Nodal rational sextics in the real projective plane

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N.B. - La thèse doit porter la déclaration précédente et remplir les conditions énumérées dans les "Informations relatives aux thèses de doctorat à l'Université de Genève".

Résumé

Dans cette thèse nous nous intéressons à la topologie des variétés algébriques réelles, et plus particulièrement à la topologie de certaines courbes algébriques dans le plan projectif réel $\mathbb{RP}^2$. L'histoire de la classification topologique de ces courbes remonte au moins jusqu'aux travaux de F. Klein et A. Harnack au XIX$^{\text{e}}$ siècle et aux questions posées dans la première partie du seizième problème de Hilbert en 1900.

Dans l'étude des variétés algébriques réelles on considère souvent deux problèmes de classification : la classification à isotopie près (plus grossière) et celle à isotopie rigide près (plus fine). Deux courbes algébriques dans $\mathbb{RP}^2$ sont dites *isotopes* si leurs ensembles de points réels sont isotopes dans $\mathbb{RP}^2$. Étant donnés un degré d et une classe de courbes de degré d (par exemple, les courbes non-singulières ou les courbes nodales ayant k points doubles), deux courbes appartenant à cette classe sont dites *rigidement isotopes* si elles peuvent être reliées par un chemin dans l'espace des courbes de degré d dans $\mathbb{RP}^2$, tel que ce chemin est entièrement contenu dans la classe considérée.

La classification à isotopie près des courbes non-singulières dans $\mathbb{RP}^2$ est connue jusqu'en degré sept. À isotopie rigide près, elle est connue jusqu'en degré six. Elle a été obtenue en degrés cinq et six respectivement par V. Kharlamov et par V. Nikulin.

Cette thèse est consacrée à la classification à isotopie rigide près de certaines classes de courbes nodales de degré six (sextiques) dans $\mathbb{RP}^2$. La classification à isotopie rigide près des courbes nodales est facile à obtenir jusqu'en degré quatre. Pour les courbes de degré 5 avec un point double, elle a été obtenue par V. Kharlamov. Très récemment, A. Jaramillo Puentes à classifié à isotopie rigide près les courbes nodales rationnelles de degré 5. La classification des sextiques avec un seul point double a été obtenue par I. Itenberg. Notre but est d'étendre cette classification aux courbes nodales ayant plusieurs points doubles.

Dans la première partie nous étudions les sextiques nodales dans dans $\mathbb{RP}^2$ sans points doubles réels, c'est-à-dire des sextiques dont les seuls singularités sont des points doubles non-réels. Le résultat principal de cette partie est la classification à isotopie rigide près de ces sextiques. Nous donnons des invariants topologiques nécessaires et suffisantes pour distinguer ces sextiques à isotopie rigide près, et nous donnons aussi une liste de tous les valeurs atteintes par ces invariants. Ce résultat peut être vu comme une généralisation de la classification des sextiques non-singulières obtenue par V. Nikulin.

La seconde partie porte sur les sextiques ayant des points doubles réels. Nous développons une méthode pour classifier les sextiques nodales séparantes, c'est-à-dire celles dont la partie réelle sépare leur complexification (l'ensemble des points complexes) en deux composantes connexes, en termes de faces de certains polytopes hyperboliques.

Cette méthode est ensuite appliqué au cas des sextiques rationnelles réelles pouvant être perturbées en des sextiques maximales ou presque maximales (dans le sens de l'inégalité de Harnack), c'est-à-dire celles ayant 11 ou 9 ovales.

L'approche retenue repose sur l'étude des périodes des surfaces K3, en utilisant le fait que le revêtement double de $\mathbb{CP}^2$ ramifié le long d'une sextique est une surface K3. Les bases de cette approche sont notamment le Théorème de Torelli Global de I. Piatetski-Shapiro et I. Shafarevich et le Théorème de Surjectivité de V. Kulikov, ainsi que sur les résultats de V. Nikulin portant sur les formes bilinéaires symétriques intégrales.

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Contents

Introduction

In the study of the topology of real algebraic varieties of low degree, there seems to be a fine line separating rather easy problems from seemingly intractable ones. The most classical problem of this kind is the first part of Hilbert's 16th problem [15], asking for a classification up to isotopy of real non-singular plane projective curves of a given degree. This problem is easy up to degree five. The classification in degree six was completed by Gudkov [13]. In degree seven it was obtained by Viro [42], and it remains open in degrees eight and higher.

Instead of classifying curves up to isotopy, one can classify them up to rigid isotopy, that is, isotopy within the space of non-singular algebraic curves of the given degree. This refined classification is known for non-singular curves in $\mathbb{RP}^2$ up to degree six. In degree six, it was obtained by Nikulin [32].

Given a real sextic in the projective plane, one can consider its set of real points $\mathbb{R}C \subset \mathbb{RP}^2$ as well as its set of complex points, also called its complexification, $C \subset \mathbb{CP}^2$. When studying real plane curves of even degree, a very fruitful idea, due to Arnol'd [1], is to deduce information about the topology of the pair $(\mathbb{RP}^2, \mathbb{R}C)$ from the complex surface X obtained as a double cover of $\mathbb{CP}^2$ branched along the curve C, together with an anti-holomorphic involution covering the complex conjugation on $\mathbb{CP}^2$. In particular, useful invariants can be extracted from the action of this involution on the second integral homology group of X. In the case of curves of degree six, the surface X is a K3 surface. Nikulin's proof uses two main ingredients: the classification of involutions on unimodular lattices, and the theory of periods of K3 surfaces, in particular the Global Torelli Theorem by Piatetski-Shapiro and Shafarevich [34] and the surjectivity of the period map, due to Kulikov [26]. This approach is very powerful, but unfortunately limited to curves of degree six.

Instead of increasing the degree, one can study curves with certain singularities in order to increase the complexity of the problem. The simplest singularities of plane curves are non-degenerate double points, also called nodes. When studying nodal curves, by a rigid isotopy one means an isotopy within the space of nodal curves of the same degree and with the same number of nodes, or equivalently, of the same degree and the same geometric genus. In his PhD thesis, Itenberg [18, 19] obtained a rigid isotopy classification of real sextics with one node.

The goal of the present thesis is to extend this classification to real sextics with several nodes. We are particularly interested in real rational sextics. These have ten nodes and can be parameterized by real rational functions.

Real rational plane curves naturally appear in different contexts. Examples are real enumerative geometry (e.g. counting real rational plane curves of degree d passing through $3d-1$ points in general position, cf. Welschinger [45]) and the study of real algebraic knots in $\mathbb{RP}^3$, where plane curves appear as projections of algebraic space curves, and can be thought of as real algebraic versions of knot diagrams (cf. Björklund [5], Mikhalkin and Orevkov [28]).

For the classification of real rational plane curves of degree four, see Gudkov [12] and the references therein, and D'Mello [7, 8]. In degree five, the isotopy classification was obtained by Itenberg, Mikhalkin, and Rau [20], and the rigid isotopy classification was obtained recently by Jaramillo Puentes [21] using real *dessins d'enfants*.

From the complex point of view, a node is a transverse intersection of two non-singular branches of the curve. Nodes of a real plane curve may either be *real*, i.e. belong to $\mathbb{RP}^2$, or *non-real*. Non-real nodes on real curves always appear in pairs, which are interchanged by complex conjugation. Among the real nodes, one may further distinguish between *hyperbolic nodes*, where both branches are real, and *solitary nodes*, where the two branches are non-real and interchanged by complex conjugation.

An important characteristic of real irreducible curves is whether they are *dividing* or not. A non-singular curve is called dividing if the set of real points $\mathbb{R}C$ divides the set of complex points C into two connected components, called *halves* of the curve. An irreducible nodal curve is called dividing if its normalization is dividing. A non-real node on a dividing curve is called *non-crossing* if the two branches intersecting at this node belong to the same half, and *crossing* otherwise.

The main results of this thesis are a rigid isotopy classification of real irreducible nodal sextics without real nodes, and a classification up to rigid isotopy for those real nodal rational sextics which can be perturbed into maximal or almost maximal (in the sense of the Thom-Smith inequality) non-singular sextics. In the following, we outline the contents of each chapter.

Chapter 1 contains preliminary material about integral lattices (Section 1.1) and hyperbolic reflection groups (Section 1.2).

In Chapter 2, we investigate the connection between nodal sextics and real K3 surfaces of degree two. Here, we allow reducible sextics. With a real nodal plane sextic, we associate a real weakly polarized K3 surface. The discrete homological invariants of these K3 surfaces form the *homological type* of the K3 surface, and by extension, of the sextic. The homological type encodes quite a lot of topological and geometrical information about the sextic, such as its decomposition into irreducible components, whether it is dividing or not, and how many crossing pairs of non-real nodes it has. However, in general the homological type does not uniquely determine the rigid isotopy class of a nodal sextic. For example, when a hyperbolic node is transformed into a solitary node through a real cusp, this does change the rigid isotopy type but not the homological type. The cornerstone for the classification of real nodal sextics is Theorem 2.18, which gives a precise description of the rigid isotopy types corresponding to a given homological type in terms of certain tilings of two hyperbolic spaces.

In Chapter 3, we classify real irreducible nodal sextics without real nodes up to rigid isotopy. This can be seen as a generalization of Nikulin's classification of non-singular sextics. We build on Nikulin's results [33] to classify the homological types of such sextics, and show that the homological type determines the rigid isotopy class in this case. The classification states that rigid isotopy classes of real irreducible sextics with m pairs of non-real nodes and without real nodes are completely determined by the isotopy type of their real part, whether they are dividing or not, and—if they are dividing—the number of pairs of non-real nodes which are crossing (Theorem 3.1). Moreover, we show that any combination of isotopy type of the real part, dividing type, and number of crossing pairs can be realized by such a sextic if this is not prohibited by classical restrictions on non-singular sextics, Harnack's inequality [14] or Rokhlin's complex orientation formula [35] for dividing curves (Theorem 3.2).

In Chapter 4, we study dividing real nodal sextics by viewing them as nodal degenerations of sextics without real nodes. With each rigid isotopy type $[C_0]$ of real nodal sextics without real nodes, we associate a pair of hyperbolic polytopes (P_+, P_-). We show that rigid isotopy classes of irreducible nodal sextics obtained by real degeneration from C_0 are in bijection with certain pairs of faces of the polytopes P_+ and P_-, considered up to certain symmetries (Theorem 4.21). In Section 4.2, we present a version of what is called *Donaldson's trick*: Using the hyperkähler structure of K3 surfaces, one can represent real vanishing cycles as holomorphic curves with respect to a different complex structure. This allows us to apply the intersection theory of complex curves on complex surfaces to real vanishing cycles.

Chapter 5 is an excursion into the realm of hyperbolic polytopes. We develop a technique – called *descent* – which allows us to describe faces of an infinite hyperbolic polytope in terms of certain combinatorial data of another, higher-dimensional hyperbolic polytope. It is a generalization of a similar method used by Itenberg [19]. We use this method in Chapter 6 for the calculation of the rigid isotopy classes of real rational sextics.

In Chapter 6, we study rigid isotopy classes of real rational sextics, using the results of Chapters 3, 4 and 5. We present a rigid isotopy classification for real nodal rational sextics which are degenerations of non-singular sextics with the maximal (eleven) or next-to-maximal (nine) number of real components. (Sextics with ten real components cannot be dividing.) We group these rigid isotopy classes by the type of their "dividing perturbation". For each group, we answer three questions: (1) How many rigid isotopy classes of real rational sextics with the given "dividing perturbation" are there? (2) How can these rigid isotopy classes be constructed, starting from explicitly constructed real reducible sextics by perturbation of real singularities and contraction of empty ovals? (3) To what extent can these rigid isotopy classes be distinguished by their position with respect to auxiliary lines and conics?

CHAPTER 1

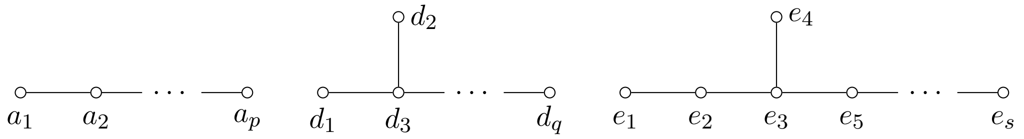
Preliminaries

1.1 Lattices

In this section, we introduce basic definitions and statements about integral lattices and their discriminant forms. All the needed results can be found in Nikulin's article [32]. Another reference we found useful is Chapter 2 of the book [10] by Degtyarev, Itenberg, and Kharlamov.

Definition 1.1. A *lattice* is a finitely generated free abelian group L endowed with an integral symmetric bilinear form $b\colon L \times L \to \mathbb{Z}$. We usually write $x \cdot y$ instead of $b(x, y)$ and x^2 instead of $b(x, x)$. A lattice L is *even* if x^2 is even for all $x \in L$. The *correlation homomorphism* of a lattice L is the homomorphism of L to its *dual lattice* $L^\vee := \mathrm{Hom}(L, \mathbb{Z})$ defined by $x \mapsto (y \mapsto x \cdot y)$. A lattice is called *non-degenerate* if its correlation homomorphism is injective, and *unimodular* if its correlation homomorphism is bijective. We denote the group of auto-isometries of a lattice L by $O(L)$.

Let us introduce notations for some commonly used lattices. Let $\langle a\rangle$ ($a \in \mathbb{Z}$) denote the rank one lattice generated by an element x with $x^2 = a$. Let U denote the *hyperbolic plane*, i.e. the rank two lattice generated by two elements u_1, u_2 with $u_1^2 = u_2^2 = 0, u_1 \cdot u_2 = 1$. Let A_p ($p \geq 1$), D_q ($q \geq 3$) and E_s ($s = 6, 7, 8$) denote the *negative definite* lattices generated by the elliptic root systems of the same name:



A_p ($p \geq 1$) $\quad$ D_p ($q \geq 3$) $\quad$ E_s ($s = 6, 7, 8$)

Let $L(k)$ denote the lattice obtained by scaling a lattice L by a factor $k \in \mathbb{Z}$, i.e., if $x_1, \ldots, x_n$ are generators for L, let $L(k)$ be the lattice generated by vectors $y_1, \ldots, y_n$ with $y_i \cdot y_j = k(x_i \cdot x_j)$.

Definition 1.2. A *finite bilinear form* is a pair (A, b), where A is a finite abelian group and $b : A \times A \to \mathbb{Q}/\mathbb{Z}$ is a symmetric bilinear form. A *finite quadratic form* is a pair (A, q), where A is a finite abelian group and $q : A \to \mathbb{Q}/2\mathbb{Z}$ is a map such that we have $q(nx) = n^2 q(x)$ for all $x \in A, n \in \mathbb{Z}$, and $b(x, y) := \frac{1}{2}(q(x + y) - q(x) - q(y))$ defines a finite bilinear form on A, which is called the *bilinear form associated with* q.

Definition 1.3. The *discriminant group* of a non-degenerate lattice L is the finite abelian group $A_L = L^\vee/L$, the cokernel of its correlation homomorphism. We may view $L^\vee$ as a subgroup of $L \otimes \mathbb{Q}$; therefore $L^\vee$ inherits a $\mathbb{Q}$-valued bilinear form, inducing a bilinear form $b_L : A_L \times A_L \to \mathbb{Q}/\mathbb{Z}$, called the *discriminant bilinear form* of L. If L is even, then the squares of elements in A_L are well-defined modulo $2\mathbb{Z}$, i.e. we have a finite quadratic form $q_L : A_L \to \mathbb{Q}/2\mathbb{Z}$, called the *discriminant (quadratic) form* of L. Note that the discriminant bilinear form can be recovered from the discriminant quadratic form via the identity $b_L(x, y) = \frac{1}{2}(q_L(x + y) - q_L(x) - q_L(y))$.

Definition 1.4. Let L be a non-degenerate even lattice. An *over-lattice* of L is an embedding of L into a non-degenerate even lattice M such that the quotient M/L is finite. In this situation, we have a chain of embeddings $L \subset M \subset M^\vee \subset L^\vee$. The quotient $S_M := M/L$ embeds into $A_L = L^\vee/L$ as an isotropic subgroup, i.e., we have $q_L|_{S_M} = 0$.

Proposition 1.5 (Nikulin [32, Proposition 1.4.1])**.** *The correspondence $M \mapsto S_M$ determines a bijection between over-lattices of L and isotropic subgroups of A_L. The discriminant form of an over lattice M is $A_M = S_M^\perp/S_M$.*

An isometry $f \in O(L)$ extends to an isometry of an over-lattice M if and only if the induced isometry $f_* \in O(q_L)$ maps $S_M \subset A_L$ to itself.

Definition 1.6. A *gluing* of two even, non-degenerate lattices L_1 and L_2 is an over-lattice L of $L_1 \oplus L_2$ in which L_1 and L_2 are primitive sublattices.

A gluing is determined by an isotropic subgroup $S \subset A_{L_1} \oplus A_{L_2}$ such that $S \cap A_{L_1} = S \cap A_{L_2} = 0$. Such a set S is the graph of an anti-isometry $\gamma : H_1 \to H_2$, where H_1 and H_2 are subgroups of A_{L_1} and A_{L_2}, respectively.

Proposition 1.7. *Let L be a gluing of two lattices L_1 and L_2, corresponding to an anti-isometry $\gamma : H_1 \to H_2$. Let f_1 be an isometry of L_1. Then there is an isometry f of L such that $f|_{L_1} = f_1$ if and only if there is an isometry f_2 of L_2 such that the induced maps $(f_i)_*$ leave the subgroups H_i invariant, and $\gamma \circ (f_1)_*|_{H_1} = (f_2)_*|_{H_2} \circ \gamma$.*

Definition 1.8. Let V be a real vector space endowed with a non-degenerate bilinear form. For any vector $v \in V$ with $v^2 \neq 0$, let $R_v \colon V \to V$ denote the reflection in the hyperplane orthogonal to v, that is, $R_v(x) = x - 2\frac{(x,v)}{(v,v)}v$.

More generally, if $U \subset V$ is a non-degenerate subspace, let $R_U : V \to V$ denote the "reflection in U", that is $R_U(u + w) = -u + w$ for $u \in U, w \in U^\perp$.

Lemma 1.9. *Let L be a non-degenerate even lattice, and let $v \in L$ be a primitive vector with $v^2 \neq 0$. The reflection R_v maps L to itself if and only if $v \in k\, L^\vee$, where $v^2 = -2k$. In this case, $[\frac{v}{k}]$ is an element of order k in A_L. In particular, if L is unimodular, then $v^2 \in \{\pm 2\}$, and if the discriminant group of L is 2-periodic, then $v^2 \in \{\pm 2, \pm 4\}$.*

Proof. Indeed,

$$R_v(x) = x + \tfrac{(x,v)}{k} v \in L \text{ for all } x \in L \Leftrightarrow (x,v) \in k\mathbb{Z} \text{ for all } x \in L \Leftrightarrow v \in kL^\vee. \qquad \square$$

Lemma 1.10. *Let L be a non-degenerate even lattice, and let $v \in L$ be a primitive vector such that R_v is an automorphism of L. Then R_v acts trivially on the discriminant group of L if and only if $v^2 \in \{\pm 2\}$.*

Proof. Let $v \in L$ be a primitive vector with $v^2 = -2k$ such that R_v maps L to itself. Then R_v acts trivially on the discriminant if and only if

$$R_v(a) - a = \tfrac{(a,v)}{k} v \in L \text{ for all } a \in L^\vee \Leftrightarrow (a,v) \in k\mathbb{Z} \text{ for all } a \in L^\vee \Leftrightarrow v \in kL.$$

Since v is primitive, this implies $k \in \{\pm 1\}$. $\square$

Lemma 1.11. *Let $f : L \to L$ be an automorphism, and let $v \in L$ be a primitive vector such that R_v defines an automorphism of L. Then $f \circ R_v = R_{f(v)} \circ f$.*

Lemma 1.12. *If L is a unimodular lattice and $S \subset L$ is a primitive non-degenerate sublattice with 2-periodic discriminant group, then R_S is an automorphism of L.*

1.2 Hyperbolic reflection groups and polytopes

In this section, we fix the notation and cite some results about hyperbolic polytopes and hyperbolic reflection groups. All the material covered in this section can be found in Vinberg's articles [40, 41], and we refer there for proofs and more details.

Let V be a finite-dimensional real vector space of dimension $n+1$, endowed with a bilinear form of signature $(1,n)$. Consider the open cone $C = \{x \in V \mid x^2 > 0\}$. It consists of two connected components; name one of them C_+ and the other one C_-. The *hyperbolic space obtained from V* is the quotient $\mathbb{H}_V = C_+/\mathbb{R}_{>0}$, where $\mathbb{R}_{>0}$ acts on C_+ by homotethy. The *isometry group* of $\mathbb{H}_V$ is the index two subgroup of $O(V)$ which maps the half-cone C_+ to itself.

A (linear) hyperplane $W \subset V$ is called *hyperbolic* if the bilinear form restricted to W has signature $(1, n-1)$. This is the case if and only if the orthogonal complement of W is a negative definite line. In this case, we view $H = \mathbb{H}_W$ as a (hyperbolic) hyperplane in $\mathbb{H}_V$.

For a vector $e \in V$ with $e^2 < 0$, let $H_e = \{[x] \in \mathbb{H}_V \mid (x,e) = 0\}$ denote the hyperplane orthogonal to e, and let $H_e^+ = \{[x] \in \mathbb{H}_V \mid (x,e) \geq 0\}$ and $H_e^- = \{[x] \in \mathbb{H}_V \mid (x,e) \leq 0\}$ denote the two half-spaces delimited by H_e.

Given two linearly independent vectors $e, f \in V$ with $e^2 = f^2 = -1$, there is the following trichotomy: If $|(e,f)| < 1$, then the hyperplanes H_e and H_f intersect at a dihedral angle of α, where $\cos\alpha = |(e,f)|$. If $|(e,f)| = 1$, then H_e and H_f *meet at infinity*, i.e. the corresponding linear hyperplanes in V intersect each other in an isotropic subspace of V. If $|(e,f)| > 1$, then H_e and H_f do not intersect, and the hyperbolic distance between them is ρ, where $\cosh \rho = |(e, f)|$.

A *convex polytope* in $\mathbb{H}_V$ is a subset $P \subset \mathbb{H}_V$ with non-empty interior which can be defined as $P = \bigcap_{i \in I} H_{e_i}^+$ where $\{e_i\}_{i \in I} \subset V$ is a set of vectors with negative square, such that each bounded subset of $\mathbb{H}_V$ intersects only finitely many of the hyperplanes $\{H_{e_i}\}_{i \in I}$. We may assume that such a presentation is non-redundant, in the sense that P is strictly contained in $\bigcap_{i \in J} H_{e_i}^+$ for every proper subset $J \subset I$. The collection of half-spaces in a minimal presentation is uniquely defined by the polytope P. Therefore, the vectors e_i are defined up to multiplication by a positive real number. They are uniquely defined once we fix a normalization condition, such as $e_i^2 = -1$ for all $i \in I$, or, if $V = L \otimes \mathbb{R}$ for some lattice L and P is defined over $L \otimes \mathbb{Q}$, that all the vectors e_i must be primitive vectors in L.

Let $P = \bigcap_{i \in I} H_{e_i}^+$ be an *acute-angled* hyperbolic polytope. This means that whenever two hyperplanes H_{e_i} and H_{e_j} intersect, they do so at a dihedral angle of at most $\pi/2$ (where we consider the angle lying inside the polytope). For a subset $J \subset I$, the intersection $\bigcap_{i \in J} H_{e_i}$ is non-empty if and only if the subspace of V spanned by $\{e_i\}_{i \in J}$ is negative definite. In this case, let the *face defined by* J be the set $F^J = P \cap \bigcap_{i \in J} H_{e_i}$. Acute-angled hyperbolic polytopes are *simple*, i.e., their polyhedral angles are simplicial (cf. Vinberg [40, p. 34]). This implies that the set of vectors J defining a face $F \subset P$ is uniquely determined by F; we have $J(F) = \{i \in I \mid F \subseteq H_{e_i}\}$. Moreover, the codimension of F^J in P is equal to the cardinality of J.

A polytope P is called a *Coxeter polytope* if whenever two hyperplanes H_{e_i} and H_{e_j} intersect, they do so at a dihedral angle of π/n for some $n \geq 2$. Coxeter polytopes can be represented graphically using Coxeter graphs. The *Coxeter graph* of a Coxeter polytope P is a decorated graph, defined as follows. There is a vertex for each facet of P. Two vertices are not connected if the hyperplanes H and H' containing the corresponding facets are orthogonal; they are connected by an n-fold edge if H and H' intersect each other at a dihedral angle of $\pi/(n+2)$, e.g. a simple edge (○—○) for an angle of $\pi/3$, a double edge (○═○) for an angle of $\pi/4$; they are connected by a heavy edge (○━○) if H and H' meet at infinity; and they are connected by a dashed edge (○- - -○) if H and H' neither intersect nor meet at infinity.

Let $P = \bigcap_{i \in I} H_{e_i}^+ \subset \mathbb{H}^n$ be a Coxeter polytope. The group $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ generated by the reflections $\{R_{e_i}\}_{i \in I}$ has a presentation of the form $\langle R_{e_i} \mid (R_{e_i} R_{e_j})^{n_{ij}} \rangle$, where H_{e_i} and H_{e_j} intersect at an angle of π/n_{ij}, and there is no relation between R_{e_i} and R_{e_j} if H_{e_i} and H_{e_j} do not intersect. The group Γ acts discretely on $\mathbb{H}_V$, and P is a fundamental domain for this action.

Theorem 1.13 (Vinberg [41])**.** *Let Γ be a discrete group of isometries of $\mathbb{H}^n$ generated by reflections in hyperplanes. Let $\mathscr{H}$ be the set of hyperplanes $H \subset \mathbb{H}^n$ for which the reflection in H belongs to Γ. Then Γ acts simply transitively on the set of chambers defined by $\mathscr{H}$, i.e. the connected components of the complement of $\mathbb{H}_V \setminus \bigcup_{H \in \mathscr{H}} H$.*

Corollary 1.14 (Vinberg [41])**.** *Let G be a discrete group of isometries of $\mathbb{H}^n$, and let Γ be a normal subgroup of G generated by reflections in hyperplanes. Let P be* *a fundamental polytope for* Γ*. Then* G *is a semi-direct product* $G = \Gamma \rtimes T$*, where* $T = \{g \in G \mid g(P) = P\}$.

Given a hyperbolic reflection group $\Gamma \subset \operatorname{Isom} \mathbb{H}^n$, the Coxeter graph of a fundamental polytope $P \subset \mathbb{H}^n$ for Γ can be computed using *Vinberg's algorithm* (see Vinberg [41]).

The reflection groups and polytopes we are interested in all arise from hyperbolic lattices. Given an even non-degenerate lattice L of signature $(1, n)$, one can extend the bilinear form defined on L to the vector space $V = L \otimes \mathbb{R}$, and consider its associated hyperbolic space $\mathbb{H}_V$. We consider reflections R_v, where $v \in L$ is a primitive vector such that either $v^2 = -2$, in which case we call v a *short root*, or $v^2 = -4$, in which case we call v a *long root*. In the Coxeter graph, we represent short roots by small circles ($\circ$) and long roots by large circles ($\bigcirc$).

CHAPTER 2

Nodal sextics, K3 surfaces and periods

In this chapter we review some facts about K3 surfaces and their periods, and in particular the bijection between rigid isotopy classes of real sextics and chambers of the corresponding real period space, up to the action of a discrete group. For details about K3 surfaces and their periods in general, we refer to the book [3] by W. Barth, K. Hulek, C. Peters and A. van der Ven. For details about "weakly polarized" K3 surfaces, see D. Morrison's article [30]. The case of K3 surfaces obtained from complex nodal sextics is also treated in D. Morrison and M. Saito's article [31]. We borrow most of the notation from A. Degtyarev [9]. The passage from the complex to the real case is done in analogy with similar situations treated by A. Degtyarev, I. Itenberg, V. Kharlamov and V. Nikulin [10, 18, 22, 32].

2.1 K3 surfaces obtained from nodal sextics

A *K3 surface* is a non-singular, compact complex surface X which is simply connected and has trivial canonical bundle. Endowed with the intersection pairing, its second cohomology group $H^2(X,\mathbb{Z})$ is an even unimodular lattice of signature $(3,19)$.

A *weak polarization* of a K3 surface X is an element $h \in H^2(X,\mathbb{Z})$ which is the class of a big and nef line bundle on X, or equivalently, a class for which $h^2 > 0$ and $h \cdot D \geq 0$ for all irreducible curves $D \subset X$. A *weakly polarized K3 surface* is a pair (X, h) formed by a K3 surface X and a weak polarization h of X. The number $h^2 \in \mathbb{N}$ is called the *degree* of h. The *nodal classes* of a weakly polarized K3 surface (X, h) are the elements $s \in H^2(X,\mathbb{Z})$ which are the classes of smooth rational curves on X for which $s \cdot h = 0$. They are always roots, i.e. they have self-intersection -2.

Let $C \subset \mathbb{CP}^2$ be a reduced (not necessarily irreducible) nodal sextic. Let $\mathrm{bl} : V \to \mathbb{CP}^2$ be the blow-up of $\mathbb{CP}^2$ in the nodes of C, and let $\tilde{C} \subset V$ be the strict transform of C. Let $\eta : X \to V$ be the double covering of V branched along $\tilde{C}$, and denote the composition $\mathrm{bl} \circ \eta$ by $\pi : X \to \mathbb{CP}^2$. Alternatively, π can be obtained by first taking the (singular) double covering $Y \to \mathbb{CP}^2$ branched along C and then taking the minimal resolution $X \to Y$, so we have the following commutative diagram:

$$\begin{tikzcd} & X \arrow[r] \arrow[d, "\eta"] \arrow[dr, "\pi"] & Y \arrow[d] & \\ \tilde{C} \subset & V \arrow[r, "\mathrm{bl}"] & \mathbb{CP}^2 & \supset C \end{tikzcd}$$

The surface X is a K3 surface. Let $h \in H^2(X, \mathbb{Z})$ be the pullback under π of the class of $\mathcal{O}_{\mathbb{P}^2}(1)$, which is Poincaré dual to the homology class of a line in $\mathbb{CP}^2$. It defines a weak polarization of degree 2 of X, and we call (X, h) the *weakly polarized K3 surface obtained from* C. Let $\sigma \subset H^2(X, \mathbb{Z})$ denote its set of nodal classes. The corresponding smooth rational curves are the fibres of π above the nodes of C. In particular, the nodal classes are pairwise orthogonal. Let $\Sigma \subset H^2(X, \mathbb{Z})$ denote the sublattice generated by σ, and let $S \subset H^2(X, \mathbb{Z})$ denote the sublattice generated by σ and h.

2.2 Real structures

Suppose that $C \subset \mathbb{CP}^2$ is a real sextic, i.e. a sextic invariant under complex conjugation, or equivalently, defined by a polynomial with real coefficients. Let (X, h) be the weakly polarized K3 surface obtained from C. As above, let $\pi : X \to \mathbb{CP}^2$ be the composition $\mathrm{bl} \circ \eta$, where $\mathrm{bl} : V \to \mathbb{CP}^2$ denotes the blow-up of $\mathbb{CP}^2$ in the nodes of C and $\eta : X \to V$ is the double covering of V branched along the strict transform of C. There are two real structures c_1, c_2 on X which are compatible with the standard real structure on $\mathbb{CP}^2$, i.e. such that $\pi \circ c_i = \mathrm{conj} \circ \pi$, where $\mathrm{conj} : \mathbb{CP}^2 \to \mathbb{CP}^2$, $[x_0 : x_1 : x_2] \mapsto [\bar{x}_0 : \bar{x}_1 : \bar{x}_2]$ denotes the complex conjugation. The involutions c_1 and c_2 commute, and their product is the deck transformation τ of the covering $\eta : X \to V$.

Let X^{c_1} and $X^{c_2} \subset X$ be the sets of fixed points of c_1 and c_2, respectively. We have $X^{c_i} = \pi^{-1}(B_i)$, where B_1 and B_2 are the two (not necessarily connected) regions delimited by $\mathbb{R}C$ inside $\mathbb{RP}^2$; more precisely, if $f \in \mathbb{R}[x_0, x_1, x_2]$ is a polynomial defining C, then we have $B_1 = \{x \in \mathbb{RP}^2 \mid f(x) \geq 0\}$ and $B_2 = \{x \in \mathbb{RP}^2 \mid f(x) \leq 0\}$ or vice versa.

It is not possible to distinguish one of the two real structures simultaneously for all irreducible real nodal sextics in a continuous way. This can be done, however, for two smaller classes of real nodal sextics which are of interest to us: real nodal sextics without real nodes and dividing real nodal sextics.

If C is an irreducible real nodal sextic *without real nodes*, then, like in the case of non-singular sextics, exactly one of the two regions B_1 and B_2 is non-orientable. The real structure c whose fixed points cover the non-orientable region of $\mathbb{RP}^2 \setminus \mathbb{R}C$ is distinguished by the following homological property (see Nikulin [32, p. 163]):

$$\text{There is an element } x \in H^2(X, \mathbb{Z}) \text{ with } c^*(x) = -x \text{ and } x \cdot h \equiv 1 \pmod{2}. \qquad (*)$$

In the special case where the real part of the curve C is empty, one real structure has no fixed points, and the fixed point set of the other real structure is a non-trivial unramified covering of $\mathbb{RP}^2$. In this case the real structure with property $(*)$ is the one without fixed points.

In the case of *dividing* irreducible real nodal sextics, there is a unique way of perturbing the real nodes so that the resulting sextic (which has only non-real nodes) is still dividing (cf. Section 4.1). We chose the real structure which leads to a real structure with property $(*)$ on the perturbed sextic.

2.3 Periods

2.3.1 Complex periods

The following proposition follows from Mayer's and Saint-Donat's results on linear systems on K3 surfaces [27, 36]. See also Degtyarev [9, Proposition 3.3.1], Morrison and Saito [31, Lemma 4.2] and Urabe [38].

Proposition 2.1. *Let (X, h) be a weakly polarized K3 surface with $h^2 = 2$, and let $\sigma \subset H^2(X, \mathbb{Z})$ be its set of nodal classes. The linear system $|h|$ determines a map $X \to \mathbb{CP}^2$ which is the minimal resolution of a double covering branched along a nodal sextic $C \subset \mathbb{CP}^2$ if and only if the following conditions hold.*

(1) *The nodal classes are pairwise orthogonal.*
(2) *If $s \in H^2(X, \mathbb{Z})$ is a root such that $2s \in \Sigma$, then $s \in \Sigma$.*
(3) *There is no root $s \in \sigma$ such that $s + h$ is divisible by 2 in $H^2(X, \mathbb{Z})$.*

We call a lattice L a *K3 lattice* if it is even, unimodular and of signature $(3, 19)$. All K3 lattices are isometric to each other.

Definition 2.2. Let L be a K3 lattice, $h \in L$ a vector with $h^2 = 2$, and $\sigma \subset L$ a set of roots orthogonal to h. Denote by Σ the sublattice of L generated by σ. The triple (L, h, σ) is called a *complex homological type* if it satisfies the following conditions.

(1) The elements of σ are pairwise orthogonal.
(2) If $s \in L$ is a root such that $2s \in \Sigma$, then $s \in \Sigma$.
(3) There is no root $s \in \sigma$ such that $s + h$ is divisible by 2 in L.

An *isomorphism* between two complex homological types (L, h, σ) and (L', h', σ') is an isometry $\psi : L \to L'$ which takes h to h' and σ to σ'.

Definition 2.3. Let (L, h, σ) be a complex homological type.

- An (L, h, σ)-*marking* of a K3 surface X is an isometry $\mu : L \to H^2(X, \mathbb{Z})$ such that $\mu(h)$ is a weak polarization of X with set of nodal classes of $\mu(\sigma)$.
- An (L, h, σ)-*marked K3 surface* is a pair (X, μ) formed by a K3 surface X and an (L, h, σ)-marking μ of X.
- A nodal reduced sextic $C \subset \mathbb{CP}^2$ is *of homological type* (L, h, σ) if the weakly polarized K3 surface obtained from C admits an (L, h, σ)-marking.

The Hodge decomposition of a K3 surface X is determined by the position of the complex line $H^{2,0}(X) \subset H^2(X; \mathbb{C})$. Every non-zero class $\omega \in H^{2,0}(X)$ is represented by a non-vanishing holomorphic 2-from on X. It satisfies $\omega^2 = 0$ and $\omega \cdot \bar{\omega} > 0$. Let $L_{\mathbb{C}}$ denote the complex vector space $L \otimes_{\mathbb{Z}} \mathbb{C}$. The bilinear form defined on L naturally extends to a $\mathbb{C}$-valued bilinear form on $L_{\mathbb{C}}$.

Definition 2.4. Let (L, h, σ) be a complex homological type. The *period space for*

(L,h,σ)*-marked K3 surfaces* is defined as $\Omega^0 = \Omega \setminus \bigcup_{v\in\Delta} H_v$, where

$$\Omega = \{\omega \in L_{\mathbb{C}} \mid \omega^2 = 0,\ \omega\cdot\bar{\omega} > 0,\ \omega \perp h,\ \omega \perp \sigma\}\big/\mathbb{C}^* \subset \mathbb{P}(L_{\mathbb{C}}),$$
$$H_v = \{[\omega] \in \Omega \mid \omega \cdot v = 0\} \quad \text{and} \quad \Delta = \{v \in L \mid v^2 = -2, v\cdot h = 0, v \notin \Sigma\}.$$

The *period* of an (L,h,σ)-marked K3 surface (X,μ) is defined as $[\mu_{\mathbb{C}}^{-1}(\omega)] \in \mathbb{P}(L_{\mathbb{C}})$ where ω is a non-zero class in $H^{2,0}(X)$ and $\mu_{\mathbb{C}}$ denotes the extension of μ to a map $L_{\mathbb{C}} \to H^2(X,\mathbb{C})$. It turns out that the period is contained in Ω^0. The *period map* is the map $\{(L,h,\sigma)\text{-marked K3 surfaces}\} \to \Omega^0$ assigning to each marked surface (X,μ) its period.

Remark 2.5. The period space Ω^0 is an open set (in the classical topology) in a quadric of dimension $19-n$, where n is the number of nodes. In particular, it is a non-compact complex manifold.

Definition 2.6. A *smooth family of complex manifolds* is a proper surjective submersion $f : \mathscr{X} \to B$ with connected fibres, where $\mathscr{X}$ and B are connected smooth manifolds, together with a complex structure defined on each fibre of f, varying smoothly along the base.

Definition 2.7. A *smooth family of* (L,h,σ)*-marked K3 surfaces* is a pair $(\mathscr{X},\mu)$, where $f : \mathscr{X} \to B$ is a smooth family of K3 surfaces and $\mu : \underline{L}_B \to R^2 f_*(\mathbb{Z})$ is an isomorphism of local systems such that $\mu_b : L \to H^2(X_b,\mathbb{Z})$ is an (L,h,σ)-marking of X_b for each $b \in B$.

The following theorem is a special case of a statement for weakly polarized marked K3 surfaces, due to Morrison [30]. It is based on the Global Torelli Theorem for K3 surfaces by Piatetski-Shapiro and Shafarevich [34] and on the surjectivity of the period map by Kulikov [26]. See also Beauville [4], Degtyarev [9, Theorem 3.4.1] and Morrison and Saito [31, Theorem 4.3].

Theorem 2.8. *The space* Ω^0 *is a fine moduli space for* (L,h,σ)*-marked K3 surfaces, that is, isomorphism classes of families of* (L,h,σ)*-marked K3 surfaces over* B *are naturally in bijection with smooth maps from* B *to* Ω^0*. In particular, there exists a universal family of* (L,h,σ)*-marked K3 surfaces over* Ω^0*.*

Remark 2.9. The equivalence between (L,h,σ)-marked K3 surfaces and nodal sextics extends to families of (L,h,σ)-marked K3 surfaces. More precisely, a family of (L,h,σ)-marked K3 surfaces $(\mathscr{X},\mu)$ determines a bundle of projective planes $\mathbb{P}(E) \to B$ and an equisingular family of nodal sextics $\mathscr{C} \subset \mathbb{P}(E)$ over B. Indeed, if $(\mathscr{X},\mu)$ is a family of (L,h,σ)-marked K3 surfaces, then $h_b = \mu_b^{-1}(h)$ is a class of Hodge type $(1,1)$ on the fibre X_b for each $b \in B$. By the variational Lefschetz theorem on $(1,1)$-classes, there is a line bundle $\mathscr{L}$ on $\mathscr{X}$ such that h_b is the class of the restriction $\mathscr{L}|_{X_b}$ for each $b \in B$. Since $\dim \Gamma(X_b, \mathscr{L}|_b) = 3$ for all $b \in B$, the direct image sheaf $\pi_*\mathscr{L}$ is locally free and defines a rank 3 vector bundle $E \to B$. The line bundle $\mathscr{L}$ defines a morphism

$\mathcal{X} \to \mathbb{P}(E)$ which is the minimal resolution of a double covering branched along an equisingular family of nodal sextics $\mathcal{C} \subset \mathbb{P}(E)$ over B.

2.3.2 Real periods

Consider a real weakly polarized K3 surface (X, h) obtained from a real nodal sextic, and let c be one of the two real structures on X lifting the complex conjugation. It induces an involutive isometry c^* on the lattice $H^2(X, \mathbb{Z})$. If a complex curve $D \subset X$ is invariant under c, then the restriction of c to D is orientation-reversing. Therefore, we have $c^*(h) = -h$ and $c^*(\sigma) = -\sigma$, where σ is the set of nodal classes of (X, h). If $s \in \sigma$ is the class of an exceptional curve over a real node, then we have $c^*(s) = -s$. If s' and s'' are the classes of the exceptional curves over a pair of non-real nodes, then we have $c^*(s') = -s''$ and $c^*(s'') = -s'$. Moreover, c^* is an anti-Hodge isometry on $H^2(X; \mathbb{C})$, i.e. it maps forms of type (p, q) to forms of type (q, p). Therefore, the map $x \mapsto c^*(\bar{x})$ defines a real structure on the complex line $H^{2,0}(X)$. Let $\omega \in H^{2,0}$ be a nonzero class such that $c^*(\bar{\omega}) = \omega$. If we write $\omega = \omega_+ + \mathrm{i}\,\omega_-$ with $\omega_\pm \in H^2(X, \mathbb{R})$, then the conditions $\omega^2 = 0$ and $\omega \cdot \bar{\omega} > 0$ imply that $\omega_+^2 = \omega_-^2 > 0$ and $c^*(\omega_\pm) = \pm\,\omega_\pm$. Therefore the three vectors h, ω_+ and ω_- are pairwise orthogonal and span a three-dimensional positive definite subspace of $H^2(X, \mathbb{R})$. In particular, the c^*-invariant sublattice of $H^2(X, \mathbb{Z})$ has one positive square.

