On the Teaching of Linear Algebra

Dorier, Jean-Luc (ed.)

This publication URL: https://archive-ouverte.unige.ch/unige:16623
Publication DOI: 10.1007/0-306-47224-4

ON THE TEACHING OF LINEAR ALGEBRA

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KLUWER ACADEMIC PUBLISHERS
NEW YORK / BOSTON / DORDRECHT / LONDON / MOSCOW

eBook ISBN: 0-306-47224-4
Print ISBN: 0-792-36539-9

Created in the United States of America

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TABLE OF CONTENTS

FOREWORD TO THE ENGLISH EDITION....................xiii

PREFACE xv

INTRODUCTION.................... xix

PART I. EPISTEMOLOGICAL ANALYSIS OF THE GENESIS OF THE THEORY OF VECTOR SPACES 1
J.-L. Dorier

1. INTRODUCTION.................... 3

2. ANALYTICAL AND GEOMETRICAL ORIGINS.................... 6

2.1. Systems of Numerical Linear Equations: First Local Theory of Linearity 6
2.1.1. Euler and the Dependence of Equations.................... 6
2.1.2. Cramer and the Birth of the Theory of Determinants.................... 8
2.1.3. The Concept of Rank 9
2.1.4. Georg Ferdinand Frobenius.................... 10
2.1.5. Further Development.................... 11

2.2. Geometry and Linear Algebra: a Two-Century-Long Process of Complex Reciprocal Exchanges.................... 11
2.2.1. Analytic Geometry.................... 13
2.2.2. Leibniz's Criticism.................... 13
2.2.3. Geometrical Representation of Imaginary Quantities 14
2.2.4. Möbius's Barycentric Calculus 15
2.2.5. Bellavitis's Calulus of Equipollences.................... 16

2.3. Hamilton's Quaternions.................... 16

2.4. Hermann Grassmann's Ausdehnungslehre 18
2.4.1. The Context.................... 18
2.4.2. The Philosophical Background.................... 20
2.4. 3. The Mathematical Content.................... 22
2.4.4. The 1862 Version.................... 24

2.5. First Phase of Unification of Finite-Dimensional Linear Problems 27
2.5.1. Analytical Origins.................... 27
2.5.2. Geometrical Context.................... 27
2.5.3. New Perspectives.................... 28

3. TOWARDS A FORMAL AXIOMATIC THEORY *30*

3.1. First Axiomatic Presentations of Linear Algebra 31
3.1.1. Giuseppe Peano 31
3.1.2. Salvatore Pincherle..33
3.1.3. Cesare Burali-Forti and Roberto Marcolongo....................................35
3.1.4. Hermann Weyl..37
3.1.5. Modern Algebra..38

3.2. Infinite-dimensional linear problems..42
3.2 1 Origins..43
3.2.2 Fredholm Integral Equation 45
3.2.3. David Hilbert 47
3.2.4. A Topological Approach..49
3.2.5. Frigyes (Frédéric) Riesz,Ernst Fisher and Erhard Schmidt 49
3.2.6. Stefan Banach, Hans Hahn and Norbert Wiener 51
3.2.7. Maurice Fréchet..53
3.2.8 John von Neumann and Quantum Mechanics 54
3.2.9. Banach's Decisive Contribution..55
3.2.10. Conclusion 55

3.3 Epilogue..56

4. CONCLUSION *59*

5. NOTES *60*

REFERENCES - PART I..73

PART 11. TEACHING AND LEARNING ISSUES **83**

CHAPTER 1. THE OBSTACLE OF FORMALISM IN LINEAR ALGEBRA 85

J.-L. Dorier, A. Robert, J. Robinet and M. Rogalski

1. INTRODUCTION *85*

2. FIRST DIAGNOSIS, A 1987 STUDY *88*

2.1. The Questionnaire 88
2.1 Rapid Analysis of the Contents of the Questionnaire 89
2.3. Relativization of the Results Obtained 89
2.4. Analysis of the Questionnaire 90
2.4.1. Question 1..90
2.4.2. Question 2.. 91
2.4.3. Question 3 91
2.4.4. Question 4.. 92
2.4.5. Question 5..93
2.4.6. Question 6 93
2.4.7. Question 7 94

