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An invitation to bounded cohomology

Nicolas Monod

Abstract. A selection of aspects of the theory of bounded cohomology is presented. The emphasis is on questions motivating the use of that theory as well as on some concrete issues suggested by its study. Specific topics include rigidity, bounds on characteristic classes, quasification, orbit equivalence, amenability.

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1. Introduction

This lecture is an invitation to explore bounded cohomology. It is not an attempt to collect all recent advances in that topic, much less an *ex cathedra* exposition of the theory. I will try to illustrate how problems from diverse origins can be translated into the framework of bounded cohomology, and how in return this theory prompts a few concrete problems. A number of questions are suggested, ranging from teasers to the ill-defined.

The terminology *bounded cohomology* proposed by M. Gromov [67] refers to the following concrete definition. Recall that the ordinary (singular) cohomology $H^*(M, \mathbf{R})$ of a manifold M can be defined by the complex of all *singular cochains* on M , which are just all real-valued functions on the set of singular simplices. The subspace of *bounded* functions yields a subcomplex and hence cohomology groups $H_b^*(M, \mathbf{R})$. Moreover, the inclusion map of this subcomplex determines a natural transformation $H_b^*(M, \mathbf{R}) \rightarrow H^*(M, \mathbf{R})$ called the *comparison map*.

This definition can be imitated for groups. One of the definitions of Eilenberg–MacLane cohomology $H^*(G, V)$ of a group G with coefficients in some module V is given by the complex of all G -equivariant V -valued functions on G^{n+1} . Considering only those functions that are bounded one obtains a complex

$$0 \longrightarrow C_b(G, V)^G \longrightarrow C_b(G^2, V)^G \longrightarrow C_b(G^3, V)^G \longrightarrow \dots$$

whose cohomology is the *bounded cohomology* $H_b^*(G, V)$ and comes again with a *comparison map* $H_b^*(G, V) \rightarrow H^*(G, V)$. One needs here to make sense of boundedness for V -valued maps; for instance, V could be a Banach space with isometric G -representation. We will usually assume that this *coefficient module* is dual (e.g.

$V = \mathbf{R}$ or a unitary representation on Hilbert space). More generally, when G is a locally compact group we (ab)use the same notations H^* , H_b^* for the *continuous* (bounded) cohomology; for present purposes, it is simply defined by requiring that all cochains be continuous¹.

I will mainly concentrate on the group case; a theorem of M. Gromov of fundamental importance states that the morphism $H_b^*(\pi_1(M), \mathbf{R}) \rightarrow H_b^*(M, \mathbf{R})$ induced by the classifying map is an isomorphism (Corollary p. 40 in [67]; see also [11]).

The above definitions, especially in the group case, might seem artificial. The next section is an attempt to challenge this perception. Nevertheless, I should point out that *there is not a single countable group G for which $H_b^*(G, \mathbf{R})$ is known, unless it is known to vanish in all degrees*. For instance, here is what is known for the free group $\mathbf{F}_2$ and trivial coefficients: $H_b^1(\mathbf{F}_2, \mathbf{R})$ vanishes, $H_b^2(\mathbf{F}_2, \mathbf{R})$ and $H_b^3(\mathbf{F}_2, \mathbf{R})$ are infinite-dimensional [11, §3], [66], [139].

However, in the case of *connected* groups, all known results seem to indicate that bounded cohomology with trivial coefficients is much better behaved than for discrete groups. The connected case can be reduced to semi-simple Lie groups, prompting the following question.

Problem A. Let G be a connected semisimple Lie group with finite centre. Is the comparison map

$$H_b^*(G, \mathbf{R}) \longrightarrow H^*(G, \mathbf{R})$$

an isomorphism?

At this time, it seems that the question of injectivity and surjectivity of this comparison map are two quite different issues; existing proofs are of a very different nature. Perhaps this will change with a better understanding of the bounded cohomology of Lie groups. But for the time being, I would like to single out the subquestion below because it is the aspect most relevant for several questions discussed in this text.

Problem A'. Is the comparison map of Problem A surjective?

A related conjecture was proposed by J. Dupont [43]; compare also the introduction of [14] and 9.3.8, 9.3.9 in [112]. One can also ask the same questions more generally for products of semisimple algebraic groups over local fields. For more general coefficient modules, even if only irreducible unitary representations are considered, both injectivity and surjectivity fail [25].

2. Three ways to stumble upon H_b^*

I will now sketch briefly three circumstances leading naturally to study bounded cohomology: (1) group algebras; (2) bounds and refinements of classical invariants; (3) quasification.

¹I emphasise that when G is not discrete, this is not the same as the cohomology of the classifying space BG .

2.1. Cohomology of group algebras. The usual Eilenberg–MacLane cohomology of a discrete group Γ can be seen as a particular case of the cohomology of algebras; namely, it is isomorphic to the cohomology of the group algebra $A = \mathbf{R}\Gamma$.² Here A is the free $\mathbf{R}$ -vector space on Γ endowed with the convolution product. In other words, it is the universal object obtained by forcing upon the group Γ an $\mathbf{R}$ -linear structure extending the multiplication in Γ . Recall that the cohomology of A is defined by Ext-functors, which means that one considers complexes of linear morphisms

$$A \otimes \cdots \otimes A \longrightarrow \mathbf{R}$$

to $\mathbf{R}$ or more generally to suitable coefficient modules.

Now, this free vector space A brings along its “free norm”, that is, the ℓ^1 -norm; the latter extends as the *projective tensor norm* on the above tensor products [72, I§1.1]. Why not, then, take this topology into account and compute the cohomology of the *topological algebra* A ? This amounts to considering complexes of *continuous* linear morphisms

$$A \otimes_{\pi} \cdots \otimes_{\pi} A \longrightarrow \mathbf{R},$$

where $\otimes_{\pi}$ is the notation indicating the choice of the projective tensor norm. The point is that *this cohomology is nothing else than the bounded cohomology of Γ* , as follows readily from the properties of $\otimes_{\pi}$. Once we deal with this *continuous cohomology* of A , we may of course replace A by its completion $\ell^1(\Gamma)$ without affecting the outcome. In conclusion, *the bounded cohomology of groups is a particular case of the cohomology of Banach algebras* as exposed in B. Johnson’s 1972 memoir [89] (see also A.Ya. Helemskiĭ [76]). I would like to point out that the group algebra case (that is, bounded cohomology) was indeed prominent in B. Johnson’s memoir. One can find therein several aspects that became intensively studied later, such as quasimorphisms, amenability and the problem of the existence of outer derivations. Nevertheless, it is M. Gromov’s paper (which also refers to ideas of W. Thurston) that gave all its impetus to the theory.

Here are two questions to conclude this outline. First, we point out that for C^* -algebras and specifically von Neumann algebras, other cohomologies have been studied, such as *completely bounded cohomology*, where the notion of boundedness is quite different from our setting; see Christensen–Effros–Sinclair [33]. Consider the von Neumann algebra $L(\Gamma)$ associated to a countable group Γ , which is the completion of A for a much weaker topology than above. Then, there is not anymore a straightforward correlation between Γ -modules and $L(\Gamma)$ -bimodules as there was for $\ell^1(\Gamma)$. However, there is a precise analogy instead: the theory of *correspondences* (see A. Connes [36] and S. Popa [130]). Whilst this will not provide a precise dictionary from the Eilenberg–MacLane cohomology of Γ to some cohomology of $L(\Gamma)$, it still

²For exact statements, one needs to give more details on how to handle the coefficients involved; for instance, A is suitable for coefficients that are real vector spaces. Moreover, Hochschild cohomology of algebras involves A -bimodules as coefficients, whilst a priori the cohomology of Γ is defined for modules. Not taking this into account would identify the Hochschild cohomology with a sum of Eilenberg–MacLane cohomology of centralisers of representatives of the conjugacy classes of Γ . We will omit all details here.

raises the interesting possibility to define for $L(\Gamma)$ an analogue of the L^2 -cohomology of Γ in the sense of M. Atiyah [4], Cheeger–Gromov [31] and W. Lück [101]. Such an analogue has been proposed by A. Connes and D. Shlyakhtenko in [38].

Here is how this programme connects to the topic of our discussion. For present purposes, let us think of the L^2 -cohomology of Γ as the Eilenberg–MacLane cohomology $H^*(\Gamma, \ell^2(\Gamma))$, where $\ell^2(\Gamma)$ is endowed with the regular representation³. The *bounded* cohomology $H_b^*(\Gamma, \ell^2(\Gamma))$ has also proved very useful in degree two (see Sections 4.1 and 4.2), even though one cannot *measure* its size as is done by means of the von Neumann dimension in the case of $H^*(\Gamma, \ell^2(\Gamma))$.

Problem B. Perform a construction analogous to Connes–Shlyakhtenko [38] but corresponding to $H_b^*(\Gamma, \ell^2(\Gamma))$ instead of $H^*(\Gamma, \ell^2(\Gamma))$. Provide non-triviality results, at least in degree two.

The second question is more indeterminate:

Problem C. Consider the cyclic cohomology of locally convex algebras $\mathcal{A}$ as in §II.5 of A. Connes’ [37]. What can be said for $\mathcal{A} = \ell^1(\Gamma)$?

The cyclic homology of the group algebra $A = \mathbf{R}\Gamma$ (devoid of any topology) has been studied notably by D. Burghelea [29]. For an example with a locally convex completion of A , smaller than $\ell^1(\Gamma)$ and closer to the spirit of [37], see R. Ji [87].

2.2. Refinements of ordinary cohomology and numerical bounds. Certain classical cohomology classes are given by explicit cocycles that happen to be bounded. This is of particular interest for characteristic classes, since explicit bounds on characteristic numbers of flat bundles carry important geometric information, such as in the Milnor–Wood inequality; we refer to J. Dupont [43] for more on such bounds.

Here is an example of this situation: The *Euler number* of a flat oriented n -vector bundle over a compact manifold M measures the obstruction to finding a non-vanishing section. It can be computed by triangulating M , choosing generic “affine” sections over the resulting simplices and then adding up obstruction signs ± 1 for each simplex – see D. Sullivan [141] and J. Smillie [138]; compare also [7, F.4]. The resulting number is clearly bounded in terms of the number of simplices, though this bound is mysterious. One obtains a nice conceptual bound by observing that the Euler class of $GL_n^+(\mathbf{R})$ has an explicit cocycle representative which is bounded, e.g. as in Ivanov–Turaev [86] (GL_n^+ refers to the group of matrices with positive determinant).

Problem D. Let $\pi = (B \rightarrow M)$ be a flat oriented n -vector bundle over a compact manifold M . Given additional structure on π , define a natural class $\mathcal{E}_b(\pi)$ in $H_b^n(M, \mathbf{R})$ whose image in $H^n(M, \mathbf{R})$ is the Euler class $\mathcal{E}(\pi)$ of π .

