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While the standard formulation of quantum theory assumes a fixed background causal structure, one can relax this assumption within the so-called process matrix framework. Remarkably, some processes, termed causally nonseparable, are incompatible with a definite causal order. We explore a form of certification of causal nonseparability in a semi-device-independent scenario where the involved parties receive trusted quantum inputs, but whose operations are otherwise uncharacterized. Defining the notion of causally nonseparable distributed measurements, we show that certain causally nonseparable processes that cannot violate any causal inequality, including the canonical example of the quantum switch, can generate noncausal correlations in such a scenario. Moreover, by imposing some further natural structure to the untrusted operations, we show that all bipartite causally nonseparable process matrices can be certified with trusted quantum inputs.

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When reasoning about quantum and classical processes alike, we usually assume a fixed causal structure. Remarkably, this turns out to be an unnecessarily restrictive assumption: there are valid processes with indefinite causal order. Such processes can be formalized within the process matrix framework, where quantum theory is taken to hold locally but no global causal structure is assumed [1]. The existence of processes incompatible with a definite causal order, termed "causally nonseparable," bears a foundational significance, but moreover can be the basis for advantages in a number of different tasks [2–4].

Some causally nonseparable process matrices can generate so-called noncausal correlations, allowing their causal nonseparability to be certified in a "device-independent" (DI) way by violating "causal inequalities" [1,5]. However, not all causally nonseparable process matrices are noncausal in this strong sense [6–8]. Indeed, it remains unclear if any physically realizable process can violate a causal inequality, and causal models have recently been formulated for a large class of quantum-realizable processes [9,10]. This notably includes the canonical "quantum switch" [11], the resource behind most known advantages arising from causal indefiniteness. At the same time, causally nonseparable process matrices can always be certified by "causal witnesses" [6,12]. This approach, however, has the drawback of being "device-dependent" (DD), as it requires one to perfectly trust the operations performed by the involved parties.

Given the obstacles towards employing a DI approach to certify particularly relevant processes, there is particular urgency in exploring intermediate, semi-DI (SDI) approaches. One possible approach recently considered is to trust only some of the parties' operations [13]. Here, we explore a different SDI regime, significantly weakening the requirements of trust on all parties while simultaneously obtaining a widely applicable certification. Inspired by recent developments in quantum nonlocality [14,15], we consider a causal game scenario where the parties receive inputs in the form of trusted quantum systems (instead of classical ones), but are otherwise untrusted or uncharacterized. We show that certain causally nonseparable processes that cannot violate any causal inequality, including the quantum switch [6,7,11], can nevertheless display some new form of noncausality in a "semi-DI with quantum inputs" (SDI-QI) scenario. We then consider a more constrained version of this scenario in which the uncharacterized operations have a specific, but rather natural structure, and we show that *all* bipartite causally nonseparable process matrices can be certified in this "measurement device and channel independent" (MDCI) scenario.

Causal (non)separability in the process matrix framework.—We focus initially on the bipartite scenario before returning, toward the end of this Letter, to the more practically pertinent scenario in which the quantum switch is formulated. Two parties, Alice and Bob, control separate labs with input and output Hilbert spaces $\mathcal{H}^{A_I}$ and $\mathcal{H}^{A_O}$ for Alice, and $\mathcal{H}^{B_I}$ and $\mathcal{H}^{B_O}$ for Bob. They may also receive some ancillary quantum states in Hilbert spaces $\mathcal{H}^{\tilde{A}}$, $\mathcal{H}^{\tilde{B}}$,

$\rho^{\tilde{A}\tilde{B}} \in \mathcal{L}(\mathcal{H}^{\tilde{A}\tilde{B}})$. (Here and throughout, we denote the space of linear operators on $\mathcal{H}^X$ as $\mathcal{L}(\mathcal{H}^X)$ and write concisely $\mathcal{H}^{XY} = \mathcal{H}^X \otimes \mathcal{H}^Y$, $\mathcal{H}^A = \mathcal{H}^{A_I A_O}$, etc., superscripts indicate on what spaces operators act.) They perform quantum operations described as quantum instruments [16], i.e., sets of completely positive maps $\mathcal{M}_a : \mathcal{L}(\mathcal{H}^{\tilde{A}A_I}) \to \mathcal{L}(\mathcal{H}^{A_O})$ and $\mathcal{M}_b : \mathcal{L}(\mathcal{H}^{\tilde{B}B_I}) \to \mathcal{L}(\mathcal{H}^{B_O})$, whose indices a, b refer to some (classical) outcomes for Alice and Bob, and whose sums $\sum_a \mathcal{M}_a$ and $\sum_b \mathcal{M}_b$ are trace-preserving.

