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Computable Qualitative Comparative Static Techniques

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COMPUTABLE QUALITATIVE COMPARATIVE STATIC TECHNIQUES[1]

BY GILBERT RITSCHARD[2]

This article is devoted to computable techniques for solving comparative static problems when only the sign of the partial derivatives of the model is considered. We first show how to extract unambiguously signed multipliers, or more generally qualitatively linked multipliers. This information then helps to reduce the size of the original system by means of a qualitative aggregation principle which we establish. As to the computation of solutions, a branch-and-bound algorithm is presented which considerably increases the efficiency of the Samuelson–Lancaster elimination principle. Finally we derive an efficient algorithm to check for signed determinants. The techniques are then applied to the analysis of an actual 20 equation model.

1. INTRODUCTION

BROADLY SPEAKING, comparative static analysis deals with the study of how, in a given economic model, endogenous variables react to given changes in exogenous variables or parameters. In the case of a fully quantified model, such an analysis can be carried out numerically, for example, by examining a quantified impact multiplier matrix. This does not allow, however, distinguishing between the conclusions which stem from the particular empirical content of the model and those which emerge from its theoretical background. The aim of qualitative methods is to make this distinction. By dealing with the model in its general formulation these methods take into account only a priori (i.e. theoretical) information in order to determine its logical implications.

For marginal changes, one way to achieve a qualitative analysis is to formally express the impact multipliers in terms of the derivatives of the equation. This allows us to see where a priori information about these derivatives is sufficient to determine the sign of some impact multipliers. However, as the size of the model, i.e. the number of equations, increases, it rapidly becomes tedious or impossible, even with the help of a computer, to formally express such impact multipliers.

An alternative way to solving comparative static problems is to make a systematic study of the implications of some specific kind of a priori information. This is the approach followed in this paper. The specific information we shall consider consists of the signs of the partial derivatives of the equations of the model. Retaining such information, one can then express the study of compara-

[1]This article reports some of the main results of the author's Doctoral Thesis [**15**]. A previous version of this article, prepared under the auspices of the LABREV while the author was guest professor at the University of Quebec at Montreal, was presented at the 6th Annual Convention of the Eastern Economic Association, Montreal, May, 1980.

[2]The author is very grateful to Daniel Royer for fruitful discussions. He also wishes to thank Professors Angus Deaton and Lise Salvas-Bronsard, as well as three anonymous referees, for constructive comments on an earlier draft of this article. Many thanks also to John Hislaire for his effort in trying to render the author's English less eccentric.

tive static problems in terms of the resolution of systems of qualitative linear equations.

The relevance of this approach was first stressed by Samuelson in 1947 [**21**]. Then, between 1962 and 1970, following Lancaster's work [**7**] quite a lot of research on qualitative linear systems was done. See [**1**] for a survey. There has been a recent renewal in this field [**6**; **11**; **12**; **13**; **14**; **18**, Ch. 7 and 8; and **19**].

Most of these studies deal with necessary and sufficient conditions for obtaining significant conclusions from sign assumptions. This includes results on the stability of qualitative matrices (see [**14**] for a survey on this topic) and on the solvability of qualitative linear systems, i.e. conditions for having a unique solution [**2**, **5**, **6**, **7**, **8**, and **13**]. Indeed, qualitative stability, as well as solvability are powerful theoretical properties. However, conditions for them to hold are very strict and unlikely to be fulfilled, except for very small systems.

The aim of this article is somewhat different. Its purpose is to provide practical techniques for the qualitative analysis of existing models. This presupposes the research of weaker properties than stability and solvability and the development of efficient algorithms. In order to achieve this goal we reason in terms of Samuelson's elimination principle [**21**, pp. 23–28]. Another interesting approach, not considered here, would be to use graph theory and to argue in terms of the principle of transmission of influence [**12**; **17**; **18**, Ch. 7].

Along the line followed, the most significant contribution so far has been Lancaster's development [**9**, **10**] of Samuelson's qualitative calculus. Despite these improvements, the applicability of the method remains very limited. Indeed, it still requires a starting set which increases exponentially with the size of the model. This article suggests then two complementary ways to overcome this difficulty: a qualitative aggregation principle and a branch-and-bound procedure. In addition the results are extended to check for signed determinants.

We should mention that as is usually the case in qualitative economics we only consider deterministic models. This ensures, even in the presence of nonlinearity, the (implicitly assumed) nonstochastic nature of the impact multipliers.

The problem to be studied is formalized in Section 2, together with a short discussion on the necessity of a previous causal ordering of the model. Section 3 introduces the concept of "qualitative link" and presents an efficient algorithm to determine such links. Section 4 discusses the improvement of the Samuelson–Lancaster elimination method by means of a branch-and-bound procedure. In Section 5, we establish an algorithm used to check for signed determinants. Finally, for illustrative purposes a qualitative analysis of the Quebec Econometric Model [**20**] is presented.

2. CAUSAL ANALYSIS AND QUALITATIVE CALCULUS

Let us consider an economic model formally represented by a system of n equations:

$$h(y,z) = 0, \tag{1}$$

which relates the equilibrium level of the n endogenous variables y to the state of the environment represented by the m exogenous variables z.

DEFINITION 1: By giving the sign (positive, negative, or zero) of each element of the matrix

$$\left[\begin{array}{cc} \frac{\partial h}{\partial y'} & \frac{\partial h}{\partial z'} \end{array}\right], \tag{2}$$

we completely characterize the *qualitative structure* of the model.

Comparative statics is mainly concerned with the study of the impact multiplier matrix, which can be written in terms of the two matrices in (2):

$$\frac{\partial y}{\partial z'} = -\left[\frac{\partial h}{\partial y'}\right]^{-1} \frac{\partial h}{\partial z'} . \tag{3}$$

Thus the problem considered is that of providing information about the content of the matrix (3) when only the knowledge of the qualitative structure is taken into account. Obviously, this information can only concern the sign pattern of the matrix $\partial y/\partial z'$.

DEFINITION 2: A multiplier $\partial y_i/\partial z_j$ is *qualitatively determined* when its sign (positive, negative, or zero) is unambiguously determined from the qualitative structure.

DEFINITION 3: A multiplier $\partial y_i/\partial z_j$ is said to be a (weak) *qualitative zero* if and only if in the model (1) y_i is determined independently from the exogenous variable z_j, i.e. if and only if z_j has no causal effect on y_i.

A causal analysis of the model, for which efficient tools related to graph theory have recently been developed (see, for instance, [**3**, **4**, and **16**] for a computer program), enables the extraction of qualitative zeros. Practically such an analysis consists in determining the block recursive decomposition of the model. This decomposition is characterized by the block triangular form into which the matrix $\partial h/\partial y'$ can be transformed through independent permutations of its rows and columns. Let

$$D = \left[\begin{array}{ccc} D_{11} & & 0 \\ \vdots & \ddots & \\ D_{p1} & \cdots & D_{pp} \end{array}\right], \tag{4}$$

where the diagonal submatrices D_{ii} are square and irreducible, be this block triangular matrix. The equations $h^k(y,z) = 0$ corresponding to the rows of a block D_{kk} are the smallest subset of equations which determines the endogenous variables y^k corresponding to the columns of D_{kk}. Indeed all variables $y^j, j < k$,

can be considered as exogenous in the interdependent block $h^k(y,z)=0$, whereas the variables y^j, $j>k$, do not appear in the equations of this block. Thus

$$\frac{\partial y^k}{\partial z_s} = -D_{kk}^{-1}\left[\frac{\partial h^k}{\partial z_s} + \sum_{j=1}^{k-1} D_{kj}\frac{\partial y^j}{\partial z_s}\right] \tag{5}$$

from which we deduce, without further proofs, the following theorem.

