On large Picard groups and the Hasse principle for curves and K3 surfaces

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On large Picard groups and the Hasse Principle for curves and K3 surfaces

by

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Let X be a proper, geometrically integral variety over a perfect field k. As usual, $\overline{k}$ will be an algebraic closure of k and we write $\mathcal{G} = \mathrm{Gal}(\overline{k}/k)$. Further, we define $\overline{X} = X \times_{\mathrm{Spec}\,k} \mathrm{Spec}\,\overline{k}$ and denote by $\mathrm{Pic}\,X$ the Picard group of linear equivalence classes of Cartier divisors on X (see for example [35], Lectures 5 and 9).

There is an obvious injective map

$$\beta : \mathrm{Pic}\,X \to (\mathrm{Pic}\,\overline{X})^{\mathcal{G}}. \tag{0.1}$$

For many diophantine questions, it is important to know that β is surjective, i.e., that every divisor class which is stable under the Galois action actually comes from a k-divisor. It seems to us that this condition is of sufficient interest to deserve a special name. So we shall denote it by BP and call it the "BigPic" condition.

DEFINITION. $\mathrm{BP}(X,k) \Leftrightarrow \mathrm{Pic}\,X = (\mathrm{Pic}\,\overline{X})^{\mathcal{G}}$.

In spite of its regular appearance in many papers (cf. e.g. [43], Chap. IV, §6, [12], §2.2, [13], Prop. 9.8(ii), [20], Prop. 7.5, and [27], Cor. 3.11), this condition does not seem to have been much investigated for its own merits. So, for instance, it is well known that

$$X(k) \neq \emptyset \Rightarrow \mathrm{BP}(X,k), \tag{0.2}$$

but what is known exactly when X has no k-rational point?

In the present paper we begin by collecting several properties of BP. Most of them are well known, but they are difficult to find all at one place in the literature. In spite of its simplicity, this study already raises several questions. In Section 2 we investigate rational curves and smooth quadrics in $\mathbb{P}^3_k$. We shall see that BP behaves in a rather unexpected fashion (Example 2.9). Over a number field, this is intimately connected with the Hasse Principle.

Then we use the condition BP as a leading thread through the arithmetic maze of projective curves and K3 surfaces. In fact, on trying to locate some explicit examples where BP does or does not hold (§3), we are naturally led to some particularly nice counterexamples to the Hasse Principle for curves, some with a point of degree 3, others with no point in any odd-degree extension of $\mathbb{Q}$.

Many examples involve some special intersections of quadrics. This feature allows us to produce various families of curves of arbitrary genus for which the Hasse Principle fails to hold, including families with fixed coefficients but varying genus (§4). At the end we exhibit some K3 surfaces with points everywhere locally, but none over $\mathbb{Q}$. In fact, we give an example in every class consisting of complete intersections (§5). Such examples appear to be new.

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1. First properties

Notation. If d is any positive integer, we define:

$\mathrm{P}_d(X,k) \Leftrightarrow \exists K/k$ with $[K:k]=d$ and $X(K)\neq\emptyset$;

$\mathrm{Z}_d(X,k) \Leftrightarrow \exists$ a (k-rational) 0-cycle of degree d on X;

$\mathrm{Z}_d^+(X,k) \Leftrightarrow \exists$ an effective (k-rational) 0-cycle of degree d on X.

Thus, $\mathrm{P}_d(X,k)$ means that X has a K-point in some (unspecified) extension of k with degree d. For simplicity, we shall write P_d, or $\mathrm{P}_d(X)$, or $\mathrm{P}_d(k)$, when no ambiguity arises. Of course, we have

$$\mathrm{P}_d \Rightarrow \mathrm{Z}_d^+ \Rightarrow \mathrm{Z}_d, \tag{1.1}$$

but neither arrow can be reversed in general, though it is true that $\mathrm{Z}_1^+ \Rightarrow \mathrm{P}_1$. Clearly, $\mathrm{Z}_1(X,k)$ holds if and only if X contains points in some extensions of k of coprime degrees. Note also:

LEMMA 1.1. *If Z_{d_1} and Z_{d_2} hold then so does Z_d, where d denotes the g.c.d. of d_1 and d_2.* ∎

An essential ingredient in this investigation is the Brauer group $\mathrm{Br}\,X$ of similarity classes of Azumaya algebras over X (see [33], Chap. IV). We denote by $\mathrm{Br}'\,X$ the cohomological Brauer group. It is known (see [33], p. 141) that $\mathrm{Br}(-)$ is a contravariant functor from schemes to abelian groups. Furthermore, $\mathrm{Br}(\mathrm{Spec}\,k)=\mathrm{Br}\,k$. Now, it is a standard fact that the BigPic condition can be reformulated as follows:

Proposition 1.2. *Let* $\psi : \mathrm{Br}\, k \to \mathrm{Br}\, X$ *be the canonical* (*constant*) *map induced by* $X \to \mathrm{Spec}\, k$. *Then* $\ker\psi$ *and* $\mathrm{coker}\,\beta$ *are isomorphic. In particular*,

$$\mathrm{BP}(X,k) \Leftrightarrow \mathrm{Br}\, k \hookrightarrow \mathrm{Br}\, X,$$

i.e., $\mathrm{BP}(X,k)$ *holds if and only if* ψ *is injective.*

Proof. From sequence (4.29) in [43] (see also [12], sequence (1.5.0) on pp. 386 and 435), and considering that X is proper and geometrically integral, we get the standard exact sequence

$$0 \to \mathrm{Pic}\, X \xrightarrow{\beta} (\mathrm{Pic}\, \overline{X})^{\mathcal{G}} \to \mathrm{Br}'\, k \xrightarrow{\varphi} \mathrm{Br}'\, X.$$

(This sequence can in fact be viewed as a special case of (5.9) in [23], Cor. 5.3, with $Y = \mathrm{Spec}\, k$.) Hence $\ker\varphi$ identifies with $\mathrm{coker}\,\beta$. Now, there is a canonical embedding of $\mathrm{Br}\, X$ in $\mathrm{Br}'\, X$ ([33], Chap. IV, Thm. 2.5), and we see from the exact sequence (4.28) in [43] that $\varphi : \mathrm{Br}'\, k \to \mathrm{Br}'\, X$ is induced by the structure morphism $X \to \mathrm{Spec}\, k$, just like ψ. So, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Br}\, X & \hookrightarrow & \mathrm{Br}'\, X \\ \uparrow{\scriptstyle\psi} & & \uparrow{\scriptstyle\varphi} \\ \mathrm{Br}\, k & \xrightarrow{\approx} & \mathrm{Br}'\, k \end{array}$$

and $\ker\psi \approx \ker\varphi$. ∎

Corollary 1.3. $\mathrm{P}_1 \Rightarrow \mathrm{BP}$.

Proof. A k-point on X defines a morphism $\mathrm{Spec}\, k \to X$ (cf. [24], Chap. II, Ex. 2.7), and therefore a retraction of the Azumaya mapping $\psi : \mathrm{Br}\, k \to \mathrm{Br}\, X$. Hence ψ is injective. ∎

Example 1.4. If k is a finite field then $\mathrm{Br}\, k = 0$, and hence $\mathrm{BP}(X,k)$ holds for any variety X.

Example 1.5. If X is a smooth projective hypersurface of dimension ≥ 3 or, more generally, a smooth complete intersection of dimension ≥ 3 then $\mathrm{Pic}\, \overline{X}$ is generated by $\mathcal{O}(1) \in \mathrm{Pic}\, X$, whence $\mathrm{BP}(X,k)$ holds over any field k.

The next result was already known to François Châtelet ([9]; cf. [10], §1.2).

Proposition 1.6. *If* X *is a Severi–Brauer variety then* $\mathrm{BP}(X) \Leftrightarrow \mathrm{P}_1(X)$.

Proof. By definition $\overline{X}$ is isomorphic to $\mathbb{P}^n_{\bar{k}}$, where $n = \dim X$. Hence $\mathrm{Pic}\, \overline{X}$ is generated by $\mathcal{O}(1)$, i.e., by the class of a hyperplane in $\mathbb{P}^n_{\bar{k}}$. This class is clearly invariant under $\mathcal{G}$, but it belongs to $\operatorname{Pic} X$ only if $X(k) \neq \emptyset$. Indeed, if $\mathcal{O}(1) \in \operatorname{Pic} X$ then there is an effective k-divisor in this class. Now, the n-fold intersection product of $\mathcal{O}(1)$ is equal to 1, and hence $X(k) \neq \emptyset$ (either by ampleness, or by a very general result of Fulton (see [22], Ex. 13.7)). ∎

Remark 1.7. In view of Proposition 1.11, this result holds also for products of Severi–Brauer varieties. However, we shall see in the next section (Theorem 2.8) that Proposition 1.6 does not hold for what is called a "generalized Severi–Brauer variety" in [15], §2.