2.4.8. Conclusion....94

3. A SECOND SURVEY MADE IN THE 1990 : IS FORMALISM A DIDACTICAL OBSTACLE RELATED TO THE LACK OF PREREQUISITES ?....95

3.1. Introduction....95

3.2. The Methodology and the Hypotheses....95

3.3. The Results....98

3.3.1. Global Analysis of the Contents of the Teaching....98

3.3.2. The Main Statistical Results....100

3.3.3. Correlations With the Pretest....100

4. THE LAST SURVEYS FROM 1991 TO 1994: THE OBSTACLE STILL HAS NOT BEEN OVERCOME....103

4.1. The Present State in an Experimental Teaching Method from Responses on Four Tests....103

4.2. Ranking....103

4.3. Vector Subspaces....105

4.3.1. Questions on the Equality or Inclusion of Subspaces....105

4.3.2. Questions on the Representation of Subspaces....107

4.4. Difficulties Stemming from Weakness in Logic and Set Theory: Inclusion and Implication, the Role of Parameters, the Notion of an Algebraic Equation of a Geometric Object....111

4.4.1. The February 1992 Exam....112

4.4.2. The 1994 Workshop....116

4.4.3. Hypotheses and Suggestions....121

5. ANNEX - MATHEMATICAL SOLUTION OF THE HYPERPLANES PROBLEM (1992, QUESTION (a))....123

6. NOTES....124

CHAPTER 2. LEVEL OF CONCEPTUALIZATION AND SECONDARY SCHOOL MATH EDUCATION....125

A. Robert

1. WHAT DO WE CALL 'LEVEL OF CONCEPTUALIZATION'?....125

1.1. Seeking Magic Squares of the Third Order....125

1.2. A different Example: Complex Numbers and their Use in Geometry....127

1.3. Outcome....128

2. LEVELS OF CONCEPTUALIZATION IN SECONDARY EDUCATION....129

3. CONCEPTUALIZATION....131

CHAPTER 3. THE TEACHING EXPERIMENTED IN LILLE....133

M. Rogalski

1. THE UNDERLYING PRINCIPLES OF THE COURSE DESIGN IN LILLE....133

1.1. Take Into Account the Formalizing, Unifying, Genaralizing, and Simplifying Nature of the Concepts of Linear Algebra....134

1.2. The Prerequisite Hypothesis....134

1.2.1. Comprehension of the True Stakes in Mathematics and a Certain Practice of Elementary Logic and Language of Set Theory are of Primary Importance 135
1.2.2. Accepting the Algebraic and Axiomatic Process 135
1.2.3. A Specific Practice of Geometry within the Sphere of Cartesian Geometry 136
1.3. Hypothesis (3) on Long-Term Course Design, Use of Meta and Changes in Point of View 136

2. THE DETAILED PLAN OF THE COURSE DESIGN 137
2.1. First Step 137
2.2. Second Step 138
2.3. Third Step 139
2.4. Fourth Step 139
2.5. Further Points 139

3. THE POSITIVE EFFECTS OF THE LILLE EXPERIMENT 142

4. ANNEXES 143
4.1. Annex 1: An Example of a Workshop on Epistemology of Axiomatic Linear Algebra 143
4.2. Annex 2: Examples of the Meaning of 'Methods' and 'Methodology' in Linear Algebra 144
4.3. Annexe 3: Chronology of the Course Design (Example in 1991) 146
4.4. Annex 4 : A Test About the Notion of Rank, (1990/91) 148
4.5. Annex 5: Capacity for Modeling by an Equation T(u) = v 148

5. NOTES 149

CHAPTER 4. THE META LEVER 151
J.-L. Dorier, A. Robert, J. Robinet and M. Rogalski
1. DEFINITIONS AND EDUCATIONAL POSITIONS 151

2. INTRODUCTION TO THE STRUCTURE OF VECTOR SPACE 155
2.1. Introduction 155
2.2 Presentation of the Teaching Sequence 156
2.3. Results 160
2.4. General Issues 161
2.5. Institutionalization 162
2.6. Long Term Evaluation 162

3. AN EXAMPLE OF THE USE OF META FOR TEACHING THE GREGORY FORMULA 163
3.1. Presentation of the Mathematical Content 163
3.2. Use of the Gregory Formula in a Classic Setting 164
3.3. Reformulating the Problem 165
3.4. Results 168