³This is correct under the finiteness assumption F_n , where n is the degree of the cohomology considered. For the general case, one should follow W. Lück [101].

In particular, given a compact orientable n -manifold M with additional structure, find a natural definition of $\mathcal{E}_b(M) \stackrel{\text{def}}{=} \mathcal{E}_b(TM \rightarrow M)$.

I should certainly be a bit more specific here, since after all one way to see the Milnor–Wood inequality is to consider an explicit cocycle for the Euler class $\mathcal{E}$ that witnesses its boundedness. However, this is quite different from having a *natural invariant* in $H_b^n(M, \mathbf{R})$, already because the map $H_b^n(M, \mathbf{R}) \rightarrow H^n(M, \mathbf{R})$ can be far from injective. The fact that additional structure on M should be required to rigidify the situation is suggested by the fact that $H_b^n(M, \mathbf{R})$ is canonically isometrically isomorphic to the bounded cohomology of the fundamental group $\Gamma = \pi_1(M)$ [67, p. 40]. By contrast, the Euler class *as class of $BGL_n^+(\mathbf{R})$* is unbounded (compare also footnote 1); thus the idea would be to transit via the *group cohomology*:

Consider for instance the setting of compact orientable manifolds supporting an affine structure. In that case the Euler class $\mathcal{E}(M)$ will come from the structure group $GL_n^+(\mathbf{R})$ by pull-back through the holonomy representation of $\pi_1(M)$ via $H^n(\pi_1(M), \mathbf{R})$ and the classifying map. In that case one would have a completely canonical choice for $\mathcal{E}_b(M)$ if, as suggested by Problem A, one proves

$$H_b^*(PSL_n(\mathbf{R}), \mathbf{R}) \cong H^*(PSL_n(\mathbf{R}), \mathbf{R}).$$

Indeed, an easy argument shows that this would imply

$$H_b^n(GL_n^+(\mathbf{R}), \mathbf{R}) \cong H^n(GL_n^+(\mathbf{R}), \mathbf{R}), \quad n \neq 1.$$

A motivation for Problem D is the following.

Problem D’. Use a natural definition of $\mathcal{E}_b(M)$ to prove that, for any compact orientable manifold M supporting an affine structure, $\mathcal{E}_b(M)$ vanishes.

This would settle the Chern–Sullivan problem of the vanishing of the Euler–Poincaré number of such manifolds M .

The example of the boundedness of the Euler class can be considerably generalised: M. Gromov proves in [67] that all primary⁴ characteristic classes are bounded, at least when viewed as classes of the structure group made discrete. A different proof avoiding the use of H. Hironaka’s resolution of singularities was provided by M. Bucher-Karlsson in her thesis [14]; it is also shown in [14, p. 60] how it follows that these primary characteristic classes are bounded already when viewed as classes of the topological group G . In other words, they lie in the image of the comparison map $H_b^*(G, \mathbf{R}) \rightarrow H^*(G, \mathbf{R})$.

Problem E. Prove the same statement for secondary characteristic classes⁵.

As pointed out in [14], this would then solve Problem A’. Indeed, J. Dupont and F. Kamber proved in [44] that, as an algebra, $H^*(G, \mathbf{R})$ is generated by primary and

⁴More precisely, the characteristic classes of flat G -bundles, where G is an algebraic subgroup of $GL_n(\mathbf{R})$.

⁵See Cheeger–Simons [32] and M. Bucher-Karlsson [14] for the definition of secondary classes.

secondary classes when G is a connected semisimple Lie group with finite centre. (The product is easily seen to preserve boundedness of cohomology classes.)

Once a natural bounded representative has been identified for a classical cohomological invariant, this opens the door to a refined invariant: Indeed, the *bounded class* of that bounded cocycle contains a priori much more information than the class one started with. This is because it is easier for cocycles to be cohomologous than to be “boundedly cohomologous”, that is, equivalent modulo *bounded coboundaries*.

A beautiful illustration of this phenomenon has been given by É. Ghys in [63] and goes as follows. Recall that if a group Γ acts by orientation-preserving homeomorphisms on the circle, it inherits an Euler class in $H^2(\Gamma, \mathbf{Z})$. This class has a canonical representative taking only values $\{0, 1\}$, thus determining a bounded class. Whilst the original class determines only the obstruction to lifting the Γ -action to an action on the line, É. Ghys proves that the bounded class completely characterises the action up to semi-conjugacy.

Another important benefit of identifying an ordinary cohomology class as being bounded is the *Gromov seminorm*. Since *homology* classes are given by cycles, i.e. formal *finite* linear combinations of simplices (singular or otherwise), there is a numerical invariant attached to every homological class, the Gromov seminorm, which is by definition the infimum of the total mass of all linear combinations representing that class. This is a method of assigning a *numerical invariant* to cohomological invariants. For instance, this number is computed for the Kähler class by Domic-Toledo [42] and Clerc-Ørsted [34].

In particular, M. Gromov [67] defines the *simplicial volume* of a closed (connected, orientable) manifold M to be the seminorm of its fundamental class (for the relative case, see also [99]). This invariant, besides its own interest, provides bounds for the minimal volume of M over all (suitably normalised) Riemannian structures, as explained by M. Gromov in [67]. The relevance of bounded cohomology is that any upper bound on a *cocycle* with non-trivial homological pairing on the given cycle provides a non-trivial lower bound on the Gromov seminorm of the cycle.

W. Thurston and M. Gromov (see the 1978 notes [142] and [67]) have shown how to use negative curvature to prove the boundedness of fundamental classes – equivalently, the positivity of the simplicial volume. The case of symmetric spaces has been solved only quite recently, with J.-F. Lafont and B. Schmidt proving in [100] that *the simplicial volume of any closed locally symmetric space of non-compact type is positive*. (A crucial ingredient of their proof is the work of C. Connell and B. Farb [35].) Actually, the case of the symmetric space of $SL_3(\mathbf{R})$ is not covered by [100]; it was claimed in [134], but the proof therein is incomplete. However a proof is provided by M. Bucher-Karlsson in [13]. M. Gromov has conjectured more generally that closed manifolds with non-positive curvature and negative Ricci curvature have positive simplicial volume.

2.3. Quasification. The word *quasification* is meant to refer to the process whereby a geometric or algebraic notion is modified to an approximate variant; typically, a

defining equality or inequality is relaxed by imposing that it hold up to some constants only. Here are a few examples of this overly vague principle.

(i) *Quasi-isometries*. Whereas a map $f: X \rightarrow Y$ between metric spaces X, Y is called isometric if $d_Y(f(x), f(x'))$ equals $d_X(x, x')$ for all $x, x' \in X$, it is said to be *quasi-isometric* if there is some constant $C \neq 0$ such that

$$C^{-1}d_X(x, x') - C \leq d_Y(f(x), f(x')) \leq Cd_X(x, x') + C \quad \text{for all } x, x' \in X.$$

A *quasi-isometry* is a quasi-isometric map whose image has finite codiameter. Here are two important motivations for this notion: (1) Geometric rigidity questions such as Mostow's strong rigidity [119] lead to consider quasi-isometries of symmetric spaces arising from a homotopy equivalence between two of its compact quotients; (2) Geometric group theory has had considerable success, following M. Gromov, in viewing finitely generated groups as metric spaces and studying their geometry [68], [71]; however, this point of view is well-defined only up to quasi-isometry.

As a matter of terminology, a *rough isometry* shall be a quasi-isometry where additive constants only are allowed:

$$d_X(x, x') - C \leq d_Y(f(x), f(x')) \leq d_X(x, x') + C.$$

(ii) *Gromov-hyperbolicity*. In the context of point (2) in (i) above, one calls a geodesic metric space *Gromov-hyperbolic* if any finite configuration of points is roughly isometric to a configuration in a tree; the additive constant is allowed to depend on the space and the number of points only. (For comparison with more usual definitions, see 2 §2, Theorem 12 (ii) in [64].) This notion is a quasi-isometry invariant even though it is a priori defined by rough isometries.

The theory of those finitely generated groups that are Gromov-hyperbolic as metric spaces is one of the major contribution to modern group theory; we refer to M. Gromov [69].

(iii) *Hyers–Ulam stability*. D. Hyers observed in [80] that whenever a mapping f between Banach spaces satisfies the additive equation upon some constant δ , then f is at finite distance (at most δ) of a truly additive mapping. A subsequent joint paper with S. Ulam [81] proposed the more delicate question of the stability of the equation defining isometries; more precisely: *If f is a rough isometry, is it close to an isometry?* After many partial results spanning half a century (starting with the Hilbertian case in [81]), the general case was solved affirmatively by P. Gruber [73] and J. Gevirtz [62] and the sharp constant provided by M. Omladič and P. Šemrl [122] in 1995.

The stability question was broadened to other contexts by Hyers–Ulam [82], [83] and a great many other authors. Within the context of isometries, three examples are: quaternionic hyperbolic spaces (P. Pansu [125]), higher rank symmetric spaces and buildings (Eskin–Farb [47] and Kleiner–Leeb [97]), hyperbolic buildings (Bourdon–Pajot [10]).

The connection between quasification and bounded cohomology appears when one considers the stability of *cocycles*. Indeed, suppose that f is “almost an n -cocycle”

for a group G in the Hyers–Ulam sense, say for a Banach G -module V . Specifically, in the model of the *inhomogeneous* bar-resolution, f is a map $G^n \rightarrow V$ with the property that

$$g_1 f(g_2, \dots, g_{n+1}) + \sum_{j=1}^n (-1)^j f(g_1, \dots, g_j g_{j+1}, \dots, g_{n+1}) + (-1)^{n+1} f(g_1, \dots, g_n)$$

is bounded independently of $g_1, \dots, g_{n+1} \in G$. The map $\delta f: G^{n+1} \rightarrow V$ defined by the above expression is therefore a bounded $(n+1)$ -cocycle; indeed it is certainly a cocycle since it is defined as a coboundary. The latter observation means that δf represents a trivial class in usual cohomology $H^{n+1}(G, V)$. But is it trivial as bounded cohomology class in $H_b^{n+1}(G, V)$? The definition of bounded cohomology gives us the answer: this class is trivial if and only if f is at finite distance of an actual n -cocycle $G^n \rightarrow V$. In conclusion, the Hyers–Ulam stability problem for $H^n(G, V)$ is exactly captured by the kernel of the comparison map

$$EH_b^{n+1}(G, V) \stackrel{\text{def}}{=} \text{Ker}(H_b^{n+1}(G, V) \longrightarrow H^{n+1}(G, V))$$

in one degree higher: EH_b^{n+1} describes “ n -quasicocycles”. This could be made formal by introducing suitably complexes of quasicocycles and defining the corresponding cohomology groups $H_{\text{quasi}}^*(G, V)$. It is then straightforward to verify that one has an infinite exact sequence

$$\begin{aligned} \dots \rightarrow H_{\text{quasi}}^{n-1}(G, V) \rightarrow H_b^n(G, V) \rightarrow H^n(G, V) \\ \rightarrow H_{\text{quasi}}^n(G, V) \rightarrow H_b^{n+1}(G, V) \rightarrow \dots \end{aligned}$$

Consider the simplest case, namely $n = 1$ and $V = \mathbf{R}$:

Definition 2.1. A *quasimorphism* is a map $f: G \rightarrow \mathbf{R}$ such that

$$\sup_{g, h \in G} |f(g) - f(gh) + f(h)| < \infty.$$

A quasimorphism is *non-trivial* if it is not a bounded perturbation of a homomorphism, or equivalently if it determines a non-zero class in $H_b^2(G, \mathbf{R})$.