Corollary 1.8 (Châtelet). *If X is a Severi–Brauer variety over a finite field k then $X(k) \neq \emptyset$.* ∎

The best known case is when X is a smooth conic. Since any smooth rational curve is k-isomorphic to some conic, we also get:

Corollary 1.9. *If X is a smooth curve of genus* 0 *then* $\mathrm{BP}(X) \Leftrightarrow \mathrm{P}_1(X)$. ∎

Example 1.10. Let n be an integer which is not a cube. Let $X \subset \mathbb{P}^2_{\mathbb{Q}}$ be the singular quartic curve with equation

$$(x_1^2 - x_0x_2)^2 + (nx_0^2 - x_1x_2)^2 + (x_2^2 - nx_0x_1)^2 = 0. \tag{1.2}$$

Then X is a geometrically integral curve of genus 0, with only one real point, namely, $(x_0, x_1, x_2) = (1, \theta, \theta^2)$, where θ denotes the real cube root of n. This is an example where $\mathrm{Z}_1(\mathbb{Q}) \not\Rightarrow \mathrm{P}_1(\mathbb{Q})$, and hence $\mathrm{BP} \not\Rightarrow \mathrm{P}_1$.

Proof. X has three double points, with coordinates $(x_0, x_1, x_2) = (1, \theta, \theta^2)$, where θ denotes any cube root of n. Together they form a 0-cycle of degree 3, so that $\mathrm{Z}_1(X, \mathbb{Q})$ holds.

Clearly, the only real solution of (1.2) is the double point corresponding to the real cube root. It follows that $X(\mathbb{Q}) = \emptyset$. Besides, $\mathrm{BP}(X, \mathbb{Q})$ holds, as a consequence of Corollary 2.3. ∎

Thus, Corollary 1.9 does not apply to singular curves of genus 0. This is not so surprising if we think that Cartier divisors have little to do with Weil divisors on a singular curve. (Cf. also [24], Chap. II, Ex. 6.9, which explains how $\operatorname{Pic} X$ can be computed from $\operatorname{Pic} \widetilde{X}$, where $\widetilde{X}$ denotes the normalization of X.)

Note also that, for the curve X of Example 1.10, $\mathrm{BP}(\widetilde{X})$ fails to hold. Indeed, $\widetilde{X}(\mathbb{Q}) = \emptyset$ and we can refer to Corollary 1.9. Hence $\mathrm{BP}(X)$ is not equivalent to $\mathrm{BP}(\widetilde{X})$. This illustrates that smoothness is an essential requirement in Proposition 1.12 below.

Proposition 1.11. *Let $p : Z \to X$ be a k-morphism of* (*proper*, *geometrically integral*) *varieties. Then* $\mathrm{BP}(Z,k) \Rightarrow \mathrm{BP}(X,k)$.

Proof. There is a commutative diagram

If ψ is injective then so is ψ_1. ■

Note that this result applies in particular when Z is a *subvariety* of X. As a matter of fact, Corollary 1.3 is the special case where $Z = \operatorname{Spec} k$.

Proposition 1.12. *Each of the conditions* P_d, Z_d, *and* BP *is a birational invariant for smooth* (*proper and geometrically integral*) *varieties.*

Proof. For P_d and Z_d, this is just Nishimura's lemma (cf. [11], Lemma 3.1.1). For BP, we use Corollary 2.6 of [33], Chap. IV: if X is any smooth integral variety then there is a canonical injective map $\operatorname{Br} X \hookrightarrow \operatorname{Br} k(X)$. Hence the Azumaya mapping $\psi : \operatorname{Br} k \to \operatorname{Br} X$ has the *same kernel* as the natural restriction homomorphism $\operatorname{Br} k \to \operatorname{Br} k(X)$. Thus it is enough to see that

$$\ker(\operatorname{Br} k \to \operatorname{Br} k(X)) = \ker(\operatorname{Br} k \to \operatorname{Br} k(Y)),$$

for any other smooth integral variety Y which is k-birationally equivalent to X. Now, this is obvious because $k(X)$ and $k(Y)$ are k-isomorphic *as fields* and the Brauer group of fields is undoubtedly a functor. ■

At first sight, it would seem that Proposition 1.12 is an easy consequence of Chow's moving lemma. However, we have not succeeded with this approach. See also Manin ([30], Chap. VI, Corollary 2.6), who deals only with the case of surfaces.

Remark 1.13. What this argument actually shows is that not only the condition BP, but the *group* $\ker \psi$ itself is a birational invariant for smooth integral varieties (not necessarily proper or geometrically integral). A deep study of this group for the class of homogeneous varieties has been undertaken recently by Merkurjev and Tignol [32].

The reason for assuming that X is also proper and geometrically integral, is that our definition of BP is in terms of $\operatorname{coker} \beta$. This looks more intuitive, but $\operatorname{coker} \beta$ does not coincide with $\ker \psi$ in general (cf. the proof of Proposition 1.2).

2. Behaviour under extensions of the ground field

LEMMA 2.1. *If K is a field containing k then there is a commutative diagram*

$$\begin{array}{ccc} \operatorname{Br} K & \xrightarrow{\psi_K} & \operatorname{Br} X_K \\ {\scriptstyle \mathrm{Res}}\uparrow & & \uparrow \\ \operatorname{Br} k & \xrightarrow{\psi} & \operatorname{Br} X \end{array}$$

where Res *is the natural restriction homomorphism, and* $X_K = X \times_{\operatorname{Spec} k} \operatorname{Spec} K$.

Proof. This is an immediate consequence of the functoriality of $\operatorname{Br}(-)$, since $\operatorname{Br} k = \operatorname{Br}(\operatorname{Spec} k)$. ∎

PROPOSITION 2.2. *Let K be a finite extension of k, and suppose* $\mathrm{BP}(X,K)$ *holds. Then* $\ker\psi \approx \operatorname{coker}\beta$ *is annihilated by* $d = \deg K/k$.

Proof. If $b \in \ker\psi$ then $\psi_K \circ \operatorname{Res}(b) = 0$, in view of Lemma 2.1. Now, ψ_K being injective, we have $\operatorname{Res}(b) = 0$ and, by corestriction, $d.\, b = \operatorname{Cor} \circ \operatorname{Res}(b) = 0$. ∎

This result can be proved just as easily from the definition of $\operatorname{coker}\beta$.

COROLLARY 2.3. $\mathrm{Z}_1 \Rightarrow \mathrm{BP}$.

Proof. If $\mathrm{Z}_1(X,k)$ holds then X contains points in some finite extensions K/k of coprime degrees. Thus, by Corollary 1.3 and Proposition 2.2, $\ker\psi$ is annihilated by 1. ∎

DEFINITION. If k is a number field, we define

$$\mathrm{BP}_{\mathrm{loc}}(X,k) \Leftrightarrow \mathrm{BP}(X_{k_v}, k_v) \text{ for every completion } k_v \text{ of } k.$$

PROPOSITION 2.4. *If k is a number field then* $\mathrm{BP}_{\mathrm{loc}} \Rightarrow \mathrm{BP}$.

Proof. By class field theory, the natural map

$$i : \operatorname{Br} k \to \bigoplus \operatorname{Br} k_v \subset \prod \operatorname{Br} k_v$$

is injective. Let us now assume $\mathrm{BP}_{\mathrm{loc}}$. Then, on applying Lemma 2.1 with $K = k_v$ and going over to the direct product, we get the commutative diagram

$$\begin{array}{ccc} \prod \operatorname{Br} k_v & \hookrightarrow & \prod \operatorname{Br} X_{k_v} \\ {\scriptstyle i}\uparrow & & \uparrow \\ \operatorname{Br} k & \xrightarrow{\psi} & \operatorname{Br} X \end{array}$$

Therefore ψ is injective. ∎

N o t a t i o n. If k is a number field, we define

$$\mathrm{P}_{\mathrm{loc}}(X,k) \Leftrightarrow X(k_v) \neq \emptyset \text{ for every completion } k_v \text{ of } k.$$

Taking Corollary 1.3 into account, we get:

COROLLARY 2.5. *If k is a number field then* $\mathrm{P}_{\mathrm{loc}} \Rightarrow \mathrm{BP}$. ■

COROLLARY 2.6 (Châtelet). *The Hasse Principle holds for Severi–Brauer varieties over a number field.*

P r o o f. This is an immediate consequence of Corollary 2.5 and Proposition 1.6. ■

R e m a r k 2.7. In the present set-up, when we talk about the *Hasse Principle*, we simply refer to the condition "$\mathrm{P}_{\mathrm{loc}}(X) \Rightarrow \mathrm{P}_1(X)$", even if X is singular. The reason is that Corollaries 1.3, 2.3, and 2.5 depend on the existence of points with no special smoothness property.

