# Cohomological and K-theoretic invariants of singularity loci

Kolokolnikova, Natalia

Université de Genève
Section de Mathématiques

Faculté des Sciences
Professeur András Szenes

# Cohomological and $K$-theoretic Invariants of Singularity Loci

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**Natalia Kolokolnikova**

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N.B. - La thèse doit porter la déclaration précédente et remplir les conditions énumérées dans les "Informations relatives aux thèses de doctorat à l'Université de Genève".

# Contents

# 1 Abstract

This thesis is devoted to the study of singularities of holomorphic maps: their geometry, as well as cohomological and $K$-theoretic invariants, their properties and computational strategies.

The main object of study of global singularity theory is the Thom polynomial, which may be defined as the $\mathrm{Gl}_m \times \mathrm{Gl}_n$-equivariant Poincaré dual of a closure of a singularity. In the 70's Damon proved that the Thom polynomial for contact singularities depends only on the relative codimension, and may be expressed in relative Chern classes. Pragacz and Weber showed that the Thom polynomial for contact singularities expressed in the relative Chern classes has positive coefficients when written in the Schur basis. In this thesis, modern proofs of these two theorems are given.

One of the main problems in global singularity theory is how to compute Thom polynomials. This proved to be very difficult, and the two main computational methods – the method of restriction equations for contact singularities and the residue formula for $A_k$-singularities – work effectively only for rather small relative codimensions. In this thesis, I show how the two methods can be combined in a different computational approach and give examples of computation.

A recent development in global singularity theory is the introduction of the $K$-theoretic invariants of singularity loci. One can define a $K$-theoretic invariant of an affine variety in two different ways: either using the algebra of functions on the variety itself, or using its smooth equivariant resolution. It is easy to show that the two invariants are equal if and only if the closure of the singularity locus has rational singularities. I prove that even for $A_2$ singularity loci, in the general case, the two invariants are different, and therefore, the $A_2$-loci may have singularities worse than rational. However, in the case of relative codimension 0, the two invariants coincide, and thus the $A_2$-loci have rational singularities.

# 2 Résumé en français

Cette thèse est consacrée à l'étude de singularités de fonctions holomorphes: leur géometrie, leurs invariants cohomologiques et $K$-théoriques, leurs propriétés et stratégies de calcul.

L'objet principal de l'étude de la théorie globale des singularités est le polynôme de Thom, qui peut être défini comme le $\mathrm{Gl}_m \times \mathrm{Gl}_n$-équivariant dual de Poincaré de l'adhérence de singularité. Dans les années 1970 Damon a montré que les polynômes de Thom des singularités contactes ne dépendent que de la codimension relative, et peuvent être exprimés en classes de Chern relatives. Pragacz et Weber ont démontré que le polynôme de Thom des singularités contactes a les coefficients positifs dans la base de Schur. Dans cette thèse, les démonstrations modernes de ces deux théorèmes sont données.

L'un des principaux problèmes de la théorie globale des singularités est de calculer les polynômes de Thom. Ce problème s'est revélé ardu, et les deux méthodes principales de calcul – la méthode d'équations de restriction pour les singularités contactes et la formule de résidues pour les singularités de type $A_k$ – ne sont efficaces qu'en cas de codimensions relatives assez petites. Dans cette thèse, je présente ces deux méthodes et je montre comment on peut combiner les deux pour obtenir une nouvelle approche de calcul. Je donne aussi les examples de ce calcul.

La récente évolution dans la théorie des singularités est l'introduction des invariant $K$-théoriques de singularités. Il y a deux stratégies pour définir l'invariant $K$-théorique de variété affine: soit on utilise l'algèbre de fonctions sur la variété, soit on utilise sa résolution équivariante. Il est facile de montrer que les deux invariants coïncident pour autant que l'adhérence de la singularité a des singularités rationnelles. Je montre que déjà pour les loci de type $A_2$, dans le cas général, les deux invariants ne sont pas égaux et donc les loci de type $A_2$ en général ont des singularités plus complexes que rationnelles. En revanche, dans le cas de codimension relative nulle, les deux invariants coïncident et donc les loci de type $A_2$ ont des singularités rationnelles.

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# 4 Organization of the thesis

This thesis consists of four parts. We begin with Section 5, where we discuss the motivation and give rigorous definitions of notions used in the rest of the thesis.

In Section 6 we formulate and give modern proofs of two fundamental theorems regarding the properties of the Thom polynomial: Damon's theorem and the Schur positivity theorem by Pragacz and Weber. These two sections are based on my paper [24].

Section 7 is devoted to the strategies of computing the Thom polynomials. We give a short introduction to the two main modern computational methods – the method of restriction equations and the residue formula. We show how one may combine the two approaches to obtain another computational strategy. We give several examples of such computations and conjecture that this new strategy in fact reduces the computation of the $Q$-polynomial for $A_d$ singularities to a finite number of substitutions. This section is based on an ongoing collaboration with Prof. András Szenes and Prof. László Fehér.

In Section 8 we give definitions of two $K$-theoretic invariants of singularity loci. We conclude that the two invariants are equal if and only if the singularity locus has rational singularities. Using the $A_2$ loci as a simple example we obtain that in the general case the two invariants are different, but they agree in case when relative codimension is equal to 0. This section is based on my paper [25].

# 5 Preliminaries

## 5.1 Motivation

Global singularity theory originates from problems in obstruction theory. Consider the following question: is there an immersion in a given homotopy class of maps between two smooth manifolds? We can reformulate this problem as follows. Suppose $M$ and $N$ are smooth real manifolds with $\dim(N) \geq \dim(M)$, and $f\colon M \to N$ is a sufficiently generic smooth map in a fixed homotopy class. The map $f$ is an immersion, if

$$\Sigma^1(f) \overset{def}{=} \{p \in M \mid \dim \operatorname{Ker}(d_p f) \geq 1\} = \varnothing.$$

The set $\Sigma^1(f)$ is called the $\Sigma^1$-*singularity locus*, or simply the $\Sigma^1$-*locus* of $f$, i.e. the points in $M$ where $f$ has a $\Sigma^1$-*singularity*: the kernel of the differential of $f$ is non-zero. In the case of $\mathbb{Z}_2$-cohomology and a sufficiently generic map $f$, the set $\Sigma^1(f)$ represents a cohomology class via Poincaré duality. Clearly, if the Poincaré dual $\mathrm{PD}[\Sigma^1(f)]$ is non-zero in $H^*(M, \mathbb{Z}_2)$, then there is no immersion in the homotopy class of $f$.

In the 50s, René Thom proved the following statement, now known as Thom's principle.

**Theorem 5.1** (Thom's principle, [33]). *Let $\Theta$ be an (appropriately defined) singularity and let $m \leq n$ be non-negative integers. Suppose $\{a_1, \ldots, a_m\}$ and $\{a'_1, \ldots, a'_n\}$ are two sets of graded variables with $\deg a_i = \deg a'_i = i$. For all smooth compact real manifolds $M$ and $N$, $\dim(M) = m$, $\dim(N) = n$, and a sufficiently generic smooth map $f\colon M \to N$,*

$$\overline{\Theta(f)} = \overline{\{p \in M \mid f \textit{ has a singularity of type } \Theta \textit{ at } p\}}$$

*is a cycle in $M$, and there exists a universal polynomial in $a_1, \ldots, a_m$ and $a'_1, \ldots, a'_n$*

$$\mathrm{Tp}[\Theta](a_1, \ldots, a_m, a'_1, \ldots, a'_n)$$

*depending only on $\Theta$, $m$ and $n$, such that*

$$\mathrm{PD}[\overline{\Theta(f)}] = \mathrm{Tp}[\Theta](w_1(TM), \ldots, w_m(TM), f^*w_1(TN), \ldots, f^*w_n(TN)) \in H^*(M, \mathbb{Z}_2),$$

*where $w_i(TM)$ and $w_j(TN)$ are the Stiefel-Whitney classes of the corresponding tangent bundles.*

This universal polynomial is called the *Thom polynomial* of $\Theta$. We will give a rigorous construction of this polynomial in the case of complex manifolds.

Thom's principle may also be translated from the real to the complex case.

**Theorem 5.2** (Thom's principle in the complex case)**.** *Let $\Theta$ be an (appropriately defined) singularity and let $m \leq n$ be non-negative integers. Suppose $\{a_1, \dots, a_m\}$ and $\{a'_1, \dots, a'_n\}$ are two sets of graded variables with $\deg a_i = \deg a'_i = i$. For all compact complex manifolds $M$ and $N$, $\dim(M) = m$, $\dim(N) = n$, and a holomorphic map $f\colon M \to N$ satisfying certain transversality conditions,*

$$\overline{\Theta(f)} = \overline{\{p \in M \mid f \text{ has a singularity of type } \Theta \text{ at } p\}}$$

*is a cycle in $M$, and there exists a universal polynomial in $a_1, \dots, a_m$ and $a'_1, \dots, a'_n$*

$$\mathrm{Tp}[\Theta](a_1, \dots, a_m, a'_1, \dots, a'_n)$$

*depending only on $\Theta$, $m$ and $n$, such that*

$$\mathrm{PD}[\overline{\Theta(f)}] = \mathrm{Tp}[\Theta](c_1(TM), \dots, c_m(TM), f^*c_1(TN), \dots, f^*c_n(TN)) \in H^*(M, \mathbb{R}),$$

*where $c_i(TM)$ and $c_j(TN)$ are the Chern classes of the corresponding tangent bundles.*

In fact, the result of Borel and Haefliger [8] implies that there are pairs of real and complex singularities for which the real Thom polynomial may be obtained by substituting the corresponding Stiefel-Whitney classes for the Chern classes in the corresponding Thom polynomial in the complex case.

Calculating Thom polynomials is difficult: some progress has been made in the works of Ronga [32], Porteous [28], Gaffney [18], Rimányi [30], Bérczi, Fehér and Rimányi [4], Fehér and Rimányi [14], and Bérczi and Szenes [5] and Kazarian [22].

## 5.2 Global singularity theory

Let $z_1, \dots, z_m$ be the standard coordinates on $\mathbb{C}^m$. Denote by $J^m$ the algebra of formal power series in $z_1, \dots, z_m$ without a constant term, i.e.

$$J^m = \{h \in \mathbb{C}[[z_1, \dots, z_m]] \mid h(0) = 0\}.$$

The space of *d-jets* of holomorphic functions on $\mathbb{C}^m$ near the origin is the quotient of $J^m$ by the ideal of series with the lowest order term of degree at least

$d+1$, i.e. the ideal generated by monomials $z_1^{i_1}\ldots z_m^{i_m}$ such that $\sum i_j = d+1$. We will denote this ideal by $I\langle\bar{z}^{d+1}\rangle$ :

$$J_d^m = J^m/I\langle\bar{z}^{d+1}\rangle.$$

As a linear space, the algebra $J_d^m$ may be identified with the space of polynomials in $z_1, \ldots, z_m$ of degree at most $d$ without a constant term. The space of $d$-jets of holomorphic maps from $(\mathbb{C}^m, 0)$ to $(\mathbb{C}^n, 0)$, or the space of *map-jets*, is denoted by $J_d^{m,n}$ and is naturally isomorphic to $J_d^m \otimes \mathbb{C}^n$. In this paper we will assume $m \leq n$.

Now let $r$ be a non-negative integer. An *unfolding* of a map-jet $\Psi \in J_d^{m,n}$ is a map-jet $\widehat{\Psi} \in J_d^{m+r,n+r}$ of the form:

$$(z_1, \ldots, z_n, y_1, \ldots, y_n) \mapsto (F(z_1, \ldots, z_n, y_1, \ldots, y_n), y_1, \ldots, y_r),$$

where $F \in J_d^{m+r,n}$ satisfies

$$F(z_1, \ldots, z_n, 0, \ldots, 0) = \Psi(z_1, \ldots, z_n).$$

The *trivial unfolding* (or a trivial suspension) is the map-jet

$$susp_r\Psi = (\Psi(z_1, \ldots, z_n), y_1, \ldots, y_r).$$

Composition of map-jets together with cancellation of terms of degree greater than $d$ gives a well-defined map

$$J_d^{m,n} \times J_d^{n,k} \longrightarrow J_d^{m,k}$$

$$(\Psi, \Phi) \mapsto \Phi \circ \Psi.$$

Consider a sequence of natural maps

$$J_d^{m,n} \to J_{d-1}^{m,n} \to \ldots \to J_1^{m,n} \cong \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n).$$

For $\Psi \in J_d^{m,n}$, the *linear part* of $\Psi$ is defined as the image of $\Psi$ in $J_1^{m,n}$ and denoted by $\mathrm{Lin}\,\Psi$.

Consider the set

$$\mathrm{Diff}_d^m = \{\Delta \in J_d^{m,m} \mid \mathrm{Lin}\,\Delta \text{ invertible}\}.$$

The previously defined operation"$\circ$"gives this set an algebraic group structure.

Let $\Delta_m \in \mathrm{Diff}_d^m$, $\Delta_n \in \mathrm{Diff}_d^n$, and $\Psi \in J_d^{m,n}$. The *left-right* action of $\mathrm{Diff}_d^m \times \mathrm{Diff}_d^n$ on $J_d^{m,n}$ is given by

$$(\Delta_m, \Delta_n)\Psi = \Delta_n \circ \Psi \circ \Delta_m^{-1}.$$

**Definition 5.3.** Left-right invariant algebraic subvarieties of $J_d^{m,n}$ are called *singularities.*

For each singularity $\Theta$ which is stratum of the $\mathrm{Diff}_d^m \times \mathrm{Diff}_d^n$-action there is a map-jet $\Phi$ defined up to left-right equivalence such that all other map-jets in $\Theta$ are left-right equivalent to a suspension of $\Phi$. Such $\Phi$ is called a *prototype* of $\Theta$.

To a given element $\Psi \in J_d^{m,n} \cong J_d^m \otimes \mathbb{C}^n$, presented as $(\Psi_1, \ldots, \Psi_n)$, $\Psi_i \in J_d^m$, we can associate an algebra $A_\Psi = J_d^m / I\langle \Psi_1, \ldots, \Psi_n \rangle$. This algebra is *nilpotent*: there exists a natural number $q$ such that $A_\Psi^q = 0$, in other words, a product of any $q$ elements of $A_\Psi$ is equal to 0. $A_\Psi$ is nilpotent because $J_d^m$ itself is nilpotent: $(J_d^m)^{d+1} = 0$.

**Definition 5.4.** Suppose $A$ is a finite-dimensional commutative nilpotent algebra. The subset

$$\Theta_A^{m,n} = \{\Psi \in J_d^{m,n} \mid A_\Psi \cong A\}$$

is called a *contact singularity.* We will omit the dependence on $d$ in the notation when the value of $d$ is clear from the context.

When clear from the context, the dependence on $m, n$ will be omitted.

In this work we will be focusing on contact singularities and some particular series of contact singularities.

**Example 5.1** (Morin singularities)**.** The main notion of this work are Morin, or $A_d$ singularities. These are the contact singularities given by the nilpotent algebra $A_d = x\mathbb{C}[x]/x^{d+1}$

The prototype of the $A_d$ singularity is given by

$$(z, y_1, \ldots, y_{d-1}) \mapsto (z^{d+1} + \sum_{i=1}^{d-1} y_i z^i, y_1, \ldots, y_{d-1}).$$

$\Theta_A$ is left-right invariant, but two map-jets with the same nilpotent algebra may be in different left-right orbits. However, there is a group acting on $J_d^{m,n}$ whose orbits are exactly the sets $\Theta_A$ for various nilpotent algebras $A$. This group is the *contact group*:

$$\mathcal{K}_d^{m,n} = \mathrm{Gl}_n(\mathbb{C} \oplus J_d^m) \rtimes \mathrm{Diff}_d^m.$$

It acts on $\mathcal{J}_d^{k,n}$ via

$$[(M, \Delta), \Psi] \mapsto (M \cdot \Psi) \circ \Delta^{-1},$$

where $M \in \mathrm{Gl}_n(\mathbb{C} \oplus J_d^m)$, $\Delta \in \mathrm{Diff}_d^m$, and " $\cdot$ " stands for matrix multiplication.

**Theorem 5.5.** [26] *Two map-jets are contact equivalent if and only if their nilpotent algebras are isomorphic.*

**Proposition 5.6.** [2] *Let $A$ be a nilpotent algebra: $A^{d+1} = 0$. For $d \geq \dim(A/A^2)$ and $n$ sufficiently large, $\Theta_A^{m,n}$ is a non-empty, left-right invariant, irreducible quasi-projective algebraic subvariety of $J_d^{m,n}$.*

## 5.3 Equivariant Poincaré dual

Suppose a topological group $G$ acts continuously on an algebraic variety $M$, and $Y$ is a closed $G$-invariant subvariety in $M$. In this section we will define an analog of a Poincaré dual of $Y$, which reflects the $G$-action: the equivariant Poincaré dual of $Y$.

Let $G$ be a topological group and let $\pi\colon EG \to BG$ be the *universal $G$-bundle*, i.e. a principal $G$-bundle such that if $p\colon E \to B$ is any principal $G$-bundle, then there is a map $\zeta\colon B \to BG$ unique up to homotopy and $E \cong \zeta^* EG$. The universal $G$-bundle exists, is unique up to homotopy equivalence and can be constructed as a principal $G$-bundle with contractible total space.

