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The quantum-state diffusion model applied to open systems

Nicolas Gisin† and Ian C Percival‡

† Group of Applied Physics, University of Geneva, 1211 Geneva 4, Switzerland
‡ Blackett Laboratory, Imperial College, Prince Consort Road, London SW7 2BZ, UK, and School of Mathematical Sciences, Queen Mary and Westfield College, University of London, Mile End Road, London E1 4NS, UK

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Abstract. A model of a quantum system interacting with its environment is proposed in which the system is represented by a state vector that satisfies a stochastic differential equation, derived from a density operator equation such as the Bloch equation, and consistent with it. The advantages of the numerical solution of these equations over the direct numerical solution of the density operator equations are described. The method is applied to the nonlinear absorber, cascades of quantum transitions, second-harmonic generation and a measurement reduction process. The model provides graphic illustrations of these processes, with statistical fluctuations that mimic those of experiments. The stochastic differential equations originated from studies of the measurement problem in the foundations of quantum mechanics. The model is compared with the quantum-jump model of Dalibard, Carmichael and others, which orginated among experimenters looking for intuitive pictures and rules of computation.

1. Introduction

In his paper on the A- and B-coefficients [1], Einstein assumed that an individual quantum system like an atom was capable of a transition or jump from one state to another with the absorption or emission of radiant energy. Although Einstein's paper stimulated the development of modern quantum mechanics, such jumps have no formal place in that theory, for which the state vector represents the properties of an ensemble of systems and not an individual system.

Despite the success of the modern theory, many physicists, particularly experimenters, have insisted on treating quantum jumps of individual systems as real, and the state vector as representing the behaviour of an individual system, as exemplified by a single run of a laboratory experiment (quantum optics provides many examples, see for instance [2]). The experimenters' picture has given them valuable physical insights [3], which have sometimes escaped the theoreticians with their relatively elaborate mathematical tools based on density operator evolution. For example Itano and co-workers based their analysis of their 'Zeno paradox' experiment on quantum jumps [4]. These quantum-jump methods give the same results as the usual theory based on density operator equations, as shown for example in [5].

Independently of these developments, and stimulated by the fundamental objections of Schrödinger [6] and particularly Einstein [7] to the usual formulations of quantum mechanics, Bohm [8], Bell [9–11] and others have insisted on the importance of alternative formulations. In particular Bohm and Bub, Pearle, Gisin, Diósi,

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and Ghirardi, Rimini and Weber [12–20] have proposed alternative quantum theories in which the state vector represents an individual sysem and follows a stochastic dynamics, such as a diffusion equation. But none of them has been used to advantage in solving specific quantum problems. In [30] it was suggested that they could.

In this paper it is shown that diffusion equations for the state vector *can* be used effectively for the analysis of a quantum system in interaction with its environment, and demonstrate that they have advantages over the usual numerical solution of the Bloch density operator equations for a wide range of problems, including the nonlinear absorber and second harmonic generation. The continuous diffusion leads naturally to sudden transitions between discrete quantum states, resembling quantum jumps, with a finite but short jumping time, determined by the equations themselves. The measurement of a quantum system becomes a simple example of the interaction of the measured system with its environment, and when treated like any other such interaction, leads to an irreversible diffusion towards one of the eigenspaces, or eigenstates, in the case of a non-degenerate spectrum.

Gisin [16, 17] and Diósi [18] have shown how to obtain diffusion or Fokker–Planck equations for the state vector from a density operator evolution equation. Percival [21] (see also Diósi [18] and Gisin and Cibils [22]) provided a natural symmetry condition under which this diffusion equation is unique, apart from a non-physical phase factor. This diffusion equation is presented here as an Itô stochastic equation for a Wiener process, as described in detail in section 2.

Given a Bloch equation for the density operator ρ

$$\dot{\rho} = -\frac{\mathrm{i}}{\hbar}[H,\rho] + \sum_m (2L_m\rho L_m^\dagger - L_m^\dagger L_m\rho - \rho L_m^\dagger L_m) \tag{1.1}$$

the differential form of the stochastic equation of motion for the state vector $|\psi\rangle$ is

$$|\mathrm{d}\psi\rangle = -\frac{\mathrm{i}}{\hbar}H|\psi\rangle\mathrm{d}t + \sum_m (2\langle L_m^\dagger\rangle_\psi L_m - L_m^\dagger L_m - \langle L_m^\dagger\rangle_\psi\langle L_m\rangle_\psi)|\psi\rangle\mathrm{d}t$$
$$+ \sum_m (L_m - \langle L_m\rangle_\psi)|\psi\rangle\mathrm{d}\xi_m \tag{1.2}$$

