Positivity for Stanley's chromatic functions

Paunov, Alexander

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Professeur András Szenes

Positivity for Stanley's Chromatic Functions

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Alexander Paunov

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Moscou (Fédération de Russie)

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Abstract

This dissertation is dedicated to the study of positivity phenomena for the coefficients of the chromatic symmetric function of a graph. This function was introduced by Stanley in 1995 as a generalization of the chromatic polynomial of a graph. Stanley considered the expansion of the chromatic symmetric function in terms of various bases of symmetric functions, and conjectured the positivity of its coefficients in the basis of the elementary symmetric functions in the case of the incomparability graphs of $(3+1)$-free posets. The conjecture has not yet been proven, but has been checked for small graphs, and proven for certain families of graphs. The strongest general result in this direction was obtained by Gasharov. He proved a weaker statement, Schur positivity of the incomparability graphs of $(3+1)$-free posets. The strongest result on the positivity of the coefficients in the basis of the elementary symmetric functions was obtained by Stanley, who proved the positivity of certain sums of these coefficients by linking them to acyclic orientations of the incomparability graph.

In this thesis we give a new proof of Gasharov's theorem, which presents a combinatorial interpretation of the Schur-coefficients in terms of planar networks. Compared to Gasharov's proof, it gives a clearer visual illustration of the cancellation procedures and is quite similar in spirit to the proof of monomial positivity of Schur functions via the Lindström–Gessel–Viennot Lemma. This construction led us to reconsider another idea of Stanley: instead of working with the chromatic symmetric function of a graph directly, we analyze certain analogs of the symmetric functions attached to graphs.

We introduce a new combinatorial object: the *correct* sequences of unit interval orders, and using these, in certain cases, we succeed to construct combinatorial models of the coefficients appearing in Stanley's conjecture. Our main result is the proof of positivity of the coefficients $c_{n-k,1^k}$, $c_{n-2,2}$, $c_{n-3,2,1}$ and $c_{2^k,1^{n-2k}}$ of the expansion of the chromatic symmetric function in terms of the basis of the elementary symmetric polynomials for the case of $(3+1)$-free posets.

Résumé en français

Cette thèse est dédiée à l'étude de phénomènes de positivité pour les coefficients de la fonction chromatique symétrique d'un graphe. Cette fonction fut introduite par Stanley en 1995 comme généralisation du polynôme chromatique d'un graphe. Stanley a considéré l'expansion de la fonction chromatique symétrique en terme de diverses bases de fonctions symétriques, et conjecturé la positivité de ses coefficients dans la base des fonctions symétriques élémentaires dans le cas du graphe d'incomparabilité des posets $(3+1)$-libres. La conjecture n'a pas encore été prouvée, mais a été vérifiée pour de petits graphes, et prouvée pour certaines familles de graphes. Le résultat général le plus fort dans cette direction a été obtenu par Gasharov. Il a prouvé un résultat plus faible, la positivité de Schur pour le graphe d'incomparabilité des posets $(3+1)$-libres. Le résultat le plus fort sur la positivité des coefficients dans la base des fonctions symétriques élémentaires a été obtenu par Stanley, qui a prouvé la positivité de certaines sommes de ces coefficients en les liant à des orientations acycliques du graphe d'incompatibilité.

Dans cette thèse nous donnons une nouvelle preuve du théorème de Gasharov, qui présente une interprétation combinatoire des coefficients de Schur en terme de réseaux planaires. Comparée à la preuve de Gasharov, elle donne une illustration visuelle plus claire des procédures d'annulations et est relativement similaire en esprit à la preuve de la positivité monomiale des fonctions de Schur via le lemme de Lindström-Gessel-Viennot. Cette construction nous a mené à reconsidérer une autre idée de Stanley : au lieu de travailler avec la fonction symétrique chromatique du graphe directement, nous analysons certains analogues des fonctions symétriques attachées aux graphes.

Nous introduisons un nouvel objet combinatoire : la bonne séquence d'ordres d'intervalle unité, et en les utilisant, dans certains cas, nous réussissons à construire des modèles combinatoires des coefficients qui apparaissent dans la conjecture de Stanley. Notre principal résultat est la preuve de la positivité des coefficients $c_{n-k,1^k}$, $c_{n-2,2}$, $c_{n-3,2,1}$ et $c_{2^k,1^{n-2k}}$ de l'expansion de la fonction chromatique symétrique en terme de la base des polynômes symétriques élémentaires pour le cas des posets $(3+1)$-libres.

1 Introduction

1.1 Stanley's chromatic function

Let G be a finite graph, $V(G)$ - the set of vertices of G, $E(G)$ - the set of edges of G.

Definition 1.1. A *proper coloring* c of G is a map

$$c : V \to \mathbb{N}$$

such that no two adjacent vertices are colored in the same color.

For each coloring c we define a monomial

$$x^c = \prod_{v \in V} x_{c(v)},$$

where $x_1, x_2, ..., x_n, ...$ are commuting variables. We denote by $\Pi(G)$ the set of all proper colorings of G, and by Λ the ring of symmetric functions in the infinite set of variables $\{x_1, x_2, ...\}$.

In [2], Stanley defined the chromatic symmetric function of a graph.

Definition 1.2. The *chromatic symmetric function* $X_G \in \Lambda$ of a graph G is the sum of the monomials x^c over all proper colorings of G:

$$X_G = \sum_{c \in \Pi(G)} x^c.$$

Definition 1.3. Denote by e_m the m-th elementary symmetric function:

$$e_m = \sum_{i_1 < i_2 < ... < i_m} x_{i_1} \cdot x_{i_2} \cdot ... \cdot x_{i_m},$$

where $i_1, .., i_k \in \mathbb{N}$. Given a non-increasing sequence of positive integers (we will call these *partitions*)

$$\lambda = (\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k),\ \lambda_i \in \mathbb{N},$$

we define the elementary symmetric function $e_\lambda = \prod_{i=1}^{k} e_{\lambda_i}$. These functions form a basis of Λ.

For a natural number k, we denote by 1^k the partition λ of length k, where

$$\lambda_1 = \lambda_2 = ... = \lambda_k = 1.$$

Definition 1.4. A symmetric function $X \in \Lambda$ is *e-positive* if it has non-negative coefficients in the basis of the elementary symmetric functions.

Definition 1.5. Denote by p_m the m-th power sum symmetric function:

$$p_m = \sum_{i \in \mathbb{N}} x_i^m.$$

Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k)$, we define the power sum symmetric function $p_\lambda = \prod_{i=1}^{k} p_{\lambda_i}$. These functions also form a basis of Λ.

Definition 1.6. Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k)$, we define the monomial symmetric function

$$m_\lambda = \sum_{i_1 < i_2 < ... < i_k} \sum_{\lambda' \in S_k(\lambda)} x_{i_1}^{\lambda'_1} \cdot x_{i_2}^{\lambda'_1} \cdot ... \cdot x_{i_k}^{\lambda'_k},$$

where the inner sum is taken over the set of all permutations of the sequence λ, denoted by $S_k(\lambda)$.

Example 1.7. The chromatic symmetric function of K_n, the complete graph on n vertices, is e-positive: $X_{K_n} = n!\, e_n$.

Definition 1.8. For a poset P, the *incomparability graph*, $\mathrm{inc}(P)$, is the graph with elements of P as vertices, where two vertices are connected if and only if they are not comparable in P.

Definition 1.9. Given a pair of natural numbers $a, b \in \mathbb{N}^2$, we say that a poset P is *(a+b)-free* if it does not contain a length-a and a length-b chain, whose elements are mutually incomparable.

Definition 1.10. A unit interval order (UIO) is a partially ordered set which is isomorphic to a finite subset of $U \subset \mathbb{R}$ with the following poset structure:

$$\text{for } u, w \in U: \ u \succ w \text{ iff } u \geq w + 1.$$

Thus u and w are incomparable precisely when $|u - w| < 1$ and we will use the notation $u \sim w$ in this case.

Theorem 1.11 (Scott-Suppes [1]). *A finite poset P is a UIO if and only if it is $(2+2)$- and $(3+1)$-free.*

1.2 Stanley's e-positivity conjecture.

Stanley [2] initiated the study of incomparability graphs of $(3+1)$-free partially ordered sets. Analyzing the chromatic symmetric functions of these incomparability graphs, Stanley [2] stated the following positivity conjecture.

Conjecture 1.12 (Stanley). *If P is a $(3+1)$-free poset, then $X_{\mathrm{inc}(P)}$ is e-positive.*

For a graph G let us denote by $c_\lambda(G)$ the coefficients of X_G with respect to the e-basis. We omit the index G whenever this causes no confusion:

$$X_G = \sum_\lambda c_\lambda e_\lambda.$$

Conjecture 1.12 has been verified with the help of computers for up to 20-element posets [6]. In 2013, Guay-Paquet [6] showed that to prove this conjecture, it would be sufficient to verify it for the case of $(3+1)$- and $(2+2)$-free posets, i.e. for unit interval orders (see Theorem 1.11). More precisely:

Theorem 1.13 (Guay-Paquet). *Let P be a $(3+1)$-free poset. Then, $X_{\mathrm{inc}}(P)$ is a convex combination of the chromatic symmetric functions*

$$\{X_{\mathrm{inc}}(P') \mid P' \text{ is a } (3+1)- \text{ and } (2+2)-\text{free poset}\}.$$

1.3 Known results.

The strongest general result in this direction is that of Gasharov [3].

Definition 1.14. For a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k)$, define the Schur functions $s_\lambda = \det(e_{\lambda^*_i + j - i})_{i,j}$, where λ^* is the conjugate partition to λ. The functions $\{s_\lambda\}$ form a basis of Λ.

Definition 1.15. A symmetric polynomial X is *s-positive* if it has non-negative coefficients in the basis of Schur functions.

Obviously, a product of e-positive functions is e-positive. This also holds for s-positive functions. Thus, the equality $e_n = s_{1^n}$ implies that e-positive functions are s-positive, and thus s-positivity is weaker than e-positivity.

Theorem 1.16 (Gasharov). *If P is a $(3+1)$-free poset, then $X_{\mathrm{inc}(P)}$ is s-positive.*

Gasharov proved s-positivity by constructing so-called P-tableau and finding a one-to-one correspondence between these tableau and s-coefficients [3]. Nevertheless, the e-positivity conjecture have not yet been proven. The strongest known result on the e-coefficients was obtained by Stanley in [2]. He showed that sums of e-coefficients over the partitions of fixed length are non-negative:

Theorem 1.17 (Stanley). *For a finite graph G and $j \in \mathbb{N}$, suppose*

$$X_G = \sum_{\lambda} c_\lambda e_\lambda,$$

and let $sink(G,j)$ be the number of acyclic orientation of G with j sinks. Then

$$sink(G,j) = \sum_{l(\lambda)=j} c_\lambda.$$

Remark 1.18. *By taking $j = 1$, it follows from the theorem that c_n is non-negative.*

Although e-positivity has not yet been proven for the general case, Stanley in [2] showed that for $n \in \mathbb{N}$ and the unit interval order $P_n = \{\frac{i}{2}\}_{i=1}^n$, the corresponding $X_{\text{inc}(P_n)}$ is e-positive. Nevertheless, e-positivity for the UIOs

$$P_{n,k} = \left\{ \frac{i}{k+1} \right\}_{i=1}^{n}$$

has not yet been proven for $k > 1$, but was checked for small n and some k (see [2]).

1.4 Main results.

In this dissertation, we give a new proof of Gasharov's theorem, which presents a combinatorial interpretation of the s-coefficients in terms of planar networks. Compared to Gasharov's proof, it gives a clearer visual illustration of the cancellation procedures and resembles the proof of monomial positivity of Schur functions using Lindström–Gessel–Viennot Lemma [9]. This allows us to look at the positivity problematics from a slightly different perspective: instead of working with the chromatic symmetric function of a graph directly, we analyze families of G-symmetric functions, described in Section 2, first time proposed by Stanley in [7].

Our next step is the introduction of *correct sequences* (abbreviated as *corrects*), defined below. These will play a major role in the dissertation.

Definition 1.19. Let U be a UIO. We will call a sequence $\vec{w} = (w_1, \dots, w_k)$ of elements of U *correct* if

- $w_i \not> w_{i+1}$ for $i = 1, 2, \dots, k-1$
- and for each $j = 2, \dots, k$, there exists $i < j$ such that $w_i \not< w_j$.

Every sequence of length 1 is correct, and sequence (w_1, w_2) is correct precisely when $w_1 \sim w_2$. The second condition (supposing that the first one holds) may be reformulated as follows: for each $j = 1, \dots k$, the subset $\{w_1, \dots, w_j\} \subset U$ is connected with respect to the graph structure $(U, \sim)$. Using this notation, we prove the following theorems.

Theorem 1.20. *Let $X_{inc(U)} = \sum_{\lambda} c_\lambda e_\lambda$ be a chromatic symmetric function of the n-element unit interval order U. Then c_n is equal to the number of corrects of length n, in which every element of U is used exactly once.*

Corollary 1.20.1. *Let $X_{inc(P)} = \sum_{\lambda} c_\lambda e_\lambda$ be a chromatic symmetric function of n-element $(3+1)$-free poset P, then c_n is a nonnegative integer.*

Indeed, positivity for the general case follows from Theorem 1.13, which presents the chromatic symmetric function of a $(3+1)$-free poset as a convex combination of the chromatic symmetric functions of unit interval orders.

Stanley [7] and Chow [5] showed the positivity of c_n for $(3+1)$-free posets using combinatorial techniques, and linked e-coefficients with the acyclic orientations of the incomparability graphs. Nevertheless, their proofs do not give visual interpretation of the cancellation procedures. The construction of correct sequences not only serves this purpose, but also creates a new approach, which allows us to obtain the following new result:

Theorem 1.21. *Let $X_{inc(P)} = \sum_{\lambda} c_\lambda e_\lambda$ be a chromatic symmetric function of the $(3+1)$-free poset P, and $k \in \mathbb{N}$. Then $c_{n-k,1^k}$, $c_{n-2,2}$, $c_{n-3,2,1}$ and $c_{2^k,1^{n-2k}}$ are non-negative integers.*

The dissertation is structured as follows: in Section 2, we describe the G-homomorphism introduced by Stanley in [7], which is essential for our approach. The new proof of Gasharov's theorem is presented in Section 3.1. The proof of Theorem 1.20 and positivity of G-power sum symmetric functions is can be found in Section 3.2. Positivity of $c_{n-k,1^k}$, $c_{n-2,2}$, $c_{n-3,2,1}$ and $c_{2^k,1^{n-2k}}$ (Theorem 1.21) is proven in Section 3.3.

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2 Stanley's G-homomorphism

For a graph G, Stanley [7, p. 6] defined G-analogues of the standard families of symmetric functions. Let G be a finite graph with vertex set $V(G) = \{v_1, ..., v_n\}$ and edge set $E(G)$. We will think of the elements of $V(G)$ as commuting variables.

Definition 2.1. For a positive integer i, $1 \leq i, \leq n$, we define the *G-analogues* of the elementary symmetric polynomials, or *the elementary G-symmetric polynomials*, as follows

$$e_i^G = \sum_{\substack{\#S=i \\ S-\text{stable}}} \prod_{v \in S} v,$$

where the sum is taken over all i-element subsets S of V, in which no two vertices form an edge, i.e. stable subsets. We set $e_0^G = 1$, and $e_i^G = 0$ for $i < 0$.

Note that these polynomials are not necessarily symmetric.

Let $\Lambda_G \subset \mathbb{R}[v_1, ..., v_n]$ be the subring generated by $\{e_i^G\}_{i=1}^n$. The map $e_i \mapsto e_i^G$ extends to a ring homomorphism $\phi_G : \Lambda \to \Lambda_G$, called the *$G$-homomorphism.* For $f \in \Lambda$, we will use the notation f^G for $\phi_G(f)$.

Example 2.2. Given a partition $\lambda = \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k,\ k \in \mathbb{N}$, we have

$$e_\lambda^G = \prod_{i=1}^{k} e_i^G,$$

$$s_\lambda^G = \det(e^G_{\lambda_i^*+j-i}).$$

For an integer function $\alpha : V \to \mathbb{N}$ and $f^G \in \Lambda_G$, let

$$v^\alpha = \prod_{v \in V} v^{\alpha(v)},$$

and $[v^\alpha]f^G$ stands for the coefficient of v^α in the polynomial $f^G \in \Lambda_G$.

Let G^α denote the graph, obtained by replacing every vertex v of G by the complete subgraph of size $\alpha(v)$: $K^v_{\alpha(v)}$. Given vertices u and v of G, a vertex of $K^v_{\alpha(v)}$ is connected to a vertex of $K^u_{\alpha(u)}$ if and only if u and v form an edge in G.