Now we can construct the space with a free $G$-action and the same homotopy type as a fixed before algebraic variety $M$, the *Borel construction*:

**Definition 5.7.** The *Borel construction* (also homotopy quotient or homotopy orbit space) for a topological group $G$ acting on a topological space $M$ is the space $EG \times_G M$, i.e. the factor of $EG \times M$ by the $G$-action: $(xg^{-1}, gy) \sim (x, y)$, where $g \in G, x \in EG, y \in M$.

**Definition 5.8.** The *equivariant cohomology* of $M$ is the ordinary cohomology for the Borel construction:

$$H_G^*(M) = H^*(EG \times_G M).$$

Note that since $(EG \times pt)/G = EG/G = BG$, the equivariant cohomology of a point is $H_G^*(pt) = H^*(BG)$.

We would like to define an analog of a Poincaré dual in the equivariant case, i.e. when a group $G$ acts on an algebraic variety $M$ and $Y \subset M$ is a closed $G$-invariant subvariety. We constructed a substitute for the orbit space of $G$-action on $M$: the Borel construction $EG \times_G M$. Now, $EG \times_G Y$ is again a $G$-invariant subvariety of $EG \times_G M$, and we want to define a dual of $EG \times_G Y$ in $H^*(EG \times_G M) = H_G^*(M)$. However, first we have to deal with the fact that $EG$ is usually infinite-dimensional by introducing an approximation.

**Lemma 5.9.** [1] *Suppose $E_1 \subset E_2 \subset \ldots$ is a sequence of finite-dimensional connected spaces with a free $G$-action compatible with the embeddings, such that $H^i(E_j) = 0$ for every fixed $i$, and $j$ large enough. Then for any $M$, any $i$ and $j$ large enough there are natural isomorphisms*

$$H^i(E_j \times_G M) \cong H^i(EG \times_G M) = H^i_G(M).$$

Let us fix $EG$, $BG$ and the finite-dimensional approximations

$$EG_1 \subset EG_2 \subset \ldots \subset EG$$

together with $BG_j = EG_j/G$. We can now consider $EG_j \times_G Y \subset EG_j \times_G M$ with $j$ large enough – two finite dimensional spaces. Let $D$ be the codimension of $EG_j \times_G Y$ in $EG_j \times_G M$.

Every irreducible closed subvariety of a non-singular variety has a well-defined Borel-Moore homology class [16], so we can define the *equivariant Poincaré dual* of $Y$ as follows:

$$\mathrm{eP}(Y) = [EG_j \times_G Y]_{BM} \in H^{2D}(EG_j \times_G M) = H^{2D}_G(M)$$

for $j$ large enough.

## 5.4 The Thom polynomial

We want to study the equivariant Poincaré dual of a closure of a singularity $\overline{\Theta} \subset J^{m,n}_d$. Since $J^{m,n}_d$ is contractible and the group $\mathrm{Diff}^n_d \times \mathrm{Diff}^n_d$ acting on it is homotopy equivalent to $\mathrm{Gl}_m \times \mathrm{Gl}_n$, the equivariant Poincaré duals of subvarieties in $J^{m,n}_d$ with respect to these groups will coincide. Therefore, in the rest of the paper we will assume $G = \mathrm{Gl}_m \times \mathrm{Gl}_n$ .

First, we need to fix $EG$, $BG$ and the corresponding approximations with an appropriate topology. Recall that $\mathbb{C}^\infty$ is defined as

$$\mathbb{C}^\infty = \{(z_1, z_2, \ldots) \mid z_i \in \mathbb{C},\ \text{only finite number of } z_i \text{ is non-zero}\}.$$

Fix $E\,\mathrm{Gl}_m = \mathrm{Fr}(m,\infty)$, the manifold of $m$-frames of vectors in $\mathbb{C}^\infty$, and $B\,\mathrm{Gl}_m = \mathrm{Gr}(m,\infty)$, the Grassmannian of $m$-planes in $\mathbb{C}^\infty$. So, in our case $EG = \mathrm{Fr}(m,\infty)\times \mathrm{Fr}(n,\infty)$ and $BG = \mathrm{Gr}(m,\infty) \times \mathrm{Gr}(n,\infty)$. The approximations are given by $EG_j = \mathrm{Fr}(m,j)\times\mathrm{Fr}(n,j)$ and $BG_j = \mathrm{Gr}(m,j)\times\mathrm{Gr}(n,j)$. With $j \to \infty$ $H^i_G(BG_j) = H^i_G(BG)$ for all $i$.

By definition,

$$\mathrm{eP}(\overline{\Theta}) \in H^*_G(J^{m,n}_d) = H^*(BG) = H^*(\mathrm{Gr}(m,\infty) \times \mathrm{Gr}(n,\infty)),$$

since $J_d^{m,n}$ is contractible.

Let $L_m$ denote the tautological vector bundle over $\mathrm{Gr}(m,\infty)$, i.e.

$$\mathrm{Gr}(m,\infty)\times\mathbb{C}^\infty\supset\{(V,p)\mid p\in V\}.$$

Then we can identify $H^*(\mathrm{Gr}(m,\infty),\mathbb{C})$ with $\mathbb{C}[c_1,\dots,c_m]$, where $c_i$ are the Chern classes of $L_m^*$ – the dual tautological bundle. This observation allows us to define the *Thom polynomial* as follows:

**Definition 5.10.** Let $d,m,n\in\mathbb{N}$ and let $m\leq n$. Let $\Theta\subset J_d^{m,n}$ be a singularity. The *Thom polynomial* of $\Theta$ is defined as

$$\mathrm{Tp}[\Theta](c,c')=\mathrm{eP}(\overline{\Theta})\in H^*(\mathrm{Gr}(m,\infty)\times\mathrm{Gr}(n,\infty))\cong\mathbb{C}[c_1,\dots,c_m]\otimes\mathbb{C}[c'_1,\dots,c'_n],$$

where $c_i$ are the Chern classes of $L_m^*$ and $c'_j$ – the Chern classes of $L_n^*$.

The notation $\mathrm{Tp}[\Theta](c,c')$ comes from the total Chern class: $c=\sum c_i$.

The Thom polynomial defined above coincides with the universal polynomial from the Thom's principle. In this paper we will think of the Thom polynomial as defined in Definition 5.10. For a detailed discussion of the relation between this definition and the Thom's principle, see [5], [14] and [22].

# 6 Structure theorems of global singularity theory

## 6.1 Damon's theorem

Before stating and proving Damon's theorem, let us first discuss the relation between Thom polynomials for different singularities.

### 6.1.1 Relation between different Thom polynomials

Suppose $A$ is a nilpotent algebra. Fix $m, n, m', n' \in \mathbb{N}$ such that $n \geq m$, $n' \geq m'$ and $n - m = n' - m'$. Consider $\Theta_A^{m,n} \subset J_d^{m,n}$ and $\Theta_A^{m',n'} \subset J_d^{m',n'}$ and the corresponding approximations for $K, K' \gg 0$ of the Borel constructions $EG_K \times_G \overline{\Theta}_A^{m,n} \subset EG_K \times_G J_d^{m,n}$ and $EG'_{K'} \times_{G'} \overline{\Theta}_A^{m',n'} \subset EG'_{K'} \times_{G'} J_d^{m',n'}$ for $G = \mathrm{Gl}_m \times \mathrm{Gl}_n$ and $G' = \mathrm{Gl}_{m'} \times \mathrm{Gl}_{n'}$.

Suppose $\varphi$ and $h$ in the following diagram are holomorphic.

$$\begin{array}{ccccc} \Sigma_1 = EG_K \times_G \overline{\Theta}_A^{m,n} & \hookrightarrow & EG_K \times_G \mathcal{J}_d^{m,n} & \xrightarrow{\pi} & \mathrm{Gr}(m, K) \times \mathrm{Gr}\,(n, K) \\ & & \downarrow h & & \downarrow \varphi \\ \Sigma_2 = EG'_{K'} \times_{G'} \overline{\Theta}_A^{m',n'} & \hookrightarrow & EG'_{K'} \times_{G'} \mathcal{J}_d^{m',n'} & \xrightarrow{\pi'} & \mathrm{Gr}(m', K') \times \mathrm{Gr}\,(n', K') \end{array}$$

If the following conditions [14] are satisfied:

- the square on the right commutes,
- $h^{-1}(\Sigma_2) = \Sigma_1$,
- $h$ is transversal to the smooth points of $\Sigma_2$,

then $h^* \,\mathrm{PD}[\Sigma_2] = \mathrm{PD}[h^{-1}(\Sigma_2)] = \mathrm{PD}[\Sigma_1]$. From the commutativity of the right square we obtain the equality

$$\mathrm{Tp}[\Theta_A^{m,n}] = \varphi^* \,\mathrm{Tp}[\Theta_A^{m',n'}].$$

Let now $m' = m + 1,\ n' = n + 1$. Define the map $\varphi$ as follows:

$$\varphi\colon\ \mathrm{Gr}(m, K) \times \mathrm{Gr}(n, K) \longrightarrow \mathrm{Gr}(m + 1, K + 1) \times \mathrm{Gr}(n + 1, K + 1)$$

$$(V_1, V_2) \mapsto (V_1 \oplus \mathbb{C}, V_2 \oplus \mathbb{C}).$$

Define $h$ in a similar way: let $(e_1, e_2, \ldots, e_{K+1})$ be a fixed orthonormal basis of $\mathbb{C}^{K+1}$, and let $(t_1, \ldots, t_m)$ be an orthonormal $m$-frame in $\mathbb{C}^K$ such that $e_{m+1} \notin \langle t_1, \ldots, t_m \rangle$, let $(\Psi_1 \ldots, \Psi_n) \in J_d^{m,n}$, i.e. $\Psi_j(z_1, \ldots, z_m) \in J_d^m$, then $h$ is given by:

$$h\colon EG_K \times_G J_d^{m,n} \longrightarrow EG'_{K+1} \times_{G'} J_d^{m+1,n+1}$$

$$((t_1, \ldots, t_m), (\Psi_1, \ldots, \Psi_n)) \mapsto ((t_1, \ldots, t_m, e_{m+1}), (\Psi_1, \ldots, \Psi_n, z_{m+1})).$$

Let us denote the set of Chern classes of the dual tautological bundle $L_m^*$ on $\mathrm{Gr}(m, K)$ by $c = c_1, \ldots, c_m$, the Chern classes of $L_n^*$ by $c' = c'_1, \ldots, c'_n$, the Chern classes of $L_{m+1}^*$ on $\mathrm{Gr}(m+1, K+1)$ by $\overline{c} = \overline{c}_1, \ldots, \overline{c}_{m+1}$ and the Chern classes of $L_{n+1}^*$ by $\overline{c}' = \overline{c}'_1, \ldots, \overline{c}'_{n+1}$. The transversality and the commutativity of the square on the right are straightforward, so the following is true:

$$\mathrm{Tp}[\Theta_A^{m,n}](c, c') = \varphi^* \,\mathrm{Tp}[\Theta_A^{m+1,n+1}](\overline{c}, \overline{c}').$$

We can also show how the pullback of $\varphi$ acts on the Chern classes $\overline{c}_i$ and $\overline{c}'_i$ :

$$\varphi^*(\overline{c}_i) = c_i \text{ for } i \le m \text{ and } \varphi^*(\overline{c}_{m+1}) = 0,$$

$$\varphi^*(\overline{c}'_i) = c'_i \text{ for } i \le n \text{ and } \varphi^*(\overline{c}'_{n+1}) = 0.$$

Using the properties of the pullback map we conclude the following.

**Lemma 6.1.** *In the above notations,*

$$\mathrm{Tp}[\Theta_A^{m,n}](c_1, \ldots, c_m, c'_1, \ldots, c'_n) = \mathrm{Tp}[\Theta_A^{m+1,n+1}](\varphi^*(\overline{c}_1), \ldots, \varphi^*(\overline{c}_{m+1}), \varphi^*(\overline{c}'_1), \ldots, \varphi^*(\overline{c}'_{n+1})) =$$

$$= \mathrm{Tp}[\Theta_A^{m+1,n+1}](c_1, \ldots, c_m, 0, c'_1, \ldots, c'_n, 0)$$

We can iterate the same procedure for $\mathrm{Tp}[\Theta_A^{m+2,n+2}]$, $\mathrm{Tp}[\Theta_A^{m+3,n+3}]$, etc, but since the Thom polynomial has a fixed degree, there will be a stabilization. This conclusion proves that the Thom polynomial depends only on the difference $n-m$ but not on $m$ and $n$, it also allows us to define the notion that generalizes the Thom polynomial.

**Definition 6.2.** Fix a nilpotent algebra $A$ and the difference between the dimensions of the source and the target of the map-jets, i.e. $n-m$ in our previous notations, denote this number by $l$. Fix $k = \mathrm{codim}(\overline{\Theta}_A^{m,n})$ in $J_d^{m,n}$. Define the *universal Thom polynomial* as

$$\mathrm{UTp}[\Theta_A^l](c_1, \ldots, c_k, c'_1, \ldots, c'_{k+l}) = \mathrm{Tp}[\Theta_A^{m,m+l}](c_1, \ldots, c_k, c'_1, \ldots, c'_{k+l})$$

for $m > k$.

For all $m, n$ such that $n - m = l$ we obtain

$$\mathrm{Tp}[\Theta_A^{m,n}](c_1, \ldots, c_m, c'_1, \ldots, c'_n) = \mathrm{UTp}[\Theta_A^l](c_1, \ldots, c_m, 0, \ldots, 0, c'_1, \ldots, c'_n, 0, \ldots, 0).$$

Let us show an important property of the universal Thom polynomial. Let

$$f\colon\ \mathrm{Gr}(m, K) \longrightarrow \mathrm{Gr}(m', K')$$

be any holomorphic map. Consider the diagram:

$$\begin{array}{ccc} EG_K \times_G \mathcal{J}_d^{m,n} & \xrightarrow{\quad\pi\quad} & \mathrm{Gr}(m, K) \times \mathrm{Gr}\,(n, K) \\ \Big\downarrow h & & \Big\downarrow \varphi \\ EG'_{K+K'} \times_{G'} \mathcal{J}_d^{m+m',n+m'} & \xrightarrow{\ \pi'\ } & \mathrm{Gr}(m + m', K + K') \times \mathrm{Gr}\,(n + m', K + K') \end{array}$$

Define $\varphi$ as

$$\varphi(V_1, V_2) = (V_1 \oplus f(V_1), V_2 \oplus f(V_1)),\ V_1 \in \mathrm{Gr}(m, K),\ V_2 \in \mathrm{Gr}(n, K).$$

Let $(e_1, \ldots, e_m)$ be the orthonormal basis for $V_1$, $(e'_1, \ldots, e'_n)$ – the orthonormal basis for $V_2$, and $(\overline{e}_1, \ldots, \overline{e}_{m'})$ – the orthonormal basis for $f(V_1)$. Let $\Psi = (\Psi_1, \ldots, \Psi_n) \in J_d^{m,n}$. Define $h$ as follows:

$$h[(e_1, \ldots, e_m, e'_1 \ldots, e'_n), \Psi] =$$

$$= [(e_1, \ldots, e_m, \overline{e}_1, \ldots, \overline{e}_{m'}, e'_1, \ldots, e'_n, \overline{e}_1, \ldots, \overline{e}_{m'}), (\Psi_1, \ldots, \Psi_n, z_{n+1}, \ldots, z_{n+m'})]$$

Let $c$ be the total Chern class of $L^*_m$, $c'$ – the total Chern class of $L^*_n$, and $d_f$ – the total Chern class of $f^*(L^*_{m'})$. We have the following formulae for the pullbacks:

$$\varphi^* c(L^*_{m+m'}) = c(L^*_m \oplus f^* L^*_{m'}) = c d_f$$

$$\varphi^* c(L^*_{n+m'}) = c(L^*_n \oplus f^* L^*_{m'}) = c' d_f$$

On the level of the universal Thom polynomials we obtain the following.

**Lemma 6.3.** *In the above notations,*

$$\mathrm{UTp}[\Theta_A^l](c, c') = \mathrm{UTp}[\Theta_A^l](c d_f, c' d_f).$$

### 6.1.2 Proof of Damon's theorem

**Theorem 6.4** (Damon, [11])**.** *Let $d, m, n \in \mathbb{N}$ and let $m \leq n$. Suppose $A$ is a finite-dimensional commutative nilpotent algebra and $\Theta_A^{m,n} \subset J_d^{m,n}$ a contact singularity. The Thom polynomial of $\Theta_A^{m,n}$ depends only on the difference $l = n - m$ and can be expressed in a single set of variables $\tilde{c}$ given by the generating series*

$$1 + \tilde{c}_1 t + \tilde{c}_2 t^2 + \ldots = \frac{\sum_{i=0}^{n} c'_i t^i}{\sum_{j=0}^{m} c_j t^j}.$$

These new variables are called the *relative Chern classes*. We will denote the Thom polynomial expressed in the relative Chern classes by $\mathrm{Tp}[\Theta_A^{m,n}](c'/c)$.

*Proof.* The previous discussion implies that if there existed a map $f$ such that $d_f = 1/c$, the Damon's theorem would be proved since

$$\mathrm{UTp}[\Theta_A^l](c, c') = \mathrm{UTp}[\Theta_A^l](1, c'/c) = \mathrm{UTp}[\Theta_A^l](c'/c).$$

In fact, such a map does not exist. The equality $c(L^*) = 1/c(Q^*)$ holds for a finite Grassmannian, so $d_f$ should be $c(Q^*)$, but the Chern classes of the dual tautological bundle can not be pulled back to $Q^*$ via a holomorphic map because $c(L^*)$ is positive (i.e. the Chern classes of $L^*$ are linear combinations with non-negative coefficients of the Poincaré duals to analytic subvarieties) and $c(Q^*)$ is not.

