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USE OF HISTORY IN A RESEARCH WORK ON THE TEACHING OF LINEAR ALGEBRA

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Linear algebra is universally recognized as a very important subject not only within mathematics but also with regards to its applications to physics, chemistry, economics, etc. In the modern organization of mathematical subjects, a vector space is one of the simplest algebraic structures. Nevertheless this simplicity is not easily recognized by students who often have great difficulty at the start of a linear algebra course in the first year of university. This paper will focus on the French context of this teaching, but several research works have drawn out similar difficulties in various countries around the world¹.

In France, the teaching of linear algebra was entirely remodeled with the "reform of modern mathematics" in the sixties. At that time, the influence of Bourbaki and a few others led to the idea -which was based on a very democratic concern- that geometry could be more easily accessible to students if it were founded on the axioms of the structure of affine spaces. Therefore the axiomatic theory of finite-dimensional vector spaces was taught in the first year of secondary school (age 15). The fate of this reform and the reaction it aroused are well known. Therefore, from the beginning of the eighties, the reform of the teaching of mathematics in French secondary schools gradually led to the removal of any subject related to modern algebra. Moreover, the teaching of geometry focused on the study of transformations on elementary figures and analytical geometry is now barely taught at secondary level. On the other hand, formal theories became unpopular and students entering university nowadays have very little practice of any formal mathematical subject. This situation created a total change in the background of students, for whom the teaching of linear algebra represented the first contact with such a "modern" approach. Of course the teaching in first year of university changed and became less theoretical. In many universities², it was decided to prepare the students for the teaching of

¹ cf. e.g. the special issue on linear algebra of *The College mathematical Journal* **24** (1) (1993).

² Unlike secondary schools, for which a national program is imposed by the ministry of education, the curricula of universities are decided locally, even though changes from one university to the other are usually superficial.

linear algebra by a preparatory course in Cartesian geometry or/and by a course in logic and set theory. Yet, in secondary school, students still learn the bases of vector geometry and the solving of systems of linear equations by Gaussian elimination. Therefore, they have some knowledge on which the teaching of linear algebra can be based. For the moment, the idea of teaching students the axiomatic elementary theory of vector spaces within the first two years of science university has not be questioned seriously, and the teaching of linear algebra, in France, remains quite formal.

In this context, A. Robert and J. Robinet (1989) showed that the main criticisms made by students toward linear algebra concern the use of formalism, the overwhelming amount of new definitions and the lack of connection with what they already know in mathematics. It is quite clear that many students have the feeling of landing on a new planet and are not able to find their way in this new world. On the other hand, teachers usually complain of their students' erratic use of the basic tools of logic or set theory. They also complain that students have no skills in elementary Cartesian geometry and consequently cannot use intuition to built geometrical representation of the basic concepts of the theory of vector spaces. These complaints correspond to a certain reality, but the few attempts of remediation -with previous teaching in Cartesian geometry or/and logic and set theory- did not seem to improve the situation substantially. Indeed, many works have shown that difficulties with logic or set theory cannot be interpreted without taking into account the specific context in which these tools are used. In my first work (1990a), I tested with statistical tools the correlation between the difficulties with the use of the formal definition of linear independence and the difficulties with the use of the mathematical implication in different contexts. Although these two types of difficulties seemed at first closely connected, the results showed clearly that no systematic correlation could be made. This means that students' difficulties with the formal aspect of the theory of vector space are not just a general problem with formalism but mostly a difficulty of understanding the specific use of formalism in the theory of vector spaces and the interpretation of the formal concepts in relation with more intuitive contexts like geometry or systems of linear equations, in which they historically emerged. We will analyze this point in more details in this paper.

In collaboration with A. Robert, J. Robinet, and M. Rogalski, we have developed a research program on the learning and teaching of linear algebra in first year of French science university. This work, which started some ten years ago, includes the elaboration and evaluation of experimental teaching but also an epistemological reflection which is built in a dialectical process on a historical analysis of the genesis of the concepts of linear algebra and a didactical analysis of the teaching and the difficulties of the students.³ In this

³ The results of this work are now gathered in a book (Dorier 1997) with the presentation of other works (in France, Canada, USA and Morocco).

paper, I will try to summarize the main issues of our research, focusing on a restricted set of concepts: linear dependence and independence, generators, basis, dimension and rank.

I. LOGIC VERSUS MEANING : THE ROLE OF FORMALISM

Linear dependence and independence, generators, basis, dimension and rank are the elementary concepts which constitute the foundations of the theory of vector spaces. For any mathematician, they seem very simple, clearly interrelated notions. Indeed, in the formal language of modern algebra they correspond to easily expressible definitions. Moreover, the logic of a hypothetico-deductive presentation induces a "natural" order between them (more or less the order given above) which reflects their intrinsic network of relations.

In a work in preparation⁴, A. Behaj has interviewed several students from second to fourth year of French and Moroccan universities. He asked them, in pairs, to build the skeleton of a course, to be addressed to first year students, on these basic notions of linear algebra. It is quite surprising to see that nearly all of them have the same initial reaction : they present the notions in their supposedly natural logical order. They do it consciously and they give justifications on the ground of logic and simplicity. Yet, most of them, when they are asked to give applications and exercises in relation with the course, gradually abandon (more or less explicitly) this "natural" order and are led to reorganize the network of relations according to a less conscious construction built within time, after using these notions in several contexts. Similar phenomena have been revealed when interviewing teachers. Not that they change their presentation, but they clearly -although it may not be really conscious- show two contradictory concerns when structuring their course : a concern for the logic of a hypothetico-deductive presentation and a concern for the applicability of their course to exercises and problems. These two concerns induce different organizations of the notions. Therefore, most of the time, the teacher's course is apparently organized according to standards of logic, proper to the rigor of a mathematical text, but the choices of examples, remarks, etc. create a secondary level of organization in relation to the solving of exercises and problems and also according to what they know of the difficulties of their students. For instance the understanding of the notion of generators is very rarely evaluated in exercises. Indeed, teachers know that once the dimension of a subspace is known (n), n independent vectors generate the sub-space; they also know that proving that vectors are independent is easier than proving that they generate a sub-space.

