial homotopies. For example

$$K(012) : \Delta[1] \times K(0) \to K(2)$$

is a simplicial homotopy between $K(02)$ and $K(12)K(01)$. Coherence is achieved by

$$K(0123) : \Delta[1]^2 \times K(0) \to K(3)$$

which is a simplicial homotopy between the two compositions

$$\begin{tikzcd} K(03) \arrow[r, "K(023)"] \arrow[d, "K(013)"'] & K(23)K(02) \arrow[d, "K(23)K(012)"] \\ K(13)K(01) \arrow[r, "K(123)K(01)"'] & K(23)K(12)K(01). \end{tikzcd}$$

A simplicially enriched functor $F : S([n]) \to \mathbf{sSet}$ is equivalent to a homotopy coherent diagram of shape $[n]$ in **sSet**. Cordier [**5**] proved in fact a more general result that a homotopy coherent diagram in a simplicially enriched category $\mathcal{D}$ of shape $\mathcal{C}$ is a simplicially enriched functor $S(\mathcal{C}) \to \mathcal{D}$, where $S(\mathcal{C})$ is the simplicially enriched category defined as the free simplicial resolution of $\mathcal{C}$, introduced by Dwyer, Kan [**13**]. This free simplicial resolution is a generalization of the construction of $S([n])$.

We are now ready to describe the gauge integration as a homotopy coherent diagram. Let **locNQMan** be a category of NQ-manifolds associated to dgcas of type

$$(\mathcal{A}_{W|_U}, Q) := (C^\infty(U) \otimes S((W^{<0})^*), Q)$$

for some W non-positively graded vector space and $U \subset W^0$ (this is a generalization of NQ-manifolds of type (38) resp. (39)). We treat it as a simplicially enriched category with the trivial enrichment. Let **locsMan** be the category of local simplicial manifolds. It has objects simplicial manifolds and morphisms are local morphisms of simplicial manifolds. We will use the following simplicial enrichment

$$\mathrm{Hom}_{\mathbf{locsMan}}(K_\bullet, L_\bullet)_n := \mathrm{Hom}_{\mathbf{locsMan}}(K_\bullet \times \Delta[n], L_\bullet).$$

For any NQ-manifold in **locNQMan** applying $K^s_\bullet$ gives a local L_∞-groupoid. If ϕ is a morphism of two such NQ-manifolds we can construct (using the local simplicial deformation retraction) a local morphism of local L_∞-groupoids. $K^s_\bullet$ is not functorial. Instead $K^s_\bullet$ is a homotopy coherent diagram in **locsMan** of shape **locNQMan**.

Let $V \to M$ be a negatively graded vector bundle with a differential on $\mathcal{A}_V$ (36) and let U be a good cover of M. We can now apply the quasi-functor $K^s_\bullet$ on the nerve $N_\bullet(\mathsf{U})$ to obtain a sort of global integral. The homomorphisms $K^s_\bullet(\phi)$ on overlaps of the covering sets (induced by isomorphisms ϕ of NQ-manifolds) are isomorphisms of local Lie ℓ-groupoids.

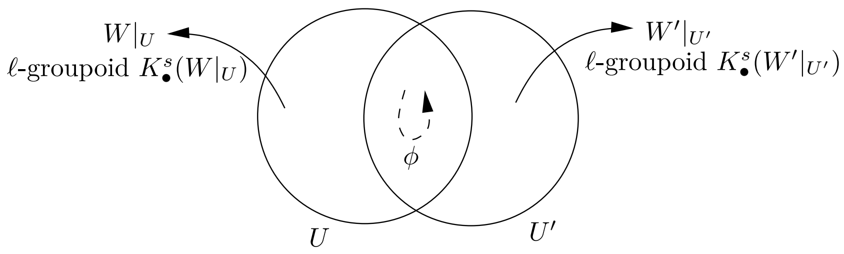

FIGURE 9. The quasi-functor $K^s_\bullet$ applied to $N_\bullet(\mathsf{U})$.

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Part 2

Article

INTEGRATION OF DIFFERENTIAL GRADED MANIFOLDS

PAVOL ŠEVERA AND MICHAL ŠIRAŇ

Abstract. We consider the problem of integration of L_∞-algebroids (differential graded manifolds) to L_∞-groupoids. We first construct a "big" Kan simplicial manifold (Fréchet or Banach) whose points are solutions of a (generalized) Maurer-Cartan equation. The main analytic trick in our work is an integral transformation sending the solutions of the Maurer-Cartan equation to closed differential forms.

Following ideas of Ezra Getzler we then impose a gauge condition which cuts out a finite-dimensional simplicial submanifold. This "smaller" simplicial manifold is (the nerve of) a local Lie ℓ-groupoid. The gauge condition can be imposed only locally in the base of the L_∞-algebroid; the resulting local ℓ-groupoids glue up to a coherent homotopy, i.e. we get a homotopy coherent diagram from the nerve of a good cover of the base to the (simplicial) category of local ℓ-groupoids.

Contents

1. Introduction

Let us recall that a Lie bracket on a finite-dimensional vector space $\mathfrak{g}$ is equivalent to a differential Q on the graded-commutative algebra $\bigwedge \mathfrak{g}^* = S((\mathfrak{g}[1])^*)$. One possible approach to the construction of the integrating simply-connected group G is due to Dennis Sullivan [14]. Consider the simplicial set of morphisms of differential graded commutative algebras

$$K_\bullet^{\mathrm{big}} = \mathrm{Hom}_{dgca}\big((\bigwedge \mathfrak{g}^*, Q), (\Omega(\Delta^\bullet), d)\big)$$

where $\Delta^\bullet$ is the Euclidean simplex and d is the de Rham differential. It can be shown that $K_\bullet^{\mathrm{big}}$ is in fact a simplicial manifold and that its simplicial fundamental group is $\pi_1^{\mathrm{simpl}}(K_\bullet^{\mathrm{big}}) \cong G$.

This can be geometrically explained as follows. Let ξ^i be a basis of $\mathfrak{g}^*$ and c^i_{jk} the structure constants of $\mathfrak{g}$ in this basis, so that $Q\xi^i = c^i_{jk}\xi^j\xi^k/2$. An n-simplex $\mu \in K_n^{\mathrm{big}}$ is determined by

$$A^i := \mu(\xi^i), \qquad i = 1, \ldots, \dim \mathfrak{g},$$

which is a collection of $1-$forms on Δ^n. Since μ respects the differentials we get

$$dA^i = \frac{1}{2} c^i_{jk} A^j A^k. \tag{1}$$

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This is the Maurer-Cartan equation so A is a flat $\mathfrak{g}$-connection on Δ^n. In other words, we can identify the elements of K_n^{big} with flat $\mathfrak{g}$-connections on Δ^n.

The integration of $\mathfrak{g}$ to G can be described as follows. A connection A on an interval (automatically flat) will give rise to an element g_A of the integrating group (the holonomy of the connection along the interval). Two such elements $g_A, g_{A'}$ are equal iff there exists a flat connection $A_{\ominus}$ on a bigon which restricts on the boundary arcs to A and A' as on the left side of Figure 1. The group multiplication is given as $g_{A_1}g_{A_2} = g_{A_3}$ iff there exists a flat connection $A_{\triangle}$ on a triangle which restricts on the boundary in the manner of right side of Figure 1.

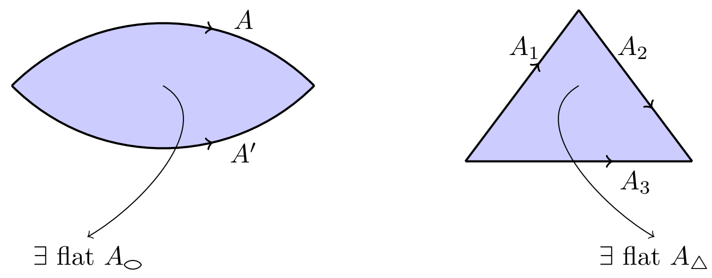

FIGURE 1. Left - two connections give the same group element. Right - group multiplication illustrated.

Let us note that the manifolds K_n^{big} are infinite dimensional. For any $g \in G$ there are infinitely many A's on the interval such that $g = g_A$. Likewise, if $g_{A_1}g_{A_2} = g_{A_3}$, there are infinitely many $A_{\triangle}$'s proving this relation. However, $K_\bullet^{\text{big}}$ can be reduced by gauge-fixing to a "smaller" (finite dimensional) simplicial submanifold $K_\bullet^s$, which is a (local) simplicial deformation retract of $K_\bullet^{\text{big}}$ and which is (locally) isomorphic to the nerve of G (which means a unique A such that $g = g_A$, and also a unique $A_{\triangle}$ restricting to A_1, A_2, A_3 on the boundary). $K_\bullet^s$ is the nerve of the local Lie group integrating $\mathfrak{g}$.

This paper generalizes this procedure to cases where $(\bigwedge \mathfrak{g}^*, Q)$ is replaced by a more general differential graded-commutative algebra. More precisely, we consider generalizations involving the introduction of generators in degrees $i \geq 0$ and allowing smooth functions of the degree-0 generators rather than just polynomials.

Let us thus consider non-negatively graded commutative algebras of the form

$$\mathcal{A}_V = \Gamma(S(V^*)),$$

where $V \to M$ is a negatively graded vector bundle. Suppose that Q is a differential on the algebra $\mathcal{A}_V$. We can say that $\mathcal{A}_V$ is the algebra of functions on the graded manifold V.

If the manifold M is a point then the differential Q is equivalent to an L_∞-algebra structure on the non-positively graded vector space $V[-1]$. In the case of a general M and of V concentrated in degree -1, a differential Q is equivalent to a Lie algebroid structure on $V[-1]$. In the general case, the differential Q is loosely called a L_∞-algebroid structure on the vector bundle $V[-1]$, or more precisely a Lie ℓ-algebroid, where $-\ell$ is the lowest degree in the negatively graded bundle V (so that $\ell \geq 1$). In this paper we will integrate these Lie ℓ-algebroids to Lie ℓ-groupoids.

Following Sullivan as above we shall study morphisms of differential graded algebras

$$(\mathcal{A}_V, Q) \to (\Omega(\Delta^n), d), \tag{2}$$

or equivalently, morphisms of differential graded manifolds

$$T[1]\Delta^n \to V.$$

We can view L_∞-algebroids as Lie algebroids with higher homotopies and the integration procedure as recovering the fundamental $\infty-$groupoid. This integration procedure was suggested in [13]. The motivation comes primarily from the problem of integration of Courant algebroids. Poisson manifolds are integrated to (local) symplectic groupoids and Courant algebroids should be integrated to symplectic 2-groupoids.

While these ideas are well known, in this work we finally overcome the long-standing analytic difficulties. Our first result (Theorem 6.2) says that the morphisms (2) form naturally a (infinite-dimensional Fréchet) Kan simplicial manifold $K_\bullet^{big}$. This result is obtained via an integral transformation (similar to a transformation appearing in the work of Masatake Kuranishi [7]) taking morphisms (2) to morphisms

$$(\mathcal{A}_V, 0) \to (\Omega(\Delta^n), d)$$

which is interesting on its own.

In more detail, in the simplest case when M is an open subset of $\mathbb{R}^n$ and $V \to M$ is a trivial graded vector bundle, we choose generators ξ^i of the algebra $\mathcal{A}_V$ (ξ^i's of degree 0 are the coordinates of $M \subset \mathbb{R}^n$ and ξ^i's of positive degree come from a basis of the fibre of V) then a morphism of dgcas (2) is equivalent to a collection of differential forms $A^i \in \Omega^{\deg \xi^i}(\Delta^\bullet)$ satisfying a generalized Maurer-Cartan equation

$$dA^i = F^i_Q(A) \tag{3}$$

where $F^i_Q := Q\xi^i$. The integral transformation is $A^i \mapsto \kappa(A)^i := A^i - hF^i_Q(A)$, where h is the de Rham homotopy operator, and is transforms the equation (3) to

$$d\kappa(A)^i = 0.$$

We prove that κ is an open embedding (of Banach or Fréchet spaces), and thus show the regularity properties of the space of the solutions of (3).