Using the Choi isomorphism [17] (see Supplemental Material (SM) [18], Sec. A), the completely positive maps $\mathcal{M}_a$, $\mathcal{M}_b$ can be represented as positive semidefinite matrices $M_a^{\tilde{A}A}$ and $M_b^{\tilde{B}B}$. Within the process matrix framework, the correlations established by Alice and Bob are then given by the probabilities

$$P(a,b) = \mathrm{Tr}\left[\left(M_a^{\tilde{A}A} \otimes M_b^{\tilde{B}B}\right)^T \left(\rho^{\tilde{A}\tilde{B}} \otimes W^{AB}\right)\right], \quad (1)$$

where $W^{AB} \in \mathcal{L}(\mathcal{H}^{AB})$ is the so-called "process matrix." To ensure that Eq. (1) always defines valid probabilities, W^{AB} must be positive semidefinite and belong to a nontrivial subspace of $\mathcal{L}(\mathcal{H}^{AB})$ [1] (see SM [18], Sec. B).

The process matrix formalism makes no *a priori* assumption of a global causal structure relating Alice and Bob. In fact, the assumption of such a structure imposes further constraints, due to the inability for a party to "signal" to the causal past. Process matrices compatible, for example, with Alice acting before Bob (denoted $A \prec B$) are of the form $W^{A\prec B} = W^{A\prec B_I} \otimes \mathbb{1}^{B_O}$, and similarly $W^{B\prec A} = W^{B\prec A_I} \otimes \mathbb{1}^{A_O}$ for Bob before Alice ($B \prec A$), with $W^{A\prec B_I}$ and $W^{B\prec A_I}$ being themselves valid process matrices [1]. Process matrices that can be written as a convex mixture of matrices compatible with $A \prec B$ and $B \prec A$, i.e., of the form

$$W^{AB} = qW^{A\prec B_I} \otimes \mathbb{1}^{B_O} + (1-q)W^{B\prec A_I} \otimes \mathbb{1}^{A_O} \quad (2)$$

with $q \in [0,1]$ are said to be "causally separable." They can be interpreted as being compatible with a definite (although probabilistic) causal order. Remarkably, there exist "causally nonseparable" process matrices that cannot be decomposed as in Eq. (2), and are thus incompatible with any definite causal order [1].

As recalled above, causal nonseparability can always be certified in a DD manner using a causal witness [6,12], while some processes can be certified in a DI way through the violation of a causal inequality [1]. Here, we consider a relaxation of the DI scenario, where rather than viewing the parties as black boxes with classical inputs and outputs, we provide them with quantum inputs. This intermediate SDI-QI scenario has previously been shown to be extremely useful for entanglement certification [14], but its applicability to causal nonseparability, where parties implement instruments rather than just measurements, remains unstudied.

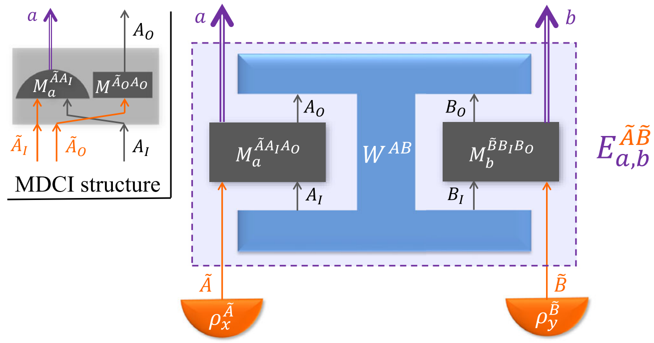

FIG. 1. SDI-QI scenario (main): A process matrix W^{AB} connects two parties who receive quantum inputs $\rho_x^{\tilde{A}}$ and $\rho_y^{\tilde{B}}$, respectively. They each perform a joint operation $[(M_a^{\tilde{A}A_I A_O})_a$ and $(M_b^{\tilde{B}B_I B_O})_b$, respectively], and produce the classical outcomes a and b. The purple box shows the D-POVM $(E_{a,b}^{\tilde{A}\tilde{B}})_{a,b}$ induced by these instruments and the process matrix. Inset: In the MDCI scenario (see later), additional structure is assumed on the quantum instruments (shown here for Alice). The quantum input is a bipartite state in $\mathcal{H}^{\tilde{A}_I\tilde{A}_O}$, a measurement is performed jointly on $\mathcal{H}^{\tilde{A}_I A_I}$ and a channel sends $\mathcal{H}^{\tilde{A}_O}$ to the process matrix through $\mathcal{H}^{A_O}$.