Definition 2.10. Let (L, h, σ) be a complex homological type. An involutive isometry ϕ on L is called *geometric* if $\phi(h) = -h$, $\phi(\sigma) = -\sigma$ and the ϕ-invariant sublattice $L_+ = \{x \in L \mid \phi(x) = x\}$ has one positive square.

Definition 2.11. A *real homological type* is a quadruple (L, h, σ, ϕ), where (L, h, σ) is a complex homological type and ϕ is a geometric involution on (L, h, σ). An *isomorphism* between two real homological types (L, h, σ, ϕ) and (L', h', σ', ϕ') is an isomorphism ψ between the complex homological types (L, h, σ) and (L', h', σ') such that $\psi \circ \phi = \phi' \circ \psi$.

Definition 2.12. Let (L, h, σ, ϕ) be a real homological type. An (L, h, σ, ϕ)*-marking* of a real K3 surface (X, c) is an (L, h, σ)-marking of X such that $\mu \circ \phi = c^* \circ \mu$. An (L, h, σ, ϕ)*-marked K3 surface* is a triple (X, c, μ) where (X, c) is a real K3 surface and μ is an (L, h, σ, ϕ)-marking of (X, c). A real sextic C is *of homological type* (L, h, σ, ϕ) if the real weakly polarized K3 surface obtained from C admits a (L, h, σ, ϕ)-marking.

Definition 2.13. Let (L, h, σ, ϕ) be a real homological type. The *period space for* (L, h, σ, ϕ)*-marked real K3 surfaces* is defined as $\Omega_\phi^0 = \Omega_\phi \setminus \bigcup_{v \in \Delta} H_v$, where

$$\Omega_\phi = \{\omega \in L_\mathbb{C} \mid \omega^2 = 0,\ \omega \cdot \bar{\omega} > 0,\ \omega \cdot h = 0,\ \omega \perp \sigma,\ \phi_*(\omega) = \bar{\omega}\} \Big/ \mathbb{R}^*,$$
$$H_v = \{[\omega] \in \Omega_\phi \mid \omega \cdot v = 0\}, \quad \text{and} \quad \Delta = \{v \in L \mid v^2 = -2, v \cdot h = 0, v \notin \Sigma\}.$$

The *period* of an (L, h, σ, ϕ)-marked real K3 surface (X, c, μ) is defined as $[\mu_\mathbb{C}^{-1}(\omega)]$, where ω is a non-zero class in $H^{2,0}(X)$ such that $c^*\omega = \bar{\omega}$, and $\mu_\mathbb{C}$ denotes the extension of μ to a map $L_{\mathbb{C}} \to H^2(X, \mathbb{C})$. It turns out that the period is contained in Ω^0_ϕ. The *real period map* is the map $\{(L, h, \sigma, \phi)\text{-marked K3 surfaces}\} \to \Omega^0_\phi$ assigning to each real marked K3 surface (X, μ, c) its period.

If ϕ is a geometric involution on a complex homological type (L, h, σ), then $x \mapsto \phi(\bar{x})$ defines a real structure on the complex period space Ω^0, and the real period space $\Omega^0_\phi \subset \Omega^0$ is the set of real points for this real structure.

Definition 2.14. A *family of real (L, h, σ, ϕ)-marked K3 surfaces* is a triple $(\mathscr{X}, \mu, c)$ where $(\mathscr{X}, \mu)$ is a family of (L, h, σ)-marked K3 surfaces and c is a real structure on $\mathscr{X}$, such that $\mu_b : L \to H^2(X_b, \mathbb{Z})$ is an (L, h, σ, ϕ)-marking of (X_b, c_b) for each $b \in B$, where c_b denotes the restriction of c to X_b.

Lemma 2.15. *Let (L, h, σ, ϕ) be a real homological type, and let $(\mathscr{X}, \mu)$ be a family of (L, h, σ)-marked K3 surfaces whose periods are contained in $\Omega^0_\phi \subset \Omega^0$. There exists a unique real structure c on $\mathscr{X}$ such that $(\mathscr{X}, \mu, c)$ is a family of (L, h, σ, ϕ)-marked real K3 surfaces.*

Proof. Consider the family $\bar{\mathscr{X}}$ obtained from $\mathscr{X}$ by inverting the complex structure on each fibre. Then $\mu \circ \phi$ defines an (L, h, σ)-marking of $\bar{\mathscr{X}}$, and the periods of the marked families $(\mathscr{X}, \mu)$ and $(\bar{\mathscr{X}}, \mu \circ \phi)$ coincide. Since Ω^0 is a fine moduli space for (L, h, σ)-marked sextics, this implies that there is a unique isomorphism $c : \mathscr{X} \to \bar{\mathscr{X}}$ such that $\mu \circ \phi = c^* \circ \mu$. In other words, c defines a real structure on $\mathscr{X}$ such that $(\mathscr{X}, \mu, c)$ is a family of (L, h, σ, ϕ)-marked real K3 surfaces. □

Corollary 2.16. *The space Ω^0_ϕ is a fine moduli space for (L, h, σ, ϕ)-marked real K3 surfaces, that is, isomorphism classes of families of (L, h, σ, ϕ)-marked K3 surfaces over B are naturally in bijection with smooth maps from B to Ω^0_ϕ. In particular, there exists a universal family of (L, h, σ, ϕ)-marked K3 surfaces over Ω^0_ϕ.*

Proposition 2.17. *The period map induces a one-to-one correspondence between rigid isotopy classes of real nodal sextics of homological type (L, h, σ, ϕ) and connected components of Ω^0_ϕ modulo the action of the automorphism group Γ of the real homological type (L, h, σ, ϕ).*

Proof. Consider the space $\mathbb{RP}\left(H^0(\mathbb{CP}^2, \mathscr{O}_{\mathbb{CP}^2}(6))^*\right) \cong \mathbb{RP}^{27}$ parameterizing real plane sextics, and let $W \subset \mathbb{RP}^{27}$ be the subset parameterizing real irreducible nodal sextics $C \subset \mathbb{CP}^2$ of homological type (L, h, σ, ϕ). Rigid isotopy classes of such sextics are connected components of W. Let U be the set of pairs (C, μ) where $C \in W$ and μ is an (L, h, σ, ϕ)-marking of the weakly polarized real K3 surface obtained from C, and let $p_1 : U \to W$ be the map forgetting the marking, i.e. $p_1(C, \mu) = C$.

Consider the universal curve $\mathscr{C}_W = \{(C, x) \in W \times \mathbb{CP}^2 \mid x \in C\}$ over W, and let $\mathscr{X}_W$ be the family of K3 surfaces obtained from $\mathscr{C}_W \subset W \times \mathbb{CP}^2$. The projection $\mathscr{X}_W \to W$ is topologically a locally trivial fibration. Therefore, once we fix an open contractible subset $N \subset W$, we have natural isomorphisms between $H^2(X_C, \mathbb{Z})$ and $H^2(X_{C'}, \mathbb{Z})$ for

$C, C' \in N$, i.e. a marking can be extended from a sextic to nearby sextics. This defines a topology on U such that the projection $p_1 : U \to W$ is a regular unramified covering with deck transformation group Γ. Therefore, p_1 induces a bijection between $\pi_0(U)/\Gamma$ and $\pi_0(W)$.

Let $p_2 : U \to \Omega_\phi^0$ denote the period map. The group $\mathrm{PGL}_3(\mathbb{R})$ naturally acts on U, and the period map is invariant with respect to this action. We claim that $p_2 : U \to \Omega_\phi^0$ is in fact a $\mathrm{PGL}_3(\mathbb{R})$-principal bundle. Assuming this claim, it follows that p_2 is a locally trivial fibration with connected fibres. In particular, it induces a bijection between the connected components of U and those of Ω_ϕ^0. Moreover, since p_2 is equivariant under the action of Γ, it gives rise to a bijection between the quotients $\pi_0(U)/\Gamma$ and $\pi_0(\Omega_\phi^0)/\Gamma$, which proves the proposition.

To prove the claim, consider the universal family $(\mathscr{X}, \mu, c)$ of real (L, h, σ, ϕ)-marked K3 surfaces over the real period space Ω_ϕ^0. Let $\mathscr{L} \to \mathscr{X}$ be the line bundle defined by the polarization and let E be the rank 3 vector bundle over Ω_ϕ^0 defined by $\pi_*\mathscr{L}$ (cf. Remark 2.9). The real structure c induces a real structure on E; let $\mathbb{R}E$ denote the real part of E, which is a real vector bundle. The elements $(C, \mu) \in p_2^{-1}([\omega])$ naturally correspond to isomorphisms between $\mathbb{P}(\mathbb{R}E_{[\omega]})$ and a fixed real projective plane. Therefore we can identify U with the projective frame bundle of $\mathbb{R}E$, which is indeed a $\mathrm{PGL}_3(\mathbb{R})$-principal bundle. $\square$

Let us reformulate Proposition 2.17 more conveniently. Fix a homological type (L, h, σ, ϕ), and define lattices K_+, K_- as follows.

$$K_+ = \{x \in L \mid x \perp h, \sigma \text{ and } \phi(x) = +x\},$$
$$K_- = \{x \in L \mid x \perp h, \sigma \text{ and } \phi(x) = -x\}$$

Both K_+ and K_- are hyperbolic lattices, i.e. they have one positive square. The group Γ of automorphisms of the real homological type (L, h, σ, ϕ) acts on the hyperboloids $\{x_+ \in K_+ \otimes \mathbb{R} \mid x_+^2 = 1\}$ and $\{x_- \in K_- \otimes \mathbb{R} \mid x_-^2 = 1\}$; let $G \subset \Gamma$ be the subgroup leaving the sheets of these hyperboloids invariant. We have $\Gamma = \{1, -\phi\} \times \{1, -R_S\} \times G$. Let $\mathbb{H}_+$ and $\mathbb{H}_-$ be the hyperbolic spaces obtained from $K_+ \otimes \mathbb{R}$ and $K_- \otimes \mathbb{R}$, respectively. Let

$$\Delta_+ = \{u \in K_+ \mid u^2 < 0 \text{ and } \exists w \in \Sigma \text{ such that } v = \tfrac{1}{2}(u + w) \in L \text{ and } v^2 = -2\},$$
$$\Delta_- = \{u \in K_- \mid u^2 < 0 \text{ and } \exists w \in \Sigma \text{ such that } v = \tfrac{1}{2}(u + w) \in L \text{ and } v^2 = -2\}.$$

The sets of hyperplanes $\{H_u\}_{u \in \Delta_+}$ and $\{H_u\}_{u \in \Delta_-}$ define tilings of the spaces $\mathbb{H}_+$ and $\mathbb{H}_-$ respectively. We call the connected components of $\mathbb{H}_+ \setminus \bigcup_{u \in \Delta_+} H_u$ *positive tiles* and the connected components of $\mathbb{H}_- \setminus \bigcup_{u \in \Delta_-} H_u$ *negative tiles*. Let $\mathscr{T}_+$ and $\mathscr{T}_-$ denote the sets of positive and negative tiles, respectively. Since the sets Δ_+ and Δ_- are stable under the action of G, the group G acts on the set $\mathscr{T}_+ \times \mathscr{T}_-$.

Theorem 2.18. *Rigid isotopy classes of sextics of homological type (L, h, σ, ϕ) are in bijection with pairs of tiles, i.e. elements of $\mathscr{T}_+ \times \mathscr{T}_-$, modulo the action of G.*

Proof. Consider the map $\Omega_\phi \to \mathbb{H}_+ \times \mathbb{H}_-$ sending an element $[\omega] \in \Omega_\phi$ to the pair $([\omega_+], [\omega_-]) \in \mathbb{H}_+ \times \mathbb{H}_-$, where $\omega = \omega_+ + \mathrm{i}\omega_-$. This map is a trivial two-sheeted covering with deck transformations $\{1, -\phi\}$. Since Δ is stable under ϕ, the union of the walls $\bigcup_{v\in\Delta} H_v$ is preserved by ϕ. Therefore, the connected components of Ω_ϕ^0 modulo Γ are in bijection with the connected components of $(\mathbb{H}_+ \times \mathbb{H}_-) \setminus \bigcup_{v\in\Delta} H_v$ modulo G, where we view H_v as a subspace of $\mathbb{H}_+ \times \mathbb{H}_-$. A subspace $H_v \subset \mathbb{H}_+ \times \mathbb{H}_-$ can be empty, of codimension one or of codimension two in $\mathbb{H}_+ \times \mathbb{H}_-$. Since we are interested in the connected components of their complement, we only need to consider the spaces H_v which are of codimension one. We can write a vector $v \in \Delta$ as $v = \frac{1}{2}(v_\Sigma + v_+ + v_-)$, where $v_\Sigma \in \tilde{\Sigma}$, $v_+ \in K_+$ and $v_- \in K_-$. Then we have $H_v = H_{v_+} \times H_{v_-}$, where $H_{v_\pm} = \{[x] \in \mathbb{H}_\pm \mid x \cdot v_\pm = 0\}$. The subspace $H_{v_\pm} \subset \mathbb{H}_\pm$ is equal to $\mathbb{H}_\pm$ if $v_\pm = 0$, it is a real hyperplane in $\mathbb{H}_\pm$ if $v_\pm^2 < 0$, and it is empty otherwise. Therefore, H_v is of codimension one in $\mathbb{H}_+ \times \mathbb{H}_-$ if and only if one of v_+ and v_- is equal to zero and the other one has negative square, i.e. we have $H_v = H_u \times \mathbb{H}_-$ with $u \in \Delta_+$, or $H_v = \mathbb{H}_+ \times H_u$ with $u \in \Delta_-$. □

2.4 Irreducible components

Our goal in this section is to determine the decomposition of C into irreducible components using the homological data of the surface X obtained from C. We do not need $C \subset \mathbb{CP}^2$ to be nodal or of degree six; let $C \subset \mathbb{CP}^2$ be a reduced plane curve of degree $2k$ with simple singularities. We generalize the notations of Section 2.1 to this situation. Let $\mathrm{bl} : V \to \mathbb{CP}^2$ be the minimal composition of blow-ups such that the components of odd multiplicity of the total transform of C are non-singular and pairwise disjoint. Let η, π, h, σ, Σ, S and τ be defined as in Section 2.1. Let $\tilde{S}$ denote the primitive closure of S in $H^2(X, \mathbb{Z})$. We would like to extract the decomposition of C into irreducible components from the quotient $\tilde{S}/S$. An important special case is covered by the following result of Morrison and Saito:

Proposition 2.19 (Morrison and Saito [31, Lemma 4.1])**.** *Let $p \subset \mathbb{CP}^2$ be a nodal sextic. Then $S \subset H^2(X, \mathbb{Z})$ is primitive if and only if C is irreducible.*

A more general result was obtained by Degtyarev:

Proposition 2.20 (Degtyarev [9, Theorem 4.3.1])**.** *Let $C \subset \mathbb{CP}^2$ be a reduced curve of degree $4m+2$ with simple singularities. Then C is irreducible if and only if $\tilde{S}/S$ contains no elements of order 2.*

We use a combination of ideas of the proofs of Morrison and Saito [31] and Degtyarev [9]. We start with the following two lemmas:

Lemma 2.21. *For every $\alpha \in \mathrm{NS}(X)$, we have $\alpha + \tau\alpha \in S$.*

Proof. We have $\alpha + \tau\alpha \equiv \eta^*(\eta_*\alpha) \pmod{\Sigma}$, and the image of η^* is contained in

$S \subset H^2(X, \mathbb{Z})$. □

Lemma 2.22. *For $\alpha \in \tilde{S}$, the following are equivalent:*

(i) $2\alpha \in S$,

(ii) $\tau\alpha - \alpha \in S$,

(iii) *α is congruent modulo S to a τ-invariant element of $\tilde{S}$,*

(iv) *α is congruent modulo S to a sum of components of the ramification curve.*

Proof. (i) ⇒ (ii): By Lemma 2.21, we have $\alpha + \tau\alpha \in S$. Subtracting 2α, it follows that $\tau\alpha - \alpha \in S$.
(ii) ⇒ (iii): We have $\tau\alpha - \alpha = s \in S_\tau$, the τ-anti-invariant part of S. Since the action of τ on Σ is induced by a permutation of σ, there is an element $t \in S$ such that $s = t - \tau t$. Then $\tau(\alpha + t) - (\alpha + t) = (\tau\alpha - \alpha) - (t - \tau t) = 0$.
(iii) ⇒ (iv): We may suppose α itself is τ-invariant. Adding a large enough multiple of h to α, we may suppose α is effective. The involution τ acts on the linear system $|\alpha|$ as a complex-linear involution, and such an involution always has fixed points. Therefore, there is an effective divisor $A \in |\alpha|$ which is invariant under τ. For an irreducible component B of the divisor A, there are four possibilities:

(a) B is contracted by π,

(b) B is a component of the ramification curve,

(c) B is a degree two cover of $\pi(B)$, or

(d) B maps birationally to $\pi(B)$, and $\tau(B) \neq B$.

In case (c), note that $\pi_*(B) = 2D$ for some curve $D \subset \mathbb{P}^2$, and $\pi^*(D) - B$ is a sum of exceptional curves. Therefore the divisor class of B is contained in S. In case (d), note that since A is invariant under τ, such curves occur only in pairs interchanged by τ and we can apply the argument of (c) to every pair $B + \tau(B)$. Thus α is congruent to a sum of components of the ramification curve modulo S.
(iv) ⇒ (i): It suffices to show that $2r \in S$ where r is the class of an irreducible component of the ramification curve. We have $2r = r + \tau r$, and hence $2r \in S$ by Lemma 2.21. □

For every singular point $p \in B$, denote by Σ_p the subgroup of Σ generated by the exceptional curves over p. For every local branch b of C at p, one can define the *associated divisor* $e_b \in \Sigma_p$ by $e_b = \pi^*(b) - 2\bar{b}$, where $\bar{b}$ is the strict transform of $\mathrm{bl}^{-1}(b)$. The following lemma was obtained by Degtyarev [9], using formulas by Yang [46].

Lemma 2.23 (Degtyarev [9, Lemma 4.2.2])**.** *For a collection of local branches at a simple plane curve singularity, the sum of the associated divisors is divisible by two if and only if the collection is either empty or contains all the branches.*

The following proposition is a consequence of this lemma.

Proposition 2.24. *If $C = C_1 + \cdots + C_n$ is the decomposition of C into irreducible components, then the 2-periodic part of $\tilde{S}/S$ is generated by elements $[C_i]$, and the only relation among them is $\sum_{i=1}^n [C_i] = 0 \in \tilde{S}/S$.*

Proof. For each component C_i, denote by $R_i \subset X$ the corresponding component of the ramification divisor. Let r_i be the class of R_i in $\mathrm{NS}(X)$. By Lemma 2.22, the elements $[r_i]$ are of order 2 in $\tilde{S}/S$ and they generate the 2-periodic part of $\tilde{S}/S$. Thus it suffices to show the following: if T is a curve which consists of some, but not all of the components of the ramification curve R, then the class of T is not contained in S.

We have $2r_i = \deg(C_i)\, h + \sum_b e_b$, where the sum ranges over all the branches of C_i at the singular points of C (see [9, Equation 4.2.1]). By Lemma 2.23, the divisor class of T lies in S if and only if the degree of $\pi_*(T) \subset \mathbb{P}^2$ is even and for every singular point p of C, $\pi_*(T)$ contains either all or none of the local branches of C at p. Clearly, this happens only if T is either empty or equal to R. □

Corollary 2.25. *If C is nodal, then the elements of $\tilde{S}/S$ are in bijection with partitions of the irreducible components of C into two sets. An element $[x] \in \tilde{S}/S$ corresponding to a decomposition $C = C_1 + C_2$, is given by*

$$x \equiv \frac{1}{2}\Big(\sum s + \deg(C_1)\, h\Big) \mod S,$$

where the sum ranges over the classes $s \in \sigma$ corresponding to nodes $p \in C_1 \cap C_2$.

2.5 Dividing type

Recall that a real non-singular curve C is called *dividing* or *of dividing type* I[1] if $C \setminus \mathbb{R}C$ consists of two connected components, and *of dividing type* II if $C \setminus \mathbb{R}C$ is connected. We extend this notion to irreducible nodal curves by declaring that an irreducible nodal curve C is dividing if its normalization $\tilde{C}$ is dividing with respect to the inherited real structure. If C is a non-singular dividing curve, then the two connected components of $C \setminus \mathbb{R}C$ are called *halves* of the curve. Note that if the real part of C is non-empty, then C is dividing if and only if the class of $\mathbb{R}\tilde{C}$ is trivial in $H_1(\tilde{C}, \mathbb{Z}/2\mathbb{Z})$.

Recall that an element α of a non-degenerate lattice L is called *characteristic* if we have $x^2 \equiv \alpha \cdot x \pmod 2$ for all elements $x \in L$. Characteristic elements always exist, and they are uniquely defined up to $2L$, i.e. the residue $\bar{\alpha} \in L/2L$ does not depend on the choice of α. A lattice is even if and only if zero is a characteristic element. Given an involutive isometry ϕ on a lattice L, we may consider the *twisted bilinear form* given by $(x, y) \mapsto x \cdot \phi(y)$. We say that an element α is *characteristic for the involution ϕ* if it is characteristic for the corresponding twisted bilinear form.

Lemma 2.26 (Arnol'd [1])**.** *Let (X, c) be a real surface such that $H^2(X, \mathbb{Z})$ has no torsion. Let $\alpha \in H^2(X, \mathbb{Z})$ be a characteristic element of the twisted bilinear form $(x, y) \mapsto x \cdot c^*(y)$, and let $\bar{\alpha}$ be the mod 2 reduction of α. Then $\bar{\alpha}$ is Poincaré dual to the mod 2 fundamental class of the set of real points.*

Let (X, h) be the weakly polarized K3 surface obtained from a real irreducible

[1] In the literature this is often simply called the *type* of a curve, but this might lead to confusion here since we also use other notions of type, such as the homological type of a curve.

nodal sextic $C \subset \mathbb{CP}^2$, and let c_1 and c_2 be the real structures on X which lift the complex conjugation of $\mathbb{CP}^2$. Let α_1 and $\alpha_2 \in H^2(X,\mathbb{Z})$ be characteristic elements for the induced involutions c_1^* and c_2^* on $H^2(X,\mathbb{Z})$, and let $\bar\alpha_1$ and $\bar\alpha_2$ denote their residues in $H^2(X;\mathbb{Z}/2\mathbb{Z})$.

By Proposition 2.19, the sublattice $S \subset H^2(X,\mathbb{Z})$ is primitive, so that we have a natural embedding $S/2S \hookrightarrow H^2(X;\mathbb{Z}/2\mathbb{Z})$. Recall that V is the blow-up of $\mathbb{CP}^2$ in the nodes of C and that $\eta : X \to V$ is the double covering branched along $\tilde C$, with deck transformation τ. The *transfer map* $\operatorname{tr} : H_2(V,\tilde C;\mathbb{Z}/2\mathbb{Z}) \to H_2(X;\mathbb{Z}/2\mathbb{Z})$ is obtained by mapping a relative 2-cell in V to the sum of its two preimages in X.

Lemma 2.27. *The transfer map* $\operatorname{tr} : H_2(V,\tilde C;\mathbb{Z}/2\mathbb{Z}) \to H_2(X;\mathbb{Z}/2\mathbb{Z})$ *is injective.*

Proof. In this proof, we assume $\mathbb{Z}/2\mathbb{Z}$ coefficients for all homology and cohomology groups. Consider the Smith exact homology sequence for the pair (X,τ) (cf. for example Degtyarev, Itenberg, and Kharlamov [10, p. 3]).

$$\cdots \to H_3(X) \xrightarrow{\eta_*} H_3(V,\tilde C) \xrightarrow{\Delta} H_2(V,\tilde C)\oplus H_2(\tilde C) \xrightarrow{\operatorname{tr}+i_*} H_2(X) \to \cdots$$

Here, i_* is the homomorphism induced by the inclusion $\tilde C \hookrightarrow X$. The second component of the connecting homomorphism Δ is the boundary map $\partial : H_3(V,\tilde C) \to H_2(\tilde C)$ which is injective because its kernel coincides with the image of $H_3(V) = 0$. To show that the transfer map is injective, suppose $x \in H_2(V,\tilde C)$ lies in its kernel. Then $(x,0)$ lies in the kernel of $\operatorname{tr}+i_*$, which coincides with the image of Δ. Because Δ is injective on the second component, this implies $x = 0$. ☐

Proposition 2.28. *The elements* $\bar\alpha_1, \bar\alpha_2 \subset H^2(X;\mathbb{Z}/2\mathbb{Z})$ *are contained in the image of* $S/2S$ *if and only if* $[\mathbb{R}\tilde C]$ *is trivial in* $H_1(\tilde C;\mathbb{Z}/2\mathbb{Z})$.

Proof. Again, we assume $\mathbb{Z}/2\mathbb{Z}$ coefficients for all homology and cohomology groups. Consider the following diagram, where the bottom row is part of the long exact homology sequence for the pair $(V,\tilde C)$:

$$\begin{tikzcd} & & H_2(X) & & \\ \cdots \arrow[r] & H_2(V) \arrow[r, "\rho"] & H_2(V,\tilde C) \arrow[u, "\operatorname{tr}"] \arrow[r, "\partial"] & H_1(\tilde C) \arrow[r] & \cdots \end{tikzcd}$$

Note that the composition $\operatorname{tr}\circ\rho$ corresponds to $\eta^* : H^2(V) \to H^2(X)$ under the identification of homology and cohomology via Poincaré duality. A basis for $H_2(V)$ is given by the strict transform of a line in $\mathbb{CP}^2$ and the exceptional classes of the blow-up. Therefore, the image of η^* is generated by the mod 2 residues of h and σ, i.e., it is $S/2S \subset H^2(X)$. Therefore, $\bar\alpha_i$ is contained in $S/2S$ if and only if $[X^{c_1}]$ is contained in $\operatorname{im}(\operatorname{tr}\circ\rho)$. Let B_1 and B_2 be the (not necessarily connected) regions delimited by $\mathbb{R}\tilde C$ in $\mathbb{RP}^2$. Recall that the fixed point sets X^{c_i} are double coverings of the regions B_i under η, and hence we have $[X^{c_i}] = \operatorname{tr}([B_i])$. Suppose that $[\mathbb{R}\tilde C]$ is trivial in $H_1(\tilde C)$. Since $\partial[B_i] = [\mathbb{R}\tilde C] = 0$ in $H_1(\tilde C)$, the classes $[B_i]$ lie in the image of ρ. Therefore,

$[X^{c_i}] = \mathrm{tr}([B_i])$ is contained in $\mathrm{im}(\mathrm{tr}\circ\rho)$ and $\bar{\alpha}_i$ is contained in $S/2S$. Now suppose that $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are contained in $S/2S$. Then the classes $[X^{c_i}]$ lie in the image of $\mathrm{tr}\circ\rho$. Because the transfer map is injective by Lemma 2.27, this implies that the classes $[B_i]$ lie in the image of ρ, and therefore $\partial[B_i] = [\mathbb{R}\tilde{C}] = 0$ in $H_1(\tilde{C})$. □

Remark 2.29. If C is dividing, then $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are in fact contained in the c_i^*-invariant part of $S/2S$. If C has only non-real nodes, with classes $s_1', s_1'', \ldots, s_m', s_m''$, then the c_i^*-invariant part of $S/2S$ is generated by the mod 2 residues of $h, s_1' + s_1'', \ldots, s_m' + s_m''$.

Lemma 2.30. *If C is an irreducible sextic with only non-real nodes, then $\bar{\alpha}_1 + \bar{\alpha}_2 = \bar{h}$.*

Proof. Note that $\mathbb{R}V \subset V$ is homologous to the preimage of a general line in $\mathbb{CP}^2$. Indeed, $\mathbb{R}V$ intersects the preimage of a non-real line in one point and does not intersect any of the exceptional curves since all the nodes are non-real. Therefore, $\eta^*([\mathbb{R}V]^{\mathrm{PD}}) = \bar{h}$, where x^{PD} denotes the class Poincaré dual to x, and $\bar{h} \in H^2(X)$ denotes the mod 2 residue of h. As noted in the proof of Proposition 2.28, the map η^* corresponds to $\mathrm{tr}\circ\rho$ under Poincaré duality. Hence, we have

$$\begin{aligned}\bar{h} &= \eta^*\left([\mathbb{R}V]^{\mathrm{PD}}\right) = (\mathrm{tr}\circ\rho\,[\mathbb{R}V])^{\mathrm{PD}} = (\mathrm{tr}([B_1]+[B_2]))^{\mathrm{PD}}\\ &= \mathrm{tr}([B_1])^{\mathrm{PD}} + \mathrm{tr}([B_2])^{\mathrm{PD}} = [X^{c_1}]^{\mathrm{PD}} + [X^{c_2}]^{\mathrm{PD}} = \bar{\alpha}_1 + \bar{\alpha}_2.\end{aligned}$$ □

Corollary 2.31. *If C is a dividing real irreducible sextic with only non-real nodes, then exactly one of the classes $\bar{\alpha}_1$ and $\bar{\alpha}_2$ is contained in $\Sigma/2\Sigma$.*

Lemma 2.32. *Let C be a dividing real irreducible sextic with only non-real nodes. The real structure with $\bar{\alpha} \in \Sigma/2\Sigma$ is the one with property $(*)$ (see Section 2.2).*

Proof. Suppose on the contrary that the real structure c with property $(*)$ has $\bar{\alpha} \notin \Sigma/2\Sigma$. Then we can write $\bar{\alpha} = \bar{h} + y$ for some $y \in \Sigma/2\Sigma$. By property $(*)$, there is an element $x \in H^2(X,\mathbb{Z})$ such that $c^*(x) = -x$ and $x \cdot h \equiv 1 \pmod 2$. By Remark 2.29, the element y is a sum of elements of the form $s_i' + s_i''$. We have $x \cdot s_i' = c^*(x)\cdot c^*(s_i') = x \cdot s_i''$, and hence $x \cdot (s_i' + s_i'') \equiv 0 \pmod 2$. This implies $0 \equiv -x^2 = x \cdot c^*(x) \equiv x \cdot \alpha \equiv x \cdot h \equiv 1$ (mod 2), a contradiction. □

Remark 2.33. Lemma 2.32 also holds for curves with empty real part. The real structure with property $(*)$ is the one without real points, and therefore $\bar{\alpha} = 0$.

2.6 Perturbation of nodes, crossing and non-crossing pairs of nodes

In this section we study how the homological data of a real nodal sextic changes (or rather, does not change) under the perturbation of a real node or a pair of non-real nodes.

We call a non-real node of a dividing curve *crossing* if the two branches intersecting at the node belong to different halves of the normalization of the curve. A non-real node is crossing if and only if this is the case for its complex conjugate node, so that we can speak about *crossing* and *non-crossing pairs* of non-real nodes.

As a corollary of the observations about the perturbation of the nodes, we show how crossing pairs can be distinguished from non-crossing pairs using only homological data.

2.6.1 The complex case

Before treating the real case, let us consider the perturbation of a node on a complex surface. Let X_0 be a compact complex surface whose only singularity is an ordinary double point at $p \in X_0$. Let $\mathrm{bl}_p : Y_0 \to X_0$ be the blow-up of X_0 at p. Then the induced homomorphism $\mathrm{bl}_p^* : H^2(X_0, \mathbb{Z}) \to H^2(Y_0, \mathbb{Z})$ is injective and its image is the orthogonal complement of $s \in H^2(Y_0, \mathbb{Z})$, the class of the exceptional curve of the blow-up. The self-intersection of s is -2.

Let $f : \mathcal{X} \to \mathbb{D}$ be a Lefschetz deformation of X_0 over the unit disk $\mathbb{D} \subset \mathbb{C}$. By this we mean that $\mathcal{X}$ is a 3-dimensional complex manifold, $f^{-1}(0) = X_0$, and f is a proper surjective holomorphic map whose differential is non-zero everywhere except at $p \in X_0 \subset \mathcal{X}$.

The space $\mathcal{X}$ retracts by deformation to the special fibre X_0. Indeed, a deformation retraction $r : \mathcal{X} \to X_0$ can be obtained by choosing a metric on $\mathcal{X}$ and using the gradient flow of the function $|f|^2$.

Let $X_\varepsilon = f^{-1}(\varepsilon)$ be a general fibre of f, and let $i \colon X_\varepsilon \hookrightarrow \mathcal{X}$ denote its inclusion. In the following proposition we collect some facts about Lefschetz deformations. A proof of these facts can be found for instance in chapter 3 of Voisin's book [44].

Proposition 2.34.

(1) *The kernel of $i_* : H_2(X_\varepsilon; \mathbb{Z}) \to H_2(\mathcal{X}; \mathbb{Z})$ is generated by the class of a two-dimensional sphere in X_ε, called a* vanishing cycle. *Let $\delta \in H^2(X_\varepsilon, \mathbb{Z})$ be the class Poincaré dual to a vanishing cycle. It is well-defined up to sign and has self-intersection number -2.*

(2) *The monodromy action on $H^2(X_\varepsilon, \mathbb{Z})$ is given by the Picard-Lefschetz map:*

$$\begin{aligned} R_\delta : H^2(X_\varepsilon, \mathbb{Z}) &\to H^2(X_\varepsilon, \mathbb{Z}) \\ x &\mapsto x + (x, \delta) \cdot \delta \end{aligned}$$

(3) *The map $i^* : H^2(\mathcal{X}, \mathbb{Z}) \to H^2(X_\varepsilon, \mathbb{Z})$ is injective and its image consists of the classes invariant under R_δ, i.e., the orthogonal complement of δ.*

Proposition 2.35. *There are two isometries $\psi : H^2(Y_0, \mathbb{Z}) \to H^2(X_\varepsilon, \mathbb{Z})$ which make the following diagram commute. One maps s to δ, the other one to $-\delta$.*

$$\begin{array}{ccc} H^2(Y_0, \mathbb{Z}) & \xrightarrow{\psi} & H^2(X_\varepsilon, \mathbb{Z}) \\ {\scriptstyle \mathrm{bl}_p^*}\uparrow & & \uparrow{\scriptstyle i^*} \\ H^2(X_0, \mathbb{Z}) & \xrightarrow{r^*} & H^2(\mathcal{X}, \mathbb{Z}) \end{array}$$

Proof. The composition $i^* \circ r^* \circ (\mathrm{bl}_p^*)^{-1}$ defines an isomorphism between the orthogonal complements $\langle s \rangle^\perp \subset H^2(Y_0, \mathbb{Z})$ and $\langle \delta \rangle^\perp \subset H^2(X_\varepsilon, \mathbb{Z})$. It can be extended to an isomorphism $\psi : H^2(Y_0, \mathbb{Z}) \to H^2(X_\varepsilon, \mathbb{Z})$ by gluing it to an isomorphism between $\langle s \rangle$ and $\langle \delta \rangle$. Such an isomorphism must map s either to δ or to $-\delta$. $\square$

Remark 2.36. This observation can also be obtained by explicitly constructing the two possible resolutions of the deformation $f : \mathscr{X} \to \mathbb{D}$, cf. Atiyah [2].

Remark 2.37. If we perturb several nodes simultaneously, the situation is very similar. Let X_0 be a surface with n nodes at $p_1, \ldots, p_n$, let $s_1, \ldots, s_n \in H^2(Y_0, \mathbb{Z})$ be the exceptional classes of the blow-up, and let $\delta_1, \ldots, \delta_n \in H^2(X_\varepsilon, \mathbb{Z})$ denote the classes of the corresponding vanishing cycles. If the sublattice generated by $s_1, \ldots, s_n$ is primitive in $H^2(Y_0, \mathbb{Z})$, then there are 2^n different isomorphisms ψ making the above diagram commute. By the gluing condition, s_i must be mapped either to δ_i or $-\delta_i$, and the signs can be chosen independently.

2.6.2 The real case

If the surface X_0 is equipped with a real structure $c : X_0 \to X_0$, then we may choose the Lefschetz deformation to be real, i.e., so that c extends to a real structure $c : \mathscr{X} \to \mathscr{X}$ which lifts the complex conjugation on $\mathbb{D}$. In this situation, we take the general fibre X_ε over a real point $\varepsilon \in \mathbb{D}^* \cap \mathbb{R}$, so that c defines a real structure on X_ε. The maps r^*, i^* and bl_p^* above are equivariant with respect to the real structures.

Proposition 2.38 (Perturbation of a real node)**.** *Let X_0 be a real surface with a real node. Choose a real Lefschetz deformation $f : \mathscr{X} \to \mathbb{D}$, and let $\psi : H^2(Y_0, \mathbb{Z}) \to H^2(X_\varepsilon, \mathbb{Z})$ be an isomorphism as in Proposition 2.35. If $\phi_0 : H^2(Y_0, \mathbb{Z}) \to H^2(Y_0, \mathbb{Z})$ and $\phi_\varepsilon : H^2(X_\varepsilon, \mathbb{Z}) \to H^2(X_\varepsilon, \mathbb{Z})$ denote the involutions induced by the respective real structures, then we have*

$$\phi_\varepsilon = \begin{cases} \psi \circ \phi_0 \circ \psi^{-1} & \text{or} \\ \psi \circ \phi_0 \circ \psi^{-1} \circ R_\delta \end{cases}$$

depending on the sign of ε, where R_δ denotes the Picard-Lefschetz map.

Proof. Since all the involved maps are equivariant with respect to the real structures, it follows from the commutative diagram above that ϕ_ε agrees with $\psi \circ \phi_0 \circ \psi^{-1}$ on the orthogonal complement of δ. Therefore ϕ_ε acts on $\langle \delta \rangle$ as ± 1. It remains to show that this sign really depends on the sign of ε. To see this, consider a path γ connecting ε with $-\varepsilon$ in the upper half plane (see Figure 2.1) and let $M(\gamma) : H^2(X_\varepsilon, \mathbb{Z}) \to H^2(X_{-\varepsilon}, \mathbb{Z})$ denote the monodromy along γ. Let $\bar{\gamma}$ denote the inverse path, i.e. $\bar{\gamma}(t) := \gamma(1-t)$. Because f is real (that is, $f \circ c = \mathrm{conj} \circ f$), we have $M(\mathrm{conj} \circ \bar{\gamma}) = \phi_\varepsilon \circ M(\bar{\gamma}) \circ \phi_{-\varepsilon}$. Moreover, since the concatenation of the paths γ and $\mathrm{conj} \circ \bar{\gamma}$ makes a full turn around 0, the composition $M(\mathrm{conj} \circ \bar{\gamma}) \circ M(\gamma)$ is the Picard-Lefschetz map R_δ. It follows that $\phi_\varepsilon = M(\gamma)^{-1} \circ \phi_{-\varepsilon} \circ M(\gamma) \circ R_\delta$. Therefore, if ϕ_ε maps the vanishing cycle δ_ε to $a\delta_\varepsilon$, $a \in \{\pm 1\}$, then $\phi_{-\varepsilon}$ maps the vanishing cycle $\delta_{-\varepsilon}$ to $-a\delta_{-\varepsilon}$. $\square$

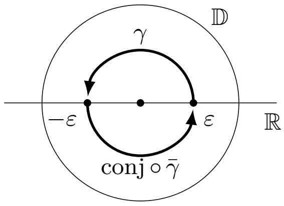


Figure 2.1

Proposition 2.39 (Perturbation of a pair of non-real nodes)**.** *Let X_0 be a real surface with a pair of non-real nodes. Choose a real Lefschetz deformation $f : \mathscr{X} \to \mathbb{D}$ of this pair. Then there is an isomorphism $\psi : H^2(Y_0, \mathbb{Z}) \to H^2(X_\varepsilon, \mathbb{Z})$ such that $\phi_\varepsilon = \psi \circ \phi_0 \circ \psi^{-1}$.*

Proof. Let $s', s'' \in H^2(Y_0, \mathbb{Z})$ denote the classes of the exceptional curves, and let $\delta', \delta'' \in H^2(X_\varepsilon, \mathbb{Z})$ be the corresponding vanishing cycles. By Proposition 2.35 and Remark 2.37, there is an isomorphism $\psi : H^2(Y_0, \mathbb{Z}) \to H^2(X_\varepsilon, \mathbb{Z})$ with $\psi(s') = \delta'$, $\psi(s'') = \delta''$, and such that ϕ_ε agrees with $\psi \circ \phi_0 \circ \psi^{-1}$ on the orthogonal complement of $\langle \delta', \delta'' \rangle$. Moreover, $\psi \circ \phi_0 \circ \psi^{-1}$ maps δ' to $-\delta''$, and ϕ_ε must map δ' either to $-\delta''$ or to δ''. In the former case we are done, and in the latter case it suffices to replace ψ by $R_{\delta'} \circ \psi$. □

Corollary 2.40. *Let C be a real irreducible sextic whose only singularities are non-real nodes, and let C' be a real non-singular sextic obtained from C be perturbing all the nodes. Let (X, h, c) and (X', h', c') be the weakly polarized real K3 surfaces obtained from C and C' respectively. Then there is an isomorphism $\psi : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ such that $\psi(h) = h'$ and $\psi \circ c^* = c'^* \circ \psi$.*

Corollary 2.41. *Let C be a dividing real irreducible sextic whose only singularities are non-real nodes, let (X, h, c) be the associated weakly polarized real K3 surface, let $\alpha \in H^2(X, \mathbb{Z})$ be a characteristic element of c^* and let $\sigma = \{s'_1, s''_1, \dots, s'_m, s''_m\}$ denote the set of nodal classes. Then we have $\alpha \equiv \sum_{i \in I} s'_i + s''_i \mod 2$, where $I \subset \{1, \dots, m\}$ is the set of indices of the crossing pairs.*

Proof. By Remark 2.29 and Lemma 2.32, we can write $\alpha \equiv \sum_{i \in I} s'_i + s''_i \mod 2$ for some set $I \subset \{1, \dots, m\}$. To see whether a particular $i \in \{1, \dots, m\}$ belongs to I, consider a curve C' obtained from C by perturbing the pair of nodes in question while keeping all the other nodes. Let (X', h', c') be the corresponding weakly polarized real K3 surface, with set of nodal classes σ'. Let $\psi : H^2(X; \mathbb{Z}) \to H^2(X'; \mathbb{Z})$ be an isomorphism as in Proposition 2.39. A characteristic element for c'^* is given by $\alpha' := \psi(\alpha)$. The i-th pair of non-real nodes is crossing if and only if C' is not dividing. By Proposition 2.28, this is the case if and only if α' is not contained (modulo 2) in the subgroup generated by σ', which is equivalent to α not being contained (modulo 2) in the subgroup generated by $\psi^{-1}(\sigma') = \sigma \setminus \{s'_i, s''_i\}$. This happens if and only if $i \in I$. □

CHAPTER 3

Real nodal sextics without real nodes

3.1 Overview and statement of the results

In this section, we consider real nodal irreducible sextics without real nodes, i.e., sextics whose only singularities are pairs of non-real nodes, and we classify them up to rigid isotopy. This classification can be seen as a generalization of Nikulin's classification of real non-singular sextics up to rigid isotopy [32].

