String Theory

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Chapter 1

String theory

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Abstract

We give an overview of the relations between matrix models and string theory, focusing on topological string theory and the Dijkgraaf–Vafa correspondence. We discuss applications of this correspondence and its generalizations to supersymmetric gauge theory, enumerative geometry and mirror symmetry. We also present a brief overview of matrix quantum mechanical models in superstring theory.

1.1 Introduction: strings and matrices

String theories are defined, in perturbation theory, by a conformal field theory (CFT) on a general Riemann surface Σ_g which is then coupled to two-dimensional gravity. If we write the CFT action as

$$S[\phi, h], \tag{1.1.1}$$

where $h_{\alpha\beta}$ is the two–dimensional metric on Σ_g and ϕ are the "matter" fields, then the basic object to compute in a string theory is the total free energy

$$F = \sum_{g=0}^{\infty} g_s^{2g-2} F_g, \qquad F_g = \int \mathcal{D}h \, \mathcal{D}\phi \, \mathrm{e}^{-S[\phi,h]} \tag{1.1.2}$$

where g_s is the string coupling constant, and the path integral in F_g is over field configurations on the Riemann surface Σ_g. We can also perturb the CFT with various operators $\{\mathcal{O}_n\}$ leading to a general action which we write

$$S[\phi, h, t] = S[\phi, h] + \sum_n t_n \mathcal{O}_n. \tag{1.1.3}$$

In this case, the free energies at genus g will depend as well on the couplings t_n, and we write $F_g(t)$ to indicate this explicit dependence.

The computation of the free energies in (1.1.2) is a phenomenal problem, and many approaches have been developed in order to solve it. In the continuum approach, one fixes diffeomorphism invariance and finds that there is a critical central charge $c = 26$ in the CFT for which the 2d metric *decouples* and one is left with an integration over a *finite* set of $3g - 3$ coordinates τ parametrizing the Deligne–Mumford moduli space $\mathcal{M}_g$ of Riemann surfaces:

$$F_g(t) \sim \int_{\mathcal{M}_g} \mathrm{d}\tau \int \mathcal{D}\phi \, \mathrm{e}^{-S[\phi,\tau,t]} \tag{1.1.4}$$

This is the so-called *critical string theory.* For $c \neq 26$ (noncritical string theories), the metric has, on top of the finite set of moduli τ, a dynamical degree of freedom–the Liouville field. Critical strings are manageable but still very hard to solve, since integrating over the moduli space remains a difficult problem. Noncritical strings for generic values of c are so far intractable.

Noncritical strings for $c < 1$ are however special, since one can not only solve for the Liouville dynamics, but also perform the resulting integrals over the moduli and obtain explicit results for $F_g(t)$ at all genera (see [DiF+95] for a review, as well as Chapter 30 of this Handbook). This is achieved by using matrix modes to *discretize* the worldsheet of the string. Nontrivial matter content with $c \leq 1$ can be implemented with a wise use of the matrix interactions. For example, the Hermitian one-matrix model

$$Z = \int \mathrm{d}M \, \mathrm{e}^{-\mathrm{Tr}\, V(M)} \tag{1.1.5}$$

where $V(M)$ is a polynomial potential, makes possible to implement the $(2, 2m-1)$ minimal models coupled to gravity by tuning the parameters of the potential. These are the famous Kazakov multicritical points. In order to recover string theory properly speaking, one has to take a *double–scaling limit.* The correlation functions and free energies of the matrix model become, in this limit, correlation functions and free energies of the string theory, and the deformations of the action by operators and couplings can be implemented by scaling appropriately the parameters of the potential. This was, historically, the first example of a closed relation between matrix models in the $1/N$ expansion and string theories, and continues to play an important role.

One of the results of the analysis of the $1/N$ expansion of matrix models is that the genus g free energies $F_g^{\mathrm{MM}}(t)$ can be calculated by using only data from the *spectral curve* of the matrix model. In [EO07], this result was formulated in a very elegant way by associating an infinite number of *symplectic invariants* $F_g^{\Sigma}(t)$ to any algebraic curve Σ. These invariants are defined recursively as residues of meromorphic forms on the curve Σ. When Σ is the spectral curve of a matrix model, the symplectic invariants F_g^{Σ} are the genus g free energies obtained in the large N expansion. The formalism provides as well recursive relations for the computation of the $1/N$ expansion of correlation functions (see Chapters 16 and 29 of this Handbook for a review of these results).

The formalism of symplectic invariants makes possible to reformulate the relationship between $c < 1$ non-critical strings and matrix models, as follows. One can take the double-scaling limit of the matrix model at the level of the spectral curve, obtaining in this way a doubly-scaled spectral curve. For example, for the $c = 0$ string, the relevant spectral curve is described by the algebraic equation

$$y^2 = (2x - \sqrt{t})^2(x + \sqrt{t}) \tag{1.1.6}$$

where t is the bulk cosmological constant. It can be shown that the formalism of [EO07] commutes with taking this limit, therefore the symplectic invariants of the double-scaled spectral curve give directly the string theory amplitudes $F_g(t) = F_g^{\Sigma}(t)$.

It turns our that the correspondence between matrix models and string theory which makes possible to solve non-critical strings is also valid for an important type of string theory models, namely *topological strings*. Topological strings can be regarded as a generalization of non-critical strings, and they describe a topological sector of physical superstring theories. In terms of mathematical complexity, they provide an intermediate class of examples in between non-critical strings with $c < 1$ and full superstring theories. Topological strings also have two important applications: in physics, they make possible to compute holomorphic quantities in a wide class of four-dimensional, supersymmetric gauge theories. They also provide a framework to formulate Gromov–Witten invariants and mirror symmetry, and therefore they have had an enormous impact in algebraic geometry.