Let $S$ be an ample line bundle over $\mathrm{Gr}(m, K)$. Then for $\alpha$ big enough, $Q_m^* \otimes S^{\otimes \alpha}$ is generated by its global holomorphic sections and thus has positive Chern classes. There exists a holomorphic map

$$f_\alpha \colon \mathrm{Gr}(m, K) \longrightarrow \mathrm{Gr}(m + m'_\alpha, K + K'_\alpha)$$

such that $f_\alpha^*(L^*_{m+m'_\alpha}) = Q_m^* \otimes S^{\otimes \alpha}$.

Let us compute the total Chern class of this twisted bundle. Denote the bundles from the splitting principle for $Q_m^*$ by $\mathcal{E}_1, \ldots, \mathcal{E}_n$ and their first Chern classes by $y_1, \ldots, y_n$, denote the first Chern class of $S$ by $z$. Then the following identity holds:

$$c(Q_m^* \otimes S^{\otimes \alpha}) = c(\mathcal{E}_1 \otimes S^{\otimes \alpha} \oplus \ldots \oplus \mathcal{E}_m \otimes S^{\otimes \alpha}) =$$

$$= \prod_{i=1}^{m}(y_i + \alpha z + 1) = \prod_{i=1}^{m}(y_i + 1) + \alpha P(\alpha) = c(Q_m^*) + \alpha \cdot P(\alpha),$$

where $\alpha \cdot P(\alpha)$ is a polynomial in $\alpha$ that contains all the summands of $\prod_{i=1}^{m}(x_i + \alpha y + 1)$ that depend on $\alpha$. Define

$$\varphi_\alpha \colon \mathrm{Gr}(m, K) \times \mathrm{Gr}(n, K) \longrightarrow \mathrm{Gr}(m + {m'}_\alpha, K + {K'}_\alpha) \times \mathrm{Gr}(n + {m'}_\alpha, K + {K'}_\alpha)$$

$$(V_1, V_2) \mapsto (V_1 \oplus f_\alpha(V_1), V_2 \oplus f_\alpha(V_1))$$

Denote the total Chern class of the dual tautological bundle $L^*_{m+m'_\alpha}$ on $\mathrm{Gr}(m + m'_\alpha, K + K'_\alpha)$ by $\bar{c}$ and the total Chern class of the dual tautological bundle $L^*_{n+m'_\alpha}$ on $\mathrm{Gr}(n + m'_\alpha, K + K'_\alpha)$ by $\bar{c}'$. Then by the previous discussion we have the following relations between the Chern classes:

$$\varphi^*(\bar{c}) = c \cdot (c(Q^*_m) + \alpha P(\alpha)) = 1 + c \cdot \alpha P(\alpha)$$

$$\varphi^*(\bar{c}') = c' \cdot (c(Q^*_m) + \alpha P(\alpha)) = c'/c + c' \cdot \alpha P(\alpha).$$

Or, on the level of the universal Thom polynomials:

$$\mathrm{UTp}[\Theta^l_A](1+c\cdot\alpha P(\alpha), c'/c+c'\cdot\alpha P(\alpha)) = \mathrm{UTp}[\Theta^l_A](1, c'/c)+\alpha P_2(\alpha) = \mathrm{UTp}[\Theta^j_A](c, c'),$$

where $\alpha P_2(\alpha)$ contains all the summands that depend on $\alpha$.

Since $\alpha P_2(\alpha) = \mathrm{UTp}[\Theta^l_A](c, c') - \mathrm{UTp}[\Theta^l_A](1, c'/c)$ their expressions in the Schur polynomial basis are also equal:

$$\alpha P_2(\alpha) = \alpha \sum W_{\lambda\mu}(\alpha) s_\lambda(c) s_\mu(c')$$

$$\mathrm{UTp}[\Theta^l_A](c, c') - \mathrm{UTp}[\Theta^l_A](1, c'/c) = \sum B_{\lambda\mu} s_\lambda(c) s_\mu(c')$$

$$\alpha \sum W_{\lambda\mu}(\alpha) s_\lambda(c) s_\mu(c') = \sum B_{\lambda\mu} s_\lambda(c) s_\mu(c')$$

This equation holds if and only if

$$B_{\lambda\mu} = \alpha W_{\lambda\mu}(\alpha)$$

for all $\lambda$ and $\mu$. However, since this is true for all sufficiently large $\alpha$, the polynomial $B_{\lambda\mu} - \alpha W_{\lambda\mu}(\alpha)$ has infinite number of roots. Thus, it is zero for all $\alpha$. This implies that $B_{\lambda\mu} = 0$ for all $\lambda$ and $\mu$, i.e. $\mathrm{UTp}[\Theta^l_A](c, c') = \mathrm{UTp}[\Theta^l_A](1, c'/c)$. □

## 6.2 Positivity

The *Schur polynomials* serve as a natural basis for the cohomology ring of Grassmannians. Given an integer partition $\lambda = (\lambda_1, \ldots, \lambda_m)$, such that $K \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m > 0$ define the conjugate partition $\lambda^* = (\lambda^*_1, \ldots, \lambda^*_k)$ by taking $\lambda^*_i$ to be the largest $j$ such that $\lambda_j \geq i$. Denote by $s_\lambda(b_1, \ldots, b_m)$ the expression of the Schur polynomials in elementary symmetric polynomials:

$$s_\lambda(b_1, \ldots, b_m) = \det\{b_{\lambda^*_i+j-i}\}^m_{i,j=1}.$$

The Schur polynomials of degree $d$ in $m$ variables form a linear basis for the space of homogeneous degree $d$ symmetric polynomials in $m$ variables.

Consider the finite Grassmannian $\mathrm{Gr}(m, K)$. The Schur polynomials indexed by $\lambda$ such that $K \geq \lambda_1 \geq \ldots \geq \lambda_m > 0$, evaluated in the Chern classes $c_1, \ldots, c_m$ of the dual tautological vector bundle $L_m^*$ are the Poincaré duals of the Schubert cycles – homological classes of Schubert varieties $\sigma_\lambda$, special varieties whose homological classes form a basis for the homology of the Grassmannian [16]:

$$s_\lambda(c_1, \ldots, c_m) = \mathrm{PD}[\sigma_\lambda].$$

The following result was first proved by Pragacz and Weber. Here we give a new proof of this result.

**Theorem 6.5** (Pragacz, Weber, [29])**.** *Let $d, m, n \in \mathbb{N}$ and let $m \leq n$. Suppose $A$ is a finite-dimensional commutative nilpotent algebra and $\Theta_A^{m,n} \subset J_d^{m,n}$ a contact singularity. The Thom polynomial of $\Theta_A^{m,n}$ expressed in the relative Chern classes is Schur-positive:*

$$\mathrm{Tp}[\Theta_A^{m,n}](c'/c) = \sum \alpha_\lambda s_\lambda(c'/c)$$

*where $\alpha_\lambda \geq 0$.*

*Proof.* By Damon's theorem, Thom polynomials for contact singularities can be written as follows:

$$\mathrm{Tp}[\Theta_A^{m,n}](c, c') = \mathrm{Tp}[\Theta_A^{m+j,n+j}](1, c'/c) =$$

$$= \sum_\lambda \alpha_{0\lambda} s_0(1) s_\lambda(c'/c) = \sum_\lambda \alpha_{0\lambda} s_\lambda(c'/c)$$

for $j$ big enough. To prove the positivity we show that $\alpha_{0\lambda} \geq 0$ for all $\lambda$.

Fix a plane $V_0 \in \mathrm{Gr}(m, K)$ and define the map

$$h\colon \mathrm{Gr}(n, K) \longrightarrow \mathrm{Gr}(m, K) \times \mathrm{Gr}(n, K)$$

$$h(V) = (V_0, V).$$

Let $\varphi$ be the unique map making the following diagram commutative:

$$\begin{array}{ccccc}
\varphi^{-1}(EG_K \times_G \overline{\Theta}_A^{m,n}) & \hookrightarrow & h^*(EG_K \times_G J_d^{m,n}) & \xrightarrow{p_2} & \mathrm{Gr}(n, K) \\
 & & \downarrow \varphi & & \downarrow h \\
\Sigma = EG_K \times_G \overline{\Theta}_A^{m,n} & \hookrightarrow & EG_K \times_G J_d^{m,n} & \xrightarrow{p_1} & \mathrm{Gr}(m, K) \times \mathrm{Gr}(n, K)
\end{array}$$

The idea of the proof is to show that

$$\sum_{\lambda} \alpha_{0\lambda} s_\lambda(c') = h^*(\mathrm{Tp}[\Theta_A^{m,n}](c,c')) = \mathrm{PD}[X],$$

where $X$ is an analytic cycle in $\mathrm{Gr}(n,K)$.

Let $\sigma_{\lambda'}$ be a homology class of a Schubert variety of dimension complementary to $\dim X$. $\mathrm{Gl}_n$ acts transitively on $\mathrm{Gr}(n,K)$, so by Kleiman's theorem [23] there exists $C \in \mathrm{Gl}_n$ such that $(CX) \cap \sigma_{\lambda'}$ is of expected dimension (so, discrete) and $CX$ is homologous to $X$.

$$\#(X \cap \sigma_{\lambda'}) = \mathrm{PD}[X] \cdot \mathrm{PD}[\sigma_{\lambda'}] = \sum_{\mu} \alpha_{0\mu} s_\mu(c') s_{\lambda'}(c') = \alpha_{0\lambda} =$$

$$= \#(CX \cap \sigma_{\lambda'}) = \sum_{x \in CX \cap \sigma_{\lambda'}} \mathrm{mult}_x \geq 0.$$

Here $\mathrm{mult}_x$ is an intersection multiplicity, which is non-negative for two analytic cycles.

Let us consider the details. We should construct the algebraic variety $X$. First, denote $EG_K \times_G J_d^{m,n}$ by $E$ and $EG_K \times_G \overline{\Theta}_A^{m,n}$ by $\Sigma$ for short. It is clear that $\varphi^{-1}(\Sigma) \subset h^*(E)$. If $\varphi$ is also transversal to $\Sigma$, then we have that

$$\varphi^* \mathrm{PD}[\Sigma] = \mathrm{PD}[\varphi^{-1}(\Sigma)].$$

By definition, we need to show that:

$$\mathrm{Im}(d_x(\varphi)) + T_{\varphi(x)}\Sigma = T_{\varphi(x)}E$$

for $x \in \varphi^{-1}(\Sigma)$. Locally

$$T_{(z,y)}E = T_z(\mathrm{Gr}(m,K) \times \mathrm{Gr}(n,K)) \oplus T_y \mathcal{J}_d^{m,n}$$

for $z \in EG_K = \mathrm{Gr}(m,K) \times \mathrm{Gr}(n,K)$ and $y \in J_d^{m,n}$. With this interpretation the transversality is obvious since $\mathrm{Im}(d_x(\varphi))$ has $T_y J_d^{m,n}$ as a direct summand and $T_{\varphi(x)}\Sigma$ has $T_z(\mathrm{Gr}(m,K) \times \mathrm{Gr}(n,K))$ as a direct summand.

Let us show that the vector bundle $h^*(E)$ has enough holomorphic sections to find a holomorphic section $s$ transversal to $\varphi^{-1}(\Sigma)$.

**Lemma 6.6.** $EG_K \times_G J_d^{m,n} = \left(\bigoplus_{i=1}^d \mathrm{Sym}^i L_m\right) \boxtimes L_n^*$.

*Proof.* An element of a fiber of $EG_K \times_G J_d^{m,n}$ is a class $[(e_m, e_n), f]$, where $f \in \mathcal{J}_d^{m,n}$, $e_m$ is a frame, i.e. a linear injective map form $\mathbb{C}^m$ to $\mathbb{C}^K$, and $e_n$ is a linear injective map form $\mathbb{C}^n$ to $\mathbb{C}^K$. We consider a class $[(e_m, e_n), f]$ with the equivalence relation

$$((e_m, e_n), f) \sim ((e_m C_m^{-1}, e_n C_n^{-1}), C_n f C_m^{-1}),$$

where $C_n \in \mathrm{Gl}_n$, $C_k \in \mathrm{Gl}_k$ .

An element of the fiber of $\left(\bigoplus_{i=1}^d \mathrm{Sym}^i L_m\right) \boxtimes L_n^*$ is a polynomial function of degree at most $d$ without a constant term between $V_m \in \mathrm{Gr}(m, K)$ and $V_n \in \mathrm{Gr}(n, K)$.

The map $[(e_m, e_n), f] \mapsto e_m \circ f \circ e_n^{-1}$ is correctly defined and is a bijection. □

We use this lemma to decompose $h^*(E)$ :

$$h^*(E) = \left(\bigoplus_{i=1}^d \mathrm{Sym}^i(\mathrm{Triv}_m)\right) \otimes L_n^* = \mathrm{Triv}_{\binom{d+m}{m}-1} \otimes L_n^*,$$

where $\mathrm{Triv}_m$ is a trivial vector bundle whose fiber is a complex vector space of dimension $m$.

We use the following theorem to show that this bundle has enough global holomorphic sections to find one transversal to $\varphi^{-1}(\Sigma)$.

**Theorem 6.7** (Parametric transversality theorem, [20])**.** *Let $M$, $N$, $Z$, $S$ be smooth manifolds. Consider $F\colon M \times S \to N \supset Z$, smooth map transversal to $Z$. Then for almost all $s \in S$ the map $F_s$ is transversal to $Z$.*

Let $D = \binom{d+m}{m} - 1$. In the notations of the Parametric transversality theorem, let

$$M = \mathrm{Gr}(n, K),\ N = \mathrm{Hom}(L_n, \mathbb{C}^D) \cong h^*(EG_K \times_G J_d^{m,n}),$$

$$Z = \varphi^{-1}(\Sigma),\ S = \Gamma(\mathrm{Hom}(L_n, \mathbb{C}^D)) = \mathrm{Hom}(\mathbb{C}^K, \mathbb{C}^D).$$

Then, the map $F$ from the theorem is the following:

$$F\colon\ \mathrm{Gr}(n, K) \times \mathrm{Hom}(\mathbb{C}^K, \mathbb{C}^D) \longrightarrow \mathrm{Hom}(L_n, \mathbb{C}^D).$$

$$(V, f) \mapsto f|_V.$$

The transversality of $F$ to $\varphi^{-1}(\Sigma)$ obviously follows from the fact that $d_{(V,f)}F$ is surjective for all $V$ and $f$.

Now, by Parametric transversality theorem, the set of holomorphic sections of $h^*(E)$ transversal to smooth points of $\varphi^{-1}(\Sigma)$ is open and dense in all holomorphic sections of this bundle. The set of holomorphic sections of $h^*(E)$ transversal to smooth points of the set of singular points of $\varphi^{-1}(\Sigma)$ is open and dense in the set of holomorphic sections transversal to smooth point of $\varphi^{-1}(\Sigma)$, and so on. Since this procedure drops the dimension of the variety, it is a finite process and the intersection of a finite number of open and dense sets is again open and dense. So, we can choose a holomorphic section $s$ transversal to $\varphi^{-1}(\Sigma)$.

The analytic subvariety $X$ from the discussion at the beginning of the proof is $s^{-1}\varphi^{-1}(\Sigma)$ :

$$\mathrm{PD}[s^{-1}\varphi^{-1}(\Sigma)] = s^* \,\mathrm{PD}[\varphi^{-1}(\Sigma)] = ({pr_2}^*)^{-1}\varphi^* \,\mathrm{PD}[\Sigma] = h^*({pr_1}^*)^{-1}\,\mathrm{PD}[\Sigma] =$$

$$= h^* \,\mathrm{Tp}[\Theta_A^{m,n}](c, c') = \sum_\lambda \alpha_{0\lambda} s_\lambda(c')$$

and the proof of positivity is complete. □

# 7 Computing Thom polynomials

Computing the Thom polynomials is difficult. So far, Thom polynomials are known only for a limited number of singularities, mostly in small relative codimensions. There are two main modern methods to compute Thom polynomials: Rimányi's method of restriction equations and the Bérczi-Szenes-Kazarian residue formula. In this section we will explain both methods and show how the combination of the two may simplify the computations.

## 7.1 Rimányi's method of restriction equations

Let us recall the method of computing the Thom polynomials introduced by Rimányi in [30].

Let $l \geq 0$ and let $\Theta$ be a singularity in the jet-space of relative codimension $l$, i.e. $\Theta \subset J_d^{m,m+l}$. Let $\theta\colon \mathbb{C}^m \to \mathbb{C}^{m+l}$ be its prototype.

**Definition 7.1.** [30] The maximal compact subgroup of the left-right symmetry group of $\theta$

$$\operatorname{Aut}\theta = \{(\Delta_m, \Delta_{m+l}) \in \operatorname{Diff}_d^m \times \operatorname{Diff}_d^{m+l} \mid \Delta_{m+l} \circ \theta \circ \Delta_m^{-1} = \theta\}$$

will be denoted by $G_\Theta$. Its representations on $\mathbb{C}^m$ and $\mathbb{C}^{m+l}$ will be $\lambda_1(\Theta)$ and $\lambda_2(\Theta)$ respectively. The vector bundles associated to the universal $G_\Theta$-bundle using the representations $\lambda_1(\Theta)$ and $\lambda_2(\Theta)$ will be called $\xi_\Theta$ and $\overline{\xi_\Theta}$. The total Chern class of $\Theta$ is defined as

$$c(\Theta) = \frac{c(\xi_\Theta)}{c(\overline{\xi_\Theta})} \in H^*(BG_\Theta, \mathbb{Z}).$$

Let the Euler class $e(\Theta) \in H^{2\operatorname{codim}\Theta}(BG_\Theta, \mathbb{Z})$ be the Euler class of the bundle $\overline{\xi_\Theta}$.