⁴ A. Behaj is a lecturer in the university of Fès (Morocco). He is preparing a doctorate supervised by G. Arsac and myself on the concept of structuration of knowledge in mathematics.

Thus the notion of generator holds a very different position within their formal presentation of the theory from the one it has within the solving of exercises.

Such observations may certainly be made about several different mathematical notions. Yet, this group of notions is particularly interesting, because each of them is somehow elementary, not in the sense that it is simple, but in the sense that it is an element which will be part of a more complex whole set of notions. In this sense, it is true that the notions of linear independence and generators are more elementary than the notion of basis, because a basis is usually defined as a set of linearly independent generators. Yet, a basis can be defined as a minimal set of generators or as a set of units, of which each element of the vector space is a unique linear combination. In these two approaches, there is no reason to say that linear independence is more elementary than the notion of basis which can be defined independently of the notion of linear independence. On the other hand, a finite-dimensional vector space may be defined as having a finite maximal set of independent vectors. Then a basis can be defined as any maximal set of independent vectors, without using the notion of generators, which has then no reason for being more elementary than the notion of basis. Although this last alternative is somehow "unnatural", the first two are sometimes chosen in textbooks or actual teaching. However all three are logically consistent.

I will draw two conclusions from these remarks :

- Any logical construction is partly arbitrary and cannot be qualified as natural, without further epistemological investigations.
- The nature and meaning of concepts is to be found beyond their logical inter-relations.

Although this may seem obvious, it is very important as far as such a theory as that of vector spaces is concerned. Indeed, in its modern axiomatic version, this theory has been highly formalized, therefore it is tempting to reduce its content to the logical network of relations between formal concepts. On the other hand, as a reaction to this extreme position, one may want to give only "practical" knowledge with reference to "practical" contexts like geometry, linear equations, differential equations, etc. But this last option reduces the meaning and general nature of linear algebra. Indeed, the formal aspect of the theory of vector spaces is the result of its general nature and a condition for its simplicity. Therefore one cannot spare the students the difficulty of formalism if the theory is to be understood with all the meaning it has now acquired. Moreover, we put forward the hypothesis that the necessity for formalism has to be understood very early in the learning of the theory. Our historical analysis (Dorier 1995a and 1997 (first part)) has been, to a great extent, conducted in order to clarify and support the preceding statement. Our epistemological reflection has now led us to understand more clearly the different stages of unification and generalization in the genesis of linear algebra but also the role played by the different contexts of origin (in geometry, linear systems and determinants, algebra, and

functional analysis). The axiomatic approach of the theory only prevailed around 1930 and was very long to be accepted by mathematicians, even in the early history of functional analysis where infinite determinants were preferred by many until Banach's 1932 *Théorie des opérations linéaire* (Dorier 1996). Yet, once accepted, the axiomatic approach was quick to replace all previous analytical approach in all fields, and its power of generalization and unification was universally recognized. In teaching, formalism should not be introduced too early and imposed without care. A formal concept has to be introduced with reference students' conceptions previously acquired in intuitively based context, as a means for generalization and simplification.

Let us take an example with the notions of linear independence and dependence.

II. THE CASE OF LINEAR DEPENDENCE AND INDEPENDENCE

In the language of the modern theory these two notions are extremely simple. They can be defined as two logically opposite notions in the language of basic set theory. Yet, even if students can be easily trained to solve standard questions like "is this set of vectors independent or not ?" in various contexts, the use of these notions in less straightforward situations may be much less easy. On the other hand, the historical elaboration of the formal definition of linear independence was not as easy as one could imagine. I will start by developing these two aspects : the students' difficulties and the historical evolution then in a third part I will draw out some conclusion on the basis of a coordinated epistemological synthesis of the first two parts.

1) The difficulties of the students

Anyone who has taught a basic course in linear algebra knows how difficult it may be for a student to understand the formal definition of linear independence and to apply it to various contexts. Moreover, once students have proved their ability to check whether a set of n -tuples, equations, polynomials or functions are independent⁵, they may not be able to use the concept of linear independence in more formal contexts.

Let us take a few examples of exercises⁶ given to beginner students to obtain a better idea of this kind of mistake.

⁵ I will not say more in this paper about the difficulties students may have in carrying out these standard tasks, not that there are none but because I will focus on another level of difficulty, less technical and more conceptual.

⁶ The three examples given here have been tested by A. Robert and J. Robinet (1989). They are not original exercises, but they reveal important recurrent mistakes of the students.