4. UNDERSTANDING LINEARITY AND NON-LINEARITY 169
4.1. Presentation of the exercise 170
4.2. Rapid analysis of the exercise 171

4.3. Situating the Exercise ... 171
4.4. The Type of Students ... 172
4.5. Analysis of the Results ... 172

5 . CONCLUSION: AN IMPORTANT METHODOLOGICAL PROBLEM, PERSPECTIVES ... 173
5.1. Evaluation of Meta-Type Teaching ... 173
5.2. Perspectives on Teacher Intervention and Meta-Lever Analyses ... 174
5.2.1. The Assumed Function of Teacher Comments Related to Teaching Phases: Various Scenarios and a Historical, Epistemological, and Pedagogical Study to Pursue ... 174
5.2.2 Form and Range of Meta-Type Comments: a More Pedagogical Problem ... 175

6 . NOTES ... 176

CHAPTER 5 . THREE PRINCIPLES OF LEARNING AND TEACHING MATHEMATICS ... 177
G. Harel
1. BACKGROUND ... 177
1.1. Proof ... 178
1.2. Time Allocated to Linear Algebra ... 178
1.3. Technology ... 179
1.4. Core Syllabus ... 179

2 . THE CONCRETENESS PRINCIPLE ... 180
2.1. Vector Space of Functions ... 181
2.2. Spatial Symbol Manipulation: The Case of Matrix Arithmetic ... 181
2.3. Evidence and Instructional Treatment ... 182
2.4. Some new Observation ... 184

3 . THE NECESSITY PRINCIPLE ... 185

4 . GENERALIZIBILITY PRINCIPLE ... 187

5 . SUMMARY ... 188

6 . NOTES ... 189

CHAPTER 6 . MODES OF DESCRIPTION AND THE PROBLEM OF REPRESENTATION IN LINEAR ALGEBRA ... 191
J. Hillel
1 . INTRODUCTION ... 191

2 . MODES OF DESCRIPTION IN LINEAR ALGEBRA ... 192
2.1. The Geometric Level: 1 -, 2-, and 3-Dimensional Spaces ... 193
2.1.1. Coordinate-Free Geometry ... 193
2.1.2. Coordinate Geometry ... 194
2.1.3. Vectors As Points ... 195

2.2. The Algebraic Mode: $\mathbb{R}^n$ ... 195
2.3. The Abstract mode: Vector Space V defined by a set of axioms ... 195
2.4. Some Students' Difficulties with the Geometric Mode ... 196
2.4.1. Arrows Versus Points ... 196
2.4.2. Standard Versus Non-Standard Basis ... 197
2.5. The General Use of a Geometric Language ... 197

3. THE PROBLEM OF REPRESENTATIONS ... 199
3.1. Moving Between Modes ... 200
3.1.1. Geometric - Algebraic Modes ... 200
3.1.2. Algebraic-Abstract Modes ... 201
3.2. The Problem of Representation in Terms of Basis ... 201
3.3. A Persistent Student Difficulty with Representation ... 202

4. APPLICABLITY OF THE GENERAL THEORY ... 205

5. CONCLUDING COMMENTS. ... 205

6. NOTES ... 206

CHAPTER 7. ON SOME ASPECTS OF STUDENTS' THINKING IN LINEAR ALGEBRA ... 209
A. Sierpinska
1. INTRODUCTION ... 209

2. PRACTICAL, AS OPPOSED TO THEORETICAL, THINKING - A SOURCE OF DIFFICULTY IN LINEAR ALGEBRA STUDENTS ... 211
2.1. Students' Practical Thinking in a 'Linear Algebra with Cabri' Course ... 212
2.1.1. Transparency of Language ... 212
2.1.2. Lack of Sensitivity to the Systemic Character of Scientific Knowledge ... 217
2.1.3. Thinking of Mathematical Concepts in Terms of their Prototypical Examples Rather than Definitions ... 222
2.1.4. Reasoning Based on the Logic of Action, and Generalization from Visual Perception ... 229
2.2. General Reflection on the Distinction Between Theoretical and Practical Thinking ... 232

