Hyperbolic spaces and bounded cohomology

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Hyperbolic Spaces and Bounded Cohomology

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voor Geertje

Résumé

Dans cette thèse nous étudions la cohomologie continue et continue bornée et quelques-unes de ces applications à des espaces hyperboliques réels et complexes. Cette thèse comprend deux parties.
Dans la première partie, nous montrons que la cohomologie continue des groupes hyperboliques réels est réalisée sur le bord, c'est-à-dire par le complexe des applications mesurables sur le bord de l'espace hyperbolique réel. En dimension 3 ceci est un résultat de Bloch. La généralisation est non-triviale. Tandis qu'en dimension 3 le stabilisateur de 3 points est trivial, en dimension supérieure à 3 ce stabilisateur est seulement compact et il n'est pas clair que ses groupes de cohomologie à coefficients dans des espaces de Fréchet non-localement convexes sont triviaux. Ce résultat peut être une première étape vers une démonstration de la conjecture de Dupont et Monod qui affirme que l'application de comparaison naturelle de la cohomologie continue bornée vers la cohomologie continue est un isomorphisme pour les groupes de Lie semi-simples connexes de centre fini. La surjectivité a déjà été montrée pour une grande classe de ces groupes. Par contre, pour l'injectivité la conjecture reste largement ouverte ; elle est seulement confirmée dans quelques cas en basse dimension. Une conséquence immédiate de notre résultat est l'injectivité de l'application de comparaison pour les groupes hyperboliques réels en degré trois.
Dans la deuxième partie, nous donnons des estimations pour la norme de Gromov de la classe de cohomologie en dimension supérieure de $\mathrm{Isom}(\mathbb{H}^2_{\mathbb{C}}, \mathbb{R})$. En conséquence nous obtenons les nouvelles bornes explicites suivantes pour le volume simplicial d'une surface fermée hyperbolique complexe :

$$\frac{2}{\pi^2}\mathrm{Vol}(M) \leq \|M\| \leq \frac{9}{\pi^2}\mathrm{Vol}(M).$$

Il existe très peu de bornes explicites connues pour le volume simplicial, un invariant introduit par Gromov pour mesurer la complexité topologique d'une variété.

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Contents

Introduction

Bounded cohomology of groups was first introduced by Trauber. It became an active field of research due to the pioneering work "Volume and bounded cohomology" by Gromov [Gro82] showing its connection with geometry and topology. Burger and Monod extended this theory to locally compact groups [Mon01]. In this thesis we study this continuous bounded cohomology and some of its applications.

The thesis falls into two parts. All results in Part I are from [Pie15] and the text of chapters 4 and 5 overlaps almost completely with this preprint. Part II concerns giving estimates on the Gromov norm of the top dimensional class in $H_c^4(\mathrm{Isom}(\mathbb{H}^2_{\mathbb{C}});\mathbb{R})$. As a consequence, we obtain an explicit upper bound for the simplicial volume of closed oriented manifolds that are locally isometric to $\mathbb{H}^2_{\mathbb{C}}$. We will now describe these two parts.

Part I

The natural comparison map

$$c : H^*_{c,b}(G;\mathbb{R}) \to H^*_c(G;\mathbb{R})$$

between continuous bounded cohomology and continuous cohomology is in general neither injective nor surjective. However, there is the following conjecture:

Conjecture 1. *[Mon06] [Dup79] Let G be a semisimple connected Lie group with finite center. Then the comparison map $c : H^*_{c,b}(G;\mathbb{R}) \to H^*_c(G;\mathbb{R})$ is an isomorphism.*

While there is a lot of evidence for the surjectivity part of Conjecture 1 (see e.g. [LafSch06], [HarOtt12]), for the injectivity part there are only a few results in low degree. For degree 2 injectivity was proven by Burger and Monod in [BurMon99]. In degree 3 and 4 it has been proven for $\mathrm{SL}_2(\mathbb{R})$ by Burger-Monod [BurMon02] and Hartnick-Ott [HarOtt15] respectively.

Let G be a semisimple connected Lie groups with finite center. Denote by the K the maximal compact subgroup of G and by P the minimal parabolic subgroup of G. While both continuous cohomology and continuous bounded cohomology can be realized by the complex

$$0 \to C_{c(,b)}(G/K;\mathbb{R})^G \to C_{c(,b)}((G/K)^2;\mathbb{R})^G \to C_{c(,b)}((G/K)^3;\mathbb{R})^G \to \dots$$

of continuous (bounded) maps defined on the symmetric space G/K, it is a special feature of the continuous bounded cohomology of semisimple Lie groups that it can be realized by the boundary resolution

$$0 \to L^\infty(G/P;\mathbb{R})^G \to L^\infty((G/P)^2;\mathbb{R})^G \to L^\infty((G/P)^3;\mathbb{R})^G \to \dots,$$

i.e. the continuous bounded cohomology of G is equal to the cohomology of the complex of measurable essentially bounded maps defined on its Furstenberg boundary G/P.

We say that the continuous cohomology of G is measurably realized on its Furstenberg boundary G/P if it is isomorphic to the cohomology of the cocomplex $(C((G/P)^{*+1};\mathbb{R})^G, \delta^*)$, where C stands for measurable cochains defined almost everywhere. If this is the case then the comparison map, since it is natural with respect to resolutions [Mon01, Proposition 9.2.3], is induced by the map $L^\infty((G/P)^2;\mathbb{R})^G \to C((G/P)^{*+1};\mathbb{R})^G$ and this may give new information concerning Conjecture 1. Therefore an intermediate step in proving Conjecture 1 can be

Conjecture 2. *Let G be a semisimple connected Lie groups with finite center. Then the continuous cohomology of G is measurably realized on its Furstenberg boundary G/P.*

There is some evidence for Conjecture 2. In [Gon93], [Gon95] Goncharov defines measurable cocycles on the space of flags $\mathcal{F}l(\mathbb{C}^m)$ representing the Borel classes in $H_c^{2n-1}(\mathrm{SL}_m(\mathbb{C});\mathbb{R})$ for $n = 2, 3$ and $m \geq 2n-1$ using the classical di- and trilogarithm. These cocycles are all bounded. We prove Conjecture 2 for $G = \mathrm{Isom}^+(\mathbb{H}^n)$:

Theorem 4.1.1. *Let $n \geq 2$. The continuous cohomology of $\mathrm{Isom}^+(\mathbb{H}^n)$ is measurably realized on the boundary $\partial\mathbb{H}^n$ of real hyperbolic space (with the natural action). Furthermore, all cocycles have essentially bounded representatives in this boundary resolution.*

This immediately implies a particular case of Conjecture 1.

Corollary 5.1.1. *The comparison map from continuous bounded cohomology to continuous cohomology for real hyperbolic space $\mathbb{H}^n$ is injective in degree 3, i.e.*

$$c : H^3_{c,b}(\mathrm{Isom}^+(\mathbb{H}^n);\mathbb{R}) \hookrightarrow H^3_c(\mathrm{Isom}^+(\mathbb{H}^n);\mathbb{R}).$$

For $n = 3$ the above theorem is a result of Bloch [Blo00, Section 7.4]. The main difficulty in the generalization from $n = 3$ to higher dimensions comes from the fact that for $n \geq 4$ the stabilizer of 3 points in $\partial\mathbb{H}^n$ is not trivial. This prevents a straightforward generalization of Bloch's proof for degree $p > 3$.

Recall that, via the van Est isomorphism, $H_c^*(\mathrm{Isom}^+(\mathbb{H}^n);\mathbb{R})$ can be identified with the de Rham cohomology of the compact dual of $\mathbb{H}^n$, which is the n-sphere. The cohomology group $H_c^*(\mathrm{Isom}^+(\mathbb{H}^n);\mathbb{R})$ is thus well-known and therefore Theorem 4.1.1 gives new information about the quotient

$$\frac{\ker\left(\delta : C((\partial\mathbb{H}^n)^{p+1};\mathbb{R})^{\mathrm{Isom}^+(\mathbb{H}^n)} \to C((\partial\mathbb{H}^n)^{p+2};\mathbb{R})^{\mathrm{Isom}^+(\mathbb{H}^n)}\right)}{\mathrm{im}\left(\delta : C((\partial\mathbb{H}^n)^{p};\mathbb{R})^{\mathrm{Isom}^+(\mathbb{H}^n)} \to C((\partial\mathbb{H}^n)^{p+1};\mathbb{R})^{\mathrm{Isom}^+(\mathbb{H}^n)}\right)}.$$

In particular, $H_c^3(\mathrm{Isom}^+(\mathbb{H}^3);\mathbb{R})$ is one dimensional and it is generated by the volume function $\mathrm{Vol} \in L^\infty((\partial\mathbb{H}^3)^4;\mathbb{R})^{\mathrm{Isom}^+(\mathbb{H}^3)}$ which sends four points in the boundary to the volume of the ideal simplex they span. Hence Bloch's result implies that, up to scalar multiplication, Vol is the only cocycle in degree 3 defined on the boundary. He used this to show that the Bloch-Wigner dilogarithm is essentially the only measurable function on $\mathbb{C}P^1$ that satisfies the five term relation. Indeed, applying such a function to the cross ratio of 4 points in $\partial\mathbb{H}^3$ gives a measurable cocycle and thus a multiple of the volume function.

One further tool for computing the continuous bounded cohomology of Lie groups can be stability results. In [Mon04] Monod proves that the continuous bounded cohomology of SL_n is stable over n. More precisely, for any local field k and $0 \leq q \leq n-1$ he shows that the standard embedding $\mathrm{GL}_{n-1}(k) \hookrightarrow \mathrm{SL}_n(k)$ induces an isomorphism $H_{c,b}^q(\mathrm{SL}_n(k)) \cong H_{c,b}^q(\mathrm{GL}_{n-1}(k))$. He proves this using nontrivial coefficients of L^∞ type and a spectral sequence argument. We prove such a stability result for the isometry group $\mathrm{Isom}(\mathbb{H}^n_{(\mathbb{C})})$ of real (or complex) hyperbolic space using simpler methods.

Part II

Simplicial volume was introduced by Gromov in [Gro82]. It gives a topological measure of the complexity of a manifold. Until now its exact value has only been computed for hyperbolic manifolds ([Gro82], [Thu78]) and for closed manifolds covered by $\mathbb{H}^2 \times \mathbb{H}^2$ [Buc08b]. Except for these results no explicit upper bounds for the simplicial volume are known. There are some more nonvanishing results. For example, in the case of negative curvature the simplicial volume is bounded from below by the Riemannian volume and therefore nonzero [Gro82], [Thu78]. Also, the simplicial volume of oriented closed connected locally symmetric spaces of non-compact type is nonzero [LafSch06]. However, in general there is also no explicit lower bound known. Let $[c_\Phi] \in H_{c,b}^2(\mathrm{PU}(2,1);\mathbb{R})$ be the Kähler class. We prove

Theorem 7.1.2.

$$\frac{2}{9}\pi^2 \leq \|[c_\Phi \cup c_\Phi]\|_\infty \leq \pi^2$$

As $c_\Phi \cup c_\Phi$ is proportional to the image under the van Est isomorphism of the volume form in $\Omega^4(\mathbb{H}^2_{\mathbb{C}};\mathbb{R})^{\mathrm{PU}(2,1)}$ applying the Gromov-Thurston proportionality principle for locally symmetric spaces of noncompact type [Buc08a] gives explicit bounds for the simplicial volume $\|M\|$ of a closed complex hyperbolic surface M:

Corollary 7.1.3. *Let M be a closed oriented manifold which is locally isometric to $\mathbb{H}^2_{\mathbb{C}}$. Then*

$$\frac{2}{\pi^2}\mathrm{Vol}(M) \leq \|M\| \leq \frac{9}{\pi^2}\mathrm{Vol}(M).$$

We furthermore obtain the following Milnor-Wood inequality relating the Euler number $\chi(\xi)$ of a $\mathrm{GL}^+(4,\mathbb{R})$-bundle ξ to the Euler characteristic of a closed complex hyperbolic surface M:

Corollary 7.1.6. *Let ξ be a flat $\mathrm{GL}^+(4,\mathbb{R})$-bundle over a closed complex hyperbolic surface M. Then*

$$|\chi(\xi)| \leq \frac{3}{2}\chi(M).$$

Outline

Let us now describe how this thesis is organized by giving a short summary of the content of each chapter.

Part I

In **Chapter 1** we introduce the classical theory of continuous cohomology of groups and the continuous bounded cohomology of groups. We describe the different resolutions that compute these groups that we use in this thesis and the natural comparison map $H^*_{c,b}(G;\mathbb{R}) \to H^*_c(G;\mathbb{R})$. In the last section we discuss some background needed for part II of this thesis, namely the Gromov-Thurston proportionality principal, Hirzebruch's proportionality principle and the Euler class.