Considering the Cauchy product [8, ch. 4.2], Stanley [7, p. 6] found a connection between the G-analogues of symmetric functions and X_G. Following Stanley [7], we set

$$T(x, v) = \sum_\lambda m_\lambda(x) e_\lambda^G(v),$$

where the sum is taken over all partitions. Then

$$[v^\alpha]T(x, v) \prod_{v \in V} \alpha(v)! = X_{G^\alpha}. \tag{1}$$

Using the Cauchy identity

$$\sum_\lambda s_\lambda(x) s_{\lambda^*}(y) = \sum_\lambda m_\lambda(x) e_\lambda(y) = \sum_\lambda e_\lambda(x) m_\lambda(y)$$

and applying the G-homomorphism, one obtains:

$$T(x, v) = \sum_\lambda m_\lambda(x) e_\lambda^G(v) = \sum_\lambda s_\lambda(x) s^G_{\lambda^*}(v) = T(v, x) = \sum_\lambda e_\lambda(x) m_\lambda^G(v). \tag{2}$$

An immediate consequence of the formulas (1) and (2) is the following result of Stanley:

Theorem 2.3 (Stanley)**.** *For every finite graph G*

1. *X_{G^α} is s-positive for every $\alpha : V(G) \to \mathbb{N}$ if and only if $s_\lambda^G \in \mathbb{N}[V(G)]$ for every partition λ.*
2. *X_{G^α} is e-positive for every $\alpha : V(G) \to \mathbb{N}$ if and only if $m_\lambda^G \in \mathbb{N}[V(G)]$ for every partition λ.*

Remark 2.4. *If $X_{G^\alpha} = \sum_\lambda c_\lambda^\alpha e_\lambda$, then $c_\lambda^\alpha = [v^\alpha] m_\lambda^G$. Hence, monomial positivity of m_λ^G is equivalent to the positivity of c_λ^α for every α.*

3 Proofs of the theorems

It follows from Theorem 2.3 that to prove that the graph G is s-positive, it is enough to show the monomial positivity of its G-Schur polynomials. On the other hand, Guay-Paquet in Theorem 1.13 showed that it is sufficient to check s-positivity for unit interval orders, in order to prove it for the general case of $(3+1)$-free posets. Therefore, in the following paragraph 3.1, we analyze the functions s_λ^G for the case $G = \mathrm{inc}(U)$, where U is UIO.

3.1 A new proof of Gasharov's theorem

Given unit interval order U, we arrange the elements of U according to its order on the real line. For instance, the incomparability graph of $U_8 = \{\frac{i}{2}\}_{i=1}^8$, the 1-chain graph with 8 vertices, has the following labeling:

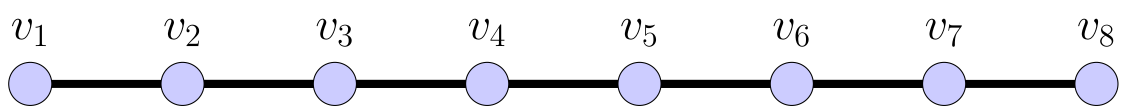

Figure 1: The incomparability graph of U_8.

Note that vertices $\{v_i\}_{i=1}^8$ represent points on the real line.

A key tool in our work is the Lindström–Gessel–Viennot lemma [9]. Let Γ be a finite directed acyclic (i.e. without directed cycles) graph with set of vertices $V(\Gamma)$ and set of edges $E(\Gamma)$. Let $w : E(\Gamma) \to R$ be a weighting of the edges with values in some commutative ring R. For every directed path ρ, denote by $w(\rho)$ the product of the weights of the edges in the path. Then, for every two vertices a and b of Γ, let

$$e(a,b) = \sum_{\rho: a \to b} w(\rho),$$

where the sum is taken over all paths from a to b.

Definition 3.1. Let $n \in \mathbb{N}$, and let us fix two ordered n-element subsets

$$A = (a_1, a_2, ..., a_n) \subset V(\Gamma) \text{ and } B = (b_1, b_2, ..., b_n) \subset V(\Gamma),$$

called *base* and *destination* vertices, correspondingly. We will call a collection $\vec{\rho} = (\rho_1, ..., \rho_n)$ of paths in Γ a *multipath* from A to B if there is a permutation σ on $\{1, 2, ..., n\}$ such that $\rho_i : a_i \to b_{\sigma(i)},\ i = 1, ..., n$. Given a multipath $\vec{\rho}$, we denote the permutation σ by $\sigma_{\vec{\rho}}$. We call $\vec{\rho}$ *non-intersecting,* if $\rho_i \cap \rho_j = \emptyset$ for $i \neq j$.

Theorem 3.2 (Lindström–Gessel–Viennot)**.** *Let Γ, $w : E(\Gamma) \to R$ be a weighted locally finite acyclic graph as above, $n \in \mathbb{N}$, $A = (a_1, a_2, ..., a_n) \subset V(\Gamma)$ and $B = (b_1, b_2, ..., b_n) \subset V(\Gamma)$. Define the matrix*

$$M_{A,B} = \begin{pmatrix} e(a_1,b_1) & e(a_1,b_2) & e(a_1,b_3) & \dots & e(a_1,b_n) \\ e(a_2,b_1) & e(a_2,b_2) & e(a_2,b_3) & \dots & e(a_2,b_n) \\ \dots & \dots & \dots & \dots & \dots \\ e(a_n,b_1) & e(a_n,b_2) & e(a_n,b_3) & \dots & e(a_n,b_n) \end{pmatrix}$$

Then, the following equality holds in the ring R:

$$\det(M_{A,B}) = \sum_{\substack{\vec{\rho}: A \to B \\ \text{non-int.}}} \mathrm{sign}(\sigma_{\vec{\rho}}) \cdot \prod_{i=1}^{n} w(\rho_i),$$

where the sum is taken over all non-intersecting multipaths.

Remark 3.3. *It follows from Theorem 1.13 that to prove Gasharov's theorem, it is sufficient to verify it for unit interval orders. Here we prove Gasharov's theorem for this case.*

Theorem 3.4. *Let $(U, \succ)$ be a unit interval order, $G = \mathrm{inc}(U)$ its incomparability graph. Then, for every partition λ, $s_\lambda^G \in \mathbb{N}[V(G)]$.*

Proof. We prove the monomial positivity of $s_{\lambda^*}^G$ by constructing a special directed graph Γ_G, the grid of G, and applying the Lindström–Gessel–Viennot theorem to Γ_G.

The vertices of Γ_G are given by the pairs (i, j), where $i \in 1, \ldots, n+1$ and $j \in \mathbb{N}$. Then, for every $i \in \{1, \ldots, n\}$, we denote by $\mathrm{next}(i) = \min\ \{j |\ v_j \succ v_i\}$. If such v_j does not exist, then we define $\mathrm{next}(i) = n+1$. From every vertex $(i,\ j),\ j < n+1$, we draw a directed edge to the vertex $(i,\ j+1)$ with the weight 1, and a directed edge to $(i+1,\ \mathrm{next}(j))$ with the weight v_i. Note that Γ_G is planar if U is a unit interval order.

For instance, for the graph U_8, mentioned above, the grid Γ_{U_8} is as follows:

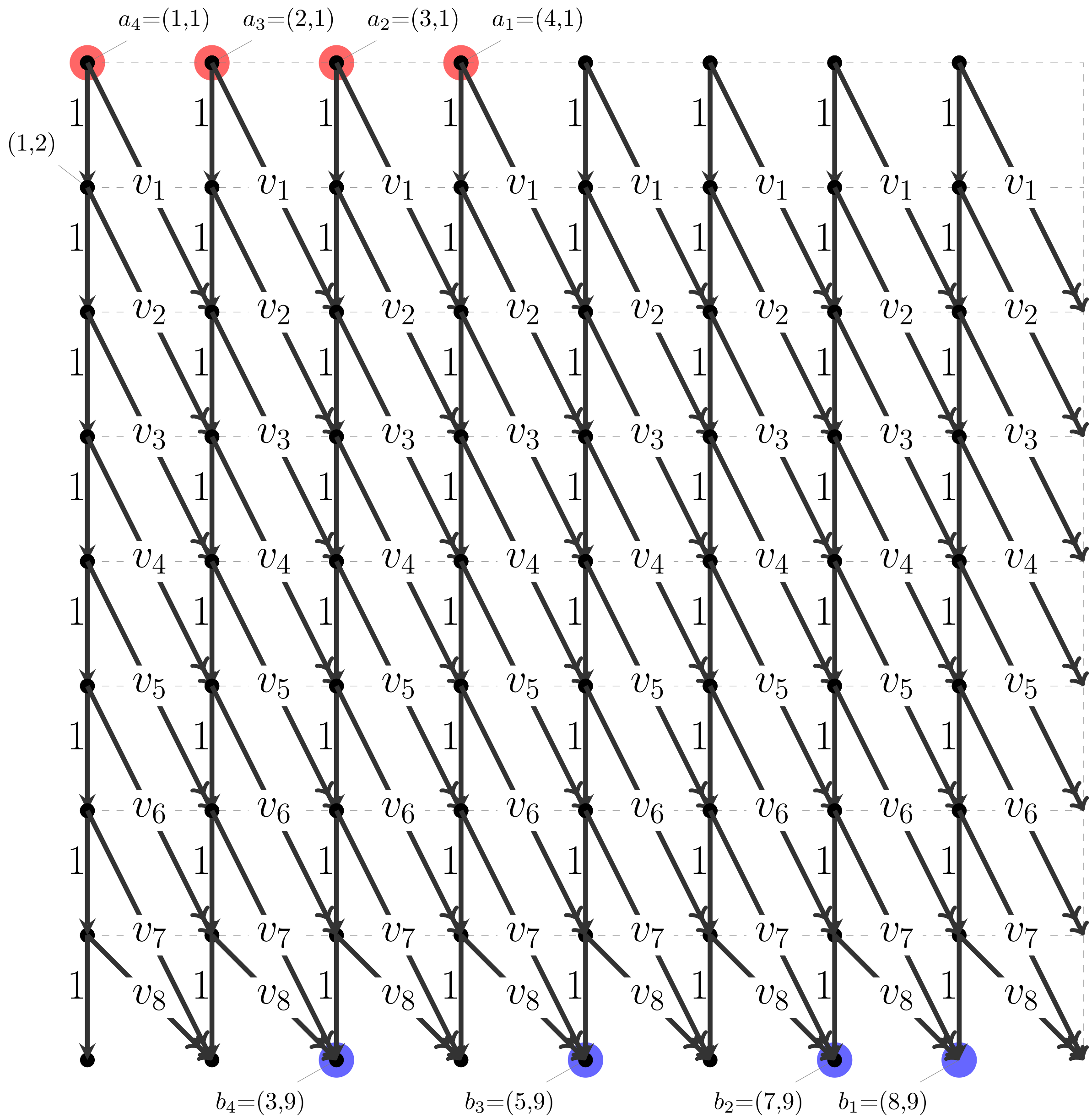

Figure 2: The grid Γ_{U_8}.

Here, the base vertices A, $(1,1), (2,1), (3,1)$, and $(4,1)$, are on the top, and are colored in red. The destination vertices B, $(3,9), (5,9), (7,9)$, and $(8,9)$, at the bottom, and are colored in blue.

It easily follows from the definition of Γ_G that, for positive integers i and j, we have

$$e((i,1),\ (i+j, n+1)) = e_j^G.$$

Note that we use the notation $e(a,b)$ for the sum over all weights of paths from vertex a to vertex b, and we use a similar notation e^G_λ for the G-elementary symmetric functions. This is not a coincidence: for the graphs we will consider in this dissertation and to which we apply Theorem 3.2, $e(a,b)$ will turn out to be the elementary G-symmetric function.

Now, let $k \in \mathbb{N}$, and fix a partition $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k$. Let

$$A = \{a_1 = (k,1),\ a_2 = (k-1,1), .., a_i = (k+1-i,1), ..,\ a_k = (1,1)\},$$

and

$$B = \{b_1 = (k+\lambda_1, n+1),\ b_2 = (k-1+\lambda_2, n+1), ...,\ b_i = (k+1-i+\lambda_i, n+1) ..., b_k = (\lambda_k+1, n+1)\}.$$

Then we have

$$e(a_i,\ b_j) = e^G_{\lambda_i+j-i}.$$

The example of $\lambda = (4,4,3,2)$ is shown on Figure 2.

Next, applying Theorem 3.2, we obtain

$$\det(e^G_{\lambda_i+j-i}) = \det(e(a_i,\ b_j)) = \sum_{\vec{\rho}:A\to B} \operatorname{sign}(\sigma_{\vec{\rho}}) \prod_{i=1}^{n} w(\rho_i), \tag{3}$$

where the sum is taken over all non-intersecting multipaths from A to B. The permutation σ must be the identity permutation for all possible non-intersecting multipaths $\vec{\rho}$ since the grid Γ_G is planar. Thus, by the definition of s^G_λ, we have

$$s^G_{\lambda^*} = \det(e^G_{\lambda_i+j-i}) = \sum_{\substack{\vec{\rho}:A\to B \\ \text{non-intersecting}}} \prod_{i=1}^{n} w(\rho_i), \tag{4}$$

where the sum is taken over all non-intersecting multipaths from A to B. This demonstrates the monomial positivity of $s^G_{\lambda^*}$. □

3.2 A new proof of monomial positivity of the G-power sum functions.

Let us repeat the definition of a central notion for our work, that of correct sequences of elements of a unit interval order.

Definition 3.5. Let $(U, \prec)$ be a unit interval order, and $G = \mathrm{inc}(U)$. We will call a sequence $\vec{w} = (w_1, \ldots, w_k)$ of elements of U *correct* if

- $w_i \not\succ w_{i+1}$ for $i = 1, 2, \ldots, k-1$
- and for each $j = 2, \ldots, k$, there exists $i < j$ such that $w_i \not\prec w_j$.

We denote by P_k^U the set of all correct sequences (abbreviated as *corrects*) of length k. Since G is uniqely defined by U, and we are working only with UIO, here and below we use the U-index instead of G. The U-analogues of symmetric functions will be analyzed.

Theorem 3.6. *Let U be a unit interval order and p_k^U the Stanley power-sum function of the corresponding incomparability graph. Then, for every natural k, we have*

$$p_k^U = \sum_{\vec{w} \in P_k^U} w_1 \cdot \ldots \cdot w_k \ \in N[U],$$

where the sum is taken over all corrects of length k.

Proof. To prove this theorem we express the power sum U-symmetric function p_k^U in terms of the elementary U-symmetric polynomials using the determinant formula:

$$p_k^U = \det \begin{vmatrix} e_1^U & 1 & 0 & \cdots & \\ 2e_2^U & e_1^U & 1 & 0 & \cdots \\ 3e_3^U & e_2^U & e_1^U & 1 & \\ \vdots & & & \ddots & \ddots \\ ke_k^U & e_{k-1}^U & \cdots & & e_1^U \end{vmatrix}. \tag{5}$$

Note that this determinant is similar to the expression for $s_{(1^k)^*}^U$ in terms of the e-basis; only the first column is different:

$$s_{(1^k)^*}^U = \det \begin{vmatrix} e_1^U & 1 & 0 & \cdots & \\ e_2^U & e_1^U & 1 & 0 & \cdots \\ e_3^U & e_2^U & e_1^U & 1 & \\ \vdots & & & \ddots & \ddots \\ e_k^U & e_{k-1}^U & \cdots & & e_1^U \end{vmatrix}. \tag{6}$$

Next, we take a partition $\lambda = 1^k$, a grid Γ_U, and vertices (see Theorem 3.2 and its use in Section 3.1)

$$A = \{a_1 = (k, 1),\ a_2 = (k-1, 1), .., a_i = (k+1-i, 1), ..,\ a_k = (1, 1)\},$$

and

$$B = \{b_1 = (k+1, n+1),\ b_2 = (k, n+1), ...,\ b_i = (k+1-i+1, n+1) ..., b_k = (2, n+1)\},$$

corresponding to the partition λ on the grid. For instance, the grid for U_5 is as follows:

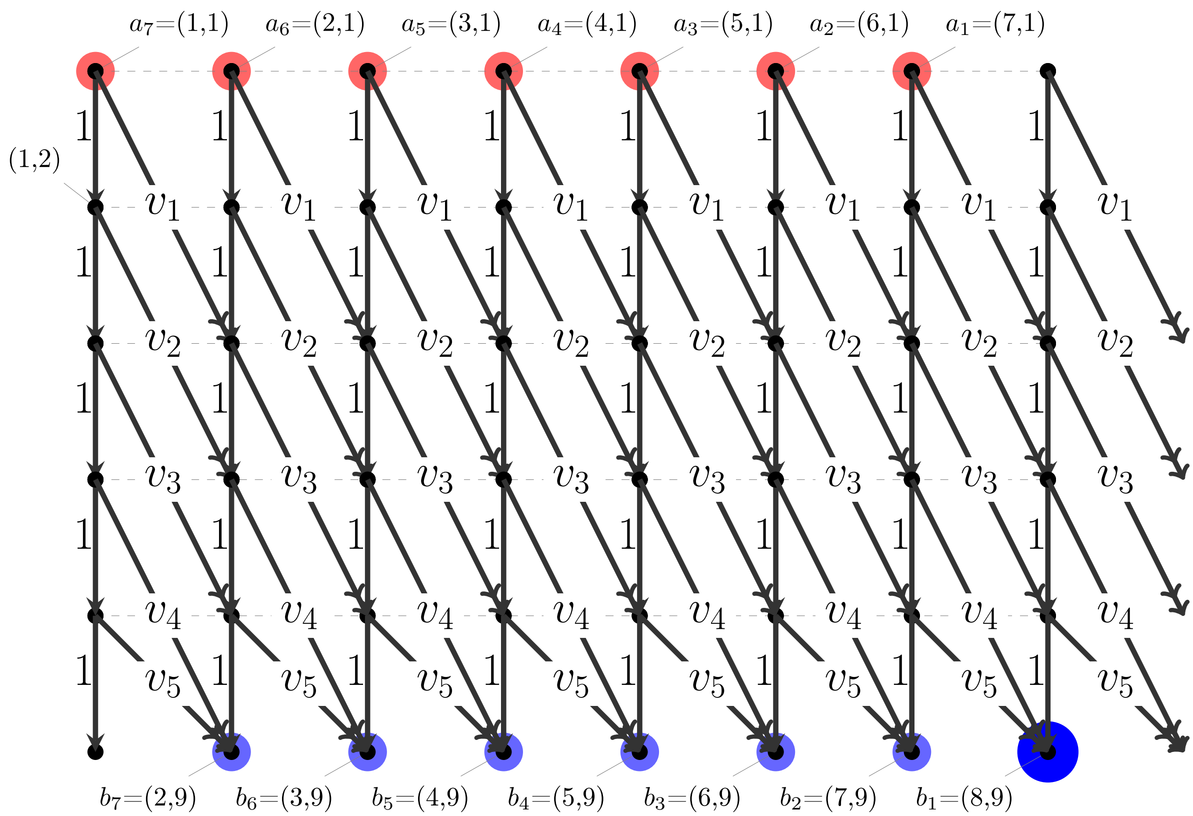

Figure 3: The grid Γ_{U_5} and vertices located with respect to partition $\lambda = 1^5$.