B. Johnson proves already in [89, 2.8] that the free group $\mathbf{F}_2$ admits a non-trivial quasimorphism. This was considerably generalised and it is now known that $EH_b^2(G, \mathbf{R})$ is infinite-dimensional for (non-elementary) free groups, surface groups, Gromov-hyperbolic groups, free products (R. Brooks [11], Brooks-Series [12], Y. Mitsuhashi [111], Barge–Ghys [5], Epstein–Fujiwara [46], K. Fujiwara [49], [50], R. Grigorchuk [66]); generalising all the previous cases, for all groups acting on a Gromov-hyperbolic metric space in a *weakly proper* way (Bestvina–Fujiwara [8]; see also U. Hamenstädt [74]). Moreover, J. Manning [102, 4.29] shows that if there is

any quasimorphism that is what he calls *bushy*, then, already $H_b^2(G, \mathbf{R})$ is infinite-dimensional. (Contrary to what has sometimes been suggested, $EH_b^2(G, \mathbf{R})$ may however be of finite non-zero dimension [115]; thus not every quasimorphism is bushy.) Very interesting quasimorphisms of a completely different nature have been constructed by Entov–Polterovich [45], Biran–Entov–Polterovich [9], Gambaudo–Ghys [58] and P. Py [131]. There, quasification is the additional freedom that allows to extend the Calabi homomorphism as a quasimorphism to larger groups that do not admit any non-zero homomorphism.

One checks that amenable groups do not have non-trivial quasimorphisms. In a completely opposed direction, it was proved in [23], [24] that irreducible lattices in semisimple Lie groups of higher rank have no non-trivial quasimorphisms. Interestingly, this property is not quasi-isometry invariant [23, 1.7].

Increasing the generality, let us consider unitary representations V . Since $H^1(G, V)$ classifies affine isometric actions on the Hilbert space V , it follows that $EH_b^2(G, V)$ contains information about *rough G -actions* on V , namely maps ϱ from G to the affine isometry group of V such that

$$\sup_{g,h \in G} \sup_{v \in V} \|\varrho(g)(\varrho(h)v) - \varrho(gh)v\| < \infty$$

(this forces the linear part of ϱ to be an actual representation). The results of [23], [24] show that any such rough action of a higher rank lattice has bounded “orbits”. Using Hyers–Ulam stability [81], we deduce the following corollary for higher rank lattices: *Every action by rough isometries (of a given constant) on a Hilbert space has bounded orbits.*

Another natural problem is to consider ε -representations (or near representations) of the group G on a Hilbert space V , that is, maps $\pi : G \rightarrow U(V)$ to the unitary group U such that

$$\sup_{g,h \in G} \|\pi(g)\pi(h) - \pi(gh)\|_{\text{op}} < \varepsilon,$$

wherein the norm is now the operator norm. The corresponding stability question is now: How close is π to an actual unitary representation? (In operator norm.)

When G is finite or more generally compact (with appropriate continuity addenda), la Harpe–Karoubi proved that for every $\delta > 0$ there is $\varepsilon > 0$ such that every ε -representation of Γ is δ -close to a unitary representation [40]. This was then established for amenable groups by D. Kazhdan in [95] using an ingenious notion of ε -cocycles. This device is however not obviously related to bounded cohomology.

Problem F. Can one reformulate in terms of bounded cohomology the problem of the stability of unitary representations?

D. Kazhdan also gives an example showing that for surface groups the phenomenon of stability of unitary representations fails to hold (Theorem 2 in [95]).

Problem F'. Prove (or disprove): Let Γ be a lattice in a connected simple Lie group of real rank at least two, e.g. $\Gamma = \mathrm{SL}_3(\mathbf{Z})$. Then for every $\delta > 0$ there is $\varepsilon > 0$ such that every ε -representation of Γ is δ -close to a unitary representation.

Notice that this conjectural stability does not follow from Kazhdan's property (T); just as for bounded cohomology, a stronger rigidity property of higher rank groups needs to be used. Indeed, any group G with a non-trivial quasimorphism $f : G \rightarrow \mathbf{R}$ lacks the stability of unitary representations: Consider for $\eta \in \mathbf{R}$ the map $\pi : G \rightarrow \mathrm{U}(\mathbf{C})$ for which $\pi(g)$ is the multiplication by $e^{i\eta f(g)}$. This is an ε -representation when η is small enough, but will not be quite close to a representation. Now recall that any non-elementary hyperbolic group admits non-trivial quasimorphisms and that there are many hyperbolic groups with property (T).

3. The rôle of amenability

The relevance of amenability to bounded cohomology has been patent ever since B. Johnson's memoir [89], where it is shown that a locally compact group G is amenable if and only if $H_b^n(G, V)$ vanishes for all $n > 0$ and all dual Banach modules V (compare also G. Noskov [120]). Since $H_b^n(G, V)$ appears in [89] as the Banach algebra cohomology of the group algebra of G , B. Johnson uses this characterisation to *define* the amenability of general Banach algebras. This suggests to consider the “bounded-cohomology dimension” of a group (or Banach algebra); more precisely:

Definition 3.1. (i) Let $\dim_b^\sharp(G) \in \mathbf{N} \cup \{\infty\}$ denote the smallest integer such that $H_b^n(G, V)$ vanishes for all $n > \dim_b^\sharp(G)$ and all dual Banach modules V .

(ii) Let $\dim_b(G) \in \mathbf{N} \cup \{\infty\}$ denote the smallest integer such that $H_b^n(G, V)$ vanishes for all $n > \dim_b(G)$ and *all* Banach modules V .

Thus G is amenable if and only if $\dim_b^\sharp(G) = 0$. There is a priori a hierarchy of increasingly weak generalisations of amenability given by $\dim_b^\sharp(G) = n$, $n \in \mathbf{N}$ (compare [89], §10.10; standard homological techniques reduce the property $\dim_b^\sharp(G) \leq n$ to showing vanishing in degree $n + 1$ only). It is not clear whether this hierarchy is really non-trivial; we refer to Section 5.4, where it is shown for instance that $\dim_b^\sharp \neq 1, 2$. The dimension $\dim_b(G)$ seems more mysterious; see [88] for some results.

3.1. Amenable actions. Just as *proper* G -spaces are relevant to compute the usual cohomology of a group G , there is a notion of *amenable* G -spaces relevant for bounded cohomology. Recall that properness is reflected in the possibility to perform finite (or compact) averaging, at least when the coefficient modules are topological vector spaces. The notion of averaging is the naïve one when the proper G -space is a homogeneous space G/K with finite or compact isotropy $K < G$, and can be carried out e.g. using Bruhat functions in the more general case.

The idea now is that *bounded* cocycles should allow averaging under more general circumstances: after all, the definition of amenable groups is that they permit equivariant averaging of bounded functions. The analogue of properness should accordingly generalise homogeneous G -spaces G/K with *amenable* isotropy group $K < G$. This is precisely the notion introduced by R. Zimmer [145], [146]: A Lebesgue space (S, ν) with non-singular G -action is called *amenable* if (i) the stabiliser of ν -almost every point is an amenable subgroup of G , (ii) the equivalence relation on S induced by the action is amenable. (This, however, is not Zimmer's original formulation.)

An important feature of this approach is that one has to work within the measurable category, because the general averaging process arising from amenability does not preserve continuity.

As suggested by the analogy with properness, one has the following result: *The G -space S is amenable if and only if the G -module $L^\infty(S)$ is relatively injective in a sense suitably adapted to bounded cohomology [24], [112].* It then follows from functorial machinery that the bounded cohomology $H_b^*(G, V)$ of a locally compact group G in a coefficient module V is canonically realised by the complex

$$0 \longrightarrow L^\infty(S, V)^G \longrightarrow L^\infty(S^2, V)^G \longrightarrow L^\infty(S^3, V)^G \longrightarrow \dots$$

Such a statement does require a functorial theory for the bounded cohomology of groups. Even though H_b^2 lacks the basic properties of cohomological functors, such a machinery has been developed; see R. Brooks [11], N. Ivanov [85], G. Noskov [120], [121] for discrete groups and [24], [112] for locally compact groups and for the connection with amenable spaces.

All this would not be very useful without interesting examples of amenable spaces; the foremost example is provided by *Poisson boundaries* of random walks. Recall that a random walk on G is given by a probability μ on G , which for simplicity we assume *full*, that is: (i) μ is absolutely continuous with respect to Haar measures, (ii) the support of μ generates G as a semi-group. To such a random walk one associates a non-singular G -space $S = \partial_\mu G$, the *Poisson boundary*; see H. Furstenberg [53], [54], [55], Kaĭmanovich–Vershik [94], V. Kaĭmanovich [90], A. Furman [52]. R. Zimmer proved that this G -space is amenable [144], [145]. (For another proof, see [92].)

One reason why the Poisson boundary $S = \partial_\mu G$ is a useful example of amenable space is that it is much “smaller” than the only obvious amenable G -space, G itself. Specifically, for μ symmetric, the diagonal action on S^2 is ergodic, as shown by L. Garnett [60, Remark p. 301] (generalising an argument which goes back to a 1939 paper of E. Hopf [79]). In fact, it satisfies even a much stronger double ergodicity property introduced in [24], [112]: *Every G -equivariant measurable map on S^2 to every continuous separable Banach G -module is constant.* (The existence of random walks with this property was established in [24], whilst the general – and nicer – proof was later provided by V. Kaĭmanovich in [91].) If we consider the above complex, we deduce that in this situation we have a canonical identification

$$H_b^2(G, V) \cong \{\text{cocycles in } L^\infty(S^3, V)^G\} / \{\text{constants}\}.$$

This concrete realisation is one of the most useful facts for studying bounded cohomology, as it allows to control explicitly whether or not a cocycle represents a non-vanishing class. Not only is this crucial to prove vanishing as well as non-vanishing theorems; it is also the main ingredient to prove cohomological statements such as the splitting

$$H_b^2(G_1 \times G_2, V) \cong H_b^2(G_1, V^{G_2}) \oplus H_b^2(G_2, V^{G_1})$$

for product groups $G = G_1 \times G_2$, see [24]. I emphasise that the occurrence of the space of G_i -invariants V^{G_i} is what distinguishes this statement from a mere Künneth formula and makes it consequential for rigidity applications (just as Y. Shalom's splitting formula for usual cohomology implies rigidity statements in [136]).