As noted in the Lie algebra case, the integration can be reduced by gauge-fixing. Following ideas of Ezra Getzler [5] we define (locally in M) a finite-dimensional locally Kan simplicial manifold $K^s_\bullet \subset K^{big}_\bullet$ by imposing a certain gauge condition $s_\bullet$ on differential forms (see Theorem 7.1)

$$K^s_\bullet = \{(A^i \in \Omega^{\deg \xi^i}(\Delta^\bullet)); dA^i = F^i_Q(A) \text{ and } s_\bullet A^i = 0\}.$$

The simplicial manifold $K^s_\bullet$ can be seen as (the nerve of) a local Lie ℓ-groupoid (Definition 7.3) integrating the Lie ℓ-algebroid structure on V ($-\ell$ is the lowest degree in V). While $K^s_\bullet$ depends on the choice of a gauge condition $s_\bullet$ (in particular, on a choice of local coordinates and of a local trivialization of V), we construct a (local) simplicial deformation retraction of $K^{big}_\bullet$ to $K^s_\bullet$, i.e. show that $K^{big}_\bullet$ and $K^s_\bullet$ are equivalent as Lie ∞-groupoids. The deformation retraction also implies that $K^s_\bullet$ is unique up to (non-unique) isomorphisms and that the local $K^s_\bullet$'s form a homotopy coherent diagram.

Let us relate our approach with the papers of Henriques [6] and Getzler [5], who solve closely related problems. André Henriques deals in [6] with the case when the base manifold M is a point. He defined Kan simplicial manifolds and Lie ℓ-groupoids, which are central definitions in our work. His constructions are based on Postnikov towers: in his case it is enough to integrate a Lie algebra to a Lie group and then to deal with Lie algebra cocycles. This approach cannot be used for non-trivial M, so there is little overlap between his and our methods.

The present paper is closer to the work of Ezra Getzler [5] who deals with nilpotent L_∞-algebras (or, from the point of view of Rational homotopy theory [14], with finitely-generated Sullivan algebras). In particular, the idea of gauge fixing is simply taken from [5] and translated from formal power series to the language of Banach manifolds.

Our paper is also closely related to the work of Crainic and Fernandes [2] on integration of Lie algebroids (corresponding to $\ell = 1$). Unlike in *op. cit.* we do not consider the truncation of $K^{big}_\bullet$ at dimension ℓ (keeping all simplices of dimension $< \ell$ intact, replacing those of dimension ℓ with their homotopy classes rel boundary, and adding higher simplices formally), as it leads, for $\ell \geq 2$, to infinite-dimensional spaces. Nonetheless our analytic results should be sufficient also for this kind of approach.

The plan of our paper is as follows. In Section 2 we recall some basic definitions concerning dg manifolds. In Section 3 we prove a technical result about homotopies of maps between dg manifolds, which is the basis for our analytic theorems. In Section 4 we define the Kuranishi map κ and show that it transforms the Maurer-Cartan equations (3) to linear equations. In this way we then prove that the spaces of solutions are manifolds. In Section 5 we prove that the spaces of the solutions of the generalized Maurer-Cartan (3) on simplices form a Kan simplicial manifold (i.e. a Lie ∞-grooupoid) and in Section 6 we globalize the results of Sections 4 and 5. (Sections 5 and 6 are not needed for the rest of the paper, but they complete the picture.) In Section 7 we show that gauge fixing produces a finite-dimensional local Lie ℓ-groupoid. In section 8 we establish a deformation retraction from the big integration to the gauge integration, which is then used in Section 9 to show that the gauge integration is functorial up to coherent homotopies.

Acknowledgment. We would like to thank to Ezra Getzler for useful discussion.

2. DG morphisms and dg manifolds

Let M be a manifold and $V \to M$ a negatively-graded vector bundle. Throughout the paper all vector bundles are finite dimensional. Let $\mathcal{A}_V$ be the graded commutative algebra

$$\mathcal{A}_V = \Gamma(S(V^*)).$$

In particular $\Omega(M) = \mathcal{A}_{T[1]M}$. We can describe morphisms of graded-commutative algebras

$$\mathcal{A}_V \to \Omega(N)$$

(where N is another manifold) in the following way.

If W is a non-positively graded vector space and N a manifold, let

$$\Omega(N,W)^m = \bigoplus\nolimits_{k\geq 0}\Omega^{m+k}(N)\otimes W^{-k}. \tag{4}$$

Abusing the notation, if $V \to M$ is a negatively graded vector bundle, let

$$\Omega(N,V)^0 = \{(f,\alpha);\, f : N \to M, \alpha \in \bigoplus\nolimits_{k>0}\Omega^k(N, f^*V^{-k})\}.$$

The two notations are compatible if we understand W as a trivial vector bundle over W^0, with the fiber $W^{<0}$.

A morphism of graded algebras

$$\mu : \mathcal{A}_V \to \Omega(N)$$

is equivalent to an element $(f,\alpha) \in \Omega(N,V)^0$ via

$$\mu|_{C^\infty(M)} = f^*,\ \mu(s) = \langle \alpha, f^*s\rangle, \qquad \forall s \in \Gamma(V^*).$$

If W is a non-positively graded vector space and $U \subset W^0$ is an open subset, let $W|_U$ be the trivial vector bundle $W^{<0} \times U \to U$, i.e.

$$\Omega(N, W|_U)^0 \subset \Omega(N,W)^0$$

is the set of those forms $A \in \Omega(N,W)^0$ whose function part (a W^0-valued function on N) is a map $N \to U \subset W^0$. Notice that

$$\mathcal{A}_{W|_U} = C^\infty(U) \otimes S\big((W^{<0})^*\big). \tag{5}$$

In this special case of $V = W|_U$ a morphism $\mu : \mathcal{A}_{W|_U} \to \Omega(N)$ corresponds to $A \in \Omega(N, W|_U)^0$ via

$$\mu(\xi) = \langle A, \xi\rangle, \qquad \forall \xi \in W^*.$$

Let now Q be a differential on the graded algebra $\mathcal{A}_{W|_U}$. If ξ^i is a (homogeneous) basis of W^*, let

$$F^i_Q := Q\,\xi^i \in \mathcal{A}_{W|_U}.$$

The identity $Q^2 = 0$ is equivalent to

$$F^i_Q \frac{\partial F^k_Q}{\partial \xi^i} = 0. \tag{6}$$

A morphism of graded algebras $\mu : \mathcal{A}_{W|_U} \to \Omega(N)$ is a differential graded (dg) morphism iff

$$dA^i = F^i_Q(A) \tag{7}$$

where $A^i := \mu(\xi^i) = \langle A, \xi^i\rangle$ and $F^i_Q(A)$ is obtained from F^i_Q by substituting A^k's for ξ^k's. This equation generalizes the Maurer-Cartan equations of the Lie algebra case (1) where F^i_Q is quadratic. We can rewrite (7) as

$$dA = F_Q(A) \tag{8}$$

where $F_Q \in \mathcal{A}_{W|_U} \otimes W$ is given by

$$\langle F_Q, \xi\rangle = Q\xi \quad \forall \xi \in W^*.$$

The set of solutions A of (8) will be denoted by

$$\Omega(N, W|_U)^{MC} \subset \Omega(N, W|_U)^0.$$

More generally, if Q is a differential on the algebra $\mathcal{A}_V$, the set of dg morphisms

$$\mathcal{A}_V \to \Omega(N)$$

will be denoted by

$$\Omega(N,V)^{MC} \subset \Omega(N,V)^0.$$

If $V \to M$ is a negatively-graded vector bundle, it is convenient to see the algebra $\mathcal{A}_V$ as the algebra of functions on the graded manifold (corresponding to) V.

Definition 2.1 (e.g. Ševera [13]). *An* N-manifold *(shorthand for* non-negatively graded manifold*) is a supermanifold Z with an action of the semigroup $(\mathbb{R}, \times)$ such that $-1 \in \mathbb{R}$ acts as the parity operator (i.e. just changes the sign of the odd coordinates).*

Let $C^\infty(Z)^k$ be the vector space of smooth functions of degree k, where the degree is the weight with respect to the action of $(\mathbb{R}, \times)$, and let $C^\infty(Z) = \bigoplus_{k\geq 0} C^\infty(Z)^k$. It is a graded-commutative algebra.

Any negatively-graded vector bundle $V \to M$ gives rise to a N-manifold via

$$C^\infty(Z) = \Gamma(S(V^*)),$$

such that $0 \cdot Z = M$. Any N-manifold (in the C^∞-category) is of this type. By abuse of notation we shall denote the N-manifold corresponding to V also by V. A morphism of graded algebras $\mathcal{A}_V \to \mathcal{A}_{V'}$ is thus equivalent to a morphism of N-manifolds $V' \rightsquigarrow V$. We shall indicate morphisms of graded manifolds with wiggly arrows ($\rightsquigarrow$). This is to prevent confusion since the category of N-manifolds contains more morphisms then the category of negatively-graded vector bundles. In particular, the following are equivalent: a morphism of graded algebras

$$\mathcal{A}_V \to \Omega(N),$$

an element of $\Omega(N,V)^0$, and a morphism of N-manifolds

$$T[1]N \rightsquigarrow V.$$

A differential Q on the graded algebra $\mathcal{A}_V$ then corresponds to the following notion.

Definition 2.2. *An* NQ-manifold *(a* differential non-negatively graded manifold*) is an N-manifold equipped with a vector field Q of degree 1 satisfying $Q^2 = 0$.*

In particular, a morphism of dg-algebras $\mathcal{A}_V \to \Omega(N)$ is equivalent to a morphism of NQ-manifolds $T[1]N \rightsquigarrow V$.

3. A homotopy lemma

Let us suppose, as above, that W is a non-positively graded finite-dimensional vector space, $U \subset W^0$ is an open subset and Q a differential on the algebra $\mathcal{A}_{W|_U}$ defined by (5). A morphism of graded algebras $\mathcal{A}_{W|_U} \to \Omega(N)$ (or a map of N-manifolds $T[1]N \rightsquigarrow W|_U$) is thus equivalent to a choice of a differential form $A \in \Omega(N, W|_U)^0$, and it is a dg morphism iff $dA = F_Q(A)$, i.e. iff $A \in \Omega(N, W|_U)^{MC}$ (see Section 2).

In this section we shall study "homotopies", i.e. solutions of the MC equation $dA = F_Q(A)$ on $N \times I$, where I is the unit interval.

If $\alpha \in \Omega(N \times I)$, let $\alpha|_{t=0} \in \Omega(N)$ (where t is the coordinate on $I = [0,1]$) be the restriction of α to $N = N \times \{0\} \subset N \times I$. Any differential form $\alpha \in \Omega(N \times I)$ can be split uniquely into a horizontal and vertical part as

$$\alpha = \alpha_h + dt\, \alpha_v$$

$$i_{\partial_t}\alpha_h = i_{\partial_t}\alpha_v = 0.$$

The components α_h and α_v are given by

$$\alpha_v = i_{\partial_t}\alpha,\ \ \alpha_h = \alpha - dt\, \alpha_v.$$

We can view α_h and α_v as forms on N parametrized by I. For any α we set

$$d_h\alpha = (d\alpha)_h.$$

Proposition 3.1 ("Homotopy Lemma"). *Let N be a manifold (possibly with corners) and let $A \in \Omega(N \times I, W|_U)^0$, where I is the unit interval. If*

$$A|_{t=0} \in \Omega(N, W|_U)^{MC} \tag{9a}$$

and if

$$i_{\partial_t}(dA - F_Q(A)) = 0, \tag{9b}$$

then

$$A \in \Omega(N \times I, W|_U)^{MC}$$

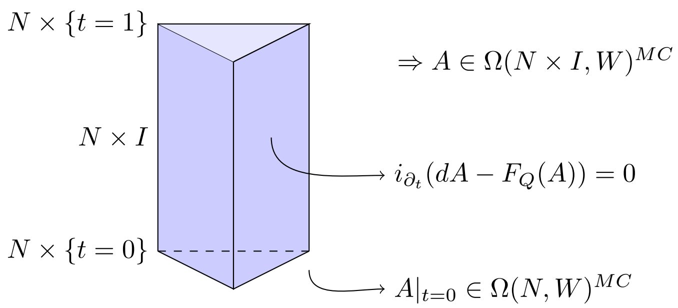

Figure 2. Homotopy Lemma.

