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UNIVERSITÉ DE GENÈVE

FACULTÉ DE SCIENCES

Section de Mathématiques

Docteur P. Ševera

On Quantization of Moduli Spaces and Poisson-Hopf Algebras

THÈSE

Présentée à la Faculté des sciences de l'Université de Genève
Pour obtenir le grade de Docteur ès sciences, mention mathématiques

Par

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DE GENÈVE**

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DOCTORAT ÈS SCIENCES, MENTION MATHÉMATIQUES

Thèse de Monsieur Jan PULMANN

intitulée :

«On Quantization of Moduli Spaces and Poisson-Hopf Algebras»

La Faculté des sciences, sur le préavis de Monsieur P. SEVERA, docteur et directeur de thèse (Section de mathématiques), Monsieur A. ALEXEEV, professeur ordinaire (Section de mathématiques), Monsieur T. Q. T. LE, professeur (School of Mathematics, Georgia Institute of Technology, Atlanta, United States of America), autorise l'impression de la présente thèse, sans exprimer d'opinion sur les propositions qui y sont énoncées.

Genève, le 13 août 2021

Thèse - 5581 -

Le Doyen

Résumé

Dans cette thèse, nous étudions deux problèmes de quantification liés à la théorie de Chern-Simons.

La première partie de la thèse est consacrée à la quantification par déformation de l'espace des phases classique de la théorie de Chern-Simons : l'espace de modules des connexions plates sur une surface. Notre point de départ est la quantification de tels espaces de modules pour les surfaces avec des points marqués sur la frontière, due à Li-Bland et Ševera [17]. En étendant les méthodes existantes de la théorie des diagrammes de cordes et de l'intégrale de Kontsevich, nous fournissons une caractérisation de la construction de [17] en termes de colonnes de surfaces (des données combinatoires liées aux triangulations).

Ensuite, nous introduisons une modification de la construction de [17], qui réduit la donnée nécessaire pour quantifier l'espace de modules à une triangulation de la surface. De plus, nous montrons que deux quantifications quelconques sont canoniquement isomorphes, ce qui donne un lien entre le groupoïde de Ptolémée de la surface et les propriétés d'un associateur de Drinfeld. Nous terminons cette partie de la thèse en explorant quelques propriétés de la modification mentionnée ci-dessus, et en présentant quelques résultats auxiliaires obtenus par des calculs informatiques, recoupant les résultats principaux.

La deuxième partie de cette thèse est constituée de l'article [59], écrit avec Pavol Ševera. Dans cet article, nous fournissons une nouvelle procédure plus générale pour quantifier les bialgèbres de Lie et leurs généralisations, les algèbres de Poisson-Hopf, en algèbres de Hopf. Cette procédure est basée sur le codage de ces algèbres de Hopf en tant que nerfs, qui sont dans ce cas des algèbres cosimpliciaux avec des morphismes supplémentaires. Les associateurs de Drinfeld jouent à nouveau un rôle important, en donnant une quantification comme une précomposition avec un foncteur construit en utilisant l'associateur.

Notre procédure est à nouveau motivée par la quantification des déformations des espaces modulaires des connexions plates, et nous expliquons cette motivation.

Summary

In this thesis, we study two quantization problems related to the Chern-Simons theory.

The first part of the thesis is devoted to deformation quantization of the classical phase space of the Chern-Simons theory: the moduli space of flat connections on a surface. Our starting point is the quantization of such moduli spaces for surfaces with marked points on the boundary, due to Li-Bland and Ševera [17]. Extending existing methods from the theory of chord diagrams and Kontsevich integral, we provide a characterization of the construction of [17] in terms of spines of surfaces: combinatorial data related to triangulations.

Then, with a novel modification of the construction of [17], we are able to simplify the datum needed to quantize the moduli space to essentially a triangulation of the surface. Moreover, we show that any two such quantizations are canonically isomorphic, giving a connection between the Ptolemy groupoid of the surface and the properties of Drinfeld associators. We finish this part of the thesis by exploring some properties of the above-mentioned modification, and by presenting some auxiliary results obtained by computer calculations, cross-checking the main results.

The second part of this thesis consists of the paper [59], joint with Pavol Ševera. In this paper, we provide a new, more general procedure to quantize Lie bialgebras and their generalizations, Poisson-Hopf algebras, to Hopf algebras. This procedure is based on encoding these Hopf algebras as nerves, which in this case are cosimplicial algebras with additional morphisms. Drinfeld associators again play an important role, giving a quantization as a precomposition with a functor constructed using the associator.

Our procedure is again motivated by the deformation quantization of moduli spaces of flat connections, and we comment on this motivation.

Dorotke a Katke

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Chapter 1

Introduction

1.1 Deformation quantization in the Drinfeld category

Chern-Simons theory, pioneered by Witten in [67], plays an important role in mathematical physics. It is constructed using a 3-manifold M and a Lie group G with an invariant pairing on its Lie algebra \mathfrak{g} . Its classical phase space is the moduli space of flat G -connections on a suitable surface $\Sigma \subset M$, and we denote this moduli space by $M_\Sigma(G)$. Being a phase space, it has a symplectic/Poisson structure, which was described by [8, 35, 33, 4].

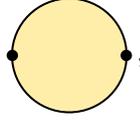
A **deformation quantization** of $M_\Sigma(G)$ is a deformation of the commutative product on the space of functions on $M_\Sigma(G)$ into an associative product in the direction given by the Poisson bracket. Such quantization of $M_\Sigma(G)$ was considered by many authors [1, 6, 41, 60, 17, 13, 47, 45]. The first, main part of this thesis, will be devoted to the study of one of these quantizations, due to Li-Bland and Ševera [17].

The symplectic structure on $M_\Sigma(G)$ was first described by Atiyah and Bott [8] as a Hamiltonian reduction of the infinite-dimensional affine space of all connections on principal G -bundles over Σ . Later, a finite-dimensional approach was developed by [4, 3], see also Chapter 2. In [4, 3], a variant of $M_\Sigma(G)$ was considered: the moduli space of flat bundles trivialized over a point $v \in \partial\Sigma$, denoted $M_{\Sigma, \{v\}}(G)$. This is a finite-dimensional smooth G -manifold, diffeomorphic to $G^{\times N}$, equipped with a so-called **\mathfrak{g} -quasi-Poisson structure**; the reduction modulo G recovers the Poisson structure on $M_\Sigma(G)$.

A quantum version of this construction was given in [17], where an associative algebra in the Drinfeld category $U\mathfrak{g}\text{-mod}_\hbar^\Phi$ was found. Here, $U\mathfrak{g}\text{-mod}_\hbar^\Phi$ is the category of $U\mathfrak{g}$ -modules with the monoidal structure twisted by a Drinfeld associator Φ . Let us describe this construction in some detail:

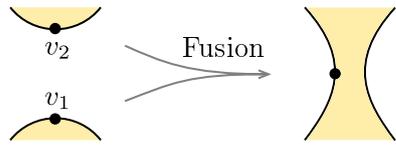
First, it is useful to consider more than a one point, i.e. $v \in \partial\Sigma$ is replaced by a discrete set $V \subset \partial\Sigma$ of **marked points**, over which the flat principal G -bundles are trivialized. The corresponding moduli space $M_{\Sigma, V}(G)$ carries commuting G actions, one for each marked point (residual gauge transformations), and a \mathfrak{g}^V -quasi-Poisson structure. The quantization of [17] can be described in the following three steps:

1. For a disk with two marked points



the moduli space of flat connections has zero quasi-Poisson structure, and its quantization is simply the usual commutative algebra $C^\infty(G)$. Similarly, for a disjoint union of N such disks, the algebra $C^\infty(G^{\times N})$ is a deformation quantization.

2. If we perform **fusion**, i.e. locally glue two marked points into one as follows



then the multiplication on the quantization changes using an action of a specific invariant element $J \in (U\mathfrak{g})^{\otimes 4}$.

3. If we remove a marked point, we have to take the algebra of functions invariants with respect to the G -action corresponding to that marked point.

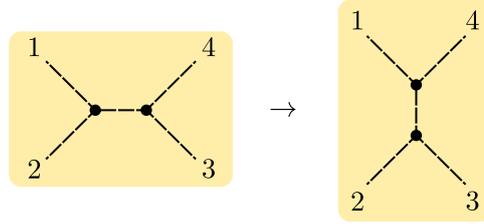
It is easy to see that any surface (with or without marked points) can be constructed starting from a collection of disks and using the two operations above. Thus, we get an associative algebra, with multiplication given by an action of the elements $J \in (U\mathfrak{g})^{\otimes 4}$ on suitable invariants in $C^\infty(G^{\times N})$; the action of J and the invariants are taken with respect to the actions determined by the combinatorics of gluing the disks together.

Our goal in this thesis is to understand this construction. Our first result is a complete specification of the datum needed to uniquely specify an algebra quantizing $M_{\Sigma, V}(G)$. As explained in Theorem 3.1.7, one obtains a quantizing algebra for a **parenthesized spine**:¹ a graph $\Gamma \subset \Sigma$ such that Σ retracts to Γ . Here, a **parenthesization** of a spine Γ is a choice of bracketings (cyclic bracketings) of half-edges of external (internal) vertices of Γ . Moreover, we show that any two such parenthesized Γ, Γ' give isomorphic algebra $A_\Gamma, A_{\Gamma'}$ and we describe the isomorphism explicitly in Theorem 3.3.8.

Then, we proceed by modifying the element $J \in (U\mathfrak{g})^{\otimes 4}$, using an another invariant element $a \in (U\mathfrak{g})^{\otimes 2}$, due to [50, Section 4]. With this modification, the datum needed to quantize $M_{\Sigma, V}(G)$ reduces to a spine with trivalent vertices in the interior of Σ and with univalent vertices sent bijectively to V , see Theorem 3.4.5. Such spines are connected by

¹There is a refinement visible on the quantum level, due to [17], requiring a sign $+$ or $-$ coloring each point in V . We can ignore this distinction for an even Drinfeld associator. For a general associator, a proper coloring of the whole spine is needed to specify the quantization.

moves generated by the **flip move**



and the isomorphism connecting the corresponding algebras is simply the action of the associator Φ twisted by $a^{1/2}$, see Theorem 3.4.7. Moreover, using a result of Penner [56], we can relate the pentagon equation for the associator with the pentagon equation between the flips. This shows that any two algebras $A_\Gamma, A_{\Gamma'}$, quantizing the same moduli space $M_{\Sigma,V}(G)$, are canonically isomorphic. In other words, this provides a deformation quantization of the moduli space $M_{\Sigma,V}(G)$ which is independent of a choice of the spine.

1.1.1 Role of the Kontsevich integral

The main tool we use to reason about the above quantization is the Kontsevich integral Z . We can think of Z as a map assigning an invariant element $Z(T) \in (U\mathfrak{g})^{\otimes n}$ to an n -component tangle T . For example, the element J mentioned above is defined as

$$J := Z \left(\left(\begin{array}{c} \uparrow \quad \downarrow \quad \uparrow \\ \downarrow \quad \uparrow \quad \downarrow \end{array} \right) \right)$$

and the element $a \in (U\mathfrak{g})^{\otimes 2}$ is

$$a := Z \left(\left(\begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \right) \right).$$

The proofs of the above theorems then use various properties of Z to prove results about a and J . For example, the compatibility of Z with strand reversal allows to connect J acting on invariants and a as in Corollary 3.2.2, and the compatibility with cabling allows us to implement the move in Corollary 3.3.7.

The element a plays an important role this work. Therefore, in Chapter 4, we explore some of its properties. It turns to be related to the coloring of marked points by \pm (Corollary 4.2.4) and to the compatibility of Z with cabling (Corollary 4.2.7). Finally, we present a few computer calculations, checking our results to low degrees.

1.2 Quantization of Poisson Hopf algebras

The second part of this thesis is the paper [59], joint with Pavol Ševera. It solves the problem of quantization of Poisson-Hopf algebras to Hopf algebras, a natural generalization of Poisson-Lie groups to arbitrary categories. This solution is based on **nerves** of Hopf algebras. These

nerves encode Poisson-Hopf algebras and Hopf algebras as functors from an enlarged versions of the simplex category: the categories \mathbf{iCom} and \mathbf{BrCom} , respectively. They are required to be (infinitesimally-)braided lax monoidal and have to satisfy the **nerve condition** [59, eq. (4)]. Since a Drinfeld associator induces a functor from \mathbf{BrCom} to (a deformation of) \mathbf{iCom} , precomposing with this functor turns a nerve of a Poisson-Hopf algebra into a nerve of a Hopf algebra, which is its quantization.

Chapter 5 consists of the paper [59], preceded by an explanation of the motivation for the nerve construction, coming from the quantization of [17]. Concretely, the nerve comes from the following fact: for a Poisson-Lie group H , one can encode the Poisson-Hopf structure on $C^\infty(H)$ into a sequence of moduli spaces of flat connections on suitably marked disks and their embeddings. Applying the quantization of [17] to this sequence, one obtains a sequence of algebras, from which it is possible to recover the Hopf algebra structure on the quantization of $C^\infty(H)$.

1.3 Organization of the thesis

In Chapter 2, we recall the necessary background on surfaces, moduli spaces of flat connections, the Drinfeld category and the Kontsevich integral, highlighting their relationship. In Chapter 3, we begin by summarizing the construction of Li-Bland and Ševera [17], after which we prove our main results about this quantization and its relationship to spines of surfaces. We relegate some proofs of a chord-diagrammatic nature into Chapter 4, which also contains additional results on the element a and some results of computer calculations.

Then, in Chapter 5, we comment on the paper [59] and reproduce it fully. We finish by presenting some possible venues for future research in Chapter 6.

Chapter 2

Background

After recalling a few useful facts about Lie groups and surfaces, we will introduce the two main tools used in this thesis: the Drinfeld category and the Kontsevich integral.

2.1 Reminder on Lie groups and Lie algebras

Let G be a Lie group and \mathfrak{g} its Lie algebra. Recall [30] that for $x \in \mathfrak{g}$, the left and right-invariant vector fields at a point $g \in G$ are given by the following equivalence classes of paths

$$\begin{aligned}x^{\text{L}} &= [t \mapsto ge^{tx}], \\x^{\text{R}} &= [t \mapsto e^{tx}g].\end{aligned}$$

In other words, x^{L} is the fundamental vector field for the action by right-multiplication, and x^{R} is the fundamental vector field for the action by left-multiplication. In terms of functions, this means

$$\begin{aligned}(x^{\text{L}}f)(g) &= (\partial_t)|_{t=0}f(ge^{tx}), \\(x^{\text{R}}f)(g) &= (\partial_t)|_{t=0}f(e^{tx}g).\end{aligned}$$

The assignments $x \mapsto x^{\text{L}}$ is a morphism of Lie algebras. For the right-invariant vector fields, one needs to use $x \mapsto -x^{\text{R}}$ to obtain a Lie algebra morphism. We will use the notation

$$x^{-\text{R}} := -x^{\text{R}}.$$

These two invariant vector fields can be related as

$$\text{Inv}_*(x^{\text{L}}) = x^{-\text{R}}, \tag{2.1}$$

where $\text{Inv}: G \rightarrow G$ is the diffeomorphism $g \mapsto g^{-1}$.

Another way to relate the left and right-invariant vector fields at $g \in G$ is via the adjoint action of G on \mathfrak{g} , i.e.

$$(x^{\text{L}})_g = ((\text{Ad}_g x)^{\text{R}})_g.$$

Extending the action of \mathfrak{g} to the action of $U\mathfrak{g}$, the above formula generalizes to

$$(w^L f)(g) = ((\text{Ad}_g S(w))^{-R} f)(g), \quad (2.2)$$

where $w = x_1 \dots x_n$ is an element of $U\mathfrak{g}$, $w^L = x_1^L \dots x_n^L$ and S is the antipode. Finally, we will use the fact that if w is \mathfrak{g} -invariant and G connected, then $\text{Ad}_g(w) = w$. We will henceforth assume that G is connected.

Finally, let us introduce a piece of notation for Lie algebra actions. We will often have multiple Lie algebra actions, with labels such as 1 or 1'. Then, we will denote the action of $x \in \mathfrak{g}$ by the action with label l by $x^{(l)}$, and similarly $w^{(l_1) \dots (l_n)}$ for $w \in U(\mathfrak{g}^{\oplus n})$.

2.2 Surfaces

For us, a surface will mean a two-dimensional smooth compact oriented manifold Σ with *non-empty boundary of each component*. Such surfaces are specified by the genus and the number of boundary components for each component.

Definition 2.2.1. A **marked surface** (Σ, V) is a surface Σ and a finite (possibly empty) subset $V \subset \partial\Sigma$.

Remark 2.2.2. Sometimes (e.g. [56]), boundary components of Σ without marked points are represented as punctures (chosen points) in the interior of Σ . Since we will consider the operation of removing the marked points, it is more useful to treat boundary components and punctures on an equal footing.

A fundamental tool for working with surfaces are triangulations, or more generally decompositions into polygons. We will work with the dual notion¹, a spine.

Definition 2.2.3. A **spine** of a marked surface (Σ, V) is a finite graph Γ embedded into Σ such that

1. Γ intersects the boundary of Σ only at V ,
2. the image of the vertices of Γ contains V ,
3. Σ deformation retracts to Γ .

The vertices of Γ mapped to V are called **external**, and the remaining vertices are called **internal**. An **oriented spine** is a spine together with a choice of an orientation of each edge of Γ .

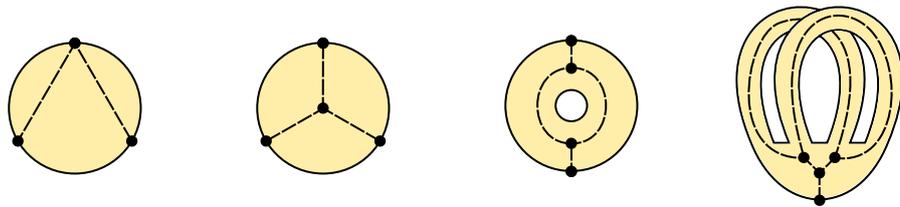
¹To be more precise, spines are dual to polygons with vertices at

1. the unmarked boundary components, seen as vertices (punctures), and at
2. the bisectors of segments between marked points lying on a boundary component.

Then, edges of the polygons are dual to the edges of the spine and an n -sided polygon is dual to an n -valent vertex of the spine.

We will consider spines up to isotopy fixing the boundary pointwise. Note that using the orientation of the surface, Γ becomes a fatgraph with cilia at the external vertices [33], together with an embedding into Σ .

We will draw the surfaces in yellow, with the counter-clockwise orientation if unambiguous. The spines will be represented by dashed lines. Examples of spines on disks, the annulus and the punctured torus are:

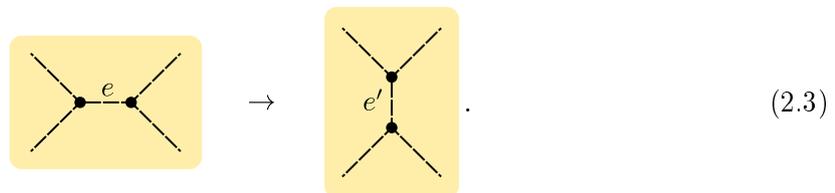


The role of triangulations will be played by the following special spines, of which all but the first picture above are an example.

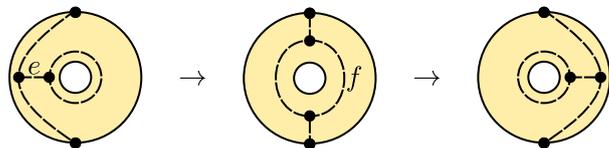
Definition 2.2.4. A **uni-trivalent spine** of a surface Σ of a marked surface (Σ, V) is a spine such that Γ is uni-trivalent graph and a vertex of Γ is univalent if and only if it is external.

Remark 2.2.5. The disk with zero or one marked point and an annulus without marked points have no uni-trivalent spines. We will exclude these degenerate cases, since also the moduli space of flat connection has trivial Poisson structure (see Section 2.3.3).

Definition 2.2.6. Let e be an edge of a uni-trivalent spine $\Gamma \subset \Sigma$ that connects two distinct trivalent vertices. A **flip** (also known as a Whitehead move, or an elementary move) is the following local change of Γ :



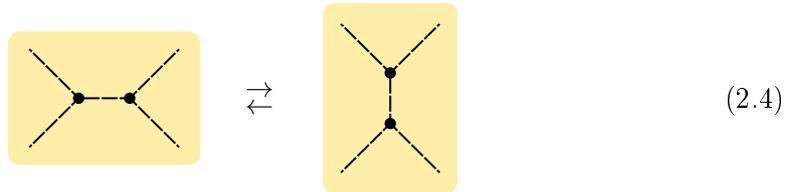
For example, the following two flips at the edges e and f connect all the uni-trivalent spines of the marked annulus.



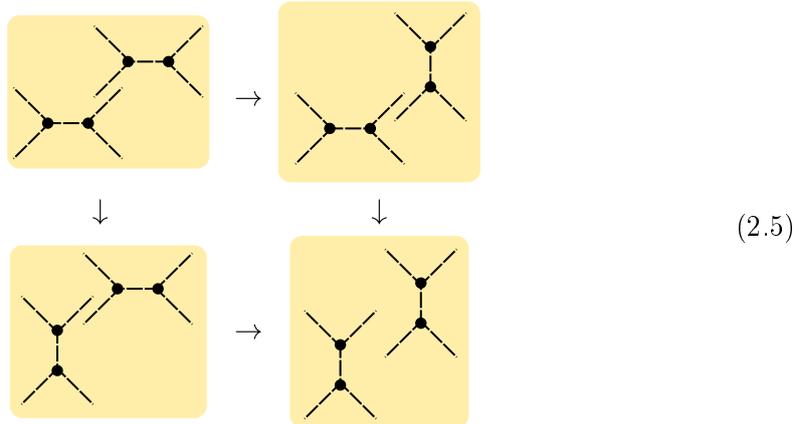
The general statement, also describing the relations between flips, is:

Proposition 2.2.7. Any two uni-trivalent spines are connected by a finite sequence of flips. Any two such sequences starting and ending at the same uni-trivalent spines are connected by a sequence of relations in the form of the following commutative diagrams:

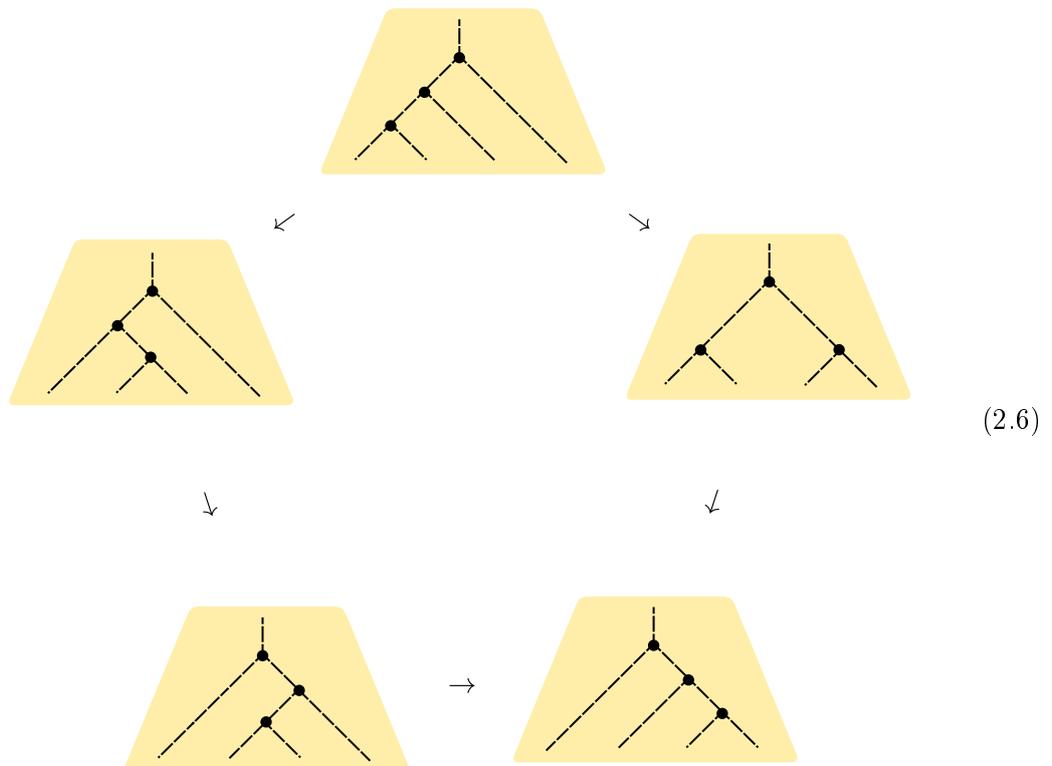
involutivity: for any edge along which one can flip:



square: for two edges which don't share a vertex:



pentagon: for two edges which share one vertex:



Proof. One can see the uni-trivalent spines of a surface as 0-cells of a cell complex, with 1-cells given by uni-trivalent spines with one four-valent vertex, corresponding to the flip move at that vertex. Next, 2-cells correspond to spines with one five-valent or two four-valent vertices, and so on. By the result of Penner ([56, Chapter 4., Theorem 5.21], with P all the unmarked punctures), this cell complex is simply connected. Thus, any two 0-cells can be connected by a path going through 1-cells, and any two such paths can be homotoped through 2-cells. The 2-cells corresponding to two four-valent vertices then correspond to the square relation, the pentagon relation follows from the 2-cells with a five-valent vertex. The first relation follows from the fact that we consider oriented paths in the fatgraph complex. \square

Remark 2.2.8. See also the short paper by Hatcher [37], where V would be the union of marked points and unmarked punctures. The relation to spines is, however, less direct in this case.

2.3 Moduli spaces of flat connections

Let (Σ, V) be a marked surface. In this and the next section, we assume that V meets every component of Σ . Recall [19, Chapter 6.] that $\Pi_1(\Sigma, V)$, the fundamental groupoid of Σ based at V , is the groupoid of homotopy classes of paths between elements of V .

Definition 2.3.1. Let G be a connected Lie group. Define the combinatorial **moduli space of flat connections** on (Σ, V) by

$$M_{\Sigma, V}(G) := \text{Hom}(\Pi_1(\Sigma, V), G).$$

This space deserves this name thanks to the following result, see e.g [16, Appendix A].

Proposition 2.3.2. *The moduli space $M_{\Sigma, V}(G)$ is diffeomorphic to the space*

$$\frac{\{\text{principal } G\text{-bundles } P \rightarrow \Sigma \text{ trivialized over } V, \text{ with a flat connection on } P\}}{\{\text{bundle isomorphisms preserving the trivialization}\}}.$$

Remark 2.3.3. Note that if Σ has non-empty boundary, it is homotopy equivalent to a wedge of circles. Thus, all principal G -bundles over such Σ are trivial, for G connected.

2.3.1 Parametrizing the moduli space

Since Σ has at least one boundary component, $\Pi_1(\Sigma, V)$ is a free groupoid [19, Chapter 9]. Thus, for any choice of a freely generating graph Γ , we get a diffeomorphism²

$$M_{\Sigma, V}(G) \cong G^{\text{edges}(\Gamma)}.$$

A spine with no internal vertices (a skeleton in [17]) gives such freely generating graph. It will be useful to generalize this parametrization to arbitrary spines.

²One can define the smooth structure on $M_{\Sigma, V}(G)$ via this isomorphism.

We use the standard notation G^S for the set of functions from a (finite) set S to G , i.e. the Lie group of tuples of elements of G labeled by S .

Definition 2.3.4. For any oriented spine Γ of (Σ, V) , the group $G^{\text{half-edges}(\Gamma)}$ acts on $G^{\text{edges}(\Gamma)}$ as follows:

1. The starting half-edge acts on the factor G corresponding to its edge e by

$$g_s \mapsto (R_{g_s^{-1}}: G_e \rightarrow G_e).$$

2. The terminal half-edge acts on the factor G corresponding to its edge e by

$$g_t \mapsto (L_{g_t}: G_e \rightarrow G_e).$$

For future reference, this convention is depicted as follows

$$\begin{array}{c} R_{g_s^{-1}} \quad L_{g_t} \\ \bullet \xrightarrow{\quad} \bullet \end{array}, \quad \text{i.e.} \quad g_e \mapsto g_t g_e g_s^{-1}. \quad (2.7)$$

If v is a vertex of Γ , let $\text{half-edges}(v)$ be the set of half-edges adjacent to v . Then, the **action of the vertex** v on $G^{\text{edges}(\Gamma)}$ is the G -action given by the diagonal inclusion $G \hookrightarrow G^{\text{half-edges}(v)}$.

Finally, define the moduli space parametrized by Γ to be the quotient of $G^{\text{edges}(\Gamma)}$ by the action of the inner vertices

$$G^\Gamma := \frac{G^{\text{edges}(\Gamma)}}{\prod_{v \in \text{internal vertices}(\Gamma)} (G \subset_{\text{diag}} G^{\text{half-edges}(v)})}.$$

This space carries a set of commuting G -actions, one for each external vertex.

Remark 2.3.5. Instead of considering oriented spines, one can also define the space G^{Γ_0} for an unoriented spine Γ_0 as consisting of elements of G^Γ for any orientation of Γ_0 , such that changing the orientation corresponds to the group inverse on that edge [33]. For concreteness, we will use oriented spines.

Proposition 2.3.6. *The map*

$$G^\Gamma \rightarrow M_{\Sigma, V}(G),$$

defined by expressing a path in $\Pi_1(\Sigma, V)$ in terms of Γ and multiplying the corresponding elements along the path, is a diffeomorphism commuting with the action of the external vertices.

For two spines Γ, Γ' , let us denote $\Psi_{\Gamma\Gamma'}: G^\Gamma \rightarrow G^{\Gamma'}$ the induced isomorphism.

Example 2.3.7. Consider the disk with three marked points on its boundary, and two oriented spines Γ, Γ' as follows:



Then³ $G^\Gamma = G^{\times 3}/G$, with $h \in G$ acting by $(g_1, g_2, g_3) \rightarrow (hg_1, hg_2, g_3h^{-1})$. Similarly, $G^{\Gamma'} = G^{\times 2}$, and the diffeomorphism $\Psi_{\Gamma\Gamma'}$ is the map

$$\Psi_{\Gamma\Gamma'} : (g_1, g_2, g_3) \mapsto (g_3g_1, g_3g_2).$$

Note that for V empty, one can also define the moduli space as a quotient

$$G^{\text{edges}(\Gamma)} / G^{\text{internal vertices}(\Gamma)},$$

which is not a manifold in general. In such cases, we will consider the subalgebra of $C^\infty(G^{\text{edges}(\Gamma)})$ invariant under the action of all vertices.

2.3.2 Quasi-Poisson geometry of moduli spaces

The moduli space has a canonical $G^{\text{external vertices}(\Gamma)}$ -action. For $\rho \in M_{\Sigma, V}(G)$ and $\gamma : v_s \rightarrow v_t$ a homotopy class of paths, the image of ρ under the action of $h \in G^{\text{external vertices}(\Gamma)}$ is $\rho^h([\gamma]) := h_{v_t}\rho([\gamma])h_{v_s}^{-1}$. For a choice of an oriented spine Γ , this is exactly the action for external vertices as described in Definition 2.3.4, Proposition 2.3.6.

Let us furthermore fix a quadratic structure on \mathfrak{g} , the Lie algebra of G .

Definition 2.3.8. A **quadratic Lie algebra**⁴ is a Lie algebra \mathfrak{g} together with an element $t \in (\text{Sym}^2 \mathfrak{g})^{\mathfrak{g}}$. For \mathfrak{g} a quadratic Lie algebra, denote $\bar{\mathfrak{g}}$ to be the same Lie algebra with $-t$ as its quadratic structure. The direct sum $\mathfrak{g} \oplus \mathfrak{h}$ is given a quadratic structure equal to the sum of the quadratic structures of \mathfrak{g} and \mathfrak{h} via $\text{Sym}^2(\mathfrak{g}) \oplus \text{Sym}^2(\mathfrak{h}) \subset \text{Sym}^2(\mathfrak{g} \oplus \mathfrak{h})$.

The **Cartan element** of \mathfrak{g} is the element $\phi \in \wedge^3 \mathfrak{g} \subset (U\mathfrak{g})^{\otimes 3}$ defined by

$$\phi = \frac{1}{4}[t_{12}, t_{23}],$$

where $t_{12} = t \otimes 1$ and $t_{23} = 1 \otimes t$ in $(U\mathfrak{g})^{\otimes 3}$.

If e_i is a basis of \mathfrak{g} , $[e_i, e_j] = f_{ij}^k e_k$ are structure constants and $t = t^{ij} e_i \otimes e_j$, then $\phi = \sum_{i,j,k} \phi^{ijk} e_i \wedge e_j \wedge e_k$ with

$$\phi^{ijk} = \frac{1}{24} t^{ia} t^{jb} f_{ab}^k.$$

The Cartan element allows us to introduce a generalization of a Poisson manifold, due to Alekseev, Kosmann-Schwarzbach and Meinrenken [3].

Definition 2.3.9. Let \mathfrak{g} be a quadratic Lie algebra and M a manifold with an action of \mathfrak{g} . A **\mathfrak{g} -quasi-Poisson structure** on M is an \mathfrak{g} -invariant bivector field π on M that satisfies

$$[\pi, \pi]/2 = \phi_M,$$

³The edges of Γ are enumerated in a counter-clockwise manner, starting from the bottom left one.

⁴This terminology is slightly non-standard, it is often required that t is invertible. The term **Casimir Lie algebra** was used in the talk [39].

where ϕ_M is the trivector field on M coming from the action of \mathfrak{g} and from the Cartan element $\phi \in \wedge^3 \mathfrak{g}$.

Similarly, a **\mathfrak{g} -quasi-Poisson algebra** is a commutative algebra A with product m_c , an action of \mathfrak{g} by derivations and an antisymmetric biderivation $\{-, -\}$ such that

$$\{\{f, f'\}, f''\} + \text{cycl.} = m_c \circ (m_c \otimes 1) \circ \phi(f \otimes f' \otimes f'')$$

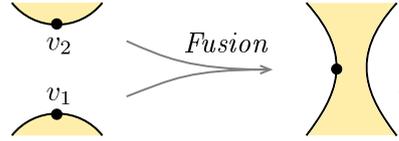
for any $f, f', f'' \in A$, where ϕ acts as an element of $\wedge^3 \mathfrak{g} \hookrightarrow \mathfrak{g}^{\otimes 3}$.

Note that the Cartan elements for \mathfrak{g} and $\bar{\mathfrak{g}}$ are equal, and thus a \mathfrak{g} -quasi-Poisson structure is also $\bar{\mathfrak{g}}$ -quasi-Poisson. A \mathfrak{g} -quasi-Poisson algebra or manifold for $\mathfrak{g} = 0$ is just a Poisson algebra or a manifold.

Let (Σ, V) be a marked surface and let $\mathfrak{g}^V \cong \mathfrak{g}^{\oplus |V|}$ be the quadratic Lie algebra with the quadratic structure given by the sum of $|V|$ terms t . The moduli space $M_{\Sigma, V}(G)$ has a canonical \mathfrak{g}^V -quasi-Poisson structure, by the works of [2, 3, 4] and [16] for $|V| > 1$. The following characterization of the bivector on the moduli space is taken from [17].

Theorem 2.3.10. *There is a unique \mathfrak{g}^V -quasi-Poisson structure π_Σ on $M_{\Sigma, V}(G)$ satisfying the following conditions*

1. *If (Σ, V) is the disk with two marked points on its boundary, the bivector field on the corresponding moduli spaces is 0.*
2. *The bivector field corresponding to a disjoint union of surfaces is the sum of their respective bivectors.*
3. *If a marked surface Σ is obtained by the corner-connected sum at marked points v_1, v_2 from a marked surface Σ_0 as follows*



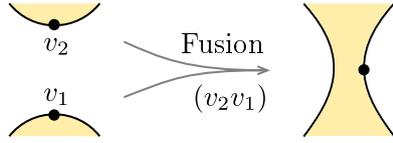
*then the bivectors are related by the so-called **fusion term***

$$\pi_\Sigma = \pi_{\Sigma_0} - \frac{1}{2} t^{ij} (e_i)^{(v_1)} \wedge (e_j)^{(v_2)} \quad (2.8)$$

where $(e_i)^{(v)}$ denotes the \mathfrak{g} -action of $e_i \in \mathfrak{g}$ corresponding to the G -action at the vertex v .

Specifically, the bivector π_Σ depends only on the topological type of the marked surface.

We note that the fusion depends on the order of the points and on the orientation of the surface: fusing v_2 to v_1 (i.e. in the opposite order) would move the new point to the other side of the “bridge”:



In this case, the additional term in π_Σ has the opposite sign.

2.3.3 Moduli spaces without marked points

The original motivation for introducing quasi-Poisson manifolds was to construct the Atiyah-Bott Poisson⁵ structure [8] on the moduli spaces of flat connections in a finite-dimensional manner. Indeed, the Cartan element for \mathfrak{g}^V decomposes into a sum over V , and on the \mathfrak{g}^V -invariant functions, the right-hand side of

$$[\pi, \pi]/2 = \sum_{v \in V} \phi_v$$

acts by zero, i.e. $C^\infty(M_{\Sigma, V}(G))^{G^V}$ is a Poisson algebra. Moreover, one can show that the Poisson bracket depends only on the topological type of the surface [33].

Sometimes, it is useful to consider a more general quotienting procedure.

Definition 2.3.11. Let \mathfrak{g} be a quadratic Lie algebra. A Lie subalgebra $\mathfrak{c} \subset \mathfrak{g}$ is called coisotropic if the image of t in $\text{Sym}^2(\mathfrak{g}/\mathfrak{c})$ is zero.

Then [17], for a $\mathfrak{g} \oplus \mathfrak{h}$ -quasi-Poisson manifold M and $\mathfrak{c} \in \mathfrak{g}$ coisotropic, the \mathfrak{c} -invariant functions on M form a \mathfrak{h} -quasi-Poisson algebra; if $\mathfrak{h} = 0$, then we get an honest Poisson algebra.

If we perform a reduction by the \mathfrak{c} -action at a marked point, the resulting moduli space will be representing by labeling the marked point by \mathfrak{c} .