**Definition 7.2.** [30] (The hierarchy of singularities.) Let $\Theta, \Xi$ be singularities in $J_d^{m,m+l}$ for $l \geq 0$. The singularity $\Theta$ will be called more complicated than $\Xi$ if $\Theta \not\subset \overline{\Xi}$. We will write $\Xi < \Theta$. Let us adapt the convention $\Theta \not< \Theta$.

**Proposition 7.3.** [30] *If* $\operatorname{codim}\Xi \geq \operatorname{codim}\Theta$, *then* $\Xi \not< \Theta$.

**Theorem 7.4.** (Rimányi's method of restriction equations, [30])

$$\operatorname{Tp}[\Theta](c(\Xi)) = \begin{cases} e(\Xi) & \text{if } \Theta = \Xi \\ 0 & \text{if } \Theta \not> \Xi \text{ and } \Theta \neq \Xi. \end{cases}$$

**Corollary 7.5.** [30]

$$\mathrm{Tp}[\Theta](c'(\Xi)) = \begin{cases} e'(\Xi) & \textit{if } \Theta = \Xi \\ 0 & \textit{if } \Theta \not> \Xi \textit{ and } \Theta \neq \Xi. \end{cases},$$

*where* $e'(\Xi)$ *and* $c'(\Xi)$ *are the Euler and the Chern classes of* $\Xi$ *corresponding to any subgroup* $G'_\Xi \leq G_\Xi$.

Often, these conditions characterize the Thom polynomial. Let us show how to compute the Thom polynomial using this method on a simple example.

**Example 7.1.** Suppose $l = 0$, and let us compute the Thom polynomial of $\Theta_{A^3} \subset J_d^{m,m}$. By Proposition 7.3, the $A_3$ singularity is more complicated than the $A_2$ and the $A_1$ singularities. The computation of the Chern and the Euler classes corresponding to singularities is described in great detail in [30], in particular, the following formulas for $A_i$ singularities are computed:

$$\begin{aligned} c(A_i) &= \tfrac{1+(i+1)x}{1+x} = 1 + ix - ix^2 + ix^3 - \dots \\ e(A_i) &= i!x^i. \end{aligned}$$

Since $\Theta_{A^3} \subset J_d^{m,m}$ is of codimension 3, its Thom polynomial is a homogeneous polynomial of degree 3 in relative Chern classes (interpreted as graded variables):

$$\mathrm{Tp}[\Theta_{A_3}^{m,m}] = Bc_1^3 + Cc_1c_2 + Dc_3.$$

We will use Rimányi's method of restriction equations to compute the unknown coefficients $B, C, D \in \mathbb{Z}$. By Theorem 7.4, we have the following equations:

1. $\mathrm{Tp}[\Theta_{A_3}^{m,m}](c(A_2)) = 0 \Leftrightarrow Bc_1^3(A_2) + Cc_1(A_2)c_2(A_2) + Dc_3(A_2) = 0$
   $\Rightarrow 4B - 2C + D = 0$

2. $\mathrm{Tp}[\Theta_{A_3}^{m,m}](c(A_1)) = 0 \Rightarrow B - C + D = 0$

3. $\mathrm{Tp}[\Theta_{A_3}^{m,m}](c(A_3)) = e(A_3) \Rightarrow 9B - 3C + D = 2,$

that is, the coefficients of $\mathrm{Tp}[\Theta_{A_3}^{m,m}]$ are given by $\begin{cases} 4B - 2C + D = 0 \\ B - C + D = 0 \\ 9B - 3C + D = 2 \end{cases} \Rightarrow \begin{cases} B = 1 \\ C = 3 \\ D = 2. \end{cases}$

This method allows us to compute the Thom polynomials when the hierarchy of singularities is known. However, the hierarchy depends on $l$ and is not known in the general case.

## 7.2 The Bérczi-Szenes-Kazarian residue formula

Another method of computing the Thom polynomials for $A_d$-singularities was presented in [5]. This formula does not depend on the hierarchy of singularities.

**Theorem 7.6.** [5] *Let $T_d \subset B_d \subset \mathrm{Gl}_d$ be the subgroups of invertible diagonal and upper-triangular matrices respectively. Denote the diagonal weights of $T_d$ by $z_1, \ldots, z_d$. Consider the $\mathrm{Gl}_d$-module of 3-tensors $\mathrm{Hom}(\mathbb{C}^d, \mathrm{Sym}^2 \mathbb{C}^d)$; identifying the weight-$(z_i - z_j + z_k)$ symbols $q_{ij}^k$ and $q_{ji}^k$, we can write a basis for this space as follows:*

$$\mathrm{Hom}(\mathbb{C}^d, \mathrm{Sym}^2 \mathbb{C}^d) = \bigoplus \mathbb{C} q_{ij}^k, \ \ 1 \le i, j, k \le d.$$

*Consider the reference element*

$$\varepsilon_{\mathrm{ref}} = \sum_{i=1}^{d} \sum_{j=1}^{d-i} q_{ij}^{i+j}$$

*in the $B_d$-invariant subspace*

$$N_d = \bigoplus_{1 \le i+j \le k \le d} \mathbb{C} q_{ij}^k \subset \mathrm{Hom}(\mathbb{C}^d, \mathrm{Sym}^2 \mathbb{C}^d).$$

*Set the notation $\mathcal{R}_d$ for the orbit closure $\overline{B_d \varepsilon_{\mathrm{ref}}} \subset N_d$, and consider its $T_d$ - equivariant Poincaré dual*

$$Q_d(z_1, \ldots, z_d) = \mathrm{eP}(\mathcal{R}_d, N_d)_{T_d},$$

*which is a homogeneous polynomial of degree $\dim(N_d) - \dim(\mathcal{R}_d)$.*

*Then for arbitrary integers $m \le n$, the Thom polynomial for the $A_d$-singularity with $m$-dimensional source space and $n$-dimensional target space is given by the following iterated residue formula:*

$$\mathrm{eP}(\Theta_{A_d}^{m,n}) = \operatorname*{Res}_{z=\infty} \frac{(-1)^d \prod_{i<k}(z_i - z_k) Q_d(z_1, \ldots, z_d)}{\prod_{k=1}^{d} \prod_{i=1}^{k-1} \prod_{j=1}^{\min(i,k-i)} (z_i + z_j - z_k)} \prod_{i=1}^{d} \mathrm{RC}\left(\frac{1}{z_i}\right) z_i^{n-m},$$

*where $\mathrm{RC}(\cdot)$ is the generating function of the relative Chern classes:*

$$\mathrm{RC}(q) = 1 + c_1 q + c_2 q^2 + \cdots = \frac{\prod_{i=1}^{n}(1 + \theta_i q)}{\prod_{j=1}^{m}(1 + \lambda_j q)},$$

*here $\theta_i$ and $\lambda_i$ denote the corresponding Chern roots.*

The only unknown ingredient in the Bérczi-Szenes-Kazarian residue formula is the $Q_d$ polynomial. While in principle it is an algebraic problem whose solution can be computed using software such as Singular or Macaulay, in reality the existing methods and the computational capacity of modern computers only allow us to find $Q_d$ for $d \le 6$.

### 7.3 The Q-polynomial

The degree of the $Q$-polynomial is the codimension of $\mathcal{R}_d$ in $N_d$. The dimension of $N_d$ may be computed by indexing the basis by the triples of $(i, j, k)$ such that $i \leq j$ and $i+j \leq k \leq d$. The dimension of the Borel orbit of the reference element is

$$\dim(\mathcal{R}_d) = \dim(B_d) - \dim(Stab_{e_{\mathrm{ref}}}) = \binom{d+1}{2} - d = \binom{d}{2}.$$

Let us compute the degree of $Q_d$ for $d \leq 7$ (the same data up to $d = 6$ may be found in [5].)

| $d$ | $\dim N_d$ | $\dim \mathcal{R}_d$ | $\deg Q_d$ |
|---|---|---|---|
| 1 | 0 | 0 | 0 |
| 2 | 1 | 1 | 0 |
| 3 | 3 | 3 | 0 |
| 4 | 7 | 6 | 1 |
| 5 | 13 | 10 | 3 |
| 6 | 22 | 15 | 7 |
| 7 | 34 | 21 | 13 |

**Example 7.2.** Since $Q_3 = 1$, we have the following formula for the Thom polynomial of $\Theta_{A_3}^{m,m} \subset J_d^{m,m}$:

$$\mathrm{Tp}(\Theta_{A_3}^{m,m}) = (-1) \operatorname*{Res}_{z_1=\infty} \operatorname*{Res}_{z_2=\infty} \operatorname*{Res}_{z_3=\infty} \frac{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)}{(2z_1 - z_2)(z_1 + z_2 - z_3)(2z_1 - z_3)} \cdot$$
$$\cdot \mathrm{RC}\left(\frac{1}{z_1}\right) \mathrm{RC}\left(\frac{1}{z_2}\right) \mathrm{RC}\left(\frac{1}{z_3}\right) dz_1 dz_2 dz_3$$

Let us focus on the case when $Q_d$ is non-trivial. Following the idea from [5], we first describe the set of equations satisfied by $\mathcal{R}_d \subset N_d$.

We will write the equations in terms of the basis dual to the $\{q_{ij}^k\}$ basis of $N_d$. The elements of this basis may be interpreted as the structure constants of the multiplication making a $d$-dimensional filtered vector space a commutative $d$-dimensional filtered algebra, i.e. let $V_1 \supset V_2 \cdots \supset V_d$ such that $V_i = \langle v_i, \ldots, v_d \rangle$ be a filtration on $\mathbb{C}^d$. The multiplication preserving the filtration is of the following form:

$$v_i \cdot v_j = \sum_{k=i+j}^{d} t_{ij}^k v_k \in V_{i+j},$$

where $t_{ij}^k \in \mathbb{Z}$ are the structure constants. Note that the reference element gives the "graded" multiplication, i.e.

$$v_i \cdot v_j = t_{ij}^{i+j} v_{i+j}.$$

Since the points in the Borel orbit of the reference element correspond to associative multiplications, $\mathcal{R}_d$ will satisfy the *associativity equations*, i.e. relations between the structure constants coming from the associative triples

$$(v_i \cdot v_j) \cdot v_k = v_i \cdot (v_j \cdot v_k).$$

**Example 7.3.** [5] The first case where a non-trivial associativity equation appears is the case $d = 4$ :

$$(v_1 \cdot v_1) \cdot v_2 = (v_1 \cdot v_2) \cdot v_1 \Leftrightarrow$$
$$t_{11}^2 t_{22}^4 = t_{12}^3 t_{13}^4.$$

The variety defined by this equation is an irreducible toric variety of the same dimension as $\mathcal{R}_d$ [5], thus they coincide. The equivariant Poincaré dual in this case is given by the sum of weights of any of the two monomials:

$$Q_4(z_1, z_2, z_3, z_4) = (2z_1 - z_2) + (2z_2 - z_4) = 2z_1 + z_2 - z_4.$$

However, in the more complicated cases the variety described by the associativity equations is not toric and, moreover, has more than one component.

**Example 7.4.** The first case where the variety given by the associativity equations has more than one component is the case $d = 6$ [5]. The following triples will give the associativity relations:

$$\begin{aligned}
(v_1 \cdot v_1) \cdot v_2 &= (v_1 \cdot v_2) \cdot v_1 \\
(v_1 \cdot v_1) \cdot v_3 &= (v_1 \cdot v_3) \cdot v_1 \\
(v_1 \cdot v_1) \cdot v_4 &= (v_1 \cdot v_4) \cdot v_1 \\
(v_1 \cdot v_2) \cdot v_3 &= (v_1 \cdot v_3) \cdot v_2 \\
(v_1 \cdot v_2) \cdot v_3 &= (v_2 \cdot v_1) \cdot v_3 \\
(v_2 \cdot v_2) \cdot v_1 &= (v_2 \cdot v_1) \cdot v_2.
\end{aligned}$$

The corresponding associativity equations are the following:

$$t_{11}^2 t_{22}^4 = t_{12}^3 t_{13}^4$$

$$t_{11}^2 t_{22}^5 + t_{11}^3 t_{23}^5 = t_{12}^3 t_{13}^5 + t_{12}^4 t_{14}^5$$

$$t_{11}^2 t_{22}^6 + t_{11}^3 t_{23}^6 + t_{11}^4 t_{24}^6 = t_{12}^3 t_{13}^6 + t_{12}^4 t_{14}^6 + t_{12}^5 t_{15}^6$$

$$t_{11}^2 t_{23}^5 = t_{13}^4 t_{14}^5$$

$$t_{11}^2 t_{23}^6 + t_{11}^3 t_{33}^6 = t_{13}^4 t_{14}^6 + t_{13}^5 t_{15}^6$$

$$t_{11}^2 t_{24}^6 = t_{14}^5 t_{15}^6$$

$$t_{12}^3 t_{33}^6 = t_{23}^5 t_{15}^6 = t_{13}^4 t_{24}^6$$

$$t_{22}^4 t_{14}^5 = t_{12}^3 t_{23}^5$$

$$t_{22}^4 t_{14}^6 + t_{22}^5 t_{15}^6 = t_{12}^3 t_{23}^6 + t_{12}^4 t_{24}^6.$$

It is easy to see that the associativity variety contains two maximal dimensional components: $\mathcal{R}_6$ and another one given by

$$\langle t_{11}^2 = 0,\ t_{12}^3 = 0,\ t_{11}^3 = 0;\ t_{14}^5 = 0,\ t_{14}^6 = 0,\ t_{15}^6 = 0,\ t_{24}^6 = 0\rangle.$$

To distinguish the $\mathcal{R}_6$ component we add an extra relation such that it is satisfied by $\mathcal{R}_6$, but not by the other component. This extra relation is computed in [5] using Macaulay:

$$t_{12}^4 t_{12}^4 t_{23}^5 t_{33}^6 + t_{22}^4 t_{13}^4 t_{12}^5 t_{33}^6 + t_{13}^4 t_{13}^4 t_{22}^5 t_{23}^6 + t_{22}^4 t_{13}^4 t_{23}^5 t_{13}^6 -$$
$$- t_{22}^4 t_{11}^4 t_{23}^5 t_{33}^6 - t_{13}^4 t_{12}^4 t_{22}^5 t_{33}^6 - t_{22}^4 t_{13}^4 t_{13}^5 t_{23}^6 - t_{13}^4 t_{13}^4 t_{23}^5 t_{22}^6 = 0.$$

The computation of the Poincaré dual $Q_6$ of a Borel orbit $\mathcal{R}_6$ is non-trivial. The computation using the description of the vanishing ideal of $\mathcal{R}_6$ by explicit relations is written in detail in [5] and is too long to recall here.

**Remark 7.7.** While we have no effective method of computing the extra relations (we can no longer use Macauley for $d = 7$), the form of the extra-relation for $\mathcal{R}_6$ suggests that the extra components appear when there exist $d$-dimensional associative algebras that admit a filtration different from the natural for $A_d$-algebras $(1, \dots, 1)$-filtration.

It is easy to see that the monomials from the extra relation for $\mathcal{R}_6$ only have $1, 2, 3$ as lower indices and $4, 5, 6$ as upper indices. That is, the extra filtration is the $(3, 3)$-filtration given by

$$V_1 = \langle v_1, v_2, v_3\rangle,\ V_2 = \langle v_4, v_5, v_6\rangle$$

$$V_1 \cdot V_1 \subset V_2,\ V_1 \cdot V_2 = V_2 \cdot V_2 = 0.$$

Or, in terms of the structure constants,

$$\langle t_{11}^2 = t_{11}^3 = t_{12}^3 = t_{14}^5 = t_{15}^6 = t_{24}^6 = 0\rangle.$$

For $d = 7$ we have found two different extra-filtrations: the $(3, 4)$- and the $(4, 3)$-filtrations.

The first one is given by

$$V_1 = \langle v_1, v_2, v_3, v_4 \rangle,\ V_2 = \langle v_5, v_6, v_7 \rangle,\ \text{or}$$

$$\langle t^2_{11} = t^3_{11} = t^4_{11} = t^3_{12} = t^4_{12} = t^4_{13} = t^4_{22} = t^6_{15} = t^7_{15} = t^7_{16} = t^7_{25} = 0 \rangle.$$

The second extra component is given by

$$V_1 = \langle v_1, v_2, v_3 \rangle,\ V_2 = \langle v_4, v_5, v_6, v_7 \rangle,\ \text{or}$$

$$\langle t^2_{11} = t^3_{11} = t^3_{12} = t^5_{14} = t^6_{14} = t^7_{14} = t^6_{15} = t^7_{15} = t^7_{16} = t^6_{24} = t^7_{24} = t^7_{25} = t^7_{34} = 0 \rangle.$$

## 7.4 Q-polynomial and the restriction equations

In this subsection we would like to show how Rimányi's method of restriction equations may be used to calculate the $Q_d$ polynomial in a different manner.

We will use a more general setup than in Theorem 7.4, following the ideas in [12]. Note that the Thom polynomial is the equivariant Poincaré dual, and the singularity is an invariant subvariety and a group orbit, so, using the fact that the normal bundle of an orbit of a group action reduces to the stabilizer group of the points of the orbit, we arrive at the following theorem.