Proof. Writing $A = A_h + dt\, A_v$ we get

$$dA - F_Q(A) = (d_h A_h - F_Q(A_h)) + dt\Big(\frac{d}{dt}A_h - d_h A_v - A_v^i \frac{\partial F_Q}{\partial \xi^i}(A_h)\Big) \tag{10}$$

and since $i_{\partial_t}(dA - F_Q(A)) = 0$, we get from here

$$\frac{d}{dt}A_h = d_h A_v + A_v^i \frac{\partial F_Q}{\partial \xi^i}(A_h). \tag{11}$$

We can now compute

$$\frac{d}{dt}(d_h A_h - F_Q(A_h))$$

using (11). We get

$$\frac{d}{dt}(d_h A_h - F_Q(A_h)) = (-1)^{\deg \xi^i - 1} A_v^i\, d_h A_h^k \frac{\partial^2 F_Q}{\partial \xi^k \partial \xi^i}(A_h) - A_v^i \frac{\partial F_Q^k}{\partial \xi^i}(A_h) \frac{\partial F_Q}{\partial \xi^k}(A_h).$$

Since Equation (6) implies

$$0 = \frac{\partial}{\partial \xi^i}\Big(F_Q^k(A_h)\frac{\partial F_Q}{\partial \xi^k}(A_h)\Big) = \frac{\partial F_Q^k}{\partial \xi^i}(A_h)\frac{\partial F_Q}{\partial \xi^k}(A_h) - (-1)^{\deg \xi^i} F_Q^k(A_h)\frac{\partial^2 F_Q}{\partial \xi^k \partial \xi^i}(A_h),$$

we get

$$\frac{d}{dt}(d_h A_h - F_Q(A_h)) = (-1)^{\deg \xi^i - 1} A_v^i (d_h A_h^k - F_Q^k(A_h)) \frac{\partial^2 F_Q}{\partial \xi^k \partial \xi^i}(A_h).$$

This is a linear differential equation for $d_h A_h - F_Q(A_h)$, which together with the initial condition (9a) implies that

$$d_h A_h - F_Q(A_h) = 0$$

and thus, in view of (9b), also

$$dA - F_Q(A) = 0.$$

This completes the proof. $\square$

Theorem 3.1. *Let N be a compact manifold (possibly with corners), let $A_0 \in \Omega(N, W|_U)^{MC}$, and let $H \in \Omega(N \times I, W)^{-1}$ be a horizonal form (i.e. $i_{\partial_t} H = 0$). Then there is $0 < \epsilon \leq 1$ and a unique $A \in \Omega(N \times [0, \epsilon], W|_U)^{MC}$ such that $A|_{t=0} = A_0$ and $A_v = H$. It is given by $A = A_h + dt\, H$, where A_h is the solution of the differential equation*

$$\frac{d}{dt} A_h = d_h H + H^i \frac{\partial F_Q}{\partial \xi^i}(A_h) \tag{12}$$

with the initial condition $A_h|_{t=0} = A_0$. We can choose $\epsilon = 1$ if the C^0-norm of $H^{(0)}$ is small enough, where $H^{(0)} : N \times I \to W^{-1}$ is the 0-form part of H.

Proof. By Proposition 3.1 and Equation (10) we see that $A \in \Omega(N \times I, W|_U)^{MC}$ iff the condition (11) holds with $A_v = H$, i.e. if the ODE (12) holds. To see that an estimate of the C^0-norm of $H^{(0)}$ implies that (12) has a solution for $t \in [0, 1]$, let us split A_h to its homogeneous parts

$$A_h = \sum_k A_h^{(k)}, \quad A_h^{(k)} \in \Omega^k(N \times I, W^{-k}).$$

In degree 0 Equation (12) is

$$\frac{d}{dt} A_h^{(0)} = (H^{(0)})^i \frac{\partial F_Q}{\partial \xi^i}(A_h^{(0)}) \tag{13}$$

with i running only over the indices with $\deg \xi^i = 1$, which has a solution under our hypothesis. Supposing that $A_h^{(m)}$'s are known for $m < k$, Equation (12) in degree k is an inhomogeneous linear ODE for $A_h^{(k)}$, and thus always has a solution. We can thus find A_h by solving (12) successively for $A_h^{(0)}$, $A_h^{(1)}$, ..., $A_h^{(\dim N)}$. □

Remark. The hypothesis on the C^0-norm of $H^{(0)}$ was used to make sure that a solution $A_h^{(0)}$ of Equation (13) exists for $t \in I$. If $W^0 = 0$, no hypothesis is needed, since $A_h^{(0)} = 0$. When $W^0 \neq 0$ then $Q : C^\infty(U) \to C^\infty(U) \otimes (W^{-1})^*$ can be seen as a linear map from W^{-1} to the space of vector fields on $C^\infty(U)$, and defines an integrable distribution on U. If $A_0 \in \Omega(N, W|_U)^{MC}$ then the image of $A_0^{(0)} : N \to U$ is contained in a leaf of this distribution. If the leaf if compact, again no hypothesis of the C^0-norm of $H^{(0)}$ is needed, since (13) is given by vector fields tangent to the leaf. The hypothesis is needed only if the closure of the leaf is non-compact.

Theorem 3.1 can be used to solve the generalized Maurer-Cartan equation $dA = F_Q(A)$ on cubes. We shall describe another method in the following section.

4. Solving the generalized Maurer-Cartan equation

Suppose now that $N \subset \mathbb{R}^n$ is a star-shaped n-dimensional submanifold with corners (typically we would take for N a n-simplex with a vertex at the origin, or a ball centered at the origin). Let

$$h : \Omega^\bullet(N) \to \Omega^{\bullet-1}(N)$$

be the de Rham homotopy operator given by the deformation retraction

$$R : N \times I \to N, \; R(x, t) = tx.$$

Let, as above, $U \subset W^0$ be an open subset and Q a differential on the algebra $\mathcal{A}_{W|_U}$.

Theorem 4.1. *A form $A \in \Omega(N, W|_U)^0$ satisfies*

$$dA = F_Q(A) \tag{14}$$

if and only if the form

$$B = A - h(F_Q(A)) \in \Omega(N, W)^0$$

satisfies

$$dB = 0. \tag{15}$$

Proof. If $dA = F_Q(A)$ then $dhF_Q(A) = dhdA = dA$, hence B is closed.

Suppose now that $dB = 0$. Let E be the Euler vector field on $N \subset \mathbb{R}^n$. By construction of h we have $i_E h = 0$, hence $i_E dh = i_E$, and thus

$$i_E(dA - F_Q(A)) = i_E(dA - dhF_Q(A)) = i_E dB = 0.$$

Since the vector field E on N and the vector field $t\partial_t$ on $N \times I$ are R-related, we get

$$i_{\partial_t}(d\,R^*A - F_Q(R^*A)) = 0.$$

As $(R^*A)|_{t=0}$ is a constant we have $(R^*A)|_{t=0} \in \Omega(N, W|_U)^{MC}$. Proposition 3.1 now implies that $d\,R^*A - F_Q(R^*A) = 0$, and therefore (by setting $t = 1$) $dA = F_Q(A)$. □

Let us show that the map $A \mapsto B = A - h(F_Q(A))$ is injective by describing explicitly its inverse.

Proposition 4.1. *Let $B \in \Omega(N, W)^0$ be such that the 0-form part of B (a map $N \to W^0$) sends $0 \in N \subset \mathbb{R}^n$ to an element of $U \subset W^0$. The equation*

$$\mathcal{L}_E a = F_Q(B + i_E a) \tag{16}$$

($\mathcal{L}_E$ is Lie derivative along E) has a solution $a \in \Omega(N', W)^1$ on $N' \subset N$ where N' is some star-shaped open neighborhood of 0 and the solution is unique if we demand N' to be maximal.

A form $A \in \Omega(N, W|_U)^0$ such that

$$B = A - h(F_Q(A))$$

exists iff $N' = N$, and in that case A is unique, $A = B + i_E a$.

Proof. We shall define a vector field $\hat{E}$ on the total space of

$$\hat{N} := (\textstyle\bigwedge T^*N \otimes W)^1$$

with the following properties:

1. A (partial) section $a : N' \to \hat{N}$ ($N' \subset N$) of the bundle $\hat{N} \to N$ is a solution of (16) iff the vector field $\hat{E}$ is tangent to the image of a.
2. $\hat{E}$ projects to the Euler vector field E on N.
3. $\hat{E}$ has a unique fixed point $P \in \hat{N}$ (lying over $0 \in N$). The fixed point is hyperbolic: the stable subspace of $T_P\hat{N}$ is the vertical subspace, and the unstable subspace projects bijectively onto $T_0 N$.

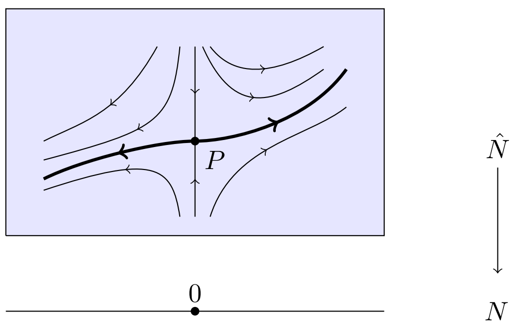

FIGURE 3. The section a is the unstable manifold of $\hat{E}$

Once $\hat{E}$ is defined, the proposition can be proven as follows. A partial section $a : N' \to \hat{N}$, where $N' \subset N$ is a star-shaped neighbourhood of $0 \in N$, is a solution of (16) iff the image of a is a local unstable manifold of $\hat{E}$. Since $\hat{E}$ projects onto E, the (full) unstable manifold of $\hat{E}$ is the image of some section a_{max}, which is the unique maximal solution of (16) we wanted to find.

The existence and uniqueness of A can then be proven as follows. If $N' = N$ then $A = B + i_E a$ satisfies $B = A - h(F_Q(A))$. To get uniqueness, notice that the operator $\mathcal{L}_E$ is invertible on $\Omega^{>0}(N)$ (its inverse can be written explicitly as an integral), and $h = i_E \mathcal{L}_E^{-1}$. If A satisfying

$B = A - h(F_Q(A))$ exists, let us set $a = \mathcal{L}_E^{-1} F_Q(A)$. As $A = B + h\mathcal{L}_E a = B + i_E a$, a is a solution of (16) with $N' = N$, and A is thus unique.

The vector field $\hat{E}$ on $\hat{N}$ is constructed as follows. Consider the natural lift E_{lift} of E to $\hat{N}$. In coordinates it is of the form

$$E_{\text{lift}} = x^i \frac{\partial}{\partial x^i} - \sum_J k_J y^J \frac{\partial}{\partial y^J}$$

where x^i's are the coordinates on N and y^J are the additional coordinates on $\hat{N}$ (corresponding to $dx^{j_1} \dots dx^{j_l}$). Notice that $k_J > 0$ for every J (all k_J's are integers).