3. SYNTHETIC-GEOMETRIC, ANALYTIC-ARITHMETIC AND ANALYTIC-STRUCTURAL MODES OF THINKING IN LINEAR ALGEBRA ... 232
3.1. A Characterization ... 233
3.2. The Tension Between the Synthetic, the Analytic-Arithmetic and the Structural Modes of Thinking in Students of Linear Algebra ... 236
3.3. Intermediate Forms of Argumentation Used by Students ... 238
3.4. General Remarks on the Students' Ways of Thinking in Linear AIgebra. 244

4. CONCLUSIONS ... 244

5. NOTES ... 246

CHAPTER 8. PRESENTATION OF OTHER RESEARCH WORKS.......... 247
M. Artigue, G. Chartier and J.-L. Dorier

1. COORDINATION OF SEMIOTIC REPRESENTATION REGISTERS.....247

2. ARTICULATION PROBLEMS BETWEEN CARTESIAN AND PARAMETRIC VIEWPOINTS IN LINEAR ALGEBRA 252

3. STUDIES WITH FIRST-YEAR UNIVERSITY STUDENTS IN VANNES.256
3.1. The 1991/92 study 256
3.2. The 1992/93 study 257

4. STRUCTURING OF KNOWLEDGE 259

5. USING GEOMETRY TO TEACH AND LEARN LINEAR ALGEBRA....262

6. NOTES..264

REFERENCES - PART II 265

CONCLUSION 273
1. Diagnosis - Epistemological Issues 273
2. Cognitive Flexibility 274
3. Long-Term And Evaluation 275
4. Perspectives 276

NOTES ON CONTRIBUTORS 277

INDEX 281

FOREWORD TO THE ENGLISH EDITION

A first version of this book was published, in French, under the title, *L'Enseignement de l'Algèbre Linéaire en Question*, by La Pensée Sauvage Édition (Grenoble) in 1997. It was the result of a collaboration between French, Brazilian, Moroccan and North American research teams for which the issue of linear algebra teaching was a central research interest.

The present English edition is not just a mere translation of the original. Most chapters have been revised according to new results, the report of a new research work has been integrated into the last chapter and Sierpinska's contribution has been totally renewed. Moreover, the introduction and the conclusion have been renewed and adapted to a more international audience.

Most contributors being French native speakers, the language issue has been quite complex.

We wish to thank here the 'équipe DIDIREM' (Paris 7 University), The IREM of Paris 7, the 'équipe d'Analyse' (Paris 6 University - CNRS), the IUFM of Versailles and Concordia University for their financial support.
We wish also to express our deep gratitude to Astrid Defence, Linda Northrup, Anna Sierpinska and Caroline West, for their precious help in translating and rereading our work.

Jean-Luc Dorier

PREFACE[1]

ANDRÉ REVUZ[2]

During the sixties, at a conference in Zürich, I made the acquaintance of a charming old man who was none other than Plancherel - of Plancherel's theorem - and who, during a very interesting conversation, insisted on the fact that of all the teaching he had done that of linear algebra seemed to be by far the most difficult for students to understand. Thirty years later the situation does not seem to have changed very much and we can assure Plancherel that he is in good company.

There is, however, in the eyes of mathematicians, a particular paradox: of all mathematical theories, and I don't think that this is an illusion, the theory of linear algebra appears to be one of the simplest and the difficulties attendant on its instruction are out of proportion with its intrinsic difficulties.

So, what is going on and what can one do about it? One radical solution to the problem is to say : anyone who can't quickly master linear algebra is incapable of doing mathematics and therefore doesn't interest us. This is certainly what many mathematicians think and what some of them don't hesitate to admit. Unfortunately, this reply is not acceptable. A first reason might be that applications of linear algebra are at the same time very varied and very significant and that it is important that outside the domain of mathematics many people know how to use it to good avail.

But there is a second more fundamental reason: mathematics is not the sole property of mathematicians who would in its name form a sect or a secret society. Fulfilling their role in society is a moral exigency and, for more selfish reasons, the best way for them to defend their discipline from attacks from those who, rightly or wrongly, have had the impression of having been kept out of it.

This amounts to saying that we need a teaching of mathematics that would be more and more efficacious, while at the same time we have to be honest enough to admit that so far it has only been so to a minor degree. (Mathematics is not the only subject in this situation but this is not to be regarded as an excuse). And it must be admitted that this is not an easy task for which all the good intentions and ideas, apparently reasonable, yet which have not been submitted to a control of reality, are of little help.