The correspondence between topological string theory and matrix models was first unveiled by Dijgkraaf and Vafa in [DV02]. It is not valid for all types of topological strings, since it only applies to a special type of target spaces, namely local (i.e. non-compact) Calabi–Yau manifolds. Since topological strings on these non-compact spaces are related to a wide class of supersymmetric gauge theories, this correspondence makes possible to use matrix model techniques to obtain exact solutions in four-dimensional quantum field theory. It also leads to a remarkable connection between matrix models, mirror symmetry, and enumerative geometry on non-compact Calabi–Yau manifolds. In this chapter we will review these developments.

The organization of this chapter is as follows. First, we will review some basic aspects of topological strings on Calabi–Yau manifolds. Then, we will state the Dijkgraaf–Vafa correspondence, we will sketch its connection to string dualities, and we will briefly explain how it can be used to compute superpotentials in certain supersymmetric gauge theories. We also explain how the correspondence extends to toric manifolds and leads to a matrix model approach to enumerative geometry, closely related to mirror symmetry. Finally, we mention applications of matrix quantum-mechanical models in superstring theory.

1.2 A short survey of topological strings

Topological strings are, as all string theories, CFTs coupled to 2d gravity. The underlying CFTs are *2d nonlinear sigma models* with a *target space* X. This means that the basic fields are maps

$$x : \Sigma_g \to X. \tag{1.2.1}$$

It turns out that, in order to construct topological strings, the target must have a very precise strucutre–it must be a *Calabi–Yau manifold*. This will guarantee among other things the conformal invariance of the model. We will then start with a very brief review of these manifolds. See [Ho+03, CoK99] for more information.

1.2.1 Calabi–Yau manifolds

Mathematically, a Calabi–Yau (CY) manifold X is a complex manifold which admits a Kähler metric with vanishing Ricci curvature. The condition of vanishing curvature is called the *Calabi–Yau condition*. Let us spell out in detail the ingredients in this definition. Since X is complex, it has complex coordinates (in local patches) that we will denote by

$$x^I, \quad x^{\overline{I}}, \qquad I = 1, \cdots, d, \tag{1.2.2}$$

where d is the complex dimension of X. X is also endowed with a Riemannian metric which is *Hermitian*, i.e. it only mixes holomorphic with antiholomorphic coordinates, and in a local patch it has the index structure $G_{I\overline{J}}$. We also need X to be *Kähler*. This means that the *Kähler form*

$$\omega = \mathrm{i} G_{I\overline{J}}\, \mathrm{d}x^I \wedge \mathrm{d}x^{\overline{J}} \tag{1.2.3}$$

which is a real two-form, is *closed*:

$$\mathrm{d}\omega = 0 \tag{1.2.4}$$

It is easy to check that for a Kähler manifold the Christoffel symbols do not have mixed indices, i.e. their only nonvanishing components are

$$\Gamma^{I}_{JK}, \qquad \Gamma^{\overline{I}}_{\overline{JK}}. \tag{1.2.5}$$

In addition, the Calabi–Yau condition requires the metric $G_{I\overline{J}}$ to be Ricci–flat:

$$R_{IJ} = 0. \tag{1.2.6}$$

It was conjectured by Calabi and then proved by Yau that, for compact Kähler manifolds, Ricci flatness is equivalent to a topological condition, namely that the first Chern class of the manifold vanishes

$$c_1(X) = 0. \tag{1.2.7}$$

The CY manifolds appearing in string theory usually have complex dimension $d = 3$, and they are called CY threefolds. Using Hodge theory, Poincaré duality and the CY condition, one can see that the Hodge diamond of a simply-connected CY threefold X, which encodes the Hodge numbers $h^{p,q}(X)$, $p, q = 1, 2, 3$, has the structure

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1}(X) & & 0 & \\
1 & & h^{1,2}(X) & & h^{1,2}(X) & & 1 \\
 & 0 & & h^{1,1}(X) & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array} \tag{1.2.8}$$

and therefore it only depends on two integers $h^{1,1}(X)$, $h^{1,2}(X)$ (we have assumed that X is simply connected). It also follows that the Euler characteristic of X is

$$\chi(X) = 2(h^{1,1}(X) - h^{1,2}(X)). \tag{1.2.9}$$

One of the most important properties of Calabi-Yau manifolds (which can actually be taken as their defining feature) is that they have a holomorphic, nonvanishing section Ω of the canonical bundle $K_X = \Omega^{d,0}(X)$. This form is unique up to multiplication by a nonzero number, and in local coordinates it can be written as

$$\Omega = \Omega_{I_1 \cdots I_d} \mathrm{d}x^{I_1} \wedge \cdots \wedge \mathrm{d}x^{I_d}. \tag{1.2.10}$$

Since the section is nowhere vanishing, the canonical line bundle is trivial, and this is equivalent to (1.2.7).

Calabi–Yau manifolds typically come in families, in the sense that once a Calabi–Yau manifold has been obtained, one can deform it by changing its parameters without violating the Calabi–Yau condition. There are two types of parameters in a Calabi–Yau family: the *Kähler parameters* and the *complex deformation parameters*. The first ones specify sizes (i.e. areas of embedded holomorphic curves) while the second ones specify the complex structure, i.e. the choice of holomorphic/anti-holomorphic splitting. It turns our that the number of Kähler parameters (which are real) is $h^{1,1}(X)$, while the number of complex deformation parameters (which is complex) is $h^{1,2}(X)$.

