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UNIVERSITÉ DE GENÈVE FACULTÉ DES SCIENCES

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On $\mathrm{Aut}\, F_n$ action on group presentations

THÈSE

présentée à la Faculté des Sciences de l'Université de Genève
pour obtenir le grade de
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par

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Curiosity... is insubordination in its purest form.

V. Nabokov

Abstract

This thesis concerns combinatorial and geometric group theory – that is, the study of the interplay between the properties of groups and those of the geometric objects on which they act. The main objective of the thesis is to investigate a classical object in group theory: the group of automorphisms of a free non-abelian group of finite rank and its action on finitely generated group presentations. For any finitely generated group, this action can be described by the Nielsen graph, also called the Product Replacement Graph. The Nielsen graph has become a subject of particular interest since the introduction of the related Product Replacement Algorithm. This is a "practical" algorithm for generating random elements in large finite groups, a highly non-trivial task in Computational Group Theory.

The main subjects of this thesis are the connectedness and the asymptotic behavior (non-amenability, to be more specific) of Nielsen graphs associated with specific groups.

Previous work on the subject was mostly focused on finite groups. The emphasis in this thesis is on investigating Nielsen graphs of infinite finitely generated groups. We consider a class of groups, which we call $\mathcal{MN}$, which includes the nilpotent groups as well as some exotic examples of infinite torsion groups acting on rooted trees. We show that for a finitely generated group in the class $\mathcal{MN}$, the Nielsen graphs can have quite different behavior: they may be connected or have infinitely many connected components. More precisely, we show that Nielsen graphs of Gupta-Sidki p-groups have infinitely many connected components; to the author's knowledge, it is the first examples of torsion groups with this property. Examples of finitely generated groups in $\mathcal{MN}$ for which Nielsen graphs are connected have already appeared in the literature; nevertheless, we exhibit new ones, namely the discrete Heisenberg groups.

The connectedness of Nielsen graphs, being an interesting question on its own, is also related to the famous Andrews-Curtis conjecture. This conjecture has been motivated by deep topological problems and remains open more than 50 years after it was formulated. In this context, we show that for a finitely generated group G in $\mathcal{MN}$, the connectedness of the Andrews-Curtis graph of G is equivalent to that of the Andrews-Curtis graph of the abelianization of G. This shows, in particular, that the class $\mathcal{MN}$ cannot provide counter-examples to the Andrews-Curtis conjecture.

The analysis of the geometry of Nielsen graphs is a rather new but fascinating and active subject. In particular, the question of amenability of infinite Nielsen graphs is closely related to the long-standing question of whether the group of automorphisms of a free non-abelian group of rank $n \geq 4$ has Property (T). We show, in collaboration with T. Smirnova-Nagnibeda, non-amenability of Nielsen graphs of indicable groups and infinite elementary amenable groups.

Résumé

La présente thèse porte sur la théorie combinatoire et géométrique des groupes, c'est-à-dire l'étude de l'interaction entre les propriétés des groupes et celles des objets géométriques sur lesquels ils agissent. L'objectif principal de ce travail est d'étudier un objet classique en théorie des groupes: le groupe des automorphismes d'un groupe libre non-abélien de rang fini et son action sur les présentations des groupes de type fini. Pour tout groupe de type fini, cette action peut être décrite par le graphe de Nielsen, aussi appelé "Product Replacement Graph". Le graphe de Nielsen est devenu un objet d'intérêt particulier depuis l'introduction du "Product Replacement algorithm". Il s'agit d'un algorithme "pratique" pour générer des éléments aléatoires dans un groupe fini – une tâche hautement non-triviale en théorie calculatoire des groupes.

Les sujets principaux de cette thèse sont la connexité et le comportement asymptotique (la non-moyennabilité, plus précisément) de graphes de Nielsen associés à des groupes spécifiques.

Les travaux antérieurs sur ces sujets ont principalement porté sur les groupes finis. Dans cette thèse, l'accent est mis sur l'étude des graphes de Nielsen de groupes infinis de type fini. Nous considérons une classe de groupes, que nous appelons la classe $\mathcal{MN}$, qui inclut les groupes nilpotents ainsi que des exemples exotiques de groupes infinis de torsion agissant sur des arbres enracinés. Nous montrons que, pour un groupe de type fini de la classe $\mathcal{MN}$, les graphes de Nielsen peuvent avoir des comportements assez différents: ils peuvent tout aussi bien être connexes qu'avoir un nombre infini de composantes connexes. Plus précisément, nous montrons que graphes de Nielsen des groupes de Gupta-Sidki ont un nombre infini de composantes connexes; à la connaissance de l'auteur, c'est le premier exemple d'un groupe de torsion avec cette propriété. Des exemples de groupes de type fini de $\mathcal{MN}$ dont les graphes de Nielsen sont connexes sont déjà apparus dans la littérature; néanmoins, on en exhibe des nouvelles, notamment les groupes de Heisenberg.

La question de la connexité des graphes de Nielsen, déjà intéressante en elle-même, est liée à la fameuse conjecture d'Andrews-Curtis. Cette conjecture a été motivée par des problèmes topologiques profonds et reste encore ouverte plus de 50 ans après sa formulation. Dans le contexte de ce problème, on montre que la classe $\mathcal{MN}$ ne peut pas fournir de contre-exemples à la conjecture d'Andrews-Curtis.

L'analyse de la géométrie des graphes de Nielsen est un sujet assez nouveau, mais fascinant et actif. En particulier, la question de la moyennabilité des graphes de Nielsen infinis est étroitement liée à la question, ouverte depuis longtemps, de savoir si le groupe des automorphismes d'un groupe libre non-abélien de rang $n \geq 4$ possède la Propriété (T). Nous montrons, en collaboration avec T. Smirnova-Nagnibeda, la non-moyennabilité des graphes de Nielsen pour les groupes indicables et les groupes élémentairement moyennables infinis.

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List of Symbols

Group theory :

G	-	finitely generated group
F_n	-	free group on n generators
$\mathrm{Aut}\, G$	-	automorphism group of G
$\langle S \rangle$	-	group generated by the set S
$\ll S \gg$ or S^G	-	normal closure of the set S in G
$\mathrm{rank}(G)$	-	rank of G, *i.e.* the minimal number of generators of G
$w(G)$	-	weight of G, *i.e.* the minimal number of normal generators of G
$G^n = \mathrm{Hom(F_n, G)}$	-	set of n-tuples of G
$\mathrm{Epi(F_n, G)}$	-	set of generating n-tuples of G
$Sch(G, M, S)$	-	Schreier graph of a group G with a generating set S, acting on a set M
$G \curvearrowright X$	-	action of G on a set X
$\mathcal{MN}$	-	class of groups with the property of having all maximal subgroups normal
$\Phi(G)$	-	Frattini subgroup of G
$[G, G]$	-	commutator subgroup of G
$St_G(\sigma)$	-	vertex stabilizer subgroup of a group G acting on a rooted tree
$St_G(n)$	-	n-th level stabilizer subgroup of a group G acting on a rooted tree
$H \wr_X G$	-	permutational restricted wreath product
Γ	-	first Grigorchuk group
G_p	-	Gupta-Sidki p-group
$\mathcal{H}_k$	-	discrete Heisenberg group of rank $2k$

Hilbert space and unitary representations :

$\mathcal{H}$	-	complex Hilbert space
$\mathcal{U}(\mathcal{H})$	-	unitary group of $\mathcal{H}$
$(\pi, \mathcal{H})$	-	unitary representation of G on $\mathcal{H}$
$l^2(X)$	-	Hilbert space of square-integrable complex valued functions on X

$(\lambda_X, l^2(X))$	-	left unitary representation of G on $l^2(X)$

Graph theory :

$R_{ij}^{\pm}, L_{ij}^{\pm}, I_j$	-	elementary Nielsen moves
$AC_{i,s}$	-	elementary Andrews-Curtis moves
$N_n(G), n \geq \text{rank}(G)$	-	Nielsen graphs of G
$AC_n(G), n \geq w(G)$	-	Andrews-Curtis graphs of G
$\partial_X(S)$	-	boundary of a vertex set S in a graph X
$h(X)$	-	Cheeger constant (isoperimetric constant) of a graph X
$\rho(X)$	-	spectral radius of a graph X

Contents

CHAPTER 1

Introduction

Several aspects of combinatorial and geometric group theory motivate the study of the action of $\operatorname{Aut} F_n$ – the automorphism group of the free group F_n on n generators – on the set $\operatorname{Epi}(F_n, G)$ of epimorphisms from F_n to some finitely generated group G. In the thesis we discuss various features of this action, including transitivity and non-amenability.

1.1 Background on Nielsen equivalence

Let G be a finitely generated group. The following transformations of the set $G^n, n \geq 1$, were introduced by J. Nielsen in [76] and are known as *elementary Nielsen moves*:

$$\begin{aligned} R_{ij}^{\pm}(g_1, \ldots, g_i, \ldots, g_j, \ldots, g_n) &= (g_1, \ldots, g_i g_j^{\pm 1}, \ldots, g_j, \ldots, g_n), \\ L_{ij}^{\pm}(g_1, \ldots, g_i, \ldots, g_j, \ldots, g_n) &= (g_1, \ldots, g_j^{\pm 1} g_i, \ldots, g_j, \ldots, g_n), \\ I_j(g_1, \ldots, g_j, \ldots, g_n) &= (g_1, \ldots, g_j^{-1}, \ldots, g_n), \end{aligned}$$

where $1 \leq i, j \leq n$, $i \neq j$.

These transformations can be seen as elements of $\operatorname{Aut} F_n$; Nielsen proved in [76] that they generate $\operatorname{Aut} F_n$. Notice that elementary Nielsen moves transform generating sets of G into generating sets. Two generating sets U and V are called *Nielsen equivalent* ($U \sim V$) if one is obtained from the other by a finite chain of elementary Nielsen moves. Notice that Nielsen equivalence is preserved by a permutation of entries of a generating set.

The rank $\operatorname{rank}(G)$ of G is the minimal number of generators of G. Fix $n \geq \operatorname{rank}(G)$.

We define *the Nielsen graph** $N_n(G)$ as follows:

- the set of vertices consists of the generating n-tuples in G, *i.e.*

$$V_{N_n(G)} = \{(g_1, ..., g_n) \in G^n \mid \langle g_1, ..., g_n \rangle = G\};$$

- two vertices are connected by an edge if one of the n-tuples is obtained from the other by an elementary Nielsen move.

Observe, that the set G^n of n-tuples in G can be identified with the set of homomorphisms from the free group F_n to G, and the set $V_{N_n(G)}$ of generating n-tuples can be identified with the set of epimorphisms $\mathrm{Epi}(F_n, G)$. Thus there is a natural action of $\mathrm{Aut}\, F_n$ on $\mathrm{Epi}(F_n, G)$ given by precomposition; this action is described by the Nielsen graph. In particular, the graph $N_n(G)$ is connected if and only if the action of $\mathrm{Aut}\, F_n$ on $\mathrm{Epi}(F_n, G)$ is transitive. The orbits of the action $\mathrm{Aut}\, F_n \curvearrowright \mathrm{Epi}(F_n, G)$ are called *Nielsen (equivalence) classes.*

An important question concerning the properties of Nielsen graphs is whether (or when) the Nielsen graph is connected. Furthermore, if it is not connected, questions about its connected components arise, such as whether all of them are isomorphic, whether they are all infinite if G is infinite, *etc.* Moreover there are questions about the geometry of these graphs, for instance, what is their growth, whether they are amenable or not, *etc.*

In recent years the connectedness of Nielsen graphs has been of particular interest since these graphs are used in a common heuristic Product Replacement Algorithm generating random group elements in a finite group. The connectedness of Nielsen graphs was extensively studied and we refer the reader to the surveys on the topic [32, 58, 81]. Historically, the question on transitivity of the action $\mathrm{Aut}\, F_n \curvearrowright \mathrm{Epi}(F_n, G)$ arose in the following framework. For a finitely generated group G consider the natural action of the group $\mathrm{Aut}\, F_n \times \mathrm{Aut}\, G$ on $\mathrm{Epi}(F_n, G)$: for $(\tau, \sigma) \in \mathrm{Aut}\, F_n \times \mathrm{Aut}\, G$ and for $\phi \in \mathrm{Epi}(F_n, G)$ define

$$\phi^{(\tau,\sigma)} = \sigma^{-1} \cdot \phi \cdot \tau.$$

Motivated by the study of presentations of finite groups, B.H. Neumann and H. Neumann defined in [75] T_n*-systems* (also called *systems of transitivity*) to be the orbits of this action.

For a given finitely generated group G, the number of T_n-systems is clearly bounded above by the number connected components of the Nielsen graph $N_n(G)$ for every $n \geq \mathrm{rank}(G)$. Let us take as an example $G = \mathbb{Z}^n$. The set of generating n-tuples of $\mathbb{Z}^n$ coincides with $GL(n, \mathbb{Z})$ and the elementary Nielsen moves induce elementary row operations on these matrices. It follows that the action of $\mathrm{Aut}\, F_n$ on $\mathrm{Epi}(F_n, \mathbb{Z}^n)$ is transitive; and in particular we conclude that there is only one T_n-system for $\mathbb{Z}^n$. On the other hand, if $G = \mathbb{Z}/p\mathbb{Z}$ for some prime $p \geq 5$ then it is

*Also called the (Extended) Product Replacement Graph.

easy to see that there are exactly $\frac{p-1}{2}$ Nielsen classes of generating 1-tuples, while the T_1-system is unique.

Related questions to those of determining T_n-systems were considered in [47] where the authors obtained a classification of epimorphisms from fundamental groups of surfaces to free groups.

One of the main conjectures concerning the connectedness of Nielsen graphs for finite groups is that the action of $\operatorname{Aut} F_n$ on generating n-tuples of a finite simple group with $n \geq 3$ is transitive[†]. This conjecture, very often attributed to Wiegold, was proven in some particular cases in [37, 31, 66, 34, 7, 81]. It is still open in full generality. Pak [80] observed that one indeed expects a large connected component in the Nielsen graph $N_n(G)$ for a finite simple group G when $n \geq 3$, more precisely the graph $\mathcal{G} = N_n(G)$ contains a connected component $\mathcal{G}'$ such that $\frac{|V(\mathcal{G}')|}{|V(\mathcal{G})|} \to 1$ when $|G| \to \infty$. In fact, even a stronger version of the Wiegold conjecture is still open: we do not know whether all Nielsen graphs $N_n(G)$ are connected for G finite and $n \geq \operatorname{rank}(G) + 1$.

For an infinite finitely generated group one does not necessarily expect the Nielsen graph $N_n(G)$ to be connected when $n \geq \operatorname{rank}(G) + 1$ (see [30]). It was first shown by Noskov ([77]) that there exists a metabelian group with non-minimal generating tuple that is not Nielsen equivalent to a generating tuple containing the trivial element. The result of Noskov also produces a negative answer to *the Waldhausen question* (see [60, page 92]); Waldhausen asked whether $\operatorname{rank}(F_n/N) < n$ *implies that N contains a primitive element of F_n*, where N is a normal subgroup of F_n.

For a given group G generated by a finite set S, and a set M with a transitive action of G on M, one can define the Schreier graph $Sch(G, M, S)$: the vertex set of the graph is M, and there is an edge connecting m_1 to m_2 for each $s \in S \cup S^{-1}$ that maps m_1 to m_2. Hence, if the action of $\operatorname{Aut} F_n$ on $\operatorname{Epi}(F_n, G)$ is transitive, then $N_n(G)$ is precisely the Schreier graph of $\operatorname{Aut} F_n$ acting on $\operatorname{Epi}(F_n, G)$ with respect to the elementary Nielsen moves. The set $\operatorname{Epi}(F_n, G)$ can also be understood as the set of left cosets of the stabilizer subgroup $St_{\operatorname{Aut} F_n}(g_1, \ldots, g_n)$ for some (any) generating n-tuple $(g_1, \ldots, g_n) \in G^n$, and $N_n(G)$ is thus the Schreier graph with respect to this subgroup in $\operatorname{Aut} F_n$. More generally, if the action is not transitive, *every connected component of $N_n(G)$ is the Schreier graph of* $\operatorname{Aut} F_n$ *with respect to the corresponding subgroup $St_{\operatorname{Aut} F_n}(g_1, \ldots, g_n)$, for any generating n-tuple $(g_1, \ldots, g_n)$ belonging to the considered connected component.* As any Schreier graph, every connected component of $N_n(G)$ comes with an orientation and a labeling of its edges by elements of the generating set. The set of elementary Nielsen moves being symmetric, orientation can be disregarded in this case.

For certain groups, for instance relatively free groups, the connected components of Nielsen graphs are actually Cayley graphs, see Theorem 3.3.2. A group is called

[†]The classification of finite simple groups implies that every finite simple group can be generated by 2 elements.

relatively free if it is free in a variety of groups (see Section 3.3 for a more detailed definition). Examples of relatively free groups include free groups, free abelian groups, free nilpotent groups $F_{d,c}$ of rank d and nilpotency class c, free solvable groups $F_{d,l}$ of rank d and derived length l, free Burnside groups $B(d,m)$ of rank d and exponent m and others.
We observe in Theorem 3.3.2 that for a relatively free group G of rank d every connected component of the Nielsen graph is isomorphic to the Cayley graph of the subgroup $T(G) \leq \operatorname{Aut} G$ of *tame automorphisms* of G. (An automorphism of a relatively free group G of rank d is *tame* if it lies in the image of the natural homomorphism $\operatorname{Aut} F_d \to \operatorname{Aut} G$). The question on tameness of automorphisms of a relatively free group has been extensively studied by Andreadakis [5], Bachmuth [8], Gupta [51], Mochizuki [9], Moriah and Shpilrain [67], Papistas [82] and many others.

1.2 Nielsen equivalence in some classes of finitely generated infinite groups

Nielsen equivalence has also been studied in finitely generated infinite groups. We are interested in the class of groups with the property of having all maximal subgroups normal. We call it the class $\mathcal{MN}$. A finite group is in $\mathcal{MN}$ if and only if it is nilpotent. Also all nilpotent groups belong to $\mathcal{MN}$ since they satisfy the normalizer condition (*i.e.* for any subgroup H, its normalizer $N_G(H)$ contains H properly). Infinite groups in $\mathcal{MN}$ also include some exotic examples of groups, such as infinite torsion groups acting by automorphisms on rooted trees. In particular, certain branch groups such as the first Grigrochuk group Γ [43, 44] and its generalisations $\{\Gamma_\omega\}$ where $\omega \in \{0, 1, 2, \}^{\mathbb{N}}$ contains all three letters $\{0, 1, 2\}$ infinitely many times, as well as Gupta-Sidki p-groups [53] belong to $\mathcal{MN}$ by [83, 85]. In [3] it was proven that all multi-edge spinal torsion groups acting on a regular p-ary rooted tree, with p odd prime, including Gupta-Sidki p-groups, belong to $\mathcal{MN}$.

Before formulating and discussing our results, we will describe what is known about Nielsen equivalence for some finitely generated groups in the class $\mathcal{MN}$.

The Nielsen equivalence in finitely generated abelian groups is fully understood (see [75, 27, 79]). Namely, if A is a finitely generated abelian group then the action of $\operatorname{Aut} F_n$ on $\operatorname{Epi}(F_n, A)$ is transitive when $n \geq \operatorname{rank}(A) + 1$. When $n = \operatorname{rank}(A)$ the number of Nielsen equivalence classes is finite and depends on the primary decomposition of A (see Theorem 3.2.1 for details). In addition, all connected components of the Nielsen graph $N_{\operatorname{rank}(A)}(A)$ are isomorphic (Corollary 3.2.3). It also follows from [75, 27, 79] that for any finitely generated abelian group there is only one T_n-system for every $n \geq \operatorname{rank}(A)$.

Further, for a finitely generated nilpotent group the action of $\operatorname{Aut} F_n$ is still transitive on $\operatorname{Epi}(F_n, G)$ when $n \geq \operatorname{rank}(G) + 1$ [30]. However when $n = \operatorname{rank}(G)$ the Nielsen equivalence classes are far from being classified. There are known examples of groups for which in case of $n = \operatorname{rank}(G)$ the unicity of Nielsen equivalence class, and even that of T_n-system, breaks down. For instance, Dunwoody [29] showed that for every pair of integers $n > 1$ and $N > 0$ there exists a finite nilpotent group of rank n and nilpotency class 2 for which there are at least N T_n-systems.

In Section 3.1 we introduce and study a natural generalisation of the class of nilpotent groups: the previously defined class $\mathcal{MN}$. Section 3.2[‡] is devoted to the study of Nielsen equivalence for finitely generated groups in $\mathcal{MN}$. We begin by generalising the observation by Evans on connectedness of the Nielsen graph for nilpotent groups to the finitely generated groups in the class $\mathcal{MN}$ when $n \geq rank(G) + 1$.

Proposition 1.2.1. *Let G be a finitely generated group in $\mathcal{MN}$. The Nielsen graph $N_n(G)$ for $n \geq \operatorname{rank}(G) + 1$ is connected.*

We then turn our attention to the case $n = \operatorname{rank}(G)$. We show that examples of

[‡]The results of Section 3.2 were published in [69, 70].

nilpotent groups for which $N_{\text{rank}(G)}(G)$ is connected include the discrete Heisenberg group and its generalisations to higher dimensions. The discrete Heisenberg group $\mathcal{H}_k$ is the group of integer upper-triangular square matrices of size $k+2$, with 1's on the diagonal and with non-zero entries only in the first row and the last column. The proof of Theorem 1.2.2 can found on the page 31.

Theorem 1.2.2. *Let $\mathcal{H}_k$ be the discrete Heisenberg group of rank $2k$. The graph $N_n(\mathcal{H}_k)$ is connected for $n \geq \text{rank}(\mathcal{H}_k) = 2k$.*

For a finitely generated group belonging to $\mathcal{MN}$, it is relevant to analyse Nielsen equivalence classes in the quotient $G/\Phi(G)$, where $\Phi(G)$ is the Frattini subgroup of G. Namely, if there are two generating n-tuples of $G/\Phi(G)$ which *are not* Nielsen equivalent, then their preimages in G are generating n-tuples of G which *are also not* Nielsen equivalent.
This becomes relevant when one considers finitely generated p-groups in the class $\mathcal{MN}$, such as branch Grigorchuk groups and Gupta-Sidki p-groups. For a finitely generated p-group G in $\mathcal{MN}$ the quotient $G/\Phi(G)$ is isomorphic to the abelian group $(\mathbb{Z}/p\mathbb{Z})^{\text{rank}(G)}$. By the classification of Nielsen equivalence classes in abelian groups, the Nielsen graph $N_{\text{rank}(G)}((\mathbb{Z}/p\mathbb{Z})^{\text{rank}(G)})$ has $\frac{p-1}{2}$ connected components for $p > 3$ and is connected for $p = 3$ or $p = 2$. We conclude that for a finitely generated p-group G in $\mathcal{MN}$ with $p > 3$ there are at least $\frac{p-1}{2}$ Nielsen classes in G. However when $p = 3$ (*e.g.* the Gupta-Sidki 3-group G_3) or $p = 2$ (*e.g.* the Grigorchuk group Γ), the question on transitivity of the action $\text{Aut}\, F_n \curvearrowright \text{Epi}(F_n, G)$ when $n = \text{rank}(G)$ is more subtle. We will show in particular that, although there is only one Nielsen class of generating pairs of $G_3/\Phi(G_3) = (\mathbb{Z}/3\mathbb{Z})^2$, the action of $\text{Aut}\, F_2$ is not transitive on $\text{Epi}(F_2, G_3)$. A natural question then is how many orbits this action has.

We show that for the Gupta-Sidki p-group, with $p \geq 3$ prime, there are infinitely many Nielsen classes when $n = \text{rank}(G_p) = 2$. There are other known examples of groups G with infinitely many Nielsen classes when $n = \text{rank}(G)$. Such examples can be found among fundamental groups of certain knots ([93, 55]), one-relator groups ([17]), and relatively free polynilpotent groups (see Section 3.2 and Section 3.3). The Gupta-Sidki groups are, to the author's knowledge, the first known examples of torsion groups that have this property. The proof of Theorem 1.2.3 can found on the page 38.

Theorem 1.2.3. *Let $p \geq 3$ be prime and G_p the Gupta-Sidki p-group. The Nielsen graph $N_2(G_p)$ has infinitely many connected components.*

It would be interesting to determine whether there are finitely or infinitely many T_2-systems of generating pairs of G_p.

The question on connectedness of the Nielsen graph $N_3(\Gamma)$ for the Grigorchuk group Γ[§] remains open. However, as we show in Chapter 1.3, if one adds another

[§]Recall that the Grigorchuk group Γ is an infinite 2-group of rank 3.

type of transformations to the elementary Nielsen moves (the elementary Andrews-Curtis moves), then the resulting graph (the Andrews-Curtis graph) with the set of vertices $\mathrm{Epi}(F_3, \Gamma)$ is connected.

In Section 3.3 we study Nielsen graphs of relatively free groups. We observe in Theorem 3.3.2 that for a relatively free group G of rank d every connected component of the Nielsen graph is isomorphic to the Cayley graph of the subgroup $T(G) \leq \mathrm{Aut}\, G$ of *tame automorphisms* of G. This implies, in particular, that the number of connected components of the Nielsen graph $N_d(G)$ is equal to the index of the subgroup $T(G)$ in $\mathrm{Aut}\, G$. Thus the question on Nielsen equivalence in a relatively free group G is reduced to the question on tameness of automorphisms of G.

Corollary 1.2.4. *Let G be a relatively free group of rank d. Then the Nielsen graph $N_d(G)$ is connected if and only if all automorphisms of G are tame. Moreover, all connected components of $N_d(G)$ are isomorphic.*

We use known results on tameness of automorphisms of relatively free groups and the criteria above to examine Nielsen graphs of various classes of relatively free groups such as free abelian groups, free nilpotent groups, free metabelian groups, free Burnside groups and others.

1.3 Andrews-Curtis equivalence

To the finitely generated group G we may associate another equivalence relation: the Andrews–Curtis equivalence. It is closely linked to the famous Andrews–Curtis conjecture, which remains open more than 50 years after it was formulated. To introduce the equivalence we need to define another type of transformations on the set of n-tuples of G.

Let S be a finite symmetric ($S = S^{-1}$) generating set of a group G. Elementary Nielsen moves together with the transformations

$$AC_{i,s}(g_1, ..., g_i, ..., g_n) = (g_1, ..., s^{-1}g_i s, ..., g_n)$$

where $1 \leq i \leq n$, $s \in S$, form the set of *elementary Andrews-Curtis moves*, or shortly AC-moves. Elementary AC-moves transform *normally generating sets* (sets which generate G as a normal subgroup) into normally generating sets. Two normally generating sets are called *Andrews-Curtis equivalent* (AC equivalent) if one is obtained from the other by a finite chain of elementary AC-moves. Note that since the group G is generated by S, a normally generating n-tuple $(g_1, \ldots, g_i, \ldots, g_n)$ is AC equivalent to $(g_1, \ldots, g^{-1}g_i g, \ldots, g_n)$ for every $g \in G$.

The Andrews–Curtis conjecture [6] is stated as follows.