1. Let U, V and W be three vectors in \mathbb{R}^3 , if they are two by two non collinear, are they independent ?

Many students think that this proposition is trivial, they are sure that it is true. If they want to prove it by using the formal definition, they are led to say more or less any kind of non sense just for the sake of giving a formal proof (as required by their teacher) to a statement of which they are convinced⁷. Nevertheless, this mistake is one aspect of a more general difficulty with the global aspect of the concept of linear (in)dependence. Indeed, many students are inclined to treat the question of linear (in)dependence by successive approximations starting with two vectors, and then introducing the others one by one. We will say that they have a local approach to a global question. Indeed, in many cases, at least if it is well controlled, this approach may be correct and actually quite efficient, yet, it is a source of mistakes in several situations. The students have built themselves what G. Vergnaud (1990) call *théorèmes-en-acte* (i.e. rules of action or theorems which are valid in some restricted situations but create mistakes when abusively generalized to more general cases). Here is a non-exhaustive list of *théorèmes-en-acte* connected with the local approach of linear (in)dependence, that we have noticed in students' activities:

- if U and V are independent of W , then U, V and W are globally independent
- if U_1 is not a linear combination of U_2, U_3, \dots, U_k , then U_1, U_2, \dots, U_k are independent
- if U_1, V_1 and V_2 are independent and if U_2, V_1 and V_2 are independent, U_1, U_2, V_1 and V_2 are independent.⁸

2.1. Let U, V and W be three vectors in \mathbb{R}^3 , and f a linear operator in \mathbb{R}^3 , if U, V and W are independent, are $f(U), f(V)$ and $f(W)$ independent ?

2.2. Let U, V and W be three vectors in \mathbb{R}^3 , and f a linear operator in \mathbb{R}^3 , if $f(U), f(V)$ and $f(W)$ are independent, are U, V and W independent ?

To answer these questions, beginners usually try to use the formal definition without first building concrete examples that would help them to obtain an idea of the result. Then, they try different combinations with the hypotheses and the conclusions, and finally answer "yes" to the first question and "no" to the second one, despite coming close to writing the correct proof for the correct answers. Here is a reconstructed proof that reflects the difficulties of the students:

$$\text{If } \alpha U + \beta V + \gamma W = 0 \text{ then } f(\alpha U + \beta V + \gamma W) = 0$$

$$\text{so } f \text{ being a linear operator: } \alpha f(U) + \beta f(V) + \gamma f(W) = 0,$$

⁷ This is what French didacticans will call "un effet de contrat".

⁸ For instance when asked what is the intersection of the two subspaces generated by U_1, U_2 , and V_1, V_2 , students prove that neither U_1 nor U_2 , are a linear combination of V_1 and V_2 , and conclude that the intersection is reduced to 0.

now as U, V and W are independent, then $\alpha = \beta = \gamma = 0$,
so $f(U), f(V)$ and $f(W)$ are independent.

In their initial analysis, Robert and Robinet concluded that this type of answer was relevant of a bad use of the mathematical implication characterized by the confusion between hypothesis and conclusion. This is indeed a serious difficulty in the use of the formal definition of linear independence. As mentioned above, I tested the validity of this hypothesis with different students. Before the teaching of linear algebra, I set up a test to evaluate the students' ability in elementary logic and particularly in the use of the mathematical implication (Dorier 1990a and b), and after the teaching of linear algebra, I gave the same exercise as above to the students. The results showed that the correlation was insignificant, in some cases it was even negative. Yet, on the whole (both tests included many questions), there was quite a good correlation between the two tests. This shows that if a certain level of ability in logic is necessary to understand the formalism of the theory of vector spaces, general knowledge, rather than specific competence is needed. Furthermore, if some difficulties in linear algebra are due to formalism, they are specific to linear algebra and have to be overcome essentially in this context.

On the other hand, some teachers may argue that, in general, students have many difficulties with proof and rigour. Several experiments that we have made with students showed that if they have connected the formal concepts with more intuitive conceptions, they are in fact able to build very rigorous proves. In the case of the preceding exercise for instance, after the test, if you ask to the students to illustrate the result with an example individually, let us say in geometry, they usually realize very quickly that there is something wrong. It does not mean that they are able to correct their wrong statement, but they know it is not correct. Therefore one main issue in the teaching of linear algebra is to give our students better ways of connecting the formal objects of the theory with their previous conceptions, in order to have a better intuitively based learning. This implies not only giving examples but also to show how all these examples are connected and what is the role of the formal concepts with regard to the mathematical activity involved.

2) Historical background

The concept of linear (in)dependence emerged historically at first in the context of linear equations (Dorier 1993, 1995 and 1997). Euler's text entitled *Sur une contradiction apparente dans la doctrine des lignes courbes* is the first one in which a question of dependence between equations was discussed. Euler's concern was to solve Cramer's paradox. This paradox first drawn out by Cramer and MacLaurin was based on two proposition commonly admitted in the beginning of the 18th century :

1. An algebraic curve of order n is uniquely determined by **Erreur !** of its points.

This was clear from elementary combinatorics by counting the coefficient of the equation of such a curve.

2. Two algebraic curves of orders n and m intersect in nm points.

It was known that some points may be multiple, infinite or imaginary, but one knew cases when all these nm points were finite real and all distinct.

Therefore for $n \geq 3$, it appears that n^2 points (common to two algebraic curves of order n) would not be sufficient to determine uniquely an algebraic curve, while the first proposition states that **Erreur!** should determine one and only one such curve. This is Cramer's paradox. Euler discussed the validity of both propositions and of their consequences and concludes that the first one, based on the fact that n linear equations determine exactly n unknown values should be restricted. At that time, the general idea that n equations determine n unknowns was so strong that nobody had taken the pain to discuss the odd case, until Euler pointed out this particularity.