In **Chapter 2** we turn to the theory of measurable cohomology. After giving the definition we discuss in some detail the subtleties that come with dealing with equivalence classes of functions. We furthermore discuss some of the properties and techniques, such as dimension shifting, that we need later on.

In **Chapter 3** we discuss the spectral sequence associated to a double complex. We will construct this spectral sequence through exact couples following [BottTu82].

In **Chapter 4** we apply the spectral sequence of Chapter 5 to the double complex $C(G^p; C((\partial\mathbb{H}^n)^{q+1};\mathbb{R}))$, with $G = \mathrm{Isom}^+(\mathbb{H}^n)$. We obtain

Theorem 4.1.1. *Let $n \geq 2$. The continuous cohomology of* $\mathrm{Isom}^+(\mathbb{H}^n)$ *is measurably realized on the boundary $\partial\mathbb{H}^n$ of real hyperbolic space (with the natural action). Furthermore, all cocycles have essentially bounded representatives in this boundary resolution.*

In **Chapter 5** we show how Theorem 4.1.1 implies the following injectivity result.

Corollary 5.1.1. *The comparison map from continuous bounded cohomology to continuous cohomology for real hyperbolic space $\mathbb{H}^n$ is injective in degree 3, i.e.*

$$c : H^3_{c,b}(\mathrm{Isom}^+(\mathbb{H}^n); \mathbb{R}) \hookrightarrow H^3_c(\mathrm{Isom}^+(\mathbb{H}^n); \mathbb{R}).$$

We also discuss how this result already follows from a more elementary argument. By similar reasoning we then obtain the following two stability results:

Lemma 5.2.1. *Let $k \leq n$ and suppose that $H^k_{c,b}(\mathrm{Isom}(\mathbb{H}^n_{(\mathbb{C})}); \mathbb{R}) = 0$. Then*

$$H^k_{c,b}(\mathrm{Isom}(\mathbb{H}^{n+1}_{(\mathbb{C})}); \mathbb{R}) = 0.$$

and

Theorem 5.2.2. *If $k + 1 \leq n$ then there exists an injection*

$$H^k_{c,b}(\mathrm{Isom}(\mathbb{H}^n_{(\mathbb{C})}); \mathbb{R}) \hookrightarrow H^k_{c,b}(\mathrm{Isom}(\mathbb{H}^{n+1}_{(\mathbb{C})}); \mathbb{R}).$$

Part II

Chapter 6 is a short introduction into complex hyperbolic geometry. We describe the different models of $\mathbb{H}^n_{\mathbb{C}}$, its boundary, and its isometry group. We end with a discussion of the Cartan angular invariant and its relation to the Kähler class.

In **Chapter 7** we turn to complex hyperbolic manifolds. We obtain a lower bound for the Gromov norm of the cup product of the Kähler class $[c_\Phi] \in H^2_{c,b}(\mathrm{PU}(2,1); \mathbb{R})$ with itself so that together with the trivial upper bound we obtain

Theorem 7.1.2.

$$\frac{2}{9}\pi^2 \leq \|[c_\Phi \cup c_\Phi]\|_\infty \leq \pi^2$$

As $c_\Phi \cup c_\Phi$ is proportional to the image under the van Est isomorphism of the volume form in $\Omega^4(\mathbb{H}^2_{\mathbb{C}}; \mathbb{R})^{\mathrm{PU}(2,1)}$ applying the proportionality principle gives estimates for the simplicial volume $\|M\|$ of a complex hyperbolic surface M:

Corollary 7.1.3. *Let M be a closed oriented manifold which is locally isometric to $\mathbb{H}^2_{\mathbb{C}}$. Then*

$$\frac{2}{\pi^2}\mathrm{Vol}(M) \leq \|M\| \leq \frac{9}{\pi^2}\mathrm{Vol}(M).$$

We furthermore obtain the following Milnor-Wood inequality relating the Euler number $\chi(\xi)$ of a $\mathrm{GL}^+(4,\mathbb{R})$-bundle ξ to the Euler characteristic of M:

Corollary 7.1.6. *Let ξ be a $\mathrm{GL}^+(4,\mathbb{R})$-bundle over a closed complex hyperbolic surface M. Then*

$$|\chi(\xi)| \leq \frac{3}{2}\chi(M).$$

Part I

Chapter 1

Continuous (bounded) cohomology

1.1 Definitions

Let G be a locally compact second countable (l.c.s.c.) group where locally compact is Hausdorff by definition. We will consider abstract groups as topological groups with respect to the discrete topology. Let A be the dual of a separable Banach space on which G acts continuously and by linear isometries. We call such a module A a *coefficient G-module*. For more information about these modules and why they are the appropriate coefficients for continuous bounded cohomology see [Mon01]. In this thesis we will only consider continuous bounded cohomology with trivial $\mathbb{R}$-coefficients. Let

$$C^p_{c,b}(G; A) := \{f : G^{p+1} \to A \mid f \text{ continuous and bounded}\},$$

with the G-action given by

$$(g \cdot f)(g_0, \ldots, g_p) := g \cdot (f(g^{-1}g_0, \ldots, g^{-1}g_p)).$$

We denote by $C^p_{c,b}(G; A)^G$ the space of G-invariant functions. Let

$$\delta : C_{c,b}(G^{p+1}; A)^G \to C_{c,b}(G^{p+2}; A)^G$$

be the standard homogeneous coboundary operator, i.e. for a cochain $\alpha \in C_{c,b}(G^{p+1}; A)^G$ and $g_0, \ldots, g_{p+1} \in G$

$$\delta\alpha(g_0, \ldots, g_{p+1}) := \sum_{i=0}^{p+1} (-1)^i \alpha(g_0, \ldots, \hat{g}_i, \ldots, g_{p+1}).$$

The complex $(C^p_{c,b}(G; A), \delta)$ is called the *bounded homogeneous resolution.* We define the *continuous bounded cohomology groups* as the cohomology of this complex, i.e.

$$H^p_{c,b}(G; A) := \frac{\ker(\delta : C^p_{c,b}(G; A)^G \to C^{p+1}_{c,b}(G; A)^G)}{\operatorname{im}(\delta : C^{p-1}_{c,b}(G; A)^G \to C^p_{c,b}(G; A)^G)}.$$

Forgetting about the boundedness condition, i.e. considering the G-modules

$$C^p_c(G; A) := \{f : G^{p+1} \to A \mid f \text{ continuous}\},$$

we obtain the *homogeneous resolution* $(C_c^p(G;A),\delta)$ and the *continuous cohomology groups*

$$H_c^p(G;A) := \frac{\ker(\delta : C_c^p(G;A)^G \to C_c^{p+1}(G;A)^G)}{\operatorname{im}(\delta : C_c^{p-1}(G;A)^G \to C_c^p(G;A)^G)},$$

where for continuous cohomology we allow as coefficients A all Fréchet spaces with a continuous G-action.

The supremum norm on $C_{c,b}^p(G;A)^G$ induces a seminorm $\|\cdot\|_\infty$ on $H_{c,b}^p(G;A)$ defined by taking the infimum over all supremum norms, i.e.

$$\|[\beta]\|_\infty := \inf\nolimits_{f\in[\beta]}\sup\nolimits_{\bar{g}\in G^{p+1}} f(\bar{g}).$$

This norm is called the *Gromov norm*.

1.2 Resolutions

We have defined the continuous (bounded) cohomology groups as the cohomology of the (bounded) homogeneous resolution. The functorial approach, which is classical for continuous cohomology and which was developed by Burger and Monod for continuous bounded cohomology, shows that these cohomology theories are also realized by other complexes. Here we will list the different resolutions we use in this thesis. For the proofs we refer to [Gui80] and [BorWal00] for continuous cohomology and [Mon01] for continuous bounded cohomology.

For both theories, instead of the homogeneous resolution defined before, one can consider the inhomogeneous resolution

$$0 \to A \to C_{c(,b)}(G;A) \to C_{c(,b)}(G^2;A) \to C_{c(,b)}(G^3;A) \to \dots,$$

with inhomogeneous coboundary operator

$$d : C_{c(,b)}(G^p;A) \to C_{c(,b)}(G^{p+1};A)$$

given by

$$\begin{aligned} d\alpha(g_1,\dots,g_{p+1}) =& g_1\cdot\alpha(g_2,\dots,g_{p+1}) + \sum_{i=1}^{p}(-1)^i\alpha(g_1,\dots,g_ig_{i+1},\dots,g_p) \\ &+ (-1)^{p+1}\alpha(g_1,\dots,g_p). \end{aligned}$$

The isomorphism between this resolution and the homogeneous resolution is induced by the map $C_{c(,b)}(G^p;A) \to C_{c(,b)}(G^{p+1};A)^G$ given by

$$\alpha \mapsto \{(g_0,\dots,g_p) \mapsto g_0\cdot\alpha(g_0^{-1}g_1, g_1^{-1}g_2,\dots,g_{p-1}^{-1}g_p)\}.$$

A particularly useful resolution for continuous bounded cohomology is obtained in [Mon01, Theorem 7.5.3]:

Theorem 1.2.1. *Let S be an amenable regular G-space and A a coefficient G-module. Then the cohomology of the complex*

$$0 \to L^\infty_{\mathrm{w}^*}(S;A)^G \to L^\infty_{\mathrm{w}^*}(S^2;A)^G \to L^\infty_{\mathrm{w}^*}(S^3;A)^G \to \dots$$

(with the standard homogeneous coboundary operator and $L^\infty_{\mathrm{w}^*}(S;A)$ the space of weak- measurable essentially bounded maps) is canonically isometrically isomorphic to $H^*_{c,b}(G;A)$*

In particular, since the minimal parabolic subgroup P of a semisimple Lie group G is amenable, this implies that $H^*_{c,b}(G;\mathbb{R})$ is isometrically isomorphic to the cohomology of the complex

$$0 \to L^\infty(G/P;\mathbb{R})^G \to L^\infty((G/P)^2;\mathbb{R})^G \to L^\infty((G/P)^3;\mathbb{R})^G \to \dots$$

Therefore the bounded continuous cohomology of a semisimple Lie group G is measurably realized on its Furstenberg boundary.

For all these resolutions we can furthermore consider the subcomplexes

$$\mathrm{Alt}^p_{c(,b)}(G;A)^G \subset C^p_{c(,b)}(G;A)^G$$

and

$$\mathrm{Alt}^\infty_{\mathrm{w}^*}(G/P;A)^G \subset L^\infty_{\mathrm{w}^*}(G/P;A)^G$$

consisting of alternating maps. The homogeneous coboundary operator restricts to these complexes and they (isometrically) realize the same cohomology groups.

Let G be a semisimple Lie group with finite center and no compact factors and let K be its maximal compact subgroup. We denote by $\mathcal{X} = G/K$ the associated symmetric space. An important resolution for continuous cohomology is given by the complex of G-invariant differential forms $(\Omega^*(\mathcal{X};\mathbb{R})^G, d)$. An isomorphism with the standard homogeneous resolution is provided by the explicit description on the cochain level of the van Est isomorphism by Dupont: Let $\Delta(g_0x,\dots,g_px)$ be the "geodesic coned simplex" with vertices $g_0x,\dots,g_px$ defined inductively as follows: The simplex $\Delta(g_0x, g_1x)$ is the geodesic segment from g_0x to g_1x and given the simplex $\Delta(g_0x,\dots,g_ix)$ the simplex $\Delta(g_0,\dots,g_ix,g_{i+1}x)$ is the union of all geodesic segments from $g_{i+1}x$ to the points of $\Delta(g_0x,\dots,g_ix)$. Then

Theorem 1.2.2 (van Est isomorphism)**.** *The continuous cohomology of G with real coefficients is isomorphic to $\Omega^*(\mathcal{X};\mathbb{R})^G$. An explicit description of this isomorphism on the cocycle level sends the differential form $\omega \in \Omega^p(\mathcal{X};\mathbb{R})^G$ to the cocycle $c_\omega \in C_c(G^{p+1};\mathbb{R})$ defined by*

$$c_\omega(g_0,\dots,g_p) = \int_{\Delta(g_0x,\dots,g_px)} \omega,$$

for any fixed basepoint x in $\mathcal{X}$.

Remark 1.2.3. *That there are no coboundaries in the above resolution follows from the fact that every G- invariant differential form on G/K is closed. Indeed, let $\omega \in \Omega^n(G/K)^G$. Fix a basepoint $0 = eK$ in G/K and let s_0 be the geodesic symmetry at this point. Then $s_0^*\omega = (-1)^n\omega$ and $s_0^* d\omega = (-1)^n d\omega$. It follows that*

$$(-1)^n d\omega = d((-1)^n\omega) = d(s_0^*\omega) = s_0^*(d\omega) = (-1)^{n+1} d\omega,$$

and thus $d\omega = 0$.

