Tropical intersection theory and surfaces

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Résumé

L'intérêt principal de la géométrie tropicale est qu'elle permet d'étudier des problèmes de géométrie algébrique classiques aux moyens d'objets plus simples. Localement ces objets sont des complexes polyhèdraux munis des structures supplèmentaires et sont appelés variétés tropicales.

Les variétés tropicales apparaîssent en dégénérant des families des variétés algébriques complexes à l'aide du logarithme, [36], [49], ou en considérant les amibes de variétés définies sur des corps non-archimèdiens. Dans ces situations, la combinatoire de la variété tropicale obtenue peut refléter la géométrie de l'espace de départ. Un des premiers exemples principales de ce phénomène est le Théorème de Correspondance de Mikhalkin pour les courbes tropicales dans le plan. Ce théorème est un des plus grandes succès de la géométrie tropicale. Cette correspondance est au départ de diverses applications en géométrie enumerative réelle et complexe dans le cas des surfaces toriques. Elle a notamment permit d'obtenir des algorithmes combinatoires pour calculer les invariants de Gromov-Witten et de Welschinger.

La géométrie tropical a aussi eu du succès avec la théorie d'intersection. Naïvement l'intersection de variétés tropicales semble bien différente de l'intersection des variétés classiques. Par exemple, deux droites dans le plan tropical peuvent s'intersecter en un infinité de point. Relativement tôt, l'intersection stable des variétés tropicales en $\mathbb{R}^n$ a été introduit [48], [38]. Dans ce cadre, de nombreux théorèmes établissent des correspondances entre intersections tropicale et classique [9], [46], [42]. Plus tard, la théorie d'intersection tropicale basée sur les diviseurs de Cartier était introduite dans [38], et dévelopée dnas [2]. En appliquant cette théorie d'intersection sur les espaces de modules des courbes rationnelles tropicales, il est possible de retrouver les invariants de Gromov-Witten et les invariants descendents [30], [47].

Dans ce travail, on défine un produit d'intersection sur les cycles tropicaux contenus dans des éventails matroïdaux. Les éventails matroïdaux peuvent se produire comme des "tropcalisations" d'espaces linéaires, cependent ils sont plus généraux [53], [54]. Ce type d'éventails correspond aux modèles locaux des variétés tropicales lisses. Le produit d'intersection défini sur ces éventails s'inspire de l'intersection stable en $\mathbb{R}^n$. Pour le définir, on utilise une technique qui ressemble aux moving lemmas en géométrie algébrique classique, à la différence que ce produit s'applique à des cycles et non à des classes d'équivalence. Malheureusement, cette machinerie ne fonctionne pas pour intersecter des cycles contenus dans le bord des variétés tropicales.

Jusqu'à présent, la majorité des résultats obtenus pour les variétés tropicales abstraites sont en dimension un. Par exemple, il existe une version tropicale du Théorème de Riemann-Roch pour les courbes, [4], [22], [40]. En plus, on peut porter un point de vue tropical sur la théorie de Brill-Noether [14] et le théorème de Torelli [13]. Comme dans le monde classique, l'étude des dimensions supérieurs pose des difficultés. Dans le cas des surfaces tropicales, différents problèmes surviennent. Le premier problème est de savoir quelles surfaces considèrer. En effet, il existe des variétés tropicales auxquelles il ne correspond aucune variété classique. Ceci se produit même localement en dimension supérieure à un. Le second problème est de savoir quelles courbes considèrer dans ces surfaces. En géométrie classique, l'étude des surfaces repose sur l'étude des courbes qu'elles contiennent. En essayant d'imiter cette approche pour les surfaces et courbes tropicales, on découvre encore que la situation est bien différente. Par exemple, Vigeland a montré l'existence de surfaces tropicales dans $\mathbb{R}^3$ de dégré $d > 3$ qui contiennent une infinité de droites tropicales. Par un théorème de Segre, une surface complexe de dégré $d > 3$ ne peut contenir qu'un nombre fini des droites [51].

La théorie d'intersection tropicale se révèle être très utile pour comprendre ce genre de phenomène. Par exemple, on peut trouver des courbes dans une surface tropicale qui s'intersectent proprement avec une multiplicité négative. En appliquant la formule d'adjonction, on remarque également qu'il existe des surfaces tropicales lisse qui contient des droites "singulières".

Cette thèse est divisée en trois chapitres. Dans le premier, on rappelle les définitions nécessaires de [38], [48], [2] avec certaines généralisations. Ensuite, on établit une relation entre les modifications tropicales d'un éventail matroïdal et les operations sur les matroïdes. À l'aide de ces notions, on définit alors ce qu'est une variété tropicale lisse. La première définition de variété tropicale abstraite a été donnée par Mikhalkin dans [38]. Ici, la différence réside dans le fait que les modèles locaux utilisés sont matroïdaux.

Dans le chapitre 2, on développe une théorie d'intersection sur les variétés tropicales lisses. Premièrement, on montre comment intersecter les cycles dans les éventails matroïdaux dans $\mathbb{R}^n$. Dans certains cas, ce produit d'intersection local s'étend aux variétés tropicales. Les problèmes surviennent lorsque deux cycles s'intersectent au bord de la variété tropicale. Il est tout de même possible de définir l'intersection au bord dans certains cas. On peut alors définir l'équivalence rationelle des cycles tropicaux. Enfin, on définit l'intersection transverse dans le cadre de l'homologie tropicale comme introduit par Itenberg, Katzarkov, Mikhalkin and Zharkov [27].

Le dernier chapitre se concentre sur le cas des surfaces. Dans un premier temps, on simplifie les produits de cycles et on décrit les intersections au bord d'une surface. On montre aussi qu'on peut définir un produit sur les classes d'homologie $(1,1)$ dans une surface compacte. On introduit ensuite la somme tropicale fibrée de deux surfaces en analogie à la somme classique des variétés lisses. En guise d'exemple, on montre qu'on peut contracter des courbes tropicales rationelles non-singulières avec auto intersection -1 dans une surface tropicale. Ensuite, on montre que la formule de Noether est préservée sous la somme tropicale.

Les surfaces décomposées en étages dans espace projectif tropical fournissent un autre instance de la somme tropicale. Pour de telles surfaces, on calcule l'homologie tropicale $(1, 1)$ et sa forme d'intersection. La dimension de ce groupe coincide avec celle du groupe $H^{1,1}$ d'une surface complexe du même dégré comme indiqué dans [27]. Cependant ces formes d'intersection diffèrent. Enfin, on montre que la formule de signature de Hirzebruch doit être modifiée pour subsister dans le cadre tropical.

Finalement, on démontre une correspondance locale entre les intersections de courbes complexes et de courbes tropicales dans les surfaces. En utilisant cette correspondance, on donne des obstructions à l'approximabilité locale des courbes tropicales. Aussi, on classifie toutes les courbes trivalentes et localement approximables. Ensuite, on applique la formule d'adjonction dans le cas local pour obtenir des obstructions générales pour l'approximabilité des courbes. Pour conclure, on donne une condition suffisante pour que le genre prédit par la formule d'adjonction soit égal au premier nombre de Betti de la courbe tropicale.

Chaque chapitre fournit une introduction plus détaillée. Une partie du contenu de cette thèse a été presentée dans [52] et [12]. Le deuxième article est une collaboration avec Erwan Brugallé et certaines parties de ce travail apparaissent dans ce texte.

Introduction

One of the main goals of tropical geometry is to study classical algebraic geometry with the help of simpler objects. Locally these objects are polyhedral complexes equipped with extra structure and are called tropical varieties.

Tropical varieties appear when degenerating families of complex algebraic varieties under the logarithm map, [36], [49], or as non-archimedean amoebas [15]. In these situations, the combinatorics of the tropical varieties can reflect the geometry the original spaces. Perhaps the best known example of this is Mikhalkin's correspondence theorem for curves in $\mathbb{R}^2$ [36]. This theorem is also one of the major successes of tropical geometry. This correspondence yielded a variety of applications to real and complex enumerative geometry of toric surfaces; namely combinatorial algorithms for computing Gromov-Witten invariants, and new results involving Welschinger invariants.

Tropical geometry has also had success with intersection theory. Naively, at first glance it seems that intersecting tropical varieties is nothing like intersecting classical ones. For example, two distinct tropical lines in the plane may intersect in an infinite number of points. Early on, stable intersection of tropical cycles in $\mathbb{R}^n$ was introduced and resolves this problem [48], [38]. Since then various correspondence theorems have been proved for such intersections, [9], [46], [42]. Also, tropical intersection theory based on Cartier divisors was introduced in [38], and developed in [2]. This has been applied to tropical moduli spaces of curves in [30], [47]. With these tropical calculations, the authors manage to recover classical Gromov-Witten invariants, and descendent invariants.

In this thesis we define an intersection product on tropical cycles contained in matroidal fans which extends to tropical manifolds. These fans may occur as the tropicalisations of linear spaces, however they are more general [53], [54]. Non-singular tropical manifolds as defined here are modelled on special types of fans known as matroidal. The intersection product on these fans is similar in a way to stable intersection in $\mathbb{R}^n$. In order to define it we use a sort of moving lemma technique, like in classical algebraic geometry, yet it has an advantage over the classical theory. In classical intersection theory we have to pass to equivalence classes of cycles to define intersection products. For the most part, in tropical intersection theory no notion of equivalence is needed. Unfortunately, this advantage of tropical geometry breaks down when we want to take self-intersections of boundary divisors.

To date most of the study of abstract tropical varieties has been in the case of curves. For example there is a tropical version of the Riemann-Roch theorem for curves, [4], [22], [40]. Also Brill-Noether theory and the Torelli theorem have been studied tropically, [14], [13]. The last chapter of this thesis is dedicated to the study of tropical surfaces. In general, the tropical world is different from the classical algebro-geometric one, for instance there are tropical spaces which have no classical counter-part. The first problem is to determine which surfaces to consider. In addition, the classical study of surfaces relies heavily on studying the curves contained in them. Trying to mimic this approach with our current definitions of tropical curves in surfaces and we immediately encounter problems. For example, Vigeland showed the existence of tropical surfaces in $\mathbb{R}^3$ of degree greater than two containing infinite families of tropical lines [56], whereas classically it is known that a surface of degree greater than two contains only finitely many lines [51]. Therefore even if the tropical surface under consideration is realisable, there may be tropical curves contained in the surface which are not.

Tropical intersection theory proves to be an indispensable tool to the study of surfaces. By intersecting curves in surfaces we observe some strange facts about the tropical world. For example, there are tropical curves in a surface which intersect properly and have a negative intersection multiplicity. Even more strange, there are non-singular tropical surfaces containing "singular" lines.

This thesis is divided into three chapters. The first chapter is a review of definitions from [38], [48], [2], with some necessary generalisations. We also establish a relation between tropical modifications of matroidal fans and operations in matroid theory. These preliminary definitions pave the way to introduce tropical manifolds. A first definition of an abstract tropical variety appeared in [38]. The definition presented here differs in that we insist that the local models be "matroidal fans".

Chapter 2 develops intersection theory on tropical manifolds. First we show how to intersect tropical cycles contained in matroidal fans in $\mathbb{R}^n$. This product is then carried over to manifolds. Intersections with boundary divisors are also given in limited cases and we extract from this a definition of tropical rational equivalence. At the end of the chapter we summarize tropical (p, q)-homology as defined by Itenberg, Katzarkov, Mikhalkin and Zharkov. Then we describe intersections of (p, q)-cycles in $\mathbb{R}^n$ in the transverse case and in matroidal fans contained in $\mathbb{R}^n$ moreover we define a cycle map proposed by Mikhalkin which identifies a tropical k-cycle with a (k, k) cycle in a compact manifold.

The final chapter focuses entirely on surfaces. First we simplify the definitions of intersection products of tropical cycles in the case of surfaces. We also show that for a compact tropical surface X, there is an intersection pairing on $(1, 1)$-homology classes. In this chapter we also give a way to glue two tropical surfaces satisfying certain conditions, which we call the tropical sum, because of its resemblance with the classical fiber sum. As examples of this construction we present tropical "blow-downs" of non-singular fan -1-curves in a tropical manifold X. Not only does the tropical fiber sum allow us to construct new tropical surfaces from old, but also to prove classical formulae for sums of surfaces, for example Noether's formula.

For a floor decomposed surface of degree d in tropical projective space we compute the tropical $(1,1)$-homology group and its intersection form. Its dimension coincides with the dimension of $H^{1,1}$ for a complex surface of the same degree, as expected by [27]. However, the intersection forms are not the same.

Finally we use tropical intersection theory to provide obstructions to approximating fan tropical curves in fan tropical planes by constant families and provide some classification theorems for locally approximable fan tropical curves. By translating the classical adjunction formula to the tropical world we obtain more general obstructions to approximating curves in the local case. We also present a sufficient condition for a tropical curve in a tropical manifold to satisfy the tropical adjunction formula with the genus the first Betti number of the curve.

Each section presents a more comprehensive introduction. Some of the contents of this thesis have been presented in [52] and [12]. The second article is a collaboration with Erwan Brugallé, and only a selection of its contents appears here near the end of Chapter 3.

Contents

Chapter 1

Tropical Manifolds

To describe a class of manifolds one first describes their local models (for example neighborhoods of $\mathbb{R}^n$ or $\mathbb{C}^n$) and then the coordinate changes (for example smooth functions or holomorphic functions). In tropical geometry the first task is slightly more difficult. Tropicalisations of the same variety considered in different spaces yield polyhedral complexes with different topology. For example, a complex line, $\mathbb{C}$ tropicalises to $\mathbb{T} = [-\infty, \infty)$. A line $\mathbb{C} \subset \mathbb{C}^2$ given by the equation $z_1 + z_2 + 1 = 0$ may tropicalise to a trivalent graph in $\mathbb{T}^2$ with a single vertex at $(0, 0)$ and the three outgoing rays in the directions $(-1, 0)$ $(0, -1)$ and $(1, 1)$, see Figure 1.2.

To start we consider standard tropical affine space which is just $\mathbb{T}^n$. Here we define tropical cycles and tropical functions. These definitions first appeared in [38], except for the definition of boundary cycles. This leads us to define tropical modifications, following [38]. We make a clear distinction between the tropical modifications that are considered here (and which appeared in [38]) and the ones presented in other works, [2], [1]. The main difference is that we work with the boundary of $\mathbb{T}^n$ and thus can obtain compact spaces as initiated in [38].

Section 1.1.4, relates **matroidal fans** or Bergman fans of matroids and tropical modifications. Bergman fans were initially defined to be the logarithmic limit sets of complex algebraic varieties [5]. For varieties defined by linear ideals, Sturmfels showed the Bergman fan depends only on the underlying matroid and he also generalized the Bergman fan construction to any loopless matroid [55]. Matroidal fans are the local models of tropical manifolds to be defined in the following section.

Finally, Section 1.2 defines tropical manifolds using matroidal fans as local models and integral affine functions as transition functions. The definitions of cycles, functions, and divisors all extend to tropical manifolds. We also introduce a notion of non-singular divisor and non-singular tropical modifications of manifolds. A proposal of Mikhalkin in [38] is to generate an equivalence relation on tropical manifolds based on these non-singular tropical modifications. Non-singular algebraic varieties are manifolds. The spaces to be introduced behave similarly to the non-singular spaces as we will see. However, these spaces are more general than non-singular classical algebraic varieties.

1.1 Local models

1.1.1 Standard tropical affine space

The tropical numbers $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ form a semi-field equipped with the following operations:

$$“x \cdot y” = x + y \text{ and } “x + y” = \max\{x, y\}.$$

As the multiplicative and additive identity we have $1_{\mathbb{T}} = 0$ and $0_{\mathbb{T}} = -\infty$ respectively. Tropical subtraction does not exist, however division in $\mathbb{T}$ corresponds to subtraction.

One way of arriving at such a semi-field is via Maslov's dequantisation of arithmetic, the reader is referred to [32] for more details. In short, there is a one-parameter family of semi-fields $(\mathbb{T}, \oplus_t, \otimes_t)$, for $t > 0$, such that,

$$\log_t : (\mathbb{R}_{\geq 0}, +, \times) \longrightarrow (\mathbb{T}, \oplus_t, \otimes_t)$$

is an isomorphism. Taking the limit, $t \mapsto \infty$, one obtains the tropical semi-field $(\mathbb{T}, \max, +)$.

From another point of view, one may consider $\mathbf{K}$ the field of generalised Puiseux series, that is locally convergent series $a = \sum_{i \in I} a_i t^i$, where $I \subset \mathbb{R}$ is well ordered and $a_i \in \mathbb{C}$, see [34]. Since the indexing set I for a series is well-ordered there is a valuation map, $val : \mathbf{K} \longrightarrow \mathbb{T}$, given by $val(\sum_{i \in I} a_i t^i) = -\min I$ and $val(0) = -\infty$. The valuation map satisfies,

$$\begin{aligned} val(ab) &= val(a) + val(b) \\ val(a + b) &\leq \max\{val(a), val(b)\}. \end{aligned}$$

If when added, the leading terms of the two series, a and b do not cancel we of course have $val(a+b) = \max\{val(a), val(b)\}$, and the resemblance with the tropical operations becomes clear.

Equip $\mathbb{T}^n = [-\infty, \infty)^n$ with the Euclidean topology, and think of it as tropical affine n-space. In $\mathbb{R}^n \subset \mathbb{T}^n$, fix the standard lattice $\mathbb{Z}^n$, with basis $e_1, \dots e_n$. The boundary of $\mathbb{T}^n$ admits a natural stratification in the following way: Let $H_i = \{x \in \mathbb{T}^n \mid x_i = -\infty\}$, be the i^{th} coordinate hyperplane and given a subset $I \subset \{1, \dots, n\}$ denote $H_I = \cap_{i \in I} H_i$, and

$$H_I^{\times} = \{x \in H_I | x \notin H_J \; I \subset J\}.$$

Then,

$$\mathbb{T}^n = \coprod_{\emptyset \subseteq I \subseteq [n]} H_I^{\times}.$$

For every $I \in [n]$, we have $H_I = \mathbb{T}^{n-|I|}$ and $H_I^{\times} = \mathbb{R}^{n-|I|}$. In a boundary stratum $H_I^{\times}$ of $\mathbb{T}^n$ fix the natural lattice isomorphic to $\mathbb{Z}^{n-|I|}$ given by $\mathbb{Z}^n / \langle e_i \mid i \in I \rangle$.

Definition 1.1.1. The sedentarity of a point $x \in \mathbb{T}^n$ is

$$S(x) = \{i \in I \mid x_i = -\infty\}.$$

The order of sedentarity of x is $s(x) = |S(x)|$.

The order of sedentarity is due to Mikhalkin and Losev. The notion of sedentarity is necessary in tropical geometry as not all points of $\mathbb{T}^n$ behave in the same way. For a first glance at this, one may think of $\mathbb{T}^n$ as the image of $\mathbb{C}^n$ by the coordinate-wise log map,

$$\begin{array}{lcl} \mathrm{Log}_t : & \mathbb{C}^n & \longrightarrow \mathbb{T}^n \\ & (z_1, \dots, z_n) & \mapsto (\log_t |z_1|, \dots, \log_t |z_n|). \end{array}$$

For a point $x \in \mathbb{T}^n$, its fiber is $\mathrm{Log}_t(x) = (S^1)^{n-k}$, where k is the order of sedentarity of x. In particular, the fiber above $(-\infty, \dots -\infty)$ is just a single point $(0, \dots, 0)$. Fixing polar coordinates r_i, θ_i on $(\mathbb{C}^*)^n = \mathbb{R}^n_{>0} \times (S^1)^n$, the sedentarity as a subset represents the cycles of the torus $(S^1)^n$ which get collapsed in the fiber over a point, i.e. if $S(x) = I$, then

$$\mathrm{Log}_t^{-1}(x) = (S^1)^n / \langle \theta_i \mid i \in I \rangle.$$

It is perhaps due to this non-uniformity of tropical affine space that early works in tropical geometry did not take into account the boundary and only worked in $\mathbb{R}^n$. Most of the necessary notions had been introduced in [38]. With the right definitions compact tropical spaces are manageable and it can be quite useful to exploit the different nature of points in $\mathbb{T}^n$.

Call $\mathbb{R}^n \subset \mathbb{T}^n$ the tropical torus, since $\mathbb{R}^n = val((\mathbf{K}^*)^n) = \mathrm{Log}_t((\mathbb{C}^*)^n)$. With the boundary removed, the points in $\mathbb{R}^n$ behave uniformly. Because of this the next round of definitions is less technical in $\mathbb{R}^n$, and so we start with this case and then generalise.

1.1.2 Tropical cycles

Tropical cycles in $\mathbb{R}^n$ have already been presented in various places, [2], [29], [38], [48]. Here these definitions are reviewed and extended to define cycles in $\mathbb{T}^n$. This generalisation is not a difficult task, but it is necessary for the consideration of compact tropical spaces.

To start we summarize some important terminology for polyhedral complexes.

- A **polyhedral complex** $P \subset \mathbb{R}^n$ is a finite collection of polyhedra in $\mathbb{R}^n$ which contains all the faces of its members and the intersection of any two polyhedra in P is a common face.

- A polyhedral complex P is **rational** if every face in P is defined by the intersection of half-spaces given by equations $\langle x, v \rangle \leq a$ where $a \in \mathbb{R}^n$ and $v \in \mathbb{Z}^n \subset \mathbb{R}^n$.

- The **support** $|P|$ of a polyhedral complex P is the union of all polyhedra in P as sets, and P is pure dimensional if $|P|$ is.
- A polyhedral complex P is **weighted** if each facet F of P is equipped with a weight $w_P(F) \in \mathbb{Z}$.
- A complex P_1 is a **refinement** of a complex P_2 if their supports are equal and every face of P_2 is a face of P_1.
- The k**-skeleton** of a polyhedral complex P, denote $P^{(k)}$, is the union of all faces of P of dimension $i \leq k$.
- The **star** of a point $p \in P$ is

$$Star_p(P) = \{v \in \mathbb{R}^n \mid \exists\ \epsilon > 0,\ p + \epsilon v \in P\}.$$

The following is the **balancing condition** well known in tropical geometry.

Definition 1.1.2. [48] [38]A pure dimensional weighted rational polyhedral complex $C \subset \mathbb{R}^n$ is balanced if it satisfies the following condition on every codimension one face $E \subset C$: Let $F_1, \dots F_s$ be the facets adjacent to E and v_i be a primitive integer vector such that for an $x \in E$, $x + \epsilon v_i \in F_i$ for some $\epsilon > 0$. Then,

$$\sum_{i=1}^{s} w_{F_i} v_i,$$

is parallel to the face E, where w_{F_i} is the weight of the facet F_i, see the left hand side of Figure 1.1.

Definition 1.1.3. [48] [38] A tropical k-cycle $C \subset \mathbb{R}^n$ is a pure k-dimensional weighted, rational, polyhedral complex satisfying the balancing condition.

Define an equivalence relation on tropical cycles by declaring a cycle with all facets of weight zero to be equivalent to the empty polyhedral complex. The set of tropical k-cycles in $\mathbb{R}^n$ modulo this equivalence will be denoted $Z_k(\mathbb{R}^n)$. Since this equivalence relation is rather trivial we will abuse our nomenclature and refer to the equivalence class of a cycle as just a tropical cycle. This set forms a group under the operation of unions of complexes and addition of weight functions denoted by $+$. See [2], [38] for more details.

Definition 1.1.4. A tropical k-cycle is **effective** if all of its facets have positive weights.

Definition 1.1.5. Given two tropical cycles $A, C \subseteq \mathbb{R}^n$ we say A is a subcycle of C if $|A| \subseteq |C|$ and given an open face F of A there is an open face of C which contains it.

Definition 1.1.6. An effective tropical cycle $L \subset \mathbb{T}^n$ is a generic tropical line if it has exactly $n+1$ unbounded edges in the directions $-e_1, \dots, -e_n$ and $e_0 = e_1 + \dots e_n$ all equipped with weight one.

See Figure 1.2 for examples of tropical lines. We will sometimes denote a tropical line with a single vertex centered at the origin by $L_{n+1} \subset \mathbb{T}^n$. A tropical line which is not "generic" may have unbounded rays in linear combinations of these vectors with coefficients one.

Remark If A is a subcycle of C, then there exists a refinement of the polyhedral structure on C so that A is a polyhedral subcomplex of C. Although we will not need to consider this refinement of C, the polyhedral structure on A as a subcycle of C will be important.

There have been two approaches to intersections of cycles $\mathbb{R}^n$. Firstly, tropical stable intersection was defined for curves in $\mathbb{R}^2$ in [48] and for general cycles by Mikhalkin in [38]. The intersection product in $\mathbb{R}^n$ of Allermann and Rau is based on intersecting with Cartier divisors and the diagonal [2]. The two definitions have been shown to be equivalent in both [47], [29]. We review the definition of stable intersection in $\mathbb{R}^n$.

Definition 1.1.7. [48], [38] Let $A \in Z_{k_1}(\mathbb{R}^n)$ and $B \in Z_{k_2}(\mathbb{R}^n)$ then their stable intersection, denoted $A.B$ is supported on the complex $(A \cap B)^k$ where $k = k_1 + k_2 - n$ with weights assigned on facets in the following way:

1. If a facet $F \subset (A \cap B)^k$ is the intersection of top dimensional facets $D \subset A$ and $E \subset B$ and D and E intersect transversely, then

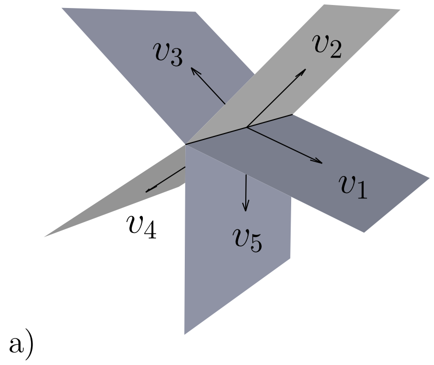
$$w_{A.B}(F) = w_A(D) w_B(E) [\mathbb{Z}^n \colon \Lambda_D + \Lambda_E],$$

where Λ_D and Λ_E are the integer lattices spanned by the faces D and E respectively.

2. Otherwise for a generic vector v with non-rational projections and an $\epsilon > 0$, in a neighborhood of F, $A_\epsilon = A + \epsilon \cdot v$ and B will meet in a collection of facets $F_1 \dots F_s$ parallel to F such that the intersection at each F_i is as in the case (1) above. Then we set,

$$w_F(A.B) = \sum_{i=1}^{s} w_{F_i}(A_\epsilon.B).$$

That the formula above is well-defined regardless of choice of the vector v follows from the balancing condition. In fact, the above weight calculation is equivalent to the fan displacement rule for intersection of Minkowski weights from [20]. For more details see [2] or [28]. By the equivalence of Mikhalkin's stable intersection and Allermann and Rau's intersection product shown in [47] [29], the following two propositions can be found in Section 9 of [2].



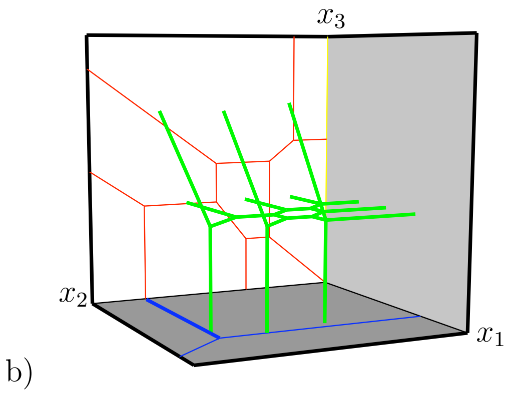


Figure 1.1: a) Balancing condition for a surface b) Cycles of sedentarity in $\mathbb{T}^3$.

Corollary 1.1.8. *[2] Given A, B tropical cycles in $\mathbb{R}^n$ the following hold,*

1. *$A.B \subset \mathbb{R}^n$ is a balanced tropical cycle.*
2. *$(A.B).C = A.(B.C)$*
3. *$A.B = B.A$*
4. *$A.(B + C) = A.B + A.C$*

Next we generalise the definition of cycles in $\mathbb{R}^n$ to $\mathbb{T}^n$ by allowing cycles contained in the boundary strata.

Definition 1.1.9. A subset $B \subseteq \mathbb{T}^n$ is said to be of sedentarity I if it is the topological closure in $\mathbb{T}^n$ of some $B^o \subset H_I^\times$. A tropical k-cycle $C \subset \mathbb{T}^n$ of sedentarity I is the closure of a tropical k-cycle $C^o \subset \mathbb{R}^{n-|I|} = H_I^\times$.

Again let $Z_{k,I}(\mathbb{T}^n)$ denote the quotient of the set of all k-cycles of sedentarity I by those with only zero weights. Given two cycles $A, B \in Z_{k,I}(\mathbb{T}^n)$ denote by $A+B$ the closure of $A^o + B^o$ as defined in $\mathbb{R}^{n-I}$. This gives a group structure on k-cycles of sedentarity I and $Z_{k,I}(\mathbb{T}^n) \cong Z_k(\mathbb{R}^{n-|I|})$.

Definition 1.1.10. The group of tropical k-cycles in $\mathbb{T}^n$ is

$$Z_k(\mathbb{T}^n) = \bigoplus_{\emptyset \subseteq I \subset [n]} Z_{k,I}(\mathbb{T}^n).$$

Just as in $\mathbb{R}^n$, a tropical cycle in $\mathbb{T}^n$ is **effective** if all of its facets are equipped with positive weights. Also, a tropical cycle $A \subset \mathbb{T}^n$ is a **subcycle** of a cycle $C \subset \mathbb{T}^n$ if the supports satisfy $|A| \subseteq |C|$ and for every open face of A there is an open face of C containing it.

Next we generalise intersections of cycles in $\mathbb{R}^n$ to intersections in $\mathbb{T}^n$ in a specific case. Recall that the vectors $e_1, \dots, e_n$ denote the standard basis of $\mathbb{Z}^n \subset \mathbb{R}^n$.

Definition 1.1.11. Let $A \subseteq \mathbb{T}^n$ be a k-cycle of sedentarity I then

- If $i \in I$, set $A.H_i = \emptyset$.
- If $I = \emptyset$ then $A.H_i$ is supported on $(A \cap H_i)^{(k-1)}$ with the weight function defined as follows: Given a facet F of $(A \cap H_i)^{(k-1)}$ it is adjacent to some facets $\tilde{F}_1, \dots, \tilde{F}_s$ of A. Then,

$$w_{A.H_i}(F) = \sum_{l=1}^{s} w_A(\tilde{F}_l)[\mathbb{Z}^n : \Lambda_{\tilde{F}_l} + \Lambda_i^\perp],$$

  where $\Lambda_i^\perp = \{x \in \mathbb{Z}^n \mid \langle x, e_i \rangle = 0\}$.
- If $I \neq \emptyset$ and $i \notin I$ then $A.H_i$ is the intersection of $A.H_{i \cup I}$ calculated in $H_I = \mathbb{T}^{n-|I|}$ as in the case above.

Every cycle $A \subset \mathbb{T}^n$ can be uniquely decomposed as a sum of its parts of different sedentarity and we extend the above definition to cycles of mixed sedentarity by linearity.

Proposition 1.1.12. *Given cycles $A, B \subset \mathbb{T}^n$ we have:*

1. *$A.H_i$ is a balanced cycle.*
2. *$(A + B).H_i = A.H_i + B.H_i$.*

Proof. For the balancing condition assume that A is of sedentarity $\emptyset$ and let $E \subset A.H_i$ be a face of codimension one which is in the interior of a face of $\mathbb{T}^n$ of sedentarity I. Let $\tilde{E}_j$ denote the faces of codimension one of A and of sedentarity $\emptyset$ which are adjacent to E. For $M >> 0$ let $L_M = \{x \in \mathbb{T}^n \mid x_i = -M\}$ then $\tilde{E}_j \cap L_M$ is in $A.L_M$ and $A.L_M$ is balanced at $\tilde{E}_j \cap L_M$. This means that

$$\sum_{\tilde{E}_j \subset \tilde{F}} w_{A.L_M}(\tilde{F} \cap L_M) v_{\tilde{F}} = \sum_{\tilde{E}_j \subset \tilde{F}} w_A(\tilde{F})[\mathbb{Z}^n : \Lambda_{\tilde{F}_l} + \Lambda_i^\perp] v_{\tilde{F}} = 0 \qquad (1.1.1)$$

where $v_{\tilde{F}}$ is the primitive integer vector orthogonal to $\tilde{E}$ generating $\tilde{F}$.

Let $\pi_I : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-|I|}$ be the linear projection with kernel $< e_i \mid i \in I >$. Then a facet $\tilde{F} \supset \tilde{E}_j$ is adjacent to a face $F \supset E$ if and only if $\pi_{I*}(v_{\tilde{F}}) = v_F$ where v_F is the primitive integer vector in $\mathbb{R}^{n-|I|}$ orthogonal to E generating F. Applying π_{I*} to 1.1.1 and taking the sum over all $\tilde{E}_j$ adjacent to E we obtain balancing at E.

When A and B are of equal sedentarity distributivity follows from the additivity of the weight function. For cycles of mixed sedentarity the intersection is defined by extending the product linearly, so it is distributive. $\square$

1.1.3 Tropical functions, modifications and divisors.

Definition 1.1.13. [38] Let U be a connected open subset of $\mathbb{T}^n$ and let $S = \bigcup_{x \in U} s(x) \subset [n]$. A tropical regular function $f : U \longrightarrow \mathbb{T}$ is a tropical Laurent polynomial

$$f(x) = \text{“}\sum_{\alpha \in \Delta} a_\alpha x^\alpha\text{”}$$

where $\Delta \subset \mathbb{Z}^n$ is such that for all $\alpha \in \Delta$, $\alpha_i \geq 0$ if $i \in S$.

A tropical regular function is a piecewise affine, integer sloped, convex function, whose graph is a finite polyhedral complex. Suppose $U \subset \mathbb{R}^n \subset \mathbb{T}^n$ is connected, then every regular function on U can be expressed as a tropical Laurent polynomial $f(x) = \text{“}\sum_{\alpha \in \Delta} a_\alpha x^\alpha\text{”}$. If U contains a point x for which $x_i = -\infty$ for some i, then “$1/x_i$” $= -x_i = \infty \notin \mathbb{T}$, and ‘$1/x_i$” is not regular on U. Distinct tropical polynomials may represent the same functions as some monomials may be redundant, for example,

$$f_1(x) = \text{“}x^2 + (-1)x + 1_{\mathbb{T}}\text{”} = \max\{2x, x - 1, 0\},$$

and

$$f_2(x) = \text{“}x^2 + 1_{\mathbb{T}}\text{”} = \max\{2x, 0\}.$$

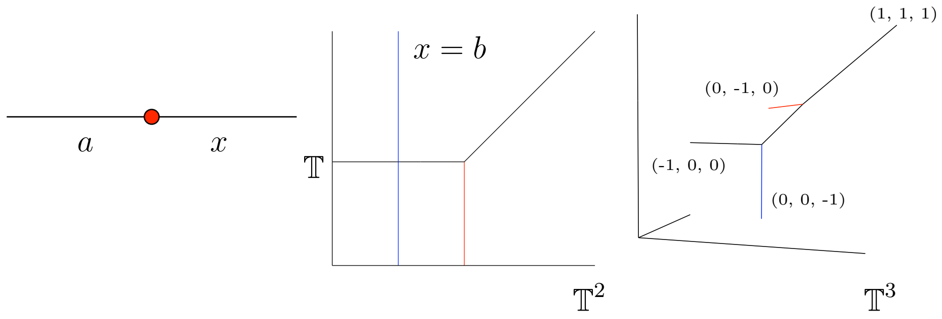
Let $\mathcal{O}_{\mathbb{T}^n}(U)$ denote the semi-ring of regular functions on U and $\mathcal{O}_{\mathbb{T}^n}$ the regular functions on $\mathbb{T}^n$.

Tropical division corresponds to subtraction and so a rational function is of the form $h = \text{“}f/g\text{”} = f - g$ where $g \neq -\infty$. On $\mathbb{R}^n \subset \mathbb{T}^n$ such a function is always defined since it is the difference of two continuous functions. At the boundary of $\mathbb{T}^n$ where the function may take values $\pm\infty$ there may be a codimension two locus where the function is not defined. For example the function $f(x) = \text{“}\frac{x_1}{x_2}\text{”}$ on $\mathbb{T}^2$ at the point $(-\infty, -\infty)$. We denote the rational functions by $\mathcal{K}_{\mathbb{T}^n}$. Given a tropical cycle $C \in Z_k(\mathbb{T}^n)$ (or $C \in Z(\mathbb{R}^n)$) regular functions and rational functions on C, denoted $\mathcal{O}_C$ and $\mathcal{K}_C$ respectively, are obtained by restriction of $\mathcal{O}_{\mathbb{T}^n}$ and $\mathcal{K}_{\mathbb{T}^n}$.

Definition 1.1.14. A function $f \in \mathcal{O}_U$ is invertible on U if “$\frac{1}{f}$” $= -f \in \mathcal{O}(U)$. For a subset $U \subset \mathbb{T}^n$ denote by $\mathcal{O}^*(U)$ the invertible functions on U.

It is not difficult to see that if $f \in \mathcal{O}^*(\mathbb{R}^n)$ then it corresponds to a tropical monomial (i.e. it is an integral affine function). Similarly, if $f \in \mathcal{O}^*(\mathbb{T}^n)$ then it corresponds to tropical multiplication by a constant in $\mathbb{T}^*$ (i.e. regular addition by a scalar in $\mathbb{R}$).

We examine the simple operation of taking the graph of a line. Let $f_1 : \mathbb{T} \longrightarrow \mathbb{T}$ be the regular function given by $f_1(x) = \max\{x, a\}$. The graph of this function is drawn in black in the middle part of Figure 1.2. Notice that the graph does not satisfy the balancing condition at the point (a, a), nor does it obtain tropical zero, $0_{\mathbb{T}} = -\infty$. There is a canonical way to fix both of these faults; add a ray in the downward vertical direction to the graph at this point, (this is drawn in red in Figure 1.2). If this ray is equipped with weight one then the resulting cycle is now balanced.



$f_1(x) = \max\{x, a\}$ $\qquad$ $f_2(x) = \max\{x, b\}$

Figure 1.2: Two modifications of a tropical line,

The right hand side of Figure 1.2 repeats this procedure for a function $f_2 : \mathbb{T}^2 \longrightarrow \mathbb{T}$, to obtain a cycle in $\mathbb{T}^3$. Both of the cycles produced after this operation happen to be tropical lines in this case.

Construction 1.1.15 Tropical modifications.
In general, given a cycle $C \subseteq \mathbb{T}^n$ we may consider the graph $\Gamma_C \subset \mathbb{T}^{n+1}$ of a function $f \in \mathcal{O}_C$. The graph $\Gamma_f(C)$ is still a rational polyhedral complex, moreover its facets inherit weights from the corresponding facets of C. But since f is only piecewise affine, the graph $\Gamma_f(C)$ is not necessarily balanced. At any unbalanced codimension one face E of $\Gamma_f(C)$ we may attach a facet generated by the vector $-e_{n+1}$, namely

$$F_E = \{(x, c) \mid x \in E, c \in (x, -\infty]\}.$$

Moreover, for each new facet there exists a unique weight $w_{F_E} \in \mathbb{N}$ balancing the resulting complex at E. Call the **undergraph** of f restricted to C the weighted complex:

$$\mathcal{U}(\Gamma_f(C)) = \bigcup_{\substack{E \subset \Gamma_f(C) \\ codim(E)=1}} F_E.$$

In Figure 1.3, the undergraph for $\Gamma_f(\mathbb{T}^2)$ where $f(x, y) = \max\{x, y, 0\}$ is drawn in red. Then at last,

$$\tilde{C} = \Gamma_f(C) \cup \mathcal{U}(\Gamma_f(C))$$

is a tropical cycle in $\mathbb{T}^{n+1}$. Let $\delta : \mathbb{T}^{n+1} \longrightarrow \mathbb{T}^n$ be the linear projection with kernel generated by e_{n+1}, then $\delta(\tilde{C}) = C$. Then $\delta : \tilde{C} \longrightarrow C$ is a regular elementary tropical modification of C along the regular function f.

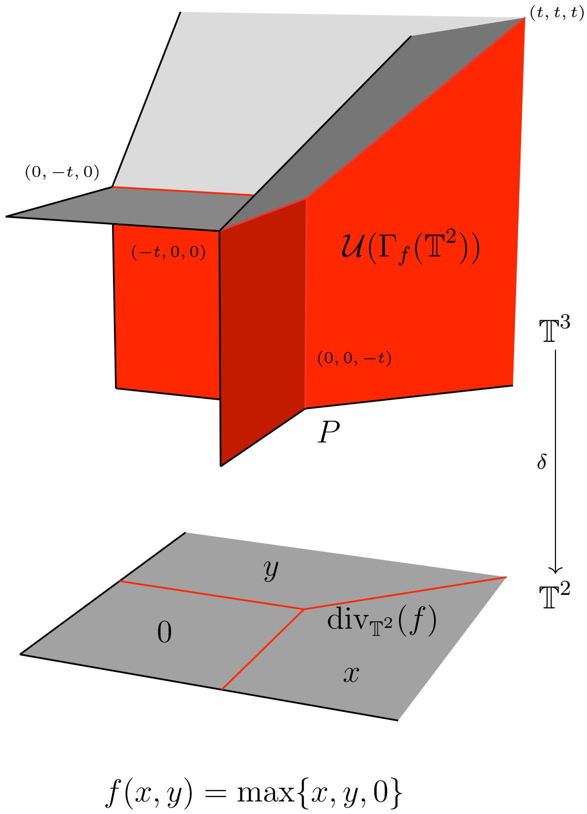


Figure 1.3: A modification of the tropical affine plane, $\delta : P \longrightarrow \mathbb{T}^2$. The undergraph consists of the three faces drawn in red and the divisor is drawn in red in $\mathbb{T}^2$. The fan P is the **standard tropical plane** in $\mathbb{TP}^3$.

Definition 1.1.16. Given a cycle $C \subseteq \mathbb{T}^n$ and a regular function $f \in \mathcal{O}_C$, the regular elementary modification of C along f is $\delta : \tilde{C} \longrightarrow C$. Where $\tilde{C}$ is given by Construction 1.1.15 and δ is the linear projection with kernel $-e_{n+1}$. Also, C is called a regular elementary contraction of $\tilde{C}$.

If the terminology Definition 1.1.16 seems confusing, the reader is encouraged to think of the notation and terminology used for the operation of blowing up in classical algebraic geometry. The blow up of a of a variety $\mathcal{X}$ refers to the map $\pi : \tilde{\mathcal{X}} \longrightarrow \mathcal{X}$, yet the term is also used to denote just the space $\tilde{\mathcal{X}}$. In this spirit, *a modification* will sometimes refer to simply the space $\tilde{C}$ where the existence of the modification map $\delta : \tilde{C} \longrightarrow C$ is implied. Continuing with this analogy, *tropical contraction* is the counter part to *blowing down* in algebraic geometry.

Definition 1.1.17. A regular modification, (respectively regular contraction), is any composition of regular elementary modifications, (respectively regular elementary contractions).

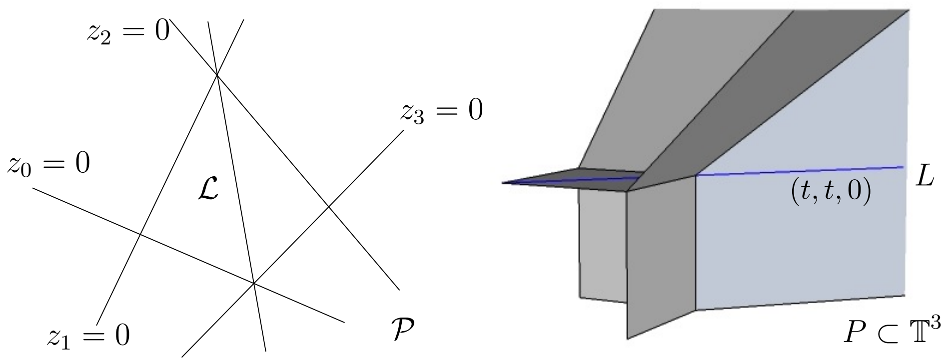


Figure 1.4: a) The line $\mathcal{L}$ drawn in the plane $\mathcal{P} \subset \mathbb{C}^3$ with respect to the coordinate axes. b) The tropicalisation (P, L) in $\mathbb{T}^3$ of $(\mathcal{P}, \mathcal{L})$ in $\mathbb{C}^3$.

Using modifications we define the divisor of a function on a cycle $C \subseteq \mathbb{T}^n$. Divisors have already been introduced in [38], and studied in depth in [2].

Definition 1.1.18. Let $f, g : \mathbb{T}^n \longrightarrow \mathbb{T}$ be regular functions and suppose $g \neq 0_{\mathbb{T}}$ and let $C \subset \mathbb{T}^n$ be a cycle and $\delta_f : \tilde{C} \longrightarrow C$ the modification of C along the function f. Then,

1. the divisor of the function f restricted to C is $\text{div}_C(f) := \delta_f(\tilde{C}.H_{n+1})$.
2. if $h =$ "f/g" then $\text{div}_C(h) := \text{div}_C(f) - \text{div}_C(g)$.

Proposition 1.1.19. *For functions $f, g \in \mathcal{K}_{\mathbb{T}^n}$ and cycles $A, B \subset \mathbb{T}^n$. We have,*

$$div_{A+B}(f) = div_A(f) + div_B(f).$$

Proof. This follows directly from Proposition 1.1.12 and Definition 1.1.18. □

Example 1.1.20
Consider the plane $\mathcal{P} \subset \mathbb{C}^3$ given by the zero set of the linear equation

$$\mathcal{F}(z_1, z_2, z_3) = z_1 + z_2 + z_3 + 1.$$

We may tropicalise this plane by taking the following limit, also known as the "logarithmic limit" set or Bergman fan [5],

$$\lim_{t \to \infty} Log_t(\mathcal{P}).$$

This is the same two dimensional fan $P \subset \mathbb{T}^3$ shown in Figure 1.3. Consider the line $\mathcal{L} \subset \mathcal{P}$ defined by the additional equation $z_3 = -1$. It is not difficult to check that $\lim_{t \to \infty} Log_t(\mathcal{L}) \subset \mathbb{T}^3$ is an affine line, $L \subset P$, passing through the origin of $\mathbb{R}^3$ in direction $(1, 1, 0)$, see Figure 1.4. We say $L \subset P$ is **approximated** by a line $\mathcal{L}$

in the plane $\mathcal{P}$. It was remarked in both [38] and [2] that $L \subset P$ is not the divisor of a tropical polynomial restricted to P, however, it can be given as the divisor of a tropical rational function, namely.

$$f(x_1, x_2, x_3) = \max\{x_1, x_2\} + \max\{x_3, 0\} - \max\{x_1, x_2, x_3, 0\}.$$

The easiest way to see that this is the correct rational function is to compute the sum of cycles,

$$\mathrm{div}_P(f) = \mathrm{div}_P(f_1) + \mathrm{div}_P(f_2) - \mathrm{div}_P(f_3),$$

where

$$f_1 = \max\{x_1, x_2\}, \quad f_2 = \max\{x_3, 0\}, \quad \text{and} \quad f_3 = \max\{x_1, x_2, x_3, 0\}.$$

Thus a curious phenomenon occurs, the complex line $\mathcal{L}$ is the divisor of a polynomial on $\mathcal{P}$, but this no longer holds for the tropicalised pair $L \subset P$.

This example motivates us to allow elementary tropical modifications along more general functions, namely tropical rational functions which have effective divisors. The construction of the modification cycle $\tilde{C}$ is the same as in the regular case. The modification $\tilde{C}$ remains an effective cycle because of the effective condition on the divisor.

Definition 1.1.21. Given an effective cycle $C \subseteq \mathbb{T}^n$ and a rational function $f \in \mathcal{K}_C$ such that $\mathrm{div}_C(f)$ is effective, the elementary modification of C along f is $\delta_f : \tilde{C} \longrightarrow C$. Where $\tilde{C}$ is given by Construction 1.1.15 and δ_f is the linear projection with kernel $-e_{n+1}$. Also, C is called an elementary contraction of $\tilde{C}$.

A *modification*, respectively *contraction*, is any composition of elementary modifications, respectively contractions. Given a rational function with effective divisor on C, it in fact defines a continuous function on all of C, including the boundary, there is no codimension two locus of indeterminacy. For a modification of a cycle $C \subset \mathbb{T}^n$ we can define pullback and pushforward maps on subcycles.

Definition 1.1.22. Let $C \subseteq \mathbb{T}^n$ be an effective cycle, $f \in \mathcal{K}_C$ be a function with effective divisor on C and $\delta : \tilde{C} \to C$ be the elementary modification along f, so that e_{n+1} generates the kernel of δ. We define the following:

1. The push-forward map of cycles is $\delta_* : Z_k(\tilde{C}) \to Z_k(C)$. For a cycle of sedentarity order 0 it is given by $\delta_* A = \delta(A)$ with weight function,

$$w_{\delta_* A}(F) = \sum_{F_i \subset A, \delta(F_i) = F} w_A(F_i)[\bar{\Lambda}_{F_i} : \Lambda_F],$$

where $\bar{\Lambda}_{F_i}$ is the image under δ of the integer lattice generated by F_i and Λ_F is the integer lattice generated by F. If A is of sedentarity I where $n+1 \notin I$, then $\delta_* A$ is given by restricting the above definition to the modification $\delta : \tilde{C} \cap H_I \longrightarrow C \cap H_I$. If $n+1 \in I$ then $\delta_* A = \delta(A)$.