where $\langle L_m\rangle_\psi = \langle\psi|L_m|\psi\rangle$ is the expectation of L_m for state $|\psi\rangle$. The first sum represents the 'drift' of the state vector and the second sum the random fluctuations, due to the interaction of the system with its environment. The $\mathrm{d}\xi_m$ are independent complex differential random variables representing a complex normalized Wiener process, and they satisfy

$$\begin{aligned} &\mathrm{M}(\mathrm{Re}(\mathrm{d}\xi_n)\,\mathrm{Re}(\mathrm{d}\xi_m)) = \mathrm{M}(\mathrm{Im}(\mathrm{d}\xi_n)\mathrm{Im}(\mathrm{d}\xi_m)) = \delta_{nm}\mathrm{d}t \\ &\mathrm{M}\mathrm{d}\xi_m = 0 \qquad \mathrm{M}(\mathrm{Re}\,\mathrm{d}\xi_n\mathrm{Im}\mathrm{d}\xi_m) = 0 \end{aligned} \tag{1.3}$$

where M represents the mean over the probability distribution.

Normally the elegant Itô and Stratonovich notations are used for such processes, in which the mean is implied without being written. Our analysis is equivalent to the Itô form, in which differentials up to the second power must be evaluated, but we write the means explicity to assist those who are unfamiliar with this notation.

Numerical solutions of such equations can be effective and efficient. The method used in the applications is described in section 3, several of these applications are

in section 4, and section 5 contains an account of the measurement of a dynamical variable as an example of the diffusion of quantum states through interaction with the measuring apparatus.

Dalibard *et al* [23], and also Carmichael [24] and Teich and Mahler [25] have independently derived an efficient method of computing the development of an open system based on stochastic evolution of the state vector, in which instantaneous quantum jumps are treated as if they were real, as mentioned at the beginning of the introduction. There are many similarities and differences between their method and ours, which are discussed in section 6, which also contains comparisons with the usual methods of numerical solution of the Bloch density operator equations and recommendations for use.

2. Evolution of density operators and quantum states

This section shows how the quantum state diffusion equation of (1.2), of the type introduced in [16–18, 20], is derived from a density operator equation (1.1).

A density operator ρ for a system with an N-dimensional state space can be expressed in many ways as a mean M over a distribution of normalized pure-state projection operators

$$\rho = \mathrm{M}|\psi\rangle\langle\psi| . \tag{2.1}$$

We seek differential equations for $|\psi\rangle$ so that

$$\dot{\rho} = \mathrm{d}\rho/\mathrm{d}t \tag{2.2}$$

is determined by a given diffential equation, such as the Bloch equation.

If an open system with density operator ρ starts at time $t = 0$ in a pure state and evolves into a mixed state as a result of the interaction, there can be no general deterministic equation for the pure states $|\psi\rangle$. But there are stochastic equations, as might be expected from the probabilistic nature of the interaction with the environment. In time $\mathrm{d}t$ the variation $|\mathrm{d}\psi\rangle$ in $|\psi\rangle$ is then given by the Itô form

$$|\mathrm{d}\psi\rangle = |v\rangle\mathrm{d}t + \sum_j |u_j\rangle\mathrm{d}\xi_j \tag{2.3}$$

where $|v\rangle\mathrm{d}t$ is the drift term and the differential stochastic fluctuations are represented by a sum over independent Wiener process $\mathrm{d}\xi_j$, which are complex, because this is quantum mechanics, with equal and independent fluctuations in their real and imaginary parts.

This complex form has simple invariance properties under unitary transformations [18, 21, 22]. As usual the mean and variance of each fluctuation are zero and $\mathrm{d}t^{1/2}$, so squares of differential stochastic fluctuations are significant, and from (1.3)

$$\mathrm{M}\mathrm{d}\xi_j = 0 \qquad \mathrm{M}\mathrm{d}\xi_j\mathrm{d}\xi_k = 0 \qquad \mathrm{M}\mathrm{d}\xi_j^*\mathrm{d}\xi_k = 2\delta_{jk}\mathrm{d}t \tag{2.4}$$

where M is used to represent a mean over both the distribution and the fluctuations due to the stochastic process.

To preserve the normalization of the state vector the fluctuations in the state must be orthogonal to that state, so

$$\langle\psi|u_j\rangle = 0\,. \tag{2.5}$$

Taking means over $|\mathrm{d}\psi\rangle$ and $|\mathrm{d}\psi\rangle\langle\mathrm{d}\psi|$,

$$\mathrm{M}|\mathrm{d}\psi\rangle = |v\rangle\mathrm{d}t \qquad \mathrm{M}|\mathrm{d}\psi\rangle\langle\mathrm{d}\psi| = 2\sum_j |u_j\rangle\langle u_j|\mathrm{d}t\,. \tag{2.6}$$