1.3 Comparison map

The inclusion

$$C^*_{c,b}(G;\mathbb{R}) \hookrightarrow C^*_c(G;\mathbb{R})$$

induces a natural comparison map

$$c : H^*_{c,b}(G;\mathbb{R}) \to H^*_c(G;\mathbb{R}),$$

which is in general neither surjective nor injective. It is conjectured ([Dup79], [Mon06]) to be an isomorphism for all semisimple connected Lie groups with finite center. There is a lot of evidence for the surjectivity part of this conjecture (see e.g. [LafSch06], [HarOtt12]). On the other hand, injectivity has so far only been established in a few cases. For degree 2 it was proven by Burger and Monod in [BurMon99]. In degree 3 and 4 it has been proven for $\mathrm{SL}_2(\mathbb{R})$ by Burger-Monod [BurMon02] and Hartnick-Ott [HarOtt15] respectively. For $\mathrm{Isom}^+(\mathbb{H}^3)$ injectivity in degree 3 follows from a result of Bloch [Blo00], our Theorem 4.1.1 in the case $n = 3$. In hyperbolic 3-space the volume function of an ideal simplex is given by a multiple of the Bloch-Wigner dilogarithm of the cross ratio of its 4 vertices in $\partial\mathbb{H}^3$. In this context, the five-term relation for the dilogarithm is the cocycle condition for the volume function. Let $\mathrm{Isom}^+(\mathbb{H}^3)$ be the group of orientation preserving isometries of $\mathbb{H}^3$. Since $H^3_c(\mathrm{Isom}^+(\mathbb{H}^3);\mathbb{R})$ is one dimensional, Theorem 4.1.1 implies that up to scalar multiplication the volume function is the only $\mathrm{Isom}^+(\mathbb{H}^3)$-invariant measurable cocycle on the boundary. As there are no coboundaries in degree 3, injectivity of the comparison map is an immediate consequence. Hence Theorem 4.1.1 implies injectivity in degree 3 for $\mathrm{Isom}^+(\mathbb{H}^n)$ for all $n \geq 2$. We will discuss this further in Chapter 5. In Bloch's case, i.e. for $n = 3$, it also implies that the space of a.e. equivalence classes of measurable functions $(\partial\mathbb{H}^3)^4 \to \mathbb{R}$ that are $\mathrm{Isom}^+(\mathbb{H}^3)$-invariant and that satisfy the cocycle condition is one-dimensional and generated by the volume function.

1.4 Proportionality principles

Let M be a closed oriented manifold of dimension n. The ℓ^1-norm $\|\cdot\|_1$ with respect to the basis of singular simplices on the space of real-valued chains $C_*(M;\mathbb{R})$ is given by

$$\left\| \sum_{j=1}^{k} a_j \sigma_j \right\|_1 = \sum_{j=0}^{k} |a_j|.$$

The *ℓ^1-seminorm* of a homology class in $\alpha \in H_*(M;\mathbb{R})$ is then defined as the infimum of the ℓ^1-norm of its representatives, i.e.

$$\|\alpha\|_1 := \inf \left\{ \sum_{j=0}^{k} |a_j| \,\middle|\, \alpha = \left[\sum_{j=1}^{k} a_j \sigma_j \right] \right\}.$$

The *simplicial volume* $\|M\|$ of M is the ℓ^1-seminorm of the the real valued fundamental class $[M] \in H_n(M;\mathbb{R})$:

$$\|M\| := \|[M]\|_1.$$

Denote by $\langle \beta, \alpha \rangle$ the canonical pairing of a cohomology class $\beta \in H^p(M;\mathbb{R})$ with a homology class $\alpha \in H_p(M;\mathbb{R})$. Recall that the *Gromov norm* $\|\beta\|_\infty$ is the semi-norm given by the infimum of the sup-norms of all cocycles representing β:

$$\|\beta\|_\infty = \inf\{\|b\|_\infty \mid [b] = \beta\} \in \mathbb{R}_{\geq 0} \cup \{+\infty\}.$$

Then we have:

Proposition 1.4.1. *[BenPet92, Proposition F.2.2] For any $\alpha \in H_p(M;\mathbb{R})$ and $\beta \in H^p(M;\mathbb{R})$*

$$|\langle \beta, \alpha \rangle| \leq \|\beta\|_\infty \cdot \|\alpha\|_1.$$

The simplicial volume and the volume of M are related by the Gromov-Thurston proportionality principal (see [Gro82, Thu78]) which is given by

$$\|M\| = \frac{\mathrm{Vol}(M)}{c(\widetilde{M})},$$

where $c(\widetilde{M})$ is a positive constant (possibly infinite) which only depends on the universal cover $\widetilde{M}$ of M. In fact, for locally symmetric spaces of non-compact type, Bucher obtains in [Buc08a] that the proportionality constant

$$c(\widetilde{M}) = \|\omega_{\widetilde{M}}\|_\infty,$$

with $\omega_{\widetilde{M}} \in H_c^n(\mathrm{Isom}_0(\widetilde{M});\mathbb{R})$ the image of the volume form under the van Est isomorphism. So we have

Proposition 1.4.2. *Let M be a locally symmetric space of noncompact type. Then*
$$\|M\| = \frac{\mathrm{Vol}(M)}{\|\omega_{\widetilde{M}}\|_\infty},$$
with $\omega_{\widetilde{M}} \in H^n_c(Isom_0(\widetilde{M});\mathbb{R})$ the image of the volume form under the van Est isomorphism.

Recall that $\chi(M)$ denotes the Euler-Poincaré characteristic of M and is defined as the alternating sum of the Betti numbers of M:
$$\chi(M) := \sum_{i=0}^{n} (-1)^i \dim(H^i(M;\mathbb{R})).$$

The volume of M is also proportional to $\chi(M)$. Indeed, by Hirzebruch's proportionality principle,

Proposition 1.4.3. *[Hir58] Let M be a closed, oriented, locally symmetric space of noncompact type of dimension n and let $\mathcal{X}_u$ be the compact dual of its universal cover $\mathcal{X} = \widetilde{M}$. Suppose that $\chi(M)$ and $\chi(\mathcal{X}_u)$ are nonzero. Then*
$$\frac{\mathrm{Vol}(M)}{\chi(M)} = (-1)^{\frac{n}{2}} \cdot \frac{\mathrm{Vol}(\mathcal{X}_u)}{\chi(\mathcal{X}_u)}.$$

As an immediate consequence, we obtain

Corollary 1.4.4. *Let M be a closed, oriented, locally symmetric space of noncompact type of dimension n and let $\mathcal{X}_u$ be the compact dual of its universal cover $\mathcal{X} = \widetilde{M}$. Suppose that $\chi(\mathcal{X}_u)$ is nonzero. Then*
$$\|M\| = (-1)^{\frac{n}{2}} \cdot \frac{\mathrm{Vol}(\mathcal{X}_u)}{\chi(\mathcal{X}_u)} \cdot \frac{\chi(M)}{\|\omega_{\mathcal{X}}\|_\infty}.$$

The *Euler class* associates to oriented vector bundles ξ over M a class $\varepsilon_n(\xi) \in H^n(M;\mathbb{R})$. The *Euler number* $\chi(\xi)$ of such a vector bundle is by definition the pairing of the Euler class with the fundamental class $[M] \in H_n(M;\mathbb{R})$ of M:
$$\chi(\xi) := \langle \varepsilon_n(\xi), [M] \rangle.$$

The Euler number is a generalization of the Euler-Poincaré characteristic.

Lemma 1.4.5. *[MilSta74, Corollary 11.12] Let M be a closed oriented smooth manifold. The Euler number of the tangent bundle of M is equal to its Euler characteristic:*
$$\chi(M) = \chi(TM).$$

Definition 1.4.6. Let M be a closed oriented smooth manifold and let ξ be a $\mathrm{GL}^+(n,\mathbb{R})$-bundle over M. A connection on ξ is said to be flat if its curvature form vanishes identically. The bundle ξ is called *flat* if it can be endowed with such a flat connection. Equivalently, this means that the bundle ξ is induced by a representation $\rho : \pi_1 M \to \mathrm{GL}^+(n,\mathbb{R})$. The manifold M *admits a flat structure* if the $\mathrm{GL}^+(n,\mathbb{R})$-bundle associated to its tangent space is flat.

Lemma 1.4.7. *[IvaTur82] Let M be a closed oriented smooth manifold and let ξ be a flat $GL^+(n,\mathbb{R})$-bundle over M. Then*

$$\|\varepsilon_n(\xi)\|_\infty \leq \frac{1}{2^n}.$$

In fact Bucher and Monod showed in [BucMon12] that $\|\varepsilon_n(\xi)\|_\infty = \frac{1}{2^n}$. Combining Lemma 1.4.7 with Proposition 1.4.1 we obtain the following inequality for the Euler number $\chi(\xi)$:

Lemma 1.4.8. *Let M be a closed oriented smooth manifold and let ξ be a flat $GL^+(n,\mathbb{R})$-bundle over M. Then*

$$|\chi(\xi)| \leq \frac{1}{2^n} \cdot \|M\|.$$

Chapter 2

Measurable cohomology

2.1 The definition

Let G again be a l.c.s.c. group and let A be a Polish Abelian G-module (to be defined below). Furthermore, let $C^p(G;A)$ denote the G-module of measurable maps $G^{p+1} \to A$ identifying those which agree almost everywhere (a.e.) endowed with the same action as the modules $C^p_{c(,b)}(G;A)$, i.e.,

$$(g \cdot f)(g_0, \ldots, g_p) := g \cdot (f(g^{-1}g_0, \ldots, g^{-1}g_p)).$$

Note that since $C^p(G;A)$ consists of equivalence classes of functions all equalities like the above only hold a.e. We will discuss this G-module in more detail in the next section. Let $\delta : C^p(G;A)^G \to C^{p+1}(G;A)^G$ be the standard homogeneous coboundary operator. The *measurable cohomology groups for G with coefficients in A* are given by

$$H^p_m(G;A) := \frac{\ker(\delta : C^p(G;A)^G \to C^{p+1}(G;A)^G)}{\operatorname{im}(\delta : C^{p-1}(G;A)^G \to C^p(G;A)^G)}.$$

In the case of measurable cohomology no functorial approach seems to exist but as before the standard inhomogeneous resolution $(C(G^p;A), d)$ realizes $H^p_m(G;A)$ and by Buchsbaum's criterion (see Section 2.3) also some other cocomplexes can be shown to realize the same cohomology group.

2.2 The G-module $C(X;A)$

Let X be a l.c.s.c. space on which G acts measurably and that is endowed with a G-invariant measure class $[\mu]$.

Remark 2.2.1. *When $X = G$ we will endow it with the class of its Haar measures $[\mu_G]$. If $X = G/H$, with $H < G$ a closed subgroup then we will endow the homogeneous space G/H with the class of the natural quasi-invariant measures.*

Definition 2.2.2. A *Polish space* is a separable topological space Y for which there exists a compatible metric ρ such that (Y, ρ) is a complete metric space. A *Polish group* is a topological group which is also a Polish space in its topology.

Example 2.2.3. *The l.c.s.c. space X is an example of a Polish space.*

Definition 2.2.4. A *Polish Abelian G-module* (A, ρ) is a Polish group A with a jointly continuous action of G and a translation invariant metric ρ. It is a *F-space* if A is furthermore a separable real topological vector space in its Polish topology and a *Fréchet space* if it is a locally convex F-space.

Let (A, ρ) be a Polish Abelian G-module. We define $C(X; A)$ to be the G-module of μ-a.e. equivalence classes of Borel maps $X \to A$, i.e. two Borel maps $f, g : X \to A$ are in the same class if they only differ on a set of μ-measure zero, where μ is any measure in the G-invariant measure class associated to X.