We proceed in two steps: In the first step (in Section 3.2), we classify the real homological types of real nodal irreducible sextics without real nodes, drawing on Nikulin's classification of "involutions with conditions" on unimodular lattices [33]. In the second step (in Section 3.3) we show, using Theorem 2.18, that to each real homological type corresponds a unique rigid isotopy type.

Let us now formulate the statement of the classification. We associate the following invariants with a real irreducible sextic $C \subset \mathbb{CP}^2$ whose only singularities are m pairs of non-real nodes:

– *Isotopy type of the real part.*
– *Dividing type.*
– *Number of crossing pairs* (only defined for dividing curves).

These three characteristics are invariant under rigid isotopies. Our first theorem states that they determine the rigid isotopy class.

Theorem 3.1. *Two real irreducible sextics with m pairs of non-real nodes are rigidly isotopic if and only if their real parts are isotopic, they are of the same dividing type and, if they are dividing, they have the same number of crossing pairs.*

By comparison, Nikulin's theorem [32] states that non-singular real sextics are rigidly isotopic if and only if they are isotopic and of the same dividing type.

The second question we consider is which combinations of isotopy type, dividing type and number of crossing pairs are realized by real irreducible sextics C with m pairs of non-real nodes. There are several geometric conditions which must be satisfied.

(1) *Existence of the smoothing.* Smoothing the non-real nodes leads to a non-singular real sextic whose real part has the same isotopy type as the original curve. The smoothed curve is dividing if and only if the original curve is dividing and without crossing pairs. Therefore, a sextic with prescribed isotopy type, dividing

type and number of crossing nodes can only exist if there exists a non-singular sextic with the same isotopy type and the appropriate dividing type. (Necessary and sufficient conditions for this are given by Nikulin [32].)

(2) *Harnack's inequality for $\tilde{C}$.* The normalization $\tilde{C}$ is a smooth real curve of genus $g = 10 - 2m$. Harnack's inequality (see Harnack [14] and Klein [23, p. 154] for a topological proof) for $\tilde{C}$ states that $l \leq g + 1 = 11 - 2m$, where l denotes the number of ovals of $\mathbb{R}C$. Moreover, the equality $l = 11 - 2m$ is only possible if C is dividing.

If C is dividing, then there are further restrictions. The set of real points of a real sextic without real nodes it topologically a union of circles, which must be embedded in $\mathbb{RP}^2$ so that they bound a disc. These circles are called *ovals*. Let l be the number of ovals. A first important restriction for dividing curves is *Klein's congruence* (see Klein [23, p. 172]), stating that $l \equiv k \pmod 2$ for curves of degree $d = 2k$. More refined restrictions come from Rokhlin's complex orientation formula (see Rokhlin [35]). The orientation of each of the halves of $\tilde{C}$ induces a boundary orientation on the real part $\mathbb{R}C$. A pair of ovals in $\mathbb{R}C$ is called *injective* if one oval is contained in the disk bounded by the other oval. An injective pair is called *positive* if the orientation of the two ovals is induced by an orientation of the annulus bounded by these ovals, and *negative* otherwise. Let Π_+ and Π_- denote the numbers of positive and negative injective pairs, respectively. In our case, Rokhlin's complex orientation formula states that

$$2(\Pi_- - \Pi_+) + l = 9 - 2r,$$

where l is the number of ovals and r is the number of crossing pairs. This implies the following relations between the topology of the real part and the number r of crossing pairs for dividing curves:

(3) *Arnold's congruence.* An oval is called *even* if it lies inside an even number of other ovals, and *odd* otherwise. Arnold's congruence states that

$$o_{\text{even}} - o_{\text{odd}} \equiv 9 - 2r \mod 4,$$

where o_{even} and o_{odd} denote the numbers of even and odd ovals, respectively.

(4) *Restriction for curves without injective pairs.* For curves without injective pairs, the complex orientation formula implies

$$l = 9 - 2r.$$

Our second theorem states that the conditions (1)–(4) are sufficient for the existence of a real sextic with prescribed isotopy type, dividing type and number of crossing pairs.

Theorem 3.2. *A triple consisting of an isotopy type, a dividing type and a number of crossing pairs (if the dividing type is I) is realized by a real irreducible sextic whose only singularities are m pairs of non-real nodes if and only if it satisfies the conditions*

(1)–(4) *above.*

Corollary 3.3. *The rigid isotopy classes of real irreducible nodal sextics without real nodes are those listed in Figures 3.1 and 3.2. In total, there are* 78 *classes of dividing sextics and* 125 *classes of non-dividing sextics.*

To describe the isotopy type of the real part, also called real scheme, we use Viro's notation: the symbol $\langle n\rangle$ stands for a collection of n empty ovals; $\langle 1\langle A\rangle\rangle$ is obtained from A by adding an oval containing all of ovals A in its interior; if A and B are collections of ovals, then $\langle A \sqcup B\rangle$ denotes the disjoint union of A and B, such that no oval of A lies in the interior of an oval of B and vice versa.

$m_{\max}$									
0	$\langle 1\sqcup 1\langle 9\rangle\rangle$				$\langle 5\sqcup 1\langle 5\rangle\rangle$				$\langle 9\sqcup 1\langle 1\rangle\rangle$
	$\{0\}$				$\{0\}$				$\{0\}$
1	$\langle 1\langle 8\rangle\rangle$	$\langle 1\sqcup 1\langle 7\rangle\rangle$	$\langle 2\sqcup 1\langle 6\rangle\rangle$	$\langle 3\sqcup 1\langle 5\rangle\rangle$	$\langle 4\sqcup 1\langle 4\rangle\rangle$	$\langle 5\sqcup 1\langle 3\rangle\rangle$	$\langle 6\sqcup 1\langle 2\rangle\rangle$	$\langle 7\sqcup 1\langle 1\rangle\rangle$	$\langle 9\rangle$
	$\{0\}$	$\{1\}$	$\{0\}$	$\{1\}$	$\{0\}$	$\{1\}$	$\{0\}$	$\{1\}$	$\{0\}$
2		$\langle 1\langle 6\rangle\rangle$	$\langle 1\sqcup 1\langle 5\rangle\rangle$	$\langle 2\sqcup 1\langle 4\rangle\rangle$	$\langle 3\sqcup 1\langle 3\rangle\rangle$	$\langle 4\sqcup 1\langle 2\rangle\rangle$	$\langle 5\sqcup 1\langle 1\rangle\rangle$	$\langle 7\rangle$	
		$\{1\}$	$\{0,2\}$	$\{1\}$	$\{0,2\}$	$\{1\}$	$\{0,2\}$	$\{1\}$	
3			$\langle 1\langle 4\rangle\rangle$	$\langle 1\sqcup 1\langle 3\rangle\rangle$	$\langle 2\sqcup 1\langle 2\rangle\rangle$	$\langle 3\sqcup 1\langle 1\rangle\rangle$	$\langle 5\rangle$		
			$\{0,2\}$	$\{1,3\}$	$\{0,2\}$	$\{1,3\}$	$\{2\}$		
4				$\langle 1\langle 2\rangle\rangle$	$\langle 1\sqcup 1\langle 1\rangle\rangle$	$\langle 3\rangle$			$\langle 1\langle 1\langle 1\rangle\rangle\rangle$
				$\{1,3\}$	$\{2,4\}$	$\{3\}$			$\{0\}$
5					$\langle 1\rangle$				
					$\{4\}$				

Figure 3.1 – Rigid isotopy classes of dividing real nodal sextics with m pairs of non-real nodes of which r are crossing. The set below each real scheme indicates the possible values for the number of crossing pairs r. The total number of pairs m can take any value between r and the upper bound $m_{\max}$ indicated on the left of each row.

3.2 Classification of the homological types

Our aim in this section is to classify the real homological types corresponding to real irreducible nodal sextics without real nodes. We fix the number $m \geq 1$ of pairs of non-real nodes. (The case $m = 0$ corresponds to non-singular sextics, for which the homological types are classified by Nikulin [32]. We exclude it here to avoid dealing with certain boundary conditions.)

3.2.1 Invariants and statement of the classification

We recall that a homological type corresponding to real nodal irreducible sextics without real nodes is a quadruple (L, h, σ, ϕ), where L is a K3 lattice, $h \in L$ is a vector of square

$m_{\max}$	
0	$\langle 1\langle 9\rangle\rangle$ $\langle 1 \sqcup 1\langle 8\rangle\rangle$ $\langle 4 \sqcup 1\langle 5\rangle\rangle$ $\langle 5 \sqcup 1\langle 4\rangle\rangle$ $\langle 8 \sqcup 1\langle 1\rangle\rangle$ $\langle 10\rangle$
0	$\langle 1\langle 8\rangle\rangle$ $\langle 1 \sqcup 1\langle 7\rangle\rangle$ $\langle 3 \sqcup 1\langle 5\rangle\rangle$ $\langle 4 \sqcup 1\langle 4\rangle\rangle$ $\langle 5 \sqcup 1\langle 3\rangle\rangle$ $\langle 7 \sqcup 1\langle 1\rangle\rangle$ $\langle 9\rangle$
1	$\langle 1\langle 7\rangle\rangle$ $\langle 1 \sqcup 1\langle 6\rangle\rangle$ $\langle 2 \sqcup 1\langle 5\rangle\rangle$ $\langle 3 \sqcup 1\langle 4\rangle\rangle$ $\langle 4 \sqcup 1\langle 3\rangle\rangle$ $\langle 5 \sqcup 1\langle 2\rangle\rangle$ $\langle 6 \sqcup 1\langle 1\rangle\rangle$ $\langle 8\rangle$
1	$\langle 1\langle 6\rangle\rangle$ $\langle 1 \sqcup 1\langle 5\rangle\rangle$ $\langle 2 \sqcup 1\langle 4\rangle\rangle$ $\langle 3 \sqcup 1\langle 3\rangle\rangle$ $\langle 4 \sqcup 1\langle 2\rangle\rangle$ $\langle 5 \sqcup 1\langle 1\rangle\rangle$ $\langle 7\rangle$
2	$\langle 1\langle 5\rangle\rangle$ $\langle 1 \sqcup 1\langle 4\rangle\rangle$ $\langle 2 \sqcup 1\langle 3\rangle\rangle$ $\langle 3 \sqcup 1\langle 2\rangle\rangle$ $\langle 4 \sqcup 1\langle 1\rangle\rangle$ $\langle 6\rangle$
2	$\langle 1\langle 4\rangle\rangle$ $\langle 1 \sqcup 1\langle 3\rangle\rangle$ $\langle 2 \sqcup 1\langle 2\rangle\rangle$ $\langle 3 \sqcup 1\langle 1\rangle\rangle$ $\langle 5\rangle$
3	$\langle 1\langle 3\rangle\rangle$ $\langle 1 \sqcup 1\langle 2\rangle\rangle$ $\langle 2 \sqcup 1\langle 1\rangle\rangle$ $\langle 4\rangle$
3	$\langle 1\langle 2\rangle\rangle$ $\langle 1 \sqcup 1\langle 1\rangle\rangle$ $\langle 3\rangle$
4	$\langle 1\langle 1\rangle\rangle$ $\langle 2\rangle$
4	$\langle 1\rangle$
5	$\emptyset$

Figure 3.2 – Rigid isotopy classes of non-dividing real nodal sextics with m pairs of non-real nodes. The number of pairs of non-real nodes m can take any value between 0 and the upper bound $m_{\max}$ indicated on the left of each row.

2, $\sigma \subset L$ is a set of $2m$ pairwise orthogonal roots orthogonal to h such that the sublattice S generated by h and σ is primitive in L, and $\phi : L \to L$ is a geometric involution (see Definition 2.10) satisfying property $(*)$ (see Section 2.2) such that there is no root $s \in \sigma$ with $\phi(s) = -s$. An *isomorphism* between two homological types $(L_1, h_1, \sigma_1, \phi_1)$ and $(L_2, h_2, \sigma_2, \phi_2)$ is an isometry $\rho : L_1 \to L_2$ such that $\rho(h_1) = h_2$, $\rho(\sigma_1) = \sigma_2$ and $\phi_2 \circ \rho = \rho \circ \phi_1$.

The following lemma can either be proved directly (see e.g. Morrison and Saito [31, Theorem 3.2]) or geometrically, using the fact that the irreducible sextics with n nodes form a connected family.

Lemma 3.4. *The complex homological type (L, h, σ) corresponding to irreducible sextics with $0 \leq n \leq 10$ nodes is unique up to isomorphism.*

Therefore, we can assume that L, h and σ are fixed, and only the involution ϕ is variable. We label the roots $\sigma = \{s'_1, s''_1, \ldots, s'_m, s''_m\}$. Let $\vartheta : S \to S$ be the involutive isometry defined by $\vartheta(h) = -h$ and $\vartheta(s'_i) = -s''_i$ for all $i \in \{1, \ldots, m\}$.

Lemma 3.5. *Each real homological type corresponding to real nodal irreducible sextics without real nodes is isomorphic to a homological type (L, h, σ, ϕ) such that the restriction of ϕ to S coincides with ϑ.*

Proof. Let (L, h, σ, ϕ) be a real homological type corresponding to real nodal irreducible sextics without real nodes. Clearly there is an isometry $\rho_0 : S \to S$ such that $\rho_0 \circ \phi = \vartheta \circ \rho_0$. Let K denote the orthogonal complement of S inside L. It is indefinite of rank $21 - 2m$, its discriminant group is $A_K \equiv (\mathbb{Z}/2\mathbb{Z})^{2m+1}$, with discriminant from $q_K = [\frac{1}{2}]^{2m} \oplus [-\frac{1}{2}]$. By Nikulin [32, Theorem 1.14.2] the homomorphism $O(K) \to O(q_K)$ is surjective. Therefore, ρ_0 extends to an isometry $\rho : L \to L$. Then the homological type $(L, h, \sigma, \rho \circ \phi \circ \rho^{-1})$ is isomorphic to (L, h, σ, ϕ) and has the desired property that

$(\rho \circ \phi \circ \rho^{-1})\big|_S = \vartheta.$ □

From now on, we assume that the involution ϕ is chosen so that $\phi\big|_S = \vartheta$.

Definition 3.6. We define invariants a, t, δ and r for a homological type (L, h, σ, ϕ) of irreducible real nodal sextics without real nodes as follows.

a : Since ϕ is an involution on a unimodular lattice, the discriminant group of the invariant sublattice L_+, which we denote by A_q, is of period 2. Let a be the length of this group, i.e. the number a such that $A_q \cong (\mathbb{Z}/2\mathbb{Z})^a$.

t : Let t be the number of negative squares of the invariant lattice L_+.

δ : Let $\alpha \in L$ be a characteristic element of the involution ϕ, and let $\bar{\alpha}$ be its residue in $L/2L$ (see section 2.5). Recall that $S/2S$ is naturally embedded in $L/2L$. Let $\delta = 0$ if $\bar{\alpha} \in S/2S$ and $\delta = 1$ otherwise.

r : The invariant r is only defined if $\delta = 0$. In that case, the characteristic element α can be expressed as a sum

$$\alpha \equiv \sum_{i \in I} s_i' + s_i'' \mod 2$$

for some subset $I \subset \{1, \ldots, m\}$, cf. Remark 2.29 and Lemma 2.32. We define r as the cardinality of the set I.

Note that a, t, δ and r only depend on the homological type up to isomorphism. The following two theorems give a complete classification of the real homological types corresponding to real nodal irreducible sextics without real nodes in terms of the invariants a, t, δ and r. The rest of this section is devoted to the proof of these theorems.

Theorem 3.7. *Two homological types of irreducible real nodal sextics without real nodes are isomorphic if and only if they have the same invariants (a, t, δ, r).*

Theorem 3.8. *A homological type (L, h, σ, ϕ) of irreducible real nodal sextics without real nodes with invariants (a, t, δ, r) exists if and only if a, t, δ and r satisfy the following conditions.*

(i) $a \leq 1 + t \leq 20 - a$.

(ii) $a \equiv 1 + t \mod 2$.

(iii) $2m \leq a$, *with equality only if* $\delta = 0$.

(iv) *If* $\delta = 0$, *then* $2r \equiv 1 - t \mod 4$.

(v) *If* $\delta = 0$ *and* $a = 1 + t$, *then* $2r \equiv a - 2 \mod 8$.

3.2.2 Nikulin's "involutions with condition"

As mentioned above, to prove Theorems 3.7 and 3.8 we rely on Nikulin's classification of "involutions with condition", which are defined as follows.

Definition 3.9 (Nikulin [33])**.** A *condition on an involution* is a triple (S, ϑ, G) formed by an even lattice S, an involution $\vartheta : S \to S$ and a normal subgroup G of the group

$\{g \in O(S) \mid g \circ \vartheta = \vartheta \circ g\}$. A *(unimodular) involution with condition* (S, ϑ, G) is a triple (L, i, ϕ) formed by a (unimodular) even lattice L, a primitive embedding $i : S \hookrightarrow L$ and an involution ϕ on L such that $\phi \circ i = i \circ \vartheta$. Two triples (L_1, i_1, ϕ_1) and (L_2, i_2, ϕ_2) are called *isomorphic* if there is an isometry $\rho : L_1 \to L_2$ such that $\rho \circ \phi_1 = \phi_2 \circ \rho$ and there is an element $g \in G$ such that $\rho \circ i_1 = i_2 \circ g$.

Given a condition (S, ϑ, G), Nikulin [33] defines invariants which uniquely determine the genus of involutions with condition (S, ϑ, G). (Roughly speaking, two triples (L_1, ϕ_1, i_1) and (L_2, ϕ_2, i_2) have the same *genus* if they are isomorphic over $\mathbb{Q}$ and over the p-adic numbers $\mathbb{Z}_p$ for all primes p.) Moreover he describes all the values these invariants can take, and he gives conditions under which the genus of an involution determines the involution up to isomorphism.

The classification of homological types of irreducible real nodal sextics without real nodes fits into this framework. The condition in our case is the triple (S, ϑ, G), where S is the lattice with orthogonal basis $\{h, s'_1, s''_1, \ldots, s'_m, s''_m\}$, the involution ϑ is given by $\vartheta(h) = -h$ and $\vartheta(s'_i) = -s''_i$ for all $i \in \{1, \ldots, m\}$, and G is the group $\{g \in O(S) \mid g(h) = h, g(\sigma) = \sigma \text{ and } g \circ \vartheta = \vartheta \circ g\}$. The group G is a semi-direct product $(\mathbb{Z}/2\mathbb{Z})^m \rtimes S_m$, where a permutation $f \in S_m$ takes s'_i to $s'_{f(i)}$ and s''_i to $s''_{f(i)}$, and the i-th basis vector $e_i \in (\mathbb{Z}/2\mathbb{Z})^m$ acts by flipping the pair $\{s'_i, s''_i\}$ and leaving the other basis vectors fixed.

Lemma 3.10. *Isomorphism classes of homological types of irreducible real nodal sextics without real nodes are in bijection with isomorphism classes of involutions with condition* (S, ϑ, G)*, say* (L, ϕ, i)*, where* L *is the fixed K3 lattice, and* ϕ *is a geometric involution with property* $(*)$*.*

In [33, Theorem 1.6.3], Nikulin gives a complete system of invariants for the genus of involutions with condition. In order to prove Theorem 3.7, it suffices to show that the isomorphism classes are determined by their genus, and that the invariants m, a, t, δ and r (see Definition 3.6) determine Nikulin's invariants for the genus.

3.2.3 Uniqueness in the genus

Given a homological type (L, h, σ, ϕ) of irreducible real nodal sextics without real nodes, we define the following sublattices of L.

$$\begin{aligned} L_+ &= \{x \in L \mid \phi(x) = x\} & L_- &= \{x \in L \mid \phi(x) = -x\} \\ S_+ &= L_+ \cap S & S_- &= L_- \cap S \\ K_+ &= L_+ \cap S^\perp & K_- &= L_- \cap S^\perp \end{aligned}$$

To show that the isomorphism classes of the involutions with condition under consideration are unique in their genera, we use the following result:

Remark 3.11 (Nikulin [33, Remark 1.6.2])**.** For an isomorphism class of an involution with condition to be unique in its genus, it is sufficient that

(1) the lattices $K_\pm$ are unique in their genera, and
(2) the natural homomorphisms $O(K_\pm) \to O(k_\pm)$ are surjective.

Note that the lattices $K_\pm$ for homological types of real nodal sextics without real nodes have one positive square and their discriminant form is of period 4.

Proposition 3.12. *Let K be an even lattice with one positive square whose discriminant form k is of period 4. Then K is unique in its genus and the homomorphism $O(K) \to O(k)$ is surjective.*

Proof. The statement clearly holds for lattices of rank one. Let us now assume that K is of rank at least two, and therefore indefinite. If the length of the discriminant group k is smaller than the rank of the lattice K, the statement follows from Nikulin [32, Theorem 1.14.2]. If the length of k is equal to the rank of K and at least 3, the statement follows from Miranda and Morrison [29, Chapter 8, Corollary 7.8]. For lattices of rank 2 with k of period 4, the statement can be verified by hand; the only such lattices are U, $\langle 2\rangle \oplus \langle -2\rangle$, U(2), $\langle 2\rangle \oplus \langle -4\rangle$, $\langle 4\rangle \oplus \langle -4\rangle$ and U(4). □

Corollary 3.13. *For a homological type (L, h, σ, ϕ) of real nodal sextics without real nodes, the lattices K_+ and K_- are unique in their genera, and the natural homomorphisms $O(K_\pm) \to O(k_\pm)$ are surjective. In particular, the homological type is unique in its genus.* □

3.2.4 Invariants for the genus

In the following we define Nikulin's invariants of the genus and show that for involutions with condition (S, ϑ, G) as in Lemma 3.10, they are determined by the invariants m, a, t, δ and r (see Definition 3.6).

Invariants of (L, ϕ).

Let A_q be the discriminant group of the lattice L_+, and let q be the discriminant form defined on it. The invariants characterizing (L, ϕ) are the rank and signature of L, the rank and signature of L_+ and the invariants of q. Since L is the fixed K3 lattice, its rank and signature are fixed. Since ϕ is a geometric involution, L_+ has one positive square. Therefore, its rank and signature are determined by t, its number of negative squares. Finally the group A_q is of period 2, and hence q is determined by the length of A_q, which is a, and by the parity of q (cf. [32, Theorem 3.6.2]). The form q is even if and only if the characteristic element $\bar{\alpha} \in L/2L$ is zero. This is the case if and only if $\delta = 0$ and $r = 0$.

The groups H_+ and H_-.

Let A_{S_+} and A_{S_-} be the discriminant groups of the lattices S_+ and S_-, respectively. Following Nikulin [33], we define subgroups $\Gamma_\pm$ and $H_\pm$ of $A_{S_\pm}$ as follows.

$$\Gamma_\pm = \{x \in S_\pm^* \mid \exists y \in S_\mp^* \text{ such that } x + y \in S\}/S_\pm \subset S_\pm^*/S_\pm = A_{S_\pm}$$
$$H_\pm = \{x \in S_\pm^* \mid \exists y \in L_\mp^* \text{ such that } x + y \in L\}/S_\pm \subset S_\pm^*/S_\pm = A_{S_\pm}$$

Note that the subgroups Γ_+ and Γ_- depend only on the condition (S, ϑ, G), whereas the subgroups $H_\pm$ depend a priori on the involution ϕ. However, it turns out that in our case H_+ and H_- do not depend on ϕ. We have $S_+ = \langle p_1, \ldots, p_m\rangle$ and $S_- = \langle n_1, \ldots, n_m, h\rangle$ where $p_i = s_i' - s_i''$ and $n_i = s_i' + s_i''$. The groups $A_{S_\pm}$ and $\Gamma_\pm$ are given as follows.

$$\begin{aligned}
A_{S_+} &= \langle [\tfrac{p_1}{4}], \ldots, [\tfrac{p_m}{4}]\rangle && \cong (\mathbb{Z}/4\mathbb{Z})^m \\
A_{S_-} &= \left\langle [\tfrac{n_1}{4}], \ldots, [\tfrac{n_m}{4}], [\tfrac{h}{2}]\right\rangle && \cong (\mathbb{Z}/4\mathbb{Z})^m \oplus \mathbb{Z}/2\mathbb{Z} \\
\Gamma_+ &= \langle [\tfrac{p_1}{2}], \ldots, [\tfrac{p_m}{2}]\rangle && \cong (\mathbb{Z}/2\mathbb{Z})^m \subset A_{S_+} \\
\Gamma_- &= \langle [\tfrac{n_1}{2}], \ldots, [\tfrac{n_m}{2}]\rangle && \cong (\mathbb{Z}/2\mathbb{Z})^m \subset A_{S_-}
\end{aligned}$$

The group H_+ contains Γ_+ and is contained in the 2-torsion subgroup of A_{S_+}. Since Γ_+ is the 2-torsion subgroup of A_{S_+}, we have $H_+ = \Gamma_-$. Similarly, H_- contains Γ_- and is contained in the 2-torsion subgroup of A_{S_-}. The group Γ_- is of index two in the 2-torsion subgroup of A_{S_-}, so H_- must be either Γ_- or $\Gamma_- \oplus \langle [\frac{h}{2}]\rangle$. Because we chose the real structure with property $(*)$, the vector h does not glue to L_+, hence we have $[\frac{h}{2}] \notin H_-$ and therefore $H_- = \Gamma_-$.

The finite quadratic form q_r.

The lattice S determines an anti-isometry $\gamma : \Gamma_+ \to \Gamma_-$. We define the lattice $H_+ \oplus_\gamma H_-$ as the gluing of H_+ and H_- along γ. More precisely, let

$$H_+ \oplus_\gamma H_- = (\Gamma_\gamma)^\perp_{H_+ \oplus H_-} \,/\, \Gamma_\gamma,$$

where $\Gamma_\gamma \subset \Gamma_+ \oplus \Gamma_-$ is the graph of γ. There is a natural embedding $\gamma_r : H_+ \oplus_\gamma H_- \hookrightarrow q$, inducing a finite quadratic form q_r on $H_+ \oplus_\gamma H_-$. In general (if $\Gamma_+ \oplus_\gamma \Gamma_-$ is strictly contained in $H_+ \oplus_\gamma H_-$) this form q_r depends on the embedding $i : S \hookrightarrow L$ and on the involution ϕ. Since in our case we have $\Gamma_\pm = H_\pm$ however, the form q_r is determined by S and θ.

Invariants of the embedding γ_r.

Let $v_q \in A_q$ be the characteristic element of q, i.e. the element for which $(v_q, x) = x^2$ mod 1 for all $x \in A_q$. We have $v_q = [\frac{\alpha}{2}] \in A_q$, where $\alpha \in L$ is a characteristic element for the involution ϕ. The embedding γ_r is characterized by whether v_q is contained in $\gamma_r(q_r)$, and if this is the case, by the orbit $G \cdot \gamma_r^{-1}(v_q) \subset q_r$. But v_q is contained in

$\gamma_r(q_r)$ if and only if $\bar{\alpha} \in L/2L$ is contained in $S/2S$, which is by definition if and only of $\delta = 0$. In this case we have $\alpha \equiv \sum_{i \in I} s_i' + s_i'' \mod 2L$, where the sum ranges over the indices of the crossing pairs. Therefore, $\gamma_r^{-1}(v_q) = \sum_i [\frac{p_i}{2}] \in q_r$ is determined (up to the action of G, which permutes the indices) by r.

3.2.5 Proof of Theorem 3.7

Proof of Theorem 3.7. Let (L, h, σ, ϕ_1) and (L, h, σ, ϕ_2) be two real homological types of real nodal irreducible sextics without real nodes which have the same invariants a, t, δ and r. We have shown in the preceding section that a, t, δ and r determine Nikulin's invariants for the genus. Therefore, by Nikulin's theorem [33, Theorem 1.6.3], the homological types (L, h, σ, ϕ_1) and (L, h, σ, ϕ_2) belong to the same genus. Since by Corollary 3.13 the genus determines the homological type up to isomorphism, it follows that (L, h, σ, ϕ_1) and (L, h, σ, ϕ_2) are isomorphic. □

3.2.6 Existence of involutions: proof of Theorem 3.8

Proof of Theorem 3.8. By Lemma 3.10, the homological types we consider correspond to involutions with condition (S, ϑ, G). Therefore, we can deduce Theorem 3.8 from Nikulin's existence theorem for involutions with conditions [33, Theorem 1.8.3]. To do this, we need to check that the conditions (i) – (v) of Theorem 3.8 are equivalent to the conditions for the existence of an involution with condition given by Nikulin in [33, Conditions 1.8.1 and 1.8.2].

For the most part, this verification is straightforward. The only more technical step, which we consider in detail, is the equivalence of condition (v) of Theorem 3.8 with *boundary condition 1* in [33, Conditions 1.8.2], in the case $\delta = 0$. This boundary condition guarantees the existence of a lattice K_+ with prescribed discriminant quadratic form k_+ and index of inertia $(1, t-m)$, using Nikulin's theorem [32, Theorem 1.10.1] on the existence of an indefinite even lattice with prescribed rank, signature and discriminant form. In our setting, *boundary condition 1* in [33, Conditions 1.8.2] is equivalent to the following condition:

$$\text{If } a = 1 + t, \text{ then } 1 - t \equiv 4\,\varepsilon_{v_+} + c_v \pmod 8, \tag{BC1}$$

where c_v and ε_{v_+} are invariants taking values in $\mathbb{Z}/8\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ respectively, whose definition we give below. We need to show that (BC1) is equivalent to condition (v) which states that

$$\text{if } a = 1 + t, \text{ then } 2r \equiv a - 2 \pmod 8.$$

The equivalence between (BC1) and (v) follows from Lemma 3.16 below. □

In the following, suppose that $\delta = 0$. Recall from Section 3.2.4 that v_q denotes the characteristic element of the finite quadratic form q.

Definition 3.14. The invariant $c_v \in \mathbb{Z}/8\mathbb{Z}$ is defined such that $\frac{1}{2} c_v = q(v_q) \mod 2$ (see

[33, p. 113]) and $c_v \in \{-1, 0, 1, 2\} \mod 8$ (see Nikulin [33, p. 118]).

Before defining ε_{v_+}, we introduce two auxiliary finite quadratic forms, γ_+ and η. Let γ_+ be the finite quadratic form

$$\gamma_+ = \begin{cases} [\frac{1}{2}] \oplus [-\frac{1}{2}] & \text{if } r \text{ is even,} \\ [\frac{1}{2}] \oplus [\frac{1}{2}] & \text{if } r \text{ is odd} \end{cases}$$

and let x_1 and x_2 be the standard generators of γ_+ (cf. Nikulin [33, p. 117, Equation 8.14]). The characteristic element of γ_+ is $v_{\gamma_+} = x_1 + x_2$, and we have $v_{\gamma_+}^2 \equiv r \pmod 2$.

Let A_{S_+} be the discriminant group of the lattice S_+, and let s_+ be the finite quadratic form defined on it. Recall that A_{S_+} is generated by the elements $e_1, \dots, e_m$, where $e_i = [\frac{s_i' - s_i''}{4}]$. Since $q_r \cong \Gamma_+ \subset A_{S_+}$, we may view $v = \gamma_r^{-1}(v_q)$ as an element of A_{S_+} (see Section 3.2.4 for the definition of q_r and γ_r). We have $v = \sum_{i \in I} 2e_i$. Let $v_\eta = v + v_{\gamma_+} \in s_+ \oplus \gamma_+$. Note that $v_\eta^2 = 0$. Hence we can define the quadratic form

$$\eta = \langle v_\eta \rangle^{\perp}_{(s_+ \oplus \gamma_+)} / \langle v_\eta \rangle,$$

(which is called "$(q_{S_+^2})_v$" in Nikulin [33, p. 117, Equation 8.15]). Let $K(\eta_2)$ be a 2-adic lattice of minimal length whose discriminant quadratic form is η. The discriminant of $K(\eta_2)$ is an element of $\mathbb{Z}_2/(\mathbb{Z}_2^*)^2$. There are two such lattices $K(\eta_2)$, and their discriminants differ by a sign.

Definition 3.15. The invariant $\varepsilon_{v_+} \in \mathbb{Z}/2\mathbb{Z}$ is defined by the equation

$$5^{\varepsilon_{v_+}} = \pm \operatorname{discr} K(\eta_2) \in \mathbb{Q}_2/(\mathbb{Q}_2^*)^2.$$

Note that the group $\mathbb{Q}_2^*/(\mathbb{Q}_2^*)^2$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$, with representatives $\{1, -1\} \times \{1, 2\} \times \{1, 5\}$ (cf. for example Serre [37, p. 18]).

Lemma 3.16. *If $\delta = 0$, then $4\,\varepsilon_{v_+} + c_v + 2r \equiv 0 \mod 8$.*

Proof. Recall that $v_q = \sum_{i \in I} [\frac{p_i}{2}] \in q_r$. We have $q(v_q) \equiv r \mod 2$. Therefore,

$$c_v \equiv \begin{cases} 0 \pmod 8 & \text{if } r \text{ is even,} \\ 2 \pmod 8 & \text{if } r \text{ is odd.} \end{cases}$$

Hence, to prove the lemma, it remains to show that

$$\varepsilon_{v_+} \equiv \begin{cases} 0 \pmod 2 & \text{if } r \equiv 0, 3 \pmod 4, \\ 1 \pmod 2 & \text{if } r \equiv 1, 2 \pmod 4. \end{cases}$$

Up to the action of G, we may suppose that $I = \{1, \dots, r\}$. Then a basis of the finite quadratic form η defined above is given by

$$\{e_i + x_1\}_{i=1}^r \cup \{e_i\}_{i=r+1}^m.$$

We define a new basis $\{f_i\}_{i=1}^m$ for η by

$$f_i = \begin{cases} [2(e_1 + \cdots + e_{i-1}) + e_i + x_1] & \text{if } i \in \{1, \dots, r\} \\ [e_i] & \text{if } i \in \{r+1, \dots, m\}. \end{cases}$$

Note that this basis is orthogonal, and we have

$$f_i^2 = \begin{cases} -\frac{5}{4} & \text{if } i \leq r \text{ and } i \text{ is odd,} \\ -\frac{1}{4} & \text{otherwise.} \end{cases}$$

Now $K(\eta_2)$ can be defined as the 2-adic lattice with orthogonal basis $b_1, \dots, b_m$, where

$$b_i^2 = \begin{cases} -5 \cdot 4 & \text{if } i \leq r \text{ and } i \text{ is odd,} \\ -1 \cdot 4 & \text{otherwise.} \end{cases}$$

We calculate the discriminant of $K(\eta_2)$ with respect to this basis. Since the basis is orthogonal, the discriminant is just the product $b_1^2 \cdot b_2^2 \cdots b_m^2$. The invariant ε_{v_+} is the number of occurrences of the factor 5 in this product, counted modulo 2. This is equal to the number of odd integers between 1 and r, and it is finally easy to check that

$$\varepsilon_{v_+} \equiv \left\lceil \frac{r}{2} \right\rceil \equiv \begin{cases} 0 & \text{if } r \equiv 0, 3 \mod 4 \\ 1 & \text{if } r \equiv 1, 2 \mod 4 \end{cases} \pmod 2. \qquad \square$$

3.3 Homological type determines rigid isotopy class

In this section, we prove the following proposition.

Proposition 3.17. *For real irreducible nodal sextics without real nodes, the rigid isotopy class is determined by the homological type.*

Proof. Recall from Section 2.3 that we denote by $\mathbb{H}_+$ and $\mathbb{H}_-$ the hyperbolic spaces obtained from $K_+ \otimes \mathbb{R}$ and $K_- \otimes \mathbb{R}$, respectively, and let

$$\Delta_+ = \{u \in K_+ \mid u^2 < 0 \text{ and } \exists w \in \Sigma \text{ such that } v = \tfrac{1}{2}(u + w) \in L \text{ and } v^2 = -2\},$$
$$\Delta_- = \{u \in K_- \mid u^2 < 0 \text{ and } \exists w \in \Sigma \text{ such that } v = \tfrac{1}{2}(u + w) \in L \text{ and } v^2 = -2\}.$$

The sets of hyperplanes $\{H_u\}_{u \in \Delta_+}$ and $\{H_u\}_{u \in \Delta_-}$ define tilings of the spaces $\mathbb{H}_+$ and $\mathbb{H}_-$ respectively. By Theorem 2.18, the rigid isotopy classes of curves with homological type (L, h, σ, ϕ) are in bijection with pairs of tiles, modulo automorphisms of the homological type. Hence, to show that there is only one rigid isotopy class for a given homological type, we need to show that the automorphisms of the homological type act transitively on the pairs of tiles. For this, it is sufficient to show the following: *For each $u \in \Delta_\pm$ there is an automorphism ρ of the homological type which acts on $\mathbb{H}_+ \times \mathbb{H}_-$ by a reflection in H_u.* Let $u \in \Delta_\varepsilon$ ($\varepsilon = \pm 1$), and let $w \in \Sigma$ be a vector such that $v = \frac{1}{2}(u + w) \in L$ with $v^2 = -2$. Since u^2 and w^2 are both non-positive and $u^2 + w^2 = (2v)^2 = -8$, we have $w^2 \in \{0, -2, -4, -6\}$. Therefore, the vector $w \in \Sigma$ is either zero or a sum of up to three elements of $\pm\sigma$. We label the nodal classes $\sigma = \{s'_1, s''_1, \ldots, s'_m, s''_m\}$, such that $\phi(s'_i) = -s''_i$ and $\phi(s''_i) = -s'_i$. Since $v - \varepsilon\phi(v) = \frac{1}{2}(w - \varepsilon\phi(w)) \in L$, a root s'_i appears in w if and only if s''_i does. This leaves two possibilities: either $w = 0$, or $w = \pm\, s'_k \pm s''_k$ for some k. In the first case ($w = 0$) we have $v = \frac{u}{2}$, and $\rho = R_v$ is an automorphism of (L, h, σ, ϕ) which acts on $\mathbb{H}_+ \times \mathbb{H}_-$ by reflection in H_u. Let us consider the second case, where $w = \pm\, s'_k \pm s''_k$ for some k. Changing the signs if necessary, we may assume $w = s'_k + \varepsilon s''_k$. Then v and $\phi(v)$ are orthogonal to each other, and $\rho_0 = R_v \circ R_{\phi(v)} = R_w \circ R_u$ is an automorphism of L which acts on $\mathbb{H}_+ \times \mathbb{H}_-$ by reflection in H_u. However, it does not necessarily map the set σ to itself. More precisely, it acts on Σ by reflection in the hyperplane orthogonal to $v = s'_k + \varepsilon s''_k$. To ensure that the set σ is mapped to itself, we compose ρ_0 with a rotation in the plane $\langle s'_k, s''_k \rangle$ if $\varepsilon = 1$: We define $\rho = \rho_0 \circ R_{s'_k} \circ R_{s''_k}$ if $\varepsilon = 1$, and $\rho = \rho_0$ if $\varepsilon = -1$. Then ρ is an automorphism of (L, h, σ, ϕ) which acts on $\mathbb{H}_+ \times \mathbb{H}_-$ by reflection in H_u. $\square$

3.4 Proof of Theorems 3.1 and 3.2

Proposition 3.18 (Geometrical interpretation of a, t, δ and r)**.** *Let $C \subset \mathbb{CP}^2$ be a real irreducible sextic whose only singularities are m pairs of non-real nodes, and let (L, h, σ, ϕ) be its homological type. Then the invariants (a, t, δ, r) have the following geometrical interpretation: if $\mathbb{R}C$ is empty, we have $a = 10$, $t = 9$, $\delta = 0$ and $r = 0$; if $\mathbb{R}C$ is not empty, we have*

$$
\begin{aligned}
&a = 11 - l, \ \textit{where } l \textit{ is the number of ovals of } C,\\
&t = 9 + \chi(B), \ \textit{where } B \textit{ denotes the non-orientable half of } \mathbb{RP}^2 \setminus \mathbb{R}C,\\
&\delta = \begin{cases} 1 & \textit{if } C \textit{ is dividing and}\\ 0 & \textit{otherwise,}\end{cases}\\
&r \quad \textit{is the number of crossing pairs of } C.
\end{aligned}
$$

Proof. The interpretations for a and t are well-known for non-singular sextics, cf. [32, Theorem 3.10.6]. If $C \subset \mathbb{CP}^2$ is a real irreducible sextic whose only singularities are m pairs of non-real nodes, consider a real non-singular sextic $C' \subset \mathbb{CP}^2$ obtained from C be perturbing all the nodes. The perturbation does not change the isotopy type of the real part, so the right hand side of the equations does not change when passing from C to C'. If the homological type of C is (L, h, σ, ϕ), then the homological type of the perturbed sextic C' is $(L, h, \emptyset, \phi)$ for a suitable marking (see Section 2.6), and thus a and t do not change when passing from C to C'. The statement about δ follows from Proposition 2.28, and the statement about r follows from Corollary 2.41. $\square$

Proof of Theorem 3.1. Let C_1 and C_2 be two real irreducible sextics whose only singularities are m pairs of non-real nodes. Suppose that C_1 and C_2 have isotopic real parts, are of the same dividing type and, if they are dividing, have the same number of crossing pairs. Then by Proposition 3.18, the invariants m, a, t, δ, r of their homological types are the same. Hence by Theorem 3.7 their homological types are isomorphic and by Proposition 3.17 the curves C_1 and C_2 are rigidly isotopic. $\square$

Given an isotopy type of a collection of ovals in $\mathbb{RP}^2$, a dividing type, and, if the dividing type is I, a number r of crossing pairs, we can define values a, t, δ and r by taking the geometric interpretation in Proposition 3.18 as a definition.