He starts by an example with two equations :

Let us just look at these two equations $3x - 2y = 5$ and $4y = 6x - 10$, one will see immediately that it is not possible to determine the two unknowns x and y , as if one eliminates x then the other unknown y disappears by itself and one gets an identical equation, from which it is not possible to determine anything. The reason for this accident is quite obvious as the second equation can be changed into $6x - 4y = 10$, which being simply the first one doubled, is thus not different.⁹

It is clear -especially by reading the end of this quotation- that Euler does not intend to fool his reader, even though he artificially hides the similarity of the two equations. Yet, it is also clear that it is not the fact that the two equations are similar that determines the dependence of the equations, but the fact that something unusual -an accident- happens in the final step of the solving process. This accident reveals the dependence of the equations, because, although there are two of them, these equations do not determine two unknowns. Mathematically speaking, the two statements are logically connected, a linear dependence between n equations in n unknowns is equivalent to the fact that the system will not have a unique solution; However the two properties correspond to two different conceptions of dependence. To be able to distinguish these two conceptions, I will call Euler's conception, *inclusive dependence*. I wish to insist on the fact that this conception is natural in the context in which Euler and all the mathematicians of his time were working, that is to say

⁹ *On n'a qu'à regarder ces deux équations : $3x - 2y = 5$ et $4y = 6x - 10$ et on verra d'abord qu'il n'est pas possible d'en déterminer les deux inconnues x et y , puisqu'en éliminant l'une x , l'autre s'en va d'elle-même et on obtient une équation identique, dont on est en état de ne déterminer rien. La raison de cet accident saute d'abord aux yeux puisque la seconde équation se change en $6x - 4y = 10$, qui n'étant que la première doublée, n'en diffère point. [Euler 1750, 226]*

with regard to the solving of equations, and not the study of equations as objects on their own.

To make my statement clearer let us see what Euler says in the case of four equations: he gives the following example:

$$\begin{aligned} 5x + 7y - 4z + 3v - 24 &= 0, \\ 2x - 3y + 5z - 6v - 20 &= 0, \\ x + 13y - 14z + 15v + 16 &= 0, \\ 3x + 10y - 9z + 9v - 4 &= 0, \end{aligned}$$

they would be worth only two as having extracted from the third the value of

$$x = -13y + 14z - 15v - 16$$

and having substituted this value in the second to get:

$$y = \frac{33z - 3v - 52}{29} \quad \text{et} \quad x = \frac{-23z + 33v + 212}{29},$$

this two values of x and y being substituted in the first and the fourth equations will lead to identical equations¹⁰, so that the quantities z and v will remain undetermined.¹¹

Here again the proof is based on the solving of the system of four equations by substitution that leads to two undetermined quantities. He does not even mention any linear relations between the equations, although they may seem rather obvious like: (1) - (2) = (4) and (1) - 2x(2) = (3) (for instance). Therefore the dependence expressed by Euler is not linear but it is a property of the equations that makes the network of constraints they impose on the unknowns equivalent to two constraints and not four, this is what we propose to call inclusive dependence.

Euler's use of terms such as *comprised* or *contained*, refers to the conception of inclusive dependence as we explained above. It does not mean that Euler was not aware of the logical equivalence with the fact that there exist linear relations between the equations, but, within his practice with linear equations, the conception of inclusive dependence is

¹⁰ Euler does not mean here that the two equations are the same but that each of them is an identity, like $0 = 0$, i.e. always true.

¹¹

$$\begin{aligned} 5x + 7y - 4z + 3v - 24 &= 0, \\ 2x - 3y + 5z - 6v - 20 &= 0, \\ x + 13y - 14z + 15v + 16 &= 0, \\ 3x + 10y - 9z + 9v - 4 &= 0, \end{aligned}$$

elles ne vaudroient que deux. car ayant tiré de la troisième la valeur de

$$x = -13y + 14z - 15v - 16$$

et l'ayant substituée dans la seconde pour avoir :

$$y = \frac{33z - 3v - 52}{29} \quad \text{et} \quad x = \frac{-23z + 33v + 212}{29},$$

ces deux valeurs de x et de y étant substituées dans la première et la quatrième équation conduiront à des équations identiques¹¹, de sorte que les quantités z et v resteront indéterminés. (ibid., 227)

more consistent and efficient. Yet, there is a difficulty for further development; indeed, the conception of inclusive dependence is limited to the context of equations and cannot be applied to other objects like n -tuples for instance. Therefore inclusive dependence is context-dependent although linear dependence is a general concept that applies to any object of a linear structure.

Another aspect of Euler's work is worth a special attention, it concerns the passage from two to three equations. Indeed, for three equations, Euler says:

[...] *The first one, being not different from the third one, does not contribute at all in the determination of the three unknowns.*

*But there is also the case when one of the three equations is contained in the two others. [...] So when it is said that to determine three unknowns, it is sufficient to have three equations, it is necessary to add the restriction that these three equations are so different that none of them is already comprised in the others.*¹²

It is important to notice that, Euler separates the case when two equations are equal from the case when the three equations are globally dependent. This points out an intrinsic difficulty of the concept of dependence which has to take all the equations in a whole into account, and not only the relations in pairs. We have seen that students have real difficulties with this point.

Moreover, in this text, Euler was able to bring out issues that can be considered in many aspects as the first consistent ideas on the concept of rank.