Process matrix scenario with quantum inputs.—We thus consider a situation where Alice and Bob are provided with quantum input states $\rho_x^{\tilde{A}}$ and $\rho_y^{\tilde{B}}$, respectively, indexed by the labels x and y. They each perform some fixed instruments $(M_a^{\tilde{A}A})_a$ and $(M_b^{\tilde{B}B})_b$. We explicitly write the dependency on the quantum inputs in the correlations $P(a,b|\rho_x^{\tilde{A}},\rho_y^{\tilde{B}})$ obtained according to Eq. (1), with $\rho^{\tilde{A}\tilde{B}} = \rho_x^{\tilde{A}} \otimes \rho_y^{\tilde{B}}$.

It will be convenient in our calculations to use the so-called "link product" $*$ [25,26], defined for any matrices $M^{XY} \in \mathcal{L}(\mathcal{H}^{XY})$, $N^{YZ} \in \mathcal{L}(\mathcal{H}^{YZ})$ as $M^{XY} * N^{YZ} = \mathrm{Tr}_Y[(M^{XY} \otimes \mathbb{1}^Z)^{T_Y}(\mathbb{1}^X \otimes N^{YZ})] \in \mathcal{L}(\mathcal{H}^{XZ})$ (where T_Y is the partial transpose over $\mathcal{H}^Y$; see also SM [18], Sec. A). Noting that a full trace $\mathrm{Tr}[(M^Y)^T N^Y]$ and a tensor product $M^X \otimes N^Z$ can both be written as a link product, and that the link product is commutative and associative, Eq. (1) can be written as

$$\begin{aligned} P(a,b|\rho_x^{\tilde{A}},\rho_y^{\tilde{B}}) &= \left(M_a^{\tilde{A}A} \otimes M_b^{\tilde{B}B}\right) * \left(\rho_x^{\tilde{A}} \otimes \rho_y^{\tilde{B}} \otimes W^{AB}\right) \\ &= E_{a,b}^{\tilde{A}\tilde{B}} * \left(\rho_x^{\tilde{A}} \otimes \rho_y^{\tilde{B}}\right) \\ &= \mathrm{Tr}\left[\left(E_{a,b}^{\tilde{A}\tilde{B}}\right)^T \left(\rho_x^{\tilde{A}} \otimes \rho_y^{\tilde{B}}\right)\right] \end{aligned} \quad (3)$$

with $E_{a,b}^{\tilde{A}\tilde{B}} = (M_a^{\tilde{A}A} \otimes M_b^{\tilde{B}B}) * W^{AB}$. According to Eq. (3), the family $\mathbb{E}^{\tilde{A}\tilde{B}} := (E_{a,b}^{\tilde{A}\tilde{B}})_{a,b}$ defines an effective, "distributed" measurement [27,28] on the quantum inputs, which we term a "distributed positive-operator-valued measure" (D-POVM); see Fig. 1.

In the SDI-QI approach, the quantum inputs $\rho_x^{\tilde{A}}$, $\rho_y^{\tilde{B}}$ and their respective spaces are taken to be trusted. However, we do not trust the instruments $(M_a^{\tilde{A}A})_a$ and $(M_b^{\tilde{B}B})_b$, and make no assumptions about the spaces $\mathcal{H}^{A_I}$, $\mathcal{H}^{A_O}$, $\mathcal{H}^{B_I}$ and $\mathcal{H}^{B_O}$. Provided we can use a tomographically complete set of trusted quantum inputs, the D-POVM elements $E_{a,b}^{\tilde{A}\tilde{B}}$ can be explicitly reconstructed via Eq. (3). The fundamental question we address here is this: if W^{AB} is causally nonseparable, can one certify its causal nonseparability by just looking at the $E_{a,b}^{\tilde{A}\tilde{B}}$'s? To tackle this question, we ask conversely whether assuming that W^{AB} is causally separable imposes any specific constraints on the $E_{a,b}^{\tilde{A}\tilde{B}}$'s.