Let us define the vector field $\hat{E}$ on $\hat{N}$ by

$$\hat{E} = E_{\text{lift}} + F_Q(B + i_E \,\cdot),$$

where

$$F_Q(B + i_E \,\cdot) : (\textstyle\bigwedge T^*N \otimes W)^1 \to (\textstyle\bigwedge T^*N \otimes W)^1$$

is understood as a vertical vector field. The vector field $\hat{E}$ has the needed properties; to see the hyperbolicity of the fixed point $P \in \hat{N}$ of N notice that the stable subspace of $T_P\hat{N}$ is the vertical subspace, with eigenvalues $-k_J$, and the unstable subspace projects bijectively onto $T_0 N$, with all eigenvalues equal to 1. □

Let us suppose that N is compact. For $r \geq 1$ let $\Omega_r(N)$ denote the Banach space of C^r-forms. The previous two results (Theorem 4.1 and Proposition 4.1) remain valid when A, B and a are C^r-forms.

Proposition 4.2. *The map*

$$\kappa : \Omega_r(N, W|_U)^0 \to \Omega_r(N, W)^0$$

$$\kappa(A) = A - h(F_Q(A)) \tag{17}$$

is a smooth open embedding of Banach manifolds.

Proof. The map κ is smooth. By Proposition 4.1 it is injective and its image is open. The inverse map (constructed in the proof of Proposition 4.1 via the unstable manifold of a vector field), defined on the image of κ, is at least C^r, by differentiable dependence of unstable manifold on parameters (Robbin [12, Theorem 4.1]). This implies that κ is a smooth open embedding. □

A map similar to (17) appears in the work of Kuranishi [7] and we will refer to it as the Kuranishi map.

Theorem 4.2. *The subset*

$$\{A \in \Omega_r(N, W|_U)^0;\, dA - F_Q(A) = 0\} =: \Omega_r(N, W|_U)^{MC} \subset \Omega_r(N, W|_U)^0,$$

is a (smooth) Banach submanifold and it is closed in the C^0-topology. The subset

$$\Omega(N, W|_U)^{MC} \subset \Omega(N, W|_U)^0$$

is a Fréchet submanifold closed in the C^0-topology.

Proof. By Theorem 4.1 we have

$$\Omega_r(N, W|_U)^{MC} = \kappa^{-1}\big(\Omega_r(N, W)^{0,cl}\big).$$

Since κ is an open embedding, the theorem follows from the fact that the space of closed forms

$$\Omega_r(N, W)^{0,cl} \subset \Omega_r(N, W)^0$$

is a C^0-closed subspace and from C^0-continuity of κ. Since the result is true for every $r \geq 1$, it also holds for C^∞-forms. □

The rest of this section is a preparation for the gauge-fixing procedure of Section 7.

It is somewhat inconvenient that $\Omega_r(N)$ is not a complex (i.e. that d is not an everywhere-defined operator $\Omega_r(N) \to \Omega_r(N)$). Following A. Henriques [6] let us consider the complex

$$\Omega_{r+}(N) := \{\alpha \in \Omega_r(N);\, d\alpha \in \Omega_r(N)\}$$

which is a Banach space with the norm

$$\|\alpha\|_{r+} := \|\alpha\|_{C^r} + \|d\alpha\|_{C^r}.$$

We can identify $\Omega_{r+}(N)$ with the closed subspace

$$\Gamma := \{(\alpha,\beta); d\alpha = \beta\} \subset \Omega_r(N) \oplus \Omega_r(N),$$

i.e. with the graph of the unbounded operator $d : \Omega_r(N) \to \Omega_r(N)$. The isomorphism $\Gamma \to \Omega_{r+}(N)$ is given by the projection $(\alpha,\beta) \mapsto \alpha$.

We have

$$\Omega_r(N, W|_U)^{MC} \subset \Omega_{r+}(N,W)^0$$

as for $A \in \Omega_r(N, W|_U)^{MC}$ we have $dA = F_Q(A) \in \Omega_r(N,W)$.

Proposition 4.3. *$\Omega_r(N, W|_U)^{MC} \subset \Omega_{r+}(N,W)^0$ is a Banach submanifold.*

Proof. Let us consider the embedding

$$e : \Omega_r(N,W)^0 \to \Omega_r(N,W)^0 \oplus \Omega_r(N,W)^1, \quad e(A) = (A, F_Q(A)).$$

For $A \in \Omega_r(N, W|_U)^{MC}$ we have $e(A) = (A, dA)$, hence e embeds $\Omega_r(N, W|_U)^{MC}$ to $\Gamma \otimes W$. The isomorphism $\Gamma \otimes W \cong \Omega_{r+}(N,W)$ then implies that $\Omega_r(N, W|_U)^{MC} \subset \Omega_{r+}(N,W)^0$ is a submanifold. □

If $A_c \in \Omega(N, W|_U)^0$ is a constant (i.e. if the 0-form component of A_c is a constant map $N \to U$ and the higher-form components of A_c vanish) then $A_c \in \Omega(N, W|_U)^{MC}$ and $\kappa(A_c) = A_c$. We shall identify constant A_c's with elements of U, i.e. we have an inclusion $U \subset \Omega(N, W|_U)^{MC}$ and $\kappa|_U = \mathrm{id}_U$.

Let us notice that the map

$$A \mapsto A - hdA$$

is a projection

$$\Omega_{r+}(N,W)^0 \to \Omega_{r+}(N,W)^{0,cl} = \Omega_r(N,W)^{0,cl}$$

which coincides with κ on $\Omega_r(N, W|_U)^{MC}$. Let us now consider more general projections (equivalently, let us replace h with another homotopy operator).

Proposition 4.4. *Let $C \subset \Omega_{r+}(N)$ be a graded Banach subspace such that $\Omega_{r+}(N) = \Omega_r(N)^{cl} \oplus C$. Then there is an neighbourhood $U \subset \mathcal{U} \subset \Omega_r(N, W|_U)^{MC}$ such that the projection w.r.t. $(C \otimes W)^0$*

$$\pi : \Omega_{r+}(N,W)^0 \to \Omega_r(N,W)^{0,cl}$$

restricts to an open embedding $\mathcal{U} \to \Omega_r(N,W)^{0,cl}$ and to the indentity on U.

Proof. Let $\pi^{res} := \pi|_{\Omega_r(N,W|_U)^{MC}}$. To show that there exists $\mathcal{U}$ with the desired property it is enough to show that the tangent map $T_{A_c}\pi^{res}$ is a linear isomorphism for each $A_c \in U$, or equivalently, that the composition

$$\Omega_r(N,W)^{0,cl} \xrightarrow{\kappa_{\mathrm{lin}}^{-1}} T_{A_c}\Omega_r(N, W|_U)^{MC} \subset \Omega_{r+}(N,W)^0 \xrightarrow{\pi} \Omega_r(N,W)^{0,cl} \tag{18}$$

is a linear isomorphism, where $\kappa_{\mathrm{lin}} = T_{A_c}\kappa$ is the linearization of κ at A_c. Explicitly,

$$\kappa_{\mathrm{lin}}(A) = A - hF_{\mathrm{Q,lin}}(A),$$

where

$$F^i_{\mathrm{Q,lin}}(A) = \frac{\partial F^i_Q}{\partial \xi^j}(A_c)A^j$$

is the linearization of F_Q at A_c.

Let us introduce a filtration $\mathcal{F}$ of the space $\Omega_{r+}(N,W)^0$ with

$$\mathcal{F}^i = \bigoplus_{k \le i} \Omega^k_{r+}(N, W^{-k}).$$

Then $hF_{Q,\mathrm{lin}} : \mathcal{F}^i \to \mathcal{F}^{i-1}$ and thus $\kappa_{\mathrm{lin}}, \kappa_{\mathrm{lin}}^{-1} : \mathcal{F}^i \to \mathcal{F}^i$.

To show that $T_{A_c}\pi^{res}$ is a linear isomorphism it is enough to verify that the filtered linear map (18) induces the identity map on the associated graded. Let $B \in \Omega_r(N,W)^{0,cl}$ and $B \in \mathcal{F}^i$. Then

$$\kappa_{\mathrm{lin}}^{-1}(B) = B + hF_{Q,\mathrm{lin}}(\kappa_{\mathrm{lin}}^{-1}(B)).$$

Since $hF_{Q,\mathrm{lin}}(\kappa_{\mathrm{lin}}^{-1}(B)) \in \mathcal{F}^{i-1}$ and B is closed, we see that the associated graded is indeed the identity. □

5. Filling horns

Let Δ^n be the n-dimensional simplex. By Theorem 4.2, the sets

$$K_n^{big}(\mathcal{A}_{W|_U}, Q) := \Omega(\Delta^n, W|_U)^{MC}$$

are naturally Fréchet manifolds. The affine maps between simplices then make the collection of K_n^{big}'s to a simplicial Fréchet manifold.

For any simplicial set $S_\bullet$ and any $0 \le k \le n$ let $S_{n,k}$ denote the corresponding horn. The k-th horn of the geometric n-simplex Δ^n will be denoted Δ^n_k.

Definition 5.1 (A. Henriques)**.** *A simplicial manifold $K_\bullet$ is* Kan *if the map $K_n \to K_{n,k}$ is a surjective submersion for any $0 \le k \le n$.*

To make the definition meaningful, one needs to check (inductively in n) that $K_{n,k}$ are manifolds. This was done by Henriques [6].

Any simplicial topological vector space $X_\bullet$ is automatically a Kan simplicial manifold. Indeed, there is an explicit continuous linear map $X_{n,k} \to X_n$ due to Moore [10] (in the proof of his theorem stating that any simplicial group is Kan) which is right-inverse to the horn map $X_n \to X_{n,k}$. Namely, supposing without loss of generality that $k = n$, if $(y_0, \dots, y_{n-1}) \in X_{n,n}$ ($y_i \in X_{n-1}$), one defines $w_0, \dots, w_{n-1} \in X_n$ via

$$w_0 = s_0 y_0, \quad w_i = w_{i-1} - s_i d_i w_{i-1} + s_i y_i \tag{19}$$

($i = 1, \dots, n-1$, d_i and s_i are the face and degeneracy maps respectively), and then $d_i w_{n-1} = y_i$ for all $0 \le i \le n-1$, i.e. w_{n-1} fills the horn $(y_0, \dots, y_{n-1}) \in X_{n,n}$.

Here is the principal result of this section (it is valid also for C^r-forms, when we get Kan simplicial Banach manifolds).

Theorem 5.1. *$K_\bullet^{big}(\mathcal{A}_{W|_U}, Q)$ is a Kan simplicial Fréchet manifold.*

Proof. Let $K_\bullet := K_\bullet^{big}(\mathcal{A}_{W|_U}, Q)$ and let $X_\bullet := \Omega(\Delta^\bullet, W)^{0,cl}$. As $X_\bullet$ is a simplicial Fréchet vector space, it is a Kan simplicial manifold.

Let us first prove that the horn maps $K_n \to K_{n,k}$ are submersion. Let us place Δ^n to $\mathbb{R}^n$ so that the k'th vertex is at $0 \in \mathbb{R}^n$. By applying the Kuranishi map κ we get the commutative square

$$\begin{array}{ccc} K_n & \xrightarrow{\kappa} & X_n \\ \downarrow & & \downarrow \\ K_{n,k} & \xrightarrow{\kappa} & X_{n,k} \end{array}$$

As the horizontal arrows are open embeddings and the vertical arrow $X_n \to X_{n,k}$ is a submersion, $K_n \to K_{n,k}$ is also a submersion.

It remains to prove that $K_n \to K_{n,k}$ is surjective. Let $A_{\mathrm{horn}} \in K_{n,k}$, let $B_{\mathrm{horn}} := \kappa(A) \in X_{n,k}$. We choose $B \in X_n$ extending B_{horn}.