Conjecture 1.3.1 (The Andrews–Curtis conjecture)**.** *Let F_n be the free group of rank $n \geq 2$. If $\{x_1, ..., x_n\}$ is a free basis and $\{y_1, ..., y_n\}$ is a normally generating set of F_n, then $(y_1, ..., y_n)$ and $(x_1, ..., x_n)$ are Andrews–Curtis equivalent.*

The Andrews-Curtis conjecture, motivated by deep topological problems, seems to be a natural extension of the theorem of Nielsen. However there are doubts about the validity of this conjecture and there have been several attempts to construct counter-examples (see Chapter 4 on Andrews–Curtis equivalence for more on that). A counter-example to the conjecture would be a normally generating n-tuple $(y_1, \ldots, y_n)$ of F_n which is not AC equivalent to the fixed basis $(x_1, \ldots, x_n)$. Only few potential counter-examples have been suggested, and none confirmed. A way to confirm such a potential counter-example is to show that its image in a quotient G of F_n and that of the basis $(x_1, \ldots, x_n)$ are not AC equivalent. This motivates the analysis of the Andrews–Curtis equivalence in finitely generated groups. One of the few positive results in this direction is that for the free solvable group G of rank n, all normally-generating n-tuples of G are Andrews–Curtis equivalent [68]. Its proof can be adapted to free nilpotent groups.

One of the potential counter-examples was proposed by Akbulut and Kirby [2] for the free group $F_2 = \langle x, y \rangle$. The pairs

$$(u, v_l) = (xyxy^{-1}x^{-1}y^{-1}, x^l y^{-(l+1)})$$

with $l \geq 4$ are still not known to be Andrews-Curtis equivalent to (x, y). There was a hope to confirm potential counter-examples in finite groups, however Borovik, Lubotzky and Myasnikov [16] showed that this is not possible by proving the following. They showed that for a finite group G two normally generating n-tuples are Andrews–Curtis equivalent if and only if they are Nielsen equivalent in the abelianization G^{ab} – the "largest" abelian quotient of G. This result, along with the observation that any two normally generating n-tuples of F_n are Nielsen equivalent in an abelian group, implies that one cannot confirm potential counter-examples to the Andrews-Curtis conjecture in finite groups.

Borovik, Lubotzky and Myasnikov in [16] further raised the question of whether the same result on Andrews-Curtis equivalence for finite groups also holds for the Grigorchuk group [43]. The Grigorchuk group is an infinite 2-group and its all proper quotients are finite. In Chapter 4 we investigate the Andrews–Curtis equivalence for finitely generated groups in the class $\mathcal{MN}$ of groups with the property that all the maximal subgroups are normal. In particular, we give an affirmative answer to the question in [16]. This result was published in the paper [69]. The proof of Theorem 1.3.2 can be found on the page 48.

Theorem 1.3.2. *Let G be a finitely generated group in $\mathcal{MN}$ and $n \geq \operatorname{rank}(G)$. Then two normally-generating n-tuples of G are Andrews–Curtis equivalent if and only if their images are Nielsen equivalent in the abelianization $G^{ab} = G/[G,G]$.*

A full description of Nielsen equivalence classes of finitely generated abelian groups is given in [27, 79]. Along with Theorem 1.3.2, this provides a characterisation of the Andrews–Curtis equivalence for finitely generated groups in $\mathcal{MN}$.

Observe that, for a finitely generated group G in $\mathcal{MN}$, a normally generating set of G is in fact a generating set. Therefore, for groups in $\mathcal{MN}$ the partition of the set of generating n-tuples into Nielsen equivalence classes is a refinement of the partition into Andrews–Curtis classes. From Theorem 1.3.2 and from the fact that for the Grigorchuk group Γ its abelianization Γ^{ab} is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$, we deduce the following.

Corollary 1.3.3. *For the Grigorchuk group Γ all generating 3-tuples of Γ are Andrews-Curtis equivalent.*

The proof of Corollary 1.3.3 can be found on the page 49.

1.4 Non-amenability of $\operatorname{Aut} F_n$ action on group presentations

The study of Nielsen graphs has become a subject of a particular interest prompted by a commonly used "practical" algorithm for generating random elements in finite groups, invented by Leedham-Green and Soicher. This algorithm, which consists of running a random walk on the Nielsen graph, has shown an excellent performance [18] although there is no rigorous justification for it. It was suggested and proven in several cases that Nielsen graphs $N_n(G)$ are expanders for n fixed and $|G| \to \infty$ [59, 33]. Expansion in the context of infinite graphs translates into non-amenability. This is what we study in Chapter 5.

We explain below some of the basic notions and the motivation for studying (non)amenability of Nielsen graphs. We then discuss non-amenability of Nielsen graphs for certain classes of finitely generated groups.

As in [26], a locally finite connected graph X of uniformly bounded degree is *amenable*, if either X is finite or

$$h(X) := \inf_{S \subset V(X)} \frac{|\partial_X(S)|}{|S|} = 0,$$

where the infimum is taken over all finite nonempty subsets S of the set of vertices $V(X)$ and $\partial_X(S)$ is the set of all edges connecting S to its complement. The number $h(X) \geq 0$ is called the *isoperimetric constant* (or the Cheeger constant) of X. A graph with several connected components is amenable if at least one of the connected components is amenable.

We will also use Kesten's characterization of amenable graphs (see *e.g.* [92, 10.3] for the extension of Kesten's criterion of amenability to all connected regular graphs). A connected m-regular graph X is *amenable* if and only if $\rho(X) = 1$, where $\rho(X) = 1/m \limsup_{k\to\infty} a_k^{1/k} \leq 1$ is the *spectral radius* of X, with $a_k(x)$ denoting the number of closed paths of length k in X, based at some (any) vertex of X.

Recall that every connected component of the Nielsen graph $N_n(G)$ is, in fact, the Schreier graph of $\operatorname{Aut} F_n$ with respect to the corresponding stabilizer subgroup $St_{\operatorname{Aut} F_n}(g_1, \ldots, g_n)$, where the generating n-tuple $(g_1, \ldots, g_n)$ belongs to the considered connected component. The question of (non)amenability of infinite Nielsen graphs is of particular interest in relation with the open problem about Property (T) for $\operatorname{Aut} F_n$, $n \geq 4$ [59] (the answer is negative for $n \leq 3$, see [49]). Namely, if a group G has Property (T) then G does not admit any amenable transitive action on an infinite countable set X; in other words, every infinite Schreier graph of G is non-amenable. This follows from the well-known amenability criterion in terms of existence of almost invariant vectors for the action of G on $l^2(X)$ (see Appendix A).

Below, we discuss the results from Chapter 5 on non-amenability of Nielsen graphs for certain families of groups. These results were published in [71].

In Chapter 5 we describe in detail the structure of Nielsen graphs $N_n(\mathbb{Z})$ for $n \geq 1$. It allows us to deduce non-amenability of all Nielsen graphs $N_n(G)$ with

$n \geq \max\{2, \operatorname{rank}(G)\}$, for finitely generated groups G that admit an epimorphism onto $\mathbb{Z}$ (such groups are called *indicable*). The proof of Theorem 1.4.1 can be found on the page 57.

Theorem 1.4.1. *Let G be a finitely generated indicable group. Then Nielsen graphs $N_n(G)$ with $n \geq \max\{2, \operatorname{rank}(G)\}$ are non-amenable.*

Further we discuss non-amenability of Nielsen graphs for infinite finitely generated elementary amenable groups. In particular we describe in detail the structure of the Nielsen graphs of the infinite dihedral group. We deduce then that the Nielsen graph $N_n(G)$ of an infinite elementary amenable group G is non-amenable for n large enough. The proof of Theorem 1.4.2 can be found on the page 63.

Theorem 1.4.2. *Let G be an infinite finitely generated elementary amenable group. Then G admits an epimorphism onto a group H that contains a normal subgroup isomorphic to $\mathbb{Z}^d$, $d \geq 1$, of finite index $i \geq 1$. As a consequence, Nielsen graphs $N_n(G)$ are non-amenable for $n \geq \operatorname{rank}(G) + \log_2 i + 1$.*

Related results in this direction appear also in a preprint [62] by Malyshev.

In the last part of Chapter 5 we obtain the following criteria of non-amenability of Nielsen graphs for relatively-free groups.

Corollary 1.4.3. *Let G be a relatively free group of rank d. Then the Nielsen graph $N_d(G)$ is non-amenable if and only if the group $T(G)$ of tame automorphisms of G is non-amenable.*

The examples considered in Section 3.3 include polynilpotent groups (which are indicable, so their Nielsen graphs are non-amenable by Theorem 1.4.1) and relatively free Burnside groups.

The free Burnside group $B(d, m)$ of rank d and exponent m is the group satisfying the law $x^m = 1$. These groups are torsion and thus cannot be indicable. By a famous result of Novikov and Adyan [78] we know that, for any $d \geq 2$ and m odd and large enough, these groups are infinite. Adyan further showed [1] that $B(d, m)$ are non-amenable for any $d \geq 2$ and odd $m \geq 665$. Hence our Theorems 1.4.1 and 1.4.2 are not applicable in this case. Nonamenability of $N_n(B(d, m))$ for all $n \geq d \geq 3$ and m odd and large enough is proven by Malyshev [62] using uniform nonamenability of $B(d, m)$. Using the work of Coulon [24] on automorphisms of free Burnside groups we deduce from Corollary 1.2.4 and Corollary 1.4.3 the following.

Corollary 1.4.4. *Let $B(d, m)$ denote the free Burnside group on d generators of exponent m. For $d \geq 3$ and m odd and large enough all connected components of $N_d(B(d, m))$ are isomorphic and non-amenable.*

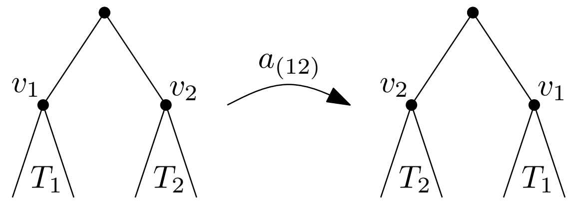
CHAPTER 2

Preliminaries on groups acting on rooted trees

In this chapter we will discuss groups acting on rooted trees. The aim of the chapter is to provide fundamental definitions and properties of these groups which will be used in further chapters. Groups acting on rooted trees include striking examples of infinite p-groups, groups of intermediate growth, amenable but not elementary amenable groups of exponential growth, *etc.*

2.1 Rooted trees and their automorphisms

Let $X = \{1, 2, \ldots, d\}$ with $d \geq 2$ be a finite set. The vertex set of the rooted tree T_d is the set of finite sequences $\{x_1 x_2 \ldots x_k : x_i \in X\}$ over X; two sequences are connected by an edge when one can be obtained from the other by right-adjunction of a letter in X. The top node (the root) is the empty sequence $\emptyset$, and the children of σ are all the σs for $s \in X$. A map $f : T_d \to T_d$ is an *automorphism of the tree* T_d if it is bijective and it preserves the root and adjacency of the vertices. An example of an automorphism of T_d is the rooted automorphism a_π, defined as follows: for the permutation $\pi \in Sym(d)$, set $a_\pi(s\sigma) := \pi(s)\sigma$. Geometrically it can be viewed as the permutation of d subtrees just below the root $\emptyset$.



Denote by $\operatorname{Aut} T_d$ *the group of automorphisms of the tree* T_d and let $G \leq \operatorname{Aut} T_d$. Denote by $St_G(\sigma)$ the subgroup of G, called *the vertex stabilizer*, consisting of the automorphisms that fix the sequence σ, *i.e.*

$$St_G(\sigma) = \{g \in G \mid g(\sigma) = \sigma\}.$$

Denote by $St_G(n)$ the subgroup of G, called *the n-th level stabilizer*, consisting of the automorphisms that fix all sequences of length n, *i.e.*

$$St_G(n) = \cap_{\sigma \in X^n} St_G(\sigma).$$

Notice an obvious inclusion $St_G(n+1) \leq St_G(n)$. Moreover, observe that for every $n \geq 0$ the subgroups $St_G(n)$ are normal and of finite index in G. We therefore have a natural epimorphism between finite groups

$$G/St_G(n+1) \to G/St_G(n), \tag{2.1}$$

for any $n \geq 0$.

The *rigid stabilizer* of a vertex $v \in T_d$ is the group $\mathrm{rist}_G(v)$ of all automorphisms acting non-trivially only on the vertices of the form vu, $u \in T_d$; in other words,

$$\mathrm{rist}_G(v) = \{g \in G \mid g(w) = w \text{ where } w \neq vu, u \in T_d\}.$$

The *n-th level rigid stabilizer*

$$\mathrm{rist}_G(n) = \langle \mathrm{rist}_G(v) \mid v \in X^n \rangle$$

is the subgroup generated by the union of the rigid stabilizers of the vertices of the n-th level.

A group $G \leq \mathrm{Aut}\, T_d$ is said to be *branch* if for every $n \geq 0$ the subgroup $\mathrm{rist}_G(n)$ is of finite index in G.

Wreath products

Let H be a group acting (on the left) by permutations on a set X and let G be an arbitrary group. Then the (*permutational restricted*) *wreath product* $H \wr_X G$ is the semi-direct product $H \ltimes G^X$, where G^X is the set of finitely supported functions $f\colon X \to G$ with H-action given by precomposition: $f^h(x) = f(hx)$.

Every element of the wreath product $H \wr_X G$ can be written in the form $h \cdot g$, where $h \in H$ and $g \in G^X$. Such a way of writing an element in $G \wr_X H$ is unique, by the definition of the semi-direct product. If we fix some indexing $\{x_1, \ldots, x_d\}$ of the set X, then $g \in G^X$ can be written as $(g_1, \ldots, g_d)$ for $g_i \in G$, where each g_i is the coordinate of g, corresponding to x_i. The multiplication rule for the elements $h \cdot (g_1, \ldots, g_d) \in H \wr_X G$ is given by the formula

$$\alpha(g_1, \ldots, g_d) \cdot \beta(f_1, \ldots, f_d) = \alpha\beta(g_{\beta(1)} f_1, \ldots, g_{\beta(d)} f_d),$$

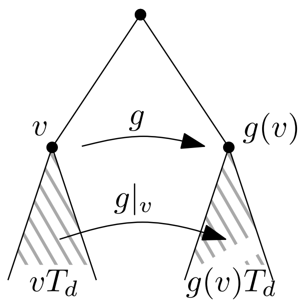
where $g_i, f_i \in G$, $\alpha, \beta \in H$ and $\beta(i)$ is the image of i under the action of β, *i.e.* such an index that $\beta(x_i) = x_{\beta(i)}$.

2.2 Self-similar actions

Our exposition on self-similar groups borrows from the book [72]. Briefly speaking, an automorphism group G of the rooted tree T_d is *self-similar* if for any element $g \in G$ the induced action of g on any subtree of T_d coincides with the action of some $h \in G$ on T_d. We will now explain precisely what it means.

For a vertex $v \in T_d$ let vT_d be the set of finite sequences $\{vx_1 \dots x_k \mid x_i \in X\}$ starting with the word v; we view vT_d as a rooted subtree of T_d. Consider an automorphism $g \in \operatorname{Aut} T_d$ and the subtrees vT_d and $g(v)T_d$. Notice that a map $g\colon vT_d \to g(v)T_d$ is an automorphism of the rooted trees. Moreover, the subtrees vT_d and $g(v)T_d$ are naturally isomorphic to the whole tree T_d. We identify vT_d and $g(v)T_d$ with T_d and obtain an automorphism $g|_v\colon T_d \to T_d$, which is uniquely determined by the condition

$$g(vw) = g(v)g|_v(w). \tag{2.2}$$



The automorphism $g|_v$ is called the *restriction* of g in v. The following properties then follow.

1. $g|_{v_1v_2} = g|_{v_1}|_{v_2}$;
2. $(g_1 \cdot g_2)|_v = g_1|_{g_2(v)} \cdot g_2|_v$.

A faithful action of a group G on T_d is said to be *self-similar* if for every $g \in G$ and every $x \in X$ there exists $h \in G$ and $y \in X$ such that

$$g(xw) = yh(w)$$

for every $w \in T_d$.

We will denote self-similar actions as pairs (G, X) where G is a group and X is an alphabet. Observe, that since the action is faithful, the pair (h, y) in the definition above is uniquely determined by the pair (g, x). If G admits a faithful action on T_d then it is isomorphic to a subgroup of $\operatorname{Aut} T_d$, with which it will be identified.

We say that an automorphism group G of the rooted tree T_d is *self-similar* if for every $g \in G$ and $v \in T_d$ we have $g|_v \in G$.

Proposition 2.2.1. *[72] Let $X = \{x_1, \dots, x_d\}$ with $d \geq 2$ be a finite set and let $Sym(X)$ be the symmetric group of all permutations of X. Then we have an isomorphism*

$$\psi : \operatorname{Aut} T_d \to Sym(X) \wr_X \operatorname{Aut} T_d,$$

given by

$$\psi(g) = \alpha(g|_{x_1}, \dots, g|_{x_d}),$$

where α is the permutation equal to the action of g on X.

Observe that the self-similarity can be reformulated in terms of wreath product of groups. Namely, we say that an automorphism group $G \leq \operatorname{Aut} T_d$ is *self-similar* if $\psi(G) \leq Sym(X) \wr_X G$.

Notation. *We will identify $g \in \operatorname{Aut} T_d$ with its image $\psi(g) \in Sym(X) \wr_X \operatorname{Aut} T_d$, and we write $g = \alpha(g|_{x_1}, \dots, g|_{x_d})$.*

2.3 Examples

We give a few examples of groups, acting on rooted trees, essential to us in further chapters.

Example 1. Let $X = \{1, 2\}$. We will be interested in the automorphisms of T_2 defined inductively by:

$$a = \sigma, b = (a, c)$$
$$c = (a, d)$$
$$d = (1, b),$$

where σ is the transposition $(1, 2) \in Sym(X)$.

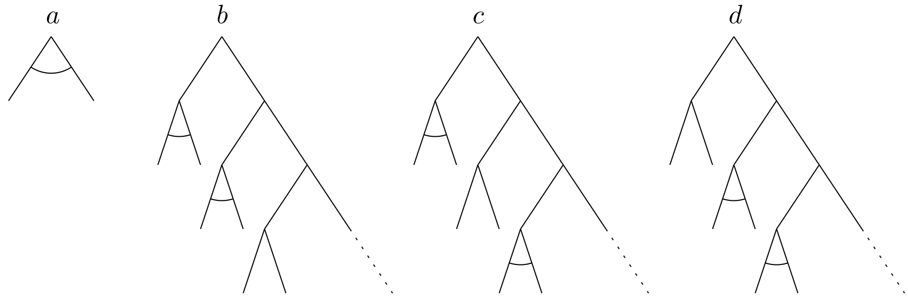


Figure 2.1: The action of the generators of the Grigorchuk group

Let the *first Grigorchuk Group* be $\Gamma = \langle a, b, c, d \rangle$. It was introduced by Grigorchuk in [43] as an example of an infinite 2-group which does not possess a finite presentation. Further study of the group Γ and its generalisations proved to be very fruitful: the group Γ is a branch group of intermediate growth [44].

Notice that the automorphisms a, b, c, d satisfy the following relations:

$$a^2 = b^2 = c^2 = d^2 = 1 \text{ and } bc = d = cb.$$

Hence the group Γ is a quotient of the free product of $\mathbb{Z}/2\mathbb{Z} = \{1, a\}$ and the Klein group (Vierergruppe) $\{1, b, c, d\}$. Using the induction on the length of elements in Γ, one can show that in fact for any $g \in \Gamma$ there exists $n \in \mathbb{N}$ such that $g^{2^n} = 1$. To see that the group Γ is infinite consider the homomorphism

$$\psi = (\phi_0, \phi_1) \colon St_\Gamma(1) \to \Gamma \times \Gamma$$

and observe that each projection $\phi_j \colon St_\Gamma(1) \to \Gamma$ is, in fact, onto.

The fact that the group Γ is branch follows from the existence of a subgroup, $K = \langle t = (ab)^2, v = (bada)^2, w = (abad)^2 \rangle$, of index 16 in Γ such that the image of $\psi \colon St_\Gamma(1) \to \Gamma \times \Gamma$ contains $K \times K$.

We also describe the family of groups $\{\Gamma_\omega\}_{\omega \in \Omega}$ generalising the construction of the the group Γ.

Let $\Omega = \{0,1,2\}^{\mathbb{N}}$ be the space of sequences over the alphabet $\{0,1,2\}$. Let $\omega = (\omega_1, \omega_2, \dots) \in \Omega$. We denote by $\tau\omega = (\omega_2, \omega_3, \dots) \in \Omega$ the shift. The group Γ_ω is generated by the automorphisms of a rooted binary tree $a, b_\omega, c_\omega, d_\omega$ defined as follows.

For any $v \in T_2$:

$$\begin{aligned} a &= \sigma \\ b_\omega(0v) &= 0\beta(\omega_1)(v) \\ b_\omega(1v) &= 1b_{\tau\omega}(v) \\ c_\omega(0v) &= 0\zeta(\omega_1)(v) \\ c_\omega(1v) &= 1c_{\tau\omega}(v) \\ d_\omega(0v) &= 0\delta(\omega_1)(v) \\ d_\omega(1v) &= 1d_{\tau\omega}(v), \end{aligned}$$

where

$$\begin{aligned} &\beta(0) = a, \quad \beta(1) = a, \quad \beta(2) = 1 \\ &\zeta(0) = a, \quad \zeta(1) = 1, \quad \zeta(2) = a \\ &\delta(0) = 1, \quad \delta(1) = a, \quad \delta(2) = a. \end{aligned}$$

If $\omega = \{0,1,2,0,1,2,\dots\}$ then Γ_ω is the first Grigorchuk group Γ. If $\omega \in \Omega$ contains all three letters $\{0,1,2\}$ infinitely many times, then Γ_ω shares many nice properties of Γ, in particular, it is a just-infinite branch 2-group [44]. We recall that a group is called just-infinite if it is infinite and its every proper quotient is finite.

Example 2. Let $X = \{1, \dots, p\}$ with p odd prime. We will be interested in the automorphisms x and y of T_p defined inductively:

$$x = \sigma, \; y = (x, x^{-1}, 1, \dots, 1, y),$$

where σ is the permutation $(1, 2, \dots, p)$ on X.
Let the *Gupta-Sidki* p-group be $G_p = \langle x, y \rangle$. The Gupta-Sidki groups were introduced in [52] and since then were extensively studied.

The generators x and y satisfy the relation $x^p = y^p = 1$, and, moreover, for any element $g \in G_p$ there exists $n \in \mathbb{N}$ such that $g^{p^n} = 1$. The Gupta-Sidki p-group is also branch; to see this consider the element $[x^{-1}yx, y] = ([y,x], 1, \dots, 1) \in [G_p, G_p]$: a subgroup generated by its conjugates by elements of $St_{G_p}(1)$ is $([G_p, G_p], 1, \dots, 1)$. It follows that the image of $\psi \colon St_{G_p}(1) \to (G_p)^p$ contains $([G_p, G_p])^p$. In addition, the abelianization $G_p^{ab} = G_p/[G_p, G_p]$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$, in particular, the subgroup $[G_p, G_p]$ is of finite index in G_p.

Example 3. Let $X = \{1, \dots, p\}$ with p an odd prime number. Consider the rooted p-ary tree T_p. Let $L = (l_n)_{n \geq 0}$ be an infinite path in T_p starting at the root. Consider, for every $n \geq 1$, immediate descendants $s_{n,k}$ for $k \in \{1, \dots, p\}$, of l_{n-1} not lying in L. We say that the doubly indexed sequence $S = (s_{n,k})_{n\geq 1,k}$ is a *multi-edge spine* of T_p. Let us choose the spine L to be associated to the rightmost infinite path.

By a denote the rooted automorphism corresponding to the cycle $(1, 2, \dots, p)$. For a given $r \in \mathbb{N}$ with $1 \leq r \leq p-1$ and a finite r-tuple $\mathbf{E}$ of $(\mathbb{Z}/p\mathbb{Z})$-linearly independent vectors let

$$e_i = (e_{i,1}, e_{i,2}, \dots, e_{i,p-1}) \in (\mathbb{Z}/p\mathbb{Z})^{p-1},\ i \in \{1, \dots, r\},$$

we recursively define directed automorphisms $b_1, \dots, b_r \in St_{\mathrm{Aut}\, T_p}(1)$ via

$$b_i = (a^{e_{i,1}}, a^{e_{i,2}}, \dots, a^{e_{i,p-1}}, b_i),\ i \in \{1, \dots, r\}.$$

The subgroup $G = G_{\mathbf{E}} = \langle a, b_1, \dots, b_r\rangle$ of $\mathrm{Aut}\, T_p$ is called the *multi-edge spinal group* associated to the defining vectors $\mathbf{E}$.

Notice that by choosing $r = 1$, $e = (1, -1, 0, \dots, 0) \in (\mathbb{Z}/\mathbb{Z})^{p-1}$ and b associated to the vector e, the group $G_{\mathbf{E}}$ generated by a and b is the Gupta-Sidki p-group.

It was proven in [46, 90] that the group $G_{\mathbf{E}}$ is an infinite p-group if and only if for every e_i we have $\sum_{j=1}^{p-1} e_{i,j} = 0 \mod p$. Moreover, if a multi-edge spinal group G acting on the rooted p-ary tree T_p is not $\mathrm{Aut}\, T_p$-conjugate to the group $G' = \langle a, b\rangle$ with $b = (a, a, \dots, a, b)$ then G is branch [3].

Example 4. Let $X = \{1, 2\}$. We will be interested in the automorphisms of T_2 defined inductively by:

$$\begin{aligned} a =& (1, b) \\ b =& (1, a)\sigma, \end{aligned}$$

where σ is the transposition $(1, 2) \in Sym(X)$.

Let the Basilica group be $G = \langle a, b\rangle$. The group G was defined for the first time by R. Grigorchuk and A. Zuk in [48] where the authors, in particular, showed that G is a torsion-free group of exponential growth without free subgroups. Further, Bartholdi and Virag showed in [11] that G is amenable. The presentation of G obtained in [48] is the following:

$$G = \langle a, b \mid \tau^{\epsilon}(\theta^{m}([a, a^{b}])) = 1, m = 0, 1, \dots, \epsilon = 0, 1\rangle,$$

where

$$\tau : \begin{cases} a \mapsto b^2 \\ b \mapsto a \end{cases} \quad \theta : \begin{cases} a \mapsto a^{b^2+1} \\ b \mapsto b \end{cases} .$$

2.4 Maximal subgroups

The analysis of maximal subgroups in finitely generated groups is important in the framework of the connectedness of Nielsen graphs (see Chapter 3 and Chapter 4). The question whether all maximal subgroups of Γ, the first Grigorchuk group, have finite index was open for a long time before it was answered positively by Pervova in [83]. This result holds for all Grigorchuk groups Γ_ω with $\omega \in \{0,1,2\}^{\mathbb{N}}$ such that each of $0,1,2$ appears infinitely many times in ω. Pervova extended this result to Gupta-Sidki p-groups in [85]. On the other hand, Bondarenko in [14] gave an example of a finitely generated branch group that does have maximal subgroups of infinite index. Recently the result of Pervova for Gupta-Sidki p-groups was generalized in [3] to torsion multi-edge spinal groups acting on the regular p-ary rooted tree, for p odd prime. The characterization of finitely generated branch groups, all maximal subgroup of which have finite index, remains open.