THEOREM 1: *Assume the model has been partitioned according to the decomposition* (4). *Then the multipliers* $\partial y^k/\partial z_s$ *are qualitative zeros if and only if*: (i) $\partial h^k/\partial z_s = 0$ *and* (ii) $\partial h^j/\partial z_s = 0$ *for all* j *for which a sequence of nonzero matrices*: $D_{kr_1}, D_{r_1r_2}, \ldots, D_{r_sj}$, *can be found.*

The application of this theorem to determine the qualitative zeros of $\partial y/\partial z'$ only requires the knowledge of the zero entries of matrix (2). Qualitative zeros are therefore more general properties of a model than those studied in the remainder of the article.

For a generic exogenous variable $\alpha = z_s$, the corresponding multiplier vector $\partial y/\partial\alpha$ is a solution of the following linear system:

$$\frac{\partial h}{\partial y'}\frac{\partial y}{\partial \alpha} = -\frac{\partial h}{\partial \alpha}. \tag{6}$$

From Theorem 1, if $\partial y^1/\partial\alpha$ is the subvector of all qualitative zeros in $\partial y/\partial\alpha$, then (6) can be written, through suitable permutations, as follows:

$$\begin{bmatrix} \dfrac{\partial h^1}{\partial y^{1'}} & 0 \\ \dfrac{\partial h^2}{\partial y^{1'}} & \dfrac{\partial h^2}{\partial y^{2'}} \end{bmatrix}\begin{bmatrix} \dfrac{\partial y^1}{\partial \alpha} \\ \dfrac{\partial y^2}{\partial \alpha} \end{bmatrix} = \begin{bmatrix} 0 \\ -\dfrac{\partial h^2}{\partial \alpha} \end{bmatrix} \tag{7}$$

where $\partial y^2/\partial\alpha$ contains no qualitative zeros. Thus in order to study the other multipliers $\partial y^2/\partial\alpha$, one only needs to consider a subsystem which has no qualitative zeros, i.e.,

$$\frac{\partial h^2}{\partial y^2}\frac{\partial y^2}{\partial \alpha} = -\frac{\partial h^2}{\partial \alpha}. \tag{8}$$

Henceforth we shall, with no loss of generality, make the following assumption.

ASSUMPTION 1: The multiplier vector $\partial y/\partial\alpha$ studied contains no qualitative zeros.

DEFINITION 4: Given a $p\times r$ matrix $B=[b_{ij}]$, the $p\times r$ matrix $H=[h_{ij}]$, with $h_{ij}=+$ if $b_{ij}>0$, $h_{ij}=-$ if $b_{ij}<0$, $h_{ij}=0$ if $b_{ij}\equiv 0$, is said to be a *qualitative matrix* associated to B.

Let us consider the $n \times n$ qualitative matrix A associated to $\partial h/\partial y'$, the $n \times 1$ qualitative vector b associated to $\partial h/\partial \alpha$, and the following obvious qualitative operations:

(9)

sum					product				
+	+	−	0	r		+	−	0	r
+	+	r	+	r	+	+	−	0	r
−	r	−	−	r	−	−	+	0	r
0	+	−	0	r	0	0	0	0	0
r	r	r	r	r	r	r	r	0	r

where an r stands for indetermination. We then have to face the problem of solving a qualitative system of the form:

$$Ax = -b. \tag{10}$$

By setting

$$H = [b \quad A], \qquad v' = [+ \quad x'], \tag{11}$$

we can, in an equivalent way, study the more convenient homogenous system: $Hv = 0$, which can be considered as the qualitative system associated to:

$$\begin{bmatrix} \frac{\partial h}{\partial \alpha} & \frac{\partial h}{\partial y'} \end{bmatrix} \begin{bmatrix} d\alpha \\ d_\alpha y \end{bmatrix} = 0 \tag{12}$$

with $d\alpha > 0$ and $d_\alpha y = (\partial y/\partial \alpha) d\alpha$.

DEFINITION 5: Assume H is $p \times r$. Then an $r \times 1$ sign vector s is a *qualitative solution* of $Hv = 0$, if and only if:

$$\sum_{j=1}^{r} h_{ij} s_j = 0 \text{ or } r \qquad \text{for all} \quad i = 1, \ldots, p. \tag{13}$$

According to (11) we shall in general only consider solutions, s, normalized such that $s_1 = +$. Let us denote by S the set of all such acceptable solutions for the qualitative system associated to (12). It then follows that a multiplier $\partial y_i/\partial \alpha$ is qualitatively determined if and only if its corresponding element s_{i+1} has the same sign in each vector s of S. Without confusion we shall also state, in this case, that the variable v_{i+1} is qualitatively determined.

DEFINITION 6: Let S be the set of qualitative solutions s of $Hv = 0$. Then two variables v_i and v_j are *qualitatively linked* ($v_i \mathcal{L} v_j$) if and only if s_i and s_j are always of the same sign, or always of opposite signs in all s of S. In the first case, v_i and v_j are said to be *positively linked* ($v_i \mathcal{L}^+ v_j$) and in the second case *negatively linked* ($v_i \mathcal{L}^- v_j$).

This definition generalizes the concept of qualitatively determined variables. From (11), qualitatively determined variables are then those variables which are qualitatively linked with v_1.

Qualitatively linked variables can easily be determined by examining the set S. However the difficulty remains in obtaining this set. Before studying this problem in Section 4 we show in Section 3 how qualitatively linked variables can be set up directly. Incidentally a qualitative aggregation procedure is suggested, which allows reduction of the size of the qualitative system being analyzed.

3. QUALITATIVELY LINKED VARIABLES AND QUALITATIVE AGGREGATION

Relation $\mathscr{L}$ being an equivalence relation (i.e. reflexive, symmetrical, and transitive) allows us to partition the set of p variables v of a qualitative system $Hv = 0$ into equivalence classes. This section provides a practical method of determining this partition.

DEFINITION 7: Assume v^1 is a subset of p_1 qualitatively linked variables. We then characterize the *qualitative link* between the variables in v^1 by a sign vector q^1 of the same size as v^1, defined by setting $q_i = q_j$ if $v_i \mathscr{L}^+ v_j$ and $q_i = -q_j$ if $v_i \mathscr{L}^- v_j$. With respect to the set S of solutions without zeros of $Hv = 0$, q^1, or $-q^1$, is the subvector s^1 corresponding to v^1 of any vector s of S.

DEFINITION 8: Assume there is a qualitative link q^1 between the variables in v^1. Then $v_1^* = q^{1'}v^1$ is said to be a *qualitative aggregate* of the variables in v^1.