1.2.2 Topological sigma models

There are two different sigma models that can be used to construct topological strings, and they are referred to as type A and type B topological sigma models [LL92, W91]. Let us first explain what is common to both of them and how they couple to gravity.

First of all, both models are nonlinear sigma models, hence they are based on a scalar, commuting field (1.2.1). Second, they are *topological field theories of the cohomological type*, i.e. they both possess a Grassmannian, scalar symmetry $\mathcal{Q}$ with the following properties:

1. Nilpotency: $\mathcal{Q}^2 = 0$.
2. The action is $\mathcal{Q}$-exact:
$$S = \{\mathcal{Q}, V\}. \tag{1.2.11}$$
3. The energy-momentum tensor is also $\mathcal{Q}$-exact,
$$T_{\alpha\beta} = \frac{\delta S}{\delta h^{\alpha\beta}} = \{\mathcal{Q}, b_{\alpha\beta}\}. \tag{1.2.12}$$

The operator $\mathcal{Q}$ is formally identical to a BRST operator, and any operator of the form $\{\mathcal{Q}, \cdot\}$ (i.e. a $\mathcal{Q}$-*exact operator*) has a vanishing expectation value. Three consequences follow from this observation:

1. The partition function does not depend on the background two-dimensional metric. Indeed,
$$\frac{\delta Z}{\delta h^{\alpha\beta}} = -\langle\{\mathcal{Q}, b_{\alpha\beta}\}\rangle = 0. \tag{1.2.13}$$
2. The operators in the cohomology of $\mathcal{Q}$, i.e. the operators $\mathcal{O}$ which satisfy
$$\{\mathcal{Q}, \mathcal{O}\} = 0, \qquad \mathcal{O} \neq \{\mathcal{Q}, \Psi\} \tag{1.2.14}$$
lead to a topological sector in the theory, since their correlation functions are metric independent. These operators correspond to *marginal deformations* of the CFT, and in both the A and the B model, turning on

a marginal deformation corresponds to a *geometric deformation* of the target CY manifold X.

3. *The semiclassical approximation is exact*: if we explicitly introduce a coupling constant $\hbar$, we have

$$Z = \int \mathcal{D}\phi \, \mathrm{e}^{-\frac{1}{\hbar}S} \Rightarrow \frac{\delta Z}{\delta \hbar^{-1}} = -\langle \{\mathcal{Q}, V\} \rangle = 0. \tag{1.2.15}$$

In particular, we can evaluate the partition function (and the correlation functions of $\mathcal{Q}$-invariant operators) in the semiclassical limit $\hbar \to 0$.

In order to obtain a string theory we have to couple these topological sigma models to two-dimensional gravity. It turns out that topological sigma models are in this respect very similar to the critical bosonic string. Indeed, what characterizes critical strings is that the two-dimensional metric field essentially decouples from the theory, except for a finite number of moduli. In the BRST quantization of the critical bosonic string, this decoupling is the consequence of having a nilpotent BRST operator, $\mathcal{Q}_{\mathrm{BRST}}$, such that the energy-momentum tensor is $\mathcal{Q}_{\mathrm{BRST}}$-exact

$$T(z) = \{\mathcal{Q}_{\mathrm{BRST}}, b(z)\}. \tag{1.2.16}$$

As we have seen, topological sigma models posses a nilpotent Grasmannian operator which plays the role of $\mathcal{Q}_{\mathrm{BRST}}$. The property (1.2.12), formally identical to (1.2.16), guarantees that a similar decoupling will occur in topological sigma models.

Topological sigma models coupled to gravity are called *topological strings*. In analogy with bosonic strings, their free energies at genus g, $F_g(t)$, can be computed by a path integral like (1.1.4), where the t denote the parameters for marginal deformations. We will now describe the A and the B models in more detail, as well as the content of their free energies. For a full exposition, see for example [Ho+03, M05].

1.2.3 The type A topological string

The type A topological sigma model includes, on top of the field x, Grassmannian fields $\chi \in x^*(TX)$, which are scalars on Σ_g, and a Grassmannian one-form ρ_μ with values in $x^*(TX)$. This one-form satisfies a self-duality condition. The relevant saddle-points or instantons in semi-classical calculations are *holomorphic maps* from Σ_g to X. These instantons are classified by topological sectors. If we consider a basis of $H_2(X, \mathbb{Z})$ denoted by Σ_i, $i = 1, \cdots, b_2(X)$, then

$$[x(\Sigma_g)] = \sum_{i=1}^{h^{1,1}(X)} n_i \Sigma_i \in H_2(X, \mathbb{Z}), \qquad n_i \in \mathbb{Z}_{\geq 0}. \tag{1.2.17}$$

In the A model, there are $b_2(X) = h^{1,1}(X)$ marginal deformations and they correspond to what are called *complexified Kähler parameters*. These can be understood as the volumes of the two-cycles Σ_i,

$$t_i^A = \int_{\Sigma_i} (\omega + \mathrm{i}B), \tag{1.2.18}$$

where $B \in H^2(X, \mathbb{Z})$ is the so–called B field and gives a natural complexification of these volumes.