Lemma 3.19. *If an isotopy type, a dividing type and a number of crossing pairs satisfy conditions* (1)–(4) *given in the introduction, then the corresponding values a, t, δ and r satisfy the conditions* (i)–(v) *of Theorem 3.8.*

Proof. Properties (i) and (ii) follow from analogous properties for non-singular sextics (see [32, Theorem 3.4.3]) using the existence of a non-singular sextic with the given isotopy type and the required dividing type (1). Using Proposition 3.18, it is straightforward to verify that the properties (iii), (iv) and (v) follow from Harnack's inequality (2) for $\tilde{C}$, Arnold's congruence (3) and the complex orientation formula for curves without injective pairs (4), respectively. $\square$

Lemma 3.20. *Different combinations of isotopy type, dividing type and number of crossing pairs, satisfying conditions* (1)–(4)*, lead to different invariants a, t, δ and r.*

Proof. This follows essentially from the fact that, for non-singular real sextics, the rigid isotopy type is determined by a, t and δ (see [32]). $\square$

Proof of Theorem 3.2. It is clear that the conditions (1)–(4) of Theorem 3.2 are necessary for the existence of a sextic with the desired invariants. To show that they are sufficient, suppose we are given an isotopy type of the real part, a dividing type, a number of pairs of nodes, and (if the dividing type is I) a number of crossing pairs, subject to the conditions (1)–(4). By Lemma 3.19, the corresponding values (a, t, δ, r) satisfy the conditions (i)–(v) of Theorem 3.8. Hence by Theorem 3.8, there is a homological type (L, h, σ, ϕ) with these invariants. By Theorem 2.18, there are sextics with this homological type. Finally, Lemma 3.20 guarantees that such sextics indeed have the desired isotopy type, dividing type and number of crossing pairs. $\square$

CHAPTER 4

Dividing real nodal sextics

4.1 Perturbations of real nodes

Before we start discussing the classification of dividing sextics, we recall some well-known facts about perturbations of real nodes on real plane curves. For more details and references, see for example Viro [43, Section 2.3].

Real nodes are either *hyperbolic*, with two real branches (with local equation $f(x,y) = x^2 - y^2 = 0$) or *solitary*, with two imaginary, complex-conjugated branches (local equation $f(x,y) = x^2 + y^2 = 0$). A perturbation of a real node is modeled by the equation $f(x,y) = \varepsilon$, where ε is a real parameter. The topology of the result of the perturbation depends on the sign of ε. From the complex point of view, a node is a transverse intersection of two smooth branches, i.e., two points on the normalization of the curve are "glued together". Perturbing a node corresponds topologically to first removing disks around both of the points which are glued together and then gluing an annulus along the boundary of these disks. When looking at real curves, different signs of ε result in different real structures on this annulus, as shown in Figure 4.1.

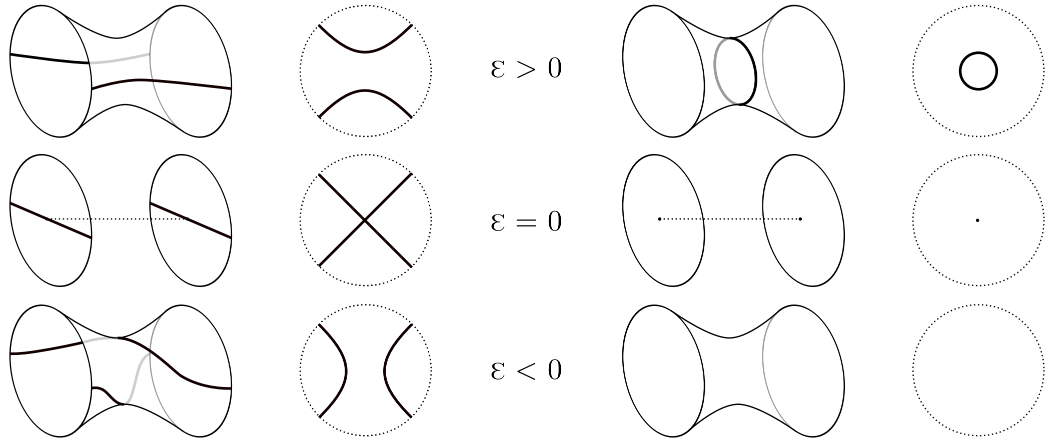
From this, one can the deduce the following important, well-known observation.

Lemma 4.1. *For a real node on a dividing curve, precisely one of the two possible perturbations results in a dividing curve. For solitary nodes, this is always the perturbation creating a new empty oval. For hyperbolic nodes, it is the perturbation which is compatible with the complex orientation.*

This leads us to the following definition.

Definition 4.2. Given a nodal dividing curve, its *dividing perturbation* is obtained by perturbing all the real nodes in such a way that the resulting curve remains dividing, while not perturbing the non-real nodes.

Note that Brusotti's theorem (see Brusotti [6]) guarantees that the signs of perturbation for the different real nodes can be chosen independently of each other.



(a) – Perturbations of a hyperbolic node (b) – Perturbations of a solitary node

Figure 4.1 – Perturbations of a real node. The left half (a) shows the possible perturbations of a hyperbolic node, and the right half (b) shows the possible perturbations of a solitary node. For both cases, the figure contains three rows. The middle row corresponds to the singular curve, while the upper and lower rows show the two possible perturbations. In each row, a part of the complex curve is depicted on the left, and a neighborhood of the singularity in the real plane is depicted on the right. The dotted lines in the middle row indicate that two points—one on each branch of the node—are identified. The real part of the curve is drawn using think lines.

4.2 Seeing degenerations as polytope faces

In this section, we study the rigid isotopy classes of dividing real nodal sextics by seeing them as nodal degenerations of sextics with only non-real nodes. When talking about dividing curves, we always suppose these curves to be irreducible. As discussed in the preceding section, if we perturb a real node on a dividing curve, then precisely one of the two possible perturbations results in a curve that is still dividing. By perturbing all the real nodes in this way, we obtain a real nodal sextic without real nodes. Hence, for each real nodal sextic C there is a unique rigid isotopy type of dividing real nodal sextics without real nodes of which C is a degeneration. This allows us to classify the rigid isotopy classes of all real nodal sextics by classifying the real nodal degenerations for each rigid isotopy class of dividing real nodal sextics without real nodes.

In Chapter 3 we classified real nodal sextics without real nodes up to rigid isotopy. In particular, we found that for each homological type of such sextics, there is a unique rigid isotopy class (Proposition 3.17). Fix a homological type (L, h, σ_0, ϕ_0) of real irreducible nodal dividing sextics without real nodes. We denote the non-real roots by $\sigma_0 = \{s_1', s_1'', \ldots, s_m', s_m''\}$ where $\phi_0(s_i') = -s_i''$. Our goal is to identify, using the period space, the rigid isotopy classes of real irreducible nodal sextics which can be obtained from sextics of homological type (L, h, σ_0, ϕ_0) by real nodal degenerations. We introduce the following notations. Let

– Γ be the group of automorphisms of the homological type (L, h, σ_0, ϕ_0),

– Σ_0 be the sublattice of L generated by σ_0,
– S_0 be the sublattice of L generated by σ_0 and h, and
– K denote the orthogonal complement of S_0 inside L.

4.2.1 Reflections in the period space

Lemma 4.3. *The only element $g \in \Gamma$ acting trivially on K is the identity.*

Proof. Suppose $g \in \Gamma$ is an automorphism whose restriction to K is the identity. A priori, g can act on the set of roots σ_0 by a permutation f. However, if g acts trivially on K, then it acts trivially on the discriminant group A_K, and therefore the permutation f of σ_0 must act trivially on the discriminant group $A_{\Sigma_0} = \langle \{[\frac{s}{2}]\}_{s\in\sigma_0}\rangle$, and this implies that f is the identity. $\square$

The (anti-)invariant transcendental lattices $K_+ = \{x \in L \mid x \perp S_0, \phi_0(x) = x\}$ and $K_- = \{x \in L \mid x \perp S_0, \phi_0(x) = -x\}$ both have one negative square; let

$$\mathbb{H}_+ = \{x \in K_+ \otimes \mathbb{R} \mid x^2 = 1\} \big/ \{\pm 1\} \text{ and}$$
$$\mathbb{H}_- = \{x \in K_- \otimes \mathbb{R} \mid x^2 = 1\} \big/ \{\pm 1\}$$

be the associated hyperbolic spaces. Let $G \subset \Gamma$ be the subgroup formed by elements which do not exchange the sheets of the hyperboloids $\{x \in K_\pm \otimes \mathbb{R} \mid x^2 = 1\}$. Proposition 4.3 implies that G acts faithfully on the product $\mathbb{H}_+ \times \mathbb{H}_-$.

Let R_+ (respectively, R_-) be the subgroup of G generated by the elements which act trivially on $\mathbb{H}_-$ (respectively, $\mathbb{H}_+$) and which act on $\mathbb{H}_+$ (respectively, $\mathbb{H}_-$) by a reflection in a hyperplane. Fix a fundamental polytope $P_+ \subset \mathbb{H}_+$ (respectively, $P_- \subset \mathbb{H}_-$) for the action of R_+ (respectively, R_-). Let T be the subgroup of G formed by the elements mapping the product polytope $P_+ \times P_-$ to itself. We call the G-translates of $P_+ \subset \mathbb{H}_+$ (respectively, $P_- \subset \mathbb{H}_-$) *chambers*.

Proposition 4.4. *The direct product $R_+ \times R_-$ acts simply transitively on all the pairs of chambers. Moreover, G is a semi-direct product $(R_+ \times R_-) \rtimes T$.*

Proof. By Theorem 1.13, the group R_+ (respectively, R_-) acts simply transitively on the set of chambers in $\mathbb{H}_+$ (respectively, $\mathbb{H}_-$). Hence, $R_+ \times R_-$ acts simply transitively on all the pairs of chambers. $\square$

Proposition 4.5. *Let $v \in K_\pm$ be a primitive vector with $v^2 < 0$. There is an element $g \in G$ which acts on K as the reflection R_v if and only if either $v^2 = -2$, or $v^2 = -4$ and $v \equiv s_i' + s_i'' \mod 2L$ for some $i \in \{1, \ldots, m\}$.*

Proof. The discriminant groups A_{S_0} and A_K are 2-periodic. Hence by Lemma 1.9, the vector v must be either of square -2 or of square -4. If $v^2 = -2$, then R_v is an element of G which acts trivially on S_0. If $v^2 = -4$, then R_v acts on A_K as $x \mapsto x + 2(x, a) \cdot a$ for all $x \in A_K$, where $a = [\frac{v}{2}] \in A_K$. Let $b \in A_{S_0}$ be the element which is glued to

$a \in A_K$. A basis for A_{S_0} is given by $\{[\frac{s}{2}]\}_{s\in\sigma_0} \cup \{[\frac{h}{2}]\}$. The reflection R_v extends to an element $g \in G$ if and only if there is a permutation f of σ_0 such that $f \circ \phi_0 = \phi_0 \circ f$ and f glues to R_v, i.e.

$$f(y) = y + 2(y, b) \cdot b \tag{4.1}$$

for all $y \in A_S$. Since the action of R_v on A_K is non-trivial (see Lemma 1.10), the permutation f cannot be trivial. Take a root $s \in \sigma_0$ for which $f(s) \neq s$. Substituting $y = [\frac{s}{2}]$ in (4.1) gives $[\frac{s+f(s)}{2}] = 2([\frac{s}{2}], b) \cdot b$, which is only possible if $b = [\frac{s+f(s)}{2}]$. Therefore, f must be a transposition. Finally, the condition $f \circ \phi_0 = \phi_0 \circ f$ implies that $b = [\frac{s_i'+s_i''}{2}]$ for some $i \in \{1, \ldots, m\}$. □

Definition 4.6. A hyperplane $H \subset \mathbb{H}_\pm$ is called a *reflection wall* if it is the orthogonal complement of a vector $v \in K_\pm$ satisfying the conditions of Proposition 4.5.

4.2.2 Face and degeneration tilings

Definition 4.7. An *admissible configuration of roots* is a pair (σ_+, σ_-), where $\sigma_+ \subset K_+$ and $\sigma_- \subset K_-$ are sets of pairwise orthogonal roots such that the sublattice $S \subset L$ generated by h, σ_0, σ_+ and σ_- is primitive, and there is no root $s \in \sigma_+ \cup \sigma_-$ for which $s + h \in 2L$. Given an admissible configuration of roots (σ_+, σ_-), we use the following notations. Let

- $\sigma = \sigma_0 \cup \sigma_+ \cup \sigma_-$,
- Σ be the sublattice on L generated by σ,
- S be the sublattice of L generated by σ and h, and
- $\phi = \phi_0 \circ R_{s_1} \circ \cdots \circ R_{s_k}$, where $\sigma_+ = \{s_1, \ldots, s_k\}$.

We call the quadruple (L, h, σ, ϕ) the *homological type corresponding to* (σ_+, σ_-).

Let (σ_+, σ_-) be an admissible configuration of roots. Denote by $\mathbb{H}_+^{\sigma_+}$ the subspace of $\mathbb{H}_+$ orthogonal to σ_+, and by $\mathbb{H}_-^{\sigma_-}$ the subspace of $\mathbb{H}_-$ orthogonal to σ_-.

Definition 4.8. A hyperplane $H \subset \mathbb{H}_+^{\sigma_+}$ is a *degeneration wall* if there is a root $v \in L$ orthogonal to h such that $H = \{[x] \in \mathbb{H}_+^{\sigma_+} \mid x \cdot v = 0\}$. A hyperplane $H \subset \mathbb{H}_+^{\sigma_+}$ is a *face wall* if it is the intersection of $\mathbb{H}_+^{\sigma_+}$ with a reflection wall in $\mathbb{H}_+$. Degeneration walls and face walls in $\mathbb{H}_-^{\sigma_-}$ are defined analogously.

Accordingly, we define two tilings of the spaces $\mathbb{H}_+^{\sigma_+}$ and $\mathbb{H}_-^{\sigma_-}$: a *face tiling*, for which tiles are delimited by face walls, and a *degeneration tiling*, whose tiles are delimited by degeneration walls.

Lemma 4.9. *Face walls are degeneration walls. Therefore, the degeneration tiling is a refinement of the face tiling.*

Proof. Let $H \subset \mathbb{H}_\pm^{\sigma_\pm}$ be a face wall, and let $v \in K_\pm$ be a primitive vector orthogonal to the corresponding reflection wall. If $v^2 = -2$, the the vector v can be used in Definition 4.8 to show that H is a degeneration wall. If $v^2 = -4$ and $v + s_i' + s_i'' \in 2L$, then $w = \frac{1}{2}(v + s_i' + s_i'')$ can be used. □

Definition 4.10. Denote by $\mathcal{T}_{\text{face}}^{\sigma+}$ (resp., $\mathcal{T}_{\text{face}}^{\sigma-}$) the set of tiles for the face tiling, and by $\mathcal{T}_{\text{deg}}^{\sigma+}$ (resp., $\mathcal{T}_{\text{deg}}^{\sigma-}$) the set of tiles for the degeneration tiling in $\mathbb{H}_{+}^{\sigma+}$ (resp., $\mathbb{H}_{-}^{\sigma-}$). Let

$$\mathcal{T}_{\text{deg}} = \bigsqcup_{(\sigma_+,\sigma_-)} \{(\sigma_+,\sigma_-)\} \times \mathcal{T}_{\text{deg}}^{\sigma+} \times \mathcal{T}_{\text{deg}}^{\sigma-},$$
$$\mathcal{T}_{\text{face}} = \bigsqcup_{(\sigma_+,\sigma_-)} \{(\sigma_+,\sigma_-)\} \times \mathcal{T}_{\text{face}}^{\sigma+} \times \mathcal{T}_{\text{face}}^{\sigma-},$$

where the unions range over all admissible configurations of roots (σ_+,σ_-).

Note that the group G acts on the set of admissible configurations of roots (σ_+,σ_-), and on the sets $\mathcal{T}_{\text{deg}}$ and $\mathcal{T}_{\text{face}}$.

Proposition 4.11. *Let (σ_+,σ_-) and (σ'_+,σ'_-) be two admissible configurations of roots, with corresponding homological types (L,h,σ,ϕ) and (L,h,σ',ϕ'). An element $g \in G$ maps (σ_+,σ_-) to (σ'_+,σ'_-) if and only if g is an isomorphism between the homological types (L,h,σ,ϕ) and (L,h,σ',ϕ').*

Proof. Suppose g is an isomorphism between the homological types (L,h,σ,ϕ) and (L,h,σ',ϕ'). First, note that g must map non-real roots to non-real roots, i.e. $g(\sigma_0) = \sigma_0$. Since $g \circ \phi = \phi' \circ g$, the isometry g maps the characteristic element of ϕ to the characteristic element of ϕ'. The characteristic elements of ϕ and ϕ' are $\alpha_\phi = \alpha_{\phi_0} + \sum_{s\in\sigma_+}[s]$ and $\alpha_{\phi'} = \alpha_{\phi_0} + \sum_{s\in\sigma'_+}[s]$, respectively. Therefore, g must map σ_+ to σ'_+ and σ_- to σ'_-, and it follows that $g \circ \phi_0 = \phi_0 \circ g$. The converse implication is straightforward. □

The following proposition is a direct application of Theorem 2.18.

Proposition 4.12. *For a fixed admissible configuration (σ_+,σ_-) with corresponding homological type (L,h,σ,ϕ), the rigid isotopy classes of curves of homological type (L,h,σ,ϕ) are in bijection with elements of $\mathcal{T}_{\text{deg}}^{\sigma+} \times \mathcal{T}_{\text{deg}}^{\sigma-}$ modulo automorphisms of the homological type (L,h,σ,ϕ).*

The following corollary is a consequence of Propositions 4.11 and 4.12.

Corollary 4.13. *The elements of the quotient $\mathcal{T}_{\text{deg}}/G$ are in bijection with rigid isotopy classes of real irreducible nodal sextics which can be obtained from sextics of homological type (L,h,σ_0,ϕ_0) by real nodal degenerations.*

4.2.3 Comparison of the two tilings

Let (σ_+,σ_-) be an admissible configuration of roots, and let H be a degeneration wall in $\mathbb{H}_{+}^{\sigma+}$ or $\mathbb{H}_{-}^{\sigma-}$. The goal of this section is to show that either H is a face wall, or there is an element $g \in G$, mapping σ to itself, which acts on $\mathbb{H}_{+}^{\sigma+} \times \mathbb{H}_{-}^{\sigma-}$ by a reflection in H.

Let $v \in L$ be a root orthogonal to h such that H is the intersection of $\mathbb{H}_{+}^{\sigma+}$ or $\mathbb{H}_{-}^{\sigma-}$ with the orthogonal complement of v. We write $v = v_\Sigma + v_K$, where $v_\Sigma \in \Sigma \otimes \mathbb{Q}$, $v_K \in K \otimes \mathbb{Q}$, $v_K^2 < 0$ and $\phi_0(v_K) = \varepsilon v_K$, where ε is $+1$ if H is a wall in $\mathbb{H}_{+}^{\sigma+}$, and -1

if H is a wall in $\mathbb{H}_{-}^{\sigma-}$. Note that $v_\Sigma \in \Sigma^\vee = \frac{1}{2}\Sigma$. Therefore we have $2v_\Sigma \in \Sigma$, with $(2v_\Sigma)^2 \in \{0, 2, 4, 6\}$. Hence, we may suppose that $2v_\Sigma$ is either zero or a sum of up to three roots in σ, i.e. we can write $v_\Sigma = \frac{1}{2}\sum_{s\in\sigma_v} s$, for some subset $\sigma_v \subset \sigma$ with at most three elements.

Definition 4.14. We call $s \in \sigma_v$ a *non-real root* if $s \in \sigma_0$, a *same-side root* if $s \in \sigma_\varepsilon$ and a *opposite-side root* if $s \in \sigma_{-\varepsilon}$.

Lemma 4.15. *Non-real roots only appear in pairs in σ_v, i.e. a non-real root s_i' belongs to σ_v if and only if s_i'' does.*

Proof. We have $v - \varepsilon\phi_0(v) = v_\Sigma - \varepsilon\phi_0(v_\Sigma) = \frac{1}{2}\sum_{s\in\sigma_v}\left(s - \varepsilon\phi_0(s)\right)$. The contribution coming from the real roots in this sum is contained in L since for real roots, we have $\phi_0(s) = \pm s$. Therefore, the sum of the contributions of the non-real roots

$$\frac{1}{2}\sum_{s\in\sigma_v\cap\sigma_0}(s - \varepsilon\phi_0(s)) \tag{4.2}$$

must also lie in L. Since we assume that curves of homological type (L, h, σ_0, ϕ_0) are irreducible, the sublattice $\Sigma_0 \subset L$ is primitive, and hence the expression (4.2) is contained in Σ_0. This implies that s_i' belongs to σ_v if and only if s_i'' does. □

Lemma 4.16. *If the homological type under consideration is dividing, then the number of opposite-side roots in σ_v is even.*

Proof. Since the homological type (L, h, σ_0, ϕ_0) is dividing, the characteristic element of ϕ_0 can be written in the from $\alpha = \sum_{i\in I}(s_i' + s_i'')$, where I is the set indexing the crossing pairs (see Section 2.5). By Lemma 4.15, the non-real roots in σ_v appear in pairs. Therefore, $v \cdot \phi_0(v) = v \cdot \left(\sum_{i\in I}(s_i' + s_i'')\right)$ is even. This implies that the square of the vector $v - \varepsilon\phi_0(v)$ is divisible by 4. Hence, the number of opposite-side roots in σ_v must be even. □

Proposition 4.17. *Let (σ_+, σ_-) be an admissible configuration of roots, and let H be a degeneration wall in $\mathbb{H}_{+}^{\sigma+}$ or $\mathbb{H}_{-}^{\sigma-}$. Then either H is a face wall, or there is an element $g \in G$, mapping σ to itself, which acts on $\mathbb{H}_{+}^{\sigma+} \times \mathbb{H}_{-}^{\sigma-}$ by a reflection in H.*

Proof. Lemmas 4.15 and 4.16 leave the eight possibilities listed in the following table, where omitted entries stand for zeroes.

		(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
number of non-real roots	$\lvert\sigma_v \cap \sigma_0\rvert$	–	–	–	–	–	–	2	2
number of opposite-side roots	$\lvert\sigma_v \cap \sigma_{-\varepsilon}\rvert$	–	–	–	–	2	2	–	–
number of same-side roots	$\lvert\sigma_v \cap \sigma_\varepsilon\rvert$	–	1	2	3	–	1	–	1

In the cases (a), (b), (c) and (d), the vector v lies in K_ε, and hence H is a face wall. In case (g), the vector $2v_K$ is of square -4 and glues to a vector of the form $s_i' + s_i''$, and therefore H is a face wall. In cases (f) and (h), the vector $2v_K$ is of square -2, and therefore H is again a face wall. The only case of a degeneration wall which is not a face wall is case (e), where σ_v consists of two opposite-side roots. Changing the signs, we may suppose that we have $v_\Sigma = s - t$ for some roots $s, t \in \sigma_{-\varepsilon}$. In this case, the isometry $g = R_v \circ R_{\phi_0(v)} = R_{s-t} \circ R_{2v_K}$ belongs to G, maps the sets σ_+ and σ_- to themselves, and acts on $\mathbb{H}_+^{\sigma_+} \times \mathbb{H}_-^{\sigma_-}$ as a reflection in the wall H. ☐

Corollary 4.18. *The map $\mathcal{T}_{\text{deg}} \to \mathcal{T}_{\text{face}}$ which maps a degeneration tile to the face tile containing it induces a bijection between the quotients $\mathcal{T}_{\text{deg}}/G$ and $\mathcal{T}_{\text{face}}/G$.*

Definition 4.19. A pair of faces $f_+ \times f_-$ of $P_+ \times P_-$ is called *admissible* if (σ_+, σ_-) is an admissible configuration of roots, where σ_+ and σ_- are the sets of roots defining the faces $f_+ \subset P_+$ and $f_- \subset P_-$, respectively.

Proposition 4.20. *The equivalence classes of elements in $\mathcal{T}_{\text{face}}$ modulo G are in bijection with equivalence classes of pairs of admissible faces of $P_+ \times P_-$ modulo T.*

Proof. For each element $(\sigma_+, \sigma_-, f_+, f_-) \in \mathcal{T}_{\text{face}}$, there is a unique pair (Q_+, Q_-) of chambers such that f_+ is a face of the polytope $Q_+ \subset \mathbb{H}_+$ defined by σ_+, and f_- is a face of the polytope $Q_- \subset \mathbb{H}_-$ defined by σ_-. This defines a G-equivariant map from $\mathcal{T}_{\text{face}}$ to the set of pairs of chambers. Then we can identify the admissible faces of $P_+ \times P_-$ with the elements of $\mathcal{T}_{\text{face}}$ mapped to the pair $P_+ \times P_-$ by this map.

Since G acts transitively on the set of pairs of chambers (Proposition 4.4), every equivalence class in $\mathcal{T}_{\text{face}}/G$ is represented by an admissible pair of faces if $P_+ \times P_-$, and this pair of faces is uniquely defined modulo T, the stabilizer of $P_+ \times P_-$. ☐

Combining Corollary 4.13, Corollary 4.18 and Proposition 4.20, we obtain the following theorem, which is the main result of this section:

Theorem 4.21. *Let (L, h, σ_0, ϕ_0) be a homological type of real irreducible nodal dividing sextics without real nodes. The pairs of admissible faces of $P_+ \times P_-$ modulo T are in bijection with rigid isotopy classes of real irreducible nodal sextics which can be obtained from sextics of homological type (L, h, σ_0, ϕ_0) by real nodal degenerations.*

4.2.4 Curves corresponding to non-admissible pairs of faces

Theorem 4.21 establishes a correspondence between *admissible* pairs of faces of $P_+ \times P_-$ and rigid isotopy classes of *irreducible nodal* degenerations. Points on non-admissible pairs of faces correspond to degenerations which are either not nodal or not irreducible. We cannot prove that there is a bijection between rigid isotopy classes and equivalence classes of pairs of faces in general. At least, Proposition 4.17 (the equivalence between the face tiling and the degeneration tiling) holds for a slightly larger class of faces, as we explain below. This will be useful in Chapter 6 for deducing the isotopy type of certain curves.

Definition 4.22. Consider a pair (σ_+, σ_-), where $\sigma_+ \subset K_+$ and $\sigma_- \subset K_-$ are sets of pairwise orthogonal roots. Let Σ denote the (not necessarily primitive) sublattice of L

generated by $\sigma = \sigma_0 \cup \sigma_+ \cup \sigma_-$, and let $\tilde{\Sigma}$ denote its primitive closure. We call (σ_+, σ_-) an *almost admissible configuration of roots* if there is no root $s \in \tilde{\Sigma}$ for which $s + h \in 2L$.

If (σ_+, σ_-) is an almost admissible configuration of roots, we define the spaces $\mathbb{H}_+^{\sigma_+}$ and $\mathbb{H}_-^{\sigma_-}$ and the face and degeneration tilings on them like in the case of admissible configurations.

Remark 4.23. Proposition 4.17 still holds if we consider an almost admissible configuration of roots (σ_+, σ_-). Consequently, one can associate a rigid isotopy class of a degeneration with each pair of faces of $P_+ \times P_-$ which is defined by an almost admissible configuration of roots.

Proof. Let (σ_+, σ_-) be an almost admissible configuration of roots, and let v be a vector defining a degeneration wall. Consider the decomposition $v = v_\Sigma + v_K$, where $v_\Sigma \in \Sigma \otimes \mathbb{Q}$, $v_K \in K \otimes \mathbb{Q}$. We have $v_\Sigma \in \tilde{\Sigma}^\vee$. Since $\tilde{\Sigma}$ is an over-lattice of Σ, we have $\tilde{\Sigma}^\vee \subset \Sigma^\vee$. Therefore, we can still conclude that $v_\Sigma \in \Sigma^\vee = \frac{1}{2}\Sigma$. The rest of the proof is not affected by the fact that Σ is not a primitive sublattice of L. $\square$

4.3 Donaldson's trick

In this section, we use the hyperkähler structure of K3 surfaces to deduce certain properties of the Coxeter graph of the polytopes P_+ and P_-. More precisely, we use the fact that on a K3 surface obtained from a real sextic one can define a different complex structure J with respect to which the real structure c turns out to be holomorphic, while the deck transformation τ becomes anti-holomorphic. This section is inspired by the very similar treatment in Chapter 15 of the book by Degtyarev, Itenberg, and Kharlamov [10], where the original idea is attributed to Donaldson [11].

4.3.1 Hyperkähler structure for smooth K3 surfaces

Theorem 4.24 (Yau [47])**.** *Let X be a K3 surface, and let κ be a Kähler class on X. There is a unique Kähler form ϕ of class κ whose associated metric g is Ricci-flat. The metric g is called the* Kähler-Einstein-metric *of (X, κ).*

For a K3 surface X with complex structure I, let ω_I denote a non-zero holomorphic $(2,0)$-form on X (which is uniquely defined up to multiplication by a scalar), and let $E(\omega_I) \subset H^2(X, \mathbb{R})$ denote the positive definite oriented two-plane with oriented basis $(\mathrm{Re}\,\omega_I, \mathrm{Im}\,\omega_I)$. The hyperkähler structure on X implies that the complex structures on X with respect to which g is Kähler are parameterized by a sphere. More precisely, we have the following theorem.

Theorem 4.25 (Barth et al. [3, Section VIII.13])**.** *Let X be a K3 surface with complex structure I, and let κ_I be a Kähler class on X. Let g be the Kähler-Einstein metric of (X, κ_I), and let $E \subset H^2(X, \mathbb{R})$ the positive definite three-dimensional subspace spanned* *by κ_I and $E(\omega_I)$.*

(1) *If J is another complex structure on X with respect to which the metric g is Kähler, then E is the direct sum of $E(\omega_J)$ and $\langle \kappa_J \rangle$, where κ_J is the Kähler class associated with g and J.*

(2) *The map sending such a complex structure J to the subspace $E(\omega_J) \subset E$ gives a bijection between complex structures on X with respect to which the metric g is Kähler, and oriented two-planes contained in E.*

Let us fix a K3 surface X with complex structure I, and let κ be a Kähler class on X. Let g be the Kähler-Einstein metric of (X, κ), and let ϕ denote the corresponding Kähler form. The following two lemmas are direct consequences of Theorems 4.24 and 4.25.

Lemma 4.26. *Let f be an (anti-)holomorphic involution on X, i.e. an involution such that $f^*I = \varepsilon I$ with $\varepsilon = \pm 1$, such that $f^*\kappa = \varepsilon\kappa$. Then $f^*\phi = \varepsilon\phi$ and hence f is an isometry for g.*

Proof. Note that $\varepsilon f^*\phi$ is a Kähler form of class κ on X whose Kähler metric f^*g is Ricci-flat. The uniqueness part of Theorem 4.24 then implies $\varepsilon f^*\phi = \phi$. It follows that $(f^*g)(u,v) = (f^*\phi)(u,(f^*I)v) = \varepsilon\phi(u, \varepsilon I v) = \phi(u, Iv) = g(u,v)$. □

Lemma 4.27. *Let f be an isometric involution on (X, g) such that $f^*\kappa = \varepsilon\kappa$, with $\varepsilon = \pm 1$. Then f is (anti-)holomorphic, i.e. we have $f^*I = \varepsilon I$.*

Proof. Since the metric g is preserved by f, it is Kähler for the complex structure εf^*I, and the corresponding Kähler polarization is κ. Then part (2) of Theorem 4.25 implies that $\varepsilon f^*I = I$. □

Let (X, c, h) be the real polarized K3 surface obtained from a non-singular real sextic $C \subset \mathbb{CP}^2$, with complex structure I and real period (ω_+, ω_-). Let g be the Kähler-Einstein metric of (X, h). Recall that $\tau : X \to X$ denotes the deck transformation of the covering $X \to \mathbb{CP}^2$. Let J and K be the complex structures on X corresponding to the oriented planes $\langle \omega_-, h \rangle$ and $\langle h, \omega_+ \rangle$ respectively (via the bijection mentioned in Theorem 4.25).

Proposition 4.28. *The involutions τ, c and $c\tau$ are (anti-)holomorphic with respect to the complex structures I, J and K according to the following table, where "+" and "−" stand for holomorphic and anti-holomorphic maps, respectively.*

complex structure	I	J	K
Kähler class	h	ω_+	ω_-
period plane	$\langle \omega_+, \omega_- \rangle$	$\langle \omega_-, h \rangle$	$\langle h, \omega_+ \rangle$
τ	$+$	$-$	$-$
c	$-$	$+$	$-$
$c\tau$	$-$	$-$	$+$

Proof. By Lemma 4.26, the involutions τ, c and $c\tau$ preserve the metric g. Then the claim follows from Lemma 4.27. □

4.3.2 Generalization to nodal K3 surfaces

We would like to apply Proposition 4.28 to K3 surfaces obtained from nodal sextics. However, such K3 surfaces are singular, or if we desingularize them, the class h is not Kähler, and therefore Theorems 4.24 and 4.25 cannot be applied. Fortunately, these theorems have been generalized to generalized K3 surfaces (i.e., compact complex surfaces with Du Val singularities whose desingularization is a K3 surface) by Kobayashi and Todorov [25].

In the following, we consider a *nodal* K3 surface Y, i.e. a generalized K3 surface whose only singularities are ordinary double points.

Remark 4.29. In differential geometric terms, nodal K3 surfaces are orbifolds: in the neighborhood of an ordinary double point, a surface looks like the quotient of $\mathbb{C}^2$ by the involution $(u, v) \leftrightarrow (-u, -v)$. This allows us to define differential geometric objects such as differential forms and metrics on nodal surfaces in an *orbifold sense*: for example, an orbifold differential form in the neighborhood of a node is defined to be a differential form on $\mathbb{C}^2$ which is invariant under the map $(u, v) \mapsto (-u, -v)$. Likewise, orbifold Riemannian metrics, orbifold Kähler forms etc. can be defined.

Definition 4.30. Let Y be a generalized K3 surface, and let $\pi : X \to Y$ be its desingularization. A class $\kappa \in H^2(Y, \mathbb{R})$ is called a *generalized Kähler class* if $\pi^*\kappa$ belongs to the boundary of the Kähler cone of X, and we have $\pi^*\kappa \cdot C = 0$ for a smooth rational curve $C \subset X$ if and only if C is contracted by π.

Theorem 4.31 (Kobayashi and Todorov [25])**.** *Let Y be a nodal K3 surface, $\pi : X \to Y$ its desingularization, and κ a generalized Kähler class on Y. There exists a unique orbifold Kähler form on Y whose associated orbifold metric g is Ricci-flat. The metric g is called the* orbifold Kähler-Einstein metric *of (Y, κ).*

Theorem 4.32 (Kobayashi and Todorov [25])**.** *Let Y be a nodal K3 surface with complex structure I, and let κ_I be a generalized Kähler class on Y. Let g be the orbifold Kähler-Einstein metric of (Y, κ_I), and let $E \subset H^2(Y, \mathbb{R})$ the positive definite three-dimensional subspace spanned by κ_I and $E(\omega_I)$.*

1. *If J is another complex structure on Y with respect to which the orbifold metric g is Kähler, then E is the direct sum of $E(\omega_J)$ and $\langle \kappa_J \rangle$, where κ_J is the Kähler class associated with g and J.*
2. *The map sending a such a complex structure J to the subspace $E(\omega_J) \subset E$ induces a bijection between complex structures on X with respect to which the orbifold metric g is Kähler, and oriented two-planes contained in E.*

Thanks to these generalizations, Lemmas 4.26 and 4.27 and Proposition 4.28 and their proofs apply to nodal K3 surfaces, and in particular to the nodal K3 surfaces obtained from nodal sextics.

4.3.3 Representing vanishing cycles by smooth rational curves

Let (X, h, c) be the real weakly polarized K3 surface obtained from a real nodal sextic $C \subset \mathbb{CP}^2$ without real nodes, and let μ be a real marking of (X, h, c). Let $P_+ \subset \mathbb{H}_+$ and $P_- \subset \mathbb{H}_-$ be the polytopes defined in Section 4.2. We may assume that the real period (ω_+, ω_-) of (X, μ) is contained in $P_+ \times P_-$. In the following, we consider degenerations of X corresponding to facets of P_+. The same applies *mutatis mutandis* to degenerations corresponding to facets of P_-.

Lemma 4.33. *For each short root $v \in K_+$ defining a facet of the polytope P_+, there is a real degeneration of X whose vanishing cycle is Poincaré dual to v.*

Proof. Since Ω^0_ϕ is a fine moduli space for (L, h, σ, ϕ)-marked K3 surfaces (Corollary 2.16), such a degeneration can be obtained by choosing a path connecting ω_+ with a point contained in the facet of P_+ defined by v. □

In the following, we suppose that C has been chosen such that the orthogonal complement of $\langle \omega_-, h\rangle$ in L is K_+. This is a very general condition, i.e., it holds for curves whose periods avoid a countable union of hyperplanes in $P_+ \times P_-$.

Proposition 4.34. *Each short root $v \in K_+$ defining a facet of the polytope P_+ is represented by a unique smooth sphere $S \subset Y$ which is holomorphic (i.e., a smooth rational curve) with respect to the complex structure J.*

Proof. Let Y be the nodal K3 surface obtained from X be contracting all the nodal classes. Let Y_J denote the nodal K3 surface Y endowed with the complex structure J. Because of the “very general” assumption above, the Néron-Severi group of Y_J is K_+. Let X_J denote the desingularization of Y_J. Its Néron-Severi group is the primitive hull of $K_+ \oplus \Sigma_0$. The walls used to define the polytope P_+ are precisely the hyperplanes which are orthogonal to roots in the Néron-Severi group of X_J (see Proposition 4.5). Therefore, the cone over the hyperbolic polytope P_+ coincides with the face of the ample cone of X_J orthogonal to Σ_0. Moreover, a root defines a facet of the ample cone if and only if it is represented by smooth rational curve (see for example Huybrechts [16, Chapter 8]). □

Proposition 4.35. *Let $S \subset X$ be a vanishing sphere as in Proposition 4.34. The sphere S is stable under the maps c and τ, and the triple (S, c, τ) is conjugate to one of the five triples $(\mathbb{CP}^1, c, \tau)$ given below, where z is an affine coordinate on $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$.*

	(a)	(b)	(c)	(d)	(e)
$c\,(z)$	z	z	$-z$	$-z$	$-z$
$\tau\,(z)$	$\bar{z}$	$-1/\bar{z}$	$\bar{z}$	$1/\bar{z}$	$-1/\bar{z}$
$(c\tau)(z)$	$\bar{z}$	$-1/\bar{z}$	$-\bar{z}$	$-1/\bar{z}$	$1/\bar{z}$
$(\mathbb{CP}^1)^c$	$\mathbb{CP}^1$	$\mathbb{CP}^1$	$\{0\} \cup \{\infty\}$	$\{0\} \cup \{\infty\}$	$\{0\} \cup \{\infty\}$
$(\mathbb{CP}^1)^\tau$	$\mathbb{R} \cup \{\infty\}$	$\emptyset$	$\mathbb{R} \cup \{\infty\}$	S^1	$\emptyset$
$(\mathbb{CP}^1)^{c\tau}$	$\mathbb{R} \cup \{\infty\}$	$\emptyset$	$\mathrm{i}\mathbb{R} \cup \{\infty\}$	$\emptyset$	S^1
$(\mathbb{CP}^1)^{\langle c,\tau\rangle}$	$\mathbb{R} \cup \{\infty\}$	$\emptyset$	$\{0\} \cup \{\infty\}$	$\emptyset$	$\emptyset$
	contraction	–	conjunction	creation	–

Proof. Since c and τ are (anti-)holomorphic with respect to J, they map the J-holomorphic sphere S to a J-holomorphic sphere. Since the class of S is (anti-)invariant under c and τ, and smooth rational curves of K3 surfaces are rigid, this implies that S is invariant under c and τ. For the classification of the triple (S, c, τ), first note that (anti-)holomorphic involutions on $\mathbb{CP}^1$ are linear. There are two conjugacy classes of holomorphic involutions: the identity and $z \mapsto -z$. There are two conjugacy classes of anti-holomorphic involutions: the standard one $z \mapsto \bar{z}$ and the antipodal map $z \mapsto -1/\bar{z}$. If c is the identity, then the two possibilities are (a) and (b), according to the conjugacy class of τ. Otherwise, we may assume that $c(z) = -z$. Since c and τ commute, the points 0 and ∞ must be either fixed or interchanged by τ. This leaves the possibilities (c), (d) and (e) up to conjugation. $\square$

Lemma 4.36. *Let $S \subset X$ be a vanishing sphere as above, and let $\{X_t\}_{t\in[0,1]}$ be a corresponding real Lefschetz degeneration. Then there is a $\langle c, \tau\rangle$-equivariant homeomorphism between the nodal K3 surface X_0 and the topological quotient X/S.*

Proof. The degeneration can be chosen such that ω_+ moves to the boundary of the polytope P_+ while ω_- stays fixed. Therefore, the complex structure J does not change along the degeneration, i.e. the family $\{X_t\}_{t\in(0,1]}$ is trivial with respect to the complex structure J, and the trivialization is unique by the Global Torelli Theorem. Hence, we have a canonical way of identifying the different fibers X_t. Also, the (anti-)holomorphic maps c and τ remain constant with respect to this trivialization. This allows us to define a canonical retraction from $X = X_1$ to X_0 which contracts the sphere S to a point. $\square$

Lemma 4.37. *If v is a vector defining a face of the polytope P_+ such that $v \not\equiv h \mod 2$, then the $\langle c, \tau\rangle$-action on the corresponding vanishing sphere S is of type* (a), (c) *or* (d) *(cf. Proposition 4.35). If it is of type* (a)*, then the corresponding degeneration is a contraction of the empty oval $\pi(S^\tau)$. If it is of type* (c)*, then the corresponding degeneration is a conjunction of the ovals containing the two points $\pi(S^c)$, along the path $\pi(S^{c\tau})$. If it is of type* (d)*, then the corresponding degeneration is a creation of an isolated node, in the region containing the point $\pi(S^c)$.*

Proof. By Proposition 4.35, the $\langle c, \tau \rangle$-action on S must be of one of the types (a)–(e). The condition $v \not\equiv h \mod 2$ implies that the vanishing of S is a nodal degeneration of the sextic (see Proposition 2.1). Using Lemma 4.36, this implies that S has to intersect the ramification curve $\pi^{-1}(C) \subset X = X^\tau$ in a circle. Therefore, types (b) and (e) can be excluded. The correspondence between types (a), (c) and (d) on one hand and contractions, conjunctions and creations on the other hand also follows from Lemma 4.36. $\square$

Lemma 4.38. *Let X be a smooth surface, and let f be a holomorphic involution on X whose fixed locus X^f is a disjoint union of smooth curves. A smooth rational curve $S \subset X$ invariant under f is either a connected component of X^f, or it intersects X^f transversely in two points.*

Proof. The restriction $f|_S$ is a holomorphic involution on $S \cong \mathbb{CP}^1$. Up to conjugation, the only such involutions are the identity and the map given by $z \mapsto -z$ in an affine chart, which has two fixed points. Let p be one of these fixed points, and let $\mathrm{d}f_p : T_pX \to T_pX$ denote the action of f on the tangent space to X in p. We have $\mathrm{d}f_p|_{T_pS} = -1$ and $\mathrm{d}f_p|_{T_pX^f} = 1$. Therefore, S and X^f intersect transversely. $\square$

Lemma 4.39. *Let X be a smooth surface, and let f be a holomorphic involution on X whose fixed locus is a disjoint union of smooth curves. The intersection number between two curves $S_1, S_2 \subset X$ which are invariant under f, but not point-wise fixed, is even.*

Proof. Let Y be the quotient surface X/f, and let $\pi : X \to Y$ denote the projection. The condition on the fixed locus guarantees that Y is non-singular. Let $T_1, T_2 \subset Y$ denote the images of S_1, S_2 under π. We have $\pi^* T_i = S_i$ and $\pi_* S_i = 2T_i$, since the restriction $\pi|_{S_i} : S_i \to T_i$ is a map of degree 2. Therefore, the intersection number $S_1 \cdot S_2 = \pi^* T_1 \cdot S_2 = T_1 \cdot \pi_* S_2 = 2(T_1 \cdot T_2)$ is even. $\square$

The following proposition summarizes the implications for the Coxeter graphs of the polytopes P_+ and P_- of dividing real irreducible sextics without real nodes.