As Cassels points out ([6], p. 256), a rational curve like the one described in Example 1.10 can, in this sense, be viewed as a counterexample to the Hasse Principle, provided n has a cube root in the field $\mathbb{Q}_2$ of 2-adic numbers (which means simply that $v_2(n) \equiv 0 \bmod 3$). Indeed, by an argument that goes back to Hilbert and Hurwitz, it suffices to consider the classical Cremona transformation in $\mathbb{P}^2_{\mathbb{Q}}$,

$$\Phi : (x_0, x_1, x_2) \mapsto (y_0, y_1, y_2),$$

where

$$y_0 = x_1^2 - x_0x_2, \quad y_1 = nx_0^2 - x_1x_2, \quad y_2 = x_2^2 - nx_0x_1. \tag{2.1}$$

We have chosen coordinates in such a way that Φ is an involution, i.e., $\Phi^2 = \mathrm{id}$. Thus it is particularly easy to check that it is birational and to compute images and preimages. Now, the inverse image of the conic C with equation $y_0^2 + y_1^2 + y_2^2 = 0$ is defined by (1.2). Further, $C(\mathbb{Q}_p) \neq \emptyset$, except for $p = 2$ and ∞. Hence $X(\mathbb{Q}_p) \neq \emptyset$ for all $p > 2$. By the assumption on n, one of the double points is defined over $\mathbb{Q}_2$. It follows that (1.2) is solvable everywhere locally.

In any other context, Proposition 2.4 would be called "the local-to-global principle" for the condition BP. However, this terminology usually implies that the *global-to-local* principle holds more or less trivially. This is not the case with the BigPic condition! In fact, we shall see (Example 2.9) that in general $\mathrm{BP}(X,k) \not\Rightarrow \mathrm{BP}(X,K)$, whether K is an algebraic extension or a completion of k.

Counterexamples already occur with smooth quadrics in $\mathbb{P}^3_k$. So we investigate this situation in some detail (cf. also [14], Thm. 2.5, where a slightly different approach is taken). For simplicity we shall assume throughout that char $k \neq 2$, the case of characteristic 2 being similar but substantially more difficult to treat explicitly. The main result is as follows:

THEOREM 2.8. *Let $X \subset \mathbb{P}^3_k$ be a smooth quadric surface, with discriminant D. Then*

$$\mathrm{BP}(X,k) \text{ fails } \textit{to hold} \quad \textit{if and only if} \quad D \textit{ is a square in } k \textit{ and } X(k)=\emptyset.$$

In this case, $\operatorname{Pic} X$ *is of index* 2 *in* $(\operatorname{Pic} \overline{X})^{\mathcal{G}} \approx \mathbb{Z} \oplus \mathbb{Z}$.

Proof. It is well known that $\operatorname{Pic} X$ has rank 2 if and only if D is a square (cf. [27], Lemma 3.5). However, we need something more precise. So, we give an explicit argument.

By assumption, char $k \neq 2$. Hence we can assume that X is given by an equation in diagonal form

$$x_1^2 - ax_2^2 - bx_3^2 + cx_0^2 = 0. \tag{2.2}$$

We denote by e and f the generators of $\operatorname{Pic} \overline{X}$, each represented by a line in one of the two rulings. We fix a square root δ of the discriminant, so that $\delta^2 = D = abc$, and also a square root α of a. Then $\overline{X}$ contains the line $\ell \in e$ (say) defined by

$$\ell = \left\{ x_1 = \alpha x_2;\ x_3 = \frac{\delta}{ab}\,\alpha x_0 \right\}. \tag{2.3}$$

Suppose $\delta \notin k$. Then for any $\sigma \in \mathcal{G}$ such that $\delta^\sigma = -\delta$, we have $\ell^\sigma \cap \ell \neq \emptyset$. So, $\ell^\sigma \not\sim \ell$. Hence any line $E \in e$ is linearly equivalent to either ℓ or ℓ^σ. Moreover, any line $E \in e$ has some conjugate in the other family f. Thus any divisor class $\lambda e + \mu f$ $(\lambda, \mu \in \mathbb{Z})$ which is invariant under $\mathcal{G}$ must satisfy $\lambda = \mu$. On the other hand, $e + f = \mathcal{O}(1)$ is in $\operatorname{Pic} X$. Thus, $\operatorname{Pic} X = (\operatorname{Pic} \overline{X})^{\mathcal{G}} = \mathbb{Z}.\,\mathcal{O}(1) \approx \mathbb{Z}$. This is a case in which BP holds.

If, on the other hand, $\delta \in k$ then it follows from (2.3) that $\ell^\sigma \sim \ell$ for any $\sigma \in \mathcal{G}$. Hence the class e of ℓ is invariant under $\mathcal{G}$. Of course, the same is true of f. It follows that $(\operatorname{Pic} \overline{X})^{\mathcal{G}} = \mathbb{Z}e \oplus \mathbb{Z}f \approx \mathbb{Z}^2$. If $X(k) \neq \emptyset$ we already know that $\mathrm{BP}(X,k)$ holds. Thus we may assume that $X(k) = \emptyset$. Then it is clear that X does not contain any k-line. In particular, we see from (2.3) that a is not a square and that ℓ has precisely one other conjugate ℓ^σ. Hence $2e$ contains an element of the form $\ell + \ell^\sigma \in \operatorname{Pic} X$. Therefore $\operatorname{Pic} X$ is the lattice of index 2 generated by $e + f$ and $2e$. In this case $\mathrm{BP}(X,k)$ does not hold, and this completes the proof of the theorem. ∎

EXAMPLE 2.9. Let X be the quadric

$$x_1^2 + x_2^2 + x_3^2 - 7x_0^2 = 0$$

over $k = \mathbb{Q}$. Then $D = -7$ is not a square; hence $\mathrm{BP}(X,\mathbb{Q})$ holds. Now, D is a square in $K = \mathbb{Q}(\sqrt{-7})$ and also in $L = \mathbb{Q}_2$. However, $X(K) \subset X(L) = \emptyset$ so that $\mathrm{BP}(X,K)$ and $\mathrm{BP}(X,L)$ both *fail* to hold.

More generally, we have:

PROPOSITION 2.10. *If D is not a square in k and $X(k)=\emptyset$ then* $\mathrm{BP}(X,k)$ *holds*, *but* not $\mathrm{BP}(X,k(\delta))$.

Proof. This follows easily from Theorem 2.8 and from a well known lemma:

LEMMA 2.11. $X(k)=\emptyset \Rightarrow X(k(\delta))=\emptyset$.

Proof. This is proved in [7] (Ex. 4.4, p. 358). We give a more geometric argument for the convenience of the reader:

Let $k'=k(\delta)$. As $\delta\in k'$, we see from the proof of Theorem 2.8 that the class e is defined over k'. We assume that $X(k')\neq\emptyset$ and choose a point $P\in X(k')$. Then there is a unique line $E\in e$ through P, and it is defined over k'.

Without loss of generality, $\delta\notin k$. Then we know that $E\in e$ has some conjugate E^σ (over k) in the other family f. Since the extension k'/k is only quadratic, there are no more than two conjugates. Hence the intersection $E^\sigma\cap E$ is a k-point of X. ∎

Remark 2.12. If we ponder over Proposition 2.10 when k is a number field, we notice something very strange. Indeed, suppose $X(k)=\emptyset$. Then, in view of Theorem 2.8, $\mathrm{BP}(X,k)$ holds only if D is not a square. So, the phenomenon of Proposition 2.10 can occur only in that case. Now, one situation in which we know for sure that $\mathrm{BP}(X,k)$ holds is when $X(k_v)\neq\emptyset$ for every completion k_v of k. But this, *a fortiori*, implies $\mathrm{P}_{\mathrm{loc}}(X,k(\delta))$. Hence, by Corollary 2.5, $\mathrm{BP}(X,k(\delta))$ holds.

The only way to avoid a conflict with Proposition 2.10 is to admit that this situation cannot occur: in other words, $X(k_v)=\emptyset$ for some valuation v. Thus we have reproved the Hasse Principle for smooth quadrics in $\mathbb{P}^3_k$!

Similarly, for $k=\mathbb{R}$, we see from Proposition 2.10 that $\mathrm{BP}(X,k)$ cannot hold if $X(k)=\emptyset$. This is connected with a more general fact:

PROPOSITION 2.13. *If k is an archimedean or nonarchimedean completion of a number field then*, *for a smooth quadric $X\subset\mathbb{P}^3_k$*, *we have*

$$\mathrm{BP}(X,k)\Leftrightarrow \mathrm{P}_1(X,k).$$

Proof. Indeed, it follows from the classification of anisotropic quadratic forms over completions of global fields (see e.g. [39], Chap. IV, Thm. 6, or [42], Prop. 6) that $X(k)=\emptyset\Rightarrow D$ is a square! ∎

This result demonstrates that Proposition 2.10 is typically a "global phenomenon".