A common preconception among mathematicians is that in order to teach mathematics well, all that is necessary is to know the subject well. The teaching of linear algebra provides a striking counter example. The theory is well developed, those who teach it know it personally very well ... yet the students do not understand.

The contrary preconception, held by educationalists who believe that with an appropriate pedagogy anything can be taught, is just as disastrous. There again linear algebra provides a counter example : it is a resistant mathematical matter whose difficulty will not be overcome by any amount of educational know how, no matter how elaborate it is, if it is not based on a sound understanding of the mathematical content; that is, the ideas, the concepts, and the methods at work in the theory.

In order to make progress in the teaching, one must first of all consider it as an object of study. That is precisely the mission of 'la didactique' of mathematics whose motto could well be : « How do students acquire a mathematical frame of mind? »

The didactic study exposed in the present work is examplary on more than one count:

- the mathematical content is present throughout.
- it is the result of team work, or several teams at work, who collaborate without losing their autonomy. This gives rise to a multiplicity of viewpoints, an enlargement of experiential fields, and criticisms of results.
- it is a carefully controlled experimental work whose results are presented with rigorous honesty.
- finally, the conclusions distinguish between results that can be reasonably considered as established and those which call for a more profound study and further experimentation. Besides which, the authors are wise enough to believe that there is more than one way to teach linear algebra and they refuse to provide recipes for teaching that some readers might be expecting from them.

Some of my fellow mathematicians have admitted to me that they would believe in the didactic theory (la didactique) were it capable of providing teachers with precise, easy to follow advice. But this is precisely what it must not do: the application of recipes without understanding their origin leads to the worst misconceptions (hundreds of examples abound). The best recommendations, the best 'instructions' in the world, are, at best, totally ineffective.

If it is true that to teach well one needs to possess a mathematical culture that is alive, it is also true that one needs to possess a vibrant didactic culture, which is neither the application of recipes, nor a routine pompously qualified as experience, but one that arises from the work of the didacticians, with whom all teachers should collaborate.

As regards the mathematical content, the first chapter highlights a very interesting historical and epistemological study of linear algebra. This study clearly demonstrates the relevance of Jacques Hadamard's remark that I never tire of citing : « Natural ideas are those which come last. » Of course 'natural' here does not mean spontaneous, but adapted to the nature of the object under study. It is an apparent paradox attached to the development of mathematics that the greatest fecund ideas are intrinsically simple, but it took a long time and a great deal of effort to bring them to light.

But this poses a formidable problem for the teacher :

Is it necessary to have the student relive the historical development until the present day? In most cases this would be a fastidious job, difficult to realize, since one does not know all the details of the evolution, and it would also mean allowing the student to follow incorrect reasoning and to get lost in dead end trails.

Should we, on the other hand, bluntly present the theory of linear algebra as a finished product in its intrinsically 'simple form'? Experience would suggest otherwise : the simplicity is not that obvious to the student.

So where does the solution lie?

To a large extent, it lies, no doubt, in what is presented in this work under the title of ‘meta lever‘, a method which it is certainly interesting to develop and further refine. There exists in mathematics courses a strange prudery which forbids one to ask questions such as, « Why are we doing this? », « At what is the objective aimed? », whereas it is usually easy to reply to such questions, to keep them in mind, and to show that one can challenge these questions and modify the objectives to be more productive or more useful. If we don‘t do this we give a false impression of a gratuitous or arbitrary interpretation of a discipline whose rules are far from being unmotivated or unfounded.

One must also consider the time aspect. Simple ideas take a long time to be conceived. Should we not therefore allow the students time to familiarize themselves with new notions? And must we not also recognize that this length of time is generally longer than that of the official length of time accorded to this teaching and that we should be counting in years? When the rudiments of linear algebra were taught at the level of the lycée (college level), the task of first year university teachers was certainly easier : for sure the student's knowledge was not very deep, however it was not negligible and it allowed them to reach a deeper understanding more quickly.

The effects of the fact that the teaching of proofs has disappeared from secondary school teaching - unbelievable, but alas true - will quickly be felt at the first year university level and subsequently in higher cycles.