Remark 2.2.5. *Note that one could also define $C(X; A)$ as consisting of μ-measurable functions up to equivalence, this will give the same space since any class of measurable functions contains a Borel function. Indeed, by Lusin's theorem if $f : X \to A$ is a measurable function there exists a sequence F_n of closed subsets of X such that $\mu(G \setminus F_n) < \frac{1}{n}$ and f restricted to F_n is continuous. Then $F = \cup F_n$ has a complement of measure zero, f is continuous on F and can be extended to a Borel function on its complement.*

Let ν be a probability measure in the measure class $[\mu]$, i.e. $\nu(X) = 1$, and let $\tilde{\rho}$ be a metric equivalent to ρ on A in which A has finite diameter. We endow $C(X; A)$ with the topology of convergence in probability with respect to ν :

Definition 2.2.6. Let $\{f_n\}$ be a sequence of measurable functions $f_n : X \to A$. Then $f_n \to f$ in probability if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that

$$\nu\left(\{x \in X \mid \tilde{\rho}(f_n(x), f(x)) \geq \epsilon\}\right) < \epsilon,$$

for all $n \geq N_\epsilon$. A corresponding metric is given by

$$d(f_1, f_2) := \inf\{r > 0 \mid \nu(\{x \in X \mid \tilde{\rho}(f_1(x), f_2(x)) > r\}) \leq r\}.$$

Remark 2.2.7. *When $A = \mathbb{R}$ this gives the usual F-space structure on $C(G; \mathbb{R})$ and this space is often denoted as $L^0(G)$.*

Remark 2.2.8. *If two functions $f, g : X \to A$ agree almost everywhere then their distance is zero in the metric of convergence in probability. Hence the space of all measurable functions from X to A endowed with the topology of convergence in probability is not Hausdorff. It is for this reason that one is led to consider maps up to a.e. equivalence.*

Proposition 2.2.9. *The G-module $C(X; A)$ is complete: If the sequence $\{f_n\}$ is a Cauchy sequence in probability of measurable functions $f_n : X \to A$ then there exists a measurable function $f : X \to A$ such that $f_n \to f$ in probability. Furthermore, there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ pointwise a.e.*

Proof. (Following the proof of Theorem 4.17 in [Men07]) Let $V_{\epsilon,n,m} = \{x \in X \mid \rho(f_n(x), f_m(x)) \geq \epsilon\}$. There exists an increasing sequence $\{n_i\}$ in $\mathbb{N}$ such that $\nu(V_{2^{-i},n_i,m}) < 2^{-i}$ for all $m \geq n_i$, in particular $\nu(V_{2^{-i},n_i,n_{i+1}}) < 2^{-i}$. Define $A_i = V_{2^{-i},n_i,n_{i+1}}$ and $B_k = \cup_{i=k}^{\infty} A_i$. Then $\nu(B_k) \leq \sum_{i=k}^{\infty} 2^{-i} = 2^{1-k}$ and if $x \notin B_k$ then for all $i \geq j \geq k$:

$$\rho(f_{n_j}(x), f_{n_i}(x)) \leq \sum_{r=j}^{i-1} \rho(f_{n_r}(x), f_{n_{r+1}}(x)) \leq \sum_{r=j}^{i-1} 2^{-r} \leq 2^{1-k}.$$

Hence $\{f_{n_i}(x)\}$ is a Cauchy sequence in A for all $x \notin B_k$. Let $B = \cap_k B_k$, so that $\nu(B) = 0$. By the above, if $x \notin B$ then the limit of $\{f_{n_i}(x)\}$ exists in A. We call this limit $f(x)$ and define the function $f : X \to A$ to be equal to $f(x)$ if $x \notin B$, for $x \in B$ we set $f(x) = 0$ so that $f_{n_i} \to f$ pointwise almost everywhere. Furthermore, $f_{n_i} \to f$ in probability since $\rho(f_{n_i}(x), f(x)) \leq 2^{1-i}$ for all $x \notin F_i$ and $\nu(F_i) \to 0$ as $i \to \infty$. Using the triangle inequality it immediately follows that also $f_n \to f$ in probability. □

Remark 2.2.10. *The converse of the second part of Proposition 2.2.9 is also true. That is, if there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \to f$ pointwise a.e. then $f_n \to f$ in probability. This is not true for the convergence in measure with respect to a non-finite measure.*

Proposition 2.2.11. *The G-module $C(X; A)$ is separable.*

Proof. (based on Remark 4.26 in [Men07]). Let $S(X; A) \subset C(X; A)$ be the subspace of simple functions, i.e. measurable functions assuming only a finite set of values. From Remark 2.2.10 it follows that this is a dense subspace of $C(X; A)$. Let $\mathcal{O}$ be a countable basis of X and let $\mathcal{O}_f$ be the countable family of finite unions of open sets $O \in \mathcal{O}$. Let B be a ν-measurable set and let $\epsilon > 0$. Since $\nu(B) < \infty$ there exist an open set $O \supset B$ and a compact set $K \subset B$ such that $\nu(O \setminus K) < \epsilon$. Thus there is a $O_f \in \mathcal{O}_f$ such that $\nu((B \setminus O_f) \cup (O_f \setminus B)) < \epsilon$. Then, since A is separable, the set $\mathcal{F} = \{f \in S(X; A) \mid f^{-1}(a) \in \mathcal{O}_f \text{ for all } a \in A\}$ is a countable dense subset of $S(X; A)$ and therefore of $C(X; A)$. □

Proposition 2.2.9 and Proposition 2.2.11 imply that $C(X; A)$ is a Polish group. Furthermore, with G-action defined by

$$(g \cdot f)(x) := g \cdot f(g^{-1}x),$$

it is a Polish Abelian G-module [Moo76a, Proposition 12]. An important property of these modules is that they satisfy the following Fubini theorem:

Theorem 2.2.12. *Let X and Y both be l.c.s.c. spaces on which G acts measurably and that are endowed with G-invariant measure classes. Let A be a Polish Abelian G-module. Then $C(X \times Y; A) \cong C(X; C(Y; A)) \cong C(Y; C(X; A))$.*

For the proof we refer to Theorem 1 in [Moo76a].

Example 2.2.13. *An important case is $X = G^{p+1}$. We write $C^p(G; A) := C(G^{p+1}; A)$. These are the G-modules in the homogeneous resolution of the measurable cohomology group $H^*_m(G; A)$ with action given by*

$$(g \cdot f)(g_0, \dots, g_p) := g \cdot f(g^{-1}g_0, \dots, g^{-1}g_p).$$

Remark 2.2.14. *The space $C(X; A)$ is not locally convex since the only convex neighborhood of 0 is the space itself. Indeed, if $C(X; A)$ contains a nonempty proper convex open subset then it follows from the geometric Hahn-Banach theorem (see e.g. [SchWol99, Section II.3.1]) that there exists a nonzero continuous linear functional on $C(X; A)$. Suppose that L is such a functional. Then there exists a $C > 0$ such that for all $f \in B(0, \delta)$ one has $|L(f)| < C$. Let $Y \subset C(X; A)$ be a measurable subset with $\mu(Y) < \delta$ and denote by χ_Y the characteristic function of this subset. Then $\lambda \cdot \chi_Y \in B(0, \delta)$ for all $\lambda > 0$ and thus*

$$|L \circ (\lambda \cdot \chi_Y)| = \lambda \cdot |L \circ \chi_Y| < M.$$

It follows that $L \circ \chi_Y = 0$ and therefore $L = 0$.

2.3 Buchsbaum's criterion and dimension shifting

Buchsbaum's criterion

Denote by $P(G)$ the category of Polish Abelian G-modules. A short exact sequence

$$0 \longrightarrow A \xrightarrow{\ i\ } B \xrightarrow{\ j\ } C \longrightarrow 0$$

in $P(G)$ is exact algebraically and such that the maps i and j are continuous homomorphisms intertwining the action of G. An *effaceable cohomological functor $H^*(G, \cdot)$ on $P(G)$* is a covariant functor from $P(G)$ to the category of Abelian groups such that

1. $H^0(G, A) = A^G$
2. Every short exact sequence $0 \to A \to B \to C \to 0$ of Polish Abelian G-modules induces a long exact sequence in cohomology
$$0 \to H^0(G; A) \to H^0(G; B) \to H^0(G; C) \to H^1(G; A) \to \dots$$
$$\dots \to H^k(G; B) \to H^k(G; C) \to H^{k+1}(G; A) \to \dots, \qquad (2.1)$$
3. $H^*(G; A)$ is *effaceable* in the category of Polish Abelian G-modules. That is, for any Polish Abelian G-module A and any $a \in H^k(G; A)$ there exists a short exact sequence
$$0 \to A \to B \to C \to 0$$
such that the image of a in $H^k(G; B)$ vanishes.

By *Buchsbaum's criterion* [Buc60] such an effaceable cohomological functor is unique (if it exists). C.C. Moore proved that measurable cohomology satisfies the above requirements and is therefore the unique effaceable cohomological functor on $P(G)$ [Moo76a, Section 4]. Let us briefly discuss why the three above conditions hold for the measurable cohomology H^*_m. The fact that $H^0_m(G; A) = A^G$ is immediate from the definition. For the third condition, Moore proves that $H^p(G; C(G; A)) = 0$ for $p > 0$ [Moo76a, Theorem 4]. Then any cohomology class $[\alpha] \in H^k_m(G; A)$ is effaced by the inclusion $\iota : A \hookrightarrow C(G; A)$ (sending $a \in A$ to the constant function $\equiv a$). Indeed, $\iota^*(\alpha) = \delta\beta$ with $\beta : G^p \to C(G; A)$ defined by

$$\beta(g_1, \ldots, g_p)(g) := (-1)^p \alpha(g_1, \ldots, g_p, g).$$

Lastly, the second condition, i.e. the existence of long exact sequences, is ensured by the existence of Borel sections: Let $H < G$ be a closed subgroup and denote by $p : G \to G/H$ the natural projection map.

Definition 2.3.1. A Borel map $f : X \to Y$ from a locally compact space to a Polish space is *locally totally bounded* if for any compact subset $K \subset X$ the image $f(K)$ is precompact in Y.

We prove below:

Lemma 2.3.2. *There exists a section $s : G/H \to G$ of the natural projection map $p : G \to G/H$ that is Borel and locally totally bounded, i.e. that sends every compact subset of G/H to a precompact subset of G.*

Now let

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

be a short exact sequence of Polish Abelian G-modules. The induced sequence

$$0 \longrightarrow C^q(G; A) \xrightarrow{i^*} C^q(G; B) \xrightarrow{j^*} C^q(G; C) \longrightarrow 0$$

is then also exact: The map $i^* : C^q(G; A) \to C^q(G; B)$ is clearly injective and the induced maps i^* and j^* are continuous. Furthermore, by Lemma 2.3.2 there is a Borel map $s : C \to B$ with $j \circ s = \mathrm{id}_C$ and thus if $\alpha \in C^q(G; C)$ then $s^* \circ \alpha \in C^q(G; B)$ is mapped to f. Hence j^* is surjective and the sequence is exact. Then a long exact sequence as in the second condition can be constructed in the standard way.

Since in general there exists no continuous cross section $G/H \to G$ continuous cohomology has no long exact sequences when we allow all Polish Abelian G-modules as coefficients. However, when restricting to Fréchet modules there do exist continuous cross sections and continuous cohomology is the unique effaceable cohomological functor on this category. In

[AusMoo13] T. Austin and C.C. Moore prove that measurable cohomology is also effaceable when restricted to Fréchet modules. We will discuss this proof in Section 2.5.

Proof of Lemma 2.3.2. We follow the proof in Section 5.1.1 in [War72]. Since G is a l.c.s.c. group it is a k_ω-space (see [FraTho77, 10)]) and therefore it is a direct limit of an ascending sequence $\{K_n\}_{n\in\mathbb{N}}$ of compact subsets. Note that this implies in particular that any compact subset of G is a subset of some K_n [FraTho77, 3)].
By the Borel selection theorem, see e.g. [Fre05, Section 423], for all $n \in \mathbb{N}$ there exists a Borel subset $B_n \subset K_n$ such that $p(B_n) = p(K_n)$ and p restricted to B_n is an injective map. The B_n can furthermore be chosen such that $B_n \subset B_{n+1}$. Indeed, by induction, if $B_1 \subset B_2 \subset \cdots \subset B_n$ then we define

$$B_{n+1} = (\tilde{B}_{n+1} \setminus \pi^{-1}(\pi(B_n))) \cup B_n,$$

where $\tilde{B}_{n+1}$ is a Borel subset of K_{n+1} on which p is one-to-one and such that $p(B_{n+1}) = p(K_{n+1})$. It follows that

$$\mathcal{B} := \bigcup_{n\in\mathbb{N}} B_n$$

is a Borel subset of G that intersects each left coset gH in exactly one point. Furthermore, if $K \subset G$ is compact then $\overline{p^{-1}(p(K)) \cap \mathcal{B}}$ is compact. Indeed, let $K \subset G$ be a compact subset. Then $K \subset K_n$ for some n. Thus, as $p(K_n) = p(B_n)$ it follows that $p^{-1}(p(K)) \cap \mathcal{B} \subset K_n$. This implies that $\mathcal{B}$ gives us a locally totally bounded Borel section s:
If $L \subset G/H$ is compact then, for example by Proposition 18 in [Bou74, IX §2 N° 10], there exists a compact $K \subset G$ such that $L \subset p(K)$. Hence $s(L) := p^{-1}(L) \cap \mathcal{B}$ is contained in the precompact subset $p^{-1}(p(K)) \cap \mathcal{B}$ and is therefore itself precompact. $\square$

Dimension shifting

Effacement together with the existence of long exact sequences allow for the technique of dimension shifting, that is we can rewrite a cohomology group as a cohomology group of lower degree (but with different coefficients). Then by induction on degree, some algebraic properties that clearly hold in lower degree may be shown to hold in higher degrees as well. We will use this technique in the proof of Theorem 4.1.1. Concretely, let $\iota : A \hookrightarrow C(G; A)$ be the embedding of the G-module A into $C(G; A)$ as the closed submodule of constant maps. Since $H^p(G; C(G; A)) = 0$ for $p > 0$ the long exact sequence obtained from short exact sequence

$$0 \to A \hookrightarrow C(G; A) \twoheadrightarrow C(G; A)/\iota(A) \to 0$$

gives isomorphisms $H^p_m(G;A) \cong H^{p-1}_m(G; C(G;A)/\iota(A))$ for all $p > 0$. The connecting map $H^p_m(G;A) \to H^{p-1}_m(G; C(G;A)/\iota(A))$ is induced by the map $Q : C^p(G;A) \to C^{p-1}(G; C(G;A))$ given by

$$(Q\alpha)(g_0, \dots, g_{p-1})(g) := (-1)^p \alpha(g_0, \dots, g_{p-1}, g), \tag{2.2}$$

for $\alpha \in C^p(G;A)$ and $g, g_0, \dots, g_{p-1} \in G$. If α is a cocycle it follows directly that $\delta(Q\alpha)(g_0, \dots, g_p)$ is the constant map $g \mapsto \alpha(g_0, \dots, g_p)$ and thus the image of $Q\alpha$ under the quotient map $C(G;A) \twoheadrightarrow C(G;A)/\iota(A)$ defines a class in $H^{p-1}_m(H; C(G;A)/\iota(A))$. Furthermore, it can be shown that this image only depends on the cohomology class of α and that Q indeed induces the connecting map. For the inhomogeneous resolution the connecting map Q on cochains is given by

$$(Q\alpha)(g_1, \dots, g_{p-1})(g) := (-1)^p \alpha(g_1, \dots, g_{p-1}, g^{-1}_{p-1} g^{-1}_{p-2} \cdots g^{-1}_1 g), \tag{2.3}$$

for $\alpha \in C(G^p;A)$ and $g, g_1, \dots, g_{p-1} \in G$.