As in the section 3.1, we have

$$e(a_i,\ b_j) = e^U_{1+j-i}.$$

Recall that (see Theorem 3.2 for more details) for every directed path ρ on Γ_U, $w(\rho)$ denotes the product of the weights of the edges in the path. Denote by $w(\vec{\rho}\ ) = (w(\rho_1), ..., w(\rho_k))$ the vector of weight products over the paths of $\vec{\rho}$.

$$s^U_k = \det(e^U_{1+j-i}) = \det(e(a_i,\ b_j)) = \tag{7}$$

$$= \sum_{\vec{\rho}=(\rho_1,\dots,\rho_k):A\to B} \operatorname{sign}(\sigma_{\vec{\rho}}) \prod_{i=1}^{k} w(\rho_i) = \sum_{\substack{(\rho_1,\dots,\rho_k):A\to B \\ \text{non-intersecting}}} \prod_{i=1}^{k} w(\rho_i). \tag{8}$$

To obtain p^U_k, we adjust the first column, multiplying every element by the number of its row:

$$p^U_k = \det \begin{vmatrix} e^U_1 & 1 & 0 & \cdots & \\ 2e^U_2 & e^U_1 & 1 & 0 & \cdots \\ 3e^U_3 & e^U_2 & e^U_1 & 1 & \\ \vdots & & & \ddots & \ddots \\ ke^U_k & e^U_{k-1} & \cdots & & e^U_1 \end{vmatrix} = \det \begin{vmatrix} e(a_1,\ b_1) & 1 & 0 & \cdots & \\ 2e(a_2,\ b_1) & e(a_2,\ b_2) & 1 & 0 & \cdots \\ 3e(a_3,\ b_1) & e(a_3,\ b_2) & e(a_3,\ b_3) & 1 & \\ \vdots & & & \ddots & \ddots \\ ke(a_k,\ b_1) & e(a_k,\ b_2) & \cdots & & e(a_k,\ b_k) \end{vmatrix} = \tag{9}$$

$$= \sum_{\vec{\rho}=(\rho_1,\dots,\rho_k):A\to B} \operatorname{sign}(\sigma_{\vec{\rho}}) \cdot \sigma^{-1}_{\vec{\rho}}(1) \prod_{i=1}^{k} w(\rho_i). \tag{10}$$

In this sum every multipath has a multiplier equal to the index of the vertex from A, from which the corresponding path goes to b_1. We mark the vertex b_1 with a larger dot on the grid (Picture 3) to emphasize this. We cannot apply Theorem 3.2 here, as we did for s^U functions, to obtain positive sum.

We will use the following notations:

- If we have a path ρ on Γ_U, which goes from a to b through z, then let us denote by $\rho|^z$ the part of ρ from a to z, and by $\rho|_z$ - the part of ρ from z to b.
- If the end of the path ρ coincides with the starting point π, then we will write $\rho * \pi$ for the concatenation of the two paths
- For a pair of paths (ρ, π), crossing in point z, we define the usual switch operation

$$\text{switch}_z(\rho, \pi) = (\rho|^z * \pi|_z, \pi|^z * \rho|_z).$$

- Given a multipath $\vec{\rho}$ with its paths ρ and π intersecting in point z, we define a multipath $\delta_z(\vec{\rho})$ by replacing (ρ, π) by $\text{switch}_z(\rho, \pi)$ in $\vec{\rho}$. Note that our map is defined correctly, because here we consider only multipaths for the partition $\lambda = 1^k$: it is obvious that 3 paths of $\vec{\rho}$ cannot intersect in one point. Note that

$$sign(\sigma(\vec{\rho}\,)) = -sign(\sigma(\delta_z(\vec{\rho}\,))).$$

- Given an intersecting multipath $\vec{\rho}$, we denote by $z(\vec{\rho}\,)$ (or just z, if it is clear which multipath is considered) its intersection point with minimum absciss and maximum ordinate, i.e. the leftmost lowest intersection point.

Next, we classify the set of multipaths in order to simplify the sum (10). Every path ρ can be uniquely defined by its weight, $w(\rho)$, which is a product over an increasing sequence (with respect to the relation $\succ$) of elements of U. Here, it is important to mention that incomparable elements of U can not be present in a weight of any path. Hence, every multipath $\vec{\rho} = (\rho_1, ..., \rho_k)$ is in one to one correspondence with its weight vector $w(\vec{\rho}\,) = (w(\rho_1), ..., w(\rho_k))$. Below, we will use the bar notation for the sets of multipaths. The corresponding sets of weight vectors will be defined using the same letters without bars.

- Let $\overline{\Omega}_k$ be the set of all multipaths $\vec{\rho} = (\rho_1, ..., \rho_k)$ from A to B.
- Let $\overline{I}_k$ be the set of all intersecting multipaths $\vec{\rho} \in \overline{\Omega}_k$, such that the two paths from $\vec{\rho}$, crossing at $z(\vec{\rho}\,)$ do not end at b_1.
- We denote by $\overline{P}_k$ the set of multipaths $\vec{\rho} \in \overline{\Omega}_k$, such that $w(\vec{\rho}\,)$ is correct:

$$\overline{P}_k = \{\vec{\rho} \in \overline{\Omega}_k |\ w(\vec{\rho}) \in P_k^U\}.$$

  Note that if $\vec{\rho} \in \overline{P}_k$, then $\vec{\rho}$ is a non-intersecting multipath, since by the definition of correct $w(\vec{\rho})$ must be a tuple with non-decreasing elements (weights) with respect to the relation $\prec$. Hence,

$$\overline{I}_k \cap \overline{P}_k = \emptyset.$$

As a consequence, the sum (10) can be rewritten as

$$\sum_{\vec{\rho}\in\overline{\Omega}_k} \text{sign}(\sigma_{\vec{\rho}}) \cdot \sigma_{\vec{\rho}}^{-1}(1) \prod_{i=1}^{k} w(\rho_i) = \sum_{\vec{\rho}\in\overline{P}_k} \text{sign}(\sigma_{\vec{\rho}}) \cdot \sigma_{\vec{\rho}}^{-1}(1) \prod_{i=1}^{k} w(\rho_i) \tag{11}$$

$$+ \sum_{\vec{\rho}\in\overline{I}_k} \text{sign}(\sigma_{\vec{\rho}}) \cdot \sigma_{\vec{\rho}}^{-1}(1) \prod_{i=1}^{k} w(\rho_i) \tag{12}$$

$$+ \sum_{\vec{\rho}\in(\overline{\Omega}_k\setminus\overline{I}_k)\setminus\overline{P}_k} \text{sign}(\sigma_{\vec{\rho}}) \cdot \sigma_{\vec{\rho}}^{-1}(1) \prod_{i=1}^{k} w(\rho_i). \tag{13}$$

Let $\vec{\rho} \in \overline{I}_k$, then it is easy to see that $\delta_z(\vec{\rho}\,) \in \overline{I}_k$, and $\delta_z(\delta_z(\vec{\rho}\,)) = \vec{\rho}$. Hence, δ_z is a sign-reversing involution on $\overline{I}_k$:

$$\text{sign}(\sigma(\vec{\rho}\,)) = -\text{sign}(\sigma(\delta_z(\vec{\rho}\,))) \text{ and } \delta_z(\overline{I}_k) = \overline{I}_k$$

On the other hand, δ_z does not change the multiplier:

$$\sigma^{-1}(\delta_z(\vec{\rho}\,))(1) = \sigma_{\vec{\rho}}^{-1}(1).$$

Hence, the term (12) vanishes, and we have:

$$\sum_{\vec{\rho}\in\overline{I}_k} \mathrm{sign}(\sigma_{\vec{\rho}})\cdot\sigma_{\vec{\rho}}^{-1}(1)\prod_{i=1}^{k} w(\rho_i) = \delta_z\left(\sum_{\vec{\rho}\in\overline{I}_k} \mathrm{sign}(\sigma_{\vec{\rho}})\cdot\sigma_{\vec{\rho}}^{-1}(1)\prod_{i=1}^{k} w(\rho_i)\right) = -\sum_{\vec{\rho}\in\overline{I}_k} \mathrm{sign}(\sigma_{\vec{\rho}})\cdot\sigma_{\vec{\rho}}^{-1}(1)\prod_{i=1}^{k} w(\rho_i) = 0.$$

Pictures 4 and 5 below illustrate this cancellation. Since neither of the 2 paths intersecting at z end at b_1, a switch at z changes the sign, but not the multiplier, and the contributions of $\vec{\rho}$ and $\delta_z(\vec{\rho}\,)$ cancel:

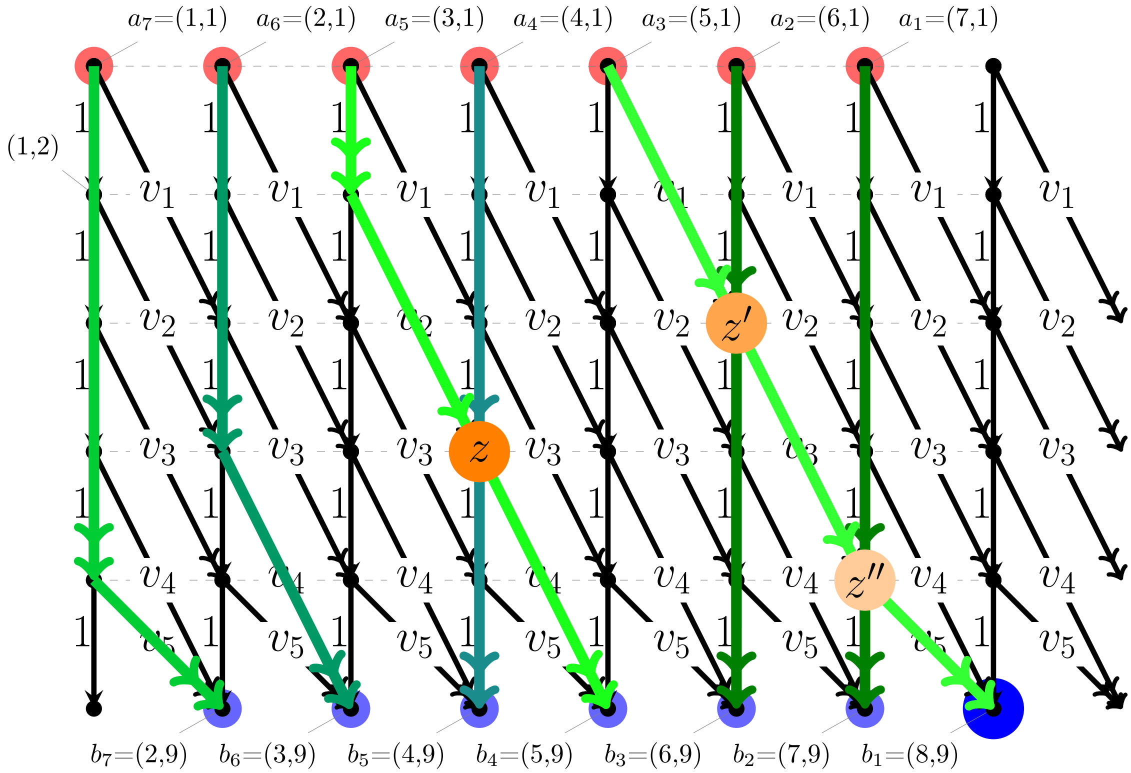

Figure 4: The grid Γ_{U_5} and multipath $\vec{\rho}$.

The path ρ_5: $a_5 \to b_4$ intersects the path ρ_4: $a_4 \to b_5$ at the point z. After the switch at z, we have the paths ρ_5': $a_5 \to b_5$ and ρ_4': $a_4 \to b_4$:

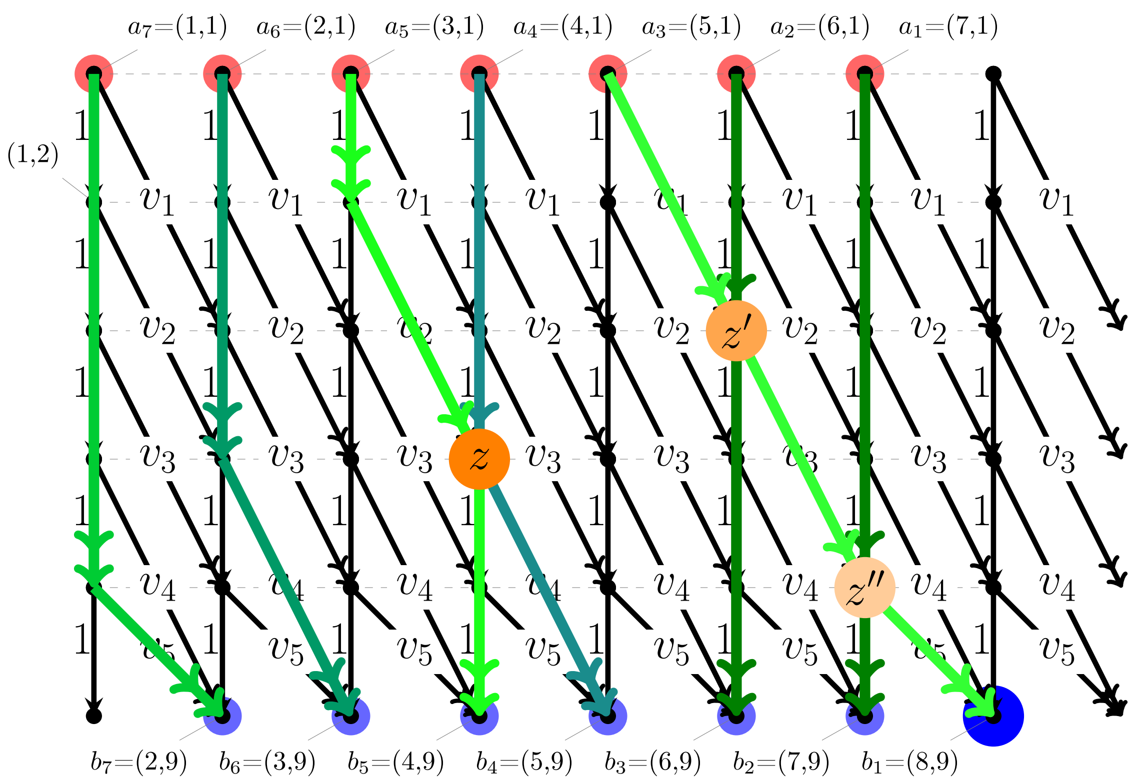

Figure 5: The grid Γ_{U_5} and multipath $\delta_z(\vec{\rho})$.

We denote by J_k the following set of weights vectors, which describe multipaths like on the Picture 6:

$$J_k = \{(1^{l-1}, v_{i_1} \cdot \ldots \cdot v_{i_l}, v_{i_{l+1}}, \ldots v_{i_k}) \,|1 \leq l \leq k;\ v_{i_j} \prec v_{i_{j+1}},\ \text{if } 1 \leq j \leq l;\ v_{i_1} \not\succ v_{i_{l+1}};\ v_{i_j} \not\succ v_{i_{j+1}},\ \text{if } l < j \leq k\}.$$

Denote by $\overline{J}_k$ the corresponding set of mutipaths, which are uniquely defined by the vectors of its weights.