Remark 2.14. The idea of considering situations like Example 2.9 has been inspired by similar results of Samuel about factorial rings (= UFD): see [36], Chap. III, §2, Ex. 3 and §3, Ex. 3, and [37], Prop. 19. Examples which can be derived from his theory are as follows:

$A = \mathbb{Q}[x,y]/(x^2+y^2-3)$ is factorial, but not euclidean.
$k = \mathbb{Q}(\sqrt{2})$ or $k = \mathbb{Q}_7 \Rightarrow A \otimes_{\mathbb{Q}} k$ is no longer factorial.
$K = k(i) \Rightarrow$ not only is $A \otimes_{\mathbb{Q}} K$ again factorial, but it is even euclidean.

Cf. also [11], Prop. 6.1 and Cor. 6.4, where $H^1(k, \operatorname{Pic} \overline{Z}) = 0$, while for some finite extension K/k we have $H^1(K, \operatorname{Pic} \overline{Z}) \neq 0$.

3. Curves of low genus and related examples. For a smooth curve of genus 0 we know from Corollary 1.9 that BP $\Leftrightarrow$ P_1. But for genus 1 any counterexample to the Hasse Principle satisfies BP (Corollary 2.5), but neither P_1 nor even Z_1. Indeed, the Riemann–Roch theorem implies that $\mathrm{Z}_1 \Leftrightarrow \mathrm{P}_1$ for curves of genus 1. This section focuses on curves of genus 2 or 3, but the discussion is more general.

Definition. Whenever we talk about a *hyperelliptic curve X defined by an equation* $s^2 = f(t)$, where f is a separable polynomial of degree $2g+2$, we mean that X is the *smooth projective model* obtained by gluing together two affine curves, with equations $s^2 = f(t)$ and $\sigma^2 = g(\tau)$, where $\tau = 1/t$, $\sigma = s \cdot \tau^{g+1}$, and $g(\tau) = \tau^{2g+2} f(1/\tau)$. By definition, a *point at infinity* on X is one of the points such that $\tau = 0$.

Notation. For a projective, geometrically integral variety X, we denote by $\operatorname{Pic}^r X$ the subset of $\operatorname{Pic} X$ made up of divisor classes of degree r.

Lemma 3.1. *Any divisor class $\delta \in \operatorname{Pic} X$ of degree d (say) induces a bijective map $\xi \mapsto \xi + \delta$ from $\operatorname{Pic}^r X$ to $\operatorname{Pic}^{r+d} X$, for any $r \in \mathbb{Z}$; and also from $(\operatorname{Pic}^r \overline{X})^{\mathcal{G}}$ to $(\operatorname{Pic}^{r+d} \overline{X})^{\mathcal{G}}$.* ∎

In other words, we can decide whether BP holds by looking simply at divisors of low degree, or—should it prove more convenient—at effective divisors of certain well-chosen degrees.

Lemma 3.2. *If X is a smooth projective curve of genus 2 then $\operatorname{Pic}^2 X = (\operatorname{Pic}^2 \overline{X})^{\mathcal{G}}$.*

Proof. Let $\delta \in (\operatorname{Pic}^2 \overline{X})^{\mathcal{G}}$. Then δ is the class of some effective divisor D, and $D \sim D^\sigma$ for all $\sigma \in \mathcal{G}$. We may without loss of generality assume that δ is not the canonical class, since this is certainly in $\operatorname{Pic}^2 X$. Hence $h^0(D) = 1 + h^0(K-D) = 1$. Thus $D = D^\sigma$ for all $\sigma \in \mathcal{G}$, and so $\delta \in \operatorname{Pic}^2 X$. ∎

Corollary 3.3. *If X is a smooth projective curve of genus* 2 *then* $\mathrm{BP}(X,k) \Leftrightarrow \mathrm{Pic}^3 X = (\mathrm{Pic}^3 \overline{X})^{\mathcal{G}}$.

P r o o f. It suffices to apply Lemma 3.1 with the canonical class for δ. From Lemma 3.2 we see that only the odd-degree components matter. The choice of degree 3 will enable us to work with effective divisors. ■

Similarly:

Lemma 3.4. *If X is a smooth projective curve of genus* 3 *then* $\mathrm{Pic}^3 X = (\mathrm{Pic}^3 \overline{X})^{\mathcal{G}}$.

P r o o f. Let $\delta \in (\mathrm{Pic}^3 \overline{X})^{\mathcal{G}}$. Then δ is the class of some effective divisor D, and $D \sim D^\sigma$ for all $\sigma \in \mathcal{G}$. If $h^0(K-D) = 0$ we conclude, as in Lemma 3.2, that $\delta \in \mathrm{Pic}^3 X$. Now, if $h^0(K-D) > 0$ then $K-D$ is equivalent to some effective divisor P of degree 1. Further, $P \sim P^\sigma$ for all $\sigma \in \mathcal{G}$. Therefore $P = P^\sigma$ for all $\sigma \in \mathcal{G}$, since X is not rational (cf. [24], Chap. II, Example 6.10.1). Hence $P \in X(k)$ and BP holds. ■

We may add that, for a curve of genus 2 or 3, $\mathrm{BP}(X)$ can fail only if $\mathrm{Pic}^3 X = \emptyset$. Indeed the degree of the canonical class is a power of 2 and $\mathrm{Z}_1 \Rightarrow \mathrm{BP}$. To illustrate Corollary 3.3, here is a concrete example showing that BP does not always hold for curves of genus 2.

Example 3.5. Let $X \subset \mathbb{P}^3_k$ be a smooth quadric whose discriminant is a square, and assume $X(k) = \emptyset$. (For example, if $k = \mathbb{Q}$, we can take $x_0^2+x_1^2+x_2^2+x_3^2 = 0$.) We fix two conjugate quadratic points $P_+, P_- \in X(\overline{k})$. Then we can find a cubic surface $G \subset \mathbb{P}^3_k$ passing through P_+ and P_- in such a way that $Y = X \cap G$ is a geometrically integral curve with P_+ and P_- as ordinary double points, and no other singularities. Let Z be the normalization of Y. Then Z is smooth of genus 2, and $\mathrm{BP}(Z,k)$ fails to hold. In fact (cf. Corollary 3.3), we have $\mathrm{Pic}^3 Z = \emptyset$, while $(\mathrm{Pic}^3 \overline{Z})^{\mathcal{G}} \neq \emptyset$.

P r o o f. We begin by proving the existence of Y as described. Let π_+ and π_- be the tangent plane to X at P_+, resp. P_-. We consider the linear system Σ of cubic surfaces passing through P_+ and P_-, and having π_+, resp. π_-, for tangent plane there. This represents no more than $2 \cdot 3 = 6$ linear constraints. Hence Σ is of dimension ≥ 13, and the linear system $\Sigma_X = \Sigma|_X$ induced on X has dimension ≥ 9.

The line d joining P_+ and P_- is defined over k. Since $X(k) = \emptyset$, it is clear that $d \not\subset X$. It follows that $P_- \notin \pi_+$ and $P_+ \notin \pi_-$; otherwise, the line d would intersect X with multiplicity at least 3 and would therefore be contained in it. Now, by considering various degenerate cases for G, like the unions of three planes containing d, or the union of π_+, π_-, and one other plane π, we see from the Bertini theorems that the general member of Σ_X is an absolutely irreducible curve, with no other singularities than P_+

and P_-. Furthermore, these are ordinary singularities, as can be seen on the intersection with $\pi_+ \cup \pi_- \cup \pi$.

It remains to prove the assertion about BP. It is clear that $\mathrm{Pic}^3 Z = \emptyset$. Otherwise, the quadric X would contain a 0-cycle of degree 3, and hence also a k-point, by Springer's theorem ([41]). On the other hand, $(\mathrm{Pic}^3 \overline{Z})^{\mathcal{G}} \neq \emptyset$. Indeed, one sees from (2.3) that X contains a line ℓ defined over some quadratic extension of k, and that its conjugate ℓ^σ is in the same linear system. Hence $Y \cap \ell \sim Y \cap \ell^\sigma = (Y \cap \ell)^\sigma$. As a consequence, $Y \cap \ell$ defines an element of $(\mathrm{Pic}^3 \overline{Z})^{\mathcal{G}}$. ∎

Another reason why $\mathrm{BP}(Z,k)$ fails to hold is simply that Y lies on X. Indeed, this defines a k-morphism $p : Z \to Y \hookrightarrow X$. So, we can apply Proposition 1.11, since we know from Theorem 2.8 that $\mathrm{BP}(X,k)$ does not hold. This suggests a way of producing many examples.

LEMMA 3.6. *Let Z/k be a (proper, geometrically integral) variety. Suppose there exists a k-morphism $\varphi : Z \to C$, where C is a smooth conic such that $C(k) = \emptyset$. Then* $\mathrm{BP}(Z,k)$ *fails to hold.*

Proof. By Proposition 1.11, if $\mathrm{BP}(Z,k)$ holds then so does $\mathrm{BP}(C,k)$. Now, in view of Corollary 1.9, this means that we have $\mathrm{P}_1(C,k)$; a contradiction. ∎

EXAMPLE 3.7. As a rather obvious illustration of this lemma, we consider, for any genus g, the (smooth, projective) hyperelliptic curve Z given in affine coordinates by

$$s^2 = 2t^{2g+2} + 3. \tag{3.1}$$

It maps into the projective conic C with equation

$$y^2 = 2x^2 + 3z^2, \tag{3.2}$$

via the morphism defined locally by $(t,s) \mapsto (t^{g+1}, s) = (x/z, y/z)$. Since $C(\mathbb{Q}_2) = C(\mathbb{Q}_3) = C(\mathbb{Q}) = \emptyset$, this is a case where $\mathrm{BP}(Z)$ fails, both locally and globally.