By Proposition 4.1, Equation (16) has a solution a on an open subset $N' \subset \Delta^n$ such that $\Delta^n_k \subset N'$. The form

$$A' = B + i_E a$$

thus satisfies

$$A' \in \Omega\big(N', W|_U\big)^{MC}, \ A'|_{\Delta^n_k} = A_{\mathrm{horn}}.$$

Let now $\phi \in C^\infty(\Delta^n)$ satisfy $0 \le \phi \le 1$, $\phi|_{\Delta^n_k} = 1$, and $\phi|_{\Delta^n \setminus N'} = 0$. Let $h : \Delta^n \to N'$ be given by $x \mapsto \phi(x)x$. Then

$$A := h^* A'$$

satisfies

$$A \in \Omega\big(\Delta^n, W|_U\big)^{MC} = K_n, \ A|_{\Delta^n_k} = A_{\mathrm{horn}}.$$

This proves surjectivity of $K_n \to K_{n,k}$, as the map sends $A \in K_n$ to $A_{\mathrm{horn}} \in K_{n,k}$. ∎

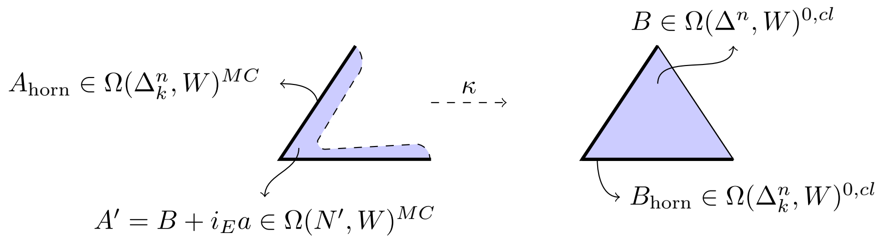

FIGURE 4. Filling horns in $K^{\mathrm{big}}_\bullet$.

6. GLOBALIZATION

So far we considered dg algebras of the form $\mathcal{A}_{W|_U}$. Let now $V \to M$ be a negatively-graded vector bundle, and let us consider the algebra $\mathcal{A}_V$ with a chosen differential Q. In this section we shall prove that Theorems 4.2 and 5.1 hold also in this more general setting.

To prove these results we need to consider the following relative situation. Let $N \subset \mathbb{R}^n$ be a compact star-shaped n-dimensional submanifold with corners. The restriction $\mathcal{A}_{T[1]\mathbb{R}^n} = \Omega(\mathbb{R}^n) \to \Omega(N)$ is a morphism of dgcas. The corresponding element $A_0 \in \Omega(N, T[1]\mathbb{R}^n)^{MC}$ is

$$A_0 = \sum_k E_k x^k + e_k dx^k$$

where x^k are the coordinates on $N \subset \mathbb{R}^n$ and E_k and e_k is the standard basis of $\mathbb{R}^n$ and of $\mathbb{R}^n[1]$ respectively. Application of κ gives

$$B_0 := \kappa(A_0) = \sum_k e_k dx^k \in \Omega(N, T[1]\mathbb{R}^n)^{0,cl}.$$

Suppose now that W is a non-positively graded vector space and $p : W \to T[1]\mathbb{R}^n$ a surjective graded linear map. Let $U \subset W^0$ be open, Q be a differential on $\mathcal{A}_{W|_U}$, and let us suppose that the pullback $p^* : \mathcal{A}_{T[1]\mathbb{R}^n} = \Omega(\mathbb{R}^n) \to \mathcal{A}_{W|_U}$ is a morphism of dgcas (i.e. that $p : W|_U \to T[1]\mathbb{R}^n$ is a morphism of NQ manifolds). Let

$$\Omega(N, W|_U)^{0,A_0} := \{A \in \Omega(N, W|_U)^0; p(A) = A_0\}$$

and similarly

$$\Omega(N, W)^{0,B_0} := \{B \in \Omega(N, W)^0; p(B) = B_0\}.$$

The elements of

$$\Omega(N, W|_U)^{MC,A_0} := \Omega(N, W|_U)^{MC} \cap \Omega(N, W|_U)^{0,A_0}$$

correspond to those dgca morphisms $\mathcal{A}_{W|_U} \to \Omega(N)$ for which the diagram

$$\begin{tikzcd} \Omega(N) & \mathcal{A}_{W|_U} \arrow[l] \\ & \Omega(\mathbb{R}^n) \arrow[ul] \arrow[u, "p^*"'] \end{tikzcd}$$

is commutative, i.e. of those morphisms $T[1]N \rightsquigarrow W|_U$ of NQ manifolds for which

$$\begin{tikzcd} T[1]N \arrow[r, rightsquigarrow] \arrow[dr, rightsquigarrow] & W|_U \arrow[d, rightsquigarrow, "p"] \\ & T[1]\mathbb{R}^n \end{tikzcd}$$

is commutative.

Proposition 6.1.

$$\Omega(N, W|_U)^{MC,A_0} = \kappa^{-1}\big(\Omega(N, W)^{0,cl,B_0}\big).$$

In particular, $\Omega(N, W|_U)^{MC,A_0} \subset \Omega(N, W|_U)^{0,A_0}$ *is a Fréchet submanifold closed in the* C^0 *topology.*

Proof. The equality follows from Theorem 4.1 and from $B_0 = \kappa(A_0)$. The fact that
$$\Omega(N, W|_U)^{MC,A_0} \subset \Omega(N, W|_U)^{0,A_0}$$
is a Fréchet submanifold closed in the C^0 topology then follows from the fact that κ is an open embedding and C^0-continuous, and from the fact that $\Omega(N, W)^{0,cl,B_0}$ is a C^0-closed affine subspace of $\Omega(N, W)^0$. □

We can now prove our first globalized result.

Theorem 6.1. *Let $N \subset \mathbb{R}^n$ be a star-shaped compact n-dimensional submanifold with corners, $V \to M$ a negatively graded vector bundle, and Q a differential on the gca $\mathcal{A}_V$. Then the subset*
$$\{\textit{dg-morphisms } \mathcal{A}_V \to \Omega(N)\} \subset \{\textit{graded algebra morphisms } \mathcal{A}_V \to \Omega(N)\},$$
i.e.
$$\Omega(N, V)^{MC} \subset \Omega(N, V)^0,$$
is a smooth Fréchet submanifold closed in the C^0-topology.

Proof. For any smooth map $f : N \to M$ we can find an open subset $U' \subset \mathbb{R}^n \times M$ containing the graph of f such that

(1) there is an open subset $U \subset \mathbb{R}^n \times \mathbb{R}^m$ and a diffeomorphism $g : U' \to U$ such that g is trivial over $\mathbb{R}^n$ (i.e. such that $\forall (x, y) \in U' \subset \mathbb{R}^n \times M$, $g(x, y) = (x, \tilde{y}(x, y))$ for some $\tilde{y}(x, y) \in \mathbb{R}^m$)
(2) the graded vector bundle $(p_M^* V)|_{U'}$ is trivial (where $p_M : \mathbb{R}^n \times M \to M$ is the projection).

As a result, there is a non-positively graded vector space $\tilde{W}$ with $\tilde{W}^0 = \mathbb{R}^m$ (a local model of $V \to M$), an open subset $U \subset W^0$ where $W = T[1]\mathbb{R}^n \oplus \tilde{W}$ (so that $W^0 = \mathbb{R}^{n+m}$) and an isomorphism of graded vector bundles
$$t : (T[1]\mathbb{R}^n \times V)|_{U'} \to W|_U$$
such that the triangle

$$\begin{tikzcd} (T[1]\mathbb{R}^n \times V)|_{U'} \arrow[rr, "t"] \arrow[dr, "p'"'] & & W|_U \arrow[dl, "p"] \\ & T[1]\mathbb{R}^n & \end{tikzcd} \tag{20}$$

commutes, where p and p' are the projections. Under the resulting isomorphism of gcas
$$t^* : \mathcal{A}_{W|_U} \cong \mathcal{A}_{(T[1]\mathbb{R}^n \times V)|_{U'}}$$
the differential $d + Q$ on $A_{(T[1]\mathbb{R}^n \times V)|_{U'}}$ is sent to a differential $\tilde{Q}$ on $\mathcal{A}_{W|_U}$ such that
$$p^* : \Omega(\mathbb{R}^n) \to \mathcal{A}_{W|_U}$$
is a dg morphism, as follows from the commutativity of (20).

By Proposition 6.1 we know that $\Omega(N, W|_U)^{MC,A_0} \subset \Omega(N, W|_U)^{0,A_0}$ is a C^0-closed submanifold. Let $\Omega(N, V)^0|_{U'} \subset \Omega(N, V)^0$ be the subset of those elements for which the graphs of their function parts $N \to M$ lie in U'. By construction t^* restricts to a diffeomorphism $\Omega(N, W|_U)^{0,A_0} \cong \Omega(N, V)^0|_{U'}$ and the image of $\Omega(N, W|_U)^{MC,A_0}$ is $\Omega(N, V)^{MC} \cap \Omega(N, V)^0|_{U'}$. As a result $\Omega(N, V)^{MC} \cap \Omega(N, V)^0|_{U'} \subset \Omega(N, V)^0|_{U'}$ is a C^0-closed submanifold, and thus also $\Omega(N, V)^{MC} \subset \Omega(N, V)^0$ is a C^0-closed submanifold, as the subsets $\Omega(N, V)^0|_{U'} \subset \Omega(N, V)^0$ form a C^0-open cover of $\Omega(N, V)^0$. □

Remark. Another proof of Theorem 6.1, avoiding Proposition 6.1, is as follows. If $\Omega(N, V)^{MC}|_{U'}$ is non-empty, one can prove that (after possibly decreasing U') there is a NQ manifold $\tilde{W}|_{\tilde{U}}$ and an open embedding of NQ manifolds
$$(T[1]\mathbb{R}^n \times V)|_{U'} \rightsquigarrow T[1]\mathbb{R}^n \times \tilde{W}|_{\tilde{U}}$$
commuting with the projections to $T[1]\mathbb{R}^n$. As a result we can identify $\Omega(N, V)^{MC}|_{U'} \subset \Omega(N, V)^0|_{U'}$ with $\Omega(N, \tilde{W}|_{\tilde{U}})^{MC} \subset \Omega(N, \tilde{W}|_{\tilde{U}})^0$, which is a submanifold by Theorem 4.2.

Our second globalized result will be proved by similar methods.

Theorem 6.2. *The Fréchet simplicial manifold $\Omega(\Delta^\bullet, V)^{MC}$ is Kan.*

Proof. Let us use the notation $K_\bullet := \Omega(\Delta^\bullet, V)^{MC}$. Let us first prove that the map $K_n \to K_{n,k}$ is surjective. Any element of $K_{n,k}$ gives us, in particular, a map $f_{\mathrm{horn}} : \Delta^n_k \to M$. Let us extend f_{horn} to a map $f : \Delta^n \to M$. As in the proof of Theorem 6.1, let $U' \subset \mathbb{R}^n \times M$ be an open subset such that U' contains the graph of f, and such that

(1) U' is diffeomorphic over $\mathbb{R}^n$ to an open subset U' of $\mathbb{R}^n \times \mathbb{R}^m$ $(m = \dim M)$
(2) the graded vector bundle $p_M^* V|_{U'} \to U'$ is trivial.

This ensures the existence of an isomorphism

$$t : (T[1]\mathbb{R}^n \times V)|_{U'} \cong (T[1]\mathbb{R}^n \times \tilde{W})|_U$$

for some non-positively graded vector space $\tilde{W}$ with $\tilde{W}^0 = \mathbb{R}^m$ and some open subset $U \subset W^0$, where $W := T[1]\mathbb{R}^n \oplus \tilde{W}$.