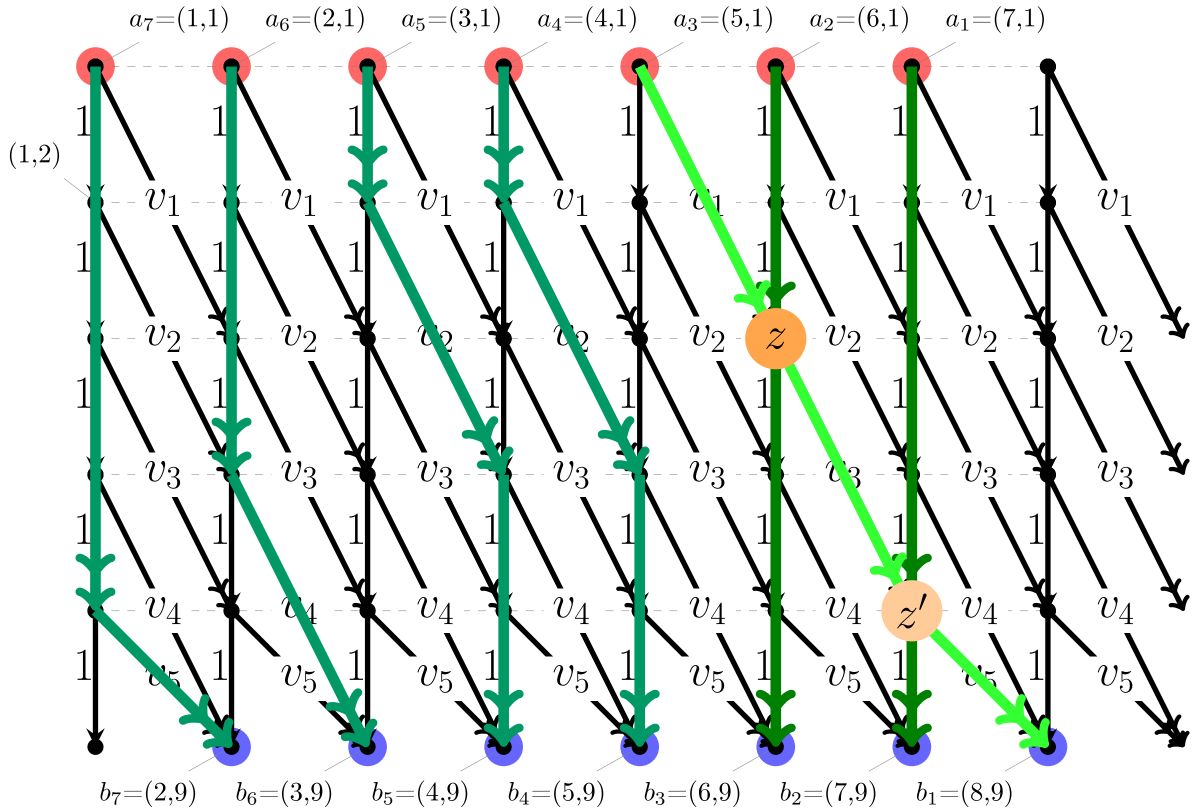

Figure 6: The grid Γ_{U_5} and a multipath without easy intersection points.

Let $\vec{\rho} \in (\overline{\Omega}_k \setminus \overline{I}_k) \setminus \overline{P}_k$.

- If $\vec{\rho}$ is intersecting, then the absolute value of the difference between the multipliers of $\vec{\rho}$ and $\delta_z(\vec{\rho}\,)$ in the sum (13) is equal to 1:

$$|\sigma_{\vec{\rho}}^{-1}(1) - \sigma(\delta_z(\vec{\rho}))^{-1}(1)| = 1,$$

because if ρ_l goes to b_1, then z could only be obtained as an intersection of ρ_l and ρ_{l-1} or ρ_{l+1}. Hence, since $\delta_z(\vec{\rho}) \in (\overline{\Omega}_k \setminus \overline{I}_k) \setminus \overline{P}_k$, we make a switch at z and eliminate one of the switched multipaths (from $(\overline{\Omega}_k \setminus \overline{I}_k) \setminus \overline{P}_k \setminus \overline{J}_k$) and the multiplier of the multipath with longer intersecting path (from $\overline{J}_k$) in the sum (13).

- If $\vec{\rho} \in (\overline{\Omega}_k \setminus \overline{I}_k) \setminus \overline{P}_k$ is non-intersecting, then its multiplier is also equal to 1. Denote the set of such multipaths by $\overline{L}_k$:

$$\overline{L}_k = \{\vec{\rho} \in (\overline{\Omega}_k \setminus \overline{I}_k) \setminus \overline{P}_k |\ \vec{\rho} \text{ is non-intersecting}\}.$$

Then,

$$L_k = \{(w_1, \ldots, w_k)|\ w_i \not\succ w_{i+1}; \exists m, \text{ s.t. } w_m \succ \max_{1 \leq q < m} w_q\}.$$

Hence, we can rewrite the sum (13) in the following way:

$$\sum_{\vec{\rho} \in \overline{\Omega}_k \setminus \overline{I}_k \setminus \overline{P}_k} \text{sign}(\sigma_{\vec{\rho}}) \cdot \sigma_{\vec{\rho}}^{-1}(1) \prod_{i=1}^{k} w(\rho_i) = \sum_{\vec{\rho} \in \overline{J}_k \sqcup \overline{L}_k} \text{sign}(\sigma_{\vec{\rho}}) \prod_{i=1}^{k} w(\rho_i). \tag{14}$$

To eliminate the sum (13), we construct a sign-reversing involution on $J_k \sqcup L_k$. Let

$$A_k = \{(1^{l-1}, w_1 \cdot \ldots \cdot w_l, w_{l+1}, \ldots w_k) \in J_k |\ l > 1 \text{ and } w_j \not\succ \max_{1 \le q < j} w_q \text{ for every } 1 \le j \le k\}.$$

$$B_k = \{(1^{l-1}, w_1 \cdot \ldots \cdot w_l, w_{l+1}, \ldots w_k) \in J_k \sqcup L_k |\ \exists m, \text{ s.t. } w_m \succ \max_{1 \le q < m} w_q\}.$$

Then, we have

$$J_k \sqcup L_k = A_k \sqcup B_k.$$

Next, we construct a sign-reversing bijection between A_k and B_k.

First, we define map $\chi : A_k \to B_k$. If $\vec{v} = (1^{l-1}, w_1 \cdot \ldots \cdot w_l, w_{l+1}, \ldots w_k) \in A_k$, then let

$$m = \max\{j \le n |\ w_m \succ w_j \text{ for } j < m\}.$$

We set

$$\chi(\vec{v}\ ) = (1^{l-1}, w_1 \cdot \ldots \cdot w_l \cdot w_m, w_{l+1}, \ldots, w_{m-1}, w_{m+1}, \ldots, w_k) \in B_k.$$

Note that χ changes the sign $\vec{v}$ by increasing its l-th weight by 1. Second, if

$$\vec{u} = (1^{l-1}, w_1 \cdot \ldots \cdot w_l, w_{l+1}, \ldots w_k) \in A_k,$$

then we set

$$m' = \max\{j \le k |\ w_l \succ w_j\},$$

and define

$$\psi(\vec{u}\ ) = (1^{l-1}, w_1 \cdot \ldots \cdot w_{l-1}, w_{l+1}, \ldots, w_{m'}, w_l, w_{m'+1}, \ldots w_k) \in B_k.$$

Note that $\psi = \chi^{-1}$. For instance, the multipath from Picture 6, which belongs to A_k, is transformed to the below mutipath (Picture 7) under the action of ψ, and vice versa Picture 6 can be obtained from Picture 7 applying direct map χ:

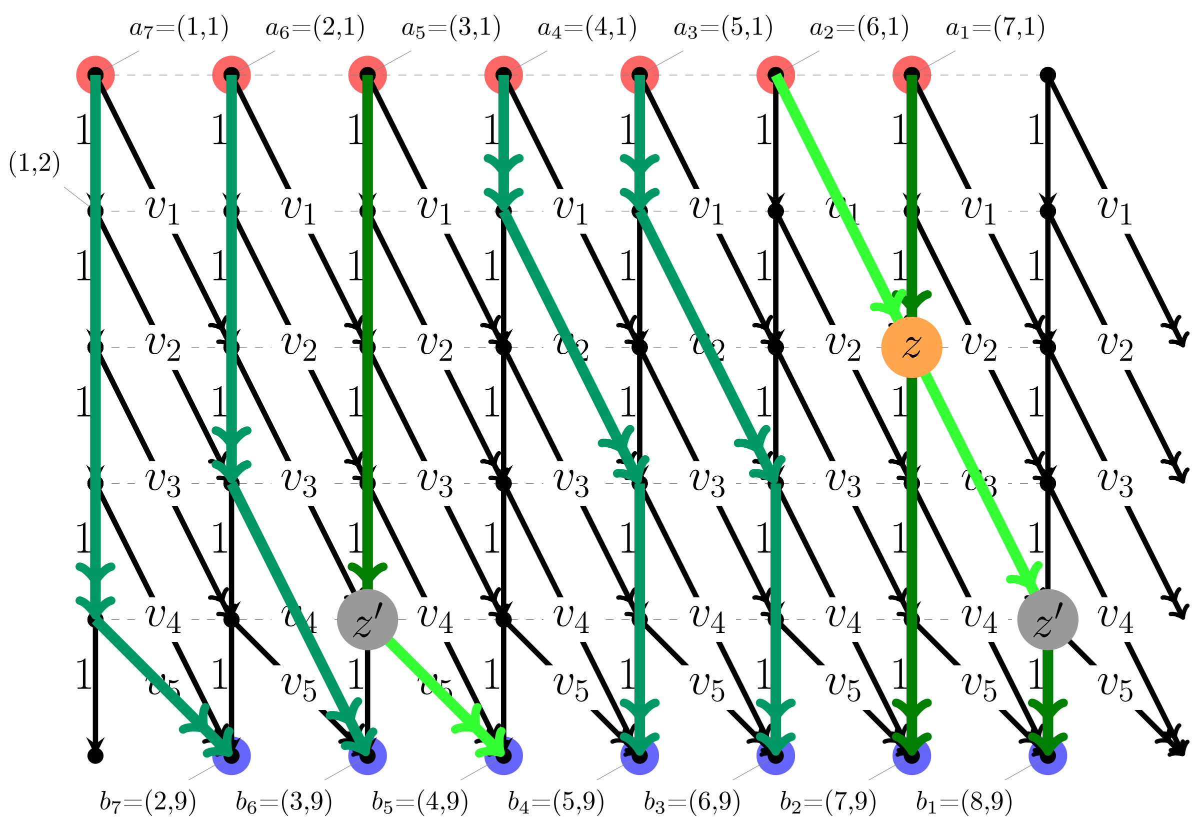

Figure 7: Image of the multipath from Picture 6 under the action of χ.

Hence, among the sums (13), (12) and (11), only the latter is non-zero, we have only the set of corrects left:

$$p_k^U = \sum_{\substack{(\rho_1,\dots,\rho_k):A\to B\\ (w(\rho_1),\dots,w(\rho_k))\in P_k^U}} \prod_{i=1}^{n} w(\rho_i).$$

□

3.3 Positivity of $m^U_{l,1^k}$, $m^U_{l,2}$, $m^U_{l,2,1}$, and $m^U_{2^l,1^k}$.

We need the following mild technical generalization of correct sequences: let $\lambda = (\lambda_1 \geq \dots \geq \lambda_k)$ be a partition of $|\lambda| = \sum_{i=1}^{k} \lambda_i$. Then, we will call sequence $(w_1, \dots, w_{|\lambda|})$ *λ-correct* if each of the subsequences $(w_1, \dots w_{\lambda_1})$, $(w_{\lambda_1+1}, \dots, w_{\lambda_1+\lambda_2})$,… $(w_{|\lambda|-\lambda_k+1}, \dots, w_{|\lambda|})$ are correct. Introduce the set

$$P_\lambda^U = \{\vec{w} = (w_1, \dots w_{\lambda_1} | w_{\lambda_1+1}, \dots, w_{\lambda_1+\lambda_2} | \dots | w_{|\lambda|-\lambda_k+1}, \dots, w_{|\lambda|}) \mid \vec{w} \text{ is } \lambda\text{-correct } \}$$

of λ-correct sequences of length-$|\lambda|$. In particular, P_l^U is the set of l-correct orderings of U. This definition is aligned with Theorem 3.6, and we have:

$$p_\lambda^U = \prod_{i=1}^{k} p_{\lambda_i}^U = \sum_{\vec{w}\in P_\lambda^U} w_1 \cdot \dots \cdot w_{|\lambda|}.$$

For $\vec{w} = (w_1, \dots w_l) \in P_l^U$ and $z \in U$ we write $z \succ \vec{w}$, if $z \succ w_i$ for every $1 \leq i \leq l$.

Theorem 3.7. *Let*

$$M_{l,1}^U = \{(\vec{w}\ |z) \in P_{l.1} |\ z \succ \vec{w} \vee z \prec w_l\},$$

then

$$m_{l,1}^U = \sum_{(\vec{w};z)\in M_{l,1}^U} w_1 \cdot \dots \cdot w_l \cdot z.$$

Remark 3.8. *According to Remark 3.3, this implies $c_{n-1,1}(U) \geq 0$.*

Proof. Since $P_{l+1}^U \subset P_l^U \times P_1^U$, using the following relation

$$m_{l,1}^U = p_l^U \cdot p_1^U - p_{l+1}^U,$$

we have

$$P_l^U \times P_1^U \setminus P_{l+1}^U = M_{l,1}^U,$$

and, as a consequence,

$$m_{l,1}^U = \sum_{(\vec{w};z)\in M_{l,1}^U} w_1 \cdot \dots \cdot w_l \cdot z.$$

□

Given a correct $\vec{w} \in P_l^U$, let

$$\theta = \max(\{i < l |\ w_i \sim w_{i+1}\}) \text{ and } J_{l-1} = (w_1, ..., w_{l-1}).$$

Theorem 3.9. *For natural $l \geq 2$, let*

$$M_{l,2}^U = \{(\vec{w}\ |q_0, q_1) \in P_{l.2}^U |\ J_{l-1} \prec q_0 \textit{ and } w_l \prec q_1 \ \vee\ w_\theta \succ q_0 \textit{ and } w_{\theta+1} \succ q_1\}.$$

Then,

$$m_{l,2}^U = \sum_{(\vec{w};q_0,q_1)\in M_{l,2}^U} w_1 \cdot \dots \cdot w_l \cdot q_0 \cdot q_1.$$

Remark 3.10. *According to Remark 3.3, this implies $c_{n-2,2}(U) \geq 0$.*

Remark 3.11. *There is a slightly more elegant version of* $M^U_{l,2}$*, which we will use in the future:*

$$M^U_{l,2} = \{(\vec{w}, q_0, q_1) \in P^U_{l,2} | \ (J_{l-1} \prec q_0 \wedge w_l \prec q_1) \ \vee \ (w_\theta \succ q_0 \wedge w_{\theta+1} \succ q_1)\}$$

Here is an illustration of an element of $M^U_{l,2}$:

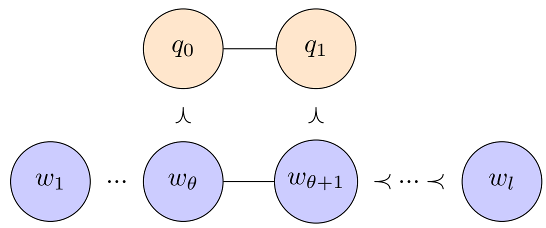

Figure 8: Illustration of $M^U_{l,2}$.

Proof. We can write

$$P^U_{l,2} \setminus M^U_{l,2} = \{(\vec{w}\ |q_0, q_1) \in P^U_{l.2} | \ (\vec{w} \prec q_0 \Rightarrow (w_l \not\prec q_1 \wedge J_{l-1} \not\prec q_1) \wedge (w_\theta \succ q_0 \Rightarrow w_{\theta+1} \not\succ q_1)\}. \quad (15)$$

The conditions in (15) have the form $A \wedge B$. We begin with a few remarks.

1. Observe that the conditions of A and B are mutually exclusive, so we can consider the two statements independently.
2. Define $\tau = \max\{i \leq l | \ q_1 \not\prec w_i \vee w_i \sim w_{i+1}\}$. Note that it could happen that $q_1 \prec \theta$, but clearly $\tau \geq \theta$.
3. For $\vec{u} \in P^U_{l+2}$, let $U_{l-1} = (u_1, ..., u_{l-1})$ and $\breve{\theta} = \max(\{i < l | \ u_i \sim u_{i+1}\})$.

To prove the theorem, we consider the following formula

$$m^U_{l,2} = p^U_l \cdot p^U_2 - p^U_{l+2}.$$

and construct two injective maps.

First, we define

$$\phi : P^U_{l,2} \setminus M^U_{l,2} \to P^U_{l+2}$$

as follows:

Let $(\vec{w}, q_0, q_1) \in P^U_{l,2} \setminus M^U_{l,2}$.

1. If $\vec{w} \prec q_0$, then we define

$$\phi(\vec{w}, \vec{q}\ ) = (...q_1, w_l, q_0),$$

which is in P^U_{l+2}, since

$$w_l \not\prec q_1 \wedge J_{l-1} \not\prec q_1.$$

Second,

$$\psi : P^U_{l+2} \to P^U_{l,2} \setminus M^U_{l,2},$$

the inverse of map ψ**:**

Let $(u_1, ..., u_{l+2}) \in P^U_{l+2}$.