Remark 2.3.12. Let us mention that there is another way to turn \mathfrak{g} -quasi-Poisson manifolds to Poisson manifolds, due to [2]. One needs an r -matrix: an element $r \in \wedge^2 \mathfrak{g}$ that satisfies $[r, r]/2 = -\phi$ in $\wedge^3 \mathfrak{g}$. Then, if π is a \mathfrak{g} -quasi-Poisson bivector on m , the bivector $\pi + r_M$ satisfies

$$[\pi + r_M, \pi + r_M]/2 = [\pi, \pi]/2 + [\pi, r_M] + [r_M, r_M]/2 = 0.$$

Here, the first and the last term cancel each other and the middle term vanishes by the invariance of π . For moduli spaces of flat connections, this recovers the construction of Fock and Rosly [33].

⁵The fact that for surfaces with boundary one obtains a Poisson structure on the moduli space is apparently a folklore result, see e.g. [9, 14].

2.4 Braided monoidal categories and Drinfeld associators

We will now introduce an important category, the braided monoidal category of $U\mathfrak{g}$ -modules twisted by the Drinfeld associator [26, 27]. For a textbook account, see Chapter XX of [42].

2.4.1 Braided monoidal categories and monoidal functors

We begin by recalling the notion of braided monoidal categories, functors between them and their relationship to algebras. Some of the standard references are [46] or [40]. Our main example will be the Drinfeld category of $U\mathfrak{g}$ -modules, see Section 2.4.3.

Definition 2.4.1. A **monoidal category** \mathcal{C} is a category equipped with an object I called the **unit**, a functor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ and natural isomorphisms

$$\begin{aligned}\alpha_{X,X',X''}: (X \otimes X') \otimes X'' &\xrightarrow{\cong} X \otimes (X' \otimes X''), \\ \lambda_X: I \otimes X &\xrightarrow{\cong} X, \\ \rho_X: X \otimes I &\xrightarrow{\cong} X\end{aligned}$$

called the **associator** and the (left and right) **unitors**. For these natural transformations, we require the following diagrams, called the **pentagon** and the **triangle**, to commute

$$\begin{array}{ccc} & ((X \otimes X') \otimes X'') \otimes X''' & \\ \alpha_{X,X',X''} \otimes 1_{X'''} \swarrow & & \searrow \alpha_{X \otimes X', X'', X'''} \\ (X \otimes (X' \otimes X'')) \otimes X''' & & (X \otimes X') \otimes (X'' \otimes X''') \\ \downarrow \alpha_{X, X' \otimes X'', X'''} & & \downarrow \alpha_{X, X', X'' \otimes X'''} \\ X \otimes ((X' \otimes X'') \otimes X''') & \xrightarrow{1_X \otimes \alpha_{X', X'', X'''}} & X \otimes (X' \otimes (X'' \otimes X''')) \end{array}$$

$$\begin{array}{ccc} (X \otimes I) \otimes X' & \xrightarrow{\alpha_{X, I, X'}} & X \otimes (I \otimes X') \\ \rho_X \otimes 1_{X'} \searrow & & \swarrow 1_X \otimes \lambda_{X'} \\ & X \otimes X' & \end{array}$$

A monoidal category where the components of α , λ and ρ are the identity maps is called **strict**.

Definition 2.4.2. A **braided monoidal category** is a monoidal category together with a **braiding**, i.e. a natural isomorphism $\beta_{X,Y}: X \otimes Y \xrightarrow{\cong} Y \otimes X$ such that the following

diagrams, called **hexagons**, commute

$$\begin{array}{ccccc}
& (X \otimes X') \otimes X'' & \xrightarrow{\beta_{X \otimes X', X''}} & X'' \otimes (X \otimes X') & \\
& \swarrow \alpha_{X, X', X''} & & \nwarrow \alpha_{X'', X, X'} & \\
X \otimes (X' \otimes X'') & & & & (X'' \otimes X) \otimes X' \\
& \searrow 1_X \otimes \beta_{X', X''} & & \swarrow \beta_{X, X''} \otimes 1_{X'} & \\
& X \otimes (X'' \otimes X') & \xrightarrow{\alpha_{X, X'', X'}^{-1}} & (X \otimes X'') \otimes X' & \\
& & & &
\end{array}$$

$$\begin{array}{ccccc}
& (X \otimes X') \otimes X'' & \xrightarrow{\beta_{X \otimes X', X''}^{-1}} & X'' \otimes (X \otimes X') & \\
& \swarrow \alpha_{X, X', X''} & & \nwarrow \alpha_{X'', X, X'} & \\
X \otimes (X' \otimes X'') & & & & (X'' \otimes X) \otimes X' \\
& \searrow 1_X \otimes \beta_{X', X''}^{-1} & & \swarrow \beta_{X, X''}^{-1} \otimes 1_{X'} & \\
& X \otimes (X'' \otimes X') & \xrightarrow{\alpha_{X, X'', X'}^{-1}} & (X \otimes X'') \otimes X' & \\
& & & &
\end{array}$$

A braided monoidal category is called **symmetric** if $\beta_{X,Y} = \beta_{Y,X}^{-1}$.

The axioms from these two definitions imply the so-called coherence theorem: two different (free) morphisms $X \rightarrow X'$ consisting of the maps $\alpha, \beta, \lambda, \rho$ are equal if they have the same underlying braid. See [40, Section 2] for a precise statement.

Definition 2.4.3. An **algebra** in a monoidal category \mathcal{C} is an object $A \in \mathcal{C}$ and two morphisms $m: A \otimes A \rightarrow A$, $u: I \rightarrow A$ that satisfy

$$\begin{array}{ccc}
(A \otimes A) \otimes A & \xrightarrow{\alpha_{A,A,A}} & A \otimes (A \otimes A) \xrightarrow{1_A \otimes m} A \otimes A \\
m \otimes 1_A \downarrow & & \downarrow m \\
A \otimes A & \xrightarrow{m} & A
\end{array} \quad , \quad (2.9)$$

$$\begin{array}{ccccc}
I \otimes A & \xrightarrow{u \otimes 1_A} & A \otimes A & \xleftarrow{1_A \otimes u} & A \otimes I \\
& \searrow \lambda_A & \downarrow m & \swarrow \rho_A & \\
& & A & &
\end{array}$$

An algebra in a braided monoidal category is called **commutative** if $m \circ \beta_{A,A} = m$.

Definition 2.4.4. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between two monoidal categories is called **lax monoidal** if it is equipped with a natural transformation

$$\mu_{X,X'}: F(X) \otimes_{\mathcal{D}} F(X') \rightarrow F(X \otimes_{\mathcal{C}} X')$$

and a morphism $\eta: I_{\mathcal{D}} \rightarrow F(I_{\mathcal{C}})$ such that the following diagrams commute (dropping the labels of the tensor products $\otimes_{\mathcal{C}}$, $\otimes_{\mathcal{D}}$ for brevity):

$$\begin{array}{ccc}
(F(X) \otimes F(X')) \otimes F(X'') & \xrightarrow{\alpha_{F(X), F(X'), F(X'')}^{\mathcal{D}}} & F(X) \otimes (F(X') \otimes F(X'')) \\
\downarrow \mu_{X, X'} \otimes 1_{F(X'')} & & \downarrow 1_{F(X)} \otimes \mu_{X', X''} \\
F(X \otimes X') \otimes F(X'') & & F(X) \otimes F(X' \otimes X'') \\
\downarrow \mu_{X \otimes X', X''} & & \downarrow \mu_{X, X' \otimes X''} \\
F((X \otimes X') \otimes X'') & \xrightarrow{F(\alpha_{X, X', X''}^{\mathcal{C}})} & F(X \otimes (X' \otimes X''))
\end{array} \quad , \quad (2.10)$$

$$\begin{array}{ccc}
I_{\mathcal{D}} \otimes F(X) \xrightarrow{\eta \otimes 1_{F(X)}} F(I_{\mathcal{C}}) \otimes F(X) & & F(X) \otimes I_{\mathcal{D}} \xrightarrow{1_{F(X)} \otimes \eta} F(X) \otimes F(I_{\mathcal{C}}) \\
\downarrow \lambda_{F(X)}^{\mathcal{D}} & & \downarrow \rho_{F(X)}^{\mathcal{D}} \\
F(X) \xrightarrow{F(\lambda_X^{\mathcal{C}})^{-1}} F(I_{\mathcal{C}} \otimes X) & & F(X) \xrightarrow{F(\rho_X^{\mathcal{C}})^{-1}} F(X \otimes I_{\mathcal{C}})
\end{array} \quad , \quad (2.11)$$

The maps μ and η are called **monoidal constraints**. A lax monoidal functor is called **strong monoidal** if μ and η are isomorphism. A (lax or strong) monoidal functor between braided monoidal categories is called **braided** if the following diagram commutes:

$$\begin{array}{ccc}
F(X) \otimes F(X') & \xrightarrow{\mu_{X, X'}} & F(X \otimes X') \\
\downarrow \beta_{F(X), F(X')}^{\mathcal{D}} & & \downarrow F(\beta_{X, X'}^{\mathcal{C}}) \\
F(X') \otimes F(X) & \xrightarrow{\mu_{X', X}} & F(X' \otimes X)
\end{array} \quad . \quad (2.12)$$

Proposition 2.4.5. *The composition of two lax monoidal functors $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ is lax monoidal, with the monoidal structure given by*

$$GF(X) \otimes GF(X') \xrightarrow{\mu_{F(X), F(X')}^G} G(F(X) \otimes F(X')) \xrightarrow{G(\mu_{X, X'}^F)} GF(X \otimes X')$$

and

$$I_{\mathcal{E}} \xrightarrow{\eta^G} G(I_{\mathcal{D}}) \xrightarrow{G(\eta^F)} GF(I_{\mathcal{C}}).$$

Definition 2.4.6. A natural transformation $\phi: F \rightarrow F'$ between two (lax or strong) monoidal functors $F, F': \mathcal{C} \rightarrow \mathcal{D}$ with monoidal structures (μ, η) and (μ', η') is called **monoidal** if the following diagrams commute:

$$\begin{array}{ccc}
F(X) \otimes F(X') & \xrightarrow{\mu_{X, X'}} & F(X \otimes X') \\
\downarrow \phi_X \otimes \phi_{X'} & & \downarrow \phi_{X \otimes X'} \\
F'(X) \otimes F'(X') & \xrightarrow{\mu'_{X, X'}} & F'(X \otimes X')
\end{array} \quad , \quad
\begin{array}{ccc}
I_{\mathcal{D}} & \xrightarrow{\eta} & F(I_{\mathcal{C}}) \\
\searrow \eta' & & \downarrow \phi_{I_{\mathcal{C}}} \\
& & F'(I_{\mathcal{C}})
\end{array} .$$

Monoidal functors are closely related to algebras.

Proposition 2.4.7.

1. If (A, m, u) is an algebra in a monoidal category \mathcal{C} and $F: \mathcal{C} \rightarrow \mathcal{D}$ is a lax monoidal functor, then $F(A)$ is an algebra in \mathcal{D} with the multiplication given by

$$F(A) \otimes F(A) \xrightarrow{\mu_{A,A}} F(A \otimes A) \xrightarrow{F(m)} F(A)$$

and the unit given by $I_{\mathcal{D}} \xrightarrow{\eta} F(I_{\mathcal{C}}) \xrightarrow{F(u)} F(A)$.

If A is commutative and F is braided monoidal, then $F(A)$ is also commutative.

2. If F' is another monoidal functor $F': \mathcal{C} \rightarrow \mathcal{D}$ and $\phi: F \rightarrow F'$ is a monoidal natural transformation, then $\phi_A: F(A) \rightarrow F'(A)$ is an algebra morphism.

Proof. If \bullet is the strict monoidal category with one object and one morphism, then the algebra $A \in \mathcal{C}$ is the same thing as a lax monoidal functor $\bullet \rightarrow \mathcal{C}$ sending the object to A . A morphism of two algebras is a monoidal transformation of such functors. \square

Remark 2.4.8 (*On terminology*). Sometimes in the literature, the name **monoidal functor** is used, most often to designate a strong monoidal functor. We will always distinguish between lax and strong monoidal functor explicitly.

A **colax** (also called **oplax**) monoidal functor is a functor with the monoidal constraint constraint going in the opposite direction $F(X \otimes_{\mathcal{C}} X') \rightarrow F(X) \otimes_{\mathcal{D}} F(X')$. Analogous results to those above then hold for coalgebras.

Monoidal categories are also called **tensor categories**; sometimes such name is reserved for linear monoidal categories (i.e. categories enriched over R -modules).

The term **monoid** is also used instead of the term algebra.

Doing calculations in monoidal categories is greatly simplified by using string diagrams [61]. Our convention is that morphisms go from bottom to top, the associator is depicted by



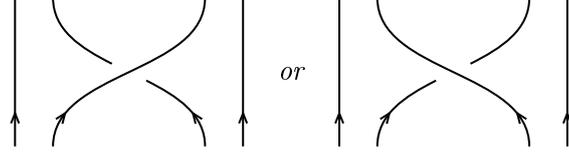
and β by an overcrossing



For \mathcal{C} monoidal, the category $\mathcal{C} \times \mathcal{C}$ is also monoidal, with component-wise monoidal structure. We want to endow the tensor functor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ with a strong monoidal structure, i.e. we are looking for natural isomorphisms

$$\mu_{(X_1, X_2), (X'_1, X'_2)} : (X_1 \otimes X_2) \otimes (X'_1 \otimes X'_2) \xrightarrow{\cong} (X_1 \otimes X'_1) \otimes (X_2 \otimes X'_2).$$

Proposition 2.4.9. *If \mathcal{C} is a braided monoidal category, then the tensor functor \otimes is monoidal with either of the following two monoidal structures*



The isomorphism $I \xrightarrow{\cong} I \otimes I$ is the inverse of $\lambda_I = \rho_I: I \otimes I \rightarrow I$.

Thus, if \mathcal{C} is a braided monoidal category, the tensor product of two algebras can be given an algebra structure.

For completeness, we write the first string diagram in terms of the associator and braiding:

$$\alpha_{X_1, X'_1, (X_2 \otimes X'_2)}^{-1} \circ 1_{X_1} \otimes \alpha_{X'_1, X_2, X'_2} \circ 1_{X_1} \otimes (\beta_{X_2, X'_1} \otimes 1_{X'_2}) \circ 1_{X_1} \otimes \alpha_{X_2, X'_1, X'_2}^{-1} \circ \alpha_{X_1, X_2, (X'_1 \otimes X'_2)}$$

The proof is a simple exercise in drawing string diagrams, see e.g. (3.2).

2.4.2 Drinfeld associators

We now define a notion central to this work: that of a Drinfeld associator. It will be a formal non-commutative power series in two variables satisfying a system of non-linear equations in the following algebra:

Definition 2.4.10. Let $n \in \mathbb{N}_{\geq 2}$. The **Drinfeld-Kohno Lie algebra** \mathfrak{k}_n is the Lie algebra with generators $\{t_{ij} \mid i, j \in 1, \dots, n, i \neq j\}$ and relations $t_{ij} = t_{ji}$, $[t_{ij} + t_{ik}, t_{jk}] = 0$ for i, j, k distinct and $[t_{ij}, t_{kl}] = 0$ for i, j, k, l distinct. The **Drinfeld-Kohno algebra** $U\mathfrak{k}_n$ is the universal enveloping algebra completed with respect to the grading given by the number of the generators.

If $\{a_1 \dots a_k\}$ and $\{b_1 \dots b_l\}$ are two disjoint subsets of $\{1, \dots, n\}$, we define

$$t_{(a_1 \dots a_k)(b_1 \dots b_l)} := \sum_{i=1 \dots k, j=1 \dots l} t_{a_i b_j}.$$

For a formal non-commutative power series $\Phi(X, Y) \in \mathbb{C}\langle\langle X, Y \rangle\rangle$ and three disjoint subsets a, b, c of $\{1, \dots, n\}$, define $\Phi^{abc} := \Phi(t_{ab}, t_{bc}) \in U\mathfrak{k}_n$. We will now define a special class of such power series, called *Drinfeld associators*, after [26, §5]; see also [52, Section 4] and [22, Chapter 10].

Definition 2.4.11. A *Drinfeld associator* is an invertible formal non-commutative power series $\Phi(X, Y) \in \mathbb{C}\langle\langle X, Y \rangle\rangle$ that satisfies the following equations

$$\Phi(t_{12}, t_{23}) \text{ is grouplike in } U\mathfrak{k}_3, \quad (2.13)$$

$$\Phi(X, Y)^{-1} = \Phi(Y, X), \quad (2.14)$$

$$\Phi^{234} \Phi^{1(23)4} \Phi^{123} = \Phi^{12(34)} \Phi^{(12)34} \text{ in } U\mathfrak{k}_4, \quad (2.15)$$

$$e^{\frac{1}{2}t_{(12)3}} = \Phi^{312} e^{\frac{1}{2}t_{13}} (\Phi^{132})^{-1} e^{\frac{1}{2}t_{23}} \Phi^{123} \text{ in } U\mathfrak{k}_3. \quad (2.16)$$

An associator is called *even* if it satisfies $\Phi(-X, -Y) = \Phi(X, Y)$.

Drinfeld constructed an explicit complex associator Φ_{KZ} as the monodromy of the Knizhnik-Zamolodchikov connection and proved that there exists an even associator with rational coefficients [26]. The only other explicitly known associator is the Alekseev-Torossian associator [5, 63], see also [31] for an overview and a proof of the irrationality of this associator.

Remark 2.4.12. What we defined is a *horizontal, grouplike Drinfeld associator with weight $\mu = 1$* ; a definition of an associator with general weight is obtained by replacing $e^{t/2}$ by $e^{\mu t/2}$ in (2.16). An associator $\Phi_\mu(X, Y)$ of a nonzero weight μ specifies an associator $\Phi_\mu(X/\mu, Y/\mu)$ of weight 1.

A non-horizontal Drinfeld associator would take values in the algebra of chord diagrams \mathcal{P}_3 defined below in Section 2.5; this algebra contains $\mathbb{C}\langle\langle X, Y \rangle\rangle$ as a subalgebra [12, Corollary 4.4], [36, Remark 16.2], [22, Theorem 12.7.1]. The axiom (2.14) gets replaced by $\Phi^{321} = \Phi^{-1}$. See Definition 2.5.4 Item 7 or [50, Section 5] for the group-like condition.

The fact that Φ is grouplike implies that its logarithm is a Lie series, i.e. an element of the completed free Lie algebra on 2 generators. One can easily show that

$$\log \Phi = \frac{1}{24}[X, Y] + (\text{terms at least cubic}). \quad (2.17)$$

The cubic term is already not uniquely specified.

The (2.15) is called the pentagon equation, (2.16) is called the hexagon equation. They correspond to the pentagon and the hexagon equations of the deformed category of representations of a quadratic Lie algebra: see Section 2.4.3. Alternatively, they can be interpreted as equalities of tangles: see Section 2.6.2.

The grouplike property (2.13) also has an interpretation in terms of representation categories for direct sums of quadratic Lie algebra, see Proposition 2.5.11. The horizontal property is important for quantization, see [17, Proposition 1].

Finally (2.14) has an interpretation in terms of tangle invariants, ensuring compatibility of the Kontsevich integral with reflections w.r.t vertical lines, see Proposition 2.6.10.

In many contexts, a second hexagon equation is included in the definition of a Drinfeld associator, in one of the two following variants:

$$\begin{aligned} e^{-\frac{1}{2}t_{(12)3}} &= \Phi^{312} e^{-\frac{1}{2}t_{13}} (\Phi^{132})^{-1} e^{-\frac{1}{2}t_{23}} \Phi^{123}, \\ e^{\frac{1}{2}t_{1(23)}} &= (\Phi^{231})^{-1} e^{\frac{1}{2}t_{13}} \Phi^{213} e^{\frac{1}{2}t_{12}} (\Phi^{123})^{-1}. \end{aligned} \quad (2.18)$$

These are equivalent by taking the inverse and renumbering $123 \mapsto 312$. For completeness, we record the relationship of this equation to the original hexagon (2.16).

Proposition 2.4.13 ([27, §3.]). *From the three equations (2.14), (2.16) and (2.18), any two imply the third one*

Proof. First, note that (2.14) is equivalent to $(\Phi^{123})^{-1} = \Phi^{321}$. This is because the algebra $U\mathfrak{t}_3$ is isomorphic to the free associative algebra on generators t_{12}, t_{23} tensored with the formal power series in the central element $t_{12} + t_{23} + t_{13}$ [26, §5].

Let us suppose that (2.14) holds. Then (2.16) and (2.18) are related by renumbering $123 \mapsto 321$.

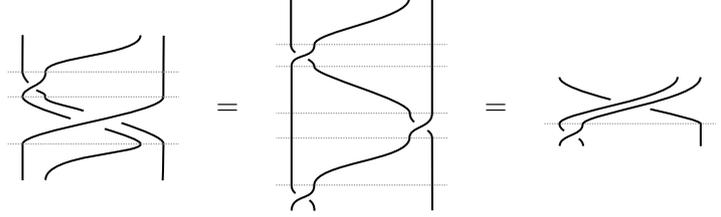
On the other hand, assume (2.16) and (2.18) (in the second form). We will prove

$$\Phi^{321} e^{\frac{1}{2}(t_{12}+t_{23}+t_{13})} \Phi = e^{\frac{1}{2}(t_{12}+t_{23}+t_{13})},$$

which implies the result, as the exponential is central:

$$\begin{aligned} \Phi^{321} e^{\frac{1}{2}(t_{12}+t_{23}+t_{13})} \Phi &= \Phi^{321} e^{\frac{1}{2}t_{23}} e^{\frac{1}{2}t_{1(23)}} \Phi \\ &\stackrel{(2.18)}{=} \Phi^{321} e^{\frac{1}{2}t_{23}} (\Phi^{231})^{-1} e^{\frac{1}{2}t_{13}} \Phi^{213} e^{\frac{1}{2}t_{12}} \\ &\stackrel{(2.16)}{=} e^{\frac{1}{2}t_{(12)3}} e^{\frac{1}{2}t_{12}}, \end{aligned}$$

where (2.16) is used after renumbering $123 \mapsto 213$. Diagrammatically (see Section 2.6.2 for a precise meaning), this is just



□

2.4.3 Infinitesimally braided categories

Let \mathcal{C} be a symmetric monoidal category with braiding σ and associator ϕ , enriched in \mathbb{C} -vector spaces. Denote \mathcal{C}_ε to be the category with the same objects, but with

$$\mathrm{Hom}_{\mathcal{C}_\varepsilon}(X, X') := \mathrm{Hom}_{\mathcal{C}}(X, X') \otimes \mathbb{C}[\varepsilon]/\langle \varepsilon^2 \rangle.$$

In other words, morphisms in \mathcal{C}_ε are of the form $f + \varepsilon g$, and the composition and the monoidal product are $\mathbb{C}[\varepsilon]/\langle \varepsilon^2 \rangle$ -linear.

Definition 2.4.14. [20] An **infinitesimal braiding** on \mathcal{C} is a natural transformation $t_{X, X'} : X \otimes X' \rightarrow X \otimes X'$ on \mathcal{C} such that

$$t_{X', X} = \sigma_{X, X'} \circ t_{X, X'} \circ \sigma_{X', X} \tag{2.19}$$

and such that the map

$$\sigma_{X, X'} \circ (1_{X \otimes X'} + \frac{\varepsilon}{2} t_{X, X'})$$

is a braiding on \mathcal{C}_ε . A category with infinitesimal braiding is called an **infinitesimally braided monoidal category**.

Looking at the ε part of the hexagons, one can characterize infinitesimal braidings as follows.

Proposition 2.4.15. *An infinitesimal braiding on \mathcal{C} is equivalently given by a natural transformation $t_{X,X'}: X \otimes X' \rightarrow X \otimes X'$ satisfying (2.19) and*

$$t_{X \otimes X', X''} = p_{X', X''}^{-1} \circ (t_{X, X''} \otimes 1_{X'}) \circ p_{X', X''} + \alpha_{X, X', X''}^{-1} \circ (1_X \otimes t_{X', X''}) \circ \alpha_{X, X', X''}. \quad (2.20)$$

Here, $p_{X', X''}: (X \otimes X') \otimes X'' \rightarrow (X \otimes X'') \otimes X'$ is the unique such universal map in a symmetric monoidal category, i.e. $p_{X', X''} = \alpha_{X, X', X''} \circ 1_X \otimes \sigma_{X', X''} \circ \alpha_{X, X'', X'}^{-1}$.

An essential example is the category of representations of a quadratic Lie algebra. This is a symmetric monoidal category, with the standard tensor product of modules and with the trivial module \mathbb{C} as the unit object. The symmetry is the usual map $X \otimes X' \rightarrow X' \otimes X$ given by $x \otimes x' \mapsto x' \otimes x$.

Example 2.4.16. Let \mathfrak{g} be a quadratic Lie algebra. Then, the symmetric monoidal category of (complex) representations⁶ of \mathfrak{g} has an infinitesimal braiding $t_{X,X'}$ given by the action of $t \in \text{Sym}^2 \mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g}$ on $X \otimes X'$.

Note that $t_{X,X'}$ is a morphism in of $U\mathfrak{g}$ -modules thanks to the invariance of t . The equation (2.19) follows from the symmetry of t and (2.20) corresponds to the \mathfrak{g} -action on a tensor product of two $U\mathfrak{g}$ -modules.

It is natural to ask if one can replace the square-zero parameter ε with a formal parameter \hbar , continuing the deformation in the direction of t . Similarly to \mathcal{C}_ε , let \mathcal{C}_\hbar be the category with Hom-spaces being formal power series in \hbar of morphisms in \mathcal{C} .

Theorem 2.4.17 ([26, 20]). *Let \mathcal{C} be as before with an infinitesimal braiding t and Φ a Drinfeld associator. Then, with the original unitors and with*

$$\begin{aligned} \beta_{X,X'} &:= \sigma_{X,X'} \circ \exp\left(\frac{\hbar}{2} t_{X,X'}\right), \\ \Phi_{X,X',X''} &:= \alpha_{X,X',X''} \circ \Phi(\hbar t_{X,X'} \otimes 1_{X''}, \hbar \alpha_{X,X',X''}^{-1} \circ 1_X \otimes t_{X',X''} \circ \alpha_{X,X',X''}), \end{aligned}$$

\mathcal{C}_\hbar is a braided monoidal category, denoted \mathcal{C}_\hbar^Φ

For a quadratic Lie algebra \mathfrak{g} , we get the category $U\mathfrak{g}\text{-mod}_\hbar^\Phi$, called the **Drinfeld category**. Its objects are $U\mathfrak{g}$ -modules, its morphisms are formal power series of $U\mathfrak{g}$ -module maps, the tensor product is the usual product of $U\mathfrak{g}$ -modules and the braiding and associativity isomorphisms are the formal power series given above.

Remark 2.4.18. There is a beautiful explanation of the above theorem, due to Bar-Natan [10] and Tamarkin [64], who used the operadic language. We refer also to the textbook of Fresse [34, Chapter 11] and the lecture notes of Willwacher [66].

Briefly, infinitesimally braided monoidal categories are algebras over an operad PaCD of Drinfeld-Kohno algebras $U\mathfrak{k}_n$, while braided monoidal categories are algebras over the operad of braid groups PaB. A Drinfeld associator induces a morphism between these operads PaB \rightarrow PaCD, and thus gives a map of algebras in the opposite direction.

⁶We are forced to take complex representations, since the coefficients of a Drinfeld associator can be complex. Alternatively, one can restrict to real associators.

2.4.4 Quantization and moduli spaces

One motivation for the category $U\mathfrak{g}\text{-mod}_\hbar^\Phi$ is that it is a natural setting in which to consider quantization of a \mathfrak{g} -quasi-Poisson algebra A , as observed by [28]. Indeed, since the bracket A does not satisfy the Jacobi identity, it cannot come from an associative star product deforming the commutative product of A . However, the notion of associativity (2.9) in the Drinfeld category $U\mathfrak{g}\text{-mod}_\hbar^\Phi$ contains the associator, which due to (2.17) contains exactly the right-hand side of the Jacobi identity for A .

Definition 2.4.19 ([28, Definition 4.4]). A **deformation quantization** of a \mathfrak{g} -quasi-Poisson algebra A , with a multiplication m_c and a bracket $\{-, -\}$, is a product $m: A \otimes A \rightarrow A$ making A into an associative algebra in the category $U\mathfrak{g}\text{-mod}_\hbar^\Phi$, such that

$$m = m_c + \hbar m_{\text{sc}} + \hbar^2 \dots$$

and such that

$$m_{\text{sc}}(f, f') - m_{\text{sc}}(f', f) = \{f, f'\}$$

for $f, f' \in A$.

For moduli spaces of flat connections on marked surfaces, a quantization in this sense was found by Li-Bland and Ševera in [17], where they constructed a power series of differential operators on the algebra⁷ $C^\infty(M_{\Sigma, V}(G))$ inducing the quantization. In this thesis, we study the properties of this quantization.

2.5 Chord diagrams

The space of chord diagrams is dual to the graded space associated to the filtered space of Vassiliev invariants, see [11]. For us, chord diagrams will be an intermediate step between tangles as in Proposition 2.4.9 and the deformation quantization from Definition 2.4.19. We will follow Le and Murakami [50] in this section.

Definition 2.5.1. Let X be a compact 1-dimensional oriented manifold with boundary. A **chord diagram** is a choice of $2n$ distinct points on X , grouped into n unordered pairs called chords, seen up to component-preserving and orientation-preserving diffeomorphisms of X . The manifold X is called the **support** of the chord diagram.

Obviously, X is a “combinatorial object”, in a sense that it is determined by the order of the chord endpoints on the components of X . We will now impose a linear relation on the space of formal linear combinations of chord diagrams.

Definition 2.5.2. The **4T relation** is the following linear combination of chord diagrams

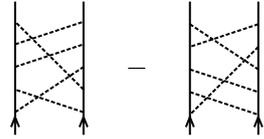
$$\text{Diagram 1} - \text{Diagram 2} - \text{Diagram 3} + \text{Diagram 4} \quad (2.21)$$

⁷Again, we either have to take complex-valued functions, or restrict to real associators. We will use the symbol $C^\infty(-)$ for both cases.

where the three solid segments are parts of X and chords are dashed; the remainder of X may contain other chords. Consider the space of all \mathbb{C} -linear combinations of chord diagrams on X modulo the subspace generated by all 4T relations. This space is graded by the number of chords: we call $\mathcal{P}(X)$ the completion of this space with respect to this grading.

We will use special notation $\mathcal{P}_n := \mathcal{P}(I^{\sqcup n})$ for chord diagrams on n intervals, and we think of the intervals as evenly spaced and oriented upwards. The elements of $\mathcal{P}(X)$ will be called chord diagrams, although they are linear combinations of chord diagrams considered modulo the 4T relations. This is at odds with the usage in combinatorics, see e.g. [7, 32].

Remark 2.5.3. One can define chord diagrams over other rings. For example, over integers, the space $\mathcal{P}_2^{(5)}$ of chord diagrams on 2 strands with 5 chords is isomorphic to $\mathbb{Z}^{148} \oplus \mathbb{Z}_2$, as found by Dogolazky by a computer experiment in his diploma thesis [25]: his order-2 element is



It is an open question whether \mathcal{P}_1 , defined over \mathbb{Z} , has torsion.

Let us now introduce various operations on the spaces of chord diagrams.

Definition 2.5.4. Let X and Y be compact oriented 1-manifolds with boundary.

1. By gluing together X and Y , we get a map

$$\mathcal{P}(X) \otimes \mathcal{P}(Y) \rightarrow \mathcal{P}(X \sqcup Y / \sim), \quad D \otimes E \mapsto DE$$

for any identification \sim of a subset of ∂X with a subset of ∂Y , compatible with the orientation.

2. Similarly, there is a map $\mathcal{P}(X) \otimes \mathcal{P}(Y) \rightarrow \mathcal{P}(X \sqcup Y)$, given by the disjoint union of two chord diagrams. The result of this map on $D \otimes E$ is denoted again by $D \otimes E$.
3. Let C be a component of X . Then define $S_C : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ by reversing the orientation of the component C and multiplying a chord diagram by $(-1)^{\text{number of chord endpoints on } C}$.
4. Similarly, define $\varepsilon_C : \mathcal{P}(X) \rightarrow \mathcal{P}(X \setminus C)$ by removing the component C and setting to 0 any diagrams that have chords ending on C .
5. If $\Delta_C X$ denotes X with 2 copies of C (i.e. $X \sqcup C$), then let

$$\Delta_C : \mathcal{P}(X) \rightarrow \mathcal{P}(\Delta_C X)$$

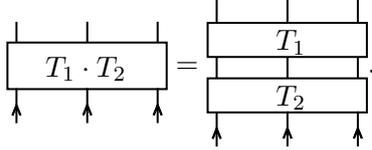
be the map given summing over all possible ways of distributing the chord endpoints on C over the 2 copies of C . In other words, if C has m chord endpoints, such chord diagram gets sent to a sum of 2^m terms.

6. If $f : X \rightarrow Y$ is an orientation-preserving embedding, then we define a map $f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ by transferring the chord endpoints using f . If f is induced by a permutation σ of the n components supporting \mathcal{P}_n , then we denote this operation by $D \mapsto D^{\sigma(1)\dots\sigma(n)}$.
7. If $D \in \mathcal{P}(X)$, a subdiagram D' of D is a chord diagram obtained by removing some of the chords of D , and D'' is the chord diagram containing only these removed chords. Then, define $\square : \mathcal{P}(X) \rightarrow \mathcal{P}(X) \otimes \mathcal{P}(X)$ by

$$\square : D \mapsto \sum_{D' \text{ subdiagram of } D} D' \otimes D''.$$

For \mathcal{P}_n , we will refer to the components just by their number, i.e. the operation of reversing the second interval is S_2 , and similarly for Δ and ε . For \mathcal{P}_1 , we drop the label completely.

The first operation turns \mathcal{P}_n into an algebra, with the convention that strands are oriented upwards:



The map \square then turns \mathcal{P}_n into a cocommutative Hopf algebra [50].

Remark 2.5.5. Le and Murakami use the opposite convention in [52, 50], i.e. $T_1 \cdot T_2$ is obtained by placing T_1 above T_2 with strands oriented downwards. This way, their product on \mathcal{P}_n is opposite to ours, and the map $S_1 \dots S_n : \mathcal{P}_n \rightarrow \mathcal{P}_n$, reversing the orientation of all strands, is an algebra isomorphism between these two algebras. See also the remarks in the proof of the Theorem 2.6.3.

Proposition 2.5.6.

1. All the above operations are compatible with the 4T relations.
2. The operation Δ_C is coassociative in the sense that $\Delta_{C_1} \circ \Delta_C = \Delta_{C_2} \circ \Delta_C$, where $C_{1,2}$ are the two components obtained by doubling C . Moreover, $\varepsilon_{C_i} \circ \Delta_C$ is the identity on $\mathcal{P}(X)$, for $i = 1, 2$.
3. On \mathcal{P}_n , the maps Δ_i , ε_i and \square are algebra maps.

Proof. The compatibility with the 4T relations is immediate for operations in Items 1, 2, 4 and 6 of Definition 2.5.4. For Item 3, the change of the number of endpoints is countered by the exchange of the two terms in the 4T relation (2.21) due to the orientation change (there are multiple cases to consider). For Item 5, there are four terms coming from splitting the segment with 2 endpoints in (2.21); of these, two cancel and the rest combines to a sum of two 4T relations. For Item 7, the terms where the two chords in 4T relation are sent to different factors of $\mathcal{P}(X) \otimes \mathcal{P}(X)$ cancel.

Both sides of the coassociativity condition are equal to the 3^m possible ways of distributing the m chord endpoints on C among the three resulting support components. The only nonzero

term after the application of ε_{C_1} is the one where all the chord endpoints are on C_2 . The compatibility with the product of \mathcal{P}_n is proven similarly. \square

Remark 2.5.7. As mentioned briefly in Remark 2.4.12, the algebra \mathcal{P}_n contains the Drinfeld-Kohno algebra $U\mathfrak{k}_n$ by sending the generator t_{ij} to the chord connecting strands i and j . The elements of this subalgebra are called **horizontal chord diagrams**. The coproduct on $U\mathfrak{k}_n$ corresponds to the map \square , and the other operations in Definition 2.5.4 have similar interpretations in terms of $U\mathfrak{k}_n$.

2.5.1 Chord diagrams and Lie algebras

Chord diagrams are very closely related to quadratic Lie algebras [57, 44, 11, 38]. Namely, we can see chord diagrams as Feynman diagrams, to be evaluated using a quadratic Lie algebra by interpreting chords as $t \in \text{Sym}^2 \mathfrak{g}$ and the strands as $U\mathfrak{g}$.

Definition 2.5.8. Let \mathfrak{g} be a quadratic Lie algebra with $t = t^{ij}e_i \otimes e_j \in (\text{Sym}^2 \mathfrak{g})^{\mathfrak{g}}$, where $\{e_i\}$ is a basis of \mathfrak{g} . Then, for $D \in \mathcal{P}_n$, define its **weight** $W^{\mathfrak{g}}(D)$ as follows:

1. Label all the chord ends by distinct labels.
2. Write an expression $e_{I^1} \otimes \cdots \otimes e_{I^n}$, where $I^i = (I_1^i \dots I_{m_i}^i)$ is the ordered set of all chord ends on the strand i starting from the top and $e_{I^i} = e_{I_1^i} \dots e_{I_{m_i}^i}$.
3. For each chord, multiply the above expression by $\hbar t^{ab}$ where a and b are the labels of the chord ends.
4. Sum over all labels, obtaining an element in $U(\mathfrak{g}^{\oplus n})$.

For example, the diagram $t_{11} \in \mathcal{P}_1$ is sent to the Casimir $\hbar t^{ij}e_i e_j \in U\mathfrak{g}$. A more complicated example is

$$\begin{array}{c} \text{Diagram: A vertical strand with two crossings. The top crossing is a loop. The bottom crossing is a chord connecting the two strands.} \end{array} \xrightarrow{W^{\mathfrak{g}}} \hbar^3 t^{ij} t^{kl} t^{mn} e_m e_n e_k e_i \otimes e_j e_l.$$

Proposition 2.5.9. *The above map $W^{\mathfrak{g}}: \mathcal{P}_n \rightarrow (U\mathfrak{g})^{\otimes n}[[\hbar]]$ is a well defined map of algebras and its image is invariant under the diagonal action of \mathfrak{g} .*

Proof. Thanks to the symmetry of t , the result is independent of which end of the chord is sent to which strand. On the 4T relation, the first two terms give

$$t^{ia} t^{jb} [e_i, e_j] \dots e_a \dots e_b = f^{kab} e_k \dots e_a \dots e_b,$$

while the last two terms give

$$t^{ia} t^{jb} e_j \dots e_a \dots [e_i, e_b] = f^{laj} e_j \dots e_a \dots e_l.$$

The tensor $f^{kab} = f_{ij}^k t^{ia} t^{jb}$ is completely antisymmetric, thanks to the invariance of t , and these terms thus cancel. The compatibility with products is ensured by the convention for

the product in \mathcal{P}_n and the definition of $W^{\mathfrak{g}}$: in both cases, the rightmost chords are on the bottom.