**Theorem 7.8.** *Let $V$ be a vector space equipped with a compact Lie group $G$ action and let $\Sigma$ be a closed $G$-invariant subvariety of $V$. If $p \in V$ does not belong to $\Sigma$, then*

$$\mathrm{eP}[\Sigma](x_1, \ldots, x_m) = 0,$$

*where $x_i$ are the diagonal weights of the Lie algebra $\mathfrak{Stab}_p$.*

Let us show how one may apply this theorem to the calculation of the $Q_d$ polynomial. Consider the space $\mathrm{Hom}(\mathrm{Sym}^2 \mathbb{C}^d, \mathbb{C}^d)$ of commutative multiplications on $\mathbb{C}^d$ compatible with the previously defined filtration. There's a torus $T_d$ acting on the dual space $N_d$, and $\mathcal{R}_d$ is a $T_d$-invariant subvariety.

The $T_d$-equivariant Poincaré dual of $\mathcal{R}_d$ is the $Q_d$ polynomial. We may write down the $Q_d$ polynomial as a general polynomial in $d$ variables of degree $\operatorname{codim} \mathcal{R}_d$ with unknown coefficients. Then, if we find a sufficient number of points outside $\mathcal{R}_d$, the equations from the theorem above will determine $Q_d$ up to multiplication, i.e. the solution will still have one parameter. There are several ways of how to get rid of it, we will return to this question later.

The most obvious way of how to find points in $N_d$ not belonging to $\mathcal{R}_d$ is to take the points corresponding to the monomials from the associativity equations, that is, to a monomial $t^k_{ij}t^{k'}_{i'j'}$ corresponds a point in $N_d$ given by

$$\begin{cases} q^k_{ij} + q^{k'}_{i'j'} \neq 0 \\ q^g_{ef} = 0 \text{ if } \{e, f, g\} \neq \{i, j, k\} \neq \{i,' j'k'\} \end{cases}$$

Since these points do not satisfy the equations satisfied by $\mathcal{R}_d$, they do not belong to $\mathcal{R}_d$.

**Example 7.5.** Let us show how to apply the method described above to the simplest case when $Q_d$ is non-trivial, that is, the case $d = 4$.

For $d = 4$, we have $\deg Q_4 = \dim N_4 - \dim \mathcal{R}_4 = 7 - 6 = 1$, so $Q_4$ is a linear polynomial in 4 variables:

$$Q_4(z_1, z_2, z_3, z_4) = a_1z_1 + a_2z_2 + a_3z_3 + a_4z_4, \ a_i \in \mathbb{Z}.$$

There is only one associative triple giving one associativity equation:

$$(v_1 \cdot v_1) \cdot v_2 = (v_1 \cdot v_2) \cdot v_1 \Rightarrow$$

$$t^2_{11}t^4_{22} = t^3_{12}t^4_{13}.$$

That means we have two monomials, so two substitutions.

1. The weights of the Lie algebra corresponding to the the stabilizer of $t^2_{11}t^4_{22}$ are given by

$$\begin{cases} x_2 = 2x_1 \\ x_4 = 2x_2 \end{cases} \Leftrightarrow \begin{cases} x_2 = 2x_1 \\ x_4 = 4x_1 \end{cases}$$

   That is, we have the following equation:

$$Q_4(x_1, 2x_1, x_3, 4x_1) = a_1x_1 + 2a_2x_1 + a_3x_3 + 4a_4x_1 = 0.$$

2. In the case of $t^3_{12}t^4_{13}$ we have

$$\begin{cases} x_3 = x_1 + x_2 \\ x_4 = 2x_1 + x_2 \end{cases}$$

   and the substitution gives us

$$Q_4(x_1, x_2, x_1 + x_2, 2x_1 + x_2) = a_1x_1 + a_2x_2 + a_3(x_1 + x_2) + a_4(2x_1 + x_2) = 0.$$

All that is left is to solve the following system of linear equations:

$$\begin{cases} a_1x_1 + 2a_2x_1 + a_3x_3 + 4a_4x_1 = 0 \\ a_1x_1 + a_2x_2 + a_3(x_1 + x_2) + a_4(2x_1 + x_2) = 0 \end{cases} \Leftrightarrow \begin{cases} a_1 + 2a_2 + 4a_4 = 0 \\ a_3 = 0 \\ a_1 + a_3 + 2a_4 = 0 \\ a_2 + a_3 + a_4 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} a_2 := t \\ a_1 = 2t \\ a_3 = 0 \\ a_4 = -t. \end{cases}$$

The final answer is $Q_4(z_1, z_2, z_3, z_4) = 2tz_1 + tz_2 - tz_4$, which agrees with the computation in [5] for $t = 1$.

The computation for $Q_5$ is similar, but can no longer be carried out by hand, the answer obtained with Maple is again a one-parameter solution. In the case of $Q_6$, however, the computation using only the restrictions coming from the associativity equations gives a two-parameter solution. This computation once again suggests that the associativity variety for $d = 6$ contains two components. Since there is an extra relation that $\mathcal{R}_6$ satisfies, we use the substitutions coming from monomials of this relation, and once again obtain a one-parameter solution. This leads to the following conjecture.

**Conjecture 7.9.** Restrictions coming from the associativity equations and from the extra equations distinguishing the $\mathcal{R}_d$ component determine $Q_d$ up to multiplication.

**Remark 7.10.** In the case of $Q_5$ there are 10 substitutions coming from the associativity equations, but the answer remains the same if we use only 6 of them. For $Q_6$ there are 37 substitutions in total, but if we take only 20 certain substitutions, we again get the correct answer. We are unable to explain the geometry related to this phenomenon yet.

## 7.5 Getting rid of the last parameter

The method described above uses the constraints that are homogeneous linear equations, so only allows us to obtain the solution up to multiplication. That is, to obtain $Q_d$, we must find a non-homogeneous equation.

**Remark 7.11.** The obvious non-homogeneous equation would be the analog of the equation

$$\mathrm{Tp}[\Theta](c(\Theta)) = e(\Theta)$$

from Rimányi's method of restriction equations, but in our case this equation is

$$Q(x, 2x, \ldots, nx) = \prod((i+j)x - ix - jx) = 0,$$

so we get no new information from it.

### 7.5.1 The coefficient of $c_1^d$

The following statement is proved by Rimányi (see [30], Corollary 5.4).

**Proposition 7.12.** [30] *The Thom polynomial of $\Theta_{A_d} \subset J_d^{m,n}$ for $m - n > 0$ in relative Chern classes is equal to*

$$\mathrm{Tp}[\Theta_{A_d}] = c_1^d + \ldots$$

The easiest way to separate the coefficient of $c_1^d$ is to use the residue formula:

$$\operatorname*{Res}_{z=\infty} \frac{(-1)^d \prod_{i<k}(z_i - z_k) Q_d(z_1, \ldots, z_d)}{\prod_{k=1}^d \prod_{i=1}^{k-1} \prod_{j=1}^{\min(i,k-i)} (z_i + z_j - z_k)} \prod_{i=1}^d \left(\frac{1}{z_i}\right) = 1$$

**Example 7.6.** Let us return to the case $d = 4$. In Example 7.5 we were able to calculate the following one-parameter solution: $Q_4(z_1, z_2, z_3, z_4) = 2tz_1 + tz_2 - tz_4$. Now, using the formula above, we can compute the value of the parameter $t$.

$$\operatorname*{Res}_{z=\infty} \frac{(2tz_1 + tz_2 - tz_4)(z_1 - z_4)(z_1 - z_3)(z_1 - z_2)(z_2 - z_4)(z_2 - z_3)(z_3 - z_4)}{(2z_1 - z_2)(2z_1 - z_3)(2z_1 - z_4)(z_1 + z_2 - z_3)(z_1 + z_2 - z_4)(z_1 + z_3 - z_4)(2z_2 - z_4) z_1 z_2 z_3 z_4}$$

$$= -\operatorname*{Res}_{z_2=0}\operatorname*{Res}_{z_3=0}\operatorname*{Res}_{z_4=0} \frac{t(z_2 - z_4)}{(2z_2 - z_4) z_2 z_3 z_4} = t = 1$$

So, the final answer is

$$Q_4(z_1, z_2, z_3, z_4) = 2z_1 + z_2 - z_4.$$

### 7.5.2 The volume of the toric orbit

In this method we use the idea from [3]. Let $w_1, \ldots w_d$ be the new variables defined by

$$z_1 = w_1,\ z_2 = 2w_1 - w_2,\ \ldots,\ z_d = dw_1 - w_2 - w_3 - \ldots w_d.$$

The $Q_d$ polynomial in these variables can be thought of as a polynomial in a distinguished variable $w_1$ whose coefficients are homogeneous polynomials in $w_2, \ldots, w_d$. Let us denote the equivariant Poincaré dual of the toric orbit $\overline{T_d \varepsilon_{\mathrm{ref}}}$ by $Q_d^0$. In [3] Bérczi proves the following theorem.

**Theorem 7.13.** [3]

$$\mathrm{coeff}_{w_1^{\mathrm{top}}}(Q_d(w_1,\dots,w_d)) = C_d Q_d^0,$$

*where the constant $C_d$ is given by*

$$C_d = \begin{cases} (-1)^{(k-2)(k-1)}(-2)^{(k-2)(k-2)}\cdot\dots\cdot(-2k+4)^1 & \textit{if } d=2k \\ (-1)^{(k-1)(k-1)}(-2)^{(k-2)(k-1)}\cdot\dots\cdot(-2k+3)^1 & \textit{if } d=2k+1. \end{cases}$$

Let us show how to use this fact when getting rid of the last parameter in the $Q_d$ polynomial.

**Example 7.7.** Let $d=5$. Suppose we have obtained a formula for the $Q_5$ polynomial up to a multiplication:

$$Q_5(z_1,\dots,z_5) = t(2z_1+z_2-z_5)(2z_1^2+3z_1z_2-2z_1z_5+2z_2z_3-z_2z_4-z_2z_5-z_3z_4+z_4z_5).$$

It is enough to compare the $\mathrm{coeff}_{w_1^{\mathrm{top}}}(Q_5(w_1,1\dots,1))$ and $C_5Q_5^0(1,\dots,1)$. Let us rewrite the one-parameter formula for the $Q_5$ polynomial using the following substitutions:

$$z_i = i\cdot w_1 - (i-1) \text{ for } i = 1..5.$$

We obtain the following:

$$Q_5(w_1,1,\dots,1) = -3tw_1 + 9t,$$

so the left hand side is $-3t$.

The constant $C_d$ on the right hand side is equal to $-1$ by the formula above.

There are several methods of computing the equivariant Poincaré dual of the toric orbit, but since we do not need the whole polynomial, only its value when evaluated at $(1,1\dots,1)$, we will compute the simplicial volume of the convex hull of the weights of $t_{ij}^{i+j}$. Let us list all the structure constants of this type:

$$t_{11}^2, t_{12}^3, t_{13}^4, t_{14}^5, t_{22}^4, t_{23}^5.$$

The weights of these vectors are:

$$(2,-1,0,0,0),$$

$$(1,1,-1,0,0),$$

$$(1,0,1,-1,0),$$

$$(1,0,0,1,-1),$$

$$(0, 2, 0, -1, 0),$$

$$(0, 1, 1, 0, -1).$$

It is easy to see that these weights lie in the codimension 2 subspace of $\mathfrak{t}_5^*$ : first, the scalar product of any of these points with $(1, 2, 3, 4, 5)$ is equal to 0, second, the scalar product of any of these points with $(1, 1, 1, 1, 1)$ is equal to 1. Let us drop the two last coordinates. Now we have 6 points in $\mathbb{R}^3$. The computation of the Poincaré dual for the corresponding toric variety goes as follows. First, we take the convex hull of these points, then we take the minimal triangulation of the convex hull. Now, the equivariant Poincaré dual will be equal to the sum over all simplices $S$ of the following products:

$$\prod_{weight(t_{ij}^k)\notin S} (z_i + z_j - z_k).$$

Note that since we are only interested in computing this sum for $z_i = 1$, the answer will be the number of simplices in the triangulation, i.e. the simplicial volume of the convex hull. This computation can be easily done with the QHull software for this case as well as for higher-dimensional cases. Here are the simplicial volumes for $n = 5, 6, 7$ computed with QHull.

| $d$ | $Q_d^0(1, \dots, 1)$ |
|---|---|
| 5 | 3 |
| 6 | 10 |
| 7 | 20 |

In our case the answer is 3, so

$$-3t = -3 \Rightarrow t = 1.$$

# 8 K-theoretic Thom polynomials

In [31] Rimányi and Szenes discussed the $K$-theoretic generalization of the Thom polynomial. As the Thom polynomial, the new invariant is the fundamental class, but not in equivariant cohomology, but in equivariant $K$-theory. However, there are two different definitions of this invariant. In this section we define both invariants and prove that they are in fact different.

## 8.1 Equivariant smooth resolution

We begin with recalling the necessary facts about smooth resolutions.

Let $X$ be an affine variety. If $Y$ is smooth and there exists a proper birational map $f\colon Y \to X$, then we say that $Y$ is a *smooth resolution* of $X$.

**Proposition 8.1.** *The cohomology groups $H^i(Y, \mathcal{O}_Y)$ do not depend on the smooth resolution $Y$, i.e. are invariants of $X$.*

This fact follows from the Elkik-Fujita Vanishing Theorem [21]. In the notations of Theorem 1-3-1 from [21], take two smooth resolutions of $X$ and a morphism between them $g\colon Z \to Y$ with $E$ equal to the support of the cokernel of the natural morphism $f^*\omega_Y \to \omega_Z$, $L$ equal to $f^*\omega_Y$, $\widetilde{L}$ equal to the structure sheaf, and $D$ and $\widetilde{D}$ – the empty divisors.

**Proposition 8.2.** $H^0(X, \mathcal{O}_X) = H^0(Y, \mathcal{O}_Y)$ *if and only if $X$ is normal.*

If $X$ is not normal, there exists a unique *normalisation* of $X$ – normal affine variety $\widetilde{X}$. In this case $H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = H^0(Y, \mathcal{O}_Y)$, but $H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \neq H^0(X, \mathcal{O}_X)$. The proof of the proposition above is based on the universal property of the normalization and Zariski's Main Theorem [27].

**Definition 8.3.** Let $X$ be a normal affine variety, then $X$ has *rational singularities* if $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$.

Suppose a Lie group $G$ acts on the affine space $\mathbb{A}^M$. Let $X \subset \mathbb{A}^M$ be a $G$-invariant subvariety. $Y$ is called an *equivariant* smooth resolution of $X$ if $Y$ is smooth, $G$ acts on $Y$, and the map $f\colon Y \to X$ is proper birational and $G$-equivariant.

Let $T$ be the maximal torus of $G$. One of the natural questions that arises in [31] is whether $\chi[H^0(X, \mathcal{O}_X)](t)$ is equal to $\chi\left[\sum(-1)^i H^i(Y, \mathcal{O}_Y)\right](t)$, $t \in T$. Note that while $X$ is an affine variety and therefore $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$, this in not necessarily true for $H^i(Y, \mathcal{O}_Y)$.

**Proposition 8.4.** *Let $G$ be a Lie group acting on $\mathbb{A}^M$. Let $X \subset \mathbb{A}^M$ be a $G$-invariant subvariety, and let $Y$ be its smooth $G$-equivariant resolution. Let $T$ be the maximal torus of $G$. The equality*

$$\chi[H^0(X, \mathcal{O}_X)](t) = \chi\left[\sum(-1)^i H^i(Y, \mathcal{O}_Y)\right](t), \ t \in T$$

*holds if and only if $X$ has rational singularities.*

In this section we study whether the $A_2$-singularity loci have rational singularities.

Let us briefly recall the necessary facts about nilpotent algebras. We will call an algebra $N$ *nilpotent* if it is finite dimensional and if there exists a natural number $k$ such that the product of each $k$ elements of the algebra vanishes, that is, $N^k = 0$. $J_d^m$ is nilpotent: $(J_d^m)^{d+1} = 0$, the algebra $J_d^1$ is often denoted by $A_d = t\mathbb{C}[t]/t^{d+1}$.

**Definition 8.5.** An algebra $C$ is $(1, 1, \ldots, 1)$-*filtered* if $C$ has an increasing finite sequence of subspaces $0 = F_{k+1} \subset F_k \subset \ldots \subset F_1 = C$ such that $F_i \cdot F_j \subset F_{i+j}$ and $\dim F_i/F_{i+1} = 1$.

Nilpotent algebras have a natural filtration: $0 = N^{k+1} \subset N^k \subset \ldots \subset N^2 \subset N$. In case of $A_d$, this filtration is a $(1, 1, \ldots, 1)$-filtration.

**Definition 8.6.** *$A_d$-singularity locus* is given by

$$\Theta_{A_d}^{m,n} = \overline{\{(P_1, \ldots, P_n) \in J_d^{m,n} \mid J_d^m / I\langle P_1, \ldots, P_n\rangle \cong A_d\}}.$$

$\Theta_{A_d}^{m,n}$ is a $\mathrm{Gl}(m) \times \mathrm{Gl}(n)$-invariant affine subvariety in $J_d^{m,n}$.

### 8.1.1 Equivariant smooth resolution of the $A_1$-locus

Let us briefly look at a simpler case, the $A_1$-locus:

$$\Theta_{A_1}^{m,n} = \overline{\{M \in \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n) \mid \operatorname{rk} M < m\}},$$

i.e. for every $M \in \Theta_{A_1}^{m,n}$ there exists a non-zero eigenvector $v \in \mathbb{C}^m$ such that $Mv = 0$.