The analysis of maximal subgroups in [83, 85, 3] starts with an observation that if a group G contains maximal subgroups of infinite index then it contains a proper dense subgroup with respect to the profinite topology. Recall, that the *profinite topology* on a group is a topology on the underlying set of the group with a basis of left cosets of subgroups of finite index. A subgroup M is *dense* in G with respect to the profinite topology if for every normal subgroup N of finite index in G

$$\epsilon_N(M) = \epsilon_N(G),$$

where $\epsilon_N\colon G \to G/N$ is the natural projection. The observation above then follows: if M is a maximal subgroup of infinite index then for any normal subgroup $N \leq G$ of finite index the difference $N \setminus M$ is non-empty and therefore $MN = G$.
The goal then is to show that there are no dense subgroups in G. The important ingredient in the papers [83, 85, 3] is the existence of a map $\theta\colon [G,G] \to [G,G]$ which decreases the length of elements in $[G,G]$. The map θ seems to be very specific to the groups considered in [83, 85, 3].
For instance, for the first Grigorchuk group Γ, the map is defined as follows:

$$[\Gamma,\Gamma] \ni z = (z_0, z_1) \mapsto \theta(z) = a^{-1}z_1 a z_0.$$

To see that the map θ is well-defined one needs to observe that z_0 and z_1 lie in the same coset of $[\Gamma,\Gamma]$, therefore $a^{-1}z_1az_0[\Gamma,\Gamma] = z_1z_0[\Gamma,\Gamma] = z_0^2[\Gamma,\Gamma] = [\Gamma,\Gamma]$.

CHAPTER 3

Connectedness of Nielsen graphs

We start this chapter by introducing and discussing properties of a class of groups which we call the class $\mathcal{MN}$. In Chapter 3.2 we analyse Nielsen graphs of finitely generated groups in the class $\mathcal{MN}$, and in Chapter 3.3 we study Nielsen graphs of relatively free groups.

3.1 Class $\mathcal{MN}$

By $\mathcal{MN}$ we denote the class of groups with the property that all maximal subgroups are normal.

It is well-known that a finite group is in $\mathcal{MN}$ if and only if it is nilpotent [86, 5.2.4]. More generally, all nilpotent groups belong to $\mathcal{MN}$, since they satisfy the normalizer condition (*i.e.* for any subgroup H, its normalizer $N_G(H)$ contains H properly).

Remark 3.1.1. *Observe that if a group G is in $\mathcal{MN}$ then every maximal subgroup M of G is of finite index. Furthermore, the quotient G/M is a cyclic group of prime order.*

Indeed, by the lattice theorem (also called the fourth isomorphism theorem) there is a one-to-one correspondence between subgroups of G containing M and those of G/M. Since there are no proper subgroups in G containing M we deduce that G/M does not have proper subgroups which implies that G/M is a cyclic group of prime order.

The converse does not hold: all maximal subgroups of the Lamplighter group $\mathcal{L} = \mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ are of finite index, however there are maximal subgroups of $\mathcal{L}$ which are not normal [41].

The property of groups to have all maximal subgroups of finite index was already considered in the literature for different classes of groups. For instance in the linear setting, Margulis and Soifer [65] showed that all maximal subgroups of a finitely generated linear group G are of finite index if and only if G is virtually solvable. The above property also was considered for branch group, however there is no such a general result as for linear groups. The question whether all maximal subgroups of Γ, the first Grigorchuk group, have finite index was open for a long time before it was answered positively by Pervova in [83]. This result holds for all Grigorchuk groups Γ_ω with $\omega \in \{0,1,2\}^{\mathbb{N}}$ such that each of $0,1,2$ appears infinitely many times in ω. Pervova extended this result to Gupta-Sidki p-groups in [85]. Recently the result of Pervova for Gupta-Sidki p-groups was generalized in [3] to torsion multi-edge spinal groups acting on the regular p-ary rooted tree, for p odd prime. On the other hand, Bondarenko in [14] gave an example of a finitely generated branch group that does have maximal subgroups of infinite index. Generally, the characterization of branch groups with the property that all maximal subgroups have finite index, remains open.

In Proposition 3.1.3 we give a criterion for a finitely generated group to be in $\mathcal{MN}$ in terms of generating and normally generating sets. This proposition implies that the free group F_n of rank $n \geq 2$ and a free product of (at least) two non-trivial groups do not belong to $\mathcal{MN}$.

We recall that the *Frattini subgroup* $\Phi(G)$ of a group G is the intersection of all maximal subgroups of G, and $\Phi(G) = G$ if G does not have maximal subgroups. Note that if a non-trivial group G is finitely generated then Zorn's Lemma implies that there exists a proper maximal subgroup of G.

Observe the following well-known properties of the Frattini subgroup:

Lemma 3.1.2. *Let G be a group.*

1. *[86, 5.2.12] Frattini subgroup $\Phi(G)$ is equal to the set of non-generators of G,* i.e. *if $g \in \Phi(G)$ and $\langle g, X\rangle = G$ then $\langle X\rangle = G$.*
2. *[30] Let $G = \langle x_1, ..., x_n\rangle$ and $\varphi_1, ..., \varphi_n \in \Phi(G)$. Then*

$$\langle x_1\varphi_1, \ldots, x_n\varphi_n\rangle = G.$$

Proposition 3.1.3. *Let G be a finitely generated group. Then G is in $\mathcal{MN}$ if and only if all normally generating sets are generating sets. Moreover, if G is in $\mathcal{MN}$, then $[G,G] \leq \Phi(G)$.*

Proof. Let G be in $\mathcal{MN}$ and assume by contradiction that there is a normally generating set S which is not a generating set. Since G is finitely generated, any proper subgroup is contained in some proper maximal subgroup [73], therefore $\langle S\rangle \leq M < G$ for some proper maximal subgroup M. Then $\ll S \gg = G \leq M^G = M < G$. It is a contradiction.

Conversely, we will prove that if all normally generating sets are generating sets then all maximal subgroups are normal. We prove the equivalent statement: if there is a maximal subgroup M which is not normal, then there exists a normally generating set which is not a generating set. We take as a normally generating set $S = M$. Then $\ll S \gg = G$ since M is maximal and $\langle S \rangle \neq G$ since M is proper.

Furthermore, if G is in $\mathcal{MN}$ then for any maximal subgroup M the quotient G/M is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ by the lattice theorem (also called the fourth isomorphism theorem), in particular, G/M is abelian. Therefore $[G,G] \leq M$ and we conclude that $[G,G] \leq \Phi(G)$. □

Notice that an obvious example of a group not belonging to $\mathcal{MN}$ is a non-abelian simple group.

Non-example 3.1.4. *Observe that the free group $F_2 = \langle x, y \rangle$ of rank 2 is not in $\mathcal{MN}$. For this notice that $\ll y^{-1}xy, x^{-1}yx \gg = F_2$, however $\langle y^{-1}xy, x^{-1}yx \rangle \neq F_2$.*

More generally, a free product of a finite number of non-trivial finitely generated groups $A_1, \dots, A_k$ does not belong to the class $\mathcal{MN}$. To see this, first suppose that $k = 2$ and suppose that A_1 is generated by $\{a_1, \dots, a_k\}$ and A_2 is generated by $\{b_1, \dots, b_m\}$. The set

$$\{b_1^{-1}a_1b_1, b_1^{-1}a_2b_1, \dots, b_1^{-1}a_kb_1^{-1}, a_1^{-1}b_1a_1, a_1^{-1}b_2a_1, \dots, a_1^{-1}b_ma_1\}$$

normally generates the group $A_1 * A_2$ but does not generate it; it follows that $A_1 * A_2$ does not belong to $\mathcal{MN}$. More generally, a free product $A_1 * \cdots * A_k$ can be seen as a free product of A_1 and $A_2 * \cdots * A_k$ and we use the argument for $k = 2$.

Notice that the infinite dihedral group D_∞, being isomorphic to $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, is not in $\mathcal{MN}$. In addition, the group $D_\infty \cong \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ is polycyclic, and therefore the class $\mathcal{MN}$ does not contain the class of finitely generated polycylic groups.

Further we give more properties of groups in $\mathcal{MN}$. In particular, we show that the class $\mathcal{MN}$ is closed under taking quotients. In particular, it shows (see Corollary 3.1.7) that if a finitely generated linear group is in $\mathcal{MN}$ then it is, in fact, nilpotent.

Proposition 3.1.5. *Let G be in $\mathcal{MN}$ and $N \lhd G$ be a normal subgroup of G. Then G/N is in $\mathcal{MN}$.*

Proof. Consider the natural projection $\pi : G \to G/N$. Observe, that for a maximal subgroup $M < G/N$ its preimage $\pi^{-1}(M)$ is a maximal subgroup of G. For this it is enough to notice that $\pi^{-1}(M)$ is a subgroup of G containing the kernel $\ker(\pi)$. Moreover, since G is in $\mathcal{MN}$ then $\pi^{-1}(M)$ is normal in G and it follows that M is normal in G/N. □

Corollary 3.1.6. *Let G be a group in $\mathcal{MN}$ and let $N \lhd G$ be a normal subgroup of G of finite index. Then G/N is nilpotent.*

Corollary 3.1.7. *If G is a finitely generated linear group in the class $\mathcal{MN}$ then G is nilpotent.*

Proof. Since G is in $\mathcal{MN}$ then all maximal subgroups of G are of finite index, as observed in the remark in the beginning of the section. The result of [65] tells us that all maximal subgroups of a finitely generated linear group G are of finite index if and only if G is virtually solvable. Furthermore, since all finite quotients of G are nilpotent, we deduce that G is solvable. Finally, to conclude that G is nilpotent we use the result from [91], saying that a finitely generated soluble group such that all of its finite homomorphic images are nilpotent is nilpotent. □

Proposition 3.1.8. *Let G be a finitely generated group in $\mathcal{MN}$, N be a normal subgroup of G contained in $\Phi(G)$ and $n \geq \text{rank}(G/N)$. Then the natural map*

$$\pi_n : Epi(F_n, G) \to Epi(F_n, G/N),$$

is surjective.

Proof. Consider the projection $\pi : G \to G/N$. Let $s_1, \ldots, s_n \in G$ such that $\pi(s_1), \ldots, \pi(s_n)$ generate G/N. Suppose $\langle s_1, \ldots, s_n \rangle \leq M$ for some proper maximal subgroup M of G; we will obtain a contradiction. Using the assumption $N \leq \Phi(G) \leq M$ we have $M/N = \pi(M) = \pi(G) = G/N$. Hence $\{e\} = \frac{G/N}{M/N} \cong G/M$. This is a contradiction with M being a proper subgroup of G. □

The following corollary is a special case of Proposition 3.1.8 and we will need it in further chapters.

Corollary 3.1.9. *Let G be a finitely generated group in $\mathcal{MN}$ and $n \geq 1$ Then the natural map*

$$\pi_n : Epi(F_n, G) \to Epi(F_n, G/[G, G])$$

is surjective.

Proof. Use Proposition 3.1.3 and Proposition 3.1.8. □

Proposition 3.1.10. *Let G be a finitely generated group such that its commutator subgroup $[G, G]$ is finitely generated. Then G is in the class $\mathcal{MN}$ if and only if $[G, G] \leq \Phi(G)$.*

Proof. We have already shown that if G is in $\mathcal{MN}$ then $[G, G] \leq \Phi(G)$. Now let G be a finitely generated group such that $[G, G]$ is finitely generated and $[G, G] \leq \Phi(G)$. Assume $G = \ll S \gg$ for some finite $S \subseteq G$. We will prove that $\langle S \rangle = G$ which will imply, by Proposition 3.1.3, that G is in $\mathcal{MN}$. By definition $\ll S \gg = \langle S^G \rangle$ where $S^G = \{g^{-1}sg \mid g \in G, s \in S\} = \{s[s, g] \mid g \in G, s \in S\}$, where $[s, g] = s^{-1}g^{-1}sg$ denotes the commutator. We have therefore $\ll S \gg = \langle s[s, g], s \in S, g \in G \rangle$ and, in particular, $\ll S \gg \subseteq \langle S, S_{[G,G]} \rangle$ where $S_{[G,G]}$ denotes a finite generating set of $[G, G]$. Since $\ll S \gg = G$ we deduce that $\langle S, S_{[G,G]} \rangle = G$. Finally, $[G, G] \leq \Phi(G)$ and by Lemma 3.1.2 we conclude that $G = \langle S \rangle$. □

Proposition 3.1.11. *Let G_1, G_2 be finitely generated groups in $\mathcal{MN}$ such that the commutator subgroups $[G_1, G_1]$ and $[G_2, G_2]$ are finitely generated. Then $G = G_1 \times G_2$ is in $\mathcal{MN}$.*

Proof. The Frattini subgroup of a direct product of finitely generated groups is the direct product of Frattini subgroups of the direct factors [28], *i.e.* $\Phi(G) = \Phi(G_1) \times \Phi(G_2)$.

Notice that $[G, G] = [G_1, G_1] \times [G_2, G_2]$ is finitely generated. By assumption G_1 and G_2 are in $\mathcal{MN}$ therefore $[G, G] \leq \Phi(G)$. We conclude that G is in $\mathcal{MN}$ by Proposition 3.1.10. □

Proposition 3.1.11 provides more examples of groups in the class $\mathcal{MN}$. For instance, the direct product $\Gamma \times \Gamma$ or $\Gamma \times G_p$, where Γ is the first Grigorchuk group and G_p is the Gupta-Sidki p-group, is also in $\mathcal{MN}$.

Remark 3.1.12. *Another implication of Proposition 3.1.5 is the following: if a finitely generated group G is in $\mathcal{MN}$ and if it is just-infinite (i.e. all proper quotients of G are finite)* or *it is just-non-solvable group (i.e. all proper quotients of G are solvable) then G is just-non-nilpotent.*

Indeed, if G is just-infinite then every proper quotient of G is nilpotent. If G is just-non-solvable, then every proper quotient of G is solvable and, moreover, belongs to $\mathcal{MN}$. To conclude we use the result in [91] saying that a finitely generated soluble group such that all of its finite homomorphic images are nilpotent is nilpotent.

Another non-example of the class $\mathcal{MN}$ is a just-non-solvable Basilica group G*.

Corollary 3.1.13. *The Basilica group G, of automorphisms of a rooted binary tree generated by automorphisms $a = (1, b)$ and $b = (1, a)\sigma$, is not in the class $\mathcal{MN}$.*

Proof. The presentation of the group G obtained in [48] (also discussed in Section 2.3) implies that the quotient G/N with $N = \ll b^2, abab \gg$ is isomorphic to the infinite dihedral group D_∞. The latter, as discussed in Nonexample 2.3, does not belong to the class $\mathcal{MN}$ and it follows that G does not belong to $\mathcal{MN}$ by Proposition 3.1.5. □

Proposition 3.1.14. *Let G be a group. If all maximal subgroups of G are of finite index and all finite quotients of G are nilpotent then G belongs to $\mathcal{MN}$.*

Proof. Let M be a maximal subgroup of G. By the assumption, M is of finite index of G. Let K be the normal core of M, that is $K = \cap_{g \in G} g^{-1} M g$. The subgroup K is normal and of finite index in G. The quotient G/K is finite and therefore nilpotent. It follows that the maximal subgroup M/K in G/K is normal. We conclude that M is normal in G. □

*The definition and the discussion concerning certain properties of G can be found in Section 2.3.

Corollary 3.1.15. *The first Grigrochuk group* Γ *and its generalisations* $\{\Gamma_\omega\}$*, where* $\omega \in \{0, 1, 2, \}^{\mathbb{N}}$ *contains all three letters* $\{0, 1, 2\}$ *infinitely many times, belong to* $\mathcal{MN}$ *[43, 44]. Also Gupta-Sidki p-groups [53] and, more generally, all multi-edge spinal torsion groups acting on a regular p-ary rooted tree, with p an odd prime, belong to* $\mathcal{MN}$*.*

Proof. It was shown by Pervova in [83] that for the Grigorchuk groups Γ_ω with $\omega \in \{0, 1, 2\}^{\mathbb{N}}$, such that each of $0, 1, 2$ appears infinitely many times in ω, maximal subgroups of Γ_ω are of finite index. Moreover, it is known that such Γ_ω are 2-groups [44] therefore their finite quotients are nilpotent.

Pervova extended her result in [85] by showing that also in Gupta-Sidki p-groups all maximal subgroups are of finite index. Recently the result for Gupta-Sidki p-groups was generalized in [3] to torsion multi-edge spinal groups acting on the regular p-ary rooted tree, for p an odd prime. In addition, the Gupta-Sidki p-groups and, more generally, torsion multi-edge spinal groups acting on the regular p-ary rooted tree, for p an odd prime, are known to be p-groups [3]; therefore their finite quotients are nilpotent. □

It was shown by Grigorchuk and Wilson in [42] that every infinite finitely generated subgroup H of Γ, the first Grigorchuk group, is (abstractly) commensurable with Γ. In the same paper, the authors showed that for a group G (abstractly) commensurable with Γ, all maximal subgroups of G are of finite index. It follows that all infinite finitely generated subgroups of Γ are in the class $\mathcal{MN}$. Notice that all finitely generated finite subgroups of Γ, being finite 2-groups and thus nilpotent, also belong to $\mathcal{MN}$.

Garrido in [35] studied the subgroup structure of Gupta-Sidki 3-group G_3 and, following results of Grigorchuk and Wilson, showed that all infinite finitely generated subgroups of G_3 have their maximal subgroups of finite index. It follows that all finitely generated subgroups of G_3 are in $\mathcal{MN}$.

3.2 Nielsen equivalence in class $\mathcal{MN}$

3.2.1 Nielsen graphs of finitely generated abelian groups

Consider a finitely generated abelian group A, the simplest example of a group in class $\mathcal{MN}$. It was first shown by B.H. Neumann and H. Neumann [75] that the Nielsen graph $N_n(A)$ is connected if $n \geq \text{rank}(A) + 1$ and A is finite; then Diaconis and Graham [27], rediscovering the result in [75], obtained the number of connected components of $N_n(A)$ when $n = \text{rank}(A)$ for finite A; and the latter result was generalized by Oancea [79] to infinite A. The following theorem combines the results on the connectedness of the Nielsen graph for finitely generated abelian groups.

Theorem 3.2.1 ([75, 27, 79]).
Let A be a finitely generated abelian group with the primary decomposition

$$A \cong \mathbb{Z}^r \times \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_s}$$

where $r, s \geq 0$, $m_s > 1$ and $m_s | m_{s-1} | \ldots | m_1$. Then $\text{rank}(A) = r + s$ and

1. *$N_n(A)$ is connected if $n > r + s$;*
2. *if $s = 0$, i.e. $A \cong \mathbb{Z}^r$, then $N_r(A)$ is connected;*
3. *otherwise if $m_s = 2$ or $m_s = 3$ then $N_{r+s}(A)$ is connected and if $m_s > 3$ then $N_{r+s}(A)$ has $\varphi(m_s)/2$ connected components,*

where φ is the Euler function ($\varphi(m_s)$ is the number of positive integers less than m_s which are coprime with m_r).

The approach taken in the paper of Diaconis and Graham (and then generalized by Oancea) can be used to show that the connected components of the Nielsen graph of a finitely generated abelian group are isomorphic. We discuss below how to do this.

For any generating n-tuple $g = (g_1, \ldots, g_n)$ of A, every g_i can be viewed as a row-vector $g_i = (a_{i,1}, \ldots, a_{i,r+s})$, where the first r coordinates of g_i belong to $\mathbb{Z}$ and $0 \leq a_{i,r+j} \leq m_j - 1$ for $j \in \{1, \ldots, s\}$.

From now on we view every generating n-tuple $(g_1, \ldots g_n)$ of A as an $n \times (r+s)$ matrix $M_{(g_1,\ldots,g_n)} = (a_{i,j})_{1\leq i\leq n, 1\leq j\leq r+s}$, which we call *a generating matrix*, such that $a_{i,j} \in \mathbb{Z}$ for $0 \leq j \leq r$ and $0 \leq a_{i,r+j} \leq m_j - 1$ for $1 \leq j \leq s$. Observe that elementary Nielsen moves induce elementary Gauss transformations on the generating matrix $M_{(g_1,\ldots,g_n)}$, such as

$$g_i \mapsto g_i + g_j \tag{3.1}$$

$$g_i \mapsto -g_i \tag{3.2}$$

where g_i and g_j are the rows in $M_{(g_1,\ldots,g_n)}$, $1 \leq i, j \leq n$ and $i \neq j$. Note, that the sum of two rows is determined by adding corresponding coordinates and taking each $(j+r)$-th coordinate modulo m_j for $1 \leq j \leq s$.

Theorem 3.2.2. *[27, 79] Let A be a finitely generated abelian group as in Theorem 3.2.1. Two generating $(r+s)$-tuples $(g_1, \ldots, g_{r+s})$ and $(h_1, \ldots h_{r+s})$ of A are Nielsen equivalent if and only if*

$$|\det M_{(g_1,\ldots,g_{r+s})}| \pmod{m_r} = |\det M_{(h_1,\ldots,h_{r+s})}| \pmod{m_r}.$$

Corollary 3.2.3. *Let A be a finitely generated abelian group. Then all connected components of the Nielsen graph $N_n(A)$ with $n \geq \operatorname{rank}(A)$ are isomorphic.*

Proof. Consider as in Theorem 3.2.1 the primary decomposition of A. For the case when the graph $N_n(A)$ is connected the statement of the corollary does not require a proof. Suppose that $s \geq 1$ and $m_s > 3$.

Let M be a generating $(r+s) \times (r+s)$-matrix of A. It follows from the proof of Theorem 3.2.2 that M can be carried by elementary Gauss transformations (3.1),(3.2) to the form

$$E = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 0 & b \end{pmatrix}$$

where $b = |\det M| \pmod{m_s}$. Notice that $gcd(m_s, b) = 1$. It follows, in particular, that the number of connected components of $N_{r+s}(A)$ is indeed $\varphi(m_s)/2$.

We deduce that for a generating $(r+s) \times (r+s)$ matrix M there exists T_M such that $M = T_M \cdot E$. Note, that the matrix T_M is unique up to summands $k_1 m_1, \ldots, k_s m_s$ in $(r+1), \ldots, (r+s)$ columns respectively, for $k_i \in \mathbb{Z}$. We will use the same notation T_M for the unique representative for such a matrix $T_M \in GL_{r+s}(\mathbb{Z})$.

Let Γ_1 and Γ_2 be two connected components of $N_{r+s}(A)$ and let M_1 and M_2 be two generating matrices belonging to Γ_1 and Γ_2 respectively. Then M_i in Γ_i can be carried by Gauss transformations (3.1), (3.2) to E_i respectively, where

$$E_i = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & b_i \end{pmatrix},$$

$1 \leq i \leq 2$ and $b_i \geq 1$ such that $gdc(b_i, m_s) = 1$.

We define a map φ between Γ_1 and Γ_2 in the following way:

$$\varphi(E_1) = E_2,$$

and for any generating matrix $M \in \Gamma_1$ such that $M = T_M \cdot E_1$ we define

$$\varphi(M) = \varphi(T_M \cdot E_1) = T_M \cdot E_2.$$

Suppose M_1 and M_2 are two generating matrices in Γ_1 connected by an edge. Then there exists a matrix T corresponding to some elementary Gauss transformation (3.1), (3.2) such that $M_1 = T \cdot M_2$. Denote by T_1 and T_2 the matrices such that $M_1 = T_1 \cdot E_1$ and $M_2 = T_2 \cdot E_1$.

Then the following equality holds:

$$\varphi(M_1) = TT_2E_2 = T \cdot \varphi(M_2).$$

It follows that $\varphi(M_1)$ and $\varphi(M_2)$ are connected by an edge in Γ_2 and hence φ is a morphism of graphs.

Observe that φ is injective. Indeed, suppose that $M_1 = T_1 \cdot E_1$ and $M_2 = T_2 \cdot E_1$ are two generating matrices of Γ_1 (for unique representatives $T_1, T_2 \in GL_{r+s}(\mathbb{Z})$), and suppose that $\varphi(M_1) = T_1 \cdot E_2 = T_2 \cdot E_2 = \varphi(M_2)$. Then the matrix $T_2^{-1}T_1 \in Stab_{GL_{r+s}(\mathbb{Z})}(E_2)$. Observe that it implies that $T_2^{-1}T_1 \in Stab_{GL_{r+s}(\mathbb{Z})}(E_1)$ to conclude that $M_1 = M_2$.

Moreover φ is surjective. Indeed, if M is a generating matrix in Γ_2 then there exists a Gauss transformation corresponding to a matrix $T \in GL_{r+s}(\mathbb{Z})$ such that $M = T \cdot E_2$. Then $\varphi(T \cdot E_1) = M$. □

We will now study connectedness of the Nielsen graph $N_n(G)$ for an arbitrary finitely generated group G in the class $\mathcal{MN}$ with $n \geq \operatorname{rank}(G)$. We split this study into two cases: when $n \geq \operatorname{rank}(G) + 1$ and $n = \operatorname{rank}(G)$.

3.2.2 Connectedness of Nielsen graphs $N_n(G)$ for $n \geq \operatorname{rank}(G) + 1$

First, we recall the following proposition by Evans.

Proposition 3.2.4. *[30]*
Let G be a finitely generated group, $\Phi(G)$ the Frattini subgroup of G and $n \geq \operatorname{rank}(G) + 1$. If for some normal subgroup $N \lhd G$ with $N \leq \Phi(G)$ the Nielsen graph $N_n(G/N)$ is connected then $N_n(G)$ is connected.

Proof. Let $\operatorname{rank}(G) = d$ and $G = \langle x_1, \ldots, x_d \rangle$. Assume that $N_n(G/N)$ is connected for some $n \geq d + 1$. Then any generating n-tuple $(g_1, \ldots, g_n)$ of G is Nielsen equivalent to $(x_1\varphi_1, \ldots, x_d\varphi_d, \varphi_{d+1}, \ldots, \varphi_n)$ for some elements $\varphi_1, \ldots, \varphi_n$ of N. Since $\varphi_1, \ldots, \varphi_d \in \Phi(G)$ then $\langle x_1\varphi_1, \ldots, x_d\varphi_d \rangle = G$ (Lemma 3.1.2). We conclude the proof with the following sequence of Nielsen moves:

$$(g_1, \ldots, g_n) \sim (x_1\varphi_1, \ldots, x_d\varphi_d, \varphi_{d+1}, \ldots, \varphi_n) \sim (x_1\varphi_1, \ldots, x_d\varphi_d, 1, \ldots, 1) \sim$$
$$(x_1\varphi_1, \ldots, x_d\varphi_d, \varphi_1, 1 \ldots, 1) \sim (x_1, x_2\varphi_2 \ldots, x_d\varphi_d, 1 \ldots, 1) \sim$$
$$(x_1, x_2, \ldots, x_d, 1 \ldots, 1).$$

□

In fact, for finitely generated groups in the class $\mathcal{MN}$ the converse is also true by Proposition 3.1.8.