The invariance follows from the fact that the element $W^{\mathfrak{g}}(D)$, seen as a map

$$\mathbb{C} \rightarrow (U\mathfrak{g})^{\otimes n}[[\hbar]],$$

is given by a composition of \mathfrak{g} -equivariant maps: t , permutations and the product of $U\mathfrak{g}$. \square

Remark 2.5.10. The usual weight systems (for links) are obtained from the above elements of $(U\mathfrak{g})^{\otimes n}[[\hbar]]$ by choosing n representations of \mathfrak{g} and taking the trace of the resulting operator, giving an element of $\mathbb{C}[[\hbar]]$. Various choices of Lie algebras and their representation then give various state-sum like rules for evaluating the chord diagrams, closely related to the skein relations [11, 49, 48].

The Drinfeld category $U(\mathfrak{g}^{\oplus n})\text{-mod}_\hbar^\Phi$ was defined using the infinitesimal braiding on $U(\mathfrak{g}^{\oplus n})$ -modules coming from $t \in \text{Sym}^2(\mathfrak{g})$. The action of the braiding and the associator can be then seen as factoring through the action of the Drinfeld-Kohno algebra $U\mathfrak{k}_n$ on $U(\mathfrak{g}^{\oplus n})$ -modules, given by sending t_{ij} to t acting on the i th and the j th factor.

However, there is more structure on $U(\mathfrak{g}^{\oplus n})\text{-mod}_\hbar^\Phi$: any chord diagram in \mathcal{P}_n acts on $U(\mathfrak{g}^{\oplus n})$, via its weight (Definition 2.5.8). The action of the Drinfeld-Kohno algebra corresponds to the horizontal chord diagrams. We will now shortly describe how the various operations on chord diagrams we introduced in Definition 2.5.4 interact with $W^{\mathfrak{g}}$ and $U\mathfrak{g}$ -modules.

Proposition 2.5.11. *Let $D \in \mathcal{P}_n$.*

1. *The algebra map $W^{\mathfrak{g}}: \mathcal{P}_n \rightarrow U(\mathfrak{g}^{\oplus n})[[\hbar]]$ intertwines the maps S_i , ε_i and Δ_i with the antipode, counit and coproduct of $U\mathfrak{g}$, acting on the i th factor of $U(\mathfrak{g}^{\oplus n})[[\hbar]]$.*
2. *For the quadratic Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$, the weight $W^{\mathfrak{g} \oplus \mathfrak{g}}(D)$ is equal to $(W^{\mathfrak{g}} \otimes W^{\mathfrak{g}})(\square D)$, after an identification $U(\mathfrak{g}^{\oplus 2n}) \cong U(\mathfrak{g}^{\oplus n})^{\otimes 2}$ grouping together the Lie algebras on the even and the odd positions.*

2.5.2 More on chord diagrams

We will now review a few useful properties of chord diagrams, which are inspired by the connection to Lie algebras. First, we introduce an operation of pasting an element of \mathcal{P}_r into a result of the action by Δ_C^{r-1} . The following proposition is due to [50, Lemma 2.1]. It is the chord-diagrammatic version of the claim that elements of \mathcal{P}_r are invariant under the diagonal action.

Proposition 2.5.12. *If $q \in C$ is not a chord endpoint, consider the map*

$$\text{insert}_C: \mathcal{P}_r \otimes \mathcal{P}(X) \rightarrow \mathcal{P}(\Delta_C^{r-1}X), \quad D_r \otimes D \mapsto \text{insert}_C(D_r \text{ in } D)$$

defined by taking $\Delta_C^{r-1}D$ and replacing the r copies of a small interval around q by the element of $D_r \in \mathcal{P}_r$. Then this map is independent of q .

This proposition implies the following useful facts about the algebras \mathcal{P}_n (see [50, Section 2.], [52, Theorem 1.]).

Corollary 2.5.13.

1. The map insert_C for $r = 1$, i.e. $\mathcal{P}_1 \otimes \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, defines an action of the algebra \mathcal{P}_1 on $\mathcal{P}(X)$, for each component C of X .
2. Let $(f_i)_* : \mathcal{P}_1 \rightarrow \mathcal{P}_n$ be the map induced by embedding into the i -th strand. Then, its image lies in the center of \mathcal{P}_n . Multiplication by such elements corresponds to the above action of \mathcal{P}_1 on the i -th strand.
3. Similarly, the image of $\Delta_{n,1} : \mathcal{P}_1 \rightarrow \mathcal{P}_n$ lies in the center of \mathcal{P}_n .
4. For $n = 1$, this implies that \mathcal{P}_1 is a commutative algebra.

Intuitively, one should think that elements of \mathcal{P}_1 can slide along the support, passing any chord endpoints. And, more generally, elements of \mathcal{P}_r can slide along the r copies of a strand coming from Δ_C^{r-1} .

By connecting the ends of the interval supporting \mathcal{P}_1 , one obtains a map $\mathcal{P}_1 \rightarrow \mathcal{P}(S^1)$. This map is actually an isomorphism of commutative algebras, where the multiplication on $\mathcal{P}(S^1)$ is given the connected sum: cutting the two circles and joining them together [11, Lemma 3.1].

We will also need the following result describing a part of the center of \mathcal{P}_n ; the claim for $n = 2$ is implicit in [52].

Proposition 2.5.14. *For $n \geq 2$, the element*

$$\sum_{1 \leq i < j \leq n} t_{ij}$$

lies in the center of \mathcal{P}_n . Specifically, $t_{12} \in \mathcal{P}_2$ is central.

Proof. If we apply the map $\Delta^{n-1} : \mathcal{P}_1 \rightarrow \mathcal{P}_n$ to the unique chord diagram with one chord $t_{11} \in \mathcal{P}_1$, the result contains n^2 terms obtained by all possible ways of placing the two chord endpoints on the n strands. Regrouping these terms gives

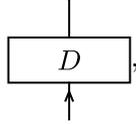
$$\Delta_{n,1}(t_{11}) = \Delta_{n,1} \left(\begin{array}{c} \text{---} \circ \text{---} \\ | \\ \text{---} \circ \text{---} \end{array} \right) = \sum_{1 \leq i \leq n} t_{ii} + 2 \sum_{1 \leq i < j \leq n} t_{ij}.$$

By Corollary 2.5.13, the LHS lies in the center, as do the elements t_{ii} . □

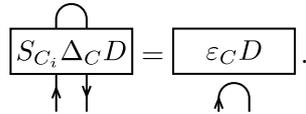
The following proposition is a diagrammatic analogue of the fact that the image of the coproduct of $U\mathfrak{g}$ acts trivially on invariants under the diagonal action.

Proposition 2.5.15. *Let C be a component of a chord diagram D diffeomorphic to an interval. Let $\cap \circ S_{C_i} \circ \Delta_C(D)$ be the result of doubling the component C via Δ_C , reversing one of the resulting intervals C_i via S_{C_i} and connecting one pair of endpoints of these two components (C_1 and C_2 denote the two copies of C). Then $\cap \circ S_{C_i} \circ \Delta_C(D)$ equals $\varepsilon_C D \otimes C$, i.e. the chord diagram obtained by multiplying it by 0 if C supports any chord endpoints.*

If we depict D with the component C by



we can depict this result by

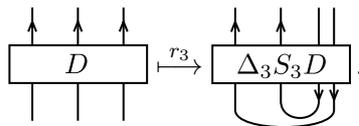


An analogous result holds for connecting with a cup \cup at the bottom.

Proof. If there are no chord endpoints on C , the proof is immediate. Otherwise, let us look at the topmost chord endpoint on C . The terms in $\Delta_C(D)$ can be grouped into two: those where this endpoint is on the first copy of C or those where it is on the second one. Applying S_{C_i} , these two terms will have opposite signs, as they differ always by the one chord endpoint. After connecting the two components C_1, C_2 , these terms cancel. \square

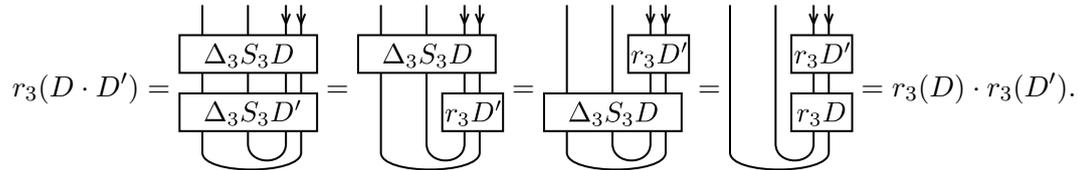
Finally, when an element of \mathcal{P}_n acts on invariants under the diagonal $\mathfrak{g} \subset \mathfrak{g}^{\oplus n}$ action, we can write one of the actions using the remaining ones. On the level of chord diagrams, this motivates the following definition.

Definition 2.5.16. Let $r_i: \mathcal{P}_n \rightarrow \mathcal{P}_{n-1}$ be the map given by reversing the strand i using S_i , doubling it repeatedly to obtain $n - 1$ copies of it and then connecting the ends of these copies with the other $n - 1$ strands. For $D \in \mathcal{P}_3$ and $r_3: \mathcal{P}_3 \rightarrow \mathcal{P}_2$, this is depicted as



Proposition 2.5.17. *The maps $r_i: \mathcal{P}_n \rightarrow \mathcal{P}_{n-1}$ are maps of algebras.*

Proof. The fact that r_i is a map of algebras is shown for $r_3: \mathcal{P}_3 \rightarrow \mathcal{P}_2$



where the middle equality follows from Proposition 2.5.12. \square

The relation to action on invariants is the following:

Proposition 2.5.18. *Let X be a $U(\mathfrak{g}^{\oplus n})$ -module and consider its subspace $X^{\mathfrak{g}}$ of invariants under the diagonal action. For $D \in \mathcal{P}_n$, the subspace $X^{\mathfrak{g}}$ is preserved by $W^{\mathfrak{g}}(D)$ and the action of $W^{\mathfrak{g}}(D)$ is equal to the action of $W^{\mathfrak{g}}(r_i D)$ on the $n-1$ actions obtained by forgetting the i th \mathfrak{g} action.*

Proof. On elements of $X^{\mathfrak{g}}$, we have for $x \in \mathfrak{g}$

$$\sum_i x^{(i)} = 0,$$

i.e.

$$x^{(i)} = \sum_{i \neq j} -x^{(j)}.$$

This corresponds to the Definition 2.5.16 of the maps r_i , with the sign change and reversal of the order of action. \square

2.6 Kontsevich Integral

The differential operators of [17] that give the quantization of the moduli space can be obtained from the Kontsevich integrals [44] of suitable tangles. We review the combinatorial construction of this integral due to Le and Murakami, mainly following [50]. For other perspectives, see the article by Bar-Natan [11] and the book by Chmutov, Duzhin and Mostovoy [22].

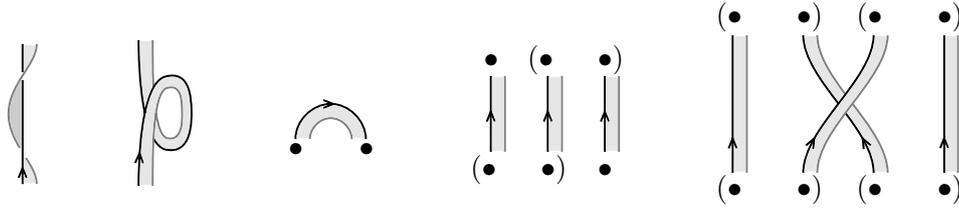
2.6.1 Tangles

Tangles are a generalization of braids (where the tangent vector is never horizontal) and links (tangles without boundary).

Definition 2.6.1. A tangle is an oriented submanifold $X \subset \mathbb{R}^2 \times [0, 1]$ such that the boundary ∂X lies in the two lines $\{(x, 0, 0) \mid x \in \mathbb{R}\} \cup \{(x, 1, 0) \mid x \in \mathbb{R}\}$. A framed tangle is a tangle together with a choice of a normal vector field along the tangle, equal to the unit vector $(\pm 1, 0, 0)$ at the tangle endpoints oriented up/down. A (framed) q-tangle is a (framed) tangle together with a choice of parenthesization of the bottom ($z = 0$) and the top ($z = 1$) endpoints. We will often use the word **tangle** to mean a framed q-tangle.

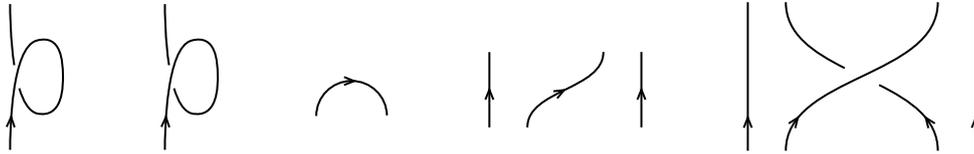
An isotopy of (framed) tangles is a path in the space of (framed) tangles, keeping the endpoints on the respective lines.

We will depict tangles using **tangle diagrams**, i.e. projections of (generic) tangles into the xz -plane with specification of over/undercrossings. The framing can be represented by drawing a ribbon instead of a strand: the gray side of the ribbon is obtained by pushing the strands along the framing:



All, with the exception of the first tangle are given the *blackboard framing*: the normal vector fields lies in the xz plane everywhere. Any framed tangle can be represented by a tangle diagram with blackboard framing, for example the first tangle is isotopic to the second one. Thus, any framed tangle can be represented by a tangle diagram (with implicit blackboard framing), and we will not distinguish isotopy classes of (framed) tangles and tangle diagrams up to Reidemeister moves [22, Theorem 1.8.2].

We will also represent the parenthesization of the tangle endpoints by spacing of the tangle ends, for example, the tangles above will be represented by:



There are multiple natural operations on (framed) q-tangle diagrams we will use, mirroring the operations for chord diagrams from Definition 2.5.4.

- Definition 2.6.2.**
1. If T and T' are two (framed) q-tangle diagrams such that the bottom of T and the top of T' agree in their parenthesizations and orientations, we define TT' to be the (framed) tangle diagram obtained by placing T on top of T' .
 2. For two (framed) q-tangle diagrams T, T' , their tensor product $T \otimes T'$ is obtained by placing T' to the right of T . The parenthesization is similarly obtained by juxtaposing the parenthesizations of T and T' .
 3. If C is a connected component of a framed q-tangle diagram T , then $\Delta_C T$ is obtained from T by adding a copy of C shifted in the direction of the framing on C . If C has boundary, the parenthesization of $\Delta_C T$ is obtained from that of T by the replacement $\bullet \mapsto (\bullet\bullet)$ at endpoints of C . Similarly, $S_C T$ is T with C reversed and $\varepsilon_C T$ is T with C removed, with the obvious parenthesization.

2.6.2 Combinatorial Kontsevich integral

Let T be a tangle diagram, with the underlying 1-manifold denoted again by T . We will now define an invariant of framed tangles $\hat{Z}(T) \in \mathcal{P}(T)$, the *normalized Kontsevich integral of T* , starting from the unnormalized Kontsevich integral $Z(T) \in \mathcal{P}(T)$.

To define Z , we require the following properties and then we specify its value on the elementary tangle diagrams, see [52].

1. If a tangle T can be written as a composition of two tangle diagrams $T_1 T_2$, then $Z(T) = Z(T_1)Z(T_2)$.

- 2. If T_1, T_2 are two tangles, then $Z(T_1 \otimes T_2) = Z(T_1) \otimes Z(T_2)$.
- 3. If C is a component of a tangle T , then $Z(S_C T) = S_C Z(T)$.

Let $t_{12} \in \mathcal{P}_2$ be the chord diagram with one chord connecting the two strands

$$t_{12} = \begin{array}{c} | \\ \hline | \\ \uparrow \quad \uparrow \end{array}$$

Then, for an overcrossing and an undercrossing, Z is defined as

$$\begin{array}{c} \text{overcrossing} \\ \downarrow \xrightarrow{Z} R := \exp(t_{12}/2) \in \mathcal{P}_2, \end{array} \tag{2.22}$$

$$\begin{array}{c} \text{undercrossing} \\ \downarrow \xrightarrow{Z} R^{-1} := \exp(-t_{12}/2) \in \mathcal{P}_2. \end{array}$$

As in Remark 2.5.7, let $t_{ij} \in \mathcal{P}_n$ be the chord diagram with one chord⁸ connecting strands i and j , i.e. for $n = 3$

$$t_{12} = \begin{array}{c} | \\ \hline | \\ \uparrow \quad \uparrow \end{array} \quad \begin{array}{c} | \\ | \\ \uparrow \end{array} \quad \begin{array}{c} | \\ | \\ \uparrow \end{array}$$

(n should be always clear from the context). Let $\Phi(x, y) \in \mathbb{C}\langle\langle x, y \rangle\rangle$ be a Drinfeld associator, as in Definition 2.4.11. Denote by the same symbol the element

$$\Phi := \Phi(t_{12}, t_{23}) = \begin{array}{c} | \\ | \\ \uparrow \end{array} \quad \begin{array}{c} | \\ | \\ \uparrow \end{array} \quad \begin{array}{c} | \\ | \\ \uparrow \end{array} + \frac{1}{24} \left(\begin{array}{c} | \\ \hline | \\ \uparrow \end{array} \begin{array}{c} | \\ | \\ \uparrow \end{array} \begin{array}{c} | \\ | \\ \uparrow \end{array} - \begin{array}{c} | \\ | \\ \uparrow \end{array} \begin{array}{c} | \\ \hline | \\ \uparrow \end{array} \begin{array}{c} | \\ | \\ \uparrow \end{array} \right) + \dots \in \mathcal{P}_3.$$

Let w_1, w_2 and w_3 be three parenthesizations, each w_i made of $|w_i|$ letters. Then, on the tangle made of straight lines going from $(w_1 w_2) w_3$ to $w_1 (w_2 w_3)$, Z is given as

$$\begin{array}{c} \begin{array}{c} w_1 \\ \vdots \\ \uparrow \end{array} \quad \begin{array}{c} (w_2 w_3) \\ \vdots \\ \uparrow \end{array} \\ \downarrow \xrightarrow{Z} \Delta_1^{|w_1|-1} \Delta_2^{|w_2|-1} \Delta_3^{|w_3|-1} \Phi \in \mathcal{P}_{|w_1|+|w_2|+|w_3|}. \\ \begin{array}{c} (w_1 w_2) \\ \vdots \\ \uparrow \end{array} \quad \begin{array}{c} w_3 \\ \vdots \\ \uparrow \end{array} \end{array}$$

On the inverse tangle, we define

$$\begin{array}{c} \begin{array}{c} (w_1 w_2) \\ \vdots \\ \uparrow \end{array} \quad \begin{array}{c} w_3 \\ \vdots \\ \uparrow \end{array} \\ \downarrow \xrightarrow{Z} \Delta_1^{|w_1|-1} \Delta_2^{|w_2|-1} \Delta_3^{|w_3|-1} \Phi^{-1} \in \mathcal{P}_{|w_1|+|w_2|+|w_3|}. \\ \begin{array}{c} w_1 \\ \vdots \\ \uparrow \end{array} \quad \begin{array}{c} (w_2 w_3) \\ \vdots \\ \uparrow \end{array} \end{array}$$

Note that for $w_1 = w_2 = w_3 = \bullet$, this is just the associator Φ and its inverse.

⁸The Drinfeld-Kohno algebra $U\mathfrak{t}_n$ injects into \mathcal{P}_n by sending the generator t_{ij} into the chord diagram t_{ij} , see Remark 2.4.12.

Finally, for caps and cups, Z is trivial

$$\begin{array}{c} \begin{array}{ccc} \text{↻} & \xrightarrow{Z} & \text{↻} \\ \text{↻} & \xrightarrow{Z} & \text{↻} \end{array} \in \mathcal{P}_1, \\ \begin{array}{ccc} \text{↻} & \xrightarrow{Z} & \text{↻} \\ \text{↻} & \xrightarrow{Z} & \text{↻} \end{array} \in \mathcal{P}_1. \end{array}$$

As defined so far, Z is not an isotopy invariant: the following value of Z is not the identity

$$\nu^{-1} := Z \left(\begin{array}{c} \text{↻} \\ \text{↻} \end{array} \right) = (1 \otimes \cap) S_{C_2}(\Phi)(\cup \otimes 1) \in \mathcal{P}_1. \quad (2.23)$$

Note that ν^{-1} equals to 1 plus terms with chords, and thus its inverse ν is well defined.

Thus, to get invariance with respect to the *zig-zag move*

$$\begin{array}{ccc} \begin{array}{c} \text{↻} \\ \text{↻} \end{array} & \rightarrow & \begin{array}{c} \text{↻} \\ \text{↻} \end{array}, \end{array} \quad (2.24)$$

we define the *normalized Kontsevich integral*. Choose $\alpha, \beta \in \mathcal{P}_1$ such that⁹ $S(\alpha) = \alpha$, $S(\beta) = \beta$ and $\alpha\beta = \nu$ (e.g. $\alpha = \beta = \nu^{\frac{1}{2}}$, see Corollary 2.6.12). Then, \hat{Z} is defined as before, with the only difference being

$$\begin{array}{c} \text{↻} \xrightarrow{\hat{Z}} \alpha \in \mathcal{P}_1, \\ \text{↻} \xrightarrow{\hat{Z}} \beta \in \mathcal{P}_1. \end{array}$$

If we want to emphasize the dependence of \hat{Z} on the choices, we will write $\hat{Z}(T; \Phi, \alpha, \beta)$.

Theorem 2.6.3 ([52], [50]). *The map \hat{Z} is a well-defined isotopy invariant of framed q -tangles.*

Proof. Let us only comment on the small differences between our approach and that of [50].

As we mentioned after Definition 2.5.4, our algebras \mathcal{P}_n are opposite to those of [50, 52]. The resulting map \hat{Z} is the same, however: we assign $\Phi(t_{12}, t_{23})$ to the “associator” tangle $(\bullet\bullet)\bullet \rightarrow \bullet(\bullet\bullet)$ with strands going upwards, while Le and Murakami associate to the same associator tangle, with strands going downwards, the element $\Phi_{LM} := \Phi(t_{12}, t_{23})$ interpreted in the opposite algebra. Using the compatibility of Z with reversing orientation, we see that Le and Murakami associate the same element $\Phi(t_{12}, t_{23}) = S_1 S_2 S_3(\Phi_{LM}) \in \mathcal{P}_3$ to the associator tangle oriented upwards. Note also that the axioms for the associator of [51, Section 4.] have the same form as ours; they are related by the “strand-reversal” algebra isomorphism $S_1 S_2 S_3: \mathcal{P}_n^{\text{op}} \rightarrow \mathcal{P}_n$; under which their associator is sent to ours.

⁹It is conjectured that $S(\alpha) = \alpha$ for all $\alpha \in \mathcal{P}_1$ [11]. If true, it would mean that the Kontsevich integral cannot distinguish a knot from its inverse.

Unlike Le and Murakami, we don't require our associator to be even, i.e. it doesn't have to satisfy $\Phi(-X, -Y) = \Phi(X, Y)$. This will affect the compatibility of Z with reflections with respect to horizontal lines, see Proposition 2.6.10.

Le and Murakami define \hat{Z} with $\alpha = \beta = \nu^{1/2}$ in [50] and with $\alpha = \nu, \beta = 1$ in [52]. Allowing more general α and β still produces a tangle invariant. This is because the only Turaev moves changing the number of such critical points involve the product $\alpha\beta$ or $\alpha S(\beta)$, see [65] and [22, Section 1.8].

Finally, Le and Murakami use the compatibility of Z with trivial extensions of tangles; we use compatibility with juxtaposition (denoted by \otimes). Both approaches are equivalent: juxtaposition $T_1 \otimes T_2$ can be expressed using composition and the trivial extensions $T_1 \otimes T_2 = T'_1 T'_2$, with T'_1 obtained by extending T_1 with vertical strands on the left, and similarly for T'_2 . \square

2.6.3 Cabling, twisting and reflections

An important result we will use is the compatibility of the Kontsevich integral with the operation of cabling. The basic case of cabling (a 0-cabling) is the removal of a component of a tangle.

Proposition 2.6.4. *Let C be a component of a tangle T . Then*

$$\varepsilon_C Z(T) = Z(\varepsilon_C T)$$

and similarly for the normalized Kontsevich integral, if $\varepsilon(\alpha) = \varepsilon(\beta) = 1 \in \mathcal{P}_1$.

Proof. It is enough to prove this proposition for elementary tangles, where it is obvious: e.g. for the associator it follows from the fact that $\Phi = 1 +$ terms with chord endpoints on all components. \square

The other interesting case is the 2-cabling or simply cabling, i.e. doubling of a component of a chord diagram. The behaviour of Z with respect to this operation was described by Le and Murakami in [50]. It is again enough to study the elementary tangles, and only the cap and cup will turn out to be non-trivial. This motivates the introduction of the following two elements, which will later play an important role.

Definition 2.6.5 ([50, Section 4.]). Define the following two elements of \mathcal{P}_2

$$a = Z \left(\left(\begin{array}{c} \curvearrowright \\ 1 \ 2 \end{array} \right) \right), \quad b = Z \left(\left(\begin{array}{c} \cup \\ 1 \ 2 \end{array} \right) \right). \quad (2.25)$$

We will later show that $a = a^{21}$ and $b = b^{21}$ in Proposition 4.1.1, i.e. it is not necessary to label the components of the tangles above. Let us also mention that, by [50, Section 4.],

$$a \cdot b = b \cdot a = \nu^{-1} \otimes \nu^{-1} \cdot \Delta \nu. \quad (2.26)$$

Theorem 2.6.6. *Let C be a component of a tangle diagram T with m_+ maxima and m_- minima. Then*

$$Z(\Delta_C T) = \text{insert}_C(a^{m_+} \cdot b^{m_-} \text{ in } Z(T)),$$

i.e. the Kontsevich integral of a tangle T with C doubled is obtained by doubling the chord diagram $Z(T)$ via Δ_C and inserting $a, b \in \mathcal{P}_2$ at maxima/minima. Moreover, if

1. $m_+ = m_-$, or
2. the associator is even and we normalize \hat{Z} with $\alpha = \beta = \nu^{1/2}$,

then

$$\hat{Z}(\Delta_C T) = \Delta_C \hat{Z}(T).$$

Proof. See Section 4 of [50]. The theorem is proven for elementary tangles, and only the caps and cups are non-trivial; their unnormalized Kontsevich integrals are trivial, but the integrals of their cables give exactly a or b . The special case follows since $a \cdot b = \nu^{-1} \otimes \nu^{-1} \cdot \Delta \nu$ for any associator, and $a = b = \nu^{-1/2} \otimes \nu^{-1/2} \cdot \Delta \nu^{1/2}$ for an even associator. \square

Le and Murakami also explain how \hat{Z} depends on the associator, using the notion of twisting by elements of \mathcal{P}_2

Definition 2.6.7 ([52, Section 7]). Let $F \in \mathcal{P}_2$ be such that $\varepsilon_1 F = \varepsilon_2 F = 1 \in \mathcal{P}_1$ and $F = F^{21}$, i.e. a **symmetric twist**. If w is a parenthesization of a word in letters \pm , define $F_w \in \mathcal{P}_{|w|}$ recursively

$$\begin{aligned} F_\emptyset &= 1 \in \mathbb{C}, \\ F_+ &= 1 \in \mathcal{P}_1, \\ F_{++} &= F \in \mathcal{P}_2, \\ F_{w_1 w_2} &= (F_{w_1} \otimes F_{w_2}) \cdot \Delta_1^{|w_1|-1} \Delta_2^{|w_2|-1} F \in \mathcal{P}_{|w_1|+|w_2|}. \end{aligned}$$

If F_w has $-$ at position i , then $F_w = S_i F_{w'}$ where $F_{w'}$ has $+$ at position i . Note that the different factors of F in F_w commute and that the assignment $F \mapsto F_w$ is a map of algebras, by Proposition 2.5.12

For a tangle $T: w_b \rightarrow w_t$, define the twisted Kontsevich integral

$$\hat{Z}^F(T) := F_{w_t} Z(T) (F_{w_b})^{-1}$$

and the twisted associator

$$\Phi^F := Z^F \left(\left(\begin{array}{c} | \\ \uparrow \end{array} \quad \begin{array}{c} \nearrow \\ \searrow \end{array} \quad \begin{array}{c} | \\ \uparrow \end{array} \right) = \Delta_2 F \cdot 1 \otimes F \cdot \Phi(\Delta_1 F \cdot F \otimes 1)^{-1}. \quad (2.27)$$

A twist F of the form $F = (u^{-1} \otimes u^{-1}) \cdot \Delta_{2,1}(u)$ for invertible $u \in \mathcal{P}_1$ is called **trivial**, since for such twist $\Phi^F = \Phi$.

For Φ an associator, Φ^F is again an associator, but it is not necessarily horizontal, nor group-like. The hexagon equation for the twisted associator follows from Proposition 2.5.14.

The following theorem is due to Drinfeld [26] and Le and Murakami [52, Theorem 8.], who proved it in the context of chord diagrams.

Theorem 2.6.8. *Any two associators in \mathcal{P}_3 are related by a symmetric twist $F \in \mathcal{P}_2$.*

This allows us to relate Kontsevich integrals calculated using different associators.

Corollary 2.6.9 ([52]). *Let $\alpha^F = \hat{Z}^F(\curvearrowright)$ and $\beta^F = \hat{Z}^F(\frown)$. Then $\alpha^F \beta^F = \nu$, α^F and β^F are invariant with respect to S and*

$$\hat{Z}(T; \Phi^F, \alpha^F, \beta^F) = \hat{Z}^F(T; \Phi, \alpha, \beta).$$

Moreover, α^F and β^F can be replaced with α and β on strands which have the same number of maximal and minimal critical points.

This implies, for example, that ν is independent of Φ , as the twisting is trivial for a tangle with one endpoint on the bottom and the top.

Proof. The twisted normalizations for the cap α^F can be represented as the following product in \mathcal{P}_1

$$\begin{array}{c} \boxed{\alpha} \\ \curvearrowright \\ \boxed{S_2 F^{-1}} \end{array} = \alpha \cdot r_2(F^{-1}),$$

using Corollary 2.5.13. Moreover, we need to use the isomorphism $\mathcal{P}_1 \cong \mathcal{P}(S^1)$ to show

$$\begin{array}{c} \curvearrowright \\ \boxed{S_2 F^{-1}} \end{array} = \begin{array}{c} \boxed{S_2 F^{-1}} \\ \curvearrowleft \end{array} = \begin{array}{c} \uparrow \\ \boxed{S_2 F^{-1}} \\ \downarrow \end{array}.$$

The fact that $\alpha^F \beta^F = \alpha \beta = \nu$ then follows from Proposition 2.5.17. This also proves that α^F and β^F on the same strand can be replaced by α and β , as the elements $r_2 F^{\pm 1}$ coming from the twist can slide along the strand and will cancel each other. The fact that $S(\alpha^F) = \alpha^F$ follows from the symmetry of F .

The equality $\hat{Z}(T; \Phi^F, \alpha^F, \beta^F) = \hat{Z}^F(T; \Phi, \alpha, \beta)$ follows from the equality on elementary tangle diagrams; for over/undercrossings, this follows from Proposition 2.5.14 and Corollary 2.5.13, as the twists will commute with $e^{t/2}$ and cancel. For caps and cups, one needs to use Proposition 2.5.15 and the property $\varepsilon_i F = 1$. \square

Let us now turn to studying the compatibility of Z with the following simple geometric operations on tangles.

Proposition 2.6.10 ([50, 23]). *Let T be a tangle.*

1. Let T' be the rotation of T by π around a vertical axis, and let $f : T \rightarrow T'$ be the diffeomorphism between the underlying 1-manifolds of T and T' . Then

$$f_*Z(T) = Z(T'),$$

where f_* is defined in Definition 2.5.4, Item 6.

2. Let \bar{T} be the reflection of T with respect to a horizontal line (this exchanges overcrossings and undercrossings), and let $g : T \rightarrow \bar{T}$ be the diffeomorphism of the underlying 1-manifolds of T and \bar{T} . Then

$$g_*Z(T) = Z(\bar{T})_{-t},$$

where, for a chord diagram D , the chord diagram D_{-t} is obtained by multiplying D by $(-1)^{\text{number of chords in } D}$.

Remark 2.6.11. The first part of the proposition is due to [50, Prop. 3.1]; the second part was considered only for even associators in [50]. Note that Le and Murakami described the two geometric operations above as reflections of tangle diagrams around vertical and horizontal axes. Unlike them, we exchange undercrossings and overcrossings in the second part of the proposition.

If the associator Φ used to define Z is even $\Phi(-X, -Y) = \Phi(X, Y)$, we recover the second part of [50, Prop 3.1]: in this case, multiplying each chord by -1 is equivalent to exchanging overcrossings and undercrossings, since only $R = e^{t/2}$ contains diagrams with an odd number of chords.

For knots, the compatibility of mirror images with Z was considered by Chmutov and Duzhin [23, Section 3.1].

Before the proof, let us give a few examples. Applying the first part of the proposition to the associator, we obtain

$$f_*Z \left(\begin{array}{c} \uparrow \\ \nearrow \\ \uparrow \end{array} \right) = Z \left(\begin{array}{c} \uparrow \\ \searrow \\ \uparrow \end{array} \right).$$

In this case, the map $f_* : \mathcal{P}_3 \rightarrow \mathcal{P}_3$ is the map exchanging the first and the third strand. Thus, this identity reduces to

$$\Phi^{-1} = \Phi^{321},$$

which is one of the axioms for the associator.

To illustrate the second part of the proposition, consider the doubled cap and cup. The map g_* is the identity on \mathcal{P}_2 in this case, and we arrive at the equality

$$Z \left(\begin{array}{c} \cup \\ \cup \end{array} \right) = Z \left(\begin{array}{c} \cap \\ \cap \end{array} \right)_{-t},$$

i.e.

$$b = a_{-t}.$$

Proof of Proposition 2.6.10. It is enough to prove this proposition for the elementary tangles. For the first part, the case of caps, cups and over/undercrossings is obvious. The associator is shown in the above example.

For the reflection w.r.t. a horizontal line, it is again the case of the associator which is non-trivial. We want to show

$$g_* Z \left(\begin{array}{c} \downarrow \\ \uparrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \downarrow \\ \uparrow \end{array} \right) = Z \left(\begin{array}{c} \downarrow \\ \downarrow \end{array} \begin{array}{c} \searrow \\ \nearrow \end{array} \begin{array}{c} \downarrow \\ \downarrow \end{array} \right)_{-t}.$$

In this case, the map g_* is the identity on \mathcal{P}_3 . The RHS is obtained from $\Phi(t_{12}, t_{23})$ by taking the inverse associator, reversing the orientation of the strands, and multiplying each chord by -1 . The second and the third operation together give the antipode of $U\mathfrak{k}_3$, i.e. the algebra antihomomorphism taking $t_{ij} \mapsto -t_{ij}$. Since Φ is grouplike in $U\mathfrak{k}_3$, this cancels the inverse.¹⁰ \square

The following useful corollary is implicit in [50].

Corollary 2.6.12. *The element $\nu \in \mathcal{P}_1$ defined in (2.23) satisfies $\nu = S(\nu)$.*

Proof. The element ν is independent of the associator by Corollary 2.6.9; and thus by choosing an even associator, we see that ν contains only chord diagrams with an even number of chords. Taking the “snake” tangle from (2.23) and applying the both the vertical and the horizontal rotation/reflection, we get the reversed tangle. However, both maps f_* and g_* , seen on \mathcal{P}_1 , are trivial.¹¹ \square

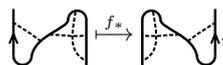
2.6.4 Tangles and Lie algebras

Finally, we can tie together the Drinfeld category and tangles. Namely, using the Kontsevich integral Z and the weight map $W^{\mathfrak{g}}$ from Definition 2.5.8, a tangle with n components, each diffeomorphic to an interval, can act on $U\mathfrak{g}^{\oplus n}$ -modules.

When necessary, we will label the components of a tangle T with the Lie algebra actions, to specify which strands acts where. Moreover, we will often drop Z and $W^{\mathfrak{g}}$, and use a hybrid notation, combining tangles and elements of \mathcal{P}_n . For example, with $a \in \mathcal{P}_2$, the following

¹⁰See the Remark 4.2.5 for a review of the algebra (anti-)homomorphisms of $U\mathfrak{k}_3$ and their effect on Φ .

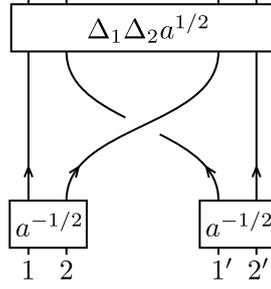
¹¹This is a delicate point, so let us illustrate it on an example. The map f_* relates the following two chord diagrams



but as an element of \mathcal{P}_1 , both are equal to



picture



will act on the tensor product $X \otimes X'$ of two $U\mathfrak{g}^{\oplus 2}$ -modules by the weight of the chord diagram

$$(\Delta_1 \Delta_2 a^{1/2})^{1324} \cdot Z \left(\begin{array}{c} \uparrow \quad \uparrow \\ \uparrow \quad \uparrow \\ \uparrow \quad \uparrow \\ \uparrow \quad \uparrow \end{array} \right) \cdot a^{-1/2} \otimes a^{-1/2}.$$

Similarly, if we have \mathfrak{g} -actions labeled by labels l_1, \dots, l_n (e.g. half-edges as in Definition 2.3.4), then we denote the action of $D \in \mathcal{P}_n$ by

$$D^{(l_1) \dots (l_n)},$$

where the i th strand acts using the \mathfrak{g} -action l_i . For example, the action of the above tangle would be written as

$$(\Delta_1 \Delta_2 a^{1/2})^{(1)(1')(2)(2')} \cdot Z \left(\begin{array}{c} \uparrow \quad \uparrow \\ \uparrow \quad \uparrow \\ \uparrow \quad \uparrow \\ \uparrow \quad \uparrow \end{array} \right)^{(1)(2)(1')(2')} \cdot (a^{-1/2})^{(1)(2)} \cdot (a^{-1/2})^{(1')(2')}.$$

Sometimes, we need to evaluate a chord diagram D using the quadratic structure of the Lie algebra \mathfrak{g} . In that case, we take the chord diagram $(D)_{-t}$ with each chord multiplied by -1 , and use the weight map $W^{\mathfrak{g}}$ associated to \mathfrak{g} .