The knowledge of the qualitative link q^1 for a class v^1, and of the sign s_i of only one component v_i of v^1, is sufficient to determine the signs s^1 of all variables in v^1. Assuming $q_i = +$, we then have: $s^1 = s_i q^1$. The procedure is then to reduce the size of the system $Hv = 0$ by considering only one variable instead of the whole vector v^1. Since all variables do not appear in all equations, one way to preserve the full qualitative information on v^1 is to replace the subvector v^1, of order p_1, by the qualitative aggregate v_1^*. Assuming without loss of generality that $v' = [v^{1'} \quad v^{2'}]$, we replace the vector v by

$$v^* = \begin{bmatrix} q^{1'}v^1 \\ v^2 \end{bmatrix} = Mv, \tag{14}$$

where the $(p - p_1 + 1) \times p$ qualitative *aggregation matrix* M is defined as

$$M = \left[\begin{array}{c:c} q^{1'} & 0' \\ \hdashline 0 & I_+ \end{array}\right], \tag{15}$$

I_+ being the qualitative matrix associated to the identity matrix I. Since the ordering of the columns of matrix H corresponds to that of the variables v, one can consider aggregating H into the following $r \times (p - p_1 + 1)$ matrix:

$$HM' = \left[H_1 \vdots H_2\right] M' = \left[H_1 q^1 \vdots H_2\right]. \tag{16}$$

For some rows $h_i' = [h_i^{1'} \vdots h_i^{2'}]$ of H, the product $h_i^{1'}q^1$ might, however, be qualitatively undetermined. According to Definition 5, the corresponding relations $h_i'v = 0$ admit as a solution any sign vector s compatible with the qualitative link q^1, i.e. all s for which $s^1 = \pm q^1$. These rows h_i' do not provide any further information and therefore can be eliminated.

DEFINITION 9: Let H^{**} be the matrix obtained after eliminating the rows h' for which $h_i^{1'}q^1$ is undetermined. We then define H^*, the qualitative matrix aggregated according to q^1, by

$$H^* = H^{**}M'. \tag{17}$$

THEOREM 2: *Assume there is a qualitative link q^1 among the variables of v^1 in $Hv = 0$, and consider the aggregated system $H^*v^* = 0$. Let S and S^* respectively be the set of qualitative solutions without zeros of $Hv = 0$ and $H^*v^* = 0$. There is then the following one to one relationship between S and S^*:*

$$S^* = \{s^*; s^* = Ms, s \in S\} \Leftrightarrow S = \{s; s = M's^*, s^* \in S^*\}. \tag{18}$$

PROOF: Let R be the set of qualitative vectors s, with $s' = [s^{1'} \quad s^{2'}]$ and $s^1 = \pm q^1$. Because of the qualitative link q^1 among the variables in v^1, the subset of vectors of R solutions of $H^{**}v = 0$ is identically equal to S. Thus, to prove the theorem, one has to establish that s^* is a solution of $H^*v^* = 0$, if and only if $s = M's^*$ is a solution of $H^{**}v = 0$. But this is obvious since $H^*s^* = H^{**}M's^* = H^{**}s$.

We can generalize the procedure to the case of several groups of qualitatively linked variables. Let q^i, $i = 1, \ldots, k$, be the qualitative links for subsets of variables v^i. To generalize the aggregation rules (14) and (17) we simply have to define the new aggregation matrix:

$$M = \begin{bmatrix} q^{1'} & & 0 & \vdots & \\ & q^{2'} & & \vdots & \\ & & \ddots & \vdots & 0 \\ 0 & & q^{k'} & \vdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & 0 & & I_+ & \end{bmatrix}. \tag{19}$$

The matrix H^{**} in (17) is obtained by eliminating from H the rows h_i' for which one of the products $h_i^{j'}q^j$, $j = 1, 2, \ldots, k$, is qualitatively undetermined.

We should mention that the elimination of rows from H can lead to a matrix H^{**} with columns of zeros. In the case where all columns of H^{**} corresponding to a class v^i of qualitatively linked variables are null, the column of H^* corresponding to the aggregate $v_i^* = v^{i'}q^i$ will also be null (see (17)).

DEFINITION 10: An aggregate v_i^* corresponding to a column of zeros in H^* can be "+" or "−" regardless of the sign of the other components of v^*. v_i^* is then called an *independent variable or aggregate*, and the class v^i, represented by v_i^*, an *independent class*.

Let $v^{*\prime} = [v^{\Delta\prime} \quad \bar{v}^{\Delta\prime}]$, where $\bar{v}^{\Delta}$ is a subvector of k independent variables. The set S^* of solutions of the aggregated system $H^*v^* = 0$ can easily be obtained from the set S^{Δ} of solutions of $H^{\Delta}v^{\Delta} = 0$, where H^{Δ} results from the elimination of the columns corresponding to $\bar{v}^{\Delta}$ in H^*. Since the k elements of $\bar{v}^{\Delta}$ are independent variables, the 2^k possible sign patterns of length k are admissible for $\bar{v}^{\Delta}$ and thus each solution s^{Δ} gives rise to 2^k solutions of the form $s^{*\prime} = [s^{\Delta\prime} \quad \bar{s}^{\Delta\prime}]$.

Let us now turn to the extraction of the qualitatively linked variables which have to be known before proceeding to a qualitative aggregation.

The method suggested is iterative and uses the qualitative aggregation principle. First a pair of qualitatively linked variables is sought. If found, the system is aggregated with respect to this pair. Next, the procedure is repeated for the aggregated system and so on until a system without qualitatively linked variables is reached.

In order to determine a couple of linked variables at each step, one can use the following sufficient condition:[3]

THEOREM 3: *Assume h_{ki} and h_{kj} are the only two nonzero entries in a row h_k' of H. Then the variables v_i and v_j are qualitatively linked in $Hv = 0$. The qualitative link in $v^{1\prime} = [v_i \quad v_j]$ is given by $q^{1\prime} = [h_{ki} \quad -h_{kj}]$.*

PROOF: The argument is straightforward since, when h_{ki} and h_{kj} are the only nonzero elements of h_k, $h_k'v = 0$ can be written $v_i = -(h_{kj}/h_{ki})v_j$.

DEFINITION 11: An $r' \times p$ qualitative matrix G is said to be *qualitatively equivalent* to the $r \times p$ matrix H, if $Gv = 0$ and $Hv = 0$ have the same set of solutions without zeros.

When Theorem 3 cannot be used directly, one may attempt to apply it to a transformation of H equivalent to H. This transformation can be obtained, for instance, by means of the following conditions:

LEMMA 1: *Assume h_{ik} is a nonzero entry of a row h_i' of H. Let $h_i^{-\prime}$ be the row obtained by reversing the sign of h_{ik}. Then the set S of solutions without zeros of $Hv = 0$ remains the same when h_{ik} is replaced by a zero in H if and only if each*

[3] For example, if we consider

$$H = \begin{bmatrix} + & + & - \\ + & + & + \end{bmatrix},$$

we have $v_1 \mathscr{L}^{-} v_2$, even if the condition of Theorem 3 is not satisfied.

sign pattern which is incompatible with $h_i^{-\prime} v = 0$ is also incompatible with at least one equation of $Hv = 0$.

THEOREM 4: *If two rows h_i' and h_j' of H can be written in the following form by joint permutations of their elements*:

$$h_i' = \left[h_i^{1\prime} \vdots h_{ik} \vdots h_i^{2\prime} \right],$$

(20)

$$h_j' = \left[h_j^{1\prime} \vdots h_{jk} \vdots 0' \right],$$

with

$$h_j^1 = h_i^1 \quad \textit{and} \quad h_{jk} = \begin{cases} 0 \\ \textit{or} \\ -h_{ik} \end{cases}$$

or

$$h_j^1 = -h_i^1 \quad \textit{and} \quad h_{jk} = \begin{cases} 0 \\ \textit{or}, \\ h_{ik} \end{cases}$$

then h_{ik} can be replaced by a zero without affecting the set S of solutions without zeros of $Hv = 0$.