1. If $u_{l+2} \succ U_{l-1}$ and $u_{l+2} \succ u_{l+1}$, then

$$\psi(\vec{u}\ ) = (u_1, u_2, ..., u_{l-1}, u_{l+1} | u_{l+2}, u_l)$$

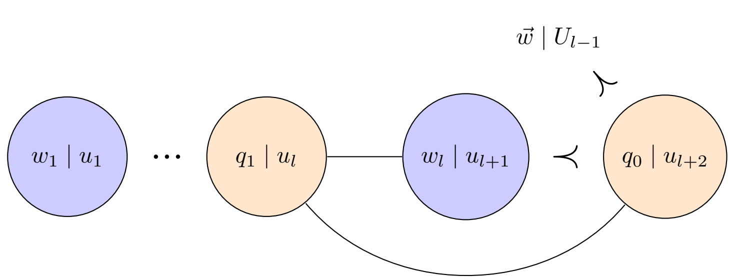

Figure 9: Illustration of $\phi_{l|2}$, if $q_0 \succ \vec{w}$.
Illustration of $\psi_{l|2}$, if $u_{l+2} \succ U_{l-1}$ and $u_{l+2} \succ u_{l+1}$.

2. $q_0 \prec w_\tau$ implies A, and implies $\tau > \theta$; then we define

$$\phi(\vec{w}, \vec{q}\,) = (...w_\tau, q_1, q_0, w_{\tau+1}, ...).$$

Here and below, $\tau + 1$ might be equal to $l + 1$, in which case we simply omit $w_{\tau+1}$.

2. if $u_{\breve\theta+1} \prec u_{\breve\theta-1}$, then

$$\psi(\vec{u}\,) = (u_1, u_2, ..., u_{\breve\theta-1}, u_{\breve\theta+2}, ..., u_{l+2}|u_{\breve\theta+1}, u_{\breve\theta}),$$

Here and below, $\breve\theta + 1$ might be equal to $l + 2$, in which case we simply omit $u_{\breve\theta+2}$.

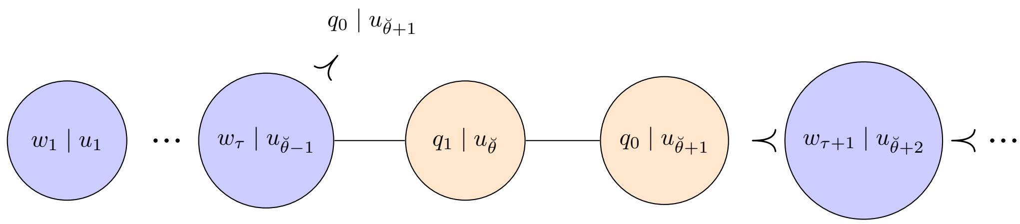

Figure 10: Illustration of $\phi_{l|2}$, if $q_0 \prec w_\tau$.
Illustration of $\psi_{l|2}$, if $u_{\breve\theta+1} \prec u_{\breve\theta-1}$.

3. Finally, if $\vec{w} \not\prec q_0$ and $q_0 \not\prec w_\tau$, then

$$\phi(\vec{w}, \vec{q}\,) = (...w_\tau, q_0, q_1, w_{\tau+1}, ...).$$

Here and below, $\tau + 1$ might be equal to $l + 1$, in which case we simply omit $w_{\tau+1}$.

3. if $u_{\breve\theta+1} \not\prec u_{\breve\theta-1}$, then

$$\psi(\vec{u}\,) = (u_1, u_2, ..., u_{\breve\theta-1}, u_{\breve\theta+2}, ..., u_{l+2}|u_{\breve\theta}, u_{\breve\theta+1}).$$

Here and below, $\breve\theta + 1$ might be equal to $l + 2$, in which case we simply omit $u_{\breve\theta+2}$.

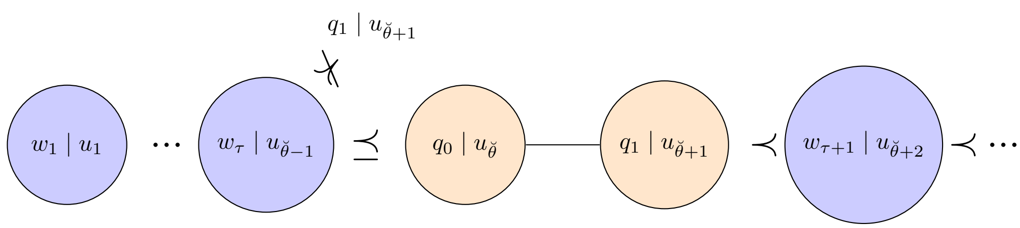

Figure 11: Illustration of $\phi_{l|2}$, if $q_0 \not\prec w_\tau$.
Illustration of $\psi_{l|2}$, if $u_{\breve\theta+1} \not\prec u_{\breve\theta-1}$.

□

To find a combinatorial interpretation of $m^U_{l,2,1}$, we construct a bijection between the right and left hand sides of the following equality:

$$p^U_l * m^U_{2,1} = m^U_{l+2,1} + m^U_{l+1,2} + m^U_{l,2,1}. \tag{16}$$

This formula and its proof are similar to the previous one.

Theorem 3.12. *For natural l, let*

$$\begin{aligned} M^U_{l,2,1} =&\{(\vec{w}\ |\vec{q}\ |z) \in P_{l,2,1}|\ (\vec{w}\ ;\vec{q}\ ) \in M^U_{l,2},\ (\vec{w}\ ;z) \in M^U_{l,1},\ (\vec{q}\ ;z) \in M^U_{2,1},\} \\ &\cup \{(\vec{w}\ |\vec{q}\ |z) \in P_{l,2,1}|\ s_l \succ z \succ \vec{q},\ \exists \gamma \in \mathbb{N},\ \textit{such that}\ \theta < \gamma < l,\ s_\gamma \sim z,\ s_\gamma \succ q_2\} \\ &\cup \{(\vec{w}\ |\vec{q}\ |z) \in P_{l,2,1}|\ s_l \succ z \succ q_2,\ \exists \gamma \in \mathbb{N},\ \textit{such that}\ \theta < \gamma < l,\ s_\gamma \sim z,\ z \sim q_1,\ s_\gamma \succ q_1\}, \end{aligned} \tag{17}$$

Then,

$$m^U_{l,2,1} = \sum_{(\vec{w};q_0,q_1;z)\in M^U_{l,2,1}} w_1 \cdot \ldots \cdot w_l \cdot q_0 \cdot q_1 \cdot z.$$

Remark 3.13. *According to Remark 3.3, this implies $c_{n-3,2,1}(U) \geq 0$.*

This theorem states that in addition to combinations of pairwise comparable corrects of lengths l, 2 and 1 we have two more cases:

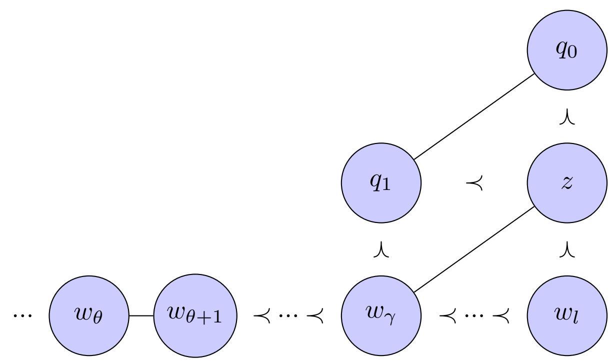

Figure 12: First exceptional element of $M^U_{l,2,1}$

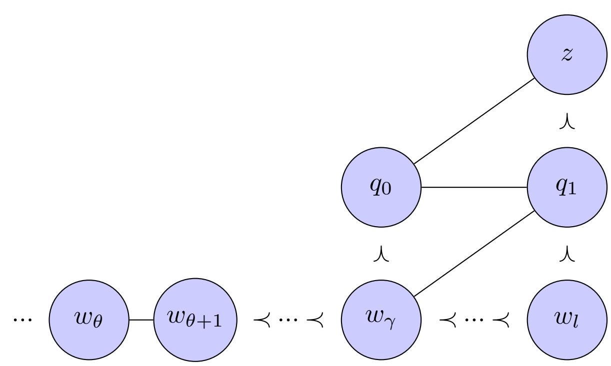

Figure 13: Second exceptional element of $M^U_{l,2,1}$

Proof. To prove the theorem 3.12 using the formula 16, we construct the maps $\varphi_{l|2,1}$ and $\psi_{l|2,1}$.

First, we construct the map from the left hand side to the right hand side

$$\phi_{l|2,1} : P_l^U \times M_{2,1}^U \to M_{l+2,1}^U \sqcup M_{l+1,2}^U \sqcup M_{l,2,1}^U.$$

Let us take

$$(\vec{w} \,|q_1, q_2 \,; z) \in P_l^U \times M_{2,1}^U.$$

Let

$$\theta = \max(\{i < l |\ w_i \sim w_{i+1}\}).$$

We will use this θ for w on the right hand side as well.

1. If $z \succ \vec{q}$.

11. If $z \succ \vec{w}$.

111. If $(\vec{w} \,; \vec{q}\,) \in M_{l,2}^U$, then

$$\phi_{l|2,1}(\vec{w} \,|\vec{q} \,; z) = (\vec{w} \,; \vec{q} \,; z) \in M_{l,2,1}^U.$$

Second, we construct

$$\psi^1_{l|2,1} : M_{l+2,1}^U \to P_l^U \times M_{2,1}^U,$$

$$\psi^2_{l|2,1} : M_{l+1,2}^U \to P_l^U \times M_{2,1}^U,$$

$$\psi^3_{l|2,1} : M_{l,2,1}^U \to P_l^U \times M_{2,1}^U.$$

We take

$$(\vec{u} \,; \xi) \in M_{l+2,1}^U;$$

$$(\vec{w}, \xi \,;\ q_0, q_1) \in M_{l+1,2}^U;$$

$$(\vec{w} \,; q_0, q_1; z) \in M_{l,2,1}^U.$$

Let $\breve{\theta} = \max(\{i < l+2 |\ u_i \sim u_{i+1}\})$.

1. If $(\vec{w} \,; q_0, q_1; z) \in M_{l,2,1}^U$ and $z \succ \vec{q}$ and $z \succ \vec{w}$, then

$$\psi^3_{l|2,1}(\vec{w} \,; q_0, q_1; z) = (\vec{w} \mid q_0, q_1 \,; z).$$

In this case we have 2 illustrations:

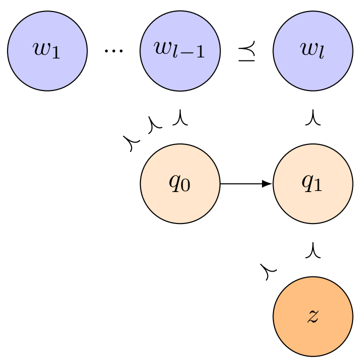

Figure 14: Illustration of $\phi_{l|2,1}$, case 111., and $\psi^3_{l|2,1}$, case 1.

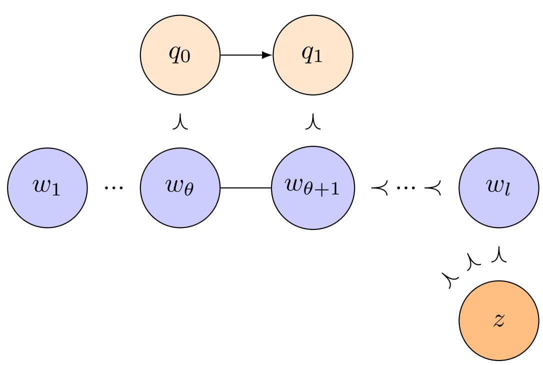

Figure 15: Illustration of $\phi_{l|2,1}$, case 111., and $\psi^3_{l|2,1}$, case 1.

112. If $(\vec{w}\ |\vec{q}\ ) \notin M^U_{l,2}$, then using

$$\phi_{l|2}(\vec{w}\ |\vec{q}\ ) \in M^U_{l+2}$$

we have

$$\phi_{l|2,1}(\vec{w}\ |\vec{q}\ |z) = (\phi_{l|2}(\vec{w}\ |\vec{q}\ )\ ; z) \in M^U_{l+2,1}$$

2. If $(\vec{u}\ ; \xi) \in M^U_{l+2,1}$ and $\xi \succ \vec{u}$, then using

$$\psi_{l|2}(\vec{u}\ ) \in M^U_{l,2}$$

we have

$$\psi^1_{l|2,1}(\vec{u}\ ; z) = (\psi_{l|2}(\vec{u}\ );\ \xi).$$

Here, we provide illustrations for the right hand side, see Theorem 3.9, where $\phi_{l|2}$ and $\psi_{l|2}$ are defined, for more details.

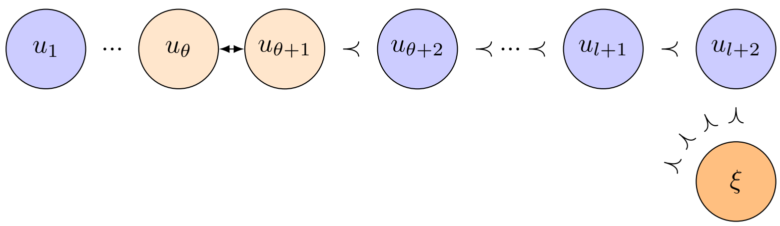

Figure 16: Illustration of $\psi^1_{l|2,1}$, when $\xi \succ \vec{u}$, general case

Note that the picture above illustrates most of the cases, except the special one, when $u_{l+2} \succ u_{l+1}$, $u_{l+2} \succ J_{l-1}$ and $u_{l+2} \sim u_l$. In this case map $\psi_{l|2}$ takes out u_{l+2} and u_l:

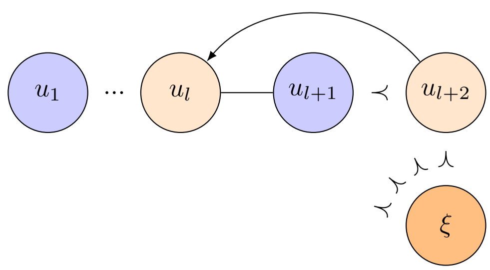

Figure 17: Illustration of $\psi^1_{l|2,1}$ (special case),
if $u_{l+2} \succ u_{l+1}$, $u_{l+2} \succ \{u_1, ..., u_{l-1}\}$ and $\xi \succ \vec{u}$.

12. If $z \not\prec w_l$ and $z \not\succ \vec{w}$, then denote

$$(\hat{w}_\theta, \hat{w}_{\theta+1}) = \begin{cases} (w_l, z), & \text{if } w_l \sim z, \\ (w_\theta, w_{\theta+1}), & z \succ w_l. \end{cases}$$

121. If $\hat{w}_\theta \succ q_0$ and $\hat{w}_{\theta+1} \succ q_1$, then

$$\phi_{l|2,1}(\vec{w}\ |q_0, q_1; z) = (\vec{w}, z\ ; q_0, q_1) \in M^U_{l+1,2}.$$

3. For $(\vec{w}, \xi\ ; q_0, q_1) \in M^U_{l+1,2}$, such that
$(\xi \succ w_l,\ w_\theta \succ q_0,\ w_{\theta+1} \succ q_1$, and $\xi \succ q_0)$,
or
$(\xi \sim w_l,\ w_l \succ q_0,\ \xi \succ q_1$, and $\xi \succ q_0)$,
we have:

$$\psi^2_{l|2,1}(\vec{w}, \xi\ ; q_0, q_1) = (\vec{w}\ |q_0, q_1; \xi).$$

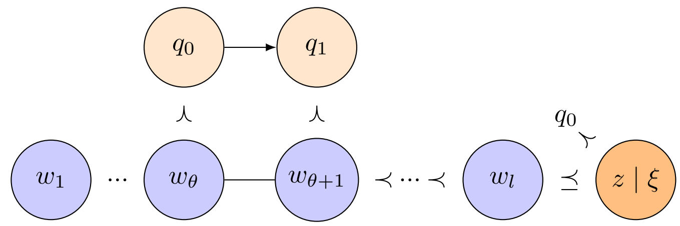

Figure 18: Illustration of 121.| 3.