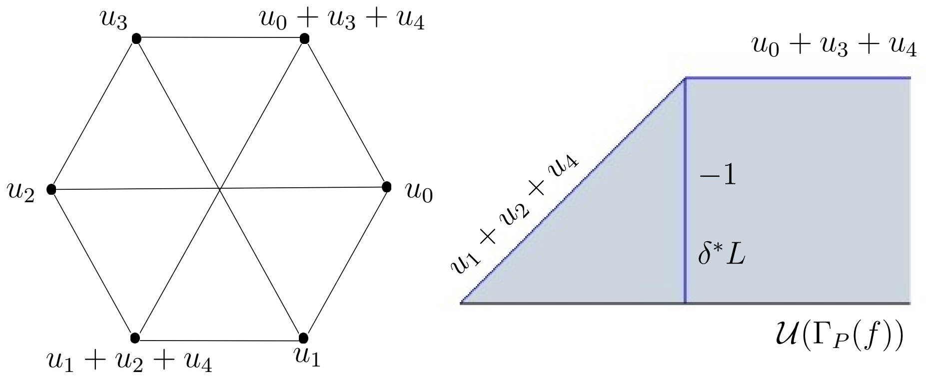


Figure 1.5: The fan $\tilde{P} \subset \mathbb{T}^4$ from Example 1.1.23 is the cone over the graph on the left. The undergraph of the modification $\delta : \tilde{P} \longrightarrow P$ is a single face shown on the right. The pullback $\delta^* L$ is not effective.

2. The pull-back map of cycles $\delta^* : Z_k(C) \to Z_k(\tilde{C})$ for a cycle $A \in Z_k(C)$, $\delta^* A$ is the cycle obtained from the graph $\{(x, f(x)) \mid x \in A\}$ by adjoining the unique collection of weighted facets in the direction $-e_{n+1}$ so that the resulting complex is balanced.

The complex $\delta_* A$ inherits its weights from A in such a way that the equilibrium of A guarantees that $\delta_* A$ is also balanced. Notice that $\delta_* \delta^* A = A$ but $\delta^* \delta_* A$ is not always equal to A. The pullback of an effective subcycle may not be effective if the modification is along a rational function.

Example 1.1.23
We return to the situation of Example 1.1.20 to examine the modification along the rational function f satisfying $\mathrm{div}_P(f) = L$. Denote this modification by $\delta : \tilde{P} \longrightarrow P$. The fan $\tilde{P} \subset \mathbb{T}^4$ is the cone over the graph drawn on the left of Figure 1.5. The undergraph of the modification consists of one face which is drawn on the right of Figure 1.5. The pullback of the cycle L, $\delta^* L$, is not an effective cycle in $\tilde{P}$, a weight of -1 must be added to the edge in direction $-e_4$ in order to satisfy the balancing condition.

All of the above definitions of regular modifications, modifications, and divisors given for cycles in $\mathbb{T}^n$ restrict to cycles in $\mathbb{R}^n$. For a cycle $C \subseteq \mathbb{R}^n$ and a function $f \in \mathcal{O}_C$, the modification is constructed in the same way as in Construction 1.1.15, and the resulting cycle $\tilde{C}$ is contained in $\mathbb{R}^n \times \mathbb{T}$. This is contrary to in [2], where the modification is contained in $\mathbb{R}^{n+1}$, however this does not change the definition of the divisor of a function f restricted $C \subseteq \mathbb{R}^n$. The reader may see [2] for this definition or simply think of it as being the part with order of sedentarity zero of $\mathrm{div}_{\overline{C}}(f)$ where $\overline{C} \subset \mathbb{T}^n$ and f is extended naturally to $\overline{C}$. We cite the following proposition which will be used later on in Section 2.1.1.

Proposition 1.1.24. *[2], [28] Given tropical rational functions $f, g \in \mathcal{K}_{\mathbb{R}^n}$ and tropical cycles $A, B \subset \mathbb{R}^n$*

1. $div(f)_{\mathbb{R}^n}.A = div_A(f)$
2. $div_{A+B}(f) = div_A(f) + div_B(f)$
3. $div_A(\text{“}f \cdot g\text{”}) = div_A(f) + div_A(g)$

Following part (1) of Proposition 1.1.24 we have:

Corollary 1.1.25. *If $f \in \mathcal{O}(\mathbb{R}^n)$ then $div(f)_{\mathbb{R}^n}.C$ is effective for every effective cycle $C \subset \mathbb{R}^n$.*

As mentioned above, the procedure of tropical modification presented by Allerman and Rau in [2], considers the resulting cycle in $\mathbb{R}^{n+1}$. To distinguish this from the modifications we presented in this section they will be called open tropical modifications.

Definition 1.1.26. An elementary **open** tropical modification of a cycle $C \subseteq \mathbb{T}^n$ (similarly $C \subseteq \mathbb{R}^n$) is

$$\delta : \tilde{C} \longrightarrow C,$$

where $\tilde{C} \subset \mathbb{T}^n \times \mathbb{R}$ (similarly $\tilde{C} \subset \mathbb{R}^{n+1}$) is given by Construction 1.1.15 by removing $\{x_{n+1} = -\infty\}$ and $\delta : \mathbb{T}^n \times \mathbb{R} \longrightarrow \mathbb{T}^n$ (similarly $\delta : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$) is a linear projection.

An elementary open tropical modification is still given by Construction 1.1.15, except that we do not take the closure of the undergraph in $\mathbb{T}^{n+1}$, (or $\mathbb{R}^n \times \mathbb{T}$).

Elementary open tropical modifications should be thought of as an embedding of the tropical cycle C with the divisor $\text{div}_C(f)$ removed, where f is the function of the modification. Similar to the case of normal tropical modifications, an elementary open tropical modification is regular if it is along a regular function f and an open tropical modification is a composition of elementary open tropical modifications.

1.1.4 Matroidal fans and modifications

This section provides a correspondence between certain types of tropical modifications and existing operations in matroid theory via the Bergman fan of a matroid. There are various ways to define a matroid, [44] all of them being equivalent. Here we will use most often the definition given by the **rank function**.

Definition 1.1.27. A matroid is a pair $M = (E, r)$, where E is a finite set and $r : 2^E \longrightarrow \mathbb{N} \cup \{0\}$ is the rank function satisfying the following,

1. For all $A \subset B \subseteq E$, $r(A) \leq r(B) \leq |B|$. (bounded and monotone)
2. For all $A, B \subseteq E$, $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$. (submodular)

Matroid theory is a very rich subject, only the necessary definitions and terminology will be reviewed here. The reader is referred to [44] for a comprehensive introduction. Here is a quick glossary of matroid terminology to be used involving a matroid $M = (E, r)$.

- The **rank of a matroid** is $r(E)$.
- A subset $I \subset E$ is **independent** if $r(A) = |A|$ and *dependent* if $r(A) < A$.
- A subset $B \subset E$ is a **basis** $r(A) = r(E) = |A|$.
- An element $e \in E$ is a **loop** if $r(e) = 0$.
- An element $e \in E$ is a **coloop** if $e \in B$ for every basis B of M.
- The matroid is said to have **double points** if there exists $i, j \in E$ such that $r(ij) = 1$.
- A subset $F \subset E$ is a **flat** if for all $e \notin F$, $r(F) < r(F \cup e)$. The flats of a matroid form a lattice Λ_M, called **the lattice of flats** in the sense of a partially ordered set.

Example 1.1.28
A projective hyperplane arrangement $\mathcal{A} = \cup_{i=0}^{n} \mathcal{H}_i$ where $\mathcal{H}_i \in \mathbb{P}^k$ are hyperplanes, gives rise to a matroid $M = (E, r)$ in the following way: Let $E = \{0, \dots, n\}$ and define the rank function by $r(I) = \operatorname{codim}_{\mathbb{P}^k}(\cap_{i \in I} \mathcal{H}_i)$. We will always assume that $\cap_{i \in E} \mathcal{H}_i = \emptyset$ so that the rank of the matroid is $k+1$, these are often called "non-central" arrangements in the literature. The independent sets correspond to collections of hyperplanes which intersect properly. Bases are collections of $k+1$ hyperplanes with empty intersection. A loop is a degenerate hyperplane given by the linear form $f(z) = 0$, and i, j are double points if and only if $\mathcal{H}_i = \mathcal{H}_j$. Flats are in one to one correspondence with the linear subspaces arising from the intersections of the hyperplanes in $\mathcal{A}$.

Given a matroid M of rank $k+1$ on a ground set $E = \{0, 1, \dots, n\}$ we will consider a projective version of the Bergman fan of M as seen in [55].

Definition 1.1.29. [38] Tropical projective space is

$$\mathbb{TP}^n = (\mathbb{T}^{n+1} \backslash (-\infty, \dots, -\infty))/(x_0, \dots, x_n) \sim (x_0 + \lambda, \dots, x_n + \lambda)$$

for $\lambda \in \mathbb{R}$.

Tropical projective space is topologically the n-simplex. We can equip $\mathbb{TP}^n$ with tropical homogeneous coordinates $[x_0 : \cdots : x_n]$ similarly to the classical setting. Also it is equipped with $n+1$ charts, $U_i = \{x \mid x_i \neq -\infty\} = \mathbb{T}^n$ which can be identified via integer affine maps. Tropical projective space is a **tropical toric variety**, as to appear in Section 1.2.1.

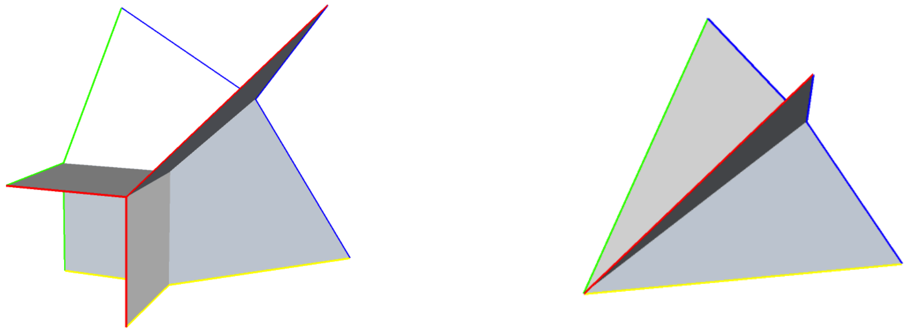


Figure 1.6: The compactifications of two tropical planes in $\mathbb{T}P^3$. On the right there is a corner point $p_{i,j,k}$ corresponding to a triple of lines.

Notice that $\mathbb{R}^n \subset \mathbb{TP}^n$ is dense. The boundary of $\mathbb{TP}^n$ is stratified in a way similar to $\mathbb{T}^n$. Given $\emptyset \neq I \subset \{0, \dots, n+1\}$ we have a face of the n-simplex corresponding to the subset of $\mathbb{TP}^n$ where $x_i = -\infty$ in homogeneous coordinates. Moreover such a face is isomorphic to $\mathbb{TP}^{n-|I|}$.

Construction 1.1.30
First assume that the matroid $M = (E, r)$ is loopless, meaning it contains no loop elements. Suppose the ground set is $E = \{0, \dots, n\}$ and M is of rank $k+1$. Let Λ_M denote the collection of flats of M. Recall that $e_1, \dots, e_n$ denotes the standard basis of $\mathbb{R}^n$. For $1 \leq i \leq n$ set $v_i = -e_i$ and $v_0 = \sum_{i=1}^n e_i$, where e_i are the standard basis directions in $\mathbb{R}^n$. For every maximal (meaning longest) chain $\emptyset \neq F_1 \subset \dots \subset F_k \neq E$ in Λ_M take the k dimensional cone given by the positive span of $\{v_{F_1}, \dots, v_{F_k}\}$ where $v_{F_l} = \sum_{i \in F_l} v_i$. Finally, $B(M)$ is the closure in $\mathbb{TP}^n$ of the union of all such polyhedral cones. This is the **fine** polyhedral structure on $B(M)$ as defined by Ardila and Klivans [3]. There is also a **coarse** structure on this set, which can be obtained from the combinatorics of the lattice of flats, see [3]. This construction is a projectivisation of the definitions given in [16], [3] up to a reflection caused by the use of the *max* convention instead of *min*.

Returning to the situation when the matroid M arises from a non-central hyperplane arrangement, as in Example 1.1.28, the complement $\mathbb{P}^k \backslash \mathcal{A}$ can be canonically embedded into $(\mathbb{C}^*)^n$ in the following way. Each hyperplane is defined by a linear form f_i up to a constant. This provides a map,

$$\begin{aligned} F : \mathbb{P}^k &\longrightarrow \mathbb{P}^n \\ x &\mapsto [f_0(x) : \dots : f_n(x)], \end{aligned}$$

Restricting F to the complement and we obtain $Fi : \mathbb{P}^k \backslash \mathcal{A} \longrightarrow (\mathbb{C}^*)^n$, call the image $\mathcal{V}$.

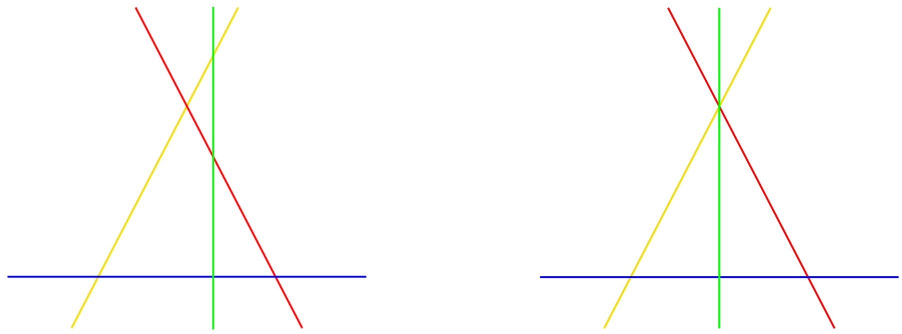


Figure 1.7: The line arrangements corresponding to the tropical planes from Figure 1.6.

When the arrangement is loopless, it was shown by Sturmfels in [55] that the *logarithmic limit set* [5] of $\mathcal{V} \subset (\mathbb{C}^*)^n$, i.e.

$$\lim_{t \to \infty} \mathrm{Log}_t(\mathcal{V})$$

is the Bergman fan of the corresponding matroid. As mentioned in Example 1.1.28, if i is a loop of a matroid corresponding to a hyperplane arrangement, then H_i is the degenerate hyperplane defined by the linear form $f_i = 0$, and so $\phi(\mathbb{P}^k)$ is contained in the i^{th} coordinate hyperplane of $\mathbb{P}^n$. A Bergman fan of a matroid with loops will be contained in the boundary strata of $\mathbb{TP}^n$.

Definition 1.1.31. Given a matroid $M = (E, r)$, let $I \subset E$ denote its collection of loops. Then the complex $B(M)$ is contained in the boundary of $\mathbb{TP}^n$ corresponding to $x_l = -\infty$ for all $l \in I$ and is equal to $B(M \backslash I) \subseteq \mathbb{TP}^{n-|I|}$.

The next two operations from matroid theory are related to tropical modifications.

Definition 1.1.32. Let $M = (E, r)$ be a matroid,

1. The deletion with respect to $e \in E$, $M \backslash e$ is the matroid $(E \backslash e, r|_{E \backslash e})$.

2. The restriction with respect to $e \in E$, $M|e$ is the matroid $(E \backslash e, r')$ where $r'(I) = r(I \cup e) - r(e)$.

3. A matroid Q is a elementary quotient of M if there exists a matroid N on a ground set $E' = E \cup e'$ such that $N \backslash e' = M$ and $N|e' = Q$, and N is called a elementary extension of M.

Again if $M_{\mathcal{A}}$ is the matroid arising from a hyperplane arrangement we can interpret the above operations geometrically. $M_{\mathcal{A}} \backslash i$ corresponds to the arrangement given by removing the e^{th} hyperplane,

$$\mathcal{A}' = \mathcal{A} \backslash H_e,$$

and the restriction $M_{\mathcal{A}}|e$ corresponds to the induced arrangement on H_e,

$$\mathcal{A}'' = \{H_i \cap H_e\}_{i \neq e}.$$

For more on this see Section 1 of [41].

We can actually perform deletions and restrictions with respect to a subset $I \subset E$, these will be denoted $M\backslash I$ and $M|I$ respectively. Also Q will be called a quotient of M if there is a matroid N with ground set $E \cup F$ such that $N\backslash F = M$ and $N|F = Q$, and N will simply be called an extension. By the following lemma all quotients of a matroid M can be represented geometrically as Bergman fans which are subfans of $B(M)$ considered with the fine subdivision.

Lemma 1.1.33. *Q is a quotient of M if and only if $B(Q) \subseteq B(M)$.*

Proof. We may assume M is loopless and that Q is a single element quotient of M, since every quotient can be formed by a sequence of single element quotients. Moreover, by Proposition 7.3.6 of [44] Q is a quotient of M if and only if $\Lambda_Q \subseteq \Lambda_M$. So supposing Q is loopless, the lemma follows immediately from the above statement and the construction of $B(M)$ in terms of the lattice of flats. If Q contains loops $L \subset E$, then $B(Q)$ is contained in the boundary stratum of $\mathbb{TP}^n$ corresponding to $x_l = -\infty$ for all $l \in I$. A face of $B(Q)$ corresponding to a chain of flats $I = F_0 \subset F_1 \subset \cdots \subset F_s \neq \{0, \dots, n\}$ of Λ_Q, is contained in the boundary of $B(M)$ if and only if the same chain is a chain in Λ_M and the lemma is proved. □

We have the following simple proposition relating tropical modifications, contractions and divisors to matroid extensions, deletions and restrictions, respectively. Recall that an element $i \in E$ is a **coloop** of a matroid $M = (E, r)$ if i is contained in every basis of M, i.e. $i \in B$ for every $B \subset E$ for which $r(B) = |B| = r(E)$. If a matroid M contains m coloops then the corresponding Bergman fan $B(M) \subset \mathbb{TP}^n$ contains an m dimensional subspace of $\mathbb{R}^n$.

Proposition 1.1.34. *Let M be a rank $k+1$ matroid on the ground set $E = \{0, \dots, n\}$. Suppose $i \in E$ is neither a loop nor a co-loop, then in every chart $U_j = \{x \in \mathbb{TP}^n \mid x_j \neq -\infty\} = \mathbb{T}^n \subset \mathbb{TP}^n$ there is an elementary tropical modification*

$$\delta_j : B(M) \cap U_j \longrightarrow B(M\backslash i) \cap U_j$$

with corresponding divisor $B(M/i) \cap U_j$.

Proof. For the lattice of flats of deletions and restrictions we have:

$$\Lambda_{M\backslash i} = \{F \subseteq E\backslash i \mid F \text{ or } F \cup i \text{ is a flat of M}\}$$
$$\Lambda_{M/i} = \{F \subseteq E\backslash i \mid F \cup i \text{ is a flat of M}\}.$$

Let $\delta_i : \mathbb{T}^n \longrightarrow \mathbb{T}^{n-1}$ be the projection in the direction of e_i. Then the image under δ_i of a k-dimensional cone of $B(M) \cap U_j$ corresponding to a chain of flats $F_1 \subset \cdots \subset F_k$

is still a k dimensional cone if and only if $i \notin F_k$. In other words, if and only if the corresponding chain is a chain of flats of $\Lambda_{M\backslash i}$. Therefore, we have

$$\delta(B(M) \cap U_j) = B(M\backslash i) \cap U_j',$$

where U_j' is a chart of $\mathbb{T}^{n-1}$. In addition, δ contracts a k-dimensional face of $B(M) \cap U_j$ if and only if $i \in F_k$. Thus the image of all contracted faces is exactly $B(M/i) \cap U_j' \subset B(M\backslash i) \cap U_j'$.

By the next lemma the codimension one cycle $B(M/i) \cap U_j'$ must be the divisor of a tropical rational function f on $B(M\backslash i) \cap U_j'$. Then up to tropical multiplication by a constant (addition) this function must satisfy

$$\Gamma_f(B(M\backslash i) \cap U_j') \subset B(M) \cap U_j$$

and so it must be the function of the modification δ. □

Lemma 1.1.35. *Let $B(M) \subset \mathbb{TP}^n$ be the Bergman fan of a matroid, and $V = B(M) \cap U_i \subset \mathbb{T}^n$ for some $i \in \{0, \dots, n\}$. If $D \subset V$ is a codimension one tropical subcycle then there exists a tropical rational function $f \in \mathcal{K}_{\mathbb{T}^n}$ such that $div_V(f) = D$.*

Proof. First suppose $V = \mathbb{T}^n$, and that D has order of sedentarity 0, then the statement is equivalent to showing that every codimension one cycle in $\mathbb{R}^n$ is the divisor of a tropical function $f \in \mathcal{K}_{\mathbb{R}^n}$. If D is effective, it is a tropical hypersurface and is given by a tropical polynomial by [33]. When D is not effective the following argument is due to an idea of Anders Jensen. Let D^- denote the collection of facets of D which have negative weights. For a face E in D^-, there exists a $v \in \mathbb{Z}^n$ and $a \in \mathbb{R}$ such that $< x, v >= a$ for all $x \in E$. Define a regular function $h_E : \mathbb{T}^n \longrightarrow \mathbb{T}$, by

$$h_E(x) = \max\{0, -w_E(< x, v > -a)\}$$

where $w_E < 0$ is the weight of E in D. The function h_E is given by the tropical polynomial $=$ "$ax^{-w_E v} + 1_{\mathbb{T}}$", and $\mathrm{div}_{\mathbb{T}^n}(h_E)$ is an affine hyperplane containing E and equipped with positive weight $-w_E$. Let $h : \mathbb{T}^n \longrightarrow \mathbb{T}$ be given by $h(x) = \sum_{E \in D^-} h_E(x)$, this corresponds to the tropical product of the tropical polynomials, so h is again a tropical polynomial. Moreover, $D + \mathrm{div}_{\mathbb{T}^n}(h)$ is an effective cycle of order of sedentarity zero in $\mathbb{T}^n$, and thus is the divisor of a tropical polynomial f. By part (3) of Proposition 1.1.24, $D = \mathrm{div}_{\mathbb{T}^n}(f - h)$, and the difference $f - h$ is a tropical rational function.

Now suppose the fan $V \subset \mathbb{T}^n$ is the closure of a k-dimensional affine subspace in $\mathbb{R}^n$ and $D \subset V$ a codimension one cycle. Therefore, the matroid corresponding to V consists of $k + 1$ coloops. There is a unique surjective linear projection $\delta : V \longrightarrow \mathbb{T}^k$ with kernel generated by standard basis directions. The image $\delta(D) \subset \mathbb{T}^k$ is isomorphic to D as an integral polyhedral complex. Moreover equipped with the weights from D, $\delta(D)$ is a balanced codimension one cycle in $\mathbb{T}^k$. Therefore, it is the divisor of a tropical rational function f on $\mathbb{T}^k$. Let $\tilde{f}$ be the pullback of this function to $\mathbb{T}^n$. It is again a tropical rational function and we have $\mathrm{div}_V(\tilde{f}) = D$.

To prove the general case we use a two step induction argument, first on the codimension of the fan V and secondly on $m = k - c(V)$ where $c(V)$ is the maximal dimension of an affine space contained in V. The base case of the two inductions were covered above, namely when $V = \mathbb{T}^n$ and when $V = \mathbb{T}^k \subset \mathbb{T}^n$. Again, we restrict to the case when D is of sedentarity order 0. First, take a linear projection $\delta : \mathbb{T}^n \longrightarrow \mathbb{T}^{n-1}$ with kernel generated by u_i where i is not a coloop of the matroid M corresponding to V. Then $\delta(V) = V'$ where V' is also a k-dimensional matroidal fan corresponding to the matroid $M\backslash i$ and let $D' \subset V'$ be the matroidal fan corresponding to M/i. By induction on the codimension of the matroidal fan there exists a rational function f on $\mathbb{T}^{n-1}$ such that $\text{div}_{V'}(f) = \delta_* D$. From f we obtain a function $\tilde{f}$ on $\mathbb{T}^n$ which is constant in the variable x_i, therefore $\tilde{f}$ is a rational function on $\mathbb{T}^n$. Moreover, $\text{div}_{\mathbb{T}^n}(\tilde{f}) = \text{div}_{\mathbb{T}^{n-1}}(f) \times \mathbb{T} \subset \mathbb{T}^n$ and $\text{div}_V(\tilde{f}) = \delta^*\delta_* D$.

We will show that $\delta^*\delta_* D - D$ is also the divisor of some rational function g on V, therefore $D = \text{div}_V(“\tilde{f}/g”)$ by part (3) of Proposition 1.1.24 and this will finish the proof. This difference, $\Delta_D = \delta^*\delta_* D - D$ is a cycle contained in the closure of the faces of V which get contracted under the projection δ, let us call the union of these faces the undergraph $\mathcal{U}$ of V as in Construction 1.1.15. Therefore, it may also be considered as a cycle in the matroidal fan $D' \times \mathbb{T} \subset \mathbb{T}^n$. By induction on $m = k - c(V)$ there exists a rational function h on $\mathbb{T}^n$ such that $\text{div}_{D'\times\mathbb{T}}(h) = \Delta_D$. Now let us restrict the function h to V. If $\text{div}_V(h) = \Delta_D$ we are done. Otherwise the cycle which is the difference of these two must be contained in $\overline{V\backslash\mathcal{U}}$, since $\text{div}_V(h)$ agrees with $\text{div}_{D'\times\mathbb{T}}(h)$ on the undergraph. Therefore, $\delta^*\delta_*(\text{div}_V(h) - \Delta_D) = \text{div}_V(h) - \Delta_D$. Again by induction on the codimension of the matroidal fan, the cycle $\delta_*(\text{div}_V(h) - \Delta_D)$ is given by a rational function h' on V'. Similarly as above h' also gives a function $\tilde{h}'$ on $\mathbb{T}^n$ and $\text{div}_V(\tilde{h}') = \text{div}_V(h) - \Delta_D$. Therefore, taking $g = h - \tilde{h}'$ we have $\text{div}_V(g) = \Delta_D$. This completes the proof when D is of sedentarity order 0.

If D does not have order of sedentarity 0 then D decomposes into a sum of cycles of sedentarity order 0 and order 1 because of its codimension, so it suffices to prove the claim when D is of sedentarity i. Then D is contained in the boundary hyperplane of $\mathbb{T}^n$ corresponding to $x_i = -\infty$ and equipped with an integer weight w. Then D is the divisor of the Laurent monomial x_i^w. This completes the proof. □

In fact even when i is a loop Proposition 1.1.34 holds, but in a particular sense where the function on $B(M\backslash i)$ producing the modification is the constant function $f = -\infty$. The divisor of such a function is all of $B(M\backslash i)$ which is equal to $B(M/i)$, if i is a loop.

Corollary 1.1.36. *Given a k-dimensional Bergman fan $B(M) \subset \mathbb{TP}^n$, every contraction $\delta : B(M) \longrightarrow \mathbb{TP}^k$ corresponds to a choice of basis of M.*

Proof. Given a basis B of M the deletion $M\backslash B^c$ produces the uniform matroid $U_{k+1,k+1}$ corresponding to $\mathbb{TP}^k$. If we delete along a set which is not the complement

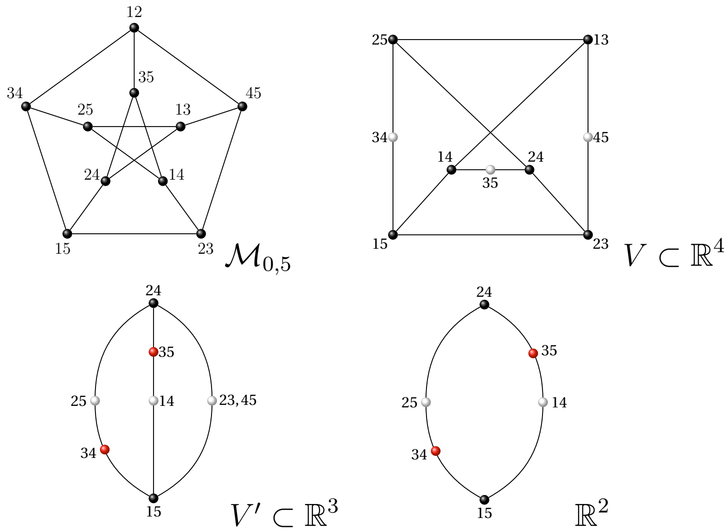


Figure 1.8: The link about the origin (or Bergman complex [55]) of the sequences of modifications producing $\mathcal{M}_{0,5}$. The divisor at each step is marked in white, the ij indicate the cone of $\mathcal{M}_{0,5}$ corresponding to a trivalent curve, see [39], [23] .

of some basis then we decrease the rank of the matroid, meaning at some step we deleted a coloop. This does not correspond to a tropical contraction. □

The next example presents $\mathcal{M}_{0,5}^{trop}$ as a composition of elementary open matroidal modifications. It is not possible to find a sequence of regular elementary matroidal modifications.

Example 1.1.37
The embedding of the moduli space of tropical rational curves with 5 marked points, $\mathcal{M}_{0,5}^{trop}$ into $\mathbb{R}^5$ (see [39], [54], [23]) is the first example of a realisable fan not obtained by a sequence of modifications along regular functions. The plane $\tilde{P} \subset \mathbb{T}^4$ can be obtained from a different sequence of modifications which are both regular. It was shown in [3] that $\mathcal{M}_{0,n}^{trop}$ corresponds to the Bergman fan of the complete graphical matroid K_{n-1}. Tropical contractions of $\mathcal{M}_{0,n}^{trop}$ correspond to the deletion of an edge of K_{n-1}. So the very first elementary tropical contraction of $\mathcal{M}_{0,5}^{trop}$ is unique by the symmetry of K_4. The link of singularity of the fans obtained by a series of elementary contractions starting from $\mathcal{M}_{0,5}^{trop}$ and finishing at $\mathbb{R}^2$ are drawn in Figure 1.8, with the divisors of each modification marked in white. In Example 3.4.18, it will be shown that the corresponding divisor of this contraction cannot be the divisor of a regular function on $\mathbb{R}^4$ restricted to V by showing that its tropical self intersection is not effective.

Definition 1.1.38. Two matroids $M = (E, r)$ and $M' = (E', r')$ are isomorphic if there is a bijection $f : E \longrightarrow E'$ such that $r(S) = r'(f(S))$.

Lemma 1.1.39. *Let M and M' be loopless matroids.*

1. *If $M \cong M'$, then there is a $T \in GL_n(\mathbb{Z})$ such that $\bar{T}(B(M)) = B(M')$ where $n = |E| = |E'|$, and $\bar{T}$ is the extension of T to $\mathbb{TP}^n$.*

2. *If $i, j \in E$ are double points of M, then $B(M)$ and $B(M\backslash i)$ are isomorphic fans, (i.e there is a one-to-one correspondence between cones), moreover the linear projection with kernel generated by e_i is invertible on $B(M\backslash i)$.*

Proof. Given a matroid isomorphism $f : M \longrightarrow M'$, it also maps L to L' their respective lattices of flats. Define $T(e_i) = e_{f(i)}$, it is an isomorphism since f is a bijection of sets. By the construction of the Bergman fan of a matroid via the lattice of flats, it is clear that $T(B(M)) = B(M')$.

If i, j are double points of M, then j is a loop of $M|i$. The tropical modification $\delta : B(M) \longrightarrow B(M\backslash i)$ is along a integer affine function f, because the divisor $B(M/i) \subset B(M\backslash i)$ is of positive sedentarity. The graph of f restricted to $B(M\backslash i)$ and δ give the homeomorphism. Combined with part 1 we have the result for an isomorphic matroid M'. $\square$

Corollary 1.1.40. *Given a loopless matroid M, the restriction matroid M/i contains loops if and only if in every chart U_j the function f_j corresponding to the modification $\delta_j : B(M) \cap U_j \longrightarrow B(M\backslash i) \cap U_j$ is the extension to $\mathbb{T}^n$ of an affine linear function on $\mathbb{R}^n$.*

In this case the linear projection in a chart $\delta : B(M) \cap U_j \longrightarrow B(M\backslash i) \cap U_j$ is invertible on the fans, and can be extended to the entire fans. Therefore, $B(M)$ and $B(M\backslash i)$ are isomorphic as polyhedral complexes.

From now on to simplify notation we will drop the use of $B(M)$ and insist that a fan be matroidal, we will only recall the underlying matroid when necessary.

Definition 1.1.41. A tropical cycle $V \subset \mathbb{T}^n$ (similarly $V \subset \mathbb{R}^n$) is a matroidal fan if there exists a matroid M on $n+1$ elements such that $|V| = |B(M) \cap U_i|$ for some coordinate chart $U_i = \{x_i \neq -\infty\} \subset \mathbb{TP}^n$, (or $|V| = |B(M) \cap \mathbb{R}^n|$).

Remark Notice that the above definition forgets the polyhedral structure of both V and $B(M)$ and considers just the support of the complexes. It is known that up to integer linear transformations the underlying matroid may not be unique. More precisely, given a matroidal fan $V \subset \mathbb{R}^n$ and a map $\mathcal{M} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ which is in $GL_n(\mathbb{Z})$, it could happen that $\mathcal{M}(V) \subset \mathbb{T}^n$ is also matroidal but the corresponding matroid may not be isomorphic to the initial M. As matroidal fans will be considered as local building blocks of tropical manifolds, which matroids give equivalent fans up to this $GL_n(\mathbb{Z})$ action this is a question which requires investigation.

Definition 1.1.42. An elementary tropical modification $\delta : \tilde{V} \longrightarrow V$ is matroidal if the fans $\tilde{V}$, V and the divisor D are all matroidal (i.e. corresponds to a quotient of the matroid corresponding to V).

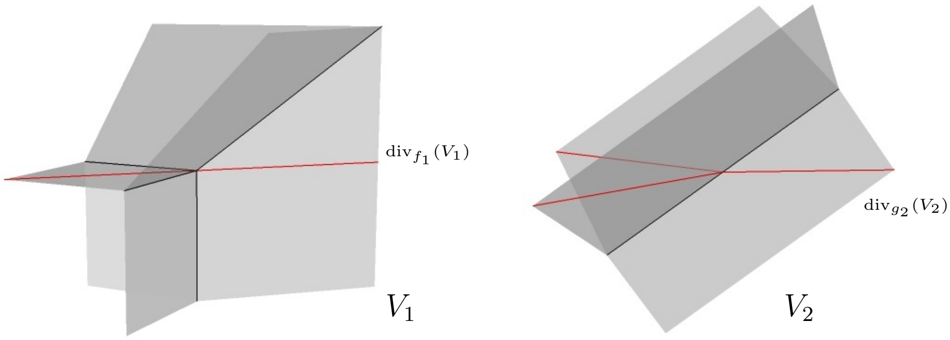


Figure 1.9: The two cycles V_1, $V_2 \subset \mathbb{R}^3$ from Example 1.1.44, the divisors $\text{div}_{f_1}(V_1) \subset V_1$, $\text{div}_{g_2}(V_2) \subset V_2$ are drawn in red in each case.

Recall the pushforward and pullback maps defined on cycles for an elementary tropical modification. The next proposition says that the maps are well-defined for compositions of elementary matroidal tropical modifications.

Proposition 1.1.43. *Given a matroidal modification $\delta : \tilde{V} \longrightarrow V$ the maps $\delta^* : Z_k(V) \longrightarrow Z_k(\tilde{V})$ and $\delta_* : Z_k(\tilde{V}) \longrightarrow Z_k(V)$ are group homomorphisms for all k, and $\delta_*\delta^* = id$.*

Proof. It was already mentioned that the pushforward and pullback maps are group homomorphisms when the modification is elementary. Therefore, we must only show that the maps δ_*, δ^* are well defined when we compose open elementary modifications. Moreover, it suffices to prove this for a cycle with order of sedentarity zero. This is because every cycle in $\mathbb{T}^n$ splits as sum of cycles of the different sedentarities, and an elementary matroidal modification restricted to a boundary stratum $\delta : \tilde{V} \cap H_I \longrightarrow V \cap H_I$, is a again a matroidal modification.

Suppose $\delta : \tilde{V} \longrightarrow V$ is the composition of two open matroidal modifications. Set $\delta_2 : \tilde{V} \longrightarrow V_2$ and $\tilde{\delta}_1 : V_2 \longrightarrow V$, so that $\tilde{\delta}_1\delta_2 = \delta$, and denote the other sequence of modifications by $\delta_1 : \tilde{V} \longrightarrow V_1$ and $\tilde{\delta}_2 : V_1 \longrightarrow V$, so that $\tilde{\delta}_2\delta_1 = \delta$, (see Example 1.1.44 for a case when the fans V_1 and V_2 differ). Without loss of generality we may suppose the kernels of $\delta_1, \delta_2 : \mathbb{R}^{n+2} \longrightarrow \mathbb{R}^{n+1}$ are generated by e_{n+1} and e_{n+2}, respectively. Then the maps $\tilde{\delta}_1$ and $\tilde{\delta}_2$ are linear projections $\mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$, with kernels e_{n+1} and e_{n+2} respectively.

For the pushforwards, the sets satisfy, $\tilde{\delta}_i\delta_j(A) = \tilde{\delta}_j\delta_i(A)$, since the δ_i's and $\tilde{\delta}_i$'s are orthogonal projections. Let $\mathcal{C}$ denote the closure of the collection of facets of C contracted by both δ_1, δ_2. If a facet F of A is outside of $\mathcal{C}$ then its contribution to the weight of $\delta(F) \subset \delta_*A$ is the same if we permute the order of contractions. So assume $F \subset \mathcal{C}$, then the lattice index may be rewritten as, $[\delta_{i*}\Lambda_F : \Lambda_{\delta_i(F)}] = [\mathbb{Z}^n : \Lambda_F + \Lambda_i^{\perp}]$ and F contributes a weight of,

$$w_A(F)[\mathbb{Z}^n : \Lambda_F + \Lambda_i^{\perp}][\mathbb{Z}^n : \Lambda_{\delta_1(F)} + \Lambda_j^{\perp}] = w_A(F)[\mathbb{Z}^n : \Lambda_F + \Lambda_i^{\perp} \cap \Lambda_j^{\perp}],$$

to $\delta(F)$. Which is independent of the order of contractions.

For the pullbacks, take a cycle A in V and let $\Gamma(A) \subset \mathbb{T}^{n+2}$ denote the graph of A along either pair of functions yielding the modification. Although the pairs of functions may differ, (see Example 1.1.44), the resulting graphs must be the same. Let $\tilde{A}$ denote the pullback of A along the composition $\tilde{\delta}_2\delta_1$ and $\tilde{A}'$ denote the pullback of A along the composition $\tilde{\delta}_1\delta_2$. Since $\tilde{A}$ and $\tilde{A}'$ are modifications of cycles in V the restriction of the linear projections δ_1 and δ_2 to $\tilde{A}$ and $\tilde{A}'$ are either one to one or send a half line to a point.

If E_A is an unbalanced codimension one face of $\Gamma(A)$, then it is unbalanced only in the e_{n+1} and e_{n+2} directions. First, if $\tilde{V}$ contains one or both of the faces:

$$\{x - te_{n+1} \mid x \in E_A \text{ and } t \in \mathbb{R}_{\geq 0}\},\ \{x - te_{n+2} \mid x \in E_A \text{ and } t \in \mathbb{R}_{\geq 0}\},$$

then these are the only facets of $\tilde{A}$ adjacent to E_A and not contained in $\Gamma(A)$, and similarly for $\tilde{A}'$. The balancing condition at E_A guarantees that the weights are the same, (remark that if $\Gamma(A)$ is already balanced in one of these directions then we do not need to add the corresponding facet).

For an unbalanced codimension one face E_A of $\Gamma(A)$ suppose the above faces do not exist. Then there is a single facet $F_{\tilde{A}}$ of $\tilde{A}$ adjacent to E_A and not in $\Gamma(A)$. Otherwise the projections δ_1 and δ_2 restricted to $\tilde{A}$ would have a finite fiber of size at least two. The same holds for $\tilde{A}'$, whose single face satisfying these conditions we call $F_{\tilde{A}'}$. Now $\tilde{A} - \tilde{A}'$ must be balanced at E_A and so the faces $F_{\tilde{A}}$ and $F_{\tilde{A}'}$ are the same and equipped with the same weights.

In this case there may be codimension one faces of $F_{\tilde{A}}$ at which there are other facets of $\tilde{A}$ adjacent. This occurs when the divisor $D_1 \subset V_1$ of the modification δ_1 is contained in the undergraph of the modification $\tilde{\delta}_2$ and $\tilde{\delta}_2^*A$ and in addition intersects $D_1 \subset V_1$ in some codimension one face. Call the resulting codimension one face G_A of $F_{\tilde{A}} \subset \tilde{A}$. Then G_A is contained in the skeleton of $\tilde{V}$ and it is also a face of $F'_{\tilde{A}} \subset \tilde{A}'$. If the cycles are unbalanced at G_A the other facets adjacent to it in $\tilde{A}$ and $\tilde{A}'$ must be:

$$\{x - te_{n+1} \mid x \in G_A \text{ and } t \in \mathbb{R}_{\geq 0}\},\ \{x - te_{n+2} \mid x \in G_A \text{ and } t \in \mathbb{R}_{\geq 0}\},$$

otherwise the projections δ_1, δ_2 would have a finite fiber of size greater than one. Again, by the balancing condition the weights of these faces in $\tilde{A}$ and $\tilde{A}'$ agree. □

The following example shows a composition of open matroidal modifications for which the intermediary fans V_1, V_2 appearing in the proof above are not the same.

Example 1.1.44
Consider the fan $V \subset \mathbb{R}^4$ obtained from $\mathbb{R}^2$ via two elementary open modifications, δ_1, δ_2. The first modification is along the function $f_2(x, y) = \max\{x, y, 0\}$ and yields the cycle $V_1 \subset \mathbb{R}^3$ shown on the left of Figure 1.9. The next modification is taken along the function $f_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}$ given by

$$f_1(x, y, z) = \max\{x, y\} + \max\{z, 0\} - \max\{x, y, z, 0\}.$$

This modification was seen already in Example 1.1.20. It may be verified that the following different sequence of modifications yields the same fan, $V \subset \mathbb{R}^4$, after a change of coordinates. If one first modifies $\mathbb{R}^2$ along the function $g_1(x,y) = \max\{x,y\}$, to obtain a cycle $C_2 \subset \mathbb{R}^3$, see the right hand side of Figure 1.9. Next, modify V_2 along the function $g_2 : \mathbb{R}^3 \longrightarrow \mathbb{R}$, given by $g_2(x,y,z) = \max\{z,0\}$. Notice on the one hand V is produced by a composition of two elementary regular modifications, and on the other by an elementary regular modification composed with an elementary modification along a rational function.

1.2 Tropical manifolds

1.2.1 Definitions

An integer affine map $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a composition of an integer linear map $\mathbb{Z}$ and a translation in $\mathbb{R}^m$. In fact such a map can be given by m tropical monomials ie. $(``a_1 \cdot z^{\alpha_1}", \cdots, ``a_m \cdot z^{\alpha_m}")$, where $(a_1, \cdots a_m) \in \mathbb{R}^m$ gives the translation and together the $\alpha_i \in \mathbb{Z}^n$ form an integer $m \times n$ matrix. Any integer affine map can be extended to $\Phi : \mathbb{T}^n \longrightarrow \mathbb{T}^m$.

A tropical manifold has coordinate changes integer affine maps and local models matroidal fans $V \subset \mathbb{T}^n$, introduced in the last subsection. We will always restrict to matroidal fans which correspond to matroids without loops or double points. The following definition is an adaptation of the one given of a tropical variety in [38].

Definition 1.2.1. A n-dimensional tropical manifold X is a Hausdorff topological space equipped with an atlas of charts $\{U_\alpha, \Phi_\alpha\}$, $\Phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{T}^n_\alpha$, such that the following hold

1. for every α there is a map $\Phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{T}^{N_\alpha}$, where V_α is n-dimensional, matroidal, **loopless** and **without double points**, such that Φ_α is a homeomorphism onto its image.

2. the overlapping maps $\Phi_{\alpha_1} \circ \Phi^{-1}_{\alpha_2} : \mathbb{T}^{N_{\alpha_2}} \to \mathbb{T}^{N_{\alpha_1}}$ are extensions of integer affine linear maps $\mathbb{R}^{N_{\alpha_2}} \longrightarrow \mathbb{R}^{N_{\alpha_1}}$.

3. X is of finite type, i.e. there is a finite collection of open sets $\{W_i\}_{i=1}^s$ such that $\bigcup_{i=1}^n W_i = X$ and $W_i \subset U_\alpha$ for some α and $\overline{\Phi_\alpha(W_i)} \subset \Phi_\alpha(U_\alpha) \subset \mathbb{T}^{N_\alpha}$.

The main difference between the above definition and that of a tropical variety appearing in [38] is that we insist that the spaces be locally matroidal, this also accounts for the removal of condition (3) appearing in [38].

Just as with smooth manifolds, we say two atlas $\{U_\alpha, \Phi_\alpha\}$, $\{U'_\beta, \Phi'_\beta\}$ on X are equivalent if their union is also an atlas, just as for smooth manifolds. Meaning, $\Phi'_\beta \circ \Phi^{-1}_\alpha$ and $\Phi_\alpha \circ \Phi'^{-1}_\beta$ are transition maps for all α, β.

Example 1.2.2 Tropical toric varieties
An n-dimensional tropical toric manifold X has local charts $\Phi : U \longrightarrow \mathbb{T}^n$ and

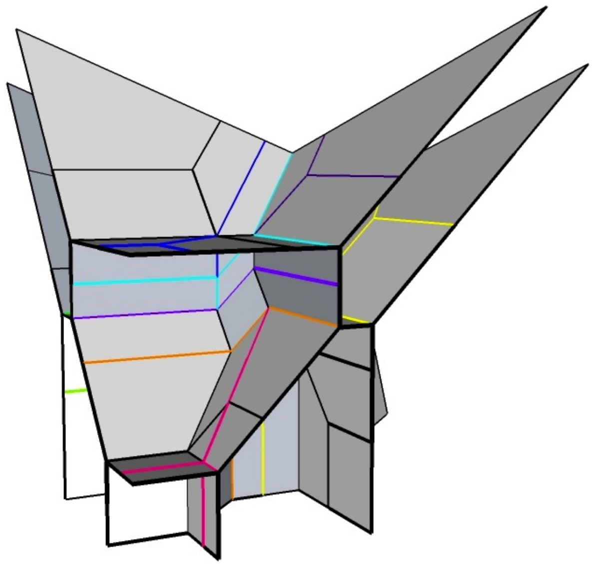


Figure 1.10: Quadric hypersurface in $\mathbb{TP}^3$.

coordinate changes given by integer linear maps. Topologically these are equivalent to the Delzant polytopes of toric manifolds from symplectic geometry, however they have a different metric as all sides of the polytope are of infinite length. Tropical projective space appeared already in the beginning of Section 1.1.4.

Example 1.2.3 Smooth tropical hypersurfaces of toric varieties
A tropical hypersurface in $\mathbb{R}^n$ is the divisors of tropical Laurent polynomials polyhedral complexes dual to regular subdivisions of lattice polytopes. If a subdivision is **primitive**, meaning each polytope in the subdivision has normalized volume equal to one, we say the hypersurface is non-singular and it is a tropical manifold. Indeed, a primitive polytope is the standard simplex up to a transformation in $GL_n(s\,\mathbb{Z})$, and dual to the standard simplex is the matroidal fan corresponding to $U_{n,n+1}$. Normally, tropical hypersurfaces are considered in $\mathbb{R}^n$, [36], [48], however we may also consider the closure of the hypersurfaces in tropical toric varieities mentioned above. In Figure 1.2.1 is a quadric hypersurface in $\mathbb{TP}^3$, with the different colored tropical curves representing some lines of the double ruling.

Example 1.2.4
Abstract tropical curves are metric graphs [38] and have appeared in many places. A neighborhood U of a k-vertex of a tropical curve C $k > 1$ is has a chart $\Phi : U \longrightarrow L_k \subset \mathbb{T}^{k-1}$ where $L_k \subset \mathbb{T}^{k-1}$ is a tropical line when $k > 1$. When $k = 1$ then the vertex is a leaf and we have $\Phi : U \longrightarrow \mathbb{T}$ which sends the vertex to $-\infty$. The integral affine transition charts give the metric on the curve C.

Example 1.2.5
As an example of a space which is not a tropical manifold, consider $\mathbb{T}\backslash\{a\}$, for

$a \in \mathbb{R}$. Equipped with the two charts $U_1 = [-\infty, a)$, $U_2 = (a, \infty)$ it is not of of finite type. In any finite cover $\{W_j\}$ there is a $W_i \subset U_i$, such that $a \in \overline{W_i}$. Similarly for any sedentarity zero cycle, $A \subset \mathbb{T}^n$ the complement $\mathbb{T}^n \setminus A$ cannot be given the structure of a tropical manifold. However, removing one of the boundary hyperplanes, e.g. from $\mathbb{T} \setminus -\infty = \mathbb{R}$ is a tropical manifold. One can use two charts to cover $\mathbb{R}$ and model each infinite open end on the open end of $\mathbb{T} = [-\infty, \infty)$. To remove the point a from $\mathbb{T}$, we may perform an **open tropical modification** along $f(x) =$ "$x + a$" $= \max\{x, a\}$, and obtain $\delta^o : L^o \longrightarrow \mathbb{T}$, where $L^o \subset \mathbb{T} \times \mathbb{R}$. Then L^o has a single trivalent vertex, where two of the three infinite rays are open.