Chapter 3

Quantization and spines

This chapter contains the main result of this thesis. We begin by recalling the quantization of moduli spaces of flat connection due to Li-Bland and Ševera [17]. Using the Kontsevich integral, we simplify their construction and describe the full set of quantizing algebras and isomorphisms between, in Theorem 3.3.8. The datum needed to describe the quantization will be a spine of the surface with additional decorations.

Then, we modify this construction using $a \in \mathcal{P}_2$ introduced in (2.25). With this modification, we can also describe the relations between the isomorphisms, obtaining quantizing algebras connected by unique isomorphisms, as explained in Theorem 3.4.7. In this setting, the datum reduces to a uni-trivalent spine and the isomorphisms between different algebras will be generated by the flip move, acting via an associator.

Some of the proofs using chord-diagrammatic techniques were moved to the next chapter, to streamline the presentation.

3.1 Quantization of Li-Bland and Ševera

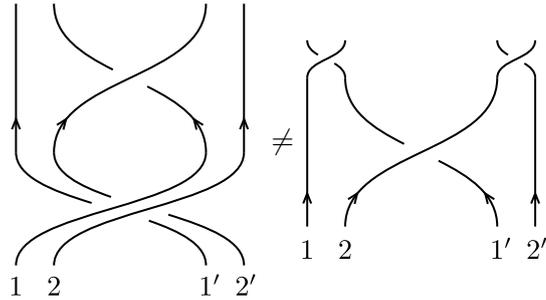
Recall from Section 2.4.3 that for any quadratic Lie algebra \mathfrak{g} and a Drinfeld associator Φ , we can consider the Drinfeld category $U\mathfrak{g}\text{-mod}_{\mathfrak{h}}^{\Phi}$. We will now introduce monoidal structure on functors corresponding to taking invariants and tensor products of representations of \mathfrak{g} .

Proposition 3.1.1 ([17, Prop. 2, Theorem 2]). *Let \mathfrak{g} and \mathfrak{h} be quadratic Lie algebras and Φ a Drinfeld associator.*

1. *Let $\mathfrak{c} \subset \mathfrak{g}$ be a coisotropic Lie subalgebra (see Definition 2.3.11). Define the **reduction functor** $(-)^{\mathfrak{c}}: U(\mathfrak{g} \oplus \mathfrak{h})\text{-mod}_{\mathfrak{h}}^{\Phi} \rightarrow U\mathfrak{h}\text{-mod}_{\mathfrak{h}}^{\Phi}$ by sending a \mathfrak{g} -module X to its subspace of \mathfrak{c} -invariants. Then, with the inclusion $X^{\mathfrak{c}} \otimes Y^{\mathfrak{c}} \rightarrow (X \otimes Y)^{\mathfrak{c}}$ as the monoidal constraint, this is a braided lax monoidal functor.*
2. *Define the **fusion functor** $F: U(\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{h})\text{-mod}_{\mathfrak{h}}^{\Phi} \rightarrow U(\mathfrak{g} \oplus \mathfrak{h})\text{-mod}_{\mathfrak{h}}^{\Phi}$ by restriction to*

The unit constraint is again the identity map $\mathbb{C} \rightarrow \mathbb{C}$. In (2.11), action on $\mathbb{C} \otimes F(X)$ corresponds to removal of the first (or last) two strands of the tangle in (3.1), by Proposition 2.6.4. Since this results in the trivial tangle, (2.11) is satisfied. \square

Remark 3.1.2. The fusion functor is not braided in general, since the two tangles corresponding to two legs of (2.12) are not equal



However, a similar diagram can be used to show that the other obvious monoidal structure of F , with an undercrossing in J , is isomorphic to the one in (3.1), via the monoidal isomorphism given by the action of $e^{-t/2}$.

The two important examples of reduction we will consider are the two coisotropic Lie subalgebras $\mathfrak{g} \subset \mathfrak{g}$ and the diagonal $\mathfrak{g} \subset \mathfrak{g} \oplus \bar{\mathfrak{g}}$.

In [17], a slight generalization of the problem of deformation quantization of moduli spaces of flat connections was proposed. Recall that a \mathfrak{g} -quasi-Poisson structure is also a $\bar{\mathfrak{g}}$ -quasi-Poisson; more generally, we can change signs of the quadratic structures of factors in $\mathfrak{g} \oplus \cdots \oplus \mathfrak{g}$ arbitrarily, since only $[t_{12}, t_{23}]$ enters into the definition of a quasi-Poisson structure.

Definition 3.1.3. Let (Σ, V) be a marked surface with a choice of a sign $+$ or $-$ for each of the marked points. Let \mathfrak{g}^V be the quadratic Lie algebra where the factor corresponding to v is $\bar{\mathfrak{g}}$ if v is marked with minus. A deformation quantization of the moduli space corresponding to such surface is then an associative algebra $A \in U(\mathfrak{g}^V)\text{-mod}_\hbar^\Phi$ satisfying the conditions of Definition 2.4.19.

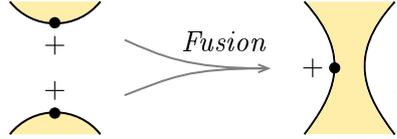
Note that for an empty V , this is a usual associative algebra in the category of vector spaces, quantizing the Poisson structure on $M_\Sigma(G)$.

Since the functors from Proposition 3.1.1 are monoidal, they take associative algebras to associative algebras. The procedure of [17] can be seen as starting with building blocks $C^\infty(G)$ and repeatedly applying these functors.

Theorem 3.1.4 ([17]). *Fix a connected Lie group G with a quadratic Lie algebra \mathfrak{g} and a Drinfeld associator Φ .*

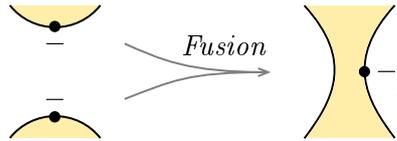
1. *The algebra $C^\infty(M_{D^2, \{+, -\}}(G)) \in U(\mathfrak{g} \oplus \bar{\mathfrak{g}})\text{-mod}_\hbar^\Phi$ with its usual commutative product of functions is a deformation quantization of the moduli space of flat connections on a disk with two points marked $+$ and $-$.*

2. Let (Σ_1, V_1) and (Σ_2, V_2) be two marked surfaces and let $A_{1,2}$ be quantizations of their moduli spaces, with $A_i \in U(\mathfrak{g}^{V_i})\text{-mod}_{\hbar}^{\Phi}$. Then the tensor product¹ $A_1 \otimes A_2 \in U(\mathfrak{g}^{V_1 \sqcup V_2})\text{-mod}_{\hbar}^{\Phi}$ quantizes the moduli space of flat connections on $\Sigma_1 \sqcup \Sigma_2$.
3. Let $A \in U(\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{h})\text{-mod}_{\hbar}^{\Phi}$ be a deformation quantization of the moduli space of flat connections on Σ , with two vertices marked $+$ corresponding to the action of $\mathfrak{g} \oplus \mathfrak{g}$. Then, $F(A) \in U(\mathfrak{g} \oplus \mathfrak{h})\text{-mod}_{\hbar}^{\Phi}$ quantizes the moduli space of flat connections on the surface obtained from Σ by the **fusion** of the two marked points, i.e. by the local replacement



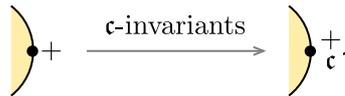
where the bottom point corresponds to the first \mathfrak{g} action (fusion in the opposite order would put the new point on the other side).

Similarly, if $A \in U(\bar{\mathfrak{g}} \oplus \bar{\mathfrak{g}} \oplus \mathfrak{h})\text{-mod}_{\hbar}^{\Phi}$ and the two vertices are marked with the sign $-$, then $F(A) \in U(\bar{\mathfrak{g}} \oplus \mathfrak{h})\text{-mod}_{\hbar}^{\Phi}$ quantizes the surface obtained by

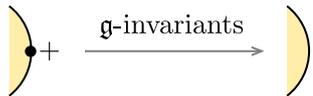


where again the bottom point corresponds to the first \mathfrak{g} action.

4. If $A \in U(\mathfrak{g} \oplus \mathfrak{h})\text{-mod}_{\hbar}^{\Phi}$ quantizes the moduli space of flat connections on Σ and $\mathfrak{c} \subset \mathfrak{g}$ is a coisotropic Lie subalgebra, then $A^{\mathfrak{c}} \in U(\mathfrak{h})\text{-mod}_{\hbar}^{\Phi}$ quantizes the moduli space of flat connections on Σ with the marked point corresponding to \mathfrak{g} decorated by \mathfrak{c} , as in Section 2.3.3.



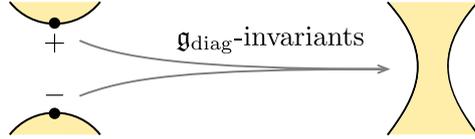
5. One special case of the previous item is the reduction by the entire Lie algebra $\mathfrak{g} \subset \mathfrak{g}$. If $A \in U(\mathfrak{g} \oplus \mathfrak{h})\text{-mod}_{\hbar}^{\Phi}$ quantizes the moduli space of flat connections on Σ , then $A^{\mathfrak{g}} \in U(\mathfrak{h})\text{-mod}_{\hbar}^{\Phi}$ quantizes the moduli space of flat connections on Σ without the marked point corresponding to \mathfrak{g} .



¹More precisely, the tensor product is completed to get all of the smooth functions on $M_{\Sigma_1, V_1}(G) \times M_{\Sigma_2, V_2}(G)$.

The case with a point marked $-$ is analogous.

6. Another special case is the reduction by the diagonal $\mathfrak{g} \subset \mathfrak{g} \oplus \bar{\mathfrak{g}}$. If there are two points marked $+$ and $-$ on Σ , with a corresponding quantizing algebra $A \in U(\mathfrak{g} \oplus \bar{\mathfrak{g}} \oplus \mathfrak{h})\text{-mod}_{\hbar}^{\Phi}$, then the subspace of invariants with respect to the diagonal action $\mathfrak{g} \subset \mathfrak{g} \oplus \bar{\mathfrak{g}}$ quantizes the moduli space on the surface obtained by the following local replacement



Proof. Let us only check the Item 3, as we use a different convention than [17]. In the first order of \hbar , the fused multiplication changes by the action of $\frac{1}{2}t^{(2)(1')}$. The bracket $\{f, f'\}$ thus changes by

$$-\frac{1}{2}t^{ij}((e_i)^{(1)}f \cdot (e_j)^{(2)}f' - (e_i)^{(1)}f' \cdot (e_j)^{(2)}f)$$

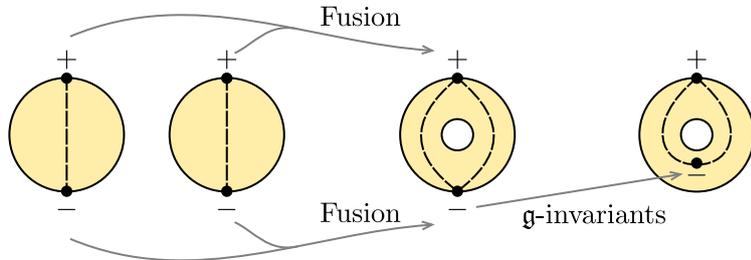
which agrees with (2.8) for fusion of bivector fields. For the fusion of points marked with $-$, the bracket changes by the opposite term. This agrees with (2.8), since fusion in opposite order gives an opposite contribution to the bivector. \square

This theorem provides a recipe for building any marked surface, or more generally any marked surface with some of the points marked by coisotropic Lie subalgebras. One starts with many copies of the disk with two points, marked $+$ and $-$. Then, a sequence of fusions and reductions is performed.

Moreover, this procedure endows the surface with a spine, with edges connecting the two marked points on the original disks. For surfaces with spines, the two special reductions by $\mathfrak{g} \subset \mathfrak{g}$ and by diagonal $\mathfrak{g} \subset \mathfrak{g} \oplus \bar{\mathfrak{g}}$ actually correspond to moving the points to the interior, as explained in Section 2.3.1. We will distinguish these cases by remembering the sign of the point for the first case:



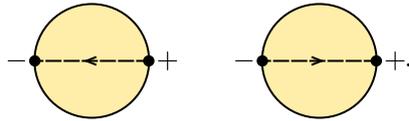
For example, the annulus with one marked point on the boundary can be constructed as follows:



We will use the reduction by $\mathfrak{g} \subset \mathfrak{g} \oplus \bar{\mathfrak{g}}$ only when the two vertices are univalent. Thus, they are both an endpoint of one of the building blocks, and this reduction glues the two disks into one. Using the canonical identification $G \times_G G \cong G$, we can then remove this unmarked bivalent vertex. The reason for introducing this operation is that it allows us to connect partially constructed surfaces at vertices with opposite signs, and it is sometimes more convenient to describe them separately, as in Proposition 3.3.4 below.

The quantum algebra resulting from Theorem 3.1.4 is given by the action of a suitable composition of the tangle J from (3.1) on functions on G^Γ , or possibly on functions invariant under the action of some coisotropic Lie subalgebras. Already from this picture, it is obvious that fusions at distinct vertices commute with each other, as the tangles J act via commuting factors of \mathfrak{g} .

Let us also mention a triviality concerning the orientation of the spine. The quantization, defined in terms of the building block $M_{D^2, \{+, -\}}(G)$, doesn't need an orientation of the spine. If we want to describe the moduli space in terms of holonomies, we need to orient the edges of a spine, e.g. for the disk with two marked points, there are two isomorphisms $M_{D^2, \{+, -\}}(G) \cong G$, coming from choices:



These two parametrizations are related by the map $\text{Inv}: g \rightarrow g^{-1}$ which is compatible with the actions at the marked points of the disk (as explained in (2.1) and (2.7)). Thus, for any two orientations of the spine, the map of moduli spaces given by Inv is actually an algebra map.

Remark 3.1.5. For even associators, an associative algebra in $U(\bar{\mathfrak{g}} \oplus \mathfrak{h})\text{-mod}_\hbar^\Phi$ is also associative in $U(\mathfrak{g} \oplus \mathfrak{h})\text{-mod}_\hbar^\Phi$ [17], and we can thus ignore the signs at the marked points (using only the positive fusion for example).

3.1.1 Description of the results

Our first goal is to simplify and completely describe the data needed to construct an algebra in $U(\mathfrak{g}^V)\text{-mod}_\hbar^\Phi$ quantizing a moduli space $M_{\Sigma, V}(G)$, using the above Theorem 3.1.4. We will now state the result, hoping that it helps the reader with the orientation in the following.

Recall from Definition 2.2.3 that a spine of a (marked) surface is a graph Γ embedded in Σ , satisfying some properties. Since our building blocks have points marked $+$ and $-$, we need to label the internal vertices of Γ as well. The second piece of data is the parenthesization of fusions. More precisely, will need the following:

Definition 3.1.6. A spine of a marked surface (Σ, V) is called² **properly colored** if each of its vertices is given a label $+$ or $-$ such that each edge connects $+$ and $-$ (recall that the external vertices are already labeled by \pm).

²In general, a proper coloring of a graph is an assignment of colors to the vertices of the graph such that each edge connects different colors [54, Chapter 3]. We will always consider only two colors, called $+$ and $-$.

A spine of a (marked) surface is called **parenthesized** if

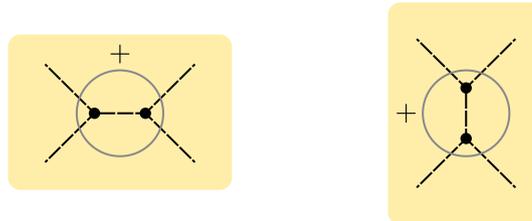
1. For each internal vertex v of valence n , we choose a non-rooted planar binary tree with n leaves connected to the n half-edges of v .
2. For each external vertex v of valence n , we choose a rooted planar binary tree with n leaves connected to the n half-edges of v .

The name **parenthesization** comes from the fact that such rooted planar binary trees are in bijection with parenthesizations. For internal vertices, the non-rooted trees describe **cyclic parenthesizations**, i.e. parenthesizations modulo moves $(AB) \rightarrow (BA)$ and $(AB)C \rightarrow A(BC)$. There are thus C_{n-1} possible parenthesizations for an n -valent external vertex and C_{n-2} parenthesizations for an n -valent internal vertex, where C_n is the n -th Catalan number [54, Chapter 14].

Parenthesized external vertices can be depicted by replacing them with a gray semi-circle with the tree drawn inside, rooted at the external vertex. For example, for an external vertex of valence 3, there are two possibilities:



Similarly, internal vertices will be replaced by gray circles with the tree inside. If the spine is colored, we will moreover put the label $+$ or $-$ next to the (semi)-circle. For example, here are the two possible trees for a four-valent vertex marked with $+$:



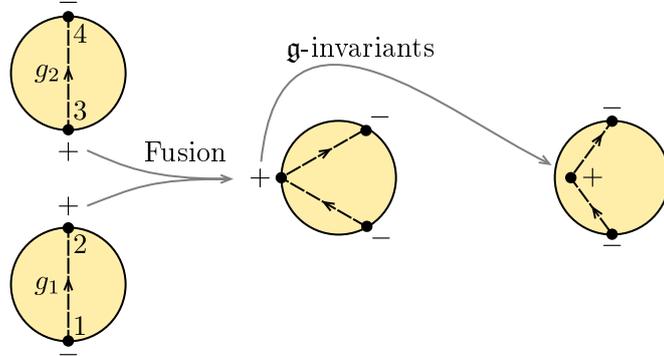
We will show that such a spine determines completely a quantization.

Theorem 3.1.7. *Let (Σ, V) be a marked surface with V possibly empty. Let $\Gamma \subset \Sigma$ a parenthesized, properly colored spine. Using this datum, the quantization of Li-Bland and Ševera defines an algebra in $U(\mathfrak{g}^V)\text{-mod}_\hbar^\Phi$, quantizing the moduli space of flat connections on (Σ, V) .*

Moreover, we will describe the possible moves connecting such spines and the induced isomorphisms of algebras, in Theorem 3.3.8. Then, we will modify the fusion, which will simplify the datum needed to a (uncolored) uni-trivalent spine, see Theorem 3.4.5. In this case, we will show that the quantization is unique in the sense that any two algebras are canonically isomorphic, see Theorem 3.4.7. These latter two theorems are the main result of this thesis.

3.2 First results on quantization

As a first example of a non-trivial quantization, let us describe the algebra in $A_1 \in U(\bar{\mathfrak{g}} \oplus \bar{\mathfrak{g}})\text{-mod}_\hbar^\Phi$ obtained by the following procedure:



If $f, f' \in C^\infty(G \times_G G)$ are two functions on the moduli space, then their deformed product is given by

$$f \star f' = m_c \circ J^{(2)(3)(2')(3')}(f \otimes f').$$

Here, m_c is the usual commutative product on $C^\infty(G \times_G G)$. The superscripts on J denote the actions³ corresponding to the half-edges as on the picture above, i.e. the action (1) is given by the left-invariant vector field on the second group (see Definition 2.3.4). The prime denotes the action on f' .

Proposition 3.2.1. *Let*

$$K := Z \left(\begin{array}{c} \left(\begin{array}{c} \uparrow \quad \downarrow \\ \downarrow \quad \uparrow \end{array} \right) \end{array} \right) \in \mathcal{P}_2. \tag{3.3}$$

Then $K = R^{-1} \cdot b$ (see (2.22), (2.25)) and the star product on A_1 equals

$$f \star f' = m_c \circ K^{(1)(1')}(f \otimes f') = m_0 \circ K^{(4)(4')}(f \otimes f').$$

We can see f and f' as functions on $C^\infty(G)$, via the identification $(g_2, g_1) \mapsto g_2 g_1$. Then the actions (1) and (4) are the left-invariant and minus the right-invariant vector fields on G .

Proof. The invariance of f means, for $x \in \mathfrak{g}$, that $x^{(2)}f = -x^{(3)}f$. Thus, if $W^\mathfrak{g}(J) = J_{(1)} \otimes J_{(2)} \otimes J_{(3)} \otimes J_{(4)} \in (U\mathfrak{g})^{\otimes 4}$ is the weight of $J \in \mathcal{P}_4$, its action is equal to the action at $(2)(2')$ of $J_{(1)}S(J_{(2)}) \otimes J_{(3)}S(J_{(4)})$, where S is the antipode of $U\mathfrak{g}$.

³It is slightly misleading to speak about \mathfrak{g} -action on $C^\infty(G \times G)^\mathfrak{g}$, since these actions in general do not preserve the invariant subspace. This subspace is, however, preserved by action of the invariant element J .

The element $J_{(1)}S(J_{(2)}) \otimes J_{(3)}S(J_{(4)})$ can be obtained using the (unnormalized) Kontsevich integral. Namely, it corresponds to the chord diagram $K \in \mathcal{P}_2$ defined above. This follows from the properties of Z : reversing the second and the fourth strand of J corresponds to the antipode of $U\mathfrak{g}$, and connecting the strands at the bottom corresponds to the multiplication $J_{(1)} \otimes S(J_{(2)}) \rightarrow J_{(1)}S(J_{(2)})$, similarly to Proposition 2.5.18. Note that we use the unnormalized Kontsevich integral.

Furthermore, we can change the action ⁽²⁾ to action ⁽¹⁾, i.e. for $w \in U\mathfrak{g}$

$$(w^{(2)}f)(g_1, g_2) = ((\text{Ad}_{(g_1)^{-1}} S(w))^{(1)}f)(g_1, g_2).$$

Applying this to the star product and using the Ad-invariance of K , we get

$$f \star f' = m_c \circ (S_1 S_2(K))^{(1)(1')} (f \otimes f').$$

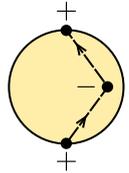
The second form of the star product, with action at ^{(4)(4')}, follows from a similar relationship between actions ⁽¹⁾ and ⁽⁴⁾.

The equality $K = R^{-1} \cdot b$ follows from the invariance of Z , as

$$Z \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right) = Z \left(\begin{array}{c} \text{Diagram 3} \end{array} \right).$$

The relation $S_1 S_2(K) = K$ is proven in the next section in Proposition 4.1.1. □

Let us also study the disk with two +’s on the boundary, i.e. the situation of Proposition 3.2.1 with + and – exchanged



Using the same convention for the four actions as above, we get

Corollary 3.2.2. *The star product of functions on the above moduli space, by exchanging +’s with –’s, is*

$$f \star f' = m_c \circ K_{-t}^{(1)(1')} (f \otimes f') = m_0 \circ K_{-t}^{(4)(4')} (f \otimes f'),$$

where $K_{-t} \in \mathcal{P}_2$ is obtained from K by multiplying each chord by -1 . Moreover,

$$K_{-t} = Z \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right) = a \cdot R.$$

Proof. Here, since we fuse at $\bar{\mathfrak{g}}$, the action of J is calculated using $-t$. This also sends the point $-$ to the opposite side of the disk. The upside-down form of K_{-t} follows from Proposition 2.6.10. \square

3.3 Further characterization of the deformation quantization

3.3.1 Associativity and cyclic property of fusion

Let us now examine the dependence of the quantization of [17] on the choices made for fusions and reductions. The first result is the associativity of the fusion, which is a general feature of monoidal categories, described in this context by [15, Theorem 2.1].

The second property is a kind of cyclic symmetry of fusion at internal vertices, and follows from the equation $K = S_1 S_2(K)$ from Proposition 4.1.1.

Let us consider the category $U(\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{h})\text{-mod}_\hbar^\Phi$. We consider two possible ways of fusing the three \mathfrak{g} actions into one keeping their order: either starting by the first two, or the last two factors. Denoting such functors $F_{(\bullet\bullet)\bullet}$ and $F_{\bullet(\bullet\bullet)}$, we have the following result

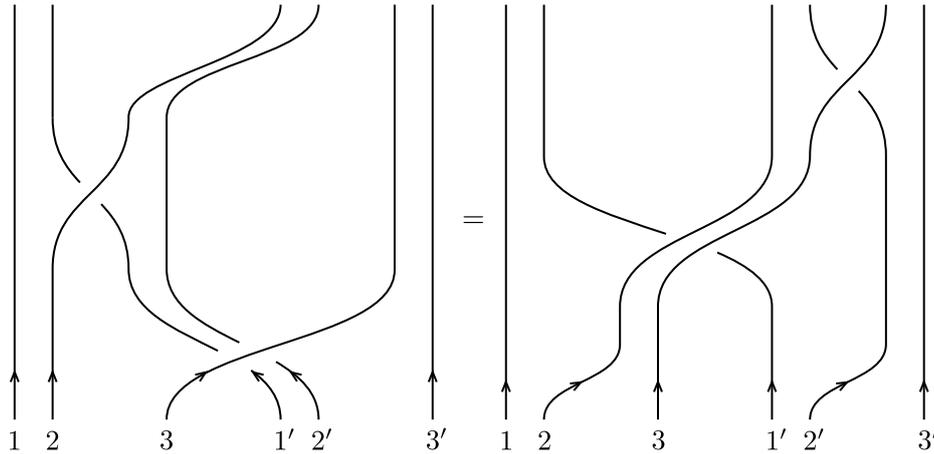
Proposition 3.3.1. *The two functors $F_{(\bullet\bullet)\bullet}, F_{\bullet(\bullet\bullet)}: U(\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{h})\text{-mod}_\hbar^\Phi \rightarrow U(\mathfrak{g} \oplus \mathfrak{h})\text{-mod}_\hbar^\Phi$ are naturally monoidally isomorphic, with the isomorphism $\alpha: F_{(\bullet\bullet)\bullet} \rightarrow F_{\bullet(\bullet\bullet)}$ given by*

$$\alpha_X = \Phi_X: F_{(\bullet\bullet)\bullet}(X) \rightarrow F_{\bullet(\bullet\bullet)}(X),$$

where the associator Φ acts on X as an element of $U(\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g})$.

A similar statement for fusion at $\bar{\mathfrak{g}}$ and the action of Φ_{-t} is also true.

Proof. Let X and X' be two objects of $U(\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{h})\text{-mod}_\hbar^\Phi$. The monoidal structure of $F_{(\bullet\bullet)\bullet}$ is given by Proposition 2.4.5 and the condition α has to satisfy to be monoidal (from Definition 2.4.6) is



Compatibility with the units follows immediately from the fact that the associator acts trivially on \mathbb{C} . \square

Let us now consider a fusion followed by the reduction of the resulting point by $\mathfrak{g} \subset \mathfrak{g}$. There are two possible orders for the fusion, which give different surfaces and different monoidal functors. However, after reduction, they correspond to the same functor:

Proposition 3.3.2. *The two functors $U(\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{h})\text{-mod}_h^\Phi \rightarrow U(\mathfrak{h})\text{-mod}_h^\Phi$, given by fusion of \mathfrak{g} in two possible orders, followed by the reduction by \mathfrak{g} , are equal as monoidal functors. A similar statement holds for $\bar{\mathfrak{g}}$.*

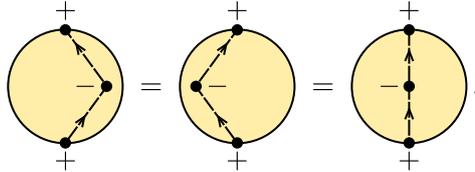
Proof. The monoidal structure of the fusion followed by the reduction is the composition

$$F(X)^\mathfrak{g} \otimes F(X')^\mathfrak{g} \longleftarrow (F(X) \otimes F(X'))^\mathfrak{g} \xrightarrow{J_{X,Y}} F(X \otimes X')^\mathfrak{g}$$

i.e. the action of $J^{(1)(2)(1')(2')}$ on invariants under the first pair $(1), (2)$ and the second pair $(1'), (2')$ of \mathfrak{g} -actions. As in the proof of Proposition 3.2.1, this action can be translated to the action of $K^{(1)(1')}$.

The opposite fusion acts by $J^{(2)(1)(2')(1')}$. Moving this action to act on (1) and $(1')$, we find the reversed tangle $(S_1 S_2(K))^{(1)(1')}$, since now the first strand has to be reversed. Thus, the result follows from Proposition 4.1.1. \square

This proposition means we don't have to distinguish which of the two possible boundary components the internal point came from. For example, for the algebra described in Corollary 3.2.2, we have schematically



Combining these results with Proposition 4.1.2, only the choice of a cyclic orientation is necessary to characterize a fusion of three points followed by the reduction.

Proposition 3.3.3. *The functors $U((\mathfrak{g}^{\oplus 3} \oplus \mathfrak{h}))\text{-mod}_h^\Phi \rightarrow U(\mathfrak{h})\text{-mod}_h^\Phi$ given by two fusions of \mathfrak{g} and reduction by \mathfrak{g} , depend only on the cyclic order of the three factors \mathfrak{g} .*

Proof. A-priori, we need to specify an order of the three Lie algebras \mathfrak{g} and a bracketing, e.g. (12)3. According to Proposition 3.3.2, after the reduction we can exchange the order of fusions; i.e. the choices (12)3 and 3(12) are equal. Moreover, before taking \mathfrak{g} -invariants, the functors (12)3 and 1(23) are isomorphic via the action Φ^{123} due to Proposition 3.3.1. However, on invariants, Φ^{123} acts by the identity, which is proven below in Proposition 4.1.2. Thus, the monoidal structures of these functors are equal. These two relations allow us to relate any two fusions in the same cyclic order. \square

This proposition allows us to finish the proof of first result described in Section 3.1.1

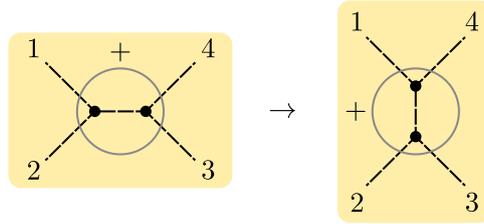
Proof of Theorem 3.1.7. Because the spine is properly colored, we can assign the disk with two points marked + and - to each edge. A parenthesization of an external vertex tells us in which order to perform the fusions; a parenthesization of an internal vertex also specifies the fusion completely, thanks to the above proposition. \square

3.3.2 Moves of parenthesizations

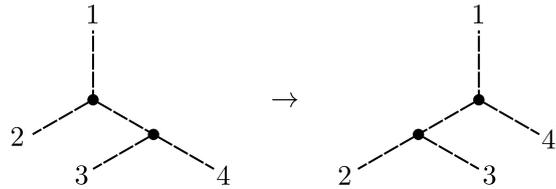
As we just proved, for any properly colored parenthesized spine $\Gamma \in (\Sigma, V)$, we get an algebra in $U(\mathfrak{g}^V)\text{-mod}_\hbar^\Phi$. Now, we would like to show that these algebras are isomorphic for different spines.

One step in this direction is Proposition 3.3.1, which tells us that changing a parenthesization or a cyclic parenthesization corresponds to an isomorphism given by the Drinfeld associator, acting on the three half-edges involved.

In the case of a cyclic parenthesization, we need to make a choice of a “root”, and the associator acts on the remaining actions. For example, for the Whitehead move at a four-valent vertex



and a choice of 1 as a root



we get the action $(\Phi^{-1})^{(2)(3)(4)}$, while with 4 as a root



we get⁴ $\Phi^{(1)(2)(3)}$.

3.3.3 Moves with + and -

Next, we would like to study the following move

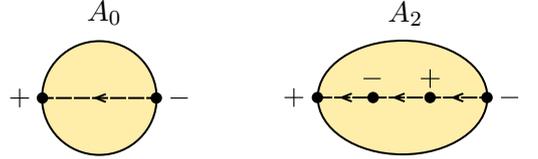
$$\begin{array}{c}
 \dots + - + - \dots \\
 \dots \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \dots
 \end{array}
 \rightarrow
 \begin{array}{c}
 \dots + \dots - \dots \\
 \dots \bullet \text{---} \bullet \text{---} \dots
 \end{array}
 \tag{3.4}$$

⁴One can check independently that these two actions are equal. Indeed, the pentagon equation (2.15) says that

$$\Phi^{234} \Phi^{1(23)4} \Phi^{123} = \Phi^{12(34)} \Phi^{(12)34}.$$

Acting on invariants under the diagonal action, the second, fourth and fifth term vanish by Proposition 4.1.2.

Let us first treat the case when the outer points are not connected to anything; the general case will follow easily. Recall that the undeformed algebra $A_0 := C^\infty(G)$ is an associative algebra in $U(\mathfrak{g} \oplus \bar{\mathfrak{g}})\text{-mod}_\hbar^\Phi$. The same surface can be obtained by two fusions and reductions on three such disks; thanks to Proposition 3.3.3, the following picture defines the algebra A_2 unambiguously.



These two algebras are isomorphic via a non-trivial isomorphism:

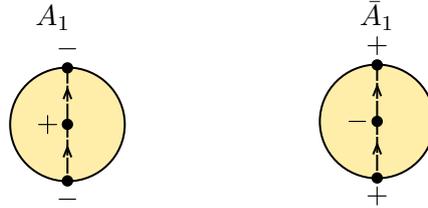
Proposition 3.3.4. *Let $A_2 := C^\infty(G^{\times 3}/G^{\times 2})$ be the algebra in $U(\mathfrak{g} \oplus \bar{\mathfrak{g}})\text{-mod}_\hbar^\Phi$ with multiplication given by the fusion as above. Let ψ_0 be the obvious isomorphism $C^\infty(G^{\times 3}/G^{\times 2}) \xrightarrow{\cong} C^\infty(G)$. Then*

$$\nu^{-1} \circ \psi_0: A_2 \rightarrow A_0$$

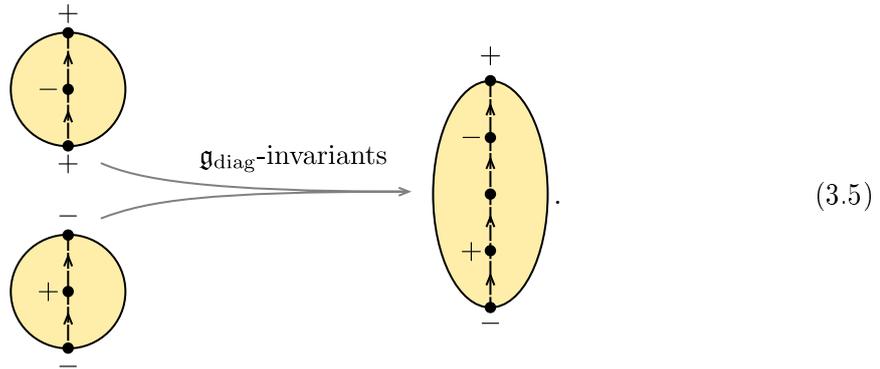
given by the action of $\nu^{-1} \in \mathcal{P}_1$ on one of the \mathfrak{g} -factors, is an algebra isomorphism.

A similar statement, with $+$ and $-$ exchanged, also holds.

Proof. The algebra A_2 can be also constructed from the two building blocks from Section 3.2.



by the reduction by the diagonal subalgebra $\mathfrak{g} \subset \mathfrak{g} \oplus \bar{\mathfrak{g}}$



In the unlabeled vertex, by Propositions 3.2.1, Corollary 3.2.2 and (2.26), we get an action of

$$K \cdot K_{-t} = R^{-1} \cdot b \cdot R \cdot a = b \cdot a = \Delta\nu \cdot \nu^{-1} \otimes \nu^{-1}.$$

We can move the element ν to act on the either end, using $S(\nu) = \nu$. The product of $f, f' \in A_2$ is thus equal to

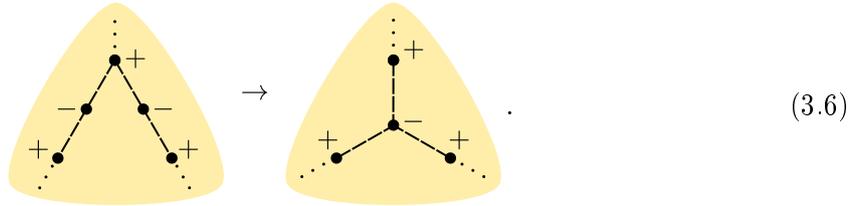
$$\nu m_c(\nu^{-1}f, \nu^{-1}f'),$$

which proves that the action of ν^{-1} gives an isomorphism between A_2 and A_0 . The case with $+$ and $-$ exchanged is proven in the same way. \square

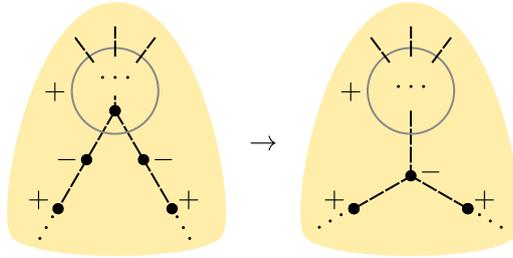
Corollary 3.3.5. *Let Γ and Γ' be two properly colored spines differing locally as in (3.4). Then their corresponding quantizing algebras are isomorphic via ν acting on one of the vertices marked $+$ or $-$, composed with the pushforward along $G^\Gamma \rightarrow G^{\Gamma'}$.*

Proof. This follows directly from the proposition above: if the top and the bottom marked points in (3.5) take part in a fusion, this does not affect the calculation which happened in the unmarked vertex in the middle. \square

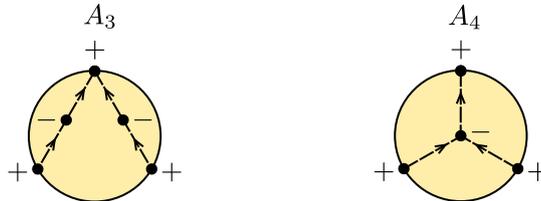
Another move we will study is the following:



We allow a situation where the top vertex has valence > 3 (> 2 for an external vertex). In that case, we require that its two half-edges on the picture are parenthesized together. The move (3.6) that reduces the valence of this vertex by one, as follows:



Again, we start by treating the following two algebras $A_3, A_4 \in U(\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g})\text{-mod}_\hbar^\Phi$:



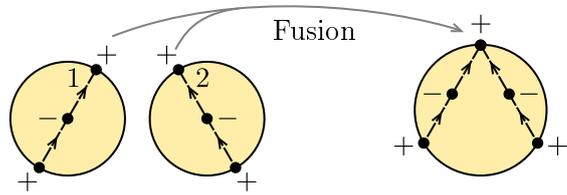
Proposition 3.3.6. *Transfer the multiplication of A_3 and A_4 to $C^\infty(G \times G)$, with the two factors G given by the following graph*



Let (1) and (2) be the two \mathfrak{g} -actions as above. Then $a^{(1)(2)}: A_3 \rightarrow A_4$ is an isomorphism of algebras.