The genus g free energies of the A model depend on the marginal deformation parameters (1.2.18). They have the structure

$$F_g^A(t^A) = \sum_{n_i} N_{g,n_i} \mathrm{e}^{-n_i t_i^A} \tag{1.2.19}$$

and they involve a sum over all saddle-points (1.2.17) (there is also a simple contribution coming from trivial instantons with $n_i = 0$, but we have not included it in (1.2.19)). In this equation, N_{g,n_i} are the *Gromov–Witten invariants* at genus g and for the two-homology class (1.2.17). They are rational numbers that "count" holomorphic maps at genus g in that class, and they can be computed by integrating an appropriate function over the space of collective coordinates of the instanton. They can be formulated in a rigorous mathematical way and they play a crucial role in modern algebraic geometry, see [CoK99, Ho+03] for an exposition.

1.2.4 The type B topological string

The type B topological sigma model includes, on top of the field x, Grassmannian fields $\eta^{\bar{I}}, \theta^{\bar{I}} \in x^*(T^{(0,1)}X)$, which are scalars on Σ_g, and a Grassmannian one-form on Σ_g, ρ_α^I, with values in $x^*(T^{(1,0)}X)$. In the B model, the only relevant saddle-points are trivial ones, i.e. constant maps x. This makes the model much easier to solve, since integration over the collective coordinates of the saddle-points becomes integration over the CY X. There are $h^{1,2}(X) = b_3(X)/2 - 1$ marginal deformations which can be interpreted as changes in the complex structure of the CY, and they are called *complex deformation parameters*. These parameters form a moduli space $\mathcal{M}$ of dimension $h^{1,2}(X)$. If the CY is presented through a set of algebraic equations, they roughly correspond to changing the parameters appearing in the equations.

The free energies F_g^B in the B model depend on the complex structures of the CY X, and mathematically they are closely related to the theory of variations of complex structures of X. A convenient parametrization of $\mathcal{M}$ is the following. Choose first a symplectic basis for $H_3(X, \mathbb{Z})$,

$$A_i, \quad B_j, \qquad i, j = 0, 1, \cdots, h^{1,2}(X). \tag{1.2.20}$$

The symplectic condition is that

$$A_i \cap B_j = \delta_{ij}. \tag{1.2.21}$$

We then define the *periods* of the Calabi-Yau manifold as

$$X_i = \int_{A_i} \Omega, \qquad \mathcal{F}_i = \int_{B_i} \Omega, \quad i = 0, \cdots, h^{1,2}(X) \tag{1.2.22}$$

where Ω is the form (1.2.10). A basic result of the theory of deformation of complex structures says that the X_i are (locally) complex projective coordinates for $\mathcal{M}$. They are called *special projective coordinates*, and since they parametrize $\mathcal{M}$ we deduce that the other set of periods must depend on them, *i.e.* $\mathcal{F}_i = \mathcal{F}_i(X)$. Moreover, one can show that there is a function $\mathcal{F}(X)$ which satisfies

$$\mathcal{F}_i(X) = \frac{\partial \mathcal{F}(X)}{\partial X_i} \tag{1.2.23}$$

and turns out to be a homogeneous function of degree two in the X_i.

Inhomogeneous coordinates can be introduced in a local patch where one of the projective coordinates, say X^0, is different from zero, and taking

$$t_i^B = \frac{X_i}{X_0}, \qquad i = 1, \cdots, h^{1,2}(X). \tag{1.2.24}$$

These coordinates on the moduli space of complex structures are called *flat coordinates*. The flat coordinates are nontrivial functions of the parameters z_i appearing in the algebraic equations defining X. One can rescale $\mathcal{F}(X)$ in order to obtain a function of the inhomogeneous coordinates t_i^B called the *prepotential*:

$$F_0^B(t^B) = \frac{1}{X_0^2}\, \mathcal{F}(X). \tag{1.2.25}$$

This is precisely the genus zero free energy of the type B topological strings on X, and it encodes all the relevant information about the B model on the sphere.

The higher genus amplitudes in the B model are also related to the variation of complex structures, but they are more difficult to describe. As first shown in [Be+94], they admit a non-holomorphic extension $F_g^B(t_B, t_B^*)$, and the original $F_g^B(t_B)$ are then recovered in the limit

$$F_g^B(t_B) = \lim_{t_B^* \to \infty} F_g^B(t_B, t_B^*). \tag{1.2.26}$$

The non-holomorphic dependence of these amplitudes is given by a recursive set of differential equations called the *holomorphic anomaly equations*. These equations do not determine the amplitudes uniquely, due to the lack of a complete set of boundary conditions. As we will see, in certain non-compact CY backgrounds, the higher genus topological string amplitudes of the type B topological string can be obtained from matrix models in a unique way.

1.2.5 Mirror symmetry

Mirror symmetry is an example of a string duality. It says that, given a CY manifold $\tilde{X}$, there is another CY manifold X such that the A model on $\tilde{X}$ is equivalent to the B model on X, and viceversa. This means that the free energies are simply equal:

$$F_g^A(t^A; \tilde{X}) = F_g^B(t^B; X), \tag{1.2.27}$$

after setting

$$t_i^B = t_i^A. \tag{1.2.28}$$

Notice that for mirror symmetry to make sense, one needs

$$h^{1,1}(\tilde{X}) = h^{1,2}(X), \tag{1.2.29}$$

so that the Hodge diamonds of X and $\tilde{X}$ are related by an appropriate reflection. Here, it is very important that the t^B in the B model are flat coordinates (1.2.24) obtained from the periods, and not the complex parameters z_i that appear in the algebraic equations defining X. For this reason, the nontrivial relation $t^A = t^B(z)$ between the complexified Kähler parameters in the A model and the complex parameters z_i is called the *mirror map*. Mirror symmetry makes possible to compute the complicated instanton expansion of the A model in terms of period integrals in the B model, and it has been extensively studied in the last fifteen years, see [Ho+03, CoK99] for a detailed exposition.