1.2.2 Boundary divisors

For a tropical manifold, the sedentarity of a point as defined as a subset in Section 1.1.1 is not independent of the choice of charts. However, since we require the matroidal fans V_α to be loopless and without parallel elements, the order of sedentarity $s(x)$ is independent of the chosen charts.

Definition 1.2.6. For a point $x \in X$ of a tropical manifold, the order of sedentarity of x is $s(x) = |S(\Phi(x))|$, where $\Phi : U \longrightarrow V \subset \mathbb{T}^N$, is a chart in a neighborhood of x.

Some tropical manifolds have a boundary, these correspond to the points with positive order of sedentarity.

Definition 1.2.7. The boundary of a tropical manifold X is

$$\partial X = \{x \in X \mid s(x) > 0\}.$$

Proposition 1.2.8. *The boundary of an n-dimensional tropical manifold X satisfies*

$$\partial X = \overline{\{x \in X \mid s(x) = 1\}}$$

and $\{x \in X \mid s(x) = 1\}$ consists of a collection of connected components D^o, such that $D = \overline{D^o}$ is a tropical manifold of dimension $n - 1$.

Proof. In each chart $\Phi : U \longrightarrow V$ the matroid M corresponding to V has no multiple elements, so $\{x \in V \mid s(x) > 1\}$ has codimension at least two in V, therefore, $\partial V = \overline{\{v \in V \mid s(x) = 1\}}$. Then by Definitions 1.2.9 and 1.2.7 we have, $\partial X = \overline{\{x \in X \mid s(x) = 1\}}$. Let $\{U_\alpha, \Phi_\alpha\}$ be an atlas for X and set $U'_\alpha = U_\alpha \cap D$. If $U'_\alpha \neq \emptyset$, let $\Phi'_\alpha = \Phi_\alpha|_{U'_\alpha}$ then,

$$\Phi'_\alpha : U'_\alpha \longrightarrow V'_\alpha = V_\alpha \cap \{x \in \mathbb{T}^{N_\alpha} \mid x_j = -\infty\} = \mathbb{T}^{N_\alpha - 1}$$

for some $j = 1, \ldots, N_\alpha$. If M_α is the matroid corresponding to the fan V_α then the matroid restriction M_α/j corresponds to $V'_\alpha \subset \mathbb{T}^{N_\alpha - 1}$ and $\{U'_\alpha, \Phi'_\alpha \mid U'_\alpha \neq \emptyset\}$ gives an atlas for D. $\square$

A **boundary divisor** of a tropical manifold X is $D = \overline{D^o} \subset X$ where D^o is a connected component of the set $\{x \in X \mid s(x) = 1\}$. Every boundary divisor is of codimension one in X. Call the set of boundary divisors $\mathcal{A}$ the **arrangement of boundary divisors** of X. Using the boundary divisors we can define a chart independent sedentarity notion of sedentarity for a tropical manifold.

Definition 1.2.9. For a point $x \in X$ we define its sedentarity,

$$S(x) = \{D \mid x \in D \in \mathcal{A}\} \subset \mathcal{A}.$$

Remark that the set of points of a given sedentarity of a tropical manifold X are in general not connected, moreover the connected components may not even be of the same dimension.

Definition 1.2.10. An n-dimensional tropical manifold X has simple normal crossing boundary divisors if the connected components of the intersection $\cup_{i \in I} D_i$ are all of dimension $n - |I|$.

In particular, if X is n dimensional and has simple normal crossings, then a neighborhood U_x of a point x of sedentarity order n will have a local neighborhood and chart $\Phi_x : U_x \longrightarrow \mathbb{T}^n$.

1.2.3 Cycles and functions on manifolds

This section extends the definitions of tropical cycles and functions to manifolds.

Definition 1.2.11. [38] Let X be a tropical manifold, $A \subset X$ is a tropical k-cycle if in every chart $\Phi_\alpha : U_\alpha \longrightarrow V_\alpha \subset \mathbb{T}^{n_\alpha}$ there exists a k-cycle $A_\alpha \subset V_\alpha$ such that $\Phi_\alpha(A \cap U_\alpha) = A_\alpha \cap \Phi_\alpha(U_\alpha)$.

A cycle $A \subset X$ is said to be of sedentarity zero in X if it is the closure in X of a set $A^o \subset X$ of sedentarity $\emptyset$. If A is irreducible and not of sedentarity zero then it is contained in the intersection of some boundary divisors of X and can be assigned a sedentarity $S(A) \subset \mathcal{A}$ as for a point in Definition 1.2.9. As in Section 1.1.2 the set of k-cycles considered modulo zero weighted complexes forms a group which we will denote $Z_k(X)$.

Definition 1.2.12. s Let X be a tropical manifold, a function $f : X \longrightarrow \mathbb{T}$, is tropically regular if in each chart $\Phi_\alpha : U_\alpha \longrightarrow V_\alpha \subset \mathbb{T}^{N_\alpha}$ there is a tropical regular function $f_\alpha \in \mathcal{O}(V_\alpha)$ such that $f|_{U_\alpha} = f_\alpha \circ \Phi_\alpha$.

Definition 1.2.13. A tropical rational function f on X is locally given by the tropical quotient of two regular functions, i.e., $f|_{U_\alpha} =$ "$g_\alpha \circ \Phi_\alpha / h_\alpha \circ \Phi_\alpha$" where g_α, h_α are tropical polynomials on $\mathbb{T}^{N_\alpha}$ and $h_\alpha \neq 0_{\mathbb{T}} = -\infty$ such that "f_α / f_β" $= 1_{\mathbb{T}} = 0$ on $U_\alpha \cap U_\beta$.

Recall that a tropical rational function on $\mathbb{T}^n$ may have an indeterminacy locus of codimension two. The same is of course true for tropical rational functions on a manifold X. Analogous to classical algebraic geometry, there are tropical Cartier divisors. These have already been introduced in [38] and [2]. Although in the later the definitions again excluded boundary considerations.

Definition 1.2.14. A Cartier divisor of a tropical manifold X is a collection of functions $f = \{f_\alpha\}$, such that f_α is a rational function on $U_\alpha \subset X$, and "f_α/f_β" is regular and invertible on $U_\alpha \cap U_\beta$.

A Cartier divisor $f = \{f_\alpha\}$ on X defines a codimension one cycle $\text{div}(f) \subset X$. In each chart this divisor is given by $\text{div}_{V_\alpha}(f_\alpha \circ \Phi_\alpha) \subset V_\alpha$. So,

$$\text{div}(f) \cap U_\alpha = \Phi^{-1}(\text{div}_{V_\alpha}(f_\alpha \circ \Phi_\alpha)),$$

and the complexes agree on the overlaps $U_\alpha \cap U_\beta$ because the functions "f_α/f_β" are regular and invertible.

Definition 1.2.15. A Cartier divisor f on a tropical manifold X is effective if $\text{div}(f) \subset X$ is an effective cycle.

Note the difference with the above tropical definition and the classical case. Classically, an effective Cartier divisor is one given by regular functions. It is because of tropical examples like 1.1.20 that we choose the above definition over the classical one. The following proposition can be interpreted as the equivalence of Weil and Cartier divisors on tropical manifolds.

Proposition 1.2.16. *Every codimension one tropical cycle D in a tropical manifold X is a Cartier divisor.*

Proof. By Lemma 1.1.35, in each chart $\Phi_\alpha(D \cap U_\alpha) \subset V_\alpha$ is given by a rational function, call this function f_α. On an overlap $U_\alpha \cap U_\beta$ the functions "f_α/f_β" are regular and invertible because $\text{div}_{U_\alpha \cap U_\beta}(\text{“}f_\alpha/f_\beta\text{”}) = 0$. □

From [38] we recall the most basic type of morphism between tropical manifolds, these are called **tropical linear morphisms**.

Definition 1.2.17 ([38])**.** A map $f : X \longrightarrow Y$ is a tropical linear morphism if for every point $x \in X$ there is a neighborhood U_x of x and U_y of $y = f(x)$ with charts $\Phi_x : U_x \longrightarrow V_x \subset \mathbb{T}^{N_x}$, $\Phi_y : U_y \longrightarrow V_y \subset \mathbb{T}^{N_y}$ such that $\Phi_y \circ f \circ \Phi_x : V_x \subset \mathbb{T}^{N_x} \longrightarrow V_y \subset \mathbb{T}^{N_y}$ is induced by an integer affine map from $\mathbb{R}^{N_x} \longrightarrow \mathbb{R}^{N_y}$.

1.2.4 Non-singular tropical modifications

First we generalise modifications from Section 1.1.3 to tropical manifolds. Given an effective Cartier divisor f of a tropical manifold X, in each chart it is possible produce an elementary tropical modification $\delta_\alpha : \tilde{V}_\alpha \longrightarrow V_\alpha$ along f_α. If in each chart the elementary modification is matroidal then the $\tilde{V}_\alpha$ form a collection of charts of a tropical manifold. We denote this $\delta : \tilde{X} \longrightarrow X$ and call it a **non-singular modification** of X.

Definition 1.2.18. A tropical linear morphism $\delta : \tilde{X} \longrightarrow X$ is an elementary non-singular modification of tropical manifolds, if in borrowing from notation of Definition 1.2.17, at every point $x \in X$ the composition $\Phi_y \circ \delta \circ \Phi_x : V_x \subset \mathbb{T}^{N_x} \longrightarrow V_y \subset \mathbb{T}^{N_y}$ is an elementary matroidal modification.

As in the local case of matroidal fans, a non-singular elementary modification is **regular** if it is the modification along a regular Cartier divisor $f \in \mathcal{O}_Y$. A tropical linear morphism $\delta : \tilde{X} \longrightarrow X$ is a **non-singular modification** if it is a composition of elementary non-singular modifications. When $\delta : \tilde{X} \longrightarrow X$ is an elementary non-singular modification of tropical manifolds, then although topologically different, $\tilde{X}$ should be considered as another model of X.

Definition 1.2.19. An effective tropical k-cycle $A \subset X$ is non-singular in X if there exists an atlas of X with charts $\Phi_\alpha : U_\alpha \longrightarrow V_\alpha$, such that $\Phi_\alpha(A \cap U_\alpha)$ corresponds to a matroid quotient of the matroid corresponding to V_α.

If f is the Cartier divisor given a non-singular modification of $\delta : \tilde{X} \longrightarrow X$, then $\text{div}(f) \subset X$ is a non-singular codimension one cycle. The next corollary follows from Proposition 1.2.16.

Corollary 1.2.20. *Given an effective codimension one non-singular cycle $D \subset X$, there exists an elementary modification of tropical manifolds $\delta : \tilde{X} \longrightarrow X$ along a Cartier divisor f with $div(f) = D$.*

Example 1.2.21 Compactifications of matroidal fans
An n-dimensional matroidal fan $V \subset \mathbb{TP}^N$ as seen in Section 1.1.4 is a global modification of $\mathbb{TP}^n$. If the corresponding matroid is without loops or double points then V has $N+1$ boundary divisors each of which is a matroidal fan of dimension $n-1$ corresponding to the matroid $M \backslash i$ for $0 \leq i \leq N$. The matroidal fan $V \cap \mathbb{R}^N$ may be compactified in other tropical toric varieties to obtain different spaces to obtain for example, $\overline{\mathcal{M}}_{0,n}^{trop}$ [28]. Although the approach to tropical compactifications is not the same in the above mentioned works it may be easily translated. Moreover, global modifications provide a way of representing the complement of a divisor as a tropical manifold. As mentioned in Example 1.2.5 the complement of a sedentarity zero divisor is not a tropical manifold. If it is possible to perform a modification sending this divisor to a boundary divisor, we may then remove it. Matroidal fans $V \cap \mathbb{R}^N$ being obtained via matroidal modifications and representing the tropicalisation of the complement of the hyperplane arrangement is an example of this.

In general it is tough to determine sequences of modifications relating equivalent spaces. Figure 1.2.1 shows a quadric surface S and $\mathbb{TP}^1 \times \mathbb{TP}^1$. Although there does not exist an elementary contraction $\delta : S \longrightarrow \mathbb{TP}^1 \times \mathbb{TP}^1$, there is another tropical manifold X and maps $X \longrightarrow S$, $X \longrightarrow \mathbb{TP}^1 \times \mathbb{TP}^1$ which are both sequences of modifications.

Recall the pullback and pushforward cycle maps defined for modifications in $\mathbb{T}^n$, in Definition 1.1.22. These definitions extend naturally to a non-singular tropical modification of manifolds $\delta : \tilde{X} \longrightarrow X$. The **pushforward** will be denoted

$$\delta_* : Z_k(\tilde{X}) \longrightarrow Z_k(X),$$

and the **pullback**

$$\delta^* : Z_k(X) \longrightarrow Z_k(\tilde{X}).$$

The boundary divisors of a non-singular modification $\tilde{X}$ of X are related to the boundary divisors of X.

Lemma 1.2.22. *Let $\delta : \tilde{X} \longrightarrow X$ be an elementary non-singular tropical modfication along Cartier divisor f with div$(f) = D \subset X$. Then the boundary divisors of $\tilde{X}$ are*

$$\mathcal{A}_{\tilde{X}} = \{\delta^* D_i \mid D_i \subset \mathcal{A}_X\} \cup \tilde{D}.$$

where $\mathcal{A}_X$ are the boundary divisors of X and $\delta_* \tilde{D} = D$.

We end this section with some remarks on possibilities for a tropical category. Tropical manifolds along with tropical linear morphisms do form a category, call it the pre-tropical category **pTrop**. In **pTrop** it would be desirable to *localise*, in the categorical sense, all the morphisms which correspond to non-singular modifications $\delta : \tilde{X} \longrightarrow X$. See for example [21] for details of localising categories. This would in turn make all tropical manifolds related by non-singular modification isomorphic. In the next sections, when we define intersection products and rational equivalence we make an effort to show where possible that these definitions behave well with respect to modifications. See for examples, Propositions 2.1.24 and 2.1.27.

Chapter 2

Intersection theory

2.1 Intersections of tropical cycles

The principle aim of this section is to describe intersections of cycles in tropical manifolds. In many situations such products may be determined "locally", similar in style to tropical stable intersection of cycles in $\mathbb{R}^n$.

As a first step we show how to intersect cycles contained in matroidal fans in $\mathbb{R}^N$. The method used here to construct this product is similar in spirit to *moving lemmas* from classical algebraic geometry. This one approach to classical intersection theory begins with a notion of equivalence of cycles (such as rational equivalence), then given two cycles $X, Y \subset W$, one shows that there exists a class X' rationally equivalent to X which intersects Y properly. Naively speaking, many tropical cycles contained in a matroidal fan may not "move" on their own, recall for Example 1.1.20. The idea is to construct a procedure which allows us to "split", instead of move, the tropical cycles into a sum in such a way that the intersection product on the components may be defined. The technique used here to construct this splitting comes from tropical modifications. Just as in the case of stable intersection, the product is defined on the level of cycles and there is no need to pass to equivalence classes.

This intersection product defined on cycles in matroidal fans can be transferred to sedentarity zero cycles intersecting away from the boundary in a tropical manifold. However, dealing with boundary intersections is more delicate and cannot always be "locally" be determined. In Subsection 2.1.2, we define intersections with boundary divisors of a tropical manifold in certain cases by extending Definition 1.1.11 which gives the intersection of a cycle with a coordinate hyperplane of $\mathbb{T}^N$. Later in this subsection we use this to define rational equivalence in a tropical manifold and the Chow groups. This is a suitable definition of tropical rational equivalence for compact spaces, the definition of which first appeared in [38]. It should be noted that a "bounded" version of tropical rational equivalence also appears in [2] where the authors consider non-compact spaces. We also show that the tropical Chow groups are preserved under global modifications of tropical manifolds.

At the end of this chapter we summarise the definitions of tropical (p, q)-homology from [27]. This first requires an introduction of the "framing" groups which are related to Orlik-Solomon algebras of matroids. Under the appropriate transversality assumptions we may define intersections of such tropical homology cycles. These definitions will serve us in Chapter 3 when we consider $(1,1)$-cycles on surfaces.

2.1.1 Intersections in matroidal fans

In this section we intersect tropical subcycles of an open matroidal fan $V \subset \mathbb{R}^n$, so throughout we restrict our attention to open matroidal tropical modifications. Set $\dim(V) = k, \dim(A) = m_1, \dim(B) = m_2$, and the expected dimension of intersection of A and B to be $m = m_1 + m_2 - k$. Also for any complex C whose support is contained in V, let $C^{(s)}$ denote the s-dimensional skeleton of C with respect to the refinement induced by the inclusion to V.

Definition 2.1.1. Let $V \subset \mathbb{R}^n$ be a matroidal fan and $A, B \subset V$ be subcycles.

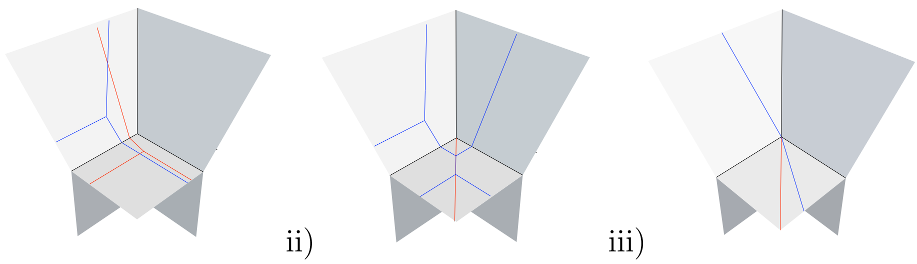
1. $A \cap B$ is proper in V if $A \cap B$ is of pure dimension m or is empty.

2. $A \cap B$ is weakly transverse in V if every facet of $(A \cap B)^{(m)}$ is in the interior of a facet of V.

3. $A \cap B$ is transverse in V if it is proper, weakly transverse and every facet of $A \cap B$ comes from facets of A and B intersecting transversely.

Example 2.1.2
The standard hyperplane $P \subset \mathbb{R}^3$ was shown in Figure 1.3, it is obtained by modifying $\mathbb{R}^2$ along the standard tropical line. Let A be the sub-cycle parameterized by $(t, t, 0)$ and B be the union of the positive span of the rays $(0, 1, 1)$, $(1-d, -d, 0)$, $(d-1, d-1, -1)$, see Figure 2.2.

The curves A and B intersect only at the vertex p of the fan. This intersection is proper but not weakly transverse. Moreover both cycles are rigid in P, meaning they cannot be moved in P by a translation. Consider the contraction $\delta : P \longrightarrow \mathbb{R}^2$,



i) ii) iii)

Figure 2.1: Cycles in the standard hyperplane in $\mathbb{R}^3$. i) Transverse intersection ii) Weakly transfer intersection iii) Neither

given by projecting in the e_3 direction. Set $\Delta_A = \delta^*\delta_*A - A$ and $\Delta_B = \delta^*\delta_*B - B$. An intersection product should of course be distributive, so we ought to have,

$$\begin{aligned} A.B &= (\delta^*\delta_*A - \Delta_A).(\delta^*\delta_*B - \Delta_B) \\ &= \delta^*\delta_*A.\delta^*\delta_*B - \delta^*\delta_*A.\Delta_B - \Delta_A.\delta^*\delta_*B + \Delta_A.\Delta_B \end{aligned}$$

Now, the cycles $\delta^*\delta_*A, \delta^*\delta_*B$ are free to move in P in the same way that δ_*A, δ_*B are free to move in $\mathbb{R}^2$. By translating $\delta^*\delta_*A, \delta^*\delta_*B$ until they intersect transversally and then translating back we can associate the weight,

$$w_{\delta^*\delta_*A.\delta^*\delta_*B}(p) = 1 = w_{\delta_*A.\delta_*B}(\delta(p)).$$

The cycles Δ_A, Δ_B are contained in the undergraph of the modification, see Figure 2.4, and are free to move in this direction. Also the cycle $\delta^*\delta_*A$ restricted to the undergraph is just $\text{div}_A(f) \times \mathbb{R}$, and similarly for $\delta^*\delta_*B$.

Now the cycles Δ_A, Δ_B may be moved by a translation into a single facet of P, see Figure 2.4. We can calculate

$$w_{\delta^*\delta_*A.\Delta_B}(p) = w_{\Delta_A.\delta^*\delta_*B}(p) = 0,$$

and

$$w_{\Delta_A.\Delta_B}(p) = 1 - d.$$

Combining all of these we obtain:

$$\begin{aligned} w_{A.B}(p) =& w_{\delta^*\delta_*A.\delta^*\delta_*B}(p) - w_{\delta^*\delta_*A.\Delta_B}(p) \\ &- w_{\Delta_A.\delta^*\delta_*B}(p) + w_{\Delta_A.\Delta_B}(p) = -d + 2. \end{aligned}$$

Our aim is to obtain a general procedure to split cycles contained in a matroidal fan V in a way so that they may be intersected. To do this we first need some technical definitions and lemmas.

Definition 2.1.3. Given $\delta : V \longrightarrow V'$ an elementary open matroidal modification, let f denote the corresponding tropical rational function and D its divisor. Let $A \subset V$ a be cycle, then denote:

1. $\Delta_A = \delta^*\delta_*A - A$.
2. $D_A = \text{div}_{\delta_*A}(f) \times \mathbb{R} \subset D \times \mathbb{R}$.

Lemma 2.1.4. *Given $\delta : V \longrightarrow V'$ an elementary open matroidal modification, let f denote the corresponding tropical rational function and D its divisor. Let A, B be subcycles of V then,*

1. $\Delta_{A+B} = \Delta_A + \Delta_B$
2. $D_{A+B} = D_A + D_B$.

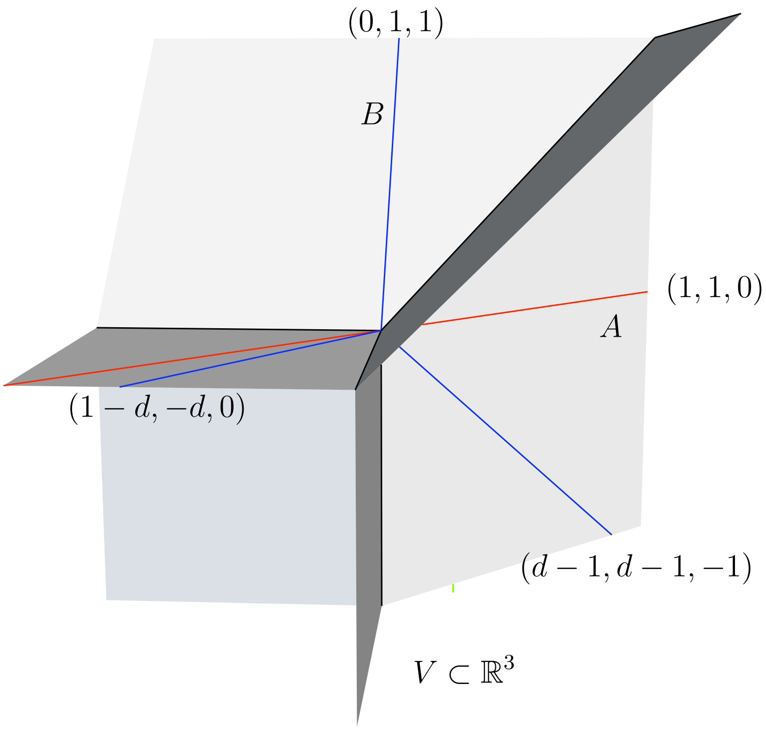


Figure 2.2: The tropical cycles in the standard hyperplane in $\mathbb{R}^3$ from Example 2.1.2

Proof. The first statement is clear since δ^*, δ_* are homomorphisms, and the second follows from $\mathrm{div}_{A+B}(f) = \mathrm{div}_A(f) + \mathrm{div}_B(f)$. $\square$

Lemma 2.1.5. *Let $\delta : V \longrightarrow V'$ be an elementary open matroidal modification along the rational function f and having divisor D. If a cycle $A \subset V$ is in Ker δ_*, then it is contained in the closure of the undergraph $\mathcal{U}(\Gamma_f(V'))$. In particular, it is also a subcycle of $D \times \mathbb{R}$ where $\mathbb{R}$ is the affine space spanned by the kernel of δ.*

Proof. Away from the divisor $D \subset V$ the map δ is one to one thus no cancellation of facets can occur in $\delta_* A$ outside of D. So $\delta(A)$ must be contained in D which implies the lemma. $\square$

A quick check shows that Δ_A is in the kernel of δ_*, since $\delta_*\delta^* = \mathrm{id}$. Therefore, for an elementary open modification of matroidal fans $\delta : V \longrightarrow V'$ and any cycle $A \subset V$ we have $\Delta_A, D_A \subset D \times \mathbb{R}$, where $D \subset V'$ is the divisor of the modification. Using this we define an intersection product on V in terms of a product on V' and $D \times \mathbb{R}$.

Definition 2.1.6. Given cycles $A, B \subset V \subset \mathbb{R}^n$ and an elementary open matroidal modification $\delta : V \longrightarrow V'$ with associated divisor D, define,

$$A.B = \delta^*(\delta_* A.\delta_* B) + C_{A.B}$$

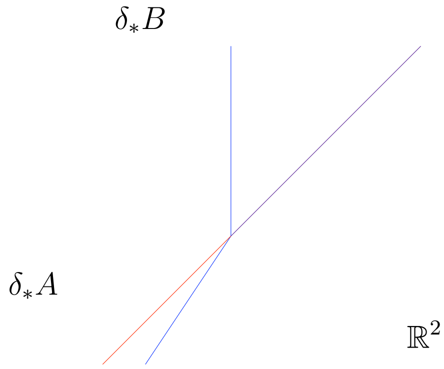


Figure 2.3: The pushforwards of the cycles $A, B \subset P$ from Example 2.1.2 where $\delta : P \longrightarrow \mathbb{R}^2$ is the linear projection in the vertical direction.

with

$$C_{A.B} = \Delta_A.\Delta_B - \Delta_A.D_B - D_A.\Delta_B,$$

where these products are calculated in the matroidal fan $D \times \mathbb{R} \subset \mathbb{R}^n$.

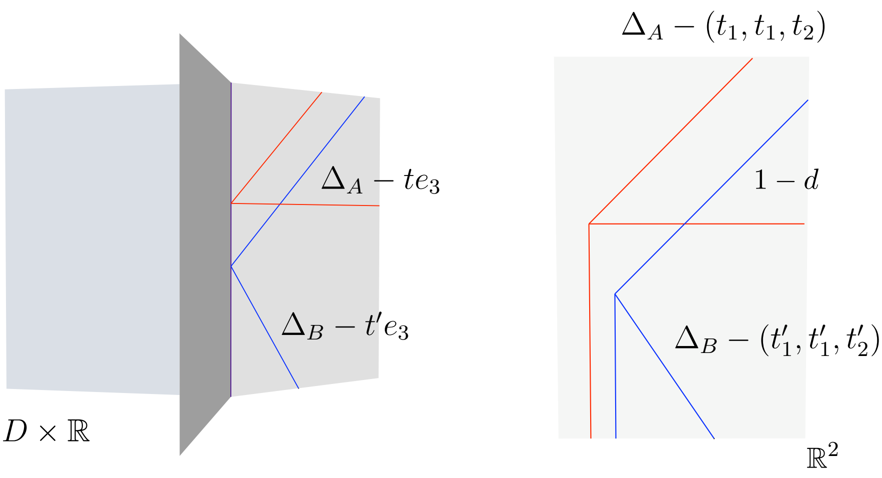


Figure 2.4: The second and then third translations of the cycles Δ_A, Δ_B from Example 2.1.2.

The above definition gives the product of two cycles A, B in V as a sum of products of cycles in fans V' and $D \times \mathbb{R}$, one of which is of lower codimension, and the other containing the linear space spanned by the kernel of δ. Continuing to apply this procedure to V' and D we continue to decrease the codimension or increase the dimension of the affine linear space contained in the fan and we can eventually reduce the intersection product in V to a sum of pullbacks of stable intersections in $\mathbb{R}^k$, where k is the dimension of V. A priori this definition depends on the choice of all contraction charts. Before showing the above definition is independent of the chosen charts in Proposition 2.1.11 we state some properties of the intersection product as defined relative to a fixed collection of open matroidal contractions.

Lemma 2.1.7. *Suppose $\delta : V \longrightarrow V'$ is an elementary open modification of matroidal fans and A, B are cycles in V'. The intersection product in V from Definition 2.1.6 calculated via the modification δ satisfies*

$$\delta^*A.\delta^*B = \delta^*(A.B).$$

Proof. In this case $\Delta_A, \Delta_B = 0$ so the term $C_{A,B}$ from Definition 2.1.6 is also 0. □

Corollary 2.1.8. *Suppose the matriodal fan $V \subset \mathbb{R}^n$ is a k-dimensional subspace of $\mathbb{R}^n$, and let $\delta : V \longrightarrow \mathbb{R}^k$ be an open matroidal contraction. For subcycles A, B in V we have,*

$$A.B = \delta^*(\delta_*A.\delta_*B).$$

Proposition 2.1.9. *Let $V \subset \mathbb{R}^n$ be a matroidal fan and A, B, C be subcycles of V. Then the intersection product given in Definition 2.1.6 relative to any choice of contraction charts satisfies the following:*

1. *$A.B$ is a balanced cycle contained in V*
2. *$A.C = C.A$*
3. *$A_1.(A_2 + A_3) = A_1.A_2 + A_1.A_3$*
4. *$A_1.(A_2.A_3) = (A_1.A_2).A_3$*
5. *$div_A(g) = div_V(g).A$*

Proof. The above properties all follow by induction. The base case being $V = \mathbb{R}^k$, where all of the above properties are satisfied. Suppose we have chosen, $\delta : V \longrightarrow V'$ as the first elementary open matroidal contraction, and let its divisor be $D \subset V'$. We may assume all of the properties stated above hold for intersections in V' and $D \times \mathbb{R}$.

For (1), the weighted balanced complex, $A.B$ is the sum of $\delta^*(\delta_*A.\delta_*B)$ and $C_{A.B}$ which are both balanced by the induction assumption, so it is balanced. Commutativity also follows immediately by induction. By Lemma 2.1.4 and distributivity for products in V' and $D \times \mathbb{R}$, we get distributivity in V.

For associativity, first notice that

$$\Delta_{A_i.A_j} = \Delta_{A_i}.D_{A_j} + D_{A_i}.\Delta_{A_j} - \Delta_{A_i}.\Delta_{A_j} \tag{2.1.1}$$
$$D_{A_i.A_j} = D_{A_i}.D_{A_j}. \tag{2.1.2}$$

The first line follows from the definition of $\Delta_{A_i.A_j}$. The statement (3) follows from Lemma 2.1.10 which follows this proposition. Then,

$$A_1.(A_2.A_3) = \delta^*(\delta_*A_1.(\delta_*A_2.\delta_*A_3)) - \Delta_{A_1}.D_{A_2.A_3} - D_{A_1}.\Delta_{A_2.A_3} + \Delta_{A_1}.\Delta_{A_2.A_3}$$

Assuming associativity in V and $D \times \mathbb{R}$ and using commutativity we can remove brackets and write:

$$\begin{aligned} A_1.(A_2.A_3) =& \delta^*(\delta_* A_1.\delta_* A_2.\delta_* A_3) + \\ & \sum_{\substack{1 \le i < j \le 3 \\ k \ne i,j}} \Delta_{A_i}.\Delta_{A_j}.D_{A_k} - D_{A_i}.D_{A_j}.\Delta_{A_k} - \Delta_{A_1}.\Delta_{A_2}.\Delta_{A_3} \end{aligned}$$

Regrouping terms and using (2) and (3) we get,

$$\begin{aligned} A_1.(A_2.A_3) &= \delta^*((\delta_* A_1.\delta_* A_2).\delta_* A_3) - \Delta_{A_1.A_2}.D_{A_3} - D_{A_1.A_2}.\Delta_{A_3} + \Delta_{A_1.A_2}.\Delta_{A_3} \\ &= (A_1.A_2).A_3. \end{aligned}$$

Lastly, given a divisor $D = \text{div}_V(g)$ we may write it as $\delta^*\delta_* D - \Delta_D$. Then $\tilde{g}(x) = g(\delta(x))$, is the function of the divisor $\delta^*\delta_* D$ where f is the function of the modification δ. So $\tilde{g} - g$ gives Δ_D by part 3 of Proposition 1.1.24. The result follows by distributivity and by applying the induction hypothesis to both parts. □

We require a final lemma before proving that the product is independent of the choice of contractions.

Lemma 2.1.10. *Let $V \subset \mathbb{R}^n$ be a matroidal fan and A, B be subcycles of V, set*

$$\tilde{A} = A \times \mathbb{R}, \quad \tilde{B} = B \times \mathbb{R}, \quad \textit{and} \quad \tilde{V} = V \times \mathbb{R}\,.$$

Then, we may choose contraction charts so that by Definition 2.1.6 we have

$$\tilde{A}.\tilde{B} = A.B \times \mathbb{R} \subset \tilde{V}.$$

Proof. The above statement holds for stable intersections in $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$. If V corresponds to a matroid M on E then $\tilde{V}$ corresponds to a matroid $\tilde{M}$ on $E \cup e$ with bases $B \cup e$ for every base B of M, in other words we have added a coloop e to the matroid M. Given an elementary open modification of matroidal fans, $\delta : V \longrightarrow V'$ with divisor D we have a corresponding elementary open modification $\tilde{\delta} : \tilde{V} \longrightarrow \tilde{V}'$ with divisor $\tilde{D} = D \times \mathbb{R}$ and $\tilde{V}' = V' \times \mathbb{R}$. In order to define the product in $A.B$, a collection of contractions are fixed. To intersect $\tilde{A}, \tilde{B}$ in $\tilde{V}$, simply choose the corresponding collection of contractions of $\tilde{V}$. Applying Definition 2.1.6 we obtain the lemma by induction. □

Theorem 2.1.11. *The intersection product from Definition 2.1.6 is independent of the choice of open matroidal contractions.*

Proof. Fix a matroidal fan $V \subset \mathbb{R}^n$ and subcycles A, B of V. We may assume by induction that the product is well-defined on $D \times \mathbb{R}$ and V' where $\delta : V \longrightarrow V'$ is any elementary open matroidal modification and D is its associated divisor.

By Corollary 1.1.36 any two open matroidal contractions $\delta, \delta' : V \longrightarrow \mathbb{R}^k$ can be related by a series of basis exchanges. So it suffices to check two things: that we may transpose the order of any two elementary open contractions to $\mathbb{R}^k$ and obtain the same intersection cycle and that if $\delta : V \longrightarrow V'$ is the composition of any two elementary open matroidal modifications, we may permute the order of the elementary contractions and obtain the same product. In other words we must show that the definition does not depend on the paths taken in the following two diagrams:

$$\begin{tikzcd} & V \arrow[dl, "\delta_1"'] \arrow[dr, "\delta_2"] & \\ V_1 \arrow[dr, "\tilde{\delta}_2"'] & & V_2 \arrow[dl, "\tilde{\delta}_1"] \\ & V' & \end{tikzcd} \qquad \begin{tikzcd} & V \arrow[dl, "\delta_1"'] \arrow[dr, "\delta_2"] & \\ \mathbb{R}^k \arrow[rr, dotted, "e_2 \mapsto e_1"'] & & \mathbb{R}^k \end{tikzcd}$$

We will start by showing the latter, let $\delta_1, \delta_2 : V \longrightarrow \mathbb{R}^k$ be two elementary open matroidal contractions. Then V is of codimension one in $\mathbb{R}^{k+1}$ and thus corresponds to a corank one matriod M. Suppose without loss of generality that the open contractions δ_i correspond to the deletion of the element i from the corresponding matroid, Then we may assume that $i = 1, 2$ are not coloops of M. If we exchange any two non coloop elements i and j of a corank one matroid M we obtain a matroid isomorphism. Also, restricting the matroid M to i or j produces isomorphic matroids $M/i \cong M/j$. Therefore the divisors $D_i, D_j \subset \mathbb{R}^k$ of the corresponding elementary open matroidal modifications $\delta_i, \delta_j : V \longrightarrow \mathbb{R}^k$ can be identified as well as the functions f_i, f_j on $\mathbb{R}^k$.

First we will construct cycles $\tilde{A}, \tilde{B} \subset V$ such that $\delta_{1*}\tilde{A} = \delta_{2*}\tilde{A} \subset \mathbb{R}^k$ and $\delta_i^*\delta_{i*}\tilde{A} = \tilde{A}$ for $i = 1, 2$ and similarly for $\tilde{B}$. Then by the above remarks concerning the two modifications the definition of the product $\tilde{A}.\tilde{B} = \delta_i^*(\delta_{i*}\tilde{A}.\delta_{i*}\tilde{B})$ does not depend on the choice of $i = 1, 2$.

To construct $\tilde{A}$ and $\tilde{B}$, let $\mathcal{C} \subset V$ denote the union of all faces of V that are not generated by the vectors v_1, v_2, where v_i generates the kernel of δ_i. Let $\tilde{A} = \delta_i^*\delta_{i*}\delta_j^*\delta_{j*}A$ and similarly for $\tilde{B}$. The cycle $\tilde{A}$ (respectively, $\tilde{B}$) is well-defined independent of the order of δ_i, δ_j since it is obtained from $A \cap \mathcal{C}$ (respectively, $B \cap \mathcal{C}$) by adding uniquely weighted facets to all codimension one faces E of A (respectively, B), parallel only to the cones spanned by E and v_i for $i = 1, 2$, so that the result satisfies the balancing condition. Similarly, in $\mathbb{R}^k$ we have $\delta_{i*}\tilde{A} = \delta_{j*}\tilde{A}$ and analogously for $\tilde{B}$, since the weighted complexes $\delta_{i*}\tilde{A} \cap \delta(\mathcal{C})$ are equal for $i = 1, 2$ and balanced in all but the $\delta_i(v_j)$ direction where $j = 1, 2$ and $i \neq j$. Adding the necessary uniquely weighted facets to the codimension one faces of this complex in the $\delta_i(v_j)$ direction gives $\delta_{i*}\tilde{A}$ for $i = 1, 2$ and similarly for $\delta_{i*}\tilde{B}$. Also by construction we have $\delta_i^*\delta_{i*}\tilde{A} = \tilde{A}$, and similarly for B.

For $i = 1, 2$, define $\Delta_A^i = \delta_i^*\delta_{i*}A - A$ and $D_A^i = \mathrm{div}_A(f_i) \times \mathbb{R} \subset D_i \times \mathbb{R}$ and similarly for B. Assume first that $A = \tilde{A} - \Delta_A^1 - \Delta_A^2$, and analogously for B. It follows that $\delta_j^*\delta_{j*}\Delta_A^i = \Delta_A^i$, and similarly for B. Then we obtain,

$$A.B = \delta_i^*(\delta_{i*}(\tilde{A} - \Delta_A^j).(\delta_{i*}(\tilde{B} - \Delta_B^j)) - D_A^i.\Delta_B^i - \Delta_A^i.D_B^i + \Delta_A^i.\Delta_B^i$$

By distributivity, Lemma 2.1.7 and the assumption that $\delta_j^*\delta_{j*}\Delta_A^i = \Delta_A^i$ and $\delta_j^*\delta_{j*}\Delta_B^i = \Delta_B^i$ we have,

$$\begin{aligned} A.B = \delta_i^*(\delta_{i*}\tilde{A}.\delta_{i*}\tilde{B}) - \tilde{A}.\Delta_B^j - \Delta_A^j.\tilde{B} + \Delta_A^j.\Delta_B^j \\ - D_A^i.\Delta_B^i - \Delta_A^i.D_B^i + \Delta_A^i.\Delta_B^i. \end{aligned}$$

The last three terms are products in $D_i \times \mathbb{R}$, and $\tilde{A}.\Delta_B^j, \Delta_A^j.\tilde{B}$ and $\Delta_A^j.\Delta_B^j$ are products in V. By applying the contraction δ_j to calculate these three products we obtain:

$$\Delta_A^j.\Delta_B^j - \tilde{A}.\Delta_B^j - \Delta_A^j.\tilde{B} = \Delta_A^j.\Delta_B^j - D_A^j.\Delta_B^j - \Delta_A^j.D_B^j.$$

Combining this with the equation above and we get,

$$\begin{aligned} A.B = \delta_i^*(\delta_{i*}\tilde{A}.\delta_{i*}\tilde{B}) - D_A^j.\Delta_B^j - \Delta_A^j.D_B^j + \Delta_A^j.\Delta_B^j \\ - D_A^i.\Delta_B^i - \Delta_A^i.D_B^i + \Delta_A^i.\Delta_B^i, \end{aligned}$$

which is symmetric in i and j except for the first term $\delta_i^*(\delta_{i*}\tilde{A}.\delta_{i*}\tilde{B})$ which was already shown to be the same for $i = 1, 2$. So $A.B$ is independent of the contraction chart chosen.

Dropping our previous assumption, for any cycle we may still write $A = \tilde{A} - \Delta_A^1 - \Delta_A^2 - \Xi_A$, where Ξ_A is a cycle contained in the kernel of both δ_{1*} and δ_{2*}. Letting $A' = A + \Xi_A$, and analogously for B, and using distributivity with respect to either contraction chart we have

$$A.B = A'.B' - A'.\Xi_B - \Xi_A.B' + \Xi_A.\Xi_B. \tag{2.1.3}$$

As seen above, the product $A'.B'$ does not depend on the choice of chart δ_{i*}. Moreover since Ξ_A, Ξ_B are in the kernels of both δ_{i*} for both $i = 1, 2$, the product Ξ_A, Ξ_B descends to $D_{ij} \times \mathbb{R}^2$ where D_{ij} is the matroid corresponding to $M/\{i, j\}$ where M is the matroid of V. This doesn't depend on the order of i and j, see Section 3.1 of [44]. The other two products also descend to $D_{ij} \times \mathbb{R}^2$ as:

$$\begin{aligned} A'.\Xi_B = (\tilde{A} - \Xi_A).\Xi_B = (D_{\tilde{A}}^i + D_{\tilde{B}}^j - \Xi_A).\Xi_B \\ \Xi_A.B' = \Xi_A.(\tilde{B} - \Xi_B) = \Xi_A.(D_{\tilde{B}}^i + D_{\tilde{B}}^j - \Xi_B) \end{aligned}$$

which are symmetric in i and j.

Now we treat the case of two elementary contractions. Let $\delta : V \longrightarrow V'$ be the composition of two elementary open matroidal contractions. First we set up notation to distinguish between the two orderings, similar to the proof of Proposition 1.1.43. We will call $\delta_i : V \longrightarrow V_i$ and $\tilde{\delta}_i : V_j \longrightarrow V'$ for $i \neq j$. Let $D_i \subset V_i$ be the divisor associated to δ_i and suppose $D_i = \mathrm{div}_{V_i}(f_i)$. Similarly, $\tilde{D}_i \subset V'$ will denote the divisor of $\tilde{\delta}_i$ and $\tilde{f}_i$ its function. Keeping the notation from the beginning of the proof for Δ_A^i and D_A^i, we also set:

$$\begin{aligned} \tilde{\Delta}_A^i =& \tilde{\delta}_i^*\tilde{\delta}_{i*}A - A \subset V_j \\ \tilde{D}_A^i =& \mathrm{div}_{\delta_*A}(\tilde{f}_i) \times \mathbb{R} \subset \tilde{D}_i \times \mathbb{R} \end{aligned}$$

Applying Definition 2.1.6 first by contracting with δ_i and then contracting with $\tilde{\delta}_j$ we obtain:

$$A.B = \delta^*(\delta_* A.\delta_* B) + C_i + \delta_j^* \tilde{C}_j$$

Where

$$C_i = \Delta_A^i.\Delta_B^i - \Delta_A^i.D_B^i - D_A^i.\Delta_B^i$$

with these three products calculated in $D_i \times \mathbb{R}$, and

$$\tilde{C}_j = \tilde{\Delta}^j_{\delta_{i*}A}.\tilde{\Delta}^j_{\delta_{i*}B} - \tilde{\Delta}^j_{\delta_{i*}A}.\tilde{D}^j_{\delta_{i*}B} - \tilde{D}^j_{\delta_{i*}A}.\tilde{\Delta}^j_{\delta_{i*}B}$$

with each product being calculated in $\tilde{D}_j \times \mathbb{R}$.

Again we first assume that $A = \delta^*\delta_* A + \Delta_A^1 + \Delta_A^2$. Then we have $\delta_i^* \tilde{\Delta}^j_{\delta_{i*}A} = \Delta_A^j$. Restricting δ_j to $D_i \times \mathbb{R}$ we get an elementary open matroidal modification $\delta_j : D_i \times \mathbb{R} \longrightarrow \tilde{D}_i \times \mathbb{R}$. This can be checked on the level of the corresponding matroids. The divisor $\tilde{D}_i$ corresponds to the matroid $M\backslash j/i$ and contracting D_i by δ_j corresponds to $M/i\backslash j$. By Proposition 3.1.26 of [44] these matroids are equal. Now applying Lemma 2.1.7 for the products in $\tilde{D}_i \times \mathbb{R}$ we have, $C_i = \delta_i^* \tilde{C}_i$ and we obtain the same cycle regardless of order.

The general case follows an argument similar to the general case of two distinct elementary contractions to $\mathbb{R}^k$. We can once again write $A = \delta^*\delta_* A - \Delta_A^1 - \Delta_A^2 - \Xi_A$ and similarly for B. The rest of the argument follows exactly as above with the products in the end being in $D_{ij} \times \mathbb{R}^2$, where again D_{ij} corresponds to the matroid $M/\{i,j\}$. $\square$

Now for weakly transverse intersections in a k-dimensional matroidal fan V we can make use of the definition of stable intersection in $\mathbb{R}^k$. For each facet F of V we can find a contraction chart $\delta : V \longrightarrow \mathbb{R}^k$ which does not collapse the face F. Recall, each facet of V corresponds to a maximal chain in the lattice of flats of the corresponding matroid. If after deleting an element i from the matroid the chain corresponding to F is still of length $k+1$, the tropical contraction δ_i of the Bergman fan does not collapse the face F. If the chain is of length $k+1$ on $n+1$ elements we can find $n-k$ elements to delete and not collapse F. Using this contraction chart to calculate the multiplicity we arrive at the following corollary.

Corollary 2.1.12. *Let $V \subset \mathbb{R}^n$ be a matroidal fan and suppose the intersection of the two subcycles $A, B \subset V$ is weakly transverse when restricted to an open facet $F \subset V$ then $A.B \cap F$ corresponds to the stable intersection of Definition 1.1.7.*

Proposition 2.1.13. *For two cycles A, B in a matroidal fan $V \subset \mathbb{R}^n$, the product $A.B$ is supported on $(A \cap B)^{(m)}$, where m is the expected dimension of intersection.*

Proof. Once again our proof goes by induction. Given a facet F of $A.B$, choose a elementary open matroidal contraction chart $\delta : V \longrightarrow V'$ which does not contract the face $E \subset V$ containing F. Again, we may take any chart which does not contract all of the facets adjacent to E. Let f be the function on V' giving the modification δ and D the corresponding divisor. Then F is contained in $\Gamma_{V'}$ the graph of f. Let

$\Gamma_D \subset \Gamma_{V'}$ be the graph of f restricted to D. If $\delta(F) \not\subset D$ then $\delta(F)$ must be a facet of $\delta_* A.\delta_* B$. By induction $\delta(F) \subset (\delta_* A \cap \delta_* B)^{(m)}$, so we must have $F \subset (A \cap B)^{(m)}$.

If on the other hand $F \subset \Gamma_D$ then $\delta(F)$ is an m dimensional face contained in $\delta_* A \cap \delta_* B \cap D$ where D the divisor of the elementary modification δ and F must be in one of the products $\Delta_A.D_B$, $\Delta_B.D_A$ or $\Delta_A.\Delta_B$ which occur in the fan $D \times \mathbb{R}$. Then assuming the statement holds on $D \times \mathbb{R}$, the facet F must be in one of $(\Delta_A \cap D_B)^{(m)} \cap \Gamma_D$, $(D_A \cap \Delta_B)^{(m)} \cap \Gamma_D$, or $(\Delta_A \cap \Delta_B)^{(m)} \cap \Gamma_D$. In any of these three cases F must be a facet of $(\Gamma_{\delta_* A} \cap \Gamma_{\delta_* B})^{(m)} \cap \Gamma_D$, and so in $(A \cap B)^{(m)}$. □

2.1.2 Intersections in manifolds

Recall the definitions of cycles in manifolds from 1.2.11. A k-cycle $A \subset X$ in every chart $\Phi_\alpha : U_\alpha \longrightarrow V_\alpha$ satisfies $A_\alpha \cap \phi(U_\alpha)$ for some k-cycle $A_\alpha \subset V_\alpha$.

Definition 2.1.14. A k-cycle $A \subset \mathbb{T}^n$ of intersects the boundary properly if $\dim(A \cap H_I) = k - |I|$ or -1 when $k - |I| \geq -1$ and $A \cap H_I = \emptyset$ otherwise for all $I \subset \{1, \dots n\}$.

Definition 2.1.15. Let $V \subset \mathbb{T}^n$ be a matroidal fan and suppose $A, B \subset \mathbb{T}^n$ are cycles intersecting the boundary of $\mathbb{T}^n$ transversally. Then define,

$$A.B = \overline{A^o.B^o},$$

where A^o, B^o are the sedentarity zero points of A and B respectively and $A^o.B^o$ is the product in the open matroidal fan $V^o \subset \mathbb{R}^n$ given by Definition 2.1.6.