3.2. Amenability degree. The remarkable properties of the Poisson boundary suggest the following notion.

Definition 3.2. Let G be a countable group (or more generally a locally compact σ -compact group). Define the *amenability degree* $a(G) \in \mathbf{N} \cup \{\infty\}$ to be the supremum of all integers n for which there is some amenable G -space (X, μ) such that the diagonal G -action on X^n has finitely many ergodic components.

For an amenable group G , the amenability degree is $a(G) = \infty$ since one can take X to be a point. On the other hand, the properties of Poisson boundaries show that $a(G) \geq 2$ for *any* G . We claim: *Every non-elementary Gromov-hyperbolic group G satisfies $a(G) = 2$.* Indeed, this holds more generally⁶ for all groups G with infinite-dimensional $H_b^2(G, \mathbf{R})$ in view of the discussion in Section 3.1:

Proposition 3.3. *If G has infinite-dimensional $H_b^n(G, \mathbf{R})$, then $a(G) \leq n$.*

Problem G. What are the possible values of $a(G)$? Can one have $3 < a(G) < \infty$?

Note that $G = \mathrm{PSL}_2(\mathbf{R})$ satisfies $a(G) \geq 3$ in view of its canonical action on the projective line. I have no example ready of a countable group G with $2 < a(G) < \infty$.

Problem H. Does $a(G) = \infty$ imply that G is amenable?

The latter question has a connection to an old question⁷ about R. Thompson's group F , namely: *is it is or is it ain't amenable?* A positive answer to Problem H would prove that F is non-amenable.

Proof. Recall first that F can be defined as the group of all piecewise affine homeomorphisms of the interval $[0, 1]$ with finitely many breakpoints at dyadic rationals and whose slopes are all powers of two [30]. The similar definition with $X = \mathbf{R}/\mathbf{Z}$ instead of $[0, 1]$ yields a group T whose diagonal action on X^n is ergodic for all n (indeed, its

⁶But the hyperbolic case can also be treated by a more geometric argument.

⁷Reportedly already considered by R. Thompson in the sixties, and then independently asked by R. Geoghegan in 1979.

action on dyadic rationals is oligomorphic). The stabiliser of any dyadic point of X is isomorphic to F , whilst the stabiliser of a non-dyadic point is an increasing union of groups isomorphic to F . Thus, if F is amenable, every stabiliser is amenable. On the other hand, the equivalence relation of the T -action on X can be seen to be hyperfinite in the Borel sense. It follows that the T -action on X would be amenable with respect to any quasi-invariant measure, and $a(T) = \infty$ would follow. On the other hand, it is well-known and easy to verify that T contains non-Abelian free subgroups, hence is non-amenable. (The above line of reasoning would actually imply that the T -action on $\mathbf{R}/\mathbf{Z}$ is amenable in the topological sense of [3].) $\square$

A first step towards Problems G and H could be the following question.

Problem I. Let B be the Poisson boundary of a symmetric (full) random walk on the group G . Suppose that the diagonal G -action on B^4 is ergodic. Does it follow that G is amenable?

One can perhaps investigate this question by considering the space of *pairs of bi-infinite random paths* on G , where a bi-infinite random path refers to a random sequence

$$(\dots, x_{-2}, x_{-1}, x_0, x_1, \dots, x_n, \dots), \quad x_i \in G, \quad i \in \mathbf{Z},$$

where all increments $x_n^{-1}x_{n+1}$ are i.i.d. according to the random walk (and, say, x_0 follows Haar measure class). The diagonal G -action commutes with the shift of indices and one can try to construct invariants of pairs of such paths under some assumption similar to non-recurrence of the random walk. For instance, consider the length in $\mathbf{N}$, or location in G , of the shortest segment in a Cayley graph connecting two paths; even equivariant G -valued “invariants” prevent higher ergodicity. Whilst there is a natural map from pairs of bi-infinite paths to B^4 , it is not clear whether Problem I can be solved in this way.

4. Rigidity

Bounded cohomology has proved to be very useful to establish rigidity results. One can distinguish roughly three settings: obstructions, invariants, superrigidity.

(i) *Obstructions.* The idea here is simply to play off vanishing against non-vanishing; that is, to prove that for certain groups Γ , H there can be no non-trivial homomorphism $\Gamma \rightarrow H$ because (a) Γ has a vanishing property for H_b^* and (b) H and certain of its subgroups have non-zero classes in H_b^* . Of course, the sense in which homomorphisms are non-trivial need to be precised and will affect which subgroups of H are considered. Interestingly, parts (a) and (b) are in general of a completely different nature and are proved by very different means (and often by different authors).

Example 4.1. This rather coarse strategy can be quite effective. For instance, if Γ is a higher rank lattice, it can be used to re-prove the following result of Farb–Kaĭmanovich–Masur [93], [48]: *Every representation of Γ into any mapping class*

group has finite image. Indeed, Γ has no non-trivial quasimorphisms [23], [24]. On the other hand, by [8] any non virtually Abelian subgroup of mapping class groups has an infinite-dimensional space of quasimorphisms. It follows that any image of Γ in a mapping class group is virtually Abelian and hence finite. (Notice that this proof does not use Margulis' normal subgroup theorem.)

(ii) *Invariants.* Sometimes one particular invariant in H_b^* classifies homomorphisms. A first example appeared in Section 2.2 with É. Ghys' study of actions on the circle up to semi-conjugacy. Another is A. Iozzi's proof [84] of Matsumoto's theorem [106]. Here is a further instance due to Burger–Iozzi [18] and Burger–Iozzi–Wienhard [20]:

Example 4.2. Let X be an irreducible Hermitian symmetric space not of tube type, and let $H = \text{Is}^0(X)$ be (the connected component of) its isometry group. By [23], $H_b^2(H, \mathbf{R})$ is generated by a bounded representative ω of the Kähler class of X . It is proved in [20] that for any finitely generated group Γ , the Zariski-dense representations $\pi: \Gamma \rightarrow H$ are classified up to conjugacy by the invariant $\pi^*\omega$ in $H_b^2(\Gamma, \mathbf{R})$. (The case $H = \text{SU}(p, q)$ was previously proved in [18].)

A more refined analysis is possible for representations of surface groups by considering the *Toledo number* associated to the pull-back of the Kähler class; see the study of maximal representations presented by Burger–Iozzi–Wienhard [22] and some of its consequences [19]. Of particular importance in this context are *tight homomorphisms*, namely those representations π for which π^* preserves the norm of the Kähler class [21], [143].

(iii) *Superrigidity.* The use of richer coefficient modules, specifically of the *regular representation* $V = \ell^2(\Gamma)$, allows in some cases to encode the entire geometric situation into $H_b^2(\Gamma, V)$. This is described in the next two sections.

Non-trivial coefficients of L^∞ type have been used in [113] to establish cohomological stabilization of the general linear groups. They have also been used by Burger–Iozzi to study representations that are maximal for yet another invariant, the *generalized Toledo number*, establishing deformation rigidity for representations into $\text{SU}(m, 1)$ of lattices in $\text{SU}(n, 1)$ ($m \geq n \geq 2$), extending the famous result of Goldman–Millson [65] to *non-uniform* lattices [15], [16], [17].

4.1. Negative curvature made cohomological. M. Gromov suggests in [71, 7.E₁] how to turn the thin triangle property of negatively curved manifolds into a cohomological invariant (and refers to Z. Sela [135]); in his construction, Kazhdan's property (T) is used to ensure non-triviality. Building on similar ideas and using the multiple ergodicity of Poisson boundaries (Section 3.1), it is shown in [117], [110] that one has $H_b^2(\Gamma, \ell^2(\Gamma)) \neq 0$ for a large class of “negatively curved” groups Γ (see also [75]).

This result can be combined with rigidity techniques (such as Furstenberg maps) and cohomological results from [24], [112] in order to obtain superrigidity theorems. Specifically, it can be fed into the splitting formula at the end of Section 3.1 to obtain

geometric information. As a first illustration, consider the result below; here H_b^2 is used as a tool to control completely all existing homomorphisms, rather than just provide an obstruction when no homomorphism exists.

Let $\Gamma < G = G_1 \times G_2$ be a lattice in a product of arbitrary locally compact groups that is *irreducible* in the sense that its projection to each G_i is dense. Let $H = \text{Isom}(X)$ be the isometry group of a metric space X that is negatively curved in the sense that it is either a proper CAT(-1) space or a Gromov-hyperbolic graph of bounded valency.

Theorem ([116, 1.5]). *Any non-elementary homomorphism $\Gamma \rightarrow H$ extends to a continuous homomorphism $G \rightarrow H$ (which must factor through some G_i), possibly after factoring out a compact normal subgroup of H .*

A similar statement holds more generally for cocycles in the sense of R. Zimmer [146].

(Compare [117], [110].) For previous results concerning algebraic groups, see Margulis [103], [104], Burger–Mozes [26], S. Adams [1] and Y. Gao [59]; for results in the setting of CAT(0) spaces, see [114]. The theorem above applies e.g. when G is a semisimple group, or when Γ is a Burger–Mozes group [27], [28], or when Γ is a Kac–Moody group [132], [133].

Problem J. Does there exist a geometric characterisation of the non-vanishing of $H_b^2(\Gamma, \ell^2(\Gamma))$? Is it a quasi-isometry invariant amongst finitely generated groups?

If this property could be reformulated e.g. in terms of quasi-actions on suitable negatively curved spaces, then such a reformulation could be construed as an analogue of J. Stallings’s famous splitting theorem [140]. Indeed, in all known cases our classes in $H_b^2(\Gamma, \ell^2(\Gamma))$ are in the kernel of the comparison map. Therefore, they appear as *quasifications* of the space $H^1(\Gamma, \mathbf{R}\Gamma)$ relevant to Stallings’s theorem.