1.3 The Dijkgraaf–Vafa correspondence

1.3.1 The correspondence

In its original formulation, the Dijkgraaf–Vafa correspondence is a statement about the genus g amplitudes of the type B topological string on some special, non-compact Calabi–Yau geometries. Such geometries are described by an equation of the form

$$u^2 + v^2 + z^2 - W'(x)^2 + R(x) = 0 \tag{1.3.1}$$

where $W(x)$, $R(x)$ are polynomials of degree $n+1$ and $n-1$, respectively. The simplest example occurs when $n = 1$, and

$$W(x) = \frac{1}{2}x^2, \qquad R(x) = \mu. \tag{1.3.2}$$

One then obtains the equation,

$$z^2 + u^2 + v^2 = \mu + x^2 \tag{1.3.3}$$

wich is the so called *deformed conifold*. It is obtained as a deformation of the so–called conifold singularity

$$z^2 + u^2 + v^2 = x^2. \tag{1.3.4}$$

by turning on the parameter μ. The noncompact manifold (1.3.1) has a holomorphic three-form:

$$\Omega = \frac{1}{8\pi^2 \mathrm{i}} \frac{\mathrm{d}x \mathrm{d}z \mathrm{d}u}{v} \tag{1.3.5}$$

which is nowhere vanishing. Therefore, this space can be regarded as a non-compact Calabi–Yau threefold.

In the geometry (1.3.1) there are n compact three-cycles A_i, $i = 1, \cdots, n$, with the topology of a three-sphere. These cycles can be regarded as $\mathbb{S}^2$ fibrations over the cuts $\mathcal{C}_i = [x_{2i-1}, x_{2i}]$ of the curve

$$y^2(x) = W'(x)^2 - R(x) = \prod_{i=1}^{2n} (x - x_i), \tag{1.3.6}$$

where we have assumed that $R(x)$ is sufficiently generic so that the x_i are all distinct.

Consider the type B topological string on the noncompact CY manifold defined by (1.3.1), and let

$$t_i^B = \int_{A_i} \Omega \tag{1.3.7}$$

be the periods of the three-cycles A_i (in a non-compact CY threefold, one has $X_0 = 1$, so that $X_i = t_i^B$). We are interested in computing $F_g^B(t^B)$. To do this, let us consider a matrix model for a Hermitian $N \times N$ matrix defined by

$$Z = \frac{1}{\mathrm{vol}(U(N))} \int \mathrm{d}M \, \mathrm{e}^{-\frac{1}{g_s} \mathrm{Tr} W(M)}, \tag{1.3.8}$$

where $W(x)$ is the potential appearing in (1.3.1). The most general solution to the loop equations of this matrix model is a *multi-cut solution*. This solution is characterized by a partition of N,

$$N = N_1 + \cdots + N_n, \tag{1.3.9}$$

where we put N_i eigenvalues near the i-th critical point of $W(x)$. At large N_i, these sets of eigenvalues fill up cuts $\mathcal{C}_i$, $i = 1, \cdots, n$. Different multi-cut solutions of the model are associated to different choices of a polynomial $R(x)$ of order $n - 1$, as in (1.3.1), and the *spectral curve* of the multi-cut solution to (1.3.8) has precisely the form (1.3.6). The *partial 't Hooft parameters*

$$S_i = g_s N_i = \frac{1}{4\pi \mathrm{i}} \oint_{\mathcal{C}_i} y(x) \mathrm{d}x, \tag{1.3.10}$$

can be computed in terms of periods of the spectral curve. The total free energy of the matrix model has a $1/N$ expansion of the form

$$F^{\mathrm{MM}}(S_i, g_s) = \log Z(N_i, g_s) = \sum_{g=0}^{\infty} F_g^{\mathrm{MM}}(S_i) g_s^{2g-2}. \tag{1.3.11}$$

According to the Dijkgraaf–Vafa correspondence, *the genus g amplitudes of the type B topological string on (1.3.1) are given by the genus g free energies of the multi-cut matrix model (1.3.8)*:

$$F_g^B(t_i^B) = F_g^{\mathrm{MM}}(S_i) \tag{1.3.12}$$

after identifying $t_i^B = S_i$. Notice that this identification implies in particular that the A-periods of Ω in the full geometry (1.3.1), as calculated in (1.3.7), must agree with the periods of the spectral curve (1.3.10). This is easily verified. Notice that, in view of the results of [EO07], we can reformulate (1.3.12) by saying that

$$F_g^B(t_i^B) = F_g^{\Sigma}(S_i) \tag{1.3.13}$$

where F_g^{Σ} are the symplectic invariants associated to the spectral curve (1.3.6), i.e. to the "non-trivial" part of the geometry (1.3.1).

The Dijkgraaf–Vafa correspondence (1.3.12) gives an answer in closed form for the perturbative amplitudes of the type B topological string, and when X is of the form (1.3.1). For $g = 0$ the correspondence follows from the equality of the special geometry solution for $F_0^B(t^B)$ and the saddle-point solution for the planar free energy of the matrix model. The equality (1.3.12) for $g = 1$ can be also verified by direct computation of both sides [KMT03, DST04]. For $g \geq 2$ there is no general proof of the equivalence, although it has been noticed that the matrix model free energies $F_g^{\mathrm{MM}}(S_i)$ can be modified in a natural way as to satisfy the holomorphic anomaly equations characterizing the type B topological string [HK07, EMO07].