I would like to see an indepth didactic study done on the time factor in the learning process. Without it ever being actually said, all teaching is organized as if pupils were immediately supposed to understand and assimiliate what they were being taught and to be able to immediately pass the periodic tests. It is not surprising, under these conditions, that ideas and culture have given way to techniques and algorithms.

It is clear that the learning time is not uniform. There are periods when nothing seems to sink in, when the teacher can have the impression that the students are marking time or even going backwards, and other periods in which everything becomes easy. It is the first of these periods which are crucial and which should form the object of an indepth didactic study because it is during these periods of time that the seeds of comprehension are sewn. To, then, (as is often the case) make students carry out series of repetitive exercises is fastidious, not in any way a positive reinforcement for those who have already understood, and a total waste of time for those who have not yet understood. It is during these dificult moments that what the authors of this book have called the ‘meta lever‘ can be the most useful; not that it will solve all problems as if by magic, but that it can act as a lever to unblock the thoughts if it is placed at the correct spot with the correct amount of leverage. With regards to this notion, the authors tread prudently, which is their way in all fields, but I, personally, am persuaded that this is a most fruitful path.

That which is clear is that the utilization of the ‘meta lever’ runs contrary to an implicit ideology which rules in teaching and which perverts it. I quote two passages from the book:

- « There is an apparent paradox: in order to improve their skills, we develop a more stringent strategy towards the students, with the risk, at least provisionally, of lowering their results in exams. »

- « The replacement of the logic of success by the logic of learning appears to be very costly; in other words, are we not likely to make learning more difficult by being more exacting? »

In reality, this is an old debate: as opposed to an ideal teaching process whose aim is the mastery of what one has learned, we have a teaching system obsessed with the results of class tests, of term exams, of contests, and which easily falls victim to last minute cramming. If the spirit which animates it is founded on the 'logic of success', it must be stated that it is only a short term success; one in which, for the majority of students, deeper understanding, assured only by a 'logic of learning', is sacrificed. It is certainly much easier for the teacher; it is wholly in accordance with the spirit of competition which corrupts our society, and by necessity the teaching that accompanies it; but finally it is a catastrophe as regards its true results. There are only a rare number of students who are able to rise to a level of 'logic of learning' by themselves. All real improvements in the teaching of mathematics must necessarily lie in the deepening and diffusion of the 'logic of learning'. If it is costly, it is because it presupposes a deep change in mentality in the teacher : he/she is no longer the prophet of truths that have to be transmitted, but a free spirit calling forth other free spirits to develop and assume their responsibilities. This presupposes the observance of seeds of ideas in the minds of the students (with a greater frequency and relevance than that with which they are usually accredited) and to give them the means in which to ripen.

Certainly, there are no strict guidelines for managing a class in this manner. This presupposes a sufficiently deep knowledge of the mathematics that we wish the students to discover and a similarly deep knowledge of the different roads to access mathematics. In other words, it presupposes that 'la didactique' of mathematics has done its job and has let its job be known. Didacticians should therefore pursue their efforts. They should not stop because of the fears expressed in the second citation above. And I say to the authors of this book : « You are on the right track. Go ahead! ».

[1] Translated by Astrid Defence and Anna Sierprinska.
[2] André Revuz is a French mathematician who played an important role during the reform of Modern Mathematics especially regarding the teaching of linear algebra (see end of the first part of this book). He also played a leading role in the development of research in math education in France.

INTRODUCTION

The introduction of the theory of vector spaces in the teaching of mathematics at secondary level education has been, in many countries, one of the main issues of the reform of modern mathematics in the 1960s. It not only changed radically the teaching of geometry, which had been for centuries the core of mathematics education, but it also represented a new approach in the teaching of algebra. During the reform, the study of linear equations, which had until then constituted the main part of linear algebra, became merely a tool. Although still a central part of linear algebra, it was overshadowed by the formal theory which was to become the model for all linear problems, even infinite-dimensional, in a unified and generalized approach. Moreover, linear models became increasingly central paradigms within mathematics as well as in other sciences.