There is nothing special about hyperelliptic curves. For instance, given any $n \in \mathbb{N}$ the smooth projective plane curve with equation

$$x_2^{2n} = 2x_1^{2n} + 3x_0^{2n} \tag{3.3}$$

also provides an example.

We are indebted to Colliot-Thélène for making the very nice remark that there is a general principle behind such examples, namely:

Proposition 3.8. *A smooth* (*proper and geometrically integral*) *variety* Z *is such that* $\mathrm{BP}(Z,k)$ *fails if and only if there exists a rational map* (*defined over* k) *from* Z *to some nontrivial Severi–Brauer variety* X.

P r o o f. Indeed, let $K = k(Z)$ be the function field of Z. Then saying that there exists a k-rational map from Z to X means that X has a K-point. Further, since Z is smooth, we saw in the proof of Proposition 1.12 that $\mathrm{BP}(Z,k)$ holds if and only if the natural restriction map $\mathrm{Br}\,k \to \mathrm{Br}\,K$ is injective.

Thus, if we assume that $\mathrm{BP}(Z,k)$ holds and that X has a K-point, we see from Lemma 2.1 that $\psi : \mathrm{Br}\,k \to \mathrm{Br}\,X$ is injective. Indeed, ψ_K is injective in view of Corollary 1.3. Then we derive from Proposition 1.6 that the Severi–Brauer variety X is trivial.

To prove the converse, let us assume that $\mathrm{BP}(Z,k)$ fails. Then we can find a nontrivial element $\alpha \in \mathrm{Br}\,k$ that vanishes in $\mathrm{Br}\,K$. By descent theory (cf. [38], Chap. X, §6), α is naturally associated with a nontrivial Severi–Brauer variety X, which splits over some finite extension of k and, of course, also over K. Hence X has a K-point, and we are done. ∎

To show that $\mathrm{BP} \not\Rightarrow \mathrm{P}_1$, it is enough to give examples over a finite field (cf. Example 1.4).

Example 3.9. Let $k = \mathbb{F}_3$ and let X be the hyperelliptic curve with affine model

$$s^2 = -t^6 + t^2 - 1. \tag{3.4}$$

Then X is smooth of genus 2, and $X(\mathbb{F}_3) = \emptyset$. Moreover, it follows from Weil's theorem that $X(K) \neq \emptyset$ for any algebraic extension K/k with more than 13 elements. As a matter of fact, $X(\mathbb{F}_9)$ is also nonempty ($s^2 = t^2 = -1$). Thus we have shown:

Lemma 3.10. *The hyperelliptic curve* X *defined by* (3.4) *has a* K*-point in every proper algebraic extension* K/k, *but not in* $k = \mathbb{F}_3$. ∎

The proof of this lemma also illustrates a general fact: over a finite field, not only does BP always hold, but also Z_1. Indeed, it suffices to replace Weil's theorem by the result of Lang and Weil [28] and use it for various coprime degrees. (For an earlier, totally different proof, see [44].) Thus we can state:

Proposition 3.11. *Let* k *be a finite field and let* X *be a geometrically integral*, *projective variety over* k. *Then*, *for any sufficiently large algebraic extension* K/k, *we have* $X(K) \neq \emptyset$. *As a corollary*, $\mathrm{Z}_1(X,k)$ *always holds.* ∎

Some time ago, the first author ([19]) produced an explicit example of a K3 surface with the properties of Lemma 3.10. Since the method of construction was completely analogous to that used for a curve in a previous paper ([18]), the details have so far not been published. Nevertheless, we may quote the result here:

EXAMPLE 3.12. Let $k = \mathbb{F}_3$ and let X be the k-variety defined as the set of zeros of the following three quadratic forms:

$$(3.5) \qquad \begin{cases} f_1 = x_1x_5 + x_2x_3 - x_2x_4 - x_3^2, \\ f_2 = x_1x_4 + x_2^2 - x_3x_5 + x_0^2, \\ f_3 = x_1^2 - x_1x_3 + x_1x_4 + x_1x_5 - x_2x_3 + x_2x_5 + x_3x_4 - x_4^2 \\ \qquad + x_4x_5 + x_5^2. \end{cases}$$

Then $X \subset \mathbb{P}_k^5$ is a smooth K3 surface. Moreover, $X(\mathbb{F}_3) = \emptyset$, but $X(K) \neq \emptyset$ for *every* proper algebraic extension $K/\mathbb{F}_3$. So, this is yet another case where $\mathrm{Z}_1 \not\Rightarrow \mathrm{P}_1$. In fact $\mathrm{P}_d(X, k)$ holds for every $d > 1$, but not for $d = 1$.

Sketch of proof. We explain briefly how this example was generated. We started from the 5 points in $\mathbb{P}_k^2$ of the form $(x, y, z) = (\omega, \omega^2, 1)$, where ω is any root of $\omega^5 + \omega^2 - 1 = 0$. These points are blown up explicitly by a change of variables:

$$x_1 = x(yz - x^2), \quad x_2 = y(yz - x^2), \quad x_3 = z(yz - x^2),$$
$$x_4 = xy^2 + yz^2 - z^3, \quad x_5 = y^3 + xyz - xz^2.$$

As expected from general theory, these variables are connected by 2 quadratic relations defining a Del Pezzo surface S in $\mathbb{P}_k^4$ ($\{x_0 = 0\}$), namely, $f_1(0, x_1, \ldots, x_5) = f_2(0, x_1, \ldots, x_5) = 0$. The third quadratic relation was obtained through a computer search after requiring that it should determine a curve on S passing through the extra points $(x_1, \ldots, x_5) = (\omega, \omega^2, 1, 0, \omega^4)$. The computer selected 15 good choices of coefficients (i.e., such that the curve has no k-point) out of 3280 possibilities. Finally it discovered that, for exactly *one* of these 15 choices, one could introduce the extra variable x_0, so as to obtain a surface in $\mathbb{P}_k^5$ such that $X(k) = \emptyset$, instead of merely a curve in $\mathbb{P}_k^4$.

Then one has to show that $X \subset \mathbb{P}_k^5$ is indeed a smooth, geometrically integral surface. This can partly be seen by a combination of geometric and arithmetic arguments, but the proof had to be completed by computer (evaluation of resultants).

Finally, it is of course easy to check that $X(\mathbb{F}_3) = \emptyset$. But to obtain the existence of K-points in every finite extension $K/\mathbb{F}_3$, we use the result of Deligne [21] (the "Riemann hypothesis"): there are points over $K = \mathbb{F}_q$ provided

$$(3.6) \qquad q^2 + q + 1 > (b_2 - 1)q = 21q.$$

Indeed, the second Betti number b_2 is equal to 22 for a K3 surface, and one can subtract 1, corresponding to the algebraic part of H^2, since the dimension is even. Thus, for $q > 19$, in particular for $q = 3^d$ with $d \geq 3$, we have $X(\mathbb{F}_q) \neq \emptyset$. Moreover, there is a solution over $\mathbb{F}_9$, with $x_0 = x_1 = x_2 = x_3 = 0$. ∎

By Hensel's lemma, Example 3.12 also yields examples (with as many as 12 variables) over the field $\mathbb{Q}_3$ of 3-adic numbers, having no point with coordinates in $\mathbb{Q}_3$, but some point in every proper unramified extension. From the construction we see that there are also some points at least in an extension of degree 5 of $\mathbb{Q}$. So, in all these cases we have Z_1, but not P_1.

On the other hand, if we wish to have examples over $\mathbb{Q}_p$ for infinitely many primes p, we cannot simply lift examples like (3.4). Indeed, as in Proposition 3.11, we see that such examples do not exist over arbitrarily large finite fields. However, we can usually get examples with bad reduction modulo p. For instance,

EXAMPLE 3.13. If $a \in \mathbb{Z}$ is not a cube modulo p, and $c \in \mathbb{Z}$ is not a square modulo p, then the hyperelliptic curve X defined by

$$s^2 = c(t^3 - a)(t^3 - a + p) \tag{3.7}$$

is such that $\mathrm{P}_3(X, \mathbb{Q}_p)$ holds, and hence $\mathrm{Z}_1(X, \mathbb{Q}_p)$, but not $\mathrm{P}_1(X, \mathbb{Q}_p)$. ∎

EXAMPLE 3.14. In Example 3.5 we can replace P_+ and P_- by the 6 intersection points of X with a twisted cubic Γ. Then one can find a surface $G \subset \mathbb{P}^3_k$ of degree 4 such that $Y = X \cap G$ is a geometrically integral curve having the six points as ordinary double points, and no other singularities. Thus Y has geometric genus 3, and the linear system of quadrics through Γ defines an embedding of the normalization of Y as a *smooth quartic* $Z \subset \mathbb{P}^2_k$. We see, as an application of Proposition 1.11, that $\mathrm{BP}(Z, k)$ fails to hold.