1.3.2 Relation to gauge/string dualities and geometric transitions

The Dijkgraaf–Vafa correspondence is an example of a *gauge/string duality*, relating a gauge theory in the $1/N$ expansion (in this case a matrix model, i.e. a zero-dimensional quantum gauge theory) to a string theory (in this case, a topological string theory). It can be also reformulated as a *open/closed string duality*, relating an open topological string theory to a closed topological string theory. We will sketch here the main argument, referring the reader to [DV02, M06] for details.

Let us consider the *singular* Calabi–Yau geometry given by

$$u^2 + v^2 + z^2 - W'(x)^2 = 0. \tag{1.3.14}$$

When $W'(x) = x$, this is the conifold singularity (1.3.4). Singularities in algebraic geometry can be avoided in two different ways. One way consists in *deforming* them, by adding lower order terms to the equation. In this way one recovers the geometry (1.3.1). Another way consists in *resolving* the singularities by "gluing" two-spheres to them. In this case, one obtains a geometry $X_{\rm res}$ with n two-cycles with the topology of $\mathbb{S}^2$. These two-spheres are in one-to-one correspondence with the critical points of $W(x)$. One can then consider the *open* type B topological string on this space. Open strings need boundary conditions, and in string theory the most general boundary conditions are provided by D-branes. Let us consider N D-branes on $X_{\rm res}$. For each partition (1.3.9) we obtain a D-brane configuration in which N_i D-branes wrap the i-th two-sphere. An open string in $X_{\rm res}$ with this D-brane configuration will have in general $h_i \geq 0$ boundaries ending on the i-th two-sphere.

We are now interested in computing open string amplitudes for this system. Let F_{g,h_i} be the open string amplitude corresponding to a Rieman surface of genus g with h_i boundaries attached to the i-th sphere. The total free energy is then given by

$$F^{B,\,\rm open}(N_i, g_s) = \sum_{g=0}^{\infty} \sum_{h_1,\cdots,h_n \geq 0} F_{g,h_1,\cdots,h_n} g_s^{2g-2+h} N_1^{h_1} \cdots N_n^{h_n}, \tag{1.3.15}$$

where g_s is the string coupling constant and $h = h_1 + \cdots + h_n$. Using the seminal results of Witten in [W95], one can show that the string field theory describing the dynamics of this system of open strings is precisely the matrix model (1.3.8). This means that

$$F^{B,\,\rm open}(N_i, g_s) = \log Z(N_i, g_s) \tag{1.3.16}$$

where $Z(N_i, g_s))$ is the partition function (1.3.8) expanded around the multi-cut vacuum characterized by the partition (1.3.9).

The open/closed string duality states that the type B topological string theory on the *resolved* geometry $X_{\rm res}$ in the presence of N D-branes is equivalent to a *closed* string field theory on the *deformed* geometry (1.3.1). This means two things. First, the periods t_i^B parametrizing the moduli space of complex parameters of (1.3.1) are given by the partial 't Hooft parameters $g_s N_i$ in (1.3.10). Second, the closed string amplitudes $F_g^{\rm closed}(t_i^B)$ are obtained by fixing the genus g of the open string theory and summing over all possible h_is:

$$F_g^{B,\,\rm closed}(t_i^B) = \sum_{h_1,\cdots,h_n \geq 0} F^{B,\,\rm open}_{g,h_1,\cdots,h_n} t_1^{h_1} \cdots t_n^{h_n}, \tag{1.3.17}$$

so that

$$F^{B,\,\rm closed}(t_i^B, g_s) = F^{B,\,\rm open}(N_i, g_s). \tag{1.3.18}$$

(1.3.12) follows from this.

We finally point out three important conceptual aspects of this open/closed string duality. First, it relates an open string theory on $X_{\rm res}$ (with N D-branes) to a closed string theory on the deformed counterpart (1.3.1), and therefore relates two different geometric backgrounds -namely, the two different desingularizations of (1.3.14). For this reason, it is sometimes called a *geometric transition*, and it is conceptually similar to other dualities of this type, like the celebrated AdS/CFT correspondence [Ah+00] or the Gopakumar–Vafa correspondence (see [M05] for a review). Second, the open string theory side is equivalent to a gauge theory (in this case, a matrix model), and to go from the open free energies to the closed free energies as in (1.3.17) one has to sum over all double-line diagrams in the gauge theory with fixed genus g, as in the large N expansion. For this reason, geometric transitions are also called *large N transitions*. Finally, in this duality the target space geometry (1.3.1) is essentially given by the spectral curve of the matrix model (1.3.6). In this sense, the Calabi–Yau geometry can be regarded as an *emergent* geometry made of a large number of gauge theory degrees of freedom, since the spectral curve encodes the distribution of eigenvalues of the matrix model in the large N limit.