Let us choose an element $A_{\mathrm{horn}} \in K_{n,k}$, i.e.[1]

$$A_{\mathrm{horn}} \in \Omega\big(\Delta^n_k, W|_U\big)^{MC,A_0}.$$

After we apply the map κ to A_{horn}, we get a closed form

$$B_{\mathrm{horn}} \in \Omega(\Delta^n_k, W)^{0,cl}.$$

and extend B_{horn} to a closed form

$$B \in \Omega\big(\Delta^n, W\big)^{0,cl}.$$

By Proposition 4.1, Equation (16) has a solution a on an open subset $N' \subset \Delta^n$ such that $\Delta^n_k \subset N'$. The form

$$A' = B + i_E a$$

thus satisfies

$$A' \in \Omega\big(N', W|_U\big)^{MC},\ A'|_{\Delta^n_k} = A_{\mathrm{horn}}.$$

Let now $\phi \in C^\infty(\Delta^n)$ satisfy $0 \le \phi \le 1$, $\phi|_{\Delta^n_k} = 1$, and $\phi|_{\Delta^n \setminus N'} = 0$. Let $h : \Delta^n \to N'$ be given by $x \mapsto \phi(x)x$. Then

$$A := h^* A'$$

satisfies

$$A \in \Omega\big(\Delta^n, W|_U\big)^{MC},\ A|_{\Delta^n_k} = A_{\mathrm{horn}}.$$

The isomorphism t then allows us to see A as an element of $\Omega\big(\Delta^n, (T[1]\mathbb{R}^n \times V)|_{U'}\big)^{MC}$. We project it to get an element of $\Omega(\Delta^n, V)^{MC} = K_n$, which finally shows that the horn map $K_n \to K_{n,k}$ is surjective.

Let us now prove that $K_n \to K_{n,k}$ is a submersion. For an open $U' \subset \mathbb{R}^n \times M$ satisfying the conditions (1) and (2) above, let $K'_n := \Omega(\Delta^n, V)^{MC}|_{U'} \subset \Omega(\Delta^n, V)^{MC} = K_n$ be the subset of those elements for which the graph of their function part lies in U'. The isomorphism of graded vector bundles gives us a diffeomorphism

$$\mathrm{id}_{\Omega(\Delta^n)} \otimes t : K'_n = \Omega(\Delta^n, V)^{MC}|_{U'} \cong \Omega(\Delta^n, W|_U)^{MC,A_0}$$

and so we have a commutative square

$$\begin{tikzcd} K'_n \arrow[r, "\kappa\circ(\mathrm{id}_{\Omega(\Delta^n)}\otimes t)"] \arrow[d] & \Omega(\Delta^n, W)^{0,cl,B_0} \arrow[d] \\ K'_{n,k} \arrow[r, "\kappa\circ(\mathrm{id}_{\Omega(\Delta^n)}\otimes t)"] & \Omega(\Delta^n_k, W)^{0,cl,B_0} \end{tikzcd}$$

where the horizontal arrows are open embeddings and the right vertical arrow is a submersion, hence also the left vertical arrow is a submersion. Thus $K_n \to K_{n,k}$ is a submersion. □

[1]By a differential form on the horn Δ^n_k we mean a differential form on each of its faces which agree on the overlaps.

7. Gauge fixing

The simplicial manifold $K_\bullet^{\mathrm{big}}$ is infinite-dimensional. We follow Getzler's [5] approach and define a finite-dimensional simplicial submanifold $K_\bullet^s \subset K_\bullet^{\mathrm{big}}$. In the case of a Lie algebra (or algebroid) $K_\bullet^s$ is the nerve of the corresponding local Lie group (or groupoid). In general $K_\bullet^s$ is a local Lie ℓ-groupoid (Definition 7.3). $K_\bullet^s$ depends on a choice of a gauge condition; different gauges lead to isomorphic (by non-canonical isomorphisms) local (weak) Lie ℓ-groupoids. This construction is local in nature. In particular, we will not need the results of Sections 5 and 6.

Let us fix an integer $r \geq 1$. Consider the cochain complex of elementary Whitney forms in $\Omega_{r+}(\Delta^n)$ on the geometric $n-$simplex

$$E(\Delta^n) = \bigoplus_{k\geq 0} E^k(\Delta^n) \subset \Omega_{r+}(\Delta^n).$$

It is given in each degree by the linear span of differential forms defined using standard coordinates $(t_0, \dots, t_n, \sum_k t_k = 1)$ on Δ^n by the formulas

$$\omega_{i_0 \dots i_k} = k! \sum_{j=0}^{k} (-1)^j t_{i_j} dt_{i_0} \dots \widehat{dt_{i_j}} \dots dt_{i_k}$$

where $\widehat{dt_{i_j}}$ means the omission of dt_{i_j}. The $k!$ factor is prescribed so that the integral over the $k-$dimensional sub-simplex given by the sequence of $k+1$ vertices $(e_{i_0}, \dots, e_{i_k})$ of Δ^n is

$$\int_{(e_{i_0}, \dots, e_{i_k})} \omega_{i_0 \dots i_k} = 1$$

(the integral of $\omega_{i_0 \dots i_k}$ over any other sub-simplex vanishes).

$E(\Delta^n)$ is a sub-complex of $\Omega_{r+}(\Delta^n)$, as

$$d\,\omega_{i_0 \dots i_k} = \sum_{i=0}^{n} \omega_{i i_0 \dots i_k}.$$

There is an explicit projection $p_\bullet$ given by Whitney [15]

$$p_n : \Omega_{r+}(\Delta^n) \to E(\Delta^n)$$

$$p_n \alpha = \sum_{k=0}^{n} \sum_{i_0 < \dots < i_k} \omega_{i_0 \dots i_k} \int_{(e_{i_0}, \dots, e_{i_k})} \alpha$$

compatible with the simplicial (cochain complex) structures of $\Omega_{r+}(\Delta^\bullet)$ and $E(\Delta^\bullet)$. $E(\Delta^\bullet)$ is furthermore isomorphic (with the isomorphism respecting the simplicial structures) to the complex of simplicial cochains on $\Delta^\bullet$.

Following Getzler [5] let us make the following Definition.

Definition 7.1. *A* gauge *on $\Omega_{r+}(\Delta^\bullet)$ is a continuous simplicial linear map*

$$s_\bullet : \Omega_{r+}(\Delta^\bullet) \to \Omega_{r+}(\Delta^\bullet)$$

of degree -1 satisfying

$$id_{\Omega_{r+}(\Delta^\bullet)} - p_\bullet = [d, s_\bullet]$$

and

$$s_\bullet{}^2 = 0, \; p_\bullet s_\bullet = s_\bullet p_\bullet = 0,$$

which is furthermore invariant under the action of the symmetric group $S_{\bullet+1}$ on $\Delta^\bullet$ (by permuting the vertices).

Restricting to $\ker s_\bullet$ can be thought of as gauge fixing. An important example of a a gauge is given by Dupont [3] in the proof of the simplicial de Rham theorem. Its construction is as follows. Let us denote h_i the de Rham homotopy operator associated to the retraction of $\Delta^\bullet$ to the i-th vertex. Then the operators

$$s_n = \sum_{k=0}^{n-1} \sum_{i_0 < \dots < i_k} \omega_{i_0 \dots i_k} h_{i_k} \dots h_{i_0}$$

where $n \geq 0$ form a gauge $s_\bullet$ (it is $\|\cdot\|_{r+}$-continuous for every r).

Notation. *We shall set*

$$\Omega_{r+}(\Delta^\bullet)^s := \ker s_\bullet \subset \Omega_{r+}(\Delta^\bullet),$$
$$\Omega_{r+}(\Delta^\bullet)^p := \ker p_\bullet \subset \Omega_{r+}(\Delta^\bullet),$$
$$\Omega_{r+}(\Delta^\bullet)^{s,p} := \ker s_\bullet \cap \ker p_\bullet \subset \Omega_{r+}(\Delta^\bullet).$$

More generally, if W is a graded vector space, we let

$$\Omega_{r+}(\Delta^\bullet, W)^s := \ker(s_\bullet \otimes \mathrm{id}_W) \subset \Omega_{r+}(\Delta^\bullet, W)$$

etc.

The meaning of a gauge is summed-up by the following proposition.

Proposition 7.1. *A gauge is equivalent to a choice of a closed and $S_{\bullet+1}$-invariant simplicial subspace*

$$\mathscr{M}(\Delta^\bullet) \subset \Omega_{r+}(\Delta^\bullet)^p$$

such that

$$d : \mathscr{M}(\Delta^\bullet) \to \Omega_r(\Delta^\bullet)^{p,cl}$$

is a bijection, i.e. such that $\Omega_{r+}(\Delta^\bullet)^p = \Omega_r(\Delta^\bullet)^{p,cl} \oplus \mathscr{M}(\Delta^\bullet)$. If $s_\bullet$ is given then

$$\mathscr{M}(\Delta^\bullet) = \Omega_{r+}(\Delta^\bullet)^{s,p}.$$

If $\mathscr{M}(\Delta^\bullet)$ is given then $s_\bullet$ is

$$\begin{tikzcd} 0 & E(\Delta^\bullet) \arrow[l, "s_\bullet"'] & \oplus & \mathscr{M}(\Delta^\bullet) \arrow[ll, bend left, "s_\bullet"] & \oplus & \Omega_r(\Delta^\bullet)^{p,cl}. \arrow[ll, bend right, "s_\bullet = d^{-1}"'] \end{tikzcd} \tag{21}$$

Proof. The map $s_\bullet$ defined by (21) clearly satisfies

$$s_\bullet p_\bullet = 0,\ p_\bullet s_\bullet = 0,\ \mathrm{id}_{\Omega_{r+}(\Delta^\bullet)} - p_\bullet = [d, s_\bullet],\ s_\bullet^2 = 0$$

so it is a gauge, and $\mathscr{M}(\Delta^\bullet) = \Omega_{r+}(\Delta^\bullet)^{s,p}$.

Conversely given a gauge $s_\bullet$ we set $\mathscr{M}(\Delta^\bullet) = \Omega_{r+}(\Delta^\bullet)^{s,p}$ and we easily see that $d : \mathscr{M}(\Delta^\bullet) \to \Omega_r(\Delta^\bullet)^{p,cl}$ is an isomorphism, and that the gauge defined by (21) coincides with the original $s_\bullet$. □

We can use $s_\bullet$ to specify the closed graded subspace $C \subset \Omega_{r+}(\Delta^\bullet)$ of Proposition 4.4. Let

$$D(\Delta^\bullet) := h_0\big(E(\Delta^\bullet)\big) \subset E(\Delta^\bullet),$$

(where h_0 is the de Rham homotopy operator given by the contraction to the vertex 0), so that $E(\Delta^\bullet) = E(\Delta^\bullet)^{cl} \oplus D(\Delta^\bullet)$. We set $C = D(\Delta^\bullet) \oplus \mathscr{M}(\Delta^\bullet)$ and consider the projection

$$\pi_\bullet : \Omega_{r+}(\Delta^\bullet) \to \Omega_r(\Delta^\bullet)^{cl}$$

w.r.t. C.

Notice that $D(\Delta^\bullet) \subset E(\Delta^\bullet)$ is not a simplicial subspace, but it is compatible with the maps between simplices that preserve the vertex 0. Likewise, $\pi_\bullet$ is not a simplicial map, but it is 0-simplicial in the following sense:

Definition 7.2. *If $X_\bullet$ and $Y_\bullet$ are simplicial sets then a sequence of maps $f_\bullet : X_\bullet \to Y_\bullet$ is a* 0-simplicial map *if it is functorial under the order-preserving maps $\{0, 1, \dots, n\} \to \{0, 1, \dots, m\}$ sending 0 to 0.*

As before let W be a finite-dimensional non-positively graded vector space, $U \subset W^0$ be an open subset, and let Q be a differential on the algebra $\mathcal{A}_{W|_U}$ and κ be the corresponding Kuranishi map given by (17). Furthermore let $\mathcal{U}_\bullet$ be the open neighborhood $U \subset \mathcal{U}_\bullet \subset \Omega_r(\Delta^\bullet, W|_U)^{MC}$ from Proposition 4.4, where $C = D(\Delta^\bullet) \oplus \mathscr{M}(\Delta^\bullet)$. We can demand $\mathcal{U}_\bullet$ to be a simplicial submanifold such that $\mathcal{U}_n$ is invariant under the action of the symmetric group S_{n+1} for every n.