PROOFS: Let E_r be the set of sign vectors s for which $h_r's = +$ or $h_r's = -$. E_r is thus the set of sign patterns for v which have to be eliminated from the equation $h_r'v = 0$. Let E be the union of the E_j, $j \neq i$, associated to the rows h_j' of H. Designate by $h_i^{0\prime}$ the row obtained by setting $h_{ik} = 0$ in row h_i'. Then if E_i^- and E_i^0 are the set of sign patterns incompatible respectively with $h_i^{-\prime}$ and $h_i^{0\prime}$ we have $E_i^0 = E_i \cup E_i^-$. The set of sign patterns incompatible with $Hv = 0$ (the complementary of S) is $E_i \cup E$. Then, obviously, the equality $E_i \cup E = E_i^0 \cup E$ is equivalent to $E_i^- \subset E$, which proves the Lemma. By construction of row h_j' in Theorem 4 we have $E_i^- \subset E_j$, which corresponds to a special (i.e. sufficient) case of Lemma 1.

The following particular cases of Theorem 4 can be mentioned: (i) If $h_{jk} = 0$ in (20), all elements of h_i^2 can be replaced by zeros. Since the transformed row will thus provide the same qualitative information as h_j', it can simply be deleted from H. (ii) If $h_i^2 = 0$, the two rows differ in only one nonzero element or in all but one. In this case both h_{ik} and h_{jk} can be replaced by zeros and the two transformed rows will provide the same qualitative information. The two rows in (20) are then replaced by the equivalent single row $[h_i^{1\prime} \quad 0']$ in H.

DEFINITION 12: A qualitative matrix in which no element can be replaced by a zero using Theorem 4 is said to be *quasi-minimal*.

Contrary to the general formulation of Lemma 1, Theorem 4 is easily applicable as it requires only the comparison of rows of H. As an illustration the reader may check that in

$$H = \begin{bmatrix} + & - & 0 & - & 0 \\ + & - & - & + & 0 \\ + & - & + & 0 & - \\ - & + & 0 & 0 & - \end{bmatrix}, \tag{21}$$

the elements h_{24}, h_{35}, h_{23}, h_{33}, h_{14}, and h_{45} can successively be set to zero. Thus the row [+ − 0 0 0] is quasi-minimal and equivalent to (21). From Theorem 3 and Definition 10, it follows that in this example v can be partitioned into four independent classes: $\{v_1, v_2\}$, $\{v_3\}$, $\{v_4\}$, $\{v_5\}$.

Using the easily checkable conditions given in Theorems 3 and 4 we build the iterative qualitative aggregation procedure illustrated by Figure 1.[4] In this procedure, the qualitative link q^1 for each class of qualitatively linked variables v^1 is built step by step. Without loss of generality suppose that a qualitative link between two aggregates v_i^* and v_j^* has been found at the kth step. Assume q^i and q^j are respectively the links between the variables v^i, represented by v_i^*, and v^j, represented by v_j^*. The link q^1 for the vector $v^{1'} = [v^{i'} \quad v^{j'}]$ is then $q^{1'} = [q^{i'} \quad q^{j'}]$ if $v_i^* \mathcal{L}^+ v_j^*$, or $q^{1'} = [q^{i'} \quad -q^{j'}]$ if $v_i^* \mathcal{L}^- v_j^*$.

Theorems 3 and 4 provide only sufficient conditions. Nevertheless the algorithm given in Figure 1 should allow us to find almost all qualitative links. Moreover qualitative links which cannot be detected by means of these conditions correspond to particular sign configurations in H which can be considered as singular cases.

Such a singular configuration is obtained, for example, by setting h_{22} equal to zero in (21). Indeed this modification does not affect the positive link between v_1 and v_2, though this link would not be detected by the procedure in Figure 1. The specificity of the configuration considered follows from the fact that, as the reader can check, the method would apply when changing to zero any other element of (21). Furthermore, replacing any zero by a nonzero term would break the linkage between v_1 and v_2.

The singularity of the configurations giving rise to nondetected linkages has been confirmed by our experiments. Among all the qualitative systems for which we were able to obtain the set S of solutions (some of them having more than 40 equations after the qualitative aggregation) we have never encountered such singular cases except, of course, for the counterexample we have built.

Since it would be excessively time consuming to check for the absence of such singular cases, it is preferable, in order to preserve efficiency, to neglect these and limit oneself to the heuristic algorithm given in Figure 1. This algorithm always works, for instance, on systems which can be put in the Lancaster [7] or more

[4]The algorithm has been successfully tested. The qualitative aggregation of systems of up to 200 equations required less than 10 seconds CPU time.

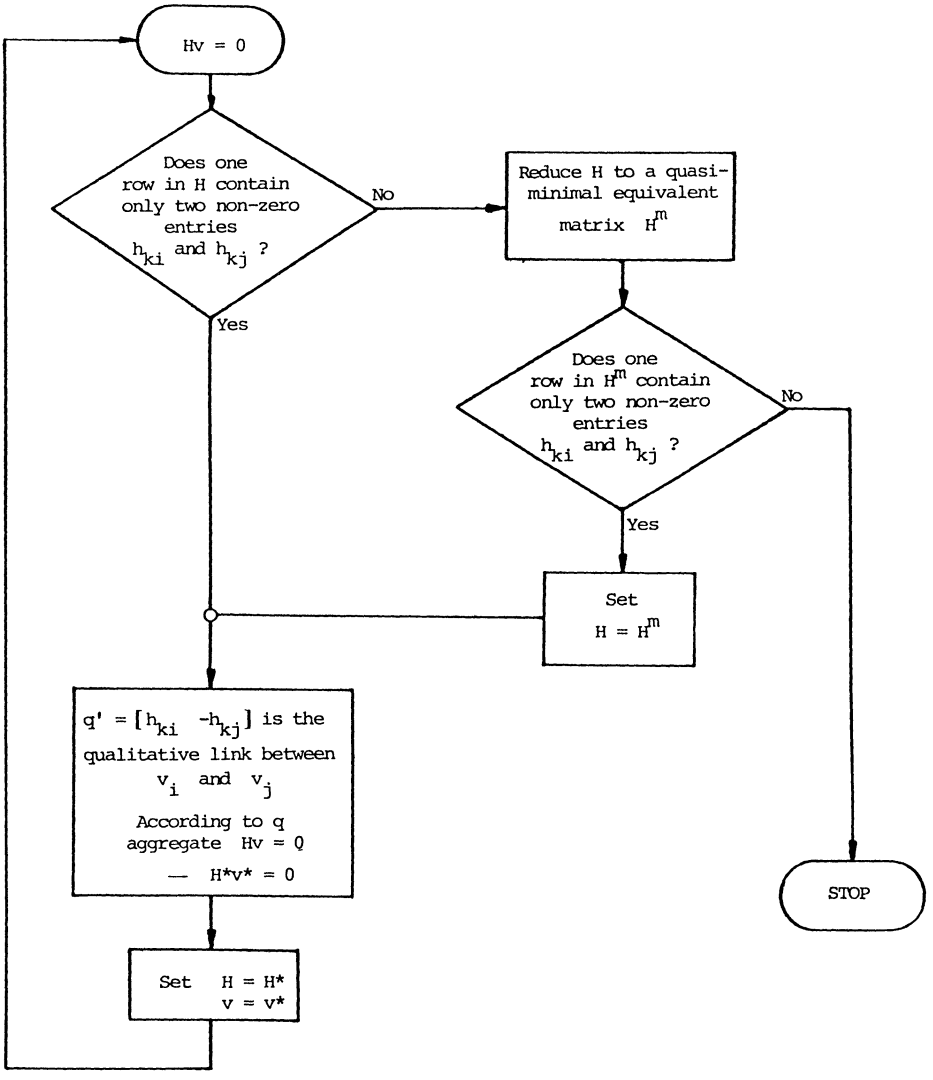

FIGURE 1—The iterative qualitative aggregation procedure.

general Gorman–Lancaster [**5**, **8**] form, i.e. when all the variables form a unique class of qualitatively linked variables.