The values of the drift and stochastic terms can be obtained from $\dot\rho$ when the initial density operator projects onto a pure state:

$$\rho_\psi = |\psi\rangle\langle\psi|\,. \tag{2.7}$$

The general form for arbitrary initial ρ then follows from the linearity of the density operator equations, such as the Bloch equation. Note that the stochastic equations for the states are nonlinear. Using (2.6), the change in the ρ is given by

$$\mathrm{d}\rho = \mathrm{M}(|\psi\rangle\langle\mathrm{d}\psi| + |\mathrm{d}\psi\rangle\langle\psi| + |\mathrm{d}\psi\rangle\langle\mathrm{d}\psi|) \qquad \dot\rho = \left(|\psi\rangle\langle v| + |v\rangle\langle\psi| + 2\sum_j |u_j\rangle\langle u_j|\right). \tag{2.8}$$

The stochastic terms are determined by the component of $\dot\rho$ in the space orthogonal to ψ:

$$(I - \rho_\psi)\dot\rho(I - \rho_\psi) = 2\sum_j |u_j\rangle\langle u_j|\,. \tag{2.9}$$

Note that although the $|u_j\rangle$ are not uniquely determined by (2.9), the operator given by the sum over their projectors *is* uniquely determined, and this is enough to determine the diffusion process uniquely, since the first and second moments of $|\mathrm{d}\psi\rangle$ given by (2.3) are the same for any $|u_j\rangle$ satisfying (2.9), and all other moments can be neglected for an Itô process.

The drift is given by

$$\dot\rho|\psi\rangle = |\psi\rangle\langle v|\psi\rangle + |v\rangle \tag{2.10}$$

$$\mathrm{Re}\langle\psi|v\rangle = \tfrac{1}{2}\langle\psi|\dot\rho|\psi\rangle \tag{2.11}$$

$$|v\rangle = \dot\rho|\psi\rangle - \left(\tfrac{1}{2}\langle\psi|\dot\rho|\psi\rangle + ic\right)|\psi\rangle \tag{2.12}$$

where ic is a non-physical imaginary phase change constant, which is determined, not by the density operator equations, but by convention, to agree with the usual Schrödinger equation in the absence of interaction with the environment.

The above theory and the results (2.9) and (2.12) apply to any linear differential equations for the density operator that are of first order in time. Now consider the Bloch equations in the Lindblad [26] form

$$\dot\rho = -\frac{\mathrm{i}}{\hbar}[H,\rho] + \sum_m (2L_m\rho L_m^\dagger - L_m^\dagger L_m\rho - \rho L_m^\dagger L_m)\,. \tag{2.13}$$

The stochastic terms are given by

$$2\sum_j |u_j\rangle\langle u_j| = \sum_m 2(I - |\psi\rangle\langle\psi|)L_m|\psi\rangle\langle\psi|L_m^\dagger(I - |\psi\rangle\langle\psi|) \tag{2.14}$$

which is satisfied by the vectors

$$|u_m\rangle = (L_m - \langle L_m\rangle_\psi)|\psi\rangle\,. \tag{2.15}$$

From (2.12) and (2.13), the drift term is

$$|v\rangle = -\frac{\mathrm{i}}{\hbar}H|\psi\rangle + \sum_m (2\langle L_m^\dagger\rangle_\psi L_m - L_m^\dagger L_m - \langle L_m^\dagger\rangle_\psi\langle L_m\rangle_\psi)\,|\psi\rangle \tag{2.16}$$

where the constant c is chosen to give the simplest form. (1.2) follows on substituting (2.15) and (2.16) into (2.3).

3. Method of numerical solution

For the numerical solutions, we used an unnormalized state vector $|\phi(t)\rangle$, which satisfies the simpler equations

$$|\mathrm{d}\phi\rangle = -\frac{\mathrm{i}}{\hbar}H|\phi\rangle\mathrm{d}t + \sum_m (2\langle L_m^\dagger\rangle_\phi L_m - L_m^\dagger L_m)\,|\phi\rangle\mathrm{d}t + \sum_m L_m|\phi\rangle\mathrm{d}\xi_m \tag{3.1}$$

obtained by omitting terms in the direction of the state vector $|\psi\rangle$ from the right-hand side of (1.2). This is justified for complex fluctuations that satisfy $\mathrm{M}\,\mathrm{d}\xi_j\mathrm{d}\xi_k = 0$ in (2.4).

The expectation value of an operator $\langle A\rangle$ is then

$$\langle A\rangle_\phi = \langle\phi|A|\phi\rangle/\langle\phi|\phi\rangle\,. \tag{3.2}$$

The norm of $|\phi\rangle$ was adjusted occasionally to keep its magnitude within the range of the computer.