For instance the quotation above, about four equations, is an illustration of the relation:

$$(number\ of\ unknowns) - (rank\ of\ the\ system) = (dimension\ of\ the\ set\ of\ solutions)$$

indeed, there are four unknowns, the equations "are worth only two" (their rank is two) and "two unknown quantities will remain undetermined" (the dimension of the set of solutions is two). Of course this result remains implicit and is not formalized, but Euler shows, through a few numerical examples never exceeding five unknowns and five equations, that he had an intuitive yet accurate and consistent conception of the relation between the size of the set of solutions and the number of relations of dependence between the equations of the system. In this sense, the specific context of Cramer's paradox helped him to elaborate such a reflection as shown in the following statement:

¹² [...] *La première ne différant pas de la troisième, ne contribue en rien à la détermination des trois inconnues.*

Mais il y a aussi le cas, où une des trois équations est contenue dans les deux autres conjointement [...] Ainsi quand on dit que pour déterminer trois inconnues, il suffit d'avoir trois équations, il y faut rajouter cette restriction, que ces trois équations diffèrent tellement entr'elles, qu'aucune ne soit déjà contenue dans les deux autres. [ibid., 226]

When two lines of fourth order¹³ intersect in 16 points, as 14 points, when they lead to equations all different among themselves¹⁴, are sufficient to determine a line of this order, these 16 points will always be such that three or even more of the equations they produce are already comprised in the others. So that these 16 points do not determine anything more than if they were 13 or 12 or even less and in order to determine the line completely we will have to add to these 16 points one or two points.¹⁵

We will see now that it took more than a century for the concept of rank to come to its maturity.

1750 is also the year when Cramer published the treatise that introduced the use of determinants which was to dominate the study of linear equations until the first quarter of the 20th century. In this context, dependence of equations but also of n -tuples was characterized by the vanishing of the determinant. However determinants produced technical tools, that were not always appropriate for approaching easily questions like the relation between the size of the set of solutions and the number of relations of dependence between the equations, as Euler had done in a very intuitive approach. On the other hand, the conception of inclusive dependence, which was still dominant prevented equations and n -tuples being treated as identical objects with regards to linearity. Thus, this was an obstacle to the use of duality reasoning.

But the concept of rank is intrinsically connected to duality as it is not only an invariant of a subspace but also of its orthogonal (i.e. its representation by equations). In the context of equations, to make all the aspects of rank explicit it is necessary to consider all the equivalent systems to a given system of homogeneous linear equations, to show that its set of solutions cannot be represented by less than a certain number (the rank of the system) of equations and this uses duality (see Dorier 1993). Therefore abandoning inclusive dependence for linear dependence was a necessary and decisive step toward the determination of the concept of rank.

¹³ This is the usual terms of the time to designate what is today known as an algebraic curve of order 4.

¹⁴ Euler uses several time of this ambiguous formulation to express the fact that the equations are independent. It is another proof of the difficulty of the global aspect of dependence as well as of the difference between linear and inclusive dependence.

¹⁵ *Quand deux lignes du quatrième ordre s'entrecoupent en 16 points, puisque 14 points, lorsqu'ils conduisent à des équations toutes différentes entr'elles, sont suffisants pour déterminer une ligne de cet ordre, ces 16 points seront toujours tels que trois ou plusieurs des équations qui en résultent sont déjà comprises dans les autres. De sorte que ces 16 points ne déterminent plus que s'il n'y en avoit que 13 ou 12 ou encore moins et partant pour déterminer la courbe entièrement on pourra encore à ces 16 points ajouter un ou deux points.* (ibid., 233)

In the second half of the 19th century many mathematicians made great progress toward this goal. But they used very elaborate technical tools within the theory of determinants and never really made anything clearly explicit. All the works that we have analyzed in this period aimed to give a rule for solving any system of linear equations in the most concise manner. The idea was to choose one of the non zero minors of maximal order (r). The r unknowns and r equations involved in this minor were called the main equations and the main unknowns¹⁶. One can apply Cramer's rule to the r main equations, the other unknowns being put on the other side of the equality with the constant terms. Therefore the value of the r main unknowns are given as functions of the other unknowns. By substitution of these values in the other equations, one gets the conditions of consistency of the system. This practical rule for solving a system made the value of r central in the relation between the size of the set of solutions and the number of relations of dependence between the equation. Yet the specific choice of the minor had to be discussed with regard to the invariance of the result. In order to abstract the concept of rank, from this technique, one had to take a more general point of view and to consider the dual problem of putting in relation a set of solutions with all systems representing this set (i.e. having this set of solutions).

The decisive improvement was made in 1875 by Frobenius. In this text about Pfaffsch problem, there is an important section about linear systems. He considers a general system¹⁷ of homogeneous linear equations :

$$a_1^{(\mu)}u_1 + a_2^{(\mu)}u_2 + \dots + a_n^{(\mu)}u_n = 0, \quad (\mu = 1, \dots, m)$$

Immediately, he gives the definition of linear independence of k solutions of this system of linear homogeneous equations:

Several particular solutions

$$A_1(\chi), A_2(\chi), \dots, A_n(\chi) \quad (\chi = 1, 2, \dots, k)^{18}$$

will be said independent or different, if $c_1A_{\alpha}^{(1)} + c_2A_{\alpha}^{(2)} + \dots + c_kA_{\alpha}^{(k)}$ cannot be all zero for $\alpha = 1, 2, \dots, n$, without c_1, c_2, \dots, c_k being all zero, in other words if the k linear forms $A_1(\chi)u_1 + A_2(\chi)u_2 + \dots + A_n(\chi)u_n$ ($\chi = 1, \dots, k$) are independent¹⁹.