In this case it is of course easy to prove that Z_1 does not hold. On the other hand, it is equally clear that Y cannot have points everywhere locally. So this example is much less interesting than the one provided by Cassels in [5], where among other properties $\mathrm{P}_{\mathrm{loc}}$ holds but not Z_1.

The whole difficulty of getting interesting examples over $\mathbb{Q}_p$ is explained by a deep result of diophantine geometry:

PROPOSITION 3.15 (Witt, Roquette, Lichtenbaum). *If X is a smooth projective curve over k, where k is an archimedean or nonarchimedean completion of a number field*, *then*

$$\mathrm{BP}(X, k) \Leftrightarrow \mathrm{Z}_1(X, k).$$

Proof. For $k = \mathbb{R}$ this result goes back to Witt [44]. The case of p-adic fields is due to Roquette and Lichtenbaum (see [29]). ∎

In view of Propositions 2.13 and 3.15, it will come as no surprise that we should study global fields in the forthcoming sections.

4. Counterexamples to the Hasse Principle for curves of any genus. So far, we have not seen any example of a curve of genus 2 for which BP holds, but not Z_1! As we have learned from the preceding discussion, the easiest way to obtain one is to assume that k is a number field and to exhibit a curve which is a counterexample to the Hasse Principle and has no point in any odd-degree extension of k. (Note, as mentioned at the beginning of Section 3, that for genus 1 any counterexample to the Hasse Principle is an instance where $\mathrm{BP} \not\Rightarrow \mathrm{Z}_1$!) We begin with a simple observation:

LEMMA 4.1. $\mathrm{Z}_1 \not\Rightarrow \mathrm{P}_{\mathrm{loc}}$ *for curves of genus* 2.

Proof. In Example 3.13, $\mathrm{Z}_1(X, \mathbb{Q})$ holds, but not $\mathrm{P}_1(X, \mathbb{Q}_p)$. ∎

Clearly, this result extends to higher genera. A consequence of this statement is that $\mathrm{Z}_1 \not\Rightarrow \mathrm{P}_1$, and also that $\mathrm{BP} \not\Rightarrow \mathrm{P}_{\mathrm{loc}}$. Similarly, the next proposition implies that $\mathrm{P}_{\mathrm{loc}} \not\Rightarrow \mathrm{P}_1$, and also that $\mathrm{BP} \not\Rightarrow \mathrm{Z}_1$. In other words, none of the logical arrows in the following diagram can be reversed.

$$\begin{array}{ccc} \mathrm{P}_1 & \Rightarrow & \mathrm{Z}_1 \\ \Downarrow & & \Downarrow \\ \mathrm{P}_{\mathrm{loc}} & \Rightarrow & \mathrm{BP} \end{array}$$

The next proposition is specially interesting because it yields a family of examples with varying genus but with fixed coefficients: only the degree of the defining equation varies.

PROPOSITION 4.2. $\mathrm{P}_{\mathrm{loc}} \not\Rightarrow \mathrm{Z}_1$ *for curves of any genus* $g \ge 1$. *In fact, let* $\{X_g\}$ *be the family of curves* (*with varying genus* g) *defined over* $\mathbb{Q}$ *by*

$$s^2 = 605 \cdot 10^6 t^{2g+2} + (18t^2 - 4400)(45t^2 - 8800). \tag{4.1}$$

For every positive integer g, *the smooth hyperelliptic curve* X_g *determined by this equation is a counterexample to the Hasse Principle. Moreover*, *if* $K/\mathbb{Q}$ *is any algebraic extension of odd degree then* $X_g(K) = \emptyset$.

Proof. Any hyperelliptic curve with affine model $Y \subset \mathbb{A}_k^2$ defined by

$$s^2 = f(t), \tag{4.2}$$

where f is a separable polynomial of degree $d = 2g + 2$, can be viewed as the smooth projective curve $X_g \subset \mathbb{P}_k^{g+2}$ described as the closure of the map

$$(1, t, t^2, \ldots, t^{g+1}, s) : Y \to \mathbb{P}_k^{g+2} \tag{4.3}$$

(see [40], Chap. II, Ex. 2.14). We shall label the coordinates of $\mathbb{P}_k^{g+2}$ in such a way that z_i corresponds to t^i ($i = 0, \ldots, g+1$), while z_{g+2} corresponds to s.

In the case of equation (4.1), we derive that X_g is smoothly embedded in the intersection of quadrics

$$\begin{cases} z_{g+2}^2 = 5 \cdot 11000^2 z_{g+1}^2 + (18z_2 - 4400z_0)(45z_2 - 8800z_0), \\ z_1^2 = z_0 z_2. \end{cases} \tag{4.4}$$

If we write

$$\begin{cases} z_0 = -\frac{1}{4400}x, \quad z_1 = \frac{1}{48}v_1, \quad z_2 = \frac{10}{9}y, \\ z_{g+1} = \frac{1}{11000}v_2, \quad z_{g+2} = u_2, \end{cases} \tag{4.5}$$

then (4.4) becomes

$$\begin{cases} u_2^2 - 5v_2^2 = 2(x+20y)(x+25y), \\ -\frac{55}{16}v_1^2 = 2xy. \end{cases} \tag{4.6}$$

It turns out that this singular Del Pezzo surface, $S \subset \mathbb{P}^4_{\mathbb{Q}}$, is a counterexample to the Hasse Principle. Indeed, if we set

$$u_1 = \tfrac{5}{4}v_1, \tag{4.7}$$

we see that S is a hyperplane section of the following 3-dimensional intersection of quadrics:

$$\begin{cases} u_2^2 - 5v_2^2 = 2(x+20y)(x+25y), \\ u_1^2 - 5v_1^2 = 2xy. \end{cases} \tag{4.8}$$

This variety, $T \subset \mathbb{P}^5_{\mathbb{Q}}$, is a known counterexample to the Hasse Principle ([11], Proposition 7.1). Thus $T(\mathbb{Q}) = \emptyset$. Moreover, by a theorem of Brumer ([4]), T does not contain any 0-cycle of odd degree over $\mathbb{Q}$.

It only remains for us to prove that $X_g(\mathbb{Q}_p) \neq \emptyset$ for every prime p. Now, let p be any prime other than 2 or 5. Clearly, (4.1) can be solved p-adically (with $t = 0$) if 2 is a square modulo p, and also if 5 is a square (with $t = \infty$). Thus we can assume that 2 and 5 are nonsquares, and hence 10 is a square modulo p. So, (4.1) can certainly be solved for every prime $p \neq 2, 5$, since the value taken by the polynomial on its right-hand side, when $t = 1$, is equal to $10 \cdot 8021^2$.

Finally, the reader will check that there are local solutions with $t = 4$ if $p = 2$, and with $t = 5$ if $p = 5$. ∎

R e m a r k 4.3. With the same argument one can produce infinitely many explicit families of examples. One such statement is as follows:

Proposition 4.4. *Let A and D be two rational numbers. We consider the family* (*with varying genus g*) *of hyperelliptic curves X_g over $\mathbb{Q}$ defined by*

$$s^2 = 5A^2t^{2g+2} + (2Dt^2 + 1)(5Dt^2 + 2). \tag{4.9}$$

Suppose D is a norm for the extension $\mathbb{Q}(\sqrt{5})/\mathbb{Q}$. Then $X_g(\mathbb{Q}) = \emptyset$, and even $\mathrm{Z}_1(X_g, \mathbb{Q})$ fails to hold, *whatever $g \geq 1$. If*, *moreover*, *D can be written in the form*

$$D = \frac{10R^2 - (5A^2 + 2)}{9 - 20R}, \tag{4.10}$$

for some $R \in \mathbb{Q}$, then $X_g(\mathbb{Q}_p) \neq \emptyset$ for all $p \neq 2, 5$. In fact, *there are infinitely many values of A and D such that X_g is a counterexample to the Hasse Principle for every g.*

We omit the details. Let us merely add that (4.10) is the precise condition for the right-hand side of (4.9) to take the form $10X^2$ at $t = 1$ (for some $X = D + R \in \mathbb{Q}$). Note that Proposition 4.2 corresponds to $A = 5/2$ and $D = -3^2/(11 \cdot 20^2)$ (while $R = 73/40$). ∎

The next proposition yields a family of nonhyperelliptic examples. Their genus is $(n-1)^2$; so, not all genera are covered.