2.4 The Eckmann-Shapiro Lemma

Let $H < G$ be a closed subgroup of the l.c.s.c. group G and let A be a Polish Abelian H-module. Define

$$\mathrm{Ind}^G_H(A) = \{ f \in C(G;A) \mid f(gh) = h^{-1} \cdot f(g) \text{ for almost all } (h,g) \in H \times G \}.$$

This is a Polish Abelian G-module with the action of G given by:

$$(g \cdot f)(g') = f(g^{-1} g').$$

Proposition 2.4.1 (Eckmann-Shapiro Lemma)**.** *Let $H < G$ be a closed subgroup of the locally compact second countable group G and let A be a Polish Abelian H-module. Then $H^k_m(G; \mathrm{Ind}^G_H(A)) \cong H^k_m(H;A)$.*

Lemma 2.4.2. *The map $\iota : A \to C(G;A)$ that embeds A into $C(G;A)$ as the submodule of constant maps, i.e.*

$$\iota(a)(g) = a, \text{ for a.e. } g \in G,$$

induces an isomorphism $(\mathrm{Ind}^G_H(A))^G \cong A^H$

Let s be a locally totally bounded Borel section of p such that $s(H) = e$.

Definition 2.4.3. The locally totally bounded map $r : G \to H$ is defined by

$$r(g) = g \cdot s(g^{-1}H), \tag{2.4}$$

for $g \in G$. Furthermore, the locally totally bounded map $\lambda : G \times G/H \to H$ is defined by

$$\lambda(g, g'H) = s(g'H)^{-1} \cdot g \cdot s(g^{-1} g' H), \tag{2.5}$$

for $g \in G$ and $g'H \in G/H$.

Proposition 2.4.4. *The map $\varrho : C(G/H; A) \to \mathrm{Ind}_H^G(A)$ defined by*

$$\varrho(f)(g) = r(g^{-1}) \cdot f(gH)$$

gives an isomorphism of topological groups $C(G/H; A) \cong \mathrm{Ind}_H^G(A)$. The action of G on $C(G/H; A)$ induced by the action of G on $\mathrm{Ind}_H^G(A)$ is

$$(g \cdot f)(g'H) = \lambda(g, g'H) \cdot f(g^{-1}g'H). \tag{2.6}$$

Proof of Proposition 2.4.1. This follows from Buchsbaum's criterion. As discussed in Section 2.3 measurable cohomology is the unique effaceable cohomological functor on $P(H)$. Thus to prove the proposition we will show that $H_m^*(G; \mathrm{Ind}_H^G(\cdot))$ also satisfies the three conditions of Buchsbaum's criterion stated in the beginning of Section 2.3:

1. By Lemma 2.4.2 we have $H^0(G; \mathrm{Ind}_H^G(A)) = A^H$.
2. If
$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$
is a short exact sequence of Polish Abelian H-modules let
$$0 \longrightarrow \mathrm{Ind}_H^G(A) \xrightarrow{i^*} \mathrm{Ind}_H^G(B) \xrightarrow{j^*} \mathrm{Ind}_H^G(C) \longrightarrow 0$$
be the induced sequence of Polish Abelian G-modules. Then clearly i^* is injective and the maps i^* and j^* are both continuous. Furthermore, if $s : G/H \to G$ and $\tilde{s} : C \to B$ are Borel sections and $f \in \mathrm{Ind}_H^G(C)$ then the map defined by $g \mapsto r(g^{-1}) \cdot \tilde{s}(f(s(gH)))$ is send to f by j^*. Hence the induced sequence is a short exact sequence and this implies the existence of long exact sequences.
3. From Proposition 2.4.4 and Theorem 2.2.12 it follows that
$$\begin{aligned} \mathrm{Ind}_H^G(C(H; A)) &\cong C(G/H; C(H; A)) \\ &\cong C(G/H \times H; A) \\ &\cong C(G; A), \end{aligned}$$
The first isomophism is given in Proposition 2.4.4. It is defined such that it intertwines with the action by G, where the action on $C(G/H; C(H; A))$ is as defined in equation 2.6:
$$(g \cdot f)(g'H)(h) = \lambda(g, g'H) \cdot f(g^{-1}g'H)(\lambda(g, g'H)^{-1}h)$$
The canonical second isomorphism intertwines with the G-action as well. For the last isomorphism, we define for $f \in C(G; A)$ the map $\tilde{f} \in C(G/H \times H; A)$ by
$$\tilde{f}(gH)(h) := s(gH)^{-1} \cdot f(s(gH)h).$$

Then

$$\begin{aligned}(g\cdot\tilde f)(g'H)(h) &= \lambda(g,g'H)\cdot\tilde f(g^{-1}g'H)(\lambda(g,g'H)^{-1}h)\\ &= \lambda(g,g'H)\cdot s(g^{-1}g'H)^{-1}f(s(g^{-1}g'H)\lambda(g,g'H)^{-1}h)\\ &= s(g'H)^{-1}\cdot g\cdot f(g^{-1}s(g'H)h),\end{aligned}$$

and

$$\begin{aligned}\widetilde{g\cdot f}(g'H)(h) &= s(g'H)^{-1}\cdot(g\cdot f)(s(g'H)h)\\ &= s(g'H)^{-1}\cdot g\cdot f(g^{-1}s(g'H)h).\end{aligned}$$

Thus this also defines an isomorphism which intertwines with the G-actions and therefore

$$H^k(G;\mathrm{Ind}_H^G(C(H;A))) = H^k(G;C(G;A)) = 0 \text{ for } k>1,$$

and any cohomology class $[\alpha]\in H^k(G;\mathrm{Ind}_H^G(A))$ is effaced by the inclusion $\iota : A\hookrightarrow C(H;A)$.

□

Proof of Lemma 2.4.2. Let $a\in A^H$ so that $h\cdot a = a$ for all $h\in H$. Then

$$\iota(a)(gh) = a = h^{-1}\cdot a = h^{-1}\cdot\iota(a)(g),$$

for all $g\in G$ and $h\in H$. So $\iota(a)$ is indeed a map in $\mathrm{Ind}_H^G(A)$. Furthermore, for all $g,g'\in G$ we have

$$(g\cdot\iota(a))(g') = \iota(a)(g^{-1}g') = a = \iota(a)(g'),$$

and hence $\iota(a)$ is a G-invariant map.
The map ι is clearly injective and continuous thus all that is left to show is that it is a surjection from A^H onto $(\mathrm{Ind}_H^G(A))^G$. Let $f\in(\mathrm{Ind}_H^G(A))^G$. Since f is G-invariant for any $g\in G$ we have $g\cdot f = f$ a.e., i.e. the set

$$E = \{g'\in G \mid f(g^{-1}g')\neq f(g')\}$$

has measure zero for all $g\in G$. Hence, by Fubini, $E_{g'} = \{g\in G\mid f(g^{-1}g')\neq f(g')\}$ has measure zero for almost every $g'\in G$. Let

$$N = \{g'\in G \mid E_{g'} \text{ is not a set of measure zero}\}.$$

Then N has measure zero and for $g'\notin N$, $f(g^{-1}g') = f(g')$ for almost every $g\in G$. It follows that $f(g') = a$ a.e. for $a\in A$ some constant. Now since $f\in\mathrm{Ind}_H^G(A)$,

$$f(g'h^{-1}) = h\cdot f(g') \text{ for almost all pairs } (g',h)\in G\times H.$$

Hence $h\cdot a = a$ for almost all $h\in H$. Since the action of H on A is continuous it then follows that in fact $h\cdot a = a$ for all $h\in H$. Thus

$$f(g) = a,\ \text{ for a.e. } g\in G \text{ and with } a\in A^H,$$

i.e. $f = \iota(a)\in(\mathrm{Ind}_H^G(A))^G$ with $a\in A^H$. □

Proof of Proposition 2.4.4. Following the proof of Proposition 17 in [Moo76a]:

- ϱ is well-defined:
  Since r is Borel the map $\varrho(f)$ is as well. Furthermore, if one changes f on a set of measure zero then $\varrho(f)$ also only changes on a set of measure zero. Hence if f_1, f_2 are in the same class in $C(G/H; A)$, i.e. they agree a.e., then $\varrho(f_1)$ and $\varrho(f_2)$ are in the same class in $C(G; A)$.
- $\varrho(f) \in \mathrm{Ind}_H^G(A)$, i.e. $\varrho(f)(gh) = h^{-1} \cdot \varrho(f)(g)$ for almost all pairs $(g, h) \in G \times H$: Indeed,

$$\begin{aligned}
\varrho(f)(gh) &= r(h^{-1}g^{-1}) \cdot f(ghH) \\
&= h^{-1}g^{-1} \cdot s(gH) \cdot f(gH) \\
&= h^{-1} \cdot r(g^{-1}) \cdot f(gH) \\
&= h^{-1} \cdot \varrho(f)(g)
\end{aligned}$$

  and hence the defining conditions hold in fact for all pairs $(g, h) \in G \times H$.
- ϱ is continuous:
  Let $f_n \to f$ in probability. Then, by Proposition 2.2.9, $\{f_n\}$ has a subsequence $\{f_{n_i}\}$ converging a.e. pointwise to f, say for $x \notin N$ with $N \subset G/H$ a set of measure zero. By the Fubini theorem the pre-image of N under the projection map, $p^{-1}(N) \subset G$, has also measure zero. We have

$$\varrho(f_{n_i})(g) = r(g^{-1}) \cdot f_{n_i}(gH) \to r(g^{-1}) \cdot f(gH) \text{ for all } g \notin p^{-1}(N),$$

  and it thus follows that the subsequence $\{\varrho(f_{n_i})\}$ of $\{\varrho(f_n)\}$ converges to $\varrho(f)$ almost everywhere. Hence $\varrho(f_n) \to \varrho(f)$ in probability.
- ϱ is bijective and therefore an isomorphism of topological groups:
  It is clear that ϱ is injective. Since a surjective morphism between two Polish topological groups is open (see for example [HofMor07]) it is enough to show that ϱ is a surjection to conclude that it gives an isomorphism of topological groups. So let $F \in \mathrm{Ind}_H^G(A)$. We will construct $F'' \in \mathrm{Ind}_H^G(A)$ such that

$$\begin{aligned}
F'' &= F \text{ a. e., and} \\
F''(gh) &= h^{-1} \cdot F''(g) \text{ for all } h \in H, g \in G.
\end{aligned}$$

  Then by the first condition F'' is equivalent to F in $\mathrm{Ind}_H^G(A)$ and by the second condition it is in the image of ϱ.
  Since $F(gh) = h^{-1} \cdot F(g)$ for almost all $(g, h) \in G \times H$ by the Fubini theorem there is a set N of measure zero in G such that if $g \notin N$, $F(gh) = h^{-1} \cdot F(g)$ holds for almost all $h \in H$. Denote by N^c the complement of N in G and define a function F' on $N^c \times H$ by

$$F'(g, h) = h^{-1} \cdot F(g) \text{ for } g \in N^c, h \in H.$$

Suppose now that $g_1 \notin N$ and $g_1 = g \cdot h_1$ for $g \notin N$. Then $h^{-1} \cdot F(g_1) = F(g_1 h)$ for almost every $h \in H$. Furthermore, $F(g_1 h) = F(gh_1 h) = h^{-1}h_1^{-1} \cdot F(g)$ for almost every $h_1 h \in H$ and hence for almost every $h \in H$. But then, since both sides are continuous in h:

$$h^{-1} \cdot F(g_1) = h^{-1}h_1^{-1}F(g) \text{ for all } h \in H,$$

and it follows that

$$F'(gh_1, h) = F'(g, h_1 h) \text{ for all } g, gh_1 \in N^c.$$

Let $L = \{y \in G \mid \exists h \in H \text{ s.t. } yh \notin N\}$ which is a subset of G that contains N^c. Let $y \in L$. Then there is a $h^{-1} \in H$ such that $yh^{-1} \notin N$. Write $g = yh^{-1}$ so that $y = gh$ with $g \notin N$. Now we can define a function F'' on L by

$$F''(y) = F''(gh) = F'(g, h).$$

Then F'' is a Borel function and

$$F''(gh) = h^{-1} \cdot F''(g).$$

The function F'' can be extended to a Borel function on the whole of G (since $L^c \subset N$ has measure zero) such that it satisfies the requirements mentioned above.