Proposition 4.40. *The short vertices of the Coxeter graphs of P_+ and P_- fall into three types:*

- *those corresponding to contractions of empty ovals ("contraction vectors", ◉),*
- *those corresponding to conjunctions ("conjunction vectors", ○), and*
- *those not corresponding to a nodal degeneration ("exceptional vectors", ○).*

Let $v_1, \ldots, v_k$ denote the contraction vectors, and let x be the class of the component of X^c which is not a sphere.

1. *The contraction vectors are pairwise orthogonal.*
2. *A conjunction vector intersects $x + v_1 + \cdots + v_k$ twice.*
3. *Conjunction vectors intersect each other evenly.*

In particular, after removing the exceptional vectors and all the multiple edges, the Coxeter graph is a bipartite graph.

Proof. The fact that there are three types follows from Lemma 4.37. (Note that creations cannot occur for dividing curves.) The statement (1) follows from the fact that the corresponding spheres are pairwise disjoint. The statement (2) follows from Lemma 4.38, and the statement (3) follows from Lemma 4.39. $\square$

CHAPTER 5

Descent

In this chapter, we study the method of *descent* which allows us in certain situations to classify the faces of a hyperbolic polytope, up to certain symmetries of the polytope, in the case where the Coxeter graph of the polytope is infinite. The method we present is inspired by and a generalization of a similar method used by Itenberg [19] to classify sextics with one node. The Sections 5.1 and 5.2 are preparations for the main part of this chapter, which starts with Section 5.3.

5.1 Polytopes emanating from a subspace

Let V be a finite-dimensional real vector space endowed with a hyperbolic bilinear form, and let $\mathbb{H}_V$ be the associated hyperbolic space. Let $\mathfrak{W}$ be a locally finite collection of hyperplanes in $\mathbb{H}_V$. (We do not require here that $\mathfrak{W}$ is invariant under reflections in hyperplanes belonging to $\mathfrak{W}$.) It divides the space $\mathbb{H}_V$ into a number of polytopes; let $\mathscr{P}_{\mathfrak{W}}$ denote the set of these polytopes. Let $E \subset V$ be a hyperbolic subspace of V such that $\mathbb{H}_E \subset \mathbb{H}_V$ is the intersection of some of the hyperplanes belonging to $\mathfrak{W}$, and let N be the orthogonal complement of E in V. The walls of $\mathfrak{W}$ containing $\mathbb{H}_E$ induce a subdivision of the Euclidean space N into a finite number of cones; let $\mathscr{C}_{\mathfrak{W}}(E)$ denote the set of these cones. The walls of $\mathfrak{W}$ intersecting $\mathbb{H}_E$ transversely define some tiling of $\mathbb{H}_E$; let $\mathscr{T}_{\mathfrak{W}}(E)$ denote the set of tiles.

Definition 5.1. We say that a polytope $P \in \mathscr{P}_{\mathfrak{W}}$ *touches* the subspace $\mathbb{H}_E$ if P intersects $\mathbb{H}_E$ in a tile, i.e., the dimension of $P \cap \mathbb{H}_E$ is equal to the dimension of $\mathbb{H}_E$. We call two polytopes *adjacent* if they intersect each other in a common face of codimension one.

Lemma 5.2. *For each tile $T \in \mathscr{T}_{\mathfrak{W}}(E)$ and each cone $C \in \mathscr{C}_{\mathfrak{W}}(E)$, there is a unique polytope $P \in \mathscr{P}_{\mathfrak{W}}$ touching $\mathbb{H}_E$ in T such that the orthogonal projection of P to N is contained in the cone $C \subset N$.*

Definition 5.3. In the situation of the previous proposition, we say that P is *the polytope emanating from T in the direction C*.

Definition 5.4. Let $P, P' \in \mathscr{P}_{\mathfrak{W}}$ be two polytopes touching $\mathbb{H}_E$. We say that P and P' are *tile adjacent* if they emanate from adjacent tiles in the same direction, and they are *direction adjacent* if they emanate from the same tile in adjacent directions.

Note that this definition only makes sense relative to a fixed subspace $\mathbb{H}_E$. Also, note that two polytopes which are tile adjacent with respect to a subspace $\mathbb{H}_E$ are not in general adjacent as polytopes in $\mathbb{H}_V$. The following statement is a consequence of Lemma 5.2.

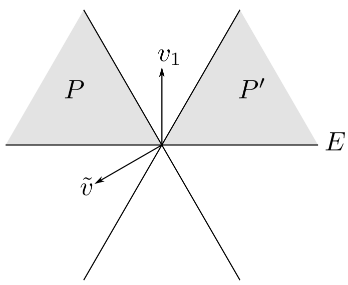
Proposition 5.5. *Any two polytopes $P, P' \in \mathscr{P}_{\mathfrak{W}}$ touching $\mathbb{H}_E$ can be connected by a sequence of polytopes $\{P_i\}_{i=0}^n$ touching $\mathbb{H}_E$, where $P_0 = P$ and $P_n = P'$, such that for each $i \in \{0, \ldots, n-1\}$, the polytopes P_i and P_{i+1} are either tile adjacent or direction adjacent.*

5.2 "Reflecting back" a chamber

Note that the considerations in the preceding subsection can be adapted to the case where the ambient space V is Euclidean instead of hyperbolic. In this section we consider the special case where $\mathfrak{W}$ is the system of walls of a finite reflection group and the subspace E is one-dimensional.

Let R be a root system in an Euclidean space V, and Let $\mathfrak{W}$ be the set of hyperplanes in V orthogonal to the roots. Let $(v_1, \ldots, v_n, \tilde{v})$ be a basis of R, defining a chamber $P \subset V$. Let $E \subset V$ be the line orthogonal to the vectors $v_1, \ldots, v_n$. There are two possibilities for a wall: either it contains the line E, or it intersects it in the origin. Hence, here the set of "tiles" $\mathscr{T}_{\mathfrak{W}}(E)$ is formed by the two rays in E emanating from the origin. Let us call the ray in which P touches E the *negative ray*, and the opposite ray the *positive ray*. Using the terminology introduced in Definition 5.3, let P' be the chamber emanating from the positive ray in the same direction in which P emanates from the negative ray. Let $\rho(v_1, \ldots, v_n; \tilde{v})$ denote the element in the Weyl group of R which maps P to P'.

Example 5.6. Let $V = \mathbb{R}^2$, and let R be a root system of type A_2, with basis $(v_1; \tilde{v})$. We have $\rho(v_1; \tilde{v}) = R_{\tilde{v}} \circ R_{v_1}$, as can be verified in the following illustration.



The way in which $\rho(v_1, \ldots, v_n; \tilde{v})$ can be written as a composition of the basic reflections $R_{v_1}, \ldots, R_{v_n}, R_{\tilde{v}}$ only depends on the Gram matrix of the basis $(v_1, \ldots, v_n, \tilde{v})$. In particular, $\rho(v_1, \ldots, v_n; \tilde{v})$ behaves well under conjugation: for an isometry g, we have $g \circ \rho(v_1, \ldots, v_n; \tilde{v}) \circ g^{-1} = \rho(g(v_1), \ldots, g(v_n); g(\tilde{v}))$.

5.3 Setting for the descent

Consider a hyperbolic space $\mathbb{H}_V$, and let $\Gamma \subset \mathrm{Isom}(\mathbb{H}_V)$ be a hyperbolic reflection group (cf. Section 1.2) acting on it. Let $P_0 \subset \mathbb{H}_V$ be a fundamental polytope for this action. Consider a face $T_0 \subset P_0$, defined by the roots $v_1, \dots, v_k$, and let $W \subset V$ denote the orthogonal complement of $\{v_1, \dots, v_k\}$, so that $\mathbb{H}_W$ is the smallest subspace of $\mathbb{H}_V$ containing T_0. Let G be the subgroup of Γ fixing each of the roots $v_1, \dots, v_k$. It acts faithfully on the hyperbolic subspace $\mathbb{H}_W \subset \mathbb{H}_V$. Let $\bar{\Gamma}$ be the subgroup of G generated by reflections in hyperplanes. Let $\bar{P} \subset \mathbb{H}_W$ be the fundamental polytope for $\bar{\Gamma}$ containing the face T_0, and let G_0 be the subgroup of G which maps $\bar{P}$ to itself. Then G is a semi-direct product $G = G_0 \ltimes \bar{\Gamma}$.

The goal of the descent is to represent equivalence classes of faces of $\bar{P}$ modulo G_0 by combinatorial data in the Coxeter graph of the polytope P_0. The details are given in the following subsections.

5.4 Tiles and tile markings

The polytope $\bar{P}$ is tessellated by faces of the Γ-translates of P_0. For each tile T, there is a unique such translate P emanating from T in the same direction as P_0 emanates from T_0, i.e. such that we have $(x, v_i) \geq 0$ for all $x \in P$ for all $i = 1, \dots, n$. Since Γ acts simply transitively on its chambers, there is a unique element $g_T \in \Gamma$ mapping P_0 to P. In particular, we can associate with each tile T a face $g_T^{-1}(T)$ of P_0.

Definition 5.7. The *tile marking* corresponding to a tile T is defined as $\mu_T = (g_T^{-1}(v_1), \dots, g_T^{-1}(v_k))$, i.e. it is the (ordered) collection of vectors defining $g_T^{-1}(T)$ as a face of P_0. We visualize a tile marking as a collection of vertices in the Coxeter graph of P_0.

Lemma 5.8. *The tile markings of two tiles $T_1, T_2 \subset \bar{P}$ coincide if and only T_1 is mapped to T_2 by an element of G_0.*

Proof. If T_1 and T_2 are two tiles whose tile markings coincide, then the composition $g = g_{T_2} \circ g_{T_1}^{-1}$ maps v_i to itself for $i \in \{1, \dots, n\}$, i.e. it belongs to G. Since g maps $T_1 \subset \bar{P}$ to $T_2 \subset \bar{P}$, it must map the polytope $\bar{P}$ to itself, and hence belong to G_0. Conversely, if $g \in G_0$ maps a tile T_1 to a tile T_2, then it must also map the polytope P_1 emanating from T_1 to the polytope P_2 emanating from T_2. Since $G_0 \subset \Gamma$ acts simply transitively on these polytopes, we have $g = g_{T_2} \circ g_{T_1}^{-1}$, and this implies that the markings of T_1 and of T_2 coincide. □

To determine which ordered collections of vertices in the Coxeter graph of P are tile markings of some tile $T \subset \bar{P}$, we use an inductive approach: We know that the collection $(v_1, \dots, v_k)$ is a tile marking, corresponding to the "base tile" T_0. Any other tile T can be connected to the base tile T_0 by a sequence of tiles $\{T_i\}_{i=0}^{n}$, with $T_n = T$, where subsequent tiles are adjacent to each other, i.e. have a facet in common. Therefore, it is sufficient to study how the marking changes as we pass from a tile T_1 to an adjacent tile

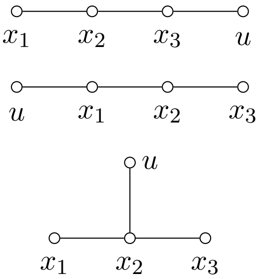
T_2. This defines some elementary transitions from one marking to another one, which we call *moves*. Then, the admissible markings are precisely those which can be reached by a sequence of moves starting from the initial collection $(v_1, \ldots, v_k)$.

To describe the allowed moves, consider an admissible marking μ_1 corresponding to a tile T_1, and let T_2 be a tile adjacent to T_1. The intersection $T_1 \cap T_2$ is a face of the polytope $P_1 = g_{T_1}(P_0)$, defined by the vectors $v_1, \ldots, v_k$ plus one additional vector, say $\tilde{v}$. There is a unique element in the finite reflection group generated by $\{R_{v_1}, \ldots, R_{v_k}, R_{\tilde{v}}\}$ which maps an interior point of $g_{T_1}(P)$ to an interior point of $g_{T_2}(P)$. Namely, this is the element $\rho(v_1, \ldots, v_k; \tilde{v})$ described in Section 5.2. Since Γ acts simply transitively on these polytopes, we must have $g_{T_2} = \rho(v_1, \ldots, v_k; \tilde{v}) \circ g_{T_1}$ and hence $\mu_2 = (g_1^{-1} \circ \rho(v_1, \ldots, v_k; \tilde{v})^{-1} \circ g_1)\,(\mu_1) = \rho(\mu_1; g_1^{-1}(\tilde{v}))^{-1}(\mu_1)$. In particular, μ_2 only depends on μ_1 and the choice of the additional vector $u = g_1^{-1}(\tilde{v})$ in the Coxeter graph of P_0.

The tile markings which can be obtained by a move from μ_1 are of the form $\mu_2 = \rho(\mu_1; u)^{-1}(\mu_1)$, where u is any vector in the Coxeter graph of P_0 such that u does not belong to μ_1, the vectors $\mu_1 \cup \{u\}$ form an elliptic subscheme of the Coxeter graph of P_0, and the root system generated by μ_1 and u does not contain a root orthogonal to μ_1. The last condition ensures that the facet of T_1 represented by u does not lie on the boundary of $\bar{P}$. In concrete examples, the possible moves can be described explicitly:

Example 5.9. Suppose that $T_0 \subset P_0$ is a face of codimension one, defined by a single vector v_1. This is the particular case of descent which was introduced by Itenberg [19] to classify sextics with one node, and it served as an inspiration for the more general framework we present here. In this case, a tile marking is just the choice of one vertex x_1 in the Coxeter graph of P_0. For a supplementary vector u, the dihedral angle between H_{x_1} and H_u could be $\pi/2$, $\pi/3$ or $\pi/4$. If it is $\pi/2$, then u is orthogonal to x_1, corresponding to a face at the boundary of $\bar{P}$. If it is $\pi/4$, then $R_{\tilde{v}}(v_1)$ is orthogonal to v_1, also corresponding to a face at the boundary of $\bar{P}$. The only remaining possibility is an angle of $\pi/3$, i.e. that u is connected to x_1 by a simple edge in the Coxeter graph of P_0. In this case, we have $\rho(x_1; u) = R_u \circ R_{x_1}$ (cf. Example 5.6). Hence, the new tile marking is $\mu_2 = \rho(x_1; u)^{-1}(\mu_1) = (R_u \circ R_{x_1})^{-1}(x_1) = R_{x_1} \circ R_u(x_1) = R_{x_1}(x_1 + u) = u$. This means that whenever a vertex u is connected by a simple edge to the marked vertex, the tile marking can move to u. Therefore, the possible tile markings are all the vertices in the Coxeter graph of P_0 which are connected to the vertex v_1 by a path of simple edges.

Example 5.10. Let $T_0 \subset P_0$ be a face of codimension 3, defined by three vectors (v_1, v_2, v_3) forming an A_3 root system. For simplicity, assume that the Coxeter graph of P_0 does not contain multiple edges, i.e. the only dihedral angles occurring are $\pi/2$ and $\pi/3$. Consider a marking $\mu_1 = (x_1, x_2, x_3)$. There are three types of moves, depending on how the additional vector u intersects x_1, x_2 and x_3:



(a) $\mu_2 = (x_2, x_3, u)$

(b) $\mu_2 = (u, x_1, x_2)$

(c) $\mu_2 = (x_3, x_2, x_1)$

5.5 Representing faces by faces of tiles

Definition 5.11. Let F be a face of $\bar{P}$, and let $T \subset \bar{P}$ be a tile touching F, i.e., such that $\dim T \cap F = \dim F$. For such a pair (T, F), let $\{f_1, \ldots, f_m\}$ be the additional roots such that the vectors $v_1, \ldots, v_k, f_1, \ldots, f_m$ define $T \cap F$ as a face of $g_T(P_0)$. The *face marking* of the pair (T, F) is the pair $(\mu_T, \nu_{(T,F)})$, where μ_T is the tile marking (cf. Definition 5.7) and $\nu_{(T,F)} = \{g_T^{-1}(f_1), \ldots, g_T^{-1}(f_m)\}$. That is, a face marking consists of an *ordered* collection μ of vertices in the Coxeter graph of P_0, describing the tile T, and an additional *unordered* collection ν of vertices in the Coxeter graph of P_0 describing the face F.

Lemma 5.12. *A face marking (μ, ν) determines a face $F \subset \bar{P}$ up to the action of the group G_0.*

Proof. By Lemma 5.8, the tile marking μ determines a tile up to the action of the group G_0. Once we choose a tile T such that $g_T^{-1}(v_1, \ldots, v_k) = \mu$, the vectors $\{v_1, \ldots, v_k\} \cup g_T(\nu)$ define a face $T \cap F$ of $g_T(P_0)$ and hence a face F of $\bar{P}$. □

Lemma 5.13. *For each face F of $\bar{P}$ there are corresponding face markings.*

Proof. It suffices to show that for each face F there is some tile T touching F. The tile F is stratified by faces of tiles, and tiles touching F correspond to full-dimensional strata. □

Lemma 5.14. *A pair (μ, ν) is a face marking of some pair (T, F) if and only if μ is a tile marking (cf. Section 5.4), and there is a set of roots $\hat{\nu}$ orthogonal to μ such that $\langle \mu, \nu \rangle_{\mathbb{Q}} = \langle \mu, \hat{\nu} \rangle_{\mathbb{Q}}$.*

Proof. Let $\mu = (x_1, \ldots, x_k)$ be tile marking of some tile $T \subset \bar{P}$, and let $\nu = \{f_1, \ldots, f_m\}$ be a set of additional roots. Let $P = g_T(P_0)$ be the polytope emanating from the tile T. Then (μ, ν) is the face marking of a pair (T, F) if and only if the face of P defined by the vectors $v_1, \ldots, v_k, g_T(f_1), \ldots, g_T(f_m)$ is contained in a face of $\bar{P}$ of codimension m. This is the case if and only if there is a set of m roots $\hat{\nu}$ orthogonal to μ such that $\langle \mu, \nu \rangle_{\mathbb{Q}} = \langle \mu, \hat{\nu} \rangle_{\mathbb{Q}}$. □

Remark 5.15. The second condition of Lemma 5.14 is obviously satisfied if the vectors $f_1, \ldots, f_m$ are already orthogonal to the vectors $x_1, \ldots, x_k$. However, this is not the only possibility. There are also situations where there is a non-trivial gluing between $\langle\mu\rangle$ and $\langle\hat{\nu}\rangle$, i.e. the cone in whose direction T emanates from $T \cap F$ can be strictly smaller than the cone in whose direction $\bar{P}$ emanates from F. For example, consider four vectors x_1, f_1, f_2, f_3 forming a graph of type D_4, with the central vertex being x_1. Then, one can choose $\hat{\nu} = \{x_2 + f_1 + f_2, x_1 + f_1 + f_3, x_1 + f_2 + f_3\}$.

5.6 Equivalences between face markings

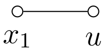
To classify the faces of $\bar{P}$ modulo the action of G_0 by means of face markings, it remains to see when two face markings (μ_1, ν_1) and (μ_2, ν_2) represent the same face modulo G_0.

To do this, fix a face F of $\bar{P}$ and consider the set of all tiles T touching F. We apply the considerations of Section 5.1 to the tiles touching F. By Proposition 5.5, any two tiles T, T' touching F can be connected be a sequence of tiles $\{T_i\}_{i=1}^n$ touching F, with $T_1 = T$ and $T_n = T'$, such that for each $i \in \{1, \dots, n-1\}$, either T_i and T_{i+1} are *direction adjacent*, i.e. they have a facet in common and $T_i \cap F = T_{i+1} \cap F$, or they are *tile adjacent*, i.e. they emanate from F in the same direction, and $T_i \cap F$ is adjacent to $T_{i+1} \cap F$.

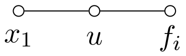
It remains to see how the face marking (μ, ν) changes as we pass from a pair (T_1, F) to a pair (T_2, F), where T_1 and T_2 are either tile adjacent or direction adjacent. Like in Section 5.4, this determines a set of moves connecting face markings representing the same face, such that two face markings represent the same face if and only if they can be connected by a sequence of such moves.

First we consider the case where T_1 and T_2 are tile adjacent. Let (μ_1, ν_1) be the face marking of the pair (T_1, F), where $\mu_1 = (x_1, \dots, x_k)$ and $\nu_1 = \{f_1, \dots, f_m\}$. The face $g_{T_1}^{-1}(F \cap T_1 \cap T_2) \subset P_0$ is defined by a set of the from $\mu_1 \cup \nu_1 \cup \{u\}$, where u is some additional root. Let $P_1 = g_{T_1}(P_0)$ and $P_2 = g_{T_2}(P_0)$. Let $\tilde{v} = g_{T_1}(u)$, and $e_i = g_{T_1}(f_i)$. There is a unique element in the finite reflection group generated by $\{R_{v_1}, \dots, R_{v_k}, R_{e_1}, \dots, R_{e_m}, R_{\tilde{v}}\}$ which maps P_1 to P_2. Using the notation introduced in Section 5.2, this element is $\rho(v_1, \dots, v_k, e_1, \dots, e_m; \tilde{v})$. It follows that $g_{T_2} = \rho(v_1, \dots, v_k, e_1, \dots, e_m; \tilde{v}) \circ g_{T_1}$. Hence, we have $\mu_2 = \rho(\mu_1, \nu_1; u)^{-1}(\mu_1)$, and likewise $\nu_2 = \rho(\mu_1, \nu_1; u)^{-1}(\nu_1)$. Starting from a marking (μ_1, ν_1), this move can be applied for every additional vector u such that $\mu_1 \cup \nu_1 \cup \{u\}$ is an elliptic subscheme and $(\mu, \nu \cup \{u\})$ is not a face marking, i.e. the face of P_1 defined by $g_{T_1}(\mu \cup \nu \cup \{u\})$ is not contained in the boundary of F.

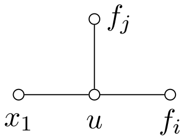
Example 5.16. Consider again the case where the face $T_0 \subset P_0$ is of codimension one, defined by a single root v_1. Let (μ_1, ν_1) be a face marking, where $\mu_1 = x_1$ and ν_1 is a set of pairwise orthogonal vectors, orthogonal to x_1. The following three moves are possible, where the diagrams only show the vectors of ν_1 not orthogonal to u.

(a) 

$\mu_2 = u,\ \nu_2 = \nu_1$

(b) 

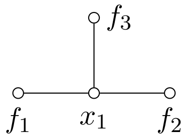
$\mu_2 = f_i,\ \nu_2 = \nu_1 \cup \{x_1\} \setminus \{f_i\}$

(c) 

$\mu_2 = x_1,\ \nu_2 = \nu_1$ (This "move" has no effect on the marking.)

Next, we consider the moves corresponding to pairs of tiles which are direction adjacent. Let T_1 and T_2 be two tiles touching F such that $T_1 \cap F = T_2 \cap F$, and such that T_1 and T_2 have a face in common. Let $\tilde{v}$ be the vector such that the face $T_1 \cap T_2 \subset P_1$ is defined by the set of roots $\{v_1, \ldots, v_k, \tilde{v}\}$, and let $u = g_{T_1}^{-1}(\tilde{v})$. Note that u must be a vector belonging to ν which is not orthogonal to μ. Here, the tile marking changes as described in Section 5.4, i.e., we have $\mu_2 = \rho^{-1}(\mu_1; u)(\mu_1)$. Moreover, $T_1 \cap F = T_2 \cap F$ implies that $\mu_1 \cup \nu_1 = \mu_2 \cup \nu_2$.

Example 5.17. We consider again the case where $T_0 \subset P_0$ is defined by a single root. Consider a face marking (μ_1, ν), where $\mu_1 = x_1$ and $\nu_1 = \{f_1, f_2, f_3\}$ intersect as in the following diagram.



There are three possible "direction adjacency" moves: the tile marking x_1 can move to f_1, f_2 of f_3.

This completes the description of the possible moves connecting face markings which are either tile adjacent or direction adjacent. Let us call two face markings *equivalent* if they can be connected by a sequence of such moves. The explanations above, together with Lemma 5.12 and Proposition 5.5, imply the following theorem.

Theorem 5.18. *Faces of $\bar{P}$ modulo the action of G_0 are in bijection with equivalence classes of face markings.*

5.7 Adding more symmetries

Let h be an isometry of $\mathbb{H}_V$ which maps walls to walls, maps the set $\{v_1, \ldots, v_k\}$ to itself and maps the polytope $\bar{P}$ to itself. Let σ be the permutation on $\{1, \ldots, k\}$ such that $h(v_i) = v_{\sigma(i)}$ for $i \in \{1, \ldots, k\}$. There is a unique element $\hat{h}$ of $h\,\Gamma = \Gamma\,h$ which maps the polytope P_0 to itself.

Consider a tile $T \subset \bar{P}$, with tile marking $\mu_T = (x_1, \ldots, x_k)$. Note that we have

$g_{h(T)}^{-1} \circ h \circ g_T = \hat{h}$. The tile marking of the tile $h(T)$ is given by

$$\begin{aligned}
\mu_{h(T)} &= g_{h(T)}^{-1}(v_1, \ldots, v_k) = (g_{h(T)}^{-1} \circ h \circ h^{-1})(v_1, \ldots, v_k) \\
&= (g_{h(T)}^{-1} \circ h)(v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(k)}) \\
&= (g_{h(T)}^{-1} \circ h \circ g_T)(g_T^{-1}(v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(k)})) \\
&= \hat{h}(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(k)}).
\end{aligned}$$

Similarly, for a face F and a tile T touching F, we have $\nu_{(h(T),h(F))} = \hat{h}\big(\nu_{(T,F)}\big)$.

Let H be a group of isometries of $\mathbb{H}_V$ which map walls to walls, map the set $\{v_1, \ldots, v_k\}$ to itself and map the polytope $\bar{P}$ to itself. Moreover, suppose that H contains G_0. As explained above, with each element $h \in H$ we can associate a pair $(\sigma, \hat{h})$, where σ is the permutation of $\{1, \ldots, k\}$ such that $h(v_i) = v_\sigma(i)$, and $\hat{h}$ is the unique element in $h\,\Gamma = \Gamma\, h$ mapping the polytope P_0 to itself. This defines a group homomorphism $H \to \mathrm{Perm}(\{1, \ldots, k\}) \times \mathrm{Sym}(P_0)$. Let $\hat{H} \subset \mathrm{Perm}(\{1, \ldots, k\}) \times \mathrm{Sym}(P_0)$ denote the image of this morphism. It acts on tile markings and face markings as explained above. Moreover, if the Coxeter graph of the polytope P_0 is finite, then the group $\hat{H}$ is finite as well.

Corollary 5.19. *Faces of $\bar{P}$ modulo the action of H are in bijection with face markings up to moves and modulo $\hat{H}$.*

CHAPTER 6

Real rational nodal sextics

6.1 Overview

Our goal in this chapter is to classify nodal real rational sextics up to rigid isotopy, using the techniques developed in the earlier chapters. It should be possible to classify all nodal rational sextics up to rigid isotopy using our approach, but we only carry out the explicit classification for the nodal real rational sextics which are close to the maximal ones—more precisely, those whose dividing perturbation is an M-curve (with 11 ovals) or an $(M–2)$-curve (with 9 ovals). In some of the other cases, there are additional technical difficulties and too many classes for an explicit classification.

By a *real rational curve* we understand a real algebraic curve of geometric genus zero such that the real structure on its normalization $\mathbb{CP}^1$ is the standard one. Let us consider a nodal real rational sextic C. Such a curve is dividing. Its dividing perturbation (see Definition 4.2) is a dividing real nodal sextic without real nodes. The rigid isotopy types of such curves are classified by the Theorems 3.1 and 3.2.

To classify nodal real rational sextics up to rigid isotopy, we first group them by the type of their dividing perturbation. To classify the nodal real rational curves with a fixed type of dividing perturbation, we use Theorem 4.21, which states that the rigid isotopy classes of such sextics are in bijection with certain pairs of faces of the hyperbolic polytopes P_+ and P_-, modulo the action of a symmetry group T.

For each type of dividing perturbation C_0, we would like to answer the following three questions:

(a) How many rigid isotopy classes of nodal real rational sextics with dividing perturbation C_0 are there?
(b) What is the topology for each of these sextics?
(c) Given two such rigid isotopy types, how can they be distinguished geometrically?

If the Coxeter graphs of the polytopes P_+ and P_- are finite, they can be computed using Vinberg's algorithm (see Section 1.2). For each vertex in the Coxeter graphs, we determine if the corresponding root is a contraction vector, a conjunction vector or an exceptional vector (cf. Proposition 4.40). We can also determine the group T explicitly: elements of T are symmetries of the polytopes P_+ and P_-. To determine T, it suffices to check which pairs of symmetries glue together to give an isometry of the $K3$ lattice L (cf. Section 1.1).

Once we know the Coxeter graphs of P_+ and P_- as well as the group T, question

(a) can be answered using the following algorithm:

1. Consider all collections formed by 10 independent vertices which correspond to contraction or conjunction vectors in the Coxeter graphs of P_+ and P_-.
2. For each of them, check if it corresponds to a nodal real rational sextic. Concretely, this is the case if and only if the sublattice S generated by h and all the nodal vectors is a primitive sublattice of L (cf. Proposition 2.19).
3. Count these collections modulo T.

Since the number of cases grows rapidly with the size of the Coxeter graphs, it is not practical to count all the cases by hand, and we automated this counting using a computer program.

If at least one of the Coxeter graphs is infinite, the number of faces corresponding to nodal real rational sextics is infinite. In these cases, we use additional symmetries (for degenerations of $\langle 1\langle 8\rangle\rangle$) or the method of descent explained in Chapter 5 (for degenerations of $\langle 7 \sqcup 1\langle 1\rangle\rangle$ and $\langle 9\rangle$).

By an *oval* of a nodal real rational curve we mean a connected component of $\mathbb{RP}^2 \backslash \mathbb{R}C$ homeomorphic to a disc which is transformed into the interior of an oval under the dividing perturbation.

With each dividing real nodal sextic we associate its *conjunction graph*: It contains a vertex for each oval, and an edge for each hyperbolic node of the curve, connecting the vertices of the ovals which meet in that node. Two ovals can be connected by several edges, and the graph may also contain loops, corresponding to conjunctions of an oval with itself. Note that the conjunction graph can be read off the corresponding collection of vectors in the Coxeter graphs of P_+ and P_-.

To answer question (b) concerning the topology of each of the rigid isotopy classes, we use an indirect approach: We consider auxiliary sextics which are reducible and which have many hyperbolic nodes. They also correspond to certain faces of the polytope $P_+ \times P_-$ (see Section 4.2.4). It is often easier to determine the topology of such curves, since their irreducible components are of lower degree, and therefore less complicated and better understood. Using Corollary 2.25, we can determine for each conjunction vector the irreducible components intersecting each other at the corresponding hyperbolic node. This information, together with the conjunction graph, turns out to be sufficient to infer the topology of these reducible sextics in all the examples we encountered. Like this we obtain a topologically correct drawing of the real part of such curves, where each crossing is identified with a conjunction vector and each oval is identified with a contraction vector.

Consider a nodal real rational curve C corresponding to a face of $P_+ \times P_-$ defined by a certain number of conjunction and contraction vectors. If the conjunction vectors are (up to a symmetry belonging to the group T) a subset of the conjunction vectors used to define the auxiliary reducible curve, then we can read the topology of the curve C off the topological drawing of the auxiliary curve. It suffices to perturb the superfluous hyperbolic nodes, and then contract the non-singular ovals corresponding to the contraction vectors.

Let us now consider question (c), how to distinguish different rigid isotopy types geometrically. The most obvious criterion to distinguish rigid isotopy types is the isotopy type. However, this is not sufficient in most of the cases. In Section 6.2 we define a refined equivalence relation called *semi-rigid isotopy*, which takes into account both the isotopy type and the order of the ovals with respect to real line pencils.

For many (but not all) types of the dividing perturbation we consider, we can show that two sextics of this type of the dividing perturbation are rigidly isotopic if and only if they are semi-rigidly isotopic.

6.2 Semi-rigid isotopies

In this section we introduce the notion of *semi-rigid isotopy* between real plane sextics, which is defined by considering the position of a sextic with respect to auxiliary lines and conics. It is well-known that auxiliary lines and conics can be used to define a reversible cyclic order on the set of empty ovals of a non-singular sextic (see Itenberg and Itenberg [17]).

Consider a non-singular real sextic with real scheme $\langle b \sqcup 1\langle a\rangle\rangle$, where $a \geq 1$ and $b \geq 0$ (for the notation see p. 29). We call the empty ovals contained in the interior of the non-empty oval *inner ovals*, and the empty ovals not contained inside another oval *outer ovals*.

Consider a line passing through a point p lying in the interior of an inner oval. Such a line must intersect both the inner oval containing p and the non-empty oval in at least two points. Since by Bézout's theorem a line intersects a sextic in at most six real points, such a line may pass through at most one more empty oval. If we let the line turn around p, it sweeps out the whole plane and will meet every empty oval at some point. Associate with each empty oval different from i the (circular) interval formed by the angles of the lines trough p which intersect this oval. This defines a cyclic order on the set of all empty ovals apart from i, where the orientation of the line pencil is induced by the fixed orientation of the non-empty oval. Note that this cyclic order cannot change when we move the point p inside i or under a rigid isotopy of the curve C, as long as p stays inside the inner oval.

If there is only one inner oval, this defines a cyclic order on the set of outer ovals. If there is more than one inner oval, we have to check to which extent the different orders are compatible, so that they can be "glued" together.

Lemma 6.1. *The cyclic orders obtained from the inner ovals separate the inner ovals from the outer ovals.*

Proof. If there are less than three inner ovals or less than two outer ovals, the statement is clearly true. Let us suppose $a \geq 3$ and $b \geq 2$. Let i, i' and i'' be three inner ovals, and let o', o'' be two outer ovals. We want to show that the cyclic order obtained from a line pencil though a point p in i separates the inner ovals i', i'' from the outer ovals o', o''. Trace a conic E through p and four more points, one inside each of the four ovals i', i'', o' and o''. By Bézout's theorem, E intersects C in at most twelve points.

However, E must intersect each of the five empty ovals as well as the non-empty oval at least twice. So there are exactly twelve real intersection points, two with each of the five empty ovals and two with the non-empty oval. Since the conic intersects the non-empty oval in two points, the intersection of their interiors is a disk, so that we can orient the conic in a way compatible with the orientation of the non-empty oval. The order in which the four ovals i', i'', o' and o'' appear along E agrees with the cyclic ordering obtained from the line pencil passing through i. Since E only intersects the non-empty oval in two points, this cyclic order must separate the inner ovals from the outer ovals. □

If there are at least two inner ovals and there is at least one outer oval, the cyclic orders break into two linear orders, one for the outer ovals and one for the inner ovals (apart from the one containing the base point).

Lemma 6.2. *Suppose there are at least two inner ovals, and at least five empty ovals in total. Then the linear orders on the outer ovals defined using line pencils through the inner ovals all agree with each other.*

Proof. Suppose we want to compare two outer ovals, say o' and o'', using the orders defined by two inner ovals i' and i''. Trace a conic through the interior of these four ovals and through a fifth empty oval. Using the same argument as above, we see that the order of o' and o'' on the line pencil defined by i' agrees with the order in which they appear on the conic. The same holds for the order on the line pencil defined by i''. □

Lemma 6.3. *Suppose that there are at least three inner ovals and there is at least one outer oval. Then there is a linear order on the inner ovals which is compatible with the partial linear orders obtained from the line pencils.*

Proof. Given two inner ovals, we can always compare them using a line pencil passing through a third inner oval. First we show that this does not depend on the choice of a third oval. We may suppose that there are at least four inner ovals, since otherwise there is no ambiguity. Let i' and i'' be the ovals we wish to compare, and let i_1 and i_2 be inner ovals we want to use to compare them, and let o be any outer oval. Again, if we trace a conic through these five ovals, we see that the orders must agree. This defines a total anti-symmetric relation on the set of inner ovals, which agrees with the partial orders obtained from the line pencils. To show that this relation defines an order, it remains to check that it is transitive. If there are at least four inner ovals, the transitivity for three ovals can be checked on the partial order obtained from the line pencil through a fourth inner oval. If there are at least five ovals in total, it can also be checked using the order coming from a conic. The only remaining case, which is $\langle 1 \sqcup 1\langle 3\rangle\rangle$, requires a separate argument. Choose three points p_1, p_2, p_3 in the three inner ovals i_1, i_2, i_3, and trace the three lines through any two of them. This divides $\mathbb{RP}^2$ into four triangles. One of them lies in the interior of the non-empty oval, and one contains the outer oval. Up to isotopy, the situation has to be as shown in Figure 6.1. We can check directly that the order is indeed transitive: if we choose the counter-clockwise

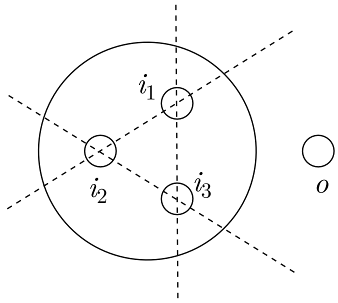


Figure 6.1 – Sextic with real scheme $\langle 1 \sqcup 1\langle 3\rangle\rangle$

orientation for the non-empty oval, we get $i_1 < i_2$, $i_2 < i_3$ and $i_1 < i_3$. □

Lemma 6.4. *Suppose there are no outer ovals and at least five inner ovals. Then there is a cyclic order on the inner ovals which in compatible with the partial cyclic orders obtained from the line pencils.*

Proof. We can use conics passing though five inner ovals to see that the different cyclic orders are compatible. Note that a conic through five inner ovals intersects the non-empty oval either in zero or in two points, but in any case the interior of the conic intersects the interior of the non-empty oval in a disk, so that we can orient the conic consistently. □

The type of order (linear or cyclic) defined on the sets of inner and outer ovals, depending on the number of inner and outer ovals, is summarized in the diagram in Figure 6.2. Since we chose an orientation of the non-empty oval in the beginning, all these orders have to be considered up to simultaneous inversion of the orders defined on the inner and outer ovals.

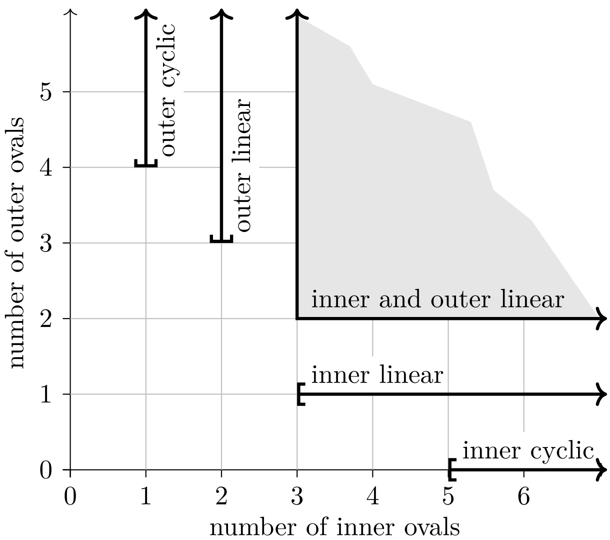


Figure 6.2 – The type of order defined on the sets of inner and outer ovals, up to simultaneous inversion, depending on the number of inner and outer ovals.

Definition 6.5. Let C_1 and C_2 be two real sextics with non-singular real parts. A *semi-rigid isotopy* between C_1 and C_2 is an isotopy which preserves the (linear or cyclic) orders defined on the sets of inner and outer ovals. We call C_1 and C_2 *semi-rigidly isotopic* if they can be connected by a semi-rigid isotopy.

We extend the notion of semi-rigid isotopy to nodal sextics by considering perturbations of the real nodes. Consider a real nodal sextic C, and choose a direction of perturbation for each of its real nodes. Under such a perturbation, some 1-cycles and some solitary nodes of $\mathbb{R}C$ are deformed into ovals. This allows us to define (linear or cyclic) orders on these "proto-ovals", i.e. 1-cycles and solitary nodes which are deformed into ovals. Consider an isotopy between two real sextics C_1 and C_2 with real nodes. Each choice of perturbation of C_1 can be pushed along the isotopy to give a perturbation of C_2, and this allows us to compare the (linear or cyclic) orders on the inner and outer "proto-ovals" of C_1 with the (linear or cyclic) orders on the inner and outer "proto-ovals" of C_2. We call the isotopy *semi-rigid* if it preserves these orders for each choice of perturbation of the real nodes.

6.3 Notations

Let us introduce some notation needed to carry out the calculations. We fix a K3 lattice $L = \mathrm{U}\oplus\mathrm{U}\oplus\mathrm{U}\oplus\mathrm{E}_8\oplus\mathrm{E}_8$. Let $\{u_1, u_2\}$, $\{w_1, w_2\}$ and $\{z_1, z_2\}$ be canonical generating sets for the hyperbolic planes and let $\{e_1, \ldots, e_8\}$ and $\{f_1, \ldots, f_8\}$ be canonical generating sets for the two lattices of type E_8. Let $\{e_1^*, \ldots, e_8^*\}$ (respectively, $\{f_1^*, \ldots, f_8^*\}$) denote the dual basis of $\{e_1, \ldots, e_8\}$ (respectively, $\{f_1, \ldots, f_8\}$). Moreover, let us fix the polarization vector $h = z_1 + z_2 \in L$, and let $y = z_1 - z_2$, so that the orthogonal complement of h in L is generated by $\{u_1, u_2, w_1, w_2, y, e_1, \ldots, e_8, f_1, \ldots, f_8\}$.