Definition 2.1.16. A cycle $A \subset X$ intersects the boundary ∂X properly if for every subset of boundary divisors $\mathcal{A}' \subseteq \mathcal{A}$ we have

$$\dim(A \cap_{D \in \mathcal{A}'} D) = k - |\,\mathcal{A}\,| \text{ or } -1,$$

when $k - |\,\mathcal{A}'\,| \geq -1$ and the intersection is empty otherwise.

For two cycles transverse to the boundary in a tropical manifold, we make use of the definition of the intersection product given in the last section.

Definition 2.1.17. For cycles $A, B \subset X$ intersecting ∂X properly, the intersection $A.B$ is given in each chart $\Phi_\alpha : U_\alpha \longrightarrow V_\alpha$ by the product $A_\alpha.B_\alpha \subset V_\alpha$ see Definition 2.1.15.

As mentioned, it may happen that a matroidal fan V can be expressed as the fan of two different matroids under a coordinate change. In [17], an equivalent intersection product is using Cartier divisors and intersecting with the diagonal. Here the authors show the intersection product to be well-defined under the integer affine coordinate changes. See Section 6 of [17] for more details. Therefore the above definition works for cycles in tropical manifolds under the provided assumption on the boundary intersections.

Back in Section 1.1.2, we gave the intersection of a k-cycle in $\mathbb{T}^n$ with a boundary hyperplane H_i, in Definition 1.1.11. Using this we define the intersection of a k-cycle in a manifold X with a boundary divisor D when the k-cycle has no component in contained D. Consider a chart $\Phi_\alpha : U_\alpha \longrightarrow V_\alpha$, and suppose without loss of generality that in this chart $\Phi_\alpha(D \cap U_\alpha)$ is contained in $H_{i_\alpha} \cap V_\alpha$, where $H_{i_\alpha} = \{x \in \mathbb{T}^{n_\alpha} \mid x_{i_\alpha} = -\infty\}$. Let $\Phi_\alpha(A \cap U_\alpha)$ be represented by the cycle A_α. Then $H_{i_\alpha}.A_\alpha$ may be intersected in $\mathbb{T}^{n_\alpha}$, moreover it is a dimension $k-1$-cycle contained in $V_\alpha \cap H_{i_\alpha}$. Then $D.A \cap U_\alpha$ is given by the inverse $\Phi^{-1}(H_{i_\alpha}.A_\alpha)$ restricted to the image.

Definition 2.1.18. Let $D \subset X$ be an irreducible boundary divisor of a tropical manifold X, and suppose $A \subset X$ is a k-cycle such that $\dim(A \cap D) < k$, then $D.A \subset X$ is the $k-1$-cycle given in each chart U_α by

$$\Phi^{-1}(H_{i_\alpha}.A_\alpha),$$

where $H_{i_\alpha}.A_\alpha$ is given by Definition 1.1.11.

Other intersections in tropical manifolds are more challenging to describe. For example, there is no "local" formula for the self intersection of a boundary divisor of a manifold. It suffices to look to $\mathbb{TP}^2$ for an example. A boundary divisor represents a line and ought to have a self-intersection equal to one. However, there is no distinguished point on this boundary divisor which may support the self-intersection like in the case of sedentarity $\emptyset$ cycles in $\mathbb{T}^2$. The self-intersection of boundary divisors requires an equivalence relation on tropical cycles. The next section presents tropical rational equivalence.

2.1.3 Rational equivalence and Chow groups

We provide the details of the definition of tropical rational equivalence given in [38]. This definition uses families over $\mathbb{TP}^1$. This is the classical approach to rational equivalence given in Chapter 1 of [18].

Recall that $\mathbb{TP}^1 = [-\infty, \infty]$, it is a tropical manifold covered by two charts $U_1, U_2 \cong \mathbb{T}$ with transition map $x \mapsto -x$ defined on $U_1 \cap U_2 = \mathbb{R}$. Given a tropical manifold X, consider the product of topological spaces: $X \times \mathbb{TP}^1$. This is also a tropical manifold, given a collection of charts $\{U_\alpha, \Phi_\alpha\}$ for X we may take

$$\{U_\alpha \times U_i, \Phi_{\alpha,i}\}$$

for $i = 1, 2$ where

$$\Phi_{\alpha,i} : U_\alpha \times U_i \longrightarrow V_\alpha \times \mathbb{T}\,.$$

The fan $V_\alpha \times \mathbb{T}$ is matroidal since it is just an extension of the matroid corresponding to V_α by a coloop. See [17] for more on products of matroidal fans in the non-projective case. The transition charts $U_\alpha \times U_i \longrightarrow U_\beta \times U_j$ are given by $A_{\alpha\beta} \times B_{ij}$, where $A_{\alpha\beta} : U_\alpha \longrightarrow U_\beta$ are the transition charts for X and $B_{12}(x) = -x$ is the coordinate change for $\mathbb{TP}^1$.

For a tropical manifold X, we distinguish two of the boundary divisors of $X \times \mathbb{TP}^1$

$$X_{-\infty} = X \times \{-\infty\} \quad \text{and} \quad X_{\infty} = X \times \{\infty\}.$$

For $i = 1, 2$ let p_i denote the projection onto the first and second factor, i.e.

$$p_1 : X \times \mathbb{TP}^1 \longrightarrow X \quad \text{and} \quad p_2 : X \times \mathbb{TP}^1 \longrightarrow \mathbb{TP}^1 .$$

A $k+1$-cycle $Y \subset X \times \mathbb{TP}^1$, will be called a **family** if $p_2(Y)$ is surjective.

Definition 2.1.19. If $Y \subset X \times \mathbb{TP}^1$ is a family, then define

$$Y(-\infty) = p_1(Y.X_{-\infty}) \quad \text{and} \quad Y(\infty) = p_1(Y.X_{\infty}).$$

The map p_1 restricted to the two above cycles is one to one, and if we equip $Y(-\infty)$ (respectively, $Y(\infty)$) with the corresponding weights in $Y.X_{-\infty}$ (respectively, $Y.X_{\infty}$) we have a subcycle of $X \cong X \times \{-\infty\}$ (respectively, $X \cong X \times \{\infty\}$.

Definition 2.1.20. A k-cycle $C \subset X$ is rationally equivalent to 0, written $C \sim_r 0$, if there exists a $k+1$ dimensional family, $Y \subset X \times \mathbb{TP}^1$ such that $C = Y(-\infty) - Y(\infty)$.

Denote the k-cycles rationally equivalent to 0 by $R_k(X)$. They form a subgroup of $Z_k(X)$ since if $C_1, C_2 \sim_r 0$ then the cycle $Y_1 - Y_2 \subset X \times \mathbb{TP}^1$ gives the family providing $C_1 - C_2 \sim_r 0$, where Y_i is the family for C_i. A $(k+1)$-dimensional family $Y \subset X \times \mathbb{TP}^1$ and any $p \in \mathbb{R} \subset \mathbb{TP}^1$ also gives a k-cycle $Y(p) \subset X$ in the following way: Consider the function on $X \times \mathbb{TP}^1$ given by $f_p(x, t) = \max\{t, p\}$, where x is a coordinate on X and t the coordinate on $\mathbb{TP}^1$, then $\text{div}_X(f_p) = X \times \{p\}$ and we define,

Definition 2.1.21. For a family $Y \subset X \times \mathbb{TP}^1$ and $p \in \mathbb{R} \subset \mathbb{TP}^1$, then

$$Y(p) = p_1(\text{div}_Y(f_p)).$$

There is another version of tropical rational equivalence defined by Allermann and Rau in [2]. This relation is the same as the one generated by $Y(p) - Y(q) \sim_r 0$ for only $-\infty < p, q < \infty$. By part (1) of the next proposition this equivalence relation is finer than the one given here. Using only this bounded version of rational equivalence yields different Chow groups for most varieties X. For example, Proposition 2.1.24 to come is not true if we use only this bounded rational equivalence from [2].

Definition 2.1.22. For a tropical manifold X we define the Chow groups to be

$$A_k(X) = \frac{Z_k(X)}{R_k(X)}.$$

Proposition 2.1.23. *Let X be a tropical manifold and $\delta : \tilde{X} \longrightarrow X$ a non-singular modification of tropical manifolds.*

1. *$C \sim_r 0$ in X if and only if there exists a cycle $Y \subset X \times \mathbb{TP}^1$ and points $p, q \in \mathbb{TP}^1$ such that $C = Y(p) - Y(q)$.*

2. *If $C_1 \sim_r C_2$ in X then their pullbacks satisfy $\delta^* C_1 \sim_r \delta^* C_2$ in $\tilde{X}$.*

3. If $C_1 \sim_r C_2$ in $\tilde{X}$ then $\delta_* C_1 \sim_r \delta_* C_2$ in X.

Proof. Given a family Y such that $C = Y(p) - Y(q)$ we construct a family Y' so that $C = Y'(-\infty) - Y'(\infty)$. Assume without loss of generality that $p < q$. Let $U = [-\infty, t)$ for $p < t < q$ be an open neighborhood of $\mathbb{TP}^1$. Consider the non-singular tropical modification $\pi_y : \tilde{U} \longrightarrow U$ given by the regular function $f_p(t) = \max\{t, p\}$. There is another contraction chart $\pi_x : \tilde{U} \longrightarrow U'$ given by projecting in the other coordinate direction of $\mathbb{T}^2$. Now, $V = X \times U$ is an open subset of $X \times \mathbb{TP}^1$, and we may take the product of the modification map on U with the identity map on X to get, $\tilde{\pi}_y : \tilde{V} \longrightarrow V$. Then the pullback $\delta^* Y$ of Y in $\tilde{V} \subset X \times \mathbb{T}^2$ intersects the $x = -\infty$ coordinate plane of $\mathbb{T}^2$ in $\mathrm{div}_{f_p}(Y) = Y(p)$. Performing the contraction $\tilde{\pi}_y : \tilde{V} \longrightarrow V'$ we obtain a family $Y' \subset X \times \mathbb{TP}^1$ with $Y'(-\infty) = Y(p)$. A symmetric sequence of modification and contraction can be done in the other coordinate chart of $\mathbb{TP}^1$ to send the point q to ∞. The pullback of Y' then gives the desired cycle.

For part (2), given a non-singular modification $\delta : \tilde{X} \longrightarrow X$ and a family $Y \subset X \times \mathbb{TP}^1$, the modification δ can be extended to the first factor giving $\tilde{X} \times \mathbb{TP}^1 \longrightarrow X \times \mathbb{TP}^1$. The pullback $\tilde{Y} = \delta^* Y \subset \tilde{X} \times \mathbb{TP}^1$ gives a family. Moreover, the modification restricts to the boundary to give: $\tilde{Y}(-\infty) = \delta^* Y(-\infty)$ and similarly $\tilde{Y}(\infty) = \delta^* Y(\infty)$.

For (3) the situation is similar to above. Now we have a family $Y \subset \tilde{X} \times \mathbb{TP}^1$, using again that the modification can be extended, we get $\delta_* Y \subset X \times \mathbb{TP}^1$ giving $\delta_* C_1 \sim_r \delta_* C_2$. □

Proposition 2.1.24. *A non-singular tropical modification $\delta : \tilde{X} \longrightarrow X$ induces group isomorphisms $\delta_* : A_k(\tilde{X}) \longrightarrow A_k(X)$ for all k.*

Proof. Taking the pullbacks of cycles in X gives us a homomorphism $\delta^* : Z_k(X) \longrightarrow Z_k(\tilde{X})$. By the last two parts of Proposition 2.1.23 this descends to an injective morphism of the Chow groups $A_k(X) \longrightarrow A_k(\tilde{X})$. It remains to see that this map is surjective. This follows from the next lemma. □

Lemma 2.1.25. *Let $\delta : \tilde{X} \longrightarrow X$ be an elementary tropical modification, and suppose C is a cycle in $\tilde{X}$, then $C \sim_r \delta^* \delta_* C$.*

Proof. Suppose that we have an atlas of charts $\tilde{U}_\alpha$ of $\tilde{X}$ and U_α of X such that $\delta_\alpha(\tilde{U}_\alpha) = U_\alpha$ and in each chart the divisor of the modification δ is $D_\alpha \subset U_\alpha$. Then in each chart $\tilde{U}_\alpha$, the cycle $C_\alpha - \delta^* \delta_* C_\alpha$ is contained in $\delta^{-1}(D_\alpha)$. Construct a family of cycles

$$B_{t_\alpha} = C - \delta_\alpha^* \delta_{\alpha *}(C) - t \cdot e_{N_\alpha} \subset \tilde{U}_\alpha \times [-\infty, 0],$$

for each α. These provide a family $B \subset X \times [0, \infty]$ such that $B_\infty = 0$ since $\delta_{\alpha *}(C_\alpha - \delta_\alpha^* \delta_{\alpha *}(C_\alpha))) = 0$. So every cycle is rationally equivalent to the pullback of a cycle in X and $\delta^* : A_k(X) \longrightarrow A_k(\tilde{X})$ is in fact an isomorphism of groups. □

Given a non-singular modification $\delta : \tilde{X} \longrightarrow X$ call $\tilde{A}$ a lift of A if $\delta_* \tilde{A} = A$.

Corollary 2.1.26. *For a non-singular tropical modification $\delta : \tilde{X} \longrightarrow X$, and a cycle $A \subset X$ any two lifts $\tilde{A}_1, \tilde{A}_2$ of A are rationally equivalent.*

Proposition 2.1.27. *Given a non-singular tropical modification $\delta : \tilde{X} \longrightarrow X$, and cycles $A, B \subset X$ such that A, B intersect ∂X properly and δ^*A, δ^*B intersect $\partial \tilde{X}$ properly then*

$$\delta^*A.\delta^*B = \delta^*(A.B).$$

Proof. By Lemma 2.1.7 from Section 2.1.1 for a primitive modification in each chart we have $\delta^*_\alpha(A_\alpha.B_\alpha) = \delta^*_\alpha(A_\alpha).\delta^*_\alpha(B_\alpha)$. □

In order to fully describe the Chow ring of a tropical manifold we must be able to intersect boundary cycles and cycles not transverse to the boundary. In Section 3.1.4 this is done for curves in surfaces. To intersect boundary divisors of a tropical manifold it is finally necessary to pass to equivalence classes as mentioned in the beginning of this section.

2.2 (p, q)-homology

2.2.1 Matroidal fans and the Orlik-Solomon algebra

The definition of (p, q)-cycles and (p, q)-homology summarized in this section are due to Itenberg, Katzarkov, Mikhalkin and Zharkov [27]. Recall that in Section 1.1.2, for a polyhedral complex $P \subset \mathbb{R}^n$ and a point $p \in P$, we defined the **star**,

$$Star_p(P) = \{v \in \mathbb{R}^n \mid \exists\, \epsilon > 0,\ p + \epsilon v \in P\}.$$

This definition can be extended naturally to polyhedral complexes in $\mathbb{T}^n$. For a point $p \in P \subset \mathbb{T}^n$ of sedentarity i, $Star_p(P) \subset \mathbb{R}^n$ will be contained in the half-space defined by $\langle x, e_i \rangle \geq 0$.

Definition 2.2.1. Given a matroidal fan $V \subset \mathbb{T}^n$ of sedentarity $\emptyset$ we associate to each point $x \in V$ a collection of groups $\mathcal{F}_k(x)$, for $k \in \mathbb{N}$. Let $\mathcal{F}_0(x) = \mathbb{Z}$, and for $k > 0$ define

- If $x \in V$ is a point of sedentarity $\emptyset$, then

  $$\mathcal{F}_k(x) = < v_1 \wedge \cdots \wedge v_k \mid v_1, \ldots, v_k \in \tau \subset Star_x(V) \cap \mathbb{Z}^n >,$$

  where τ is any face of $Star_x(V)$.

- If $x \in V$ is a point of sedentarity I, $\mathcal{F}_\bullet(x)$ is a quotient of the above construction. More precisely,

  $$\mathcal{F}_k(x) = \frac{< v_1 \wedge \cdots \wedge v_k \mid v_1, \ldots, v_k \in \tau \subset Star_x(V) \cap \mathbb{Z}^n >}{< e_i \mid i \in I >},$$

  where e_i is the standard basis in $\mathbb{R}^n$.

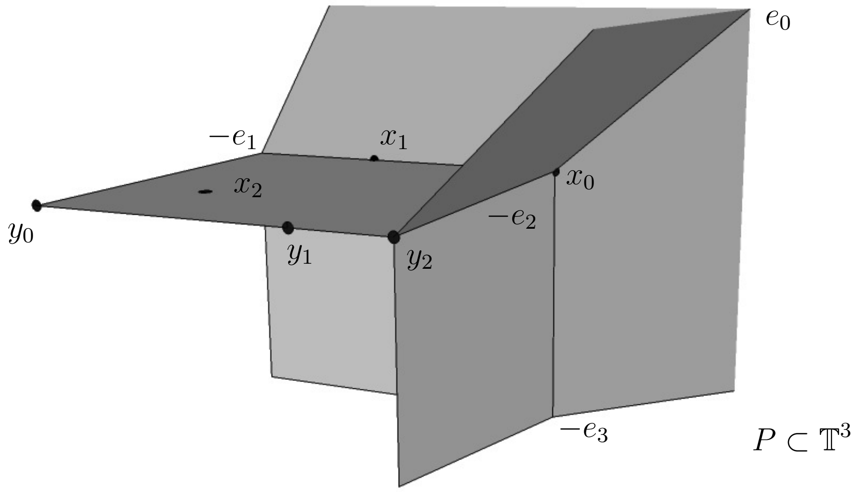


Figure 2.5: The matroidal fan $P \subset \mathbb{T}^3$ with showing the six points from Example 2.2.2.

Example 2.2.2
Consider the tropical fan $P \subset \mathbb{T}^3$ seen previously in Chapter 1, in Examples 1.1.20 and 1.1.23. The fan is drawn again in Figure 2.5, for each of the points on P in the figure, we give $\mathcal{F}_\bullet(x)$. Starting with the points of sedentarity we have:

$$\begin{array}{lll}
\mathcal{F}_0(x_0) = \mathbb{Z} & \mathcal{F}_0(x_1) = \mathbb{Z} & \mathcal{F}_0(x_2) = \mathbb{Z} \\
\mathcal{F}_1(x_0) = \Lambda^1(\mathbb{Z}^3) & \mathcal{F}_1(x_1) =< e_1, e_2 > & \mathcal{F}_1(x_2) =< e_1, e_2 > \\
\mathcal{F}_2(x_0) = \Lambda^2(\mathbb{Z}^3) & \mathcal{F}_2(x_2) =< e_1 \wedge e_2, e_1 \wedge e_3 > & \mathcal{F}_2(x_2) =< e_1 \wedge e_2 >
\end{array}$$

For the points of positive sedentarity we have:

$$\begin{array}{lll}
\mathcal{F}_0(y_0) = \mathbb{Z} & \mathcal{F}_0(y_1) = \mathbb{Z} & \mathcal{F}_0(y_2) = \mathbb{Z} \\
\mathcal{F}_1(y_0) = 0 & \mathcal{F}_1(y_1) =< e_1 > & \mathcal{F}_1(y_2) =< e_1, e_3 > \\
\mathcal{F}_2(y_0) = 0 & \mathcal{F}_2(y_1) = 0 & \mathcal{F}_2(y_2) = 0.
\end{array}$$

If σ is an open cone of V then $\mathcal{F}_\bullet$ is constant along σ, meaning $\mathcal{F}_\bullet(x) = \mathcal{F}_\bullet(y)$ for $x, y \in \sigma$. Because of this we will sometimes use the notation $\mathcal{F}_\bullet(\sigma)$. When we speak of $\mathcal{F}_\bullet(M)$ for a matroid M we mean the algebra associated to the origin which is contained in $B(M)$.

Definition 2.2.3. If $\tau \subset \bar{\sigma}$ define the map $i_{\sigma,\tau} : \mathcal{F}_\bullet(\sigma) \longrightarrow \mathcal{F}_\bullet(\tau)$, induced by the inclusion map and the quotient by all e_j for $j \in s(\tau)\backslash s(\sigma)$.

We choose the terminology "inclusion map" for $i_{\sigma,\tau}$, despite the fact that these maps are not always injective, (if $\sigma \subset \bar{\tau}$ is a face of greater sedentarity the map is the quotient). Denote the dual algebra by $\mathcal{F}^\bullet(x) = \mathrm{Hom}(\mathcal{F}_\bullet(x), \mathbb{Z})$, then we have restriction homomorphisms $r_{\tau,\sigma} : \mathcal{F}^\bullet(\tau) \longrightarrow \mathcal{F}^\bullet(\sigma)$.

The Orlik-Solomon algebra of a matroid, $OS^\bullet(M)$ is defined as a quotient of the free algebra $\Lambda^\bullet \mathbb{Z}^n$. We refer the reader to [41], particularly Sections 3.1 and 5.4 for the definitions and details. When the matroid M arises from a complex hyperplane arrangement $\mathcal{A}$, the next theorem was proved for braid arrangements and conjectured for general arrangements by Arnol'd. Later following work of Brieskorn [8] it was proved for general arrangements using the Thom isomorphism theorem by Orlik-Solomon [41].

Theorem 2.2.4 (Theorem 5.89 [41])**.** *Given a non-central arrangement $\mathcal{A}$ of hyperplanes in $\mathbb{CP}^k$ there is a ring isomorphism,*

$$H^\bullet(\mathcal{C}(\mathcal{A}), \mathbb{Z}) \cong OS^\bullet(M)$$

where M is the matroid of the arrangement $\mathcal{A}$ and $\mathcal{C}(\mathcal{A}) = \mathbb{CP}^k \setminus \mathcal{A}$.

Even without the definition of $OS^\bullet(M)$ the above theorem is interesting as it implies that the cohomology of the complement is a combinatorial invariant. It does not depend on the choice of hyperplanes in $\mathcal{A}$ only on their intersection properties.

The key ingredient to the proof of Theorem 2.2.4 is the existence of the following exact sequence relating the Orlik-Solomon algebras of deletions and restrictions of a matroid from Definition 1.1.32.

Proposition 2.2.5 (Theorem 3.65 [41])**.** *Let $M = (E, r)$ be a matroid and $i \in E$ then*

$$0 \longrightarrow OS^m(M\backslash i) \longrightarrow OS^m(M) \longrightarrow OS^{m-1}(M|i) \longrightarrow 0$$

is an exact sequence.

We include the next theorem simply because its proof is a nice application of tropical modifications. An algebraic proof can be found in [27], their proof also shows that there is an isomorphism of algebras.

Theorem 2.2.6. *For a matroid M, there is a group isomorphism $\mathcal{F}^m(M) \cong OS^m(M)$ for all $m \geq 0$.*

We first prove the following lemma:

Lemma 2.2.7. *Let M be a matroid on the ground set E and suppose $i \in E$ is not a coloop, then for each k we have the short exact sequence,*

$$0 \longrightarrow \mathcal{F}_{k-1}(M|i) \xrightarrow{\gamma} \mathcal{F}_k(M) \xrightarrow{\delta} \mathcal{F}_k(M\backslash i) \longrightarrow 0.$$

Proof. Set,

$$\begin{aligned} V &= B(M) \cap U_j \subset \mathbb{T}^n, \\ V' &= B(M\backslash i) \cap U_j \subset \mathbb{T}^{n-1}, \\ D &= B(M/i) \cap U_j \subset \mathbb{T}^{n-1}. \end{aligned}$$

By Proposition 1.1.34 $\delta : V \longrightarrow V'$ is a tropical modification with divisor D where δ is the linear map with kernel e_i. Let $f : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ be the integer affine piecewise linear function which extends to $\mathbb{T}^{n-1}$ to give the above modification. For a vector $v \in V'$ let $\gamma(v) = (v, f(v)) \in \mathbb{R}^n$ be its lift under the modification.

From 1.1.34, the function f is integer affine if and only if the divisor D is of sedentarity meaning $\mathcal{F}_\bullet(M|i) = 0$. From Lemma 1.1.39,

$$\mathcal{F}_k(M) \cong \mathcal{F}_k(M\backslash i)$$

and we have short exact sequence.

Otherwise, given a collection of integral vectors $v_1, \dots, v_{k-1}$ in a cone $\sigma \in D \subset \mathbb{R}^{n-1}$ the collection $e_i, \gamma(v_1), \dots, \gamma(v_{k-1})$ is contained in a cone $\tilde{\sigma} \in V$, generated by $\gamma(\sigma)$ and e_i. Although f is not linear, the map given by

$$v_1 \wedge \cdots \wedge v_{k-1} \mapsto e_i \wedge \gamma(v_1) \wedge \cdots \wedge \gamma(v_{k-1})$$

is still well defined, since $(v + w, f(v + w)) = \gamma(v) + \gamma(w) + me_i$, where $m \in \mathbb{Z}$. Taking the exterior product with e_i kills any indeterminacy. This gives the injective morphism $\mathcal{F}_{k-1}(M|i) \xrightarrow{\gamma_*} \mathcal{F}_k(M)$.

Given a collection of integral vectors $v_1, \dots, v_k$ in a cone of V, $\delta(v_1), \dots, \delta(v_k)$ are contained in a cone of V'. So $\mathcal{F}_k(M) \xrightarrow{\delta_*} \mathcal{F}_k(M\backslash i)$ given by

$$v_1 \wedge \cdots \wedge v_k \mapsto \delta(v_1) \wedge \cdots \wedge \delta(v_k),$$

is also a surjection. Finally, $Ker(\gamma_*) = Im(\delta_*)$, since $v \in Ker(\delta_*)$ if and only if $v = v' \wedge e_i$, but $v' \wedge e_i \in \mathcal{F}_k(M)$ if and only if $v' \in \mathcal{F}_{k-1}(M|i)$. This completes the proof. □

All the groups are free, so the Hom functor preserves exactness and we also get:

$$0 \longrightarrow \mathcal{F}^k(M\backslash i) \longrightarrow \mathcal{F}^k(M) \longrightarrow \mathcal{F}^{k-1}(M|i) \longrightarrow 0$$

Proof. Theorem 2.2.6 The Orlik-Solomon algebra and $\mathcal{F}^\bullet$ have similar exact sequences by Proposition 2.2.5 and Lemma 2.2.7. If M is the uniform matroid $U_{r,r}$ there are graded isomorphisms:

$$\mathcal{F}^\bullet(U_{r,r}) \cong OS^\bullet(U_{r,r}) \cong \Lambda^\bullet \mathbb{Z}^r .$$

For every rank r matroid there is a sequence of matroidal deletions along non-coloop elements ending at the uniform matroid $U_{r,r}$. By an induction on the rank and corank of the matroids, the vertical arrows at the right and left of the following diagram are isomorphisms.

$$\begin{tikzcd}
0 \arrow[r] & OS^k(M\backslash i) \arrow[r] \arrow[d, "\cong"] & OS^k(M) \arrow[r] \arrow[d] & OS^{k-1}(M|i) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathcal{F}^k(M\backslash i) \arrow[r, "\gamma"] & \mathcal{F}^k(M) \arrow[r, "\delta"] & \mathcal{F}^{k-1}(M|i) \arrow[r] & 0
\end{tikzcd}$$

By the five lemma the vertical arrow in the middle is also an isomorphism. □

2.2.2 (p, q)-cycles

The following is a summary of definitions of tropical (p, q)-homology which are to appear in [27]. To get an idea of the objects we start by describing (p, q)-cells contained in a matroidal fan $V \subset \mathbb{T}^n$. Consider the **coarse** polyhedral structure on a matroidal fan $V \subset \mathbb{T}^n$, and let $\sigma \subset V$ be a closed face. Consider the singular q-cells contained in σ such that their interior is contained in $\text{Int}(\sigma)$. Each such q-cell may be equipped with a **framing** coefficient, this is an element in $\mathcal{F}_p(\sigma)$. Let $C_q(\sigma)$ denote the group of singular q-chains formed by taking integer linear combinations of such singular q-cells. Then the (p, q)-chains in σ, denoted $C_{p,q}(\sigma)$ are the formal linear combinations over $\mathbb{Z}$ of p-framed q-cells, i.e.

$$C_{p,q}(\sigma) = C_q(\sigma) \otimes_{\mathbb{Z}} \mathcal{F}_p(\sigma).$$

The (p, q)-chains of V is the direct sum of all groups $C_{p,q}(\sigma)$ for σ a face of V, i.e. $C_{p,q}(V) = \oplus_{\sigma \subset V} C_{p,q}(\sigma)$.

For faces $\tau \subset \bar{\sigma} \subset V$ there is a homomorphism between the groups, $i_{\tau,\sigma}$: $\mathcal{F}_\bullet(\tau) \longrightarrow \mathcal{F}_\bullet(\sigma)$ from Definition 2.2.3. This allows us to define a boundary map on (p, q)-chains:

$$\partial : C_{p,q} \longrightarrow C_{p,q-1}$$

It is easy to see that $\partial^2 = 0$ since it acts only on the singular cells.

The above definitions may be extended for a tropical manifold X [27]. Then, tropical (p, q)-homology is the homology of the chain complex of $C_{p,q}$ groups and the above boundary operator. The group $H_{p,q}(X)$ may be thought of as homology with coefficients in $\mathcal{F}_p(X)$. The (p, q)-chains on a manifold are denoted $C_{p,q}(X)$. As usual we call a (p, q)-chain σ a cycle if it is in the kernel of ∂, denote the (p, q)-cycles by $Z_{p,q}(X)$. A **boundary** is a (p, q)-chain contained in the image of ∂, denote the (p, q)-boundaries by $Im_{p,q}(X)$.

Definition 2.2.8. The (p, q)-tropical homology groups of a tropical manifold X are

$$H^{trop}_{p,q}(X) := \frac{Z_{p,q}(X)}{Im_{p,q}(X)}.$$

The singular part of a (p, q)-chain may not be closed in ordinary singular homology however equipped with a coefficient in $\mathcal{F}_p$ it may become closed. Take for example a chain whose boundary is of greater sedentarity. Then, the map $\mathcal{F}_p(\sigma) \longrightarrow \mathcal{F}_p(\tau)$ is a quotient and the framing coefficient may be sent to zero. Also a q-chain homologous to zero in singular homology may no longer be trivial when equipped with a framing.

Example 2.2.9 Tropical toric surfaces
As an exercise we compute the (p, q) homology groups of a tropical toric surface X. Following the argument given by Mikhalkin in [35] for $\mathbb{TP}^n$. To start, for $q < 2$ every (p, q)-cycle in X is homologous to a (p, q)-cycle supported on the boundary ∂X of X. This is because the groups

$$H_q(X, \partial X; \Lambda_p(\mathbb{Z}^n)) = 0$$

for all (p,q), so for every (p,q)-cycle there is a cycle in ∂X such that their sum is the boundary of a $(p,q+1)$-cycle.

Now that any (p,q)-cycle can be moved to a q dimensional part of the boundary, if $p>q$ then the framing must vanish, since the framing coefficient here is contained in

$$\Lambda^\bullet(\mathbb{Z}^n)/ < v_i \mid i \in I > .$$

For $p=0$ we just have homology with $\mathbb{Z}$ coefficients of X. This space is contractible so $H_{0,0}(X)=\mathbb{Z}$ and $H_{0,q}=0$ if $q\neq 0$.

If $p=1, q=2$, there are no non-trivial closed cycles. If the boundary of a 2-cell is not of sedentarity then the framing coefficient cannot vanish, so the 2-cell must be all of X. Even still the framing only becomes zero on the boundary when it is the normal direction to a divisor. Not all divisors can have parallel normal vectors,

If $p=q=2$ then the 2-cell must be supported on all of X since the 2-framing could only vanish on the boundary if it is contained in ∂X. The framing $\phi\in\Lambda^2(\mathbb{Z}^2)$ can be expressed as $ke_1\wedge e_2$, for $k\in\mathbb{Z}$, so $H_{2,2}(X)=\mathbb{Z}$.

Finally the interesting case when $p=q=1$. Any cycle is equivalent to a sum of boundary divisors which must have a parallel 1-framing, meaning orthogonal to normal vector of a face. The surface X induces an orientation on the boundary divisors. Let $\alpha_i=(D_i,\phi_i)$ where ϕ is the framing with the same orientation as D_i. Then $H_{1,1}$ is generated by the α_i's with two relations coming from the boundaries of the $(2,1)$-cells $\tau_1=(X,e_1)$ and $\tau_2=(X,e_2)$, these boundaries are determined by the combinatorics of the dual fan of X.

$$H_{1,1}=\frac{\bigoplus_{i=1}^N \alpha_i\,\mathbb{Z}}{\langle \partial\tau_1,\partial\tau_2\rangle}.$$

Example 2.2.10 Abstract tropical curves
An abstract tropical curve is a graph equipped with a complete inner metric [38]. A neighborhood U of a non-leaf vertex $v\in C$ has a chart $\Phi: U\longrightarrow L_k\subset\mathbb{T}^{k-1}$ where k is the valency of the vertex and L_k is the 1-dimensional fan in $\mathbb{T}^{k-1}$ with directions

$$-e_1,\dots,-e_k,\ \text{and}\ e_0=e_1+\dots,e_k.$$

These charts provide the framing groups, $\mathcal{F}_\bullet(x)$, (only $\mathcal{F}_0(x)$ and $\mathcal{F}_1(x)$ may be non-zero). Since C is one dimensional, connected and $\mathcal{F}_0(x)=\mathbb{Z}$, $\mathcal{F}_2(x)=0$ for all $x\in C$ we have

$$H_{0,0}(C)=\mathbb{Z},\quad H_{0,2}(C)=H_{2,0}=0,\quad\text{and}\quad H_{p,q}(X)=0\ \text{if}\ p+q>2.$$

Let g denote the first Betti number of the graph. Then, we also have

$$H_{0,1}(C)=H_1(C;\mathbb{Z})\cong\mathbb{Z}^g\,.$$

The cycles $Z_{1,0}(C)$ are 1-framed points. Consider a collection of breaking points of the graph $\{x_1,\dots,x_g\}$, and let $\{l_1,\dots,l_g\}$ be a collection of cycles in $\mathbb{Z}_{0,1}(C)$ such that $x_i\in l_j$ if and only if $i=j$. Assign a primitive integer framing to x_i so that it is in the same direction as the oriented tangent vector of l_i. This defines a bilinear pairing on the cycle groups and it is easy to check that it descends to homology, so

$$H_{0,1}(C) \times H_{1,0}(C) \longrightarrow \mathbb{Z}\,.$$

This form is diagonalised over $\mathbb{Z}$ by the framed points x_i and the cycles l_i, so $H_{1,0}(C)$ is dual to $H_{0,1}(C)$.

Finally, if α is a $(1,1)$-cycle then the boundary of its supporting 1-simplicial chain must be on the vertices of the curve. For $\partial\alpha$ to be zero at an interior vertex, which recall is modeled on $L_k \subset \mathbb{T}^n$, all adjacent edges must be contained in α. Since C is connected α in fact contains all edges of the graph. Choosing an orientation on each of the edges of C and equipping every edge with a primitive integer vector that is consistently oriented with the edge we obtain a generator of $H_{1,1}(C) \cong \mathbb{Z}$. We may also choose a primitive integer vector with opposite orientation and obtain the negative of this .

2.2.3 Intersection of (p,q)-cycles

For the following definitions, we fix the standard orientation on $\mathbb{R}^n$, namely the standard basis vectors, $\{e_1, \dots, e_n\}$ form a positively oriented basis. Suppose, $\phi \in \Lambda^p(\mathbb{Z}^n)$ and $\varphi \in \Lambda^{p'}(\mathbb{Z}^n)$, then we may assume:

$$\phi = w_1 \wedge \cdots \wedge w_{n-p'} \wedge v_1 \wedge \cdots \wedge v_{p+p'-n}$$

$$\varphi = v_1 \wedge \cdots \wedge v_{p+p'-n} \wedge u_1 \wedge \cdots \wedge u_{n-p}.$$

In particular, if ϕ and φ correspond respectively to p and p'-dimensional lattices which intersect transversally, then $v_1, \dots, v_{p+p'-n}$ is a basis for the intersection (we do not need to bother with whether this basis is oriented).

Definition 2.2.11. Let $\alpha = (\phi, a)$ be a (p,q)-cell in $\mathbb{R}^n$ and $\beta = (\varphi, b)$ be a (p',q')-cell in $\mathbb{R}^n$. Suppose the underlying q-cell a and the q'-cell b intersect transversally in $\mathbb{R}^n$. Then the product of (ϕ, a) and (φ, b) is the $(p+p'-n, q+q'-n)$-cell

$$w(\phi \frown \varphi, a \frown b)$$

where,

1. the $(q+q'-n)$-cell $a \frown b$, is the oriented intersection of the simplicial cells;

2. the weight is the determinant,

$$w = \mathrm{Det}(w_1, \dots, w_{n-p'}, v_1, \dots, v_{p+p'-n}, u_1, \dots, u_{n-p}),$$

   where,

$$\phi = w_1 \wedge \cdots \wedge w_{n-p'} \wedge v_1 \wedge \cdots \wedge v_{p+p'-n}$$
$$\varphi = v_1 \wedge \cdots \wedge v_{p+p'-n} \wedge u_1 \wedge \cdots \wedge u_{n-p};$$

3. the $(p+p'-n)$-framing is
$$\phi \frown \varphi = v_1 \wedge \cdots \wedge v_{p+p'-n}.$$

We pause to make some remarks about parts two and three of the above definition. First, notice that if the lattices $\Lambda_\phi, \Lambda_\varphi \subset \mathbb{Z}^n$ corresponding to ϕ, φ do not intersect properly, (i.e. $\dim(\Lambda_\phi \cap \Lambda_\varphi) > p+p'-n$), then the determinant from part (2) is zero, making the entire intersection zero. Moreover, the above definition does not depend on whether $\{v_1, \dots v_{p+p'-n}\}$ is oriented or not, since an opposite choice of orientation changes the sign of w and also of $\phi \frown \varphi$ in part (3). Also notice that Definition 2.2.11 is in general not commutative, there may be a sign difference when we change the order of intersections.

Fixing the opposite orientation on $\mathbb{R}^n$ then in fact the total intersection product does not change since the change in orientation will produce a sign change in both the intersection of the singular cell and a sign change in the framing.

Lastly, there is topological reasoning behind the definition of the framing $\phi \frown \varphi$ and weight w in parts two and three above. Let $T_n = (S^1)^n = \mathbb{R}^n \setminus \mathbb{Z}^n$ denote the n-dimensional torus,
$$H_p(T_n;\mathbb{Z}) = \mathbb{Z}^{\binom{n}{p}} \quad \text{and} \quad H^p(T_n;\mathbb{Z}) = \Lambda^p(\mathbb{Z}^n).$$
By Poincaré duality we have an intersection pairing on $H_\bullet(T_n;\mathbb{Z})$. A p-framing ϕ represents a class in $H_p(T_n;\mathbb{Z})$ and intersecting with a p' framing φ gives a class in $H_{p+p'-n}(T_n;\mathbb{Z})$. The $p+p'-n$ framing: $w\phi \frown \varphi$ defined above represents this intersection.

Example 2.2.12
Consider the two cycles α, β in $\mathbb{R}^2$ on the left of Figure 2.6. At the intersection point x, the cycle α is given by the cell (e_1+e_2, a) and β is given by the cell (e_1, b). The positively oriented tangent vector to a being also e_1+e_2 and to b it is e_2. Therefore $a \frown b = x$ but the weight w from part 2 is equal to -1. Therefore, the intersection is $-x$.

Let us suppose we were to consider (p,q)-cycles in an n-dimensional matroidal fan $V \subset \mathbb{T}^N$. Then a (p,q)-cell (ϕ, a) and (p',q')-cell (φ, b) intersect transversally in V, if $\text{Int}(a)$, $\text{Int}(b)$ are contained in an open facet $F^o \subset \mathbb{R}^n \subset \mathbb{R}^N$of V, and they intersect transversally in $\mathbb{R}^n$. Then, Definition 2.2.11 can be extended to transverse intersections in V.

Definition 2.2.13 (Transversal intersections)**.** Let $V \subset \mathbb{T}^N$ be a matroidal fan of dimension n and $\alpha \in Z_{p,q}(V)$ and $\beta \in Z_{p',q'}(V)$, then α and β intersect transversally in V if the following hold,

1. the q-chain supporting α and the q'-chain supporting β intersect transversally;

2. the interior of each cell of dimension $q+q'-n$ in the intersection is contained in the interior of a facet of V.

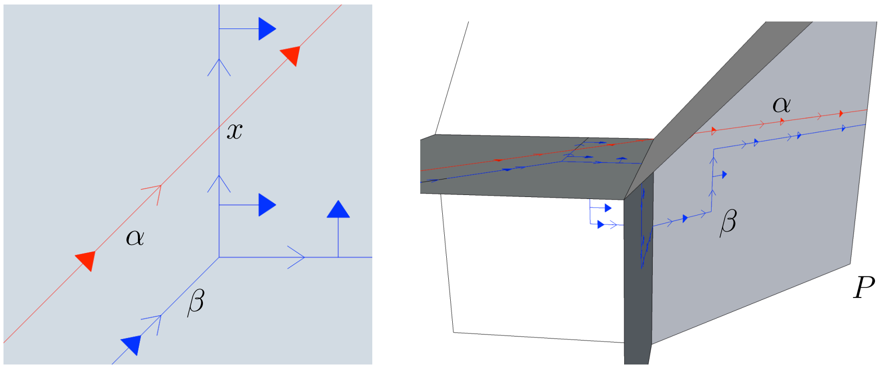


Figure 2.6: a) Two $(1, 1)$ homology cycles in a plane. b) A close up of their only point of intersection. Solid arrows denote their framings and the other arrows the orientations of the cells.

Example 2.2.14
On the right side of Figure 2.6 there are two $(1, 1)$-cycles α, β intersecting transversally in a facet of the matroidal fan $P \subset \mathbb{T}^3$. The point of intersection is in the interior of a facet of V, moreover zooming into the intersection point in the facet, the cycles are exactly the α and β drawn on the left. Therefore, the weight of the intersection is -1.

In the next chapter on a smooth compact tropical surface X we construct an intersection product on homology classes in $H_{1,1}(X)$. We end with a way of relating the previously encountered k-cycles to tropical (p, q)-cycles.

2.2.4 The cycle map

Recall the definition of a tropical cycle $A \in Z_k(X)$ from Section 1.2.3. If X is a **compact** tropical manifold, there is a *cycle map*

$$Cyc : Z_k(X) \longrightarrow Z_{k,k}(X)$$

for every k. Just as in complex geometry, see [18]. A facet of a tropical cycle $A \in Z_k(\mathbb{T}^n)$ is almost a k-cell, except it lacks an orientation and framing. For a facet F choose any orientation, there is a $\mathbb{Z}_2$ choice, (in the end this choice will not make a difference). Since A is rational, each facet F is parallel to a k-dimensional lattice Λ_F. Choose a basis $\{v_1, \dots v_k\}$ for Λ_F, that is positively oriented with respect to the chosen orientation of F. The framing ϕ_F will be $v_1 \wedge \dots \wedge v_k \in \Lambda^k \mathbb{Z}^n$. Then F becomes $w_F(F, \phi_F)$, where w_F is the integer weight of F in A. This does not depend on the initial choice of orientation since $(F, \phi_F) \in C_q \otimes \mathcal{F}_p$.

$$Cyc(A) = \sum_{F \text{ facet of } A} w_F(F, \phi_F).$$

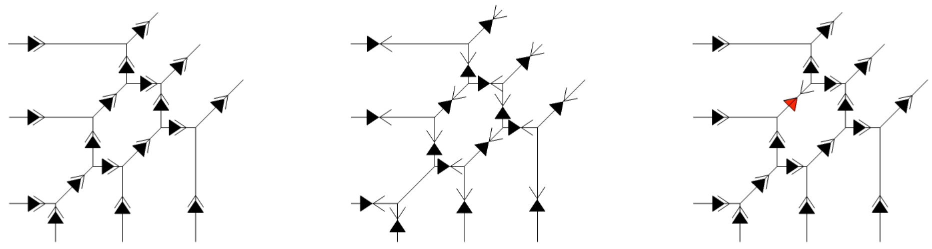


Figure 2.7: a) A framing given by Cyc. b) A framing given by $\overline{Cyc}$ c) An inconsistent framing giving a cycle that is not closed.

Alternatively, one may take a negatively oriented basis for the framing of each face, this would give $-Cyc(A)$.

Proposition 2.2.15. *Let X be a compact tropical manifold and $A \subset X$ a tropical k-cycle then, $\partial Cyc(A) = 0$.*

Proof. When a codimension one face of A is of sedentarity $\emptyset$ this follows from the balancing condition on A. This is because the coefficient of this face in $\partial Cyc(A)$ is the sum

$$\sum_{F_i \supset E} \pm w(F_i)\phi_{F_i} \in \Lambda^k \mathbb{Z}^{N_i},$$

where the sign depends on the orientation of the faces. But because we have chosen a parallel framing, for each F_i we may write $v_{F_i} = u_i \wedge v_E$, where $\phi_E \in \Lambda^{k-1} \mathbb{Z}^{N_i}$ is a positively oriented basis of Λ_E and u_i is the outward pointing normal vector F_i relative to E. The sum above becomes

$$(\sum_{F_i \supset E} u_i) \wedge v_E.$$

By the balancing condition $\sum_{F_i \supset E} u_i$ is parallel to the face E, and so the above wedge product is zero.

Now if E is of positive sedentarity, it follows from above if A has the same sedentarity. Otherwise, take a chart $\Phi : U \longrightarrow V \subset \mathbb{T}^N$, then E must be in the closure of $\Phi(U \cap F)^o \subset \mathbb{R}^N$, where F is some face of A therefore the framing of $\Phi(U \cap F)$ becomes zero when we take the boundary. $\square$

Chapter 3

Tropical surfaces

The study of tropical surfaces presented here relies on the intersection theory of curves and of $(1,1)$-cycles. The first part of Section 3.1 simplifies the intersection products defined in Chapter 2 in the case of surfaces. We also show that the product for transverse (p,q)-cycles in a tropical manifold can be defined on $(1,1)$-homology classes of a compact tropical manifold and use this to define the self-intersection of boundary curves.

Section 3.2 describes a procedure called the tropical sum, used to construct new tropical surfaces from old. As an example we use the sum construction to describe blow downs of submatroidal -1-rational fan curves in tropical surfaces. In 3.2.2 we consider tropical versions of some equalities for complex algebraic surfaces, namely Noether's formula and Hirzebruch's signature theorem. Noether's formula is proved for a certain class of surfaces described in Proposition 3.2.17.

In Section 3.3 we compute the $(1,1)$-homology of a floor decomposed tropical hypersurfaces in $\mathbb{TP}^3$ and also the intersection form on $H_{1,1}$. It follows from Theorem 3.3.5, that the tropical homology of such a surface does not satisfy an analogue of the Hodge Index Theorem. However, the signature of the form on $H_{1,1}$ is the signature of the intersection form on H^2 of a non-singular complex hypersurface in $\mathbb{CP}^3$ of the same degree.

The final part of this section considers local approximability of tropical curves in surfaces. To do this we establish a relation between tropical and complex intersection multiplicities for curves. This is then applied throughout the rest of Section 3.4 to completely classify approximable fan tropical curves contained in the intersection of the standard tropical hyperplane in $\mathbb{R}^3$ and an rational affine hyperplane. This is the case considered by Bogart and Katz in [7]. Furthermore, in 3.4.4 this classification is generalised to cover all trivalent tropical fan curves contained in any fan plane $P \subset \mathbb{R}^N$.

Section 3.2.2 presents the canonical class, K_X, of a tropical surface X as a tropical 1-cycle following [38]. Theorem 3.4.41 establishes the following adjunction formula for tropical curves which are locally submatroidal (**non-singular**),

$$b_1(C) = \frac{K_X.C + C^2}{2} + 1.$$

The dramatic result of this section is the discovery of irreducible tropical curves in surfaces for which the right hand side of the above equation is negative. We establish a correspondence between the classical adjunction formula and the above tropical one for constant families which leads to a general local obstruction to lifting tropical curves in surfaces given in Theorem 3.4.36. An interesting conclusion to be deduced from this section is highlighted in Example 3.4.40, which shows that there are singular lines in smooth tropical surfaces. This leads to the curious phenomenon that being non-singular is not an intrinsic property of tropical varieties.

The results in presented in Sections 3.4.1, 3.4.5 and parts of 3.4.3 are joint work with Erwan Brugallé from [12].

Before we begin we make some global definitions. A **tropical curve** in a surface is an effective 1-cycle. A **fan cycle** (respectively curve) in a matroidal fan $V \subset \mathbb{R}^n$ is a tropical cycle (respectively curve) with a single vertex coinciding with the vertex of V. The terms **boundary curve** and boundary divisor will be used interchangeable to denote a boundary divisor equipped with weight one. We call points which are intersections of two or more boundary divisors of X **corner points**. The intersection $A.B$ of two 1-cycles is a well defined collection of points in $(A \cap B)^{(0)}$, in Section 3.2 and onwards we will be interested in numerical properties of intersections. Therefore, after summarizing and simplifying the local definitions we will usually just mean by $A.B$ the sum of the local multiplicities, i.e

$$A.B = \sum_{x \in (A \cap B)^{(0)}} (A.B)_x.$$

This should be clear from context. Lastly, before the precise definition is given in Section 3.4 one should think of the tropicalisation of a complex curve $\mathcal{C} \subset (\mathcal{C}^*)^N$ as the logarithmic limit set, $\text{Trop}(\mathcal{C}) = \lim_{t \to \infty} \text{Log}_t(\mathcal{C})$ equipped with weights. The tropicalisation of a curve will only be used sparingly in examples until the precise definition is provided.