¹⁶ The terms "*équations principales*" and "*inconnues principales*" used by Rouché and Fontenay is very popular in France. Yet it seems that no English equivalent has been used systematically.

¹⁷ In fact he starts with a system of independent equations and then generalizes to any kind of system.

¹⁸ These are k solutions. We would maybe note today $A(\chi) = (A_1(\chi), A_2(\chi), \dots, A_n(\chi))$ each of them. Therefore: $a_1^{(\mu)}A_1(\chi) + a_2^{(\mu)}A_2(\chi) + \dots + a_n^{(\mu)}A_n(\chi) = 0$, for each $\mu = 1, \dots, m$ and each $\chi = 1, 2, \dots, k$.

¹⁹ *Mehrere particuläre Lösungen*

$$A_1(\chi), A_2(\chi), \dots, A_n(\chi) \quad (\chi = 1, 2, \dots, k)$$

Not only this is a definition quite similar to the modern definition of linear independence but it also explicitly shows the similarity between n-tuples of solutions and equations with regard to their linearity. This simple idea allows Frobenius to give in a couple of pages a full overview of the properties of the rank of a system (yet still defined implicitly²⁰ as the maximal order of non zero minor). The main idea is to use the concept of associated (*zugeordnet* oder *adjungiert*) system, which is, in modern terms, the representation by equations of the orthogonal subspace.

Let us consider the system²¹ :

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = 0 \\ \dots \\ a_{p1}x_1 + \dots + a_{pn}x_n = 0 \end{cases} \quad (\text{I})$$

if $(A_1(\chi), A_2(\chi), \dots, A_n(\chi))$ ($\chi = 1, 2, \dots, n-r$) -r being the maximal order of non-zero minors- is a basis of solutions of (I), the associated system is:

$$\begin{cases} A_1^{(1)}x_1 + \dots + A_n^{(1)}x_n = 0 \\ \dots \\ A_1^{(n-r)}x_1 + \dots + A_n^{(n-r)}x_n = 0 \end{cases} \quad (\text{I}^*)$$

again if $(B_1^{(v)}, B_2^{(v)}, \dots, B_n^{(v)})$ ($\chi = 1, 2, \dots, q$) is a basis of solutions of (I*), then the associated system is:

$$\begin{cases} B_1^{(1)}x_1 + \dots + B_n^{(1)}x_n = 0 \\ \dots \\ B_1^{(q)}x_1 + \dots + B_n^{(q)}x_n = 0 \end{cases} \quad (\text{I}^{**})$$

Frobenius proves that, whatever the choices of base are, (I**) is equivalent to (I) and $q = r$.

This first result on duality in finite-dimensional vector spaces shows the double level of invariance connected to rank both for the system and for the set of solutions. Moreover, Frobenius' approach allows a system to be seen as part of a class of equivalent systems having the same set of solutions: a fundamental step toward the representation of subspaces by equations.

This short analysis²² shows how adopting a formal definition (here of linear dependence and independence) may be a fundamental step in the construction of a theory, and is therefore an essential intrinsic constituent of this theory and not only a change of style. It also points out several epistemological difficulties attached to the concepts of linear dependence and independence.

sollen daher unabhängig oder verschieden heissen, wenn $c_1A_\alpha^{(1)} + c_2A_\alpha^{(2)} + \dots + c_kA_\alpha^{(k)}$ nicht für $\alpha = 1, 2, \dots, n$, verschwinden kann, ohne dass c_1, c_2, \dots, c_k sämtlich gleich Null sind, mit andern Worten wenn die k linearen Formen $A_1(\chi)u_1 + A_2(\chi)u_2 + \dots + A_n(\chi)u_n$ ($\chi = 1, \dots, k$) unhanabhängig sind [Frobenius 1875, 255].

²⁰ The term of rank (*Rang*) will be introduced for the first time in [Frobenius 1879, 1]

²¹ This is a summary of Frobenius' ideas with adapted notations and vocabulary.

²² For more detail see (Dorier 1993 and 1997).

3) Epistemological synthesis

In his work, R. Ousman (1996) gave a test to students in their final year of *lycée* (just before entering university). Through this test, he wanted to analyze the conception of students on dependence in the context of linear equations and in geometry before the teaching of the theory of vector spaces. He gave several examples of systems of linear equations and asked the students whether the equations were independent or not. The answers showed that the students justify their answer through the solving of the system and very rarely by mentioning linear relations between the equations. In other words they very rarely give a justification in terms of linear combinations but most of time in terms of equations vanishing or unknowns remaining undetermined. Their conception of (in)dependence is, like Euler's, that of inclusive dependence and not linear dependence. Yet, this is not surprising, as these students, like Euler and the mathematicians of his time, are only concerned by the solving of the system, therefore inclusive dependence is more natural and more relevant for them.

This established fact and our previous historical analysis lead us to a didactical issue. When entering university, students already have conceptions of concepts like linear (in)dependence in several contexts²³; when they learn the formal concept, they have to understand the connection with their previous conceptions. If not, they may have two conceptions of the same concept and yet not know clearly that they refer to the same concept. Moreover, making the connection helps in giving the formal concept a better intuitive foundation. Yet, the students must also understand the role of the formal concept and have an idea of the improvement it brings. In the case of linear algebra, and more specifically of linear (in)dependence, the formal concept is the only means to comprehend all the different types of "vectors" in the same manner with regard to their linearity. In other words, students must be aware of the unifying and generalizing nature of the formal concept. In our research, we used what we called the *meta lever*. Therefore we build teaching situations leading students to reflect on the epistemological nature of the concepts with explicit reference to their previous knowledge (Dorier 1992, 1995b and 1997 and Dorier et al. 1994a and b). In this approach, the historical analysis is a source of inspiration as well as a means of control. Yet, these activities must not be only a speech of the teacher, nor a reconstruction of the historical development, they must reconstruct an

²³ We only mentioned the context of equations here, but it is clear that they have conceptions in other contexts. In geometry, if they know the vectors (as it is the case in France) their conception will be close to the formal concept, yet, they had a conception prior to vectorial geometry, connected to the ideas of alignment and coplanarity.

epistemologically controlled genesis taking into account the specific constraints of the teaching context.