The practical method is based on a finite sample of s_m pure states, in which the density operator is approximated by a finite sum over projection operators onto these states:

$$\rho \approx \sum_s \frac{1}{s_m}\rho_s = \sum_s \frac{1}{s_m}|\phi_s\rangle\langle\phi_s|/\langle\phi_s|\phi_s\rangle\,. \tag{3.3}$$

For each state of the sample the state-differential diffusion equation (3.1) was iterated as a finite-difference equation. The stochastic term was added directly; the Hamiltonian and drift terms were integrated using a Runge–Kutta method with overall step length $\mathrm{d}t$. At chosen intervals, the mean values of any 'output' operators required for the output were computed.

General-purpose programs have been written in Pascal and FORTRAN for the solution of any density operator equations of the form (1.1) in which the Hamiltonian,

Lindblad and ouput operators are expressed as finite sums over finite products of creation and annihilation operators with one or two degrees of freedom. The examples presented here were executed on a notebook computer.

As in the recent quantum-jump method of Dalibard, Carmichael and others [23–25], one advantage of the stochastic differential equation method lies in the fact that for a state space of dimension N, a state vector needs N complex numbers to define it, whereas the density operator requires the equivalent of $N^2/2$.

A disadvantage is that, as in a laboratory experiment, the results are given by statistical distributions, with errors that depend on the size of the sample. But this similarity between stochastic theory and stochastic experiment is obviously an advantage too; it provides a graphic illustration of the processes that take place, and thus insight into them.

For speed of computation the advantages of the method outweigh the disadvantage when the sample size is smaller than approximately twice the complex dimension of the state vector. We have found in practice that quite good results are often obtained with small sample sizes, as shown in the next section. The computer memory required by the stochastic method is less, by a factor $C_0 N$, than with direct integration of the Bloch equations, where C_0 is a constant, and this allows the stochastic-diffusion method to be used where the direct-integration method cannot.

Since the theory uses the well developed Itô stochastic calculus, and the solutions are continuous in time, any of the existing tools [27] for numerical integration of such stochastic equations could be used to improve the method of integration.

4. Examples

Figure 1 illustrates the simplest example of the forced and damped linear oscillator in interaction representation, for which the Hamiltonian and Lindblad operators of (1.2) are

$$H = 2\mathrm{i}(a^\dagger - a) \qquad L = a \tag{4.1}$$

and the initial state is $|\psi_0\rangle = |8\rangle$. The solutions of the Bloch equations are well known, but the method provides a graphic illustration for each run of the reduction towards a coherent state, which evolves like the classical motion and has no stochastic fluctuations. This is an example in the quantum domain of localization in phase space, which is characteristic of our everyday experience in the classical domain.

The next example is the nonlinear absorber, with

$$H = 0.1\mathrm{i}(a^\dagger - a) \qquad L = a^2 . \tag{4.2}$$

The system starts in the ground state, and is weakly excited on resonance, as shown by the interaction representation hamiltonian. This is chosen because it is a tractable example, which is a good test of the method because the stochastic fluctuation is large in comparison with the hamiltonian and nonlinear dissipation. It has also been analysed with the same parameters using the positive P-representation, which is an extension of the Wigner function representation, and for which there are runaway problems [28].

For a single run, figure 2 shows the strong oscillations in the expectation value of the mean photon number with time, figure 3 shows the distribution over photon number in time and figure 4 over the sample at a given time, indicating the equivalence of

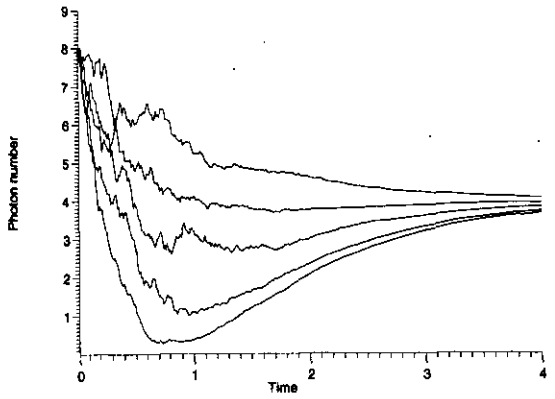

Figure 1. The forced damped linear oscillator, showing reduction towards a coherent state which has no stochastic fluctuations.

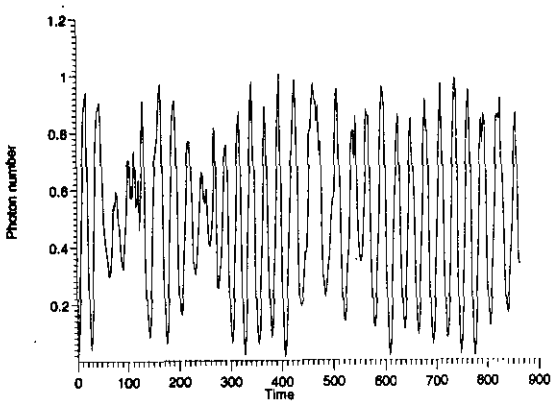

Figure 2. Oscillations in the expectation value of the mean photon number $\langle a^\dagger a \rangle$ for a single run of the nonlinear absorber.

