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Turbulence Effects on Kinetic Equations

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It is well known that fluid dynamics can be derived from a kinetic (Boltzmann equation) framework. Here we propose that the variance of a fluctuating kinetic relaxation time be linked to turbulent time scales. It is further proposed that this connection be explored by direct numerical simulation of turbulence.

KEY WORDS: Kinetic equation; turbulence; Boltzmann; direct numerical simulation; relaxation time.

1. INTRODUCTION

In recent years, there have been two developments that have made it attractive to consider kinetic (Boltzmann equation) approaches to fluid simulations. First, lattice Boltzmann methods in which both space and velocity are discretized have been shown to be an efficient approach to computational fluid dynamics [10]. Second, it has been shown that a suitably defined "effective" Boltzmann equation describing the kinetic evolution of turbulent quasi-particles (eddies) may be fruitful to address the physics of turbulent flows [6, 7]. In particular, it has been observed that the kinetic approach offers potential advantages in dealing with non-perturbative aspects of fluid turbulence, since these effects can be likened to high-Knudsen number excitations on top of local hydrodynamic equilibria associated with the large eddies.

Analogies between kinetic theory and turbulent flows have been pursued for more than a century; indeed, one of the earliest and most productive ideas, namely the notion that small eddies act upon the large ones

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through an effective viscosity, was prompted by direct analogy with kinetic theory. The Saint-Venant–Boussinesq–Reynolds–Kelvin idea of eddy viscosity hinges on the assumption of a clear-cut scale separation between small and large eddies, an assumption which is likely to break down for fully developed turbulence where no such clear scale separation exists. This basic problem is potentially solved by the kinetic approach by shifting the focus from effective viscosity to an effective relaxation time. Perhaps, the most far-reaching observation of the kinetic approach to turbulence is that *relaxation* to local equilibrium is the most fundamental mechanism, while the notion of *viscosity* only emerges in the limit where relaxation times are short enough for small eddies to be enslaved to the local equilibria produced by the large ones (see, e.g., [9] in which kinetic equations are shown to be derivable using renormalization group ideas across scales). Once this is realized, modeling efforts are shifted from effective viscosities in the framework of hydrodynamic equations, to effective relaxation within the Boltzmann kinetic equation. These ideas have been implemented in widely available numerical tools (based on the Lattice Boltzmann equation) and have received confirmation by recent simulations of fully developed turbulent flows around real-life body shapes, such as cars and airplanes [7].

More recently, the analogies between (effective) kinetic theory and turbulent flows have been extended to the case of non-linear turbulence models [5, 7]. In addition, it has been shown that kinetic theory imposes precise constraints on the filtering procedures used to developed coarse-grained equations for fluid turbulence [1]. In this paper, we propose that the effective kinetic equation for turbulence flows is accessible by study of the fluctuations of the local kinetic relaxation times, whose variance then determines the turbulent eddy viscosity.

2. BOLTZMANN EQUATION FOR FLUID DYNAMICS

The Boltzmann equation expressed in so-called BGK form is [2, 4]

$$Df \equiv (\partial_t + v_a \partial_a) f = (g - f)/\tau, \tag{1}$$

where $f \equiv f(\mathbf{x}, \mathbf{v}, t)$ is the kinetic distribution function describing the probability of finding a 'particle' at position $\mathbf{x}$ with speed $\mathbf{v}$ at time t. In Eq. (1), g is a local equilibrium to which the distribution f relaxes on a timescale τ under collisional relaxation. It is assumed that g is a Maxwell distribution at temperature T

$$g = M(u, v) = \sqrt{\beta/\pi}\, e^{-\beta(v-u)^2} \tag{2}$$

with $\beta = 1/2T$ (in units of the Boltzmann constant $k_B = 1$). Equation (1) holds at the finest level of resolution, namely from the integral scale L all the way down to the Kolmogorov scale [3] l_k, the latter being the *de facto* molecular scale of the effective kinetic theory.

By projecting out the velocity degrees of freedom from the discrete version of Eq. (1), and assuming weak departure from local equilibrium, there results the Navier-Stokes description of fluid flow with viscosity $\nu = T(\tau - \frac{1}{2}dt)$.

3. EFFECTS OF TURBULENCE ON THE KINETIC RELAXATION TIME

In order to explore the effect of macroscopic turbulent flow fluctuations to obtain an effective Boltzmann kinetic description of turbulence, it is useful to begin by expressing the kinetic equation (1) in its Second-order differential form in which the kinetics is evaluated with delay-time dt,

$$\left[1 + \tau\left(z + \frac{dt}{2}z^2\right)\right] f = g, \tag{3}$$

where the operator z is the directional derivative along v, namely $z \equiv \partial_t + v_a \partial_a$, and g is the local equilibrium. Formally the solution of Eq. (3) is given by

$$f = \left(1 + \tau z + \frac{1}{2}\tau z^2 dt\right)^{-1} g \equiv G(z, \tau)\, g \tag{4}$$

The basic idea that we propose here is motivated by direct numerical simulation (DNS) data [8], namely that the effective kinetic description of turbulence can be obtained by averaging over the fluctuations of the local relaxation time τ, that is,

$$\langle f \rangle = \langle G(z, \tau) \rangle\, g, \tag{5}$$

where brackets stand for average, namely $\langle h \rangle = \int p(\tau) h(\tau) d\tau$, where $p(\tau)$ is the probability density of τ. Here we have assumed, as a first approximation, a 'maximal randomness' relationship in which G and g are statistically independent and that the fluctuations of g may be neglected compared to the fluctuations of f.