The teaching of vector space theory was therefore, in many senses, emblematic of the reform of modern mathematics. One of the challenges of this reform was to make mathematics accessible more directly to more students. In this sense, the theory of vector spaces appeared as a model of simplicity. It seemed, for instance, easy and powerful to interpret the solution set of a system of numerical or differential linear equations as the inverse image of a point by an affine transformation. However, it did not take very long, after the reform was implemented, to realize that what was regarded as so easy was in fact a source of serious cognitive difficulties. More or less rapidly, depending on the country, the reform had progressively been abandoned in the early 1980s, and the theory of vector spaces disappeared from secondary education. It is now taught only in first year of university and mostly in science-orientated curricula. The content of the course may be very formal, limited to $\mathbb{R}^n$ and matrix calculus, or only introduced within a geometrical setting. However, teachers in charge of a linear algebra course are very often frustrated and disarmed when faced with the inability of their students to cope with ideas that they consider to be so simple. Usually, they incriminate the lack of practice in basic logic and set theory or the impossibility for the students to use geometrical intuition. On the other hand, students are overwhelmed by the number of new definitions and the lack of connection with previous knowledge.

The purpose of this book is not to give a miraculous solution to overcome these difficulties. It will present a substantial overview of research works, which consist in diagnoses of students' difficulties, epistemological analyses and experimental teaching, offering local remediation. Nevertheless, these works' main results are new questions, problems and difficulties. Yet, this should not be interpreted as a failure. Improving the teaching and learning of mathematics cannot consist in one remediation valid for all. Cognitive processes and mathematics are far too complex for such an idealistic simplistic view. It is a deeper knowledge of the nature of the concepts, and the cognitive difficulties they enclose, that helps teachers make their teaching richer and more expert; not in a rigid and dogmatic way, but with flexibility. Therefore, the purpose of this book is to inform mathematics teachers in charge of linear algebra courses, as well as researchers in mathematics education, about the main results of research works on the teaching and learning of linear algebra. In France, like in North America, research in mathematics education at university level focused, in a first stage, mostly on teaching and learning of Calculus. Research about the teaching and learning of linear algebra started at the beginning of the 1980s, and gradually became a major issue in the 1990s. In this book, we will try to expose, in a maximum of detail, research works which have played an initiating role in the field. We will also give an overview of more recent developments. Of course, it is impossible to be exhaustive, thus we apologize for research works that we have not mentioned.

The first part of this book is not directly devoted to the teaching of linear algebra. It presents, on the basis of a historical survey, an epistemological analysis on the nature of linear algebra. Through the study of original works, Dorier analyzes the evolution of linear algebra from the first theoretical results on systems of linear equations (around 1750), until the final elaboration of the axiomatic theory of vector spaces and the first attempts at teaching these concepts. In particular, the author points out and analyzes different phases in the process of unification and generalization that led to the modern theory. He shows the difficulties, sometimes important, that each new phase had to overcome before being accepted, especially regarding the axiomatic approach.

This historical part is in interaction with the didactical research, not only because the historical material nourishes the didactical analysis but also because the didactical concern gives some specific orientation in the epistemological analysis of the historical context.

The second part of the book, devoted to educational issues, is divided into 8 chapters.

The first four chapters present a synthesis of a research program led by Dorier, Robert, Robinet and Rogalski, who have been working, as a team, on the issue of linear algebra, since 1987.

The first chapter is devoted to the first investigations made by the group, their evolution and the first conclusions and perspectives of research to which they led to. It is mostly a diagnosis, made in ordinary conditions, of the teaching of linear algebra in French science universities. It gives an overview of the main errors and difficulties of the students and, in relation with the historical analysis, it allows for a better understanding of how the unifying and generalizing nature of linear algebra is a source of the learning and teaching difficulties. In this sense, the main issue raised in this chapter concerns what the authors have named *the obstacle of formalism.*

In the second chapter, Robert introduces the concept of *level of conceptualization.* This theoretical approach allows a new type of analysis of the difficulties encountered by the students in the learning of linear algebra.

In the third chapter, Rogalskí gives a global description of the *teaching project* with which he has experimented for several years. This project, corresponding to the experimental aspect of the research, is based on the first diagnoses, but it also tests new hypotheses.

The fourth chapter is devoted to the presentation of the *meta lever*: a new teaching tool, elaborated in order to try to make students overcome the obstacle of formalism, which is central in the experimentation. Three types of experimental use of this teaching tool are presented as illustrations. The issue concerning the difficulties encountered in the evaluation of these experimentations is addressed in the conclusion of this chapter, which also presents a synthesis of the results of the whole research project, the difficulties that remain unsolved, and the perspectives.