Causally separable D-POVMs.—Suppose that $W^{AB} = W^{A\prec B_I} \otimes \mathbb{1}^{B_O}$ is compatible with the order $A \prec B$. Then one can easily show (see SM [18], Sec. C) that

$$\sum_b E_{a,b}^{\tilde{A}\tilde{B}} = E_a^{\tilde{A}} \otimes \mathbb{1}^{\tilde{B}} \tag{4}$$

with $E_a^{\tilde{A}} = M_a^{\tilde{A}A} * \mathrm{Tr}_{B_I} W^{A\prec B_I} \geq 0$ defining a (single-partite) POVM $(E_a^{\tilde{A}})_a$. Equation (4) can be interpreted as a no-signalling condition from Bob to Alice [27]: indeed, it implies that Alice's marginal probability distribution does not depend on Bob's quantum input. A D-POVM satisfying $\sum_b E_{a,b}^{\tilde{A}\tilde{B}} = E_a^{\tilde{A}} \otimes \mathbb{1}^{\tilde{B}}$ for all a is thus compatible with the causal order where Alice receives her quantum input and acts before Bob ($\tilde{A} \prec \tilde{B}$); we generically denote such a D-POVM $\mathbb{E}^{\tilde{A}\prec\tilde{B}} = (E_{a,b}^{\tilde{A}\prec\tilde{B}})_{a,b}$. Similarly, for the order $B \prec A$, the resulting D-POVM must satisfy $\sum_a E_{a,b}^{\tilde{A}\tilde{B}} = \mathbb{1}^{\tilde{A}} \otimes E_b^{\tilde{B}}$ for all b; we generically denote such a D-POVM $\mathbb{E}^{\tilde{B}\prec\tilde{A}} = (E_{a,b}^{\tilde{B}\prec\tilde{A}})_{a,b}$.

In analogy with the corresponding definition for process matrices [cf. Eq. (2)], we introduce the following:

Definition 1: A bipartite D-POVM $\mathbb{E}^{\tilde{A}\tilde{B}}$ that can be decomposed as a convex mixture of D-POVMs compatible with the causal orders $\tilde{A} \prec \tilde{B}$ and $\tilde{B} \prec \tilde{A}$, i.e., of the form

$$\mathbb{E}^{\tilde{A}\tilde{B}} = q\mathbb{E}^{\tilde{A}\prec\tilde{B}} + (1-q)\mathbb{E}^{\tilde{B}\prec\tilde{A}}, \tag{5}$$

where $q \in [0,1]$ is said to be "causally separable."

Clearly, it follows from the previous discussion that a causally separable process matrix can only generate causally separable D-POVMs. It turns out (see SM [18], Sec. E) that the converse also holds: any causally separable D-POVM can be realized by appropriate local operations on a causally separable process matrix.

SDI-QI certification of causal nonseparability.—Let us note already that one can verify whether a given D-POVM is causally nonseparable with semidefinite programming. Just as for process matrices [6,12], one can indeed construct "witnesses of causal nonseparability for D-POVMs" that certify any causally nonseparable D-POVM $\mathbb{E}^{\tilde{A}\tilde{B}}$ (see SM [18], Sec. H). Concretely, a witness provides a family $\mathbb{S}^{\tilde{A}\tilde{B}} = (S_{a,b}^{\tilde{A}\tilde{B}})_{a,b}$ of operators such that $\sum_{a,b} S_{a,b}^{\tilde{A}\tilde{B}} * E_{a,b}^{\tilde{A}\tilde{B}} < 0$ only if $\mathbb{E}^{\tilde{A}\tilde{B}}$ is causally nonseparable. Taking $\{\rho_x^{\tilde{A}}\}_x$ and $\{\rho_y^{\tilde{B}}\}_y$ to be tomographically complete sets and writing $S_{a,b}^{\tilde{A}\tilde{B}} = \sum_{x,y} s_{a,b}^{(x,y)} \rho_x^{\tilde{A}} \otimes \rho_y^{\tilde{B}}$, one can thus reconstruct the witness from the correlations $P(a,b|\rho_x^{\tilde{A}},\rho_y^{\tilde{B}})$ and certify the causal nonseparability of $\mathbb{E}^{\tilde{A}\tilde{B}}$ by observing

$$\sum_{a,b} S_{a,b}^{\tilde{A}\tilde{B}} * E_{a,b}^{\tilde{A}\tilde{B}} = \sum_{a,b,x,y} s_{a,b}^{(x,y)} P(a,b|\rho_x^{\tilde{A}},\rho_y^{\tilde{B}}) < 0. \tag{6}$$

To certify the causal nonseparability of a process matrix in a SDI-QI manner, the key problem is thus to find some ancillary systems $\mathcal{H}^{\tilde{A}}$, $\mathcal{H}^{\tilde{B}}$ and some instruments $(M_a^{\tilde{A}A})_a$ and $(M_b^{\tilde{B}B})_b$ such that the D-POVM $\mathbb{E}^{\tilde{A}\tilde{B}}$ introduced in Eq. (3) is causally nonseparable.