For the analogous move with $+$ and $-$ exchanged, the two algebras are isomorphic via the action of $b = a_{-t}$.

Proof. Let us start with A_3 . It can be constructed in the following way



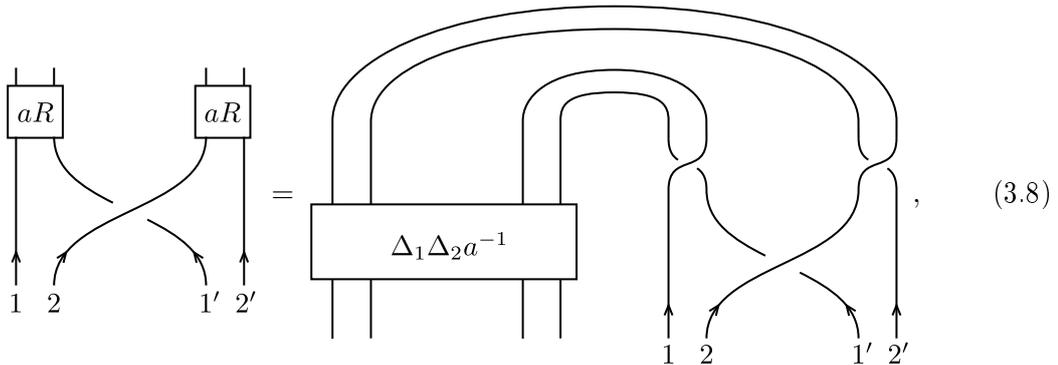
Thus, the multiplication on A_3 is given by

$$m_c \circ (K_{-t}^{(1)(1')} K_{-t}^{(2)(2')} J^{(1)(2)(1')(2')})$$

Our strategy is again to write the above element as a Kontsevich integral of some tangle. Rewriting

$$K_{-t}^{(1)(1')} K_{-t}^{(2)(2')} J^{(1)(2)(1')(2')} = (aR)^{(1)(1')} (aR)^{(2)(2')} J^{(1)(2)(1')(2')},$$

we can represent R as an overcrossing. The elements a can be represented by the doubled cap

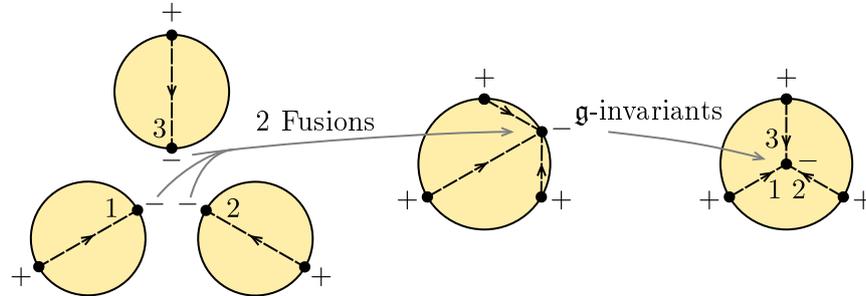


where the parenthesization of the bottom is $((\bullet\bullet)(\bullet\bullet))((\bullet\bullet)(\bullet\bullet))$. To include the doubled cap into the tangle, we used the equality

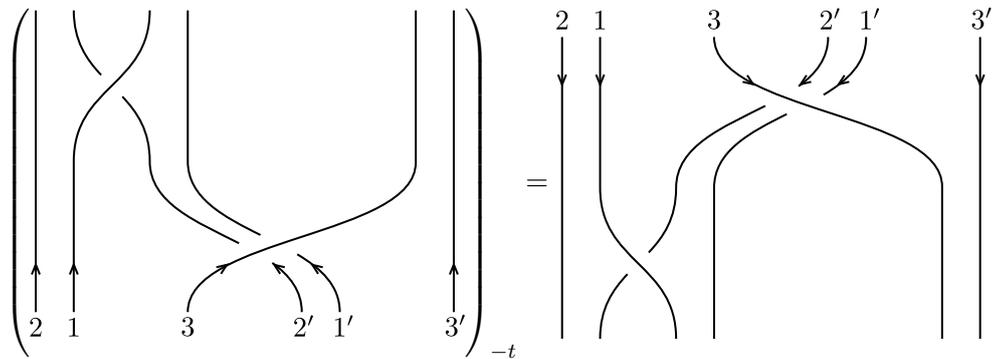
$$Z \left(\begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array} \right) = \Delta_1 \Delta_2 a \cdot a \otimes a,$$

which follows from Theorem 2.6.6.

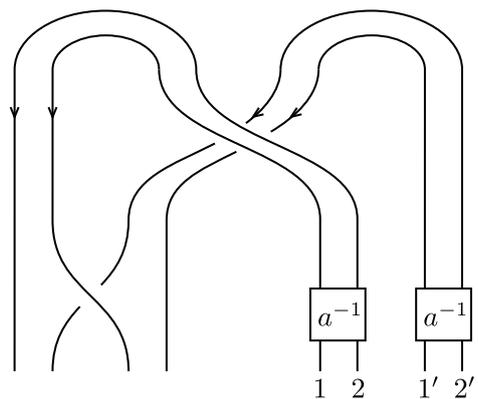
Let us now look at the other algebra A_4 . We construct it by two fusions and a reduction as follows



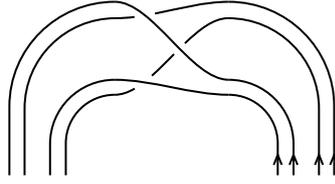
The order of the fusions is (21)3, i.e. first we fuse together points 2 and 1. We choose this cyclic order since we want to match the labels in (3.7), and fusion of points with $-$ has an opposite convention. The multiplication is given by the following chord diagram



where we used Proposition 2.6.10. Since we act on invariants under the diagonal action, we can change the action ⁽³⁾ for the other two, and transfer the multiplication to $C^\infty(G \otimes G)$ according to (3.7). This adds two doubled caps, so we have to divide by a , obtaining



The two tangles are both isotopic to



and thus the two algebras are isomorphic via a .

The case with $+$ and $-$ exchanged is analogous. Keeping the above notation, the algebra \bar{A}_3 is obtained via fusion 21, and the algebra \bar{A}_4 is obtained via fusion (12)3. Using the same reasoning, one obtains that, apart from twisting with $b \in \mathcal{P}_2$, both multiplications are given by the upside-down version of the above tangle. \square

Corollary 3.3.7. *Let Γ and Γ' be two properly colored spines differing locally as in (3.6) or with $+$ and $-$ exchanged. If the top vertex has valence > 3 (> 2 if external), we require the two incident half-edges to be parenthesized together. Then the corresponding quantizing algebras $A_\Gamma, A_{\Gamma'}$ are isomorphic via a (respectively b) acting the two half-edges as on (3.7), composed with the pushforward along the diffeomorphism $G^\Gamma \rightarrow G^{\Gamma'}$.*

Proof. Similarly to Corollary 3.3.5, the calculation in the above proposition does not interact with fusions at other parts of the surface. We assumed that if the top vertex of (3.6) has higher valence, then its the two half-edges on (3.6) are parenthesized together. Thus, in the diagram describing the multiplication, the part described in (3.8) appears on the top. \square

Now we can summarize the previous results.

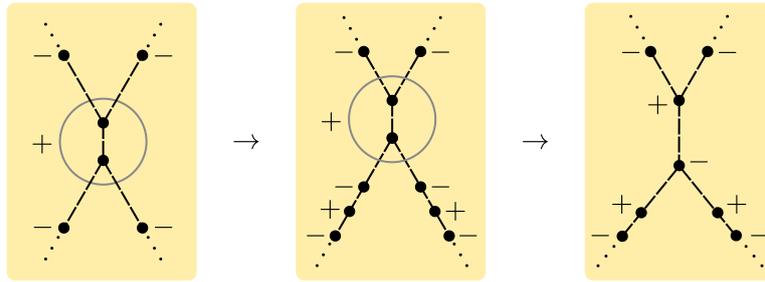
Theorem 3.3.8. *Let (Σ, V) be a marked surface and $\Gamma \subset \Sigma$ a properly colored parenthesized spine. Let A_Γ be the algebra in $U(\mathfrak{g}^V)\text{-mod}_\hbar^\Phi$, quantizing the moduli space of flat connections on (Σ, V) described in Theorem 3.1.7.*

Then, any two parenthesized colored spines Γ, Γ' are connected by a finite sequence of moves:

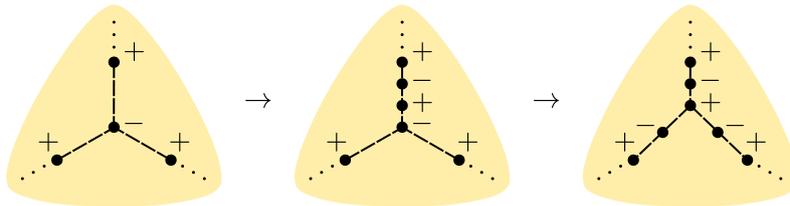
1. *the change of a parenthesization of a four-valent internal vertex via a flip,*
2. *the move (3.4),*
3. *the move (3.6),*

and their versions with $+$ and $-$ exchanged. Each of this moves induces an isomorphism between the quantizing algebras, as described in Proposition 3.3.1, Corollary 3.3.5 and Corollary 3.3.7, giving an isomorphisms $A_\Gamma \cong A_{\Gamma'}$.

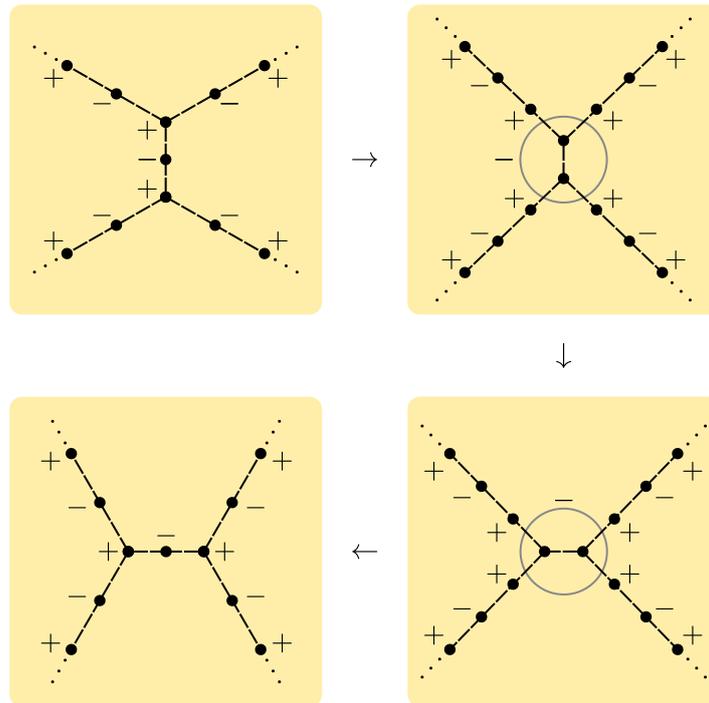
Proof. First, using moves (3.4) and (3.6), we can resolve any internal vertex of valence > 3 into multiple internal trivalent vertices. For example, for a four-valent internal vertex decorated with $+$, we have



Similarly, the moves (3.4) and (3.6) can resolve any external vertex into a tree with a (uni-valent) root at that marked point. Furthermore, these moves can ensure that all the trivalent vertices are decorated with +:



Using the move (3.4), we can freely change the number of bivalent vertices, and it is only the uni-trivalent spine structure which is important. Thus by Proposition 2.2.7, we only need to implement the flip move to relate all such uni-trivalent spines. This is done, using the associator as in Section 3.3.2, as follows:



□

3.4 Modified fusion

Our next goal is to describe the relations between these isomorphisms. However, we will not do this for the colored parenthesized spines. Instead, to decrease the complexity, we will first modify the fusion of [17] using $a^{1/2} \in \mathcal{P}_2$.

Definition 3.4.1. Define a monoidal functor $F^m: U(\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{h})\text{-mod}_\hbar^\Phi \rightarrow U(\mathfrak{g} \oplus \mathfrak{h})\text{-mod}_\hbar^\Phi$ to be the fusion functor of Proposition 3.1.1 with the monoidal constraint given by the action of

(3.9)

The map $\mathbb{C} \rightarrow F^m(\mathbb{C})$ is unchanged.

The fusion at $\bar{\mathfrak{g}}$ is defined with the above morphism using $-t$ to obtain an element of $(U\mathfrak{g})^{\otimes 4}$.

Using Proposition 2.6.10 and $(a^{1/2})_{-t} = b^{1/2}$, the fusion at $\bar{\mathfrak{g}}$ can be also described by the tangle (3.9) turned upside-down with a replaced by b .

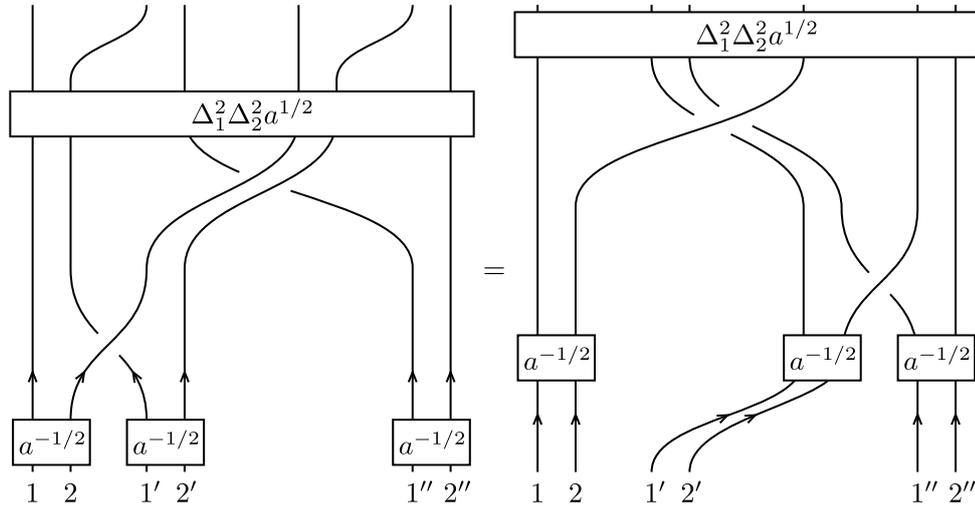
Note that the modified fusion is isomorphic to the original one via a monoidal natural transformation given by the action of $a^{1/2}: F \rightarrow F^m$.

Proposition 3.4.2.

1. *The modified fusion F^m is again strong monoidal.*
2. *The Item 3 of Theorem 3.1.4 remains true with the modified fusion as well, i.e. the modified fusion of algebras quantizes the fusion of the moduli spaces of flat connections.*
3. *As in Proposition 3.3.1, the modified fusion is associative via the natural transformation given by the action of $\Phi^{a^{1/2}}$, i.e. the associator twisted by $a^{1/2}$, as defined in (2.27)*
4. *The modified fusion has the cyclic symmetry at internal vertices, as in Proposition 3.3.3.*

Proof.

1. The monoidal constraint is in this case the following equality

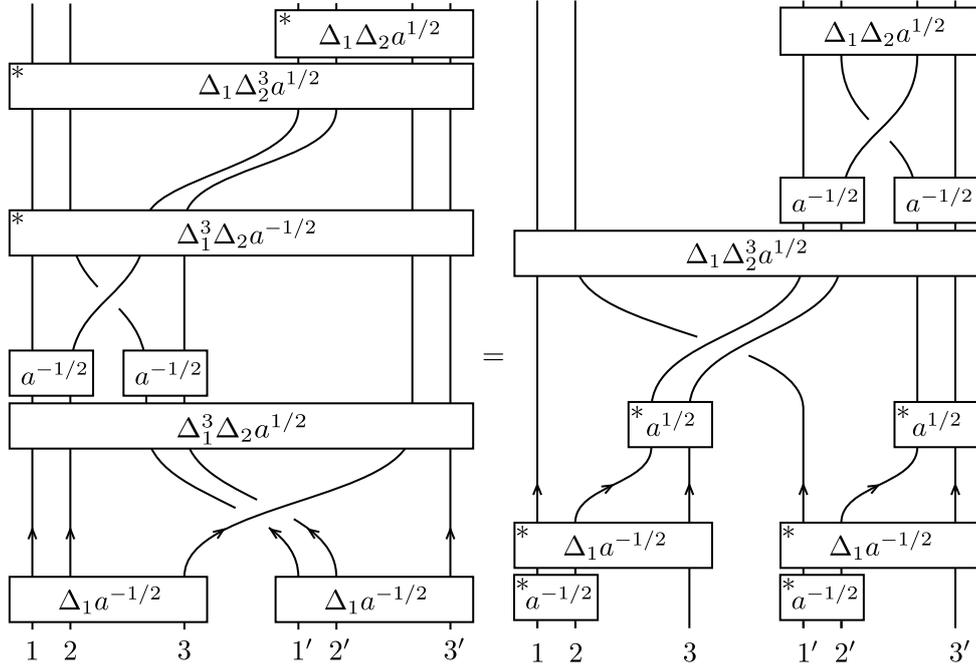


where the terms $\Delta_1 \Delta_2 a^{\pm 1/2}$ in the middle of the tangle, coming from the two modified fusions, canceled. Due to Proposition 2.5.12, the factor $a^{\pm 1/2}$ can slide to the top and the bottom of the tangle through the associators, and the equality holds since the tangles are equal. Since deleting a strand from a gives the unit of \mathcal{P}_1 by Proposition 2.6.4, the modified fusion is also compatible with units.

Alternatively, this follows from the remark that the two fusions are monoidally isomorphic.

2. Since a is made out of the associator, it starts with the unit of \mathcal{P}_2 and the next term contains 2 chord diagrams. Thus, to the first order in \hbar , the original and the modified fusion coincide.

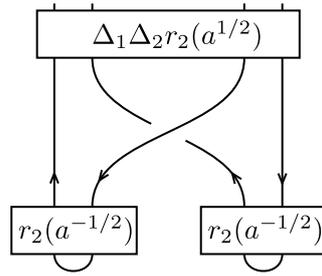
3. The proof is similar as in the first case:



Here, the terms coming from the twisting of the associator are marked by *. The terms $\Delta_1 \Delta_2 a^{\pm 1/2}$ which canceled directly are omitted. Using Proposition 2.5.12, we can move some of the boxes around. This allows to cancel $\Delta_1^3 \Delta_2 a^{\pm 1/2}$ on the LHS and $a^{\pm 1/2}$ on the RHS. The rest organizes to the same tangle sandwiched between the same factors of a on the top and the bottom.

Compatibility of this natural transformation with units is immediate from the fact that a acts by the identity on the trivial representation.

4. First, let's consider the analogue of Proposition 3.3.2, saying that fusions in two orders followed by a reduction are equal. The action of the modified fusion on invariants is given by



i.e.

$$\Delta r_2(a^{1/2}) \cdot K \cdot r_2(a^{-1/2}) \otimes r_2(a^{-1/2}) = K \cdot \Delta \nu^{-1/2} \cdot \nu^{1/2} \otimes \nu^{1/2},$$

where we used Proposition 4.1.3. This chord diagram is again invariant with respect to switching the orientation of the two strands. Note that this chord diagram describes

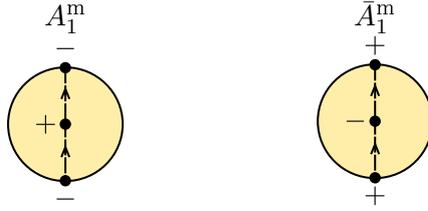
the quantization of the moduli space on disk with two points marked minus, which will be useful later.

Finally, to show that $\Phi^{a^{1/2}}$ acts trivially on invariants, we need to prove $r_3(\Phi^{a^{1/2}}) = 1 \in \mathcal{P}_2$. This is a rather technical calculation, delegated to Proposition 4.1.6. \square

Now, let us turn to the moves described in Section 3.3.3. Indeed, the whole point of the modified fusion was that the algebra isomorphisms associated to these moves become trivial.

Proposition 3.4.3. *Using the modified fusion, the algebras described in Corollary 3.3.5 and Corollary 3.3.7 are isomorphic via the pushforward along the diffeomorphism $G^\Gamma \cong G^{\Gamma'}$.*

Proof. Let us first prove the analogue of Proposition 3.3.4. The multiplications on the two building blocks



were computed already in the proof of Proposition 3.4.2; for the first one it is given by the action of

$$K \cdot \Delta \nu^{-1/2} \cdot \nu^{1/2} \otimes \nu^{1/2}. \quad (3.10)$$

For the second one, it is the above expression with chords multiplied by -1 , i.e.

$$K_{-t} \cdot \Delta \nu^{-1/2} \cdot \nu^{1/2} \otimes \nu^{1/2}, \quad (3.11)$$

since ν contains only terms with an even number of chords. The product of these two elements is

$$K \cdot K_{-t} \cdot \Delta \nu^{-1} \cdot \nu \otimes \nu = a \cdot b \cdot \Delta \nu^{-1} \cdot \nu \otimes \nu = 1,$$

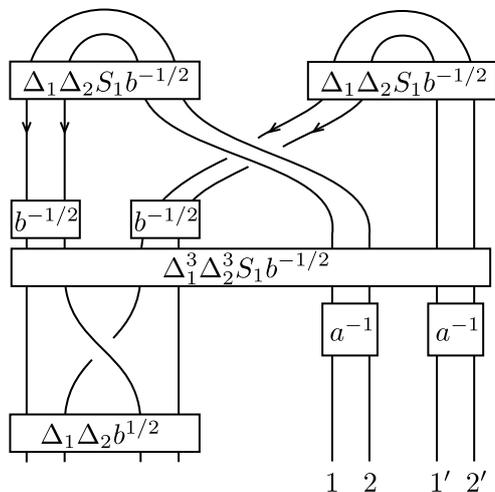
using (2.26).

For the Proposition 3.3.6, the algebra A_3^m computed as A_3 but with the modified fusion

has multiplication given by

$$\begin{array}{c}
 \begin{array}{c} \sqrt{\nu} \quad \sqrt{\nu} \\ \Delta \nu^{-1/2} \\ aR \\ \Delta_1 \Delta_2 a^{1/2} \\ \begin{array}{c} \uparrow \quad \uparrow \\ a^{-1/2} \quad a^{-1/2} \\ 1 \quad 2 \quad 1' \quad 2' \end{array} \end{array} \\
 = \\
 \begin{array}{c}
 \Delta_1 \Delta_2 a^{-1/2} \\
 \begin{array}{c} \sqrt{\nu} \quad \sqrt{\nu} \\ \Delta \nu^{-1/2} \\ a^{-1/2} \quad a^{-1/2} \\ 1 \quad 2 \quad 1' \quad 2' \end{array}
 \end{array}
 \end{array} \quad (3.12)$$

For the algebra A_4^m , we have to use $b^{\pm 1/2}$ for the modified fusion at internal vertices, getting



by the same reasoning with Proposition 2.6.10. Let us treat the boxes from top to bottom. The terms $\Delta_1 \Delta_2 b^{-1/2}$ are equal to $\Delta r_2(b^{-1/2}) = \Delta \nu^{1/2}$, one on the strands 12 and one on the strands 1'2'. Next, the terms $b^{-1/2}$ can slide over to a^{-1} . The term $\Delta_1^3 \Delta_2^3 b^{1/2}$ is equal to $\Delta^3 r_2(b^{1/2}) = \Delta^3 \nu^{-1/2}$ and it can slide around freely. Let us combine it with the bottom box

$\Delta_1 \Delta_2 b^{1/2}$. In other words, we get

Using $ab = \nu^{-1} \otimes \nu^{-1} \cdot \Delta \nu$, the product on the left is equal to

$$\Delta_1 \Delta_2 b^{1/2} \cdot \Delta^3 \nu^{-1/2} = \Delta_1 \Delta_2 a^{-1/2} \cdot \Delta \nu^{-1/2} \otimes \Delta \nu^{-1/2}.$$

The product of the elements on the right is

$$\Delta \nu^{1/2} \cdot b^{-1/2} \cdot a^{-1} = \nu^{1/2} \otimes \nu^{1/2} \cdot a^{-1/2}.$$

Sliding the terms $\nu^{1/2}$ to the left, we get the same chord diagram as in (3.12). \square

The following remark is a continuation of Remark 3.1.5.

Remark 3.4.4. It is interesting to look at the case when the associator is even. In this case, $a = b = \nu^{-1/2} \otimes \nu^{-1/2} \cdot \Delta \nu^{1/2}$ by [50]. The twisted associator $\Phi^{a^{1/2}}$ is equal to Φ and the multiplication on disk with two minuses from the above proposition simplifies to

$$K \cdot \Delta \nu^{-1/2} \cdot \nu^{1/2} \otimes \nu^{1/2} = R^{-1},$$

which is equal to $e^{t_{11}/4} \otimes e^{t_{11}/4} \cdot \Delta e^{-t_{11}/4}$. In other words, the quantization of a disk with two points marked $+$ is isomorphic to the usual commutative algebra $C^\infty(G)$ via the action of $e^{t_{11}/4}$. The square of this element defines a ribbon structure on $U\mathfrak{g}\text{-mod}_\hbar^\Phi$ [43], which hints at a possible extension of our construction.

3.4.1 Uni-trivalent spines

We are now in a position to prove the main results of this thesis, giving a complete description of the algebras coming from the modified fusion. The first result reduces the datum needed to specify an algebra to a uni-trivalent spine.

Theorem 3.4.5. *Let A^m be an algebra built by a sequence of modified fusions and reductions, specified by an isotopy class of a parenthesized colored spine in Σ as in Theorem 3.3.8. Then A^m depends only on the underlying uni-trivalent spine obtained by forgetting the coloring and resolving the parenthesized vertices into planar trees.*

Proof. Using Proposition 3.4.3, we can forget the signs of the vertices in the interior: any two choices of signs can be related by the two moves (3.4), (3.6), and the moves induce trivial isomorphisms between the algebras. \square

Obviously, there are many ways to construct such algebra, since there are many properly colored parenthesized spines giving the same uni-trivalent spine. Let us give, for concreteness, one such way, which comes from coloring all the trivalent vertices by $-$ and adding a bivalent vertex colored $+$ in the middle of an edge connecting two minuses.

Proposition 3.4.6. *The algebra A_{Γ}^m assigned to a uni-trivalent spine $\Gamma \in (\Sigma, V)$ can be computed as follows:*

1. *To any vertex, assign the algebra with multiplication defined by (3.12) acting on any two consecutive half-edges in the cyclic order of the vertex*
2. *Fuse and reduce together the vertices according to the adjacency data of the uni-trivalent spine. This has an effect of acting by*

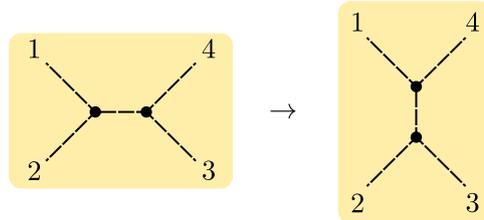
$$K \cdot \Delta \nu^{-1/2} \cdot \nu^{1/2} \otimes \nu^{1/2} = b \cdot \Delta \nu^{-1/2} \cdot \nu^{1/2} \otimes \nu^{1/2} \cdot R^{-1}. \quad (3.14)$$

on the fused half-edges

3. *Points marked with $+$ can be turned to points marked with $-$ by fusing and reducing at $+$ with the undeformed algebra $A_0 = C^\infty(G)$, represented by the disk with two points marked, by $+$ and $-$.*
4. *Finally, if some of the marked points are marked with coisotropic Lie subalgebras, restrict to the respective invariants.*

Finally, let us relate the algebras corresponding to the same surface with different spines.

Theorem 3.4.7. *If a spine Γ' is obtained from Γ by the flip move (2.3), the corresponding algebras are isomorphic via $\Phi^{a^{1/2}}$, acting on three half-edges at the four-valent vertex as described in Section 3.3.2:*



$$A_{\Gamma}^m \xrightarrow[\cong]{(\Phi^{a^{1/2}})_{(1)(2)(3)}} A_{\Gamma'}^m$$

Any two uni-trivalent spines are related by a sequence of flip moves, and thus any two algebras are isomorphic. Moreover, two different sequences of flip moves relating the same uni-trivalent spines induce the same isomorphism of algebras.

Proof. The only non-trivial relation between the flips is the pentagon, see Proposition 2.2.7. This relation is implied by the pentagon equation for the associator (2.15), which holds also for the twisted associator. \square

Remark 3.4.8. Note that $\Phi^{a^{1/2}} = (\Phi_{-t})^{b^{1/2}}$ by Corollary 4.2.2; this is another manifestation of the fact the modification of the fusion allows us to forget about the signs of the internal vertices.

We can now define a canonical algebra associated to a surface via the following general nonsense. This is similar in spirit to factorization homology of [13].

Definition 3.4.9. Let (Σ, V) be a marked surface. The algebras A_{Σ}^m and their isomorphisms form a diagram in $U(\mathfrak{g}^V)\text{-mod}_{\hbar}^{\Phi}$, where the source category has one morphisms between any two objects, which are indexed by uni-trivalent spines of (Σ, V) . Then, let $A_{\Sigma, V}^m$ be the colimit of this diagram.

In other words, an element of $A_{\Sigma, V}^m$ is a collection of elements $a_{\Gamma} \in A_{\Gamma}^m$, one for each Γ , such that the isomorphism $A_{\Gamma}^m \rightarrow A_{\Gamma'}^m$ takes a_{Γ} to $a_{\Gamma'}$.

Proposition 3.4.10. *The mapping class group of Σ fixing the boundaries acts on $A_{\Sigma, V}^m$.*

Proof. An element of the mapping class group ψ gives an embedding $\Sigma \rightarrow \Sigma$ sending the spine Γ to $\psi\Gamma$. The identity map on $C^{\infty}(G^{\Gamma})$ is compatible with the action at all the half-edges, and we get that it is an algebra morphism $A_{\Gamma}^m \rightarrow A_{\psi\Gamma}^m$. Define the (left) action of ψ on $A_{\Sigma, V}^m$ by sending $a \in A_{\Gamma}^m$ to $a \in A_{\psi\Gamma}^m$. Such sequence of elements is again compatible with the isomorphisms $A_{\Gamma}^m \rightarrow A_{\Gamma'}^m$, since the same sequence of flips that takes Γ to Γ' also takes $\psi\Gamma$ to $\psi\Gamma'$. \square

3.4.2 Embeddings of surfaces

We will end this chapter by giving a partial result on morphisms of algebras induced by embeddings of surfaces. Note that embeddings of surfaces induce maps of moduli spaces in the opposite direction and maps of algebras of functions in the original direction.

The compatibility with embeddings was discussed by Li-Bland and Ševera in [15], with applications to deformation quantization of Poisson-Lie groups and more (see also Chapter 5). We extend their results from skeletons to spines and discuss the modified fusion as well.

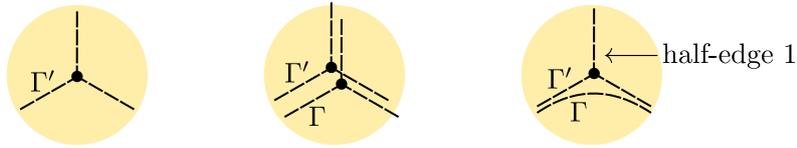
Proposition 3.4.11. *Let $(\Sigma, V, \Gamma) \rightarrow (\Sigma', V', \Gamma')$ be an embedding of surfaces sending Γ to a subgraph of Γ' . Then the map of algebras $A_{\Gamma}^m \rightarrow A_{\Gamma'}^m$ induced by $G^{\Gamma'} \rightarrow G^{\Gamma}$ is an algebra map in $U(\mathfrak{g}^V)\text{-mod}_{\hbar}^{\Phi}$, where we see A_{Γ}^m as a $\mathfrak{g}^{V'}$ -module by extending the action to $\mathfrak{g}^{V' \setminus V}$ trivially.*

Proof. As explained in Proposition 3.4.6, the multiplication m' on $A_{\Gamma'}$ is given by the factor (3.12) at the trivalent vertices and (3.14) at the edges connecting trivalent vertices (or trivalent vertices with external points marked $-$, or two external points marked $-$). This second action can be seen as adding bivalent vertices marked with $+$ to these edges.

Let us study a product in $A_{\Gamma'}^m$ of two functions f, f' from A_{Γ}^m . We will collect various terms of the product of f, f' coming from the bivalent and trivalent vertices of Γ' , showing that this equals to the product of f, f' in A_{Γ}^m .

If Γ' has a bivalent vertices above an edge of Γ , the action of (3.14) on f, f' is as if Γ had a bivalent vertex of the same sign on the edge e .

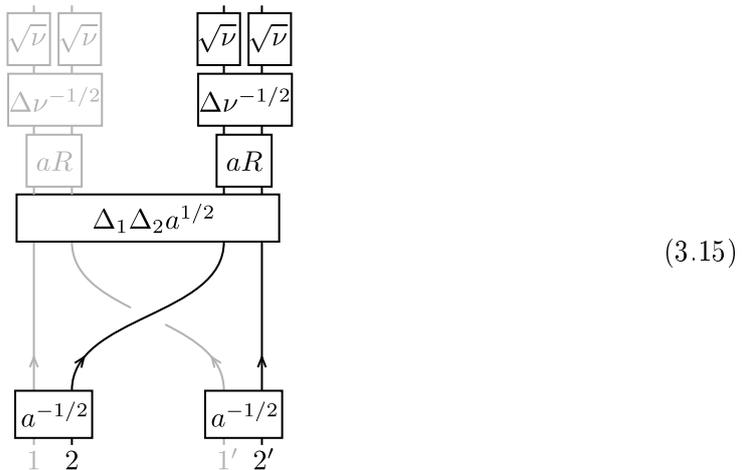
Internal trivalent vertices of Γ' either contain zero or three incident half-edges of Γ , or lies on an edge of Γ :



In the cases of zero half-edges, the relevant part of the product m' is the identity, and in the case of three half-edges, the action of m' agrees with the action of the product on m in this trivalent vertex.

Thus, only the case with an internal vertex v' of Γ' being traversed by an edge e of Γ remains. In other words, two of the three half-edges of v' lie on e . Let us choose the action of (3.12) such that the Lie algebra action 1 is the third edge, i.e. it acts by zero on functions pulled back from G^{Γ} . This is because a pullback of a function from G^{Γ} to a function on $G^{\Gamma'}$ does not depend on the holonomies along edges in $\Gamma' \setminus \Gamma$.

An action of a chord diagram D such that its component C acts trivially is the same as the action of $\varepsilon_C D$. Thanks to the compatibility of the Kontsevich integral with a deletion of an edge as in Proposition 2.6.4, we get an action of the following chord diagram, obtained from (3.12) by removing the strands 1 and 1':



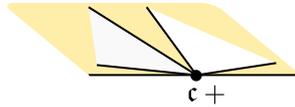
Moreover, again from Proposition 2.6.4, $\varepsilon_1 a = 1 \in \mathcal{P}_1$ and thus we obtain the action of

$$a \cdot R \cdot \Delta \nu^{-1/2} \cdot \nu^{1/2} \otimes \nu^{1/2}.$$

Such trivalent vertex therefore acts on the functions pulled back from G^Γ as a bivalent vertex labeled with minus, see (3.11).

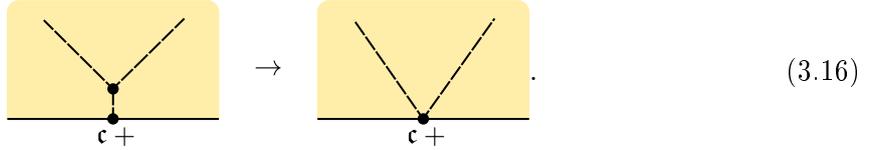
Collecting the terms described above, we get the action of (3.12) in trivalent vertices of Γ , and the action of bivalent vertices with alternating signs on edges of Γ . Thus, by Theorem 3.4.5, we get the product on A_Γ^m . \square

The algebra morphisms from Proposition 3.4.11 obviously descend to surfaces with some points decorated with coisotropic subalgebras. We will now consider a **once-degenerate embedding**: a map which is an embedding everywhere apart from one point marked \mathfrak{c} , which has two preimages marked by the same sign and the same coisotropic Lie subalgebra $\mathfrak{c} \subset \mathfrak{g}$. Locally, such an embedding looks as follows:



where the embedded surface is in white.

To describe the compatibility with skeletons, we will change the skeleton on the target surface slightly:



The multiplication is still the same, we just change the parametrization of the moduli space by removing one of the internal points. After such move, it is meaningful to ask for a once-degenerate embedding to be compatible with spines, i.e. Γ should be subgraph of Γ' , with the two vertices marked by \mathfrak{c} collapsed together.

Proposition 3.4.12. *Let $(\Sigma, V, \Gamma) \rightarrow (\Sigma', V', \Gamma')$ be a once-degenerate embedding compatible with spines. Then the action of $a^{1/2}$ at the two preimages, composed with the pushforward along $G^{\Gamma'} \rightarrow G^\Gamma$, is an algebra map between the corresponding quantizing algebras $A_\Gamma^m \rightarrow A_{\Gamma'}^m$. If the preimages are marked with $-$, the action is by $b^{1/2}$.*

Proof. The proof is adapted from [15, Theorem 2.2]. The difference between the products on A_Γ^m and $A_{\Gamma'}^m$ is the modified fusion at the two preimages. The action of $a^{1/2}$ cancels the normalization of the modified fusion and the remaining horizontal chord diagram J acts on invariants under $\mathfrak{c}^{\oplus 4}$ by the identity, as \mathfrak{c} is coisotropic. \square

This follows from the first part of Proposition 2.6.10: the induced map f_* on \mathcal{P}_2 is the identity in this case.

Alternatively, using the first part of Proposition 2.6.10 on the tangle J (from (3.1)), we get $J = J^{4321}$ in the notation of Definition 2.5.4. For the element K , this implies $K = J_4S(J_3) \otimes J_2S(J_1) = S_1S_2(K^{21})$, and it is enough to prove $K = K^{21}$. As in the first part, this reduces to $b = b^{21}$. Deforming the tangle, we get

$$Z \left(\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right) = Z \left(\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right)$$

i.e. $R^{-1} \cdot b = b^{21} \cdot R^{-1}$, where we labeled the strands of the chords for identification with \mathcal{P}_2 . The result follows since R^{-1} is central. \square

Note that we showed $a^{21} = S_1S_2(a) = a$ and similarly for b . Thus, all possible labellings and orientations of the doubled cap/cup give a/b .