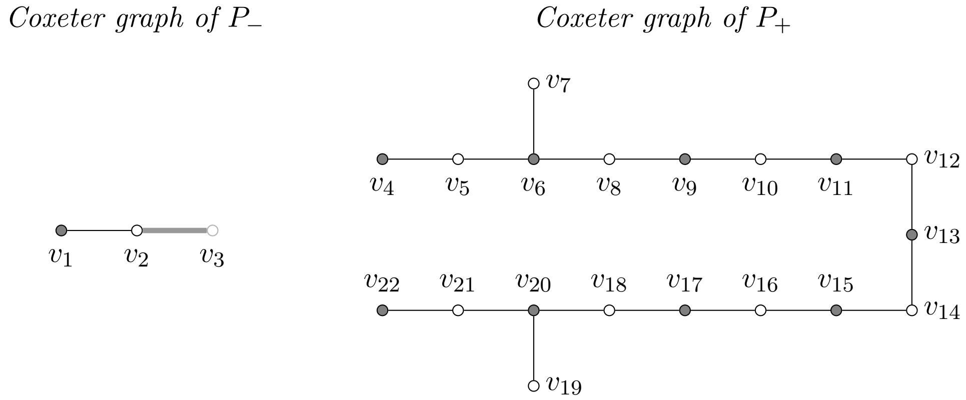
Let us introduce the following notation for sublattices of L: If $x_1, \ldots, x_k \in L$ form a canonical generating set for a lattice S which is one of the standard lattices $\langle a\rangle, \mathrm{U}, \mathrm{A}_p, \mathrm{D}_q, \mathrm{E}_s$ defined in Section 1.1, then we use the symbol $S^{x_1,\ldots,x_k}$ to denote this lattice —for example $\mathrm{U}^{u_1,u_2}, \langle -2\rangle^y, \mathrm{E}_7^{e_1,\ldots,e_7}, \langle -2\rangle^{e_8^*}$, etc. In some of the cases treated in the following subsections, one of the E_8 lattices is split into two D_4 lattices. For convenience, we define the following vectors:

$$\begin{aligned} c_1 &= f_2^* - f_4^* & \qquad d_1 &= f_4^* - f_5^* + f_7^* \\ c_2 &= f_1 & d_2 &= f_6 \\ c_3 &= f_2 & d_3 &= f_7 \\ c_4 &= f_2^* - f_5^* & d_4 &= f_8 \end{aligned}$$

Both $\{c_1, \ldots, c_4\}$ and $\{d_1, \ldots, d_4\}$ generate a lattice of type D_4, and one is the orthogonal complement of the other inside $\mathrm{E}_8^{f_1,\ldots,f_8}$.

6.4 Degenerations of $\langle 1 \sqcup 1\langle 9\rangle\rangle$

Let $L_+ = \mathrm{U}^{w_1,w_2} \oplus \mathrm{E}_8^{e_1,\dots,e_8} \oplus \mathrm{E}_8^{f_1,\dots,f_8}$, and $L_{-h} = \mathrm{U}^{u_1,u_2} \oplus \langle -2\rangle^y$. The lattice L_+ is unimodular, while the discriminant from of L_{-h} is the two-element finite quadratic form $[\frac{1}{2}]$, which has no non-trivial automorphisms. The Coxeter graphs of the polytopes P_+ and P_- are both finite; they are shown in Figure 6.3.



$v_1 = u_1 - u_2$	$v_6 = e_3$	$v_{11} = e_8$	$v_{16} = f_7$	$v_{21} = f_2$
$v_2 = u_2 - y$	$v_7 = e_4$	$v_{12} = e_8^* + w_2$	$v_{17} = f_6$	$v_{22} = f_1$
$v_3 = y$	$v_8 = e_5$	$v_{13} = w_1 - w_2$	$v_{18} = f_5$	
$v_4 = e_1$	$v_9 = e_6$	$v_{14} = f_8^* + w_2$	$v_{19} = f_4$	
$v_5 = e_2$	$v_{10} = e_7$	$v_{15} = f_8$	$v_{20} = f_3$	

Figure 6.3

The polytope P_- has no symmetries. The polytope P_+ has a $\mathbb{Z}/2\mathbb{Z}$ symmetry. Since the discriminant forms of L_+ and L_{-h} don't have non-trivial automorphisms, this symmetry extends to L, i.e. we have $T \cong \mathbb{Z}/2\mathbb{Z}$. The only exceptional vector is v_3.

Lemma 6.6. *The contraction vectors are* $\{v_1, v_4, v_6, v_9, v_{11}, v_{13}, v_{15}, v_{17}, v_{20}, v_{22}\}$.

Proof. Since v_6 and v_{20} have degree 3 in the Coxeter graph, they must be contraction vectors. Then it follows from Proposition 4.40 that among the vectors defining P_+, the contraction vectors are $\{v_4, v_6, v_9, v_{11}, v_{13}, v_{15}, v_{17}, v_{20}, v_{22}\}$. Either v_1 or v_2 must be a contraction vector, and the other one must correspond to a conjunction of the outer oval with the non-empty oval. It remains to decide which one is which. Let $x \in K_-$ be the class of the genus 9 component of the real part. We can write $x = a_1u_1 + a_2u_2 + by$, with $a_1, a_2, b \in \mathbb{Z}$. We have $x^2 = 2g - 2 = 16$. There are two possibilities: If the contraction vector is v_1, then $(x, v_1) = 0$ and $(x, v_2) = 1$. The only solution for x in this case is $x = 3(u_1 + u_2) - y$. If the contraction vector is v_2, then $(x, v_2) = 1$ and $(x, v_1) = 0$. The only solution for x in this case is $x = -4u_1 - 3u_2 + 2y$. In the second case, we have $(x, v_3) = -4 < 0$. This is not possible, since both x and v_3 are represented by holomorphic curves with respect to the twisted complex structure J (see Section 4.3). Hence, v_1 must be the contraction vector. $\square$

Lemma 6.7. *The face of $P_+ \times P_-$ orthogonal to all conjunction vectors corresponds to a reducible nodal curve C consisting of a real conic and a real rational quartic, where the vectors corresponding to intersections between the conic and the quartic are v_7, v_8, v_{10}, v_{12}, v_{14}, v_{16}, v_{18} and v_{19}. The isotopy type of C is shown in Figure 6.4.*

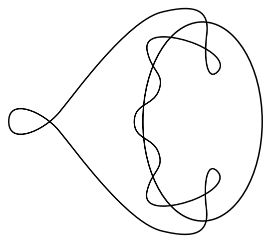


Figure 6.4 – A reducible degeneration C of a curve of type $\langle 1 \sqcup 1\langle 9\rangle\rangle$.

Proposition 6.8. (1) *All rigid isotopy types of real rational curves obtained by real nodal degeneration from a curve of type $\langle 1 \sqcup 1\langle 9\rangle\rangle$ can be obtained from the reducible sextic C by first perturbing some of the nodes and then contracting empty ovals.*

(2) *Two such real rational sextics are rigidly isotopic if and only if they are semi-rigidly isotopic.*

(3) *There are* 124 *rigid isotopy types for such curves.*

Proof. (1) Since C contains all the possible conjunctions, this is clear.
(2) For the curve $\langle 1 \sqcup 1\langle 9\rangle\rangle$, there is a linear order on the set of inner ovals, defined up to inversion (see Section 6.2). Given a rational curve obtained by real nodal degeneration from a curve of type $\langle 1 \sqcup 1\langle 9\rangle\rangle$, one can identify the inner ovals with the contraction vectors in the Coxeter graph, up to the $\mathbb{Z}/2\mathbb{Z}$ symmetry. Once such an identification is chosen, the conjunction vector corresponding to each hyperbolic node of the curve is uniquely determined by the pair of ovals it connects, since for each pair of ovals, there is at most one corresponding conjunction vector in the Coxeter graph.
(3) This count is obtained by enumerating the admissible pairs of faces of $P_+ \times P_-$ modulo T. □

6.5 Degenerations of $\langle 5 \sqcup 1\langle 5\rangle\rangle$

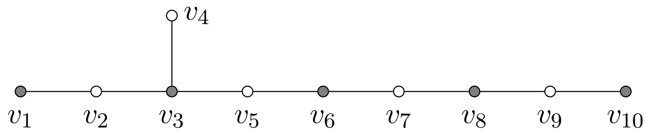
Let $L_+ = \mathrm{U}^{u_1,u_2} \oplus \mathrm{E}_8^{e_1,\dots,e_8}$, and $L_{-h} = \mathrm{U}^{w_1,w_2} \oplus \mathrm{E}_8^{f_1,\dots,f_8} \oplus \langle -2\rangle^y$. The lattice L_+ is unimodular, while the discriminant from of L_{-h} is the $[\frac{1}{2}]$. The Coxeter graphs of the polytopes P_+ and P_- are both finite; they are shown in Figure 6.5.

The polytopes P_+ and P_- have no symmetries. The only exceptional vector is v_{22}.

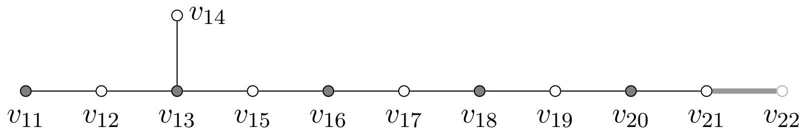
Lemma 6.9. *The contraction vectors are* $\{v_1, v_3, v_6, v_8, v_{10}, v_{11}, v_{13}, v_{16}, v_{18}, v_{20}\}$.

Proof. Since v_3 and v_{13} have degree 3 in the Coxeter graphs, they must be contraction vectors. Then it follows from Proposition 4.40 that the contraction vectors are

Coxeter graph of P_+



Coxeter graph of P_-

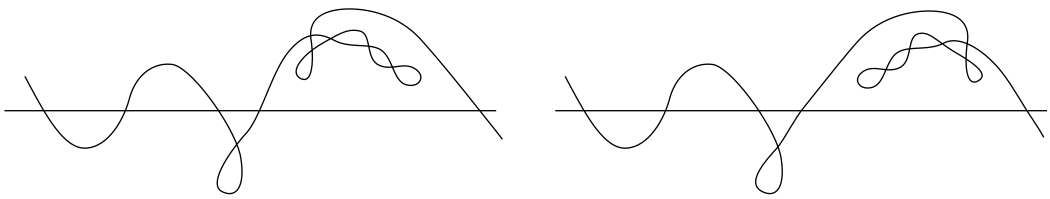


$v_1 = e_1$	$v_6 = e_6$	$v_{11} = f_1$	$v_{16} = f_6$	$v_{21} = w_2 - y$
$v_2 = e_2$	$v_7 = e_7$	$v_{12} = f_2$	$v_{17} = f_7$	$v_{22} = y$
$v_3 = e_3$	$v_8 = e_8$	$v_{13} = f_3$	$v_{18} = f_8$	
$v_4 = e_4$	$v_9 = u_2 + e_8^*$	$v_{14} = f_4$	$v_{19} = f_8^* + w_2$	
$v_5 = e_5$	$v_{10} = u_1 - u_2$	$v_{15} = f_5$	$v_{20} = w_1 - w_2$	

Figure 6.5

$\{v_1, v_3, v_6, v_8, v_{10}, v_{11}, v_{13}, v_{16}, v_{18}, v_{20}\}$. □

Lemma 6.10. *The face of $P_+ \times P_-$ orthogonal to all conjunction vectors corresponds to a reducible nodal curve C consisting of a line and a rational quintic, where the vectors corresponding to intersections between the line and the quintic are v_{14}, v_{15}, v_{17}, v_{19} and v_{21}. The isotopy type of C is shown in Figure 6.6 (a).*



(a) – correct inner orientation (b) – wrong inner orientation

Figure 6.6 – For the reducible degeneration C of a curve of type $\langle 5 \sqcup 1\langle 5\rangle\rangle$ (cf. Lemma 6.10), there are a priori two possible isotopy types, (a) and (b). It turns out that (a) is the correct one.

Proof. From the conjunction graph and the information how the irreducible components intersect each other, one can deduce that the isotopy type of C has to be as shown in Figure 6.6 (a) or (b). To decide which of the two is correct, we use the classification up to isotopy of real rational quintics by Itenberg, Mikhalkin, and Rau, see [20, Table 3]. □

Proposition 6.11. (1) *All rigid isotopy types of real rational curves obtained by real*

nodal degeneration from a curve of type $\langle 5 \sqcup 1\langle 5\rangle\rangle$ can be obtained from the reducible sextic C by first perturbing some of the nodes and then contracting empty ovals.

(2) *Two such real rational sextics are rigidly isotopic if and only if they are semi-rigidly isotopic.*

(3) *There are 234 rigid isotopy types for such curves.*

Proof. (1) Since C contains all the possible conjunctions, this is clear.
(2) For the curve $\langle 5 \sqcup 1\langle 5\rangle\rangle$, there are linear orders on the set of inner and outer ovals, defined up to simultaneous inversion (see Section 6.2). There is a unique rigid isotopy type corresponding to the curve with 10 isolated nodes. If not all empty ovals are contracted, then there must be a conjunction between an empty oval and the non-empty oval, corresponding to one of the vectors v_4 and v_{14}. This conjunction breaks the symmetry and allows us to identify the ovals with the corresponding contraction vectors. The conjunction vector corresponding to each hyperbolic node of the curve is then uniquely determined by the pair of ovals it connects, since for each pair of ovals, there is at most one corresponding conjunction vector in the Coxeter graph.
(3) This count is obtained by enumerating the admissible pairs of faces of $P_+ \times P_-$ modulo T. □

6.6 Degenerations of $\langle 9 \sqcup 1\langle 1\rangle\rangle$

Let $L_+ = \mathrm{U}^{u_1,u_2}$ and $L_{-h} = \mathrm{U}^{w_1,w_2} \oplus \mathrm{E}_8^{e_1,\dots,e_8} \oplus \mathrm{E}_8^{f_1,\dots,f_8} \oplus \langle -2\rangle^y$. The lattice L_+ is unimodular, while the discriminant from of L_{-h} is the two-element finite quadratic form $[\frac{1}{2}]$, which has no non-trivial automorphisms. The Coxeter graphs of the polytopes P_+ and P_- are both finite; they are shown in Figure 6.7. The Coxeter graph of P_- was first calculated by Vinberg [41].

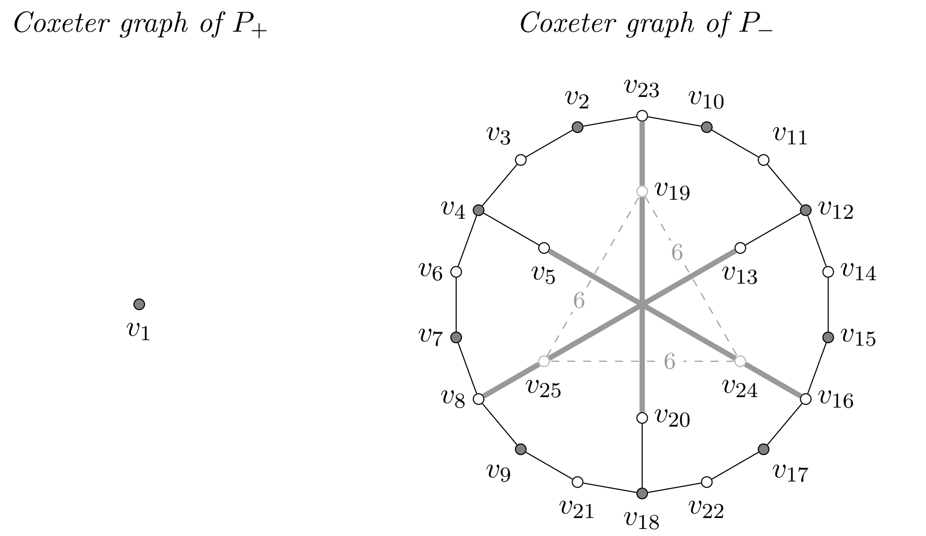
The symmetry group of the polytope P_- is S_3. Since the discriminant forms of L_+ and L_{-h} don't have non-trivial automorphisms, all symmetries extend to L, i.e. we have $T \cong S_3$. The vectors not corresponding to conjunctions or contractions are v_{19}, v_{24}, v_{25}.

Lemma 6.12. *The contraction vectors are $\{v_1, v_2, v_4, v_7, v_9, v_{10}, v_{12}, v_{15}, v_{17}, v_{18}\}$.*

Proof. Since the vertices v_4, v_{12} and v_{18} have degree 3 in the Coxeter graph, they must be contraction vectors. Then it follows from Proposition 4.40 that the contraction vectors are $\{v_1, v_2, v_4, v_7, v_9, v_{10}, v_{12}, v_{15}, v_{17}, v_{18}\}$. □

Lemma 6.13. *The face of $P_+ \times P_-$ orthogonal to all conjunction vectors corresponds to a reducible nodal curve C consisting of three lines L_1, L_2, L_3 and a nonsingular cubic E. The isotopy type of C is shown in Figure 6.8.*

Proof. To determine the decomposition of C into irreducible components, we explicitly calculate the quotient $\tilde{S}/S$ and apply Proposition 2.24. To do this, we write down a matrix for the embedding $S \hookrightarrow L$ in standard bases, reduce the coefficients modulo



$v_1 = u_1 - u_2$	$v_6 = e_5$	$v_{11} = f_2$	$v_{16} = f_7$	$v_{21} = e_8^* + w_2$
$v_2 = e_1$	$v_7 = e_6$	$v_{12} = f_3$	$v_{17} = f_8$	$v_{22} = f_8^* + w_2$
$v_3 = e_2$	$v_8 = e_7$	$v_{13} = f_4$	$v_{18} = w_1 - w_2$	$v_{23} = e_1^* + f_1^* - y + 2(w_1 + w_2)$
$v_4 = e_3$	$v_9 = e_8$	$v_{14} = f_5$	$v_{19} = y$	$v_{24} = 2e_4^* + 2f_7^* - 3y + 6(w_1 + w_2)$
$v_5 = e_4$	$v_{10} = f_1$	$v_{15} = f_6$	$v_{20} = w_2 - y$	$v_{25} = 2e_7^* + 2f_4^* - 3y + 6(w_1 + w_2)$

Figure 6.7

2 and compute its null-space (still over $\mathbb{F}_2$). The result of this computation is that $\tilde{S}/S \cong (\mathbb{Z}/2\mathbb{Z})^3$, generated by the elements $l_1 := \frac{1}{2}(v_5 + v_6 + v_8 + v_{20} + v_{21} + h)$, $l_2 := \frac{1}{2}(v_{13} + v_{14} + v_{16} + v_{20} + v_{22} + h)$ and $l_3 := \frac{1}{2}(v_3 + v_5 + v_{11} + v_{13} + v_{23} + h)$. In particular, it follows that C is nodal. Applying Corollary 2.25, it follows that C consists of three lines L_1, L_2, L_3 and a residual cubic E, and the conjunction vectors corresponding to the intersections between the irreducible components are as follows:

$\cap$	L_2	L_3	E
L_1	v_{20}	v_5	$v_6,\ v_8,\ v_{21}$
L_2		v_{13}	v_{14}, v_{16}, v_{22}
L_3			$v_3,\ v_{11}, v_{23}$

The isotopy type of C can now be deduced from this table together with the conjunction graph, which can be read off the Coxeter graph. We start by drawing three lines L_1, L_2, L_3 and mark their intersection points v_5, v_{13} and v_{20}. Then we choose one region next to the intersection marked v_{20} and decide that it belongs to the oval corresponding to the vector v_{18}. From there, we can trace the non-contractible part of the cubic E and at each step, the conjunction graph determines how it must intersect the three lines. In the end, it can be seen that the oval of E must be inserted as shown in Figure 6.8. to make sure that C can be perturbed into a curve of type $\langle 9 \sqcup 1\langle 1\rangle\rangle$. $\square$

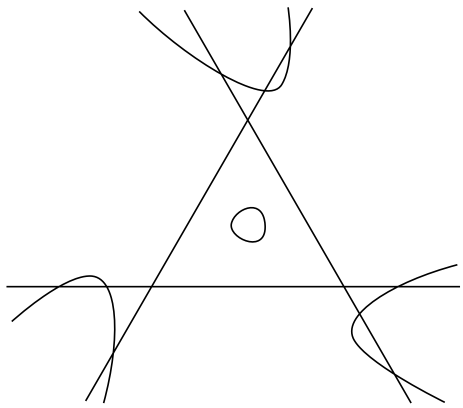


Figure 6.8 – A reducible degeneration C of a curve of type $\langle 9 \sqcup 1\langle 1\rangle\rangle$.

Proposition 6.14. (1) *All rigid isotopy types of real rational curves obtained by real nodal degeneration from a curve of type $\langle 9 \sqcup 1\langle 1\rangle\rangle$ can be obtained from C by first perturbing some of the nodes and then contracting empty ovals.*

(2) *Two such real rational sextics are rigidly isotopic if and only if they are semi-rigidly isotopic.*

(3) *There are* 136 *rigid isotopy types for such curves.*

Proof. (1) Since C contains all the possible conjunctions, this is clear.
(2) For the curve $\langle 9 \sqcup 1\langle 1\rangle\rangle$, there is a cyclic order on the set of outer, defined up to inversion (see Section 6.2). There is a unique rigid isotopy type corresponding to the curve with 10 isolated nodes. If not all empty ovals are contracted, then there must be a conjunction between an empty oval and the non-empty oval, corresponding to one of the vectors v_5, v_{13}, v_{20}. This conjunction allows to identify the tree ovals which can conjunct with the non-empty oval, and therefore allows us to identify the ovals with the corresponding contraction vectors, up to the symmetry group S_3. Once such an identification is chosen, the conjunction vector corresponding to each node of the curve is uniquely determined by the pair of ovals it connects, since for each pair of ovals, there is at most one corresponding conjunction vector in the Coxeter graph.
(3) This count is obtained by enumerating the admissible pairs of faces of $P_+ \times P_-$ modulo T. □

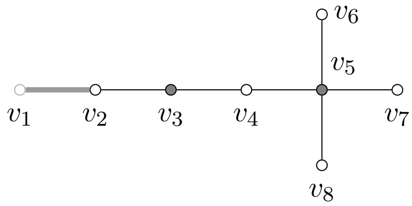
6.7 Degenerations of $\langle 2 \sqcup 1\langle 6\rangle\rangle$

6.7.1 Curves with only real nodes

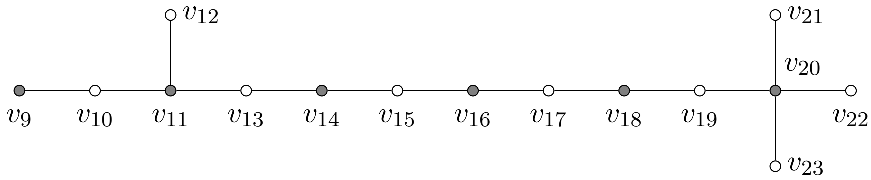
Let $L_+ = \mathrm{U}^{w_1,w_2} \oplus \mathrm{E}_8^{e_1,\ldots,e_8} \oplus \mathrm{D}_4^{c_1,\ldots,c_4}$ and $L_{-h} = \mathrm{U}^{u_1,u_2} \oplus \mathrm{D}_4^{d_1,\ldots,d_4} \oplus \langle -2\rangle^y$. The Coxeter graphs of the polytopes P_+ and P_- are both finite; they are shown in Figure 6.9.

Lemma 6.15. *The symmetry group is $T = S_3$, permuting the set of pairs $\{(v_6, v_{21}), (v_7, v_{22}), (v_8, v_{23})\}$. The only exceptional vector is v_1. The contraction vectors are v_3, v_5, v_9, v_{11}, v_{14}, v_{16}, v_{18} and v_{20}.*

Coxeter graph of P_-



Coxeter graph of P_+



$v_1 = y$	$v_6 = d_1$	$v_{11} = e_3$	$v_{16} = e_8$	$v_{21} = c_1$
$v_2 = u_2 - y$	$v_7 = d_2$	$v_{12} = e_4$	$v_{17} = w_2 + e_8^*$	$v_{22} = c_2$
$v_3 = u_1 - u_2$	$v_8 = d_4$	$v_{13} = e_5$	$v_{18} = w_1 - w_2$	$v_{23} = c_4$
$v_4 = u_2 + d_3^*$	$v_9 = e_1$	$v_{14} = e_6$	$v_{19} = w_2 + c_3^*$	
$v_5 = d_3$	$v_{10} = e_2$	$v_{15} = e_7$	$v_{20} = c_3$	

Figure 6.9

Proof. The discriminant form of L_+ is the discriminant form of the lattice D_4, which is the finite quadratic form $\mathcal{V}_2$ defined as follows: The underlying group is $(\mathbb{Z}/2\mathbb{Z})^2$, and the quadratic form has value 1 on the three non-zero elements (cf. [10, p. II.5.2]). The automorphism group of $\mathcal{V}_2$ is S_3, permuting the three non-zero elements. Given a root system of type D_4 with basis d_1, d_2, d_3, d_4, it is easy to see that there is an isomorphism between symmetries of this basis and automorphisms of the discriminant form. Hence, a permutation of $\{c_1, c_2, c_4\}$ glues to a unique permutation of $\{d_1, d_2, d_4\}$. We defined $c_1, \ldots, c_4$ and $d_1, \ldots, d_4$ so that the symmetry $c_i \mapsto c_{\sigma(i)}$ glues to $d_i \mapsto d_{\sigma(i)}$, where σ is a permutation of the set $\{1, 2, 4\}$. □

Lemma 6.16. *Let C be a curve corresponding to the face defined by all conjunction vectors. It has a D_6 singularity and eight nodes, and its irreducible components are a line, a conic and a rational cubic. Its isotopy type is shown in Figure 6.10 on the left hand side. Let C_a, C_b and C_c be curves defined by the sets of vectors $S_{\mathrm{com}} \cup \{v_6, v_7, v_8, v_{21}\}$, $S_{\mathrm{com}} \cup \{v_6, v_7, v_{21}, v_{23}\}$ and $S_{\mathrm{com}} \cup \{v_6, v_{21}, v_{22}, v_{23}\}$ respectively, where $S_{\mathrm{com}} = \{v_2, v_4, v_{10}, v_{12}, v_{13}, v_{15}, v_{17}, v_{19}\}$. The curves C_a, C_b and C_c can be obtained from C by perturbing the D_6 singularity as shown in Figure 6.10 on the right hand side.*

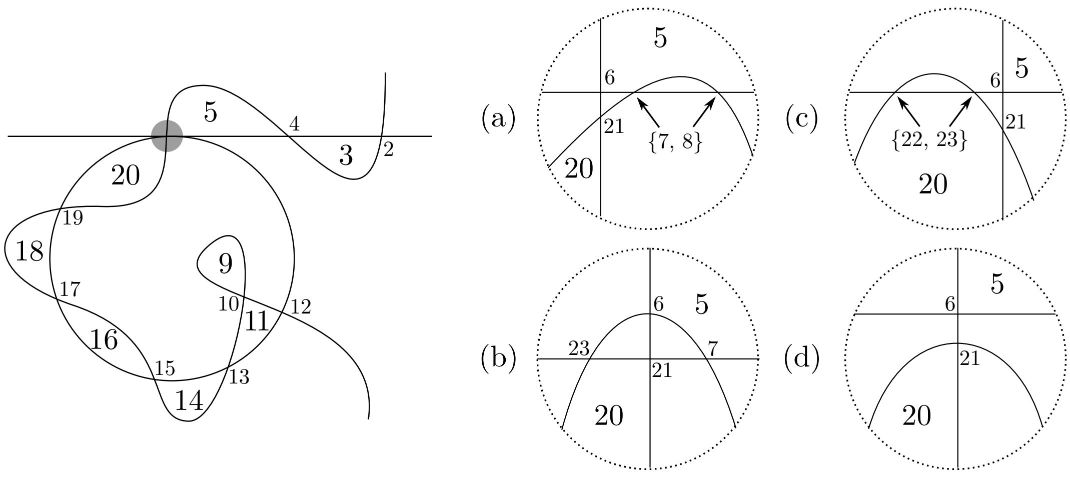


Figure 6.10 – Reducible degenerations of a curve of type $\langle 2 \sqcup 1\langle 6\rangle\rangle$. The isotopy type of the curve C is shown on the left, with the D_6 singularity marked in gray. The small figures (a), (b), (c) and (d) on the right show four different ways of perturbing the D_6 singularity, resulting in reducible nodal curves C_a, C_b, C_c and C_d, respectively. The curves C_a, C_b and C_c have twelve real nodes, while the curve C_d has ten real nodes and a pair of non-real nodes.

Proposition 6.17. (1) *All rigid isotopy types of real rational curves obtained by real nodal degeneration from a curve of type $\langle 2 \sqcup 1\langle 6\rangle\rangle$ without real nodes can be obtained from one of the curves C_a, C_b, C_c by first perturbing some of the nodes and then contracting empty ovals.*

(2) *Two such real rational sextics are rigidly isotopic if and only if they are semi-rigidly isotopic.*

(3) *There are* 460 *rigid isotopy types for such curves.*

Proof. (1) A set of pairwise orthogonal vectors in the Coxeter graph corresponds to a nodal curve if and only if it contains no two of the three pairs (v_6, v_{21}), (v_7, v_{22}) and (v_8, v_{23}). This implies that up to symmetry, the conjunction vectors in such a set are contained in one of the sets corresponding to the curves C_a, C_b and C_c.
(2) Each set of vectors corresponding to an irreducible nodal curve contains either a conjunction of an inner oval with the non-empty oval or at least two conjunctions of one outer oval with the non-empty oval. This determines the identification between empty ovals and contraction vectors. If the conjunction multi-graph is the same, but the corresponding faces of $P_+ \times P_-$ are not equivalent, then the order of the conjunctions on the non-empty oval is different. This can be checked case by case, using Figure 6.10 (a), (b), (c). The only such cases (up to equivalence) are $\{v_6, v_{21}\}$ vs. $\{v_6, v_{22}\}$, $\{v_6, v_7, v_{21}\}$ vs. $\{v_6, v_7, v_{23}\}$ and $\{v_6, v_{21}, v_{22}\}$ vs. $\{v_6, v_{22}, v_{23}\}$.
(3) This count is obtained by enumerating the admissible pairs of faces of $P_+ \times P_-$ modulo T. ☐

6.7.2 Curves with a pair of non-real nodes

Using the notations of the previous subsection, we let the vectors corresponding to the non-real roots be $\frac{1}{2}(l_1 \pm l_2)$, where $l_1 = v_8 - v_7$ and $l_2 = v_{23} - v_{22}$. The Coxeter graphs of the polytopes orthogonal to l_1, l_2 are shown in Figure 6.11.

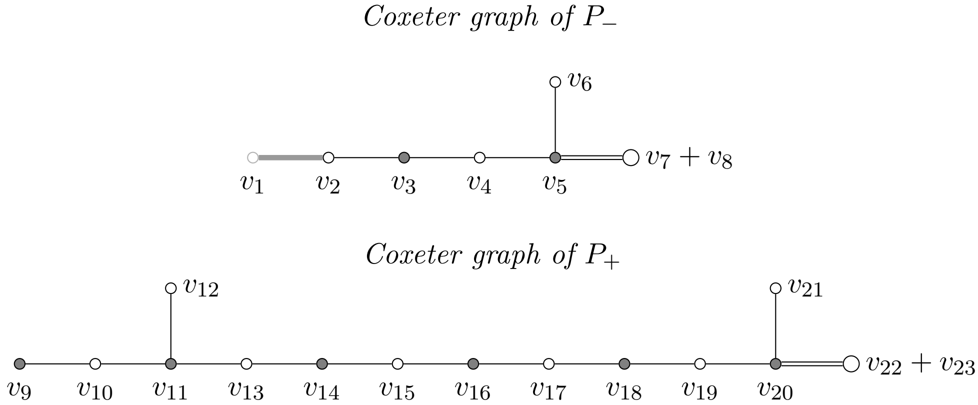


Figure 6.11

The polytopes P_+ and P_- have no symmetries. Therefore, the group T is trivial.

Lemma 6.18. *Let C_d be a curve corresponding to the face orthogonal to the vectors $\{l_1, v_2, v_4, v_6, l_2, v_{10}, v_{12}, v_{13}, v_{15}, v_{17}, v_{19}, v_{21}\}$. It is nodal, and its irreducible components are a line, a conic and a rational cubic. It can be obtained from the curve C defined in Lemma 6.16 by perturbing the* D_6 *singularity into two real and two non-real nodes, as shown in Figure 6.10 (d).*

Proposition 6.19. (1) *All rigid isotopy types of real rational curves obtained by real*

nodal degeneration from a curve of type $\langle 2 \sqcup 1\langle 6\rangle\rangle$ with a pair of non-real nodes are obtained from the curve C_d by first perturbing some of the real nodes and then contracting the empty ovals.

(2) *There are 222 rigid isotopy types for such curves.*

Proof. (1) Since C_d contains all the possible conjunctions, this is clear.
(2) This count is obtained by enumerating the admissible pairs of faces of $P_+ \times P_-$. □

Remark 6.20. Not all these curves can be distinguished by semi-rigid isotopy. For example, consider a curve corresponding to the face orthogonal to $\{v_2, v_4, v_9, v_{11}, v_{14}, v_{16}, v_{18}, v_{20}\}$, and another curve corresponding to the face orthogonal to $\{v_4, v_6, v_9, v_{11}, v_{14}, v_{16}, v_{18}, v_{20}\}$. These two curves are semi-rigidly isotopic, but not rigidly isotopic. Their isotopy type is shown in Figure 6.12.

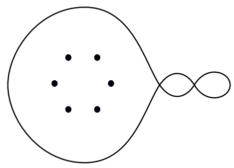


Figure 6.12

6.8 Degenerations of $\langle 6 \sqcup 1\langle 2\rangle\rangle$

6.8.1 Curves with only real nodes

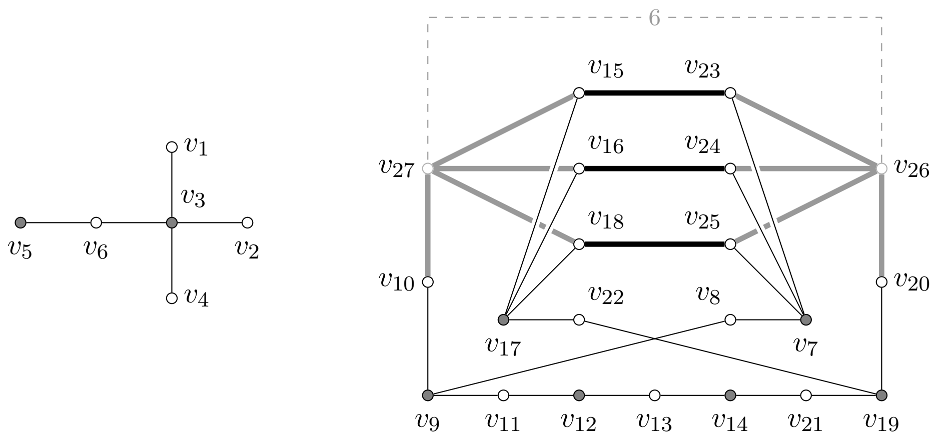
Let $L_+ = \mathrm{U}^{u_1,u_2} \oplus \mathrm{D}_4^{d_1,\dots,d_4}$ and $L_{-h} = \mathrm{U}^{w_1,w_2} \oplus \mathrm{E}_8^{e_1,\dots,e_8} \oplus \mathrm{D}_4^{c_1,\dots,c_4} \oplus \langle -2\rangle^y$. The Coxeter graphs of the polytopes P_+ and P_- are both finite; they are shown in Figure 6.13.

Lemma 6.21. *The symmetry group is $T = S_3 \times \mathbb{Z}/2\mathbb{Z}$, where the factor S_3 permutes the set of ordered triples $\{(v_1, v_{15}, v_{23}), (v_2, v_{16}, v_{24}), (v_4, v_{18}, v_{25})\}$, and the factor $\mathbb{Z}/2\mathbb{Z}$ acts on the Coxeter graph of P_- as a reflection in a vertical axis in the representation given in Figure 6.13. The exceptional vectors are v_{26} and v_{27}. The contraction vectors are v_3, v_5, v_7, v_9, v_{12}, v_{14}, v_{17} and v_{19}.*

Proof. One can check that the $\mathbb{Z}/2\mathbb{Z}$ symmetry acts trivially on the discriminant. For the S_3 factor, the reasoning is analogous to the one given in the proof of Lemma 6.15 □

Lemma 6.22. *Let C_1 be a curve corresponding to the face of $P_+ \times P_-$ defined by the vectors $S_{\mathrm{com}} \cup \{v_1, v_2, v_{15}, v_{16}\}$, where $S_{\mathrm{com}} = \{v_4, v_6, v_8, v_{10}, v_{11}, v_{13}, v_{20}, v_{21}, v_{22}, v_{25}\}$. Its singularities are an ordinary triple point and nine nodes, and its irreducible components are three lines and a rational cubic. Its isotopy type is as shown in Figure 6.14 (1). Let C_{1a} and C_{1b} be curves corresponding to the faces defined by the sets of vectors $S_{\mathrm{com}} \cup \{v_1, v_{15}, v_{16}\}$ and $S_{\mathrm{com}} \cup \{v_1, v_2, v_{15}\}$, respectively. They are both nodal with 12*

Coxeter graph of P_+ *Coxeter graph of P_-*



$v_1 = d_1$	$v_8 = e_2$	$v_{15} = c_1$	$v_{22} = w_2 + c_3^*$
$v_2 = d_2$	$v_9 = e_3$	$v_{16} = c_2$	$v_{23} = e_1^* + 2c_1^* + 2(w_1 + w_2) - y$
$v_3 = d_3$	$v_{10} = e_4$	$v_{17} = c_3$	$v_{24} = e_1^* + 2c_2^* + 2(w_1 + w_2) - y$
$v_4 = d_4$	$v_{11} = e_5$	$v_{18} = c_4$	$v_{25} = e_1^* + 2c_4^* + 2(w_1 + w_2) - y$
$v_5 = u_1 - u_2$	$v_{12} = e_6$	$v_{19} = w_1 - w_2$	$v_{26} = y$
$v_6 = u_2 + d_3^*$	$v_{13} = e_7$	$v_{20} = w_2 - y$	$v_{27} = 2e_4^* + 2(c_1^* + c_2^* + c_4^*)$
$v_7 = e_1$	$v_{14} = e_8$	$v_{21} = w_2 + e_8^*$	$\quad + 6(w_1 + w_2) - 3y$

Figure 6.13

nodes. They can be obtained from C_1 by perturbing the triple point into three real nodes, as shown in Figure 6.14 (1.a) and (1.b) respectively.

Lemma 6.23. *Let C_2 be a curve corresponding to the face of $P_+ \times P_-$ defined by the vectors $\{v_1, v_6, v_8, v_{10}, v_{11}, v_{13}, v_{15}, v_{16}, v_{18}, v_{20}, v_{21}, v_{22}\}$. It is nodal, and its irreducible components are two lines and a rational quartic. Its isotopy type is as shown in Figure 6.14 (2).*

Proposition 6.24. (1) *All rigid isotopy types of real rational curves obtained by real nodal degeneration from a curve of type $\langle 6 \sqcup 1\langle 2\rangle\rangle$ without real nodes can be obtained from one of the curves C_{1a}, C_{1b}, C_2 by first perturbing some of the nodes and then contracting empty ovals.*

(2) *Two such real rational sextics are rigidly isotopic if and only if they are semi-rigidly isotopic.*

(3) *There are* 803 *rigid isotopy types for such curves.*

Proof. (1) A set of pairwise orthogonal vectors in the Coxeter graph corresponds to a nodal curve if and only if it contains no two of the three pairs (v_1, v_{15}), (v_2, v_{16}) and (v_4, v_{18}), and no two of the three pairs (v_1, v_{23}), (v_2, v_{24}) and (v_4, v_{25}). This implies that up to symmetry, the conjunction vectors in such a set are contained in one of the sets corresponding to the curves C_{1a}, C_{1b} and C_2.
(2) We use an argument analogous to the one used to prove Proposition 6.17 (2), by considering both the graph of conjunctions and the order of the conjunctions on the non-empty oval. Among the vectors $\{v_1, v_2, v_4, v_{15}, v_{16}, v_{18}, v_{23}, v_{24}, v_{25}\}$, the ambiguous pairs that we need to distinguish are, up to symmetry:

- $\{v_1, v_{15}\}$ vs. $\{v_1, v_{16}\}$: distinguished using C_{1a}
- $\{v_1, v_{15}, v_{25}\}$ vs. $\{v_2, v_{15}, v_{25}\}$: distinguished using C_{1b}
- $\{v_1, v_{15}, v_{16}\}$ vs. $\{v_1, v_{16}, v_{18}\}$: distinguished using C_2
- $\{v_1, v_{15}, v_{16}, v_{25}\}$ vs. $\{v_4, v_{15}, v_{16}, v_{25}\}$: distinguished using C_{1a}
- $\{v_1, v_2, v_{15}\}$ vs. $\{v_2, v_4, v_{15}\}$: see below
- $\{v_1, v_{15}, v_{25}\}$ vs. $\{v_2, v_{15}, v_{25}\}$: distinguished using C_{1b}

The only pair which cannot be distinguished using the order of the conjunctions on the non-empty oval is $\{v_1, v_2, v_{15}\}$ vs. $\{v_2, v_4, v_{15}\}$. However, it turns out that there are no rational nodal curves whose corresponding face is orthogonal to the vectors $\{v_2, v_4, v_{15}\}$.
(3) This count is obtained by enumerating the admissible pairs of faces of $P_+ \times P_-$ modulo T. □

6.8.2 Curves with a pair of non-real nodes

Using the notations of the previous subsection, we let the vectors corresponding to the non-real roots be $\frac{1}{2}(l_1 \pm l_2)$, where $l_1 = v_2 - v_1$ and $l_2 = v_{16} - v_{15}$. The Coxeter graphs of the polytopes orthogonal to l_1, l_2 are shown in Figure 6.15.