The simplest case is if a bipartite process matrix can generate noncausal correlations—i.e., if it is "noncausal," or even "not extensibly causal" [7]—then it is fairly easy to see that it can generate a causally nonseparable D-POVM. Indeed, these processes can be certified in a fully DI manner through the violation of a causal inequality using classical, rather than quantum, inputs (cf. SM [18], Sec. F).

Conceptually, it is more interesting to determine whether some "causal" process matrices can generate causally nonseparable D-POVMs. One such bipartite process was formulated by Feix *et al.* [8]. We were again able to find simple instruments that directly generate a causally nonseparable D-POVM from this process (see SM [18], Sec. I). In contrast, the alternative SDI approach of Ref. [13] in which only some parties are trusted was unable to certify the causal nonseparability of this process. This highlights the potential power of our SDI-QI approach.

Certifying all bipartite causally nonseparable process matrices with trusted quantum inputs.—The fact that the nonseparability of some specific causal processes can be nontrivially certified in a SDI-QI way leads one to wonder whether there is a systematic way to obtain a causally nonseparable D-POVM from any causally nonseparable process matrix. Indeed, in the study of entanglement one can certify *any* entangled state with trusted quantum inputs in a "measurement-device-independent" (MDI) manner, and a general recipe is known to construct MDI entanglement witnesses [14,15]. Currently this remains an open question with the general SDI-QI approach introduced above.

Interestingly, the answer turns out to be positive, in the bipartite case, if one makes a further, physically motivated, assumption on the structure of the instruments used by Alice and Bob. In particular, let us now consider a modified scenario, which we term "measurement device and channel independent" (MDCI) and where we assume that Alice and Bob's (trusted) ancillary Hilbert spaces have a bipartite structure of the form $\mathcal{H}^{\tilde{A}} = \mathcal{H}^{\tilde{A}_I\tilde{A}_O}$ and $\mathcal{H}^{\tilde{B}} = \mathcal{H}^{\tilde{B}_I\tilde{B}_O}$, and that their instruments have the following structure (here, e.g., for Alice; see also Fig. 1 inset): (i) Alice performs a joint quantum measurement (i.e., a POVM) on the subsystem of her quantum input in $\mathcal{H}^{\tilde{A}_I}$ and the (untrusted) system in $\mathcal{H}^{A_I}$ she receives from the process matrix; (ii) the part of the quantum input in $\mathcal{H}^{\tilde{A}_O}$ is sent (independently from the joint measurement on $\mathcal{H}^{\tilde{A}_I A_I}$) to the process matrix in the (untrusted) output space $\mathcal{H}^{A_O}$ via a quantum channel (i.e., a completely positive trace-preserving map). The Choi maps of the instruments then factorize accordingly as

$$M_a^{\tilde{A}A} = M_a^{\tilde{A}_I A_I} \otimes M^{\tilde{A}_O A_O}, \qquad M_b^{\tilde{B}B} = M_b^{\tilde{B}_I B_I} \otimes M^{\tilde{B}_O B_O}, \tag{7}$$

with $\sum_a M_a^{\tilde{A}_I A_I} = \mathbb{1}^{\tilde{A}_I A_I}$ and $\mathrm{Tr}_{A_O} M^{\tilde{A}_O A_O} = \mathbb{1}^{\tilde{A}_O}$, and similarly for Bob. Importantly, in this MDCI scenario, we make no assumption about the POVMs and completely positive trace-preserving maps themselves, so they may be completely uncharacterized. We only assume the specified bipartite structure of the instruments, a natural assumption that can be physically justified if the quantum input is provided as two physically distinct systems (e.g., photons in two separate fibers) and distinct operations performed on these inputs.