**Proposition 8.7.** *The space*

$$\{(M, v) \mid Mv = 0, \ M \in \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n), \ v \in \mathbb{C}^m\} \subset \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n) \times \mathbb{P}^{m-1}$$

*is an equivariant smooth resolution of $\Theta_{A_1}^{m,n}$.*

This space can be understood as follows: let us fix an element $v \in \mathbb{P}^{m-1}$ and describe the set $\{M \in \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n) \mid Mv = 0\}$.

There is a tautological sequence of vector bundles on $\mathbb{P}^{m-1}$ :

$$\begin{array}{ccccc} \mathcal{O}(-1) = L & \longrightarrow & \mathbb{C}^m & \longrightarrow & Q \\ & & \downarrow & & \\ & & \mathbb{P}^{m-1} & & \end{array}$$

We can apply $\mathrm{Hom}(*, \mathbb{C}^n)$ to it and obtain the following sequence:

$$\begin{array}{ccccc} \mathrm{Hom}(Q, \mathbb{C}^n) & \longrightarrow & \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n) & \longrightarrow & \mathrm{Hom}(L, \mathbb{C}^n) \\ & & \downarrow & & \\ & & \mathbb{P}^{m-1} & & \end{array}$$

The map $\mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n) \to \mathrm{Hom}(L, \mathbb{C}^n)$ can be interpreted as the evaluation map $M \mapsto Mv$ for a fixed $v \in \mathbb{P}^{m-1}$. Its kernel is exactly $\mathrm{Hom}(Q, \mathbb{C}^n)$.

The equivariant smooth resolution of $\Theta^{m,n}_{A_1}$ defined above may be presented as the following vector bundle:

$$\begin{array}{ccc} \mathrm{Hom}(Q, \mathbb{C}^k) & \longrightarrow & \Theta^{n,k}_{A_1} \\ \downarrow & & \\ \mathbb{P}^{n-1} & & \end{array}$$

It is well-known that $\Theta^{m,n}_{A_1}$ has rational singularities. In this paper we study the rationality of the singularities of $\Theta^{m,n}_{A_2}$ and prove the following theorems.

**Theorem 8.8.** $\widetilde{\Theta^{m,n}_{A_2}}$ *in general can have singularities worse than rational.*

**Theorem 8.9.** $\widetilde{\Theta^{m,m}_{A_2}}$ *has rational singularities.*

Before proving the main theorems, we recall the explicit construction for the equivariant smooth resolution of $\Theta^{m,n}_{A_2}$, the Borel-Weil-Bott theorem, and demonstrate the spectral sequences technique that will allow us to study the rationality of the singularities of the $A_2$-loci.

## 8.2 Equivariant smooth resolution of the $A_2$-locus

In this section we recall an explicit construction for the equivariant smooth resolution of the $A_2$-locus following [22]. The general case is discussed in [22] and [5].

Before we present the equivariant smooth resolution of $\Theta^{m,n}_{A_2}$, we need to introduce some preliminary notions.

**Definition 8.10.** The *curvilinear Hilbert scheme* of order 2 is defined as follows:

$$\mathrm{Hilb}_{A_2}(\mathbb{C}^m) \cong \overline{\{I \subset J_2^m \mid J_2^m/I \cong A_2\}}.$$

Each ideal $I \in \mathrm{Hilb}_{A_2}(\mathbb{C}^m)$ comes with the tautological sequence:

$$I \longrightarrow J_2^m \longrightarrow N \cong J_2^m/I$$

To construct a smooth equivariant resolution of $\Theta^{m,n}_{A_2}$ we start with the following vector bundle:

$$\begin{array}{ccc} \mathrm{Hom}(\mathbb{C}^n, I) & \longrightarrow & \Theta^{m,n}_{A_2} \\ \downarrow & & \\ \mathrm{Hilb}_{A_2}(\mathbb{C}^m) & & \end{array}$$

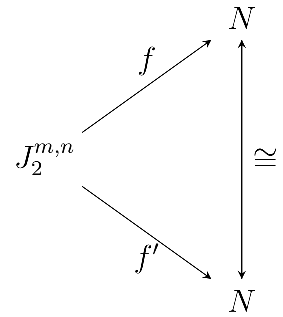
The fiber over $I \in \mathrm{Hilb}_{A_2}(\mathbb{C}^m)$ is the space of all $n$-tuples of elements of $I$. The set of $n$-tuples of elements of $I$ that generate $I$ is Zariski open in $\mathrm{Hom}(\mathbb{C}^n, I)$ and the projection $\mathrm{Hom}(\mathbb{C}^n, I) \twoheadrightarrow J_d^{m,n} \supset \Theta^{m,n}_{A_2}$ is proper.

This vector bundle is not a smooth equivariant resolution of $\Theta^{m,n}_{A_2}$ because $\mathrm{Hilb}_{A_2}(\mathbb{C}^m)$ is not smooth. The next step is to find a smooth equivariant resolution of $\mathrm{Hilb}_{A_2}(\mathbb{C}^m)$.

Since every $I \in \mathrm{Hilb}_{A_2}(\mathbb{C}^m)$ is equipped with the tautological sequence mentioned above, we can rewrite $\mathrm{Hilb}_{A_2}(\mathbb{C}^m)$ as

$$\mathrm{Hilb}_{A_2}(\mathbb{C}^m) = \{f \colon J_2^m \to N \mid \dim N = 2,\ f - \text{surj. alg. homomorphism}\} /_{\sim}$$

The equivalence relation is defined as follows: $f \sim f'$ if the diagram commutes:




We will be interested in $(1,1)$-filtered 2-dimensional nilpotent algebras. There are two different types of them:

- $A_2$ with the natural $(1,1)$-filtration: $A_2^2 \subset A_2$,
- algebra $N$ generated by two elements, such that the product of any two elements of $N$ is 0. This algebra does not have a natural $(1,1)$-filtration, so we introduce an artificial $(1,1)$-filtration $F_1 \subset N$, where $F_1$ is any line in $N$.

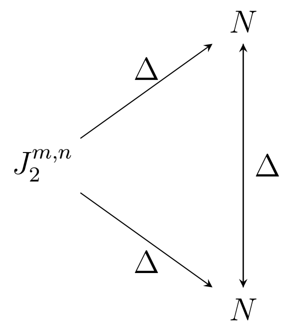
Let us introduce the notation for filtered algebra homomorphisms. Suppose $N$ and $C$ are filtered algebras. We will denote a homomorphism compatible with the filtrations on $N$ and $C$ by

$$f \colon N \xrightarrow{\Delta} C$$

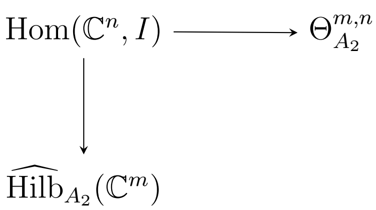
**Proposition 8.11.** *The smooth equivariant resolution of* $\mathrm{Hilb}_{A_2}(\mathbb{C}^m)$ *is given by*

$$\widehat{\mathrm{Hilb}}_{A_2}(\mathbb{C}^m) = \{f \colon J_2^m \xrightarrow{\Delta} N \mid N \text{ – 2-dim. } (1,1)\text{-filt., } f \text{ – surj.}\} /_{\sim},$$

*The equivalence is taken up to a filtered algebra isomorphism:*




The following vector bundle is a smooth equivariant resolution of the $A_2$-locus:



Now we need to find a simpler interpretation of this resolution.

Let $g$ be the inverse of the canonical map $J_2^m \to J_2^m/(J_2^m)^2 \cong \mathbb{C}^m$:

$$g\colon \mathbb{C}^m \to J_2^m$$

Let us denote its image by $Im(g) = E^*$. $E^*$ is the linear part of $J_2^m$.

Let $A^\Delta$ be a 2-dimensional algebra equipped with the $(1,1)$-filtration and $f \in \widehat{\mathrm{Hilb}}_{A_2}(\mathbb{C}^m)$. We can define two natural maps

$$\psi_1\colon E \to A^\Delta,\ \psi_1 = f\big|_{E^*}$$

$$\psi_2\colon \operatorname{Sym}^2 A^\Delta \to A^\Delta$$

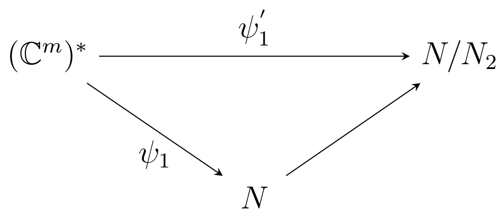
**Proposition 8.12.** *The linear map $\psi_1 \oplus \psi_2\colon E^* \oplus \operatorname{Sym}^2 A^\Delta \to A^\Delta$ is surjective.*

**Proposition 8.13.** *Let $N$ be a 2-dimensional filtered vector space.*

$\widehat{\mathrm{Hilb}}_{A_2}(\mathbb{C}^m)$ *is in one-to-one correspondence with the set of isomorphism classes of pairs $(\psi_1, \psi_2)$, where $\psi_2\colon \operatorname{Sym}^2 N \to N$ is a map giving $N$ an associative commutative algebra structure and $\psi_1\colon (\mathbb{C}^m)^* \to N$ is a linear map such that $\psi_1 \oplus \psi_2$ is surjective. Pairs $(\psi_1, \psi_2)$ are taken up to filtered algebra isomorphism.*

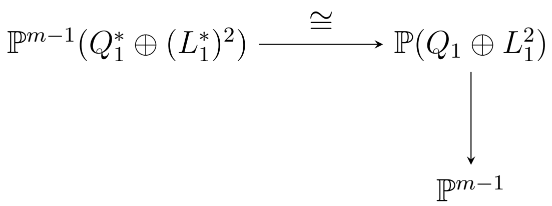
Let us describe $\widehat{\mathrm{Hilb}}_{A_2}(\mathbb{C}^m)$ using this correspondence.

Suppose $N$ be a 2-dimensional vector space with a filtration $N_2 \subset N$, where $N_2$ is a line in $N$.




The kernel of this map is defined by $Ker(\psi_1') = \{V \subset (\mathbb{C}^m)^* \mid \dim V = m-1\} = \mathbb{P}^{m-1}(\mathbb{C}^m)^* \cong \mathbb{P}^{m-1}$. Let us denote $\mathcal{O}(-1)$ over $\mathbb{P}^{m-1}$ by $L_1$ and the quotient bundle by $Q_1$.

The kernel of $\psi_1 \oplus \psi_2$ is then a codimension 2 subspace in $\operatorname{Sym}^2 L_1 \oplus (\mathbb{C}^m)^* \cong L_1^2 \oplus (\mathbb{C}^m)^*$, such that it's projection is of codimension 1 in $(\mathbb{C}^m)^*$, that is:



Let us fix a point $a$ in $\mathbb{P}^{m-1}$. The fiber over this point is $\mathbb{P}((Q_1 \oplus L_1^2)|_a) = \mathbb{P}V_a$. Let $V$ be an $m$-dimensional complex vector space. We have the following tautological sequence on $\mathbb{P}V_a$ :

$$\begin{array}{ccccc} \mathcal{O}(-1) = L_2 & \longrightarrow & V_a & \longrightarrow & Q_2 \\ & & \downarrow & & \\ & & \mathbb{P}V_a & & \end{array}$$

This description allows us to present the smooth equivariant resolution of the $A_2$-locus in the following form:

$$\begin{array}{ccc} \mathrm{Hom}\left(\mathbb{C}^n, \frac{\mathrm{Sym}^2\,\mathbb{C}^m \oplus Q_1}{L_2}\right) & \longrightarrow & \Theta_{A_2}^{m,n} \\ \downarrow & & \\ \mathbb{P}(Q_1 \oplus L_1^2) & & \\ \downarrow & & \\ \mathbb{P}^{m-1} & & \end{array}$$

## 8.3 The Borel-Weil-Bott theorem

Let $V$ be an $m$-dimensional complex vector space. In this paper we use the Borel-Weil-Bott theorem to compute the cohomology of $\mathrm{Gl}(V)$-equivariant vector bundles on $\mathbb{P}V$.

The irreducible representations of $\mathrm{Gl}(V)$ are parametrized by their highest weights – non-increasing integer partitions $\lambda$ of length $m$ (we allow the entries to be equal to 0): $\lambda_1 \geq \lambda_2 \geq \lambda_m \geq 0$. We will denote the irreducible representation of $\mathrm{Gl}(V)$ of highest weight $\lambda$ by $\Sigma^\lambda V$.

Consider the canonical sequence of vector bundles on $\mathbb{P}V$ :

$$\begin{array}{ccccc} \mathcal{O}(-1) = L & \longrightarrow & V & \longrightarrow & Q \\ & & \downarrow & & \\ & & \mathbb{P}V & & \end{array}$$

We will be interested in computing the cohomology of $\mathrm{Gl}(V)$-equivariant vector bundles of the form $\Sigma^\lambda Q \otimes L^k$ on $\mathbb{P}V$. Following the argument in [15], a vector bundle of this form may be presented as a pushforward of the corresponding line bundle on the flag variety of $\mathrm{Gl}(V)$. Thus, we may compute its cohomology using the following interpretation of the Borel-Weil-Bott theorem.

**Theorem 8.14** (The Borel-Weil-Bott theorem, [15])**.** *Consider an irreducible* $\mathrm{Gl}(V)$*-equivariant vector bundle* $\Sigma^\lambda Q \otimes L^k$ *on* $\mathbb{P}V$*. Denote by* $(\lambda, k)$ *the concatenation of* $\lambda = (\lambda_1, \dots, \lambda_{m-1})$ *and* $k$*, and by* $\rho = (m, m-1, \dots, 1)$ *the half-sum of the positive roots of* $\mathrm{Gl}(V)$*.*

*Consider* $(\lambda, k) + \rho = (\lambda_1 + m, \lambda_2 + m - 1, \dots, \lambda_{m-1} + 2, k + 1)$*.*

*If two entries of* $(\lambda, k) + \rho$ *are equal, then*

$$H^i(\mathbb{P}V, \Sigma^\lambda Q \otimes L^k) = 0 \textit{ for all } i.$$

*If all entries of* $(\lambda, k) + \rho$ *are distinct, then there exists a unique permutation* $\sigma$ *such that* $\sigma((\lambda, k) + \rho)$ *is strictly decreasing, i.e. dominant. The length of this permutation,* $l(\sigma)$*, is the number of strictly increasing pairs of elements of* $(\lambda, k) + \rho$*.*

$$\textit{Then } H^i(\mathbb{P}V, \Sigma^\lambda Q \otimes L^k) = \begin{cases} \Sigma^{\sigma((\lambda,k)+\rho)-\rho} V & \textit{if } i = l(\sigma) \\ 0 & \textit{otherwise.} \end{cases}$$

**Example 8.1.** Let us compute $H^i(\mathbb{P}^3, Q \otimes \mathrm{Sym}^2 Q \otimes L^5)$.

First, we need to decompose $Q \otimes \mathrm{Sym}^2 Q$ into the direct sum of irreducible representations. The algorithm is the same as in decomposing the product of two corresponding Schur polynomials into a sum of Schur polynomials, for the details see [16] or [17].

In the case of $Q \otimes \mathrm{Sym}^2 Q$, we obtain the following:

$$Q \otimes \mathrm{Sym}^2 Q = \Sigma^{(1,0,0)} Q \otimes \Sigma^{(2,0,0)} Q = \Sigma^{(3,0,0)} Q + \Sigma^{(2,1,0)} Q.$$

To compute the cohomology groups of the initial sheaf, we compute the cohomology groups of both irreducible summands:

$$H^i(\mathbb{P}^3, Q \otimes \mathrm{Sym}^2 Q \otimes L^5) = H^i(\mathbb{P}^3, \Sigma^{(3,0,0)} Q \otimes L^5) \oplus H^i(\mathbb{P}^3, \Sigma^{(2,1,0)} Q \otimes L^5).$$

Applying the Borel-Weil-Bott theorem to $\Sigma^{(3,0,0)} Q \otimes L^5$, we first construct the sequence $(\lambda, k)$: here $\lambda = (3, 0, 0)$ and $k = 5$. We see that $(\lambda, k) + \rho = (3, 0, 0, 5) + (4, 3, 2, 1) = (7, 3, 2, 6)$ has no repetitions. The unique permutation making $(7, 3, 2, 6)$ decreasing is $\sigma = (2, 3, 4)$. Since there are two increasing pairs in $(7, 3, 2, 6)$, namely, $\{3, 6\}$ and $\{2, 6\}$, $l(\sigma)$ – the length of $\sigma$ – is 2. Finally,

$\sigma((\lambda, k) + \rho) - \rho = (7,6,3,2) - (4,3,2,1) = (3,3,1,1)$, so the only non-zero cohomology group is

$$H^2(\mathbb{P}^3, \Sigma^{(3,0,0)}Q \otimes L^5) = \Sigma^{(3,3,1,1)}\mathbb{C}^4.$$

The second irreducible summand is $\Sigma^{(2,1,0)}Q \otimes L^5$. Here we obtain $(\lambda, k) + \rho = (2,1,0,5) + (4,3,2,1) = (6,4,2,6)$ – there are repetitions, so

$$H^i(\mathbb{P}^3, \Sigma^{(2,1,0)}Q \otimes L^5) = 0 \text{ for all } i.$$

The final answer is $H^i(\mathbb{P}^3, Q \otimes \mathrm{Sym}^2 Q \otimes L^5) = \begin{cases} \Sigma^{(3,3,1,1)}\mathbb{C}^4 \text{ if } i = 2 \\ 0 \text{ if } i \neq 2 \end{cases}$.