- The action of G on $f \in C(G/H; A)$ induced by the action of G on $\mathrm{Ind}_H^G(A)$ via ϱ is defined by

$$\varrho(g \cdot f)(g') = \varrho(f)(g^{-1}g')$$

and thus

$$r(g'^{-1}) \cdot (g \cdot f)(g'H) = r(g'^{-1}g) \cdot f(g^{-1}g'H).$$

This implies

$$\begin{aligned}(g \cdot f)(g'H) &= r(g'^{-1})^{-1} \cdot r(g'^{1}g) \cdot f(g^{-1}g'H) \\ &= \lambda(g, g'H) \cdot f(g^{-1}g'H).\end{aligned}$$

□

The proof of the Eckmann-Shapiro Lemma is complicated by the fact that the maps in $C^p(G; A)$ are only defined almost everywhere. However, sometimes it is possible to write down the isomorphism explicitly.

Example 2.4.5. *If cochains in $H^q_m(G, Ind^G_H(A))$ are continuous, as for example in degree 1, we can give explicit maps on the cochain level which induce the isomorphism of the Eckmann-Shapiro lemma. Define*

$$u^n : C_c(G^{q+1}; A)^H \to C_c(H^{q+1}; A)^H$$

by

$$u^n(\sigma)(h_0, \ldots, h_q) = \sigma(h_0, \ldots, h_q),$$

for $h_0, \ldots, h_q \in H$. Note that this is well defined because σ is continuous in G^{q+1}. The map

$$v^n : C_c(G^{q+1}; C(G/H; A))^G \to C_c(G^{q+1}; A)^H$$

can be defined as follows: Let $\beta \in C_c(G^{q+1}; C(G/H; A))$ and $g \in G$. Define $F_g(\beta) \in C_c(G^{q+1}; A)$ by

$$F_g(\beta)(g_0, \ldots, g_q) := \beta(s(gH)g_0, \ldots, s(gH)g_q)(gH),$$

for $g_0, \ldots, g_q \in G$. Then if β is G-invariant, $F_g(\beta)$ is independent of g and gives an element in $C_c(G^{q+1}; A)^H$. We define $v^n(\beta)$ to be this element and then we can define

$$\phi = u^n \circ v^n : C_c(G^{q+1}; C(G/H; A))^G \to C_c(H^{q+1}; A)^H.$$

Its inverse in cohomology is given by

$$\psi(\alpha)(g_0, \ldots, g_q)(gH) = \alpha(\lambda(g_0, gH), \ldots, \lambda(g_q, gH)),$$

with λ as defined in equation 2.5, $\alpha \in C_c(H^{p+1}; A)^G$, and $g, g_0, \ldots, g_q \in G$.

2.5 Isomorphism with continuous cohomology

Just as for continuous bounded cohomology and continuous cohomology there is a natural inclusion of cochains

$$C^*_c(G; A) \hookrightarrow C^*(G; A)$$

which induces a comparison map

$$H^*_c(G; A) \to H^*_m(G; A)$$

from measurable cohomology to continuous cohomology. Austin and Moore prove in [AusMoo13] that this map is an isomorphism for all Fréchet coefficients A:

Theorem 2.5.1. *[AusMoo13, Theorem A] Let G be a l.c.s.c. group and let A be a Fréchet G-module. Then $H^*_m(G; A) \cong H^*_c(G; A)$.*

Their proof is based on applying Buchsbaum's criterion. Since in general there exists no continuous cross section $G/H \to G$, continuous cohomology has no long exact sequences when we allow all Polish G-modules as coefficients. However, when restricting to Fréchet modules there do exist continuous cross sections and continuous cohomology is the unique effaceable cohomological functor on this category. Austin and Moore prove that measurable cohomology is also effaceable when restricted to Fréchet modules. The crucial step in the proof is [AusMoo13, Proposition 33] which states that for any cohomology class in $H_m^*(G;A)$ there exists a locally totally bounded representative. The main ingredients of its proof are dimension shifting and the fact that a locally totally bounded measurable cocycle $\bar{\alpha} : G^p \to C(G;A)/\iota(A)$ can be lifted to a locally totally bounded measurable map $\alpha : G^p \to C(G;A)$. That there exist such a lift follows from the Borel selection theorem, i.e. Lemma 2.3.2. Then, as the two theories also agree in degree zero, Buchsbaum's criterion applies. Below we give the details of this proof of which we will use techniques in section 4.5.

Proposition 2.5.2. *Any cohomology class in $H_m^p(G;A)$ can be represented by a measurable cocycle $\psi : G^p \to A$ that is locally totally bounded.*

Proof. The proof is by induction on the degree p of $H_m^p(G;A)$. In degree $p = 1$ cohomology classes have a continuous representative by Proposition 2.5.5 below and thus in degree 1 the proposition holds for any Polish Abelian G-module A.
Now let $p \geq 2$ and let $\alpha : G^p \to A$ be a cocycle. Let $\beta := Q\alpha \in C^{p-1}(G;C(G;A))$, with Q the connecting map for the inhomogeneous resolution as defined in equation 2.3 in Section 2.3. Denote by $\bar{\beta}$ the corresponding cocycle in the quotient $C^{p-1}(G;C(G;A)/\iota(A))$. By induction, there exists a locally totally bounded representative $\bar{\kappa}$ of the class $[\bar{\beta}] \in H_m^{p-1}(G;C(G;A)/\iota(A))$ and thus $\bar{\beta} = \bar{\kappa} + d\bar{\lambda}$ for some measurable map $\bar{\lambda} : G^{p-2} \to C(G;A)/\iota(A)$.
From the Borel selection theorem, i.e. Lemma 2.3.2, it follows that there exists measurable lifts κ and λ of $\bar{\kappa}$ and $\bar{\lambda}$ such that κ is still locally totally bounded. Thus

$$\beta = \kappa + d\lambda + \eta,$$

for some $\eta : G^{p-1} \to \iota(A) \subset C(G;A)$ and therefore $\alpha = d\kappa + d\eta$ (when we identify the constant map $(d\kappa + d\eta)(g_1, \ldots, g_p)$ with its constant value in A). Now

$$d\kappa = \alpha - d\eta$$

is a locally totally bounded representative of $[\alpha]$:
For any compact $K \subset G$:

$$d\kappa(K^p) \subset \bigcup_{g \in K} (g \cdot \kappa(K^{p-1})) - \kappa(K^{p-1}) + \cdots + (-1)^{p+1}\kappa(K^{p-1}).$$

Therefore, as κ is locally totally bounded it follows that $d\kappa(K^p)$ is precompact as a subset of $C(G;A)$. Since the topology of the closed subgroup $\iota(A) \subset C(G;A)$ agrees with the subspace topology this implies that $d\kappa$ is locally totally bounded when considered as a cocycle $G^p \to A$. □

Now we can prove Theorem 2.5.1:

Proof of Theorem 2.5.1. Both theories are equal to A^G in degree 0 and they also agree in degree 1 as measurable cocycles $\alpha : G \to A$ in the inhomogeneous resolution always have a continuous representative (see Proposition 2.5.5 below). Furthermore, if $p > 1$ one can show that any cohomology class in $H^p_m(G;A)$ has a representative cocycle ψ that is effaced by the inclusion

$$\iota : A \hookrightarrow C_c(G;A) \tag{2.7}$$

That is, there exists a $\kappa : G^{p-1} \to C_c(G;A)$ such that $\delta\kappa = \psi$ where ψ is viewed as a function $G^p \to C_c(G;A)$ taking values in $\iota(A) \subset C_c(G;A)$. It then follows that the measurable cohomology H^*_m is effaceable if restricted to the category of Fréchet modules (In general, i.e. when A is a Polish module, H^*_m is effaced by the inclusion $A \hookrightarrow C(G;A)$ but $C(G;A)$ is in general not locally convex so in particular not a Fréchet module which is why we have to restrict to $C_c(G;A)$ here). Then both theories are effaced by the inclusion $A \hookrightarrow C_c(G;A)$, have long exact sequences (when restricted to the category of Fréchet G-modules) and agree in degree 1. It follows by Buchsbaum criterion that the two theories agree in all degrees.
The map $\kappa : G^{p-1} \to C_c(G;A)$ can be defined by

$$\begin{aligned}\kappa(g_1, \dots, g_{p-1})(g) :=& \\ &(-1)^p \int_G \psi(g_1, \dots, g_{p-1}, g_{p-1}^{-1} \cdots g_1^{-1} g h)\eta(h) d\mu_G(h),\end{aligned} \tag{2.8}$$

where $\eta : G \to \mathbb{R}_{\geq 0}$ is a compactly-supported continuous function with $\int_G \eta d\mu_G = 1$, with μ_G the Haar measure on G. Then κ indeed takes values in the G-module $C_c(G;A)$ of continuous functions since η is continuous and the Haar measure μ_G is left-invariant.

Now the usual calculation gives:

$$
\begin{aligned}
& d\kappa(g_1,\dots,g_p)(g) \\
&= (g_1\cdot\kappa(g_2,\dots,g_p))(g) + \sum_{i=1}^{p-1}(-1)^i\kappa(g_1,\dots,g_ig_{i+1},\dots,g_p)(g) \\
&+(-1)^p\kappa(g_1,\dots,g_{p-1})(g) \\
&= g_1\cdot\kappa(g_2,\dots,g_p)(g_1^{-1}g) + \sum_{i=1}^{p-1}(-1)^i\kappa(g_1,\dots,g_ig_{i+1},\dots,g_p)(g) \\
&+(-1)^p\kappa(g_1,\dots,g_{p-1})(g) \\
&= (-1)^p g_1\cdot\int_G \psi(g_2,\dots,g_p,g_p^{-1}\cdots g_2^{-1}(g_1^{-1}gh))\eta(h)d\mu_G(h) \\
&+\sum_{i=1}^{p-1}(-1)^{p+i}\int_G \psi(g_1,\dots,g_ig_{i+1},\dots,g_p,g_p^{-1}\cdots g_1^{-1}gh)\eta(h)d\mu_G(h) \\
&+(-1)^{2p}\int_G \psi(g_1,\dots,g_{p-1},g_{p-1}^{-1}\cdots g_1^{-1}gh)\eta(h)d\mu_G(h) \\
&= \int_G \left[(-1)^p d\psi(g_1,\dots,g_p,g_p^{-1}\cdots g_1^{-1}gh)+\psi(g_1,\dots,g_p)\right]\eta(h)d\mu_G(h) \\
&= \int_G \psi(g_1,\dots,g_p)\eta(h)d\mu_G(h) \\
&= \psi(g_1,\dots,g_p)
\end{aligned}
$$

□

Remark 2.5.3. *The integral in equation 2.8 in the above proof is the Bochner integral. Because ψ is locally totally bounded and η is compactly supported it takes its values in a compact set. It follows that the integral is absolutely summable, see e.g. [Tho75, Theorem 3], and therefore that the integral exists.*

Remark 2.5.4. *If G is compact then one can integrate over the whole group so in the proof above we do not need the function η. Hence $\kappa : G^{p-1} \to A$ and $\psi = d\kappa$ is a coboundary. Therefore $H^p_m(G;A) = 0$.*

In the inhomogeneous resolution, the cocycles in degree 1 are crossed homomorphisms. That is, measurable maps $f : G \to A$ that satisfy $f(gg') = g\cdot f(g') + f(g)$ for almost every pair $(g,g') \in G\times G$. Such a crossed homomorphism is equal to a continuous crossed homomorphism a.e.:

Proposition 2.5.5. *Let $f \in C(G;A)$ be a cocycle. Then f has a continuous representative.*

Proof. For the proof we refer to [Moo76a, Theorem 3]. Here Moore proves that a cocycle $f \in C(G;A)$ is equal a.e. to a (unique) crossed homomorphism

$\tilde{f} : G \to A$ which is defined everywhere and such a crossed homomorphism turns out to be continuous. For a discussion of the proof of the continuity of crossed homomorphisms also see [Min14]. $\square$

We will use this fact for the proof of Proposition 4.2.2 in Section 4.4.