122. If $\hat{w}_{\theta+1} \not\succ q_1$ or $\hat{w}_\theta \not\succ q_0$, we take

$$\tau = \max(\{i < l | w_i \not\succ q_1 \vee w_i \sim w_{i+1} \vee w_i \sim z\})$$

(note that $\tau \geq \theta$), and insert q_0 or q_1 after it:

1221. If $q_0 \prec w_\tau$, then

$$\phi_{l|2,1}(\vec{w}\ |\vec{q}\ ; z) =$$
$$= (w_1, ..., w_\tau, q_1, w_{\tau+1}, ..., w_l, z\ ; q_0) \in M^U_{l+2,1}$$

4 For $(\vec{u}\ ; \xi) \in M^U_{l+2,1}$, such that $\xi \prec u_{l+2}$, define

$$\eta = \max(0, \{i \geq \breve{\theta} | u_i \sim \xi\}).$$

41. If $\eta > 0$, then we take out u_η, u_{l+2} and ξ:

411. If $\eta = \breve{\theta} + 1$ and $u_{\breve{\theta}} \succ \xi$, then

$$\psi^1_{l|2,1}(\vec{u}\ ; \xi) = (u_1, ..., u_{\eta-1}, u_{\eta+1}, ..., u_{l+1}\ |\xi, u_\eta; u_{l+2}).$$

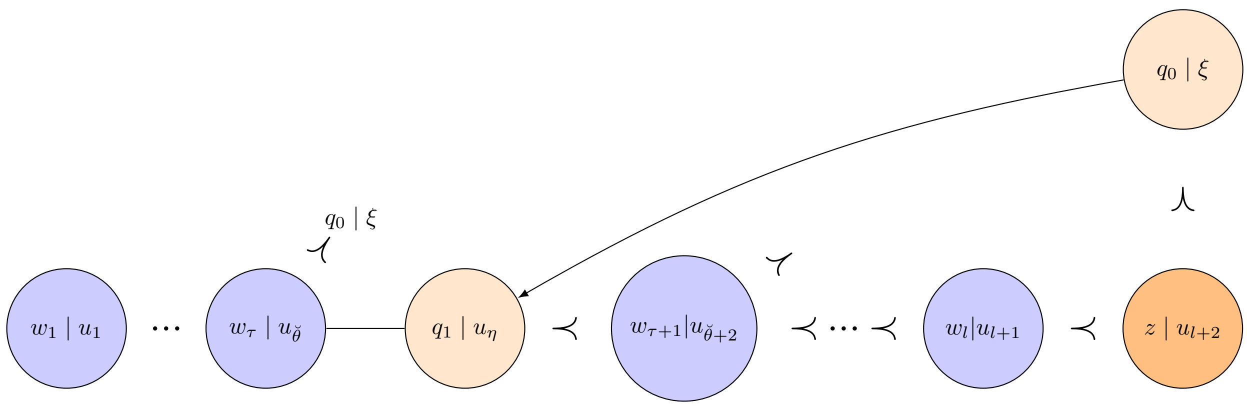

Figure 19: Illustration of 1221.|411.

1222. If $q_0 \not\prec s_\tau$, then

$$\phi_{l|2,1}(\vec{w}\ |\vec{q}\ ; z) =$$
$$= (w_1, ..., w_\tau, q_0, s_{\tau+1}, ..., w_l, z\ ; q_1) \in M^U_{l+2,1}.$$

412. If ($\eta = \theta$ and $\xi \prec u_{\theta+1}$) or ($\eta = \theta + 1$ and ($u_\theta \sim u_{\breve{\theta}+2}$ or $u_{\breve{\theta}} \not\succ \xi$) or $\eta > \breve{\theta} + 1$, then

$$\psi^1_{l|2,1}(\vec{u}\ ; \xi) = (u_1, ..., u_{\eta-1}, u_{\eta+1}, ..., u_{l+1}\ |u_\eta, \xi\ ; u_{l+2}).$$

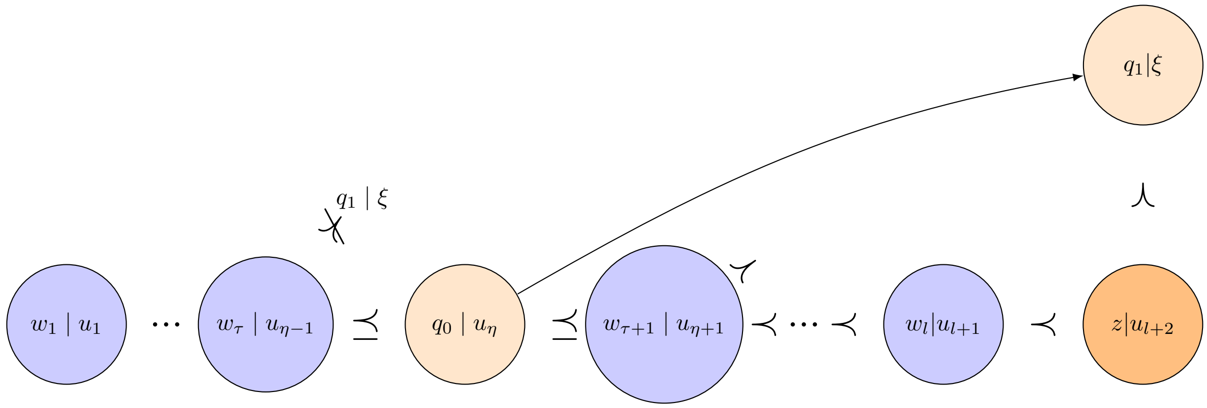

Figure 20: Illustration of 1222.|412.

Here we have a special case when $\tau = \theta$. Note that it is possible that $w_\tau \succ q_1$:

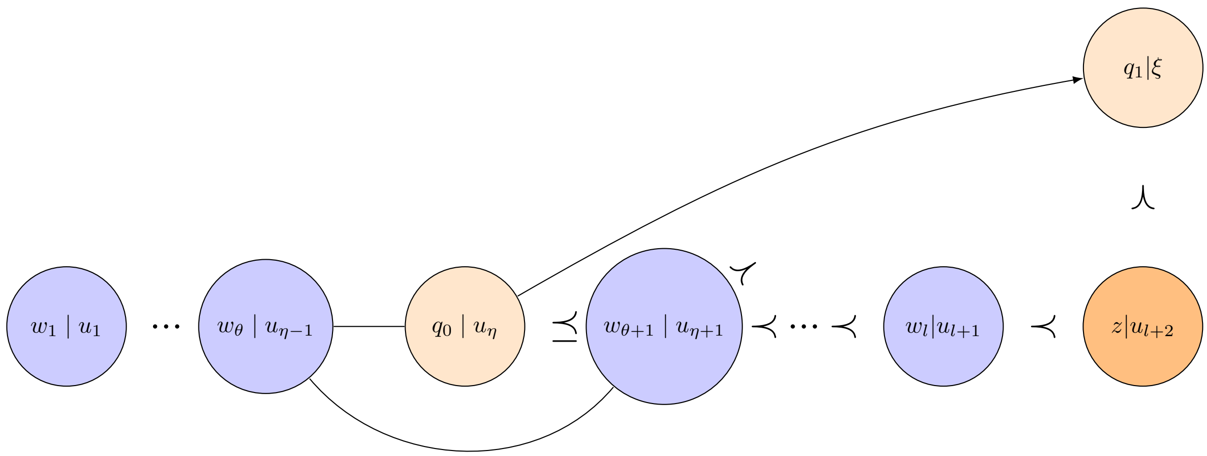

Figure 21: Illustration of 1222.|412.

13. If $z \prec w_l$.

131. If $w_{\theta+1} \succ q_1$ and $w_\theta \succ q_0$, then

$$\phi_{l|2,1}(\vec{w}\ |\vec{q}\ ; z) = (\vec{w}\ ; \vec{q}\ ; z) \in M^U_{l,2,1}.$$

5. If $(\vec{w}\ ; q_0, q_1; z) \in M^U_{l,2,1}$ and $z \succ \vec{q}$, and $z \prec w_l$, and $w_{\theta+1} \succ q_1$ and $w_\theta \succ q_0$, then

$$\psi^3_{l|2,1}(\vec{w}\ ; q_0, q_1; z) = (\vec{w}\ |q_0, q_1; z).$$

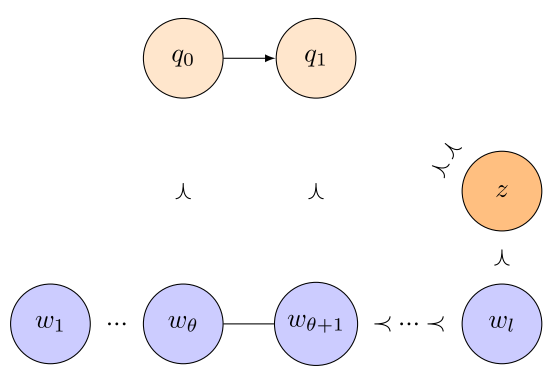

Figure 22: Illustration of 131.|5.

132. If $w_{\theta+1} \not\succ q_1$ or $w_\theta \not\succ q_0$, then Let

$$\gamma = \max(0, \{\theta < i < l | w_i \sim z\}).$$

1321. if $\gamma > \theta$ and $\gamma > \tau$ (i.e. $\exists\, w_\gamma \sim z$), then

$$\phi_{l|2,1}(\vec{w}\ |\vec{q}\ ; z) = (\vec{w}\ ; \vec{q}\ ; z) \in M^U_{l,2,1}.$$

This is the first type exceptional element, shown before on the Figure 12 and on the picture below:

6. If $(\vec{w}\ ; q_0, q_1; z) \in M^U_{l,2,1}$ is the first type exceptional element , then

$$\psi^3_{l|2,1}(\vec{w}\ ; q_0, q_1; z) = (\vec{w}\ |q_0, q_1; z).$$

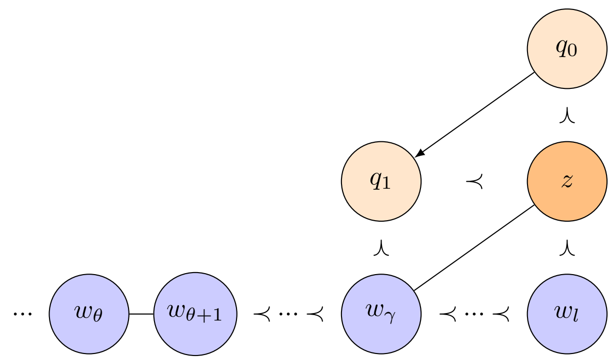

Figure 23: First exceptional element of $M^U_{l,2,1}$

1322. Otherwise (i.e. if $\gamma = 0$ or $\gamma < \tau$), we have:

$$\phi_{l|2,1}(\vec{w}\ |\vec{q}\ ; z) = (\phi_{l|2}(\vec{w}\ |\vec{q}\ ); z).$$

7. For $(\vec{u}\ ; \xi) \in M^U_{l+2,1}$, such that

$$\xi \prec u_{l+2} \text{ and } \eta = 0,$$

what implies

$$(\xi \succ u_{\theta+1} \text{ and } \xi \succ u_\theta),$$

we have

$$\psi^2_{l|2,1}(\vec{u}\ ; \xi) = (\psi_{l|2}(\vec{u}\ ); \xi).$$

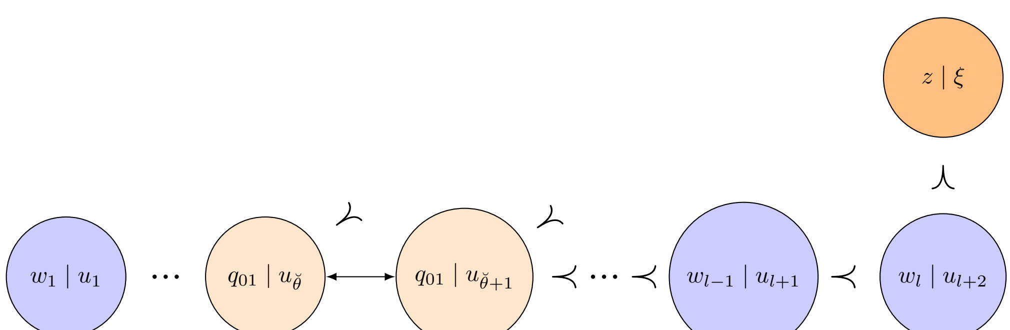

Figure 24: Illustration of 332.|19. (general case)

2. If $q_1 \succ z$ and $q_0 \sim z$.

21. If $q_0 \succ \vec{w}$

211. If $z \succ \vec{w}$, then

$$\phi_{l|2,1}(\vec{w}\ |\vec{q}\ ; z) = (\vec{w}\ ; \vec{q}\ ; z) \in M^U_{l,2,1}.$$

8. If For $(\vec{w}\ ; q_0, q_1; z) \in M^U_{l,2,1}$, such that

$$q_1 \succ z \text{ and } q_0 \sim z \text{ and } z \succ \vec{w},$$

we have

$$\psi^3_{l|2,1}(\vec{w}\ ; q_0, q_1; z) = (\vec{w}\ |q_0, q_1; z).$$

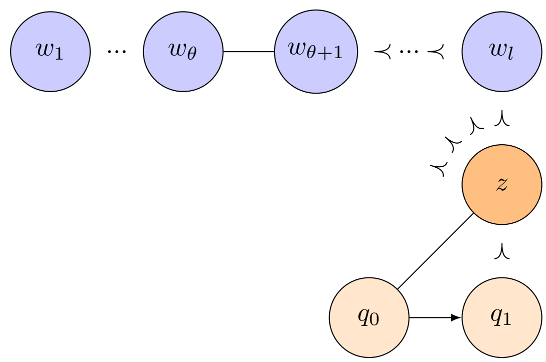

Figure 25: Illustration of 211.|8.

212. If $z \not\succ \vec{w}$ and $z \not\prec w_l$, then

$$\phi_{l|2,1}(\vec{w}\ |\vec{q}\ ; z) = (\vec{w}, z\ ; \vec{q}\ ) \in M^U_{l+1,2}.$$

9. For $(\vec{w}, \xi\ ; q_0, q_1) \in M^U_{l+1,2}$, such that $q_0 \succ \vec{w}$, $q_0 \sim \xi$ and $q_1 \succ \xi$, we have

$$\psi^2_{l|2,1}(\vec{w}, \xi\ ; q_0, q_1) = (\vec{w}\ |q_0, q_1; \xi).$$

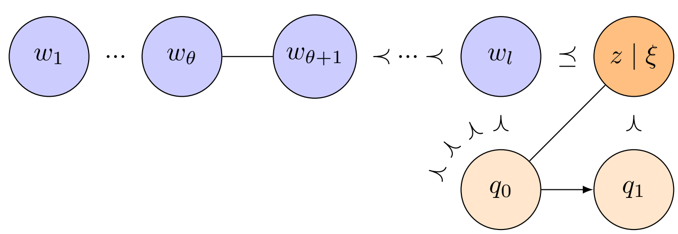

Figure 26: Illustration of 212.|9.

22. If $q_0 \not\succ \vec{w}$ and $q_0 \not\prec w_l$, then

$$\phi_{l|2,1}(\vec{w}\ |\vec{q}\ ; z) = (\vec{w}, q_0, q_1\ ; z) \in M^U_{l+2,1}.$$

10. For $(\vec{u}\ ; \xi) \in M^U_{l+2,1}$, such that $u_{l+1} \sim \xi$, $u_{l+1} \sim u_{l+2}$, and $u_{l+2} \succ \xi$, we have

$$\psi^2_{l|2,1}(\vec{u}\ ; \xi) = (u_1, ..., u_l|u_{l+1}, u_{l+2}\ \xi).$$

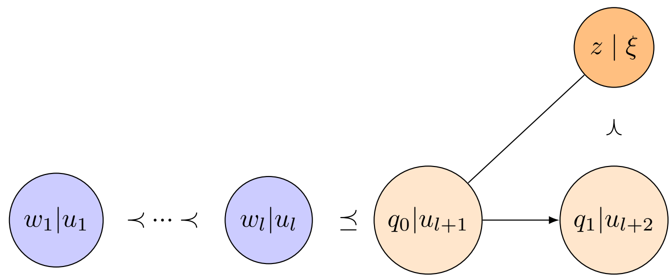

Figure 27: Illustration of 22.|10.

23. If $q_0 \prec w_l$.

231. If $q_1 \sim w_l$, then

$$\phi_{l|2,1}(\vec{w}\ |\vec{q}\ ; z) = (\vec{w}, q_1; q_0, z) \in M^U_{l+1,2}.$$

11. For $(\vec{w}, \xi\ ; q_0, q_1) \in M^U_{l+1,2}$, such that

$$\xi \sim w_l \text{ and } \xi \sim q_0 \text{ and } q_0 \prec w_l \text{ and } q_1 \prec \xi,$$

we have

$$\psi^2_{l|2,1}(\vec{w}, \xi\ ; q_0, q_1) = (\vec{w}\ |q_0, \xi\ ; q_1).$$

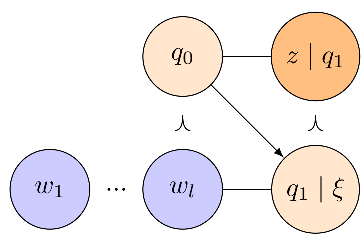

Figure 28: Illustration of 231.|11.