3.1 Intersections in surfaces

3.1.1 Intersection with a boundary curve

All points of a boundary divisor in a tropical manifold are of sedentarity greater than zero. A boundary curve D and a non-boundary cycle C in a tropical surface always intersect in a finite collection of points of sedentarity one or two.

Definition 3.1.1. Let $C \subset X$ be a cycle of sedentarity zero and $D \subset X$ an irreducible boundary divisor. We define,

$$C.D = \sum_{x \in C \cap D} (C.D)_x x.$$

Where

(1) If x is a point of sedentarity one in $C \cap D$ adjacent to an edge e of C then $(C.D)_x = w_e$, where w_e is the weight of e.

(2) If x is a point of sedentarity two, choose a neighborhood U of x and take a chart $\Phi : U \longrightarrow V \subset \mathbb{T}^N$, so that x is the point of sedentarity two and the image of $D \cap U$ is contained in the divisor of V corresponding to $x_i = -\infty$ in the coordinates of $\mathbb{T}^N$. Let $v_1, \dots v_s$ denote the primitive integer directions of all rays of $\Phi(C \cap U)$ which end at the corner of V and $w_1, \dots w_s$ their respective weights. Then

$$(C.D)_x = \sum_{j=1}^{s} w_j \langle e_i, v_j \rangle.$$

In different charts the weights are preserved and under a integer affine transformation and so is the inner product is preserved, so the intersection multiplicity defined in part (2) of Definition 3.1.1 is independent of the chart chosen. Any boundary divisor can be decomposed into a sum of irreducible boundary divisors, and we extend the product by linearity.

Definition 3.1.2. Given two distinct irreducible boundary curves D_1, D_2 in a tropical manifold X, then

$$D_1.D_2 = \sum_{x \in D_1 \cap D_2} x.$$

3.1.2 Intersections at corner points.

Two non-boundary tropical cycles may still intersect at the boundary of X. Given a point $x \in (C_1 \cap C_2)^{(0)}$ in the interior of a boundary divisor, we define the intersection multiplicity to be zero.

We now consider corner points. Figure 3.2 shows two cycles, which are both affine lines, meeting at the corner of $\mathbb{T}^2$. Their primitive integer directions are (p, q) and (r, s), where we insist that $p, q, r, s < 0$. Let w, v be the weights of the edges of these two cycles. After a small deformation of one of the two curves, tropical stable intersection gives intersection multiplicity $wv \min\{ps, qr\}$.

Example 3.1.3
In Figure 3.1 is the real drawing of two complex curves

$$\mathcal{C}_1 = \{Y - X^3 = 0\} \quad \text{and} \quad \mathcal{C}_2 = \{Y + X^2 = 0\}.$$

The curve $\mathcal{C}_1$ has an inflection point at $(0, 0)$ to the line $\{Y = 0\}$, and the curve $\mathcal{C}_2$ has a tangency at the same point also to the line $\{Y = 0\}$. In this simple example it is easy to check that the curves have intersection multiplicity 2 at $(0, 0)$ despite the fact that the product of the multiplicities of the curves at this point is 1. The tropicalisations of the curves are dual to the Newton polytopes so, $\text{Trop}(\mathcal{C}_1) = C_1$ is

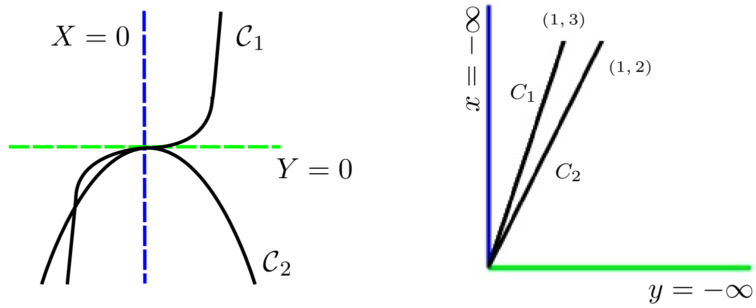


Figure 3.1: The intersection of two curves at an inflection point and a tangency, along with the tropicalisation.

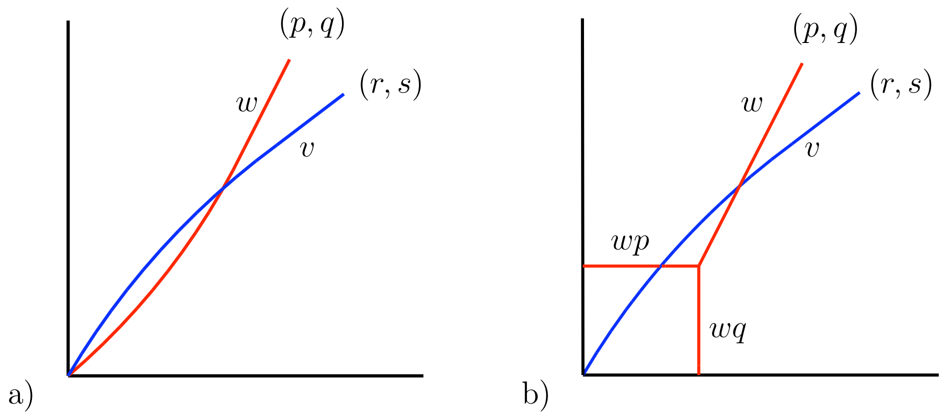


Figure 3.2: a) Two cycles intersecting in a corner of $\mathbb{T}^2$ b) A small deformation showing their intersection multiplicity.

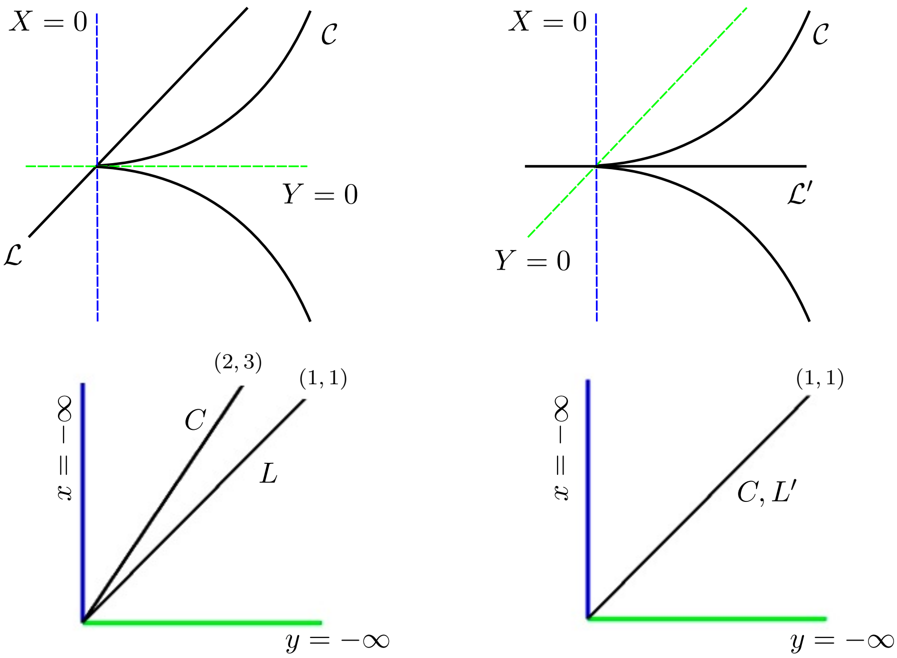


Figure 3.3: On top are the two configurations of curves with respect to the coordinate axes which appear in Example 3.1.5. Below are their respective tropicalisations.

a ray in the direction $(1,3)$, and $\mathrm{Trop}(\mathcal{C}_2) = C_2$ is a ray in the direction $(1,2)$. The tropical multiplicity at the corner is

$$(C_1.C_2)_{(-\infty,-\infty)} = 2.$$

Definition 3.1.4. Let $C_1, C_2 \subset V \subset \mathbb{T}^N$ be two tropical cycles of sedentarity zero in a two-dimensional matroidal fan V. Suppose V contains the corner point $x = (-\infty, \dots, -\infty)$ and the cycles each have exactly one ray converging to x.

1. If $P = \mathbb{T}^2$ suppose the ray of C_1 has primitive integer direction (p, q) with weight w and C_2 has primitive direction (r, s) with weight u then define,

$$(C_1.C_2)_{(-\infty,-\infty)} = wu\min\{ps, qr\}.$$

2. Otherwise, fix a projection $\pi : \mathbb{T}^N \longrightarrow \mathbb{T}^2$ which does not contract the faces of V containing the two rays of the cycle then define,

$$(C_1.C_2)_x = (\pi(C_1).\pi(C_2))_{(-\infty,-\infty)}.$$

When the cycles have multiple rays converging to the corner point extend by distributivity.

Example 3.1.5
At the top left of Figure 3.3 is a cuspidal cubic $\mathcal{C} = \{X^3 - Y^2 = 0\}$ and a line $\mathcal{L} = \{X - Y = 0\}$ drawn with respect to the coordinates X, Y. Again the tropical curves are dual to the Newton polytopes, so $\text{Trop}(\mathcal{C}) = C$ is a ray in direction $(2, 3)$ and $\text{Trop}(\mathcal{L}) = L$ a ray in direction $(1, 1)$. Their tropical intersection multiplicity at $(-\infty, -\infty)$ is 2 and corresponds with the complex one at $(0, 0)$.

The right hand side of Figure 3.3 depicts and the tropicalisations $\text{Trop}(\mathcal{C}) = C$ and $\text{Trop}(\mathcal{L}') = L$ are both supported on the ray of $(1, 1)$, however in C this ray is equipped with weight 2. From the definition, $(C.L')_{(-\infty,-\infty)} = 2$ whereas $(\mathcal{C} \,.\, \mathcal{L})_{(0,0)} = 3$. The complex intersection multiplicity at $(0, 0)$ of two curves is not always equal to the multiplicity of their tropicalisations at $(-\infty, -\infty)$.

3.1.3 Intersection at points of sedentarity zero

Given two cycles: $C_1, C_2 \subset X$, let x be a point in $(C_1 \cap C_2)^{(0)}$ of sedentarity zero. The intersection multiplicity at such a point has been given in Section 2.1.1 for general tropical manifolds. Here we give an alternative and simpler definition in the case of surfaces. To begin, choose a neighborhood U of x and a chart $\Phi : U \longrightarrow V \subset \mathbb{R}^N$ where V is a matroidal fan and for $i = 1, 2$, $C_i \cap U$ is a fan with vertex v the vertex of V. Throughout, $\overline{V} \subset \mathbb{TP}^N$ will denote the compactification of $V \subset \mathbb{R}^N$ given by the unimodular basis $u_1, \dots, u_N$ used to construct the fan V as in Section 1.1.4. Then $\overline{V}$ is a modification of $\mathbb{TP}^2$. We will assume as usual that the matroid corresponding to V contains no loops nor double points.

The compactification of the plane $V \subset \mathbb{R}^N$ to $\overline{V} \subset \mathbb{TP}^N$ determines an arrangement of tropical lines
$$L_i = \overline{V} \cap \{x_i = -\infty\} \subset \mathbb{TP}^N,$$
for $0 \leq i \leq N$ where $[x_0 : \cdots : x_N]$ are tropical homogeneous coordinates on $\mathbb{TP}^N$. The arrangement will be denoted $A = \{L_0, \dots, L_N\}$. A **point** of A is a point contained in at least two lines of A, denote the collection of points $p(A)$. For a point $p \in p(A)$ we may associate to it the maximal subset $I \subset \{0, \dots, N\}$ such that $p \in \cap_{i \in I} L_i$, therefore we may index the points by p_I. . The **size** of a point p_I is $|I|$. The points $p(A)$ are in correspondence with flats of the matroid corresponding to V that are of size greater than 1 but rank less than 3. A point $p_I \in p(A)$ has a neighborhood U and a chart $\Phi : U_p \longrightarrow \mathbb{TP}^{|I|}$, such that $\Phi(p_I) = (-\infty, \dots, -\infty)$. Given two cycles $C_1, C_2 \subset V$ their closures $\overline{C}_1, \overline{C}_2 \subset \overline{V}$ may intersect at points $p_I \in p(A)$. The intersection multiplicity being given by Definition 3.1.4. Using the intersection multiplicities at the points $p(A)$ we will give an alternative formula for the intersection of two fan cycles $C_1, C_2 \subset V \subset \mathbb{R}^N$ at v the vertex of V. First we define the degree of a fan cycle in a matroidal fan V.

First suppose that a vector $v \in \mathbb{Z}^N$ is contained in V. By the construction of V described in Section 1.1.4, v is contained in a cone generated by u_i, u_I for some $I \in p(A)$ and $i \in I$, where $u_I = \sum_{i \in I} u_i$. Therefore, there is a unique expression, $v = \rho_i(v)u_i + \rho_I(v)u_I$ where $\rho_i(v), \rho_I(v)$ are non-negative integers. Set $r_i(v) = \rho_i(v) + \rho_I(v)$ if the direction v is contained in a cone generated by u_i and u_I

for some $I \ni i$, and $r_i(v) = 0$ otherwise. Given an edge $e \in \mathrm{Ed}(C)$, we denote by v_e the primitive integer vector of e pointing outward from the vertex of C.

Definition 3.1.6. Let $V \subset \mathbb{R}^N$ be a 2-dimensional matroidal fan with respect to the unimodular basis $u_1, \dots, u_N$ and $C \subset V$ be a fan tropical cycle. For an edge $e \in \mathrm{Ed}(C)$, let u_e denote the primitive integer vector in the direction of e and w_e the weight of e. Choose a u_i and define,

$$\deg(C) = \sum_{e \in \mathrm{Ed}(C)} w_e r_i(v_e). \tag{3.1.1}$$

It follows from the balancing condition that the above definition is independent of the choice of u_i.

Theorem 3.1.7. *Let $V \subset \mathbb{R}^N$ be a matroidal fan, given two tropical fan cycles $C_1, C_2 \subset V$, let $\overline{C}_1, \overline{C}_2$ denote the closures of the cycles in $\overline{V} \subset \mathbb{TP}^N$ and let A denote the arrangement defined by $\overline{V}$. The intersection multiplicity of the two cycles C_1, C_2 at the vertex v of V is,*

$$(C_1.C_2)_v = \deg(C_1)\deg(C_2) - \sum_{x \in p(A)} (\overline{C}_1.\overline{C}_2)_x.$$

Proof. If $V = \mathbb{R}^2$ for two curves C_1, C_2 the statement follows immediately by calculating the difference in the mixed volumes of the polytopes dual to the curves C_1, C_2 and the simplicies of size $\deg(C_1)$, $\deg(C_2)$. For general cycles we may sum with curves to obtain effective cycles and deduce the result by linearity.

Suppose $V \subset \mathbb{R}^N$ is of codimension at least one. Choose an elementary open contraction $\delta : V \longrightarrow V'$, and denote the associated divisor $D \subset V'$. Suppose without loss of generality that the kernel of δ is generated by u_N. By induction we may assume that the statement holds for the cycles $\delta_*C_1, \delta_*C_2 \subset V'$. Moreover, by Definition 3.1.6 it is mere linear algebra that $\deg(C_i) = \deg(\delta_*C_i)$. The statement reduces to:

$$\sum_{p' \in p(A')} (\overline{\delta_*C}_1.\overline{\delta_*C}_2)_{p'} - \sum_{p \in p(A)} (\overline{C}_1.\overline{C}_2)_p = (\Delta_1.\Delta_2)_v \tag{3.1.2}$$

where Δ_i is the correction cycle for C_i from Section 2.1.1 and v is the vertex of the fan. For a point $p_I \in p(A)$, if $N \notin I$ then $p_I \in p(A')$ and moreover,

$$(\overline{\delta_*C}_1.\overline{\delta_*C}_2)_{p_I} = (\overline{C}_1.\overline{C}_2)_{p_I}.$$

If $|I \backslash N| = 1$ then $I \backslash N$ does not correspond to a point in $p(A')$ so declare

$$(\overline{\delta_*C}_1.\overline{\delta_*C}_2)_{p_{I\backslash N}} = 0.$$

So Equation 3.1.2 becomes

$$\sum_{\substack{p_I \in p(A) \\ \text{s.t.} N \in I}} (\overline{\delta_*C}_1.\overline{\delta_*C}_2)_{p_{I\backslash N}} - (\overline{C}_1.\overline{C}_2)_{p_I} = (\Delta_1.\Delta_2)_v.$$

For a point $p_I \in p(A)$ such that $N \in I$ and a cycle $C \subset V$ let,

$$\text{Edge}^o_C(p_I) = \{e \subset \text{Edge}(C), \mid p_I \in \overline{e},\ e \not\subset V^{(1)}\}.$$

For each edge $e \in \text{Edge}^o_C(p_I)$ define a trivalent cycle $C(e)$. This cycle consists of an edge in the direction of e of weight w_e, an edge in the direction u_N, and the third edge in the direction of $\delta^*\delta_*(e)$, i.e. it is supported on the 1-skeleton of V. The weights on the last two edges are determined by the balancing condition. The cycle Δ_i splits into the sum:

$$\Delta_i = -\sum_{\substack{p_I \in p(A) \\ \text{s.t.} N \in I}} \sum_{e \in \text{Edge}^o_{C_i}(p_I)} C_i(e),$$

If $I = \{s, N\}$, and $e \in \text{Edge}^o_{C_1}(p_I)$ and $f \in \text{Edge}^o_{C_2}(p_I)$. Denote the primitive integer vectors in the directions of e, f by u_e, u_f respectively, and suppose $u_e = pu_s + qu_N$ and $u_f = ru_s + su_N$. Then,

$$(C_1(e).C_2(f))_v = -(\overline{e}.\overline{f})_{p_I} = -w_e w_f \min\{ps, qr\}. \tag{3.1.3}$$

For I such that $|I| > 2$ and $N \in I$, and an edge $e \in \text{Edge}^o_C(p_I)$ the cycle $C(e)$ has rays in directions u_e, u_N, u_I and suppose $u_e = pu_I + qu_N$ and $u_f = ru_I + qu_N$.

$$(\delta_* e.\delta_* f)_{p_{I \setminus N}} = w_e w_f pr.$$

If $e \in \text{Edge}^o_{C_1}(p_l)$ and $f \in \text{Edge}^o_{C_2}(p_J)$ then $(C_1(e).C_2(f))_v = 0$ and the product of the correction cycles is:

$$(\Delta_1.\Delta_2)_v = \sum_{\substack{p_I \in p(A) \\ \text{s.t.} N \in I}} \sum_{\substack{e \in \text{Edge}^o_{C_1}(p_l) \\ f \in \text{Edge}^o_{C_2}(p_l)}} (C_1(e)C_2(f))_v.$$

The theorem now follows from the distributivity of the intersection products. □

This alternative definition of the local intersection product will serve us in Section 3.4.1 where we relate complex and tropical intersection numbers.

3.1.4 Intersection form on $(1,1)$-cycles

Recall that we defined in Section 2.2.3 the intersection product of a (p,q) and a (p',q')-cycle intersecting *transversally* in $\mathbb{R}^n$ and in $V \subset \mathbb{T}^n$. Recall that the choice of orientation on $\mathbb{R}^n$ did not effect the product of transversal $(1,1)$-cycles. So the intersection can be extended to $(1,1)$ cycles in X intersecting transversally in every chart. The intersection of two transversely meeting $(1,1)$-cycles is a collection of $\mathbb{Z}$ weighted points. Throughout we let $\alpha.\beta$ denote the sum of the weights of these points, (i.e. $\alpha.\beta$ is a class in $H_{0,0}(X)$). We will prove that this product descends to $(1,1)$-homology classes on a smooth compact tropical manifold X.

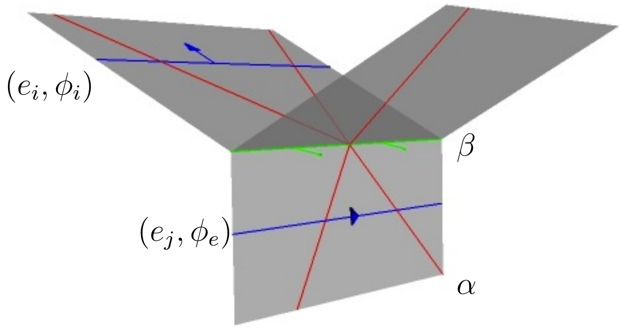


Figure 3.4: Intersection on an edge of a matroidal fan $V \subset \mathbb{T}^N$ and the splitting from Equation 3.1.5.

First we extend the product in the following case: Let α and β be $(1,1)$-cycles of X which intersect properly. Let x be a point in their intersection (i.e. a point in the intersection of the underlying simplicial 1-cycles); moreover, suppose that x lays in the interior of a framed edge (e, ϕ) of β where e is contained in a codimension one face of X, (e may be in the boundary of X). See Figure 3.4. Label the facets of X adjacent to e, by $F_0, \dots, F_k$. Then there exists a splitting of the framing,

$$\phi = \phi_e + \sum_{i=0}^{k} \phi_i \tag{3.1.4}$$

where ϕ_e is parallel to the edge e, and for all $0 \leq i \leq k$, there exists a small $\epsilon > 0$ such that $x \pm \epsilon\phi_i \in F_i$ and $< \phi_i, \phi_e >= 0$. Let $e_i = e \pm \epsilon\phi_i$ be a translation of e contained in the facet F_i. Then (e_i, ϕ_i) is a $(1,1)$-cell which intersects α transversally in the facet F_i in a finite collection of points in a neighborhood of x. Define

$$(\alpha.\beta)_x := \alpha.(e_j, \phi_e) + \alpha.\sum_{i=1}^{k}(e_i, \phi_i). \tag{3.1.5}$$

See the right hand side of Figure 3.4.

Notice that the first term of the above expression requires a choice of $e_i \subset F_i$. We must verify that the above definition does not depend on this choice of e_i nor on the choice of splitting of the framing ϕ from 3.1.4. Firstly, for every $(1,1)$-chain in α adjacent to x, we may suppose that the 1-chain f is oriented outward from x as a singular chain and has coefficient one. This is simply by transferring any change of orientation or multiplicity to the framing vector ϕ_f. In addition, for a $(1,1)$-cell (f, ϕ_f) of α contained in a facet F_i, we have $\phi_f = pv_i + qv_e$ for some $p, q \in \mathbb{Z}$. Here, v_e is the primitive integer vector in the direction of e, and v_i is the primitive integer direction orthogonal to v_e and parallel to F_i. Thus, $(f, \phi_f).(e_i, \phi_e) = p$. Moreover, the vectors $v_0, \dots v_k$ satisfy $\sum_{i=0}^{k} v_i = 0$. Then the fact that $(\alpha.\beta)_x$ does not depend on the choice of face F_i follows from

$$\sum_{\substack{f\in\alpha \\ x\in f}} \phi_f = 0. \tag{3.1.6}$$

This is the condition that α is closed.

To show that Definition 3.1.5 does not depend on the splitting it suffices to consider the case when the initial edge of β is (e, v_i), where v_i are one of the vectors from above. Without loss of generality we may assume $i = 0$. Here the other representation of the framing is given by $v_0 = -\sum_{i=1}^k v_i$. So the corresponding edge of β may be simply moved to (e_0, v_0) or split as $-\sum_{i=1}^k (e, v_i)$, and then each (e, v_i) translated to (e_i, v_i). A $(1,1)$-cell (f, ϕ_f) of α, adjacent to x and contained in the facet F_i again satisfies $\phi_f = pv_i + qv_e$, so that $(f, \phi_f).(e_i, v_i) = q$. Once again the two splittings give the same result for $(\alpha.\beta)_x$ because α satisfies the Equation 3.1.6.

Notice that in showing the intersection product in the above case is well defined we used that X is locally matroidal. Say a $(1,1)$-cycle α in a surface X **properly intersects the skeleton** of X if

$$\alpha \cap X^{(0)} = \emptyset \quad \text{and} \quad \dim(\alpha \cap X^{(1)}) = 0.$$

Then we have the following lemma.

Lemma 3.1.8. *Suppose X is a compact tropical surface, and suppose α, β, β' are $(1,1)$-cycles such that*

1. *$\beta \sim \beta'$;*

2. *α and β intersect transversally in X, similarly for α and β';*

3. *α, β, β' all properly intersect the skeleton of X;*

then $\alpha.\beta = \alpha.\beta'$.

Proof. It suffices to consider the case when $\beta - \beta'$ is the boundary of a single $(1,2)$-cell τ. By subdividing the $(1,2)$-cell if necessary, we may suppose that $\tau \in U$ where U has a chart $\Phi : U \longrightarrow V \subset \mathbb{T}^n$. We identify τ and all of the $(1,1)$-cycles with their images in V. This $(1,2)$-cell must be contained in a closed facet of V, and $\partial\tau$ is equipped with a constant framing ϕ. First suppose that τ is contained in the interior of a facet F of V, then in particular τ contains no points of positive sedentarity, and $\partial\tau$ is also in the interior of F, in this case we may also suppose that α and $\partial\tau$ intersect transversally. The cycle α intersects $\partial\tau$ in a finite collection of points, in the interior of cells (f, ϕ_f) of α. Once again we may assume that all 1-cells f are oriented to point outwards from τ and have multiplicity one. Then since α is closed we have,

$$\sum_{\substack{(f,\phi_f)\in\alpha \\ f\cap\partial\tau\neq\emptyset}} \phi_f = 0.$$

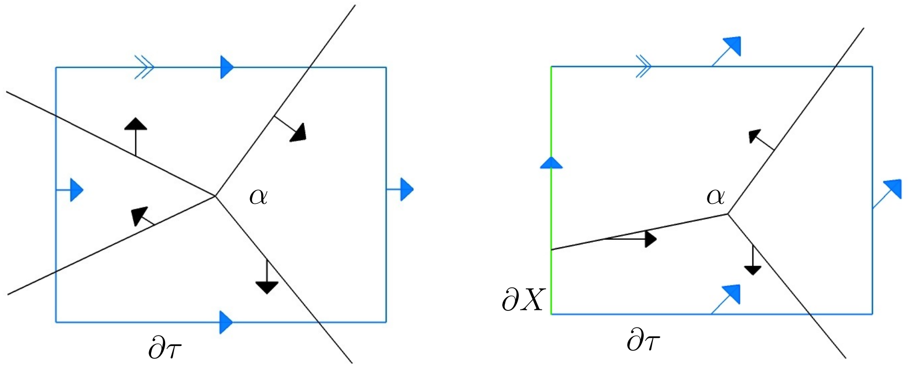


Figure 3.5: Intersection with a boundary $\partial\tau$ when the boundary is of sedentarity $\emptyset$ and when it is of mixed sedentarity.

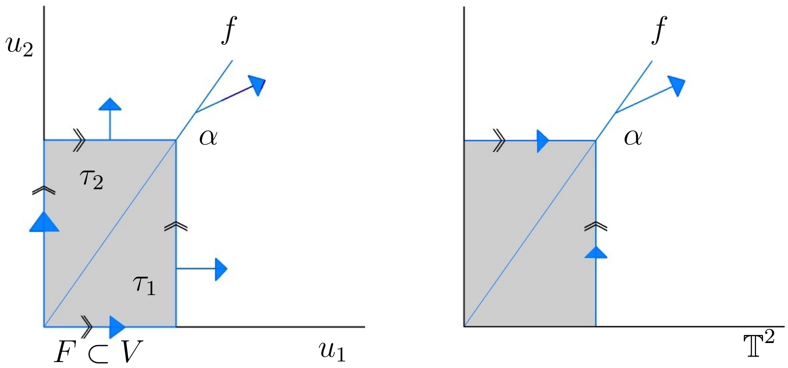


Figure 3.6:

The framing on $\partial\tau$ is constant, say ϕ_τ, so we obtain

$$\partial\tau.\alpha = \pm \sum_{\substack{(f,\phi_f)\in\alpha \\ f\cap\partial\tau\neq\emptyset}} \phi_\tau \wedge \phi_f = 0.$$

See the right hand side of Figure 3.5.

If τ is not contained in an open face, then $\partial\tau$ has edges on the one skeleton, and the cycles β and $\partial\tau$ may intersect properly at such an edge. If τ contains no points of positive sedentarity, then $\alpha.\partial\tau = 0$ follows from Definition 3.1.5 and the above argument by translating each edge of $\partial\tau$ into the interior of F. Otherwise we may assume without loss of generality that τ contains a single edge e of sedentarity $\{1\}$. Also we may suppose for simplicity that τ intersects no other edges of V. Let v_1, v_2 denote a primitive integer basis of the facet of V containing τ where $< v_1, v_2 >= 0$ and v_1 is the direction orthogonal to the stratum of the boundary containing e. Also, suppose τ is equipped with framing $pv_1 + qv_2$ for $p, q \in \mathbb{Z}$ then all edges of $\partial\tau$ have framing $pv_1 + qv_2$ except e which is equipped with framing qv_2.

For edges of α intersecting $\partial\tau$ we have

$$\sum_{\substack{(f,\phi_f)\in\alpha\\ f\cap\partial\tau\neq\emptyset}} \phi_f - rv_1 = 0,$$

for some $r \in \mathbb{Z}$. The term rv_1 appears since the boundary map takes the quotient of the framing of an edge of α intersecting the edge of $\partial\tau$ contained in the boundary of V. Then,

$$\begin{aligned}
0 &= \langle \phi_\tau, \sum_{\substack{(f,\phi_f)\in\alpha\\ f\cap\partial\tau\neq\emptyset}} \phi_f - rv_1\rangle \\
0 &= \langle \phi_\tau, \sum_{\substack{(f,\phi_f)\in\alpha\\ f\cap\partial\tau\neq\emptyset, f\cap e=\emptyset}} \phi_f\rangle + \langle \phi_\tau, \sum_{\substack{(f,\phi_f)\in\alpha\\ f\cap\partial\tau\neq\emptyset, f\cap e\neq\emptyset}} \phi_f - rv_1\rangle \\
0 &= \langle \phi_\tau, \sum_{\substack{(f,\phi_f)\in\alpha\\ f\cap\partial\tau\neq\emptyset, f\cap e=\emptyset}} \phi_f\rangle + \langle qv_2, \sum_{\substack{(f,\phi_f)\in\alpha\\ f\cap\partial\tau\neq\emptyset, f\cap e\neq\emptyset}} \phi_f\rangle \\
0 &= \alpha.\partial\tau
\end{aligned}$$

This proves the lemma. □

Lemma 3.1.9. *Let α be a $(1,1)$-cycle in X, then there exists a $(1,1)$-cycle α' such that $\alpha \sim \alpha'$ and α' intersects the skeleton of X properly.*

Proof. If α contains a $(1,1)$-cell (e,ϕ) where e is in the one skeleton of X, we can choose a splitting of ϕ as in 3.1.4, and then move the cells (e,ϕ_i), and (e,ϕ_e) into the interior of the facets of X. So we may suppose that $\dim(\alpha \cap X^{(1)}) = 0$ and let $x \in \alpha \cap X^{(0)}$ be a point of sedentarity order 0. Take a chart $\Phi : U \longrightarrow V \subset \mathbb{R}^n$ and identify a neighborhood of x in X with a the neighborhood of a vertex of a matroidal fan $V \in \mathbb{R}^n$, similarly identify α with its image in the fan V. Let $L_{N+1} \subset \mathbb{T}^N$ denote the 1-dimensional fan with rays $u_0, \dots, u_N$ where as usual $u_i = -e_i$ for $1 \leq i \leq N$ and $u_0 = e_1 + \cdots + e_N$. Since $V \subset \mathbb{T}^N$ contains no loops, $L_{N+1} \subset V$.

We first show that we may find $\alpha_1 \sim \alpha$ such that in a neighborhood of x, α_1 is supported on L_{N+1} and is equipped with weight d on all of its edges. If the first statement is true the second statement about the weights is trivial by linear algebra since α_1 is closed. To prove the claim, we first move all of the edges of α that are adjacent to x to the 1-skeleton of V, this is Step 1. Denote the $(1,1)$-cells of α adjacent to x by (ϕ_f, f). Once again suppose that the 1-cell f is oriented outwards from x and has coefficient one. If f is in the interior of a facet F of V, generated by integer vectors v_1, v_2 then we may write $\phi_f = p_1v_1 + p_2v_2$ for some $p_1, p_2 \in Z$ since V is unimodular. In the facet F consider the $(1,2)$-cell with framing p_iv_i and supported on a two cell contained in a ball of radius ϵ about x, whose boundary contains x and has components laying on f and the boundary of F generated by v_i. Then $\alpha - \partial\tau_1 - \partial_2$ is equivalent to α. Repeating this procedure for all f contained in the interior of facets of V and we obtain a cycle α_1 supported on the 1-skeleton of V in a neighborhood of x. See the left hand side of Figure 3.6. Step 2 is to move the cells passing through the vertex to be supported on edges of $L_k \subset V \subset \mathbb{T}^n$. If the skeleton contains a ray which is not in the direction u_i it must be $u_I = \sum_{i \in I} u_I$ for some $I \subset \{0, \dots, N\}$. Moreover for every $i \in I$ there is a facet of V spanned by u_i and u_I. Now we may apply the same procedure used to move the rays of α to the skeleton to move any edges of α_1 supported on the rays in direction u_I to the rays in direction u_i. Therefore $\alpha \sim \alpha_1$ and α_1 is supported on L_{N+1} in a neighborhood of x.

Finally Step 3 moves α_1 away from the vertex. This is easy to do if we can deform L_{N+1} to another tropical line L' contained in V. To see that this is always the case, consider a full contraction $\delta : \mathbb{T}^n \longrightarrow \mathbb{T}^2$, then $\delta_* L_{N+1}$ is the tropical line with vertex at the origin and consisting of three rays. Let $L = \delta_* L_{N+1} + \epsilon v \subset \mathbb{T}^2$ be a translation for a generic vector v and a small $\epsilon > 0$. Then $L' = \delta^* L \subset V$ not containing the vertex of V. Now, the cells of α_1 contained on the edges of L_{N+1} may be deformed in V to the edges L'. See Figure 3.7.

If $x \in V$ is a point of positive sedentarity I, then a similar procedure near the boundary produces a cycle α' whose only cell intersecting x is supported on the ray of the skeleton in direction u_I. Once again we may split this cycle in the directions u_i for $i \in I$ and obtain a cycle homologous to α which no longer passes through x. See the righthand side of Figure 3.6.

The only case left to consider is when α is contained in a boundary divisor of X, this is proved in the following lemma. □

Lemma 3.1.10. *Let $\alpha \in Z_{1,1}(X)$ be supported on an irreducible boundary divisor $D \subset X$, then there exists a $(1,1)$-cycle $\alpha' \in Z_{1,1}(X)$ such that $\alpha \sim \alpha'$ and α' intersects the skeleton of X properly.*

Proof. In each chart $\Phi : U \longrightarrow V \subset \mathbb{T}^N$ it is clear that we may move $\Phi(\alpha \cap U)$ so that it intersects the boundary in at most a finite collection of points. So it suffice to match the images up in the overlaps of two open sets $U \cap U'$. As 1-cells this is simple, however the result may not be closed. Its boundary will be a collection of points with framing orthogonal to $\Phi(D \cap U)$, call this direction v. At all such points add a ray in the direction v with the appropriate weight. Doing this in all overlaps we obtain a cycle α' which is homologous to α and intersecting the boundary in a finite collection of points. Now we may apply Lemma 3.1.9 to obtain a cycle intersecting the skeleton of X properly. □

The above lemma along with Lemma 3.1.8 allows us to calculate self-intersections of boundary divisors of a surface X. The next lemma proves that on a compact tropical surface, the product on 1-cycles A, B reviewed in the last section gives the same result as the $(1,1)$ product on $Cyc(A), Cyc(B)$.

Theorem 3.1.11. *Suppose A, B are two tropical 1-cycles on a smooth compact tropical surface X, then*

$$Cyc(A.B) = Cyc(A).Cyc(B).$$

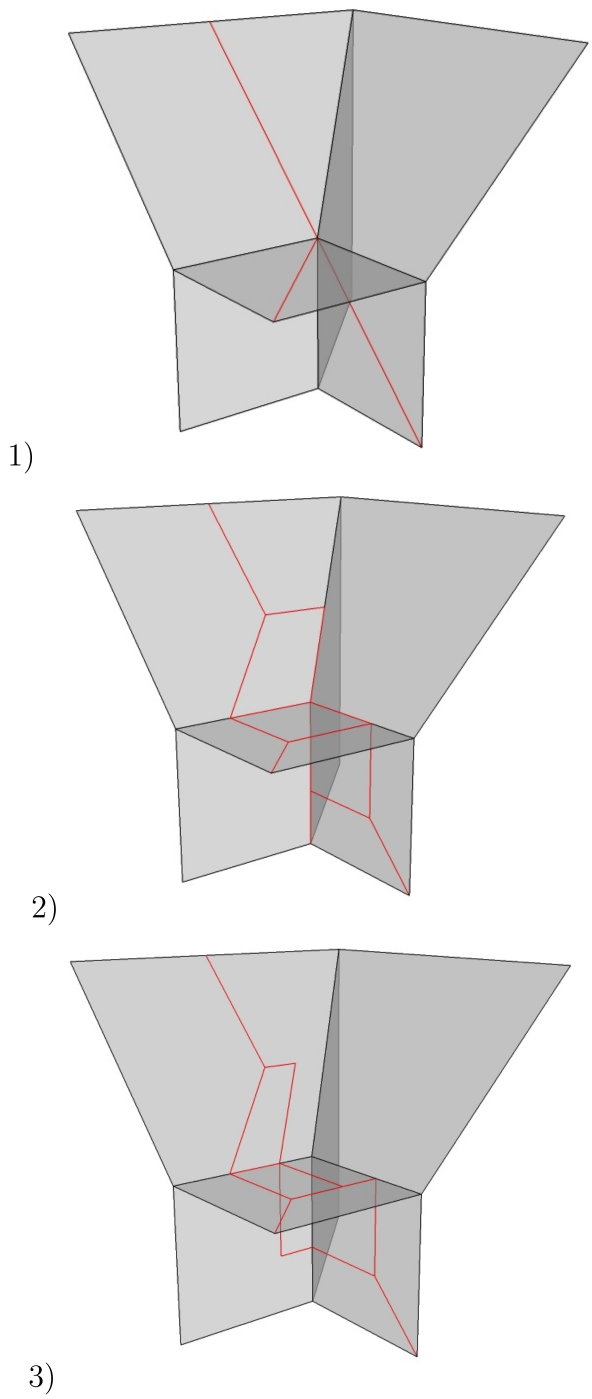


Figure 3.7: The three steps of moving a $(1,1)$-cycle away from a vertex of $V \subset \mathbb{T}^n$.

Proof. Let x be a point of $(A \cap B)^{(0)}$, and let $(A.B)_x$ denote its multiplicity in $A.B$. Choose a neighborhood of $x \in X$ such that $U \cap A$, $U \cap B$ are both contained in fan cycles in $V \subset \mathbb{T}^n$ under a chart $\Phi : U \longrightarrow V \subset \mathbb{T}^n$. We identify x, A, B and U with their images in V, and set $\mathcal{D} = \overline{U} \backslash U$. Then, $Cyc(A) \cap \overline{U}$ and $Cyc(B) \cap \overline{U}$ are relative $(1, 1)$ homology cycles, i.e. they are in $H_{1,1}(\overline{U}, \mathcal{D})$. Let K be a small compact set containing a ball of size ϵ about x. By the last lemma we can find cycles α, β homologous to $Cyc(A), Cyc(B)$ respectively, which intersect transversally in K and agree with $Cyc(A), Cyc(B)$ outside of K. Moreover, the total intersection multiplicity of α, β in K is the same as $(A.B)_x$. If x is in a facet, or on an edge this is trivial. If x is a corner point it follows from the definition of corner multiplicities and the splitting given above. Moreover, if x is a vertex of V then by using Definition 3.1.7 and the splitting from the above lemma we can easily compare the intersection multiplicities.

It remains to show that away from the points $x \in (A \cap B)^{(0)}$ there is no contribution to the intersection multiplicity. Outside of the compact sets K about every point $x \in (A \cap B)^{(0)}$ if the cycles α, β intersect then they must do so in parallel edges of rational slope, moreover the framing coincides with the rational direction. Therefore α, β have parallel framings, we may move the cycles to meet transversally while keeping their framings constant. Then every additional intersection point will contribute 0 since by Definition 2.2.11 the weight $w = 0$. This completes the proof. □

For tropical 1-cycles in a surface we now have two equivalent ways of defining the intersection product (assuming we do not need to take self-intersections of boundary divisors). Because of the equivalence of the two products we will not make a distinction and apply which either one is most convenient for the given situation.

3.2 Tropical sums

Given two tropical surfaces X, X' each with isomorphic boundary divisors D, D' satisfying certain conditions we construct a new tropical surface $X \# X'$ by "glueing" the two original surfaces. The resulting surface, called the tropical sum, is analogous to the fiber sum for manifolds and symplectic sum in the symplectic category [24]. With the tropical sum we may construct new surfaces from old. Examples which will be presented are instances of tropical blow downs and "floor decomposed" surfaces, previously introduced by Mikhalkin [37].

3.2.1 The construction

As a warm up, the next example describes the sum of two tropical curves.

Example 3.2.1
Given two abstract tropical curves C_1, C_2 suppose D_i is a finite collection of leaves

of C_i and $f : D_1 \longrightarrow D_2$ a bijection. As a graph, the tropical sum is simply:

$$C_1\#C_2 = \frac{C_1 \sqcup C_2}{\{x_i \sim f(x_i) \mid x_i \in D_1\}}.$$

An abstract tropical curve is also equipped with a metric. On the sum $C_1\#C_2$ we may impose lengths on the k new bounded edges where $k = |D_i|$, therefore we obtain a k-parameter family of abstract tropical curves.

Definition 3.2.2. An isomorphism of tropical curves is an isometry $f : C_1 \longrightarrow C_2$.

Recall that an irreducible boundary divisor D is said to have **simple normal crossings** with the rest of the boundary divisors of X if $D \cap D_i \cap D_j = \emptyset$ for all other distinct boundary divisors D_i, D_j not equal to D. For such a divisor we define the normal bundle of D in X. Tropical vector bundles have already been treated in the "open" case by Allermann in [1].

Definition 3.2.3. Let $D \subset X$ be a boundary divisor having simple normal crossings with the other boundary divisors of X, let $\{U_i\}$ be a covering of D in X, such that $U_i' = D \cap U_i$ is simply connected for all i. The normal bundle $N_X(D)$ is the tropical manifold given by the open sets $U_i' \times \mathbb{T}$ with charts $\Phi_i' : U_i' \times \mathbb{T}$ the extension of the charts $\Phi_i : U_i \longrightarrow V_i \times \mathbb{T} \subset \mathbb{T}^{N_i}$ from X.

Rational sections were also defined in the open case in [1].

Definition 3.2.4. Let $D \subset X$ be a non-singular tropical curve in surface X and $\pi : N_X(D) \longrightarrow D$ be its normal bundle. A continuous function $\sigma : D \longrightarrow N_X(D)$ is a rational section if $\pi \circ \sigma = id$ and in every chart σ is a piecewise integer affine function.

Let $N_X^o(D) = N_X(D) \backslash \sigma_{-\infty}$, where $\sigma_{-\infty}$ is the $-\infty$ section. Then, $N_X^o(D)$ is an $\mathbb{R}$ bundle. By Definition 3.2.3 it is tautological that a neighborhood of D in X may be identified with a neighborhood of the zero section in $N_X(D)$. For simplicity we will restrict ourselves to the case when the restriction of the normal bundle to D^o is trivial. In particular, this is the case if $b_1(D) = 0$. In this case, we can easily find a non-zero rational section of $N_X(D)$, where by zero we mean $0_{\mathbb{T}} = -\infty$.

For an irreducible boundary divisor let $B(D) \subset D$ denote the bounded edges of D. If $D = [\infty, -\infty]$ then set $B(D) = 0 \in D$. Then $D \backslash B(D)$ is a collection of edges adjacent to the leaves of D.

Lemma 3.2.5. *Let $D \subset X$ be an irreducible boundary divisor and suppose D has simple normal crossings with the other boundary divisors of X. If $N_X(D^o)$ is trivial, then there is a non-zero section $\sigma : D \longrightarrow N_X(D)$ which is constant on $B(D)$ and integer affine linear on each leaf of D.*

Proof. By assumption $N_X(D^o) = D^o \times \mathbb{T}$. However, the closure in $N_X(D)$ of a constant section of $N_X(D^o)$ may zero or undefined on the boundary. For an edge e of D adjacent to a leaf l, let v denote the unit tangent vector to e pointing away from $B(D)$. Then there is a unique integer k, such that if $d\sigma(v) = k$ then $\sigma(l) \in \mathbb{R}$. By continuity this defines a section σ. $\square$

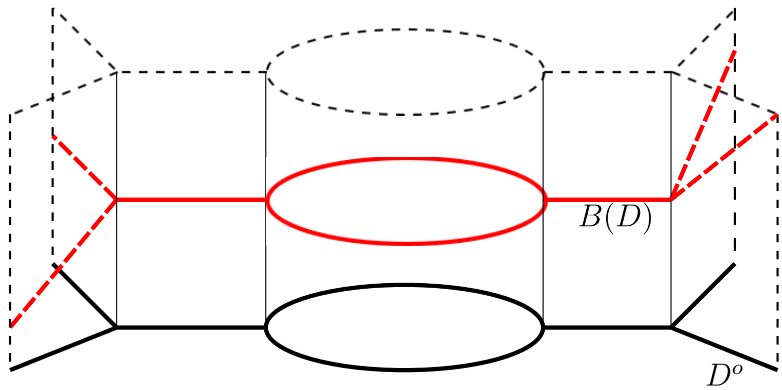


Figure 3.8: A section σ from Lemma 3.2.5 of the normal bundle $N_X(D)$ where $N_X(D^o)$ is trivial.

In the case that the normal bundle restricted to D^o is trivial, the section from Lemma 3.2.5 gives the self-intersection of D.

Corollary 3.2.6. *Let $D \subset X$ be an irreducible boundary divisor having simple normal crossings with the other boundary divisors of X. For every edge e in $B(D)\backslash D$ let v_e denote a unit tangent vector to e pointing away from $B(D)$. If $N_X(D^o)$ is trivial and σ is a section from Lemma 3.2.5 then,*

$$D.D \sim \sum_{e \in B(D)\backslash D} d\sigma(v_e).$$

Proof. Suppose the section $\sigma(D) \subset N_X(D)$ is contained in a neighborhood of the $-\infty$ section which can be identified with a neighborhood of $D \subset X$, (if not we may translate substract from σ a constant $0 << M \in \mathbb{R}$). The section, $\sigma(D)$ can be completed to a balanced tropical cycle $\overline{\sigma}(D) \subset X$ simply by adding to every leaf l of $B(D)$ a vertical ray with weight

$$\sum_{l \in e \in B(D)\backslash D} d\sigma(v_e).$$

It is easy to see that $\overline{\sigma}(D) \sim D$ as $(1, 1)$-cycles. This proves the corollary. $\square$

In Section 1.2.3, we defined tropical linear maps of manifolds.

Definition 3.2.7. Suppose D and D' are irreducible boundary divisors in tropical surfaces X and X' respectively and having simple normal crossings with the other boundary divisors of X and X' respectively. An invertible tropical linear map $g : N^o_X(D) \longrightarrow N^o_X(D')$ is an isomorphism if there is an isomorphism of curves $f : D \longrightarrow D'$ such that $f\pi = \pi' g$. It is *orientation reversing* if it reverses the orientation of each fiber $\approx \mathbb{R}$.

In each fiber a bundle isomorphism $g : N^o_X(D) \longrightarrow N^o_X(D')$ must be integer affine linear and invertible. If g also reverses orientation then restricted to a fiber it is given by $x \mapsto -x + M$ where $M \in \mathbb{R}$.

To form the tropical sum we assume the following conditions:

1. The boundary divisor $D \subset X$ (respectively $D' \subset X'$) has simple normal crossings with the other boundary divisors of X and (respectively X');

2. The normal bundles $N_X(D^o)$ and $N_{X'}(D'^o)$ are both trivial;

3. There is an isomorphism $f : D \longrightarrow D'$ and an orientation reversing bundle isomorphism $g : N^o_X(D) \longrightarrow N^o_{X'}(D')$ such that $f\pi = \pi' g$;

4. Let $U \subset N_X(D)$ be a neighborhood of the $-\infty$ section such that there is an inclusion $i : U \longrightarrow X$, where $i(U)$ is a neighborhood of D in X. Similarly let $U' \subset N'_X(D')$ be a neighborhood of the $-\infty$ section such that there is an inclusion $i' : U' \longrightarrow X'$. Suppose $\sigma \subset U$ is a section of $N_X(D)$ given by Lemma 3.2.5 such that $\sigma' = g \circ \sigma \circ f^{-1} \subset U'$.

Under these assumptions $X \backslash i(\sigma)$ consists of two connected components, let X_σ denote the closure in X of the component not containing D, and analogously for $X'_{\sigma'}$ where $\sigma' = g \circ \sigma \circ f^{-1}$.

Definition 3.2.8 (**Tropical sum**)**.** Let X and X' be tropical surfaces and suppose $D \subset X$, $D' \subset X'$ are irreducible boundary divisors, $f : D \longrightarrow D'$ an isomorphism, and $g : N^o_X(D) \longrightarrow N^o_{X'}(D')$ an orientation reversing bundle isomorphism satisfying conditions (1) – (4) above. Then the tropical sum of X and X' with respect to g and σ is,

$$X \# X' = \frac{X_\sigma \sqcup X'_{\sigma'}}{i \circ \sigma(x) \sim i' \circ g \circ \sigma(x)}.$$

In general, different choices of σ and g will amount to a one-parameter family of tropical manifolds. However changing the isomorphism $f : D \longrightarrow D'$ may produce completely different spaces.