## 8.4 Rationality of the singularities of the $A_2$-loci

In this section we show that $\widetilde{\Theta^{m,m}_{A_2}}$, the normalization of $\Theta^{m,m}_{A_2}$, has rational singularities, and give an example, where $\widetilde{\Theta^{m,n}_{A_2}}$ has singularities worse than rational.

Consider the quasi-projective variety $Y$ – Kazarian's smooth resolution of $\Theta^{m,n}_{A_2}$ :

$$\begin{array}{ccc} Y = \mathrm{Hom}\left(\mathbb{C}^n, \frac{\mathrm{Sym}^2 \mathbb{C}^m \oplus Q_1}{L_2}\right) & \longrightarrow & \Theta^{m,n}_{A_2} \\ \downarrow p_1 & & \\ \mathbb{P}(Q_1 \oplus L_1^2) & & \\ \downarrow p_2 & & \\ \mathbb{P}^{m-1} & & \end{array}$$

By definition, $\widetilde{\Theta^{m,n}_{A_2}}$ has rational singularities if $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$. We will compute these cohomology groups step by step, by pushing forward along the tower.

Fix a point $a$ in $\mathbb{P}^{m-1}$, the fiber over this point is $p_2^{-1}(a) = \mathbb{P}((Q_1 \oplus L_1^2)|_a) \cong \mathbb{P}V_a$, where $V_a$ is an $m$-dimensional complex vector space. Let us also denote the constant sheaf $(\mathrm{Sym}^2 \mathbb{C}^m \oplus Q_1)|_a$ on $\mathbb{P}V_a$ by $W$.

Since the fiber over a point $b$ in $\mathbb{P}V_a$, $\left(\mathrm{Hom}\left(\mathbb{C}^n, \frac{\mathrm{Sym}^2 \mathbb{C}^m \oplus Q_1}{L_2}\right)\right)\Big|_b$, is affine, we have $H^i(Y, \mathcal{O}_Y) = H^i(\mathbb{P}V_a, (p_1)_*\mathcal{O}_Y)$. Moreover, the $\mathbb{C}^*$-action on the fiber allows us to decompose $(p_1)_*\mathcal{O}_Y$ into homogeneous components:

$$(p_1)_*\mathcal{O}_Y = \mathcal{O}_Y|_{p_1^{-1}(b)} \cong \bigoplus_l \mathrm{Sym}^l \left(\frac{W \otimes \mathbb{C}^n}{L_2 \otimes \mathbb{C}^n}\right).$$

This decomposition leads to the following identity on the level of cohomology:

$$H^i(Y, \mathcal{O}_Y) = H^i(\mathbb{P}V_a, (p_1)_*\mathcal{O}_Y) = \bigoplus_l H^i\left(\mathbb{P}V_a, \mathrm{Sym}^l\left(\frac{W \otimes \mathbb{C}^n}{L_2 \otimes \mathbb{C}^n}\right)\right).$$

Let us compute $H^i\left(\mathbb{P}V_a, \mathrm{Sym}^l\left(\frac{W\otimes\mathbb{C}^n}{L_2\otimes\mathbb{C}^n}\right)\right)$. We start with the Koszul resolution [9] of $\mathrm{Sym}^l\left(\frac{W\otimes\mathbb{C}^n}{L_2\otimes\mathbb{C}^n}\right)$:

$$\Lambda^l(L_2 \otimes \mathbb{C}^n) \to \Lambda^{l-1}(L_2 \otimes \mathbb{C}^n) \otimes \mathrm{Sym}^1(W \otimes \mathbb{C}^n) \to \dots$$

$$\dots \to \Lambda^{l-i}(L_2 \otimes \mathbb{C}^n) \otimes \mathrm{Sym}^i(W \otimes \mathbb{C}^n) \to \dots$$

$$\dots \to \Lambda^1(L_2) \otimes \mathrm{Sym}^{l-1}(W \otimes \mathbb{C}^n) \to \mathrm{Sym}^l(W \otimes \mathbb{C}^n) \to \mathrm{Sym}^l\left(\frac{W \otimes \mathbb{C}^n}{L_2 \otimes \mathbb{C}^n}\right)$$

We are interested in the case when $l$ is sufficiently large. Note that since $L_2$ is a line bundle, $\Lambda^i(L_2 \otimes \mathbb{C}^n)$ vanishes for $i > n$. Using these facts we can rewrite the resolution as follows.

**Resolution 1:**

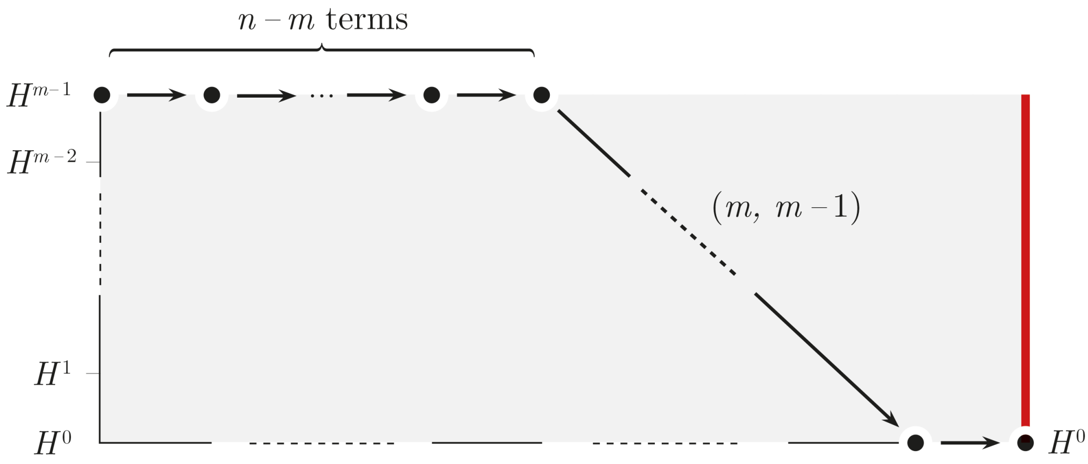
$$L_2^n \otimes \Lambda^n(\mathbb{C}^n) \to \dots \to L_2^{l-i} \otimes \Lambda^{l-i}(\mathbb{C}^n) \otimes \mathrm{Sym}^i(W \otimes \mathbb{C}^n) \to \dots$$

$$\dots \to \mathrm{Sym}^l(W \otimes \mathbb{C}^n) \to \mathrm{Sym}^l\left(\frac{W \otimes \mathbb{C}^n}{L_2 \otimes \mathbb{C}^n}\right)$$

According to the Borel-Weil-Bott theorem,

- $H^{m-1}(\mathbb{P}V_a,\ \mathcal{O}(-k)) \cong \mathrm{Sym}^{k-m} V_a \otimes \det V_a$ if $k - m \geq 0$,
- $H^{m-1}(\mathbb{P}V_a,\ \mathcal{O}(-k)) \cong 0$ if $k - m < 0$,
- $H^i(\mathbb{P}V_a,\ \mathcal{O}(-k)) \cong 0$ if $i \neq m - 1$.

This knowledge allows us to write down the Leray spectral sequence, which is a collection of indexed pages, i.e. tables with arrows pointing in the direction $(m, m-1)$ on the $m$-th page. The Leray spectral sequence allows us to obtain the cohomology groups of $\mathrm{Sym}^l\left(\frac{W\otimes\mathbb{C}^n}{L_2\otimes\mathbb{C}^k}\right)$ by computing successive approximations. On the first page of the Leray spectral sequence, to each sheaf in the resolution above corresponds a column of its cohomology groups:




According to Leray's theorem, the spectral sequence for the exact sequence converges to zero. The only term in the first column that can be cancelled by the other terms in the spectral sequence is the term in the 0-th line. This means that $H^i\left(\mathbb{P}V_a, \mathrm{Sym}^l\left(\frac{W\otimes\mathbb{C}^n}{L_2\otimes\mathbb{C}^n}\right)\right)$ vanishes for $i>0$.

Applying the pushforward $(p_2)_*$, we obtain

$$H^i(Y,\mathcal{O}_Y)=\bigoplus_l H^i\left(\mathbb{P}^{m-1}, H^0\left(\mathbb{P}V_a, \mathrm{Sym}^l\left(\frac{W\otimes\mathbb{C}^n}{L_2\otimes\mathbb{C}^n}\right)\right)\right).$$

Let us construct the resolution of $H^0\left(\mathrm{Sym}^l\left(\frac{W\otimes\mathbb{C}^n}{L_2\otimes\mathbb{C}^n}\right)\right)$. In the spectral sequence above, whatever remains in the line number $m-1$ after the first page goes exactly to $\mathrm{Sym}^l(W\otimes\mathbb{C}^n)$ in the line number 0 on the $m$-th page. This allows us to write down the following resolution:

$$\det V_a\otimes\mathrm{Sym}^{n-m}V_a\otimes\Lambda^n\mathbb{C}^n\otimes\mathrm{Sym}^{l-n}(W\otimes\mathbb{C}^n)\to\dots$$

$$\dots\to\det V_a\otimes\mathrm{Sym}^{n-m-i}V\otimes\Lambda^{n-i}\mathbb{C}^n\otimes\mathrm{Sym}^{l-(n-i)}(W\otimes\mathbb{C}^n)\to\dots$$

$$\dots\to\det V_a\otimes\Lambda^m\mathbb{C}^n\otimes\mathrm{Sym}^{l-m}(W\otimes\mathbb{C}^n)\to\mathrm{Sym}^l(W\otimes\mathbb{C}^n)\to H^0\left(\mathrm{Sym}^l\left(\frac{W\otimes\mathbb{C}^n}{L_2\otimes\mathbb{C}^n}\right)\right)$$

Which can be presented in the following form.

**Resolution 2:**

$$\det Q\otimes L_1^2\otimes\mathrm{Sym}^{n-m}(Q_1\oplus L_1^2)\otimes\Lambda^n\mathbb{C}^n\otimes\mathrm{Sym}^{l-n}((\mathrm{Sym}^2\mathbb{C}^m\oplus Q_1)\otimes\mathbb{C}^n)\to\dots$$

$$\dots\to\det Q\otimes L_1^2\otimes\mathrm{Sym}^{n-m-i}(Q_1\oplus L_1^2)\otimes\Lambda^{n-i}\mathbb{C}^n\otimes\mathrm{Sym}^{l-(n-i)}((\mathrm{Sym}^2\mathbb{C}^m\oplus Q_1)\otimes\mathbb{C}^n)\to\dots$$

$$\dots\to\det Q\otimes L_1^2\otimes\Lambda^m\mathbb{C}^n\otimes\mathrm{Sym}^{l-m}((\mathrm{Sym}^2\mathbb{C}^m\oplus Q_1)\otimes\mathbb{C}^n)\to$$

$$\to\mathrm{Sym}^l((\mathrm{Sym}^2\mathbb{C}^m\oplus Q_1)\otimes\mathbb{C}^n)\to H^0\left(\mathrm{Sym}^l\left(\frac{(\mathrm{Sym}^2\mathbb{C}^m\oplus Q_1)\otimes\mathbb{C}^n}{L_2\otimes\mathbb{C}^n}\right)\right)$$

This allows us to formulate our first result.

**Theorem 8.15.** $\widetilde{\Theta_{A_2}^{m,m}}$ *has rational singularities.*

*Proof.* If $m = n$ then Resolution 2 may be rewritten as follows:

$$\begin{aligned} \det Q \otimes L_1^2 \otimes \Lambda^n \mathbb{C}^n \otimes \operatorname{Sym}^{l-n}((\operatorname{Sym}^2 \mathbb{C}^n \oplus Q_1) \otimes \mathbb{C}^n) \to \\ \to \operatorname{Sym}^l((\operatorname{Sym}^2 \mathbb{C}^n \oplus Q_1) \otimes \mathbb{C}^n) \to \\ \to H^0\left(\operatorname{Sym}^l\left(\frac{(\operatorname{Sym}^2 \mathbb{C}^n \oplus Q_1) \otimes \mathbb{C}^n}{L_2 \otimes \mathbb{C}^n}\right)\right) \end{aligned} \qquad (\star)$$

We will prove that, in the corresponding spectral sequence, there are no non-trivial terms above the 0-th line.

**Lemma 8.16.**

$$\operatorname{Sym}^N((\operatorname{Sym}^2 \mathbb{C}^n \oplus Q_1) \otimes \mathbb{C}^n) =$$
$$= \bigoplus_{i=0}^{N}\left(\left(\operatorname{Sym}^{N-i}\left(\operatorname{Sym}^2 \mathbb{C}^n \otimes \mathbb{C}^n\right)\right) \otimes \bigoplus_{(i_1,\dots,i_n)}^{i_1+\dots+i_n=i} \operatorname{Sym}^{i_1} Q_1 \otimes \dots \otimes \operatorname{Sym}^{i_n} Q_1\right).$$

Setting $N = l$, the lemma provides the decomposition of $\operatorname{Sym}^l((\operatorname{Sym}^2 \mathbb{C}^n \oplus Q_1) \otimes \mathbb{C}^n)$. The only non-constant sheaves here are the sheaves of the form

$$\operatorname{Sym}^{i_1} Q_1 \otimes \dots \otimes \operatorname{Sym}^{i_n} Q_1.$$

We decompose this tensor product into a sum of irreducible representations:

$$\operatorname{Sym}^{i_1} Q_1 \otimes \dots \otimes \operatorname{Sym}^{i_k} Q_1 = \bigoplus_{\lambda} a_\lambda \Sigma^\lambda Q_1,$$

where $\lambda = (\lambda_1, \dots, \lambda_m)$, $\sum \lambda_k = \sum i_j$, and $a_\lambda$ are non-negative integers.

Since there is no multiplication by a power of $L_1$ and $\lambda$ is already dominant, i.e. strictly decreasing, by the Borel-Weil-Bott theorem $H^i(\mathbb{P}^{m-1}, \operatorname{Sym}^{i_1} Q_1 \otimes \dots \otimes \operatorname{Sym}^{i_n} Q_1) = 0$ for $i > 0$.

This proves that the term in the second line of the resolution $(\star)$ does not have any higher cohomology.

However, the term in the first line of the resolution $(\star)$ has $L_1^2$ as a multiplier. As before, we use the lemma above for $N = l - n$ to find the decomposition of this term. The non-trivial part in this case is the following:

$$\det Q_1 \otimes L_1^2 \otimes \bigoplus_{\lambda} a_\lambda \Sigma^\lambda Q_1 = \det \mathbb{C}^m \otimes L_1 \otimes \bigoplus_{\lambda} a_\lambda \Sigma^\lambda Q_1.$$

Let us apply the Borel-Weil-Bott theorem to $\Sigma^\lambda Q_1 \otimes L_1$ :

$$(\lambda_1, \dots, \lambda_{m-1}, 1) + (m, \dots, 1) = (\nu_1 + m, \dots, \nu_{m-1} + 2, 2).$$

Since $\nu_{m-1} \geq 0$, we either have a dominant sequence if $\nu_{m-1} > 0$, or a repetition if $\nu_{m-1} = 0$. In both cases there is no higher cohomology.

So, there are no non-trivial entries in the corresponding Leray spectral sequence above the 0-th line, so $H^i(Y, \mathcal{O}_Y) = 0$ for $i > 0$, and $\widetilde{\Theta^{m,m}_{A_1}}$ has rational singularities. $\square$

**Theorem 8.17.** $\widetilde{\Theta^{m,n}_{A_2}}$ *in general has singularities worse than rational.*

*Proof.* Consider the case $m = 5,\ n = 7,\ l = 7.$

We prove that $H^1\left(\mathbb{P}^4,\ \mathrm{Sym}^7\left(\frac{(\mathrm{Sym}^2\mathbb{C}^5 \oplus Q_1)\otimes\mathbb{C}^7}{L_2\otimes\mathbb{C}^7}\right)\right) \ncong 0$. In this particular case Resolution 2 is the following:

$$\det Q_1 \otimes L_1^2 \otimes \mathrm{Sym}^2(Q_1 \oplus L_1^2) \to$$
$$\to \det Q_1 \otimes L_1^2 \otimes (Q_1 \oplus L_1^2) \otimes \Lambda^6\mathbb{C}^7 \otimes ((\mathrm{Sym}^2\mathbb{C}^5 \oplus Q_1) \otimes \mathbb{C}^7) \to$$
$$\to \det Q_1 \otimes L_1^2 \otimes \Lambda^5\mathbb{C}^7 \otimes \mathrm{Sym}^2((\mathrm{Sym}^2\mathbb{C}^5 \oplus Q_1) \otimes \mathbb{C}^7) \to$$
$$\to \mathrm{Sym}^7((\mathrm{Sym}^2\mathbb{C}^5 \oplus Q_1) \otimes \mathbb{C}^7) \to$$
$$\to H^0\left(\mathrm{Sym}^7\left(\frac{(\mathrm{Sym}^2\mathbb{C}^5 \oplus Q_1) \otimes \mathbb{C}^7}{L_2 \otimes \mathbb{C}^7}\right)\right)$$

Consider the term in the first line of the resolution above.