We will denote the simplicial set of gauge-fixed dg-morphisms in $\mathcal{U}_\bullet$ by

$$K^s_\bullet(\mathcal{A}_{W|_U}, Q) := \Omega_r(\Delta^\bullet, W|_U)^{MC,s} := \mathcal{U}_\bullet \cap \Omega_{r+}(\Delta^\bullet, W)^{0,s}.$$

Its elements are (sufficiently small) forms $A \in \Omega_r(\Delta^\bullet, W|_U)^0$ satisfying the equations

$$dA = F_Q(A), \quad s_\bullet A = 0.$$

Let us now recall the definition of higher Lie groupoids. As is usual we actually define higher groupoids as nerves.

Definition 7.3. *(Getzler [5], Henriques [6], Zhu [17]) A Kan simplicial manifold $K_\bullet$ is called a* Lie ℓ-groupoid *if the maps $K_n \to K_{n,k}$ are diffeomorphisms for all $n > \ell$ and all $0 \le k \le n$. A finite dimensional simplicial manifold $K_\bullet$ is called a* local Lie ℓ-groupoid *if the maps $K_n \to K_{n,k}$ are submersions for all $n \ge 1$, $0 \le k \le n$ and open embeddings for all $n > \ell$, $0 \le k \le n$.*

Theorem 7.1. *The simplicial set $K^s_\bullet(\mathcal{A}_{W|_U}, Q)$ is a finite-dimensional local Lie ℓ-groupoid with $-\ell$ being the lowest degree of W.*

Proof. Let

$$\pi_\bullet : \Omega_{r+}(\Delta^\bullet, W)^0 \to \Omega_r(\Delta^\bullet, W)^{0,cl}$$

be the projection w.r.t. $(C \otimes W)^0$ (where $C = D(\Delta^\bullet) \oplus \mathscr{M}(\Delta^\bullet)$) and let

$$\pi^{res}_\bullet : \mathcal{U}_\bullet \to \Omega_r(\Delta^\bullet, W)^{0,cl}$$

be the restriction of $\pi_\bullet$ to $\mathcal{U}_\bullet$. By Proposition 4.4, $\pi^{res}_\bullet$ is an open embedding. Let us recall that $\pi^{res}_\bullet$ is a 0-simplicial map.

Let $K^s_\bullet := K^s_\bullet(\mathcal{A}_{W|_U}, Q)$. We have

$$\begin{aligned} K^s_\bullet &= \mathcal{U}_\bullet \cap \Omega_{r+}(\Delta^\bullet, W)^{0,s} \\ &= (\pi^{res}_\bullet)^{-1}\big(\Omega_r(\Delta^\bullet, W)^{0,cl} \cap \Omega_{r+}(\Delta^\bullet, W)^{0,s}\big) = (\pi^{res}_\bullet)^{-1}\big(E(\Delta^\bullet, W)^{0,cl}\big) \end{aligned}$$

as $\Omega_r(\Delta^\bullet, W)^{0,cl} \cap \Omega_{r+}(\Delta^\bullet, W)^{0,s} = E(\Delta^\bullet, W)^{0,cl}$. This implies that each degree of the simplicial set $K^s_\bullet$ is a finite-dimensional smooth manifold and that

$$\pi^{res}_\bullet : K^s_\bullet \to E(\Delta^\bullet, W)^{0,cl} \tag{22}$$

is an open embedding.

It remains to prove that it is a local Lie ℓ-groupoid. Since $E(\Delta^\bullet, W)^{0,cl}$ is a Lie ℓ-groupoid and $\pi^{res}_\bullet$ a 0-simplicial map, $K^s_\bullet$ satisfies the required conditions for the projections $K^s_n \to K^s_{n,0}$. The action of the symmetric group S_{n+1} on K^s_n then ensures that the conditions are satisfied for all horn projections $K^s_n \to K^s_{n,k}$, i.e. that $K^s_\bullet$ is a local Lie ℓ-groupoid. □

The simplicial manifold $K^s_\bullet(\mathcal{A}_{W|_U}, Q) = \Omega_r(\Delta^\bullet, W|_U)^{MC,s}$ was constructed using infinite-dimensional techniques. We can now see it as a simplicial submanifold of a finite-dimensional simplicial vector space:

Theorem 7.2. *The projection*

$$E(\Delta^\bullet, W)^0 \oplus \mathscr{M}(\Delta^\bullet, W)^0 \to E(\Delta^\bullet, W)^0$$

restricts to an emdedding of simplicial manifolds

$$K^s_\bullet(\mathcal{A}_{W|_U}, Q) \to E(\Delta^\bullet, W)^0.$$

Proof. The projection is a simplicial map. It restricts to an embedding since (22) is an (open) embedding. □

We can thus identify $K^s_\bullet(\mathcal{A}_{W|_U}, Q)$ with

$$\{A \in E(\Delta^\bullet, W)^0; (\exists A' \in \mathscr{M}(\Delta^\bullet, W)^0)\ d(A + A') = F_Q(A + A')\}$$

intersected with a suitable open subset of $E(\Delta^\bullet, W)^0$.

8. Deformation retraction

In this section we shall prove that $K^s_\bullet$ and $K^{big}_\bullet$ are equivalent as local Lie ∞-groupoids, namely we shall construct a local simplicial deformation retraction of $K^{big}_\bullet$ onto $K^s_\bullet$.

If $A_0 \in \Omega_r(\Delta^\bullet, W|_U)^{MC}$ and $G \in \Omega_r(\Delta^\bullet, W)^{0,cl,p}$ so that $d(s_\bullet G) = G$, and if the C^0-norm of $s_\bullet G$ is small enough, then Theorem 3.1 gives us a form

$$\tilde{A} \in \Omega_r(\Delta^\bullet \times I, W|_U)^{MC}$$

such that $\tilde{A}|_0 = A_0$ and $\tilde{A}_v := i_{\partial_t}\tilde{A} = q^* s_\bullet G$, where $q : \Delta^\bullet \times I \to \Delta^\bullet$ is the projection. It is given by Equation (12) with $H = q^* s_\bullet G$, i.e. by

$$\frac{d}{dt}\tilde{A}_h = G + (s_\bullet G^i)\frac{\partial F_Q}{\partial \xi^i}(\tilde{A}_h).$$

Let us use the notation $\tilde{A}(A_0, G)$ for the form $\tilde{A}$ constructed in this way.

Proposition 8.1. *There is an open neighbourhood $\mathcal{V}_\bullet$ of $U \times \{0\}$ in*

$$\Omega_r(\Delta^\bullet, W|_U)^{MC,s} \times \Omega_r(\Delta^\bullet, W)^{0,cl,p}$$

such that the map $(A_0, G) \mapsto \tilde{A}(A_0, G)|_{t=1}$ is an open embedding $\psi : \mathcal{V}_\bullet \to \Omega_r(\Delta^\bullet, W|_U)^{MC}$.

Proof. Since ψ restricts to the identity on $U \times \{0\}$, it's enough to show that for each $A_c \in U$ the tangent map $\psi_{lin} := T_{(A_c,0)}\psi$ is invertible. We have

$$\psi_{lin}(A_0, G) = A_0 + G + (s_\bullet G^i)\frac{\partial F_Q}{\partial \xi^i}(A_c)$$

(for A_0 tangent to $\Omega_r(\Delta^\bullet, W|_U)^{MC,s}$ at A_c and $G \in \Omega_r(\Delta^\bullet, W)^{0,cl,p}$).

Let us recall that the projection $\pi_\bullet$ w.r.t. $(C \otimes W)^0$ (where $C = D(\Delta^\bullet) \oplus \mathscr{M}(\Delta^\bullet)$) gives us isomorphisms

$$T_{A_c}\Omega_r(\Delta^\bullet, W|_U)^{MC,s} \cong E(\Delta^\bullet, W)^{0,cl}$$

and

$$T_{A_c}\Omega_r(\Delta^\bullet, W|_U)^{MC} \cong \Omega_r(\Delta^\bullet, W)^{0,cl} = E(\Delta^\bullet, W)^{0,cl} \oplus \Omega_r(\Delta^\bullet, W)^{0,cl,p}.$$

Since the term $(s_\bullet G^i)\frac{\partial F_Q}{\partial \xi^i}(A_c) \in \mathscr{M}(\Delta^\bullet, W)^0$ is removed by this projection, we see that ψ_{lin} is indeed an isomorphism. $\square$

Theorem 8.1. *There is an open neighbourhood $\mathcal{W}_\bullet$ of U in*

$$\Omega_r(\Delta^\bullet, W|_U)^{MC}$$

which is a simplicial submanifold, and a (smooth) simplicial map

$$\Psi^s_\bullet : \mathcal{W}_\bullet \to \Omega_r(\Delta^\bullet \times I, W|_U)^{MC}$$

with these properties:

(1) $\Psi^s_\bullet(A)|_{t=1} = A$ *(for every $A \in \mathcal{W}_\bullet$)*
(2) $\Psi^s_\bullet(A)|_{t=0} \in \Omega_r(\Delta^\bullet, W|_U)^{MC,s}$
(3) *if $A \in \Omega_r(\Delta^\bullet, W|_U)^{MC,s}$ then $\Psi^s_\bullet(A) = q^*A$ where $q : \Delta^\bullet \times I \to \Delta^\bullet$ is the projection.*

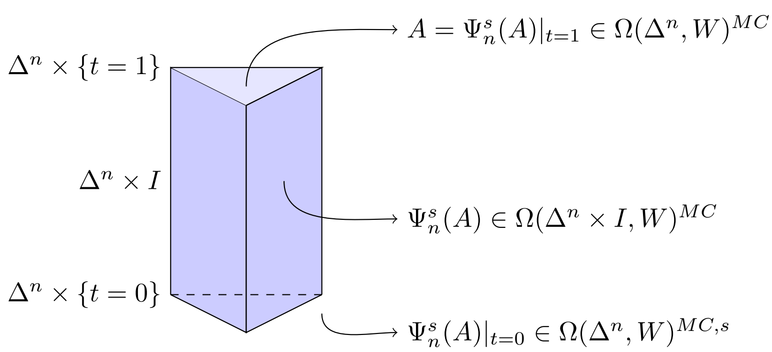

Figure 5. Deformation retraction

Proof. We set $\mathcal{W}_\bullet := \psi(\mathcal{V}_\bullet)$ (we might then have to decrease $\mathcal{W}_\bullet$ to ensure that it is a simplicial submanifold), and $\Psi^s_\bullet(A) := \tilde{A}(\psi^{-1}(A))$. □

Let us recall that a simplicial homotopy is a simplicial map $X_\bullet \times \mathrm{I}_\bullet \to Y_\bullet$ where the simplicial set $\mathrm{I}_\bullet$ (the simplicial interval) is the set of non-decreasing maps $f : \{0, \dots, \bullet\} \to \{0, 1\}$. We can now formulate the main result of this section.