Lemma 3.2.9. *Suppose X is a tropical fiber sum of X_1 and X_2 along the irreducible boundary divisors $D \subset X_1$ and $D' \subset X_2$, for some σ and g. Then,*

1. $(K^o_{X_1})^2 + (K^o_{X_2})^2 = (K^o_X)^2$

2. $D^2 = -D'^2$

Proof. The intersection $(K^o_X)^2$ is supported on points of $X^{(0)}$ that are corner points of X of size larger than two or are points with sedentarity $\emptyset$. Such a point is in exactly one of $X_1^{(0)}$ or $X_2^{(0)}$. Therefore the statement of (1) follows from local intersection multiplicities and the assumption that D, D' have simple normal crossings with the other boundary divisors of X_1 and X_2 respectively.

Part (2) follows from Corollary 3.2.6 and that $g(\sigma) = \sigma'$ where g is orientation reversing. □

Example 3.2.10 Contracting fan -1-curves
Let $E \subset X$ be an irreducible boundary divisor of a surface X with simple normal crossings to the other boundary divisors. Suppose that $E^2 = -1$ and that E is a fan with the valency of its interior vertex k. Then, the normal bundle restricted to E^o is trivial and

$$N_X(E^o) = L_k^o \times \mathbb{T} \subset \mathbb{R}^{k-1} \times \mathbb{T}\,.$$

Where L_k^o is the fan tropical line in $\mathbb{R}^{k-1}$. Without loss of generality we may assume that a section σ is the closure in $N_X(E)$ of a fan with vertex $(0, \dots, 0, -M) \in L_k^o \times \mathbb{T}$ for some $M >> 0$ and k outgoing rays in primitive integer directions: $-e_1, \dots, -e_{k-1},$ and $e_1 + \cdots + e_{k-1} - e_k$. This is due to the fact that $E^2 = -1$.

To "contract" E we perform a tropical sum with the right modification of $\mathbb{TP}^2$. Let $V = L_k \times \mathbb{T} \subset \mathbb{T}^k$ and let $\overline{V} \subset \mathbb{TP}^k$, so that $\overline{V}$ is a non-singular modification of $\mathbb{TP}^2$. Then $\overline{V}$ has exactly one boundary divisor L which is k valent (the rest are all $[-\infty, \infty]$). The normal bundle to L^o is

$$N_{\overline{V}}(L^o) = L_k^o \times \mathbb{T} \subset \mathbb{R}^{k-1} \times \mathbb{T},$$

and a section σ' of $N_{\overline{V}}(L)$ is the closure of a fan with outgoing rays in primitive integer directions: $-e_1, \dots, -e_{k-1},$ and $e_1 + \cdots + e_{k-1} + e_k$. The map $L_k^o \times \mathbb{R} \longrightarrow L_k^o \times \mathbb{R}$ given by $(x, t) \longrightarrow (x, -t)$ induces an orientation reversing isomorphism of $g : N_X^o(E) \longrightarrow N_{\overline{V}}^o(L)$. The sum $X' = X \# \overline{V}$ is the **contraction** of the -1-curve E.

If $E \subset X$ is a non-singular fan curve with $E^2 = -1$ and not necessarily a boundary curve, we may perform a non-singular modification, $\delta : \tilde{X} \longrightarrow X$ along a Cartier divisor corresponding to E. Then $\tilde{X}$ has boundary divisor $\tilde{E}^2 = -1$. Now it is possible to sum $\tilde{X}$ and $\overline{V}$ along $\tilde{E}$ and L as above. This gives a way of contracting E up to modification of X.

Remark The above gives a partial tropical version of the Castelenovo-Enriques criterion for blowing down -1 rational curves in classical algebraic geometry. It is clear that it is not complete since it only works for fan curves. Even still there are minus -1-fan curves that do not fit our definition of non-singular, and we are not able to perform a modification sending them to the boundary. These curves appear in Example 3.4.26 in the section on locally approximable curves.

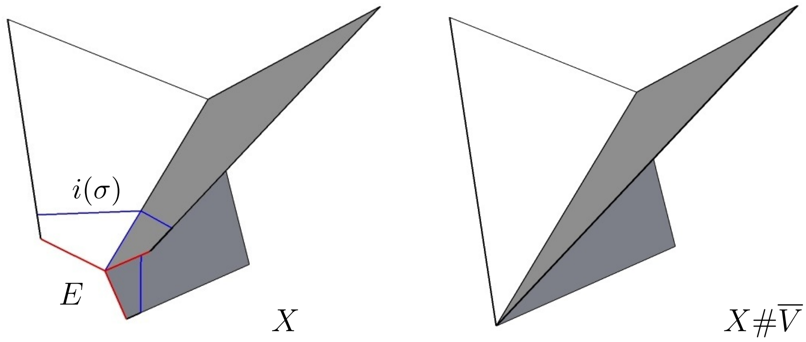


Figure 3.9: A neighborhood of a -1-fan curve E in a surface X, and the result after summing with $\overline{V}$.

3.2.2 Noether's formula for sums and modifications

In this subsection we show that a tropical version of Noether's formula can be proved combinatorially for tropical surfaces constructed from toric varieties and abelian varieties via modifications and tropical sums. Classically Noether's formula states that for a compact complex surface $\mathcal{X}$ [25]:

$$\chi(\mathcal{O}_{\mathcal{X}}) = \frac{K_{\mathcal{X}}^2 + c_2(\mathcal{X})}{12}.$$

Where $\chi(\mathcal{O}_{\mathcal{X}})$ is the holomorphic Euler characteristic of $\mathcal{X}$, $K_{\mathcal{X}}$ is its canonical class of $\mathcal{X}$ and $c_2(\mathcal{X})$ is its second Chern class. This formula is one of the steps to proving Riemann-Roch for line bundles on surfaces [25] which states that for a line bundle $\mathcal{L}$ on a compact complex surface $\mathcal{X}$:

$$\chi(\mathcal{L}) = \chi(\mathcal{O}_{\mathcal{X}}) + \frac{\mathcal{L}\,.\,\mathcal{L} - \mathcal{L}\,.K_{\mathcal{X}}}{2}.$$

We first define a 1-cycle, K_X, in a tropical surface X representing its canonical class as given by Mikhalkin in [38]. This is a cycle supported on the codimension one skeleton of X with weights determined by the local structure of X. The following definition is for all tropical manifolds.

Definition 3.2.11. [38] Let X be a tropical manifold, the canonical class K_X is supported on the codimension one skeleton of X, with the weight of a face E of codimension one equal to $w_{K_X}(E) = v(E) - 2$, where $v(E)$ is the number of facets of X adjacent to E.

Notice that the canonical class of X decomposes as $K_X = K_X^0 - \sum_{i=1}^d D_i$, where K_X^0 is a cycle of sedentarity zero supported on the one skeleton of X and D_i are the boundary divisors of X.

Lemma 3.2.12. *For $\delta : \tilde{X} \longrightarrow X$ a tropical modification we have: $K_{\tilde{X}} \sim_r \delta^* K_X$.*

Proof. Let D be the divisor of the modification δ and $\tilde{D}$ be the corresponding boundary divisor of $\tilde{X}$. We have

$$\delta^* K_X = \delta^* K_X^0 - \sum_{i=1}^{d} \tilde{D}_i,$$

where $\tilde{D}_i = \delta^* D_i$ along with $\tilde{D}$ are the boundary divisors of X. Moreover, $K^0_{\tilde{X}} = \delta^* K^0_X + \delta^* D$. By, Lemma 2.1.26 $\delta^* D \sim_r \tilde{D}$. Combining all this we obtain $K_{\tilde{X}} \sim_r \delta^* K_X$. □

As also mentioned by Mikhalkin in [38] it is possible to describe a cycle representing the Chern class $c_k(X)$ of a tropical manifold. In general, the class $c_k(X)$ is supported on the codimension k skeleton with weights determined by the local geometry of the manifold. To describe the multiplicities of points $x \in X^{(0)}$ in $c_2(X)$ for a surface, we first give the multiplicity of a vertex of a plane $P \subset \mathbb{R}^N$.

Definition 3.2.13. Let $P \subset \mathbb{R}^N$ be a plane and A be the arrangement defined by P. For each tropical line $L_i \in A$ let A_i denote the point arrangement on L_i. Let v be the vertex of P, then,

$$m_v = [3 - |p(A)| + \sum_{i=0}^{N} (|p(A_i)| - 2)] \tag{3.2.1}$$

and

$$c_2(P) = m_v v.$$

Definition 3.2.14. For X a surface define,

$$c_2(X) = \sum_{x \in X^{(0)}} m_x x$$

where,

1. if $x \in D_1 \cap D_2$ for D_1, D_2 boundary divisors of X then $m_x = 1$.

2. if $x \in D_1^o$ where D is a boundary divisor of X then, $m_x = 2 - val(x)$, where $val(x)$ is the number edges of D adjacent to x.

3. if $x \in X \backslash \partial X$ then for a neighborhood $U \ni x$ fix a chart $\Phi : U \longrightarrow P \subset \mathbb{T}^N$ so that $\Phi(x)$ is the vertex of P. Then, $m_x = m_v$ as defined in Equation 3.2.1.

Notice that for the compactification of a plane $\overline{P} \subset \mathbb{TP}^N$ we have $\deg(c_2(\overline{P})) = 3$. This follows from the fact that $|p(A)|$ is the number of sedentarity ≥ 2 points of $\overline{P}$ and $|p(A_i)|$ is the valency of a point of sedentarity 1.

Remark Part (3) of definition 3.2.14 has a fault, namely it is not a priori clear that this definition does not depend on the choice of chart $\Phi : U \longrightarrow P$ and compactification of P. It is necessary to find a definition for the multiplicity in $c_2(X)$ of a point of sedentarity zero which depends only on the combinatorics of the fan $P \subset \mathbb{R}^N$. Up until now we have yet to uncover one.

The next lemma is a hint that the Hirzebruch's signature formula may hold also for sums of tropical surfaces. Hirzebruch's signature formula (see [50]) states that for a compact complex surface $\mathcal{X}$,

$$3\mathbf{Sign}(\mathcal{X}) = K_{\mathcal{X}}^2 - 2c_2(\mathcal{X}).$$

Where $\mathbf{Sign}(\mathcal{X})$ is the signature of the intersection form on $H^2(\mathcal{X})$. By Novikov additivity, the signature for real manifolds is additive under cobordism, [31]. The next lemma shows the right hand side is additive under taking tropical sums. We return to Hirzebruch's formula at the end of Section 3.3 in the case of floor decomposed surfaces.

Lemma 3.2.15. *Suppose X is a tropical sum of X_1, X_2 along boundary divisors $D \subset X_1$, $D' \subset X_2$. Let $\mathcal{A}$ denote the boundary arrangement of X and $\mathcal{A}_i$ the boundary arrangement of X_i for $i = 1, 2$. Then,*

$$(\sum_{\tilde{D} \in \mathcal{A}} \tilde{D})^2 = (\sum_{D_1 \in \mathcal{A}_1} D_1)^2 + (\sum_{D_2 \in \mathcal{A}_2} D_2)^2 - 4|l(D)|,$$

where $l(D)$ is the set of leaves of D.

Proof. The divisors $\tilde{D}$ of $\mathcal{A}$ give a partition of $\mathcal{A}_1 \cup \mathcal{A}_2$ the size of $\tilde{D}$. For every $\tilde{D} \subset \mathcal{A}$ we can take a non-boundary $(1, 1)$ cycle α in X homologous to $\tilde{D}$ intersecting $\sigma \subset X$ only on unbounded rays of σ with framing orthogonal to σ. If a divisor $\tilde{D} \in \mathcal{A}$ is a result of combining divisors $D_{i_1}, \dots, D_{i_k}$ of $\mathcal{A}_1$ and divisors $D_{j_1}, \dots D_{j_l}$ of $\mathcal{A}_2$ along some collection of leaves $l_1, \dots l_p$ of D, then $\alpha \cap X_\sigma = (D_{i_1} + \dots + D_{i_k}) \cap X_\sigma$. So, $\tilde{D}^2 = D_I^2 + D_J^2$, where

$$D_I = D_{i_1} + \dots + D_{i_k} \text{ and } D_J = D_{j_1} + \dots + D_{j_l}.$$

Similarly, $\tilde{D}.\tilde{D}' = D_I.D_{I'} + D_J.D_{J'}$. Moreover by Lemma 3.2.9, $D^2 = -D'$. So the difference

$$(\sum_{\tilde{D} \in \mathcal{A}} \tilde{D})^2 - (\sum_{D_1 \in \mathcal{A}_1} D_1)^2 - (\sum_{D_2 \in \mathcal{A}_2} D_2)^2$$

is exactly $-4|l(D)|$ as claimed. ☐

Lemma 3.2.16. *Let X_1, X_2 be compact tropical surfaces and $D \subset X_1, X_2$ a boundary divisor of each such that the tropical sum along D exists. Suppose $X = X_1 \# X_2$, then*

$$K_X^2 - 2c_2(X) = K_{X_1}^2 + K_{X_2}^2 - 2[c_2(X_1) + c_2(X_2)].$$

Proof. Notice first that

$$\begin{aligned} c_2(X) &= c_2(X_1) + c_2(X_2) - 2\sum_{v\in V(D)}(2 - val(v))v \\ &= c_2(X_1) + c_2(X_2) + 2c_1(D). \end{aligned}$$

Here $c_1(D) = \sum_{v\in V(D)}(val(v) - 2)v$ is the canonical class of the curve, $V(D)$ is the vertex set of the curve and $val(v)$ is the valency of a vertex.

Applying the equalities from Lemmas 3.2.9 and 3.2.15 to obtain:

$$\begin{aligned} K_X^2 - (K_{X_1}^2 + K_{X_2}^2) &= -2[\sum D_i.D + \sum D_i'.D' - K_{X_1}^o.D - K_{X_2}^o.D'] \\ &= 4\sum_{v\in V(D)}(val(v) - 2)v \\ &= 4c_1(D) \end{aligned}$$

Thus, $K_X^2 - (K_{X_1}^2 + K_{X_2}^2) = 4c_1(D) = [c_2(X) - (c_2(X_1) + c_2(X_2))]$ and the proof is complete. ☐

Noether's formula for an algebraic surface $\mathcal{X}$ relates its holomorphic Euler characteristic with its first and second Chern classes. By the conjectured tropical Hodge decomposition [27] the holomorphic Euler characteristic $\chi(\mathcal{X}) = \sum_{p=0}^n h^{0,n}(\mathcal{X})$, should be replaced by the topological Euler characteristic of the tropical surface X. As not every tropical surface may arise from a degeneration of an algebraic surface, it is not clear whether Noether's formula holds for all tropical surfaces. However it holds under sums and modifications as the next proposition shows.

Proposition 3.2.17. *Given X a compact tropical surface, let $\chi(X)$ be the Euler characteristic of the complex X. Then Noether's formula*

$$\chi(X) = \frac{K_X^2 + c_2(X)}{12}$$

holds if

1. *If X is a tropical toric variety.*
2. *If X is a tropical abelian surface i.e. $X = \mathbb{R}^2/\Lambda$ where Λ is an integral lattice, see [40].*
3. *If $\delta : \tilde{X} \longrightarrow X$ is a non-singular modification and the formula holds for X.*
4. *If $X = X_1\#X_2$ and the formula holds for X_1, X_2.*

Proof. The first two are simple, in both cases $\chi(X) = 0$ since on one hand X is a closed polygon and in the other case X is a torus. A combinatorial proof of Noether's formula for toric varieties using only the polygon, can be found in [45]. For the abelian surface both K_X and $c_2(X)$ are zero and this is trivial.

In part (3), we have $\chi(\tilde{X}) = \chi(X)$ since X is a retraction of $\tilde{X}$. For $K_X^2 = K_{\tilde{X}}^2$, first apply Lemma 3.2.12. Then it is simple to check that, we have $\delta^* D_i.\delta^* D_j = D_i.D_j$ and $\delta^* K_X^o.\delta^* D_i = K_X^o.D_i$. To see that $c_2(X) = c_2(\tilde{X})$, suppose $D \subset X$ is the divisor of the modification δ. If $x \notin D^{(0)}$, then the contribution of x to $c_2(X)$ is the same as the contribution of $\delta^* x$ to $c_2(\tilde{X})$. Otherwise, we may check that

$$m_x(X) = m_{\delta^* x}(X') + m_{x'}(X'),$$

where x' is the point corresponding to $x \in D^{(0)} \subset X$ in the corresponding boundary divisor $\tilde{D} \subset \tilde{X}$. Therefore, $c_2(X) = c_2(\tilde{X})$ and the statement of (3) holds.

For the sum formula, apply Lemma 3.2.16 to obtain:

$$\begin{aligned} K_X^2 + c_2(X) &= K_{X_1}^2 - 2c_2(X_1) + K_{X_2}^2 - 2c_2(X_2) + 3c_2(X) \\ &= 12\chi(K_{X_1}) + 12\chi(K_{X_2}) + 3[c_2(X) - c_2(X_1) - c_2(X_2)] \\ &= 12\chi(K_{X_1}) + 12\chi(K_{X_2}) - 6c_1(D) \\ &= 12\chi(K_{X_1}) + 12\chi(K_{X_2}) - 12\chi(D) \end{aligned}$$

By the formula for Euler characteristics of non-disjoint unions Noether's formula holds. $\square$

3.3 Signature of floor decomposed surfaces

3.3.1 Floor decomposed tropical surfaces

Floor decompositions were first introduced for tropical curves by Brugallé and Mikhalkin in [11]. Here the authors recover recursive formulas for counting tropical curves in toric surfaces which reduces to counting certain kinds of graphs with multiplicities. Floor decomposed surfaces in $\mathbb{R}^3$ were also introduced by Mikhalkin [37]. Such a surface is dual to a special type of subdivision of its Newton polytope, allowing it to be separated into "floors".

For d a natural number fix,

$$\Delta_d = \text{Conv}\{(0,0,0),(d,0,0),(0,d,0),(0,0,d)\}.$$

A tropical surface $X \subset \mathbb{TP}^3$ of degree d is the closure in $\mathbb{TP}^3$ of a tropical hypersurface given by a tropical polynomial

$$f(x) = \text{“}\sum_{\alpha \in \Delta_d} a_\alpha x^\alpha,$$

where $x \in \mathbb{R}^3$, [48]. Such a tropical polynomial defines a subdivision $\mathcal{S}$ of Δ_d and this subdivision is dual to the tropical surface X, see [48]. A subdivision $\mathcal{S}$ of a Newton polytope Δ_d is **primitive** if for every $\Delta' \in \mathcal{S}$, we have $\text{vol}(\Delta') = \text{vol}(\Delta_1)$. A tropical surface of degree d is a tropical manifold, if it is dual to a primitive subdivision of Δ_d.

Definition 3.3.1. A smooth tropical surface of degree d, $X \subset \mathbb{TP}^3$, is **floor decomposed** if its corresponding dual primitive subdivision contains the hyperplanes $\{(x,y,z) \mid z = k\}$ for $1 \leq k \leq d-1$, (see Figure 3.10).

Removing all open faces containing the vertical z direction from a floor decomposed surface $X \subset \mathbb{R}^3$ there are d connected components, where d is the degree of the surface. These are called the **floors**. We denote the floor dual to the part of the subdivision $\mathcal{S}$ laying between hyperplanes $z = d-i$ and $z = d-i-1$ by $F_{i+1,i}$. Two adjacent floors, $F_{i,i-1}$ and $F_{i+1,i}$, are joined by **walls**. A wall of X is a connected component of the complement of the floors. In Figure 3.11 are the floors and walls of the quardric surface from Figure 3.10.

Topologically, a wall is a cylinder over a tropical curve. Denote by $C_i \subset \mathbb{R}^2$ the curve corresponding to the projection of the wall joining the adjacent floors $F_{i,i-1}$ and $F_{i+1,i}$. Then C_i is dual to the subdivision $\mathcal{S}$ restricted to the two dimensional simplex,

$$\text{Conv}\{(0,0,d-i),(i,0,d-i),(0,i,d-i)\}.$$

Moreover, the tropical polynomial defining the curve C_i is defined by a tropical polynomial $f_i(x,y)$, obtained by restricting the polynomial $f(x,y,z)$ defining the surface X to the above 2-simplex and substituting $z = 1_{\mathbb{T}} = 0$.

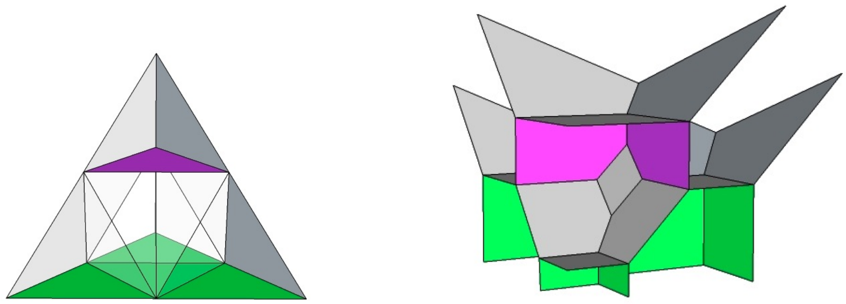


Figure 3.10: A subdivision of the size two simplex and a dual floor decomposed tropical surface of degree 2.

Definition 3.3.2. A **floor plan** for a surface is a collection of tropical plane curves $\{C_1, \dots, C_d\}$, $C_i \subset \mathbb{R}^2$, such that:

1. C_i is dual to a primitive subdivision of $\mathrm{Conv}\{(0,0),(i,0),(0,i)\}$ for $1 \leq i \leq d$.

2. for $1 \leq i \leq d-1$, C_i intersects C_{i+1} in $i(i+1)$ points contained in the interior of edges of both C_i and C_{i+1}.

Part a) of Figure 3.11, shows two curves of the floor plan $\{C_1, C_2\} \subset \mathbb{R}^2$. A floor decomposed surface $X \subset \mathbb{R}^3$ determines a floor plan. Each curve $C_i \subset \mathbb{R}^2$ is the image of a wall joining floors $F_{i,i-1}$ and $F_{i+1,i}$ under the linear projection in the vertical direction, as mentioned above. Conversely, given a floor plan $\{C_1, \dots, C_d\}$, using a pair of curves $C_i, C_{i+1} \subset \mathbb{R}^2$, we may construct a floor $F_{i+1,i} \subset \mathbb{R}^3$ of a floor decomposed surface X, up to a translation in the vertical direction. A curve $C_i \subset \mathbb{R}^2$ is a tropical hypersurface and so it is given by a tropical polynomial f_i. The difference $f_{i+1} - f_i$ gives a tropical rational function, (recall Section 1.1.3). For a real constant $a \in \mathbb{R}$, the floor $F_{i+1,i}$ is simply the graph of $\mathbb{R}^2$ along the function $a + f_{i+1} - f_i$. If the real constants are properly chosen, the graphs corresponding to adjacent floors may be joined via vertical faces and the result is a floor decomposed tropical surface of degree d.

In fact, we may view a floor decomposed surface as a result of the tropical fiber sum of surfaces from Section 3.2. From a floor $F_{i+1,i}$, we may construct another compact tropical surface $X_{i+1,i}$ by adding a unique collection of faces at any unbounded codimension one face of $F_{i+1,i}$. The procedure is identical to adding the undergraph of a tropical modification, except that for unbalanced edges of $F_{i+1,i}$ corresponding to C_i faces must be added in the upward vertical direction. Denote by $X_{i+1,i}$ the closure of this tropical cycle in the tropical toric variety given by the polytope $\overline{\Delta}$ which has outward pointing primitive integer vectors of its facets given by:

$$(-1,0,0), \quad (0,-1,0), \quad (0,0,-1), \quad (0,0,1) \quad \text{and} \quad (1,1,1).$$

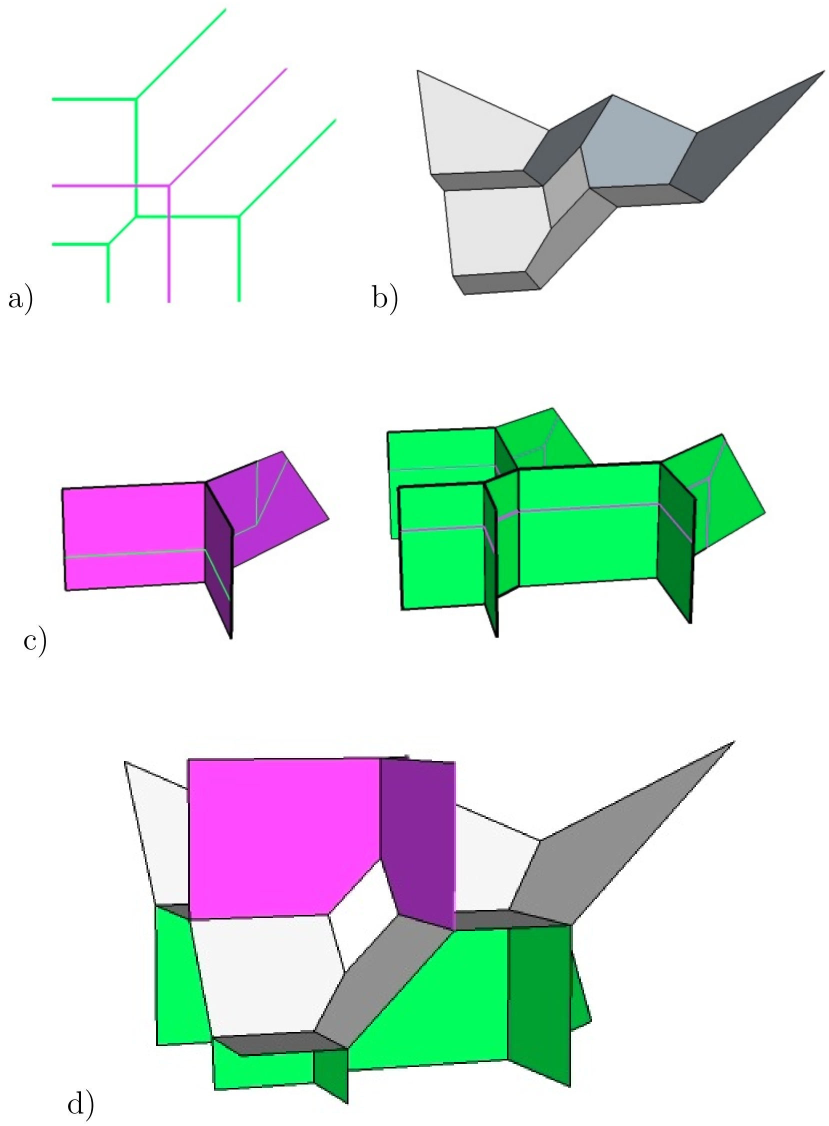


Figure 3.11: a) The floor plan of the degree two floor decomposed surface from Figure 3.10. b) The floor corresponding floor $F_{2,1}$. c) The walls corresponding to C_1 and C_2 along with sections. d) The surface $X_{2,1}$.

Equivalently $X_{i+1,i}$ is the closure in $\overline{\Delta}$ of the tropical hypersurface $X^o_{i+1,i}$ obtained by restricting coefficients of the tropical polynomial f of X to be in

$$\Delta_{i+1,i} = \{(x,y,z) \in \Delta_d \mid d-i-1 \leq z \leq d-i\}.$$

The surface $X_{i+1,i}$ contains C_i and C_{i+1} as boundary curves and the self-intersections of these boundary curves in $X_{i+1,i}$ are:

$$C_i^2 = -i \quad \text{and} \quad C_{i+1}^2 = i+1.$$

This can be seen by taking piecewise integer affine sections σ_i, σ_{i+1} of the normal bundles, $N_{X_{i+1,i}}(C_i)$ and $N_{X_{i+1,i}}(C_{i+1})$ by Lemma 3.2.5 and Corollary 3.2.6. Such sections are drawn in the walls for $X_{2,1}$ in Figure 3.11. For all $1 \leq i \leq d$, the surfaces $X_{i,i-1}$ and $X_{i+1,i}$ may be glued along the common boundary curve C_i by Definition 3.2.8. Moreover, we have proved the following lemma.

Lemma 3.3.3. *A floor decomposed tropical surface $X \subset \mathbb{TP}^3$ of degree d is the tropical sum of d surfaces $X_{0,1}, \dots, X_{d,d-1}$ such that*

1. *each $X_{i+1,i}$ is the closure in $\mathbb{T}(\overline{\Delta})$ of the tropical hypersurface given by restricting f to $\Delta_{i+1,i}$;*

2. *the surfaces $X_{i,i-1}$ and $X_{i+1,i}$ are summed along the common boundary curve C_i.*

3.3.2 Tropical $(1,1)$-cycles in floor decomposed surfaces.

In this section we calculate tropical $H_{1,1}$ for a floor decomposed surface in $\mathbb{TP}^3$. The dimension of $H_{1,1}$ corresponds to the dimension of the classical Hodge group $H_{1,1}(\mathcal{X})$ for $\mathcal{X}$ a complex surface in $\mathbb{TP}^3$ of the same degree. In addition an explicit set of generators of $H_{1,1}$ can be given based on the combinatorics of the floor decomposition. However, by calculating the intersection form on these $(1,1)$-cycles we find that for $d > 3$ they do not satisfy the Hodge Index Theorem from complex algebraic geometry. We state this theorem in the case of surfaces below.

Theorem 3.3.4 (Hodge Index Theorem)**.** *For a smooth projective surface $\mathcal{X} \subset \mathbb{CP}^N$, let $\mathcal{H} \subset \mathcal{X}$ denote the hyperplane section. Then,*

$$H^{1,1}(\mathcal{X}) = \langle \mathcal{H} \rangle \oplus \langle \mathcal{H} \rangle^{\perp}$$

where the intersection form $Q(\mathcal{X})$ is negative definite on $\langle \mathcal{H} \rangle^{\perp}$.

It follows directly from the above theorem, that the signature of the intersection form on $H^{1,1}(\mathcal{X})$ is surface $\mathcal{X}$ is $(1, h^{1,1}(X) - 1)$. For smooth floor decomposed tropical surfaces in $\mathbb{TP}^3$ we will prove the following theorem.

Theorem 3.3.5. *A smooth tropical floor decomposed surface $X_d \subset \mathbb{TP}^3$ of degree d has $H_{1,1}(X_d) = \mathbb{Z}^N$ where*

$$N = 1 + 2b_2(X_d) + \sum_{i=1}^{d-1} i^2 + i - 1.$$

Moreover, the signature of the intersection form on $H_{1,1}(X_d)$ is:

$$(1 + b_2(X_d), N - 1 - b_2(X_d)).$$

Following immediately from this theorem we have:

Corollary 3.3.6. *The Hodge Index Theorem does not hold on $H_{1,1}(X)$ for a smooth tropical floor decomposed surface $X \subset \mathbb{TP}^3$.*

To describe a set of generators for $H_{1,1}(X)$, we use the description of X as a sum of the surfaces $X_{i,i+1}$ and a Mayer-Vietoris sequence. When taking the tropical fiber sum of two surfaces X_1, X_2 along a common boundary curve C to obtain $X = X_1 \# X_2$, the tropical homology groups $H_{1,1}$ are related by the following long exact sequence:

$$\begin{aligned} \cdots \longrightarrow H_{1,2}(X) \longrightarrow H_{1,1}(\tilde{C}) \longrightarrow H_{1,1}(X_1^o) \oplus H_{1,1}(X_2^o) \longrightarrow \\ \longrightarrow H_{1,1}(X) \longrightarrow H_{1,0}(\tilde{C}) \longrightarrow H_{1,0}(X_1^o) \oplus H_{1,0}(X_2^o) \longrightarrow \cdots \end{aligned} \tag{3.3.1}$$

Above X_i^o denotes $X_i \backslash C$ for $i = 1, 2$ and $\tilde{C}$ is a surface constructed from the curve C which will be described below. The inclusion and boundary maps are the same standard maps from the long exact sequence in topology with constant coefficients, (see Chapter 2 of [26]). Notice that the boundary map $H_{1,k}(X) \longrightarrow H_{1,k-1}(\tilde{C})$ only decreases in the p-index.

To prove Theorem 3.3.5 will require some preparation. First let us describe a collection of cycles on the surfaces $X_{i+1,i}$.

Lemma 3.3.7. *The surface $X_{i,i+1} \subset \mathbb{T}(\overline{\Delta})$ has*

$$H_{1,1}(X_{i,i+1}) = \mathbb{Z}^{i(i+1)+1}$$

and intersection form of signature $1 - i(i + 1)$.

Proof. Consider the projection map $\pi : X_{i+1,i} \longrightarrow \mathbb{TP}^2$. We claim that $H_{1,1}(X_{i+1,i})$ is generated by the tropical 1-cycles $E_k = \pi^{-1}(x_k)$ such that $x_k \in C_i \cap C_{i+1}$, and a cycle $\tilde{L}$. First we describe $\tilde{L}$, let $L \subset \mathbb{TP}^2$ be a generic tropical line and consider $\pi^{-1}L \cap F_{i+1,i}$ this is a $(1, 1)$-chain whose boundary is a collection of vertically framed points, which occur when L intersects either C_i or C_{i+1}. Attach to these points of $\pi^{-1}L \cap F_{i+1,i}$ positively weighted rays in the $\pm e_3$ direction to obtain a closed balanced 1-cycle $\tilde{L}$. Performing the same procedure to any translation of L, $L' \subset \mathbb{TP}^2$ and we obtain a homologous cycle $\tilde{L}'$, and so $\tilde{L}^2 = 1$. Moreover, $E_k.\tilde{L} = 0$ and $E_k.E_j = 0$.

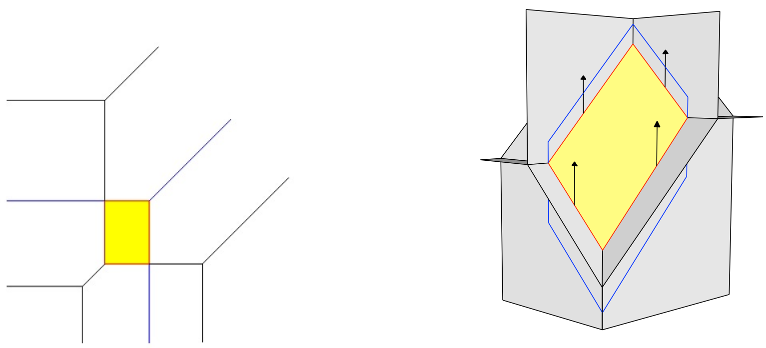


Figure 3.12: A -2 cycle on a floor (L, C)

The self-intersection E_k^2 is supported at a single point which is a vertex of $X_{i+1,i}$. A calculation shows $E_k^2 = -1$ for all k.

For any $(1,1)$-cycle $\alpha \in H_{1,1}$ choose a representative such that all edges not contained on the floor have vertical framing and all edges contained on the floor have framing parallel to the edges of $F_{i+1,i}$. Then, a projection $\pi(\alpha) \subset \mathbb{TP}^2$, is a $(1,1)$ cycle if we equip and edge $\pi(f)$ with framing $\pi(\phi_f)$. Now consider the lift of $\pi(\alpha)$ just as we did for $L \subset \mathbb{TP}^2$. Call this lift $\tilde{\alpha}$. If $\alpha \sim \tilde{\alpha}$ then $\alpha = k\tilde{L}$. Otherwise, $\alpha - \tilde{\alpha}$ is a non-trivial cycle, equipped only with the vertical framing. This implies $\alpha - \tilde{\alpha}$ is a combination of $Cyc(E_k)$. This proves the lemma. □

Remark The map $\pi : X_{i,i+1} \longrightarrow \mathbb{TP}^2$ is a tropical blowup at $i(i+1)$ points of sedentarity $\emptyset$. Indeed it is the graph of a rational function, and so it is the blow up along the common zeros of the curves $C_i, C_{i+1} \subset \mathbb{R}^2$. The 1-cycles E_k are the exceptional divisors, $E_k^2 = -1$ and $\tilde{L}$ is the proper transform of a line. Of course the points of the blow up are not in generic position as soon as $i > 1$.

We can describe another set of generators of $H_{1,1}(X_{i+1,i})$ that consist of "floor cycles"

Definition 3.3.8. Let $X_{i+1,i}$ be a smooth compact tropical surface dual to a primitive subdivision of $\Delta_{i+1,i}$. A **floor cycle** on $X_{i+1,i}$ is any simplicial 1-cycle supported on the 1-skeleton $F^{(1)}_{i+1,i} \subset X_{i+1,i}$ and equipped with a constant vertical framing.

As convention we choose the vertical framing on floor cycles to always be upwards, i.e. in the direction $+e_3$. Thus for a floor cycle α, the negation $-\alpha$ is obtained by reversing the orientation of the simplicial 1-cycle and keeping the upward vertical framing.

An example of a floor cycle on $X_{2,1}$ is drawn in red in Figure 3.12. A floor cycle is indeed closed since its base is a simplicial cycle and it is equipped with a constant framing. The 1-skeleton $F^{(1)}_{i+1,i}$ is topologically the union of the two curves C_i, C_{i+1} defining the two floors. Therefore any simplicial 1-cycle in $C_i \cup C_{i+1}$ gives a $(1,1)$-floor cycle.

Let $C_i, C_{i+1} \subset \mathbb{R}^2$ be two curves defining the surface $X_{i+1,i}$ (and thus also the floor $F_{i+1,i}$. Call an intersection point $x \in C_i \cap C_{i+1}$ a **junction**. Every $(1,1)$ floor cycle α determines a function on the junctions.

$$\delta_\alpha : C_i \cap C_{i+1} \longrightarrow \{-1, 0, 1\},$$

where

$$\delta_\alpha(x) = \begin{cases} 0 & \text{if } x \notin \alpha \text{ or } \alpha \text{ goes from } C_j \text{ to } C_j \text{ through } x \\ 1 & \text{if } \alpha \text{ is oriented from } C_{i+1} \text{ to } C_i \text{ through } x \\ -1 & \text{if } \alpha \text{ is oriented from } C_i \text{ to } C_{i+1} \text{ through } x \end{cases}$$

It can be verified that the intersection of any two floor cycles is

$$\alpha.\alpha' = -\sum_{x \in C_i \cap C_{i+1}} \delta_\alpha(x)\delta_{\alpha'}(x).$$

Moreover, $\tilde{L}.\alpha = 0$.

Proposition 3.3.9. *For a surface $X_{i+1,i} \subset \mathbb{T}(\overline{\Delta})$, the tropical homology group $H_{1,1}(X_{i+1,i})$ is also generated by the cycles $\tilde{L}, C_i, \alpha_1, \dots, \alpha_m$ for $m = i^2 + 1 - 1$ where α_k are floor cycles (recall Definition 3.3.8). Moreover,*

$$\tilde{L} \perp \langle \alpha_1 \dots \alpha_k \rangle \quad \textit{and} \quad \alpha_k^2 = -2$$

for all $1 \leq k \leq i(i+1) - 1$ and the intersection pairing is negative definite on the floor cycles.

Proof. The difference of any two exceptional divisors $E_i - E_j$ is equivalent to a floor cycle α having $\delta_\alpha(x_i) = 1, \delta_\alpha(x_j) = -1$ and $\delta_\alpha(x_k) = 0$ for $k \neq i, j$. It is clear that for such a cycle $\alpha^2 = -2$ and $H.\alpha = C.\alpha = 0$. □

Therefore, the floor cycles supported on a floor $F_{i+1,i} \subset X$ form a $i^2 + i - 1$ dimensional vector subspace and the intersection form of $H_{1,1}(X)$ restricted to this space is negative definite.

Example 3.3.10
A quadric has two floors, the top having no such cycles and the bottom having just one. Together with the hyperplane section H they give the intersection form

$$\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Just as for $H^{1,1}(\mathbb{CP}^1 \times \mathbb{CP}^1) = H^2(\mathbb{CP}^1 \times \mathbb{CP}^1)$ taken with generators $\mathcal{L}_1 + \mathcal{L}_2$ and $\mathcal{L}_1 - \mathcal{L}_2$.

Example 3.3.11
For a floor decomposed cubic there is one floor cycle on $X_{2,1}$ as in the example above. By Proposition 3.3.9 the floor $F_{3,2}$ supports 5 floor cycles with a negative definite intersection form. Together with the hyperplane section H, these cycles form a 7 dimensional vector space with intersection form

$$Q_X = \begin{pmatrix} 3 & & & & & & \\ & -2 & & & & & \\ & & -2 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & -2 \end{pmatrix}.$$

This agrees with the intersection form on a smooth complex cubic surface $\mathcal{X}$.

A floor decomposed tropical surface of degree larger than or equal to 4, has $b_2(X) > 0$. In this case there are additional $(1,1)$-cycles obtained from joining the floors.

Proposition 3.3.12. *A smooth floor decomposed tropical surface $X \subset \mathbb{TP}^3$ of degree d has independent $(1,1)$-cycles*

$$\{\beta_1, \gamma_1, \ldots, \beta_{b_2(X)}, \gamma_{b_2(X)}\}$$

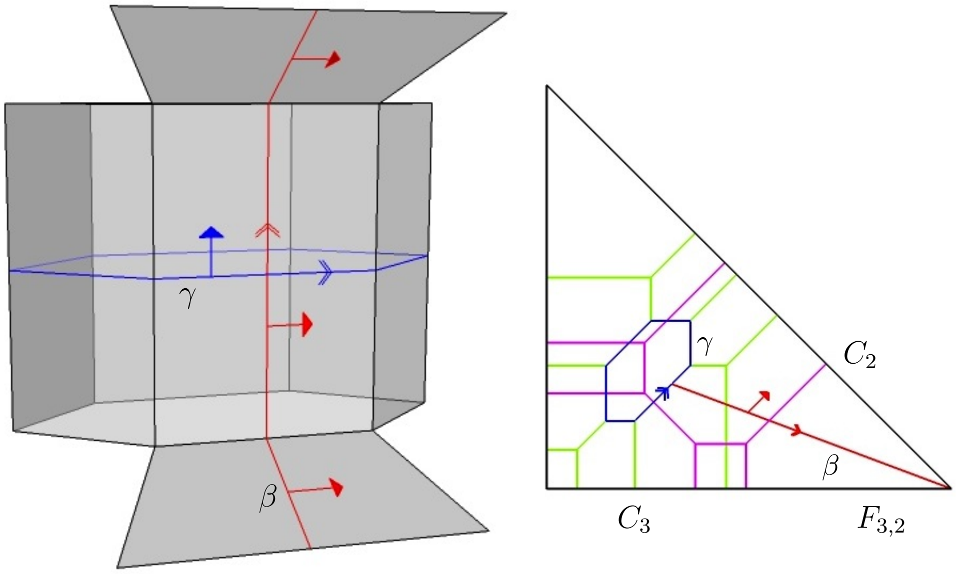
such that,

1. $\beta_i.H = \gamma_i.H = 0$ *for all* $1 \leq i \leq b_2(X)$.
2. $\beta_i.\gamma_j = \delta_{ij}$
3. $\gamma_i.\alpha = 0$ *where α is a floor cycle from Definition 3.3.8 supported on any floor of X.*

Proof. The surface $X \subset \mathbb{TP}^3$ is the sum of a surface X_{d-1} of degree $d-1$ and $X_{d,d-1}$. The proposition holds for X_1, so suppose by induction it holds on X_{d-1}. By the Mayer-Vietoris sequence with constant coefficients we have:

$$b_2(X_d) = b_2(X_{d-1}) + b_1(C_{d-1}). \tag{3.3.2}$$

So we must describe a pair of $(1,1)$-cycles α, β for each simplicial 1-cycle of C_{d-1}. Recall Example 2.2.10 described $H_{1,0}(C_{d-1})$ and $H_{0,1}(C_{d-1})$, and gave dual bases of 1-cycles: $l_1, \ldots, l_{b_1(C_{d-1})}$ and framed points: $x_1, \ldots, x_{b_1(C_{d-1})}$. The cycles l_i can be identified with 1-cycles in X on a wall. Equipping this cycle with a constant vertical framing and we obtain a $(1,1)$-cycle $\gamma_i = (e_3, l_i)$ for all $1 \leq i \leq b_1(C_{d-1})$. See Figure **??**.



Now a framed point $x_i \in C_{d-1}$ corresponds to framed points $y_i \in F_{d,d-1}$ and $y_i' \in F_{d-1,d-2}$. Moreover, $y_i' - y_i$ bounds a framed one cell on the wall joining $F_{d,d-1}$ and $F_{d-1,d-2}$. In addition, the framed point y_i is a boundary in $F_{d,d-1}$, and y_i' is a boundary in $F_{d-1,d-2}$. To see this, consider a path from $x_i \in C_{d-1} \subset \mathbb{TP}^2$ to a corner, as in the righthand side of Figure **??**. Lifting this path to both $F_{d-1,d-2}$ and $F_{d,d-1}$ it bounds a collection of vertically framed points which are all homologous to zero in X. This produces a cycle β_i.

Now to prove the claimed intersection properties. We can choose a hyperplane section H such that $H \cap \gamma_i, H \cap \beta_i = \emptyset$ therefore statement (1) is true. For γ_i and β_j coming from the same curve C_l statement (2) follows easily from the duality of the 1-cycles and framed points. Otherwise, the cycles are disjoint and the statement is clear. Finally, any cycle γ_i can be translated along the wall joining $F_{i,i-1}$ and $F_{i+1,i}$, thus not intersecting the floors or any floor cycles at all. This completes the proof. □

The cycles β_i, γ_i are exactly analogous to the new cycles appearing in the homology of a glueing of two 4-manifolds along a boundary, [31].

Lemma 3.3.13. *For a floor decomposed surface X, we have*

1. *the three groups $H_{1,2}(X)$, $H_{1,0}(X^o_{d-1})$, and $H_{1,0}(X^o_{d-1,d})$ all vanish.*
2. $H_{1,0}(\tilde{C}_{d-1}) = \mathbb{Z}^{g(C_{d-1})+1}$
3. $H_{1,1}(\tilde{C}_{d-1}) = \mathbb{Z}^{g(C_{d-1})}$

Proof. For the first statement, $H_{1,2}(X) = 0$ for dimension reasons the same as in Example 2.2.9. For the next two groups first notice that $H_{1,0}(\mathbb{TP}^2) = 0$. Now given $(\phi, p) \in Z_{1,0}(X^o_{d-1,d})$ if the framing ϕ is vertical then it is clearly homologous to zero. Therefore we may suppose $< e_3, \phi >= 0$, where $e_3 = (0, 0, 1)$. Extend the projection in the vertical direction in an affine chart to obtain: $\pi : X^o_{d,d-1} \longrightarrow \mathbb{TP}^2$. The framed point $(\pi(\phi), \pi(p))$ is homologous to zero in $\mathbb{TP}^2$. We may lift the $(1,1)$-chain which has $(\pi(\phi), \pi(p))$ as its boundary back to $X^o_{d,d-1}$ by taking the graph along the rational function $f_{i+1} - f_i$ from which $X^o_{d,d-1}$ is constructed to a $(1,1)$-chain τ in $X^o_{d,d-1}$. Then the boundary is:

$$\partial \tau = (\phi, p) + \sum_{i=1}^{k} (\pm e_3, p_i),$$

where the points p_i lay on the skeleton of $X^o_{d,d-1}$. Since all vertically framed points are homologous to 0, we have $(p, \phi) = \partial \tau'$, and so it is also homologous to 0.

The same claim for $H_{1,0}(X^o_{d-1})$ follows by induction and the isomorphism

$$H_{1,0}(X^o_d) \cong H_{1,0}(X_{d-1,d-2}) \oplus H_{1,0}(X_{d-2}),$$

obtained again from Mayer-Vietoris.

For the second statement, first notice that a vertically framed point on $\tilde{C}_{d-1}$ represents a class in $H_{1,0}(\tilde{C}_{d-1})$. A 1-framed point (ϕ, x) where $< \phi, e_3 >= 0$, represents a framed point on the underlying curve C_{d-1}. Therefore we have $H_{1,1}(\tilde{C}_{d-1}) = \mathbb{Z}^{g(C_{d-1})+1}$ by Example 2.2.10.

Finally for $H_{1,1}(\tilde{C}_{d-1})$, since $\tilde{C}_{d-1}$ is open, we may assume every $(1,1)$-class has a representative which is bounded. Therefore, it must be a framed bounded class in $H_1(\tilde{C}_{d-1})$. The only such classes come from cycles in C_{d-1} equipped with a vertical framing, thus $H_{1,1}(\tilde{C}_{d-1}) = \mathbb{Z}^{g(C_{d-1})}$. □

Finally we give the proof of the main theorem of this section.

Proof of Theorem 3.3.5. Combining Lemma 3.3.13 and the Mayer-Vietoris sequence gives:

$$H_{1,1}(X) = \frac{H_{1,1}(X^o_{d-1}) \oplus H_{1,1}(X^o_{d,d-1})}{i_* H_{1,1}(\tilde{C})} \oplus H_{1,0}(\tilde{C}).$$

Therefore, $H_{1,1}(X)$ is a free group. By a dimension count the floor cycles span all but $1 + 2b_2(X)$ dimensions of $H_{1,1}(X_d)$. Moreover, there are an additional $2b_2(C)$ cycles which we have seen. These are the vertically framed loops of the curves C_i living on the walls joining two floors and their duals constructed in Proposition 3.3.12.