Corollary 3.2.5. *Let G be a finitely generated group in $\mathcal{MN}$ and $n \geq \operatorname{rank}(G) + 1$. Then $N_n(G)$ is connected if and only if $N_n(G/\Phi(G))$ is connected.*

We now generalise Evans' result to finitely generated groups in the class $\mathcal{MN}$.

Proposition 3.2.6. *Let G be a finitely generated in $\mathcal{MN}$. The Nielsen graph $N_n(G)$ for $n \geq \operatorname{rank}(G) + 1$ is connected.*

Proof. It follows from Proposition 3.1.10 that $[G, G] \leq \Phi(G)$. The group $G/[G, G]$ is abelian with $\operatorname{rank}(G/[G, G]) = \operatorname{rank} G$. By Theorem 3.2.1 the Nielsen graph $N_n(G/[G, G])$ is connected for $n \geq \operatorname{rank}(G) + 1$. The conclusion follows from Proposition 3.2.4. □

3.2.3 Nielsen graphs $N_{\mathrm{rank}(G)}(G)$: from one to infinitely many connected components

It turns out that even though the connectedness result for $N_n(G)$ holds for every finitely generated group G in the class $\mathcal{MN}$ and $n \geq \mathrm{rank}(G)+1$, the situation for $N_{\mathrm{rank}(G)}(G)$ can vary depending on the group G: from $N_{\mathrm{rank}(G)}(G)$ being connected to $N_{\mathrm{rank}(G)}(G)$ having infinitely many connected components.

We have already discussed the Nielsen graph and its connected components for a finitely generated abelian group in Section 3.2.1. Further, in Theorem 1.2.2, we obtain connectedness of the Nielsen graph $N_{\mathrm{rank}(G)}(G)$ for the family of Heisenberg groups $\mathcal{H}_k$. Then we will show in Theorem 1.2.3 that the graph $N_{\mathrm{rank}(G)}(G)$ has infinitely many connected components if G is the Gupta-Sidki p-group.

Connectedness of Nielsen graphs of discrete Heisenberg groups $\mathcal{H}_k$

The discrete Heisenberg group $\mathcal{H}_k$, $k \geq 1$, is the group of integer matrices of the form

$$v = \begin{pmatrix} 1 & \mathbf{x} & z \\ 0 & I_k & \mathbf{y} \\ 0 & 0 & 1 \end{pmatrix}$$

with $\mathbf{x} = (x_1, \ldots, x_k)$ a row vector of length k, $\mathbf{y} = (y_1, \ldots, y_k)^T$ a column vector of length k, and I_k is the $k \times k$ identity matrix.

Theorem (Theorem 1.2.2)**.** *The graph $N_n(\mathcal{H}_k)$ is connected when $n \geq$* $\mathrm{rank}(\mathcal{H}_k) = 2k$.

Proof of Theorem 1.2.2. We will prove that $N_{\mathrm{rank}(\mathcal{H}_k)}(\mathcal{H}_k/\Phi(\mathcal{H}_k))$ is connected and deduce connectedness for $N_{\mathrm{rank}(\mathcal{H}_k)}(\mathcal{H}_k)$. We refer to Corollary 3.2.5 for the case $n \geq \mathrm{rank}(\mathcal{H}_k)+1$.

Each element of $\mathcal{H}_k$ can be written as $(x_1, \ldots, x_k, y_1, \ldots, y_k, z)$. The identity element of $\mathcal{H}$ is $(0, 0, \ldots, 0)$ and

$$(x_1, \ldots, x_k, y_1, \ldots, y_k, z)^{-1} = (-x_1, \ldots, -x_k, -y_1, \ldots, -y_k, x_1y_1 + \cdots + x_ky_k - z).$$

The group multiplication is then given by the following rule:

$$(x_1, \ldots, x_k, y_1, \ldots, y_k, z)(x'_1, \ldots, x'_k, y'_1, \ldots, y'_k, z') = \\ (x_1 + x'_1, \ldots, x_k + x'_k, y_1 + y'_1, \ldots, y_k + y'_k, z + z' + x_1y'_1 + \cdots + x_ky'_k).$$

Calculations show that $[\mathcal{H}_k, \mathcal{H}_k] = Z(\mathcal{H}_k) \cong \mathbb{Z}$ and $\mathrm{Ab}(\mathcal{H}_k) = \mathbb{Z}^{2k}$.
Observe that $\mathcal{H}_k = \langle (1,0,\ldots,0,0), (0,1,\ldots,0,0), \ldots, (0,\ldots,1,0) \rangle$ and $\mathrm{rank}(\mathcal{H}_k) = 2k$. Let $\pi : \mathcal{H}_k \to \mathcal{H}_k/[\mathcal{H}_k, \mathcal{H}_k]$ be the natural epimorphism.

Since $\mathrm{Aut}\, F_{2k}$ acts transitively on generating $2k$-tuples of $\mathcal{H}_k/[\mathcal{H}_k, \mathcal{H}_k] \cong \mathbb{Z}^{2k}$, then any generating $2k$-tuple of $\mathcal{H}_k$ is Nielsen equivalent to

$$\Big((1, 0, \ldots, 0, m_1), (0, 1, \ldots, 0, m_2), \ldots, (0, \ldots, 1, m_{2k})\Big) \tag{3.3}$$

for some integers $m_1, \dots, m_{2k}$. Observe that the $2k$-tuple in (3.3) generates $\mathcal{H}_k$ for any choice of $m_1, \dots, m_{2k} \in \mathbb{Z}$: indeed, since $\mathcal{H}_k$ is nilpotent group then $[\mathcal{H}_k, \mathcal{H}_k] \leq \Phi(\mathcal{H}_k)$ by Proposition 3.1.10. To obtain connectedness of $N_{2k}(\mathcal{H}_k)$ it is sufficient to prove that there exists $\sigma \in \operatorname{Aut} F_{2k}$ which transforms

$$\Big((1,0,\dots,0,0),(0,1,\dots,0,0),\dots,(0,\dots,1,0)\Big)$$

into

$$\Big((1,0,\dots,0,m_1),(0,1,\dots,0,m_2),\dots,(0,\dots,1,m_{2k})\Big)$$

for any $m_i \in \mathbb{Z}$.

We calculate

$$(L_{1,k+1}^{-1}R_{1,k+1})^{m_1}\Big((1,0,\dots,0,0),(0,1,\dots,0,0),\dots,(0,\dots,1,0)\Big) = \Big((1,0,\dots,0,m_1),(0,1,\dots,0,0),\dots,(0,\dots,1,0)\Big).$$

More generally $\forall i : 1 \leq i \leq k$

$$(L_{i,k+i}^{-1}R_{i,k+i})^{m_i}\Big((1,0,\dots,0,0),\dots,(0,\dots,1,\dots,0,0),\dots,(0,\dots,1,0)\Big) =$$

$$\Big((1,0,\dots,0,0),\dots,(0,\dots,1,\dots,0,m_i),\dots,(0,\dots,1,0)\Big)$$

and $\forall k : k+1 \leq i \leq 2k$

$$(L_{i,i-k}R_{i,i-k}^{-1})^{m_i}\Big((1,0,\dots,0,0),\dots,(0,\dots,1,\dots,0,0),\dots,(0,\dots,1,0)\Big) =$$

$$\Big((1,0,\dots,0,0),\dots,(0,\dots,1,\dots,0,m_i),\dots,(0,\dots,1,0)\Big).$$

□

Infinitely many connected components in Nielsen graphs of Gupta-Sidki p-groups

Observe, that the following result holds for any finitely generated group in the class $\mathcal{MN}$.

Proposition 3.2.7. *Let G be a finitely generated group in $\mathcal{MN}$ and $\Phi(G)$ be the Frattini subgroup. If the Nielsen graph $N_{\text{rank}(G)}(G/\Phi(G))$ is not connected then $N_{\text{rank}(G)}(G)$ is not connected.*

Proof. If two generating tuples of $N_{\text{rank}(G)}(G/\Phi(G))$ are not Nielsen equivalent, then their preimages by Proposition 3.1.8 are generating tuples of $N_{\text{rank}(G)}(G)$, which are not Nielsen equivalent. □

The result of Pervova [85] states that all maximal subgroups of G_p are of finite index. It follows, in particular, that they are normal. Since all maximal subgroups of G_p are normal, it follows that the quotient $G_p/\Phi(G_p)$ is abelian. In addition, any generating set of the quotient $G_p/\Phi(G_p)$ can be lifted up to the generating set of G_p by Proposition 3.1.8. Moreover, in [84] it was shown that the abelianization $G_p^{ab} = G_p/[G_p, G_p]$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$. Therefore $G_p/\Phi(G_p)$ is a quotient of $(\mathbb{Z}/p\mathbb{Z})^2$ of rank 2; we deduce that $G_p/\Phi(G_p) \cong (\mathbb{Z}/p\mathbb{Z})^2$.

Using Proposition 3.2.7 and also the fact that for $p > 3$ there are, by Theorem 3.2.1, $\frac{p-1}{2}$ Nielsen classes of generating pairs of the quotient $G_p/\Phi(G_p) = (\mathbb{Z}/p\mathbb{Z})^2$, we conclude that there are at least $\frac{p-1}{2}$ Nielsen classes in G_p for $p > 3$.

For $p = 3$ the question on the transitivity of the action of $\operatorname{Aut} F_2$ on $\operatorname{Epi}(F_2, G_3)$ is more subtle. We will show in particular that, although there is only one Nielsen class of generating pairs of $G_3/\Phi(G_3) \cong (\mathbb{Z}/3\mathbb{Z})^2$, the action of $\operatorname{Aut} F_2$ is not transitive on $\operatorname{Epi}(F_2, G_3)$. A natural question then is how many orbits this action has.

Theorem (Theorem 1.2.3)**.** *Let $p \geq 3$ be prime and G_p the Gupta-Sidki p-group. Then there are infinitely many Nielsen equivalence classes of generating pairs of G_p.*

To prove Theorem 1.2.3 we use an observation by Nielsen (sometimes also attributed to Higman, see lemma 3.2.8) as well as an analysis on conjugacy classes in the Gupta-Sidki p-group.

Lemma 3.2.8 (Nielsen)**.** *Let (u, v) and (u', v') be two Nielsen equivalent generating pairs of a group G. Then the commutator $[u, v]$ is conjugate either to $[u', v']$ or to $[u', v']^{-1}$.*

The proof of this lemma is a straightforward calculation of commutators of the pairs obtained from (u, v) by the elementary Nielsen moves.

In order to show that two elements are not conjugate in G_3, the Gupta-Sidki 3-group, sometimes we use the finite quotients $G_3/St_{G_3}(n)$ by the n-th level stabilizers. Consider a natural epimorphism

$$\pi\colon G_3 \to G_3/St_{G_3}(4).$$

The finite quotient $G_3/St_{G_3}(4)$ can be seen as a subgroup of $Sym(81)$ with

$$\pi(x) = (1, 28, 55)(2, 29, 56) \dots (27, 54, 81)$$

and

$$\pi(y) = (1, 10, 19) \dots (9, 18, 27)(28, 46, 37) \dots (36, 54, 45)(55, 58, 61)\cdot$$
$$(56, 59, 62)(57, 60, 63)(64, 70, 67)(65, 71, 68)(66, 72, 69)(73, 74, 75)(76, 78, 77).$$

Recall that two elements are conjugate in the symmetric group if and only if their cycle types are the same. Therefore if for two elements $g, h \in G_3$ their images $\pi(g)$ and $\pi(h)$ have different cycle types in $Sym(81)$ then, in particular, they are not conjugate in G_3. Below all computations in $Sym(81)$ were done using GAP.

Example 3.2.9. *The elements $yx^{-1}y^{-1}xy$ and y are not conjugate in G_3. Indeed,*

$$\pi(yx^{-1}y^{-1}xy) = (1, 22, 10, 3, 24, 12, 2, 23, 11)(4, 25, 13, 5, 26, 14, 6, 27, 15)\cdot$$
$$(7, 19, 16)(8, 20, 17)(9, 21, 18)(55, 64, 79)(56, 65, 80)(57, 66, 81)\cdot$$
$$(58, 67, 74, 60, 69, 73, 59, 68, 75)(61, 70, 78, 62, 71, 76, 63, 72, 77),$$

and its cycle type differs from the one of $\pi(y)$.

Let G_3 be the Gupta-Sidki 3-group. Set $z_1 = [x, y] \in [G_3, G_3]$ and for all $n > 1$ set $z_n = (1, 1, z_{n-1})$. The fact that $z_n \in G_3$ follows from [54].

Proposition 3.2.10. *The elements $[x, yz_k]$, $[x, yz_j]^{\pm 1}$ and $z_1^{\pm 1}$ are not pairwise conjugate in G_3 for any $k, j > 2$ such that $k \neq j$.*

Proof. We prove the following two claims in order to conclude the proposition:

Claim 1. *$[x, yz_n]$ is not conjugate to $z_1^{\pm 1}$ for any $n > 2$.*

Claim 2. *$[x, yz_k]$ and $[x, yz_j]^{\pm 1}$ are not conjugate for $k, j > 2$ and $k \neq j$.*

The claims will be proved by contradiction. We compute that $z_1 = [x, y] = (y^{-1}x, x, xy)$ and $[x, yz_n] = (z_{n-1}^{-1}y^{-1}x, x, xyz_{n-1})$.

Proof of Claim 1. Assume that $[x, yz_n]$ and $z_1^{\pm 1}$ are conjugate, then there exists $g = x^i(g_1, g_2, g_3) \in G_3$ for some integer $i \in [0, 2]$, such that $[x, yz_n] = g^{-1}z_1^{\pm 1}g = (g_1^{-1}, g_2^{-1}, g_3^{-1})x^{-i}(y^{-1}x, x, xy)^{\pm 1}x^i(g_1, g_2, g_3)$. Observe that $i = 0$ because x is not conjugate neither to $(y^{-1}x)^{\pm 1}$ nor to $(xy)^{\pm 1}$. Moreover x is not conjugate to x^{-1} in G_3 therefore $[x, yz_n]$ can be conjugate only to z_1. We will prove that it is not the case. *For this it is enough to show that xyz_{n-1} and xy are not conjugate in G_3.* We will show it by induction assuming that

$$(*)\ yz_n \text{ and } y \text{ are not conjugate in } G_3 \text{ for any } n \geq 1$$

and then will show that (*) is indeed the case.

Suppose that xyz_{n-1} and xy are conjugate in G then there exists $g = x^i(g_1, g_2, g_3)$ for some integer $i \in [0, 2]$ such that

$$xyz_{n-1} = x(x, x^{-1}, yz_{n-2}) = (g_1, g_2, g_3)^{-1}x^{-i}(xy)x^i(g_1, g_2, g_3).$$

- If $i = 0$ then $(x, x^{-1}, yz_{n-2}) = (g_3^{-1}, g_1^{-1}, g_2^{-1})(x, x^{-1}, y)(g_1, g_2, g_3)$ and it follows that $g_2 y z_{n-2} g_2^{-1} = y$.
- If $i = 1$ then $(x, x^{-1}, yz_{n-2}) = (g_3^{-1}, g_1^{-1}, g_2^{-1})(y, x, x^{-1})(g_1, g_2, g_3)$ and it follows that $x g_2 y z_{n-2} g_2^{-1} x^{-1} = y$.
- If $i = 2$ then $(x, x^{-1}, yz_{n-2}) = (g_3^{-1}, g_1^{-1}, g_2^{-1})(x^{-1}, y, x)(g_1, g_2, g_3)$ and it follows that $g_2 y z_{n-2} g_2^{-1} = y$.

By assumption (*) elements yz_{n-2} and y are not conjugate in G_3 and we deduce that xyz_{n-1} is not conjugate to xy in G_3 modulo assumption (*).

Proof of the assumption (*): yz_n and y are not conjugate in G_3 for any $n \geq 1$.

1. The assumption holds for $n = 1$. To see this, look at the action of yz_1 and y on the 4th level of the tree, see Example 3.2.9.
2. Suppose (*) is true for $n - 1$.
3. Consider $yz_n = (x, x^{-1}, yz_{n-1})$ and suppose it is conjugate to $y = (x, x^{-1}, y)$. Then there exists $g = x^i(g_1, g_2, g_3) \in G_3$ with $0 \leq i \leq 2$ such that

$$(g_1, g_2, g_3)^{-1} x^{-i} (x, x^{-1}, y) x^i (g_1, g_2, g_3) = (x, x^{-1}, yz_{n-1}).$$

   Since x is not conjugate neither to x^{-1} nor to y in G_3 then $i = 0$. Therefore $(g_1^{-1} x g_1, g_2^{-1} x^{-1} g_2, g_3^{-1} y g_3) = (x, x^{-1}, yz_{n-1})$. We obtain the contradiction with the step of induction.

Proof of Claim 2. We will prove Claim 2 modulo Assumption (*) and (**) below and then in the end prove that both assumptions indeed hold.

Assumption (*): for any $k, j \geq 1$ such that $k \neq j$ the elements yz_k and yz_j are not conjugate in G_3.

Assumption (**): for any $n \geq 2$ the element x is not conjugate to xyz_n or $z_n^{-1} y^{-1} x$ in G_3.

We prove Claim 2 by contradiction. Suppose that there exists $g = x^i(g_1, g_2, g_3) \in G_3$ such that

$$[x, yz_k] = g^{-1} [x, yz_j]^{\pm 1} g$$

or equivalently

$$\begin{gathered}(z_{k-1}^{-1} y^{-1} x, x, xyz_{k-1}) = \\ (g_1^{-1}, g_2^{-1}, g_3^{-1}) x^{-i} (z_{j-1}^{-1} y^{-1} x, x, xyz_{j-1})^{\pm 1} x^i (g_1, g_2, g_3).\end{gathered} \tag{3.4}$$

Observe that x is not conjugate to x^{-1}, $z_{j-1}^{-1} y^{-1} x^{-1}$ and $x^{-1} y z_{j-1}$. To see this, look at the quotient $G_3 / St_{G_3}(1) \cong \mathbb{Z}/3\mathbb{Z}$ and notice that the images of x and x^{-1} are not conjugate in $\mathbb{Z}/3\mathbb{Z}$. Therefore $[x, yz_k]$ cannot be conjugate to $[x, yz_j]^{-1}$. Moreover, it follows from Assumption (**) that $i = 0$ in equation (3.4).

To obtain the contradiction it is sufficient to show that xyz_{k-1} is not conjugate to xyz_{j-1}. Suppose they are conjugate, then there exists $g = x^i(g_1, g_2, g_3) \in G_3$ with $0 \leq i \leq 2$ such that

$$x(x, x^{-1}yz_{k-2}) = (g_1^{-1}, g_2^{-1}, g_3^{-1})x^{-i}x(x, x^{-1}, yz_{j-2})x^i(g_1, g_2, g_3).$$

- If $i = 0$ then $(x, x^{-1}, yz_{k-2}) = (g_3^{-1}xg_1, g_1^{-1}x^{-1}g_2, g_2^{-1}yz_{j-2}g_3)$ and it follows that $yz_{k-2} = g_2^{-1}yz_{j-2}g_2$.
- If $i = 1$ then $(x, x^{-1}, yz_{k-2}) = (g_3^{-1}yz_{j-2}g_1, g_1^{-1}xg_2, g_2^{-1}x^{-1}g_3)$ and it follows that $yz_{k-2} = g_2^{-1}x^{-1}yz_{j-2}xg_2$.
- If $i = 2$ then $(x, x^{-1}, yz_{k-2}) = (g_3^{-1}x^{-1}g_1, g_1^{-1}yz_{j-2}g_2, g_2^{-1}xg_3)$ and it follows that $yz_{k-2} = g_2^{-1}yz_{j-2}g_2$.

By Assumption (*), elements yz_{k-2} and yz_{j-2} are not conjugate in G_3 and we deduce that xyz_{k-1} and xyz_{j-1} are not conjugate in G_3 modulo assumptions (*) and (**).

Proof of the assumption (*) Without loss of generality suppose that $j > k$. Suppose $yz_k = (x, x^{-1}yz_{k-1})$ and $yz_j = (x, x^{-1}, yz_{j-1})$ are conjugate. Then there exists $g = x^i(g_1, g_2, g_3) \in G_3$ with $0 \leq i \leq 2$ such that

$$(x, x^{-1}, yz_{k-1}) = (g_1, g_2, g_3)^{-1}x^{-i}(x, x^{-1}, yz_{j-1})x^i(g_1, g_2, g_3).$$

Since x is not conjugate to x^{-1} or to yz_{j-1} we conclude that $i = 0$ and hence yz_{k-1} and yz_{j-1} are conjugate. Continuing in the same way, we deduce that the elements $yz_1 = (xy^{-1}x, 1, yxy)$ and $yz_{j-k+1} = (x, x^{-1}, yz_{j-k})$ are conjugate. We obtain a contradiction since x is not conjugate to $xy^{-1}x$ or to yxy (to see this it is enough to look at the action of these elements on the 4th level of the tree) or to 1.

Proof of the assumption (**) To see that x is not conjugate to xyz_2 or $z_2^{-1}y^{-1}x$, it is enough to look at the action of these elements on the third level of the tree and to see that they have different cycle types, hence they are not conjugate in the quotient $G_3/St_{G_3}(3)$. And for $n \geq 3$, the action of z_n on the third level is trivial therefore it is enough to look at the action of x, xy and $y^{-1}x$ on the third level to see that they have different cycle types and therefore not conjugate in $G_3/St_{G_3}(3)$. □

Let G_p be the Gupta-Sidki p-group for $p \geq 5$ prime. Set $z_1 = [x, y]$ and for $n > 1$ set $z_n = (1, \ldots, 1, z_{n-1})$. The fact that $z_n \in G_p$ follows from [54].

Proposition 3.2.11. *For any $k, j > 2$ and $k \neq j$ the elements $[x, yz_k]$ and $[x, yz_j]^{\pm 1}$ are not conjugate in G_p.*

Proof. By contradiction, suppose that there exists an element

$$g = x^i(g_1, \ldots, g_p) \ \in \ G_p$$

with $0 \leq i \leq p - 1$ such that

$$[x, yz_k] = g^{-1}[x, yz_j]^{\pm 1}g$$

or, in other words,

$$\begin{gathered}(z_{k-1}^{-1}y^{-1}x, x^{p-2}, x, 1, \dots, 1, yz_{k-1}) = \\ (g_1^{-1}, \dots, g_p^{-1})x^{-i}(z_{j-1}^{-1}y^{-1}x, x^{p-2}, x, 1, \dots, 1, yz_{j-1})^{\pm 1}x^i(g_1, \dots, g_p)\end{gathered} \tag{3.5}$$

Suppose $i \neq 0$. Observe that x is not conjugate to 1, x^{-1}, $x^{-1}yz_{j-1}$, x^{p-2}, x^2, and to $(yz_{j-1})^{\pm 1}$. To see this, look at the quotient $G_p/St_{G_p}(1) \cong \mathbb{Z}/pZ$, and notice that the image of x is not conjugate to the images of the elements above. Therefore x must be conjugate to $z_{j-1}^{-1}y^{-1}x$, in other words there exists $h = x^m(h_1, \dots, h_p) \in G_p$ with $0 \leq m \leq p-1$ such that

$$x = (h_1, \dots, h_p)^{-1}x^{-m} \cdot (a_1, \dots, a_p)x \cdot x^m(h_1, \dots, h_p),$$

where $a_1 = x^{-1}$, $a_2 = x$, $a_p = z_{j-2}^{-1}y^{-1}$ and $a_k = 1$ otherwise.

It follows that the following system of equations holds:

$$\begin{cases} h_p^{-1}a_{\pi^{m+1}(1)}h_1 & = 1 \\ h_1^{-1}a_{\pi^{m+1}(2)}h_2 & = 1 \\ \dots \\ h_{p-1}^{-1}a_{\pi^{m+1}(p)}h_p & = 1, \end{cases}$$

where π^{m+1} is the mth power of the permutation $(1, 2, \dots, p)$ and, for each $1 \leq r \leq p$, $\pi^{m+1}(r)$ denotes the image of r under π^{m+1}.

After solving the system one obtains that

$$h_p^{-1}a_{\pi^{m+1}(1)}a_{\pi^{m+1}(2)} \dots a_{\pi^{m+1}(p)}h_p = 1,$$

which gives us a contradiction to $i \neq 0$.

In view of equation (3.5) and that $i = 0$, in order to obtain a contradiction to the initial assumption that $[x, yz_k]$ is conjugate to $[x, yz_j]^{\pm 1}$, it is enough to show that yz_{k-1} is not conjugate to yz_{j-1}. Without loss of generality suppose that $k > j$.

Suppose by contradiction that yz_{k-1} is conjugate to yz_{j-1}, i.e. there exists $h = (h_1, \dots, h_p)x^l \in G_p$ with $0 \leq l \leq p-1$ such that

$$\begin{gathered}(x, x^{-1}, 1, \dots, 1, yz_{k-2}) = \\ (h_1, \dots, h_p)^{-1}x^{-l}(x, x^{-1}, 1, \dots, 1, yz_{j-2})x^l(h_1, \dots, h_p).\end{gathered}$$

Observe that x is not conjugate to 1, x^{-1} and yz_{j-2}. Hence $l = 0$ and therefore yz_{k-2} is conjugate to yz_{j-2}. We repeat the same arguments $j-2$ times to conclude that $yz_{k-j+1} = (x, x^{-1}, 1, \dots, 1, yz_{k-j})$ and $yz_1 = (xy^{-1}x, x^{p-3}, x, 1, \dots, 1, y^2)$ are conjugate. Observe that x^{-1} is not conjugate to 1, $xy^{-1}x$, x^{p-3}, x and y^2. The contradiction then follows and we deduce that yz_{k-1} is not conjugate to yz_{j-1} which concludes the proof.

□

We are now able to deduce that there are infinitely many Nielsen equivalence classes of generating pairs of the Gupta-Sidki p-group for any $p \geq 3$ prime.

Proof of Theorem 1.2.3. Fix $p \geq 3$ prime. Let $z_1 = [x, y] \in [G_p, G_p]$ and for all $n > 1$ let $z_n = (1, \ldots, 1, z_{n-1}) \in G_p$. It follows from Theorem 4.1.1 [54] that $z_n \in [G_p, G_p]$. Since $[G_p, G_p] = \Phi(G_p)$ and $\langle x, y\rangle = G_p$ then by Proposition 3.1.2 we deduce that $\langle x, yz_n\rangle = G_p$. We conclude by Lemma 3.2.8, Proposition 3.2.10 and Proposition 3.2.11 that there are infinitely many orbits of the action $\operatorname{Aut} F_2 \curvearrowright \operatorname{Epi}(F_2, G_p)$. □

The Gupta-Sidki p-group being a subgroup of $\operatorname{Aut} T_p$, the group of automorphisms of the regular p-ary rooted tree T_p, has natural quotients by $St_G(n)$, the level stabilizer subgroups. These quotients are finite nilpotent 2-generated groups with growing nilpotency class. The latter is true since the limit of these quotients in the space of marked 2-generated groups is the Gupta-Sidki p-group itself, which is not finitely presentable [87].