The teaching of linear algebra in North America has different characteristics than in France. It is usually less formal, and the model of $\mathbb{R}^n$ and matrix calculus are more central. Nevertheless, this teaching presents some difficulties, which may be partly different from those in France, but are globally quite similar. Three chapters of this book are devoted to North American research works about the teaching and learning of linear algebra.

In the fifth chapter, Harel presents the recommendations made by the *Linear Algebra Curriculum Study Group.* He gives his personal interpretation of these recommendations through three teaching/learning principles: the concreteness principle, the necessity principle and the generalizibility principle. Although this approach is presented in the context of a North American perspectives, it bears several similarities with the issue raised in the preceding work concerning the unifying and generalizing nature of the theory of vector space.

In the sixth chapter, Hillel distinguishes several modes of description (or language) in use in linear algebra: the abstract mode, the algebraic mode and the geometric mode. He analyzes, not only how these different modes of description function and can be used, but also how they interact, and especially how it is possible (or not) to move from one mode to the other. Moving between the algebraic and the formal modes of representation has been noticed as one specific dificulty. Hillel analyzes this problem in activities concerning the matrix representations of an operator in different bases.

In the sixth chapter, Sierpinska analyzes some aspects of students' reasoning in linear algebra. She bases her work on several teaching experiments at Concordia University, in Montreal, between 1993 and 1999. In some sessions the students used the dynamic geometry software, Cabri Geometry II. Some tutoring sessions were also analyzed. From her analysis of these sequences, and of the history of linear algebra, she is led to distinguish between practical and theoretical thinking. Students think in practical rather than theoretical ways and Sierpinska points out several cases in which this is a source of difficulties in the learning of linear algebra. In the second stage of her work, Sierpinska distinguishes between three modes of reasoning in linear algebra, corresponding to its three interacting languages : the visual geometric language, the arithmetic language and the structural language. The author illustrates with examples students' reluctance to enter into the structural mode of thinking and, in particular, their inability to move flexibly between the three modes.

In the eight chapter (the final one), five different works are presented. Four are doctoral dissertations (one being still in progress) and are all more or less connected with the research program presented in the first four chapters.

The first two are quite closely connected and are also related to Hillel's and Sierpinska's contributions. In the first, Pavlopoulou, using Duval's approach, distinguishes three registers of semiotic representations in linear algebra : graphic, tabular and formal. Her hypothesis is that the ability of operating in each register and, even more, of translating from one into the other, is essential to the understanding of concepts in linear algebra. This work is closely connected with Hillel's approach.

The second work, by Alves-Dias, addresses the general issue of cognitive flexibility through the question of change of register but also mathematical setting and viewpoint. Alves-Dias's hypothesis is that some complex cognitive processes cannot be reduced to problems of conversions. She analyzes in detail the question of change of representations for vector subspaces between parametric and Cartesian equations. In her experimentation, she shows that, even if changes of register and setting are important in this mathematical task, the cognitive activity cannot be reduced to them.

The third work presents the results of several surveys led over two years by a research team in Rennes. These surveys present a diagnosis of students' difficulties and some possible remediations. This work is complementary to the first chapter.

In the fourth work, Behaj is interested in the way knowledge is structured in the learner's mind in order to be memorized. He also wants to see in which way the structure of a course proposed by a teacher can influence the structuring in the students' mind. He made several interviews with teachers and pairs of students at least in second year of university. The mathematical subject he chose was linear algebra. Therefore, his work gives a valuable material about teachers' practice and the evolution of students' knowledge in linear algebra after two, three or four years at university.

Finally, in the fifth work, Chartier analyzes the role and the place of geometry in the teaching of linear algebra. As a theoretical framework, she uses Fischbein's work on mathematical intuition (i.e., distinguishing between analogical and paradigmatic models) to interpret the position of geometry towards linear algebra in the historical context, in textbooks, and in teachers' and students' practice.

The structure chosen for the book results from our concern to preserve the identity of each research work. Nevertheless, this choice may give the reader the feeling of a collection of isolated works without connection. Most of the authors, however, have collaborated over several years, but it is not always easy to reflect, in the details of each presentation, the interactions they had. In the conclusion, we try to give a synthetic overview of all the works presented, focusing on the common issues.

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