232. If $q_1 \prec w_l$ and $w_\theta \succ q_0$ and $w_{\theta+1} \succ q_1$, then

$$\phi_{l|2,1}(\vec{w}\ |\vec{q}\ ; z) = (\vec{w}\ ; \vec{q}\ ; z) \in M^U_{l,2,1}.$$

12. For $(\vec{w}\ ; q_0, q_1; z) \in M^U_{l,2,1}$, such that $w_\theta \succ q_0$ and $w_{\theta+1} \succ q_1$ and $q_0 \sim z$ and $q_1 \succ z$, we have

$$\psi^3_{l|2,1}(\vec{w}\ ; q_0, q_1; z) = (\vec{w}\ |q_0, q_1; z).$$

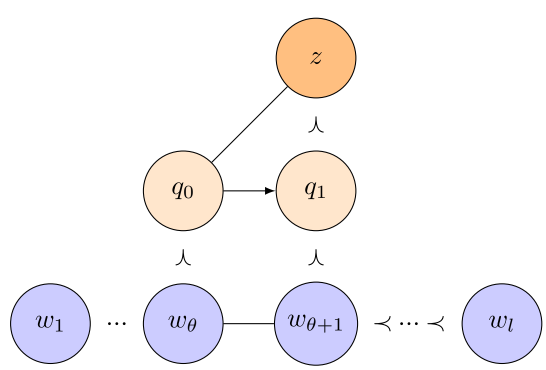

Figure 29: Illustration of 232.|12.

233. If $q_1 \prec w_l$ and ($w_\theta \not\succ q_0$ or $w_{\theta+1} \not\succ q_1$), let

$$\tau = \max(\{i < l | w_i \not\succ q_1 \vee w_i \sim w_{i+1}\}).$$

2331. If $q_0 \prec w_\tau$ ($\Rightarrow$ $q_1 \sim w_\tau$), then

$$\phi_{l|2,1}(\vec{w}\ |\vec{q}\ ; z) = (\vec{w}\ ; \vec{q}\ ; z) \in M^U_{l,2,1}.$$

This case is isomorphic to the second exceptional element type, shown below:

13. If $(\vec{w}\ ; q_0, q_1; z) \in M^U_{l,2,1}$ has the second exceptional element type, then

$$\psi^3_{l|2,1}(\vec{w}\ ; q_0, q_1; z) = (\vec{w}\ |q_0, q_1; z).$$

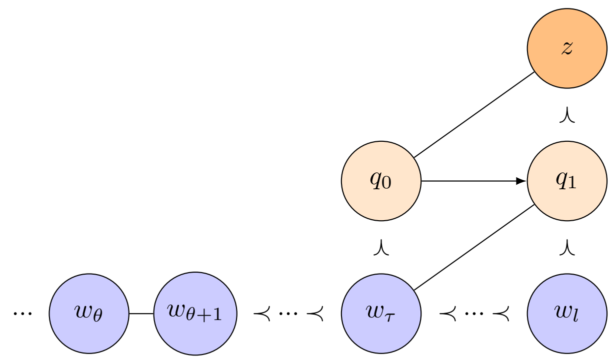

Figure 30: Illustration of 2331.|13.
(Second exceptional element from $M^U_{l,2,1}$).

2332. If $q_0 \not\prec w_\tau$, then

$$\phi_{l|2,1}(\vec{w}\ |\vec{q}\ ; z) = (\phi_{l|2}(\vec{w}\ |q_0, z)\ ; q_1) =$$
$$= (w_1, ..., w_\tau, q_0, z, w_{\tau+1}, ..., w_l\ ; q_1) \in M^U_{l+2,1}.$$

The following 3 pictures illustrate this case:

14. For $(\vec{u}\ ; \xi) \in M^U_{l+2,1}$, such that $\xi \prec u_{l+2}$, $\eta = \theta$ and $\xi \succ u_{\theta+1}$, we have

$$\psi^1_{l|2,1}(\vec{u}\ ; \xi) = (u_1, ..., u_{\theta-1}, u_{\theta+2}, ..., u_{l+2}\ |u_\theta, \xi\ ; u_{\theta+1}).$$

As a reminder,

$$\breve{\theta} = \max(\{i < l + 2|\ u_i \sim u_{i+1}\}),$$
$$\eta = \max(0, \{i \geq \theta | u_i \sim \xi\}).$$

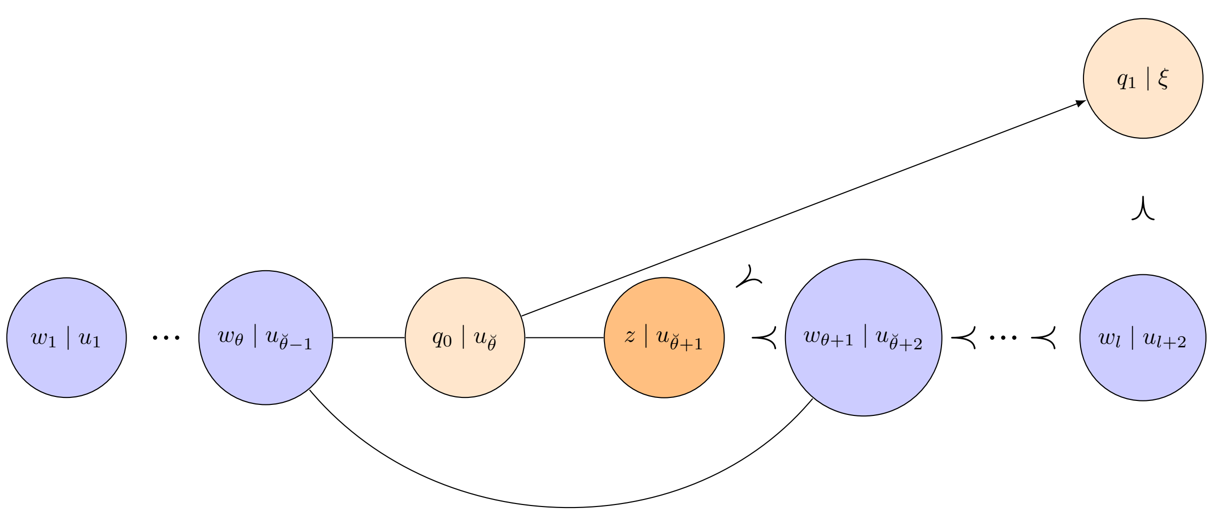

Figure 31: Illustration of 2332.|14.
($\phi_{l|2,1}$, if $\tau = \theta$. Note that it may happen that $w_\tau \succ q_1$).

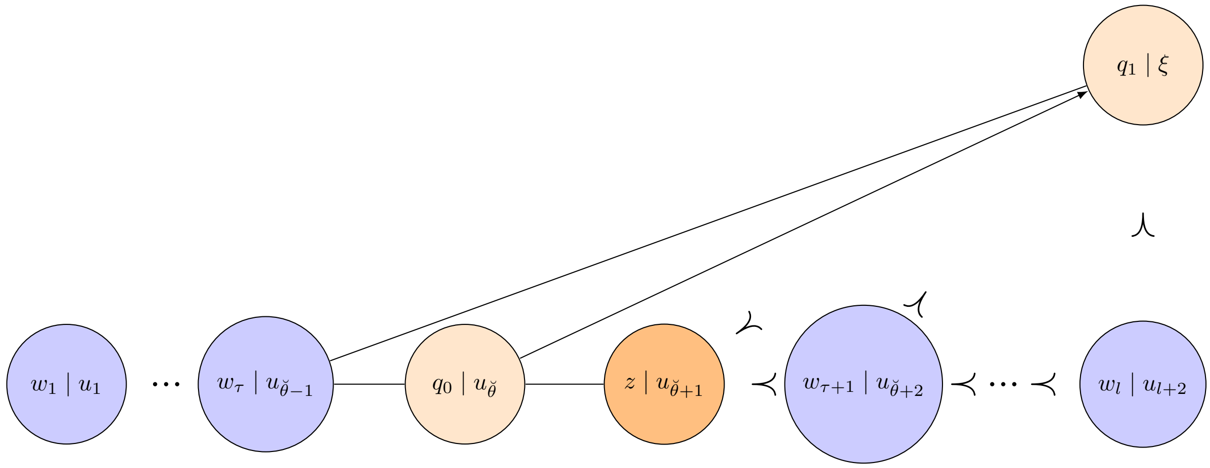

Figure 32: Illustration of 2332.|14.
($\phi_{l|2,1}$, if $q_1 \sim w_\tau$ and $w_\tau \sim q_0$.)

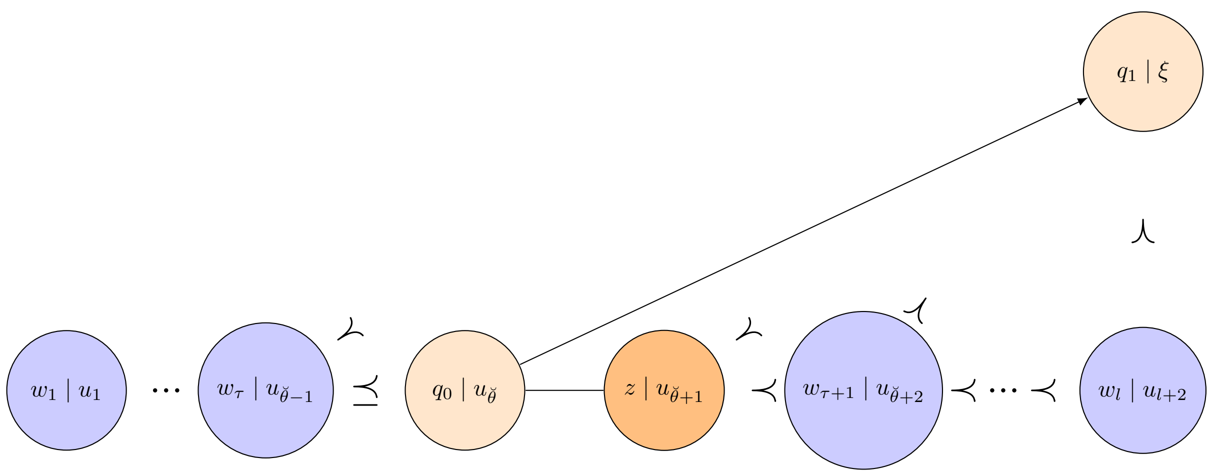

Figure 33: Illustration of 2332.|14.
($\phi_{l|2,1}$, if $q_1 \succ w_\tau$.)

3. If $q_1 \succ z$ and $q_0 \succ z$.

   This is the easiest case, since we insert $\vec{q}$ and z independingly.

31. If $z \succ \vec{w}$, then

$$\phi_{l|2,1}(\vec{w}\ |\vec{q}\ ; z) = (\vec{w}\ |\vec{q}\ ; z) \in M^U_{l,2,1}.$$

15. For $(\vec{w}\ ; q_0, q_1; z) \in M^U_{l,2,1}$, such that

    $z \succ \vec{w}$, $q_0 \succ z$ and $q_1 \succ z$,

    we have

$$\psi^3_{l|2,1}(\vec{w}\ ; q_0, q_1; z) = (\vec{w}\ |q_0, q_1; z).$$

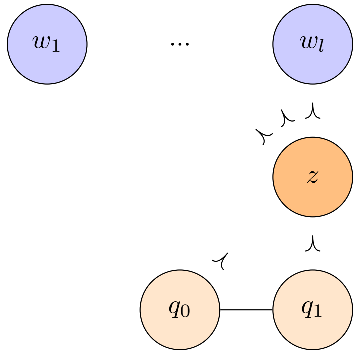

Figure 34: Illustration of 31.|15.

32. If $z \not\prec w_l$ and $z \not\succ \vec{w}$ ($\Rightarrow q_0 \not\prec w_l$)

321. If $q_0 \succ \vec{w}$, then

$$\phi_{l|2,1}(\vec{w}\ |\vec{q}\ ; z) = (\vec{w}, z\ ; \vec{q}\ ) \in M^U_{l+1,2}.$$

16. If $(\vec{w}, \xi\ ; q_0, q_1) \in M^U_{l+1,2}$,
and $q_0 \succ \vec{w}$, and $q_1 \succ \xi$, then

$$\psi^2_{l|2,1}(\vec{w}, \xi\ ; q_0, q_1) = (\vec{w}\ |q_0, q_1; \xi).$$

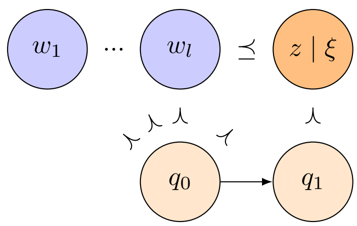

Figure 35: Illustration of 321.|16.

322. If $q_0 \not\succ \vec{w}$, then

$$\phi_{l|2,1}(\vec{w}\ |\vec{q}\ ; z) = (\vec{w}\ , \vec{q}\ ; z) \in M^U_{l+2,1}.$$

17. For $(\vec{u}\ ; \xi) \in M^U_{l+2,1}$, such that

$$\xi \prec u_{l+2} \text{ and } \xi \prec u_{l+2},$$

what implies

$$(\xi \prec u_{\theta+1} \text{ and } \xi \prec u_{\theta}),$$

we have

$$\psi^2_{l|2,1}(\vec{u}\ ; \xi) = (\psi_{l|2}(\vec{u}\ ); \xi) = (u_1, ..., u_l | u_{l+1}, u_{l+2}; \xi).$$

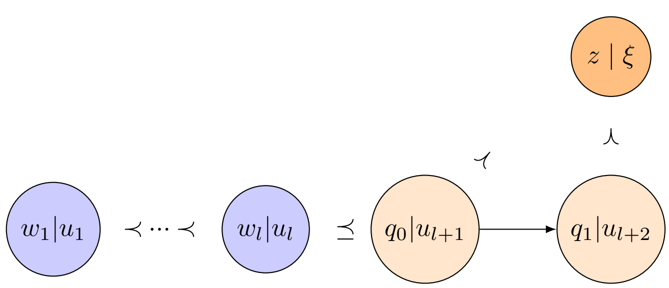

Figure 36: Illustration of 322.|17.

33. If $z \prec w_l$

331. If $(\vec{w}\,;\vec{q}\,) \in M^U_{l,2}$, then

$$\phi_{l|2,1}(\vec{w}\,|\vec{q}\,;z) = (\vec{w}\,;\vec{q}\,;z) \in M^U_{l,2,1}.$$

18. If $(\vec{w}\,;q_0,q_1;z) \in M^U_{l,2,1}$, and

$$q_0 \succ z,\; q_1 \succ z \text{ and } w_l \succ z, \text{ then}$$

$$\psi^3_{l|2,1}(\vec{w}\,;\vec{q}\,;z) = (\vec{w}\,|\vec{q}\,;z).$$

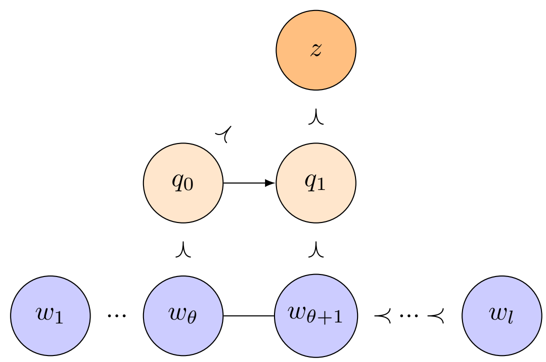

Figure 37: Illustration of 331.|18.

332. If $(\vec{w}\,|\vec{q}\,) \notin M^U_{l,2}$, then

$$\phi_{l|2,1}(\vec{w}\,|\vec{q}\,;z) = (\phi_{l|2}(\vec{w}\,|\vec{q}\,);z) \in M^U_{l+2,1}$$

19. If $(\vec{u}\,|\xi) \in M^U_{l+2,1}$, $\xi \prec u_{l+2}$ and $\eta = 0$, what imlies

$$(\xi \prec u_{\theta+1} \text{ and } \xi \prec u_\theta),$$

then we have

$$\psi^2_{l|2,1}(\vec{u}\,;\xi) = (\psi_{l|2}(\vec{u}\,);\xi).$$

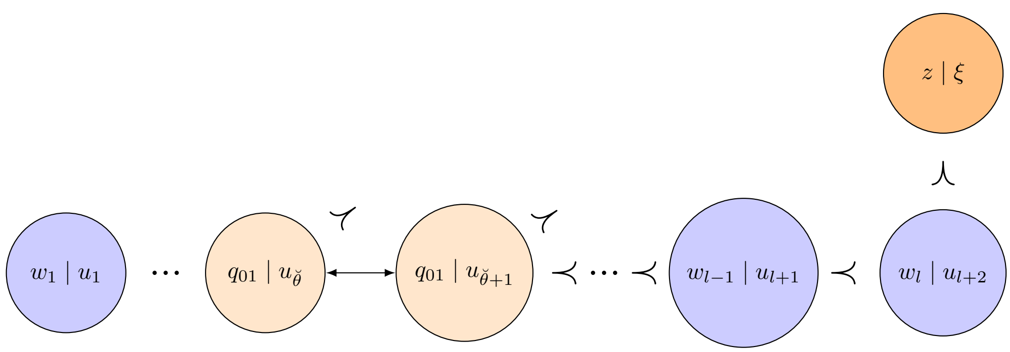

Figure 38: Illustration of 332.|19. (general case)

It is easy to see that by construction the left hand side fully describes the set $P^U_l \times M^U_{2,1}$. Now, we check that the right hand side coincide with $M^U_{l+2,1} \sqcup M^U_{l+1,2} \sqcup M^U_{l,2,1}$:

- Let $(\vec{u}\,;\xi) \in M^U_{l+2,1}$. Then the following cases from the right hand side clearly describe $M^U_{l,2,1}$:

  2. $\xi \succ \vec{u}$;
  4. $\xi \prec u_{l+2}$ and $\eta > 0$;
  7. $\xi \prec u_{l+2}$, $\eta = 0$ and $(\xi \succ u_{\theta+1}$ and $\xi \succ u_\theta)$;
  19. $\xi \prec u_{l+2}$, $\eta = 0$ and $(\xi \prec u_{\theta+1}$ and $\xi \prec u_\theta)$.