We show that for each $n \geq 1$ the quotient group $G^{(n)} = G_3/St_{G_3}(n+3)$ of the Gupta-Sidki 3-group has the property that the action $\operatorname{Aut} F_2 \curvearrowright \operatorname{Epi}(F_2, G^{(n)})$ is not transitive. Note that there is only one Nielsen equivalence class in $(\mathbb{Z}/3\mathbb{Z})^2$, the abelianization of each $G^{(n)}$. It would be interesting to realize whether the number of Nielsen classes grows with n but this for the moment remains an open question. An affirmative answer on this question, in particular, would imply that there were infinitely many Nielsen equivalence classes in G_3. Notice, however, that the proof of Theorem 1.2.3 does not rely on Proposition 3.2.12.

Proposition 3.2.12. *Let G_3 be the Gupta-Sidki 3-group and $St_{G_3}(n)$ the level stabilizer subgroups of G_3. Set $G^{(n)} = G_3/St_{G_3}(n+3)$. Then the action $\operatorname{Aut} F_2 \curvearrowright \operatorname{Epi}(F_2, G^{(n)})$ is not transitive for every $n \geq 1$.*

Proof of Proposition 3.2.12. First, we show that the graph $N_2(G_3/St_{G_3}(4))$ is not connected. Consider two pairs $(u, v) = (x, y)$ and $(u', v') = (x^{-1}y^{-1}xy \cdot x, y)$ in G_3. Since $\langle x, y\rangle = G_3$ and $[G_3, G_3] = \Phi(G_3)$, it follows that (u', v') is also a generating pair of G_3 by Lemma 3.1.2.

Denote the images of (u, v) and (u', v') in the finite quotient $G_3/St_{G_3}(4)$ by $(\overline{u}, \overline{v})$ and $(\overline{u'}, \overline{v'})$. Clearly the pairs $(\overline{u}, \overline{v})$ and $(\overline{u'}, \overline{v'})$ are generating. If they are Nielsen equivalent then by Nielsen criterion (Lemma 3.2.8) their commutators $[\overline{u}, \overline{v}]$ and $[\overline{u'}, \overline{v'}]^{\pm 1}$ must be conjugate in $Sym(81)$ and, in particular, their cycle types must be the same. We will obtain the contradiction with the latter.

We calculate the commutators respectively :

$[\overline{u}, \overline{v}] = (1, 16, 19, 3, 18, 21, 2, 17, 20)(4, 10, 22, 5, 11, 23, 6, 12, 24)(7, 13, 25)(8, 14, 26)$
$(9, 15, 27)(28, 37, 46)(29, 38, 47)(30, 39, 48)(31, 40, 49)(32, 41, 50)(33, 42, 51)(34, 43, 52)$
$(35, 44, 53)(36, 45, 54)(55, 70, 79)(56, 71, 80)(57, 72, 81)$
$(58, 64, 74, 59, 65, 75, 60, 66, 73)\ (61, 67, 78, 63, 69, 77, 62, 68, 76)$,

$[\overline{u'}, \overline{v'}] = (1, 10, 25, 2, 11, 26, 3, 12, 27)(4, 15, 21)(5, 13, 19)(6, 14, 20)(7, 17, 23, 9, 16,$
$22, 8, 18, 24)\ (28, 41, 53, 29, 42, 54, 30, 40, 52)(31, 45, 47, 32, 43, 48, 33, 44, 46)(34, 37, 50$
$, 35, 38, 51, 36, 39, 49)\ (55, 70, 79)(56, 71, 80)(57, 72, 81)(58, 64, 74, 59, 65, 75, 60, 66, 73)$
$(61, 67, 78, 63, 69, 77, 62, 68, 76)$.

The cycle types of $[\overline{u}, \overline{v}]$ and $[\overline{u'}, \overline{v'}]^{\pm 1}$ are different therefore $(\overline{u}, \overline{v})$ and $(\overline{u'}, \overline{v'})$ are not Nielsen equivalent.

For any $l \geq 4$, there exists an epimorphism from $G_3/St_{G_3}(l)$ to $G_3/St_{G_3}(4)$. We will show that the Nielsen graph $N_2(G_3/St_{G_3}(l))$ is not connected using Gaschütz lemma [36]. Gaschütz lemma asserts that if there exists an epimorphism between finite groups $f\colon G \to H$ and $m \geq \operatorname{rank}(G)$ then for any generating m-tuple $(h_1, \ldots, h_m)$ of H there exists a generating m-tuple $(g_1, \ldots, g_m)$ of G with $f(g_i) = h_i$ for $i = 1, \ldots, m$. Hence the generating pairs $(\overline{u}, \overline{v})$ and $(\overline{u'}, \overline{v'})$ of $G_3/St_{G_3}(4)$ have preimages, generating pairs in $G_3/St_{G_3}(l)$, which are not Nielsen equivalent. The proof is completed.

□

3.3 Nielsen equivalence in relatively free groups

In this section we will introduce a notion of a relatively free group G in some variety $\mathcal{B}$ and discuss connectedness of the Nielsen graph $N_n(G)$. We will mostly deal with the case $n = \operatorname{rank}(G)$.

A *variety of groups* $\mathcal{B}$ is a class of groups that satisfy a fixed system of relations

$$\{v = 1\}_{v \in \mathcal{V}}$$

where v runs through a set $\mathcal{V}$ of finite length freely reduced words over some alphabet X and the inverse X^{-1}, called the laws of the variety. In other words, a group G is in $\mathcal{B}$ if and only if all laws $\{v = 1\}_{v \in \mathcal{V}}$ hold in G when elements of G are substituted for the letters.

Examples of varieties of groups include the variety of all groups defined by the empty set of laws, the variety of abelian groups defined by the commutative law $xy = yx$, nilpotent groups of a given nilpotency class, solvable groups of a given derived length and so on. Another example is the "Burnside" variety of groups of exponent p defined by the law $x^p = 1$. By a theorem of Birkhoff [13], a class of groups is a variety if and only if it is closed under taking subgroups, homomorphic images and unrestricted direct products.

Let $\mathcal{B}$ be a variety of groups with the set of laws $\{v = 1\}_{v \in \mathcal{V}}$. For an arbitrary group G denote by $\mathcal{V}(G)$ the subgroup of G generated by all values of words $v \in \mathcal{V}$ when elements of G are substituted for letters. The subgroup $\mathcal{V}(G)$ is called the *verbal subgroup* of G defined by $\mathcal{V}$. It is easy to see that $G \in \mathcal{B}$ if and only if $\mathcal{V}(G) = \{1\}$. Verbal subgroups are fully invariant (i.e., invariant by all endomorphisms of the group), in particular characteristic.

Every variety $\mathcal{B}$ of groups with the set of laws $\mathcal{V}$ contains for all $d \geq 1$ the "*relatively free group*" of rank d, which is the factor of the free group F_d by its verbal subgroup $\mathcal{V}(F_d)$. Examples of relatively free groups include free groups, free abelian groups, free nilpotent groups $F_{d,c}$ of rank d and nilpotency class c, free solvable groups $F_{d,l}$ of rank d and derived length l, free Burnside groups $B(d, m)$ of rank d and exponent m and so on.

Let F_d be the free group of rank $d \geq 2$ and let $\mathcal{V}$ be a verbal subgroup of F_d. Denote by G the corresponding relatively free group $F_d/\mathcal{V}$. As $\mathcal{V}$ is characteristic, the natural mapping $\pi : F_d \to G$ induces a homomorphism

$$\rho : \operatorname{Aut} F_d \to \operatorname{Aut} G. \tag{3.6}$$

Elements of the image of ρ are called *tame automorphisms* of G. We denote by $T(G)$ the subgroup of tame automorphisms in $\operatorname{Aut} G$. Note that $T(G) \cong \operatorname{Aut} F_d / \operatorname{Ker} \rho$. Note also that the set

$$S = \{\rho(R_{ij}^{\pm}), \rho(L_{ij}^{\pm}), \rho(I_j), 1 \leq i, j \leq d, i \neq j)\}$$

of images of elementary Nielsen moves is a finite generating set of $T(G)$.

Lemma 3.3.1. *For a relatively free group G of rank d there is a bijection between* $\operatorname{Aut} G$ *and* $\operatorname{Epi}(F_d, G)$.

Proof. Notice that for every d-generated group $G = \langle x_1, \ldots, x_d \rangle$ there is a natural action of $\operatorname{Aut} G$ on $\operatorname{Epi}(F_d, G)$ by composition, and this action is free. So by fixing a generating d-tuple $(x_1, \ldots, x_d)$ in G we can map $\operatorname{Aut} G$ bijectively on the $\operatorname{Aut} G$-orbit of $(x_1, \ldots, x_d)$.

We will now show that if G is relatively free, then any element of $\operatorname{Epi}(F_d, G)$ belongs to this orbit. First, observe that since $\mathcal{V}$ is a verbal subgroup of F_d, G has the same presentation $G = \langle g_1, \ldots, g_d \mid v \in \mathcal{V} \rangle$ for any generating d-tuple $(g_1, \ldots, g_d)$. Second, recall that two groups having the same presentation are isomorphic (see [61], Theorem 1.1). From this we deduce that any generating d-tuple $(g_1, \ldots, g_n)$ is the image of $(x_1, \ldots, x_d)$ by an automorphism of G. □

We now have the following description of the graph $N_d(G)$.

Theorem 3.3.2. *Let G be a relatively free group of rank d. Denote by $i \in \mathbb{N} \cup \{\infty\}$ the index of the subgroup $T(G)$ of tame automorphisms in the full group of automorphisms* $\operatorname{Aut} G$. *Then the Nielsen graph $N_d(G)$ consists of i connected components, each of them isomorphic to the Cayley graph $Cay(T(G), S)$ of $T(G)$ with respect to the set S determined by the elementary Nielsen moves.*

Proof. Let $(g_1, \ldots, g_d)$ be a generating d-tuple of G. Think about it as $\pi(x_1) = g_1, \ldots, \pi(x_d) = g_d$ for a free basis $x_1, \ldots, x_d$ of F_d and the projection $\pi : F_d \to G$. Then for any $\sigma \in \operatorname{Aut} F_d$ the action of $\rho(\sigma)$ is given by $\rho(\sigma)(g_k) = \pi(\sigma(x_k))$ for $1 \leq k \leq d$, with ρ defined by (3.6).

We consider the action of $\operatorname{Aut} F_d$ on $\operatorname{Epi}(F_d, G)$ and prove that every connected component of the Nielsen graph $N_d(G)$ is $Cay(T(G), S)$. For this we show that $St_{\operatorname{Aut} F_d}(g_1, \ldots, g_d) = \operatorname{Ker} \rho$. Assume that $\sigma \in St_{\operatorname{Aut} F_d}(g_1, \ldots, g_d)$. It then defines a trivial map on generators and therefore a trivial automorphism of G. Hence $\sigma \in \operatorname{Ker} \rho$. Conversely, if $\sigma \in \operatorname{Ker} \rho$ then $\rho(\sigma)(g_1, \ldots, g_d) = (\pi(\sigma(x_1), \ldots, \pi(\sigma(x_d)) = (\pi(x_1), \ldots, \pi(x_d))$ and, by definition of the action of $\sigma \in \operatorname{Aut} F_d$ on $\operatorname{Epi}(F_d, G)$, as explained in Section 1.1, $\sigma \in St_{\operatorname{Aut} F_d}(g_1, \ldots, g_d)$. Since the subgroup $\operatorname{Ker} \rho$ is normal in $\operatorname{Aut} F_d$, we conclude that every connected component of $N_d(G)$ is the Cayley graph $Cay(T(G), S)$.

Assume that two generating d-tuples U_1 and U_2 lie in different connected components of $N_d(G)$, i.e. $\forall \sigma \in \operatorname{Aut} F_d$ we have $U_1^\sigma \neq U_2$. By Lemma 3.3.1 the tuples U_1 and U_2 define automorphisms of G, namely, $U_1 = \varphi_1(g_1, \ldots, g_d)$, $U_2 = \varphi_2(g_1, \ldots, g_d)$ for some $\varphi_1, \varphi_2 \in \operatorname{Aut} G$. Since U_1 and U_2 are not Nielsen equivalent we have $\rho(\sigma)\varphi_1(g_1, \ldots, g_d) \neq \varphi_2(g_1, \ldots, g_d)$ for all $\sigma \in \operatorname{Aut} F_d$. Therefore two automorphisms define two different connected components if and only if they lie in different right cosets of the subgroup $T(G)$ in $\operatorname{Aut} G$. We conclude that the number of connected components is equal to the index $[\operatorname{Aut} G : T(G)]$. □

We deduce the following corollary.

Corollary (Corollary 1.2.4)**.** *Let G be a relatively free group of rank d. Then the Nielsen graph $N_d(G)$ is connected if and only if all automorphisms of G are tame.*

Examples

Recall that a group G is called *polynilpotent* ([88]) if it admits a finite normal series $G \geq G_{m_1} \geq G_{m_1,m_2} \geq ... \geq 1$ where G_{m_1} is the m_1-th member of its lower central series, G_{m_1,m_2} is the m_2-th member of the lower central series of the group G_{m_1} and so on.

We consider separately the cases of free abelian, free nilpotent, free (nilpotent of class 2)-by-abelian, free metabelian and free center-by-metabelian groups to describe what is known about connectedness of Nielsen graphs of free polynilpotent groups.

It is well known that the map $\operatorname{Aut} F_d \to GL_d(\mathbb{Z})$ is onto, so that all automorphisms of the free abelian groups are tame. Moreover, not only $N_d(\mathbb{Z}^d)$ but all Nielsen graphs of free abelian groups are connected. Indeed, if $n > d$ then for any generating n-tuple $(v_1, \dots, v_n)$, the vectors $v_1, \dots, v_n$ are linearly dependent in $\mathbb{Z}^d$ viewed as a $\mathbb{Z}$-module. Without loss of generality let $v_1, \dots, v_d$ be a linearly independent set of vectors that generates $\mathbb{Z}^d$ and deduce that $(v_1, \dots, v_n) \sim (e_1, \dots, e_d, 1, \dots, 1)$ where $e_1, \dots, e_d$ is the standard basis for $\mathbb{Z}^d$.

Let $F_{d,c}$ be the free nilpotent group $F_{d,c}$ of rank $d \geq 1$ and nilpotency class $c \geq 1$. If $n \geq d+1$ then the Nielsen graph $N_n(F_{d,c})$ is connected ([30]).

We will further discuss the connectedness of $N_n(F_{d,c})$ for $n = d$. Andreadakis [5] showed that all automorphisms of $\operatorname{Aut} F_{d,c}$ are tame when $c = 1$ and $c = 2$. Note the particular cases: if $c = 1$ then $F_{d,1} = \mathbb{Z}^d$; if $c = 2$ and $d = 2$ then $F_{2,2}$ is the Heisenberg group $\mathcal{H}_1 = \langle x, y \mid [x,[x,y]], [y,[x,y]] \rangle$ (see the previous section for connectedness of Nielsen graphs of Heisenberg groups).

It has been shown however that when $c \geq 3$, the group $\operatorname{Aut} F_{d,c}$ contains non-tame automorphisms [5, 8]. Using non-tame automorphisms produced in [5] we show that $N_2(F_{2,3})$ contains infinitely many connected components.

Proposition 3.3.3. *For the free nilpotent group $F_2(3)$ there are infinitely many Nielsen equivalence classes of generating pairs of $F_2(3)$.*

Proof. Andreadakis [5] showed that for $F_2(3) = \langle x, y \rangle$, the central automorphism of $F_2(3)$

$$\left\{ \begin{array}{lll} x & \mapsto & x[y,x,x]^{\lambda_1}[y,x,y]^{\lambda_2} \\ y & \mapsto & y[y,x,x]^{\mu_1}[y,x,y]^{\mu_2} \end{array} \right. \tag{3.7}$$

is tame only if $\lambda_1 = \mu_2$ and $\lambda_2 = \mu_1 = 0^{\dagger}$. We consider a non-tame automorphism of $F_2(3)$:

$$\alpha : \left\{ \begin{array}{lll} x & \mapsto & x[y,x,x] \\ y & \mapsto & y \end{array} \right. ,$$

†[a,b,c] means the commutator [[a,b],c].

and show that there is no $\sigma \in \operatorname{Aut} F_2$ such that $\alpha^i = \sigma\alpha^j$, for $i > j$. Equivalently, there is no $\sigma \in \operatorname{Aut} F_2$ such that $\alpha^k = \sigma$ for an arbitrary positive integer k. By the criterion stated above the automorphism α^k is tame if it is trivial. Consider

$$\alpha^2(x) = x[y,x,x]\Big[y, x[y,x,x], x[y,x,x]\Big] = x[y,x,x]^2.$$

More generally,

$$\alpha^k : \begin{cases} x & \mapsto & x[y,x,x]^k \\ y & \mapsto & y \end{cases}$$

Hence α^k is not trivial for all $k > 0$. □

For free (nilpotent of class 2)-by-abelian groups $G_d = F_d/[F_d^{'}, F_d^{'}, F_d^{'}]$, Gupta and Levin [51] proved that the group $\operatorname{Aut} G_4$ contains non-tame automorphisms. Papistas [82] extended their result to $d \geq 4$ and also showed that in the case $d = 2$ and $d = 3$ the group $\operatorname{Aut} G_d$ is not finitely generated. Therefore $N_2(G_2)$ and $N_3(G_3)$ have infinitely many connected components by Corollary 1.2.4.

For free metabelian groups $M_d = F_d/[\gamma_2(F_d), \gamma_2(F_d)]$, where $\gamma_2(F_d)$ is the second derived subgroup, Bachmuth and Mochizuki [9, 10] proved that M_2 and M_d, $d \geq 4$, have only tame automorphisms. However Chein [19] showed that M_3 has non-tame automorphisms, and moreover $\operatorname{Aut} M_3$ is not finitely generated [9]. Corollary 1.2.4 then implies that there are infinitely many connected components in $N_3(M_3)$.

For free center-by-metabelian groups $G_d = F_d/[\gamma_2(F_d), F_d]$, Stöhr [89] proved that $\operatorname{Aut} G_d$ is not finitely generated for $d = 2$ and $d = 3$, so we can again conclude by Corollary 1.2.4 that there are infinitely many connected components of $N_d(G_d)$ for $d = 2$ or $d = 3$. For $d \geq 4$, the group $\operatorname{Aut} G_d$ is generated by tame automorphisms and at most one additional automorphism [89], but the question whether all automorphisms are tame remains open.

Let us now consider the free Burnside group $B(d,m) = F_d/F_d^m$ where F_d^m is the verbal subgroup of F_d generated by the law $x^m = 1$, $m \geq 2$, $d \geq 2$.

Moriah and Shpilrain observed in [67] that when $d = 2$ and $m \geq 5$ then the graph $N_2(B(2,m))$ is not connected. The argument goes as follows. Pick q such that $1 < q < m-1$ and $gcd(q,m) = 1$ and let (x_1, x_2) be a generating pair of $B(2,m)$. The map $x_1 \mapsto x_1^q$, $x_2 \mapsto x_2$ can be extended to an automorphism φ of $B(2,m)$ which is not tame. Indeed, if φ can be lifted to $\varphi' \in \operatorname{Aut} F_2$ then φ' induces an automorphism on $\mathbb{Z}^2$ given by the following matrix:

$$\begin{pmatrix} q + pk_1 & pk_2 \\ pk_3 & 1 + pk_4 \end{pmatrix},$$

for some $k_1, k_2, k_3, k_4 \in \mathbb{Z}$. The determinant of this matrix is not equal to ± 1 therefore it does not belong to $GL_2(\mathbb{Z})$. We obtain a contradiction with φ' being an automorphism of $\operatorname{Aut} F_2$.

More generally, for the free Burnside group on d generators $x_1, \ldots, x_d$, for $m \geq 5$ odd, the graph $N_d(B(d,m))$ is not connected. Indeed, the arguments above can be applied to show that the following automorphism defined on the generators $x_1 \mapsto x_1^2, x_2 \mapsto x_2, \ldots, x_d \mapsto x_d$ is not tame.

CHAPTER 4

Andrews-Curtis equivalence

The famous *Andrews-Curtis conjecture* asserts that, for a free group F_n of rank $n \geq 2$ and a free basis $(x_1, \ldots, x_k)$ of F_n, every normally generating n-tuple of $(y_1, \ldots, y_n)$ of F_n can be transformed into $(x_1, \ldots, x_n)$ by a sequence of elementary Andrews-Curtis moves. Note, that very often this conjecture is formulated in terms of balanced presentations of the trivial group. A *balanced presentation* is a presentation of a group that uses a finite number of generators and an equal number of relations. The Andrews-Curtis conjecture stated above is equivalent to the following: for $n \geq 2$ every balanced presentation $\langle x_1, \ldots, x_n \mid w_1, \ldots, w_n \rangle$ of the trivial group can be transformed using elementary Andrews-Curtis moves on $\{w_1, \ldots, w_n\}$ to the trivial presentation $\langle x_1, \ldots, x_n \mid x_1, \ldots, x_n \rangle$.

Similarly, as the Nielsen graph is associated to the Nielsen equivalence, we define below the Andrews-Curtis graph associated to the Andrews-Curtis equivalence.

As in [57], the *weight* $w(G)$ of a group G is the minimal number of normal generators of G. The weight and the rank of an abelian group A clearly coincide. Notice, that if a set S normally generates G then its image in the abelianization $G^{ab} = G/[G, G]$ generates the abelianization G^{ab}. It follows, in particular, that for the free group F_n we have $w(F_n) = \text{rank}(F_n)$.

Given an integer $n \geq w(G)$ and a finitely generated group G, the *Andrews-Curtis graph* (AC-graph) $AC_n(G)$ of the group G with respect to the generating set S is defined as follows:

- the set of vertices consists of normally generating n-tuples, *i.e.*

$$V_{AC_n}(G) = \{(g_1, ..., g_n) \in G^n \mid \ll g_1, ..., g_n \gg = G\};$$

- two vertices are connected by an edge if one of them is obtained from the other by an elementary AC-move.

The graph $AC_n(G)$ is a regular graph which admits loops and multiple edges. The connectedness of the Andrews-Curtis graph of the group G does no depend on the choice of the generating set. Further, when we discuss connectedness of the Andrews-Curtis graph of the group G, we will not specify the generating set.

The Andrews-Curtis conjecture can be reformulated in terms of graphs.

Conjecture 4.0.4 (The Andrews-Curtis conjecture)**.** *For a free group F_n of rank $n \geq 2$, the Andrews-Curtis graph $AC_n(F_n)$ is connected.*

There are doubts as to whether the Andrews-Curtis conjecture is true. The way to disprove it would be to find two normally generating systems of F_n that are not Andrews-Curtis equivalent.

Akbulut and Kirby [2] suggest a series of potential counter-examples for $F_2 = \langle x, y \rangle$, *i.e.* normally generating pairs which are not known to be AC equivalent to (x, y):

$$(u, v_l) = (xyxy^{-1}x^{-1}y^{-1}, x^l y^{-(l+1)}),\ l \geq 4. \tag{4.1}$$

Note, that these normally generating pairs arise from the presentation of the trivial group $\{x, y | xyx = yxy, x^l = y^{l+1}\}$. This presentation is seen to be the trivial group as follows: $xyx = yxy$ implies that $x = (yx)x(yx)^{-1}$, so $x^{l+1} = yxy^{l+1}(yx)^{-1} = yx^l y^{-1} = y^{l+1} = x^l$ and so $x = 1$ and $y = 1$.

A way to confirm a potential counter-example is to show that its image in a quotient G of F_n and that of the basis $(x_1, \ldots, x_n)$ are not connected in $AC_n(G)$. The easiest quotient to consider is an abelian group A: the Andrews-Curtis graph of A coincides with its Nielsen graph and the Nielsen equivalence for finitely generated abelian groups is fully understood, see Theorem 3.2.1. Notice that the images of (u, v_l) in any abelian group of rank 2 satisfy the following equivalence:

$$(xyxy^{-1}x^{-1}y^{-1}, x^l y^{-(l+1)}) \sim_{AC} (x, y)$$

so for every homomorphism $\phi : F_2 \to A$ into an abelian group A, the images of the pairs (4.1) are AC equivalent.

Observe that any two normally generating n-tuples of the free group F_n are Nielsen equivalent in the abelian quotient A of F_n. Indeed, let us fix the basis $(x_1, \ldots, x_n)$ of F_n and let $(y_1, \ldots, y_n)$ be a normally generating n-tuple of F_n. Then for the natural projection $\pi\colon F_n \to \mathbb{Z}^n$ the generating matrix associated to $(\pi(y_1), \ldots, \pi(y_n)) \in GL_n(\mathbb{Z})$ has determinant ± 1. Consider the natural projection $\tau\colon \mathbb{Z}^n \to A$: the determinant of the generating matrix of $\tau(\pi(y_1)), \ldots, \tau(pi(y_n))$ has its absolute value 1 and it follows from Theorem 3.2.2 that the projection of $(y_1, \ldots, y_n)$ and that of $(x_1, \ldots, x_n)$ in any abelian quotient A are Nielsen equivalent.

It was suggested in [15] that one could confirm one of the proposed potential counter-examples by showing that for some homomorphism $\phi : F_2 \to G$ to a finite group G, the images of the pairs (4.1) are not Andrews-Curtis equivalent.

In view of the latter, [16] considered the class of groups with the following property: for any $n \geq \max\{w(G), 2\}$, two normally generating n-tuples U, V are AC equivalent in G if and only if their images are Nielsen equivalent in the abelianization G^{ab}. Therefore groups from this class will not confirm the potential counter-examples (4.1). Borovik, Lubotzky and Myasnikov [16] proved that all finite groups belong to this class. They also ask whether it is true for the Grigorchuk group which is just-infinite, *i.e* all proper quotients of the group are finite. Our Theorem 1.3.2 provides a characterization of the connected components of the Andrews-Curtis graph for finitely generated groups in $\mathcal{MN}$, giving in particular an affirmative answer on their question.

Theorem (Theorem 1.3.2)**.** *Let G be a finitely generated group in $\mathcal{MN}$ and $n \geq w(G) = \operatorname{rank}(G)$. Then two normally generating n-tuples U, V are AC equivalent if and only if they are Nielsen equivalent in the abelianization $G^{ab} = G/[G,G]$. In other words, the connected components of the AC-graph $AC_n(G)$ are precisely the preimages of the connected components of the Nielsen graph $N_n(G^{ab})$.*

Before proving the theorem, we will show the following technical lemma.