Proposition 4.5. *For any odd integer n such that Fermat's Last Theorem holds*, *the smooth curve $\Gamma \subset \mathbb{P}^3_{\mathbb{Q}}$ defined by*

$$\begin{cases} x^2 - 9y^2 - 8z^2 + 25t^2 = 0, \\ y^n + z^n + t^n = 0 \end{cases} \tag{4.11}$$

is a counterexample to the Hasse Principle.

Proof. To show that $\Gamma(\mathbb{Q}) = \emptyset$, it suffices to check that the first equation has no rational solution with $y + z = t = 0$, etc. On the other hand, Γ contains the real point $(\sqrt{17}, 1, -1, 0)$. This solution is also good over $\mathbb{Q}_2$. If p is any other prime then there is a solution with $(y, z, t) = (1, 0, -1)$ provided -1 is a square, in particular for $p = 17$. Further, there is a solution with $(y, z, t) = (1, -1, 0)$ if 17 is a square, and with $(y, z, t) = (0, 1, -1)$ if -17 is a square. This covers all possibilities for p. Finally, it is easy to check that the curve is smooth. ∎

Many more examples can be constructed according to this scheme, including some of higher dimension (see Proposition 5.2).

Lemma 4.1 and Proposition 4.2 indicate that there is no direct relationship between Z_1 and $\mathrm{P}_{\mathrm{loc}}$, two conditions under which we know BP is guaranteed to hold. Nevertheless, one question remains: if we assume that *both* Z_1 *and* $\mathrm{P}_{\mathrm{loc}}$ hold, is this enough to secure P_1? In other words, does Z_1 imply the Hasse Principle? In fact, Bremner, Lewis, and Morton ([3], see also [2]) have already shown that this is not true for genus 3. Similarly, for genus 2 we have:

PROPOSITION 4.6. *The Hasse Principle is not a consequence of* Z_1 *for curves of genus* 2. *In fact*, *the hyperelliptic curve* X *over* $\mathbb{Q}$ *defined by*

$$s^2 = 2(t^3 + 7)(t^3 - 7) \tag{4.12}$$

is a counterexample to the Hasse Principle which satisfies Z_1.

Proof. This curve clearly contains some points of degree 3 (with $s = 0$), and the canonical class has degree 2. Moreover, there are points everywhere locally. Indeed, by Weil's theorem, it is enough to check that there are p-adic points for $p \leq 13$. Now, there are solutions with $s = 0$ for $p = 2$, 5, and 11. Further, (4.12) has a solution with $t = 0$ for $p = 3$, and with $t = 2$ for $p = 7$ and 13.

On the other hand, $X(\mathbb{Q}) = \emptyset$. Indeed, there is no solution at infinity, since 2 is not a square in $\mathbb{Q}$. But X maps into the elliptic curve E with equation

$$y^2 = x^3 - 392, \tag{4.13}$$

via the morphism defined by $(t, s) \mapsto (2t^2, 2s) = (x, y)$. Now, we read from Table 2a) in [1] that the only rational point on this curve is the point at infinity: $E(\mathbb{Q})$ has rank 0 and no torsion point (cf. also [25], Chap. 1, Thm. 3.3). Since X has no rational point above the point at infinity of E, it follows that $X(\mathbb{Q}) = \emptyset$, as contended. ∎

Some further counterexamples to the Hasse Principle for curves of genus 2 can be found in [34], Chap. 17, and in the appendix of [31].

Remark 4.7. It is worth noting that the proof of Proposition 4.2 differs very much from the other two. Indeed the argument proceeds by showing that the curve X lies on a variety T for which the *Brauer–Manin obstruction* (cf. [12], §3) is known to hold ([11], §7). This easily implies that the Brauer-Manin obstruction holds also for X. Indeed:

LEMMA 4.8. *Let* $X \xrightarrow{\alpha} T$ *be a* k-*morphism of* (*proper*) k-*varieties*, *where* k *is a number field. Suppose that the Brauer–Manin obstruction to the Hasse Principle holds for* T. *Then it holds also for* X.

Proof. If $\{P^v\} \in \prod_v X(k_v)$ then $\{Q^v = \alpha(P^v)\} \in \prod_v T(k_v)$. By assumption, there exists $\mathcal{B} \in \mathrm{Br}\, T$ such that $\sum \mathrm{inv}_v \mathcal{B}(Q^v) \neq 0$. Then $\mathcal{A} = \alpha^*(\mathcal{B}) \in \mathrm{Br}\, X$ is such that $\sum \mathrm{inv}_v \mathcal{A}(P^v) = \sum \mathrm{inv}_v \alpha^*(\mathcal{B})(P^v) = \sum \mathrm{inv}_v \mathcal{B}(\alpha(P^v)) = \sum \mathrm{inv}_v \mathcal{B}(Q^v) \neq 0$. ∎

On the other hand, Propositions 4.5 and 4.6 are proved in quite a different fashion: one does find a map $X \to T$, but T is just a variety *with "very few" rational points*. Hence it is not at all clear whether these examples can be interpreted by means of the Brauer–Manin obstruction, since $T(k) \neq \emptyset$. (Of course this may become evident from some other argument.)

5. Counterexamples to the Hasse Principle for K3 surfaces. It is shown in the general classification that among K3 surfaces three classes of complete intersections exist. For each of them, we exhibit a counterexample to the Hasse Principle.

PROPOSITION 5.1. *Let $S \subset \mathbb{P}^5_{\mathbb{Q}}$ be the K3 surface defined as the smooth intersection of quadrics*

$$\begin{cases} U^2 = XY + 5Z^2, \\ V^2 = 13X^2 + 950XY + 32730Y^2 + 670Z^2, \\ W^2 = -X^2 - 134XY - 654Y^2 + 134Z^2. \end{cases} \tag{5.1}$$

It is a counterexample to the Hasse Principle. Moreover, S *does not satisfy* $\mathrm{Z}_1(\mathbb{Q})$. *Hence it is also a case where* $\mathrm{BP} \not\Rightarrow \mathrm{Z}_1$.

Proof. We begin by proving that $S(\mathbb{Q}) = \emptyset$. This will also explain how this example was produced. In fact, it suffices to change coordinates as follows:

$$U = u_1,\ V = 6u_2,\ W = 6v_2,\ X = 2x,\ Y = y,\ Z = v_1. \tag{5.2}$$

Then we see that S lies on the singular threefold T given by (4.8). Hence $S(\mathbb{Q}) = \emptyset$ and by Brumer's theorem, as in Proposition 4.2, S does not contain any 0-cycle of odd degree over $\mathbb{Q}$.

The choice of the third equation was dictated by two necessities: S had to be smooth and we wanted to have $S(\mathbb{Q}_p) \neq \emptyset$ for all primes p. As remarked in Example 3.12, the latter condition can be achieved relatively easily, according to Deligne, provided that we control all primes up to 19. For instance, writing out that (4.8) has a solution with $x = -2$, $u_1 = v_1 = y = 1$, and $v_2 = 0$, whenever 23 is a square, we could deal at once with the primes $p = 7, 11, 13$, and 19. It was enough to require that the third equation be of the form

$$au_1^2 + bu_2^2 + cv_1^2 + dv_2^2 + ex^2 + fy^2 = 0$$

for the specified values of the variables (and $u_2^2 = 6^2 \cdot 23$). Adding similar conditions corresponding to the other primes $p \leq 19$, we obtained a linear system with solutions like the one adopted here:

$$67u_1^2 - 402v_1^2 + 18v_2^2 + 2x^2 + 327y^2 = 0.$$

The rest of the proof is somewhat laborious, because we have to check that S is smooth and at the same time to beware of bad primes. Indeed, for such primes, the reduction of S is not a K3 surface and the estimates coming from Deligne's theorem have to be modified. So, for the convenience of the reader we give a different approach. First of all, it is clear that $S(\mathbb{R}) \neq \emptyset$, for instance by choosing $XY > 0$ and Z sufficiently large. Then we have $S(\mathbb{Q}_2) \neq \emptyset$, by considering $(X, Y, Z) = (4, 1, 3)$. Indeed, 33 is a square in $\mathbb{Q}_2$. For all other primes, it is enough to look at congruences modulo p. So we proceed as follows:

If -1 is a square then we have the solution $(X, Y, Z) = (20, 1, 1)$. (This is good in particular for $p = 5$.) If -5 is a square then we have the solution $(X, Y, Z) = (-130, 1, 1)$. (This deals also with $p = 3$ or 7.) Hence, without loss of generality, we can assume that 5 is a square and that $p > 11$. (For $p = 11$ there is the solution $(X, Y, Z) = (-4, 1, 1)$.) Then, if 22 is also a square, we have the solution $(X, Y, Z) = (-40, 1, 3)$. Now, as mentioned above, if 33 is a square then we have the solution $(X, Y, Z) = (4, 1, 3)$. Thus we can assume that 22 and 33 are both nonsquares, which implies that 6 is a square. Then if 7 is a square there is the solution $(X, Y, Z) = (-90, 23, 27)$, and if 7 is a nonsquare there is the solution $(X, Y, Z) = (-230, 43, 47)$. Hence, in all cases there is a solution.