4. IMPROVEMENT OF THE SAMUELSON–LANCASTER ELIMINATION PRINCIPLE

This section discusses the computation of the set S of qualitative solutions for a qualitative system: $Hv = 0$, with r equations and p variables. We shall assume

only that no variable v_i is a qualitative zero, so that S contains at least one solution s without zeros.

Here one may use Samuelson's elimination principle [**21**, pp. 23–29]. The technique consists in eliminating from the set of all sign patterns s of length p those for which (13) does not hold. This leads us to compare 2^p a priori possible sign vectors s with each row of H.[5] In order to improve this procedure, Lancaster [**10**] suggested representing sign vectors by binary numbers, according to which the comparison reduces simply to that of integer numbers. Under this form the efficiency of the elimination principle remains limited, however, due to the size 2^p of the *reference set*, i.e. the set of all a priori sign patterns, which increases exponentially with p, the number of variables. For example, one would have to test more than 10^6 possibilities for $p = 20$, and more than 10^{15} for $p = 50$.[6]

With respect to this limitation a gain in efficiency will be achieved if, before computing S, we can reduce the number p of variables by qualitative aggregation. From Theorem 2, this can be done without loss of generality. Nevertheless, the method will fail when the number of classes of qualitatively linked variables remains too large.

In order to overcome this difficulty, we now present a branch-and-bound algorithm which renders the elimination principle operational even for a large number, p, of variables.

THEOREM 5: *Assume* $Hv = 0$ *can be written in the form*:

$$(22) \qquad \begin{bmatrix} H_{11} & 0 \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = 0$$

where the partition of v *is compatible with that of the columns of* H (H_{11} *and* H_{22} *need not be square.*) *Then for any solution* $s' = [s^{1'} \quad s^{2'}]$ *of* $Hv = 0$, s^1 *has to be a solution of* $H_{11}v^1 = 0$.

PROOF: By Definition 5, if s is a solution of $Hv = 0$, then it is also a solution of $[H_{11} \quad 0]v = 0$. In this subsystem, variables in v^2 are independent. Thus s^1 has to be a solution of $H_{11}v^1 = 0$.

In order to write H in the form (22), it is sufficient that this matrix has some zero entries. It is thus almost always possible to compute the set S sequentially. First, using the elimination principle, compute S^1 the set of solutions for $H_{11}v^1 = 0$. Then, search for the solutions of $[H_{21} \quad H_{22}]v = 0$ among the set of all possible sign vectors s for which s' belongs to S^1. From Theorem 5, the solutions thus obtained are all the acceptable solutions of $Hv = 0$. The advantage

[5] Note that there are 2^p a priori possibilities if only sign vectors without zero are considered. Otherwise this number would be 3^p.

[6] Examples of analyses of simple macromodels with Lancaster's algorithm can be found in [**10**, and **22**, Ch. 5].

of such a decomposition is that it enables us to compute the set S without having to test all 2^p a priori sign sequences.

The above two-step method can easily be generalized to a more efficient l step procedure by writing $Hv = 0$ in the form:

$$(23) \qquad \begin{bmatrix} H_{11} & & & 0 \\ H_{21} & H_{22} & & \\ \vdots & & \ddots & \\ H_{l1} & & & H_{ll} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^l \end{bmatrix} = 0$$

where l is assumed to be as large as possible. This implies, for instance, that the last column of each $r_i \times p_i$ submatrix H_{ii} contains only nonzero elements.

The solutions of $Hv = 0$ are sought recursively by seeking at each kth step the solution of

$$(24) \qquad [H_{k1} \quad \dots \quad H_{kk}] \begin{bmatrix} v^1 \\ \vdots \\ v^k \end{bmatrix} = 0,$$

among the sign vectors $[s^{1'} \quad \dots \quad s^{k-1'} \quad s^{k'}]$ where $[s^{1'} \quad \dots \quad s^{k-1'}]$ is one of the solutions obtained at the previous step.

Practically this recursive procedure leads to a branch-and-bound construction of the set S. This is shown in the following example:

$$(25) \qquad \left[\begin{array}{ccc:c:c} - & + & - & 0 & 0 \\ 0 & + & + & 0 & 0 \\ \hdashline - & 0 & + & + & 0 \\ \hdashline 0 & - & 0 & + & - \end{array}\right] \left[\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ \hdashline v_4 \\ \hdashline v_5 \end{array}\right] = 0$$

for which the determination of the solutions is described in Figure 2. The number alongside each neglected branch is that of the equation with which it is incompatible.

It must be noted here that the efficiency of the procedure is not independent of the ordering of the variables. Efficiency is optimized if, when writing the system in the form (23), the number p_i of columns of the matrices H_{ii}, $i = 1, \dots, l$, is minimized.

To illustrate the gain in efficiency of the proposed recursive procedure over the basic Samuelson–Lancaster elimination principle, we mention that, for the five 20 equation systems studied in Section 6, the computation time without previous qualitative aggregation was reduced from 30 minutes to 10 seconds CPU time on a UNIVAC 1108 computer. This gain in efficiency of the recursive procedure (the same being true for the aggregation algorithm) is due to a systematic accounting for the knowledge of the zero entries. For this reason the gain over

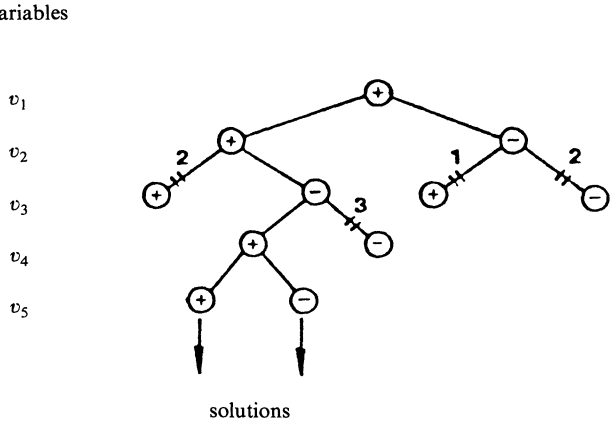

FIGURE 2.

the basic elimination principle increases with the proportion of zero entries in the matrix, which itself generally increases with the size of the system to be analyzed. Moreover, this renders a more systematic accounting of the causal outline of the system useless.

5. SIGNED DETERMINANTS

From Samuelson's correspondence principle it is legitimate, in a comparative static analysis, to impose that the determinant of the Jacobian be qualitatively defined (i.e. stability conditions require a given sign for $|\partial h/\partial y'|$). The purpose of this section is to show how to include this information efficiently in the qualitative calculus.

For the qualitative system $Ax = -b$, we have under Assumption 1: $|A|x_i = |A_i|$, $i = 1, \ldots, n$, where A_i is the matrix obtained by substituting $-b$ to the ith column of A. The determinant $|A|$ being qualitatively defined, all variables x_i for which the sign of $|A_i|$ is unambiguously determined from the qualitative structure are consequently qualitatively defined.

In order to apply this property one must be able to efficiently check whether the determinants $|A_i|$, $i = 1, \ldots, n$, are signed. As shown by Lancaster [**10**, p. 292] this can be done, for a generic matrix $F = A_i$, by checking whether the homogenous system, $Fw = 0$, admits a nontrivial qualitative solution.[7]

DEFINITION 13: For an $r \times p$ qualitative system $Hv = 0$, a variable v_i is said to be a *strong qualitative zero* if and only if $s_i = 0$ in each qualitative solution.