Lemma 4.0.5. *Let G be a group generated by two elements x and y. Denote by $[G,G]$ the commutator subgroup of G. Then for any $\varphi \in [G,G]$ the normally generating pairs $(x\varphi, y)$ and (x, y) are AC equivalent. Similarly, the normally generating pairs $(x, y\varphi)$ and (x, y) are AC equivalent.*

Proof. Denote by $\operatorname{ord}_G(g)$ the order of g in G. One computes that

$$\{x^{n_1}y^{n_2} | n_1 \in (-\operatorname{ord}_G(x), \operatorname{ord}_G(x)), n_2 \in (-\operatorname{ord}_G(y), \operatorname{ord}_G(y))\}$$

is a right coset representative system for G mod $[G,G]$. Using the Reidemeister-Schreier rewriting process we find a set S of generators for $[G,G]$:

$$S = \{x^{n_1}y^{n_2}x(x^{n_1+1}y^{n_2})^{-1}, n_1 \in (-\operatorname{ord}_G(x), \operatorname{ord}_G(x)), n_2 \in (-\operatorname{ord}_G(y), \operatorname{ord}_G(y))\}.$$

We will proceed by induction on the length of $\varphi \in [G,G]$ in generators of S to prove that (x,y) and $(x\varphi, y)$ are AC equivalent. Let $\varphi = s\varphi'$ with $s \in S$ and $\varphi' \in [G,G]$ such that $l_S(\varphi') < l_S(\varphi)$, where l_S denotes the length of an element in the alphabet S. Then

$$\begin{gathered}(x\varphi, y) = (xs\varphi', y) = (x \cdot x^{n_1}y^{n_2}xy^{-n_2}x^{-n_1-1}\varphi', y) \sim_{I_2,AC} \\ (x \cdot x^{n_1}y^{n_2}xy^{-n_2}x^{-n_1-1}\varphi', x^{n_1+1}y^{-1}x^{-n_1-1}) \sim_{L_{12}\ |n_1|+1 \text{ times}} \\ (x^{n_1+2}y^{-n_2}x^{-n_1-1}\varphi', x^{n_1+1}y^{-1}x^{-n_1-1}) \sim_{AC,I_2} (x^{n_1+2}y^{-n_2}x^{-n_1-1}\varphi', y) \sim_{AC} \\ (x^{n_1+2}y^{-n_2}x^{-n_1-1}\varphi', x^{n_1+2}yx^{-n_1-2}) \sim_{L_{12}\ |n_2| \text{ times}} (x\varphi', x^{n_1+2}yx^{-n_1-2}) \sim_{AC} \\ (x\varphi', y).\end{gathered}$$

By induction we conclude that $(x\varphi, y)$ and (x, y) are AC equivalent.

Similarly, we will proceed by induction on the length of $\varphi \in [G,G]$ in generators of S to prove that (x,y) and $(x, y\varphi)$ are AC equivalent. We have:

$$(x, y\varphi) = (x, yx^{n_1}y^{n_2}x(x^{n_1+1}y^{n_2})^{-1}\varphi') \sim_{I_1,AC}$$
$$(yx^{n_1}y^{n_2}x^{-1}y^{-n_2}x^{-n_1}y^{-1}, yx^{n_1}y^{n_2}x(x^{n_1+1}y^{n_2})^{-1}\varphi') \sim_{L_{21},AC,I_1} (x, yx^{-1}\varphi') \sim_{AC}$$
$$(yxy^{-1}, yx^{-1}\varphi') \sim_{L_{21},AC} (x, y\varphi').$$

By induction (x, y) and $(x, y\varphi)$ are AC equivalent. □

Remark. Lemma 4.0.5 can be generalised to the group G generated by n elements $x_1, \dots, x_n$. In this case, $[G, G] = \langle S \rangle$, with

$$S = \{x_1^{m_1}...x_n^{m_n}x_l(x_1^{m_1}...x_l^{m_l+1}...x_n^{m_n})^{-1}, m_i \in (-\operatorname{ord}_G(x_i), \operatorname{ord}_G(x_i)), m_n \neq 0, 1 \leq l < n\}.$$

One can repeat the arguments above to show that for any $\varphi \in [G, G]$ the normally generating n-tuple $(x_1, \dots, x_i\varphi, \dots, x_n)$ is AC equivalent to $(x_1, \dots, x_n)$ for all i: $1 \leq i \leq n$.

Proof of Theorem 1.3.2. Let G be a finitely generated group in $\mathcal{MN}$. By Proposition 3.1.3, normally generating sets coincide with generating sets of G and $\operatorname{rank}(G) = w(G)$. Consider two generating n-tuples U, V in G.

Assume that U and V are AC equivalent. Then for any normal subgroup $N \lhd G$ their images in G/N are AC equivalent. In particular, they are Nielsen equivalent when $N = [G, G]$.

Assume now that the images of U, V in G^{ab} are Nielsen equivalent.

Let us first consider the case $n = \operatorname{rank}(G) = 2$. To prove that $U = (u_1, u_2)$ and $V = (x, y)$ are AC equivalent, it is sufficient to prove that (x, y) and $(x\varphi_1, y\varphi_2)$ are AC equivalent for every φ_1 and φ_2 in $[G, G]$. By lemma 4.0.5 we have that for every $\varphi_1 \in [G, G]$ the generating pair (x, y) is AC equivalent to $(x\varphi_1, y)$. Now let $\hat{x} = x\varphi_1$ and observe that $G = \langle \hat{x}, y \rangle$. By lemma 4.0.5 we conclude that $(\hat{x}, y)$ is AC equivalent to $(\hat{x}, y\varphi_2)$ for any φ_2 in $[G, G]$. We deduce that (x, y) and $(x\varphi_1, y\varphi_2)$ are AC equivalent.

Let $n = \operatorname{rank}(G) \geq 3$ and $G = \langle x_1, \dots, x_n \rangle$. As indicated in the remark above, Lemma 4.0.5 can be generalised to the case of G generated by $x_1, \dots, x_n$ and the proof above can be repeated with more similar cases to consider.

Finally, assume $n > \operatorname{rank}(G)$, and for simplicity $\operatorname{rank}(G) = 2$. Let us fix a system of generators $\{x, y\}$ of G.

Since $G/[G, G]$ is abelian and $n > \operatorname{rank}(G/[G, G])$, the

$$AC_n(G/[G, G]) = N_n(G/[G, G])$$

is connected by Theorem 3.2.1. We need to prove that $AC_n(G)$ is connected. It is sufficient to show that $(x\varphi_1, y\varphi_2, \varphi_3, ..., \varphi_n) \sim_{AC} (x, y, 1, ..., 1)$, for all $\varphi_1, ..., \varphi_n$ in $[G, G]$. We use that $(x\varphi_1, y\varphi_2) \sim_{AC} (x, y)$ for all φ_1, φ_2 in $[G, G]$, and conclude with the following: $(x\varphi_1, y\varphi_2, \varphi_3, ..., \varphi_n) \sim_{AC} (x, y, \varphi_3, ..., \varphi_n) \sim_{AC} (x, y, \varphi_3 \cdot (\varphi_3)^{-1}, ..., \varphi_n \cdot (\varphi_n)^{-1}) = (x, y, 1, ..., 1)$.

If $n > \operatorname{rank}(G) > 2$ then one concludes that $AC_n(G)$ is connected with the same type of arguments. □

There is a different proof for the fact that if $\langle x_1, ..., x_n \rangle = G$ then $\forall c \in [G, G]$: $(x_1, ..., x_n) \sim_{AC} (x_1 c, x_2, ..., x_n)$ (see [68, Property 2]).

Corollary 4.0.6. *Let G be a finitely generated group in $\mathcal{MN}$. Then the Andrews-Curtis graph $AC_n(G)$ is connected for $n \geq \operatorname{rank}(G) + 1$.*

Proof. Connectedness of $AC_n(G)$ follows from the fact that for a group in the class $\mathcal{MN}$ the set of normally generating n-tuples coincides with the set of generating n-tuples (Proposition 3.1.3) and that the Nielsen graph $N_n(G)$ is connected for $n \geq \operatorname{rank}(G) + 1$ (see Corollary 3.2.6). □

Together with Theorem 1.3.2 we obtain connectedness of the Andrews-Curtis graph $AC_3(\Gamma)$ of the first Grigorchuk group Γ. Note, that the question whether the Nielsen graph $N_3(\Gamma)$ is connected is still open.

Corollary (Corollary 1.3.3)**.** *For the Grigorchuk group Γ the Andrews-Curtis graph $AC_n(\Gamma)$ is connected for $n \geq 3$.*

Proof. Connectedness of $AC_n(\Gamma)$ for $n \geq 4$ follows from Corollary 4.0.6. For $n = 3$ we use Theorem 1.3.2. The quotient $\Gamma/[\Gamma, \Gamma] \cong (\mathbb{Z}/2\mathbb{Z})^3$ (see, for example, [25], VIII.22) and the graph $N_3((\mathbb{Z}/2\mathbb{Z})^3)$ is connected by Theorem 3.2.1. This concludes the proof by Theorem 1.3.2. □

CHAPTER 5

Non-amenability of infinite Nielsen graphs

In Section 1.4 we discussed the motivation for studying non-amenability of infinite Nielsen graphs. The main goal of this chapter is to show the results on non-amenability of Nielsen graphs of indicable and elementary amenable groups. In particular, we will describe in details the Nielsen graphs of $\mathbb{Z}$ and D_∞ (the infinite dihedral group). Moreover, in the end of the chapter, using Theorem 3.3.2, we will obtain a criterion for non-amenability of the Nielsen graph of relatively free groups.

Recall that a locally finite connected graph X of uniformly bounded degree is *amenable* if either X is finite or

$$h(X) := \inf_{S \subset V(X)} \frac{|\partial_X(S)|}{|S|} = 0,$$

where the infimum is taken over all finite nonempty subsets S of the set of vertices $V(X)$ and $\partial_X(S)$ is the set of all edges connecting S to its complement. A graph with several connected components is amenable if at least one of the connected components is amenable.

We also recall the Kesten's characterization of amenable graphs (see *e.g.* [92, 10.3] for the extension of Kesten's criterion of amenability to all connected regular graphs). A connected m-regular graph X is *amenable* if and only if $\rho(X) = 1$, where $\rho(X) = 1/m \limsup_{k\to\infty} a_k^{1/k} \leq 1$ is the *spectral radius* of X, with $a_k(x)$ denoting the number of closed paths of length k in X, based at some (any) vertex of X.

We begin by proving a few lemmas about non-amenability of subgraphs and graph coverings that will be used to deduce the main theorems of this chapter.

Lemma 5.0.7. *Let X be an infinite connected graph with uniformly bounded degree. Let $X^{'}$ be a subgraph of X and suppose that there exists $D \geq 0$ such that for every vertex $x \in V(X)$ there exists a vertex $x^{'} \in V(X^{'})$ at distance at most D. If $X^{'}$ is non-amenable, then X is non-amenable.*

Proof. Let $S \subset V(X)$ be a finite subset of the vertex set of X. Denote by $B_D(S) = \{x \in V(X) \mid d_X(x, S) \leq D\}$ the D-neighborhood of the set S in X.

By assumption on the graph X', for every vertex $s \in S$ there is at least one vertex $v \in B_D(S) \cap V(X')$. Set

$$N := \max_{v \in B_D(S) \cap V(X')} |S \cap B_D(v)|.$$

Then $|B_D(S) \cap V(X')| \geq |S|/N$.

Let d be the uniform bound on the vertex degree of the graph X, then for each $v \in B_D(S) \cap V(X')$ we can estimate $N \leq d + d(d-1) + \cdots + d \cdot (d-1)^{D-1} \leq d^{D+1}$ since there are at most d vertices at distance 1 from v, $d(d-1)$ vertices at distance 2 from v, ..., $d \cdot (d-1)^{D-1}$ vertices at distance D from v. We conclude that

$$|B_D(S) \cap V(X')| \geq |S|/d^{D+1}.$$

Now we can estimate

$$\begin{aligned} |B_{D+1}(S)| \geq & |B_D(S)| + |\partial_{X'}(B_D(S) \cap V(X'))| \\ \geq & |B_D(S)| + h(X')|B_D(S) \cap V(X')| \geq |S| + h(X')|S|/d^{D+1}. \end{aligned}$$

By the same rough count as above, we have $|\partial S| \geq |B_1(S) \setminus S| \geq |B_{D+1}(S) \setminus S|/d^{D+1}$. Putting all the estimates together we get

$$\frac{|\partial S|}{|S|} \geq \frac{|B_{D+1}(S) \setminus S|}{d^{D+1}|S|} \geq h(X')/d^{2D+2}$$

for any finite subset $S \subset V(X)$. Hence X is non-amenable.

□

Recall, that a graph X *covers* a graph X' if there is a surjective graph morphism $\varphi \colon X \to X'$ that is an isomorphisms when restricted to the star (a small open neighborhood) of any vertex of X. In this case the map φ is called a *covering map.*

Lemma 5.0.8. *If a graph covers a non-amenable graph then it is itself non-amenable.*

Proof. Let $\varphi \colon X \to X'$ be a graph covering map. Since φ is a covering, closed paths in X are mapped onto closed paths in X'. We deduce therefore that $a_k^{X'}(\varphi(x)) \geq a_k^X(x)$ for any $x \in V(X)$, where $a_k^X(x)$ is the number of closed paths of length k starting from a point x in X; and consequently $\rho(X) \leq \rho(X')$. In particular if $\rho(X') < 1$ then $\rho(X) < 1$. □

Lemma 5.0.9. *Let $\pi \colon G \to H$ be an epimorphism between finitely generated groups and $n \geq \operatorname{rank}(G)$. If $N_n(H)$ is connected then $N_n(G)$ covers $N_n(H)$.*

Proof. Let us consider the map

$$\varphi\colon N_n(G) \to N_n(H),$$

$$\varphi((g_1, \dots, g_n)) = (\pi(g_1), \dots, \pi(g_n))$$

and prove that it is a covering map.

First, observe that φ maps the star of a vertex $(g_1, \dots, g_n)$ of $N_n(G)$ bijectively onto the star of $\varphi((g_1, \dots, g_n))$ in $N_n(H)$ because the map φ commutes with the action of Aut F_n.

Second, the map φ is surjective. To see this we consider a generating n-tuple $(h_1, \dots, h_n)$ of H and show that there exists a generating n-tuple of G which is mapped by φ onto $(h_1, \dots, h_n)$. By assumption $N_n(H)$ is connected, therefore for any $(s_1, \dots, s_n) \in V(N_n(G))$ its image $\varphi(s_1, \dots, s_n)$ is connected to $(h_1, \dots, h_n)$ by a sequence of elementary Nielsen moves. As φ commutes with the elementary Nielsen moves we conclude that $(h_1, \dots, h_n)$ is the image under φ of some n-tuple in G^n that belongs to the orbit of $(s_1, \dots, s_n)$ under automorphisms of F_n, thus is generating. □

Remark. Observe that if we drop the condition that $N_n(H)$ is connected, in Lemma 5.0.9, we are still able to conclude that each connected component of $N_n(G)$ covers some connected component of $N_n(H)$.

5.1 Non-amenability of Nielsen graphs of finitely generated indicable groups

We further give a detailed analysis of the Nielsen graphs $N_n(\mathbb{Z})$, $n \geq 1$. A description of the graph $N_2(\mathbb{Z})$ appears as Example 1.3 in [63].

Proposition 5.1.1. *The Nielsen graph $N_n(\mathbb{Z})$ is finite if $n = 1$ and non-amenable if $n \geq 2$. In addition, $N_n(\mathbb{Z})$ is connected for $n \geq 1$.*

Remark. For a finitely generated infinite group G, the graph $N_n(G)$ for $n \geq$ rank(G) is finite if and only if $G \cong \mathbb{Z}$ and $n = 1$.

Indeed, suppose that rank$(G) \geq 2$. Take $n \geq$ rank(G). If $N_n(G)$ is finite then in particular the group of automorphisms Aut G is finite which is equivalent to G being a finite and central extension of $\mathbb{Z}$ [4]. Since any such group has infinite abelianization, it admits an epimorphism onto $\mathbb{Z}$. Then $N_n(G)$ covers the infinite graph $N_n(\mathbb{Z})$, which is in contradiction with our assumption rank$(G) \geq 2$. Thus $G \cong \mathbb{Z}$ and $n = 1$.

Proof. Notice that the set of vertices $V(N_1(\mathbb{Z})) = \{1, -1\}$ and $I_1(1) = -1$, and therefore $N_1(\mathbb{Z})$ is finite and connected.

From now on suppose $n \geq 2$. The set of vertices of the Nielsen graph $N_n(\mathbb{Z})$ is $V(N_n(\mathbb{Z})) = \{(x_1, \ldots, x_n) \mid \langle x_1, \ldots, x_n \rangle = \mathbb{Z}\} = \{(x_1, \ldots, x_n) \mid gcd(x_1, \ldots, x_n) = 1\}$. By the Euclid's algorithm $N_n(\mathbb{Z})$ is connected.

To prove non-amenability of $N_n(\mathbb{Z})$, $n \geq 2$, we will exhibit a rooted subforest Γ in $N_n(\mathbb{Z})$ of vertex degree at least 3 everywhere except in the roots of its components. This subforest spans all but $2n$ vertices of $N_n(\mathbb{Z})$. Non-amenability of $N_n(\mathbb{Z})$ will then follow from non-amenability of the subforest by Lemma 2.1.

The subforest Γ is described by its components: $\Gamma = \cup_{A,B}\Gamma_{A,B}$ where A and B are disjoint subsets of $\{1, \ldots, n\}$ (including the empty set) and $|B| \leq n-2$.

Let us first describe the component $\Gamma_{\emptyset,\emptyset}$ of Γ. The vertex set of $\Gamma_{\emptyset,\emptyset}$ is

$$V(\Gamma_{\emptyset,\emptyset}) = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \mid \langle x_1, \ldots, x_n \rangle = \mathbb{Z} \text{ and } x_i > 0, 1 \leq i \leq n\}.$$

At every vertex $(x_1, \ldots, x_n) \in V(\Gamma_{\emptyset,\emptyset})$, consider all the edges $\{e_{ij}(x_1, \ldots, x_n)\}_{1 \leq i,j \leq n}$ that correspond to $R^+_{ij}(x_1, \ldots, x_n)$. Some of them will have to be deleted so that the graph $\Gamma_{\emptyset,\emptyset}$ has no cycles, loops or multiple edges.

Here is one way to define the set of edges to be deleted.

- ◇ if $R^+_{12}(x_1, \ldots, x_n) = R^+_{ij}(x_1, \ldots, x_n)$, $(i,j) \neq (1,2)$, delete $e_{ij}(x_1, \ldots, x_n)$;
- ◇ if $R^+_{21}(x_1, \ldots, x_n) = R^+_{ij}(x_1, \ldots, x_n)$, $(j,i) \neq (2,1)$, delete $e_{ij}(x_1, \ldots, x_n)$.

Notice that $R^+_{12}(x_1, \ldots, x_n) \neq R^+_{21}(x_1, \ldots, x_n)$. Indeed, if they were equal, then $x_1 + x_2 = x_1$ and $x_2 + x_1 = x_2$, therefore $x_1 = x_2 = 0$.

- ◇ If $R^+_{12}(x_1, \ldots, x_n) = R^+_{ij}(y_1, \ldots, y_n)$, $(i,j) \neq (1,2)$, delete $e_{ij}(y_1, \ldots, y_n)$;
- ◇ if $R^+_{21}(x_1, \ldots, x_n) = R^+_{ij}(y_1, \ldots, y_n)$, $(i,j) \neq (2,1)$, delete $e_{ij}(y_1, \ldots, y_n)$.

Notice that $R^+_{12}(x_1, \ldots, x_n) \neq R^+_{21}(y_1, \ldots, y_n)$. Indeed, if they were equal then $x_1 + x_2 = y_1$ and $y_2 + y_1 = x_2$, therefore $x_1 + y_2$=0, we obtain a contradiction with $x_i, y_i > 0$.

- ◇ Otherwise, if there exist (i_1, j_1), …, (i_k, j_k) with $k \geq 2$ such that

  $$R^+_{i_l j_l}(x_1, \ldots, x_n) = R^+_{i_m j_m}(x_1, \ldots, x_n)$$

  for $1 \leq l, m \leq k$, $l \neq m$, and neither of indices (i_l, j_l) or (i_m, j_m) is equal to $(1,2)$ or $(2,1)$ then keep only the edge with the largest in the lexicographical order index and keep it in the graph. The same rule applies when there exist (i_1, j_1), …, (i_k, j_k) for $k \geq 2$ such that $R^+_{i_l j_l}(x_1, \ldots, x_n) = R^+_{i_m j_m}(y_1, \ldots, y_n)$ for $1 \leq l, m \leq k$, $l \neq m$ and neither of indices (i_l, j_l) or (i_m, j_m) is equal to $(1,2)$ or $(2,1)$: only the edge with the index largest in the lexicographical order remains in the graph $\Gamma_{\emptyset,\emptyset}$.

We conclude that the graph $\Gamma_{\emptyset,\emptyset}$ with the given structure of edges does not have cycles, loops or multiple edges. Let the vertex $(1, \dots, 1) \in V(\Gamma_{\emptyset,\emptyset})$ be the root of this graph. There are at least two edges, $e_{12}(1, \dots, 1)$ and $e_{21}(1, \dots, 1)$, coming out of $(1, \dots, 1)$, therefore it is of degree at least 2. Any other vertex $(x_1, \dots, x_n)$ in $\Gamma_{\emptyset,\emptyset}$ is of degree at least 3:

- if $x_1 > x_2$ then $(x_1 - x_2, x_2, \dots, x_n)$ is connected to $(x_1, \dots, x_n)$ by $e_{12}(x_1 - x_2, x_2, \dots, x_n)$, moreover there are at least two edges coming out of $(x_1, \dots, x_n)$: $e_{12}(x_1, \dots, x_n)$ and $e_{21}(x_1, \dots, x_n)$.
- if $x_2 > x_1$ then $(x_1, x_2 - x_1, \dots, x_n)$ is connected to $(x_1, \dots, x_n)$ by $e_{21}(x_1, x_2 - x_1, \dots, x_n)$, moreover there are at least two edges coming out of $(x_1, \dots, x_n)$: $e_{12}(x_1, \dots, x_n)$ and $e_{21}(x_1, \dots, x_n)$.
- if $x_1 = x_2$, then since $(x_1, \dots, x_n) \neq (1, \dots, 1)$ there exists $x_i \neq 0$, $1 \leq i \leq n$, such that $x_i \neq x_1$. If $x_1 > x_i$ then $(x_1 - x_i, \dots, x_i, \dots, x_n)$ has to be connected to $(x_1, \dots, x_n)$ by $e_{1i}(x_1 - x_i, \dots, x_i, \dots, x_n)$ unless $e_{1i}(x_1 - x_i, \dots, x_i, \dots, x_n)$ is in $F_{\emptyset,\emptyset}$, which means that there is another edge coming in $(x_1, \dots, x_n)$. We deduce that there is at least one edge coming in $(x_1, \dots, x_n)$ when $x_1 > x_i$. Assume now that $x_i > x_1$ then $(x_1, \dots, x_i - x_1, \dots, x_n)$ is connected to $(x_1, \dots, x_n)$ by $e_{i1}(x_1, \dots, x_i - x_1, \dots, x_n)$ unless $e_{i1}(x_1, \dots, x_i - x_1, \dots, x_n)$ was deleted, which means that there is another edge coming in $(x_1, \dots, x_n)$. We deduce that there is at least one edge coming in $(x_1, \dots, x_n)$ when $x_i > x_1$. Moreover, there are always at least two edges coming out of $(x_1, \dots, x_n)$: $e_{12}(x_1, \dots, x_n)$ and $e_{21}(x_1, \dots, x_n)$.

Consider any point $(x_1, \dots, x_n) \in V(\Gamma_{\emptyset,\emptyset})$. By construction of $\Gamma_{\emptyset,\emptyset}$ there exists a path from $(x_1, \dots, x_n)$: $(x_1, \dots, x_n) \to (x_1^{(1)}, \dots, x_n^{(1)}) \to (x_1^{(i)}, \dots, x_n^{(i)}) \to ...$ such that $x_1^{(i+1)} + \dots + x_n^{(i+1)} < x_1^{(i)} + ... x_n^{(i)}$. This sequence terminates at $(1, \dots, 1)$ since $x_i > 0$, $1 \leq i \leq n$, and we conclude that any point in $\Gamma_{\emptyset,\emptyset}$ is connected to $(1, \dots, 1)$. Therefore $\Gamma_{\emptyset,\emptyset}$ is a connected graph without cycles, i.e., a tree, every vertex of which, except for the root, is of degree at least 3.

More generally, for any $A \subseteq \{1, \dots, n\}$ we define the subgraph $\Gamma_{A,\emptyset}$ of $N_n(\mathbb{Z})$ with the set of vertices

$$V(\Gamma_{A,\emptyset}) = \{(x_1, \dots, x_n) \in \mathbb{Z}^n \mid \langle x_1, \dots, x_n \rangle = \mathbb{Z},\ x_i < 0 \text{ if } i \in A \text{ and } \\ x_i > 0 \text{ otherwise}\}$$

and the set of edges $E(\Gamma_{A,\emptyset})$ defined symmetrically to $E(\Gamma_{\emptyset,\emptyset})$. The same arguments show that it is a tree, every vertex of which, except for the root, is of degree at least 3.

Next, we "lower the dimension" and define, for all disjoint subsets A and B of $\{1, \dots, n\}$ (including the empty set), $|B| \leq n - 2$, subgraphs $\Gamma_{A,B}$ of $N_n(\mathbb{Z})$ with

$$V(\Gamma_{A,B}) = \{(x_1, \dots, x_n) \mid \langle x_1, \dots, x_n \rangle = \mathbb{Z},\ x_i < 0 \text{ if } i \in A,$$

$$x_i = 0 \text{ if } i \in B, \text{ and } x_i > 0 \text{ otherwise}\}.$$

At every vertex $(x_1, \dots, x_n) \in V(\Gamma_{A,B})$, consider all the edges $\{e_{ij}(x_1, \dots, x_n)\}_{1 \leq i,j \leq n}$ corresponding to $R^+_{ij}(x_1, \dots, x_n)$ and then delete a subset of them so the graph $\Gamma_{A,B}$ has no cycles, loops or multiple edges.

For $A = \emptyset$ the set of edges to be deleted can be defined in the similar way as for $\Gamma_{\emptyset,\emptyset}$, but instead of using R^+_{12} and R^+_{21}, we use $R^+_{i_1 j_1}$ and $R^+_{j_1 i_1}$ such that $i_1, j_1 \notin B$. And for $A \neq \emptyset$ we define the edges of $\Gamma_{A,B}$ symmetrically to the edges of $\Gamma_{\emptyset,B}$.

Let the vertex $(\epsilon_1, \dots, \epsilon_n) \in V(\Gamma_{A,B})$ be the root of $\Gamma_{A,B}$ for $\epsilon_i = -1$ if $i \in A$, $\epsilon_i = 0$ if $i \in B$, and $\epsilon_i = 1$ otherwise. As before, the graph $\Gamma_{A,B}$ is a tree, and every vertex, except for the root, is of degree at least 3. This completes the description of Γ.

Observe that

$$V(N_n(\mathbb{Z})) = V(\Gamma) \cup (\pm 1, 0, \dots, 0) \cup \dots \cup (0, \dots, 0, \pm 1).$$

Non-amenability of $N_n(\mathbb{Z})$ follows from Lemma 5.0.7. □

Figure 5.1 represents a finite fragment of the (infinite) Nielsen graph $N_2(\mathbb{Z})$ constructed using Mathematica 9.