We conclude this section with a dictionary of the relations obtained between matrix models and type B topological strings in the geometries (1.3.1).

matrix model	type B topological string
't Hooft parameters	flat coordinates
spectral curve	target CY geometry
planar free energy	prepotential
$1/N$ expansion	genus expansion

1.3.3 Applications to supersymmetric gauge theories

One of the most important applications of the Dijkgraaf–Vafa correspondence has been the calculation of exact superpotentials in $\mathcal{N}=1$ supersymmetric gauge theories. The canonical example, as discussed in [CIV01, DVb02], is $\mathcal{N}=1$, $U(M)$ Yang–Mills theory with a chiral superfield in the adjoint representation Φ. When there is no tree-level superpotential for Φ, this is in fact $U(M)$, $\mathcal{N}=2$ Yang–Mills theory. A tree-level superpotential for Φ of degree $n+1$, $W(\Phi)$, will break supersymmetry down to $\mathcal{N}=1$. The classical vacua of the theory are determined by the distribution of the eigenvalues of the matrix Φ over the n critical points of $W(\Phi)$. A partition

$$M = M_1 + \cdots + M_n, \tag{1.3.19}$$

where M_i eigenvalues are at the i-th critical point, corresponds to the classical gauge symmetry breaking pattern

$$U(M) \to U(M_1) \times \cdots \times U(M_n). \tag{1.3.20}$$

Quantum mechanically, these vacua are characterized by a gaugino condensate and confinement in each of the $SU(M_i)$ factors, and by dynamically generated scales Λ_i. Let

$$S_i = \frac{1}{32\pi^2}\mathrm{Tr}_{SU(M_i)}W_\alpha^2 \tag{1.3.21}$$

be the gaugino superfield in the $SU(M_i)$ factor. The effective superpotential describing the quantum vacua associated to the classical pattern (1.3.20) can be computed as a function of the superfields S_i, $W_{\mathrm{eff}}(S_i)$. In [DV02, CIV01, DVb02] the following formula was proposed for $W_{\mathrm{eff}}(S_i)$:

$$W_{\mathrm{eff}}(S_i) = \sum_{i=1}^{n}\left\{M_i S_i \log(S_i/\Lambda_i^3) - 2\pi\mathrm{i}\tau S_i + M_i \frac{\partial F_0^{\mathrm{MM}}(S_i)}{\partial S_i}\right\}. \tag{1.3.22}$$

In this formula,

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi\mathrm{i}}{g^2} \tag{1.3.23}$$

is the classical, complexified gauge coupling. $F_0^{\mathrm{MM}}(S_i)$ is the genus zero free energy of the Hermitian matrix model (1.3.8), corresponding to a multi-cut solution with partial 't Hooft parameters S_i (notice that the rank of the matrix model, N, is *a priori* unrelated to the rank M of the gauge group in the supersymmetric gauge theory).

This application of the Dijkgraaf–Vafa correspondence follows by relating the topological string to the *physical* type IIB superstring theory [Be+94, DVb02]. When the type IIB superstring is compactified on X_{res}, in the background of M D5-branes wrapping the two-spheres as in (1.3.19), one obtains an "engineering" of the four-dimensional gauge theory we have just described. The effective superpotential, as a function of the gaugino superfields, can be computed from the genus zero open string amplitudes of the *topological* string on X_{res}. Since these amplitudes are computed by the matrix model, we obtain the relation (1.3.22). See [DVb02] for more details.

The relation (1.3.22) between matrix models and supersymmetric gauge theories can be also derived by using field-theory techniques [D+03, C+02, AFH04] and it has led to an extensive literature. In particular, it can be extended to many other $\mathcal{N} = 1$ supersymmetric gauge theories, and it leads to powerful derivations of gauge theory dualities and exact solutions, like for example the Seiberg–Witten solution or the duality properties of softly broken $\mathcal{N} = 4$ supersymmetric gauge theories.

1.4 Matrix models and mirror symmetry

The Dijkgraaf and Vafa correspondence, as it was originally formulated in [DV02], gives a matrix model description only for the type B topological string on geometries of the form (1.3.1). However, one can find a matrix model-like description for a different, important class of noncompact Calabi–Yau geometries. These geometries come in mirror pairs. In the type A topological string, they are *toric* Calabi–Yau manifolds. These are manifolds which contain an algebraic torus $\mathbb{T} \subset (\mathbb{C}^*)^r \subset X$ as an open set, and they admit an action of $\mathbb{T}$ which acts on this set by multiplication. Important examples of toric Calabi–Yau threefolds are the *resolved conifold*

$$\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1 \tag{1.4.1}$$

and the spaces

$$K_S \to S, \tag{1.4.2}$$

where S is an algebraic surface and K_S is its canonical bundle. A simple example occurs when $S = \mathbb{P}^2$. The resulting Calabi–Yau

$$\mathcal{O}(-3) \to \mathbb{P}^2, \tag{1.4.3}$$

is also known as *local* $\mathbb{P}^2$. Toric Calabi–Yau geometries have a mirror description given by

$$u^2 + v^2 = P(x, y), \qquad x, y \in \mathbb{C}^* \tag{1.4.4}$$

where $P(x, y)$ is a polynomial in x, y. The locus

$$P(x, y) = 0 \tag{1.4.5}$$

defines an algebraic curve $\Sigma \subset \mathbb{C}^* \times \mathbb{C}^*$. Notice that this curve is the analogue of (1.3.6) in the geometries of the form (1.3.1). For example, the mirror curve to local $\mathbb{P}^2$ (1.4.3) is the cubic curve in $\mathbb{C}^* \times \mathbb{C}^*$ given by

$$y(y + x + 1) + zx^3 = 0 \tag{1.4.6}$$

where z is a complex deformation parameter. The flat coordinates parametrizing the moduli space of complex structures of the Calabi–Yau manifolds (1.4.4) reduce to periods on the curve (1.4.5), but since the variables belong now to $\mathbb{C}^* \times \mathbb{C}^*$ the meromorphic form to integrate is different, and one has

$$t_i^B = \oint_{\mathcal{C}_i} \log y \frac{\mathrm{d}x}{x}. \tag{1.4.7}$$

As we pointed out in the introduction, given any algebraic curve Σ one can construct a series of matrix-model-like quantities $F_g^\Sigma(t)$ associated to it, which are called the symplectic invariants of the curve. We can now regard the curve (1.4.5) as an algebraic curve Σ and calculate its symplectic invariants following the formalism of [EO07][1]. An extension of the Dijkgraaf–Vafa correspondence