An equivalent formulation of the above result is as follows:

[7]Note however that this nontrivial solution need not be a solution vector without zeros as Lancaster implicitly assumes. It would be so only for matrices F which cannot be put in block triangular form.

THEOREM 6: *Assume F is a square qualitative matrix. Then its determinant $|F|$ is qualitatively determined if and only if all the variables of $Fw = 0$ are strong qualitative zeros.*

The problem considered can thus be solved by seeking the strong qualitative zeros of $Fw = 0$. This can be done using the following properties:

LEMMA 2: *Assume the matrix H of an $r \times p$ qualitative system $Hv = 0$ contains a row h_i' with $h_{ik} \neq 0$ and $h_{ij} = 0$ for all $j \neq k$. The variable v_k is then a strong qualitative zero.*

THEOREM 7: *An $r \times p$ qualitative system $Hv = 0$ has a strong qualitative zero if and only if it can be aggregated into a system $H^*v^* = 0$ for which at least one row $h_i^{*\prime}$ of H^* has one and only one nonzero entry.*

PROOF: The argument for Lemma 2 is obvious since $h_i'v = h_{ik}v_k = 0$ implies $v_k = 0$. The sufficiency in Theorem 7 follows directly from Lemma 2. The necessity of the condition is proven by the following argument. Let $H^*v^* = 0$ be a system in which no variables are qualitatively linked, such that it cannot be further aggregated. Assume that each row of the $r^* \times p^*$ matrix H^* has at least two nonzero entries. Then $H^*v^* = 0$ admits at least $p^* + 1$ solutions without zeros. Then no variable in v^*, nor in v, can be a strong qualitative zero.

The algorithm given in Figure 3 is derived from Theorems 6 and 7 and Lemma 2. The successive deletion of the strong qualitative zeros is done without loss of generality. Indeed if we denote w^0 and F^0 the vector and matrix obtained by eliminating the strong qualitative zeros from w and the corresponding columns of F, we have: $Fw = F^0w^0$. It is important, however, to apply this elimination procedure to the original system, and not to the aggregated one $F^*w^* = 0$. Indeed, if the equations eliminated during the aggregation procedure are redundant with respect to the solutions without zeros they may, however, remain essential for the extraction of strong qualitative zeros.

Note that it is not necessary to apply the algorithm to each matrix A_i, $i = 1, \ldots, n$. Indeed, when $|A_i|$ is signed, each variable qualitatively linked to x_i will also have a signed determinant associated with it. Moreover, each matrix A_i differs from another by only one column, apart from permutations of columns. We then have the following property.

THEOREM 8: *Assume the k ($< n$) variables in w^1 are the strong qualitative zeros of the $n \times n$ system $A_iw = 0$. Let $w' = [w^{1'} \quad w^{2'}]$ and partition $A_i = [A_i^1 \quad A_i^2]$ compatibly. Then the k matrices A_j which, apart from permutations of columns, differ from A_i by one column of A_i^1, all have an unsigned determinant.*

PROOF: Since w^2 are not strong qualitative zeros, $A_iw = 0$ admits a nontrivial solution $[0' \quad s^{2'}]$. But it is then also a solution of any system $Fw = 0$, with

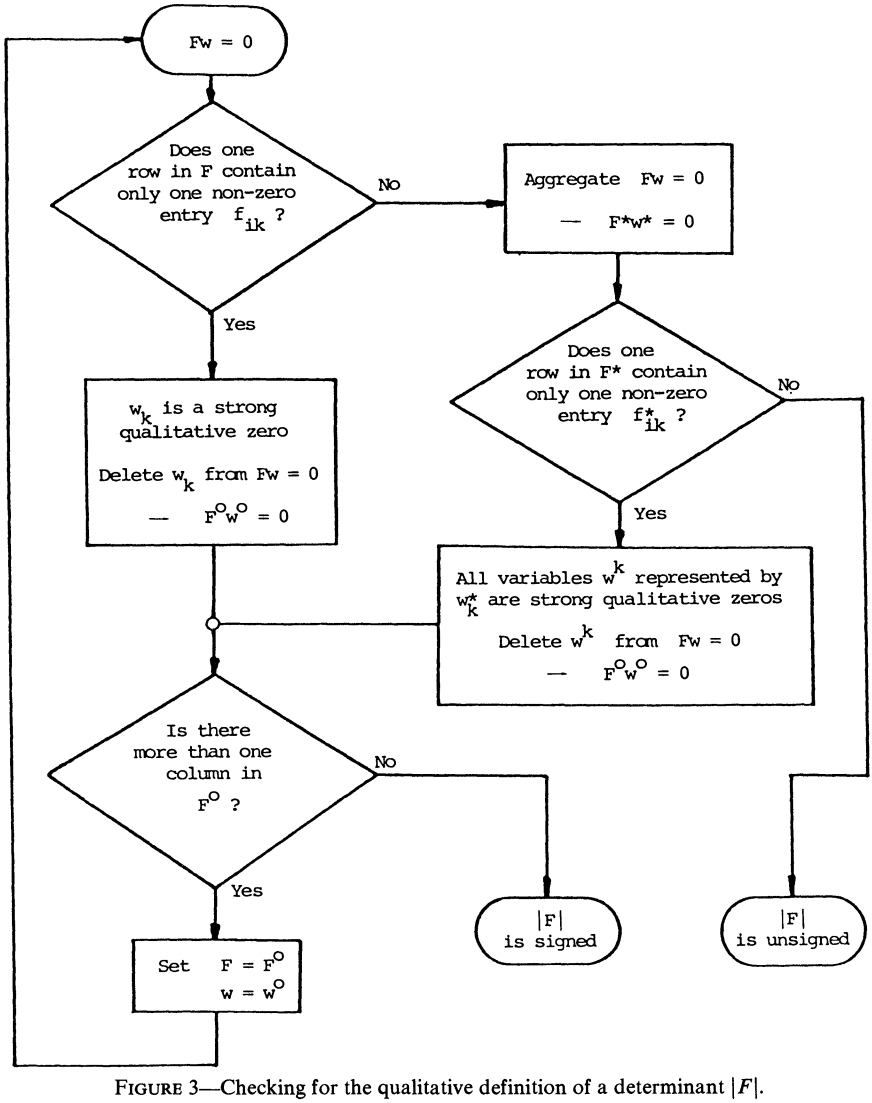

FIGURE 3—Checking for the qualitative definition of a determinant $|F|$.

$F = [F^1 \quad A_i^2]$. Thus, each matrix A_j which can be put into this form by permutations of its columns has an unsigned determinant.

The problem remains, then, in finding the signs of qualitatively defined determinants. For a signed determinant $|F|$, all nonzero terms in its expansion are of the same sign. This sign can thus simply be obtained by computing only one such term.

The method proposed requires the qualitative aggregation of the system

$Fw = 0$ at each step. This is done efficiently by means of the iterative aggregation procedure. Since the latter is a heuristic algorithm it will then also be the case of the procedure in Figure 3. Note, however, that the possibly nondetected qualitative links will not necessarily lead to a wrong conclusion about the determinant. Undetected signed determinants should thus be even more rare than missed links.[8]

Besides its efficiency, another advantage of the method is that it permits checking for the sign determinacy of a determinant under side conditions. Indeed, as is shown in the next section, side conditions often can be expressed in terms of additional qualitative equations. Taking these into account, the study of the determinant requires then the analysis of a rectangular matrix. But the algorithm in Figure 3 applies obviously to such nonsquare matrices, which solves the problem.