For instance, with regard to linear (in)dependence, French students entering university normally have a good practice of Gaussian elimination for solving systems of linear equations. It is therefore possible in the beginning of the teaching of linear algebra to make them reflect on this technique not only as a tool but also as a means to investigate the properties of the systems of linear equations. This does not conform to the historical development, as the study of linear equations was historically mostly held within the theory of determinants. Yet, Gaussian elimination is a much less technical tool and a better way for showing the connection between inclusive dependence and linear dependence as identical equations (in the case when the equations are dependent) are obtained by successive linear combinations of the initial equations. Moreover, this is a context in which such question as "what is the relation between the size of the set of solutions of a homogeneous system and the number of relations of dependence between the equations ?" can be investigated with the students as a first intuitive approach for the concept of rank.²⁴ M. Rogalski has experimented teaching sequences illustrating these ideas (Rogalski 1991, Dorier et al. 1994a and b and Dorier 1992 and 1997).

Here is for instance an exercise given to the students to illustrate this idea:

A magic square of order 4 and of sum zero is a square matrix of order 4 with real coefficients such that the sum along each column, each row and both diagonals is zero. Without any calculation, give an evaluation as precise as possible of the number of entries that you can freely chose in any magic square of order 4 and of sum zero.

The number that the students have to evaluate is of course the dimension of the space of magic square of order 4 and sum zero. It is less than 16 (the number of entries in a square of side 4) and more than $16 - 10 = 6$, because the coefficients are solutions of a system of 10 equations. To be more precise one must know the number of dependences, i. e. the rank, of the equations. Without calculation, it is easy to see that at least three equations are independent (for instance the three equations expressing that the sum along each row is zero), so that the rank is at least 3 and the dimension is therefore less than $16 - 3 = 13$. It would not be very difficult to be even more precise.

This exercise is interesting because it operates with large dimensions, yet the equations are quite simple. Moreover its concrete framework makes the question more accessible to the students, even if they do not know the formal concepts of dimension and rank. It is important though to prevent tedious calculations and to emphasize intuitive reasoning even if the outcome is less accurate.

²⁴ As far as we know Gauss himself has never come near such an investigation.

On another level, the historical and didactical analyses confirm the fact that there is an epistemological difficulty in treating the concept of linear (in)dependence as a global property (remember the distinctions made by Euler). It follows special care must be taken in the teaching regarding this point. For instance exercises such as the first one analyzed above can be discussed with the students. Moreover, for the teacher, knowing the type of *théorèmes-en-acte*, that students may have built, must help them in understanding the mistakes of the students and therefore in correcting them more efficiently.

Finally we will give the scheme of a teaching experiment that we have set up for the final step in the teaching when introducing the formal theory after having made as many connections as possible with previous knowledge and conceptions in order to build better intuitive foundations.

After the definition of a vector space and sub-space and linear combination, the notion of generator is defined. A set of generators gathers all the information we have on the sub-space, it is therefore interesting to reduce it to the minimum. Therefore, the question is to know when it is possible to take away one generator, the remaining vectors still being generators for the whole sub-space. The students easily find that the necessary and sufficient condition is that the vector to be taken away must be a linear combination of the others. This provides the definition of linear dependence: "a vector is linearly dependent on others if and only if it is a linear combination of them". This definition is very intuitive, yet it is not completely formal, and it needs to be specified for sets of one vector. It induces without difficulty the definition of a set of independent vectors as a set of which no vector is a linear combination of the others. To feel the need for a more formal definition, one just has to reach the application of this definition. Indeed, students must answer the question: "are these vectors independent or not?". With the definition above, they need to check that each vector, one after the other, is a linear combination of the others. After a few examples, with at least three vectors, it is easy to explain to the students that it would be better to have a definition in which all the vectors play the same role (it is also interesting to insist on the fact that this is a general statement in mathematics). One is now ready to transform the definition of linear dependence into : "vectors are linearly dependent if and only if there exists a zero linear combination of them, whose coefficients are not all zero." The definition of linear independence being the negation of this, it is therefore a pure problem of logic to reach the formal definition of linear independence. A pure problem of logic, but in a precise context, where the concepts have made sense to the students with an intuitive background.

This approach has been proven to be efficient with regard to the student's ability to use the definitions of linear dependence and independence, even in formal contexts such as in the three exercises discussed in section II.1.