4.1.2 Proposition 3.3.3

In the proof of Proposition 3.3.3, we used the fact that the associator Φ acts by the identity on the space of invariants under the diagonal $\mathfrak{g} \subset \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$ action. Using Proposition 2.5.18, this follows from the following statement:

Proposition 4.1.2. *The identity*

$$r_3(\Phi) = r_3(\Phi^{-1}) = 1$$

holds in \mathcal{P}_2 .

This statement was proven for the KZ associator in [53, Lemma 2.]

Proof. The claim is true for any element Φ given as an exponential of a Lie series with no linear terms.

Since Φ is an exponential of a Lie series in t_{12}, t_{23} , a general term in its expansion can be written as

$$A \cdot [B, t_{12}]$$

or

$$A' \cdot [B', t_{23}].$$

Recall that r_3 is a map of algebras, and $r_3(t_{12}) = t_{12}$, $r_3(t_{23}) = -t_{12} - t_{22}$ are central in \mathcal{P}_2 . Thus, r_3 applied on such commutators vanishes and the only nonzero term of $r_3(\Phi)$ is the identity. \square

4.1.3 Proposition 3.4.2

First, in Item 4 of the proof of Proposition 3.4.2, we used $r_2(a) = \nu^{-1}$.

Proposition 4.1.3. *The equality*

$$r_1(a) = r_2(a) = r_1(b) = r_2(b) = \nu^{-1}$$

holds in \mathcal{P}_1 .

Proof. Let us calculate $r_1(a)$ using the Kontsevich integral

$$r_1(a) = Z \left(\text{diagram} \right) = \nu^{-3/2} \cdot \hat{Z} \left(\text{diagram} \right) = \nu^{-3/2} \cdot \hat{Z} \left(\text{diagram} \right) = \nu^{-1}.$$

The other cases are analogous. □

The other fact used in the proof of Proposition 3.4.2 was the equality $r_3(\Phi^{a^{1/2}}) = 1 \in \mathcal{P}_2$. We start by a preparatory result

Proposition 4.1.4. *The elements $a, b \in \mathcal{P}_2$ can be computed as*

$$a = Z \left(\text{diagram} \right) = Z \left(\text{diagram} \right), \tag{4.3}$$

$$b = Z \left(\text{diagram} \right) = Z \left(\text{diagram} \right). \tag{4.4}$$

Proof. Let us prove the first equality. Looking closely at the definition of a , we get a sequence of two associators

$$a = Z \left(\text{diagram} \right).$$

The top associator, together with the cap, is what we want to compute in (4.3). As an element of \mathcal{P}_2 , it can slide through the doubled leg of the bottom associator. The bottom associator is identity, by Proposition 4.1.2. □

Remark 4.1.5. Apparently, this result disagrees with Lemma 3.1 of [53]. This is probably due to some issue with the normalization of Z_f of [53]: if Z_f were the unnormalized Kontsevich integral, their result would not be compatible with removing the component with a bend. The Lemma 10.2 of [55] agrees with our result in Proposition 4.1.4. We also checked this result via a computer script, see Section 4.3.

Now we can show the desired identity.

Proposition 4.1.6. *It is true that $r_3(\Phi^{a^{1/2}}) = 1 \in \mathcal{P}_2$, i.e. the associator Φ twisted by $a^{1/2}$ acts trivially on invariants under the diagonal action.*

integral going through the normalized Kontsevich integral:

$$\begin{aligned} Z \left(\begin{array}{c} \text{Diagram 1} \\ 12 \end{array} \right) &= \hat{Z} \left(\begin{array}{c} \text{Diagram 2} \\ 12 \end{array} \right) \cdot \nu^{-1/2} \otimes \nu^{-3/2} \\ &= Z \left(\begin{array}{c} \text{Diagram 3} \\ 12 \end{array} \right) \cdot 1 \otimes \nu^{-1} = 1 \otimes \nu^{-1} \cdot a. \end{aligned}$$

Here, the last equality is again Proposition 4.1.4. Thus, together we get

$$r_3(1 \otimes a) = b^{-1}1 \otimes \nu^{-1} = \nu \otimes 1 \cdot \Delta \nu^{-1} \cdot a,$$

and for the square root

$$r_3(1 \otimes a)^{1/2} = r_3(1 \otimes a^{1/2}) = \nu^{1/2} \otimes 1 \cdot \Delta \nu^{-1/2} \cdot a^{1/2}.$$

3rd term: In this case, $r_3(\Phi) = 1 \in \mathcal{P}_2$ by Proposition 4.1.2.

4th term: Here, $r_3(a^{-1/2} \otimes 1) = a^{-1/2}$, since we are doubling and connecting a strand with no chord endpoints

5th term: For $r_3(\Delta_1 a^{-1/2})$, we get

$$r_3(\Delta_1 a^{-1/2}) = \Delta \left(\begin{array}{c} \text{Diagram 4} \\ \Delta_1 \Delta_2 S_2 a^{-1/2} \end{array} \right) = \Delta \nu^{1/2}$$

Putting these terms together, we get

$$r_3(\Phi^{a^{1/2}}) = (\nu^{-1/2} \otimes 1) \cdot (\nu^{1/2} \otimes 1 \cdot \Delta \nu^{-1/2} \cdot a^{1/2}) \cdot a^{-1/2} \cdot \Delta \nu^{1/2} = 1.$$

□

Remark 4.1.7. Note that for an even associator, $a^{1/2} = \nu^{-1/2} \otimes \nu^{-1/2} \cdot \Delta \nu^{1/2}$ is a trivial twist and the previous proposition reduces to Proposition 4.1.2.

Twisting by a results again in the horizontal associator $(\Phi)_{-t}$, see Theorem 4.2.1 below. Thus, $r_3(\Phi^a) = 1$ again by Proposition 4.1.2. Denoting $x = r_3(\Delta_2 a^{1/2} \cdot 1 \otimes a^{1/2})$ and $y = r_3(a^{-1/2} \otimes 1 \cdot \Delta_1 a^{-1/2})$ the two terms coming from the twist, the equation $r_3(\Phi^a) = 1$ shows that $x^2 y^2 = 1$. Thus x^2 and y^2 commute, and so do their square roots, which implies that $xy = (x^2 y^2)^{1/2} = 1$, giving an alternative proof of the above Proposition 4.1.6.

4.2 The element a and applications

The elements $a, b \in \mathcal{P}_2$ from Definition 2.6.5 played an important role in our description of the deformation quantization of moduli spaces of connections. This is not the first time they appeared in the literature: they were used in the definition of the LMO invariant [53, 55] and in the extension of the Kontsevich integral to graphs by Dancso [24]. We will now recall some of their properties and describe a few further applications.

Recall that a and b were shown to be symmetric $a^{21} = a$, $b^{21} = b$ and invariant under reversal of both strands $S_1 S_2 a = a$, $S_1 S_2 b = b$. They are related by $a_{-t} = b$ and by [50],

$$ab = \nu^{-1} \otimes \nu^{-1} \cdot \Delta\nu.$$

Moreover, if the associator is even, then

$$a = b = \nu^{-1/2} \otimes \nu^{-1/2} \cdot \Delta\nu^{1/2}.$$

4.2.1 Twisting by a

Since a is symmetric and $\varepsilon_i a = 1 \in \mathcal{P}_1$ by Proposition 2.6.4, we can consider twisting the associator by a as in (2.27).

Theorem 4.2.1. *The associator Φ twisted with a is equal to Φ_{-t} .*

The theorem immediately implies the following

Corollary 4.2.2. *We have an equality in \mathcal{P}_3*

$$\Phi^{a^{1/2}} = (\Phi_{-t})^{b^{1/2}},$$

and this chord diagram is even, i.e. $\Phi^{a^{1/2}} = (\Phi^{a^{1/2}})_{-t}$.

Proof. First, note that twisting by a is the same as twisting twice by $a^{1/2}$, i.e.

$$(\Phi^{a^{1/2}})^{a^{1/2}} = \Phi_{-t},$$

Twisting this equation by $a^{-1/2}$ and noticing that $a^{-1/2}$ is equal to $b^{1/2}$ up to the trivial twist $\nu^{1/2} \otimes \nu^{1/2} \cdot \Delta\nu^{-1/2}$, we arrive at the result. The fact that this associator is even follows from the formula $a_{-t} = b$. \square

Remark 4.2.3. There are two important groups, GT_1 and GRT_1 , acting freely and transitively on the set of associators [26]. The element a thus implements as a twist the element of GRT_1 which sends $\Phi(X, Y)$ to $\Phi(-X, -Y)$, considered already by Drinfeld [26, Proposition 6.3] for the Knizhnik-Zamolodchikov associator.

Another easy corollary is the following

Corollary 4.2.4. *The identity functor*

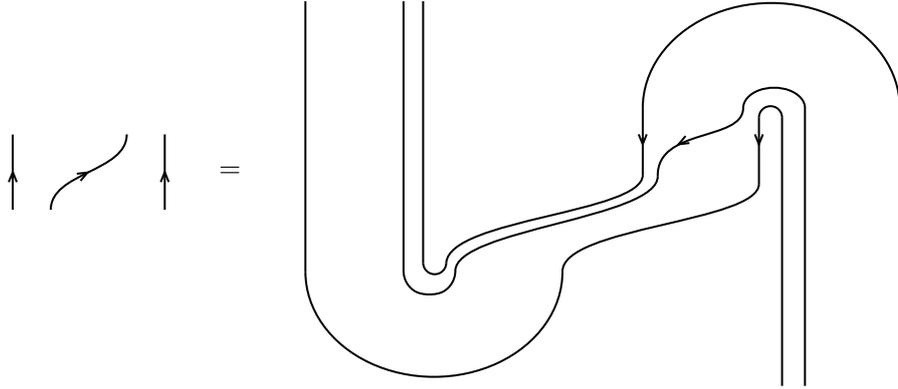
$$U(\bar{\mathfrak{g}} \oplus \mathfrak{h})\text{-mod}_\hbar^\Phi \rightarrow U(\mathfrak{g} \oplus \mathfrak{h})\text{-mod}_\hbar^\Phi$$

is monoidal, with the monoidal structure given by the action of a on the two factors of \mathfrak{g} .

Proof. The equation (2.10) is exactly the condition that the original associator is a twist of the new one. \square

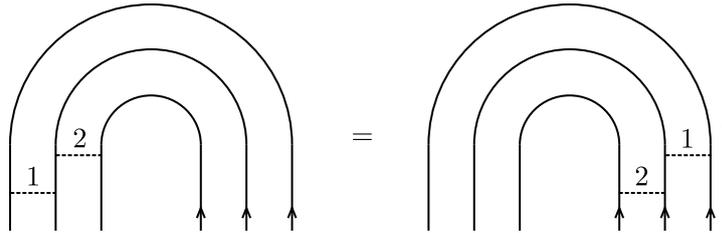
In this formulation, Theorem 4.2.1 gets an explanation via moduli spaces. Indeed, take a marked surface Σ with a point v marked with $-$. We can turn this into $+$ by adding one more disk marked $+$ and $-$ and performing fusion and reduction at $-$ with the point v . From Corollary 3.2.2, we know this is equivalent to acting by $a \cdot R$, and $R = e^{-t_{11}/2} \otimes e^{-t_{11}/2} \cdot \Delta e^{t_{11}/2}$ is the trivial twist. This gives one possibility of the proof of the Theorem 4.2.1; we present a more direct proof.

Proof of Theorem 4.2.1. Let us study the following equality of tangles, the **siphon identity**



Let us act with \hat{Z} on both sides of this identity. The left hand side is equal to $\Phi(t_{12}, t_{23})$. The right hand side has commuting terms $\nu \otimes \nu \otimes \nu$ coming from the normalization of the caps and cups and $\Delta^2(\nu^{-1})$ coming from the twice-cabled associator (using Proposition 2.5.12). Moreover, there is $\Delta_2 b \cdot 1 \otimes b$ in the triple cup and $\Delta_1 a \cdot a \otimes 1$ in the triple cap, via Theorem 2.6.6. Finally, between these two terms, we have the “small” associator.

Let us move the chords in this associator to the bottom right corner. This exchanges t_{12} with t_{23} and reverses the order of the multiplication:



In other words, we get the image of $\Phi(X, Y)$ under the anti-automorphism ϕ of $\mathbb{C}\langle\langle X, Y \rangle\rangle$ given by $X \mapsto Y$ and $Y \mapsto X$. However, $S_{\mathfrak{S}_3} \Phi(Y, X) = \Phi(X, Y)$ by (2.13), (2.14), i.e. the anti-automorphism $X \mapsto -Y$ and $Y \mapsto -X$ is the identity on the associator, and this anti-automorphism differs from ϕ by the automorphism $X \mapsto -X, Y \mapsto -Y$. This implies $\phi(\Phi(X, Y)) = \Phi(-X, -Y)$.

Thus, the equality of tangles above implies the following equality of chord diagrams:

$$\Phi = \nu \otimes \nu \otimes \nu \cdot \Delta^2(\nu^{-1}) \cdot (\Delta_2 b \cdot 1 \otimes b) \cdot \Phi_{-t} \cdot (\Delta_1 a \cdot a \otimes 1).$$

Using $b = a^{-1} \cdot \nu^{-1} \otimes \nu^{-1} \cdot \Delta\nu$, the factors of ν cancel and we get

$$\Phi = (\Delta_2 a^{-1} \cdot 1 \otimes a^{-1}) \cdot \Phi_{-t} \cdot (\Delta_1 a \cdot a \otimes 1)$$

or

$$\Phi_{-t} = (\Delta_2 a \cdot 1 \otimes a) \cdot \Phi \cdot (\Delta_1 a^{-1} \cdot a^{-1} \otimes 1).$$

□

Remark 4.2.5. For completeness, we include a table with the actions of the following (anti)-automorphisms on associators. To describe their effect completely, it is enough to know how they act on homogeneous Lie words of even or odd polynomial degree:

pol. degree	automorphism		anti-automorphism	
	even	odd	even	odd
$X \mapsto X$ $Y \mapsto Y$	+	+	−	+
$X \mapsto Y$ $Y \mapsto X$	−	−	+	−
$X \mapsto -X$ $Y \mapsto -Y$	+	−	−	−
$X \mapsto -Y$ $Y \mapsto -X$	−	+	+	+

The first and the third row are true for any Lie series; the second and the fourth row follow from (2.14). The anti-automorphism ϕ from the above proof is the second map in the right column, which has the same effect as the third map in the first column.

4.2.2 Obstruction to cabling

The Theorem 2.6.6 of Le and Murakami [50] states that for an even associator and normalization $\alpha = \beta = \nu^{1/2}$, the normalized Kontsevich integral is compatible with cabling. We will now prove that for an associator that is not even, no such normalization is possible. Note that for general α and β , we suppose that $S(\alpha) = \alpha$ and $S(\beta) = \beta$.

As explained in [50, Section 4.], the only non-trivial compatibility is for caps and cups. For a cap and general normalization, this gives the condition

$$\hat{Z} \left(\text{cap}; \Phi, \alpha, \beta \right) = \Delta \hat{Z} \left(\text{cup}; \Phi, \alpha, \beta \right),$$

i.e.

$$a \cdot \alpha \otimes \alpha = \Delta\alpha,$$

and similarly

$$b \cdot \beta \otimes \beta = \Delta\beta.$$

Indeed, for an even associator, it is enough to take $\alpha = \beta = \nu^{1/2}$.

Remark 4.2.6. Note that from the three equations we demand from α, β for \hat{Z} to be compatible with cabling

$$\begin{aligned}\alpha \cdot \beta &= \nu, \\ a &= \alpha^{-1} \otimes \alpha^{-1} \cdot \Delta\alpha, \\ b &= \beta^{-1} \otimes \beta^{-1} \cdot \Delta\beta,\end{aligned}$$

any two imply the third one.

Recall from Section 2.6.3 that a trivial twist $u^{-1} \otimes u^{-1} \cdot \Delta u$, with $u \in \mathcal{P}_1$, leaves the associator unchanged. Theorem 4.2.1 thus immediately implies the following:

Corollary 4.2.7. *If an associator Φ is not even $\Phi(X, Y) \neq \Phi(-X, -Y)$, then there exists no $\alpha \in \mathcal{P}_1$ such that $a = \alpha^{-1} \otimes \alpha^{-1} \cdot \Delta\alpha$.*

Remark 4.2.8. The equation $a = \alpha^{-1} \otimes \alpha^{-1} \cdot \Delta\alpha$ has an interpretation in the Drinfeld category $U\mathfrak{g}\text{-mod}_h^\Phi$. As explained in [18], if we restrict to a subcategory with duals (e.g. finite-dimensional modules), we can normalize evaluation and coevaluation maps using α and β to satisfy the snake identity. The spaces $(V \otimes W)^*$ and $W^* \otimes V^*$ are both naturally dual to $V \otimes W$, and thus they are canonically isomorphic. This isomorphism $(V \otimes W)^* \rightarrow W^* \otimes V^*$ turns out to be given by the action of

$$\Delta\alpha \cdot \Delta\nu^{-1} \cdot (\beta \otimes \beta) \cdot b = a \cdot \alpha^{-1} \otimes \alpha^{-1} \cdot \Delta\alpha. \quad (4.5)$$

Thus, for a non-even associator, there are no universal normalizations $\alpha, \beta \in \mathcal{P}_1$ making this isomorphisms to be the identity. This disproves the Proposition 3.4 of [18], which confounds the antipode of $U\mathfrak{k}_3$ with the one of $(U\mathfrak{g})^{\otimes 3}$.

4.3 Computer experiments

We will close this chapter by presenting a few results of a computer script written to perform calculations in the algebras \mathcal{P}_n . We first check against known results and previous computer calculations and then verify some of the results presented in this and the previous chapter.

This script, written in Python, is available online at [58]. It finds an explicit basis of the first few spaces $\mathcal{P}_n^{(m)}$ of n -strands chord diagrams with m chords. This is done by listing all the possible chord diagrams and all the 4T relations and solving this system of equations over rational numbers. The size of the problem grows exponentially with the number of chords, and on a modern computer with a few minutes of computing time, the following dimensions are obtainable:

dim	\mathcal{P}_1	\mathcal{P}_2	\mathcal{P}_3	\mathcal{P}_4	\mathcal{P}_5
1 chord	1	3	6	10	15
2 chords	2	9	28	69	145
3 chords	3	23	111	394	1130
4 chords	6	60	449	2035	7781
5 chords	10	148	—	—	—
6 chords	19	—	—	—	—

The program actually outputs dimensions of the subspaces of $\mathcal{P}_n^{(m)}$ of chord diagrams where all components of the support have a chord endpoint lying on them. The numbers in the table are calculated from these dimensions. The first column agrees with [11, Section 6.1], the second column agrees with [25, Kapitel VI]. The first row is $n + \binom{n}{2} = n(n+1)/2$.

Let us now perform a few calculations with the script. For a complete code, see [58, thesis.py]. We are able to perform the calculations up to 4 chord diagrams.

We begin by checking the pentagon equation (2.15) for the Knizhnik-Zamolodchikov associator:

```

1 >>> lhs = sim.simplify((tensor(units[0], Phi)
2                       * Phi.double(1)
3                       * tensor(Phi, units[0])).up_to_order(maxorder))
4 >>> rhs = sim.simplify((Phi.double(2) * Phi.double(0)).up_to_order(maxorder))
5 >>> print(sim.simplify(lhs-rhs).lc)
6 []

```

The empty list [] represents the empty linear combination of chord diagrams, i.e. 0. The operations `tensor` and `.double(i-1)` are the operations \otimes and¹ Δ_i from Definition 2.5.4. We don't show the two sides of the pentagon equation, the full output is available at [58, results_maxorder3.pdf] and [58, results_maxorder4.pdf].

We also check that our value of ν^{-1} is equal to the one in [22, p. 345]. The method `close_under(i, j)` reverses the strand $i+1$ and connects it with the strand $j+1$ on the bottom. Since we don't want to reverse the last strand of the associator, it is necessary to counter this reversal by S_2 , i.e. `.S(1)`:

```

1 >>> KZnuinv = Phi.close_under(1, 0).S(1).close_under(1, 0)
2 >>> print(sim.simplify(nuinvCDM.up_to_order(maxorder)
3                   - nuinvKZ.up_to_order(maxorder)).lc)
4 []

```

We also check Proposition 4.1.2:

```

1 >>> Phi0nInvariants = sim.simplify(Phi.double(2)
2                               .close_under(2, 1)
3                               .close_under(2, 0)
4                               .up_to_order(4)
5                               - units[1])
6 >>> print(Phi0nInvariants.lc)
7 []

```

Let us now turn to the element a . It is calculated by

```

1 >>> a = (tensor(units[0], Phiinv)*Phi.double(2)).up_to_order(maxorder)\
2                               .close_off(2, 1, maxorder)\
3                               .close_off(2, 0, maxorder)

```

For the KZ associator and up to order 3, it is equal to

¹The strands are indexed starting from 0 in the script.

$$a = \downarrow \downarrow + \frac{1}{24} \left[\text{diagram 1} \right] - \frac{1}{24} \left[\text{diagram 2} \right] + \frac{i\zeta(3)}{8\pi^3} \left[\text{diagram 3} \right] + \frac{i\zeta(3)}{8\pi^3} \left[\text{diagram 4} \right] - \frac{i\zeta(3)}{4\pi^3} \left[\text{diagram 5} \right].$$

We check Proposition 4.1.3:

```
1 >>> print(sim.simplify((a.close_under(1, 0) - nuinvKZ).up_to_order(maxorder))
2                                     .lc)
3 []
```

and Proposition 4.1.4:

```
1 >>> print(sim.simplify((a - Phiinv.close_off(0, 1)).up_to_order(maxorder))
2                                     .lc)
3 []
```

Finally, we check the Theorem 4.2.1:

```
1 >>> Phia = sim.simplify(Twist(Phi, a, maxorder))
2 >>> print(sim.simplify((Phia - Phi.changed_signs()).up_to_order(maxorder))
3                                     .lc)
4 []
```


Chapter 5

Quantization of Poisson Hopf algebras

We now come to the second part of this thesis, the joint paper with Pavol Ševera [59]. We start by giving an informal overview and explain the relation to the first part of the thesis. Then, we reproduce the paper [59].

5.1 Overview

The paper [59], continuing the work [62] of Ševera, provides a deformation quantization of Poisson-Hopf algebras, which are a generalization of Lie bialgebras and Poisson-Lie groups. Lie bialgebras were first quantized by Etingof and Kazhdan in [29]. Apart from giving a more general result, our methods make the actual quantization an easy consequence of the work of Drinfeld [26], using the formalism described in Section 2.4.3.

This is achieved by describing Poisson Hopf algebras in an infinitesimally-braided monoidal category \mathcal{C} [59, Section 5.1] via their **nerves**: functors N from a certain infinitesimally-braided monoidal category \mathbf{iCom} to \mathcal{C} . These functors, motivated by nerves of groups [59, Section 3.2], have to be lax monoidal, compatible with the infinitesimal braiding and they have to satisfy the so-called nerve condition [59, eq. (4)]. Concretely, the category \mathbf{iCom} has non-negative integers as objects and the nerve condition says that a natural map $N(2)^{\otimes n} \rightarrow N(n+1)$ is an isomorphism.

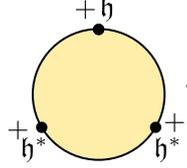
Similarly, we prove that braided lax monoidal functors $\mathbf{BrCom} \rightarrow \mathcal{D}$ satisfying the nerve condition are equivalent to Hopf algebras in a braided monoidal category \mathcal{D} . Then, a choice of a Drinfeld associator Φ gives the following sequence of functors

$$\mathbf{BrCom} \xrightarrow{\Phi} \mathbf{iCom}_{\hbar}^{\Phi} \xrightarrow{N_{\hbar}} \mathcal{C}_{\hbar}^{\Phi} ,$$

where the first functor is described in Section 2.4.3 and the second functor is a trivial \hbar -linear extension of N . Their composition then corresponds to a Hopf algebra quantizing the Poisson-Hopf algebra encoded by N .

Now, we will comment on the relation between [59] and the quantization of moduli spaces of flat connections on surfaces [17]; see also [17, Section 6.1] and [15, Section 4].

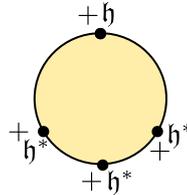
Recall [21, Chapter 1] that a Poisson-Lie group H is a Lie group with a Poisson structure such that the group product is a Poisson map. Linearizing this bivector at the group unit, the Lie algebra¹ \mathfrak{h} acquires a cobracket $\mathfrak{h} \rightarrow \mathfrak{h} \wedge \mathfrak{h}$, making \mathfrak{h} into a Lie bialgebra. Let us denote by $\mathfrak{g} := \mathfrak{h} \oplus \mathfrak{h}^*$ its Drinfeld double, which is a quadratic Lie algebra with $\mathfrak{h}, \mathfrak{h}^* \subset \mathfrak{g}$ coisotropic. Let G be a connected integration of \mathfrak{g} and consider the moduli space of flat G -connections associated to the following surface



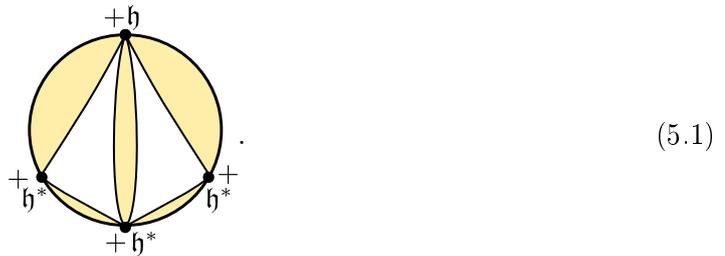
If the product $H \times H^* \rightarrow G$ is a diffeomorphism², this moduli space is diffeomorphic to $(G \times G)/(H \times H^* \times H^*) \cong H$, and the Poisson structure described in Theorem 2.3.10 is the original Poisson-Lie structure of H .

Using [17], we can find an associative algebra quantizing the Poisson algebra $C^\infty(H)$. Let us denote this algebra $N(\bullet\bullet)$. The two bullets represent the marked points on the bottom of the disk as well as the objects of iCom and BrCom . Our goal is to describe the Hopf algebra structure of $N(\bullet\bullet)$ from the perspective of surfaces.

For example, it is possible to recover the coproduct of $N(\bullet\bullet)$ from the compatibility of the deformation quantization with (degenerate) embeddings. Consider the following surface, with a quantization denoted³ $N(\bullet\bullet\bullet)$



and the following degenerate embedding (the source is a disjoint union of two disks, which are filled with white color):



On the level of moduli spaces, this embedding is the identity map on $H \times H$, and therefore the map of algebras $n: N(\bullet\bullet) \otimes N(\bullet\bullet) \rightarrow N(\bullet\bullet\bullet)$, induced by the embedding, is an isomorphism.

¹In this chapter, \mathfrak{h} will *not* be quadratic.

²See [15] for the general case.

³For simplicity, we don't draw the spines and don't discuss their compatibility with the embeddings.

Thus, the following composition with another embedding

$$\begin{array}{c}
 \begin{array}{c}
 \text{+}\mathfrak{h} \\
 \circ \\
 \diagup \quad \diagdown \\
 \circ \quad \circ \\
 \text{+}\mathfrak{h}^* \quad \text{+}\mathfrak{h}^* \\
 \circ \\
 \text{+}\mathfrak{h}^*
 \end{array} \\
 N(\bullet\bullet) \longrightarrow N(\bullet\bullet\bullet) \xrightarrow{n^{-1}} N(\bullet\bullet) \otimes N(\bullet\bullet)
 \end{array}$$

is a map of algebras.

To prove coassociativity of this map, giving a coproduct on $N(\bullet\bullet)$, it is useful to introduce an algebra $N(\bullet^k)$ as the quantization of a disk with k points labeled \mathfrak{h}^* on the bottom and one point labeled \mathfrak{h} on top. This sequence organizes into a functor from the simplex category to the category of associative algebras, where the non-decreasing maps $\{0, \dots, n-1\} \rightarrow \{0, \dots, m-1\}$ describe where the bottom points are mapped:

$$N \left(\begin{array}{cccc}
 0 & 1 & 2 & 3 \\
 \bullet & \bullet & \bullet & \bullet \\
 \uparrow & \nearrow & \nearrow & \uparrow \\
 \bullet & \bullet & \bullet & \bullet \\
 0 & 1 & 2 & 3
 \end{array} \right) := \begin{array}{c}
 \text{+}\mathfrak{h} \\
 \circ \\
 \diagup \quad \diagdown \\
 \circ \quad \circ \\
 \text{+}\mathfrak{h}^* \quad \text{+}\mathfrak{h}^* \\
 \circ \\
 \text{+}\mathfrak{h}^* \quad \text{+}\mathfrak{h}^*
 \end{array} .$$

Moreover, this functor is lax monoidal, with e.g the monoidal constraint $N(\bullet\bullet) \otimes N(\bullet\bullet) \rightarrow N(\bullet^4)$ given by

$$\begin{array}{c}
 \text{+}\mathfrak{h} \\
 \circ \\
 \diagup \quad \diagdown \\
 \circ \quad \circ \\
 \text{+}\mathfrak{h}^* \quad \text{+}\mathfrak{h}^* \\
 \circ \\
 \text{+}\mathfrak{h}^* \quad \text{+}\mathfrak{h}^*
 \end{array} .$$

However, to fully capture the Hopf algebra, it is necessary to add morphisms to both the source and the target category of the functor N . The correct enhancement of the source category is BrCom [59, Example 2.2], which has additional morphisms where strands are allowed to braid:

$$\begin{array}{cccc}
 0 & 1 & 2 & 3 \\
 \bullet & \bullet & \bullet & \bullet \\
 \uparrow & \nearrow & \nearrow & \uparrow \\
 \bullet & \bullet & \bullet & \bullet \\
 0 & 1 & 2 & 3
 \end{array} .$$

Images of such morphisms will not be algebra maps in general, and thus the target of N will be the category of vector spaces, instead of associative algebras. With these changes, one

arrives at one of the main results of [59], which is true as well if we replace the category of vector spaces with any braided monoidal category \mathcal{C} .

Theorem 5.1.1 ([59, Theorem 3.4]). *The category of Hopf algebras with an invertible antipode in \mathcal{C} is equivalent to the category of braided lax monoidal functors $\mathbf{BrCom} \rightarrow \mathcal{C}$ satisfying the nerve condition.*

This also gives an interpretation of the nerve condition via (5.1): it comes from the fact that the moduli spaces of flat connections above are completely described by the collection of holonomies “between” the consecutive points on the bottom.

Remark 5.1.2. The more general morphisms in \mathbf{BrCom} are obviously not formally captured by the deformation quantization of [17] studied in the first part of this thesis. However, they should come from a more general topological field theory, with morphisms between surfaces given not just by embeddings, but by cobordisms with marked points lying on strands knotted inside the cobordisms.

The second important result of [59] is an analogous characterization of Poisson Hopf algebras as functors from the category \mathbf{iCom} [59, Theorem 5.2]. The objects of \mathbf{iCom} are again natural numbers, while the morphisms are linear combinations of permutations decorated with horizontal chords. The main difficulty is constructing an infinitesimal braiding acting on tensor powers of this Poisson Hopf algebra [59, Section 6.2].

Then, with these two results, the problem of quantization reduces to precomposition with the functor $\mathbf{BrCom} \rightarrow \mathbf{iCom}_\hbar^\Phi$, coming from a Drinfeld associator Φ .

QUANTIZATION OF POISSON HOPF ALGEBRAS

JÁN PULMANN AND PAVOL ŠEVERA

ABSTRACT. We describe a method for quantization of Poisson Hopf algebras in \mathbb{Q} -linear symmetric monoidal categories. It is compatible with tensor products and can also be used to produce braided Hopf algebras. The main idea comes from the fact that nerves of groups are symmetric simplicial sets. Nerves of Hopf algebras then turn out to be braided rather than symmetric and nerves of Poisson Hopf algebras to be infinitesimally braided. The problem is thus solved via the standard machinery of Drinfeld associators.

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1. INTRODUCTION

One of the most natural deformation quantization problems is quantization of Poisson Hopf algebras [3], i.e. of commutative Hopf algebras with a compatible

Poisson bracket. Given a Poisson Hopf algebra with product m_0 , coproduct Δ_0 , antipode S_0 , Poisson bracket p , unit η and counit ϵ , the problem is to deform m_0 , Δ_0 , and S_0 to

$$m = \sum_{n=0}^{\infty} \hbar^n m_n \quad \Delta = \sum_{n=0}^{\infty} \hbar^n \Delta_n \quad S = \sum_{n=0}^{\infty} \hbar^n S_n$$

so that the result is still a Hopf algebra and so that

$$m - m^{op} = \hbar p + O(\hbar^2).$$

The deformation should be given by universal formulas with rational coefficients, i.e. as a morphism of props

$$\text{Hopf} \rightarrow \text{PoisHopf}(\mathbb{Q})_{\hbar},$$

and thus usable for quantization of Poisson Hopf algebras in arbitrary \mathbb{Q} -linear symmetric monoidal categories and functorial in a rather strong sense. Furthermore, it is natural to demand the quantization to be compatible with tensor products of Hopf algebras; equivalently it means that the morphism of props is compatible with a suitable cocommutative coalgebra enrichment.

Quantization of Poisson Hopf algebras includes, in particular, quantization of Lie bialgebras, which was solved in the seminal work of Etingof and Kazhdan [6]. Nonetheless, the problem of quantization of Poisson Hopf algebras is stronger and appears more natural.

We solve this problem using nerves of Hopf algebras. The nerve of a group is not just a simplicial set, but a symmetric simplicial set. A similar statement is true for nerves of commutative Hopf algebras. If we replace the symmetric structure with a braided one, we get nerves of (possibly braided) Hopf algebras. An infinitesimally braided structure corresponds to nerves of Poisson Hopf algebras, and their quantization can thus be obtained by Drinfeld associators (which are machines producing braided structures out of infinitesimally braided ones).

The natural setup for our method is slightly more general: quantization of Poisson Hopf algebras in infinitesimally braided categories to braided Hopf algebras. At the end of the paper we also describe the outcome of our method when applied to quantization of suitable Poisson Hopf algebroids.

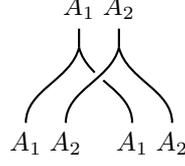
2. PRELIMINARIES

2.1. Algebras, coalgebras, Hopf algebras. An *algebra* (or a *monoid*) in a monoidal category \mathcal{C} is an object $A \in \mathcal{C}$ equipped with an associative product $m: A \otimes A \rightarrow A$ and with a unit $\eta: 1_{\mathcal{C}} \rightarrow A$ of m . Similarly a *coalgebra* $C \in \mathcal{C}$ is equipped with a coassociative coproduct $\Delta: C \rightarrow C \otimes C$ and with a counit $\epsilon: C \rightarrow 1_{\mathcal{C}}$.

An algebra A in a braided monoidal category (BMC) \mathcal{C} is *commutative* if the product $m: A \otimes A \rightarrow A$ satisfies $m = m \circ \beta_{A,A}$, where $\beta_{X,Y}: X \otimes Y \rightarrow Y \otimes X$ denotes the braiding, i.e.

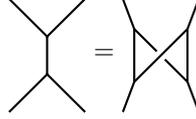
(with morphisms going upwards and braiding depicted by \times).

If A_1, A_2 are algebras in a BMC \mathcal{C} , then so is $A_1 \otimes A_2$ with the product [9]



and with the unit $1_e \cong 1_e \otimes 1_e \xrightarrow{\eta_{A_1 \otimes A_2}} A_1 \otimes A_2$. In this way algebras in \mathcal{C} form a monoidal category. The tensor product of two commutative algebras in a symmetric monoidal category (SMC) is commutative, but in a BMC it is, in general, not true.

A coalgebra H in the monoidal category of algebras in a BMC \mathcal{C} is a (braided) *bialgebra*. Explicitly this means that H is both an algebra and a coalgebra in \mathcal{C} and that we have the identities



and

$$\epsilon \circ \eta = \text{id}_{1_e}, \quad \Delta \circ (\eta \otimes \eta) = \eta, \quad \epsilon \circ m = \epsilon \otimes \epsilon$$

(where we tacitly identify $1_e \otimes 1_e$ with 1_e via the canonical isomorphism).

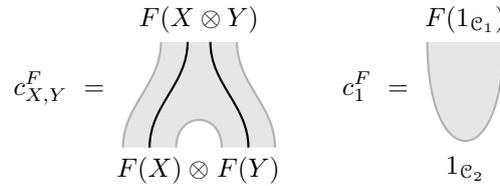
A (braided) *Hopf algebra* in a BMC \mathcal{C} is a bialgebra H with an additional morphism $S: H \rightarrow H$ (the *antipode*) such that

$$m \circ (\text{id} \otimes S) \circ \Delta = m \circ (S \otimes \text{id}) \circ \Delta = \eta \circ \epsilon.$$

2.2. Lax monoidal functors. A *lax monoidal functor* is a functor $F: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ between two monoidal categories, equipped with natural transformations (called *coherence morphisms*)

$$c_{X,Y}^F: F(X) \otimes F(Y) \rightarrow F(X \otimes Y), \quad c_1^F: 1_{\mathcal{C}_2} \rightarrow F(1_{\mathcal{C}_1})$$

making F compatible with the associativity and the unit constraints of \mathcal{C}_1 and \mathcal{C}_2 (see e.g. [8] for details). Coherence morphisms can be graphically depicted as



A *strong monoidal functor* is a lax monoidal functor whose coherence morphisms are isomorphisms. A *strict monoidal functor* is a strong monoidal functor whose coherence morphisms are identities.

A natural transformation $\alpha_X: F(X) \rightarrow G(X)$ between two lax monoidal functors $F, G: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ is *monoidal* if

$$\alpha_{X \otimes Y} \circ c_{X,Y}^F = c_{X,Y}^G \circ (\alpha_X \otimes \alpha_Y) \quad \text{and} \quad \alpha_{1_{\mathcal{C}_2}} \circ c_1^F = c_1^G.$$

Lax monoidal functors form a category with monoidal natural transformations as morphisms.