6. AN APPLICATION: SOME QUALITATIVE PROPERTIES OF A 20-EQUATION ECONOMETRIC MODEL[9]

Our purpose here is to show how the techniques developed in the previous sections can practically be used in the analysis of an actual fair sized model. The point is that, except for very simple models, the sign assumptions which define the qualitative structure are generally not sufficient to obtain significant results. Removing qualitative indeterminacies will thus require the introduction of side conditions. The crux of the matter is that the qualitative techniques considered permit dealing with such further information. This is made possible by the fact that the algorithms presented apply whatever the dimensions $r \times p$ of the qualitative system.

[8]The following matrix (pointed out by a referee) provides an example of such an undetected determinant:

$$\begin{bmatrix} + & 0 & - & - \\ + & + & + & 0 \\ 0 & - & + & - \\ + & - & 0 & + \end{bmatrix}.$$

However this counterexample does not invalidate the necessity in Theorem 7. Indeed, this qualitative matrix can be aggregated though the proposed aggregation procedure does not work in this case. The difficulty arises here because of the absence of necessity of the condition of Theorem 4 on which the aggregation algorithm is based. It is easily shown that the necessary and sufficient (but nonoperational!) condition given in Lemma 1 is satisfied for every nonzero entry in the above matrix. Any (but one at time) nonzero element can thus be changed to zero. The procedure then works without further problems.

On the other hand the reader can easily check that the algorithm detects a signed determinant for any matrix A_i extracted from a system written in the Gorman–Lancaster [**5**, **8**] standard form. These matrices A_i, are indeed reducible, each irreducible diagonal block of which can be put in the Lancaster form [7]: $A = [a_{ij}]$, with $a_{ij} = +$ for $i \leq j$, $a_{ij} = 0$, for $i > j + 1$, and $a_{j(j-1)} = -$, $j = 2, \ldots, n$.

[9]The analysis reported here has been carried out by means of the computer program ANAS [**4**, **16**]. All the algorithms in this paper have been introduced in the latter version of ANAS.

To illustrate this point we study some qualitative properties of a regional economic policy model: the Quebec Econometric Model [**20**]. The relations and the variables are given in the Appendix together with the basic qualitative assumptions taken into account for the analysis.

Impact multipliers are studied for five exogenous variables: 1. Public Consumption ($\overline{GP}$), 2. Public Investment ($\overline{IG}$), 3. Federal Tax Rate ($\overline{RF}$), 4. Wage Rate ($\overline{W}$), 5. Interest Rate on Mortgage ($\overline{IH}$). Since the model is dynamic, it is important to mention that the analysis relates only to direct (one period) multipliers.

The qualitative structure analyzed is summarized in Table I which also gives the causal outline of the model. As each exogenous variable considered appears in the first interdependent block, none of the 100 multipliers studied is a qualitative zero.

From the basic qualitative information, we determine the following classes of qualitatively linked variables for each exogenous variable considered:

$$
\begin{aligned}
&\{C, ICR, TI, YD\},\\
&\{TPP, YP\},\\
&\{YW, MO, EP, Y\},\\
&\{IM, IB, PS, DIV, TC\}.
\end{aligned}
\tag{26}
$$

All links are positive. If we only consider $\overline{IG}$, $\overline{RF}$, $\overline{W}$, and $\overline{IH}$, the variables U and EG are also positively linked. Among the effects on the other endogenous variable we can note that the impact on TPF is qualitatively independent in all five cases. The computation of the qualitative solutions gives 100 possibilities when $\overline{GP}$ is the exogenous variable considered, 80 for $\overline{IG}$, 64 for $\overline{RF}$, 116 for $\overline{W}$, and 112 for $\overline{IH}$.

These results are not very relevant. This, however is not surprising because of the generality of the assumptions considered. In order to obtain more significant results, we must consider further general information.

The determinant of the matrix A associated to the Jacobian $\partial h/\partial y'$ of the model is qualitatively undetermined. Assuming the model represents the equilibrium relations of some underlying dynamic process, stability conditions imply a given sign for the determinant $|\partial h/\partial y'|$. Our first additional constraint is therefore the following:

CONSTRAINT 1: $|A| = \operatorname{sign}[(-1)^{20}] = +$.

Positive determinants $|A_i|$ have been obtained for $\partial YP/\partial\overline{GP}$ and $\partial YP/\partial\overline{IG}$

TABLE I

CAUSAL AND QUALITATIVE STRUCTURE OF THE MODEL

	IH	W	IG	GP	RF	C	YD	YP	TPP	TPF	Y	TI	ICR	IM	IB	PS	U	MO	EP	EG	YW	YNI	DIV	TC	YR
101	·	·	·	·	·	−	+	·	·	·	·	·	·	·	·	·	·	·	·	·					
519	·	·	·	·	·	·	−	+	−	−	·	·	·	·	·	·	·	·	·	·					
518	·	·	·	·	·	·	·	−	·	·	+	−	·	·	·	·	·	·	·	·					
413	·	·	·	·	·	·	·	+	−	·	·	·	·	·	·	·	·	·	·	·					
414	·	·	·	·	+	·	·	+	−	−	·	·	·	·	·	·	·	·	·	·					
517	·	·	+	+	·	+	·	·	·	·	−	·	+	+	+	·	·	·	·	·					
415	·	·	·	·	·	·	+	·	·	·	·	−	·	·	·	·	·	·	·	·					
102	−	·	·	·	·	·	+	·	·	·	·	·	−	·	·	·	·	·	·	·					
103	·	·	·	·	·	·	·	·	·	·	·	·	·	−	·	+	·	·	·	·					
104	·	·	·	·	·	·	·	·	·	·	·	·	·	·	−	+	·	·	·	·					
205	·	·	·	·	·	·	·	·	·	·	+	·	·	·	·	−	−	·	·	·					
520	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	−	+	−	−					
310	·	·	·	·	·	·	·	·	·	·	+	·	·	·	·	·	·	−	·	·					
311	·	−	·	·	·	·	·	·	·	·	+	·	·	·	·	·	·	·	−	·					
312	·	·	·	+	·	·	·	·	·	·	·	·	·	·	·	·	+	·	·	−					
206	·	+	·	·	·	·	·	·	·	·	+	·	·	·	·	·	·	·	·	·	−				
208	·	·	·	·	·	·	·	·	·	·	+	·	+	·	·	·	·	·	·	·	·	−			
209	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	+	·	·	·	·	·	·	−		
416	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	+	·	·	·	·	·	·	·	−	
207	·	·	+	·	·	·	·	·	·	·	·	·	+	·	+	·	·	·	·	·	·	·	+	·	−

and negative ones for $\partial Y/\partial\overline{RF}$, $\partial YP/\partial\overline{W}$, and $\partial YP/\partial\overline{IH}$.[10] Imposing Constraint 1 leads to the results:[11]

$$(27)\qquad \frac{\partial YP}{\partial GP}>0,\qquad \frac{\partial YP}{\partial IG}>0,\qquad \frac{\partial Y}{\partial RF}<0,\qquad \frac{\partial YP}{\partial W}<0,\qquad \frac{\partial YP}{\partial IH}<0.$$

This information is taken into account in the analysis by adding a qualitative equation to each of the five qualitative systems. For example, adding the equation

$$(28)\qquad c'v=(+)v_{GP}+(-)v_{YP}=0$$

ensures a positive link between the variations of $\overline{GP}$ and YP.

According to Lancaster's linear combination technique [**10**, p. 288], quantitative a priori information sometimes allows definition of additional qualitative equations. This method, together with the knowledge of the exact value (-1) of the Jacobian diagonal elements, as well as that of the coefficients in definitional equation 519, enables us to take into account the hypothesis.