Moreover, it is quite a discovery for the student to realize that a formal definition may be more practical than an "intuitive" one. In Behaj's work, quoted above, it was clearly shown that this fact is not clear for many students and even for some of their teachers. Most of them keep seeing the fact that a vector is a linear combination of the others as a consequence of the definition of linear dependence. Therefore they believe that this consequence is the practical way of proving that vectors are or are not independent, even if that goes contrary to their use of these definitions. Yet, imagine that one has to check whether three vectors u , v and w are independent. If one proves that u is not a linear combination of v and w , there is still a chance that v and w would be collinear in which case the three vectors are dependent. Many students would conclude from the first step that the three vectors are independent. Doing so they would not be easily contradicted because in most of the cases the result is true. One can easily imagine that starting by proving that one vector is not a linear combination of the others will get even more dangerous with more than three vectors. In theory, if one applies this definition, one has to check that each vector is not a linear combination of the others. There are shortcuts (like in our example, if you check that v and w are not collinear) but they require students to have a good control of the meaning of linear dependence. Therefore the formal definition is a good means to prevent false justifications. On the other hand, it is useful when one knows that some vectors are dependent to use it through the fact that one vector is a linear combination of the others because this is what is meaningful.

4) Conclusions

Formalism is what students themselves confess to fear most in the theory of vector spaces. One didactical solution is to avoid formalism as far as possible, or at least to make it appear as a final stage gradually. Because we think that formalism is essential in this theory (our historical analysis has confirmed this epistemological fact), we give a different answer: formalism must be put forward in relation to intuitive approaches as the means of understanding the fundamental role of unification and generalization of the theory. This has to be an explicit goal of the teaching. This is not incompatible with a gradual approach toward formalism, but it induces a different way of thinking out the previous stages. Formalism is not only the final stage in a gradual process in which objects become more and more general, it must appear as the only means of comprehending different aspects within the same language. The difficulty here is to give a functional aspect to formalism while approaching it more intuitively.

Linear dependence is a formal notion that unifies different types of dependence which interact with various previous intuitive conceptions of the students. It has been shown

above how in the historical development of linear algebra the understanding of this fact was essential for the construction of the concepts of rank and duality. In the teaching, this questioning has to be made explicit -by using the meta lever-, if we do not want misunderstanding to persist. Therefore even at the lowest levels of the theory the question of formalism has to be raised in interaction with various contexts where previous intuitive conceptions have been built by the students. The construction of a formal approach right from the beginning is a necessary condition for the understanding of the profound epistemological nature of the theory of vector spaces. In this sense, formalism has to be introduced as the answer to a problem that students are able to understand and to make their own, in relation to their previous knowledge in fields where linear algebra is relevant. These include at least geometry and linear equations but may also include polynomials or functions, although in those fields one may encounter more difficulties.

III. CONCLUSION

From the example of linear (in)dependence, we can now draw out some conclusions about the epistemological reflection we have conducted on the bases of our didactical and historical research works. In the experimental teaching, we did not use historical texts directly with the students, even though we refer sometimes to historical facts (for instance when introducing a new concepts). Moreover, in the case of linear algebra, our analysis shows that teaching may gain from taking some distance from the historical order of development. Indeed, the unifying and generalizing aspects of the theory of vector spaces is not only a fundamental character of the theory but is also a very recent one -it was really only used in the 1930s. Moreover, the final developments were linked essentially with functional analysis and brought out issues which are beyond the mathematical background of first year university students. All the subjects we can modelize with vector-spaces at this level of teaching have historically been solved with other tools, mostly the theory of determinants. Yet, for reasons we have explained above, we think that Gaussian elimination is a much better adapted tool than determinants to study linear equations and introduce basic notions of linear algebra. Thus the artificial genesis that our work led us to build for the teaching of linear algebra differs in many ways from the historical genesis. It is generally admitted that teaching should not and cannot reproduce all the historical aspects of the development of a mathematical discipline. Yet, if sometimes the constraints are only due to the shortening of time and cognitive or institutional aspects of the teaching situation; in the case of linear algebra there is a more complex epistemological constraint. Nevertheless, although artificial, the genesis induced by the teaching has to take the historical development into account. It cannot be only based on logical constraints, as it may have been at the beginning of modern mathematics. Yet, it must also take into account

the whole history of linear algebra as far as possible, even in its latest transformations. This implies that the formal aspect of the theory must appear as a final stage of maturity, in a context where it makes sense. We have tried to show how this can be done concerning the concept of linear (in)dependence. The historical analysis is then a fundamental tool at least on two levels :

- it provides a source of inspiration and an epistemological control for the building of an artificial genesis.

- it helps in understanding and analyzing the mistakes of the students. An erratic use of formal tools (using logic and set theory) can then be interpreted as a missing connection in the ontological process. This is fundamental, as an erratic mistake is usually evacuated by the teacher without further comment, and is therefore likely to reappear. But, if the teacher can locate the missing connection in the ontological process leading to the mistake, a much better remediation is possible.

Moreover, the historical analysis interacts with the didactical analysis on a more global level. For instance, in the case of linear algebra, it shows the necessity of interactions between different frameworks and registers of representation. In our experimental teaching, we have tried to make the student more aware of this possibility and its importance for a better understanding of formal concepts. In a work in preparation, M. Dias (1995) has shown the lack of use of changes of framework, register of representation and point of view in traditional teaching. She is now building and evaluating the effects of teaching sequences in order to encourage a better *cognitive flexibility* of the students. The analysis of the historical role of the different uses of changes of framework, register of representation, point of view or style (Granger 1995) may be very interesting work to support this type of didactical analysis. We have already analyzed the role of the geometrical framework in the emergence of vector spaces of functions (Dorier 1996), and are planning to enlarge this type of analysis to different aspects of the genesis of linear algebra.

We think that there are many means of interaction between historical and didactical analyses. Both of them provide epistemological reflections which are complementary and there is a lot to gain both for historical and didactical research works to try to emphasize the similarity and complementarity of approach. We hope that this paper has shown the relevance of this statement in a very specific context.

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