A lax monoidal functor $F: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ between two BMCs is *braided* if

$$F(\beta_{X,Y}^{\mathcal{C}_1}) \circ c_{X,Y}^F = c_{Y,X}^F \circ \beta_{F(X),F(Y)}^{\mathcal{C}_2}.$$

Lax monoidal functors send algebras to algebras: If $A \in \mathcal{C}_1$ is an algebra then $F(A) \in \mathcal{C}_2$ is an algebra with the product

$$F(m) \circ c_{A,A}^F = \begin{array}{c} F(A) \\ \text{---} \\ \text{---} \\ \text{---} \\ F(A) \otimes F(A) \end{array}$$

and with the unit $F(\eta) \circ c_1^F$. Braided lax monoidal functors send commutative algebras to commutative algebras.

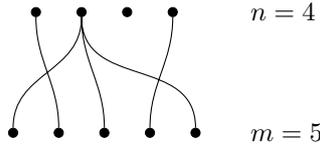
2.3. Props. A *prop* is a strict SMC whose objects are symbols \bullet^n , $n \geq 0$, with the tensor product of objects $\bullet^m \otimes \bullet^n = \bullet^{m+n}$ and with the unit object $1 = \bullet^0$. A *braided prop* is a strict BMC with the same property.

A morphism of (braided or ordinary) props $F: \mathcal{P}_1 \rightarrow \mathcal{P}_2$ is a strict braided monoidal functor such that $F(\bullet) = \bullet$.

If \mathcal{P} is an ordinary/braided prop, a \mathcal{P} -*algebra* in a strict SMC/BMC \mathcal{C} is an object $A \in \mathcal{C}$ together with a strict braided monoidal functor $F: \mathcal{P} \rightarrow \mathcal{C}$ such that $F(\bullet) = A$. (If \mathcal{C} is not strict, F is rather a strong braided monoidal functor, unique up to monoidal isomorphisms which are the identity at the object \bullet . We can specify a particular F by demanding $F(\bullet^n) = A^{\otimes n}$ for a chosen parenthesization of the tensor power $A^{\otimes n}$, and demanding $c_1^F = \text{id}$ and $c_{\bullet^m, \bullet^n}^F: A^{\otimes m} \otimes A^{\otimes n} \rightarrow A^{\otimes(m+n)}$ to be the corresponding composition of the associativity (or unit) constraints of \mathcal{C} .)

Example 2.1. Let Com be the prop of commutative algebras. It is generated by morphisms $\underline{m}: \bullet\bullet \rightarrow \bullet$ and $\underline{\eta}: 1 \rightarrow \bullet$ modulo the relations saying that \underline{m} is commutative and associative and that $\underline{\eta}$ is a unit for \underline{m} . A Com -algebra in a SMC is a commutative algebra.

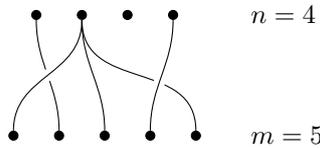
A morphism $\bullet^m \rightarrow \bullet^n$ in Com can be represented graphically as



with joining lines corresponding to the product $\bullet\bullet \rightarrow \bullet$ and missing lines to the unit $1 \rightarrow \bullet$, i.e. as a map $\{1, \dots, m\} \rightarrow \{1, \dots, n\}$. Composition of morphisms is then equal to the composition of these maps.

Example 2.2. Let BrCom be the braided prop of commutative algebras. It is again generated by two morphisms $\underline{m}: \bullet\bullet \rightarrow \bullet$ (the product) and $\underline{\eta}: 1 \rightarrow \bullet$ (the unit). A BrCom -algebra in a BMC is a commutative algebra.

The morphisms $\bullet^m \rightarrow \bullet^n$ in BrCom are braids with m strands, non-bijectively attached at the top:



3. NERVES OF HOPF ALGEBRAS

3.1. Symmetric simplicial spaces. An (augmented) symmetric simplicial object in a category \mathcal{S} is a functor

$$\mathbf{Com}^{\text{op}} \rightarrow \mathcal{S}.$$

Informally speaking, we want to study the case when \mathcal{S} is a “category of spaces”. We take the dual point of view and replace “spaces” with commutative algebras. Hence, if \mathcal{C} is a symmetric monoidal category and $\text{ComAlg}(\mathcal{C})$ the category of commutative algebras in \mathcal{C} , our object of interest is the category of functors

$$\mathbf{Com} \rightarrow \text{ComAlg}(\mathcal{C}),$$

i.e. (augmented) symmetric cosimplicial commutative algebras in \mathcal{C} .

Let us observe that every object \bullet^n of \mathbf{Com} is a commutative algebra, being a tensor power of the commutative algebra \bullet , and also that every morphism in \mathbf{Com} is a morphism of algebras. This gives us the following result.

Proposition 3.1. *Suppose \mathcal{C} is a symmetric monoidal category. The category of functors*

$$\mathbf{Com} \rightarrow \text{ComAlg}(\mathcal{C})$$

is isomorphic to the category of symmetric lax monoidal functors

$$\mathbf{Com} \rightarrow \mathcal{C}$$

with monoidal natural transformations as morphisms.

Namely, if $F: \mathbf{Com} \rightarrow \mathcal{C}$ is symmetric lax monoidal then $F(\bullet^n)$ is a commutative algebra in \mathcal{C} as \bullet^n is a commutative algebra in \mathbf{Com} , and thus F becomes a functor $\mathbf{Com} \rightarrow \text{ComAlg}(\mathcal{C})$.

Proof. To get the isomorphism in the opposite direction, if $F: \mathbf{Com} \rightarrow \text{ComAlg}(\mathcal{C})$ is a functor then we can define a symmetric lax monoidal structure on F as follows: $c_{X,Y}^F$ is the composition,

$$F(X) \otimes F(Y) \rightarrow F(X \otimes Y) \otimes F(X \otimes Y) \rightarrow F(X \otimes Y)$$

where the first arrow comes from

$$X = X \otimes 1 \xrightarrow{\text{id}_X \otimes \eta_Y} X \otimes Y \quad \text{and} \quad Y = 1 \otimes Y \xrightarrow{\eta_X \otimes \text{id}_Y} X \otimes Y$$

and the second arrow is the product in $F(X \otimes Y)$, and c_1^F is the unit $1_{\mathcal{C}} \rightarrow F(1)$ of the commutative algebra $F(1)$. \square

3.2. Nerves of groups and groupoids. If S is a set, let Pair_S denote the pair groupoid of S , i.e. the groupoid with the set of objects S and with a unique morphism between any two objects.

The (symmetric augmented) *nerve* of a groupoid G is the functor

$$N_G: \mathbf{Com}^{\text{op}} \rightarrow \text{Set}$$

given on objects by

$$N_G(\bullet^n) = \text{Hom}(\text{Pair}_{\{1,2,\dots,n\}}, G).$$

On morphisms it is given by seeing them as maps $\{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, n\}$.

If G is a group, this means

$$N_G(\bullet^n) = \{g: \{1, 2, \dots, n\}^2 \rightarrow G \mid g(i, i) = 1 \text{ and } g(i, j)g(j, k) = g(i, k)\}$$

and we have a bijection

$$(1) \quad N_G(\bullet^n) \cong G^{n-1}, \quad g \mapsto (g(i, i+1) \mid 1 \leq i \leq n-1).$$

We can recognize nerves of groups among all functors $N: \mathbf{Com}^{\text{op}} \rightarrow \mathbf{Set}$ using (1) as follows. Let us consider the morphisms $\phi_i: \bullet\bullet \rightarrow \bullet^n$

$$\phi_i = \begin{array}{ccccccc} & & 1 & \cdots & i & i+1 & \cdots & n \\ & & \bullet & \cdots & \bullet & \bullet & \cdots & \bullet \\ & & | & & | & | & & \\ & & \bullet & & \bullet & \bullet & & \end{array} \quad (1 \leq i \leq n-1)$$

and the resulting map

$$(2) \quad (N(\phi_1), \dots, N(\phi_{n-1})): N(\bullet^n) \rightarrow N(\bullet\bullet)^{n-1}.$$

We then have the following elementary and well known result.

Proposition 3.2. *The assignment $G \mapsto N_G$ is an equivalence of categories between the category of groups and the category of functors $N: \mathbf{Com}^{\text{op}} \rightarrow \mathbf{Set}$ such that (2) is a bijection for every $n \geq 2$ and such that $|N(1)| = |N(\bullet)| = 1$.*

The group corresponding to N is $G = N(\bullet\bullet)$, the product is

$$N\left(\begin{array}{c} \bullet & \bullet \\ \diagdown & \diagup \\ \bullet & \bullet \end{array}\right): G \times G \rightarrow G$$

(where we identify $N(\bullet^3)$ with $G \times G$ via (2)) and the unit and the inverse are

$$N\left(\begin{array}{c} \bullet \\ \diagup & \diagdown \\ \bullet & \bullet \end{array}\right) \quad \text{and} \quad N\left(\begin{array}{c} \bullet & \bullet \\ \diagup & \diagdown \\ \bullet & \bullet \end{array}\right).$$

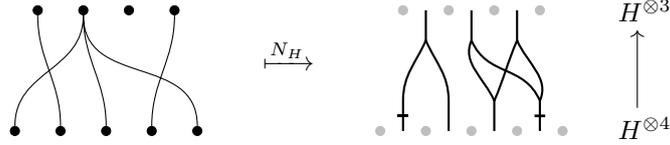
3.3. Nerves of commutative Hopf algebras. By analogy with groups one can define the nerve of a commutative Hopf algebra $H \in \mathcal{C}$ where \mathcal{C} is a symmetric monoidal category (SMC). It is a functor

$$N_H: \mathbf{Com} \rightarrow \mathbf{ComAlg}(\mathcal{C})$$

or equivalently, in view of Proposition 3.1, a symmetric lax monoidal functor

$$N_H: \mathbf{Com} \rightarrow \mathcal{C}.$$

On objects it is given by $N_H(\bullet^n) = H^{\otimes(n-1)}$, $N_H(1) = 1_{\mathcal{C}}$ (to simplify the notation we suppose that \mathcal{C} is strict monoidal; if not, one needs to choose a parenthesization of $H^{\otimes(n-1)}$ and use the associativity constraint of \mathcal{C} accordingly). The graphical algorithm for computing $N_H(\phi)$ for $\phi: \bullet^m \rightarrow \bullet^n$ is:



- ▷ put one H between every two consecutive \bullet 's
- ▷ for every consecutive pair $\bullet\bullet$ in \bullet^m , if the order of the pair is reversed in the image, apply S (depicted by \dagger) to the corresponding H
- ▷ if the distance of the two \bullet 's in the image is k , apply iterated Δ to get a morphism $H \rightarrow H^{\otimes k}$ (if $k = 0$, apply ϵ)
- ▷ finally multiply H 's arriving between consecutive \bullet 's in \bullet^n .

The coherence morphisms of N_H (obtained from Proposition 3.1 and from the commutative algebra structure of $N_H(\bullet^n)$) are as follows: for $m, n > 0$ the morphism $N_H(\bullet^m) \otimes N_H(\bullet^n) \rightarrow N_H(\bullet^{m+n})$ is

$$(3) \quad \text{id}_H^{\otimes(m-1)} \otimes \eta \otimes \text{id}_H^{\otimes(n-1)}: H^{\otimes(m-1)} \otimes H^{\otimes(n-1)} \rightarrow H^{\otimes(m+n-1)}$$

and the remaining coherence morphisms are identities.

If $\mathcal{C} = \mathbf{Set}^{\text{op}}$ and $\otimes = \times$ then H is a group and N_H is its nerve, with the identification $N_H(\bullet^n) = H^{n-1}$ given by (1).

One can recognize nerves of commutative Hopf algebras among all symmetric lax monoidal functors $\mathbf{Com} \rightarrow \mathcal{C}$ by an analogue of the map (2). We shall say that

a braided lax monoidal functor $N: \mathbf{Com} \rightarrow \mathcal{C}$ or $N: \mathbf{BrCom} \rightarrow \mathcal{C}$ satisfies the nerve condition if the morphism $N(\bullet\bullet)^{\otimes(n-1)} \rightarrow N(\bullet^n)$ given by

$$(4) \quad \begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \end{array} \quad (n = 5)$$

is an isomorphism for every $n \geq 2$, and if the two morphisms

$$1e \xrightarrow{c_1^N} N(1) \xrightarrow{N(\eta)} N(\bullet)$$

are also isomorphisms. We then have the following minor generalization of Proposition 3.2 (which corresponds to the case $\mathcal{C} = \mathbf{Set}^{\text{op}}$, $\otimes = \times$).

Proposition 3.3. *Let \mathcal{C} be a SMC. The category of commutative Hopf algebras in \mathcal{C} is equivalent to the category of symmetric lax monoidal functors*

$$N: \mathbf{Com} \rightarrow \mathcal{C}$$

satisfying the nerve condition.

The Hopf algebra corresponding to N is $H = N(\bullet\bullet)$. The commutative algebra structure of H comes from the commutative algebra $\bullet\bullet$, i.e. the product is

$$m = \begin{array}{c} \bullet \quad \bullet \\ \downarrow \quad \downarrow \\ \downarrow \quad \downarrow \\ \bullet \quad \bullet \end{array}$$

and the unit is the composition $1e \rightarrow N(1) \rightarrow N(\bullet\bullet)$. The coproduct, the counit, and the antipode are given by

$$\Delta = \begin{array}{c} \bullet \quad \bullet \\ \downarrow \quad \downarrow \\ \downarrow \quad \downarrow \\ \bullet \quad \bullet \end{array} \quad \epsilon = \begin{array}{c} \bullet \quad \bullet \\ \downarrow \quad \downarrow \\ \downarrow \quad \downarrow \\ \bullet \quad \bullet \end{array} \quad S = \begin{array}{c} \bullet \quad \bullet \\ \downarrow \quad \downarrow \\ \downarrow \quad \downarrow \\ \bullet \quad \bullet \end{array}$$

where we implicitly use the isomorphisms $N(\bullet^3) \cong N(\bullet\bullet)^{\otimes 2}$ and $N(\bullet) \cong 1e$ given by the nerve condition.

The proof is the same as for Proposition 3.2 and we leave its details to the reader.

3.4. Nerves of braided Hopf algebras. The previous proposition has a straightforward generalization to the world of noncommutative Hopf algebras (announced in [12, Remark 4]; a similar result appeared in [7]):

Theorem 3.4. *Let \mathcal{C} be a BMC. The category of Hopf algebras with invertible antipodes in \mathcal{C} is equivalent to the category of braided lax monoidal functors*

$$N: \mathbf{BrCom} \rightarrow \mathcal{C}$$

satisfying the nerve condition.

The Hopf algebra corresponding to N is $H = N(\bullet\bullet)$. The algebra structure of H comes from the algebra structure of $\bullet\bullet$ (which is the tensor product of two algebras \bullet), i.e. the product of H is

$$m = \begin{array}{c} \bullet \quad \bullet \\ \downarrow \quad \downarrow \\ \downarrow \quad \downarrow \\ \bullet \quad \bullet \end{array}$$

and the unit is the composition $1_e \rightarrow N(1) \rightarrow N(\bullet\bullet)$. The coproduct, the counit, and the antipode are given by

$$\Delta = \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \end{array} \quad \epsilon = \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} \quad S = \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array}$$

where we implicitly use the isomorphisms $N(\bullet^3) \cong N(\bullet\bullet)^{\otimes 2}$ and $N(\bullet) \cong 1_e$ given by the nerve condition.

The proof can be found in §6.1. Its main part is the construction of a suitable N out of H , which is a braided version of the above-given construction $H \mapsto N_H$ for commutative Hopf algebras.

4. INFINITESIMAL BRAIDINGS

4.1. Infinitesimally braided categories. If R is a commutative ring, an R -linear category is a category enriched over R -modules, i.e. a category such that $\text{Hom}(X, Y)$ is an R -module for any two objects X and Y and such that the composition of morphisms is R -bilinear.

An R -linear monoidal category is a category which is both R -linear and monoidal and such that the tensor product of morphisms is R -bilinear.

If \mathcal{C} is an R -linear category, let \mathcal{C}_ϵ be the $R[\epsilon]/(\epsilon^2)$ -linear category obtained by extension of scalars, i.e. \mathcal{C}_ϵ has the same objects as \mathcal{C} and

$$\text{Hom}_{\mathcal{C}_\epsilon}(X, Y) = \text{Hom}_{\mathcal{C}}(X, Y)[\epsilon]/(\epsilon^2).$$

An R -infinitesimally braided monoidal category (R -iBMC) is a symmetric R -linear monoidal category \mathcal{C} with a natural transformation $t_{X,Y}: X \otimes Y \rightarrow X \otimes Y$ called *infinitesimal braiding* such that \mathcal{C}_ϵ with the modified braiding

$$\beta_{X,Y} := \sigma_{X,Y} \circ (1 + \epsilon t_{X,Y})$$

(where $\sigma_{X,Y}: X \otimes Y \rightarrow Y \otimes X$ is the symmetry in \mathcal{C}) is a braided monoidal category. Moreover, $t_{X,Y}$ is required to satisfy the symmetry condition

$$t_{Y,X} = \sigma_{X,Y} \circ t_{X,Y} \circ \sigma_{Y,X}.$$

The morphism $t_{X,Y}$ will be graphically represented by a horizontal chord

$$\begin{array}{c} X \quad Y \\ | \quad | \\ \text{---} \\ | \quad | \\ X \quad Y \end{array}$$

An i -braided lax/strong monoidal functor $\mathcal{C}_1 \rightarrow \mathcal{C}_2$ between two R -iBMCs is an R -linear symmetric lax/strong monoidal functor such that

$$F(t_{X,Y}^{\mathcal{C}_1}) \circ c_{X,Y}^F = c_{X,Y}^F \circ t_{F(X),F(Y)}^{\mathcal{C}_2}.$$

Let us notice that any R -linear SMC becomes an R -iBMC if we set $t_{X,Y} = 0$.

Example 4.1 ([4]). If \mathfrak{g} is a Lie algebra over R and $t \in \mathfrak{g} \otimes \mathfrak{g}$ is symmetric and \mathfrak{g} -invariant then the symmetric monoidal category $U\mathfrak{g}\text{-mod}$ is R -i-braided via

$$t_{X,Y} = \rho_X \otimes \rho_Y(t) \in \text{End}(X \otimes Y).$$

4.2. Drinfeld associators. An iBMC is a SMC with a first order deformation of the symmetry to a braiding. A natural question is whether one can extend this first order deformation to a formal deformation.

If \mathcal{C} is an R -linear category, let \mathcal{C}_\hbar be the $R[[\hbar]]$ -linear category with $\text{Ob}(\mathcal{C}_\hbar) = \text{Ob}(\mathcal{C})$ and with

$$\text{Hom}_{\mathcal{C}_\hbar}(X, Y) = \text{Hom}_{\mathcal{C}}(X, Y)[[\hbar]].$$

Likewise, if $F: \mathcal{C} \rightarrow \mathcal{C}'$ is an R -linear functor, let $F_\hbar: \mathcal{C}_\hbar \rightarrow \mathcal{C}'_\hbar$ be its (continuous) $R[[\hbar]]$ -linear extension, i.e.

$$F_\hbar\left(\sum f_n \hbar^n\right) = \sum F(f_n) \hbar^n, \quad f_n \in \text{Hom}_{\mathcal{C}}(X, Y).$$

Theorem/Definition 4.2 (Drinfeld [4]). *There is an element $\Phi \in \mathbb{Q}\langle\langle x, y \rangle\rangle$ (a Drinfeld associator), $\Phi = 1 + [x, y]/24 + \dots$, with the following property. If $\mathbb{Q} \subset R$ and if \mathcal{C} is a R -iBMC then \mathcal{C}_\hbar with the new braiding $\beta_{X,Y}^{new} = \sigma_{X,Y} \circ \exp(\hbar t_{X,Y}/2)$ and the new associativity constraint*

$$\gamma_{X,Y,Z}^{new} = \gamma_{X,Y,Z} \circ \Phi(\hbar t_{X,Y} \otimes \text{id}_Y, \hbar \gamma_{X,Y,Z}^{-1} \circ (\text{id}_X \otimes t_{Y,Z}) \circ \gamma_{X,Y,Z}),$$

(where $\gamma_{X,Y,Z}: (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$ is the associativity constraint of \mathcal{C}) is a BMC which we shall denote by \mathcal{C}_\hbar^Φ .

If $F: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ is an R -i-braided lax/strong monoidal functor then

$$F_\hbar: (\mathcal{C}_1)_\hbar^\Phi \rightarrow (\mathcal{C}_2)_\hbar^\Phi,$$

with the same coherence morphisms, is braided lax/strong monoidal.

Notice that

$$\beta_{X,Y}^{new} = \sigma_{X,Y} \circ (1 + \frac{\hbar}{2} t_{X,Y}) + O(\hbar^2), \quad \gamma_{X,Y,Z}^{new} = \gamma_{X,Y,Z} + O(\hbar^2).$$

Also notice that if $t_{X,Y} = 0$ for all objects $X, Y \in \mathcal{C}$ then $\mathcal{C}_\hbar^\Phi = \mathcal{C}_\hbar$ as a symmetric monoidal category, as then $\beta_{X,Y}^{new} = \sigma_{X,Y}$ and $\gamma_{X,Y,Z}^{new} = \gamma_{X,Y,Z}$.

4.3. Poisson algebras in infinitesimally braided categories. An (infinitesimally braided) *Poisson algebra* in an iBMC \mathcal{C} is a commutative algebra $A \in \mathcal{C}$ together with a biderivation

$$p: A \otimes A \rightarrow A$$

(the ‘‘Poisson bracket’’) satisfying the modified skew-symmetry

$$(5) \quad p + p \circ \sigma_{A,A} = m \circ t_{A,A}$$

(where $m: A \otimes A \rightarrow A$ is the product) and the modified Jacobi identity

$$(6) \quad \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} - \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array}$$

(using the notation $p = \blacklozenge$ and $m = \blacktriangleright$).

A commutative algebra $A \in \mathcal{C}$ is *strongly commutative* if $m \circ t_{A,A} = 0$. Then A with $p = 0$ is a Poisson algebra.

If $A \in \mathcal{C}$ is a Poisson algebra and if $F: \mathcal{C} \rightarrow \mathcal{C}'$ is an i-braided lax monoidal functor then $F(A) \in \mathcal{C}'$ is a Poisson algebra too.

If $A_1, A_2 \in \mathcal{C}$ are Poisson algebras then so is $A_1 \otimes A_2$, with the Poisson bracket

$$(7) \quad \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$$

Under this tensor product, Poisson algebras in \mathcal{C} form a monoidal category.

If \mathcal{C} is an R -linear SMC with $t_{X,Y} = 0$ (for all objects X, Y) then these notions reduce to the usual definition of Poisson algebras in a linear SMC, and to the usual definition of their tensor products.

Example 4.3. If $\mathcal{C} = U\mathfrak{g}\text{-mod}$ is as in Example 4.1 and if $1/2 \in R$ then Poisson algebras in \mathcal{C} are the same as \mathfrak{g} -quasi-Poisson algebras defined in [1]. Namely if we split a Poisson bracket p on $A \in \mathcal{C}$ to its anti-symmetric and symmetric parts

$$p = \{, \} + \frac{1}{2}\rho_A \otimes \rho_A(t)$$

then $\{, \}$ is a \mathfrak{g} -invariant biderivation of the commutative algebra A satisfying

$$\{a, \{b, c\}\} + c.p. = -\frac{1}{4}\phi_A(a \otimes b \otimes c)$$

where $\phi_A: A^{\otimes 3} \rightarrow A$ is given by $\phi_A = m^{(3)} \circ \rho_A^{\otimes 3}(\phi)$, where $m^{(3)}: A^{\otimes 3} \rightarrow A$ is the product of 3 elements and $\phi = [t \otimes 1, 1 \otimes t] \in \bigwedge^3 \mathfrak{g} \subset (U\mathfrak{g})^{\otimes 3}$.

Tensor product of Poisson algebras then corresponds to the fusion product from *op. cit.*

The definition of Poisson algebras and of their tensor product come from the following fact. Suppose that $\mathbb{Q} \subset R$ and let A be an algebra in \mathcal{C}_\hbar^Φ with a product

$$m = \sum_{n=0}^{\infty} \hbar^n m_n \quad (m_n \in \text{Hom}_{\mathcal{C}}(A \otimes A, A))$$

such that m_0 is commutative. If we define $p \in \text{Hom}_{\mathcal{C}}(A \otimes A, A)$ via the braided commutator

$$(8) \quad m - m \circ \beta_{A,A}^{-1} = \hbar p + O(\hbar^2), \text{ i.e. } p = m_1 - m_1 \circ \sigma_{A,A} + \frac{1}{2}m_0 \circ t_{A,A}$$

then m_0 and p make A to a Poisson algebra in the iBMC \mathcal{C} . We shall then say that m is a *quantization* of the Poisson bracket p .

Moreover, if A_1 and A_2 are two such algebras in \mathcal{C}_\hbar^Φ then the resulting Poisson bracket on $A_1 \otimes A_2$ is given by (7).

If $A \in \mathcal{C}$ is strongly commutative then it remains, with the same product (i.e. with $m = m_0$) and unit, a commutative algebra in \mathcal{C}_\hbar^Φ .

4.4. Infinitesimally braided props. An *R-infinitesimally braided prop* (or *R-i-braided prop*) is a strict *R-iBMC* with objects \bullet^n , $n \geq 0$, with the tensor product of objects $\bullet^m \otimes \bullet^n = \bullet^{m+n}$, and with the unit $1 = \bullet^0$.

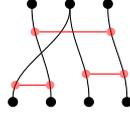
Let \mathcal{A}_n be the Drinfeld-Kohno algebra

$$\mathcal{A}_n := R \langle t_{ij}, 1 \leq i, j \leq n, i \neq j \mid t_{ij} = t_{ji}, [t_{ij} + t_{ik}, t_{jk}] = 0, \\ [t_{ij}, t_{kl}] = 0 \text{ if } i, j, k, l \text{ are all different} \rangle$$

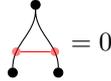
on which S_n acts by permuting the indices. For any *R-i-braided prop* \mathcal{P} we have a canonical map of *R-algebras* $S_n \times \mathcal{A}_n \rightarrow \text{End}(\bullet^n)$, with S_n coming from the symmetric monoidal category structure and t_{ij} being the chord connecting the i 'th and j 'th \bullet in \bullet^n .

Example 4.4. Let $\text{iCom}(R)$ be the *R-i-braided prop* of strongly commutative algebras (initial among strict *R-iBMCs* with a chosen strongly commutative algebra.) It is generated by morphisms $\underline{m}: \bullet\bullet \rightarrow \bullet$ and $\underline{\eta}: 1 \rightarrow \bullet$ modulo the relations saying that \underline{m} is strongly commutative and associative and that $\underline{\eta}$ is its unit. Morphisms

are then R -linear combinations of “maps with chords”



modulo the strong commutativity relation



and the Drinfeld-Kohno relations (the defining relations of \mathcal{A}_n).

5. POISSON HOPF ALGEBRAS AND THEIR QUANTIZATION

5.1. Poisson Hopf algebras in infinitesimally braided categories. An (infinitesimally braided) *Poisson Hopf algebra* in an iBMC \mathcal{C} is a commutative Hopf algebra $H \in \mathcal{C}$ together with a Poisson bracket $p: H \otimes H \rightarrow H$ such that the coproduct $\Delta: H \rightarrow H \otimes H$ is a morphism of Poisson algebras. In other words, p has to satisfy

$$(9) \quad \begin{array}{c} \diagup \\ | \\ \diagdown \end{array} = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ | \quad | \\ \diagdown \quad \diagup \end{array} + \begin{array}{c} \diagdown \\ | \\ \diagup \end{array} + \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \diagdown \quad \diagup \end{array}$$

Suppose that $\mathbb{Q} \subset R$ and fix an associator Φ . A *quantization* of a Poisson Hopf algebra

$$(H, m_0, \Delta_0, \eta, \epsilon, S_0, p)$$

in \mathcal{C} is a deformation of its product, coproduct, and antipode

$$m = \sum_{n=0}^{\infty} \hbar^n m_n \quad \Delta = \sum_{n=0}^{\infty} \hbar^n \Delta_n \quad S = \sum_{n=0}^{\infty} \hbar^n S_n$$

s.t. $(H, m, \Delta, \eta, \epsilon, S)$ is a Hopf algebra in the BMC $\mathcal{C}_{\hbar}^{\Phi}$ and such that (8) holds.

Example 5.1. An example of Poisson Hopf algebras in nontrivial iBMCs is given by Manin quadruples [13], generalizing the standard link between Poisson Hopf algebras, Lie bialgebras and Manin triples [3].

A *Manin quadruple* is a quadruple $(\mathfrak{d}, \mathfrak{h}, \mathfrak{h}^*, \mathfrak{g})$, where \mathfrak{d} is a finite-dimensional (or suitably topological) quadratic Lie algebra and $\mathfrak{h}, \mathfrak{h}^*, \mathfrak{g} \subset \mathfrak{d}$ are Lie subalgebras such that

$$\mathfrak{d} = \mathfrak{h} \oplus \mathfrak{h}^* \oplus \mathfrak{g}$$

as a vector space and such that

$$\mathfrak{h}^{\perp} = \mathfrak{h} \oplus \mathfrak{g}, \quad \mathfrak{h}^{*\perp} = \mathfrak{h}^* \oplus \mathfrak{g}.$$

As an example, \mathfrak{d} can be semisimple, $\mathfrak{p}, \mathfrak{p}^* \subset \mathfrak{d}$ a pair of opposite parabolic subalgebras, \mathfrak{h} and \mathfrak{h}^* their nilpotent radicals, and $\mathfrak{g} = \mathfrak{p} \cap \mathfrak{p}^*$.

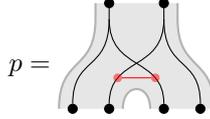
The Lie subalgebra $\mathfrak{g} \subset \mathfrak{d}$ is quadratic (i.e. the restriction of the quadratic form to \mathfrak{g} remains non-degenerate), and thus the category $\mathcal{C} = U\mathfrak{g}\text{-mod}$ is infinitesimally braided. $U\mathfrak{h}$ is naturally a Poisson Hopf algebra in \mathcal{C}^{op} . See [13] for details and for the corresponding generalization of Poisson-Lie groups.

5.2. Nerves of Poisson Hopf algebras. For Poisson Hopf algebras we have the following analogue of Theorem 3.4.

Theorem 5.2. *Let \mathcal{C} be a R -iBMC. The category of Poisson Hopf algebras in \mathcal{C} is equivalent to the category of i -braided lax monoidal functors*

$$N: \text{iCom}(R) \rightarrow \mathcal{C}$$

satisfying the nerve condition. The commutative Hopf algebra corresponding to N is $H = N(\bullet\bullet)$, given by Proposition 3.3 (using the inclusion $\text{Com} \subset \text{iCom}(R)$), and the Poisson bracket on H comes from the Poisson bracket on $\bullet\bullet = \bullet \otimes \bullet$ (where \bullet has $p = 0$), i.e.



The proof can be found in §6.2.

5.3. The “universal quantization functor”.

Proposition 5.3. *There is a braided strong monoidal functor*

$$U_\Phi: \text{BrCom} \rightarrow \text{iCom}(\mathbb{Q})_{\hbar}^\Phi$$

whose reduction mod \hbar is the projection $\text{BrCom} \rightarrow \text{Com}$.

Proof. The algebra $\bullet \in \text{iCom}(\mathbb{Q})$ is strongly commutative, and thus it remains, with the same product and unit, a commutative algebra in $\text{iCom}(\mathbb{Q})_{\hbar}^\Phi$. This gives us a braided strong monoidal functor $\text{BrCom} \rightarrow \text{iCom}(\mathbb{Q})_{\hbar}^\Phi$ sending \bullet to \bullet , and its reduction mod \hbar has clearly the required property. \square

5.4. Quantization of Poisson Hopf algebras. Poisson Hopf algebras can be quantized as follows.

Theorem 5.4. *Let H be a Poisson Hopf algebra in an R -iBMC \mathcal{C} , and let*

$$N: \text{iCom}(R) \rightarrow \mathcal{C}$$

be its nerve. Suppose that $\mathbb{Q} \subset R$, and let Φ be a Drinfeld associator. Then the composed braided lax monoidal functor

$$\text{BrCom} \xrightarrow{U_\Phi} \text{iCom}(\mathbb{Q})_{\hbar}^\Phi \subset \text{iCom}(R)_{\hbar}^\Phi \xrightarrow{N_{\hbar}} \mathcal{C}_{\hbar}^\Phi$$

satisfies the nerve condition. The resulting Hopf algebra structure on $H \in \mathcal{C}_{\hbar}^\Phi$ (cf. Theorem 3.4) is a quantization of the Poisson Hopf algebra $H \in \mathcal{C}$.

Proof. To see that the composed functor \tilde{N} satisfies the nerve condition we reduce it mod \hbar , and get the composition

$$\text{BrCom} \xrightarrow{\text{projection}} \text{Com} \xrightarrow{N} \mathcal{C}$$

which satisfies the nerve condition because N does. Since \tilde{N} is its \hbar -deformation, it also satisfies the nerve condition (which says that some morphisms should be invertible).

We need to check that we get a quantization of the Poisson Hopf algebra. If m is the product on $H \in \mathcal{C}_{\hbar}^\Phi$, we have

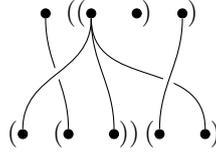
$$m - m \circ \beta_{H,H}^{-1} = \text{diagram 1} - \text{diagram 2} = \hbar \text{diagram 3} + O(\hbar^2)$$

and thus indeed $m - m \circ \beta_{H,H}^{-1} = \hbar p + O(\hbar^2)$ where p is the Poisson bracket of the Poisson Hopf algebra $H \in \mathcal{C}$. \square

If $t_{X,Y} = 0$ for all objects $X, Y \in \mathcal{C}$ then the previous theorem gives us a quantization of H to a Hopf algebra in the SMC \mathcal{C}_\hbar .

Remark 5.5. To see how the quantization depends on choices it is better to use parenthesized versions of the props BrCom and iCom .

Following [2], if \mathcal{P} is a (ordinary or braided or R -i-braided) prop, let $\text{Pa}\mathcal{P}$ be its parenthesized version. It is a (symmetric or braided or R -i-braided) non-strict monoidal category equivalent to \mathcal{P} : its objects are fully parenthesised words in \bullet (i.e. expressions built out of \bullet using a non-associative product), and its morphisms are the morphisms of \mathcal{P} forgetting the parenthesization. A typical morphism in PaBrCom looks as



Theorems 3.4 and 5.2 remain true with PaBrCom and PaiCom in place of BrCom and iCom . The advantage of this setup is that now we have a *unique* strict braided monoidal functor $\text{Pa}U_\Phi: \text{PaBrCom} \rightarrow \text{PaiCom}_\hbar^\Phi$ given by the commutative algebra structure on \bullet and thus the resulting quantization of Poisson Hopf algebras depends only on the associator Φ and not on the choice of U_Φ (which depends on a choice of a parenthesization). A similar argument also gives us an action of the Grothendieck-Teichmueller Lie algebra grt on the prop of Poisson Hopf algebras by derivations.

5.5. Cocommutative coalgebra enrichment. Let us now consider categories enriched over the category of cocommutative R -coalgebras; we shall call them *R -cc enriched categories*.

In an *R -cc enriched BMC* the tensor product of morphisms has to be compatible with the enrichment, and the braiding and the associativity and unit constraints are required to be grouplike, i.e. to be R -cc enriched natural transformations.

An *R -cc enriched iBMC* is an R -cc enriched SMC with an infinitesimal braiding such that $t_{X,Y}$ is primitive for every X, Y (which is equivalent to the deformed braiding $\sigma_{X,Y} \circ (1 + \epsilon t_{X,Y})$ being grouplike).

Example 5.6. The R -i-braided prop $\text{iCom}(R)$ is R -cc enriched, if we demand its generating morphisms $\underline{m}: \bullet\bullet \rightarrow \bullet$ and $\underline{\eta}: 1 \rightarrow \bullet$ to be grouplike.

Example 5.7. Let $\text{iPoissHopf}(R)$ be the R -i-braided prop of Poisson Hopf algebras. It is naturally R -cc enriched, with all the generating morphisms (product, coproduct, unit, counit, antipode) being grouplike, except for the Poisson bracket \underline{p} which satisfies $\Delta \underline{p} = \underline{p} \otimes \underline{m} + \underline{m} \otimes \underline{p}$.

Example 5.8. Let $\text{PoissHopf}(R)$ be the R -linear prop of Poisson Hopf algebras. It is naturally R -cc enriched, in the same way as $\text{iPoissHopf}(R)$.

The R -cc enrichment of the prop $\text{PoissHopf}(R)$ is equivalent to the fact that if H_1 and H_2 are Poisson Hopf algebras in an R -linear SMC \mathcal{C} , then so is $H_1 \otimes H_2$. Let us explain it in a way which works also for braided and i-braided props.

If \mathcal{C}_1 and \mathcal{C}_2 are R -linear categories, let $\mathcal{C}_1 \boxtimes_R \mathcal{C}_2$ be the R -linear category with

$$\text{Ob}(\mathcal{C}_1 \boxtimes_R \mathcal{C}_2) = \text{Ob}(\mathcal{C}_1) \times \text{Ob}(\mathcal{C}_2),$$

$$\text{Hom}_{\mathcal{C}_1 \boxtimes_R \mathcal{C}_2}((X_1, X_2), (Y_1, Y_2)) = \text{Hom}_{\mathcal{C}_1}(X_1, Y_1) \otimes_R \text{Hom}_{\mathcal{C}_2}(X_2, Y_2).$$

If \mathcal{C}_1 and \mathcal{C}_2 are braided R -linear monoidal then so is $\mathcal{C}_1 \boxtimes_R \mathcal{C}_2$: \otimes is given componentwise, i.e. $(X_1, X_2) \otimes (Y_1, Y_2) = (X_1 \otimes Y_1, X_2 \otimes Y_2)$ and similarly for morphisms, $1_{\mathcal{C}_1 \boxtimes_R \mathcal{C}_2} = (1_{\mathcal{C}_1}, 1_{\mathcal{C}_2})$, and we have

$$\gamma^{\mathcal{C}_1 \boxtimes_R \mathcal{C}_2} = \gamma^{\mathcal{C}_1} \otimes_R \gamma^{\mathcal{C}_2} \quad \text{and} \quad \beta^{\mathcal{C}_1 \boxtimes_R \mathcal{C}_2} = \beta^{\mathcal{C}_1} \otimes_R \beta^{\mathcal{C}_2}$$

and likewise for the unit morphisms. Moreover, if \mathcal{C}_1 and \mathcal{C}_2 are R -iBMCs then so is $\mathcal{C}_1 \boxtimes_R \mathcal{C}_2$, with the infinitesimal braiding

$$t^{\mathcal{C}_1 \boxtimes_R \mathcal{C}_2} = t^{\mathcal{C}_1} \otimes_R \text{id} + \text{id} \otimes_R t^{\mathcal{C}_2}.$$

If \mathcal{C} is an R -cc enriched BMC or iBMC then the functor

$$\Delta_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C} \boxtimes_R \mathcal{C},$$

given on objects by $\Delta_{\mathcal{C}}(X) = (X, X)$ and on morphisms by the coalgebra coproduct, is braided (or infinitesimally braided) strictly monoidal.