[1] The fact that the variables x, y live now in $\mathbb{C}^*$ leads to some modifications of this formalism, however. See [Bo+09] for details.

was conjectured in [M08, Bo+09] and states that the genus g free energies of the type B topological string theory on the CY (1.4.4) are given by the symplectic invariants of the curve Σ given by (1.4.5),

$$F_g^B(t_i^B) = F_g^\Sigma(t_i^B), \tag{1.4.8}$$

where t_i^B are the periods (1.4.7). By mirror symmetry, this computes the type A topological string amplitudes on toric manifolds, and in particular generating functionals of Gromov–Witten invariants of these manifolds. Extensive checks of this proposal were performed in [M08, Bo+09]. An important aspect of this proposal is that one can also compute *open* string amplitudes on the geometries (1.4.4). These are given by the generating functionals associated to Σ which correspond to matrix model correlation functions. When applied to (1.4.4), and after using mirror symmetry, they can be seen to compute *open* Gromov–Witten invariants associated to an important class of D-branes in the mirror toric manifolds.

To summarize, the Dijkgraaf–Vafa correspondence can be extended to a large family of non-compact Calabi–Yau manifolds given by toric manifolds and their mirrors. The mirror geometry leads naturally to a spectral curve. The symplectic invariants and generating functionals associated to this curve by using matrix model techniques give topological string amplitudes and in particular they calculate Gromov–Witten invariants of toric Calabi–Yau's.

We conclude with some remarks on this correspondence between matrix models and topological strings on toric manifolds and their mirrors. First of all, this correspondence (in contrast to the Dikgraaf–Vafa correspondence) does not provide explicit matrix integral realizations of topological string theory. These are not needed to compute the genus g amplitudes, since they only depend on the spectral curve, which is provided by the mirror geometry (1.4.4). However, for certain toric Calabi–Yau manifolds (like the resolved conifold) there are explicit matrix models whose spectral curve coincides with the one featuring in their mirror geometry. These matrix models are called *Chern–Simons matrix models* and were introduced in [M04] to describe the partition function of Chern–Simons gauge theory on certain three-manifolds, see [A+04, M06, M05] for more details on these models. Second, the symplectic invariants of (1.4.5) give, through mirror symmetry, the genus g free energies of the A model on toric Calabi–Yau manifolds. These free energies can be also obtained as sums over partitions using the topological vertex formalism [A+05]. Indeed, it seems that the spectral curves (1.4.5) describe the saddle point of these sums in the limit of large partitions, and that the formalism of [EO07] can be applied as well to obtain an analogue of the topological $1/N$ expansion [E08]. This makes possible to verify the correspondence of [M08, Bo+09], at least in some cases, by using the theory of the topological vertex. Finally, there are other, simpler enumerative problems which can be solved as well in terms of the matrix-model like formalism of [EO07], like for example the calculation of Hurwitz numbers of the sphere [BoM08].

1.5 String theory, matrix quantum mechanics, and related models

Matrix quantum mechanics, a close cousin of matrix models, also plays an important role in string theory. In matrix quantum mechanics the fundamental variables are time-dependent matrices, and the theory can be regarded as a one-dimensional quantum gauge theory. One of the most important applications of matrix quantum mechanics is the nonperturbative description of string theories in one dimension. For example, the $c = 1$ noncritical bosonic string can be defined by the double-scaling limit of a matrix quantum mechanics model with a single, Hermitian matrix in an appropriate potential, see [GM93] for a review and a list of references.

With the advent of D-branes, a more crucial role has been advocated for certain matrix quantum mechanical models, in the context of M-theory. The low-energy dynamics of a system of N D0-branes in type IIA theory is described by the dimensional reduction of $\mathcal{N} = 1$ supersymmetric Yang–Mills theory to $0 + 1$ dimensions. The resulting gauge-fixed Lagrangian can be written as

$$L = \frac{1}{2}\mathrm{Tr}\Big\{\dot{X}^a\dot{X}^a + \frac{1}{2}[X^a, X^b]^2 + \theta^T(\mathrm{i}\dot{\theta} - \Gamma_a[X^a, \theta])\Big\}. \tag{1.5.1}$$

In this Lagrangian, the X^a are nine $N \times N$ matrices, θ are sixteen $N \times N$ Grasmann matrices forming a spinor of $SO(9)$, and Γ_a are the corresponding gamma matrices. According to a conjecture in [B+97], the large N limit of this Lagrangian should describe all of M-theory in light-front coordinates. There is evidence that certain aspects of eleven-dimensional supergravity are appropriately captured by this matrix quantum-mechanical model, see for example [T01] for a detailed review. In particular, linearized gravitational interactions, as well as nonlinear corrections, can be obtained from this model. This is an important example of how gravitational space-time dynamics emerges as a large distance phenomenon from a well-defined quantum-mechanical system with matrix degrees of freedom.

Other important matrix quantum-mechanical models of D-brane dynamics include the so-called ADHM model, describing bound states of D4 and D0 branes. This model has important applications in instanton and BPS state counting in supersymmetric gauge theory and string theory, see for example [N05] for a review of the applications to instanton counting.

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