It remains only to prove the above expression for the signature. Let, β, γ, be a pair of dual $(1,1)$-cycles arising from a cycle of the curve C_{d-1} in the floor plan of X_d. The intersection form restricted to this pair is:

$$\begin{pmatrix} 0 & 1 \\ 1 & \star \end{pmatrix}.$$

So the intersection pairing on $H_{1,1}(X_d)$ has the form:

	A_1	A_2	v	B	C
A_1	$\star$	0	$\star$	$\star$	0
A_2	0	$\star$	$\star$	$\star$	0
v	$\star$	$\star$	$\star$	$\star$	0
B	$\star$	$\star$	$\star$	$\star$	$1 \ \ddots \ 1$
C	0	0	0	$1 \ \ldots \ 1$	0

Where:

- A_1 is the subspace generated by cycles contained entirely in X^o_{d-1} and not in $H_{1,1}(\tilde{C}_{d-1})$.
- A_2 is the subspace generated by the floor cycles in $X^0_{d,d-1}$.
- C is the subspace generated by the cycles γ and B is the subspace generated by the duals β from Proposition 3.3.12.

We have seen already that the subspaces A_1, A_2, B and C are all orthogonal to the hyperplane section H and $H^2 > 0$. Using this and the above matrix of the intersection form we have:

$$Sign(X) = 1 + Sign(A_1) + Sign(A_2),$$

similar to the proof of Novikov additivity in [31]. The form is negative definition on A_2 and since it has basis the floor cycles, we have

$$Sign(A_2) = d(d-1) - 1.$$

Now the space A_1 with intersection product is the same as the orthogonal complement of the hyperplane section in $H_{1,1}(X_{d-1})$. By induction we have:

$$Sign(A_1) = 2 + 2b_2(X_{d-1}) - h_{1,1}(X_{d-1}).$$

Combining the above three equalities we obtain

$$Sign(X) = 2 + 2b_2(X_{d-1}) - h_{1,1}(X_{d-1}) + d(d-1), \tag{3.3.3}$$

which reduces to $Sign(X) = 2+2b_2(X_d)-h_{1,1}(X_d)$ after substituting Equation 3.3.2 along with

$$h_{1,1}(X_{d-1}) = h_{1,1}(X_d) - d(d-1) - 2g(C) + 1.$$

This completes the proof of the theorem. □

It also follows from the above theorem that the signature of the form on $H_{1,1}(X_d)$ is additive under the sum $X_d = X_{d-1}\#X_{d,d-1}$.

Corollary 3.3.14. *Let $X_d \subset TP^3$ be a smooth tropical floor decomposed surface and let $X_d = X_{d-1} \# X_{d,d-1}$ be its decomposition. Then,*

$$Sign(X_d) = Sign(X_{d-1}) + Sign(X_{d,d-1}).$$

By a direct comparison with the complex case, for a smooth floor decomposed surface of degree d, $X_d \subset \mathbb{TP}^3$ we obtain:

Corollary 3.3.15. *Let $X_d \subset \mathbb{TP}^3$ be a smooth floor decomposed tropical surface of degree d, then*

$$Sign(X_d) = \boldsymbol{Sign}(\mathcal{X}_d)$$

*where $\mathcal{X}_d \subset \mathbb{CP}^3$ is any smooth complex degree d surface, and **Sign** denotes the index of the intersection form on $H^2(\mathcal{X}_d)$.*

Finally we return to the Hirzebruch Signature Formula, as mentioned in Section 3.2.2. For a **complex** surface

$$3\mathbf{Sign}(\mathcal{X}) = K_{\mathcal{X}}^2 - 2c_2(\mathcal{X}),$$

where $\mathbf{Sign}(\mathcal{X})$ is the signature of the intersection form on $H^2(\mathcal{X})$. Lemma 3.2.16 shows that the tropical version of the right hand side of the above formula is additive under a tropical fiber sum. By this lemma and Corollary 3.3.14 we obtain.

Proposition 3.3.16. *A smooth floor decomposed surface $X \subset \mathbb{TP}^3$ satisfies*

$$3Sign(X) = K_X^2 - 2c_2(X), \tag{3.3.4}$$

where $Sign(X)$ is the signature of the intersection form on $H_{1,1}(X)$.

Proof. It remains only to show that the formula holds for $X \subset \mathbb{TP}^3$ of degree 1, as well as for the surfaces $X_{d,d-1}$. When X is of degree one we verify that $K_X^2 = 9$ and $c_2(X) = 3$ and $Sign(X) = 1$, so the formula holds. For $X_{d,d-1}$,

$$Sign(X_{d,d-1}) = 1 - d(d-1),$$

by Proposition 3.3.9. For $X_{d,d-1}$ we calculate:

$$c_2(X_{d,d-1}) = 3 + d(d-1) \quad \text{and} \quad K^2_{X_{d,d-1}} = 9 - d(d-1).$$

This verifies that Equation 3.3.4 holds for $X_{d,d-1}$ and the proposition is proved. □

3.4 Local approximation of curves

With the exception of parts of 3.4.2, the next few sections are results from joint work with Erwan Brugallé which appear in [12].

A tropical curve in a surface $C \subset X \subset (\mathbb{C}^*)^N$ is said to be globally approximable if there exists families $\mathcal{C}_t \subset \mathcal{X}_t \subset (\mathbb{R}^*)^N$ such that,

$$C = \lim_{t\to\infty} \mathrm{Log}(\mathcal{C}_t), \quad \text{and} \quad X = \lim_{t\to\infty} \mathrm{Log}(\mathcal{X}_t),$$

(along with a way of determining the weights of the limit so that they coincide with C and X). As usual, we mean here the Hausdorff limit of sets. When such families exist the pair (X, C) is said to be approximable.

There are known examples of non-approximable tropical pairs (X, C). Perhaps the first such examples were provided by Vigeland, who constructed in [56] examples of non-singular tropical surfaces in $\mathbb{R}^3$ of degree $d \geq 3$ containing infinitely many tropical lines. It is a well known fact that a surface $X \subset \mathbb{CP}^3$ of degree 3 contains exactly 27 lines, since it is the blow up of $\mathbb{CP}^2$ in 6 points. Moreover, it is a theorem of Segre that a non-singular general surface of degree 4 contains no lines [51]. Up until recently, it was not possible to prohibit Vigeland's lines without appealing to this theorem of complex geometry. In [7], Bogart and Katz are able to prohibit the trivalent members of the families of lines given explicitly by Vigeland by using only the tropical data. In [12], we prohibit all possible families on a surface of degree greater than or equal to 3 and in addition are able to remaining cases and the problem of Vigeland's lines is completely resolved. To summarize we obtained the following theorem.

Theorem 3.4.1. *[12] Let $X \subset \mathbb{R}^3$ be a smooth tropical hypersurface.*

- *If* $\deg(X) = 3$ *then there are at most a finite number of tropical lines which are approximable in X.*
- *If* $\deg(X) \geq 4$ *and X is generic, then there are no tropical lines which are approximable in X.*

In both cases the obstructions are based on a technique of tropical geometry known as **initial degeneration** or **localisation**. Given a tropical variety $X \subset \mathbb{R}^N$ for any point $x \in X$ let

$$Star_x(X) = \{v \in \mathbb{R}^N \mid x + v\epsilon \in X \text{ for } \epsilon > 0\}.$$

If X is approximated by $\mathbf{Log}_t(\mathcal{X}_t)$ for a family $\mathcal{X}_t \subset (\mathbb{C}^*)^N$, then for every point $x \in X$, $Star_x(X)$ may be approximated by a constant family, (see [6], [33]). The situtation is the same for pairs, if the pair (X, C) is approximable then $Star_x(C) \subset Star_x(X)$ is approximable by a pair of constant families. In other words, if a pair $C \subset X \subset \mathbb{R}^N$ is globally approximable it is everywhere locally approximable by constant families.

Here we will consider local approximability of tropical curves in tropical surfaces. It should be noted that there are also non-trivial global obstructions to approximating tropical curves in $\mathbb{R}^3$ and also of pairs of surfaces and curves. Mikhalkin constructed an example of a non-planar tropical elliptic cubic curve in $\mathbb{R}^3$ which is everywhere locally approximable. However, by the Riemann-Roch theorem a complex spatial elliptic cubic is always planar. Mikhalkin and Brugallé also constructed the following example of a curve in the standard hyperplane which is everywhere locally approximable but not globally approximable [10].

In this section $\mathcal{P} \subset (\mathbb{C}^*)^N$ will be a plane, (i.e. a two dimensional linear space). Equivalently, $\mathcal{P} \subset (\mathbb{C}^*)^N$ is defined by $N-2$ equations of degree one. Here we will always compactify the plane $\mathcal{P} \subset (\mathbb{C}^*)^N$ to $\overline{\mathcal{P}} = \mathbb{CP}^2 \subset \mathbb{CP}^N$, where $\mathbb{CP}^N$ is the toric compactification of $(\mathbb{C}^*)^N$ given by the standard simplex $\Delta \subset \mathbb{R}^N$. In general, there may be other toric compactifications of $\mathcal{P} \subset (\mathbb{C}^*)^N$ to $\mathbb{CP}^2 \subset \mathbb{CP}^N$. See [12] for a more general presentation without a fixed compactification and when $\mathcal{P}$ is a plane up to the action of $GL_n(\mathbb{Z})$ on the torus.

A fan tropical plane in $\mathbb{R}^n$ is a two dimensional matroidal fan. A fan tropical plane $P \subset \mathbb{R}^n$ is approximated by a plane $\mathcal{P} \subset (\mathbb{C}^*)^N$ if

$$P = \text{Trop}(\mathcal{P}) = \lim_{t \to \infty} \text{Log}_t(\mathcal{P}) \subset \mathbb{R}^N .$$

The tropical plane P is the **logarithmic limit set** or **Bergman fan** of $\mathcal{P}$ as mentioned in Section 1.1.4. As a reminder, $\text{Trop}(\mathcal{P})$ can be constructed using only the matroid underlying the plane $\mathcal{P}$. We fix the same vectors as in Section 1.1.4, $u_1, \dots u_N$, and set $u_0 = -\sum_{i=1}^N u_i$. These are the outward pointing primitive integer vectors of the facets of the standard simplex Δ.

Recall in Section 1.1.4, we considered the Bergman fan of a matroid in $\mathbb{TP}^N$. By declaring, $\log_t(0) = -\infty$, the coordinate-wise logarithm, Log_t, can be extended to $\mathbb{CP}^N$. The operation of taking the closure in $\mathbb{TP}^N$ and tropicalising in $\mathbb{CP}^N$ commute so that,

$$\text{Trop}(\overline{\mathcal{P}}) = \overline{\text{Trop}(\mathcal{P})}.$$

The compactification $\overline{\text{Trop}(\mathcal{P})}$ will often be denoted by $\overline{P}$. In Section 3.1.3 we described an arrangement A of tropical lines defined by $\overline{P}$ and the set of points $p(A)$.

Given a plane $\mathcal{P} \subset (\mathbb{C}^*)^N$ the compactification $\overline{\mathcal{P}}$ also defines an arrangement of lines $\mathcal{A} = \{\mathcal{L}_0, \dots, \mathcal{L}_N\}$, where $\mathcal{L}_i = \overline{\mathcal{P}} \cap \{z_i = 0\}$, where the z_i. A **point** of the complex arrangement $\mathcal{A}$ is a point $\mathbf{p}$ contained in at least two lines of $\mathcal{A}$, denote the collection of points by $\mathbf{p}(\mathcal{A})$. For a point $\mathbf{p} \in \mathbf{p}(\mathcal{A})$ we may associate to it the maximal subset $I \subset \{0, \dots, N\}$ such that $\mathbf{p} \in \cap_{i \in I} L_i$, therefore we may index the points by $\mathbf{p}_I$. The points $\mathbf{p}(\mathcal{A})$ are in correspondence with flats of the matroid corresponding to $\mathcal{P}$ that are of size greater than 1 but rank less than 3. They are also in correspondence with the points $p(A)$ of the arrangement A defined by $\text{Trop}(P)$. Again the size of a point $\mathbf{p}_I$ is $|I|$. So for a plane $\mathcal{P} \subset (\mathbb{C}^*)^N$ and a tropical plane $P \subset \mathbb{R}^n$ there is a correspondence between A and $\mathcal{A}$ and $p(A)$ and $\mathbf{p}(\mathcal{A})$.

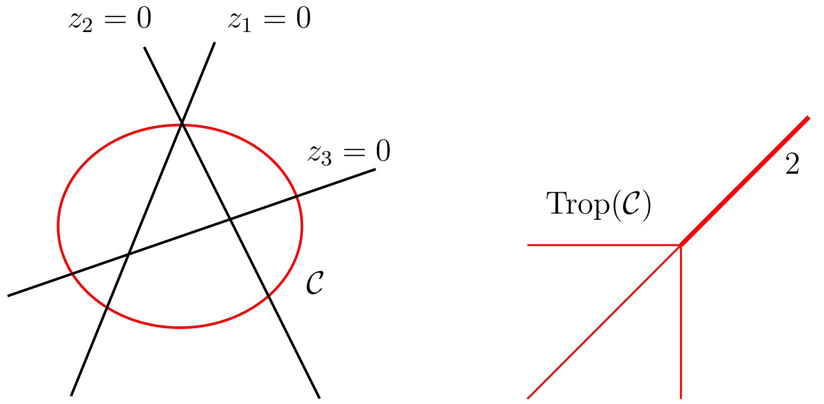


Figure 3.13: The curve $\mathcal{C} \subset (\mathbb{C}^*)^2$ from Example 3.4.3 with respect to the three coordinate lines along with its tropicalisation $\mathrm{Trop}(\mathcal{C}) \subset \mathbb{R}^2$.

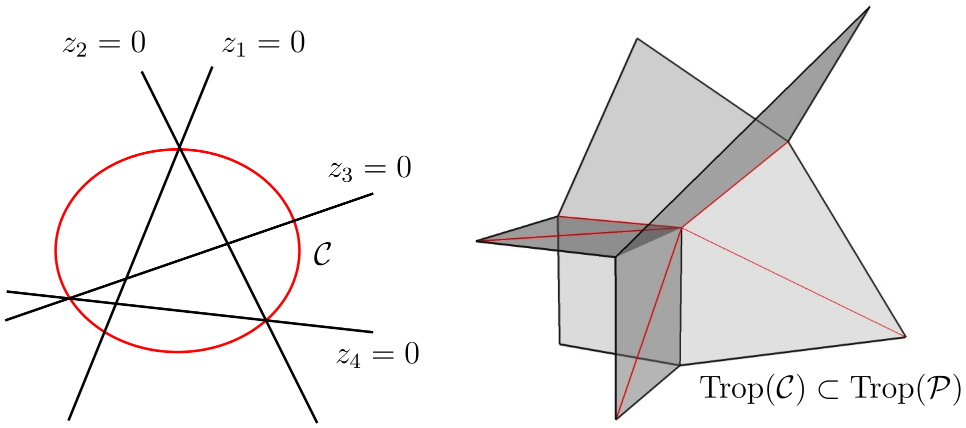


Figure 3.14: The curve $\tilde{C} \subset \mathcal{P}$ drawn with respect to the four lines in the arrangement determined by $\mathcal{P}$ along with its tropicalisation $\mathrm{Trop}(\mathcal{C}) \subset \mathrm{Trop}(\mathcal{P}) \subset \mathbb{R}^3$.

Given a complex algebraic curve $\mathcal{C} \subset \mathcal{P} \subset (\mathbb{C}^*)^N$, its tropicalisation is supported on

$$C = \mathrm{Trop}(\mathcal{C}) = \lim_{t\to\infty} \mathrm{Log}_t(\mathcal{C}) \subset P$$

and comes equipped with weights assigned to the edges. To obtain these weights for a curve $\mathcal{C} \subset (\mathbb{C}^*)^N$, take a non-singular compactification $\mathcal{X}$ of $\mathcal{P}$ such that the curve $\mathcal{C}$ intersects no $\mathcal{D}_i \cap \mathcal{D}_j$, where $\mathcal{D}_i, \mathcal{D}_j$ are boundary divisors of $\mathcal{X}$. Such a compactification of $\mathcal{C}$ in $(\mathbb{C}^*)^N$ is called **compatible**. Let $\overline{\mathcal{C}}$ denote the compactification of $\mathcal{C}$ in $\mathcal{X}$. An edge e of $\mathrm{Trop}(\mathcal{C})$ corresponds to a boundary divisor $\mathcal{D}_e$ of $\mathcal{X}$. The weight w_e assigned to the edge e is the intersection multiplicity of $\overline{\mathcal{C}}$ and $\mathcal{D}_e$ in $\mathcal{X}$.

Definition 3.4.2. The tropicalisation $\mathrm{Trop}(\mathcal{C})$ of a curve $\mathcal{C} \subset (\mathbb{C}^*)^N$ is

$$\lim_{t \to \infty} \mathrm{Log}_t(\mathcal{C})$$

equipped with weights on edges as described above.

Example 3.4.3
Consider the curve $\overline{\mathcal{C}} \subset \mathbb{CP}^2$ given by the homogenous equation

$$f(z_1, z_2, z_3) = z_1 z_3 + z_2 z_3 - 2z_1^2 + z_2^2 + z_1 z_2$$

and let $\mathcal{C} = \overline{\mathcal{C}} \cap (\mathbb{C}^*)^2$. See Figure 3.13 for the complex conic with respect to the three coordinate lines of $\mathbb{CP}^2$ and its tropicalisation to $\mathbb{R}^2$. Let $\mathcal{L} \subset \mathbb{CP}^2$ be the line with equation $z_1 + z_2 - z_3 = 0$. Let $\mathcal{P} \subset (\mathbb{C}^*)^3$ be the zero set of $z_4 - z_1 - z_2 + z_3 = 0$ and let $\tilde{C} \subset \mathcal{P}$ be defined by both equations $f(z_1, z_2, z_3)$ and $z_1 + z_2 - z_3 = 0$. The curve $\tilde{\mathcal{C}}$ is drawn with respect to the four lines determined by $\mathcal{P}$ and its tropicalisation $\mathrm{Trop}(\mathcal{C}) \subset \mathrm{Trop}(\mathcal{P})$ are drawn in Figure 3.14.

The tropicalisation as defined above is the tropical limit of a **constant** family. Such tropical limits may also be referred to as the **constant coefficient** case due to their appearance as images under the valuation map of varieties over $\mathbb{K}$ defined by equations with coefficients in $\mathbb{C}$ [6]. If a tropical curve, $\mathrm{Trop}(\mathcal{C})$, is the limit of a constant family as above, it is always a fan curve with vertex v, the vertex of the plane $\mathrm{Trop}(\mathcal{P})$. Throughout this section we will entertain the following question.

Question 3.4.4. *Fix $\mathcal{P} \subset (\mathbb{C}^*)^N$ a plane, given a fan tropical curve $C \subset Trop(\mathcal{P}) \subset \mathbb{R}^N$ does there exist a complex algebraic curve $\mathcal{C} \subset \mathcal{P} \subset (\mathbb{C}^*)^N$ such that $Trop(\mathcal{C}) = C$?*

When the answer is yes, the tropical curve C is said to be **coarsely approximable**. When in addition the curve $\mathcal{C}$ is reduced and irreducible we say that C is **finely approximable** or simply **approximable**. We will mostly be interested in the case of fine approximability, hence the shortening to "approximable".

In general, the answer to Question 3.4.4 depends on the fixed plane $\mathcal{P}$ and not just its tropicalisation $\mathrm{Trop}(\mathcal{P})$. This was noticed by Bogart and Katz in [7]. Here, Example 3.4.19 of the next section is a simple case of the examples provided in [7]. This phenomenon also reappears in the complete classification of trivalent curves to be given in Subsection 3.4.4. We may also refine Questions 3.4.4 to consider parameterised curves and their tropicalisations. This is done in [12].

To tackle these questions our main technique is to apply tropical intersection theory of curves along with its relation to complex intersection multiplicities established in the next section.

3.4.1 Realisability of intersections.

Here we relate the tropical and complex intersection products for two curves of sedentarity zero in surfaces in the local case. To relate the intersection multiplicity of two fan tropical curves $\text{Trop}(\mathcal{C}_1)$, $\text{Trop}(\mathcal{C}_2) \subset \text{Trop}(\mathcal{P})$ at the vertex v, with the intersection of their respective approximations in $\mathcal{P} \subset \mathbb{C}^N$ we must define an appropriate compactification of the plane $\mathcal{P}$. This is done shortly in Definition 3.4.6 but first we relate the tropical degree of a fan curve as given in Definition 3.1.6 with the degree of its approximation.

Lemma 3.4.5. *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a plane and suppose the tropical fan curve $C \subset$ Trop($\mathcal{P}$) is coarsely approximable by the curve $\mathcal{C} \subset \mathcal{P}$, then* $\deg(\overline{\mathcal{C}}) = \deg(C)$*, where $\overline{\mathcal{C}}$ is the closure of $\mathcal{C}$ in the compactification $\overline{\mathcal{P}} = \mathbb{CP}^2 \subset \mathbb{CP}^N$.*

Proof. Consider the tropical hyperplane $H \subset \mathbb{R}^N$ given by the tropical polynomial,

$$\text{“}u_1 + \cdots + u_N + 0\text{”} = \max\{u_1, \ldots, u_N, 0\}.$$

For every $1 \leq i \leq N$ there is a translation $H' = H + v$ such that H' intersects the tropical curve $C \subset P$ only in the facet of H whose normal vector is u_i and in edges e of C such that $\langle u_i, u_e \rangle > 0$. Each such edge e that intersects the tropical hypersurface H' does so with multiplicity $w_e\langle u_i, u_e \rangle$. Therefore by Definition 3.1.6 $\deg(C) = \deg(H'.C) = \deg(H.C)$, where $H'.C$ is the tropical stable intersection.

The translation H' is approximated by the family

$$\mathcal{H}_t = \{(t^{v_1}z_1, \ldots t^{v_N}z_N \mid (z_1, \ldots, z_N) \in \mathcal{H}\}$$

in the sense that $\lim_{t\to\infty} \text{Log}_t(\mathcal{H}_t) = H'$. The family $\mathcal{H}_t$ can be considered as a hypersurface defined over the field of Puisseux series, and $\mathcal{C}$ as a curve defined over the Puisseux series but with constant coefficients. The intersection of H' and C is proper, so by Theorem 8.8 of [29], it is the intersection number of $\mathcal{H}_t$ and $\mathcal{C}$. Compactifying $\mathbb{R}^N$ to $\mathbb{TP}^N$, the closures $\overline{H'}$ and $\overline{C}$ do not intersect at the boundary for a generic translation. Therefore $\deg(C) = \deg(H'.C) = \deg(\overline{\mathcal{H}}.\mathcal{C}) = \deg(\overline{\mathcal{C}})$, and the lemma is proved. □

We will now describe the compactification of the open space $\mathcal{P}$ needed to relate the tropical and complex intersection numbers. Recall that before Definition 3.4.2, we described compatible compactifications of a curve $\mathcal{C} \subset \mathcal{P} \subset (\mathbb{C}^*)^N$.

Definition 3.4.6. Given two curves in a plane $\mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{P} \subset (\mathbb{C}^*)^N$, a compactification $\mathcal{X}$ of $(\mathbb{C}^*)^N$ is **compatible** with $\mathcal{C}_1, \mathcal{C}_2, \mathcal{P} \subset (\mathbb{C}^*)^N$ if it is compatible with both curves and the compactification $\tilde{\mathcal{P}} \subset \mathcal{X}$ of $\mathcal{P}$ is a non-singular toric surface.

Example 3.4.7
To construct a compatible compactification of two curves $\mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{P} \subset (\mathbb{C}^*)^N$, first let $\Sigma \subset \mathbb{R}^N$ be the complete unimodular fan yielding $\mathbb{CP}^N$ as its toric variety with respect to the basis $u_1, \ldots, u_N$, therefore $\tilde{\mathcal{P}} = \mathbb{CP}^2 \subset \mathcal{X}(\Delta)$. Let $\tilde{\Sigma}$ be a unimodular completion of

$$\Sigma \cup \text{Trop}(\mathcal{C}_1) \cup \text{Trop}(\mathcal{C}_2) \subset \mathbb{R}^N .$$

Then, $\mathcal{X} = \mathcal{X}(\tilde{\Sigma})$ is a compatible compactification of $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{P}$. This compactification $\tilde{\mathcal{P}}$ of $\mathcal{P}$ can be obtained from the compactification $\overline{\mathcal{P}} = \mathbb{CP}^2$ by blowing up points in $\mathbf{p}(\mathcal{A})$ and points above them which are the intersections of boundary divisors.

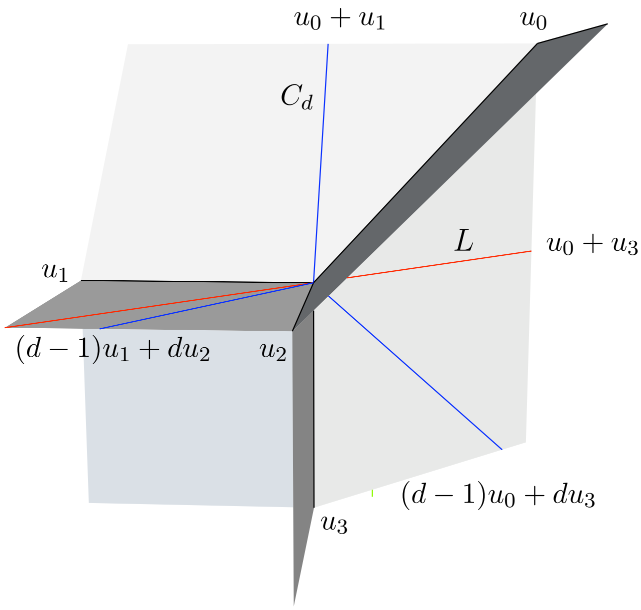


Figure 3.15: The curve C_d and the line L from Example 3.4.10.

The following theorem states the correspondence between the complex and tropical intersection numbers for curves.

Theorem 3.4.8. *If $\mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{P} \subset (\mathbb{C}^*)^N$ approximate $C_1, C_2 \subset P \subset \mathbb{R}^N$ respectively, then*

$$\tilde{\mathcal{C}}_1.\tilde{\mathcal{C}}_2 = C_1.C_2$$

where $\tilde{\mathcal{C}}_i$ is the compactification of $\mathcal{C}_i$ in the compatible toric compactification $\tilde{\mathcal{P}} \subset \mathcal{X}$ from Example 3.4.7.

We postpone the proof of Theorem 3.4.8 to present the some corollaries and examples.

Example 3.4.9
Let P be the standard tropical hyperplane in $\mathbb{R}^3$ and let $L \subset P$ be the affine line in the direction $(1,1,0)$. It is the red curve depicted in Figure 3.15, and also the curve encountered in Example 1.1.20 in Section 1.1.3. Using Theorem 3.1.7, to compute the self-intersect of L, we find, $L.L = -1$. As shown in Example 1.1.20, L is approximated by a line $\mathcal{L} \subset \mathcal{P}$, see again Figure 1.4 for a real drawing of $\mathcal{L}$. The blow up of $\mathbb{CP}^2$ at the two points $\mathcal{L}_i \cap \mathcal{L}_j$ and $\mathcal{L}_k \cap \mathcal{L}_l$ gives the desired compactification $\tilde{\mathcal{P}}$. The proper transform of $\mathcal{L}$ in $\tilde{\mathcal{P}}$ is indeed a curve of self intersection -1.

Example 3.4.10
Once again let $P \subset \mathbb{R}^3$ be the standard tropical hyperplane, L be the line from

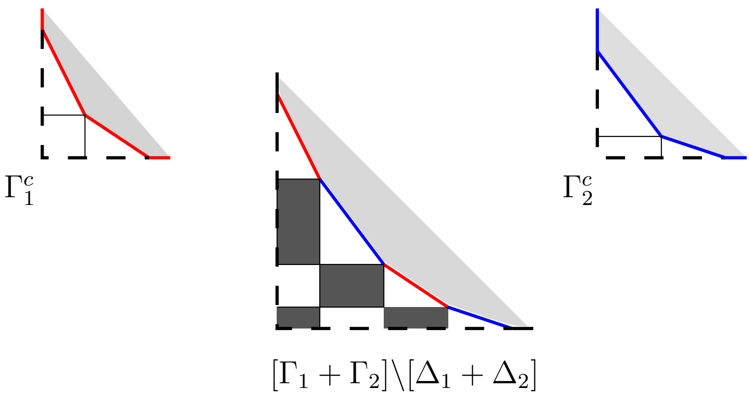


Figure 3.16: The polygons of the proof of Lemma 3.4.13.

Example 3.4.9 and C_d to be the trivalent tropical curve of degree d centered at the vertex of P with weight one rays in the directions

$$u_0 + u_1, \quad (d-1)u_0 + du_3, \quad \text{and} \quad (d-1)u_1 + du_2,$$

see Figure 3.15. These curves intersect at two corner points of $\overline{P} \subset \mathbb{TP}^3$. Locally at these two corners the curves appear as rays converging at the corner of $\mathbb{T}^2$, L in the direction $(-1,-1)$ and C_d in the direction $(-d, 1-d)$. Using Theorem 3.1.7 we have

$$C_d.L = d \cdot 1 - 2(d-1) = -d + 2.$$

Two distinct irreducible algebraic curves in a compact non-singular surface always have non-negative intersection multiplicity. Calling upon this well-known fact we obtain the following two corollaries to Theorem 3.4.8.

Corollary 3.4.11. *Suppose that $C, D \subset P \subset \mathbb{R}^N$ are two irreducible tropical curves such that $C \neq D$ and $C.D < 0$. If C is approximable then D is not approximable.*

It follows from this corollary that the curve C_d from Example 3.4.10 is not approximable for $d \geq 3$. An earlier proof of this appeared in [52] but made use of tropical modifications instead of a correspondence of intersection numbers.

Corollary 3.4.12. *If $C \subset P$ is approximable and $C^2 < 0$ then C is approximated by a unique curve $\mathcal{C} \subset \mathcal{P}$.*

Before giving the proof of Theorem 3.4.8 we first introduce some notation. Given $\mathcal{C}$ an algebraic curve in affine space $\mathbb{C}^2$ defined by a polynomial $P(z,w) = \sum a_{i,j} z^i w^j$, we denote by $\Delta(\mathcal{C}) = Conv\{(i,j) \in \mathbb{Z}^2 \mid a_{i,j} \neq 0\}$ its Newton polygon, and we define (see Figure 3.16)

$$\Gamma(\mathcal{C}) = Conv(\Delta(\mathcal{C}) \cup \{(0,0\}), \quad \text{and} \quad \Gamma_0(\mathcal{C}) = \Gamma(\mathcal{C}) \setminus \Delta(\mathcal{C}).$$

Once a coordinate system is fixed in $\mathbb{C}^2$, the equation of an algebraic curve is defined up to a non-zero multiplicative constant. In particular the polygons $\Delta(\mathcal{C})$, $\Gamma(\mathcal{C})$, and $\Gamma_0(\mathcal{C})$ do not depend on the particular choice of the defining polynomial $P(z,w)$.

The latter definition translates literaly to tropical curves in $\mathbb{T}^2$. If C is the tropicalisation of a projective plane curve $\mathcal{C}$ in the coordinates (z,w), then we have

$$\Delta(\mathcal{C}) = \Delta(C), \quad \Gamma(\mathcal{C}) = \Gamma(C), \quad \text{and} \quad \Gamma_0(\mathcal{C}) = \Gamma_0(C).$$

The tropicalisation $\mathrm{Trop}(\mathcal{C}) \subset \mathbb{T}^2$ determines the Newton polytope with respect to the fixed coordinate system of a complex curve $\mathcal{C}$ such that $\mathrm{Trop}(\mathcal{C}) = C$.

Lemma 3.4.13. *Let $\Delta(\mathcal{C}_1), \Delta(\mathcal{C}_2)$ be the Newton polygons of two affine algebraic curves $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{C}^2$ with respect to a fixed coordinate system and set $C_i = Trop(\mathcal{C}_i)$ for $i = 1,2$. Then,*

$$(C_1.C_2)_{(-\infty,-\infty)} = MV(\Gamma(\mathcal{C}_1), \Gamma(\mathcal{C}_2)) - MV(\Delta(\mathcal{C}_1), \Delta(\mathcal{C}_2)).$$

Where MV denotes the mixed volume of the two Newton polytopes.

Proof. To shorten notation we will denote $\Delta(\mathcal{C}_i)$ by Δ_i and analogously for Γ_i and Γ_i^c. When $\Delta_i = \Gamma_i$ for $i = 1$ or 2 we have $(C_1.C_2)_{(-\infty,-\infty)} = 0$ and also $MV(\Delta_1, \Delta_2) = MV(\Gamma_1, \Gamma_2)$. Otherwise, $\Gamma_i^c = \Gamma_i \backslash \Delta_i \neq \emptyset$ for both $i = 1,2$. Figure 3.16 shows the non-convex polygons, $[\Gamma_1 + \Gamma_2] \backslash [\Delta_1 + \Delta_2]$, Γ_1^c and Γ_2^c. Observe that

$$MV(\Gamma_1, \Gamma_2) - MV(\Delta_1, \Delta_2) = A([\Gamma_1 + \Gamma_2] \backslash [\Delta_1 + \Delta_2]) - A(\Gamma_1^c) - A(\Gamma_2^c),$$

where A denotes the lattice area. The intersection $\Delta_1 \cap \Gamma_1^c$ consists of a collection of edges which will be called outward edges of Δ_1 and we will denote by σ_i. Similarly the edges of $\Delta_2 \cap \Gamma_2^c$ will be called the outward edges of Δ_2 and denoted τ_j. Since the curve C is a fan tropical curve each such outward edge σ_i is dual to an edge e_i of the tropical curve which converges to $(-\infty, -\infty)$ and analogously for the outward edges of Δ_2.

Subdividing the polygons $[\Gamma_1 + \Gamma_2] \backslash [\Delta_1 + \Delta_2]$, Γ_1^c and Γ_2^c as in Figure 3.16, it is clear that the above difference in areas is the sum of the areas of all the shaded rectangles in $[\Gamma_1 + \Gamma_2] \backslash [\Delta_1 + \Delta_2]$ in Figure 3.16. Each such shaded rectangle is formed from a pair of outward edges $\sigma_i \subset \Delta_1 \cap \Gamma_1^c$, $\tau_j \subset \Delta_2 \cap \Gamma_2^c$. Suppose the primitive outward vectors of σ_i, τ_j have directions (p_i, q_i), (r_j, s_j) respectively, and also that σ_i and τ_j have integer lengths w_i and u_j respectively. Then the area of such a rectangle is given by $w_i u_j \min\{p_i s_j, q_i r_j\}$. By duality, the tropical curve C_1 has an edge e_i of weight w_i with primitive integer direction (p_i, q_i) converging to $(-\infty, -\infty)$, and similarly C_2 has an edge f_j with primitive integer direction (r_j, s_j) and weight u_j. By Definition 3.1.4 these rays contribute exactly $w_i u_j \min\{p_i s_j, q_i r_j\}$ to the tropical intersection multiplicity at the corner $(-\infty, -\infty)$. The difference in the mixed volumes

$$MV(\Gamma_1, \Gamma_2) - MV(\Delta_1, \Delta_2)$$

is distributive amongst the outward edges of Δ_1 and Δ_2 and so is the tropical intersection multiplicity at the corner, thus the lemma is proved. □

Corollary 3.4.14. *Let $C \subset \mathbb{T}^2$ be a tropical curve, then*

$$(C^2)_{(-\infty,-\infty)} = A(\Gamma^c(C)),$$

where $A(\Gamma^c(C))$ is the normalized area of $\Gamma^c(C)$.

Together with the next corollary, the above lemma relates the intersection product of two curves after blowing up the necessary points above a single $\mathbf{p}_I$. Recall that in Example 3.4.7 a compatible compactification $\mathcal{C}_1, \mathcal{C}_2 \subset \tilde{\mathcal{P}} \subset \mathcal{X}$ is obtained from $\overline{\mathcal{P}} = \mathbb{CP}^2$ by performing a sequence of blowups at points starting with the points $\mathbf{p}_I \in \mathbf{p}(\mathcal{A})$ and then continuing at points infinitely close to $\mathbf{p}_I$ which are intersections of the boundary divisors. For two curves $\mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{P}$ let $\tilde{\mathcal{P}}_I$ be the surface obtained from $\overline{\mathcal{P}} = \mathbb{C}\,P^2$ by making all necessary blowups only at and above the point $\mathbf{p}_I$. Applying the relation of mixed volumes and intersection numbers from toric geometry, (see Section 5.4 of [19]), when $I = \{i, j\}$ we obtain the following corollary to Lemma 3.4.13.

Corollary 3.4.15. *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a plane with associated line arrangement $\mathcal{A}$ and let $\mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{P}$ be two curves. Suppose $\boldsymbol{p}_{i,j} \in \boldsymbol{p}(\mathcal{A})$ and let $\tilde{\mathcal{C}}_k$ denote the closure of $\mathcal{C}_k$ in $\tilde{\mathcal{P}}_{i,j}$ and $C_k = Trop(\mathcal{C}_k)$ then*

$$\tilde{\mathcal{C}_1}.\tilde{\mathcal{C}_2} = \deg(\overline{\mathcal{C}}_1)\deg(\overline{\mathcal{C}}_2) - (\overline{C}_1.\overline{C}_2)_{p_{i,j}}.$$

Proof of Theorem 3.4.8. In $\overline{\mathcal{P}} = \mathbb{CP}^2$, Bézout's Theorem gives

$$\overline{\mathcal{C}}_1.\overline{\mathcal{C}}_2 = \deg(\overline{\mathcal{C}}_1)\deg(\overline{\mathcal{C}}_2).$$

We claim that after the sequence of blowups starting at $\mathbf{p}_I$, the degree of the intersections of the curves after the blow up decreases by the tropical multiplicity $(C_1.C_2)_{p_I}$ at the corresponding point.

When $I = \{i, j\}$, the claim follows directly from Corollary 3.4.15. When $|I| = m > 2$, suppose the tropicalisations $\overline{C}_1, \overline{C}_2$ each have a single ray converging to the point $p_I \in \overline{P}$ and that the ray of $\text{Trop}(\mathcal{C}_1)$ is contained in the face F_i and the ray of $\text{Trop}(\mathcal{C}_2)$ is contained in the face F_j.

If $i \neq j$ then after blowing up at $\mathbf{p}_I$ the proper transforms of $\overline{C}_1$ and $\overline{C}_2$ do not intersect at any points $\mathcal{E}_I \cap \mathcal{L}_{i'}$ for $i' \in I$, and further blow ups do not affect the intersection number of the curves. In a chart given by the projection $\pi_{i,j}$ the blowup at $\mathbf{p}_I$ is toric, therefore after the blowup the intersection of the curves decreases by $(\pi_{i,j}(C_1).\pi_{i,j}(C_2))_{(-\infty,-\infty)}$, which by Definition 3.1.4 is $(C_1.C_2)_{p_I}$, see Figure 3.2.

If $i = j$ then after blowing up at $\mathbf{p}_I$ the proper transforms $\mathcal{C}_1', \mathcal{C}_2'$ will contain $\mathcal{E}_I \cap \mathcal{L}_{i'}$ if and only if $i' = i$. Therefore, in a chart $\pi_{i,i'}$ for any $i' \in I$ all further blowups at points above $\mathbf{p}_I$ are toric and by applying Corollary 3.4.15 the claim is proved.

The claim holds in the case when multiple rays of the tropical curves C_1, C_2 converge to p_I, since the multiplicity of the curves $\mathcal{C}_1, \mathcal{C}_2$ at the point $\mathbf{p}_I$ is a distributive function of the rays of the tropical curve converging to the corresponding point p_I. Continuing the process at each point $\mathbf{p}_I \in \mathbf{p}(\mathcal{A})$ we obtain the theorem. $\square$

3.4.2 Approximable lines in planes

Here we further simplify intersections in the case of lines in a tropical plane $P \subset \mathbb{R}^N$ and describe them combinatorially using their corresponding matroids. Given two tropical lines $L_1, L_2 \subset P \subset \mathbb{R}^N$ where P is a tropical plane. Let A denote the arrangement of tropical lines defined by P and let A_i denote the arrangement of points on a line defined by L_i. As usual we suppose that P does not contain loops nor double points, however the lines L_i may contain double points.

Lemma 3.4.16. *Let $L_1, L_2 \subset P \subset \mathbb{R}^N$ be two tropical fan lines in a tropical fan plane, then their intersection multiplicity at the vertex v is given by,*

$$(L_1.L_2)_v = 1 - |\{p_I \mid p \in p(A_1) \cap p(A_2) \text{ and } |I| \geq 2\}|.$$

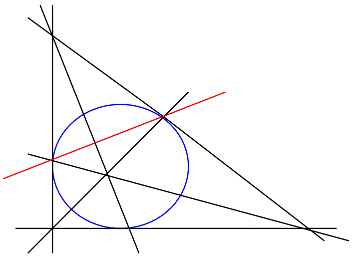
Proof. The degree of a line $L \subset P$ is one, so Theorem 3.1.7, gives

$$(L_1.L_2)_v = 1 - \sum_{x \in P(A)} (\overline{L}_1.\overline{L}_2)_x.$$

Moreover, for points $x \in P(A)$, we have $x \in \overline{L}_i$ if and only if $x \in p(A_i)$. By applying the definition of corner intersections in 3.1.4 we have $(\overline{L}_1.\overline{L}_2)_x = 1$ if $x \in p(A_1), p(A_2)$. This proves the lemma. $\square$

Corollary 3.4.17. *Let $L \subset P \subset \mathbb{R}^N$ be a fan tropical line in a fan tropical plane, then*

$$(L^2)_v = 1 - |\{p_I \in p(A) \mid |I| \geq 2\}|.$$

a)


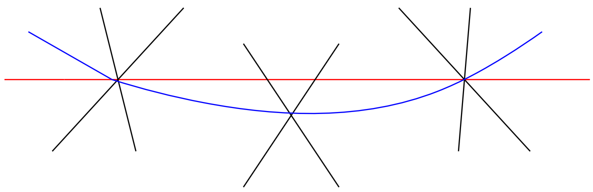
b)


Figure 3.17: a) The arrangement corresponding to $\mathcal{M}_{0,5}$ in black with the Fano and Anti-Fano divisors drawn in blue and red respectively. b) The line arrangements from Example 3.4.19.

Example 3.4.18
The moduli space of five marked points on $\mathbb{CP}^2$, $\mathcal{M}_{0,5}$, is the complement of a hyperplane arrangement $\mathcal{A}$ whose underlying matroid is the graphical matroid given by K_4, [3]. This fan also previously appeared here in Section 1.1.4. We also exhibited the tropical moduli space $M_{0,5} \subset \mathbb{R}^5$ as a result of three modifications of $\mathbb{R}^2$ and showed that all elementary contractions $\delta : M_{0,5} \longrightarrow V \subset \mathbb{R}^4$ lead to isomorphic fans V and divisors D. Moreover, the arrangement corresponding to the divisor D has three points of size 2. By Corollary 3.4.17, $(D.D)_x = -1$ in V. This along with Corollary 1.1.25 proves the claim delivered in Section 1.1.4, that $M_{0,5} \subset \mathbb{R}^N$ can not be obtained via a sequence of modifications along tropical regular functions.

Next we consider a pair of fan tropical lines $L_1, L_2 \subset M_{0,5}$. The line L_1 consists of 3 rays with directions:

$$u_0 + u_1, \quad u_2 + u_3, \quad \text{and} \quad u_4 + u_5.$$

The line L_2 has 4 rays in directions:

$$u_0 + u_1, \quad u_2 + u_3, \quad u_4, \quad \text{and} \quad u_5.$$

A quick verification shows these lines are contained in the fan $M_{0,5} \subset \mathbb{R}^5$. By Lemma 3.4.16, $(L_1.L_2)_x = -1$. Therefore, L_1, L_2 may not be simultaneously approximable. Indeed the matroid extension along L_1 yields the Fano matroid realisable only over a field of characteristic two, and the extension L_2 the anti-Fano matroid, realisable over any field not of characteristic two.

Example 3.4.19
This example presents a tropical line in a plane, $L \subset P \subset \mathbb{R}^5$, for which the answer to the approximation problem depends on the complex plane $\mathcal{P} \subset (\mathbb{C}^*)^5$ approximating P, such examples were already presented in Section 8 of [7]. It also follows from this that that the relative lifting problem cannot be solved using only tropical data.

Choose a hyperplane arrangement $\mathcal{A}$ on $\mathbb{CP}^2$ with corresponding matroid $U_{3,6}$. There are 15 point flats p_{ij} corresponding to the intersection $\mathcal{L}_i \cap \mathcal{L}_j$ for any pairs of lines in $\mathcal{A}$. Let $\mathcal{P} \subset (\mathbb{C}^*)^5$ be a linear embedding of the complement and $P \subset \mathbb{R}^5$ be its tropicalisation. Suppose three point flats corresponding to three disjoint pairs of lines happen to be collinear, say $\mathbf{p}_{12}, \mathbf{p}_{34}, \mathbf{p}_{56}$. Then the line containing these three points tropicalises to a trivalent line $L \subset P$ with $L^2 = -2$. However, choosing a generic configuration $\mathcal{A}'$ with matroid $U_{3,6}$, $\mathcal{P}' \subset (\mathbb{C}^*)^5$ the three points $\mathbf{p}_{12}, \mathbf{p}_{34}, \mathbf{p}_{56}$ would not be collinear, and joining any two produces a line which tropicalises to $L_{ijkl} \subset P$ with $L^2_{ijkl} = -1$. Moreover, $L.L_{ijkl} = -1$ confirms that the two lines cannot be simultaneously approximable.

The following simple lemma describes when a line $L \subset \text{Trop}(\mathcal{P})$ is approximable by a line $\mathcal{L} \subset \mathcal{P}$.

Lemma 3.4.20. *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a complex plane, and let $u_0, \dots, u_N$ denote the unimodular basis used to construct Trop($\mathcal{P}$). Then there is a k-valent fan tropical* *line $L \subset$ Trop($\mathcal{P}$) approximable in $\mathcal{P}$ if and only if for some $0 \leq m \leq k$ the arrangement determined by $\mathcal{P}$ contains m collinear points $\boldsymbol{p}_{I_1}, \ldots, \boldsymbol{p}_{I_m}$ such that the sets $I_1, \ldots, I_m$ are disjoint and $I_1 \cup \cdots \cup I_m = \{0, \ldots, N\} \backslash J$ where $|J| = k - m$. Moreover, L is the tropical line with rays in directions:*

$$u_{I_1}, \ldots, u_{I_m}, \quad \text{and} \quad u_j \quad \text{for } j \in J,$$

where $u_{I_i} = \sum_{s \in I_i} u_s$. This line is approximated by the line passing through the m collinear points $\boldsymbol{p}_{I_i}$.

Proof. Suppose a curve $\mathcal{L}$ approximates L, since $\deg(L) = 1$, $\overline{\mathcal{L}}$ must be a line in the compactification of $\mathcal{P}$ to $\overline{\mathcal{P}} = \mathbb{CP}^2$. Therefore, $\overline{\mathcal{L}}$ intersects each line in the arrangement determined by $\mathcal{P}$ exactly once, and because L is k-valent and each edge is of weight one, $\mathcal{L}$ may intersect the arrangement in only k points. As the $N+1$ lines of the arrangement are indexed by $\{0, \ldots, N\}$, there must be a partition of this set into k subsets, $I_1, \ldots, I_k$. The subsets I_i of size greater than one correspond to points $\mathbf{p}_{I_i}$ of the arrangement through which the line $\mathcal{L}$ passes, therefore these points must be collinear. This completes the proof. $\square$

Definition 3.4.21. Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a plane and $C \subset \text{Trop}(\mathcal{P})$ be a fan tropical curve, define the approximation space of C in $\mathcal{P}$ to be

$$\text{App}(\mathcal{P})_C = \{\mathcal{C} \subset \mathcal{P} \mid \text{Trop}\, \mathcal{C} = C\}.$$

Corollary 3.4.22. *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a plane and $L \subset$ Trop($\mathcal{P}$) $\subset \mathbb{R}^N$ then,*

1. $(L.L)_x = 1 \Longleftrightarrow \dim(App(\mathcal{P})_L) = 2$
2. $(L.L)_x = 0 \Longleftrightarrow \dim(App(\mathcal{P})_L) = 1$
3. $(L.L)_x = -1 \Longrightarrow |App(\mathcal{P})_L| = 1$
4. $(L.L)_x \leq -2 \Longrightarrow |App(\mathcal{P})_L| = 0, 1$
5. $App(\mathcal{P})L = \emptyset \Longrightarrow (L.L)_x \leq -2$

3.4.3 Classification of affine planar curves in the standard hyperplane

In this section we fix $P_{st} \subset \mathbb{R}^3$ the standard hyperplane, and $\mathcal{P}_{st} \subset (\mathbb{C}^*)^3$ a plane such that $\text{Trop}(\mathcal{P}_{st}) = P_{st}$, any generic plane in $(\mathbb{C}^*)^3$ will do. Recall that $\mathcal{P}_{st} = \mathbb{CP}^2 \setminus \mathcal{A}$ where $\mathcal{A} = \{\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3\}$ is an arrangement of four generic lines. Up to automorphism of $\mathbb{CP}^2$ there is only one such arrangement.

For a fan tropical curve C let Aff(C) denote the affine span of the curve C and aff_C denote the dimension of the affine span of the curve. In this section we will suppose that $\text{aff}_C \leq 2$ so that C is contained in an affine hyperplane $H \subset \mathbb{R}^3$ of rational slope. The plane H is the tropicalisation of a toric surface in $(\mathbb{C}^*)^3$. In [7], Bogart and Katz provide some necessary conditions to approximate such a curve.