Remark 4.3. In a different direction, there is indeed a characterisation of Gromov-hyperbolic groups due to I. Mineyev [109]; the main ingredient therein is the surjectivity of the comparison map in degree two, see M. Gromov [69, 8.3.T] and I. Mineyev [108]. It is unclear whether one could obtain interesting definitions of a *rank* by postulating surjectivity in a given higher degree. Using another type of exotic cohomology, namely ℓ^∞ cohomology, S. Gersten also provided a characterisation of hyperbolicity (see [61] and compare [2]).

4.2. Orbit equivalence. Consider ergodic free measure-preserving actions of a countable group Γ on a probability space (X, μ) . The quotient space X/Γ is completely singular, but can nevertheless be investigated by shifting the focus to the type Π_1 *measured equivalence relation* $\mathcal{R} \subseteq X \times X$ induced by the action. Accordingly, one calls two actions (by possibly different groups) *orbit equivalent* (OE) if the resulting relations are isomorphic. The Ornstein–Weiss theorem [123] implies that *all*

such action of all amenable countable groups are OE. It is therefore of interest to find obstructions to OE or better yet rigidity results.

The context of Section 4.1 comes into play as follows. Let $\Gamma = \Gamma_1 \times \Gamma_2$ be a product of torsion-free groups with $H_b^2(\Gamma_i, \ell^2(\Gamma_i)) \neq 0$, e.g. non-elementary hyperbolic groups. Consider a Γ -space (X, μ) that is *irreducible* in the sense that each Γ_i -action is ergodic.

Theorem ([118, 1.6]). *Any Γ -space that is OE to X is actually conjugated to X , possibly twisting the action by an automorphism of Γ .*

Theorem ([118, 1.9]). *If any mildly mixing action of any torsion-free group Λ is OE to X , then Λ is isomorphic to Γ and the actions are conjugated.*

Compare with Hjorth–Kechris [78]. One can also use our techniques to show [118, 1.14]: *There exists a continuum of mutually non weakly isomorphic relations of type Π_1 with trivial outer automorphism group.*

5. Randomorphisms

The notion of *randomorphism* between two groups is proposed below. It is closely connected to orbit equivalence and related ideas; thus, not much originality is claimed, except perhaps for the language proposed, which I believe has its own appeal.

5.1. Random maps. The space G^H of all maps $f: H \rightarrow G$ between the countable groups H, G has a natural structure of Polish space, given by the product uniform structure (with G viewed discrete). Therefore, it makes sense to think of “random maps” $f: H \rightarrow G$ simply as probability measures on this nice Polish space. To avoid redundancy coming from the free G -action(s), we define the Polish space

$$[H, G] \stackrel{\text{def}}{=} \{f: H \rightarrow G : f(e) = e\}$$

(a closed subspace of G^H). There is a natural H -action on $[H, G]$ defined by

$$(h.f)(x) \stackrel{\text{def}}{=} f(xh)f(h)^{-1} \quad \text{for } f \in [H, G], h, x \in H.$$

The basic observation is that a homomorphism $H \rightarrow G$ is nothing but an H -fixed point for this action.

Definition 5.1. A *randomorphism* from H to G is an H -invariant probability measure on $[H, G]$.

Notice that the subset of injective maps is closed in $[H, G]$. This suggest a naïve notion of injectivity for randomorphisms:

Definition 5.2. A randomorphism is a *randembedding* if it is supported on the injective maps. We say that H is a *random subgroup* of G if it admits a randembedding into G .

Compare with the notion of *placement* proposed by M. Gromov [70, 4.5] and with Y. Shalom's related viewpoint on uniform embeddings [137]. One can verify that a point in $[H, G]$ is almost periodic (i.e. its H -orbit relatively compact) if the corresponding map is Lipschitz in the appropriate sense.

Definition 5.3. A randomorphism, randembedding or random subgroup is *geometric* if the corresponding measure is compactly supported in $[H, G]$.

Problem K. Which groups admit the non-Abelian free group $\mathbf{F}_2$ as a geometric random subgroup?

The following has been proved by D. Gaboriau and independently R. Lyons (private communication); Gaboriau's proof uses percolation techniques, relying among other things on [124] and [77].

Theorem 5.4. *Every non-amenable group admits $\mathbf{F}_2$ as a random subgroup.*

5.2. Back to orbit equivalence. If the two countable groups G, H have OE actions as in Section 4.2 on (X, μ) and (Y, ν) respectively, then there is a measure space isomorphism $F: X \rightarrow Y$ such that $F(H.x) = G.F(x)$ almost everywhere. One can also consider the more general situation where $F(H.x) \subseteq G.F(x)$; for instance, H could be a subgroup of a group having an action orbit equivalent to the G -action. By freeness, there is a measurable map

$$\alpha: H \times X \longrightarrow G$$

defined almost everywhere by $F(h.x) = \alpha(h, x).F(x)$. This map is a *cocycle* in that it satisfies

$$\alpha(hk, x) = \alpha(h, k.x)\alpha(k, x) \quad \text{for } h, k \in H, \mu - \text{a.e. } x \in X.$$

The cocycle α yields a measurable map $\hat{\alpha}: X \rightarrow [H, G]$ defined by $\hat{\alpha}(x)(h) = \alpha(h, x)$. This map is H -equivariant, and therefore the measure $\hat{\alpha}_*\mu$ is a randomorphism from H to G .

Observe that regardless of any measure, there is a *tautological cocycle* $E: H \times [H, G] \rightarrow G$ with respect to the H -action on $[H, G]$ defined by $E(h, f) = f(h)$. In the above construction, the map $\hat{\alpha}$ intertwines the cocycle α to the tautological cocycle. Therefore, $[H, G]$ together with its tautological cocycle has a way to reflecting all OE cocycles within the space of all invariant probability measures on $[H, G]$.

The Ornstein–Weiss theorem [123] shows in particular that every amenable group is a random subgroup of $\mathbf{Z}$. Therefore, one has the following striking “Random Tits Alternative”:

Any countable group is either a random subgroup⁸ of $\mathbf{Z}$ or has $\mathbf{F}_2$ as a random subgroup.

⁸actually, *measure equivalent* to $\mathbf{Z}$; compare Section 5.5.

The viewpoint of measured relations and OE allows to formulate a related question: Does *any* non-amenable type II_1 relation contains the orbits of a $\mathbb{F}_2$ -action? (See Kechris–Miller [96, 28.14] and D. Gaboriau [57, 5.16]). This is only known to hold for relations of non-trivial *cost* (M. Pichot [126], Kechris–Miller [96, 28.8]; see [56] for the notion of cost).

5.3. Modules. Just as a homomorphism $H \rightarrow G$ yields a pull-back functor from G -modules to H -modules, we can define pull-backs through randommorphisms:

Definition 5.5. Let V be a coefficient G -module, μ a randommorphism from H to G and $1 \leq p \leq \infty$. The L^p -pull-back of V through μ is the Banach space $L^p(\mu, V)$ endowed with the H -action

$$(h \cdot \varphi)(f) \stackrel{\text{def}}{=} f(h^{-1})^{-1} \varphi(h^{-1} \cdot f), \quad h \in H, \varphi \in L^p(\mu, V), f \in [H, G].$$

We are mostly interested in $p = 2, \infty$. For instance, analysing the case $V = \ell^2(G)$ shows:

Lemma 5.6. *A random subgroup of an amenable group is itself amenable.*

Recall that in the case of injective homomorphisms, the pull-back has an adjoined functor called *(co-)induction*. There is again an analogue for randembeddings. Consider G^H with precomposition by right H -translation and postcomposition by right G -translation. This is isomorphic to the E -twisted (cf. [146, p.65]) product H -space $[H, G] \times G$ endowed with an additional G -action by right multiplication. Therefore, it inherits an invariant σ -finite regular Borel measure $\tilde{\mu}$ defined as the product of μ with the counting measure. The following generalises induction (compare [70, 4.5.C] and [118, 4.1]).

Definition 5.7. Let V be a coefficient H -module, μ a randembedding from H to G and $1 \leq p \leq \infty$. The L^p -induced module of V through μ is the Banach space $L^p(\tilde{\mu}, V)^H$ of H -equivariant maps endowed with the G -action by right translations.

5.4. Application to bounded cohomology. The induction methods used in [118] can be seen to yield the following.

Proposition 5.8. *Let H be a random subgroup of G and V a coefficient H -module. Let W be the L^∞ -induced module. For every $n \geq 0$ there is an injection $H_b^n(H, V) \hookrightarrow H_b^n(G, W)$.*

(Due to the unwieldy nature of L^∞ spaces, it is essential in [118] to have a similar injectivity statement for the L^p -induced module with $p < \infty$, e.g. $p = 2$. However, the latter is only known to hold when $n \leq 2$.)

Corollary 5.9. *If H is a random subgroup of G , then $\dim_b^\sharp(H) \leq \dim_b^\sharp(G)$.*

Appealing to Theorem 5.4, we conclude:

Corollary 5.10. *No group can have $\dim_b^\sharp = 1, 2$.*

Problem L. Is $\dim_b^\sharp(\mathbf{F}_2)$ infinite? If so, is even $H_b^n(\mathbf{F}_2, \mathbf{R})$ non-zero for all $n \geq 2$?

If $\dim_b^\sharp(\mathbf{F}_2) = \infty$, then it follows that the hierarchy proposed by B. Johnson (cf. beginning of Section 3) collapses completely, since we then have $\dim_b^\sharp(G) = 0$ or ∞ for every group G , according to whether it is amenable or not.

5.5. Categorical approach. As we have seen, orbit equivalences yield randomorphisms. However orbit equivalence is symmetrical; the intuition is that these randomorphism should more precisely be “isomorphisms” in an appropriate category whose morphisms are represented by randomorphisms. A related problem is that the notion of randembedding of Definition 5.2 does not follow the usual categorical pattern that should define mono-randomorphisms. A third issue is that it is unclear when a randomorphism should be considered to be an epimorphism.

In order to address these points, it is necessary to define the composition of two randomorphisms. It seems that there are (at least) two natural composition products:

(i) The independent product. Let G, H, L be countable groups. The composition map

$$[L, H] \times [H, G] \longrightarrow [L, G], \quad (f, f') \longmapsto f' \circ f$$

is continuous. Given probability measures μ, ν on $[L, H]$ respectively $[H, G]$, denote by $\nu \circ \mu$ the image of the product measure $\mu \times \nu$ under this map. Thus $\nu \circ \mu$ is the product of *independently* chosen random maps. If both μ, ν are randomorphisms, then so is $\nu \circ \mu$. The main defect of the independent product is the scarcity of randomorphisms that are invertible for this product.