Using this additional structure, we prove in the SM (Sec. G) [18] that every element $E_{a,b}^{\tilde{A}\tilde{B}}$ of a D-POVM obtained from a causally separable process matrix W^{AB} necessarily decomposes as

$$E_{a,b}^{\tilde{A}\tilde{B}} = q E_{a,b}^{\tilde{A} \prec \tilde{B}_I} \otimes \mathbb{1}^{\tilde{B}_O} + (1-q) E_{a,b}^{\tilde{B} \prec \tilde{A}_I} \otimes \mathbb{1}^{\tilde{A}_O} \tag{8}$$

for some $E_{a,b}^{\tilde{A} \prec \tilde{B}_I}, E_{a,b}^{\tilde{B} \prec \tilde{A}_I} \geq 0$. Remarkably, this structure is sufficient to certify the causal nonseparability of any causally nonseparable process matrix by looking at a single D-POVM element in a systematic way. In particular, by taking ancillary spaces isomorphic to $\mathcal{H}^{A_I A_O}$ and $\mathcal{H}^{B_I B_O}$ and appropriately chosen instruments $M_a^{\tilde{A}A}$, $M_b^{\tilde{B}B}$, when Alice and Bob observe $a = b = 0$ their operations effectively "teleport" W^{AB} to the ancillary spaces so that $E_{0,0}^{\tilde{A}\tilde{B}}$ is (up to normalization) formally the same as W^{AB}. One can then show that if W^{AB} cannot be decomposed as in Eq. (2), then the D-POVM element $E_{0,0}^{\tilde{A}\tilde{B}}$ generated in this way can also not be decomposed as in Eq. (8). Full details of the argument are given in Sec. G of the SM [18].

Since matrices of the form of Eq. (8) can be characterized via semidefinite programming, one can once again use techniques similar to causal witnesses to certify that a D-POVM is *not* of this form (see SM [18], Sec. H). Just as in the SDI-QI scenario, we can then compute the observed witness "value" and thereby certify any bipartite causally nonseparable process matrix in an MDCI way, including those that cannot violate causal inequalities.

We note that an analogous result and systematic construction is also known for MDI entanglement witnesses [14,15]. In contrast to that result, however, the extra MDCI structure assumed in Eq. (7) is crucial here: in the standard SDI-QI case where it is not assumed, no specific structure is imposed in general on a single D-POVM element generated by a causally separable process matrix (see SM [18], Sec. G), and the SDI-QI certification of the previous section thus required considering the full D-POVM.

Generalization to the quantum switch scenario.—A causally nonseparable process that has received significant interest is the "quantum switch" [11], a tripartite process in which the order of Alice and Bob's operations on some "target system" is coherently controlled by the state of a "control qubit," given to a third party, Fiona, at the end. The quantum switch provides advantages in several tasks [2–4] and, unlike any known bipartite causally nonseparable process, has a clear physical interpretation. Indeed, several experimental realizations have been performed [29–33].

The quantum switch can be described as a process matrix $W_{\mathrm{QS}} \in \mathcal{L}(\mathcal{H}^{ABF})$ in a restricted tripartite scenario—which we call the "$(2+F)$-partite scenario"—in which Fiona has no output Hilbert space and simply performs a measurement. In this scenario, the only relevant causal orders are $A \prec B \prec F$ and $B \prec A \prec F$ [6], and the generalization of Eq. (1), as well as the definitions of causally separable process matrices and D-POVMs is straightforward (see SM [18], Sec. D for details). W_{QS} is known to be causally nonseparable but to only generate causal correlations [6,7]. Its importance as a resource in many tasks makes certifying its causal nonseparability a key problem, and multiple experiments have done this in a DD way [30,31].

Can this important process be certified in a SDI-QI or MDCI way, despite being extensibly causal [7]? We find that in both scenarios the response is positive. Indeed, in the SDI-QI case (i.e., without assuming any structure on the instruments used) and taking a qubit target system and qubit ancillary systems (quantum inputs) for Alice and Bob, and without any quantum input for Fiona, the instruments

$$\begin{aligned} M_a^{\tilde{A}A} &= |a\rangle\langle a|^{A_I} \otimes |\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|^{\tilde{A}A_O}, \\ M_b^{\tilde{B}B} &= |b\rangle\langle b|^{B_I} \otimes |\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|^{\tilde{B}B_O}, \end{aligned} \qquad M_\pm^F = |\pm\rangle\langle\pm|^F, \tag{9}$$