The group T is of order two. The only symmetry is the one which acts in the Coxeter

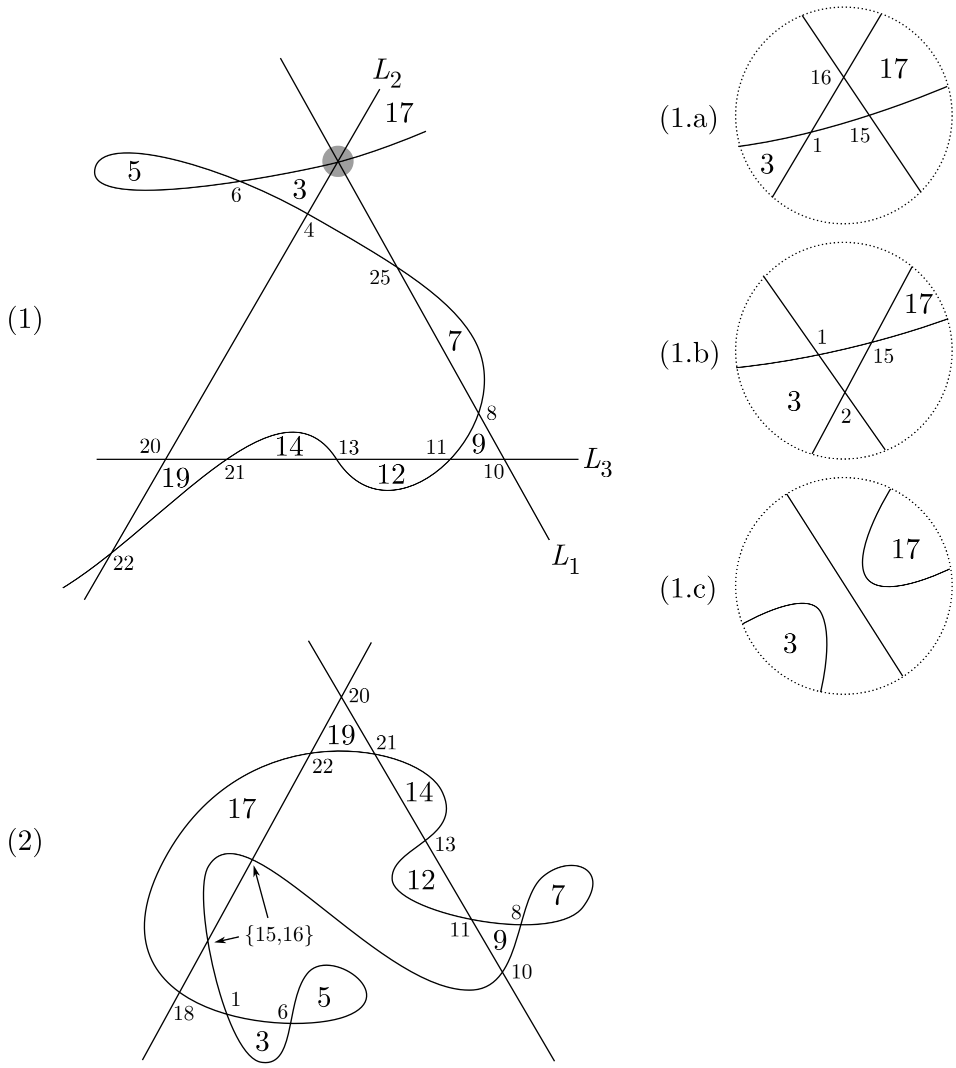


Figure 6.14 – Reducible degenerations of a curve of type $\langle 6 \sqcup 1\langle 2\rangle\rangle$. (1)—The isotopy type of the curve C_1 is shown on the left, with the triple point marked in gray. The small figures (1.a), (1.b), (1.c) on the right show three different ways of perturbing the triple point, resulting in reducible nodal curves C_{1a}, C_{1b} and C_{1c}, respectively. The curves C_{1a} and C_{1b} have 13 real nodes, while the curve C_{1c} has ten real nodes and a pair of non-real nodes. (2)—Isotopy type of the nodal curve C_2.

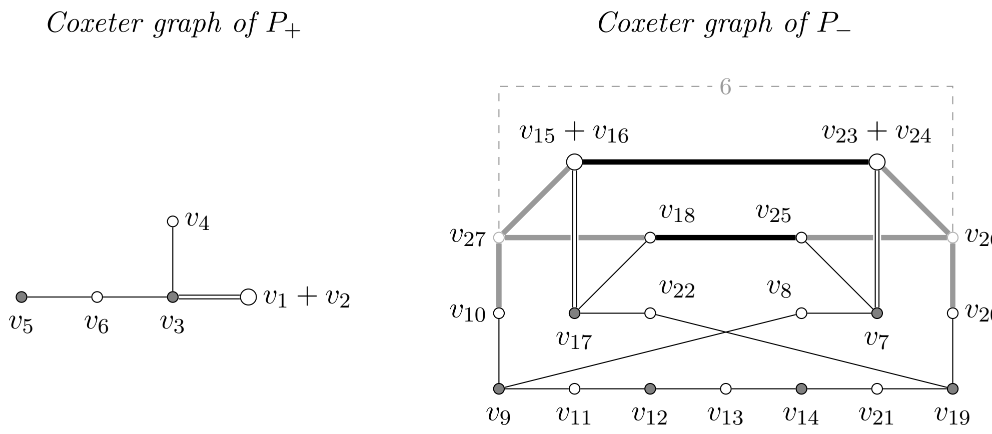


Figure 6.15

graph of P_- as a reflection in a vertical line in the representation given in Figure 6.15.

Lemma 6.25. *Let C_{1c} be a curve corresponding to the face orthogonal to the vectors $\{l_1, v_4, v_6, l_2, v_8, v_{10}, v_{11}, v_{13}, v_{20}, v_{21}, v_{22}, v_{25}\}$. It is nodal, and its irreducible components are two lines and a rational quartic. It can be obtained from the curve C_1 defined in Lemma 6.22 by perturbing the triple point into a pair of non-real nodes, as shown in Figure 6.14 (1.c).*

Proposition 6.26. (1) *All rigid isotopy types of real rational curves obtained by real nodal degeneration from a curve of type $\langle 6 \sqcup 1\langle 2\rangle\rangle$ with a pair of non-real nodes are obtained from the curve C_{1c} by first perturbing some of the real nodes and then contracting the empty ovals.*

(2) *Two such real rational sextics are rigidly isotopic if and only if they are semi-rigidly isotopic.*

(3) *There are* 204 *rigid isotopy types for such curves.*

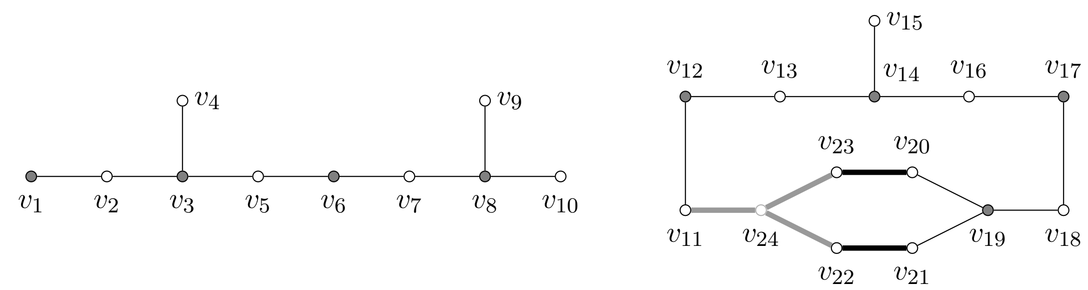
Proof. (1) Since the vectors v_{18} and v_{25} cannot be chosen at the same time, we may assume that, up to symmetry, the conjunction vectors involved are a subset of $\{v_4, v_6, v_8, v_{10}, v_{11}, v_{13}, v_{20}, v_{21}, v_{22}, v_{25}\}$.
(2) Either the inner ovals are both contracted, or they can be identified with the corresponding contraction vectors, since only one of them can conjunct with the non-empty oval. For the outer ovals, we can choose an identification between the ovals and the contraction vectors in P_-. The two possible choices are equivalent, since there is an symmetry in T which reverses the order of the ovals. Once such an identification is chosen, the conjunction vector corresponding to each node of the curve is uniquely determined by the pair of ovals it connects, since for each pair of ovals, there is at most one corresponding conjunction vector in the Coxeter graph.
(3) This count is obtained by enumerating the admissible pairs of faces of $P_+ \times P_-$. □

6.9 Degenerations of $\langle 4 \sqcup 1\langle 4\rangle\rangle$

6.9.1 Curves with only real nodes

Let $L_+ = \mathrm{U}(2)^{u_1+w_1,u_2+w_2} \oplus \mathrm{E}_8^{e_1,\ldots,e_8}$ and $L_{-h} = \mathrm{U}(2)^{u_1-w_1,u_2-w_2} \oplus \mathrm{E}_8^{f_1,\ldots,f_8} \oplus \langle -2\rangle^y$. The Coxeter graphs of the polytopes P_+ and P_- are both finite; they are shown in Figure 6.16.

Coxeter graph of P_+ *Coxeter graph of P_-*



$v_1 = e_1$	$v_6 = e_6$	$v_{11} = f_1^* + u_1 - w_1$	$v_{15} = f_4$	$v_{20} = f_8^* + u_1 - w_1$
$v_2 = e_2$	$v_7 = e_7$	$\quad + u_2 - w_2 - y$	$v_{16} = f_5$	$v_{21} = f_8^* + u_2 - w_2$
$v_3 = e_3$	$v_8 = e_8$	$v_{12} = f_1$	$v_{17} = f_6$	$v_{22} = u_2 - w_2 - y$
$v_4 = e_4$	$v_9 = e_8^* + u_2 + w_2$	$v_{13} = f_2$	$v_{18} = f_7$	$v_{23} = u_1 - w_1 - y$
$v_5 = e_5$	$v_{10} = e_8^* + u_1 + w_1$	$v_{14} = f_3$	$v_{19} = f_8$	$v_{24} = y$

Figure 6.16

Lemma 6.27. *The symmetry group is $T = \mathbb{Z}/2\mathbb{Z}$, exchanging each of the pairs (v_9, v_{10}), (v_{20}, v_{21}) and $(v_{22}, v_{23})\}$. The only exceptional vector is v_{24}. The contraction vectors are v_1, v_3, v_6, v_8, v_{12}, v_{14}, v_{17} and v_{19}.*

Lemma 6.28. *Let C_1 be a curve corresponding to the face of $P_+ \times P_-$ defined by the vectors $S_{\mathrm{com}} \cup \{v_9, v_{10}, v_{20}, v_{21}\}$, where $S_{\mathrm{com}} = \{v_2, v_4, v_5, v_7, v_{11}, v_{13}, v_{15}, v_{16}, v_{18}\}$. Its singularities are an ordinary triple point and nine nodes, and its irreducible components are a line, a conic and a rational cubic. Its isotopy type is as shown in Figure 6.17 (1). Let C_{1a} be a curve corresponding to the face defined by the set of vectors $S_{\mathrm{com}} \cup \{v_9, v_{20}, v_{21}\}$. It is nodal with 12 nodes, and it can be obtained from C_1 by perturbing the triple point into three real nodes, as shown in Figure 6.17 (a).*

Let C_2 be a curve corresponding to the face of $P_+ \times P_-$ defined by the vectors $S_{\mathrm{com}} \cup \{v_9, v_{10}, v_{20}, v_{22}\}$. Its is nodal, and its irreducible components are three lines and a rational cubic. Its isotopy type is as shown in Figure 6.17 (2).

Let C_3 be a curve corresponding to the face of $P_+ \times P_-$ defined by the vectors $S_{\mathrm{com}} \cup \{v_9, v_{22}, v_{23}\}$. Its is nodal, and its irreducible components are two lines and a rational quartic. Its isotopy type is as shown in Figure 6.17 (3).

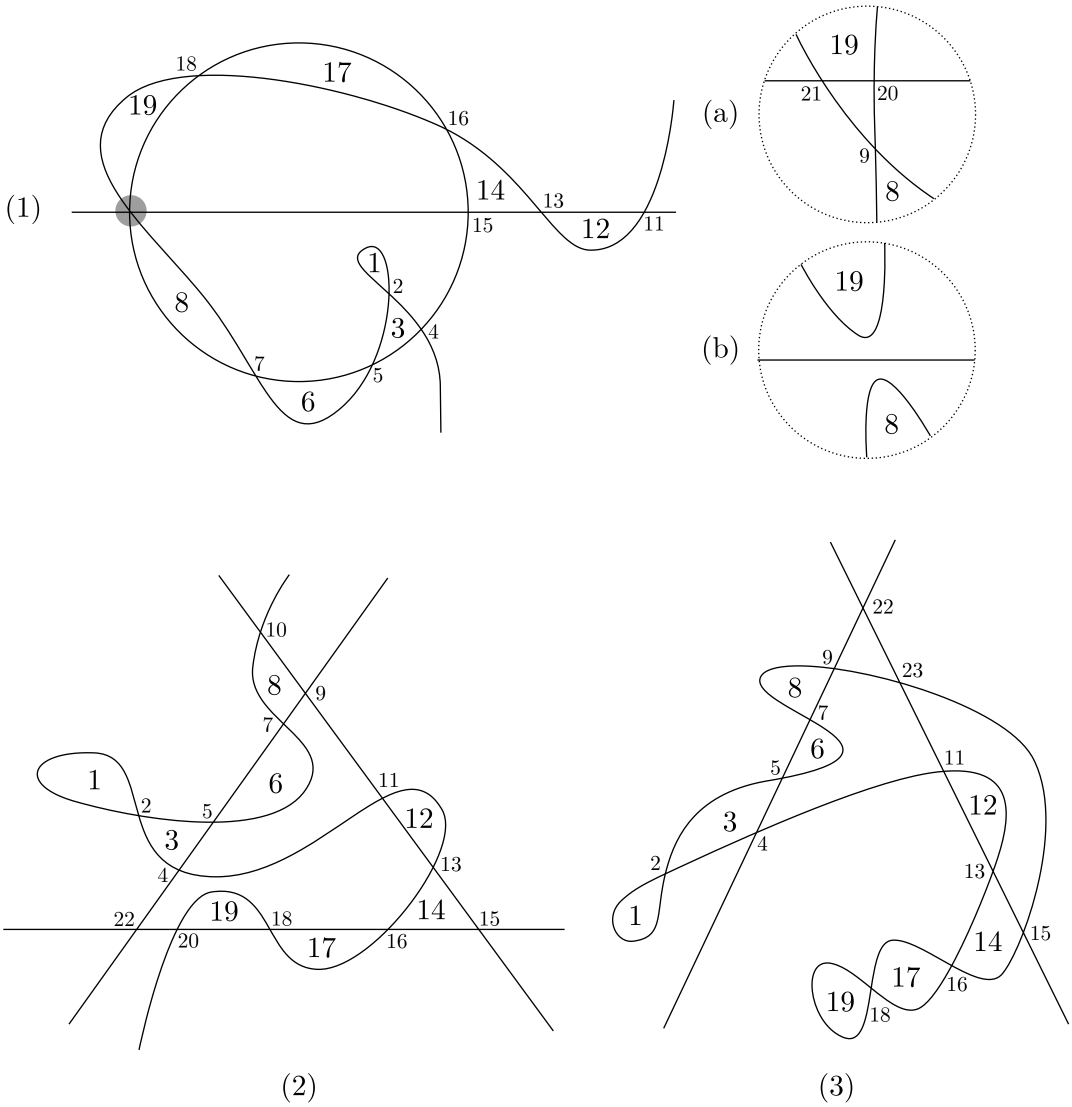


Figure 6.17 – Reducible degenerations of a curve of type $\langle 4 \sqcup 1\langle 4\rangle\rangle$. (1)—The isotopy type of the curve C_1 is shown on the left, with the triple point marked in gray. The small figures (a), and (b) on the right show two different ways of perturbing the triple point, resulting in reducible nodal curves C_{1a} and C_{1b} respectively. The curve C_{1a} has 12 real nodes, while the curve C_{1b} has nine real nodes and a pair of non-real nodes. (2), (3)—Isotopy type of the nodal curves C_2 and C_3.

Proposition 6.29. (1) *All rigid isotopy types of real rational curves obtained by real nodal degeneration from a dividing curve of type $\langle 4 \sqcup 1\langle 4\rangle\rangle$ without real nodes can be obtained from one of the curves C_{1a}, C_2, C_3 by first perturbing some of the nodes and then contracting empty ovals.*

(2) *Two such real rational sextics are rigidly isotopic if and only if they are semi-rigidly isotopic.*

(3) *There are* 1088 *rigid isotopy types for such curves.*

Proof. (1) A set of pairwise orthogonal vectors in the Coxeter graph corresponds to a nodal curve if and only if it does not contain one of the quadruples $\{v_9, v_{10}, v_{20}, v_{21}\}$, $\{v_9, v_{10}, v_{22}, v_{23}\}$. This implies that up to symmetry, the conjunction vectors in such a set are contained in one of the sets corresponding to the curves C_{1a}, C_2 and C_3.
(2) We use an argument analogous to the one used to prove Proposition 6.17 (2), by considering both the graph of conjunctions and the order of the conjunctions on the non-empty oval. Among the vectors $\{v_9, v_{10}, v_{20}, v_{21}, v_{22}, v_{23}\}$, the ambiguous pairs that we need to distinguish are, up to symmetry:

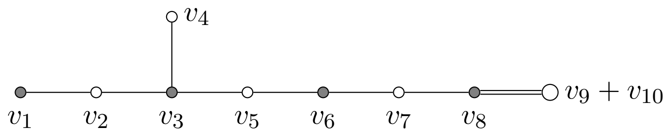
- $\{v_9, v_{20}\}$ vs. $\{v_9, v_{21}\}$: distinguished using C_{1a};
- $\{v_9, v_{22}\}$ vs. $\{v_9, v_{23}\}$: distinguished using C_2;
- $\{v_9, v_{20}, v_{22}\}$ vs. $\{v_{10}, v_{20}, v_{22}\}$: distinguished using C_3.

(3) This count is obtained by enumerating the admissible pairs of faces of $P_+ \times P_-$ modulo T. □

6.9.2 Curves with a pair of non-real nodes

Using the notations of the previous subsection, we let the vectors corresponding to the non-real roots be $\frac{1}{2}(l_1 \pm l_2)$, where $l_1 = v_9 - v_{10}$ and $l_2 = v_{20} - v_{21}$. The Coxeter graphs of the polytopes orthogonal to l_1, l_2 are shown in Figure 6.18.

Coxeter graph of P_+



Coxeter graph of P_-

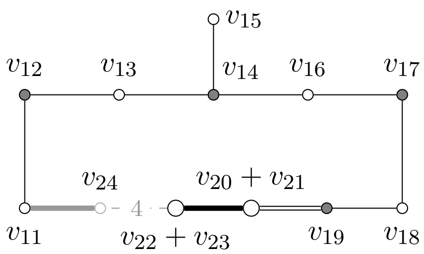


Figure 6.18

Lemma 6.30. *Let C_{1b} be a curve corresponding to the face orthogonal to the vectors $\{l_1, v_2, v_4, v_5, v_7, l_2, v_{11}, v_{13}, v_{15}, v_{16}, v_{18}\}$. It is nodal, and its irreducible components are a line and a rational quintic. It can be obtained from the curve C_1 defined in Lemma 6.28 by perturbing the triple point into a pair of non-real nodes, as shown in Figure 6.17 (b).*

Proposition 6.31. (1) *All rigid isotopy types of real rational curves obtained by real nodal degeneration from a dividing curve of type $\langle 4 \sqcup 1\langle 4\rangle\rangle$ with a pair of non-real nodes are obtained from the curve C_{1b} by first perturbing some of the real nodes and then contracting the empty ovals.*

(2) *Two such real rational sextics are rigidly isotopic if and only if they are semi-rigidly isotopic.*

(3) *There are* 98 *rigid isotopy types for such curves.*

Proof. (1) Since C_{1b} contains all the possible conjunctions, this is clear.
(2) Either all the empty ovals are contracted, or they can be identified with the contraction vectors, since all the possible conjunctions of empty ovals with the non-empty oval are asymmetrical. Then, the conjunction vector corresponding to each node of the curve is uniquely determined by the pair of ovals it connects, since for each pair of ovals, there is at most one corresponding conjunction vector in the Coxeter graph.
(3) This count is obtained by enumerating the admissible pairs of faces of $P_+ \times P_-$. □

6.10 Degenerations of $\langle 5 \sqcup 1\langle 3\rangle\rangle$

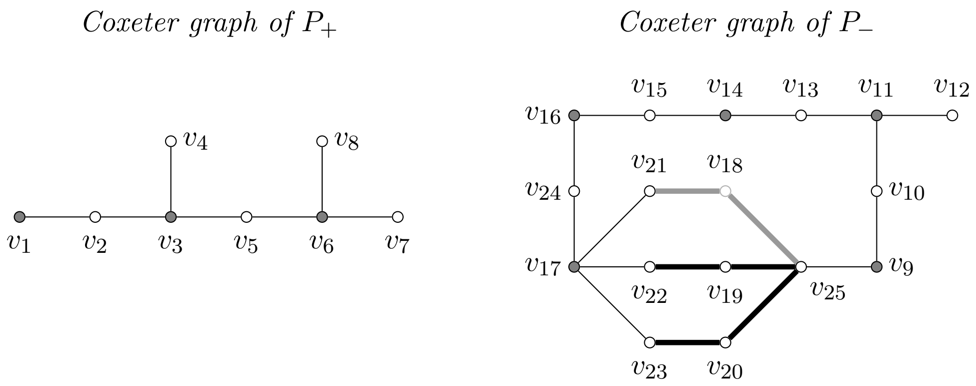
We first calculate the Coxeter graphs for curves of type $\langle 5 \sqcup 1\langle 3\rangle\rangle$ without non-real nodes. Let $L_+ = \langle 2\rangle^{u_1+u_2} \oplus \mathrm{E}_7^{e_1,\dots,e_7}$ and $L_{-h} = \mathrm{U}^{w_1,w_2} \oplus \mathrm{E}_8^{f_1,\dots,f_8} \oplus \langle -2\rangle^{e_8^*} \oplus \langle -2\rangle^{u_1-u_2} \oplus \langle -2\rangle^{y}$. The Coxeter graphs of the polytopes P_+ and P_- are both finite; they are shown in Figure 6.19.

We let the vectors corresponding to the non-real roots be $\frac{1}{2}(l_1 \pm l_2)$, where $l_1 = v_8 - v_7$ and $l_2 = v_{20} - v_{19}$. The Coxeter graphs of the polytopes orthogonal to l_1, l_2 are shown in Figure 6.20.

The polytopes P_+ and P_- have no symmetries. The contraction vectors are $v_1, v_3, v_5, v_9, v_{11}, v_{14}, v_{16}$ and v_{17}.

Lemma 6.32. *Let C_0 be a curve corresponding to the face defined by the vectors $\{v_2, v_4, v_5, v_7, v_8, v_{10}, v_{12}, v_{13}, v_{15}, v_{19}, v_{20}, v_{21}, v_{24}, v_{25}\}$, and let C be a curve corresponding to the face defined by the vectors $\{l_1, v_2, v_4, v_5, l_2, v_{10}, v_{12}, v_{13}, v_{15}, v_{21}, v_{24}, v_{25}\}$. The singularities of C_0 are an ordinary triple point and ten nodes, and its irreducible components are three lines and a rational cubic. Its isotopy type is shown in Figure 6.21. The curve C is nodal with ten real nodes and a pair of non-real nodes. Its irreducible components are two lines and a rational quartic. It can be obtained from C_0 by perturbing the triple point into a pair of non-real nodes, as shown in Figure 6.21.*

Proposition 6.33. (1) *All rigid isotopy types of real rational curves obtained by real*



$$
\begin{array}{llll}
v_1 = e_1 & v_8 = e_7 + 2e_8 + e_8^* & v_{14} = f_6 & v_{21} = w_2 - y \\
v_2 = e_2 & \quad + (u_1 + u_2) & v_{15} = f_7 & v_{22} = w_2 - e_8^* \\
v_3 = e_3 & v_9 = f_1 & v_{16} = f_8 & v_{23} = w_2 - (u_1 - u_2) \\
v_4 = e_4 & v_{10} = f_2 & v_{17} = w_1 - w_2 & v_{24} = w_2 + f_8^* \\
v_5 = e_5 & v_{11} = f_3 & v_{18} = y & v_{25} = f_1^* + 2(w_1 + w_2) - e_8^* \\
v_6 = e_6 & v_{12} = f_4 & v_{19} = e_8^* & \quad - (u_1 - u_2) - y \\
v_7 = e_7 & v_{13} = f_5 & v_{20} = u_1 - u_2 &
\end{array}
$$

Figure 6.19

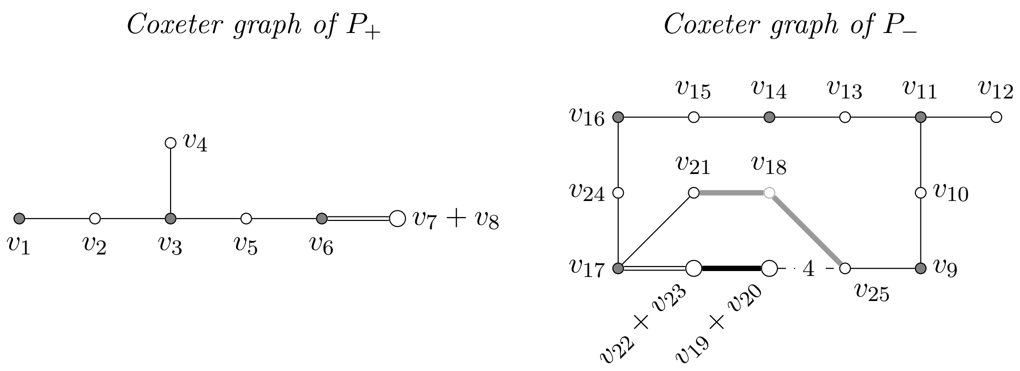


Figure 6.20

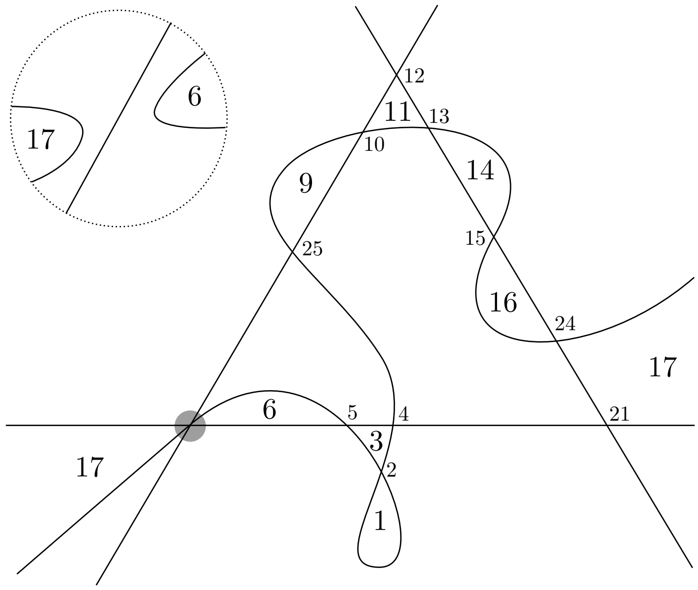


Figure 6.21 – Reducible degenerations of a curve of type $\langle 5 \sqcup 1\langle 3\rangle\rangle$. The large picture shows a reducible curve C_0 with a triple point and nine real nodes. The detail shows how the triple point can be perturbed into a pair of non-real nodes, resulting in the curve nodal C with nine real nodes and a pair of non-real nodes.

nodal degeneration from a curve of type $\langle 5 \sqcup 1\langle 3\rangle\rangle$ *with a pair of non-real nodes are obtained from the curve* C *by first perturbing some of the real nodes and then contracting the empty ovals.*

(2) *There are* 255 *rigid isotopy types for such curves.*

Proof. (1) Since C contains all the possible conjunctions, this is clear.
(2) This count is obtained by enumerating the admissible pairs of faces of $P_+ \times P_-$. □

Remark 6.34. Not all these curves can be distinguished by semi-rigid isotopy. For example, consider a curve corresponding to the face orthogonal to $\{v_1, v_4, v_5, v_9, v_{11}, v_{14}, v_{16}, v_{17}\}$, and another curve corresponding to the face orthogonal to $\{v_2, v_4, v_6, v_9, v_{11}, v_{14}, v_{16}, v_{17}\}$. These two curves are semi-rigidly isotopic, but not rigidly isotopic. Their isotopy type is shown in Figure 6.22.

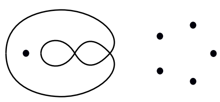
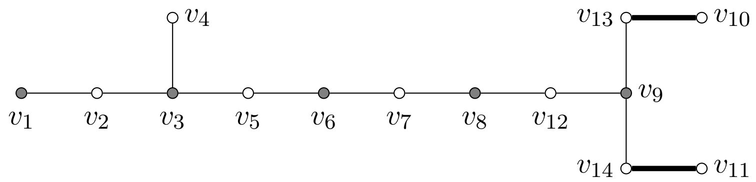


Figure 6.22

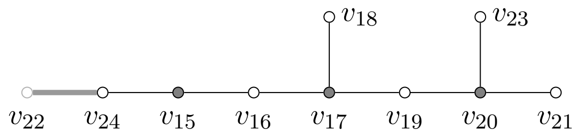
6.11 Degenerations of $\langle 3 \sqcup 1\langle 5\rangle\rangle$

We first calculate the Coxeter graphs for curves of type $\langle 3 \sqcup 1\langle 5\rangle\rangle$ without non-real nodes. Let $L_+ = \mathrm{E}_8^{e_1,\ldots,e_8} \oplus \mathrm{U}^{w_1,w_2} \oplus \langle -2\rangle^{u_1-u_2} \oplus \langle -2\rangle^{f_8^*}$ and $L_{-h} = \mathrm{E}_7^{f_1,\ldots,f_7} \oplus \langle 2\rangle^{u_1+u_2} \oplus \langle -2\rangle^{y}$. The Coxeter graphs of the polytopes P_+ and P_- are both finite; they are shown in Figure 6.23.

Coxeter graph of P_+



Coxeter graph of P_-



$v_1 = e_1$	$v_7 = e_7$	$v_{13} = w_2 - e_8^*$	$v_{19} = f_5$
$v_2 = e_2$	$v_8 = e_8$	$v_{14} = w_2 - (u_1 - u_2)$	$v_{20} = f_6$
$v_3 = e_3$	$v_9 = w_1 - w_2$	$v_{15} = f_1$	$v_{21} = f_7$
$v_4 = e_4$	$v_{10} = f_8^*$	$v_{16} = f_2$	$v_{22} = y$
$v_5 = e_5$	$v_{11} = u_1 - u_2$	$v_{17} = f_3$	$v_{23} = f_7 + 2f_8 + f_8^* + (u_1 + u_2)$
$v_6 = e_6$	$v_{12} = e_8^* + w_2$	$v_{18} = f_4$	$v_{24} = u_1 + u_2 - y + f_1^* - f_8^*$

Figure 6.23

We let the vectors corresponding to the non-real roots be $\frac{1}{2}(l_1 \pm l_2)$, where $l_1 = v_{10} - v_{11}$ and $l_2 = v_{23} - v_{21}$. The Coxeter graphs of the polytopes orthogonal to l_1, l_2 are shown in Figure 6.24.

The polytopes P_+ and P_- have no symmetries. The contraction vectors are $v_1, v_3, v_6, v_8, v_9, v_{15}, v_{17}$ and v_{20}.

Lemma 6.35. *Let C_0 be a curve corresponding to the face defined by the vectors $\{v_2, v_4, v_5, v_7, v_{12}, v_{13}, v_{14}, v_{16}, v_{18}, v_{19}, v_{21}, v_{23}, v_{24}\}$, and let C be a curve corresponding to the face defined by the vectors $\{l_1, v_2, v_4, v_5, v_7, v_{12}, l_2, v_{16}, v_{18}, v_{19}, v_{24}\}$. The singularities of C_0 are an ordinary triple point and nine nodes, and its irreducible components are a line, a conic and a rational cubic. Its isotopy type is shown in Figure 6.25. The curve C is nodal with nine real nodes and a pair of non-real nodes. Its irreducible components are a line and a rational quintic. It can be obtained from C_0 by perturbing the triple point into a pair of non-real nodes, as shown in Figure 6.25.*

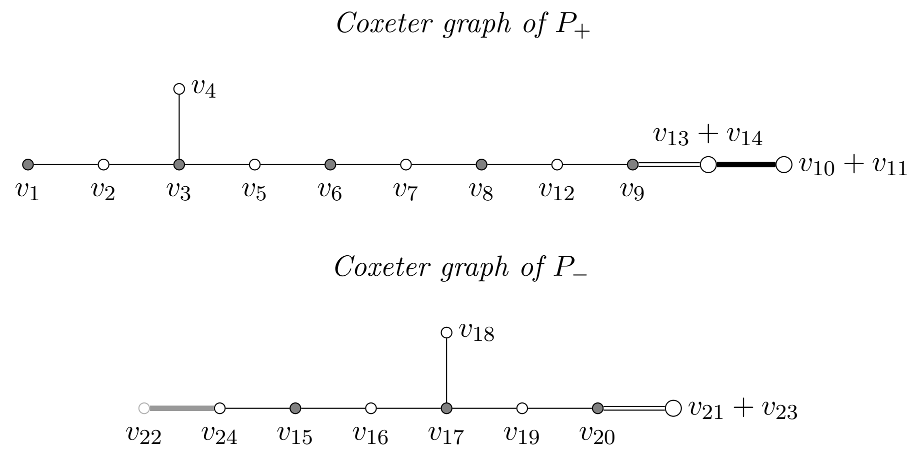


Figure 6.24

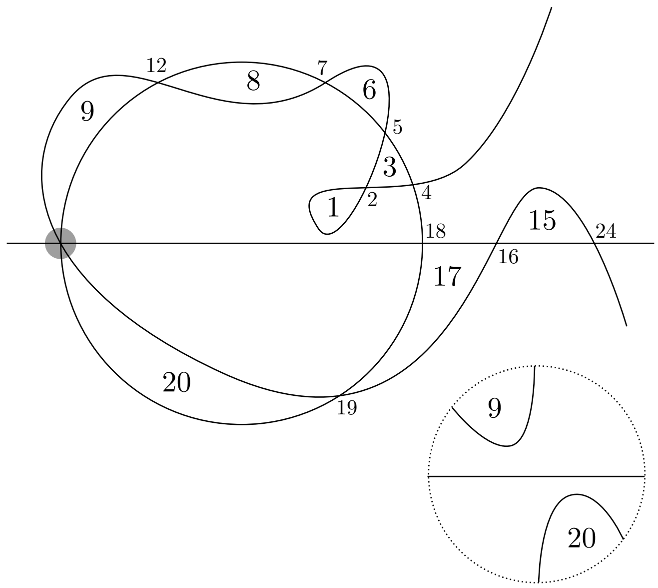


Figure 6.25 – Reducible degenerations of a curve of type $\langle 3 \sqcup 1\langle 5\rangle\rangle$. The large picture shows a reducible curve C_0 with a triple point and nine real nodes. The detail shows how the triple point can be perturbed into a pair of non-real nodes, resulting in the curve nodal C with nine real nodes and a pair of non-real nodes.

Proposition 6.36. (1) *All rigid isotopy types of real rational curves obtained by real nodal degeneration from a curve of type $\langle 3 \sqcup 1\langle 5\rangle\rangle$ with a pair of non-real nodes are obtained from the curve C by first perturbing some of the real nodes and then contracting the empty ovals.*

(2) *There are* 90 *rigid isotopy types for such curves.*

Proof. (1) Since C contains all the possible conjunctions, this is clear.
(2) This count is obtained by enumerating the admissible pairs of faces of $P_+ \times P_-$. ☐

Remark 6.37. Not all these curves can be distinguished by semi-rigid isotopy. For example, consider a curve corresponding to the face orthogonal to $\{v_1, v_3, v_6, v_8, v_9, v_{15}, v_{18}, v_{19}\}$, and another curve corresponding to the face orthogonal to $\{v_1, v_3, v_6, v_8, v_9, v_{16}, v_{18}, v_{20}\}$. These two curves are semi-rigidly isotopic, but not rigidly isotopic. Their isotopy type is shown in Figure 6.26.

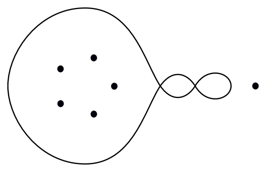


Figure 6.26

6.12 Degenerations of $\langle 1 \sqcup 1\langle 7\rangle\rangle$

Let $L_+ = \langle 2\rangle^{w_1+w_2} \oplus \mathrm{E}_8^{e_1,\dots,e_8} \oplus \mathrm{E}_7^{f_1,\dots,f_7}$, and $L_{-h} = \mathrm{U}^{u_1,u_2} \oplus \langle -2\rangle^{f_8^*} \oplus \langle -2\rangle^{w_1-w_2} \oplus \langle -2\rangle^{y}$. Let $l_1 = w_1 + w_2 + 2f_8 + f_8^*$, and $l_2 = w_1 - w_2 - f_8^*$. The Coxeter graphs of the polytopes P_+ and P_- are both finite; they are shown in Figure 6.27.

The symmetry group T is $\mathbb{Z}/2\mathbb{Z}$, acting as a reflection in a horizontal line in the representation of the Coxeter graph of P_+ given in Figure 6.27. The only exceptional vector is v_{20}. The contraction vectors are $\{v_1, v_3, v_6, v_8, v_{10}, v_{12}, v_{15}, v_{22}\}$.

Lemma 6.38. *Let C be a curve corresponding to the face defined by the vectors $\{l_1, v_2, v_4, v_5, v_7, v_9, v_{11}, v_{13}, v_{14}, l_2, v_{21}\}$. It is nodal, and its irreducible components are a conic and a rational quartic. Its isotopy type is shown in Figure 6.28.*

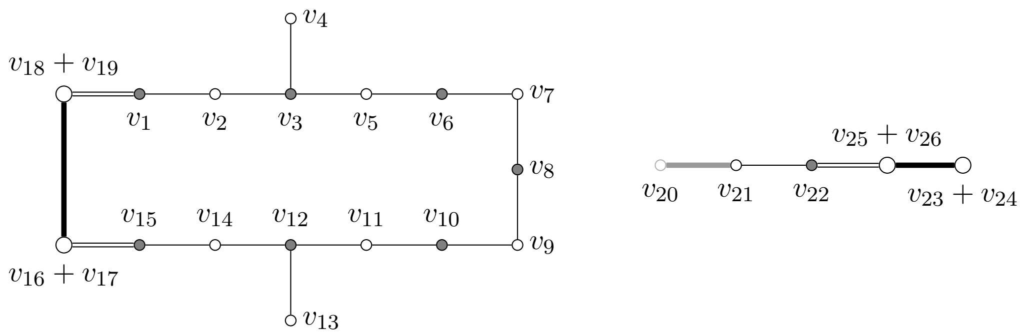
Proposition 6.39. (1) *All rigid isotopy types of real rational curves obtained by real nodal degeneration from a curve of type $\langle 1 \sqcup 1\langle 7\rangle\rangle$ can be obtained from C by first perturbing some of the nodes and then contracting empty ovals.*

(2) *Two such real rational sextics are rigidly isotopic if and only if they are semi-rigidly isotopic.*

(3) *There are* 130 *rigid isotopy types for such curves.*

Proof. (1) Since C is obtained by degenerating *all* the conjunction vectors, this is clear.

Coxeter graph of P_+ *Coxeter graph of P_-*



$v_1 = e_1$
$v_2 = e_2$
$v_3 = e_3$
$v_4 = e_4$
$v_5 = e_5$
$v_6 = e_6$
$v_7 = e_7$
$v_8 = e_8$

$v_9 = w_1 + w_2 + e_8^* + f_1^* - f_8^*$
$v_{10} = f_1$
$v_{11} = f_2$
$v_{12} = f_3$
$v_{13} = f_4$
$v_{14} = f_5$
$v_{15} = f_6$

$v_{16} = f_7$
$v_{17} = f_7 + 2f_8 + f_8^* + w_1 + w_2$
$v_{18} = w_1 + w_2 + e_1^*$
$v_{19} = 2(w_1 + w_2) + e_1^* + f_8^* + 2f_8$
$v_{20} = y$
$v_{21} = u_2 - y$

$v_{22} = u_1 - u_2$
$v_{23} = w_1 - w_2$
$v_{24} = f_8^*$
$v_{25} = u_2 - (w_1 - w_2)$
$v_{26} = u_2 - f_8^*$

Figure 6.27

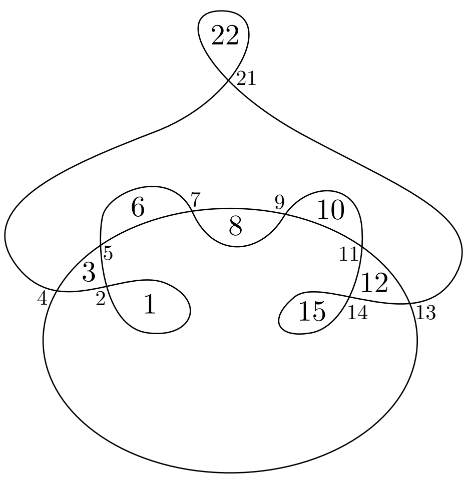


Figure 6.28 – Reducible nodal degeneration C of a curve of type $\langle 1 \sqcup 1\langle 7\rangle\rangle$.

(2) For curves of type $\langle 1 \sqcup 1\langle 7\rangle\rangle$, there is a linear order on the set of inner ovals, defined up to inversion. Given a rational curve obtained by real nodal degeneration from a curve of type $\langle 1 \sqcup 1\langle 7\rangle\rangle$, one can identify the inner ovals with the contraction vectors in the Coxeter graph, up to the $\mathbb{Z}/2\mathbb{Z}$ symmetry. Once such an identification is chosen, the conjunction vector corresponding to each node of the curve is uniquely determined by the pair of ovals it connects, since for each pair of ovals, there is at most one corresponding conjunction vector in the Coxeter graph.
(3) This count is obtained by enumerating the admissible pairs of faces of $P_+ \times P_-$ modulo T. $\square$

6.13 Degenerations of $\langle 1\langle 8\rangle\rangle$

6.13.1 Curves with only real nodes

Let $L_+ = \mathrm{U}(2)^{u_1+w_1,u_2+w_2} \oplus \mathrm{E}_8^{e_1,\ldots,e_8} \oplus \mathrm{E}_8^{f_1,\ldots,f_8}$, and $L_{-h} = \mathrm{U}(2)^{u_1-w_1,u_2-w_2} \oplus \langle -2\rangle^{y}$. The Coxeter graph of the polytope P_- is finite; it is shown in Figure 6.29. The Coxeter graph of the polytope P_+ is infinite. Let $P_+^0 \subset P_+$ be a smaller polytope delimited by walls orthogonal to vectors $v \in K_+$ of square -2 as well as vectors $v \in K_+$ of square -4 such that $v \in 2L^\vee$. This smaller polytope is finite. Its Coxeter graph is shown in Figure 6.29.