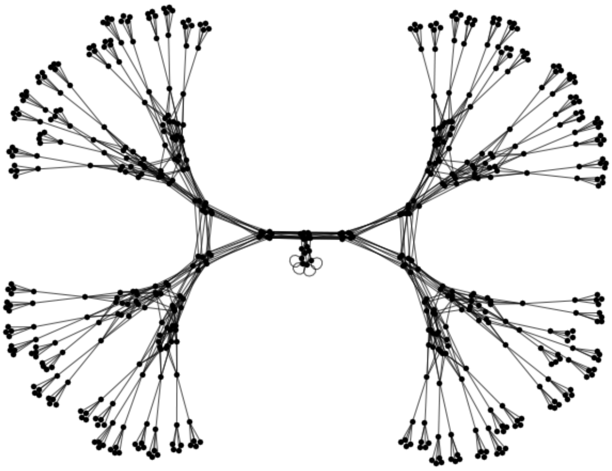


Figure 5.1: A finite fragment of $N_2(\mathbb{Z})$.

Remark. The proof of Proposition gives us moreover an explicit estimate of the Cheeger constant of $N_n(\mathbb{Z})$. Let S be a finite subset of vertices of $N_n(\mathbb{Z})$. If $S \subseteq \{(\pm 1, 0, \dots, 0) \cup \dots \cup (0, \dots, 0, \pm 1)\}$ then $|\partial_{N_n(\mathbb{Z})}(S)| \geq (n-1)|S|$. Otherwise, let $A = S \cap \{(\pm 1, 0, \dots, 0) \cup \dots \cup (0, \dots, 0, \pm 1)\}$. Then $|\partial_{N_n(\mathbb{Z})}(S)| \geq |\partial_\Gamma(S \setminus A)|$ and

$$\frac{|\partial_{N_n(\mathbb{Z})}(S)|}{|S|} \geq \frac{|\partial_\Gamma(S \setminus A)|}{|S \setminus A|} \frac{|S \setminus A|}{|S|} \geq \frac{|\partial_\Gamma(S \setminus A)|}{|S \setminus A|} \frac{1}{2n+1}.$$

Therefore, $h(N_n(\mathbb{Z})) \geq \min\{(n-1), \frac{1}{2n+1} h(\Gamma)\} \geq \frac{1}{2n+1}$.

Remark. Note that non-amenability of $N_n(\mathbb{Z})$ for $n \geq 3$ also follows from the fact that $GL_n(\mathbb{Z})$, $n \geq 3$, has property (T). Indeed, suppose that A is a finitely generated abelian group, and $n \geq \operatorname{rank}(A)$. Then A is a quotient of the free abelian group $\mathbb{Z}^n$. Consider the natural projection $\pi\colon\ F_n \to\ F_n/[F_n, F_n]$ which induces a homomorphism

$$\rho : \operatorname{Aut} F_n \to \operatorname{Aut}(F_n/[F_n, F_n]) = GL_n(\mathbb{Z}).$$

Every Nielsen move defines an automorphism of $\mathbb{Z}^n$ which belongs to $\rho(\operatorname{Aut} F_n)$. Therefore every connected component of the Nielsen graph $N_n(A)$ is the Schreier graph

$$Sch(\rho(\operatorname{Aut} F_n), St_{\rho(\operatorname{Aut} F_n)}(a_1, \ldots, a_n), \{\text{Nielsen moves}\}) \tag{5.1}$$

with respect to the generating n-tuple $(a_1, \ldots, a_n)$ which belongs to the connected component. Moreover observe that ρ is an epimorphism [61, 3.5.1]. We conclude that

$$(1) = Sch(GL_n(\mathbb{Z}), St_{GL_n\mathbb{Z}}(a_1, \ldots, a_n), \{\text{Nielsen moves}\}).$$

Apply this to $A = \mathbb{Z}$ to conclude

$$N_n(\mathbb{Z}) = Sch(GL_n(\mathbb{Z}), St_{GL_n\mathbb{Z}}(x_1, \ldots, x_n), \{\text{Nielsen moves}\})$$

for some generating n-tuple $(x_1, \ldots, x_n)$ of $\mathbb{Z}$. As mentioned in the introduction, every connected infinite Schreier graph of a Property (T)-group is non-amenable. The graph $N_n(\mathbb{Z})$ is connected and infinite and therefore non-amenable. This argument does not apply to the case $n = 2$.

Recall, that a group G is called *indicable* if it admits an epimorphism onto $\mathbb{Z}$. Proposition 5.1.1 allows to conclude that all Nielsen graphs $N_n(G)$ of a finitely generated indicable group G, $n \geq \max\{2, \operatorname{rank}(G)\}$, are non-amenable.

Theorem (Theorem 1.4.1)**.** *Let G be a finitely generated indicable group. Then all Nielsen graphs $N_n(G)$, $n \geq \max\{2, \operatorname{rank}(G)\}$, are non-amenable.*

Proof of Theorem 1.4.1. Consider an epimorphism $\pi : G \to \mathbb{Z}$. The corresponding graph morphism $N_n(G) \to N_n(\mathbb{Z})$, $n \geq \max\{2, \operatorname{rank}(G)\}$, is a covering map by Lemma 5.0.9. We conclude by Lemma 5.0.8 and Proposition 5.1.1. □

5.2 Non-amenability of Nielsen graphs of infinite finitely generated elementary amenable groups

We begin by describing the Nielsen graphs of the infinite dihedral group $D_\infty = \langle r, s \mid s^2, srs = r^{-1} \rangle = \langle x, y \mid x^2, y^2 \rangle$.

Corollary 5.2.1. *The Nielsen graph $N_n(D_\infty)$ is infinite connected for $n \geq 2$ and is non-amenable for $n \geq 3$.*

Proof. Recall that $D_\infty \cong \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$, so that $D_\infty/\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$. For any $(x_1, \ldots, x_n) \in N_n(D_\infty)$ consider its image $(\overline{x}_1, \ldots, \overline{x}_n)$ in $N_n(D_\infty/\mathbb{Z})$. Obviously $(\overline{x}_1, \ldots, \overline{x}_n)$ is at bounded distance from $(\overline{1}, \overline{0}, \ldots, \overline{0})$ in $N_n(D_\infty/\mathbb{Z})$. The same Nielsen moves which carry $(\overline{x}_1, \ldots, \overline{x}_n)$ to $(\overline{1}, \overline{0}, \ldots, \overline{0})$ will carry $(x_1, \ldots, x_n)$ to $(x, y_1, \ldots, y_{n-1})$ in $N_n(D_\infty)$ for some $x \in D_\infty$ and $y_i \in \mathbb{Z}$, $1 \leq i \leq n-1$, such that at least one of y_i is not equal to 0 in $\mathbb{Z}$ (because D_∞ is of rank 2).

For each $x \in D_\infty$ and $y_1, \ldots, y_{n-1} \in \mathbb{Z}$ such that $\langle x, y_1, \ldots, y_{n-1} \rangle = D_\infty$, denote by ϕ_x the map from $N_{n-1}(\mathbb{Z})$ to $N_n(D_\infty)$ induced by the map on the vertices that sends $(y_1, \ldots, y_n)$ to $(x, y_1, \ldots, y_n)$.

Denote by X' the subgraph $\sqcup_x \phi_x(N_{n-1}(\mathbb{Z}))$ of $N_n(D_\infty)$. Every vertex in $N_n(D_\infty)$ is at uniformly bounded distance from some vertex in X' by the remark above. It follows from Proposition 5.1.1 that X' is non-amenable for $n \geq 3$. By Lemma 5.0.7 we conclude that $N_n(D_\infty)$, $n \geq 3$, is non-amenable.

To show connectedness of $N_n(D_\infty)$ for $n \geq 2$ we recall that $D_\infty \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ and evoke Grushko-Neumann's theorem [50, 74] about Nielsen graphs of free products. □

The picture below represents a finite fragment of the (infinite) Nielsen graph $N_2(D_\infty)$ constructed using Mathematica 9.

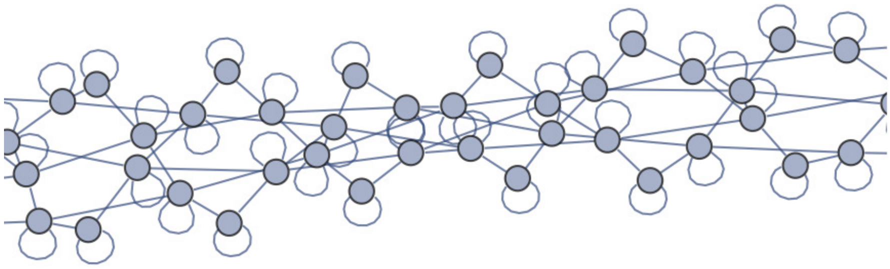


Figure 5.2: A finite fragment of $N_2(D_\infty)$.

Proposition 5.2.2. *The Nielsen graph $N_2(D_\infty)$ is quasi-isometric to a line. In particular, it is amenable.*

Proof. Let $D_\infty = \langle a, b \mid a^2, aba = b^{-1} \rangle$. We consider the subgraph Γ of $N_2(D_\infty)$ whose vertex set coincides with the vertex set of $N_2(D_\infty)$, keeping only the edges labeled by $R_{ij}(=R_{ij}^{+})$ and I_j, $i \neq j, 1 \leq i, j \leq 2$. Γ is a directed graph of vertex degree 8 with loops. Since $\mathrm{Aut}\, F_2 = \langle \{R_{ij}, I_j, i \neq j, 1 \leq i, j \leq 2\} \rangle$, then

$$\Gamma = Sch(\mathrm{Aut}\, F_2, \mathrm{Epi}(F_2, D_\infty), \{R_{ij}, I_j, i \neq j, 1 \leq i, j \leq 2\}).$$

It follows that Γ is quasi-isometric to $N_2(D_\infty)$.

Observe that the infinite strip on Figure 3 is a subgraph of Γ. Notice that

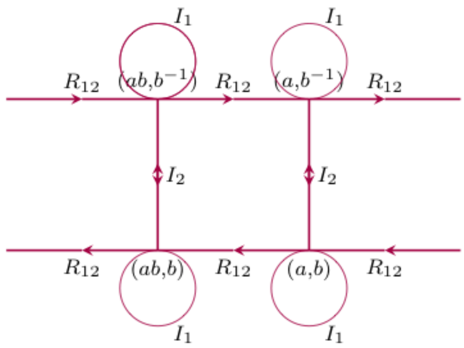


Figure 5.3:

all vertices on this strip are of the form $(ab^n, b^{\pm 1})$, $n \in \mathbb{Z}$, and each vertex has a loop labeled I_1. Indeed, $(ab)^2 = abab = b^{-1}b = 1$ and by induction $(ab^n)^2 = ab^{n-1}aabab^n = ab^{n-1}ab^{n-1} = (ab^{n-1})^2$.

Observe that also the infinite strip on Figure 4 is a subgraph of Γ. Notice that

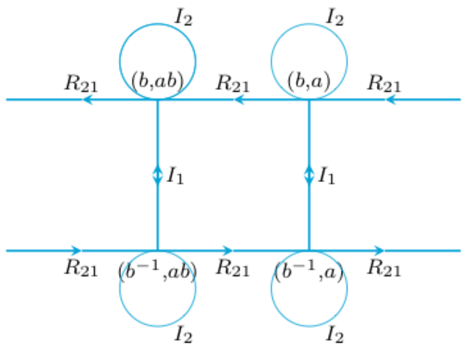


Figure 5.4:

all vertices on this strip are of the form $(b^{\pm 1}, ab^n)$, $n \in \mathbb{Z}$, and each vertex has a loop labeled I_2.

For any $n \in \mathbb{Z}$, $n \geq 0$, the following equalities hold:

$$\begin{aligned}
R_{12}R_{21}R_{12}^n(a,b) &= R_{21}I_1R_{21}^n(b,a),\\
R_{21}R_{12}R_{21}I_1R_{21}^n(b,a) &= R_{12}^n(a,b),\\
R_{12}I_2R_{12}^n(a,b) &= R_{21}R_{12}R_{21}^n(b,a),\\
R_{21}^n(b,a) &= R_{12}R_{21}R_{12}I_1R_{12}^n(a,b),\\
R_{12}R_{21}(I_2R_{12}I_2)^n(a,b) &= R_{21}I_1(I_1R_{21}I_1)^n(b,a),\\
R_{21}R_{12}R_{21}I_1(I_1R_{21}I_1)^n(b,a) &= (I_2R_{12}I_2)^n(a,b),\\
R_{12}I_2(I_2R_{12}I_2)^n(a,b) &= R_{21}R_{12}(I_1R_{21}I_1)^n(b,a),\\
(I_1R_{21}I_1)^n(b,a) &= I_1R_{12}R_{21}(I_2R_{12}I_2)^{n-1}(a,b)
\end{aligned}$$

We are now able to see how these two strips are connected in Γ (see Figure 5).

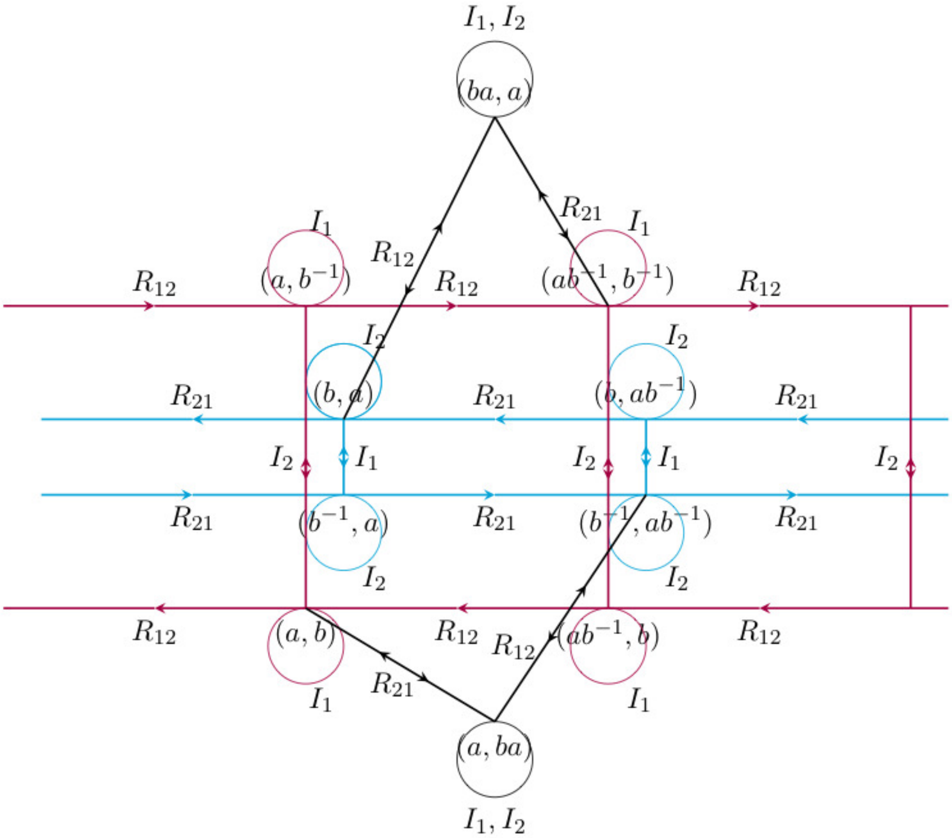


Figure 5.5:

Observe that for any $n \in \mathbb{Z}$, $n \geq 0$, the vertices $R_{21}R_{12}^n(a,b)$, $R_{12}R_{21}^n(b,a)$, $R_{21}(I_2R_{12}I_2)^n(a,b)$

and $R_{12}(I_1R_{21}I_1)^n(b,a)$ have loops labeled I_1, I_2. The calculations above show that the graph spanned by the vertices $(ab^{\pm n}, b^{\pm 1})$, $(b^{\pm 1}, ab^{\pm n})$, $R_{21}R_{12}^n(a,b)$, $R_{12}R_{21}^n(b,a)$, $R_{21}(I_2R_{12}I_2)^n(a,b)$ and $R_{12}(I_1R_{21}I_1)^n(b,a)$ for $n \in \mathbb{N} \cup \{0\}$, is regular of degree 8. Since it is a subgraph of Γ of full degree and Γ is connected we conclude that it coincides with Γ.

Notice that the vertices

$$R_{21}R_{12}^n(a,b),\ R_{12}R_{21}^n(b,a),\ R_{21}(I_2R_{12}I_2)^n(a,b) \text{ and } R_{12}(I_1R_{21}I_1)^n(b,a)$$

are at distance 1 from either $(ab^{\pm n}, b^{\pm 1})$ or $(b^{\pm 1}, ab^{\pm n})$. Thus we deduce that the graph Γ is quasi-isometric to the line.

□

The following Proposition will be used to show that the Nielsen graph $N_n(G)$ is non-amenable when G is an infinite elementary amenable group for sufficiently large n.

Proposition 5.2.3. *Let H be a group that contains a normal subgroup isomorphic to $\mathbb{Z}^d$, $d \geq 1$, of finite index $i > 1$. Then all Nielsen graphs $N_n(H)$ are non-amenable for $n \geq \operatorname{rank}(H) + \log_2 i + 1$.*

Proof. Denote by Q a normal subgroup of H isomorphic to $\mathbb{Z}^d$, $d \geq 1$, and denote by $F = H/Q$ the finite quotient of H. Denote also by $r = \operatorname{rank}(F) \leq \operatorname{rank}(H)$. For any $(x_1, \ldots, x_n) \in N_n(H)$ consider its image $(\overline{x}_1, \ldots, \overline{x}_n)$ in $N_n(F)$. The Nielsen graph $N_n(F)$ is obviously finite, and it is connected [81, Prop. 2.2.2] for $n \geq r + \log_2 i$. Hence $(\overline{x}_1, \ldots, \overline{x}_n)$ is at bounded distance from $(\overline{a_1}, \overline{a_2}, \ldots, \overline{a_r}, \overline{1}, \ldots, \overline{1})$ in $N_n(F)$ for $a_1, \ldots a_r \in H$, such that $\langle \overline{a_1}, \overline{a_2}, \ldots, \overline{a_r} \rangle = F$. The Nielsen moves that carry $(\overline{x}_1, \ldots, \overline{x}_n)$ to $(\overline{a_1}, \overline{a_2}, \ldots, \overline{a_r}, \overline{1}, \ldots, \overline{1})$ in $N_n(F)$ will carry $(x_1, \ldots, x_n)$ to

$$(a_1y_1, a_2y_2, \ldots, a_ry_r, y_{r+1}, \ldots, y_n)$$

in $N_n(H)$ for some $(y_1, \ldots, y_n) \in Q^n$.

If $y_m = 1$ for all $r+1 \leq m \leq n$ then $\langle a_1y_1, a_2y_2, \ldots, a_ry_r \rangle = H$. A sequence of elementary Nielsen moves $R_{r+1,1}, \ldots, R_{r+1,r}$ applied to

$$(a_1y_1, a_2y_2, \ldots, a_ry_r, 1, \ldots, 1)$$

corresponds to a path in the Cayley graph of H with generators

$$\{a_1y_1, a_2y_2, \ldots, a_ry_r\}.$$

Since Q is of index i in H, the ball of radius i around any vertex in this Cayley graph contains at least one vertex representing an element of Q. We can therefore conclude that $(a_1y_1, a_2y_2, \ldots, a_ry_r, 1, \ldots, 1)$ is within at most i steps from $(a_1y_1, a_2y_2, \ldots, a_ry_r, z, 1, \ldots, 1)$ in $N_n(H)$, with $z \in Q$ and $z \neq 1$.

For each $\hat{h} = (h_1, \ldots, h_r) \in H^r$ such that $\langle h_1, \ldots, h_r, y_{r+1}, \ldots, y_n \rangle = H$ and $y_m \neq 1$ for some $r+1 \leq m \leq n$, denote by $\phi_{\hat{h}}$ the morphism from the graph

$N_{n-r}(\langle y_{r+1},\dots,y_n\rangle)$ to the graph $N_n(H)$ induced by the map on the vertices that sends $(z_{r+1},\dots,z_n)$ to $(h_1,\dots,h_r,z_{r+1},\dots,z_n)$.

As a nontrivial subgroup of Q, $\langle y_{r+1},\dots,y_n\rangle$ is isomorphic to $\mathbb{Z}^s$ for some $s\geq 1$. Notice that $n-r\geq 2$ and clearly $n-r\geq \operatorname{rank}(\langle y_{r+1},\dots,y_n\rangle)$. We use Lemma 5.0.9 and Proposition 5.1.1 to deduce that $N_{n-r}(\langle y_{r+1},\dots,y_n\rangle)$ is non-amenable.

Denote by X' the subgraph $\sqcup_{\hat{h}}\phi_{\hat{h}}(N_{n-r}(\langle y_{r+1},\dots,y_r\rangle))$ of $N_n(H)$. Every vertex in $N_n(H)$ is at uniformly bounded distance from some vertex in X' by the first paragraph of the proof. Moreover X' is non-amenable since each $\phi_{\hat{h}}(N_{n-r}(\langle y_{r+1},\dots,y_r\rangle))$ is non-amenable. By Lemma 5.0.7 we conclude that $N_n(H)$ is non-amenable for $n\geq \operatorname{rank}(H)+\log_2 i+1$.

□

Denote by EG the class of *elementary amenable* groups, i. e., the smallest class of groups containing finite groups and abelian groups, which is closed with respect to taking subgroups, quotients, extensions and direct limits.

For each ordinal α define inductively a subclass EG_α of EG in the following way. EG_0 consists of finite groups and abelian groups. If α is a limit ordinal then

$$EG_\alpha = \bigcup_{\beta<\alpha} EG_\beta.$$

Further, $EG_{\alpha+1}$ is defined as the class of groups which are extensions of groups from the set EG_α by groups from the same set. Each of the classes EG_α is closed with respect to taking subgroups and taking quotients [20]. *The elementary complexity* of a group $G\in EG$ is the smallest α such that $G\in EG_\alpha$.

Recall that a group is *just-infinite* if it is infinite and all its non-trivial normal subgroups are of finite index. A just-infinite group G is *hereditary just-infinite* if it is residually finite and every subgroup $M<G$ of finite index is just-infinite. The proof of Theorem 1.4.2 is based on the following trichotomy for finitely generated just-infinite groups:

Theorem 5.2.4 ([45])**.** *Any finitely generated just-infinite group is either branch, or contains a normal subgroup of finite index which is isomorphic to the direct product of a finite number of copies of a group L, where L is either simple or hereditary just infinite.*

Branch groups are the groups that have a faithful level transitive action on an infinite spherically homogeneous rooted tree $T_{\bar{m}}$ defined by a sequence $\{m_n\}_{n=1}^{\infty}$ of natural numbers $m_n\geq 2$ (determining the branching number of vertices of level n) with the property that the rigid stabilizer $rist_G(n)$ has finite index in G for each $n\geq 1$. Here $rist_G(n)$ denotes the product $\prod_{v\in V_n} rist_G(v)$ of rigid stabilizers $rist_G(v)$ of all vertices on the n-th level of the tree, where $rist_G(v)<G$ is the subgroup of elements fixing the vertex v and acting trivially outside the full subtree rooted at v. For more on branch groups see [46]. The statement of the next Proposition appeared already in [39] but there is no proof of it in the literature, that is why we include a proof here.

Proposition 5.2.5 (R. I. Grigorchuk). *Let G be a finitely generated branch just-infinite group. Then it does not belong to the class EG of elementary amenable groups.*

Proof. Suppose G is branch. Let α be a minimal ordinal with the property $G \in EG_\alpha$. Then G is an extension of a subgroup $N \lhd G$, $N \in EG_{\alpha-1}$ (G cannot be presented as a direct limit of subgroups of smaller elementary complexity since G is finitely generated). As G is just-infinite, N has finite index. It turns out that N is not necessarily a branch group. However it can be shown that N satisfies the definition of a branch group with a single relaxation, namely, that the number of orbits of the action on the levels is uniformly bounded (instead of being equal to 1 in the original definition). Proposition 5.2.5 will be proven by induction on α for any group satisfying the relaxed branch condition.

It is proven in Theorem 4 [46], that for each nontrivial normal subgroup K of a branch group G there is n such that K contains the commutator subgroup $(rist_G(n))'$. The same proof essentially works for groups satisfying this relaxed branch condition, one just needs to "decompose" the tree $T_{\bar{m}}$ on which the group acts into finitely many invariant subtrees on each of which the action is level transitive and so the restriction of the action to each component is a branch group. As each class EG_β is closed with respect to taking subgroups or quotients, the group $(rist_G(n))'$ belongs to the class $EG_{\alpha-1}$.

Consider the decomposition $rist_N(n) = \prod_{v\in V_n} rist_N(n)$. For each $v \in V_n$ the corresponding group $M_v = rist_N(v)$ satisfies the relaxed branch condition for the action on a rooted subtree T_v of $T_{\bar{m}}$. Indeed, for each level k of T_v the number of orbits for the action of $rist_N(v)$ is uniformly bounded by the same constant which bounds the number of orbits of the action of N on $T_{\bar{m}}$. Rigid stabilizer $rist_{M_v}(k)$ is a subgroup of finite index in M_v as it contains the product $\prod_{u\in V_k(T_v)} rist_N(u)$ where $V_k(T_v)$ denotes the set of vertices of level k in the subtree T_v.

Moreover, each M_v is just-infinite. Indeed, let us suppose that $P_v \lhd M_v$ is a normal subgroup. The group $Q := \prod_{w\in V_n} P_v^{g_w}$ where elements $g_w \in G$ are chosen in such a way that $P_v^{g_w}$ is a subgroup of $rist_G(w)$, is normal not only in N but also in G and has infinite index. Contradiction.

Therefore M_v is a finitely generated (as a quotient of the finitely generated group $rist_N(n)$) just-infinite group from the class $EG_{\alpha-1}$ that satisfies the relaxed branch condition, which gives us the final contradiction.

□

Theorem (Theorem 1.4.2). *Let G be an infinite finitely generated elementary amenable group. Then G admits an epimorphism onto a group H that contains a normal subgroup isomorphic to $\mathbb{Z}^d$, $d \geq 1$, of finite index $i \geq 1$. All Nielsen graphs $N_n(G)$ are non-amenable for $n \geq \operatorname{rank}(G) + \log_2 i + 1$.*

Proof of Theorem 1.4.2. Let G be an infinite finitely generated elementary amenable group. As any infinite finitely generated group, it can be epimorphically mapped onto a finitely generated just-infinite group $\bar{G}$. The property of being elementary amenable is preserved in homomorphic images, so $\bar{G}$ is also elementary amenable.

We now use the classification of Theorem 5.2.4. A finitely generated just-infinite branch group cannot be elementary amenable by Proposition 5.2.5. An infinite finitely generated simple group cannot be elementary amenable [20], therefore G cannot contain a normal subgroup of finite index which is isomorphic to the direct product of a finite number of copies of a simple group.