By Taylor expanding the Green function $G(z, \tau)$ we immediately obtain

$$\langle f \rangle = \sum_{n=0}^{\infty} \langle \tau \rangle^n (-Z)^n\, g, \tag{6}$$

where

$$Z = z + \frac{1}{2} z^2 \, dt.$$

From (6), it is clear that depending on the probability distribution of τ, the effective Green's function can be quite different than the bare one. Next, we truncate (6) to second order in Z, that is

$$\langle f \rangle = (1 - \langle \tau \rangle Z + \langle \tau^2 \rangle Z^2) \, g + O(Z^2) \tag{7}$$

leading to

$$\langle f \rangle = [1 - \langle \tau \rangle z + (\langle \tau^2 \rangle - \tfrac{1}{2} \langle \tau \rangle dt) z^2] \, g + O(z^2)]. \tag{8}$$

Finally, we want to express this in the same form as laminar kinetics in Eq. (3) using an effective relaxation time. It is readily checked that the resulting effective kinetic equation is

$$\left[1 + \langle \tau \rangle z + (\langle \tau \rangle^2 + \frac{1}{2} \langle \tau \rangle dt - \langle \tau^2 \rangle) z^2 \right] f = g. \tag{9}$$

This τ-averaged kinetic equation has an effective viscosity

$$\langle \nu \rangle = T \left(\langle \tau \rangle - \frac{1}{2} dt + \frac{\langle \theta^2 \rangle}{\langle \tau \rangle} \right), \tag{10}$$

where $\langle \theta^2 \rangle = \langle \tau^2 \rangle - \langle \tau \rangle^2$ is the variance of the fluctuations of τ. This result implies that the mean value of τ affects only the usual microscale (collisional) viscosity while the variance of τ contributes an additional 'eddy' viscosity.

4. COMPARISON WITH DNS DATA

In this section, we compare the results presented above with data from a DNS of a homogeneous isotropic flow. The idea is to fully resolve the turbulence at moderate Reynolds number (Re) by a Lattice Boltzmann numerical simulation, without use of a turbulence model. The resulting lattice kinetic variables are spatially averaged and reinterpreted as quantities resulting from coarse-grained dynamics with subgrid-viscosity modeling. As it has been shown in [8], the effective local relaxation time of this coarse-grained dynamics is measured in terms of the ratio τ^* between elements of the Reynolds stress tensor

$$\sigma_{\alpha\beta} = -\left\langle u'_\alpha u'_\beta \right\rangle, \quad \text{where } u'_\alpha = u_\alpha - \langle u_\alpha \rangle \tag{11}$$

and the strain rate tensor

$$S_{\alpha\beta} = \frac{1}{2}(\partial_\alpha u_\beta + \partial_\beta u_\alpha). \tag{12}$$

The latter can be computed from local kinetic quantities and does not require the application of finite differences to evaluate the velocity gradients.

The simulation is performed on a cubic lattice of size $128 \times 128 \times 128$ with periodic boundary conditions, and a Reynolds number of $Re = 1513$. The turbulence is driven by a random forcing at each time step, which excites the two highest wavenumbers of the system.

In Fig. 1 we plot the probability distribution of τ^*/τ for the off-diagonal components of the strain rate. Spatial averages are computed over cubes of size h^3, with an increasing value of h.

The mean value and the variance of the distributions are shown on Fig. 2. Although a lack of sufficient statistics becomes apparent at a size of $h = 8$, we conclude that both the mean value and the variance show a correct (increasing) trend as a function of h.

Finally, the quality of the data is verified by fitting the Reynolds stress tensor with the Smagorinsky and $k - \varepsilon$ turbulence models. The Smagorinsky model predicts a value of

$$\sigma_{\alpha\beta} = \eta\delta_{\alpha\beta} + c_S|S|S_{\alpha\beta} \tag{13}$$

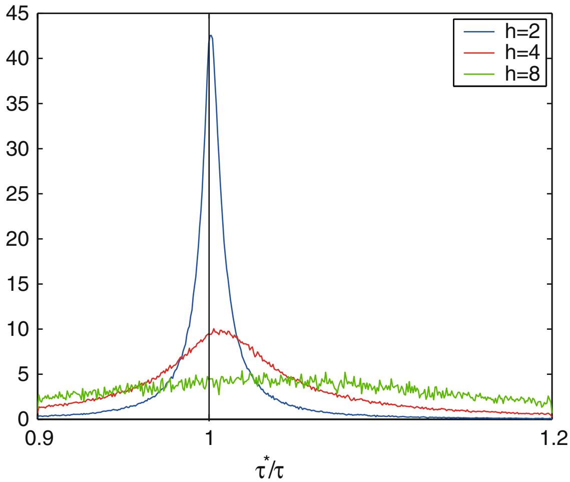

Fig. 1. Spatial distribution of the effective relaxation time τ^*/τ based on the off-diagonal components of σ and S.