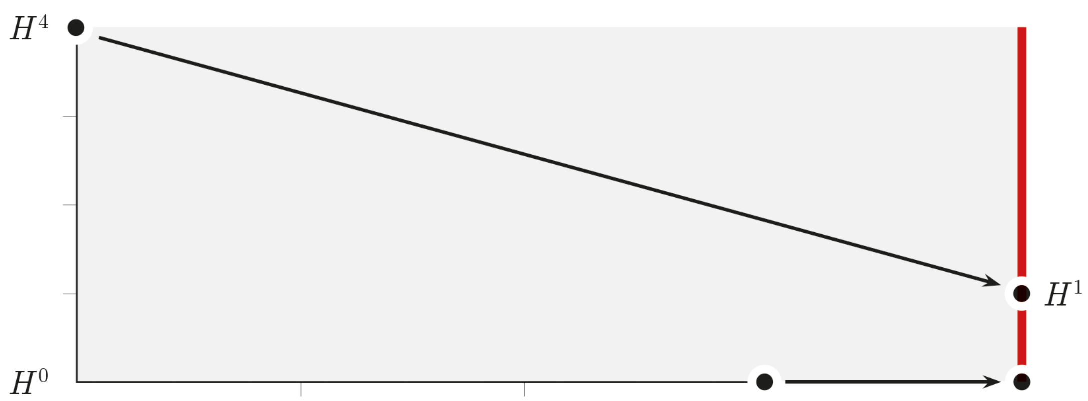
$$\det Q_1 \otimes L_1^2 \otimes \mathrm{Sym}^2(Q_1 \oplus L_1^2) = \det Q_1 \otimes L_1^2 \otimes \left(\mathrm{Sym}^2 Q_1 \oplus Q_1 \otimes L_1^2 \oplus L_1^4\right) =$$
$$= \det Q_1 \otimes L_1^6 \oplus \det Q_1 \otimes L_1^2 \left(\mathrm{Sym}^2 Q_1 \oplus Q_1 \otimes L_1^2\right).$$

Using the Borel-Weil-Bott theorem, one can easily check that

$$H^4\left(\mathbb{P}^4,\ \det Q_1 \otimes L_1^6\right) \ncong 0,$$
$$H^0(\mathbb{P}^4,\ \mathrm{Sym}^7((\mathrm{Sym}^2\mathbb{C}^5 \oplus Q_1) \otimes \mathbb{C}^7)) \ncong 0,$$

but all other terms of the resolution do not have any cohomology.

The corresponding Leray spectral sequence is the following:



Thus, we proved that

$$H^1\left(\mathbb{P}^4,\ \operatorname{Sym}^7\left(\frac{\left(\operatorname{Sym}^2 \mathbb{C}^5 \oplus Q_1\right) \otimes \mathbb{C}^7}{L_2 \otimes \mathbb{C}^7}\right)\right) \not\cong 0,$$

and therefore $\widetilde{\Theta_{A_2}^{5,7}}$ has singularities worse than rational. □

According to Boutot [10], the GIT quotient of a smooth variety with respect to a reductive group has rational singularities. Thus, we have the following corollary of the Theorem 8.17.

**Corollary 8.18.** $\Theta_{A_2}^{m,n}$ *can not be presented as a reductive quotient of a smooth variety.*

For the recent results on the GIT quotient with respect to non-reductive groups, see the works of Kirwan and Bérczi [7], and Bérczi, Doran, Hawes and Kirwan [6].

**Remark 8.19.** In both Theorem 8.15 and Theorem 8.17 we consider the normalizations of the $A_2$-loci. Let us show that the normalization is not redundant, i.e. that $\Theta_{A_2}^{m,n}$ is not always normal.

Let $V$ be a complex vector space equipped with the action of a compact Lie group $G$, and let $X$ be a closed $G$-invariant subvariety of $V$. Suppose $Y$ is a smooth $G$-equivariant resolution of $X$.

Consider the following diagram:

$$\begin{array}{ccc} & & H^0(Y, \mathcal{O}_Y) \\ & \overset{f}{\nearrow} & \uparrow h \\ H^0(V, \mathcal{O}_V) = \bigoplus_l \operatorname{Sym}^l V^* & & \\ & \underset{g}{\searrow} & \\ & & H^0(X, \mathcal{O}_x) \end{array}$$

We know that $g$ is always surjective, and, according to Proposition 8.2, $h$ is an isomorphism if and only if $X$ is normal. Now, if $f$ is not surjective, then $h$ can not be an isomorphism, and therefore in this case $X$ is not a normal variety.

Let $V = J_2^{m,n}$, $G = \mathrm{Gl}(m) \times \mathrm{Gl}(n)$, $X = \Theta_{A_2}^{m,n}$, and let $Y$ be the Kazarian's smooth equivariant resolution of $\Theta_{A_2}^{m,n}$.

Consider Resolution 2 in the general case:

$$\det Q \otimes L_1^2 \otimes \operatorname{Sym}^{n-m}(Q_1 \oplus L_1^2) \otimes \Lambda^n \mathbb{C}^n \otimes \operatorname{Sym}^{l-n}((\operatorname{Sym}^2 \mathbb{C}^m \oplus Q_1) \otimes \mathbb{C}^n) \to \dots$$
$$\dots \to \det Q \otimes L_1^2 \otimes \operatorname{Sym}^{n-m-i}(Q_1 \oplus L_1^2) \otimes \Lambda^{n-i} \mathbb{C}^n \otimes \operatorname{Sym}^{l-(n-i)}((\operatorname{Sym}^2 \mathbb{C}^m \oplus Q_1) \otimes \mathbb{C}^n) \to \dots$$
$$\dots \to \det Q \otimes L_1^2 \otimes \Lambda^m \mathbb{C}^n \otimes \operatorname{Sym}^{l-m}((\operatorname{Sym}^2 \mathbb{C}^m \oplus Q_1) \otimes \mathbb{C}^n) \to$$
$$\to \operatorname{Sym}^l((\operatorname{Sym}^2 \mathbb{C}^m \oplus Q_1) \otimes \mathbb{C}^n) \to H^0\left(\mathbb{P}^{m-1},\ \operatorname{Sym}^l\left(\frac{(\operatorname{Sym}^2 \mathbb{C}^m \oplus Q_1) \otimes \mathbb{C}^n}{L_2 \otimes \mathbb{C}^n}\right)\right).$$

Recall that

$$H^0(Y, \mathcal{O}_Y) = \bigoplus_l H^0\left(\mathbb{P}^{m-1},\ \operatorname{Sym}^l\left(\frac{(\operatorname{Sym}^2 \mathbb{C}^m \oplus Q_1) \otimes \mathbb{C}^n}{L_2 \otimes \mathbb{C}^n}\right)\right) \text{ and}$$

$$H^0(V, \mathcal{O}_V) = \bigoplus_l \operatorname{Sym}^l((\operatorname{Sym}^2 \mathbb{C}^m \oplus \mathbb{C}^m) \otimes \mathbb{C}^n) = \bigoplus_l H^0(\mathbb{P}^{m-1}, \operatorname{Sym}^l((\operatorname{Sym}^2 \mathbb{C}^m \oplus Q_1) \otimes \mathbb{C}^n)).$$

Since the map $f$ from the diagram above preserves the graded components, it is enough to prove that

$$f_l\colon\ \operatorname{Sym}^l((\operatorname{Sym}^2 \mathbb{C}^m \oplus \mathbb{C}^m) \otimes \mathbb{C}^n) \longrightarrow H^0\left(\mathbb{P}^{m-1},\ \operatorname{Sym}^l\left(\frac{(\operatorname{Sym}^2 \mathbb{C}^m \oplus Q_1) \otimes \mathbb{C}^n}{L_2 \otimes \mathbb{C}^n}\right)\right)$$

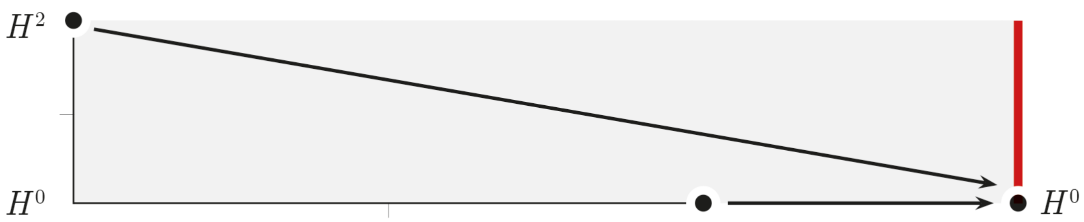
is not surjective for some fixed $l$.

Note that $f_l$ is the right arrow in the line $H^0$ of the first page of the Leray spectral sequence corresponding to Resolution 2. That is, if we can find an example of a spectral sequence with a non-horizontal arrow pointing to the term $H^0\left(\mathbb{P}^{m-1},\ \operatorname{Sym}^l\left(\frac{(\operatorname{Sym}^2 \mathbb{C}^m \oplus Q_1) \otimes \mathbb{C}^n}{L_2 \otimes \mathbb{C}^n}\right)\right)$, we prove that $f$ is not surjective.

Let $m = 3,\ n = 4,\ l = 4$. In this case Resolution 2 is the following:

$$\det Q \otimes L_1^2 \otimes (Q_1 \oplus L_1^2) \otimes \Lambda^4 \mathbb{C}^4 \to$$
$$\to \det Q \otimes L_1^2 \otimes \Lambda^3 \mathbb{C}^4 \otimes ((\operatorname{Sym}^2 \mathbb{C}^3 \oplus Q_1) \otimes \mathbb{C}^4) \to$$
$$\to \operatorname{Sym}^4((\operatorname{Sym}^2 \mathbb{C}^3 \oplus Q_1) \otimes \mathbb{C}^4) \to H^0\left(\mathbb{P}^2,\ \operatorname{Sym}^4\left(\frac{(\operatorname{Sym}^2 \mathbb{C}^3 \oplus Q_1) \otimes \mathbb{C}^4}{L_2 \otimes \mathbb{C}^4}\right)\right).$$

A straightforward computation using the Borel-Weil-Bott theorem shows that the corresponding Leray spectral sequence is the following.



We see that there is a non-horizontal arrow pointing to $H^0\left(\mathbb{P}^2,\ \mathrm{Sym}^4\left(\frac{(\mathrm{Sym}^2\mathbb{C}^3\oplus Q_1)\otimes\mathbb{C}^4}{L_2\otimes\mathbb{C}^4}\right)\right)$, thus $\Theta^{3,4}_{A_2}$ is not a normal variety.

**Remark 8.20.** Since the equivariant resolutions for the $A_3$-loci given in [5] and [22] are smooth, the computational methods presented in this paper may be used to check the rationality of the singularities of $\Theta^{m,n}_{A_3}$.

## 8.5 Kazarian's model for $A_d$ singularities

Let us recall the construction of Kazarian's resolution [22] for $A_d$ singularities (we have already seen this construction for the case of $A_2$ and $A_1$ singularities in the previous section).

As in the case of $A_2$ singularity, we construct the resolution of the locus $\Theta_A \subset J^{m,n}_d$ using the Hilbert scheme. Recall the following notations:

$$\mathrm{Hilb}_d(\mathbb{C}^m) = \{I \subset J^m_d \mid \dim(J^m_d/I) = d\},$$

$$\mathrm{Hilb}_{A_d}(\mathbb{C}^m) = \overline{\{I \subset J^m_d \mid J^m_d/I \cong A_d\}}.$$

As discussed before, $\mathrm{Hilb}_{A_d}(\mathbb{C}^m)$ is not smooth and not convenient for the future computations.

Let us fix a filtration on on a $d$-dimensional vector space $V$ :

$$V = V_0 \supset V_1 \supset \cdots \supset V_d = 0,\ \dim V_i/V_{i+1} = 1.$$

We may define the Hilbert scheme remembering the filtration:

$$\widetilde{\mathrm{Hilb}}_{A_d}(\mathbb{C}^m) = \{(I, \Delta) \mid (J^m_d/I)^\Delta \cong A_d\}.$$

It is clear that there exists a birational map

$$f\colon \widetilde{\mathrm{Hilb}}_{A_d}(\mathbb{C}^m) \to \mathrm{Hilb}_{A_d}(\mathbb{C}^m).$$

In the general case, Kazarian's resolution [22] is a smooth compact variety $M_d$ defined as the moduli space of the following flags. Take $V$ – a $d$-dimensional vector space with the filtration defined above, together with a surjective linear map $V \twoheadleftarrow (\mathbb{C}^m)^* \oplus \mathrm{Sym}^2 V$ such that

$$W_i = V/V_i \twoheadleftarrow (\mathbb{C}^m)^* \oplus S_i,\ i = 1 \ldots d,$$

where $S_i \twoheadleftarrow \mathrm{Sym}^2(W_i) \twoheadleftarrow \mathrm{Sym}^2 V$ is generated by $W_k \otimes W_j$ for $k + j \le i$.

The variety $M_d$ can be constructed by induction. For $d = 1$ we have $S_1 = 0$, $W_1 \twoheadleftarrow (\mathbb{C}^m)^*$ and $M_1 = \mathrm{Gr}(1, m) = \mathbb{P}^{m-1}$ together with the tautological line bundle $L_1$.

Suppose we have constructed $M_{d-1}$ with the sequence of maps

$$W_1 \twoheadleftarrow W_2 \twoheadleftarrow \cdots \twoheadleftarrow W_{d-1}$$

and the tautological bundles $L_i$ over $M_i$ and with the surjective linear map $W_{d-1} \twoheadleftarrow (\mathbb{C}^m)^* \oplus S_{d-1}$. Since $S_d$ is determined by $W_1, \ldots, W_{d-1}$, it can also be interpreted as a bundle over $M_{d-1}$.

$M_d$ parametrizes subspaces $W_d \twoheadleftarrow (\mathbb{C}^m)^* \oplus S_d$ such that $W_{d-1} \twoheadleftarrow W_d$, so let us define $M_d$ as the bundle over $M_{d-1}$ :

$$M_d = \mathbb{P}((\mathbb{C}^m)^* \oplus S_d)/W_{d-1}).$$

The construction of the manifold $M_r$ can be presented as the following diagram:

$$M_d \xrightarrow{\mathbb{P}(E_d/W_{d-1})} M_{d-1} \xrightarrow{\mathbb{P}(E_{d-1}/W_{d-2})} \cdots \longrightarrow \ldots \xrightarrow{\mathbb{P}((\mathbb{C}^m)^*)} pt,$$

where $E_i = (\mathbb{C}^m)^* \oplus S_i$.

**Proposition 8.21.** [22] *$M_d$ is smooth and compact.*

The manifold $M_d$ is defined together with the projection $V \twoheadleftarrow (\mathbb{C}^m)^* \oplus S_d$. The restriction $V \twoheadleftarrow (\mathbb{C}^m)^*$ gives the linear map and $V \twoheadleftarrow S_d$ defines the filtered commutative algebra structure on $V$. The dual picture determines a filtered commutative coalgebra structure.

Let us summarize the previous discussion in the form of a diagram:

$$\begin{array}{ccccccc}
\mathrm{Hom}(\mathrm{Sym}^2 V, V) \supset \mathcal{R}_d & & & & & & \\
\downarrow & & & & & & \\
M_d & \longrightarrow & M_{d-1} & \longrightarrow & \cdots & \longrightarrow & pt \\
\uparrow & & & & & & \\
\widetilde{\mathrm{Hilb}}_{A_d}(\mathbb{C}^m) & \longrightarrow & \mathrm{Hilb}_{A_d}(\mathbb{C}^m) & \hookrightarrow & \mathrm{Hilb}_d(\mathbb{C}^m) & &
\end{array}$$

**Lemma 8.22.** [22] *Suppose $\gamma$ is a generic section of* $\mathrm{Hom}(\mathrm{Sym}^2 V, V) \to M_d$. *Then* $\widetilde{\mathrm{Hilb}}_{A_d}(\mathbb{C}^m) = \gamma^{-1}(\mathcal{R}_d)$

**Example 8.2.** For $d = 1$ we have already shown that $M_1 = \mathbb{P}^{m-1}$. Let us denote the tautological sequence over $M_1$ by $\mathcal{O}(-1) = L_1 \to \mathbb{C}^m \to Q_1$.

For $d = 2$, $S_2 = W_1 \otimes W_1$, so $M_2 = \mathbb{P}(((\mathbb{C}^m)^* \oplus S_2)/W_1) = \mathbb{P}(Q_1 \oplus L_1^2)$. Let us denote the bundles from the tautological sequence over $M_2$ by $Q_2$ and $L_2$.

For $d = 3$, $S_3 = W_1 \otimes W_1 \oplus W_1 \otimes W_2$, so $M_3 = \mathbb{P}(((\mathbb{C}^m)^* \oplus S_3)/W_2) = \mathbb{P}(Q_2 \oplus (L_1 \otimes L_2))$.

In the general case,

$$S_d = \bigoplus_{i+j \leq d} W_i \otimes W_j, \text{ and } M_d = \mathbb{P}\left(Q_{d-1} \oplus \left(\bigoplus_{i=1}^{\left[\frac{d-1}{2}\right]} L_i \otimes L_{d-i}\right)\right).$$

**Remark 8.23.** Starting from $d = 4$ there will be points in $M_d$ such that the canonical commutative filtered algebra structure defined by $W_d \twoheadleftarrow (\mathbb{C}^m)^* \oplus S_d$ in the corresponding fiber is not associative. Moreover, the bundle $\mathrm{Hom}(\mathbb{C}^n, I)$ from Kazarian's resolution is not defined over $M_d$ for $d \geq 4$, since the definition of this bundle requires a choice of the map on the right:

$$I \to \bigoplus_{i=1}^{d} \mathrm{Sym}^i(\mathbb{C}^m) \to A,$$

which is not unique for $d \geq 4$. However, this vector bundle is defined over the sublocus where the canonical algebra structure in the fiber is associative.

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