Theorem 8.2. *There is a local simplicial deformation retraction*

$$R^s_\bullet : \Omega_r(\Delta^\bullet, W|_U)^{MC} \times \mathrm{I}_\bullet \to \Omega_r(\Delta^\bullet, W|_U)^{MC}$$

of $\Omega_r(\Delta^\bullet, W|_U)^{MC}$ *to* $\Omega_r(\Delta^\bullet, W|_U)^{MC,s} \subset \Omega_r(\Delta^\bullet, W|_U)^{MC}$*, where "local" means that it is defined on an open neighbourhood of* U *in* $\Omega_r(\Delta^\bullet, W|_U)^{MC}$*.*

Proof. If $f \in \mathrm{I}_\bullet$, $f : \{0, \dots, \bullet\} \to \{0, 1\}$, let $g : \Delta^\bullet \to \Delta^\bullet \times I$ be the affine map given on the vertices by $g(v_i) = (v_i, 1 - f(i))$ $(i = 0, \dots, \bullet)$. Given $A \in \mathcal{W}_\bullet \subset \Omega(\Delta^\bullet, W|_U)^{MC}$ we set $R^s_\bullet(A, f) := g^* \Psi^s_\bullet(A)$. □

9. Functoriality and naturality

If $\phi : V \rightsquigarrow V'$ is a morphism of NQ manifolds, i.e. if we have a morphism of dg algebras $\phi^* : \mathcal{A}_{V'} \to \mathcal{A}_V$, by composition we get a morphism of Banach simplicial manifolds

$$(\phi_*)_\bullet : \Omega_r(\Delta^\bullet, V)^{MC} \to \Omega_r(\Delta^\bullet, V')^{MC}$$

(or of Fréchet simplicial manifolds if we remove the subscript r). In this way $V \mapsto \Omega_r(\Delta^\bullet, V)^{MC}$ is a functor from the category of NQ manifolds to the category of Banach simplicial manifolds.

The situation is more complicated when we consider the finite-dimensional simplicial manifolds $K^s_\bullet(W|_U) := \Omega_r(\Delta^\bullet, W|_U)^{MC,s}$. To avoid mentioning new and new open subsets, let us make the following definition.

Definition 9.1. *If* $X_\bullet$ *and* $Y_\bullet$ *are simplicial manifold, a* local homomorphism $X_\bullet \to Y_\bullet$ *is a smooth simplicial map* $U_\bullet \to Y_\bullet$*, where* $U_\bullet \subset X_\bullet$ *is an open simplicial submanifold containing all fully degenerate simplices. Two local homomorphisms are declared equal if they coincide on some* $U_\bullet$*.*

Let

$$i^s_\bullet : \Omega_r(\Delta^\bullet, W|_U)^{MC,s} \to \Omega_r(\Delta^\bullet, W|_U)^{MC}$$

be the inclusion, and

$$p^s_\bullet : \Omega_r(\Delta^\bullet, W|_U)^{MC} \to \Omega_r(\Delta^\bullet, W|_U)^{MC,s}$$

the projection given by $p^s_\bullet(A) = \Psi^s_\bullet(A)|_{t=0}$ ($p^s_\bullet$ is a local homorphism).

If $\phi : W|_U \rightsquigarrow W'|_{U'}$ is a map of dg manifolds, i.e. if we have a morphism $\phi^* : \mathcal{A}_{W'|_{U'}} \to \mathcal{A}_{W|_U}$ of dg algebras, we get a local morphism

$$K^s_\bullet(\phi) : \Omega_r(\Delta^\bullet, W|_U)^{MC,s} \to \Omega_r(\Delta^\bullet, W'|_{U'})^{MC,s}$$

of local Lie ℓ-groupoids, defined as the composition

$$\Omega_r(\Delta^\bullet, W|_U)^{MC,s} \xrightarrow{i_s} \Omega_r(\Delta^\bullet, W|_U)^{MC} \xrightarrow{(\phi_*)_\bullet} \Omega_r(\Delta^\bullet, W'|_{U'})^{MC} \xrightarrow{p_s} \Omega_r(\Delta^\bullet, W'|_{U'})^{MC,s}.$$

Let us observe that

$$K^s_\bullet(\phi \circ \phi') \neq K^s_\bullet(\phi) \circ K^s_\bullet(\phi')$$

in general, i.e. $K^s_\bullet$ is not a functor. It is, however, a "functor up to homotopy", i.e. a homotopy coherent diagram in the sense of Vogt [16], or, using a more recent terminology, a quasi-functor. Let us describe this quasi-functor in pedestrian terms. An n-simplex f in the nerve of the category of dg manifolds of the type $W|_U$ is a chain of composable morphisms

$$f := \Big(W_0|_{U_0} \xrightarrow{\phi_0} W_1|_{U_1} \xrightarrow{\phi_1} \dots \xrightarrow{\phi_{n-1}} W_n|_{U_n}\Big).$$

Combining the maps $(\phi_{i*})_\bullet$, $i = 0, \dots, n-1$ with the local simplicial deformation retractions $R^s_\bullet$ we obtain simplicial maps

$$K^s_\bullet(f) : K^s_\bullet(W_0|_{U_0}) \times \mathrm{I}^{n-1}_\bullet \to K^s_\bullet(W_n|_{U_n})$$

defined (in the case of $n=3$, as an illustration) as the composition

$$\begin{tikzcd}
K^s_\bullet(W_0|_{U_0})\times(\mathrm{I}_\bullet)^2 \arrow[d, "i^s_\bullet\times id"'] & K^{big}_\bullet(W_1|_{U_1})\times(\mathrm{I}_\bullet)^2 \arrow[d, "R^s_\bullet\times id"] & K^{big}_\bullet(W_2|_{U_2})\times\mathrm{I}_\bullet \arrow[d, "R^s_\bullet"] & K^{big}_\bullet(W_3|_{U_3}) \arrow[d, "p^s_\bullet"] \\
K^{big}_\bullet(W_0|_{U_0})\times(\mathrm{I}_\bullet)^2 \arrow[ur, "(\phi_{0*})_\bullet\times id"] & K^{big}_\bullet(W_1|_{U_1})\times\mathrm{I}_\bullet \arrow[ur, "(\phi_{1*})_\bullet\times id"] & K^{big}_\bullet(W_2|_{U_2}) \arrow[ur, "(\phi_{2*})_\bullet"] & K^s_\bullet(W_3|_{U_3})
\end{tikzcd}$$

By construction, they define a homotopy coherent diagram. To explain what it means, let us use Vogt's notation for $K^s_\bullet(f)$

$$K^s_\bullet(\phi_0,t_1,\phi_1,t_2,\dots,t_{n-1},\phi_{n-1}):K^s_\bullet(W_0|_{U_0})\to K^s_\bullet(W_n|_{U_n})$$

where $t_i\in\mathrm{I}_\bullet$, i.e. t_i's are non-decreasing maps $\{0,\dots,\bullet\}\to\{0,1\}$. Then

$$K^s_\bullet(\phi_0,t_1,\dots,t_{n-1},\phi_{n-1})=$$
$$=\begin{cases}K^s_\bullet(\phi_1,t_2,\dots,t_{n-1},\phi_{n-1}) & \text{if } \phi_0=id\\ K^s_\bullet(\phi_0,t_1,\dots,\max(t_i,t_{i+1}),\dots,t_{n-1},\phi_{n-1}) & \text{if } \phi_i=id,\ 0<i<n-1\\ K^s_\bullet(\phi_0,t_1,\dots,t_{n-2},\phi_{n-2}) & \text{if } \phi_{n-1}=id\\ K^s_\bullet(\phi_0,t_1,\dots,\phi_i\circ\phi_{i-1},\dots,t_{n-1},\phi_{n-1}) & \text{if } t_i=0 \text{ identically}\\ K^s_\bullet(\phi_i,t_{i+1},\dots,\phi_{n-1})\circ K^s_\bullet(\phi_0,t_1,\dots,\phi_{i-1}) & \text{if } t_i=1 \text{ identically}\end{cases}$$

More compactly, a homotopy coherent diagram can be described as follows (an introduction to the subject can be found in Porter [11]). If $\mathcal{C}$ is a category (in our case the category of NQ manifolds of the form $W|_U$) and if $\mathcal{D}$ is a simplicially enriched category (in our case the category of simplicial manifolds with local morphisms) then a homotopy coherent diagram from $\mathcal{C}$ to $\mathcal{D}$ is a simplicially enriched functor $S(\mathcal{C})\to\mathcal{D}$, where $S(\mathcal{C})$ is the simplicially enriched category defined as the free simplicial resolution of $\mathcal{C}$, introduced by Dwyer and Kan [4].

We thus have the following result.

Theorem 9.1. *$K^s_\bullet$ is a homotopy coherent diagram from the category of dg manifolds of type $W|_U$ to the simplicially enriched category of local simplicial manifolds with the simplicial enrichment given by*

$$Hom(L,M)_n:=Hom(L\times\Delta[n],M)$$

where $\Delta[n]$ is the simplicial set representing the n-simplex. Here "local" means that morphisms are defined to be local homorphisms of simplicial manifolds.

For a general dg manifold given by a negatively graded vector bundle $V\to M$ (not of the form $W|_U$) one can apply the quasi-functor $K^s_\bullet$ on the nerve $N_\bullet(\mathsf{U})$ of a good cover U of M. We can say a bit more in this situation: the homomorphisms $K^s_\bullet(\phi)$ on overlaps are actually isomorphisms.

Indeed, for any dg-map $\phi:W|_U\rightsquigarrow W'|_{U'}$ the linearization $(\phi_*)_{\bullet,\mathrm{lin}}$ of $(\phi_*)_\bullet$ commutes with $s_\bullet,p_\bullet$. Therefore the linearization $(\phi_*)^s_{\bullet,\mathrm{lin}}$ of $(\phi_*)^s_\bullet$ is functorial, i.e. if $\phi':W'|_{U'}\rightsquigarrow W''|_{U''}$ is another map of dg manifolds then $(\phi'_*\circ\phi_*)^s_{\bullet,\mathrm{lin}}=(\phi'_*)^s_{\bullet,\mathrm{lin}}\circ(\phi_*)^s_{\bullet,\mathrm{lin}}$. For $\mathrm{id}:W|_U\rightsquigarrow W|_U$ we have $(\mathrm{id}_*)^s_\bullet=\mathrm{id}_{\Omega_r(\Delta^\bullet,W|_U)^{MC,s}}$. Together with functoriality this implies that if ϕ is an isomorphism of dg manifolds then $K^s_\bullet(\phi)$ is an isomorphism of local Lie ℓ-groupoids.

10. Concluding remarks

Let us conclude with two open problems.

For a general NQ manifold one cannot expect to globalize the gauge-integration to obtain Lie ℓ-groupoids. Lie's third theorem fails already in the case of Lie algebroids; the integrability condition was discovered by Crainic and Fernandes [2]. Their obstructions are formulated in terms of the monodromy groups. There seems to be a generalization of this result in terms of a sequence of higher monodromy groups at each point of the body of the higher degree dg manifolds. It would we interesting to make it precise, and to find a better relation with the gauge integration procedure.

The second problem concerns dg manifolds with additional structure. An important case is a symplectic structure ω of degree ℓ. This is a Poisson manifold resp. Courant algebroid for $\ell=1$ resp. $\ell=2$. How is this structure reflected in $K^{\mathrm{big}}_\bullet$ and $K^s_\bullet$? It is well known that for $\ell=1$

we get a symplectic groupoid and for general ℓ the result should be a symplectic Lie ℓ-groupoid. This is sketched in Ševera [13] and proven for exact Courant algebroids by Li-Bland, Ševera [8] and Mehta, Tang [9]. This question is related to the AKSZ construction [1] (with boundary) if we consider a field theory with $T[1]\Delta^\bullet$ as the worldvolume and the target manifold (Z,ω) a dg symplectic manifold of degree $\bullet - 1$. Then the formal graded manifold $\mathrm{Hom}(T[1]\Delta^\bullet, Z)$ is a formal dg symplectic manifold of degree -1.

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