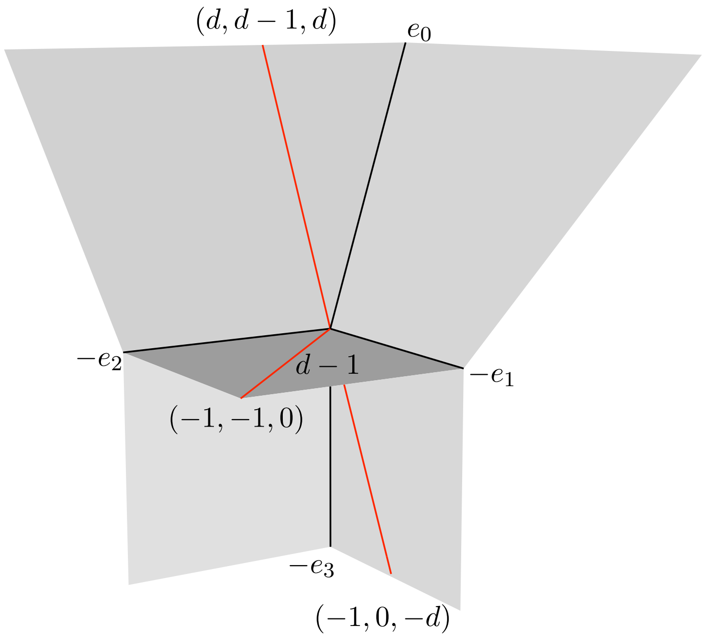


Figure 3.18: The trivalent tropical curve $C_d \subset P$ in from Example 3.4.26.

Theorem 3.4.23. *[7] Let C be a fan tropical curve in $P_{st} \subset \mathbb{R}^3$ the standard tropical hyperplane, such that C is also contained in $H \subset \mathbb{R}^3$ an affine rational plane. Suppose C is finely approximated by $\mathcal{C}$ in $\mathcal{P}_{st} \subset (\mathbb{C}^*)^3$. Then one of the following must hold:*

1. *C is equal to $H.P_{st}$, the stable intersection of P_{st} and H in $\mathbb{R}^3$, or*

2. *$H.P_{st}$ contains one of the affine bisecting lines contained in P_{st}.*

This theorem is based on the two following observations:

Lemma 3.4.24 (Bogart-Katz, [7])**.** *If H is a rational affine plane in $\mathbb{R}^3$ containing C and if C is approximable by a reduced and irreducible complex algebraic curve $\mathcal{C} \subset \mathcal{P}_{st}$, then there exists a non-singular binomial algebraic surface $\mathcal{H} \subset (\mathbb{C}^*)^3$ such that $H = Trop(\mathcal{H})$ and $\mathcal{C} \subset \mathcal{H}$.*

Lemma 3.4.25 (Bogart-Katz, [7])**.** *Let $\mathcal{H} \subset (\mathbb{C}^*)^3$ be a non-singular binomial surface. If the intersection of $\mathcal{P}_{st}$ and $\mathcal{H}$ in $(\mathbb{C}^*)^3$ is non-complete, then the curve $\mathcal{P}_{st} \cap \mathcal{H}$ has a unique singular point which is a node. Moreover, if $\mathcal{P}_{st} \cap \mathcal{H}$ has two irreducible components $\mathcal{C}_1$ and $\mathcal{C}_2$, then the two embedded tropical curves $Trop(\mathcal{C}_1)$ and $Trop(\mathcal{C}_2)$ are at most 3-valent, and at least one of them is 2-valent.*

By Theorem 5.3.2 of [43] all curves which are stable intersections of P_{st} and H are coarsely approximable. However, condition (2) is not sufficient for approximability. For example, the non-approximable curves presented in Example 3.4.10 happen to satisfy condition (2) of the above theorem of Bogart and Katz. The next example presents the only family of curves satisfying part (2) which are in fact approximable.

Example 3.4.26
Let u_0, u_1, u_2, and u_3 denote the primitive integer directions of the four rays of the standard hyperplane $P_{st} \subset \mathbb{R}^3$. For $d > 1$ consider the tropical curve $C_d \subset P_{st}$ with three rays in the directions:

$$u_i + u_j, \quad du_k + u_i, \quad \text{and} \quad du_l + u_j.$$

Where the edge in direction $u_i + u_j$ is of weight $d-1$, and the two other rays are of weight one. For $d = 1$, let C_d be the bivalent line with rays of weight one in directions:

$$u_k + u_i, \quad \text{and} \quad u_l + u_j.$$

The curve C_d is of degree d and an example is drawn in Figure 3.18.

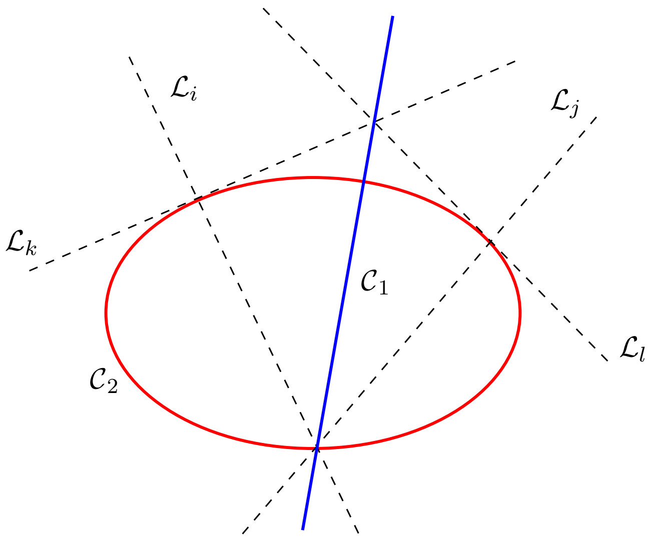


Figure 3.19: The plane $\mathcal{P}$ drawn as $\mathbb{CP}^2$ minus the four lines $\mathcal{L}_i, \mathcal{L}_j, \mathcal{L}_k, \mathcal{L}_l$, with the curves $\mathcal{C}_d$ drawn for $d = 1$ and 2.

For every $d \geq 1$ we will construct an irreducible rational curve $\mathcal{C}_d \subset \mathcal{P} \subset (\mathbb{C}^*)^3$, such that $\operatorname{Trop} \mathcal{C}_d = C_d$. This curve has multiplicity $d-1$ at the point $\mathcal{L}_i, \mathcal{L}_j$, and tangencies of order d to the lines $\mathcal{L}_k$ and $\mathcal{L}_l$ at the points $\mathcal{L}_i \cap \mathcal{L}_k$ and $\mathcal{L}_j \cap \mathcal{L}_l$. Up to automorphism of $\mathbb{CP}^2$ we are free to fix the 4 generic lines of the arrangement corresponding to $\mathcal{P}$ as we wish. Let $\mathcal{L}_i$ be the line at infinity and

$$\mathcal{L}_k = \{y = 0\}, \quad \mathcal{L}_j = \{x = 0\}, \quad \text{and} \quad \mathcal{L}_l = \{x + y - 1 = 0\}.$$

The existence of an approximation is equivalent to the existence of an irreducible complex algebraic curve $\mathcal{C}$ in $\mathbb{C}^2$ defined by the equation $yP(x) - 1 = 0$ where $P(x)$

is a complex polynomial of degree $d-1$ with no multiple root, and such that $\mathcal{C}$ has order of contact d at $(0,1)$ with the line $\mathcal{L}_l$. Now it is not hard to check that one has to have $P(x) = 1 + x + \ldots + x^{d-1}$. The curve $\mathcal{C}$ has the correct order of contact with each of the lines $\mathcal{L}_i$ of $\overline{\mathcal{P}} \setminus \mathcal{P}$ in order to tropicalise to $C \subset P$. A quick verification shows that $\mathcal{C}$ is parameterised by a rational curve with $d+1$ punctures.

Theorem 3.4.27 (Brugallé, S., [12])**.** *If $C \subset P_{st} \cap H$ then C is approximable by an irreducible curve if and only if C is one of the following:*

1. *C is the stable intersection of P_{st} and H.*
2. *up to symmetry of P_{st}, $C = C_d$ from Example 3.4.26, where $d = \deg(C)$.*

As noticed by Bogart and Katz, Lemma 3.4.25 implies immediately that if $C \subset P_{st}$ is a fan tropical curve with $\text{aff}_C \leq 2$ which is finely approximable in $\mathcal{P}_{st}$ and not equal to the tropical stable intersection of P_{st} and H, then C is either 2 or 3-valent. In addition to this we now extract further information from the Lemmas 3.4.24 and 3.4.25, in order to prove Theorem 3.4.27. Suppose that $C \subset H \cap P_{st}$ is approximable by an irreducible curve $\mathcal{C}$, and let $\mathcal{H} \subset (\mathbb{C}^*)^3$ be the binomial hypersurface given by Lemma 3.4.24, which contains $\mathcal{C}$ and tropicalises to H. Let Δ be the Newton polytope of the surface $\mathcal{P}_{st} \cup \mathcal{H} \subset (\mathbb{C}^*)^3$, and $\mathcal{X}(\Delta)$ the associated toric variety. Denote the compactifications of $\mathcal{P}_{st}$ and $\mathcal{H}$ in $\mathcal{X}(\Delta)$ by $\tilde{\mathcal{P}}_{st}$ and $\tilde{\mathcal{H}}$ respectively, and let $\tilde{\mathcal{C}}_0 = \tilde{\mathcal{H}} \cap \tilde{\mathcal{P}}_{st}$. Note that $\tilde{\mathcal{H}}^2 = 0$, which implies that $\tilde{\mathcal{C}}_0^2 = 0$.

Lemma 3.4.28. *Let $\mathcal{P} \subset (\mathbb{C}^*)^3$ be a uniform plane, and let $\mathcal{H} \subset (\mathbb{C}^*)^3$ be a reduced and irreducible binomial surface. We denote by $\Delta_{\mathcal{P},\mathcal{H}}$ the Newton polytope of the tropical surface Trop($\mathcal{P}$) $\cup$ Trop($\mathcal{H}$), and by $\mathcal{X}(\Delta_{\mathcal{P},\mathcal{H}})$ the toric variety defined by $\Delta_{\mathcal{P},\mathcal{H}}$. Let $\overline{\mathcal{P}}$ and $\overline{\mathcal{H}}$ be respectively the closure of $\mathcal{P}$ and $\mathcal{H}$ in $\mathcal{X}(\Delta_{\mathcal{P},\mathcal{H}})$, and let $\overline{\mathcal{C}} = \overline{\mathcal{P}} \cap \overline{\mathcal{H}}$. Then the curve $\overline{\mathcal{C}}$ is reduced and $\overline{\mathcal{C}}^2 = 0$ in $\overline{\mathcal{P}}$. Moreover, if $\overline{\mathcal{C}}$ is reducible, then $\overline{\mathcal{C}}$ has exactly two irreducible components $\overline{\mathcal{C}}_1$ and $\overline{\mathcal{C}}_2$, and $\overline{\mathcal{C}}_1^2 = \overline{\mathcal{C}}_2^2 = -1$ in $\overline{\mathcal{P}}$.*

In particular, Trop$(\mathcal{C}_1)^2$ = Trop$(\mathcal{C}_2)^2 = -1$ in Trop($\mathcal{P}$).

Proof. We define $\mathcal{C} = \mathcal{P} \cap \mathcal{H} = \overline{\mathcal{C}} \cap (\mathbb{C}^*)^3$. According to Lemma 3.4.25, the curve $\mathcal{C}$ has at most one singular point, so it has to be reduced and cannot have more than two irreducible components. Hence the same is true for $\overline{\mathcal{C}}$. Since $\overline{\mathcal{H}}^2 = 0$, we also have $\overline{\mathcal{C}} = 0$ in $\overline{\mathcal{P}}$. Suppose that $\overline{\mathcal{C}}$ has two irreducible components $\overline{\mathcal{C}}_1$ and $\overline{\mathcal{C}}_2$. Since $\overline{\mathcal{H}}^2 = 0$ we have

$$\overline{\mathcal{C}}_1.\overline{\mathcal{C}} = \overline{\mathcal{C}}_2.\overline{\mathcal{C}} = 0$$

which implies that

$$\overline{\mathcal{C}}_1^2 + \overline{\mathcal{C}}_1.\overline{\mathcal{C}}_2 = \overline{\mathcal{C}}_2^2 + \overline{\mathcal{C}}_1.\overline{\mathcal{C}}_2 = 0.$$

So we are left to show that $\overline{\mathcal{C}}_1.\overline{\mathcal{C}}_2 = 1$. Since the curve $\overline{\mathcal{C}}$ is reducible, it follows from Lemma 3.4.25 that $\mathcal{C}$ has a unique singular point, which is a node, in $(\mathbb{C}^*)^3$. Hence the result will follow from the fact that $\overline{\mathcal{C}}$ intersects the boundary $\mathcal{X}(\Delta_{\mathcal{P},\mathcal{H}}) \setminus (\mathbb{C}^*)^3$ transversally at non-singular points of $\overline{\mathcal{C}}$.

To prove this last claim, we may assume that $\mathcal{H}$ is a subtorus of $(\mathbb{C}^*)^3$. In this case, there is a surjection $\phi : \mathrm{Hom}((\mathbb{C}^*)^3, \mathbb{C}^*) \otimes \mathbb{R} \to \mathrm{Hom}(\mathcal{H}, \mathbb{C}^*) \otimes \mathbb{R}$. Moreover, if $Q \in Hom((\mathbb{C}^*)^3, \mathbb{C}^*)$ is an equation of $\mathcal{P}$ in $(\mathbb{C}^*)^3$, then $\phi(Q)$ is an equation of $\mathcal{C}$ in $\mathcal{H}$. In particular, the Newton polygon of $\phi(Q)$ is dual to the tropical curve $\mathrm{Trop}(\mathcal{C})$, seen as a tropical curve in $\mathrm{Trop}(\mathcal{H})$. According to Lemma 3.4.25, the tropical curve $\mathrm{Trop}(\mathcal{C})$ is 4-valent, so the Newton polygon of $\phi(Q)$ is a quadrangle. The polynomial Q has exactly 4 monomials and $\phi(Q)$ has no few monomials than Q, so we get that $\phi(Q)$ also has exactly 4 monomials. In particular, the only non-zero coefficients of $\phi(Q)$ are the vertices of its Newton polygon. This implies that $\overline{\mathcal{C}}$ intersects the boundary $\overline{\mathcal{H}} \setminus \mathcal{H}$ transversally at non-singular points of $\overline{\mathcal{C}}$. □

To complete the proof of Theorem 3.4.27, it remains to list all possible 3-valent fan tropical curves C with $\mathrm{aff}_C \leq 2$ and to compute C^2 for each of them. This is the content of the next lemma.

Lemma 3.4.29. *Let $C \subset P_{st}$ be an irreducible 3-valent fan tropical curve with $\mathrm{aff}_C \leq 2$. Let H be the non-singular binomial tropical surface containing C, and e_1, e_2, and e_3 the edges of C. Let u_0, u_1, u_2, u_3 denote the primitive integer directions of the four rays of P_{st}. Then the curve C is one of the following types:*

1. *There exists $0 \leq \alpha, \beta$ with $\gcd(d, \alpha, \beta) = 1$ and $\alpha + \beta \leq d$, such that*

   $$w_{e_1}u_{e_1} = du_i+\alpha u_j, \quad w_{e_2}u_{e_2} = \beta u_j+du_k, \quad \textit{and} \quad w_{e_3}u_{e_3} = (d-\alpha-\beta)u_j+du_l,$$

   see Figure 3.20 (1). In this case the curve C is the tropical intersection of P_{st} and H, and $C^2 = 0$;

2. *There exists $0 \leq \alpha, \beta \leq d$ with $\gcd(d, \alpha, \beta) = 1$ such that*

   $$w_{e_1}u_{e_1} = du_i+\alpha u_j, \quad w_{e_2}u_{e_2} = (d-\alpha)u_j+(d-\beta)u_k, \quad \textit{and} \quad w_{e_3}u_{e_3} = \beta u_k+du_l,$$

   see Figure 3.20 (2). In this case, $C^2 = -\alpha\beta$;

3. *There exists $0 \leq \alpha < \beta \leq d$ with $\gcd(d, \alpha, \beta) = 1$ such that*

   $$w_{e_1}u_{e_1} = \alpha u_i+\beta u_j, \quad w_{e_2}u_{e_2} = (d-\alpha)u_i+(d-\beta)u_j, \quad \textit{and} \quad w_{e_3}u_{e_3} = du_k+du_l,$$

   see Figure 3.20 (3). In this case, $C^2 = -d^2 + \beta d - \alpha\beta$.

Note that cases (1) and (2) when $\alpha = \beta = 0$ (and consequently $d = 1$) coincide with the case (3) when $\alpha = 0$ and $\beta = d = 1$.

Proof. The intersection numbers follow from a direct computation using Theorem 3.1.7. In case (1), we have to prove in addition that the curve C is the tropical intersection of P_{st} and H, which is non-trivial only for $\alpha \neq 0$ and $\beta \neq 0$. Let us denote by C' the tropical intersection of $P_{st} \cap H$. In this case it is clear that C and C' have the same underlying sets. Since C is irreducible, it remains to prove that C' is also irreducible.

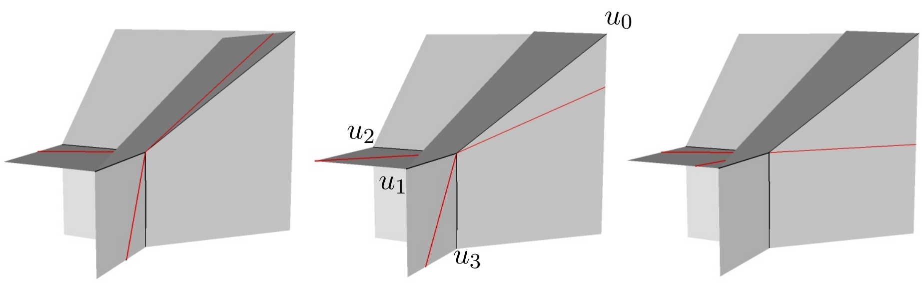


Figure 3.20: The three types of curves from Lemma 3.4.29, listed in order from left to right.

Without loss of generality, we can assume that

$$u_0 = (1, 1, 1, 1), \quad u_1 = (-1, 0, 0), \quad u_2 = (0, -1, 0), \quad \text{and} \quad u_3 = (0, 0, -1).$$

The surface H is given by a classical linear equation of the form

$$ax + by + cz = 0 \quad \text{with} \quad \gcd(a, b, c) = 1.$$

Let us denote by $w_{1,2}$ (resp. $w_{2,3}$) the weight of the edge of C' lying in the convex cone spanned by u_1 and u_2 (resp. u_3 and u_2). A computation gives $w_1 = \gcd(a, b)$, and $w_2 = \gcd(b, c)$. Hence w_1 and w_2 are relatively prime and C' is irreducible and $C' = C$. This completes the proof. □

Remark Note that the same proof gives that the tropical stable intersection of *any* tropical surface of degree 1 in $\mathbb{R}^3$ made of an edge and 3 faces with *any* non-singular binomial tropical surface in $\mathbb{R}^3$ is *always* irreducible.

Proof of Theorem 3.4.27. Combining Lemmas 3.4.28 and 3.4.29 we see that the only possible approximable curves C are of type (1) in the list given in Lemma 3.4.29, or of type (2) with $\alpha = \beta = 1$. As mentioned in the statement of Lemma 3.4.29, the curve of type (3) when $\alpha = 0, \beta = 1$ coincides with curves of type (1) and (2). A curve of type (1) is always approximable as it is a stable intersection of P_{st} and H, see Remark 3.4.3 . The curves of type (2) with $\alpha = \beta = 1$ already appeared in Example 3.4.26, where they were shown to be approximable in every degree by a rational curve. This completes the proof of the theorem. □

Combining Lemma 3.4.29 and Theorem 3.4.8, we obtain immediately the following simple criteria for approximating trivalent fan curves in P_{st}.

Corollary 3.4.30. *Let $C \subset P_{st}$ be a trivalent fan tropical curve. Then C is finely approximable in $\mathcal{P}_{st}$ if and only if $C^2 = 0$ or -1.*

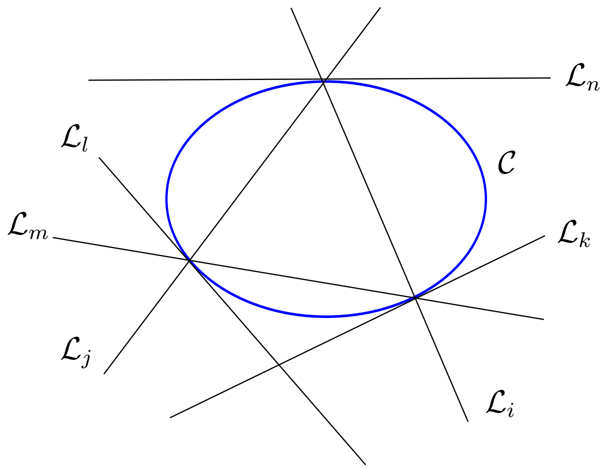


Figure 3.21: The curve $\mathcal{C}$ from Lemma 3.4.31 with respect to the lines indexed by i, j, k, l, m, n.

3.4.4 Trivalent curves in planar fans

Combining the complete classification of curves C in the standard plane $P_{st} \subset \mathbb{R}^3$ with $\text{aff}_C \leq 2$ given in the previous subsection, along with the case of lines presented in Subsection 3.4.20 we may classify all approximable trivalent fan tropical curves in planes of any codimension.

Given two line arrangements on $\mathbb{CP}^2$ such that $\mathcal{A} \subset \mathcal{A}'$ there is a natural inclusion of their respective planes $i : \mathcal{P}' \hookrightarrow \mathcal{P}$. Moreover, if $\mathcal{C} \subset \mathcal{P}$ is a curve, there is a curve $\mathcal{C}' \subset \mathcal{P}'$ such that the closure of $i(\mathcal{C}')$ in $\mathcal{P}$ is $\mathcal{C}$. This is used in the next lemma.

Before considering the general case we remark that not every tropical plane $P \subset \mathbb{R}^N$ contains trivalent fan tropical curves. In fact, in order for P to contain a trivalent curve there must exist three sets I_1, I_2, I_3 satisfying: $I_1 \cup I_2 \cup I_3 = \{0, \dots, N\}$ and if $|I_i| > 1$ then p_{I_i} is a point of the corresponding line arrangement.

Given two line arrangements $\mathcal{A} \subset \mathcal{A}'$ in $\mathbb{CP}^2$, there is a natural inclusion of their respective planes $i : \mathcal{P}' \hookrightarrow \mathcal{P}$.

Lemma 3.4.31. *Let $\mathcal{P} \subset (\mathbb{C}^*)^3$ be a uniform plane, and denote the lines of the associated arrangement by $\mathcal{L}_i, \mathcal{L}_j, \mathcal{L}_k$, and $\mathcal{L}_l$. In addition, let $\mathcal{C}_2 \subset \mathcal{P}$ be the degree 2 curve from part (2) of Theorem 3.4.27. Then the further fan tropical curves are finely approximable in the plane $\mathcal{P}'$ in following cases (see Figure 3.21):*

1. *the plane $\mathcal{P}' \subset (\mathbb{C}^*)^4$ corresponds to the arrangement of 5 lines obtained by adding to the arrangement of $\mathcal{P}$ the unique line $\mathcal{L}_m$ which passes through the two points $\boldsymbol{p}_{i,k}$, $\boldsymbol{p}_{j,l}$; the three rays of $C \subset Trop(\mathcal{P}')$ are of weight one with primitive integer directions*

$$u_i + u_j, \quad u_i + 2u_k + u_m, \quad \textit{and} \quad u_j + 2u_l + u_m.$$

2. *the plane* $\mathcal{P}' \subset (\mathbb{C}^*)^4$ *corresponds to the arrangement of* 5 *lines obtained by adding to the arrangement of* $\mathcal{P}$ *the unique line* $\mathcal{L}_n$ *which is tangent to* $\mathcal{C}_2$ *at the point* $\boldsymbol{p}_{i,j}$*; the three rays of* $C_2 \subset$ *Trop*$(\mathcal{P}')$ *are of weight one with primitive integer directions*

$$u_i + u_j + 2u_n, \quad u_i + 2u_k, \text{ and } \quad u_j + 2u_l.$$

3. *the plane* $\mathcal{P}' \subset (\mathbb{C}^*)^5$ *corresponds to the arrangement of* 6 *lines obtained from the arrangement of* $\mathcal{P}$ *by adding the lines* $\mathcal{L}_m$ *and* $\mathcal{L}_n$ *from parts* (1) *and* (2)*; the three rays of* $C \subset$ *Trop*$(\mathcal{P}')$ *are of weight one and with primitive integer directions*

$$u_i + u_j + 2u_m, \quad u_i + 2u_k + u_n, \quad \text{and} \quad u_j + 2u_l + u_n.$$

Proof. In each of the three above cases the tropical curves are approximated by the curve $\mathcal{C} = \mathcal{C}_2 \cap \mathcal{P}'$. □

Theorem 3.4.32. *Let* $N \geq 3$*, let* $\mathcal{P} \subset (\mathbb{C}^*)^N$ *be a non-degenerate plane, and let* $C \subset$ *Trop*$(\mathcal{P})$ *be an irreducible 2 or 3-valent fan tropical curve. Then the curve* C *is finely approximable in* $\mathcal{P}$ *if and only if we are in one of the following cases:*

1. $\deg(C) = 1$ *and* C *and* $\mathcal{P}$ *satisfy Lemma 3.4.20;*

2. C *and* $\mathcal{P}$ *satisfy one of the three situations described in Lemma 3.4.31;*

3. *the plane* $\mathcal{P} \subset (\mathbb{C}^*)^3$ *is non-uniform and* C *is any irreducible trivalent fan tropical curve;*

4. *the plane* $\mathcal{P} \subset (\mathbb{C}^*)^3$ *is uniform and* C *is a trivalent curve from part* (2) *of Theorem 3.4.27 or part* (1) *of Lemma 3.4.29.*

As a remark we mention that the trivalent lines with $N \geq 6$ in case (1) of Theorem 3.4.32 and the curves of case (2) and (3) of Lemma 3.4.31 are **exceptional**, in the sense that for a generic choice of plane $\mathcal{P}$ which tropicalises to the fans in each of these cases (i.e. whose line arrangement has the right intersection lattice), the corresponding tropical curve will not be approximable.

Proof. All of the above tropical curves were shown to be approximable in the corresponding plane in Lemmas 3.4.20, 3.4.31, Remark 3.4.3, and Theorem 3.4.27.

Let C be an irreducible 2 or 3-valent fan tropical curve which is finely approximable by a curve $\mathcal{C}$ in some plane $\mathcal{P}$. It remains to show that the pair $(\mathcal{P}, \mathcal{C})$ is one of those described in the theorem. According to Lemma 3.4.20, this is true if $\deg(C) = 1$, so let us suppose that $\deg(C) \geq 2$.

Suppose first that the arrangement determined by $\mathcal{P}$ contains a uniform sub-arrangement $\mathcal{A}_0$ of 4 lines $\mathcal{L}_i, \mathcal{L}_j, \mathcal{L}_k$, and $\mathcal{L}_l$ yielding a plane $\mathcal{P}_0$. Then there is a natural inclusion $\mathcal{P} \hookrightarrow \mathcal{P}_0$. If $\mathcal{C} \subset \mathcal{P}$ approximates C, let $\mathcal{C}_0 \subset \mathcal{P}_0$ be the closure of

$\mathcal{C}$ in $\mathcal{P}_0$ and $C_0 \subset \text{Trop}(\mathcal{P}_0)$ its tropicalisation. Then $\deg(C_0) = \deg(C) \geq 2$, which implies by Theorem 3.4.27 that the curve C_0 is trivalent. Therefore C_0 is either of type (1) from Lemma 3.4.29 or from case (2) of Theorem 3.4.27.

Suppose it is the former. Since $\deg(C_0) \geq 2$, up to relabeling the four lines in $\mathcal{A}_0$, it follows from Lemma 3.4.29 that $\mathcal{C}_0 \subset \mathcal{P}_0$ intersects this uniform arrangement in the three points $\mathbf{p}_{i,j}, \mathbf{p}_{i,k}, \mathbf{p}_{i,l}$. Let $\mathcal{L}$ be another line of the arrangement $\mathcal{A}$ determined by $\mathcal{P}$. Since C is trivalent, the line $\mathcal{L}$ must intersect $\mathcal{C}$ with multiplicity $\deg(C)$ at one of these three points, which is impossible according to Lemma 3.4.29.

If $C_0 \subset \text{Trop}(\mathcal{P}_0)$ satisfies part (2) of Theorem 3.4.27, then $\mathcal{C}_0$ is the unique curve given in the proof of Theorem 3.4.27 (up to relabelling the four lines in $\mathcal{A}_0$). This curve intersects the arrangement $\mathcal{A}_0$ in the points $\mathbf{p}_{i,j}, \mathbf{p}_{i,k}$, and $\mathbf{p}_{j,l}$. Note that the only singular point of $\overline{\mathcal{C}} \subset \overline{\mathcal{P}}$ may be at the point $\mathbf{p}_{i,j}$ and that the tangent line to $\overline{\mathcal{C}}_0$ at the points $\mathbf{p}_{i,k}$ and $\mathbf{p}_{j,l}$ is already contained in the arrangement $\mathcal{A}_0$. Therefore, any other line passing through $\mathbf{p}_{i,k}$ or $\mathbf{p}_{j,l}$ has intersection multiplicity 1 with $\overline{\mathcal{C}}_0$ at this point. Let $\mathcal{L}$ be a line in $\mathcal{A} \setminus \mathcal{A}_0$. Since the tropical curve C is trivalent, we are in one of the following two situations:

1. the line $\mathcal{L}$ passes through the points $\mathbf{p}_{i,k}, \mathbf{p}_{j,l}$, and the sum of the intersection multiplicities of $\overline{\mathcal{C}}$ and $\mathcal{L}$ at these two points is equal $\deg(C)$; since the intersection multiplicity of $\overline{\mathcal{C}}$ and $\mathcal{L}$ is 1 at these points, this is possible only if $d = 2$;

2. the line $\mathcal{L}$ passes through the point $\mathbf{p}_{i,j}$, $\mathbf{p}_{i,k}$, or $\mathbf{p}_{j,l}$, and intersects $\overline{\mathcal{C}}$ with multiplicity $\deg(C)$ at this point; since the intersection multiplicity of $\overline{\mathcal{C}}$ and $\mathcal{L}$ is 1 at $\mathbf{p}_{i,k}$ and $\mathbf{p}_{j,l}$, the line $\mathcal{L}$ necessarily passes through $\mathbf{p}_{i,j}$, which is an ordinary point of multiplicity $d-1$ of $\overline{\mathcal{C}}$; since C is 3-valent, the line $\mathcal{L}$ must have the same intersection multiplicity with all local branches of $\overline{\mathcal{C}}$ at $\mathbf{p}_{i,j}$, which is possible only if $d = 2$.

Hence if we are not in cases (1) or (4) from the statement of the theorem, we are necessarily in case (2).

If the arrangement $\mathcal{A}$ does not contain a uniform subarrangement of 4 lines then according to Lemma 3.4.33 below, all but one line of $\mathcal{A}$ must belong to the same pencil. The arrangement $\mathcal{A}$ contains a subarrangement $\mathcal{A}'_0$ of 4 lines, 3 of which belong to the same pencil, and defining a plane $\mathcal{P}'_0$. As previously, if $\mathcal{C} \subset \mathcal{P}$ approximates C, let $\mathcal{C}'_0 \subset \mathcal{P}'_0$ be the closure of $\mathcal{C}$ in $\mathcal{P}'_0$ and $C'_0 \subset \text{Trop}(\mathcal{P}'_0)$ its tropicalisation. Since $\deg(C'_0) \geq 2$, the curve C'_0 is trivalent. Hence according to Remark 3.4.3, C'_0 is the tropical stable intersection of $\text{Trop}(\mathcal{P}'_0)$ and $\text{Aff}(C)$. If $\mathcal{C}'_0$ does not pass through the triple point of $\mathcal{A}'_0$, then since $\text{Trop}(\mathcal{C})$ is trivalent we must have $\mathcal{A} = \mathcal{A}'_0$. If $\mathcal{C}'_0$ passes through the triple point of $\mathcal{A}'_0$, then there exist $|I| - 2$ lines of $\mathcal{A}$ such that each branch of $\mathcal{C}$ at $\mathbf{p}_I$ has order of contact $\deg(C)$ with these lines. This implies that $|I| = 3$, which completes the proof. $\square$

Lemma 3.4.33. *If $\mathcal{A}$ is a line arrangement not containing a uniform subarrangement of 4 lines then all but one of the lines in $\mathcal{A}$ are contained in the same pencil.*

Proof. By assumption not all lines of $\mathcal{A}$ belong to the same pencil, so there is a subarrangement of 3 lines $\mathcal{L}_i, \mathcal{L}_j$, and $\mathcal{L}_k$, that is uniform. Every other line in $\mathcal{A}$ must belong to the pencil determined by a pair of lines in this subarrangement, otherwise there would be 4 lines forming a uniform subarrangement. If two of the additional lines indexed by l, m belong to different pencils given by say $\mathbf{p}_{i,j}$ and $\mathbf{p}_{i,k}$ then the subarrangement given by j, k, l, m is uniform, and we obtain a contradiction. □

As a remark we mention that curves of Case (4) and (5) are exceptional, in the sense that for a generic choice of plane $\mathcal{P}$ which tropicalises to the fans in these two cases the curves mentioned above are not approximable.

Corollary 3.4.34. *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a non-degenerate generic plane approximating $P = Trop(\mathcal{P})$, if $N > 5$ there are no trivalent tropical curves $C \subset P$ with $\deg(C) > 1$ approximable in $\mathcal{P}$.*

For case (1), when $\deg(C) = 1$, the above corollary does not always hold. Pappus's Hexagon theorem gives an arrangement of 9 lines containing 3 points which are always collinear over any field. This yields an example of a trivalent tropical line $L \subset P \subset \mathbb{R}^7$ which is approximable in every $\mathcal{P} \subset (\mathbb{C}^*)^7$ such that $\mathrm{Trop}(\mathcal{P}) = P$.

Corollary 3.4.35. *The tropical curves of Theorem 3.4.32 are all approximated by rational curves with three punctures.*

3.4.5 Obstructions from the adjunction formula

In algebraic geometry the adjunction formula relates the canonical class of a smooth hypersurface $\mathcal{D}$ to the canonical class of the smooth ambient variety $\mathcal{X}$ by,

$$K_{\mathcal{D}} = (K_{\mathcal{X}} + \mathcal{D})|_{\mathcal{D}},$$

see for example [25]. The canonical class of a Riemann surface of genus g is $2g - 2$, so for a smooth curve $\mathcal{C}$ in a projective surface $\mathcal{X}$ the formula reduces to,

$$g(\mathcal{C}) = \frac{K_{\mathcal{X}}.\mathcal{C} + \mathcal{C}^2}{2} + 1, \tag{3.4.1}$$

where $g(\mathcal{C})$ is the genus of $\mathcal{C}$. If $\mathcal{C} \subset \mathcal{X}$ is not a smooth curve the right hand side of Equation 3.4.1 defines the arithmetic genus of the curve and we denote it by g_a. Let g_b denote the geometric genus of a curve, which is a birational invariant and is the genus of a resolution $\mathcal{C}' \longrightarrow \mathcal{C}$, where $\mathcal{C}'$ is a smooth curve, see [25]. If $\mathcal{C}$ is irreducible then $g_b(\mathcal{C}) \geq 0$. Then Equation 3.4.1 yields the inequality

$$0 \leq g_b(\mathcal{C}) \leq g_a(\mathcal{C}) = \frac{K_{\mathcal{X}}.\mathcal{C} + \mathcal{C}^2 + 2}{2}, \tag{3.4.2}$$

and $g_b(\mathcal{C}) = g_a(\mathcal{C})$ if and only if $\mathcal{C}$ is smooth. In the case fan tropical curves in planes, Theorem 3.4.36 interprets the above inequality using combinatorial data from the tropical curve and the arrangement defined by the plane P to give a general obstruction to approximation.

Beforehand, we introduce some more notation. Recall that if the arrangement $\mathcal{A}$ determined by a plane $\mathcal{P} \subset (\mathbb{C}^*)^N$ contains a point $\mathbf{p}_I$, then the fan tropical plane $P = \text{Trop}(\mathcal{P}) \subset \mathbb{R}^N$ contains a ray in the direction $u_I = \sum_{i \in I} u_i$, where $u_1, \dots, u_N$ is a unimodular basis used to construct the fan P. Given a fan tropical curve $C \subset P$, let w_I denote the weight of the edge of C in the direction u_I, with the convention that $w_I = 0$ if C does not contain a ray in this direction.

Theorem 3.4.36. *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a non-degenerate plane, and $C \subset Trop(\mathcal{P})$ be a fan tropical curve. If C is finely approximable by a complex curve $\mathcal{C} \subset \mathcal{P}$ then*

$$2g(\mathcal{C}) \leq C^2 + (N-2)\deg(C) - \sum_{e_i \subset Ed(C)} w_{e_i} - \sum_{\boldsymbol{p}_I \in \boldsymbol{p}(\mathcal{A})} (|I|-2)w_I + 2, \qquad (3.4.3)$$

with equality if and only if $\mathcal{C}$ is non-singular.

Proof. Again let $\tilde{\mathcal{P}}$ be the surface compatible with $\mathcal{C}_1, \mathcal{C}_2$ obtained from $\overline{\mathcal{P}} = \mathbb{CP}^2$ by blowing up at the points $\mathbf{p}_I$ and points above them which are intersections of boundary divisors. Let $\pi : \tilde{\mathcal{P}} \longrightarrow \mathbb{CP}^2$ denotes the contraction map. The boundary $\partial\tilde{\mathcal{P}} = \tilde{\mathcal{P}} \setminus \mathcal{P}$ is a collection of non-singular divisors consisting of the proper transforms of the $N+1$ lines in $\overline{\mathcal{P}} \setminus \mathcal{P}$ along with all exceptional divisors. Given $\mathbf{p}_I \in \mathbf{p}(\mathcal{A})$, we denote by $\mathcal{E}_I$ the proper transform in $\tilde{\mathcal{P}}$ of the exceptional divisor of the blowup of $\mathbb{CP}^2$ at the point $\mathbf{p}_I$, by $\partial\tilde{P}$ the sum of all divisors in $\tilde{\mathcal{P}} \setminus \mathcal{P}$, and by $\mathcal{L}$ the divisor class of a line in $\mathbb{C}P^2$. Note that the divisors $\mathcal{E}_I$ are contained in the support of $\partial\tilde{P}$. We will prove in Lemma 3.4.38 that the canonical class of $\tilde{\mathcal{P}}$ can be written in the following way:

$$K_{\tilde{\mathcal{P}}} = (N-2)\pi^* \mathcal{L} - \partial\tilde{\mathcal{P}} - \sum_{\mathbf{p}_I \in \mathbf{p}(\mathcal{A})} (|I|-2)\, \mathcal{E}_I,$$

With $K_{\tilde{\mathcal{P}}}$ written this way, we may calculate $K_{\tilde{\mathcal{P}}}.\mathcal{C}$ using just the tropical curve $C = \text{Trop}(\mathcal{C})$. Firstly, $\pi^* \mathcal{L} . \mathcal{C} = \deg(C)$. By definition of the weights of the edges of $\text{Trop}(C)$ we have

$$\partial \mathcal{P} . \tilde{\mathcal{C}} = \sum_{e \in \text{Ed}(C)} w_e,$$

and

$$K_{\tilde{\mathcal{P}}}. \mathcal{C} = (N-2)d - \sum_{e \in \text{Ed}(C)} w_e - \sum_{\mathbf{p}_I \in \mathbf{p}(\mathcal{A})} (|I|-2)w_I.$$

By Theorem 3.4.8 we have $\tilde{\mathcal{C}}^2 = C^2$. Applying the adjunction formula for $\tilde{\mathcal{C}} \subset \tilde{\mathcal{P}}$ we obtain the claimed inequality. $\square$

Not only does Theorem 3.4.36 provide information about the genus of a possible parameterisation of a curve $\mathcal{C} \subset \mathcal{P}$ approximating $C \subset P$, it also has the following corollary when we insist that C is finely approximable.

Corollary 3.4.37. *Let $P \subset \mathbb{R}^N$ be a non-degenerate fan tropical plane and $C \subset P$ a fan tropical curve. If the pair (P, C) is finely approximable, then*

$$C^2 + (N-2)\deg(C) - \sum_{e \in Ed(C)} w_e - \sum_{\boldsymbol{p}_I \in \boldsymbol{p}(\mathcal{A})} (|I|-2)w_I + 2 \geq 0.$$

The following lemma completes the proof of Theorem 3.4.36.

Lemma 3.4.38. *Using the same notations as in the proof of Theorem 3.4.36, we have*

$$K_{\tilde{\mathcal{P}}} = (N-2)\pi^* \mathcal{L} - \partial\tilde{\mathcal{P}} - \sum_{\boldsymbol{p}_I \in \boldsymbol{p}(\mathcal{A})} (|I|-2)\, \mathcal{E}_I\,.$$

Proof. To see that the canonical class can be expressed as claimed we first start with

$$K_{\mathbb{C}P^2} = -3\,\mathcal{L} = -\sum_{i=0}^{N} \mathcal{L}_i + (N-2)\,\mathcal{L}$$

where the $\mathcal{L}_i$'s are the lines in $\overline{\mathcal{P}} \setminus \mathcal{P}$. If $\pi' : \mathcal{P}' \longrightarrow \mathbb{CP}^2$ is the blowup of $\mathbb{CP}^2$ at the point $\mathbf{p}_I$, the canonical classes are related as follows:

$$K_{\mathcal{P}'} = \pi'^* K_{\mathbb{CP}^2} + \mathcal{E}_I$$

Then

$$\begin{aligned} K_{\mathcal{P}'} &= (N-2)\pi'^* \mathcal{L} - \pi'^*(\sum_{i=0}^{N} \mathcal{L}_i) + \mathcal{E}_I \\ &= (N-2)\pi'^* \mathcal{L} - \sum_{i=0}^{N} \tilde{\mathcal{L}}_i - |I|\, \mathcal{E}_I + \mathcal{E}_I, \end{aligned}$$

where $\tilde{\mathcal{L}}_i$ is the proper transform of $\mathcal{L}_i$. Moreover $\partial\,\mathcal{P}' = \sum_{i=0}^{N} \tilde{\mathcal{L}}_i + \mathcal{E}_I$, so

$$K_{\mathcal{P}'} = (N-2)\pi'^* \mathcal{L} - \partial\,\mathcal{P}' - (|I|-2)\, \mathcal{E}_I\,.$$

Blowing up further at points above $\mathbf{p}_I$ that are the intersection of two boundary divisors, the exceptional divisor is again a boundary divisor of the new surface. Continuing the process at each $\mathbf{p}_I$ to obtain $\tilde{\mathcal{P}}$ we obtain:

$$K_{\tilde{\mathcal{P}}} = (N-2)\pi^* \mathcal{L} - \partial\tilde{\mathcal{P}} - \sum_{\mathbf{p}_I \in \mathbf{p}(\mathcal{A})} (|I|-2)\, \mathcal{E}_I,$$

which completes the proof. □

We conclude this part with some examples and remarks surrounding Theorem 3.4.36. Notice that the Corollary 3.4.37 makes no distinction between approximability in different complex planes which have the same tropicalisation. As such, a curve

$C \subset P$ which is approximable in some $\mathcal{P}$ tropicalising to P will never be obstructed by the condition given in this corollary.

The inequality given in Corollary 3.4.37 is by no means a sufficient condition for approximability. As the next example shows there are non-approximable curves $C \subset X$ for which the adjunction formula holds. An interesting question to ask is which such curves may be approximated by pseudo-holomorphic curves for example.

Example 3.4.39
Consider the fan tropical curve $C \subset P_{st} \subset \mathbb{R}^3$ with three rays of weight one in the primitive integer directions:

$$(1-d, 1, 1), \quad (d-1, d-2, d-1) \quad \text{and} \quad (0, 1-d, d).$$

The degree of this curve is d, and its self-intersection at the vertex of P_{st} is

$$(C)^2_v = d^2 - (d-1) - d - d(d-1) = 1 - d.$$

Therefore the right hand side of Equation 3.4.3 is equal to 0, and the curve is not obstructed by Corollary 3.4.37.

It should be noted that by applying an integer affine transformation to $\mathbb{R}^3$ the pair (P, C) may be transformed to the pair (S, L) where L is a line and S is a tropical surface of degree d. This line and surface along with many others were presented by Vigeland in [56]. Here Vigeland presented generic tropical surfaces of degree greater than three containing infinite families of lines. However, a generic complex surface in $\mathbb{CP}^3$ of degree three contains no families of lines, and a general surface of degree greater than three contains no lines, see [25]. Therefore, the above tropical curve C is not approximable in a generic surface $\mathcal{S}$ for $d \geq 4$. However, it was not until the results of Bogart in Katz [7] that an obstruction based only on tropical data was presented. These curves are not approximable for $d \geq 3$ by their proposition cited here as Theorem 3.4.23.

This section is concluded with another example which demonstrates a strange phenomenon of tropical subvarieties.

Example 3.4.40
Consider the curves $C_d \subset P_{st}$ from Example 3.4.10. For $d \geq 3$ there are many ways to see that such a curve is not approximable. For $d \geq 3$ these curves are obstructed by Corollary 3.4.11 and Theorem 3.4.23. In addition a quick calculation shows that, $(C)^2_v = -d^2+2d-1$, and the right hand side of Equation 3.4.3 is equal to $-d^2+3d-2$ which is less than zero for $d \geq 3$. So for $d \geq 3$ the curves are again obstructed by Corollary 3.4.37.

What is interesting is that there exists an integer affine linear map which sends the pair (P_{st}, C_d) to a pair (S_d, L), where $S_d \subset \mathbb{R}^3$ is a tropical surface of degree d and L a tropical line. So separately the line L and the surface S_d are both tropically smooth, however L considered as a subvariety of the surface S_d is singular since it does not satisfy the adjunction formula. Thus being tropically smooth is not an intrinsic property!

3.4.6 A sufficient condition for the tropical adjunction formula

Tropically, the situation is this, given a tropical curve C in a surface X, we may naively define the tropical genus to be the first Betti number of C viewed as a graph. Yet with this definition there are many examples of complex curves whose tropicalisations are too coarse and thus may "hide" some genus of the original curve. For example, the tropicalisation as defined in the beginning of Section 3.4 always results in a fan, so that $b_1(\text{Trop}(\mathcal{C})) = 0$ and is strictly less then $g(\mathcal{C})$ for curves of positive genus. In addition, the next subsection explores more examples of tropical curves for which the adjunction formula fails more dramatically. Imposing the condition that a tropical curve C is locally submatroidal in X, meaning that C is locally given by a Bergman fan of a rank two matroid which is the quotient of the local matroid defining X the following proposition holds.

Theorem 3.4.41. *For any irreducible non-singular curve $C \subset X$ for a compact tropical surface X we have*

$$b_1(C) = \frac{K_X.C + C^2}{2} + 1. \tag{3.4.4}$$

Proof. Suppose C is a boundary divisor of the surface X. Then we may write:

$$K_X = K_X^0 - C - \sum_{i=1}^{d} D_i$$

where the boundary divisors of X are the set $\{C, D_1, \dots, D_d\}$. So we must verify:

$$g = \frac{K_X^0.C - \sum_{i=1}^{d} D_i.C}{2} + 1.$$

This equality is verified by a simple Euler characteristic computation. For every leaf l of the curve C, there is a collection of boundary divisors D_i which meet C at this point. We construct a new graph G from C by adding to each leaf l an edge for each boundary divisor which meets C at l. Let $V(G)$ and $E(G)$ denote the edge and vertex sets of a graph G, and $v_G(x)$ denote the valency of $x \in V(G)$ in G. By subsection 3.1.1 we obtain,

$$K_X^0.C = \sum_{x \in V(G)} v_G(x) - 2. \tag{3.4.5}$$

Subsection 3.1.1 gives:

$$\sum_{i=1}^{d} D_i.C = \sum_{l \in L(C)} v_G(l) - 2.$$

So that,

$$K_X^0.C - \sum_{i=1}^{d} D_i.C = \sum_{x \in v(C)} v_C(x) - 2 = 2(|E(C)| - |V(C)|) = 2g - 2.$$

When C is not a boundary divisor we modify X along the Cartier divisor corresponding to C and obtain a new tropical manifold $\tilde{X}$ which has the curve $\tilde{C}$ corresponding to C at the boundary. Then $\tilde{C} \subset \tilde{X}$ is a boundary divisor thus it satisfies the adjunction formula. Moreover, $\tilde{C}^2 = \delta^*(C)^2 = C^2$. By Lemma 3.2.12, $K_{\tilde{X}}.\tilde{C} = \delta^* K_X . \delta^* C = K_X . C$. Moreover, $b_1(C) = b_1(\tilde{C})$ so that $C \subset X$ satisfies the the formula 3.4.4 as well. □

The condition that C be non-singular (meaning locally sub-matroidal) in X is not a necessary condition for the adjunction formula to hold. As an example, it may be checked that the approximable fan curves $C_d \subset P_{st}$ satisfy the adjunction formula 3.4.4 as stated in Theorem 3.4.41. This is expected since such curves are approximated by rational curves $\mathcal{C}_d \subset \mathcal{P}_{st}$.

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