An elementary finitely generated amenable hereditary just-infinite group is isomorphic to either $\mathbb{Z}$ or to D_∞. See Theorem 5.5 in [40] for a proof of this fact by Y. de Cornulier. Hence, any infinite finitely generated elementary amenable group is mapped onto a just-infinite group H that contains a normal subgroup of finite index isomorphic either to $\mathbb{Z}^d$ or to D_∞^d, $d \geq 1$. Moreover D_∞ contains $\mathbb{Z}$ as a subgroup of index 2, so the second case is reduced to the first. The proof is concluded via Lemma 5.0.8, Lemma 5.0.9 and Proposition 5.2.3.

□

5.3 Non-amenability of Nielsen graphs of relatively free groups

For a relatively free group G of rank d we described explicitly the Nielsen graph $N_d(G)$ (Theorem 3.3.2). In particular, we showed that every connected component of the Nielsen graph is isomorphic to the Cayley graph of the subgroup $T(G) \leq \operatorname{Aut} G$ of *tame automorphisms* of G. This implies in particular that all connected components of the Nielsen graph $N_d(G)$ are isomorphic. Their number is equal to the index of the subgroup $T(G)$ in $\operatorname{Aut} G$. We deduce the following criteria.

Corollary (Corollary 1.4.3)**.** *Let G be a relatively free group of rank d. Then the Nielsen graph $N_d(G)$ is non-amenable if and only if the group $T(G)$ of tame automorphisms of G is non-amenable.*

Let now H be a quotient of a relatively free group G of rank d. Then every connected component of the Nielsen graph $N_d(H)$ is the Schreier graph

$$Sch(\rho(\operatorname{Aut} F_d), St_{\rho(\operatorname{Aut} F_d)}(h_1, \dots, h_d), \{\text{Nielsen moves}\})$$

for some generating d-tuple that belongs to the connected component. (This has been observed in [59, Prop.1.10] for finite groups.) For infinite Nielsen graphs we get the following sufficient condition of non-amenability that replaces the criterion in Corollary 1.4.3 for quotients of relatively free groups – recall the discussion from the introduction about the link between Property (T) of a group and non-amenability of its infinite Schreier graphs.

Corollary 5.3.1. *Let H be a finitely generated group in some variety of groups $\mathcal{B}$. For $d \geq \operatorname{rank}(H)$ denote by G the relatively free group in $\mathcal{B}$ of rank d. If the subgroup $T(G) < \operatorname{Aut} G$ of tame automorphisms of G has Property (T), then every infinite component of the Nielsen graph $N_d(H)$ is non-amenable.*

Moreover, Lemma 5.0.8 and Lemma 5.0.9 imply the following Corollary (generalizing Theorem 1.4.1 for $n \geq 3$).

Corollary 5.3.2. *Let K be a finitely generated group that admits an epimorphism onto a group H belonging to some variety of groups $\mathcal{B}$. Let $d \geq \operatorname{rank}(K)$ and denote by G the relatively free group of rank d in $\mathcal{B}$. If $T(G)$ has Property (T), and if $N_d(H)$ is infinite and connected, then every connected component of $N_d(K)$ is non-amenable.*

Note, that most of examples considered in Section 3.3 are indicable and non-amenability of their Nielsen graphs can be deduced from Theorem 1.4.1. We will turn our attention to free Burnside groups. In [24], Coulon proved the following theorem.

Theorem 5.3.3. *[24] Let $d \geq 3$. There exists an integer m_0 such that for all odd m larger than m_0, the group* $\operatorname{Out} B(d,m)$ *of outer automorphisms of $B(d,m)$ contains a subgroup isomorphic to F_2.*

It follows from Coulon's proof that the free subgroup that he finds in $\operatorname{Out} B(d,m)$ is in fact a subgroup of induced tame automorphisms. Indeed, the injective homomorphism $F_2 \hookrightarrow \operatorname{Out} F_d/F_d^m$ that he constructs is induced by a homomorphism $F_2 \to \operatorname{Out} F_d$. In particular we can conclude that $T(B(d,m))$ is non-amenable.

Corollary (Corollary 1.4.4)**.** *Let $B(d,m)$ denote the free Burnside group on d generators of exponent m. For $d \geq 3$ and m odd and large enough all connected components of $N_d(B(d,m))$ are isomorphic and non-amenable.*

CHAPTER 6

Open questions and further research directions

In this chapter we will briefly discuss open questions appearing as continuations of the topics described in the thesis.

Andrews-Curtis group

As explained in Section 1.3, the Nielsen graph describes the action of $\operatorname{Aut} F_n$ on the set $\operatorname{Epi}(F_n, G)$. It would be interesting to find the "Andrews–Curtis group" AC_n which corresponds to the Andrews–Curtis graph. In particular, one observes that $\operatorname{Aut} F_n$ and F_n^n are subgroups of the group AC_n and moreover they generate AC_n. On the other hand, AC_n itself is a subgroup of $\operatorname{Aut} F_{2n}$. The study of the structure of AC_n as a group can give new tools for investigating the Andrews–Curtis conjecture.

Connectedness of Nielsen graphs of finitely generated groups in class $\mathcal{MN}$ for $n = \operatorname{rank}(G)$

We are far from having a complete understanding of Nielsen graphs $N_{\operatorname{rank}(G)}(G)$ for finitely generated groups in class $\mathcal{MN}$. For instance, for the Grigorchuk group Γ and $n \geq \operatorname{rank}(\Gamma) = 3$, Theorem 1.3.2 implies that the Andrews–Curtis graph $AC_n(\Gamma)$ is connected. Yet, the question on connectedness of the Nielsen graph $N_3(\Gamma)$ "in the rank" is open.

T_2-systems of Gupta-Sidki p-groups

For the Gupta-Sidki p-groups, the Nielsen graph $N_2(G_p)$ has infinitely many connected components, see Theorem 1.2.3. An interesting question in this direction is whether there are infinitely many T_2-systems of generating pairs for the Gupta-Sidki p-group?

Class $\mathcal{MN}$

A problem of a different nature, but in the same framework, is to understand the class $\mathcal{MN}$ itself.

As mentioned before, groups belonging to $\mathcal{MN}$ include nilpotent groups, branch Grigorchuk groups, Gupta-Sidki groups and, more generally, all multi-edge spinal torsion groups acting on a regular p-ary rooted tree, with p odd prime. Also, if two finitely generated groups G_1 and G_2 belong to $\mathcal{MN}$ and their respective commutator subgroups are finitely generated, then the direct product $G_1 \times G_2$ also belongs to $\mathcal{MN}$.
It is well known that nilpotent groups have polynomial growth; the Grigorchuk groups belonging to $\mathcal{MN}$ have intermediate growth. On the other hand, it is an open question whether the growth of Gupta-Sidki p-groups is intermediate. A natural question is whether there are finitely generated groups in $\mathcal{MN}$ of exponential growth?

Non-amenability of Nielsen graphs

It would be interesting to answer the following challenging question which is a weaker version of the problem about Property (T) for $\operatorname{Aut} F_n$, $n \geq 4$: *is every infinite connected component of $N_n(G)$ for $n \geq \max\{3, \operatorname{rank}(G)\}$ non-amenable, for a finitely generated group G?* Showing that all Nielsen graphs and even, more generally, all Schreier graphs of $\operatorname{Aut} F_n$ are either finite or non-amenable is not sufficient for proving that $\operatorname{Aut} F_n$ has Property (T). However the property of a group having its all Schreier graphs either finite or non-amenable is interesting on its own and was considered by Cornulier in [23].

It was observed in Section 5.2 that, for the infinite dihedral group D_∞ (which is of rank 2), the Nielsen graph $N_2(D_\infty)$ is quasi-isometric to the line, and therefore amenable. For the moment it is the only known example of an infinite amenable Nielsen graph.

Nonamenability of Nielsen graphs was also investigated in a recent preprint [62] by Malyshev, where the author proves, among other things, that the Nielsen graphs $N_n(G)$ with $n \geq \operatorname{rank}(G)$ are non-amenable for uniformly non-amenable groups G. It would be interesting to analyse what happens when G is not in the class of groups already considered, *e.g.* when G is of intermediate growth, say, the first Grigorchuk group.

APPENDIX A

Schreier graphs of Kazhdan groups are either finite or non-amenable

The aim of this appendix is to prove the statement in the title.

A.1 Kazhdan Property (T)

Below we give a definition of Property (T) of a topological group, introduced by Kazhdan in [56]. This property was used, for instance, by Margulis in [64] to obtain an explicit construction of a family of expanders. In similar vein, in Section A.3 we will explain how Property (T) of a given group G implies that the isoperimetric constant of any infinite Schreier graph of G is bounded away from 0.

Let $\mathcal{H}$ be a complex Hilbert space. The *unitary group* $\mathcal{U}(\mathcal{H})$ of $\mathcal{H}$ is the group of all surjective bounded linear operators $U\colon \mathcal{H} \to \mathcal{H}$ such that, for all $\xi, \eta \in \mathcal{H}$,

$$\langle U\xi, U\eta \rangle = \langle \xi, \eta \rangle,$$

or equivalently such that $U^*U = UU^* = I$ where U^* denotes the adjoint of U and I the identity operator on $\mathcal{H}$.

Let G be a topological group. A *unitary representation* of G on $\mathcal{H}$ is a group homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$ such that the mapping

$$G \to \mathcal{H}, \ g \mapsto \pi(g)\xi$$

is continuous for every $\xi \in \mathcal{H}$. We denote the unitary representation by $(\pi, \mathcal{H})$.

For a subset K of G and real number $\epsilon > 0$, a unit vector $\xi \in \mathcal{H}$ is (K, ϵ)-invariant if

$$\sup_{x \in K} ||\pi(x)\xi - \xi|| < \epsilon.$$

The representation $(\pi, \mathcal{H})$ *almost has invariant vectors* if it has (K, ϵ)-invariant vectors for every compact subset K of G and every $\epsilon > 0$. The representation $(\pi, \mathcal{H})$ has *non-zero invariant vectors* if there exists $\xi \neq 0$ in $\mathcal{H}$ such that $\pi(g)\xi = \xi$, for all $g \in G$. A group G has *Kazhdan's Property (T)* if there exists a compact subset K of G and $\epsilon > 0$ with the following property: every unitary representation $(\pi, \mathcal{H})$ of G which has a (K, ϵ)-invariant vector also has a non-zero invariant vector. Such a group is also called a *Kazhdan group*.

Observe the following proposition.

Proposition A.1.1. *[12] Let G be a topological group. The following statements are equivalent:*

1. *G has Kazhdan's Property (T);*
2. *If a unitary representation $(\pi, \mathcal{H})$ of G almost has invariant vectors then it has non-zero invariant vectors.*

Examples of groups having Property (T) include, in particular, $SL_n(\mathbb{K})$ with $n \geq 3$ where $\mathbb{K}$ is a local field (for example, $\mathbb{K} = \mathbb{R}$). Moreover, a lattice in a locally compact group G has Property (T) if and only if G has Property (T); in particular, the group $SL_n(\mathbb{Z})$ with $n \geq 3$ has Property (T).

For the detailed exposition on Kazhdan's Property (T) we refer the reader to [12].

A.2 Amenable actions of groups

In this section we will prove the equivalence between several definitions of amenable actions of groups.

Let X be a set and G be a group acting on X. An action of G on X is *amenable* [38] if there exists a G-invariant mean on X, *i.e.* a map

$$\mu : 2^X = \mathcal{P}(X) \to [0, 1]$$

such that

- $\mu(X) = 1$;
- $\mu(A \cup B) = \mu(A) + \mu(B)$ for every pair of disjoint subsets A, B of X;
- $\mu(gA) = \mu(A)$ for all $g \in G$ and $A \subseteq X$.

A group G is *amenable* if its action on itself by left multiplication is amenable. It is easy to see that every finite group is amenable. Observe, using theorem below, that the infinite cyclic group $\mathbb{Z}$ is amenable. More generally, all solvable groups are amenable.

We will formulate an equivalent definition of an amenable action of G on X, using the left unitary representation defined below. For each $g \in G$ define an operator $\lambda_X(g)$ on $l^2(X)$, the Hilbert space of square-integrable complex valued functions on X, by

$$\lambda_X(g)\xi(x) = \xi(g^{-1}x),$$

for all $\xi \in l^2(X)$ and $x \in X$. Then $(\lambda_X, l^2(X))$ is a unitary representation of G on $l^2(X)$, called *the left unitary representation of G on $l^2(X)$*.

The following theorem gives us equivalent definitions of an amenable action of a group G on a set X. The proof below is based on [12], where these well-known reformulations of amenability are proven for the case $X = G$.

Theorem A.2.1. *Let X be a set and G be a group acting on X. Then the following are equivalent:*

1. *The action of G on X is amenable.*
2. *There exists a G-invariant state on $l^\infty(X)$,* i.e. *there exists a linear functional M on $l^\infty(X)$ with $M(1_X) = 1$, $M(\varphi) \geq 0$ for any $\varphi \geq 0$ in $l^\infty(X)$, and for any $\varphi \in l^\infty(X)$ and $g \in G$ we have $M(\varphi_g) = M(\varphi)$ where $\varphi_g(x) = \varphi(gx)$.*
3. *(Reiter's Property) For every finite subset K of G and every $\epsilon > 0$, there exists $\xi \in l^1(X)^{1,+}$ such that*
$$\sup_{x\in K} ||\lambda_X(g)\xi - \xi||_1 \leq \epsilon,$$
where $l^1(X)^{1,+} = \{\eta \in l^1(X) \mid ||\eta||_1 = 1 \text{ and } \eta(x) \geq 0 \text{ for all } x \in X\}$.

4. *The left unitary representation* $(\lambda_X, l^2(X))$ *of* G *almost has invariant vectors.*

5. *(Følner condition) For every finite subset* $L \subset G$ *and every* $\epsilon > 0$*, there exists a finite subset* $A \subset X$ *such that* $|gA \triangle A| < \epsilon |A|$ *for every* $g \in L$.

Remark A.2.2. *The set of all states on* $l^\infty(X)$ *is a weak* closed and, hence, compact convex subset of the unit ball of* $l^\infty(X)^*$.

Proof. $1 \Leftrightarrow 2$. If there exists a G-invariant state M on $l^\infty(X)$, we let $\mu(A) = M(\chi_A)$ for any $A \subset X$, where $\chi_A \in l^\infty(X)$ is the characteristic function of A.

On the other hand, suppose there exists a G-invariant mean μ on X. Then for any simple function $\varphi = \sum_{i=1}^m \alpha_i \chi_{A_i} \in l^\infty(X)$, with $\alpha_i \in \mathbb{C}$ and $A_i \subseteq X$, we let $M(\varphi) = \sum_{i=1}^m \alpha_i \mu(A_i)$. More generally, let us take $\varphi \in l^\infty(X)$. There exists a sequence of simple functions $(\varphi_n)_n$ in $l^\infty(X)$ such that $(\varphi_n)_n$ converges uniformly to φ. Notice that $(M\varphi_n)_n$ is a Cauchy sequence and its limit does not depend on the choice of $(\varphi_n)_n$. We let $M(\varphi) = \lim_{n\to\infty} M(\varphi_n)$.

$2 \Rightarrow 3$. Let M be a G-invariant state on $l^\infty(X)$ and let $K = \{g_1, \dots, g_n\}$. Consider the Banach space $(l^1(X))^n$ and the convex subset C formed of all n-tuples $(\lambda_X(g_1)f - f, \dots, \lambda_X(g_n)f - f)$ for all $f \in l^1(X)^{1,+}$.

The weak topology on C is the product of the weak topologies. Since $l^1(X)^{1,+}$ is weak* dense in the set of all states on $l^\infty(X)$ [22, Chapter 5, Proposition 4.1], there exists a net $(f_k)_k$ in $l^1(X)^{1,+}$ converging to M in the weak* topology. Hence for such a net $(f_k)_k$ in $l^1(X)^{1,+}$, the net $(\lambda_X(g_i)f_k - f_k)_k$ converges to 0 in the weak topology (for any $g_i \in K$).

Since C is convex then its closure in the weak topology coincides with the closure of C in the norm topology ([22, Chapter 5, Theorem 1.4]). It follows that for any $\epsilon > 0$ there exists $\xi \in l^1(X)^{1,+}$ such that

$$||\lambda_X(g_i)\xi - \xi||_1 < \epsilon,$$

for any $g_i \in K$.

$3 \Rightarrow 2$. Suppose 3 holds. Then there is a net $(f_i)_i$ in $l^1(X)^{1,+}$ such that

$$\lim_i ||\lambda_X(g)f_i - f_i||_1 = 0,$$

for all $g \in G$. We let M to be a weak* limit point of $(f_i)_i$ in the set of all states on $l^\infty(G)$. It follows that M is G-invariant.

$4 \Rightarrow 3$. Assume that 4 holds. Then, for a given finite subset K in G and $\epsilon > 0$, there exists a unit vector $\xi_{K,\epsilon} \in l^2(X)$ such that

$$\sup_{g\in K} ||\lambda_X(g)\xi_{K,\epsilon} - \xi_{K,\epsilon}||_2 < \epsilon.$$

Let $f_{K,\epsilon} = |\xi_{K,\epsilon}|^2$. Then $f_{K,\epsilon} \in l^1(X)^{1,+}$ and, by Cauchy-Schwarz inequality,

$$||\lambda_X(g)f_{K,\epsilon} - f_{K,\epsilon}||_1 \leq ||\lambda_X(g)\overline{\xi_{K,\epsilon}} + \overline{\xi_{K,\epsilon}}||_2 \cdot ||\lambda_X(g)\overline{\xi_{K,\epsilon}} - \overline{\xi_{K,\epsilon}}||_2 \leq$$

$$2||\xi_{K,\epsilon}||_2 \cdot ||\lambda_X(g)\overline{\xi_{K,\epsilon}} - \overline{\xi_{K,\epsilon}}||_2 < 2\epsilon,$$

and the Reiter's Property follows.

$3 \Rightarrow 4$. Suppose that G has the Reiter's Property. Then for a finite subset K of G and $\epsilon > 0$ we let $\xi \in l^1(X)^{1,+}$ be such that

$$\sup_{g\in K} ||\lambda_X(g)\xi - \xi||_1 < \epsilon.$$

Let $f = \sqrt{\xi}$. Then f is a unit vector in $\in l^2(X)$. We use the inequality $|a-b|^2 \leq |a^2 - b^2| = |a-b|(a+b)$ for all non-negative real numbers a and b to obtain

$$||\lambda_X(g)f - f||_2^2 \leq \sum_{x\in X} |f^2(g^{-1}x) - f^2(x)| = ||\lambda_X(g)\xi - \xi||_1 < \epsilon,$$

for all $g \in K$. The conclusion then follows.

$5 \Rightarrow 3$. Assume that 5 holds. Let $\epsilon > 0$ and K be a finite subset of G. We take the appropriate finite subset $A \subset X$ in 5 and let $\xi = \frac{\chi_A}{|A|} \in l^1(X)^{1,+}$. Then for each $g \in K$ we have

$$\lambda_X(g)\xi(x) = \frac{\chi_A(g^{-1}x)}{|A|} = \frac{\chi_{gA}(x)}{|A|},$$

for all $x \in X$.

It follows that

$$||\lambda_X(g)\xi - \xi||_1 = \frac{\sum_{x\in X} |\chi_{gA}(x) - \chi_A(x)|}{|A|} = \frac{|gA \triangle A|}{|A|}.$$

$3 \Rightarrow 5$. The proof uses "Namioka's trick“ adapted by Connes [21].

For each $a > 0$ we let $E_a = \chi_{]a,\infty[}$ to be the characteristic function of the interval $]a,\infty[\subset \mathbb{R}^+$. For arbitrary $s, t \in \mathbb{R}^+$ we get

$$\int_0^\infty |E_a(s) - E_a(t)| da = |s-t|.$$

Therefore for arbitrary $f, g \in l^1(X)^+$ we have

$$\int_0^\infty |E_a(f) - E_a(g)| da = |f-g|$$

and hence

$$\int_0^\infty ||E_a(f) - E_a(g)||_1 da = ||f-g||_1.$$

In particular,

$$\int_0^\infty ||E_a(f)||_1 da = ||f||_1.$$

From this we can deduce the following estimate. Let $f_1, \ldots, f_n$ be n elements of $l^1(x)^+$ and $\epsilon > 0$ with

$$||f_i - f_1||_1 < \epsilon ||f_1||_1 \text{ for } i = 1, \ldots, n.$$

Then there exists $a > 0$ such that $E_a(f_1)(x) \neq 0$ for some $x \in X$ and that

$$\frac{1}{n} \sum_{1}^{n} ||E_a(f_i) - E_a(f_1)||_1 < \epsilon ||E_a(f_1)||_1.$$

Let $L \subset G$ be a finite subset of G and let $\epsilon > 0$. Condition 3 implies the existence of a vector $\xi \in l^1(X)^{1,+}$ such that for all $g \in L$:

$$||\lambda_X(g)\xi - \xi||_1 < \frac{\epsilon}{|L|} ||\xi||_1.$$

Applying the above fact, there is a number $a > 0$ such that

$$\frac{1}{|L|} \sum_{g \in L} ||E_a(\lambda_X(g)\xi) - E_a(\xi)||_1 < \frac{\epsilon}{|L|} ||E_a(\xi)||_1.$$

Observe, that in particular, $E_a(\xi)(x) \neq 0$ for some $x \in X$. Let

$$A = \{x \in X \mid \xi(x) > a\};$$

the set A is a non-empty finite set with $||E_a(\xi)||_1 = |A|$. We finish with the following:

$$||E_a(\lambda_X(g)\xi) - E_a(\xi)||_1 = |gA \triangle A| < \epsilon |A|,$$

for any $g \in L$.

□

Lemma A.2.3. *Let G be a discrete Kazhdan group acting transitively on a countable set X. The action $G \curvearrowright X$ is amenable if and only if the set X is finite.*

Proof. The action of G on X is amenable if and only if the left unitary representation λ_X of G on $l^2(X)$ almost has invariant vectors by Theorem A.2.1. Since G has Property (T) then there is a non-zero vector $\xi \in l^2(X)$ such that $\lambda_X(g)\xi(x) = \xi(g^{-1}x) = \xi(x)$ for all $g \in G$ and $x \in X$. Since the action of G is transitive, it implies that $\xi \in l^2(X)$ is a constant on X therefore X must be finite. □

A.3 Schreier graphs of Kazhdan groups

A locally finite connected graph X of uniformly bounded degree is *amenable* if

$$h(X) := \inf_{S \subseteq V(X)} \frac{|\partial_X(S)|}{|S|} = 0,$$

where the infimum is taken over all finite nonempty subsets S of the set of vertices $V(X)$ and $\partial_X(S)$ is the set of all edges connecting S to its complement. The number $h(X) \geq 0$ is called the *isoperimetric constant* (or the Cheeger constant) of X.

In other words, a locally finite connected graph X of uniformly bounded degree is *amenable* if for any $\epsilon > 0$ there exists a finite subset of vertices $S \subset X$ such that $\frac{|\partial_X(S)|}{|S|} < \epsilon$.

Lemma A.3.1. *Let G be a discrete group, let S be its finite symmetric set of generators ($S = S^{-1}$) and H be a subgroup of G. Denote by $\Gamma = Sch(G, H, S)$ the Schreier graph of G with respect to H.*

The action of G on the set of vertices of Γ is amenable if and only if the graph Γ is amenable.

Proof. Suppose that the action of G on Γ is amenable. By the Følner condition for any $\epsilon > 0$ there exists a finite subset of vertices $F \subset \Gamma$ such that

$$|sF \triangle F| < \frac{\epsilon}{|S|} \cdot |F|$$

for every $s \in S$. In particular,

$$\frac{|\partial_\Gamma(F)|}{|F|} = \frac{\sum_{s \in S} |sF \setminus F|}{|F|} < \epsilon$$

and thus the graph Γ is amenable.

Suppose now that the graph Γ is amenable. First we will prove the following.

Claim. For any finite subset $L \subset G$ and $\epsilon > 0$ there exists a finite subset of vertices $F \subset \Gamma$ such that

$$\frac{| \cup_{l \in L} lF \setminus F|}{|F|} < \epsilon. \tag{A.1}$$

We will then show that this claim implies the Følner condition for the action of G on the set of vertices of Γ.

Proof of the claim. Let $m > 0$ be big enough such that $L \subseteq S^m$ where S^m is the set of words of length m over the alphabet S. Let $\epsilon > 0$. As Γ is amenable there exists a finite subset of vertices $F \subset \Gamma$ such that for any $s \in S$ we have $|sF \setminus F| < \epsilon \frac{|F|}{c(m)}$ where $c(1) = |S|$ and $c(m) = |S^m|(1 + c(m-1) \cdot |S^{m-1}|)$.

Consider $\cup_{l \in L} lF \setminus F$. Notice that $\cup_{l \in L} lF \setminus F \subseteq \cup_{g \in S^m} gF \setminus F$. For any $g \in S^m$ we have

$$gF \setminus F \subseteq (gF \setminus \cup_{g' \in S^{m-1}} g'F) \cup (\cup_{g' \in S^{m-1}} g'F \setminus F) \subseteq (g'sF \setminus g'F) \cup (\cup_{g' \in S^{m-1}} g'F \setminus F)$$

where the last inclusion holds since g can be written as a product of some element $g' \in S^{m-1}$ and $s \in S$. Inductively, we obtain that $\cup_{g \in S^m} gF \setminus F$ is a subset of the finite union of $g(sF \setminus F)$ for some $g \in G$ and $s \in S$. The number of summands of this finite union is bounded by $c(m)$. Then

$$|\cup_{l \in L} lF \setminus F| \leq |\cup_{g \in S^m} gF \setminus F| \leq c(m)\epsilon \frac{|F|}{|c(m)|} = \epsilon |F|.$$

Now we will show that the claim above implies the Følner condition for the action of G on the set of vertices of Γ. Let L be a finite subset of G. Without loss of generality suppose that $e \in L$ and L is symmetric. Let $\epsilon > 0$. By the inequality (A.1) there exists a finite set $F \subset \Gamma$ such that $\frac{|\cup_{l \in L^2} lF \setminus F|}{|F|} < \epsilon/2$.

For any $l \in L$ we have

$$\cup_{l' \in L} ll'F \setminus \cup_{l' \in L} l'F \subseteq \cup_{l'' \in L^2} l''F \setminus F.$$

On the other hand

$$\cup_{l' \in L} l'F \setminus \cup_{l' \in L} ll'F \subseteq \cup_{l'' \in L^2} l''F \setminus F.$$

Hence for any $l \in L$,

$$\frac{|l(\cup_{l' \in L} l'F) \triangle \cup_{l' \in L} l'F|}{|\cup_{l' \in L} l'F|} \leq 2 \frac{|\cup_{l'' \in L^2} l''F \setminus F|}{|F|} < \epsilon.$$

For $F \subset \Gamma$ chosen above, the set $\cup_{l \in L} lF$ satisfies the Følner condition for the finite subset $L \subset G$. □

Proposition A.3.2. *Let G be a countable discrete Kazhdan group, let S be its finite symmetric set of generators ($S = S^{-1}$) and H be a subgroup of G. Denote by $\Gamma = Sch(G, H, S)$ the Schreier graph of G with respect to H. If the graph Γ is amenable then it is finite.*

Proof. The conclusion follows from Lemma A.2.3 and Lemma A.3.1. □

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