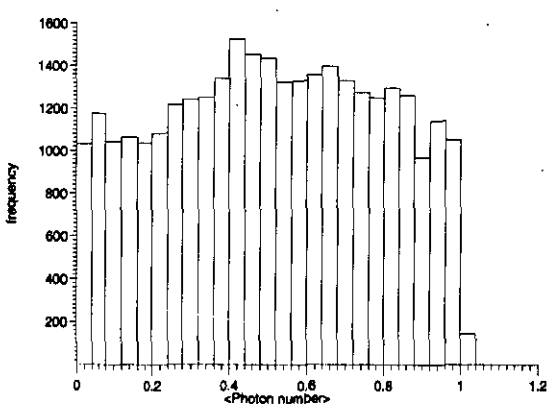

Figure 3. Distribution of mean photon number for 31 000 times t uniformly distributed in t for a single run of the nonlinear absorber. Note the nearly uniform spread over the range from 0 to 1.

the two for sufficiently long times. Figure 5 shows the mean over 100 samples, compared with a numerical solution using the direct integration of the density operator equations.

The next example is a quantum cascade with emission only, also in interaction

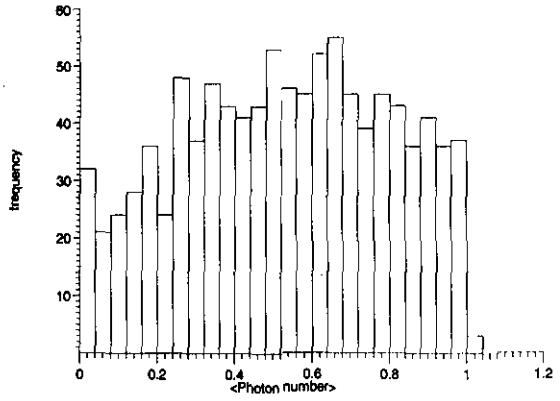

Figure 4. Distribution of photon number for 1000 different runs at time $t = 200$. Note the similarity to the previous distribution.

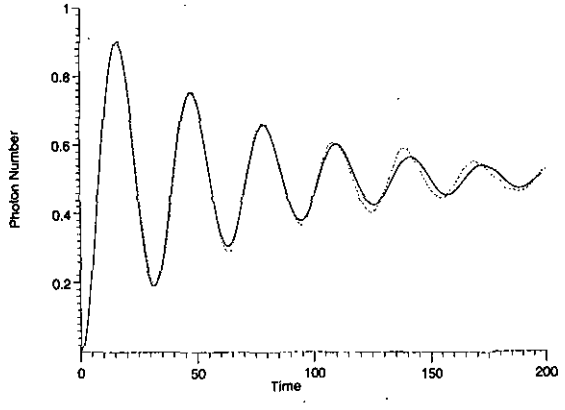

Figure 5. Density-operator (full curve) and stochastic-diffusion (broken curve) methods compared for the nonlinear absorber. The time increment is $dt = 0.02$ and 100 samples are used for approximating the average.

representation. The operator L_1 represents the interaction required to measure the system in a given state, whereas L_2 represents the damping due to photon emission:

$$H = 0 \qquad L_1 = 6a^\dagger a \qquad L_2 = 0.1a\,. \tag{4.3}$$

Figure 6 shows that the continuous-state diffusion automatically produces sudden transitions between states in a single run. Clearly these are not instantaneous jumps, but their sharpness depends on the nature of the interaction of the system with its environment, through L_1. Their frequency is determined by L_2.

Figure 7 illustrates a similar example in which there are only two states, but there is also absorption and stimulated emission, so the operators are given by

$$H = 0.1(a^\dagger + a) \qquad L_1 = a^\dagger a \qquad L_2 = 0.1a\,. \tag{4.4}$$

The final example of this section is the representation of a system for second-harmonic generation, a realistic problem of two degrees of freedom with a very simple nonlinear interaction. In the following equations the first term in the Hamiltonian is the forcing term on one oscillator, and the second term is the nonlinear interaction. The operators L_1 and L_2 provide linear damping for each of the oscillators:

$$H = 20\mathrm{i}(a_1^\dagger - a_1) + 0.2\mathrm{i}(a_1^{\dagger 2}a_2 - a_1^2 a_2^\dagger) \qquad L_1 = a_1 \qquad L_2 = a_2\,. \tag{4.5}$$

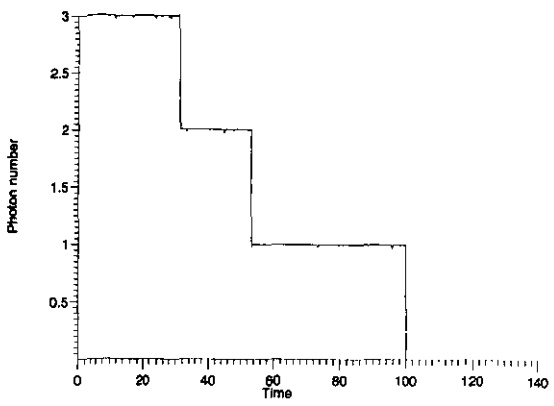

Figure 6. A quantum cascade with emission only. The continuous-state diffusion automatically produces sudden transitions between quantum states in a single run, but these are not instantaneous jumps.