Chapter 3

Spectral sequence associated to a double complex

In this chapter we will discuss the construction of a spectral sequence of a double complex. This is a special case of the spectral sequence of a filtered complex.

A *graded filtered chain complex* is a chain complex (K, D) of modules with grading $K = \bigoplus_{k\in\mathbb{Z}} C^k$, differential $D : C^k \to C^{k+1}$ and a filtration

$$K = K_0 \supset K_1 \supset K_2 \supset \dots$$

consisting of subcomplexes, i.e. submodules K_i such that $DK_i \subset K_i$. The filtration determines an *associated graded complex*

$$GK = \bigoplus_{p=0}^{\infty} K_p/K_{p+1}.$$

Usually the filtration is extended to negative indices by defining $K_p = K$ for $p < 0$.

Let

$$K = \bigoplus_{p,q\in\mathbb{Z}} K^{p,q}$$

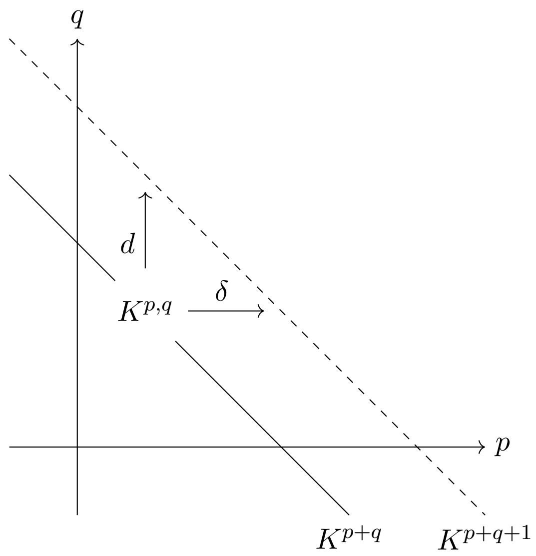
be a double complex with horizontal differential operator δ and vertical differential operator d that commute with each other, i.e. the following diagram commutes

$$\begin{tikzcd}
K^{p,q+1} \arrow[r, "\delta"] & K^{p+1,q+1} \\
K^{p,q} \arrow[u, "d"] \arrow[r, "\delta"] & K^{p+1,q} \arrow[u, "d"]
\end{tikzcd}$$

This is a graded chain complex with grading

$$K^m = \bigoplus_{p+q=m} K^{p,q},$$

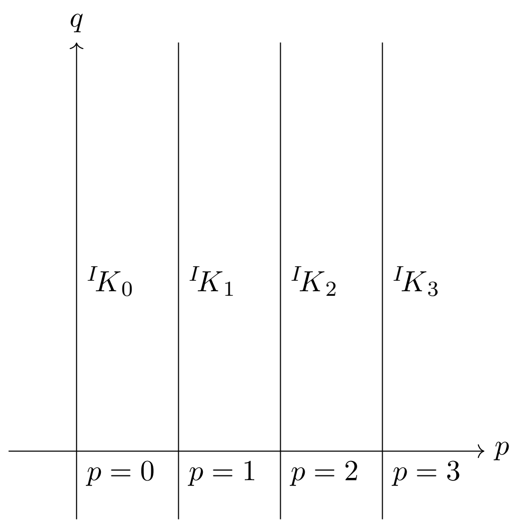
and differential $D : K^m \to K^{m+1}$ given by $D = \delta + (-1)^p d$. Note that from $d^2 = \delta^2 = 0$ and $d\delta = \delta d$ it immediately follows that $D^2 = 0$.



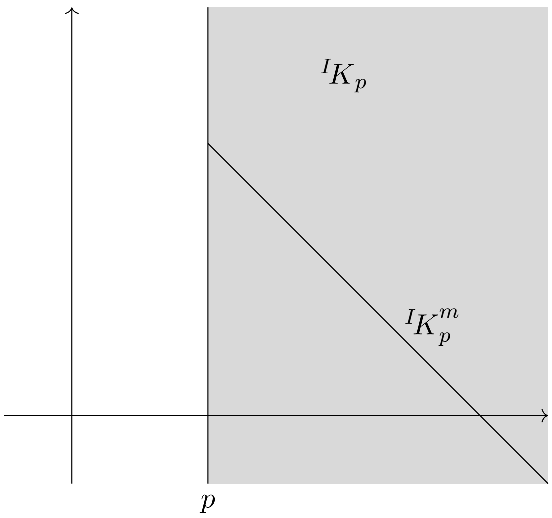
We define the *vertical filtration* on K by:

$$ {}^I K_p = \bigoplus_{i \geq p} \bigoplus_{q \geq 0} K^{i,q}. $$

So $K = {}^I K_0 \supset {}^I K_1 \supset {}^I K_2 \supset \ldots$:



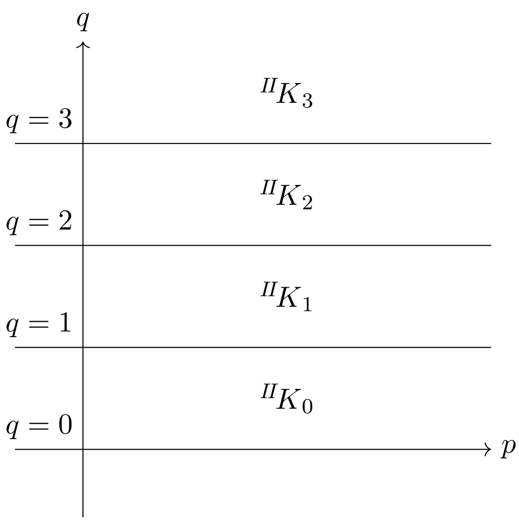
To distinguish the filtration degree from the grading degree, the latter will be called the dimension. The filtration $\{{}^I K_p\}$ on K induces a filtration in each dimension, i.e. ${}^{I}K_p^m = {}^{I}K_p \cap K^m$ gives a filtration on K^m:



By symmetry, we can just as well define the *horizontal filtration*

$${}^{II}K_q = \bigoplus_{i \geq q} \bigoplus_{p \geq 0} K^{p,i},$$

and we obtain $K = {}^{II}K_0 \supset {}^{II}K_1 \supset {}^{II}K_2 \supset \dots$:



which also gives a filtration ${}^{II}K_p^m = {}^{II}K_p \cap K^m$ on K^m.

Definition 3.0.1. A *spectral sequence* (of cohomological type) consists of

1. A module $E_r^{p,q}$ defined for each $p, q \in \mathbb{Z}$ and each integer $r \geq r_0$, where r_0 is some nonnegative integer.

2. Differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that $d_r^2 = 0$ and E_{r+1} is the cohomology of (E_r, d_r), i.e.

$$E_{r+1}^{p,q} = H^{p,q}(E_r^{*,*}, d_r) = \frac{\ker(d_r : E_r^{p,q} \to E_r^{p+r,q-r+1})}{\operatorname{im}(d_r : E_r^{p-r,q+r-1} \to E_r^{p,q})}.$$

The r^{th} stage of the spectral sequence is its E_r*-term* or its r^{th} *page.*

A spectral sequence *converges* if for every p, q there is a $r = r(p,q)$ such that for $r \geq r(p,q)$ the differential d_r is zero on $E_r^{p,q}$ and $E_r^{p+r,q-r+1}$. Then $E_r^{p,q} = E_{r+k}^{p,q}$ for all $k > 0$. We denote this common module by $E_\infty^{p,q}$ and write

$$E_r^{p,q} \Longrightarrow E_\infty^{p,q}.$$

The simplest situation in which we can determine convergence is when $d_s = 0$ for all $s \geq r$, where r does not depend on p and q. In this case we say that the spectral sequence *degenerates at the* E_r *term.*

A spectral sequence $\{E_r, d_r\}$ *converges to a graded module* H^* if H^* has a filtration $H^* = F_0 \supset F_1 \supset F_2 \supset \dots$ such that E_∞ is isomorphic to $\bigoplus F_p/F_{p+1}$ as graded modules, i.e.

$$E_\infty^m = \bigoplus_{p+q=m} E_\infty^{p,q} \cong F_p^m / F_{p+1}^m,$$

where $F_p^m = H^m \cap F_p$.

In this chapter we will prove

Theorem 3.0.2. *Let* (K, d, δ) *be a double complex. Then there exist two spectral sequences* $\{{}^I E_r, {}^I d_r\}$ *and* $\{{}^{II} E_r, {}^{II} d_r\}$ *whose first two pages are given by*

$$\begin{aligned} {}^I E_1^{p,q} &= H_d^q(K^{p,*}), \\ {}^I E_2^{p,q} &= H_\delta^p H_d^q(K^{*,*}), \\ {}^{II} E_1^{p,q} &= H_\delta^p(K^{*,q}), \\ {}^{II} E_2^{p,q} &= H_d^q H_\delta^p(K^{*,*}), \end{aligned}$$

and that both converge to $H_D^*(K)$.

As suggested by the notation these two spectral sequences correspond to the vertical and the horizontal filtration. Following [BottTu82], §14 we will construct them through exact couples. This construction is due to Massey.

3.1 Exact couples

Definition 3.1.1. An *exact couple* $\mathcal{C} = \{A, B, i, j, k\}$ consists of two modules A, B and homomorphisms $i : A \to A$, $j : A \to B$ and $k : B \to A$ such that the diagram

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & B \arrow[ul, "k"] & \end{tikzcd}$$

is exact.

Let $\mathcal{C} = \{A, B, i, j, k\}$ be such an exact couple and define $d : B \to B$ by $d = j \circ k$. Then $d^2 = j \circ (k \circ j) \circ k = 0$ since $\mathrm{im}(j) = \ker(k)$. Thus the cohomology group $H_d(B) = \frac{\ker(d)}{\mathrm{im}(d)}$ is well defined. Furthermore, from the exact couple $\mathcal{C}$ a new exact couple can be formed, the derived couple $\mathcal{C}'$:

Definition 3.1.2. Let $\mathcal{C} = \{A, B, i, j, k\}$ be an exact couple. Then the *derived couple* is $\mathcal{C}' = \{A', B', i', j', k'\}$:

$$\begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & B', \arrow[ul, "k'"] & \end{tikzcd}$$

where $A' = i(A)$, $B' = H_d(B)$ and the homomorphisms i', j' and k' are defined as follows:

1. i' is induced from i, i.e. for $a' = i(a) \in A'$:
$$i'(a') = i(i(a)).$$
2. For $a' = i(a) \in A'$:
$$j'(a') = [j(a)] \in H_d(B) = B'.$$
3. k' is induced from k, i.e. for $[b] \in H_d(B)$:
$$k'([b]) = k(b).$$

It is straightforward to check that this gives again an exact couple. For completeness, we will prove this in detail by the following two lemma's.

Lemma 3.1.3. *The derived couple $\mathcal{C}'$ is well defined.*

Proof. A', B' and i' are clearly well defined. Furthermore, note that $k(b) \in A' = i(A)$ since $d(b) = j(k(b)) = 0$ implies that $k(b) \in \ker(j) = \mathrm{im}(i)$. Furthermore, if $[b_1] = [b_2] \in B'$ then $b_1 = b_2 + d(b_3)$ for some $b_3 \in B$. It follows that $k(b_1) = k(b_2) + k \circ d(b_3) = k(b_2) + k \circ j \circ k(b_3) = k(b_2)$ and thus k' does not depend on the choice of b. Hence k' is also well defined. The only thing left to check is that j' is well defined. This follows from:

1. $j(a)$ is a cycle: $d(j(a)) = j \circ (k \circ j(a)) = 0$.
2. j' does not depend on the choice of $a \in A$:
   Suppose that $a' = i(a_1) = i(a_2)$ with $a_1, a_2 \in A$. Then $a_1 - a_2 \in \ker(i) = \mathrm{im}(k)$. So there is a $b \in B$ such that $a_1 - a_2 = k(b)$ and it follows that
   $$j(a_1) - j(a_2) = j \circ k(b) = d(b).$$
   So indeed $[j(a_1)] = [j(a_2)]$.

It follows that the derived couple $\mathcal{C}' = \{A', B', i', j', k'\}$ is indeed well defined. □

Lemma 3.1.4. *The derived couple $\mathcal{C}'$ is again an exact couple.*

Proof.