with $|\mathbb{1}\rangle\rangle^{\tilde{A}A_O} = \sum_i |i\rangle^{\tilde{A}} \otimes |i\rangle^{A_O}$ and similarly for $|\mathbb{1}\rangle\rangle^{\tilde{B}B_O}$ (cf. SM [18], Sec. A) give a causally nonseparable D-POVM (see [18], Sec. I). These instruments can be interpreted as Alice and Bob performing computational basis measurements on the untrusted systems they receive from the process (in $\mathcal{H}^{A_I}$ and $\mathcal{H}^{B_I}$, respectively) while sending their quantum inputs to the process via identity channels; Fiona then measures in the basis $\{|\pm\rangle = (1/\sqrt{2})(|0\rangle \pm |1\rangle)\}_\pm$.

To understand how robust this certification is, we can consider the robustness of causal nonseparability to noise. Let us consider the "depolarized" quantum switch

$$W_{\rm QS}(r) = \frac{1}{1+r}(W_{\rm QS} + r\mathbb{1}^{ABF}/8) \tag{10}$$

parametrized by $r \geq 0$; it is known that $W_{\rm QS}(r)$ is causally nonseparable for $r \lesssim 1.576$ [12]. With the instruments [Eq. (9)], it is readily checked that $W_{\rm QS}(r)$ generates a causally nonseparable D-POVM for $r \lesssim 0.367$ (see SM [18], Sec. I). Despite extensive numerical searches, we were unable to find instruments allowing us to certify the causal nonseparability of $W_{\rm QS}(r)$ for $0.367 \lesssim r \lesssim 1.576$ with our SDI-QI approach. It thus seems that this approach cannot certify all causally nonseparable processes (we found a similar "robustness gap" for the bipartite process of Ref. [8] discussed above), in contrast to the MDI certification of entanglement and the MDCI certification of causal nonseparability in the bipartite case. Nevertheless, the fact our approach provides a noise robust SDI-QI certification of the quantum switch is of significant relevance, given that it is responsible for most known applications of causal nonseparability and yet cannot be certified in a fully DI manner.

One may wonder whether the bipartite results on MDCI witnesses generalize straightforwardly to the $(2+F)$-partite case. Surprisingly, this turns out not to be the case. Nonetheless, one can show that MDCI certification is possible for some important classes of processes in this scenario: the "TTU-" and "TUU-noncausal" processes of Ref. [13]. These include, in particular, the depolarized quantum switch $W_{\rm QS}(r)$ of Eq. (10) for $r \lesssim 1.319$ [13], significantly improving the noise tolerance obtained above for SDI-QI certification without the additional MDCI assumption, showing how robustly the quantum switch can be certified with only rather weak assumptions about the performed operations. Nonetheless, there remains a gap for $1.319 \lesssim r \lesssim 1.576$ where it is open whether $W_{\rm QS}(r)$ can be certified in a MDCI way. A detailed study and discussion of this is given in Sec. G of the SM [18].

Discussion.—In this contribution we significantly relaxed the assumptions required to certify the causal nonseparability of many processes, investigating both SDI-QI and MDCI scenarios. Notably, we showed how the quantum switch can be certified in a SDI-QI way, and that *all* bipartite causally nonseparable process matrices can be certified in a MDCI manner.

One key open question is to understand precisely which causally nonseparable processes can be certified in a SDI-QI way. Our inability to find instruments generating a causally nonseparable D-POVM from $W_{\rm QS}(r)$ for $0.367 \lesssim r \lesssim 1.576$ indeed leads us to conjecture that *some* such processes cannot be certified in this way.

Beyond understanding fully the bipartite case, an important future direction is the generalization to multipartite process matrices, where the definition of causal (non)separability is more subtle [7,34]. One may wonder, for example, whether one can provide a SDI-QI or MDCI certification of more general quantum circuits with quantum control of causal order than just the quantum switch, which can also not violate causal inequalities [9]. Another interesting direction is whether our SDI-QI approach can be combined with self-testing techniques to construct fully DI witnesses (as, e.g., in Refs. [35,36] for the case of entanglement). More broadly, we believe that the notion of causally nonseparable D-POVMs we introduced may be of independent interest to study in its own right; this also suggests that new types of causal nonseparability could be defined, for other kinds of objects beyond process matrices and D-POVMs. Finally, the idea of imposing extra structure on the instruments used (as in the MDCI scenario) could be adapted to a wide range of quantum resources, opening up new approaches for their certification and exploitation.

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