- Let $(\vec{w}\ ,\xi;q_0,q_1) \in M^U_{l+1,2}$. Then the following cases from the right hand side clearly describe $M^U_{l,2,1}$.:

    3. $(\xi \succ w_l,\ w_\theta \succ q_0,\ w_{\theta+1} \succ q_1$, and $\xi \succ q_0)$, or $(\xi \sim w_l,\ w_l \succ q_0,\ \xi \succ q_1$, and $\xi \succ q_0)$;
    11. $(\xi \sim w_l,\ w_l \succ q_0,\ \xi \succ q_1$, and $\xi \sim q_0)$;

    9. $q_0 \succ \vec{w},\ q_1 \succ \xi$ and $q_0 \sim \xi$;
    16. $q_0 \succ \vec{w},\ q_1 \succ \xi$ and $q_0 \succ \xi$.

- Let $(\vec{w}\ ;q_0,q_1;z) \in M^U_{l,2,1}$. Then the following cases from the right hand side clearly describe $M^U_{l,2,1}$.:

    1. $z \succ \vec{q}$ and $z \succ w$;
    5. $z \succ \vec{q}$ and $z \prec w_l$;

    8. $q_0 \sim z,\ q_1 \succ z$ and $z \succ \vec{w}$;
    12. $q_0 \sim z,\ q_1 \succ z$ and $(w_\theta \succ q_0$ and $w_{\theta+1} \succ q_1)$;

    15. $q_0 \succ z,\ q_1 \succ z$ and $z \prec w_l$;
    18. $q_0 \succ z,\ q_1 \succ z$ and $z \prec w_l$;

    6. First exceptional type element;
    13. Second exceptional type element.

Since every number from 1 to 19 was used exactly once, this completes the proof. □

Next, we introduce the set

$$E^U_k = \{\vec{\varepsilon} = (\varepsilon_1,\dots,\varepsilon_k)|\ \varepsilon_i \prec \varepsilon_{i+1},\ \text{for } 1 \le i < k\}.$$

Theorem 3.14. *For natural numbers l and k, let*

$$M^U_{l,1^k} = \{(\vec{w}\ |\vec{\varepsilon}) \in P^U_l \times E^U_k\,|\ \varepsilon \prec w_l\ \vee\ \varepsilon \succ J_l,\ \textit{for every } \varepsilon \in \vec{\varepsilon}\},$$

Then,

$$m^U_{l,1^k} = \sum_{(\vec{w};\vec{\varepsilon})\in M^U_{l,1^k}} w_1 \cdot \ldots \cdot w_l \cdot \varepsilon_1 \cdot \ldots \cdot \varepsilon_k.$$

Remark 3.15. *According to Remark 3.3, this implies $c_{n-k,1^k}(U) \ge 0$.*

Proof. We prove this by induction on k. Note that for $k=1$, the definition of $M^U_{l,1^k}$ coincides with $M^U_{l,1}$ from the Theorem 3.7. Thus, the case $k=1$ is implied by the Theorem 3.7.

Assume the statement is true for k, and consider

$$p^U_l * e^U_{k+1} = m^U_{l,1^{k+1}} + m^U_{l+1,1^k}.$$

Below, we construct two mutually inverse maps, $\phi_{l,1^{k+1}}$ and $\psi_{l,1^{k+1}}$:

First, we define

$$\phi_{l,1^{k+1}} : P^U_l \times E^U_{k+1} \to M^U_{l,1^{k+1}} \sqcup M^U_{l+1,1^k}$$

as follows:

Let $(\vec{w}\ |\vec{\varepsilon}\ ) \in P^U_l \times E^U_{k+1}$

1. If $\varepsilon_i \prec w_l\ \vee\ \varepsilon_i \succ \vec{w}$ for $1 \le i \le k+1$, then

$$\phi_{l,1^{k+1}}(\vec{w}\ |\vec{\varepsilon}\ ) = (\vec{w}\ ;\vec{\varepsilon}\ ) \in M^U_{l,1^{k+1}}.$$

Second, the inverse of map $\psi_{l,1^{k+1}}$:

$$\psi^1_{l,1^{k+1}} : M^U_{l,1^{k+1}} \to P^U_l \times E^U_{k+1};$$
$$\psi^2_{l,1^{k+1}} : M^U_{l+1,1^k} \to P^U_l \times E^U_{k+1}.$$

Let $(\vec{w}\ ;\vec{\varepsilon}\ ) \in M^U_{l,1^{k+1}}$ and $(\vec{w}\ ,z;\vec{\nu}\ ) \in M^U_{l+1,1^k}$.

1. For $(\vec{w}\ ;\vec{\varepsilon}\ ) \in M^U_{l,1^{k+1}}$, we have:

$$\psi^1_{l,1^{k+1}}(\vec{w}\ ;\vec{\varepsilon}\ ) = (\vec{w}\ |\vec{\varepsilon}\ ) \in P^U_l \times E^U_{k+1}.$$

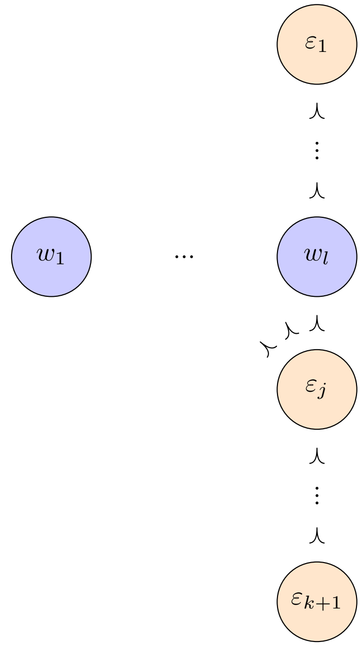

Figure 39: Illustration of 1.|1.

2. If $\exists\, i$, s.t. $\varepsilon_i \not\prec w_l \;\wedge\; \varepsilon_i \not\succ \vec{w}$, then define

$$m = \max(i \,|1 \le i \le k{+}1, \varepsilon_i \not\prec w_l \wedge \varepsilon_i \not\succ \vec{w}),$$

then we have

$$\phi_{l,1^{k+1}}(\vec{w}\ |\vec{\varepsilon}\ ) =$$
$$= (\vec{w}\ , \varepsilon_m; \varepsilon_1, ..., \varepsilon_{m-1}, \varepsilon_{m+1}, .., \varepsilon_{k+1}) \in M^U_{l+1,1^k}.$$

2. For $(\vec{w}\ , z; \vec{\nu}\ ) \in M^U_{l+1,1^k}$, we define

$$j = \min(i \,|1 \le i \le k, z \prec \nu_i),$$

then we have

$$\psi^2_{l,1^{k+1}}(\vec{w}\ , z; \vec{\nu}\ ) =$$
$$= (\vec{w}\ |\nu_1, ..., \nu_{j-1}, z, \nu_j, .., \nu_k) \in P^U_l \times E^U_{k+1}.$$

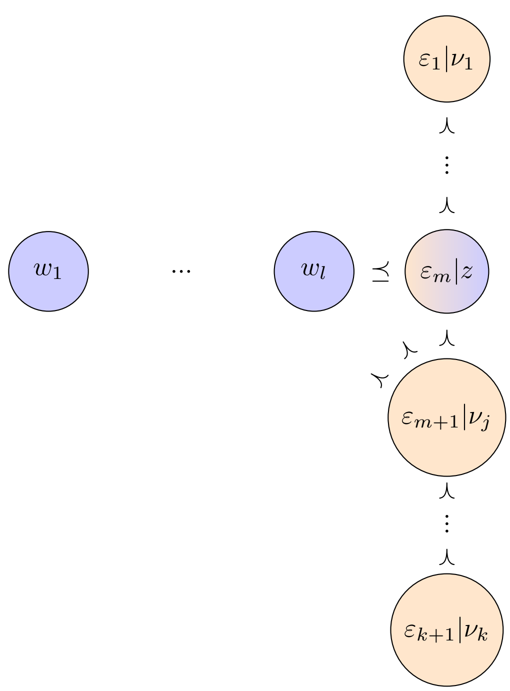

Figure 40: Illustration of 2.|2.

This completes the proof. ◻

Theorem 3.16. *For natural numbers l and k, let*

$$M^U_{2^l,1^k} = \{(\vec{\xi},\vec{\varepsilon}\,) \in E^U_l \times E^U_{k+l} |\ \exists \{i_j\}_{j=1}^l,\ 0 < i_1 < i_2 < ... < i_l < k+l,\ s.t.\ \xi_j \sim \varepsilon_{i_j}\ for\ 1 \le j \le l\},$$

Then

$$m^U_{2^l,1^k} = \sum_{(\vec{\xi},\vec{\varepsilon})\in M^U_{2^l,1^k}} \xi_1 \cdot ... \cdot \xi_l \cdot \varepsilon_1 \cdot ... \cdot \varepsilon_{l+k}.$$

Remark 3.17. *According to Remark 3.3, this implies $c_{2^k,1^{n-2k}}(U) \ge 0$.*

We begin with the following definitions.

Definition 3.18. For a nonnegative integer l, let $(\vec{\nu},\vec{\mu}\,) \in E^U_l \times E^U_{l+1}$. We say that $\vec{\nu}$ is *covered* by $\vec{\mu}$ if $\nu_i \sim \mu_i$ and $\nu_i \sim \mu_{i+1}$ for $1 \le i \le l$.

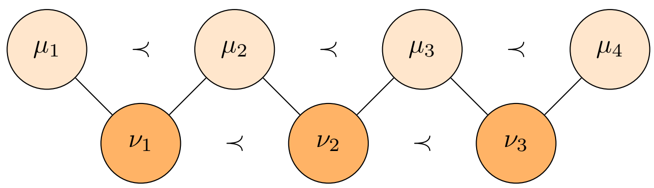

Figure 41: $\vec{\mu}$ *covers* $\vec{\nu}$ $(l = 3)$.

Here we allow $l = 0$. In this case $\vec{\mu}$ has length 1, while $\vec{\nu}$ is a null-vector. We consider μ as an *upper layer* and ν as a *lower layer*, as shown on the picture.

Definition 3.19. For $\vec{\nu} \in E^U_l$ and $\vec{\mu} \in E^U_m$, such that $\nu_i \nsim \mu_j$, where $1 \le i \le l$ and $1 \le j \le m$, we define the vector

$$\vec{\nu} \uplus \vec{\mu} \in E^U_{l+m}$$

as an increasing (with respect to $\prec$) sequence of the elements of $\vec{\nu}$ and $\vec{\mu}$, where every element is used exactly one time.

Definition 3.20. For $\vec{\nu} \in E^U_l$ and a tuple $1 \le i_1 < ... < i_k \le l$, we define the vector

$$\vec{\nu} \setminus (\nu_{i_1}, ..., \nu_{i_k}) \in E^U_{l-k}$$

as an increasing (with respect to $\prec$) sequence of the elements of $\vec{\nu}$, which are not in $\{\nu_{i_1}, ..., \nu_{i_k}\}$, where every element is used exactly once.

Proof. We prove the theorem by induction. According to the Theorem 3.14, the statement is true for $l = 1$. Then, we show that if

$$M^U_{2^{l'},1^k} = \{(\vec{\xi},\vec{\varepsilon}\,) \in E^U_{l'} \times E^U_{k+l'} |\ \exists \{i_j\}_{j=1}^{l'},\ 0 < i_1 < i_2 < ... < i_{l'} < k+l',\ \text{such that}\ \xi_j \sim \varepsilon_{i_j}\ \text{for every}\ 1 \le j \le l'\},$$

for every $l' \le l$, then this formula also works for $l = l + 1$.

Note that since U is UIO, it is 3+1-free and it is not possible that for some i, ξ_i is incomparable to 3 elements of $\vec{\varepsilon}$, or ε_i is incomparable to 3 elements of $\vec{\xi}$.

Following the analogous formula for symmetric polynomials, we have

$$e^U_{l+1} \cdot e^U_{l+1+k} = m^U_{2^{l+1},1^k} + \binom{1}{k+2} \cdot m^U_{2^l,1^{k+2}} + \binom{2}{k+4} \cdot m^U_{2^{l-1},1^{k+4}} + ... + \binom{l}{2l+k} \cdot m^U_{2,1^{2l+k}} + \binom{l+1}{2l+k+2} \cdot m^U_{1^{2l+k+2}}.$$

To prove the theorem we will construct a surjective map

$$\phi : E^U_{l+1} \times E^U_{l+k+1} \to \coprod_{j=0}^{l+1} M^U_{2^{l+1-j},1^{k+2j}}.$$

Let $(\vec{\xi} \mid \vec{\varepsilon}\,) \in E^U_{l+1} \times E^U_{l+k+1}$. Then, we define

$$\Gamma(\vec{\xi} \mid \vec{\varepsilon}\,) = \{(\vec{\mu}\,,\,\vec{\nu}) = ((\xi_{i_1}, ..., \xi_{i_m}), (\varepsilon_{j_1}, ..., \varepsilon_{j_{m-1}}))\,|\; 1 \le m \le l+1, \text{ s.t. } \vec{\mu} \text{ covers } \vec{\nu} \text{ and } \xi_{i_1} \nsim \varepsilon_{j_1-1},\; \xi_{i_m} \nsim \varepsilon_{j_{m-1}+1}\},$$

the set of all subvectors of $\vec{\xi}$, which cover some subvectors of $\vec{\varepsilon}$. Due to the last condition,

$$\xi_{i_1} \nsim \varepsilon_{j_1-1} \text{ and } \xi_{i_m} \nsim \varepsilon_{j_{m-1}+1},$$

these subvectors are of maximum possible lengths, and different elements of $\Gamma(\vec{\xi} \mid \vec{\varepsilon}\,)$ do not have common elements of U. Note that ξ_i, which is comparable to every element of $\vec{\varepsilon}$, is treated as a subvector which covers 0-subvector, and $((\xi_i),\ \vec{0}\,) \in \Gamma$. Denote by $|\Gamma|$ the number of elements of Γ. Here and below we omit $(\vec{\xi} \mid \vec{\varepsilon}\,)$ if it is clear what Γ is considered.

Then, ϕ is defined by switching covered subvectors of $\vec{\varepsilon}$ and their coverings from $\vec{\xi}$:

$$\phi(\vec{\xi} \mid \vec{\varepsilon}\,) = \left(\Big(\vec{\xi} \setminus \biguplus_{(\vec{\mu},\vec{\nu})\in\Gamma} \vec{\mu}\Big) \uplus \biguplus_{(\vec{\mu},\vec{\nu})\in\Gamma} \vec{\nu}\ ;\ \Big(\vec{\varepsilon} \setminus \biguplus_{(\vec{\mu},\vec{\nu})\in\Gamma} \vec{\nu}\Big) \uplus \biguplus_{(\vec{\mu},\vec{\nu})\in\Gamma} \vec{\mu} \right).$$

It is clear that

$$\phi(\vec{\xi} \mid \vec{\varepsilon}\,) \in E^U_{l+1-|\Gamma|} \times E^U_{l+k+1+|\Gamma|},$$

since every switch decreases the length of the first component (upper layer) by 1, and increases the second vector (lower layer) by 1. For instance, the Picture 41 will be transformed into the one below:

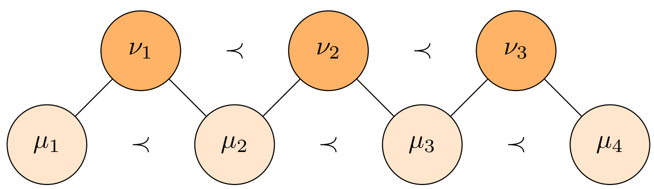

Figure 42: Picture 41 under the action of ϕ.

Next, after the switch, for every element of the first vector of $\phi(\vec{\xi} \mid \vec{\varepsilon}\,)$ there is a correspondent incomparable element from the second component (lower layer), so

$$\phi(\vec{\xi} \mid \vec{\varepsilon}\,) \in M^U_{2^{l+1-|\Gamma|},1^{k+2|\Gamma|}}.$$

On the other hand, we have exactly $\binom{|\Gamma|}{k+2|\Gamma|}$ possibilities to make the inverse switch for $\phi(\vec{\xi} \mid \vec{\varepsilon}\,)$, since there are

$$(l+k+1+|\Gamma|) - (l+1-|\Gamma|) = k+2|\Gamma|$$

"free" elements on the lower layer, from which we have to choose $|\Gamma|$ elements. Hence, there are exactly $\binom{|\Gamma|}{k+2|\Gamma|}$ elements of $E^U_{l+1} \times E^U_{l+k+1}$, which are maped into $\phi(\vec{\xi} \mid \vec{\varepsilon}\,)$ under the action of ϕ. Using the same principle, one can say that ϕ is a surjective map. This completes the proof.

□

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