(ii) The fibred product. Another product has the flavour of groupoids and is defined as follows. Two randomorphisms μ from L to H and ν from H to G are *composable* if there is an isomorphism of Lebesgue spaces

$$F: ([L, H], \mu) \xrightarrow{\cong} ([H, G], \nu)$$

which is equivariant with respect to the tautological cocycle $L \times [L, H] \rightarrow H$; that is, $F(\ell.f) = f(\ell).F(f)$ μ -a.e. In that case we define the *fibred product* $\nu \cdot_F \mu$ as the image of μ under the map

$$[L, H] \longrightarrow [L, G], \quad f \longmapsto F(f) \circ f.$$

Both products are a particular case of the construction that associates a randomorphism from L to G to the data of a randomorphism from L to H together with a measurable map from $[L, H]$ to probability measures on $[H, G]$ that is equivariant with respect to the tautological cocycle. In all cases the verification follows from the formula $\ell.(f' \circ f) = (f(\ell).f') \circ (\ell.f)$.

Problem M. (i) Subsume both product in one categorical construction, for instance by defining a suitable equivalence relation on randommorphisms. (ii) Produce an interesting definition of a group of auto-randommorphisms of a given group G . (iii) Reformulate A. Furman's results [51] as a determination of this group for higher rank lattices.

5.6. Random forests. Let G be a countable group and $A \subseteq G$ a non-empty finite subset. Define the compact G -space of unoriented⁹, labelled 4-regular graphs as

$$\mathcal{G}_A \stackrel{\text{def}}{=} \{(x_{i,g}) \in \prod_{g \in G} (gA)^{\mathbb{Z}/4\mathbb{Z}} : x_{h,i+2} = g \text{ when } h = x_{g,i}\}.$$

The topology is the product topology and $k \in G$ acts by $(kx)_{g,i} = k(x_{k^{-1}g,i})$.

Definition 5.11. The space $\mathcal{F}_A$ of 4-forests is the closed G -invariant subspace $\mathcal{F}_A \subseteq \mathcal{G}_A$ of those elements $(x_{i,g})$ for which every connected component is acyclic.

An immediate application of Tarski's theorem on paradoxical decompositions (or of appropriate proofs of it) yields:

Proposition 5.12. *The group G is non-amenable if and only if $\mathcal{F}_A \neq \emptyset$ for A large enough.*

Problem K'. For which groups does there exists an invariant probability measure on $\mathcal{F}_A$ for some A ?

Notice that there is a canonical continuous map $\mathcal{F}_A \rightarrow [\mathbf{F}_2, G]$ which is $\mathbf{F}_2$ -equivariant when the left hand side is endowed with the natural $\mathbf{F}_2$ -action defined by the labelling of the forests. Moreover, this map ranges in the space of injections. Therefore, groups with a measure as in Problem K' will have a geometric random free subgroup as in Problem K.

6. Additional questions

Here is the motivation that prompted me to consider randommorphisms to begin with. It is an old observation of J. Dixmier [41] and M. Day [39] that every uniformly bounded representation of an amenable group is *unitarizable*, i.e. conjugated to a unitary representation. The problem of the converse to this statement proposed in J. Dixmier's 1950 article [41] is still open, despite remarkable work most notably by G. Pisier (see [128], [129] and [127]).

It is nevertheless possible to show very explicitly that any group containing $\mathbf{F}_2$ has uniformly bounded representations that are not unitarizable (see e.g. Theorem 2.1 and Lemma 2.7 in [128]). On the other hand, one can also induce such representations from *random subgroups* exactly as explained above and still obtain uniformly bounded representations.

⁹We make the usual convention that an unoriented edge consists of two opposed oriented edges.

Problem N. (i) Is the uniformly bounded representation induced as above non-unitarizable? (ii) Is this true at least for random embeddings of $\mathbf{F}_2$ arising from an invariant measure on the space of forests?

We have seen that the vanishing of H_b^* with general dual coefficients characterises amenability; however, it is not enough to consider trivial coefficients $\mathbf{R}$ even in all degrees. Indeed, refining the method of J. Mather [105], Matsumoto–Morita [107] have proved that the (non-amenable) group of compactly supported homeomorphisms of $\mathbf{R}^n$ has vanishing H_b^* with trivial coefficients. Is the situation nicer for linear groups?

Problem O. (i) Let $\Gamma < GL_d(\mathbf{C})$ be a subgroup that is not virtually soluble (equivalently, Γ is non-amenable). Is $H_b^n(\Gamma, \mathbf{R})$ non-zero for some $1 < n < d^2$? (ii) If not, is there at least a *separable* dual Banach Γ -module V with $H_b^n(\Gamma, V)$ non-zero for some $n \geq 2$?

One way to approach the problem would be to prove first the conjecture proposed as Problem A'. This done, one needs to deduce geometric finiteness properties from the annihilation of the pull-back $H_b^*(G, \mathbf{R}) \rightarrow H_b^*(\Gamma, \mathbf{R})$. The latter step is not at all hopeless, especially in view of the results of B. Klingler for usual cohomology in [98].

One hint to the difficulties that could arise (in addition to Problem A') is the fact that B. Klingler needs non-trivial coefficient modules. However, as mentioned in the introduction, the analogous statement to Problems A and A' fails already for unitary representations. Therefore, either one needs to construct cohomology classes for such representations that are genuinely new (as in [25]) and prove results similar to those of [98] for such classes, or one needs to argue that even trivial coefficients suffice because pull-back in bounded cohomology tends to be injective in cases where the usual pull-back is not. (For example, the volume form of the hyperbolic plane restricts non-trivially in H_b^2 to free lattice subgroups of $PSL_2(\mathbf{R})$, whilst it vanishes when sent to usual cohomology.)

An alternative approach for linear groups could be to consider more generally polynomially bounded cocycles.

Problem P. Let $G = G(k)$ be a simple group of k -rank $r > 0$ over a local field k . Quasify B. Klingler's cocycles [98] in order to obtain new classes in degree $r + 1$ for cohomology with polynomial growth degree $r - 1$ (in a suitable module).

I suspect that $(r + 1)$ -cohomology with polynomial growth degree $r - 1$ is indeed the right place to look for “rank r phenomena”.

Ch. Bavard proves [6] that a group has non-trivial quasimorphisms if and only if its *stable commutator length* is non-zero.

Problem Q (M. Abért). Let k be a countable field of infinite transcendence degree over its prime field. Does the group $SL_3(k[X])$ have non-trivial quasi-morphisms?

The interest of $SL_3(k[X])$ in connection with Ch. Bavard's result is that for fields k as above this group is known to have infinite commutator width, which is a priori not enough to control the stable length.

Finally a question from [113]. For a prime p , denote by $v_p: \mathbf{Q} \setminus \{0\} \rightarrow \mathbf{Z}$ the p -adic valuation (normalised by $v_p(p^n) = -n$). If q is another prime, define $D_{p,q}: \mathbf{Q} \setminus \{0, 1\} \rightarrow \mathbf{Z}$ by

$$D_{p,q}(x) = v_p(x)v_q(1-x) - v_q(x)v_p(1-x).$$

This function is obviously unbounded; on the other hand, one can form arbitrary linear combinations of such $D_{p,q}$ by varying the primes p, q .

Problem R. Is the function $\sum_{p < q} \alpha_{p,q} D_{p,q}$ unbounded on $\mathbf{Q} \setminus \{0, 1\}$ for every family of real numbers $\{\alpha_{p,q}\}$ (unless they are all zero)?

It was observed in [113] that a positive answer would imply $H_b^3(\mathrm{GL}_2(\mathbf{Q}), \mathbf{R}) = 0$. Moreover, it follows from the stabilisation results of [113] that the latter vanishing would imply $H_b^3(\mathrm{GL}_n(\mathbf{Q}_p), \mathbf{R}) = 0$ for all $n \in \mathbf{N}$ and all primes p .

References

- [1] Adams, S., Reduction of cocycles with hyperbolic targets. *Ergodic Theory Dynam. Systems* **16** (6) (1996), 1111–1145.
- [2] Allcock, D. J., and Gersten, S. M., A homological characterization of hyperbolic groups. *Invent. Math.* **135** (3) (1999), 723–742.
- [3] Anantharaman-Delaroche, C., and Renault, J., *Amenable groupoids*. Monogr. Enseign. Math. 36, L’Enseignement Mathématique, Genève, 2000.
- [4] Atiyah, M. F., Elliptic operators, discrete groups and von Neumann algebras. In *Colloque “Analyse et Topologie” en l’Honneur de Henri Cartan; Astérisque* **32–33** (1976), 43–72.
- [5] Barge, J., and Ghys, É., Surfaces et cohomologie bornée. *Invent. Math.* **92** (3) (1988), 509–526.
- [6] Bavard, C., Longueur stable des commutateurs. *Enseign. Math.* (2) **37** (1–2) (1991), 109–150.
- [7] Benedetti, R., and Petronio, C., *Lectures on hyperbolic geometry*. Universitext, Springer-Verlag, Berlin 1992.
- [8] Bestvina, M., and K. Fujiwara, K., Bounded cohomology of subgroups of mapping class groups. *Geom. Topol.* **6** (2002), 69–89.
- [9] Biran, P., Entov, M., and Polterovich, L., Calabi quasimorphisms for the symplectic ball. *Commun. Contemp. Math.* **6** (5) (2004), 793–802.
- [10] Bourdon, M., and Pajot, H., Rigidity of quasi-isometries for some hyperbolic buildings. *Comment. Math. Helv.* **75** ((4)) (2000), 701–736.
- [11] Brooks, R., Some remarks on bounded cohomology. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference*, Ann. of Math. Stud. 97, Princeton University Press, Princeton, N.J., 1981, 53–63.
- [12] Brooks, R., and Series, C., Bounded cohomology for surface groups. *Topology* **23** (1) (1984), 29–36.