CONSTRAINT 2: $0<(\partial f_{13}/\partial YP)+(\partial f_{14}/\partial YP)<1.$

This constraint postulates that the marginal change in Personal Taxes ($TPP+TPF$) induced by a marginal change in Personal Income YP is less than unity. By subtracting, in the Jacobian (Table I), rows 413 and 414 from row 519, we obtain the qualitative equation:

$$(29)\qquad (-)v_{RF}+(-)v_{YD}+(+)v_{YP}+(+)v_{TPP}=0.$$

CONSTRAINT 3: $\partial f_{14}/TPP\geq -1.$

Analogously for this constraint, subtracting row 414 from row 519 leads to the following qualitative equation:

$$(30)\qquad (-)v_{RF}+(-)v_{YD}+(+)v_{YP}+(-)v_{TPP}=0.$$

The qualitative analysis under Constraints 1, 2, and 3 is done by adding the

[10] As a referee did point out to me the five corresponding matrices A_i are reducible to block triangular matrices, the greatest irreducible block, of size 6×6, being in A_{GP}. Elementary matrix properties indicate that the entries of off-diagonal blocks in a block triangular matrix do not affect the determinant of such a matrix. For this reason the more reducible a matrix is the more chance it has of having a signed determinant. It is, however, worthwhile to notice that the reducibility is not a necessary condition for this. For instance, from what has been said in footnote 8, the algorithm works for any $n\times n$ irreducible matrix that can be put in the Lancaster standard form [7]: $A=[a_{ij}]$, with $a_{ij}=+$ for $i\leq j$, $a_{ij}=0$ for $i>j+1$, and $a_{j(J-1)}=-$, $j=2,\ldots,n$. In terms of zeros the only necessary condition for having a signed determinant is that the matrix contains at least $(n-1)(n-2)/2$ null entries.

[11] The same signs have been obtained numerically in [**20**, pp. 375–377] for a quantified version of the model.

TABLE II

MULTIPLIERS QUALITATIVELY DETERMINED UNDER CONSTRAINTS 1 AND 2

		1 *GP*	2 *IG*	3 *RF*	4 *W*	5 *IH*
101	*C*	+	+	−[b]	−	−
102	*ICR*	+	+	−[b]	−	−
103	*IM*	·	·	·	·	·
104	*IB*	·	·	·	·	·
205	*PS*	·	·	·	·	·
206	*YW*	+	+	−[b]	·	−
207	*YR*	·	·	·	·	·
208	*YNI*	+	+	−[b]	−	−
209	*DIV*	·	·	·	·	·
310	*MO*	+	+	−[b]	−	−
311	*EP*	+	+	−[a]	−	−
312	*EG*	·	·	·	·	·
413	*TPP*	+[a]	+[a]	·	−[a]	−[a]
414	*TPF*	·	·	·	·	·
415	*TI*	+	+	−[b]	−	−
416	*TC*	·	·	·	·	·
517	*Y*	+	+	−[a]	−	−
518	*YP*	+[a]	+[a]	·	−[a]	−[a]
519	*YD*	+	+	−[b]	−	−
520	*U*	·	·	·	·	·

[a] Signs determined under Constraint 1 only.
[b] Signs determined under Constraints 1, 2, and 3.

above equations to the qualitative system studied for each exogenous variable. We thus obtain the 52 signed multipliers given in Table II.

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APPENDIX

THE QUEBEC ECONOMETRIC MODEL

(a) *Equations and Related Endogenous Variables*:

1. *Expenditures*:

101 Private Consumption:

$$C = f_1(YD, C_{-1}).$$
$$(-) \quad (+)$$

102 Investment in Housing Construction:

$$ICR = f_2(YD, \overline{IH}, ICR_{-1}).$$
$$(-) \quad (+) \ (-)$$

103 Equipment Investment:

$$IM = f_3(PS, \Delta Y_{-1}, IB_{-1}).$$
$$(-) \quad (+)$$

104 Plant Investment:

$$IB = f_4(PS, \overline{A}, \overline{R}).$$
$$(-) \quad (+)$$

2. *Income*:

205 Corporate Profits before Tax:

$$PS = f_5(Y, \ U).$$
$$(-) \quad (+)(-)$$

206 Wage Bill:

$$YW = f_6(Y, \overline{\Delta IPM}, \overline{W}).$$
$$(-) \quad (+) \ (-) \ (+)$$

207 Interests and other Capital Incomes:

$$YR = f_7(ICR, IB, IM, \overline{IG}, \overline{OPR}).$$
$$(-) \quad (+)(+)(+)(+)$$

208 Unincorporated Enterprises Net Income:

$$YNI = f_8(Y, ICR).$$
$$(-) \quad (+) \ (+)$$

209 Dividends:

$$DIV = f_9(PS, DIV_{-1}).$$
$$(-) \quad (+)$$

3. *Labor Market*:

310 Potential Labor Supply:

$$MO = f_{10}(Y, \overline{PQ}).$$
$$(-) \quad (+)$$

311 Private Employment:

$$EP = f_{11}(Y, \ \overline{W}).$$
$$(-) \quad (+)(-)$$

312 Public Employment:

$$EG = f_{12}(U, \overline{GP}, \overline{GFM}).$$
$$(-) \quad (+)(+)$$

4. *Taxes*:

413 Provincial Personal Taxes:

$$TPP = f_{13}(YP, \overline{RP}).$$
$$(-) \quad (+)\ (+)$$

414 Federal Personal Taxes:

$$TPF = f_{14}(YP, \overline{RF}, TPP).$$
$$(-) \quad (+)\ (+)\ (-)$$

415 Indirect Taxes:

$$TI = f_{15}(YD, TI_{-1}, \overline{RI}).$$
$$(-) \quad (+) \quad (+)$$

416 Taxes on Corporate Profits:

$$TC = f_{16}(PS, \overline{RC}).$$
$$(-) \quad (+)\ (+)$$

5. *Accounting and Definitional Equations*:

517 Gross National Product:

$$Y = C + ICR + IB + IM + \overline{GP} + \overline{GFM} + \overline{IG} - \overline{\Delta H}.$$

518 Personal Income:

$$YP = Y - TI - \overline{A} + \overline{TR} + \overline{IDP} + \overline{BSNP}.$$

519 Disposal Income:

$$YD = YP - TPP - TPF.$$

520 Unemployment:

$$U = MO - EP - EG.$$

(b) *Analyzed Exogenous Variables*:

1. $\overline{GP}$ Provincial Public Spending.
2. $\overline{IG}$ Public Investment.
3. $\overline{RF}$ Average Federal Personal Tax Rate.
4. $\overline{W}$ Wage Rate.
5. $\overline{IH}$ Interest Rate on Mortgage

(c) *Other Exogenous Variables*:

$\overline{A}$ Depreciation.
$\overline{BSNP}$ Undistributed Profits.
$\overline{GFM}$ Federal and Municipal Public Spending.
$\overline{\Delta H}$ Stock Adjustment.
$\overline{IDP}$ Interest on Public Debt.

$\overline{IPM}$ Consumption Price Index in Montreal.
$\overline{OPR}$ Debt Capital Ratio.
$\overline{RQ}$ Quebec Population.
$\overline{R}$ Interest Rate.
$\overline{RC}$ Corporate Profit Tax Rate.
$\overline{RI}$ Indirect Tax Rate.
$\overline{RP}$ Average Provincial Personal Tax Rate.
$\overline{TR}$ Net Transfers.

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