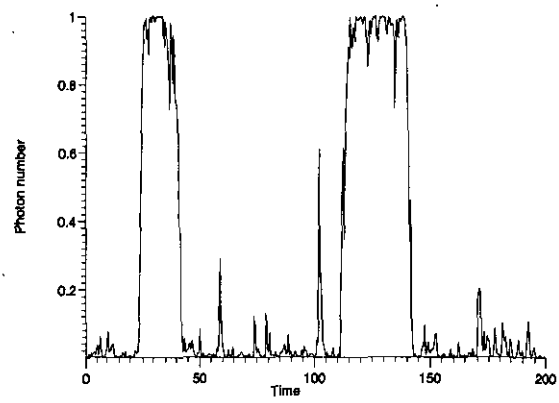

Figure 7. Transitions between a pair of states with absorption and stimulated emission.

In practice the quantum numbers are much higher than in the previous examples. For our case the amplitudes only became insignificant for $n_1 = 140$ and $n_2 = 80$, so that if the density operator equation (1.1) had been directly integrated, then about $140^2 \times 80^2 \approx 1.25 \times 10^8$ real numbers would have been needed for storage and computation at each time t.

It would appear that for the state-diffusion method the state vector would need $2 \times 140 \times 80$ real numbers to represent it, but in fact the state vector at a given time on an individual run does not span the full space, and it was possible to follow the mean photon number with a moving photon number representation that never included more than 55 states for the first ocillator and 40 for the second, giving 4400 real numbers to represent a state, as against about 5 million for the density operator.

Figure 8 shows the results of a single run, with the mean photon number of each oscillator, and the variance

$$\sigma(a^\dagger a) = (\langle (a^\dagger a)^2 \rangle - \langle a^\dagger a \rangle^2)^{1/2} \tag{4.6}$$

of the expectation value, as a function of time. The fluctuations in this example were relatively small, so 20 runs already provided a good approximation, as shown in figure 9.

This can be compared with the positive P-representation calculations of Dörfle and Schenzle [29] and of Schack and Schenzle [28].

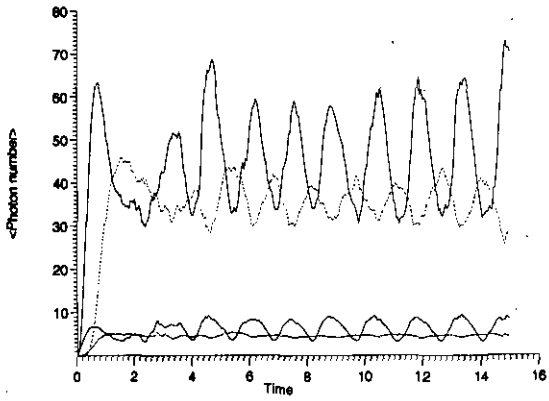

Figure 8. A single run of the second-harmonic generation system represented by the operators of (4.5). The mean photon number is plotted against time for first oscillator, represented by the upper full curve, and the second oscillator, represented by the upper broken curve. The lower curves represent the variance in each case.

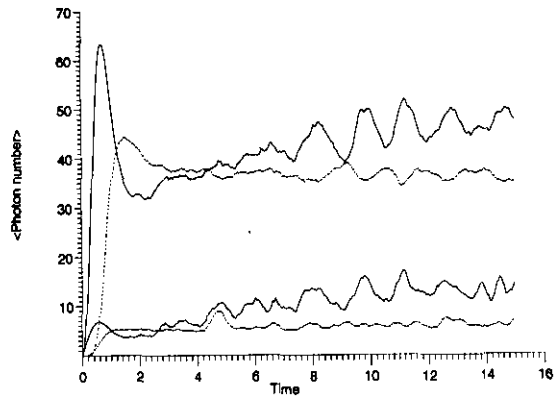

Figure 9. A mean over 20 runs for second-harmonic generation, with the same meaning for each curve. The variance shown is a mean over the variances for each run, so that it can be compared directly with figure 8.