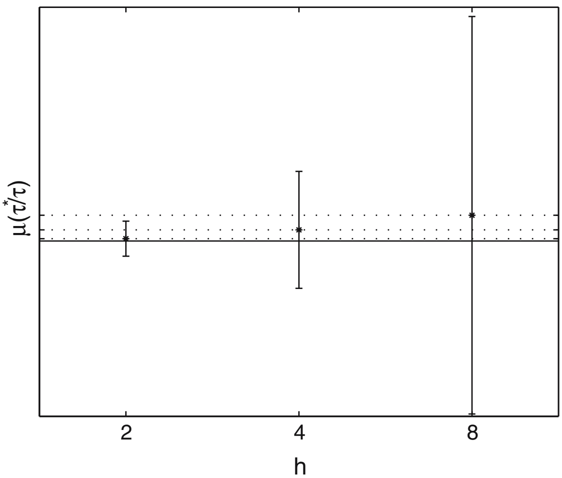

Fig. 2. Dependence of the mean values $\mu(\tau^*/\tau)$ on the size h of the coarse-graining.

and the $k-\varepsilon$ model a value of

$$\sigma_{\alpha\beta} = -\frac{2}{3}k\delta_{\alpha\beta} + c_\mu \frac{k^2}{\varepsilon} S_{\alpha\beta}. \tag{14}$$

Here, c_S and c_μ are empirical parameters of the models, and the $k-\varepsilon$ model depends on the turbulent kinetic energy

$$k = \frac{1}{2}\left\langle |u'|^2 \right\rangle \tag{15}$$

and the dissipation rate

$$\varepsilon = \frac{\nu}{2}\left\langle |\partial_\alpha u'_\beta + \partial_\beta u'_\alpha|^2 \right\rangle. \tag{16}$$

A fit of Eq. (13) using numerical data for off-diagonal parts of the tensors is shown on Fig. 3. The fit predicts a value of $c_S = 0.022 \pm 0.003$. The quality of the fit is characterized by an R-square value (ratio of the sum of squares of the regression to the total sum of squares) of $R^2 = 0.080$.

The result of the fit with Eq. (14) is shown in Fig. 4. Although the distribution of the data is dominated by the occurrence of small values of the turbulent kinetic energy, the data has a weak correlation with the commonly accepted value $c_\mu \approx 0.09$.

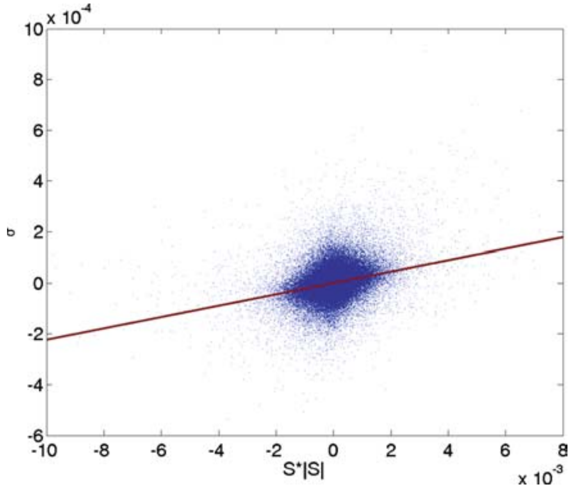

Fig. 3. Fit of the Smagorinsky turbulence model using DNS data.

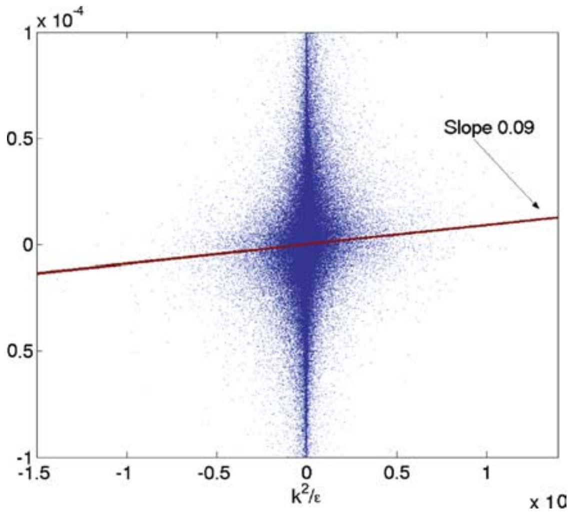

Fig. 4. Fit of the $k - \varepsilon$ turbulence model using DNS data.

5. CONCLUSION

We conclude that kinetic models for turbulent flows can be obtained from the basic Boltzmann kinetics by averaging over fluctuating relaxation times. We also hypothesize that the effect of turbulent fluctuations is dominated by the variance of the relaxation time, not the possible change in the mean relaxation time. Finally, we propose here that this picture of turbulence be tested and explored using a variety of DNS methods based on detailed kinetic models using lattice BGK techniques.

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