- [13] Bucher-Karlsson, M., Simplicial volume of locally symmetric spaces covered by $SL_3(\mathbb{R})/SO(3)$. Preprint.
- [14] Bucher-Karlsson, M., *Characteristic classes and bounded cohomology*. PhD thesis, ETH Zürich, Diss. Nr. 15636, 2004.
- [15] Burger, M., and Iozzi, A., Bounded cohomology and deformation rigidity in hyperbolic geometry. Preprint.
- [16] Burger, M., and Iozzi, A., Bounded differential forms, generalized Milnor-Wood inequality and an application to deformation rigidity. Preprint, 2005.
- [17] Burger, M., and Iozzi, A., A useful formula in bounded cohomology. Preprint, 2005.
- [18] Burger, M., and Iozzi, A., Bounded Kähler class rigidity of actions on Hermitian symmetric spaces. *Ann. Sci. École Norm. Sup. (4)* **37** (1) (2004), 77–103.
- [19] Burger, M., Iozzi, A., Labourie, F., and Wienhard, A., Maximal representations of surface groups: Symplectic Anosov structures. *Pure Appl. Math. Q.* **1** (3) (2005), 555–601.
- [20] Burger, M., Iozzi, A., and Wienhard, A., Hermitian symmetric spaces and Kähler rigidity. *Transform. Groups*, to appear.
- [21] Burger, M., Iozzi, A., and Wienhard, A., Tight embeddings. Preprint.
- [22] Burger, M., Iozzi, A., and Wienhard, A., Surface group representations with maximal Toledo invariant. *C. R. Math. Acad. Sci. Paris* **336** (5) (2003), 387–390.
- [23] Burger, M., and Monod, N., Bounded cohomology of lattices in higher rank Lie groups. *J. Eur. Math. Soc. (JEMS)* **1** (2) (1999), 199–235.
- [24] Burger, M., and Monod, N., Continuous bounded cohomology and applications to rigidity theory *Geom. Funct. Anal.* **12** (2) (2002), 219–280.
- [25] Burger, M., and Monod, N., On and around the bounded cohomology of SL_2 . In *Rigidity in dynamics and geometry*, Springer-Verlag, Berlin 2002, 19–37.
- [26] Burger, M., and Mozes, S., CAT(-1)-spaces, divergence groups and their commensurators. *J. Amer. Math. Soc.* **9** (1) (1996), 57–93.
- [27] Burger, M., and Mozes, S., Groups acting on trees: from local to global structure. *Inst. Hautes Études Sci. Publ. Math.* **92** (2001), 113–150, 2000.
- [28] Burger, M., and Mozes, S., Lattices in product of trees. *Inst. Hautes Études Sci. Publ. Math.* (**92**) (2000), 151–194.
- [29] Burghel, D., The cyclic homology of the group rings. *Comment. Math. Helv.* **60** (3) (1985), 354–365.
- [30] Cannon, J. W., Floyd, W. J., and Parry, W. R., Introductory notes on Richard Thompson’s groups. *Enseign. Math.* (2) **42** (3–4) (1996), 215–256.
- [31] Cheeger, J., and Gromov, M., L_2 -cohomology and group cohomology. *Topology* **25** (2) (1986), 189–215.
- [32] Cheeger, J., and Simons, J., Differential characters and geometric invariants. In *Geometry and topology*, Lecture Notes in Math. 1167, Springer-Verlag, Berlin 1985, 50–80.
- [33] Christensen, E., Effros, E. G., and Sinclair, A., Completely bounded multilinear maps and C^* -algebraic cohomology. *Invent. Math.* **90** (2) (1987), 279–296.
- [34] Clerc, J.-L., and Ørsted, B., The Gromov norm of the Kaehler class and the Maslov index. *Asian J. Math.* **7** (2) (2003), 269–295.

- [35] Connell, Christopher, Farb, Benson, The degree theorem in higher rank. *J. Differential Geom.* **65** (1) (2003), 19–59.
- [36] Connes, A., On the classification of von Neumann algebras and their automorphisms. In *Convegno sulle Algebre C^* e loro Applicazioni in Fisica Teorica* (INDAM, Rome 1975), Symposia Math. XX, Academic Press, London 1976, 435–478.
- [37] Connes, A., Noncommutative differential geometry. *Inst. Hautes Études Sci. Publ. Math.* **62** (1985), 257–360.
- [38] Connes, A., and Shlyakhtenko, D., L^2 -homology for von Neumann algebras. *J. Reine Angew. Math.* **586** (2005), 125–168.
- [39] Day, M. M., Means for the bounded functions and ergodicity of the bounded representations of semi-groups. *Trans. Amer. Math. Soc.* **69** (1950), 276–291.
- [40] de la Harpe, P., and Karoubi, M., Représentations approchées d’un groupe dans une algèbre de Banach. *Manuscripta Math.* **22** (3) (1977), 293–310.
- [41] Dixmier, J., Les moyennes invariantes dans les semi-groupes et leurs applications. *Acta Sci. Math. Szeged* **12** (1950), 213–227.
- [42] Domic, A., and Toledo, D., The Gromov norm of the Kaehler class of symmetric domains. *Math. Ann.* **276** (3) (1987), 425–432.
- [43] Dupont, J. L., Bounds for characteristic numbers of flat bundles. In *Algebraic topology, Aarhus 1978*, Lecture Notes in Math. 763, Springer-Verlag, Berlin 1979, 109–119.
- [44] Dupont, J. L., and Kamber, F. W., On a generalization of Cheeger-Chern-Simons classes. *Illinois J. Math.* **34** (2) (1990), 221–255.
- [45] Entov, M., and Polterovich, L., Calabi quasimorphism and quantum homology. *Internat. Math. Res. Notices* (**30**) (2003), 1635–1676.
- [46] Epstein, D. B. A., and Fujiwara, K., The second bounded cohomology of word-hyperbolic groups. *Topology* **36** (6) (1997), 1275–1289.
- [47] Eskin, A., and Farb, B., Quasi-flats and rigidity in higher rank symmetric spaces. *J. Amer. Math. Soc.* **10** (3) (1997), 653–692.
- [48] Farb, B., and Masur, H., Superrigidity and mapping class groups. *Topology* **37** (6) (1998), 1169–1176.
- [49] Fujiwara, K., The second bounded cohomology of a group acting on a Gromov-hyperbolic space. *Proc. London Math. Soc.* (3) **76** (1) (1998), 70–94.
- [50] Fujiwara, K., The second bounded cohomology of an amalgamated free product of groups. *Trans. Amer. Math. Soc.* **352** (3) (2000), 1113–1129.
- [51] Furman, A., Gromov’s measure equivalence and rigidity of higher rank lattices. *Ann. of Math.* (2) **150** (3) (1999), 1059–1081.
- [52] Furman, A., Random walks on groups and random transformations. In *Handbook of dynamical systems*, Vol. 1A, North-Holland, Amsterdam 2002, 931–1014.
- [53] Furstenberg, H., A Poisson formula for semi-simple Lie groups. *Ann. of Math.* (2) **77** (1963), 335–386.
- [54] Furstenberg, H., Random walks and discrete subgroups of Lie groups. In *Advances in Probability and Related Topics*, Vol. 1, Dekker, New York 1971, 1–63..
- [55] Furstenberg, H., Rigidity and cocycles for ergodic actions of semisimple Lie groups (after G. A. Margulis and R. Zimmer). In *Bourbaki Seminar, Vol. 1979/80*, Lecture Notes in Math. 842, Springer-Verlag, Berlin 1981, 273–292.

- [56] Gaboriau, D., Coût des relations d'équivalence et des groupes. *Invent. Math.* **139** (1) (2000), 41–98.
- [57] Gaboriau, D., Arbres, groupes, quotients. Mémoire d'habilitation, ÉNS-Lyon, 2002.
- [58] Gambaudo, J.-M., and Ghys, É., Commutators and diffeomorphisms of surfaces. *Ergodic Theory Dynam. Systems* **24** (5) (2004), 1591–1617.
- [59] Gao, Y., Superrigidity for homomorphisms into isometry groups of CAT(−1) spaces. *Transform. Groups* **2** (3) (1997), 289–323.
- [60] Garnett, L., Foliations, the ergodic theorem and Brownian motion. *J. Funct. Anal.* **51** (3) (1983), 285–311.
- [61] Gersten, S. M., A cohomological characterization of hyperbolic groups. Unpublished, <http://www.math.utah.edu/~gersten>.
- [62] Gevirtz, J., Stability of isometries on Banach spaces. *Proc. Amer. Math. Soc.* **89** (4) (1983), 633–636.
- [63] Ghys, É., Groupes d'homéomorphismes du cercle et cohomologie bornée. In *The Lefschetz centennial conference*, Part III, Amer. Math. Soc., Providence, RI, 1987, 81–106.
- [64] Ghys, É., and de la Harpe, P. (eds.), *Sur les groupes hyperboliques d'après Mikhael Gromov*. Progr. Math. 83, Birkhäuser, Boston 1990.
- [65] Goldman, W. M., and Millson, J. J., Local rigidity of discrete groups acting on complex hyperbolic space. *Invent. Math.* **88** (1987), 495–520.
- [66] Grigorchuk, R. I., Some results on bounded cohomology. In *Combinatorial and geometric group theory*, London Math. Soc. Lecture Note Ser. 204, Cambridge University Press, Cambridge, 1995, 111–163.
- [67] Gromov, M., Volume and bounded cohomology. *Inst. Hautes Études Sci. Publ. Math.* (**56**) (1982), 5–99.
- [68] Gromov, M., Infinite groups as geometric objects. In *Proceedings of the International Congress of Mathematicians* (Warszawa, 1983), Vol. 1, PWN, Warsaw 1984, 385–392.
- [69] Gromov, M., Hyperbolic groups. In *Essays in group theory*, Math. Sci. Res. Inst. Publ. 8, Springer-Verlag, New York 1987, 75–263.
- [70] Gromov, M., Rigid transformations groups. In *Géométrie différentielle* (Paris, 1986), Travaux en Cours 33, Hermann, Paris 1988, 65–139.
- [71] Gromov, M., Asymptotic invariants of infinite groups. In *Geometric group theory* (Sussex, 1991), Vol. 2, London Math. Soc. Lecture Note Ser. 182, Cambridge University Press, Cambridge, 1993, 1–295.
- [72] Grothendieck, A., Produits tensoriels topologiques et espaces nucléaires. *Mem. Amer. Math. Soc.* **1955** (16) (1955), 140pp.
- [73] Gruber, P. M., Stability of isometries. *Trans. Amer. Math. Soc.* **245** (1978), 263–277.
- [74] Hamenstädt, U., Bounded cohomology and isometry groups of hyperbolic spaces. Preprint.
- [75] Hamenstädt, U., Isometry groups of proper hyperbolic spaces. Preprint.
- [76] Helemskiĭ, A. Y., *The homology of Banach and topological algebras*. Math. Appl. (Soviet Ser.) 41, Kluwer Academic Publishers Group, Dordrecht 1989.
- [77] Hjorth, G., A lemma for cost attained. Preprint.