5. Reduction to an eigenstate by a measurement

In the usual picture, quantum measurement is represented by the projection of the state $|\psi\rangle$ onto an eigenstate of the measured dynamical variable with Hermitian operator A. This satisfies neither the exact Schrödinger dynamics of isolated quantum systems nor the approximate Schrödinger dynamics used for systems in interaction with their environment.

In stochastic reduction theories measurement is just a simple example of the interaction of a system with its environment, and is treated like any other such interaction. The standard-state vector equation (1.2) is used, with zero Hamiltonian and a single Lindblad operator A to represent the interaction.

The following analysis shows that the change in mean square deviation is a negative continuous function of $|\psi\rangle$, except at the eigenstates, where it is zero, as shown in [17]. The state vectors of the ensemble thus diffuse towards the eigenstates, a process similar to Pearle's model of 'gambler's ruin' [13].

The mean value is

$$a = \langle\psi|A|\psi\rangle \tag{5.1}$$

and the mean square deviation is

$$\mathrm{M}(\langle A^2\rangle - a^2) \tag{5.2}$$

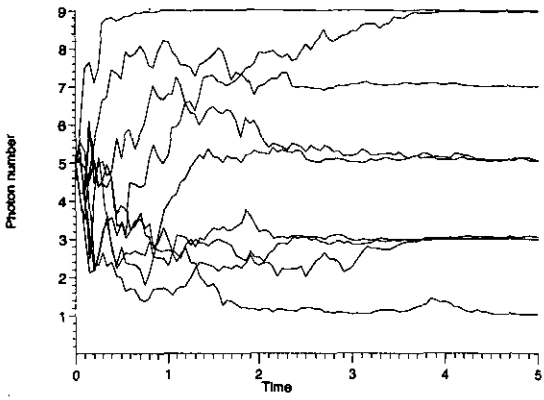

Figure 10. Mean photon number as a function of time for the measurement process. The stochastic convergence to the eigenstates can be clearly seen.

where all expectations $\langle\cdot\rangle$ in this section are for the initial state $|\psi\rangle$.

From (1.2), because A is Hermitian,

$$|\mathrm{d}\psi\rangle = -(A-a)^2|\psi\rangle\mathrm{d}t + (A-a)|\psi\rangle\mathrm{d}w \tag{5.3}$$

so the change in the expectation of A in time $\mathrm{d}t$ is

$$\begin{aligned}\mathrm{d}a &= \langle\mathrm{d}\psi|A|\psi\rangle + \langle\psi|A|\mathrm{d}\psi\rangle + \langle\mathrm{d}\psi|A|\mathrm{d}\psi\rangle \\ &= -2\langle\psi|(A-a)^2A|\psi\rangle\mathrm{d}t + 2\langle\psi|(A-a)A|\psi\rangle\mathrm{d}w_{\mathrm{R}} + 2\langle\psi|(A-a)^2A|\psi\rangle\mathrm{d}t \\ &= 2(\langle A^2\rangle - a^2)\mathrm{d}w_{\mathrm{R}}\end{aligned} \tag{5.4}$$

and consequently

$$\mathrm{M}\mathrm{d}a^2 = \mathrm{M}(2a\mathrm{d}a + \mathrm{d}a\mathrm{d}a) = 4(\langle A^2\rangle - a^2)^2\mathrm{d}t \tag{5.5}.$$

Since measurement does not affect the mean of the expectation of any commuting operator, we have

$$\mathrm{M}\,\mathrm{d}\langle A^2\rangle = 0 \tag{5.6}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{M}(\langle A^2\rangle - a^2) = -4(\langle A^2\rangle - a^2)^2 \leqslant 0 \tag{5.7}$$

which is negative.

The example of the measurement of the photon number of an oscillator is illustrated in figure 10. The Hamiltonian and Lindblad operators are

$$H = 0 \qquad L = 0.4a^{\dagger}a \tag{5.7}$$

and the intial state had the same amplitude for every odd level between 1 and 9 inclusive for each of nine samples. The stochastic attraction towards the eigenstates can be clearly seen. In the laboratory this attraction occurs at an early stage in the interaction of the system with the measuring apparatus.

6. Discussion and prospects

We have demonstrated a new numerical model for the motion of a quantum system in interaction with its environment, in which the state vector represents an individual system and diffuses continuously in the space of quantum states. The diffusion equation is derived directly from the density operator equation, such as the Bloch equation, and is consistent with it. Programs have been written in Pascal and FORTRAN for any system of one or two degrees of freedom in which all the operators can be represented as finite sums over finite products of creation and annihilation operators.

As in the usual representation by a density operator, the state-diffusion model depends on the cut that is made between the system and its environment, although beyond a certain point, the precise position of the cut makes no significant difference to the final result that is compared with experiment. The model shows that the cut can be very close to the microworld of quantum systems.

The model possesses the following advantages over the usual direct numerical solution of the Bloch equation.