Finally, we have to show that S is smooth. This is clear for all points such that $UVW \neq 0$. By looking more carefully at the Jacobian matrix, one sees almost immediately that S is smooth if either $VW \neq 0$ or $UV \neq 0$ or $UW \neq 0$. But, if we assume that these three expressions vanish then we must have either $V = W = 0$ or $U = V = 0$ or $U = W = 0$. In each case we find 4 triples (X, Y, Z). For instance, if $V = W = 0$ then $V^2 - 5W^2 = 18(X + 40Y)(X + 50Y) = 0$. We see from the Jacobian matrix that the corresponding points are nonsingular. ∎

The following example is even simpler. However, it is not known whether or not $\mathrm{Z}_1(S, \mathbb{Q})$ holds (cf. [17] and [26]).

PROPOSITION 5.2. *The K3 surface $S \subset \mathbb{P}^4_{\mathbb{Q}}$, defined as the smooth intersection*

$$\begin{cases} U^2 = 3X^2 + Y^2 + 3Z^2, \\ 5V^3 = 9X^3 + 10Y^3 + 12Z^3, \end{cases} \tag{5.3}$$

is a counterexample to the Hasse Principle.

Proof. The second equation is a famous example due to Cassels and Guy [8]. It is a counterexample to the Hasse Principle for diagonal cubic surfaces. Hence $S(\mathbb{Q}) = \emptyset$, since a rational solution could occur only for $X = Y = Z = 0$, and then $U = V = 0$.

It is quite easy to see that S is smooth. In fact, a mere look at the Jacobian matrix shows that a singular point on S would satisfy $U = V = 0$. Because of the first equation, such a point could not be real. But, as one can check, the condition that the other minors should vanish defines only some real points.

Then it is clear that $S(\mathbb{R}) \neq \emptyset$. Furthermore, the first equation can be solved globally if (X, Y, Z) is any of the following triples: $(0, 1, 0)$, $(-1, 1, 0)$, $(0, -1, 1)$, or $(2, -1, -1)$. Thus it is enough to show that, for any prime p, one of these triples gives rise to a p-adic solution of the second equation. Now, the first triple is a solution whenever 2 is a cube modulo p; and the second one is a solution whenever 5 is a cube. This deals in particular with all primes $p \equiv -1 \bmod 3$. Suppose now that $p \equiv +1 \bmod 3$. Then the multiplicative group $\mathbb{F}_p^*/(\mathbb{F}_p^*)^3$ has 3 elements. Thus, if neither 2 nor 5 is a cube modulo p, then 2 and 5 can either be in the same class modulo cubes, in which case $\frac{2}{5}$ is a cube. Or they are in two different classes, and then $10 = 2 \cdot 5$ is a cube. In the former case, the third triple is a solution. In the latter, we can make use of the fourth triple. This works also over $\mathbb{Q}_3$, where 10 is a cube. ∎

Our next example is a simple application of a technique introduced by Swinnerton-Dyer. It was shown in [16] (Appendix 1) how this method can furnish some counterexamples to the Hasse Principle for smooth surfaces in $\mathbb{P}^3_{\mathbb{Q}}$ of any degree ≥ 3. Although an example of degree 5 was worked out explicitly, the case of degree 4 received no special attention then.

PROPOSITION 5.3. *Let $K = \mathbb{Q}(\zeta)$ be the abelian extension of $\mathbb{Q}$ generated by a primitive fifth root of unity ζ. Define $\theta = 1-\zeta$ and consider the equation*

$$t(t+x)((t+x)^2 + 13t(t+x) - t^2) = N_{K/\mathbb{Q}}(x + \theta y + \theta^2 z). \tag{5.4}$$

Then (5.4) *is the equation of a smooth quartic surface $S \subset \mathbb{P}^3_{\mathbb{Q}}$, which is a counterexample to the Hasse Principle.*

Proof. The proof depends on standard facts about the abelian field K. Its discriminant is a power of 5. In fact, θ is a root of the Eisenstein polynomial $x^4 - 5x^3 + 10x^2 - 10x + 5$. So, the prime 5 is totally ramified and we have $(5) = (\theta)^4$. Moreover, a prime p splits completely if and only if $p \equiv 1 \bmod 5$. More generally, its residue class degree is the least integer $f \geq 1$ such that $p^f \equiv 1 \bmod 5$ (see [7], Chap. 3, Lemmas 3 and 4). Thus it is quite easy to show that $S(\mathbb{Q}) = \emptyset$, and even that the threefold with equation

$$t(t+x)((t+x)^2 + 13t(t+x) - t^2) = N_{K/\mathbb{Q}}(x + \theta y + \theta^2 z + \theta^3 w) \tag{5.5}$$

has no rational point.

Indeed, the right-hand side of (5.5) does not vanish over $\mathbb{Q}$, since this would imply $x = y = z = w = 0$, and hence $t = 0$. Thus we may assume that x and t are coprime integers (while $y, z, w \in \mathbb{Q}$). Now, 5 does not divide the right-hand side. Otherwise, each factor of $N(x + \theta y + \theta^2 z + \theta^3 w)$ would have the same valuation ≥ 1 at $\mathfrak{q} = (\theta)$. Since x, θy, $\theta^2 z$, and $\theta^3 w$ have *distinct* valuations—respectively congruent to 0, 1, 2, 3 modulo 4—the only possibility is that x is a multiple of 5: $v_{\mathfrak{q}}(x) = 0 \Rightarrow v_{\mathfrak{q}}(x+\theta y+\theta^2 z+\theta^3 w) \leq 0$.

But, if 5 divides the left-hand side of (5.5) and $5 \mid x$ then $5 \mid t$; so x and t would not be coprime.

Then we see that t, $t+x$, and $(t+x)^2+13t(t+x)-t^2$ are coprime in pairs. So, if a prime $p \neq 5$ divides the left-hand side of (5.5) then either it is a norm (i.e., $p \equiv 1 \bmod 5$) or it occurs in each factor to a power which is a multiple of its residue class degree f (but $p^f \equiv 1 \bmod 5$). Hence, taking signs into account, each factor on the left-hand side must be congruent to ± 1 modulo 5. This is impossible, for if we assume this for the first two factors then the third one satisfies the congruence $(t+x)^2+13t(t+x)-t^2 \equiv +1 \pm 3-1 \equiv \pm 3 \bmod 5$. This contradiction shows that (5.5) has no rational solutions.

On the other hand, (5.4) has solutions everywhere locally, even on the curve defined by $z = 0$:

$$(5.6) \quad 13t^4 + 28t^3x + 16t^2x^2 + tx^3 = x^4 + 5x^3y + 10x^2y^2 + 10xy^3 + 5y^4.$$

Indeed, there is a real point with $x = 0$. Over $\mathbb{Q}_2$ we can set $t = y = 1$ and use Hensel's lemma to solve for x: with $x = 0$ we get a congruence modulo 8, while the x-derivative has valuation 1 only. Over $\mathbb{Q}_p$ for $p > 2$, it is enough to find nonsingular solutions modulo p. Apart from $(x, y, t) = (1, 1, 1)$ for $p = 3$ and $(1, 1, 8)$ for $p = 29$, there are solutions with $t = 0$ for $p = 11$ and 31, with $x = 0$ for $p = 7$ and 13, and with $y = 0$ for all other primes $p \leq 31$. It follows from Weil's theorem that (5.6) has a p-adic solution for every prime p, up to a discussion of possible bad reduction at certain primes.

Finally, we have to show that the surface $S \subset \mathbb{P}^3_{\mathbb{Q}}$ is smooth. To begin with, we notice that there is no singularity in the plane defined by $t = 0$. Indeed, the derivative with respect to t yields the following condition for a point to be singular:

$$(5.7) \qquad 52t^3 + 84t^2x + 32tx^2 + x^3 = 0.$$

If $t = 0$, this implies $x = 0$. Now, a singular point of S would also be singular on the curve cut out by the plane $t = 0$. But this curve splits into a union of four lines. So, the point would lie in the intersection of two of them. Hence $x + \theta_i y + \theta_i^2 z = 0$ for two distinct conjugates of θ. Since $x = 0$, this implies $y = z = 0$, a contradiction.

Thus we can set $t = 1$. Then, according to (5.7), any singular point of S lies in one of the three planes defined by the roots of the equation $x^3 + 32x^2 + 84x + 52 = 0$. The rest of the discussion is better performed with the help of a computer. However, we notice that we can replace 13 in (5.4) by any integer m satisfying certain congruence conditions, like $m > 0$, $m \equiv 13 \bmod 20$, and maybe a few others. For general m, it follows from one of the Bertini theorems that a singularity can occur only on the fixed part of the corresponding linear system. Hence $t(t+x) = 0$. But we already know that there is no singular point with $t = 0$ or, in view of (5.7), with $t = 1$

and $x = -1$. For all suitable m, the associated surface is a counterexample to the Hasse Principle. Almost every one of them is smooth. ∎

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