- [78] Hjorth, G., and Kechris, A. S., Rigidity theorems for actions of product groups and countable Borel equivalence relations. *Mem. Amer. Math. Soc.* **177** (833) (2005), 109pp.
- [79] Hopf, E., Statistik der geodätischen Linien in Mannigfaltigkeiten negativer Krümmung. *Ber. Verh. Sächs. Akad. Wiss. Leipzig* **91** (1939), 261–304.
- [80] Hyers, D. H., On the stability of the linear functional equation. *Proc. Nat. Acad. Sci. U. S. A.* **27** (1941), 222–224.
- [81] Hyers, D. H., and Ulam, S. M., On approximate isometries. *Bull. Amer. Math. Soc.* **51** (1945), 288–292.
- [82] Hyers, D. H., and Ulam, S. M., Approximately convex functions. *Proc. Amer. Math. Soc.* **3** (1952), 821–828.
- [83] Hyers, D. H., and Ulam, S. M., On the stability of differential expressions. *Math. Mag.* **28** (1954), 59–64.
- [84] Iozzi, A., Bounded cohomology, boundary maps, and rigidity of representations into $\text{Homeo}^+(\mathbb{S}^1)$ and $\text{SU}(1, n)$. In *Rigidity in Dynamics and Geometry*. Springer-Verlag, Berlin 2002.
- [85] Ivanov, N. V., Foundations of the theory of bounded cohomology. *J. Soviet Math.* **37** (1987), 1090–1115.
- [86] Ivanov, N. V., and Turaev, V. G., The canonical cocycle for the Euler class of a flat vector bundle. *Dokl. Akad. Nauk SSSR* **265** (3) (1982), 521–524.
- [87] Ji, R., Nilpotency of Connes’ periodicity operator and the idempotent conjectures. *K-Theory* **9** (1) (1995), 59–76.
- [88] Johnson, B. E., Approximate diagonals and cohomology of certain annihilator Banach algebras. *Amer. J. Math.* **94** (1972), 685–698.
- [89] Johnson, B. E., Cohomology in Banach algebras. *Mem. Amer. Math. Soc.* **127** (1972), 96pp.
- [90] Kaĭmanovich, V. A., The Poisson formula for groups with hyperbolic properties. *Ann. of Math.* (2) **152** (3) (2000), 659–692.
- [91] Kaĭmanovich, V. A., Double ergodicity of the Poisson boundary and applications to bounded cohomology. *Geom. Funct. Anal.* **13** (4) (2003), 852–861.
- [92] Kaĭmanovich, Vadim A. Amenability and the Liouville property. Probability in mathematics. *Israel J. Math.* **149** (2005), 45–85.
- [93] Kaĭmanovich, V. A., and Masur, H., The Poisson boundary of the mapping class group. *Invent. Math.* **125** (2) (1996), 221–264.
- [94] Kaĭmanovich, V. A., and Vershik, A. M., Random walks on discrete groups: boundary and entropy. *Ann. Probab.* **11** (3) (1983), 457–490.
- [95] Kazhdan, D., On ε -representations. *Israel J. Math.* **43** (4) (1982), 315–323.
- [96] Kechris, A. S., and Miller, B. D., *Topics in orbit equivalence*. Lecture Notes in Math. 1852, Springer-Verlag, Berlin 2004.
- [97] Kleiner, B., and Leeb, B., Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings. *Inst. Hautes Études Sci. Publ. Math.* (86) (1997), 115–197.
- [98] Klingler, B., Volumes des représentations sur un corps local. *Geom. Funct. Anal.* **13** (5) (2003), 1120–1160.
- [99] Kuessner, T., *Relative simplicial volume*. PhD thesis, Universität Tübingen, 2001.

- [100] Lafont, J.-F., and Schmidt, B., Simplicial volume of closed locally symmetric spaces of non-compact type. *Acta Math.*, to appear.
- [101] Lück, W., Dimension theory of arbitrary modules over finite von Neumann algebras and L^2 -Betti numbers. I. Foundations. *J. Reine Angew. Math.* **495** (1998), 135–162.
- [102] Manning, J. F., Geometry of pseudocharacters. *Geom. Topol.* **9** (2005), 1147–1185..
- [103] Margulis, G. A., On the decomposition of discrete subgroups into amalgams. *Selecta Math. Soviet.* **1** (2) (1981), 197–213.
- [104] Margulis, G. A., *Discrete subgroups of semisimple Lie groups*. Ergeb. Math. Grenzgeb. 17, Springer-Verlag, Berlin 1991.
- [105] Mather, J. N., The vanishing of the homology of certain groups of homeomorphisms. *Topology* **10** (1971), 297–298.
- [106] Matsumoto, S., Some remarks on foliated S^1 bundles. *Invent. Math.* **90** (1987), 343–358.
- [107] Matsumoto, S., and Morita, S., Bounded cohomology of certain groups of homeomorphisms. *Proc. Amer. Math. Soc.* **94** (3) (1985), 539–544.
- [108] Mineyev, I., Straightening and bounded cohomology of hyperbolic groups. *Geom. Funct. Anal.* **11** (4) (2001), 807–839.
- [109] Mineyev, I., Bounded cohomology characterizes hyperbolic groups. *Q. J. Math.* **53** (1) (2002), 59–73.
- [110] Mineyev, I., Monod, N., and Shalom, Y., Ideal bicomings for hyperbolic groups and applications. *Topology* **43** (6) (2004), 1319–1344.
- [111] Mitsumatsu, Y., Bounded cohomology and l^1 -homology of surfaces. *Topology* **23** (4) (1984), 465–471.
- [112] Monod, N., *Continuous bounded cohomology of locally compact groups*. Lecture Notes in Math. 1758, Springer-Verlag, Berlin 2001.
- [113] Monod, N., Stabilization for SL_n in bounded cohomology. In *Discrete geometric analysis*, Contemp. Math. 347, Amer. Math. Soc., Providence, RI, 2004, 191–202.
- [114] Monod, N., Superrigidity for irreducible lattices and geometric splitting. *J. Amer. Math. Soc.*, to appear 2006.
- [115] Monod, N., and Rémy, B., Boundedly generated groups with pseudocharacter(s). Appendix to: J. F. Manning, Quasi-actions on trees and Property (QFA), *J. London Math. Soc.* **73** (1) (2006), 104–108.
- [116] Monod, N., and Shalom, Y., Negative curvature from a cohomological viewpoint and cocycle superrigidity. *C. R. Acad. Sci. Paris Sér. I Math.* **337** (10) (2003), 635–638.
- [117] Monod, Nicolas, and Shalom, Yehuda, Cocycle superrigidity and bounded cohomology for negatively curved spaces. *J. Differential Geom.* **67** (3) (2004), 395–455.
- [118] Monod, N., and Shalom, Y., Orbit equivalence rigidity and bounded cohomology. *Ann. of Math.*, to appear, 2006.
- [119] Mostow, G. D., The rigidity of locally symmetric spaces. In *Actes du Congrès International des Mathématiciens* (Nice, 1970), Tome 2, Gauthier-Villars, Paris 1971, 187–197.
- [120] Noskov, G. A., Bounded cohomology of discrete groups with coefficients. *Leningr. Math. J.* **2** (5) (1991), 1067–1084.
- [121] Noskov, G. A., The Hochschild-Serre spectral sequence for bounded cohomology. In *Proceedings of the International Conference on Algebra*, Part 1, Contemp. Math. 131, Amer. Math. Soc., Providence, RI, 1992, 613–629.

- [122] Omladič, M., and Šemrl, P., On nonlinear perturbations of isometries. *Math. Ann.* **303** (4) (1995), 617–628.
- [123] Ornstein, D. S., and Weiss, B., Ergodic theory of amenable group actions. I. The Rohlin lemma. *Bull. Amer. Math. Soc.* **2** (1) (1980), 161–164.
- [124] Pak, I., and Smirnova-Nagnibeda, T., On non-uniqueness of percolation on nonamenable Cayley graphs. *C. R. Acad. Sci. Paris Sér. I Math.* **330** (6) (2000), 495–500.
- [125] Pansu, P., Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un. *Ann. of Math.* (2) **129** (1) (1989), 1–60.
- [126] Pichot, M., Quasi-périodicité et théorie de la mesure. PhD thesis, ÉNS-Lyon, 2005.
- [127] Pisier, G., Simultaneous similarity, bounded generation and amenability. Preprint.
- [128] Pisier, G., *Similarity problems and completely bounded maps*, Lecture Notes in Math. 1618, expanded edition, Springer-Verlag, Berlin 2001.
- [129] Pisier, G., Are unitarizable groups amenable? In *Infinite groups: geometric, combinatorial and dynamical aspects*, Progr. Math. 248, Birkhäuser, Basel 2005, 323–362.
- [130] Popa, S., Correspondences. INCREST preprint, 1986.
- [131] Py, P., Quasi-morphisme de Calabi sur les surfaces de genre supérieur. *C. R. Math. Acad. Sci. Paris* **341** (1) (2005), 29–34.
- [132] Rémy, B., Groupes de Kac-Moody déployés et presque déployés. *Astérisque* **277** (2002), 348pp.
- [133] Rémy, B., Kac-Moody groups as discrete groups. To appear in the proceedings of *Geometric Group Theory* (2002), Hindustan Book Agency.
- [134] Savage, R. P., Jr., The space of positive definite matrices and Gromov’s invariant. *Trans. Amer. Math. Soc.* **274** (1) (1982), 239–263.
- [135] Sela, Z., Uniform embeddings of hyperbolic groups in Hilbert spaces. *Israel J. Math.* **80** (1–2) (1992), 171–181.
- [136] Shalom, Y., Rigidity of commensurators and irreducible lattices. *Invent. Math.* **141** (1) (2000), 1–54.
- [137] Shalom, Y., Harmonic analysis, cohomology, and the large-scale geometry of amenable groups. *Acta Math.* **192** (2) (2004), 119–185.
- [138] Smillie, J., An obstruction to the existence of affine structures. *Invent. Math.* **64** (3) (1981), 411–415.
- [139] Soma, T., The zero-norm subspace of bounded cohomology. *Comment. Math. Helv.* **72** (4) (1997), 582–592.
- [140] Stallings, J., On torsion-free groups with infinitely many ends. *Ann. of Math.* (2) **88** (1968), 312–334.
- [141] Sullivan, D., A generalization of Milnor’s inequality concerning affine foliations and affine manifolds. *Comment. Math. Helv.* **51** (2) (1976), 183–189.
- [142] Thurston, W. P., *Three-dimensional geometry and topology*. Princeton Math. Ser. 35, Princeton University Press, Princeton, NJ, 1997.
- [143] Wienhard, A., Bounded cohomology and geometry. PhD thesis, Universität Bonn, 2004.
- [144] Zimmer, R. J., Amenable ergodic actions, hyperfinite factors, and Poincaré flows. *Bull. Amer. Math. Soc.* **83** (5) (1977), 1078–1080.

- [145] Zimmer, R. J., Amenable ergodic group actions and an application to Poisson boundaries of random walks. *J. Functional Analysis* **27** (3) (1978), 350–372.
- [146] Zimmer, R. J., *Ergodic theory and semisimple groups*. Monogr. Math. 81, Birkhäuser Verlag, Basel 1984.

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