(1) The model provides insight into the behaviour of individual systems and processes, which are represented in a form that can be compared directly with experimental and observational systems and processes.

(2) The computer store required is of order $C_1 N$, where N is the dimension of the space of quantum states. This compares with the $C_2 N^2$ required for the density operator equation, which can rule out this method altogether. The C_i are constants.

(3) For speed of computation the advantages of the method outweigh the disadvantages when the sample size is smaller than approximately twice the complex dimension of the state vector.

(4) The process of measurement is represented just like any other interaction of a quantum system and its environment, and needs no special treatment. This is the original reason for introducing stochastic equations for state vectors.

There are also disadvantages of the diffusion model:

(i) Where high precision is required, the sample required may be too large for advantage (3), so the method may be slower than direct integration of the density-matrix equations, sometimes much slower. These situations are likely to arise in practice where the dimension of the state space is small and the corresponding experimental or observational statistical errors are very small.

(ii) Analytic solutions of the stochastic equations are rarely available (though some do exist [18, 31, 32]).

(iii) The model is unfamiliar.

The disadvantages (ii) and (iii) are likely to become less severe with time and greater familiarity with application of the diffusion model.

The first three advantages are shared with the stochastic quantum-jump model of Dalibard, Carmichael and others mentioned in the introduction, although for advantage (1), their model provides a *different* insight, and it remains to be seen which, if any, is preferable. The continuous stochastic-diffusion model has the following further advantages over the quantum-jump model.

(5) Given a density operator equation, equation (1.2) for diffusion of quantum states is derived explicitly and uniquely. The uniqueness of the diffusion equation provides a one-to-one relation between a probability distribution over pure states at an initial time, and the distribution at later times.

(6) Since its solutions are continuous in time, and it uses the well developed Itô stochastic calculus, all the existing tools [27] for numerical integration of such stochastic equations can be used.

(7) Although the solutions are continuous in time, fast transitions resembling quantum jumps appear naturally.

The advantages of the stochastic differential equation method outweigh the disadvantages in a very wide variety of situations, which occur in many fields, including quantum optics, quantum chemistry, and the physics of condensed matter.

Obviously the similarities between the continuous-diffusion and quantum-jump methods outweigh the differences. Although at the moment the latter have not been used to represent a measurement process, the work of Diósi [18] suggests that they might be used in this way (advantage (4)), and that it might also be possible to derive unique quantum-jump equations from density-operator equations (advantage (5)). It is not clear at the moment which method is preferable as a practical tool, or whether each might have its advantages in different situations.

But it is clear that the two models have completely different origins. There is thus a remarkable convergence of two trends in physics that have previously been quite distinct: the quantum measurement 'problem' as considered by physicists concerned about the foundations of quantum physics, and the quantum measurement process as treated pragmatically by experimenters looking for intuitive pictures and rules of computation.

The stochastic reduction model of quantum mechanics provides both insight and practical tools for the solution of a wide variety of physical problems.

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Appendix

The Lindblad form of the Bloch equations is not unique. If L_m is a set of m_1 operators which represent the time evolution of the density operator in (1.1), then the same evolution is given by the j_1 operators K_j, where

$$L_m = \sum_j u_{mj} K_j \tag{A1}$$

and

$$\sum_m u_{mj} u^*_{mj'} = \delta_{jj'} \tag{A2}$$

so that u_{mj} is unitary when the dimensions m_1 and j_1 are equal. Substitution in (1.1) gives the same equation with K_j in place of L_m. For the uniqueness of the

state-diffusion equation the two forms of the Bloch equation must produce equivalent state-diffusion equations. By substitution in (1.2)

$$|\mathrm{d}\psi\rangle = -\frac{\mathrm{i}}{\hbar}H|\psi\rangle\mathrm{d}t + \sum_{mjj'}(2\langle K_{j'}^{\dagger}\rangle_{\psi}K_j - K_{j'}^{\dagger}K_j - \langle K_{j'}^{\dagger}\rangle_{\psi}\langle K_j\rangle_{\psi})|\psi\rangle u_{mj}u_{mj'}^{*}\mathrm{d}t$$
$$+ \sum_{m}(K_j - \langle K_j\rangle_{\psi})|\psi\rangle\mathrm{d}\xi_j^0 \tag{A3}$$

where

$$\mathrm{d}\xi_j^0 = \sum_j u_{mj}\mathrm{d}\xi_m \qquad \mathrm{M}\,\mathrm{d}\xi_{j'}^0\mathrm{d}\xi_j^0 = 0 \qquad \mathrm{M}\,\mathrm{d}\xi_{j'}^{0*}\mathrm{d}\xi_j^0 = \delta_{j'j} \tag{A4}$$

and the products are to be taken in the Itô sense as mean values. The two state-diffusion equations are therefore equivalent, as required.

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