In particular, if \mathcal{P} is an R -cc enriched prop (or braided prop, or i-braided prop) and if $\mathcal{P} \rightarrow \mathcal{C}_{1,2}$ are two R -linear symmetric (or braided or i-braided) strong monoidal functors, then by composition with $\Delta_{\mathcal{P}}: \mathcal{P} \rightarrow \mathcal{P} \boxtimes_R \mathcal{P}$ we get a symmetric (or braided or i-braided) strong monoidal functor $\mathcal{P} \rightarrow \mathcal{C}_1 \boxtimes_R \mathcal{C}_2$. In other words, if $A_1 \in \mathcal{C}_1$ and $A_2 \in \mathcal{C}_2$ are \mathcal{P} -algebras, then so is $(A_1, A_2) \in \mathcal{C}_1 \boxtimes_R \mathcal{C}_2$.

Finally, if $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}$ is a SMC, the tensor product functor $\otimes: \mathcal{C} \boxtimes_R \mathcal{C} \rightarrow \mathcal{C}$ is a symmetric strong monoidal functor, and so $A_1 \otimes A_2 \in \mathcal{C}$, being the \otimes -image of the \mathcal{P} -algebra (A_1, A_2) , is a \mathcal{P} -algebra too.

Let us conclude with the compatibility of Drinfeld's construction of the BMC \mathcal{C}_h^Φ with the product of categories \boxtimes_R and with R -cc enrichment. Drinfeld [4] showed that Φ can be chosen grouplike (wrt. the coproduct where x and y are primitive); from now on we shall assume that Φ is grouplike. This implies that

$$(\mathcal{C}_1 \boxtimes_R \mathcal{C}_2)_h^\Phi = (\mathcal{C}_1)_h^\Phi \boxtimes_{R[[\hbar]]} (\mathcal{C}_2)_h^\Phi \quad (\text{as } R[[\hbar]]\text{-linear BMCs})$$

and also that if \mathcal{C} is an R -cc enriched iBMC then \mathcal{C}_h^Φ is an $R[[\hbar]]$ -cc enriched BMC. (Here we abuse the notation slightly - our $R[[\hbar]]$ -modules are of the form $M[[\hbar]]$ for some R -module M , with the tensor product $M[[\hbar]] \otimes_{R[[\hbar]]} M'[[\hbar]] := (M \otimes_R M')[[\hbar]]$.)

5.6. Quantization in terms of props and compatibility with products. If \mathcal{D} an R -cc enriched BMC, let $\mathcal{D}^{gl} \subset \mathcal{D}$ be the its sub-BMC with all the objects but only the grouplike morphisms. If \mathcal{C} is a BMC we shall say that a braided monoidal functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is *grouplike* if its image and coherence morphisms are in \mathcal{D}^{gl} .

Example 5.9. The braided strong monoidal functor $U_\Phi: \text{BrCom} \rightarrow \text{iCom}(\mathbb{Q})_h^\Phi$ is grouplike. This is because the commutative algebra $\bullet \in \text{iCom}(\mathbb{Q})_h^\Phi$ is a commutative algebra in the subcategory $(\text{iCom}(\mathbb{Q})_h^\Phi)^{gl}$, as \underline{m} and $\underline{\eta}$ are grouplike morphisms.

Let us now recast Theorem 5.4 into the language of props. Let Hopf be the prop of Hopf algebras and BrHopf the braided prop of Hopf algebras.

Theorem 5.10. *There is a grouplike braided strong monoidal functor*

$$Q_\Phi^{br}: \text{BrHopf} \rightarrow \text{iPoissHopf}(\mathbb{Q})_h^\Phi, \quad Q_\Phi^{br}(\bullet^n) = \bullet^n \quad (\forall n)$$

such that the resulting Hopf algebra structure on $\bullet \in \text{iPoissHopf}(\mathbb{Q})_h^\Phi$ is a quantization of the Poisson Hopf algebra structure on $\bullet \in \text{iPoissHopf}(\mathbb{Q})$, and a grouplike morphism of props

$$Q_\Phi: \text{Hopf} \rightarrow \text{PoissHopf}(\mathbb{Q})_h$$

with the same property.

Proof. Since \bullet is a Poisson-Hopf algebra in $\text{iPoissHopf}(\mathbb{Q})$, by Theorem 5.4 we get its quantization in $\text{iPoissHopf}(\mathbb{Q})_{\hbar}^{\Phi}$, and thus a braided strong monoidal functor

$$Q_{\Phi}^{br}: \text{BrHopf} \rightarrow \text{iPoissHopf}(\mathbb{Q})_{\hbar}^{\Phi}, \quad Q_{\Phi}^{br}(\bullet^n) = \bullet^n \quad (\forall n).$$

To see that Q_{Φ}^{br} is grouplike, let us notice that the nerve functor corresponding to the Poisson Hopf algebra $\bullet \in \text{iPoissHopf}(\mathbb{Q})$,

$$N: \text{iCom}(\mathbb{Q}) \rightarrow \text{iPoissHopf}(\mathbb{Q}),$$

is R -cc enriched i-braided lax monoidal. As a result

$$N_{\hbar}: \text{iCom}(\mathbb{Q})_{\hbar}^{\Phi} \rightarrow \text{iPoissHopf}(\mathbb{Q})_{\hbar}^{\Phi}$$

is R -cc enriched braided lax monoidal, and thus its composition with U_{Φ} is a group-like braided lax monoidal functor $\text{BrCom} \rightarrow \text{iPoissHopf}(\mathbb{Q})_{\hbar}^{\Phi}$. As a result, all the defining operations of the resulting braided Hopf algebra $\bullet \in \text{iPoissHopf}(\mathbb{Q})_{\hbar}^{\Phi}$ are grouplike, and thus Q_{Φ}^{br} is grouplike.

Applying the same construction and reasoning to $\text{PoissHopf}(\mathbb{Q})$ we get a grouplike morphism of props $Q_{\Phi}: \text{Hopf} \rightarrow \text{PoissHopf}(\mathbb{Q})_{\hbar}$. \square

Remark 5.11. As explained in [5], quantization of Lie bialgebras is equivalent to a suitable morphism of props

$$\text{Hopf} \rightarrow \overline{\text{PoissHopf}(\mathbb{Q})}_{\hbar}$$

where $\overline{\text{PoissHopf}(\mathbb{Q})}_{\hbar}$ is a certain completion of $\text{PoissHopf}(\mathbb{Q})$. As we have shown, such a morphism exists also without a completion.

The previous theorem puts our quantization of Poisson Hopf algebras to the following form. If $H \in \mathcal{C}$ is a Poisson Hopf algebra in an R -iBMC with $\mathbb{Q} \subset R$, i.e. if we have a \mathbb{Q} -i-braided strong monoidal functor $F: \text{iPoissHopf}(\mathbb{Q}) \rightarrow \mathcal{C}$ with $F(\bullet) = H$, then we compose F_{\hbar} with Q_{Φ}^{br} and thus make H to a Hopf algebra in $\mathcal{C}_{\hbar}^{\Phi}$. If the infinitesimal braiding of \mathcal{C} vanishes then we can work with PoissHopf and Q_{Φ} in place of iPoissHopf and Q_{Φ}^{br} .

If $H \in \mathcal{C}$ is a Poisson Hopf algebra, let $H_{\hbar}^{\Phi} \in \mathcal{C}_{\hbar}^{\Phi}$ denote the same object H with its new Hopf algebra structure. The fact that Q_{Φ}^{br} and Q_{Φ} are grouplike gives us the following.

Corollary 5.12. *Suppose that $H_1 \in \mathcal{C}_1$ and $H_2 \in \mathcal{C}_2$ are Poisson Hopf algebras. Then also $(H_1, H_2) \in \mathcal{C}_1 \boxtimes_R \mathcal{C}_2$ is a Poisson Hopf algebra, and we have the equality of Hopf algebras*

$$(H_{1\hbar}^{\Phi}, H_{2\hbar}^{\Phi}) = (H_1, H_2)_{\hbar}^{\Phi} \in \mathcal{C}_{1\hbar}^{\Phi} \boxtimes_{R[\hbar]} \mathcal{C}_{2\hbar}^{\Phi} = (\mathcal{C}_1 \boxtimes_R \mathcal{C}_2)_{\hbar}^{\Phi}.$$

If \mathcal{C} has vanishing infinitesimal braiding, and thus $\mathcal{C}_{\hbar}^{\Phi} = \mathcal{C}_{\hbar}$ is a SMC, and if $H_{1,2} \in \mathcal{C}$ are Poisson Hopf algebras, then also

$$(H_1 \otimes H_2)_{\hbar}^{\Phi} = H_{1\hbar}^{\Phi} \otimes H_{2\hbar}^{\Phi}.$$

6. PROOFS OF THE NERVE THEOREMS

6.1. Nerves of braided Hopf algebras (proof of Theorem 3.4). We can suppose that \mathcal{C} is strict monoidal. We shall prove the theorem in the following form: there is an *isomorphism* of categories between Hopf algebras with invertible antipodes in \mathcal{C} and braided lax monoidal functors $N: \text{BrCom} \rightarrow \mathcal{C}$ which satisfy the *strict nerve condition*: the isomorphisms in the nerve condition are required to be identities.

Let us construct $N = N_H: \text{BrCom} \rightarrow \mathcal{C}$ out of a Hopf algebra H . By the strict nerve condition we have

$$N_H(\bullet^n) = H^{\otimes(n-1)}.$$

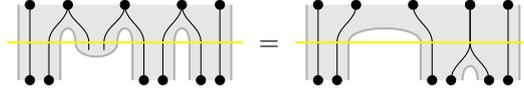
The strict nerve condition also implies that the coherence morphisms

$$N_H(\bullet^m) \otimes N_H(\bullet^n) \rightarrow N_H(\bullet^{m+n})$$

for $m, n > 0$ are

$$\text{id}_H^{\otimes(m-1)} \otimes \eta \otimes \text{id}_H^{\otimes(n-1)} : H^{\otimes(m-1)} \otimes H^{\otimes(n-1)} \rightarrow H^{\otimes(m+n-1)}$$

where $\eta: 1_e \rightarrow H$ is the unit of H , as explained for the case $m = 2, n = 3$:



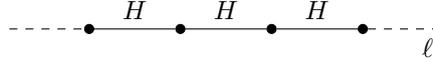
The remaining coherence morphisms are identities.

The main part of the proof is to describe the values of N_H on the morphisms of BrCom. Let us do it and illustrate it with the example

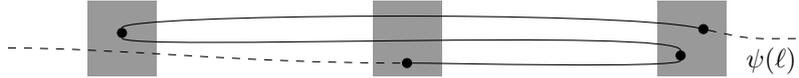
$$N_H(\text{braid}): H^{\otimes 3} \rightarrow H^{\otimes 2}.$$

If $\phi: \bullet^m \rightarrow \bullet^n$, we get $N_H(\phi): H^{\otimes(m-1)} \rightarrow H^{\otimes(n-1)}$ as follows:

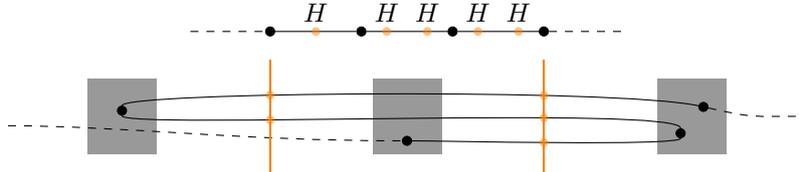
- ▷ We start by drawing m points on a horizontal line ℓ in the plane. There are $m - 1$ intervals between these points, each of them carrying one copy of H (all together representing $H^{\otimes(m-1)}$).



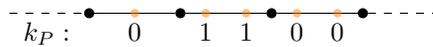
- ▷ The morphism (braid) ϕ can be interpreted as an isotopy of the plane between the identity and a diffeomorphism ψ of \mathbb{R}^2 , such that the isotopy stays the identity outside a large disk. The diffeomorphism ψ brings the m points to n different landing pads, again situated along a horizontal line.



- ▷ Now we draw $n - 1$ vertical lines between the landing pads. We suppose that they intersect $\psi(\ell)$ transversely. If the ψ -image of an interval meets the vertical lines k times, we apply the iterated coproduct $H \rightarrow H^{\otimes k}$ to the copy of H associated to that interval (if $k = 0$, we use the counit). At this point we have one H for each intersection point, i.e. we produced a morphism $H^{\otimes(m-1)} \rightarrow H^{\otimes M}$ where M is the number of the intersections.

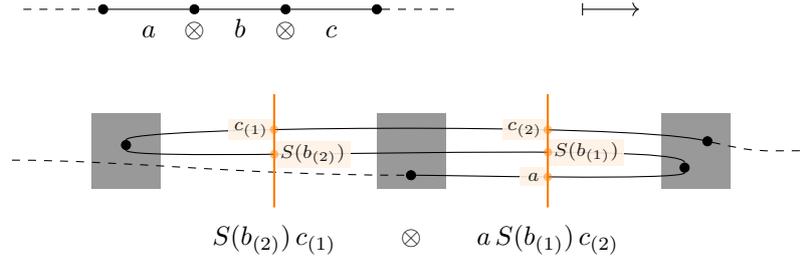


- ▷ For each of the intersection points P we find the total number of half-turns k_P when moving along $\psi(\ell)$ (turns in the positive direction are counted positively, in the negative direction negatively). Then we apply S^{k_P} to the H corresponding to P .

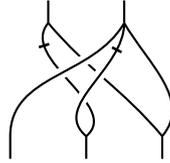


- ▷ We move each of the M H 's by the isotopy (using the braiding in \mathcal{C}), and finally multiply them along the vertical lines from the bottom to the top. Composing all these morphisms we get $N_H(\phi): H^{\otimes(m-1)} \rightarrow H^{\otimes(n-1)}$. (The multiplication is done by first moving the intersection points to a horizontal position, with the bottom-most points on the left and the top-most on the right, and then multiplying the corresponding H 's.)

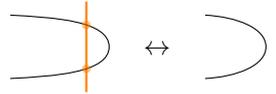
In our example we thus have (suppressing the braiding in \mathcal{C} in the formula)



i.e. $N_H(\phi): H^{\otimes 3} \rightarrow H^{\otimes 2}$ is



To see that $N_H(\phi)$ is well defined, i.e. independent of the details of the isotopy and of the transversals, we need to verify that it is invariant under moves of the type



This invariance follows from the defining property of the antipode.

The fact that N_H is a functor, i.e. that $N_H(\phi_1 \circ \phi_2) = N_H(\phi_1) \circ N_H(\phi_2)$, follows from H being a bialgebra. Namely, let $m^{(p)}: H^{\otimes p} \rightarrow H$ denote the iterated product

$$m^{(0)} = \eta, \quad m^{(p+1)} = m \circ (m^{(p)} \otimes \text{id}_H).$$

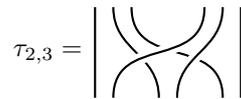
and $\Delta^{(q)}: H \rightarrow H^{\otimes q}$ the iterated coproduct

$$\Delta^{(0)} = \epsilon, \quad \Delta^{(q+1)} = (\Delta^{(q)} \otimes \text{id}_H) \circ \Delta.$$

Then for each p, q we have

$$(10) \quad (m^{(q)})^{\otimes p} \circ \tau_{p,q} \circ (\Delta^{(p)})^{\otimes q} = \Delta^{(p)} \circ m^{(q)}$$

where $\tau_{p,q}: H^{\otimes pq} \rightarrow H^{\otimes pq}$ is given by the braid which reshuffles q groups, each made of p strands, into p groups, each made of q strands, by taking the first element of each group together, then second etc., using overcrossings only.



The expression for $N_H(\phi_1 \circ \phi_2)$ differs from the expression for $N_H(\phi_1) \circ N_H(\phi_2)$ only by replacing RHSs of (10) with LHSs, and so they are equal. (If we use the

“ Δ is horizontal and m vertical” convention as in the construction of N_H then the identity (10) is the commutativity of the diagram

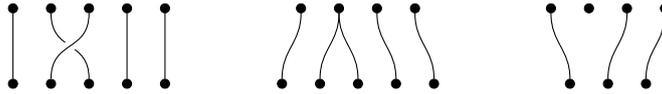
$$\begin{array}{ccc}
 \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} & \xrightarrow{(\Delta^{(p)})^{\otimes q}} & \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \\
 m^{(q)} \downarrow & & \downarrow (m^{(q)})^{\otimes p} \\
 \bullet & \xrightarrow{\Delta^{(p)}} & \bullet
 \end{array} \quad (p = 2, q = 3)$$

which is much more enlightening in our context.)

Finally, the fact that N_H is braided lax monoidal and that it satisfies the strict nerve condition is evident. Moreover the construction $H \mapsto N_H$ is clearly functorial.

If $N: \mathbf{BrCom} \rightarrow \mathcal{C}$ satisfies the strict nerve condition then checking that $H_N := N(\bullet\bullet)$ (with the operations given in the theorem) is a Hopf algebra is a simple manipulation with diagrams; it is also a special case of Theorem 1 from [12]. Again the construction $N \mapsto H_N$ is functorial.

One easily checks that $H_{N_H} = H$ (as Hopf algebras). To finish the proof, we need to verify that $N_{H_N} = N$, and it is sufficient to do it for the morphisms of the type



as they generate \mathbf{BrCom} . Checking it is again a straightforward calculation.

Remark 6.1. The functor N_H can alternatively be constructed as follows. If \mathcal{D} is the category of H -dimodules in \mathcal{C} (the Yetter-Drinfeld category) then H is naturally a commutative algebra in \mathcal{D} , and thus we have a braided strict monoidal functor $F: \mathbf{BrCom} \rightarrow \mathcal{D}$, $F(\bullet^n) = H^{\otimes n}$. H -coinvariants (of the H -coaction on dimodules) is then a braided lax monoidal functor $\mathcal{D} \rightarrow \mathcal{C}$ (in general this functor might not be defined on whole of \mathcal{D} , but it is defined on the image of F), and we can define N_H as the composition of these two braided lax monoidal functors.

In more detail, H -dimodules are objects X of \mathcal{C} equipped with a (left) H -action and H -coaction, such that for any H -module Y the \mathcal{C} -morphism

$$\tilde{\beta}_{X,Y}: X \otimes Y \rightarrow Y \otimes X, \quad \tilde{\beta}_{X,Y} = \begin{array}{c} Y \quad X \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ X \quad Y \end{array}$$

is a morphism of H -modules. The category \mathcal{D} of H -dimodules is braided via $\tilde{\beta}$. H is an H -comodule via Δ and there is a unique (adjoint) H -module structure on H such that the action $H \otimes Y \rightarrow Y$ is an H -module morphism for every H -module Y . This makes H to an object of \mathcal{D} , and its algebra structure makes it to a commutative algebra in \mathcal{D} . We have

$$\tilde{\beta}_{H,H} = \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array}$$

and for any morphism $\phi: \bullet^p \rightarrow \bullet^q$ in \mathbf{BrCom} we get $N_H(\phi) = \kappa_q \circ F(\phi) \circ \iota_p$ where the morphisms ι_p and κ_q are defined below (see Equation (12)).

6.2. Nerves of Poisson Hopf algebras (proof of Theorem 5.2). As in §6.1, we shall suppose that \mathcal{C} is strict monoidal and prove the theorem in the form saying that there is an isomorphism of categories between the category of Poisson Hopf

algebras in \mathcal{C} and the category of i -braided lax monoidal functors $N: i\text{Com}(R) \rightarrow \mathcal{C}$ satisfying the strict nerve condition.

The main part of the proof is again a construction of a suitable $N_H: i\text{Com}(R) \rightarrow \mathcal{C}$ out of a Poisson Hopf algebra $H \in \mathcal{C}$. We already know the restriction of N_H to the subcategory $\text{Com} \subset i\text{Com}(R)$, which is the nerve of the commutative Hopf algebra H , and also its coherence morphisms (3).

We shall build the functor N_H in two steps. Let $r \in \text{End}_{\mathcal{C}}(H \otimes H)$ be

$$r = - \begin{array}{c} | \\ \bullet \\ | \end{array} \begin{array}{c} / \\ \backslash \end{array} \begin{array}{c} | \\ | \end{array}$$

and let

$$\tau = r + r^{op} + t_{H,H}$$

where $r^{op} = \sigma_{H,H} \circ r \circ \sigma_{H,H}$.

Lemma 6.2. *There is a unique R -linear symmetric strict monoidal functor*

$$F_H: i\text{Com}(R) \rightarrow \mathcal{C}$$

satisfying $F_H(\bullet) = H$, $F_H(\underline{m}) = m$, $F_H(\underline{\eta}) = \eta$, $F_H(t_{\bullet,\bullet}) = \tau$, where m and η are the product and the unit of H .

On objects the functor F_H is given by $F_H(\bullet^n) = H^{\otimes n}$ and on a typical morphism by

using the notation $\tau = \begin{array}{c} | \\ \text{---} \\ | \end{array}$.

Out of F_H we now build N_H as follows. On objects it is defined by

$$(11) \quad N_H(\bullet^n) = H^{\otimes(n-1)}, \quad N_H(1) = 1_{\mathcal{C}}.$$

To define N_H on morphisms, let $\iota_n: N_H(\bullet^n) \rightarrow F_H(\bullet^n)$ be the morphisms

$$\iota_0 = \text{id}: 1_{\mathcal{C}} \rightarrow 1_{\mathcal{C}}, \quad \iota_1 = \eta: 1_{\mathcal{C}} \rightarrow H$$

and for $n > 1$

$$\iota_n = \begin{array}{c} \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ | \quad | \quad | \quad | \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \end{array} \quad (n = 4).$$

Then N_H is determined by the requirement that ι_n 's form a natural transformation $N_H \rightarrow F_H$. If $\kappa_n: F_H(\bullet^n) \rightarrow N_H(\bullet^n)$ satisfy $\kappa_n \circ \iota_n = \text{id}$ then for any morphism $\phi: \bullet^p \rightarrow \bullet^q$ in $i\text{Com}(R)$ we have

$$(12) \quad N_H(\phi) = \kappa_q \circ F_H(\phi) \circ \iota_p.$$

For the morphisms κ_n we can take $\kappa_0 = \text{id}$ and for $n > 0$

$$\kappa_n = \begin{array}{c} | \quad | \quad | \quad | \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ | \quad | \quad | \quad | \end{array} \quad (n = 5)$$

where we use the notation $\epsilon = \mathbf{1}$.

As promised, we have the following result.

Lemma 6.3. *N_H , defined on objects by (11) and on morphisms by (12), is an i -braided lax monoidal functor satisfying the strict nerve condition.*

Let us now prove the two lemmas.

Proof of Lemma 6.2. We need to verify the identities

$$\begin{array}{c} \text{A cup with a red wavy line} = 0, \quad \text{A cup with a red wavy line and a vertical line} = \text{A cup with a red wavy line and a vertical line} + \text{A cup with a red wavy line and a vertical line} \end{array}$$

and that τ_{ij} 's satisfy the Drinfeld-Kohno relations, where $\tau_{ij} \in \text{End}(H^{\otimes n})$ is τ acting on the i 'th and j 'th H in $H^{\otimes n}$.

The first identity follows from (5). The second identity, saying that τ is a biderivation, follows from the fact that both $t_{H,H}$ and r are biderivations.

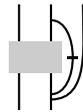
To show the Drinfeld-Kohno relations, let us first prove that τ is a morphism of H -comodules, i.e. that

$$(13) \quad \begin{array}{c} \text{A crossing with a red wavy line} = \text{A crossing with a red wavy line} \end{array}$$

This follows from

$$\begin{array}{c} \text{A crossing with a blue dot} = \text{A crossing with a blue dot} + \text{A crossing with a blue dot} + \text{A crossing with a red dot} \end{array}$$

which, in turn, is the identity (9) plugged into the gray rectangle of



Let us introduce the notation

$$\begin{array}{c} \text{A square with a crossing} = \text{A square with a blue dot} \quad \text{so that} \quad r = - \text{A square with a crossing} \end{array}$$

The Jacobi identity (6) implies after a straightforward calculation

$$\begin{array}{c} \text{A square with a crossing} - \text{A square with a crossing} = \text{A square with a blue dot} \end{array}$$

which, in turn, implies

$$(14) \quad \begin{array}{c} \text{A square with a crossing and a red wavy line} + \text{A square with a crossing and a red wavy line} = \text{A square with a crossing and a red wavy line} + \text{A square with a crossing and a red wavy line} \end{array}$$

We can finally prove the non-trivial Drinfeld-Kohno relation $[\tau_{12} + \tau_{13}, \tau_{23}] = 0$. It is the sum of the identities

$$\begin{aligned} [r_{12} + r_{13}, \tau_{23}] &= 0 \\ [r_{12}^{op} + r_{13}^{op}, \tau_{23}] &= 0 \\ [t_{12} + t_{13}, \tau_{23}] &= 0. \end{aligned}$$

The first one is

$$\begin{array}{c} \text{A square with a crossing and a red wavy line} = \text{A square with a crossing and a red wavy line} \end{array}$$

and so it follows from (13). The second one follows from (14), and the third one from $t_{12} + t_{13} = t_{H,H \otimes H}$ and from the fact that t is a natural transformation. \square

Proof of Lemma 6.3. Let us make H to a left H -comodule via the coaction $\Delta: H \rightarrow H \otimes H$. Then $F_N(\bullet^n) = H^{\otimes n}$ becomes an H -comodule as well, and (13) implies that for any morphism $\phi: \bullet^p \rightarrow \bullet^q$ in $\text{iCom}(R)$ the morphism $F_H(\phi)$ is a morphism of H -comodules.

The morphism ι_n is the equalizer of

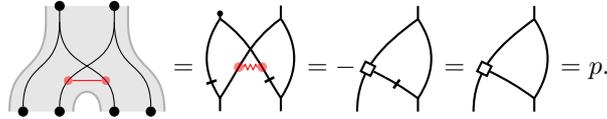
$$F_H(\bullet^n) \begin{array}{c} \xrightarrow{\text{coaction}} \\ \xrightarrow{\eta \otimes \text{id}} \end{array} H \otimes F_H(\bullet^n)$$

(i.e. it gives us the coinvariants of the coaction). This shows that N_H is indeed a functor.

The fact that N_H is symmetric lax monoidal with the coherence morphisms c_{\bullet^m, \bullet^n} given by (3) follows from the identity $\iota_{m+n} \circ c_{\bullet^m, \bullet^n} = \iota_m \otimes \iota_n$, which then means that ι_n 's form a monoidal natural transformation $N_H \rightarrow F_H$. The fact that N_H satisfies the strict nerve condition is evident. Finally, the fact that N_H is i-braided follows easily from seeing ι_n as the above-mentioned equalizer. \square

Once N_H is constructed, the rest of the proof is as in §6.1: The construction $H \mapsto N_H$ is functorial in H . If $N: \text{iCom}(R) \rightarrow \mathcal{C}$ satisfies the strict nerve condition then checking that $H_N := N(\bullet\bullet)$ (with the operations given in the theorem) is a Poisson Hopf algebra, i.e. that (9) is satisfied, is a simple manipulation with diagrams. Again the construction $N \mapsto H_N$ is functorial.

Let us verify that $H_{N_H} = H$ as Poisson Hopf algebras, i.e. that from N_H we get back the Poisson bracket p on H . By the definition of N_H and its coherence morphisms we have



It remains to check that $N_{H_N} = N$. The equality is true for morphisms from Com by Proposition 3.3 and one easily checks the equality for a single chord. These morphisms generate $\text{iCom}(R)$, and so the two functors are equal.

7. SEMICOMMUTATIVE HOPF ALGEBROIDS

The main idea of this paper was that groups (or commutative Hopf algebras) have symmetric nerves, and that deforming this symmetric structure to a braided structure provides their quantization. A natural idea is to extend it to other objects having symmetric nerves. The simplest option is to generalize groups to groupoids. The quantum object that we get is “quantum groupoids with a classical base”.

Definition 7.1 (Maltsonitis [10]). *A semicommutative Hopf algebroid over a commutative ring R is a pair of R -algebras B and H , with B commutative, with the following additional structure:*

- ▷ two R -algebra homomorphisms $\eta_L, \eta_R: B \rightarrow Z(H)$ (the center of H); this makes H to a B - B -bimodule (via $b_1 \otimes h \otimes b_2 \mapsto \eta_L(b_1) h \eta_R(b_2)$)
- ▷ B - B -bimodule morphisms $\Delta: H \rightarrow H \otimes_B H$ and $\epsilon: H \rightarrow B$ making H to a coalgebra in the monoidal category of B - B -bimodules, which are also algebra morphisms
- ▷ an invertible R -algebra anti-homomorphism $S: H \rightarrow H$ such that

$$S \circ \eta_L = \eta_R \quad S \circ \eta_R = \eta_L$$

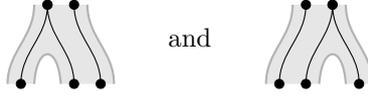
$$m \circ (\text{id}_H \otimes_B S) \circ \Delta = \epsilon \circ \eta_L, \quad m \circ (S \otimes_B \text{id}_H) \circ \Delta = \epsilon \circ \eta_R$$

where m is the product on H .

Remark 7.2. For simplicity we gave the definition in the category of R -modules. The definition and also the rest of this section can be generalized to BMCs where $H \otimes_B H$ and its iterations are well defined and behaved. The only change that needs to be done is that $\eta_L: B \rightarrow H$ should be a left-central and $\eta_R: B \rightarrow H$ a right central morphism of algebras.

A *commutative Hopf algebroid* corresponds to the case when H is commutative. A *semicommutative Poisson Hopf algebroid* is a commutative Hopf algebroid together with a Poisson bracket on H with the properties that the maps $\eta_{L,R}$ send B to the Poisson center of H and $\Delta: H \rightarrow H \otimes_B H$ and $\epsilon: H \rightarrow B$ are Poisson algebra morphisms, where the Poisson bracket on B is zero. The *quantization problem* is to deform m , Δ , and S (the rest of the structure is not deformed) so that we obtain a semicommutative Hopf algebroid and so that the deformed m is a quantization of the Poisson bracket on H .

This problem can again be solved using nerves. If $N: \text{BrCom} \rightarrow R\text{-mod}$ or $N: \text{Com} \rightarrow R\text{-mod}$ is a braided lax monoidal functor then $N(\bullet) \in R\text{-mod}$ is a commutative algebra and $N(\bullet\bullet)$ is its bimodule via the $N(\bullet)$ -actions



The map (4) is easily seen to factor through

$$(15) \quad \underbrace{N(\bullet\bullet) \otimes_{N(\bullet)} N(\bullet\bullet) \otimes_{N(\bullet)} \cdots \otimes_{N(\bullet)} N(\bullet\bullet)}_{n-1} \rightarrow N(\bullet^n).$$

We shall say that N satisfies the *Segal condition* [11] (or *groupoid nerve condition*) if (15) is an isomorphism for every n and if $c_1^N: R \rightarrow N(1)$ is an isomorphism.

Theorem 7.3. *The category of semicommutative Hopf algebroids over R with invertible antipodes is equivalent to the category of braided lax monoidal functors*

$$N: \text{BrCom} \rightarrow R\text{-mod}$$

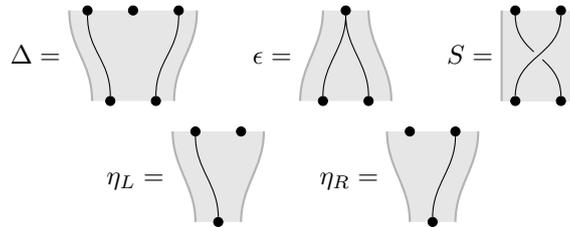
satisfying the Segal condition. The same is true for commutative Hopf algebroids and symmetric lax monoidal functors

$$N: \text{Com} \rightarrow R\text{-mod}.$$

The Hopf algebroid corresponding to N is

$$H = N(\bullet\bullet) \quad B = N(\bullet).$$

The algebra structure on H and B comes from the algebra structure of $\bullet\bullet$ and \bullet . The remaining operations are given by



where we implicitly use the isomorphism $N(\bullet^3) \cong N(\bullet\bullet) \otimes_{N(\bullet)} N(\bullet\bullet)$ given by the Segal condition.

Finally, semicommutative Poisson Hopf algebroids over R are equivalent to i -braided lax monoidal functors

$$N: i\text{Com}(R) \rightarrow R\text{-mod}$$

satisfying the Segal condition. The Poisson bracket on H comes from the Poisson bracket on $\bullet\bullet$.

The theorem can be proven by a suitable modification of the proofs in §6; we leave the details to the reader. Let us just mention a few hints. The algebra $B = N(\bullet)$ must be commutative since $\bullet \in \text{BrCom}$ is commutative and N is braided lax monoidal. Similarly one checks that η_L and η_R send $B = N(\bullet)$ to the center of $H = N(\bullet\bullet)$. The proof of Theorem 3.4 (§6.1) generalizes without any problems, if we set

$$N(\bullet^n) = \underbrace{H \otimes_B H \otimes_B \cdots \otimes_B H}_{n-1}$$

where \otimes_B is given in terms of the B -bimodule structure of H given by η_L and η_R . Likewise, the proof of Theorem 5.2 (§6.2) goes through by putting

$$F(\bullet^n) = \left(\bigotimes_B \right)_{i=1}^n H$$

where this time \otimes_B is defined using η_L only.

If $\mathbb{Q} \subset R$, we get from Theorem 7.3 immediately a solution of the quantization problem: If $N: \text{iCom}(R) \rightarrow R\text{-mod}$ is the nerve of a semicommutative Poisson Hopf algebroid then the composition of braided lax monoidal functors

$$\text{BrCom} \xrightarrow{U_\Phi} \text{iCom}(\mathbb{Q})_{\hbar}^\Phi \subset \text{iCom}(R)_{\hbar}^\Phi \xrightarrow{N_{\hbar}} R\text{-mod}_{\hbar}$$

satisfies the Segal condition and thus gives us a semicommutative Hopf algebroid.

An interesting question remains whether our method can be applied for quantization of other objects with symmetric nerves (higher groups or groupoids).

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Chapter 6

Further directions

Let us close this thesis by mentioning a few possible directions for future research.

- One question is about the relationship of the quantization of Li-Bland and Ševera [17] and the quantization of Ben-Zvi, Brochier and Jordan [13] using the factorization homology.

It should not be difficult to relate these two approaches: for $\mathcal{A} = U\mathfrak{g}\text{-mod}_\hbar^\Phi$, the algebra $T^R(\mathbb{C})$ of [13] should be the quantization of the moduli space on the disk with two marked points as described in Proposition 3.2.1. Moreover, in both papers, fusions are given by the tensor product functor.

If a relationship is precisely established, it would lead to a simplification of [13], since they use the annulus with one marked point as a building block, which needs to be “cut open” when making e.g. the punctured torus [13, Remark 5.9].

- Continuing in the same spirit, the quantizations of [6, 60] in terms of links in $\Sigma \times [0, 1]$ should also be related with [17]. For example, the approach of Andersen, Mattis and Reshetikhin [6] cuts the surface into squares and hexagons, which is essentially the same datum as a uni-trivalent spine. Moreover, they need to choose an even associator, or distinguish upper and lower parts of squares, which should correspond to the coloring by \pm of [17].

Thus, it is natural to conjecture that the product described in Proposition 3.4.6 (or a product given by other natural choice of coloring), acting on two functions given by holonomies along two curves on Σ , is the Kontsevich integral of [6] associated to the stacking of these two curves.

- As mentioned in Remark 3.4.4, the category $U\mathfrak{g}\text{-mod}_\hbar^\Phi$ also carries a natural transformation $e^{t_{11}/2}$ corresponding to a double-twist of the framing. It would be interesting to understand its role in the quantization.

- Turning to the element

$$a^{1/2} = Z \left(\left(\text{diagram} \right)^{1/2} \right),$$

it would be interesting to find its knot-theoretic or physical origin. The element a itself is related to the reflection around a horizontal line or to multiplication of chords by -1 (see Theorem 4.2.1), but the square root $a^{1/2}$ does not have such a straightforward interpretation.

- One concrete question about $a^{1/2}$ is the following: The associator $\Phi^{a^{1/2}}$ is even and satisfies an equation $r_3(\Phi^{a^{1/2}}) = 1 \in \mathcal{P}_2$ (Proposition 4.1.6), which is always satisfied by horizontal associators. Moreover, the twisting by a gives $\Phi^a = \Phi_{-t}$ (Theorem 4.2.1), which is horizontal. The question is whether $\Phi^{a^{1/2}}$ is horizontal, which would provide a horizontal associator “half-way” between Φ and Φ_{-t} (see also [31]).
- Coming to the second part of the thesis, in [59], we quantized Poisson-Hopf algebras by expressing them as their nerves and then precomposing these nerves with a functor coming from the Drinfeld associator.

These nerves of Poisson-Hopf algebras are infinitesimally-braided monoidal functors satisfying the nerve condition. If one drops the nerve condition, such functors should correspond to higher groups or higher groupoids (the case of groupoids is treated in [59]). It would be interesting to understand what is an analogue of a Poisson-Lie structure on a higher group, and what kind of object is its quantization, induced by the Drinfeld associator.

- Finally, as mentioned in Remark 5.1.2, the quantization of Poisson-Hopf algebras described in [59] hints at a more general topological field theory, where morphisms between surfaces are cobordisms with the marked points tracing curves inside of them. Moreover, the spines described in Chapter 3 should extend to triangulations of cobordisms. A flip move is then non-trivial triangulation of a cylinder, giving a three-dimensional interpretation to the associator.

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