The symmetry group of P_- is $\mathbb{Z}/2\mathbb{Z}$, exchanging v_{24} and v_{25}. The symmetry group of P_+ can be decomposed into two parts: the symmetries of the smaller polytope P_+^0, and the reflection group generated by the reflections in the hyperplanes orthogonal to the long roots v_{21} and v_{22}. The latter is a parabolic reflection group of type $\tilde{A}_1$, that is, an infinite dihedral group D_∞. The former is the symmetry group of the square, D_4. The complete symmetry group of the polytope P_+ is the semi-direct product $D_4 \ltimes D_\infty$. To determine the group T, we need to see how the symmetries of P_+ and those of P_- glue together, i.e. how they act on the discriminant group. The discriminant form of the lattice $\mathrm{U}(2)$ has two automorphisms: the identify and the one induced by exchanging two vectors forming a hyperbolic basis.

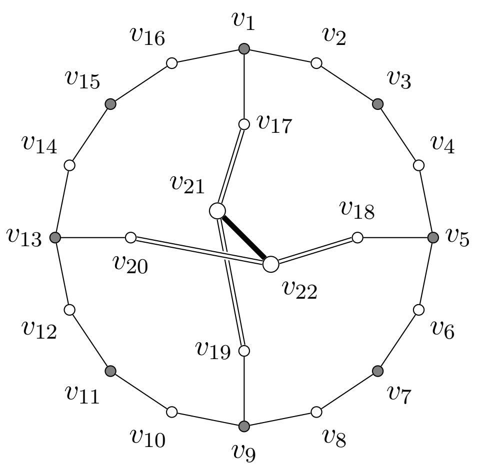
The non-trivial symmetry of P_- acts non-trivially on the discriminant group. Therefore, any symmetry of P_+ glues either to the identity of P_- or to the non-trivial symmetry. Hence, the group T is isomorphic to $D_4 \ltimes D_\infty$.

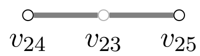
The only exceptional vector is v_{23}. The contraction vectors are $\{v_1, v_3, v_5, v_7, v_9, v_{11}, v_{13}, v_{15}\}$.

Consider a face F of P_+ defined by a set of pairwise orthogonal roots. Applying an element of the group D_∞ generated by the reflections in hyperplanes orthogonal to the vectors v_{21} and v_{22}, we may assume that F intersects the polytope P_+^0. As a face of P_+^0, the intersection $F \cap P_+^0$ is either defined by a set of pairwise orthogonal short roots, or it is defined by a set of pairwise orthogonal short roots plus a long root (v_{21} or v_{22}) intersecting one of the short roots. In the former case, we say that the face F is *of the first kind*, and in the latter case, we say that it is *of the second kind*.

Given a face F of the second kind, up to a symmetry of P_+^0, we may assume that

Coxeter graph of P_+^0 *Coxeter graph of P_-*





$v_1 = f_8$
$v_2 = u_1 + w_1 + u_2 + w_2 + e_1^* + f_8^*$
$v_3 = e_1$
$v_4 = e_2$
$v_5 = e_3$
$v_6 = e_5$
$v_7 = e_6$
$v_8 = e_7$
$v_9 = e_8$
$v_{10} = u_1 + w_1 + u_2 + w_2 + e_8^* + f_1^*$
$v_{11} = f_1$
$v_{12} = f_2$
$v_{13} = f_3$
$v_{14} = f_5$
$v_{15} = f_6$
$v_{16} = f_7$
$v_{17} = u_2 + w_2 + f_8^*$
$v_{18} = e_4$
$v_{19} = u_2 + w_2 + e_8^*$
$v_{20} = f_4$
$v_{21} = u_1 + w_1 - (u_2 + w_2)$
$v_{22} = 3(u_1 + w_1) + 5(u_2 + w_2) + 2e_4^* + 2f_4^*$
$v_{23} = y$
$v_{24} = u_2 - w_2 - y$
$v_{25} = u_1 - w_1 - y$

Figure 6.29

the short root and the long root intersecting it are v_{17} and v_{21}, respectively. Then, as a face of P_+, the face F is defined by the short roots defining $F \cap P_+^0$ as a face of P_+^0 together with the vector $v_{26} := R_{v_{21}}(v_{17}) = v_{17} + v_{21}$.

Lemma 6.40. *Let C_1 be a curve corresponding to the face of $P_+ \times P_-$ defined by the vectors $\{v_2, v_4, v_6, v_8, v_{10}, v_{12}, v_{14}, v_{16}, v_{17}, v_{18}, v_{20}, v_{24}, v_{26}\}$, and let C_2 be a curve corresponding to the face defined by the vectors $\{v_2,\ v_4,\ v_6,\ v_8,\ v_{10},\ v_{12},\ v_{14},\ v_{16},\ v_{17},\ v_{18},\ v_{19},\ v_{20},\ v_{24},\ v_{25}\}$. The curves C_1 and C_2 are both nodal. The irreducible components of C_1 are two lines and two conics. The irreducible components of C_2 are four lines and a conic. The isotopy types of C_1 and C_2 are shown in Figure 6.30 (1) and (2), respectively.*

Proposition 6.41. (1) *All rigid isotopy types of real rational curves obtained by real nodal degeneration from a dividing curve of type $\langle 1\langle 8\rangle\rangle$ with only real nodes can be obtained from one of the curves C_1 and C_2 by first perturbing some of the nodes and then contracting empty ovals.*

(2) *Two such real rational sextics are rigidly isotopic if and only if they are semi-rigidly isotopic.*

(3) *There are 477 rigid isotopy types for such curves.*

Proof. (1) Curves corresponding to faces of the first kind can be obtained from C_2, since C_2 contains all the possible conjunctions. For curves corresponding to faces of the second kind, we may assume that the short root and the long root intersecting it are v_{17} and v_{21}, respectively. Moreover, there may be at most one of the vectors v_{24} and v_{25}, since otherwise the corresponding curve is not nodal. We can assume that this vector, if it is present, is v_{24}. Indeed, the composition $R_{v_{21}} \circ R_{v_{24}-v_{25}}$ defines an element of T which interchanges the pairs $\{v_{24}, v_{25}\}$ and $\{v_{17}, v_{26}\}$ and leaves the other vectors invariant. (Note that v_{19} cannot appear because it is not orthogonal to v_{26}.)

(2) Note that a curve corresponds to a face of the second kind if and only if there is an empty oval which conjuncts twice with the non-empty oval. We treat curves corresponding to faces of the first kind and those corresponding to faces of the second type separately.

Let us first consider curves corresponding to faces of the second kind. We may assume that the oval which conjuncts twice with the nonempty oval corresponds to the vector v_1. The cyclic order on the empty ovals allows us to identify the remaining ovals with the remaining contraction vectors, up to a $\mathbb{Z}/2\mathbb{Z}$ symmetry. After choosing one of the two identifications, we can also identify the conjunctions with the conjunction vectors, since for each pair of ovals there is at most vector corresponding to a conjunction between those ovals. The only remaining question is, if the non-empty oval conjuncts with itself, if this self-conjunction corresponds to the vector v_{24} or v_{25}. However, we have shown above that these vectors are equivalent, using the symmetry $R_{v_{21}} \circ R_{v_{24}-v_{25}}$. Therefore, the isotopy type together with the order of the ovals determines the face of $P_+ \times P_-$ modulo T.

Now consider a curve corresponding to a face of the first kind. If not all the empty ovals are contracted, then there must be a conjunction of an empty oval with the non-empty

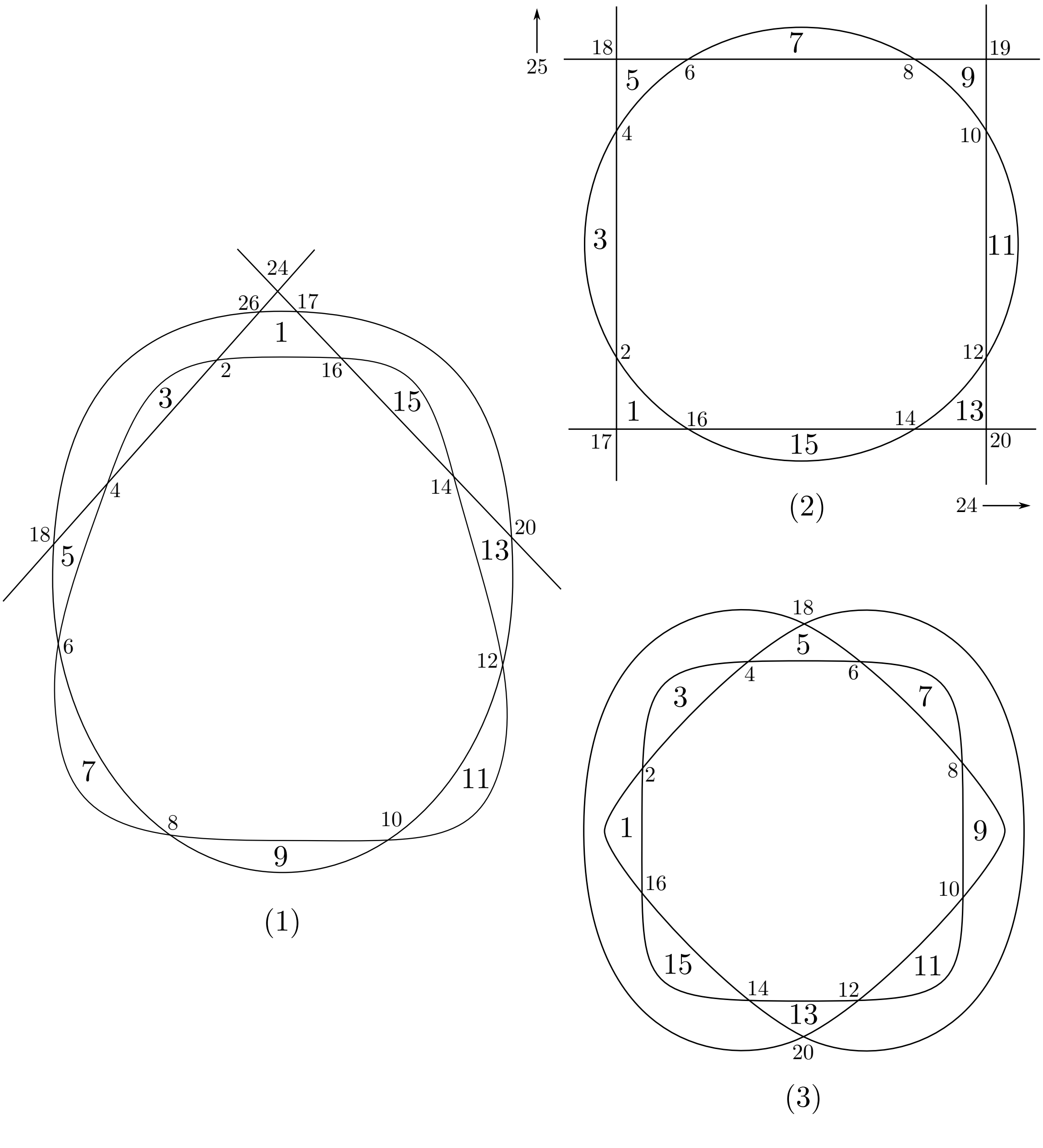


Figure 6.30 – Reducible nodal degenerations C_1, C_2, C_3 of a curve of type $\langle 1\langle 8\rangle\rangle$. The curves C_1 and C_2 have only real nodes, while C_3 has a pair of non-real nodes. In figure (2), the labels 24 and 25 mark the intersection points of the parallel lines at infinity.

oval. This allows us to identify the ovals with the corresponding contraction vectors, up to a symmetry in D_4. Such an identification also determines the vectors corresponding to the conjunctions, using the same argument as above. It remains to see how to distinguish the two vectors v_{24} and v_{25} which correspond to self-conjunctions of the non-empty oval. If there is no such self conjunction or if there are two, no distinction is necessary. Therefore, we may assume that the non-empty oval has one self-conjunction. Since there are eight ovals and only real nodes, the graph of conjunctions must contain two cycles. Since we assume that there is one self-conjunction of the non-empty oval, this leaves one cycle among the conjunctions corresponding to vectors in P_+. Such a cycle must contain precisely two conjunctions between empty ovals and the non-empty oval. There are two cases: either the two corresponding conjunction vectors lie opposite to each other with respect to the cyclic order (like v_{17} and v_{19}), or they are adjacent (like v_{17} and v_{18}). In the former case, the vectors v_{24} and v_{25} are equivalent, using a symmetry like $R_{v_{21}} \circ R_{v_{24}-v_{25}}$. In the latter case, the two choices are different, and they can be distinguished looking at the cyclic order of the conjunctions along the non-empty oval, which can be read off Figure 6.30 (2).
(3) This count is obtained by enumerating the admissible pairs of faces of $P_+^0 \times P_-$ modulo T. □

6.13.2 Curves with a pair of non-real nodes

Using the notations of the previous subsection, we let the vectors corresponding to the non-real roots be $\frac{1}{2}(l_1 \pm l_2)$, where $l_1 = v_{21}$ and $l_2 = v_{24} - v_{25}$. The Coxeter graphs of the polytopes orthogonal to l_1, l_2 are as follows, where $v_{27} := 2v_{17} + v_{21}$ and $v_{28} := v_{19} + v_{21}$.

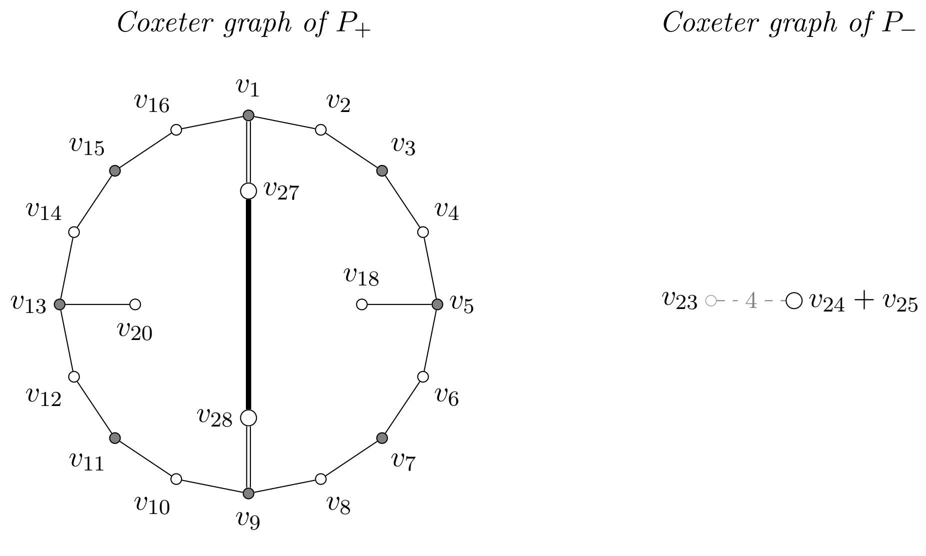


Figure 6.31

The group T is the symmetry group of the polytope P_+, which is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Lemma 6.42. *Let C_3 be a curve corresponding to the face of $P_+ \times P_-$ defined by the vectors $\{l_1, v_2, v_4, v_6, v_8, v_{10}, v_{12}, v_{14}, v_{16}, v_{18}, v_{20}, l_2\}$. It is nodal, and its irreducible components* *are three conics. The isotopy type of C_3 is shown in Figure 6.30 (3).*

Proposition 6.43. (1) *All rigid isotopy types of real rational curves obtained by real nodal degeneration from a dividing curve of type $\langle 1\langle 8\rangle\rangle$ with a pair of non-real nodes can be obtained from C_3 by first perturbing some of the real nodes and then contracting empty ovals.*

(2) *Two such real rational sextics are rigidly isotopic if and only if they are semi-rigidly isotopic.*

(3) *There are* 52 *rigid isotopy types for such curves.*

Proof. (1) Since C_3 contains all the possible conjunctions, this is clear.
(2) There is only one face corresponding to a contraction of all the empty ovals. If not all ovals are contracted, then an empty oval must conjunct with the non-empty oval. Using the cyclic order on the set of empty ovals, this allows us to identify the empty ovals with the corresponding contraction vectors, up to a symmetry of the polytope P_+. Since for each pair of ovals there is at most one possible conjunction between them, this also determines the conjunction vector corresponding to each conjunction. Hence, the isotopy type together with the order of the ovals determines the corresponding face of $P_+ \times P_-$ modulo T.
(3) This count is obtained by enumerating the admissible pairs of faces of $P_+ \times P_-$ modulo T. □

6.14 Degenerations of $\langle 7 \sqcup 1\langle 1\rangle\rangle$

Consider the lattice $N = \mathrm{U}^{w_1,w_2} \oplus \mathrm{E}_8^{e_1,\ldots,e_8} \oplus \mathrm{E}_8^{f_1,\ldots,f_8} \oplus \langle -2\rangle^y$. Let $P_0 \subset \mathbb{H}_N$ be a polytope delimited by hyperplanes orthogonal to short roots. (This is the polytope P_- for curves of type $\langle 9 \sqcup 1\langle 1\rangle\rangle$.) The Coxeter graph for P_0 is shown in Figure 6.7. It contains a cycle of length 18, and its symmetry group is S_3. Consider an A_3 subscheme of this Coxeter graph, formed by three vertices a_1, a_2, a_3, that is, a_1 and a_3 are orthogonal, and a_2 is connected to both a_1 and a_3 by a single edge. Let K_- be the orthogonal complement of $\langle a_1, a_2, a_3\rangle$ in N, and let $K_+ = \mathrm{U}^{u_1,u_2} \oplus \langle -4\rangle^{a_1+a_3}$. Let the vectors corresponding to the non-real nodes be $s' = a_1 + a_2$ and $s'' = a_2 + a_3$. The Coxeter graph of the polytope P_+ is shown in Figure 6.32. It has no symmetries. It contains only one vertex corresponding to a short root. This root must be a contraction vector, corresponding to the inner oval. Since there are no other short roots, there are no conjunctions possible, i.e., the inner oval cannot conjunct with the non-empty oval. Therefore, the inner oval must be contracted.

Coxeter graph of P_+

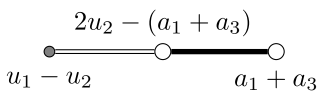


Figure 6.32

To obtain all the admissible faces of P_- modulo symmetries, and therefore the rigid isotopy classes of real rational curves obtained by real nodal degeneration from a curve of type $\langle 7 \sqcup 1\langle 1\rangle\rangle$ with a pair of non-real nodes, we use descent (see Chapter 5), starting from the polytope $P_0 \subset \mathbb{H}_N$. Note that all the walls defining $P_- \subset \mathbb{H}_-$ are intersections of $\mathbb{H}_-$ with hyperplanes in $\mathbb{H}_N$ defined by a short root. The tile marking $\mu = (x_1, x_2, x_3)$ can move as shown in Figure 6.33 (a). Therefore, all the possible subschemes of type A_3 in the Coxeter graph of P_0 are admissible tile markings.

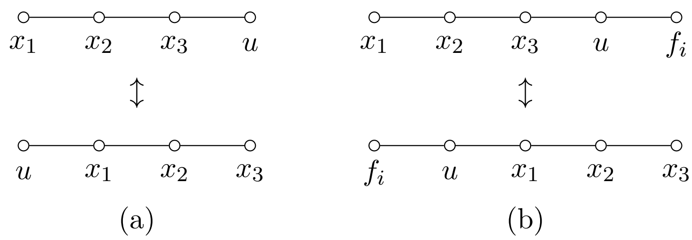


Figure 6.33 – The possible moves in the A_3 descent

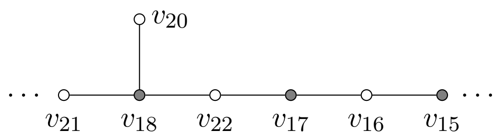
The face markings corresponding to admissible faces are of the form (μ, ν), where $\mu = (x_1, x_2, x_3)$ is a tile marking (see above), and $\nu = \{f_1, \ldots, f_7\}$ is a set of seven pairwise orthogonal vectors in the Coxeter graph of P_0 which are orthogonal to μ. The possible moves of the face markings are those shown in Figure 6.33 (a) and (b). Each symmetry of the polytope $\bar{P}$ can be extended to an automorphism of N which maps the set $\{a_1, a_2, a_3\}$ to itself. Let H be the group consisting of all automorphisms of N which map the set $\{a_1, a_2, a_3\}$ to itself and which maps the polytope $\bar{P}$ to itself. The corresponding group $\hat{H} \subset \mathrm{Perm}(\{a_1, a_2, a_3\}) \times \mathrm{Sym}(P_0)$ is isomorphic to $\mathrm{Sym}(P_0) \cong S_3$, where the odd elements of S_3 interchange a_1 and a_3. By Corollary 5.19, the faces of $\bar{P}$ modulo symmetries are in bijection with face markings up to moves modulo $\hat{H}$. This allows us to enumerate all the admissible faces of $P_+ \times P_-$ modulo T.

Let C be a curve whose rigid isotopy type corresponds to the face defined by the marking (μ_1, ν), where $\mu_1 = (v_{22}, v_{17}, v_{16})$ and $\nu = \{v_3, v_5, v_6, v_8, v_{11}, v_{13}, v_{14}, v_{20}, v_{21}, v_{23}\}$, using the notation of Figure 6.7.

Lemma 6.44. *The curve C is nodal, and its irreducible components are three lines and a cubic. Its isotopy type is as shown in Figure 6.34.*

Lemma 6.45. *Every face marking (μ, ν) is equivalent (up to moves and symmetries) to a face marking (μ', ν') where μ' is either $\mu_1 = (v_{22}, v_{17}, v_{16})$ or $\mu_2 = (v_{17}, v_{16}, v_{15})$.*

Proof. For the convenience of the reader, we reproduce a part of the Coxeter graph of P_0 here. (The full Coxeter graph is given in Figure 6.7.)



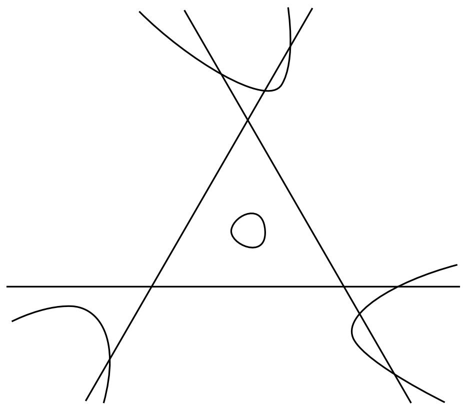


Figure 6.34 – Reducible nodal degeneration C of a curve of type $\langle 7 \sqcup 1\langle 1\rangle\rangle$ with a pair of non-real nodes.

The possible tile markings are, up to symmetry, $\mu_1 = (v_{22}, v_{17}, v_{16})$, $\mu_2 = (v_{17}, v_{16}, v_{15})$, $\mu_3 = (v_{18}, v_{22}, v_{17})$, $\mu_4 = (v_{21}, v_{18}, v_{22})$, $\mu_5 = (v_{20}, v_{18}, v_{22})$. A face marking (μ_3, ν) can either move to (μ_1, ν) (by a move of type (a), if $v_{15} \notin \nu$), or to $(\mu_2, \nu \cup \{v_{18}\} \setminus \{v_{15}\})$ (by a move of type (b), if $v_{15} \in \nu$). Similarly, a face marking (μ_4, ν) can either move to (μ_3, ν) or to $(\mu_1, \nu \cup \{v_{21}\} \setminus \{v_{16}\})$, and likewise, a face marking (μ_5, ν) can either move to (μ_3, ν) or to $(\mu_1, \nu \cup \{v_{20}\} \setminus \{v_{16}\})$. □

Proposition 6.46. (1) *All rigid isotopy types of real rational curves obtained by real nodal degeneration from a curve of type $\langle 7 \sqcup 1\langle 1\rangle\rangle$ with a pair of non-real nodes is obtained from the reducible curve C by first perturbing some of the real nodes and then contracting empty ovals.*

(2) *Two such real rational sextics are rigidly isotopic if and only if they are semi-rigidly isotopic.*

(3) *There are 82 rigid isotopy types for such curves.*

Proof. (1) Each rigid isotopy type corresponds to a face of P_- modulo symmetries, and each such face is represented by a face marking (μ, ν) as described above, which is uniquely determined up to moves and symmetries. By Lemma 6.45, we may suppose that μ is either μ_1 or μ_2. If we disregard the vectors of ν corresponding to contractions of empty ovals, we may assume that $\mu = \mu_1$ (using the argument used in the proof of Lemma 6.45). Then, ν must be a subset of $\{v_3, v_5, v_6, v_8, v_{11}, v_{13}, v_{14}, v_{20}, v_{21}, v_{23}\}$, and hence the corresponding rigid isotopy type can be obtained by deforming some of the nodes of the curve C and then contracting empty ovals.
(2) We associate with each rigid isotopy type its conjunction graph. Then we check (case by case, using a computer) that the conjunction graph together with the cyclic order of the ovals determines the face marking up to moves and symmetries.
(3) This number is obtained by counting (using a computer) all the face markings, modulo moves and symmetries, which correspond to rigid isotopies of real rational sextics. □

6.15 Degenerations of $\langle 9\rangle$

6.15.1 Curves with only real nodes

Let $L_+ = \mathrm{U}(2)^{u_1+w_1,u_2+w_2}$ and $L_{-h} = \mathrm{U}(2)^{u_1-w_1,u_2-w_2} \oplus \mathrm{E}_8^{e_1,\ldots,e_8} \oplus \mathrm{E}_8^{f_1,\ldots,f_8} \oplus \langle -2\rangle^y$.

The lattice $L_+ = \mathrm{U}(2)$ contains no vectors of square -2. Therefore, we have $P_+ = \mathbb{H}_+$. Its "symmetry group" (the group of automorphisms of L_+ which map $\mathbb{H}_+$ to itself) is $\mathbb{Z}/2\mathbb{Z}$.

Note that all automorphisms of the discriminant form of L_{-h} fix the element $[\frac{y}{2}]$. Hence, all the automorphisms of the lattice L_{-h} can be glued to an automorphism of the lattice K_+. Therefore, the group T consists of all the automorphisms of the negative polytope P_-. Let $\bar{P} \subset P_-$ be a polytope delimited by all hyperplanes H such that the reflection in H defines an automorphism of L_+. The faces of P_- modulo symmetries of P_- are in bijection with those faces of $\bar{P}$ which are contained in the boundary of P_-, modulo symmetries of $\bar{P}$.

To compute the faces of $\bar{P}$ modulo symmetries, we use the descent explained in Chapter 5. Note that the lattice L_{-h} is isomorphic to $\langle 2\rangle \oplus \mathrm{E}_8 \oplus \mathrm{E}_8 \oplus \langle -2\rangle \oplus \langle -2\rangle$. Therefore, it can be seen as the orthogonal complement of a short root in the lattice $M = \mathrm{U}\oplus\mathrm{E}_8\oplus\mathrm{E}_8\oplus\langle -2\rangle\oplus\langle -2\rangle$. Let $P_0 \subset \mathbb{H}_M$ be a polytope delimited by all hyperplanes H such that the reflection in H defines an automorphism of M. The Coxeter graph of the polytope P_0 is finite. It has been computed by Vinberg and Kaplinskaja [39]. Following [39], we distinguish between three types of vertices in the Coxeter graph of P_0: *basic vertices*, *vertices of the first kind* and *vertices of the second kind*. Basic vertices correspond short roots v (i.e. with $v^2 = -2$) such that v glues non-trivially to its orthogonal complement, vertices of the first kind correspond to long roots (i.e. with square -4), and vertices of the second kind correspond to short roots v such that there is no gluing between $\langle v\rangle$ and its orthogonal complement. There are 25 basic vectors, 5 vectors of the first kind and 20 vectors of the second kind. The basic vertices form a subdivision of the Petersen graph; the symmetry group of the Coxeter graph if S_5. It acts transitively on the vertices of the first kind and on the vertices of the second kind. For a complete description of the Coxeter graph of P_0, we refer to [39].

We distinguish six different types of face markings (μ, ν) which define faces of P_- corresponding to rigid isotopy classes of real nodal rational sextics. In each case, the tile marking μ is a basic vector x. In detail, the six types are explained below.

Type A: The set ν consists of 10 basic vectors which are orthogonal to x and pairwise orthogonal.

The two possible moves are those listed in Example 5.16. In the first move, the vertex x moves to a vertex u, where x and u are connected by a simple edge, and both x and u are orthogonal to ν. In the second move, the vector x can be swapped with a vector $f \in \nu$ if there is a vector u which is connected to both x and f by a simple edge and orthogonal to the other vertices of ν.

Type B: The set ν consists of 9 basic vectors which are orthogonal to x and pairwise orthogonal, and one vector of the second kind, orthogonal to x and the 9 basic vectors.

The possible moves are the same as for type A. In particular, the vector of the second kind cannot move. Since the symmetry group S_5 acts transitively on the set of vectors of the second kind, we may fix the vector of the second kind. Then, we have to consider the face markings up to moves and up to the actions of the stabilizer of the chosen vector of the second kind, which is isomorphic to S_3.

Type C: The set ν consists of 8 pairwise orthogonal basic vectors, a vector of the first kind a and a vector of the second kind b, where $(a, b) = 2$, i.e. the corresponding vertices are connected by a double edge in the Coxeter graph.

Again, the possible moves are the same as for types A and B. In particular, the vectors of the first and second kind cannot move. The symmetry group S_5 acts transitively on the pairs (a, b) formed by a vector of the first kind and a vector of the second kind which are connected by a double edge. Therefore, we may assume that a and b are fixed, and consider the face markings of type C up to the allowed moves and up to the action of the stabilizer of a and b, which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Type D: The set ν consists of 9 pairwise orthogonal basic vectors and one vector of the first kind, which is orthogonal to 8 of the basic vectors and connected to the remaining basic vector by a double edge in the Coxeter graph.

Again, the possible moves are the same as for types A and B, but only the 8 basic vectors orthogonal to the vector of the first kind can move. The vector of the first kind and the basic vector attached to it cannot move. The symmetry group S_5 acts transitively on the pairs (a, b) formed by a vector of the first kind a and a basic vector b which are connected by a double edge. Therefore, we may assume that a and b are fixed, and consider the face markings of type C up to the allowed moves and up to the action of the stabilizer of a and b, which is isomorphic to the dihedral group of order eight.

Type E: The set ν consists of 10 pairwise orthogonal basic vectors. Three of them are connected to the vector x by a simple edge, while the other seven are orthogonal to it.

For this type of face marking, we have to consider "tile adjacency moves" as well as "direction adjacency moves" (cf. Example 5.17). The direction adjacency moves allow to exchange the tile marking vector x with one of the three adjacent vectors belonging to ν. Up to such moves, we can assume that the tile marking vector is always the one central one in the D_4 subscheme. There are no tile adjacency moves.

Since the symmetry group S_5 acts transitively on the subschemes of type D_4 in the Coxeter graph, we may assume that the position of the D_4 subscheme is fixed, and consider the choices of the remaining seven basic vectors up to the action of the stabilizer of the D_4 subscheme, which is isomorphic to dihedral group of order 12.

Type F: The set ν consists of nine pairwise orthogonal basic vectors and one vector of the second kind. Three of the basic vectors are connected to the vector x by a simple edge, while the other six are orthogonal to it. The vector of the second kind is orthogonal to x and the basic vectors.

As for type E, there are no tile moves, and we can assume that the tile marking vector x lies at the center of the D_4 subscheme. The symmetry group S_5 acts transitively on the choices of pairs formed by a D_4 subscheme and a vector of the second kind.

Remark 6.47. There are face markings other than the ones of types A, B, C, D, E and F described above which correspond to faces of P_- defined by a set of pairwise orthogonal roots. For example, the tile marking vector x might be connected to a vector of the first kind, or there might be a subscheme of type F_4. However, the corresponding configurations of roots are not homological types corresponding to irreducible sextics, since the sublattice spanned by h and the roots is not primitive.

Proposition 6.48. *There are* 565 *rigid isotopy types of real rational curves obtained by real nodal degeneration from a curve of type* $\langle 9 \rangle$ *with only real nodes. The number of rigid isotopy types corresponding to each type of face marking is as follows:*

Type	A	B	C	D	E	F
Number of rigid isotopy types	46	41	135	174	78	91

Proof strategy. The description of the six types of face markings together with the possible moves and symmetries allows us to generate, with the help of a computer program, a list of all the possible faces of P_- defined by a set of pairwise orthogonal roots modulo symmetry. For each face, we check (again using a computer) if the corresponding homological type corresponds to a real rational sextic. □

Proposition 6.49. *Two real rational curves obtained by real nodal degeneration from a curve of type* $\langle 9 \rangle$ *with only real nodes are rigidly isotopic if and only if they are isotopic.*

Proof strategy. With each rigid isotopy type, we can associate its *graph of conjunctions* —a graph whose vertices correspond to the empty ovals, and whose edges correspond to conjunctions between the respective ovals. Multiple edges and loops are allowed. The graph of conjunctions is determined (up to isomorphism) by the isotopy type of a curve. Therefore, it is sufficient to show that different rigid isotopy types have different graphs of conjunctions. To do this, we proceed in two steps: First, we show how the graph of conjunctions determines the type of the face marking (A up to F) of the corresponding face. Then, we check, again with the help of a computer, that within each of the six types of face markings, no two equivalence classes of face markings have isomorphic graphs of conjunctions. To distinguish the types A to F, we look at short cycles in the graph of conjunctions. Note that the shortest cycle made up from pairwise orthogonal basic vectors in the Coxeter graph of P_0 is of length 4. Shorter cycles can be created in different ways: Vectors of the second kind correspond to self-conjunctions, i.e. cycles of length one. A vector of the first kind a connected to a vector b by a double edge results in a second conjunction vector intersecting the contraction vectors in the same way as b. If b is a vector of the second kind, this given tow self-conjunctions of the same ovals. If

b is a basic vector, this gives a cycle of length two. Finally, the subschemes of type D_4 appearing in types E and F lead to cycles of length three. The numbers of short cycles for each type are summarized in the following table:

Type	A	B	C	D	E	F
Cycles of length 1	–	1	2	–	–	1
Cycles of length 2	–	–	–	1	–	–
Cycles of length 3	–	–	–	–	1	1

Therefore, the conjunction graph determines the type of the face marking. It remains to check that within each type, the equivalence classes of face markings are distinguished by their conjunction graphs. We verify this case by case, using a computer program. □

Proposition 6.50. *There are ten reducible nodal sextics, C_a, C_b, C'_b, C_c, C'_c, C''_c, C_d, C'_d, C_e and C_f, such that each real rational sextic obtained by real nodal degeneration from a dividing curve of type $\langle 9 \rangle$ with only real nodes can be obtained from one of these ten curves by first perturbing some nodes and then contracting the empty ovals. More precisely, curves whose face marking is of type A can be obtained from C_a, curves whose face marking is of type B can be obtained from C_b or C'_b, and so on. The isotopy types of the ten curves are shown in Figure 6.35.*

Proof startegy. For each of the reducible curves we fix a corresponding face marking. Then we check (using the computer) that for each face marking corresponding to a rigid isotopy type of a real rational sextic, the conjunction vectors are, up to equivalence by moves and symmetries, a subset of the vectors appearing in the marking corresponding of one of the ten curves. To deduce the isotopy types of the reducible curves, we use the conjunction graphs and the data determining which irreducible components intersect each other at each node, obtained using Corollary 2.25. □

6.15.2 Curves with a pair of non-real nodes

Let L_+ and L_{-h} be defined as in the preceding subsection. Let the vectors corresponding to the non-real nodes be $s' = u_1 - u_2$ and $s'' = w_1 - w_2$. Then the lattice K_+ is of type $\langle 4 \rangle$, so the polytope P_+ is of dimension zero, i.e. a single point. The lattice K_+ is isomorphic to the orthogonal complement of a short root in the lattice $N = \mathrm{U} \oplus \mathrm{E}_8 \oplus \mathrm{E}_8 \oplus \langle -4 \rangle$. Therefore, we can use a descent from N to describe the polytope P_+. Let P_1 be a fundamental polytope in $\mathbb{H}_N$ delimited by hyperplanes H such that the reflection in H defines an automorphism of N. The polytope P_1 can be seen as a slice of the polytope $P_0 \subset \mathbb{H}_M$ defined in the preceding subsection orthogonal to a vector of the first kind. The Coxeter graph of P_1 was also calculated by Vinberg and Kaplinskaja [39]. Again, following [39], we distinguish basic vertices, vertices of the first kind and vertices of the second kind. The Coxeter graph consists of 22 basic vertices, 3 vertices f the first kind and 12 vertices of the second kind. The symmetry group of P_1 is S_4.

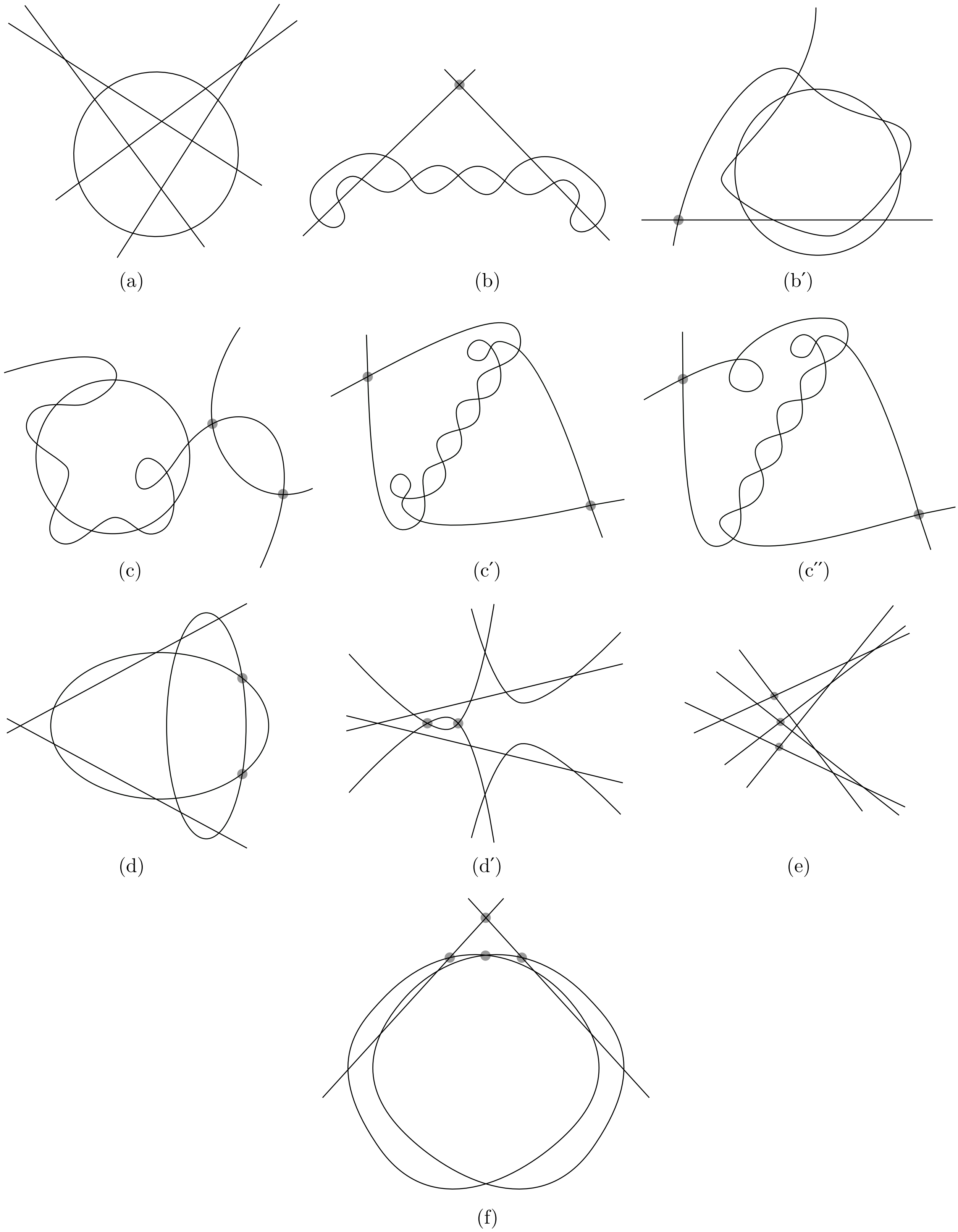


Figure 6.35 – Isotopy types of reducible nodal degenerations of a sextic of type $\langle 9 \rangle$ with only real nodes. The nodes marked with small gray circles belong to the short cycles in the graph of conjunctions (see Proposition 6.49).

Face markings (μ, ν) corresponding to rigid isotopy classes of real rational sextics consist of a singe basic vector $\mu = x$ and a collection ν of 8 pairwise orthogonal basic vectors, which are all orthogonal to x. The possible moves between the face markings are the same as those for type A in the case with only non-real nodes.

Consider the nodal sextic C, whose irreducible components are two conics and two lines, shown in Figure 6.36.

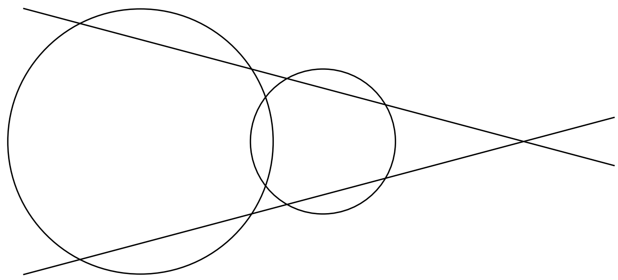


Figure 6.36 – Isotopy type of the reducible sextic C whose dividing perturbation is of type $\langle 9 \rangle$ with a pair of non-real nodes.

Proposition 6.51. (1) *All rigid isotopy types of real rational curves obtained by real nodal degeneration from a dividing curve of type $\langle 9 \rangle$ with a pair of non-real nodes can be obtained from C by first perturbing some of the real nodes and then contracting empty ovals.*

(2) *There are* 53 *rigid isotopy types for such curves, but only* 37 *different isotopy types.*

Proof strategy. (1) We fix a face marking corresponding to C. Then we check (using a computer) that for all the face markings corresponding to real rational nodal sextics, the conjunction vectors are (up to moves and symmetries of the face marking) contained in the fixed face marking corresponding to the curve C. To ensure that the curve C corresponding to the fixed face marking indeed looks as depicted above, we again use the conjunction graph in combination with information about the intersection between the different irreducible components obtained using Corollary 2.25.
(2) The number of rigid isotopy classes is obtained by enumerating (using a computer) the number of admissible face markings modulo moves and symmetries. For each rigid isotopy class, we can calculate its graph of conjunctions. In each case where the graph of conjunction does not determine the rigid isotopy class, it is a tree with at most one vertex of degree at least three. Therefore, it determines the isotopy type of the corresponding curves. $\square$

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