1. $\mathrm{im}(j') = \ker(k')$:
   (a) $\mathrm{im}(j') \subset \ker(k')$: Let $a' = i(a) \in A'$ with $a \in A$. Then
   $$k' \circ j'(a') = k' \circ j'(i(a)) = k'([j(a)]) = k \circ j(a) = 0.$$
   (b) $\ker(k') \subset \mathrm{im}(j')$: Suppose $[b] \in \ker(k')$, i.e. $k'([b]) = k(b) = 0$. Then from $\ker(k) = \mathrm{im}(j)$ it follows that $b = j(a)$ for some $a \in A$ and thus
   $$[b] = [j(a)] = j'(i(a)) \in \mathrm{im}(j').$$
2. $\mathrm{im}(k') = \ker(i')$:
   (a) $\mathrm{im}(k') \subset \ker(i')$: Let $[b] \in H(B)$. Then
   $$i'(k'([b])) = i'(k(b)) = i(k(b)) = 0.$$
   (b) $\ker(i') \subset \mathrm{im}(k')$: Suppose that $a' = i(a) \in \ker(i')$ with $a \in A$, i.e. $i'(i(a)) = i(i(a)) = 0$. Then $i(a) \in \ker(i) = \mathrm{im}(k)$. Hence
   $$a' = i(a) = k(b) = k'([b])$$
   for some $b \in \ker(d)$.
3. $\mathrm{im}(i') = \ker(j')$:
   (a) $\mathrm{im}(i') \subset \ker(j')$: Let $a' = i(a) \in A'$ with $a \in A$. Then
   $$j'(i'(a')) = j'(i(i(a))) = [j(i(a))] = 0.$$
   (b) $\ker(j') \subset \mathrm{im}(i')$: Let $a' = i(a) \in \ker(j')$. Then
   $$j'(i(a)) = [j(a)] = 0,$$
   and therefore it follows that there is a $b \in B$ such that $j(a) = d(b)$. Hence $a \in j^{-1}(d(b)) = j^{-1}(j \circ k(b))$ and thus $a = k(b) + a_1$ with $a_1 \in \ker(j) = \mathrm{im}(i)$. It follows that
   $$a' = i(a) = i(k(b)) + i(a_1) = i(a_1) \in i(\mathrm{im}(i)) = \mathrm{im}(i').$$

These three equalities imply that $\mathcal{C}'$ is again an exact couple. □

3.2 The spectral sequence of a filtered complex

Let (K, D) be a graded filtered chain complex such that for each dimension n the filtration $\{K_p^n\}$ has finite length. To any such complex we will associate a sequence of exact couples $\mathcal{C}^{(r)} = \{A_r, E_r, i_r, j_r, k_r\}$, with $C^{(r)} = (C^{(r-1)})'$ the derived couple of $\mathcal{C}^{(r-1)}$, such that $\{E_r, d_r = j_r \circ k_r\}$ is a spectral sequence converging to $H^*(K, D)$. In particular, for a double complex with the vertical respectively horizontal filtration this construction will give the spectral sequences of Theorem 3.0.2.

Let

$$A = \bigoplus_{p \in \mathbb{Z}} K_p.$$

This is again a differential complex with operator D, since $D : K_p^m \to K_p^{m+1}$. Define $i : A \to A$ to be given by the inclusions $i : K_{p+1} \hookrightarrow K_p$ which restrict to

$$i : A^m \cap K_{p+1} \to A^m \cap K_p,$$

and let B be the quotient

$$0 \longrightarrow A \xrightarrow{i} A \xrightarrow{j} B \longrightarrow 0,$$

i.e. $B = \oplus K_p / K_{p+1}$. Then B is also a differential complex with differential operator induced by D. The derived long exact sequence

$$\ldots \longrightarrow H^k(A) \xrightarrow{i_1} H^k(A) \xrightarrow{j_1} H^k(B) \xrightarrow{k_1} H^{k+1}(A) \longrightarrow \ldots \quad (3.1)$$

gives an exact couple

$$\begin{tikzcd} H(A) \arrow[rr, "i_1"] & & H(A) \arrow[dl, "j_1"] \\ & H(B). \arrow[ul, "k_1"] & \end{tikzcd}$$

which we will write as

$$\mathcal{C}^{(1)} = \qquad \begin{tikzcd} A_1 \arrow[rr, "i"] & & A_1 \arrow[dl, "j_1"] \\ & E_1. \arrow[ul, "k_1"] & \end{tikzcd}$$

We obtain a sequence of exact couples

$$\mathcal{C}^{(r)} = \qquad \begin{tikzcd} A_r \arrow[rr, "i"] & & A_r \arrow[dl, "j_r"] \\ & E_r, \arrow[ul, "k_r"] & \end{tikzcd}$$

where $\mathcal{C}^{(r)} = (\mathcal{C}^{(r-1)})'$.

Remark 3.2.1. *We define a bigrading by setting* $A^{p,q} = A^{p+q} \cap K_p$ *and* $B^{p,q} = B^{p+q} \cap K_p/K_{p+1}$. *Note that for a bicomplex* $\bigoplus K^{p,q}$ *with the vertical filtration* ${}^I K_p = \bigoplus_{i\geq p} \bigoplus_{j\geq 0} K^{i,j}$ *this gives*

$$B^{p,q} = {}^I K_p^{p+q} / {}^I K_{p+1}^{p+q} = \frac{\bigoplus_{i\geq p} K^{i,p+q-i}}{\bigoplus_{i\geq p+1} K^{i,p+q-i}} \cong K^{p,q},$$

while with the horizontal filtration ${}^{II} K_p = \bigoplus_{i\geq p} \bigoplus_{j\geq 0} K^{j,i}$ *we obtain*

$$B^{p,q} = {}^{II} K_p^{p+q} / {}^{II} K_{p+1}^{p+q} = \frac{\bigoplus_{i\geq p} \bigoplus_{j\geq 0} K^{p+q-i,i}}{\bigoplus_{i\geq p+1} \bigoplus_{j\geq 0} K^{p+q-i,i}} \cong K^{q,p}.$$

Furthermore, $i : A^k \cap K_{p+1} \to A^k \cap K_p$ *has bidegree* $(-1, 1)$, $j : A^k \cap K_p \to B^k \cap K_p/K_{p+1}$ *has bidegree* $(0, 0)$ *and* $k : B^k \cap K_p/K_{p+1} \to A^{k+1} \cap K_{p+1}$ *has bidegree* $(1, 0)$. *From the definitions it follows that* j_r *has bidegree equal to bidegree*$(j) - (r-1) \times$ *bidegree*$(i) = (r-1, -r+1)$, *while the bidegree of* k_r *stays the same as for* k. *Thus* $d_r = j_r \circ k_r$ *has bidegree* $(r, -r+1)$. *Furthermore, by definition* $E_r = H(E_{r-1}, d_{r-1})$ *and hence* $\{E_r, d_r\}$ *is a spectral sequence.*

Proposition 3.2.2. *Let* $K = \bigoplus_{n\in\mathbb{Z}} K^n$ *be a graded filtered complex such that in each dimension* n *the filtration* $\{K_p^n\}$ *has finite length. Then the spectral sequence* $\{E_r, d_r\}$ *defined above converges to* $H_D^*(K)$.

Proof. Recall that $A_1 = H_D(A) = H_D(\bigoplus K_p)$. Since D preserves the filtration, i.e. $D : K_p^m \to K_p^{m+1}$, in fact $A_1 = \bigoplus H(K_p)$. Let $l(m)$ be the length of the filtration $\{K_p^m\}$. Then A_1^m is the direct sum of the terms in the following (non-exact) sequence

$$\ldots \xleftarrow{\cong} H^m(K) \xleftarrow{i} H^m(K_1) \xleftarrow{i} H^m(K_2) \xleftarrow{i} \ldots \xleftarrow{i} H^m(K_{l(m)}).$$

Thus

$$A_r^m = i^{r-1}(A_1^m) = \bigoplus_{p=-\infty}^{l(m)} i^r(H^m(K_p)),$$

In particular, $A_2^m = i(A_1^m) = \bigoplus i(H^m(K_p))$ is the direct sum of the terms in the sequence

$$\ldots \xleftarrow{\cong} H^m(K) \xleftarrow{i} i(H^m(K_1)) \xleftarrow{i} i(H^m(K_2)) \xleftarrow{i} \ldots \xleftarrow{i} i(H^m(K_{l(m)})).$$

Note that since $i(H^m(K_1)) \subset H^m(K)$ the map $i : i(H^m(K_1)) \to H^m(K)$ is an inclusion. Continuing like this, if $r > l(m) + 1$ then $i^r(H^m(K_p))$ is included in $H^m(K)$ for all p and A_r^m is the direct sum of

$$\ldots \xleftarrow{\cong} H^m(K) \supset i(H^m(K_1)) \supset i^2(H^m(K_2)) \cdots \supset i^{l(m)} H^m(K_{l(m)}).$$

Denote by F_p^m this image by i^p of $H^m(K_p)$ in $H^m(K)$. This gives a filtration of $H^m(K)$:

$$H^m(K) = F_0^m \supset F_1^m \supset F_2^m \supset \cdots \supset F_{l(m)}^m,$$

(called the induced filtration) and we will see that the spectral sequence converges to the graded complex associated to this filtration. Note that by the above A_r^m becomes stationary with limit term

$$A_\infty^m = \bigoplus_{p=-\infty}^{l(m)} F_p^m.$$

Likewise, for $r > l(m+1)+1$ the map $i : A_r^{m+1} \to A_r^{m+1}$ is an inclusion and it follows that $k_r : E_r^m \to A_r^{m+1}$ is the zero map. Note that $i : A_\infty^m \to A_\infty^m$ sends $F_{p+1}^m \hookrightarrow F_p^m$. Hence E_r^m also becomes stationary and

$$E_\infty^m = \bigoplus_{p\in\mathbb{Z}} F_p^m / F_{p+1}^m.$$

This is the associated graded complex of $H^*(K)$ so the spectral sequence $\{E_r, d_r\}$ indeed converges to $H^*(K)$. □

Example 3.2.3. *If $E_r^{p,q} = 0$ for $p < 0$ and $q < 0$ then $E_r^{p,q}$ is a so-called first quadrant spectral sequence. Then for r large enough $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ has codomain 0 and $d_r : E_r^{p-r,q+r-1} \to E_r^{p,q}$ has domain 0. Thus in this case the spectral sequence always converges.*

3.3 Proof of convergence

In this section we give the proof of Theorem 3.0.2. Since

$${}^I K_p^m = K^m \cap {}^I K_p = \bigoplus_{i\geq p} K^{i,m-i}$$

the vertical filtration $\{{}^I K_p^m\}$ has length m, i.e. ${}^I K_p^m = 0$ for $p > m$:

$$\cdots = {}^I K_{-1}^m = {}^I K_0^m \supset {}^I K_1^m \supset {}^I K_2^m \supset \cdots \supset {}^I K_{m-1}^m \supset {}^I K_m^m \supset 0,$$

where $K_0^m = K^m$. Clearly, the horizontal filtration $\{{}^{II} K_q^m\}$ has length m as well. It thus follows from Proposition 3.2.2 that these two filtrations correspond to spectral sequences ${}^I E_r$ and ${}^{II} E_r$ that converge to $H_D^*(K)$. It remains to show that their first two pages are as claimed. We prove this for ${}^I E_r$. By symmetry, it then also follows for ${}^{II} E_r$.
Recall that

$$B^m = \bigoplus_{p\in\mathbb{Z}} {}^I K_p^m / {}^I K_{p+1}^m$$

and $\delta : K^{p,q} \to K^{p+1,q}$. Thus $\delta : {}^I K_p^m \to {}^I K_{p+1}^{m+1}$ vanishes on B^m and therefore D is equal to $(-1)^p d$ on B. It follows that

$$ {}^I E_1 = H_D(B) = H_d(K). $$

Furthermore, ${}^I E_2 = H({}^I E_1, {}^I d_1)$ with ${}^I d_1 = j_1 \circ k_1$, where the map $k_1 : H^k(B) \to H^{k+1}(A)$ is the coboundary map in the long exact sequence (3.1).

$$
\begin{tikzcd}
 & {} & {} & {} & \\
0 \arrow[r] & A^{k+1} \cap {}^I K_{p+1} \arrow[r, "\mathrm{i}"] \arrow[u] & A^{k+1} \cap {}^I K_p \arrow[r, "\mathrm{j}"] \arrow[u] & B^{k+1} \cap {}^I K_p / {}^I K_{p+1} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & A^k \cap {}^I K_{p+1} \arrow[r, "\mathrm{i}"] \arrow[u, "\mathrm{D}"'] & A^k \cap {}^I K_p \arrow[r, "\mathrm{j}"] \arrow[u, "\mathrm{D}"'] & B^k \cap {}^I K_p / {}^I K_{p+1} \arrow[r] \arrow[u, "\mathrm{D}"'] & 0 \\
 & {} \arrow[u] & {} \arrow[u] & {} \arrow[u] &
\end{tikzcd}
$$

Let $b \in A^k \cap {}^I K^p$ represent a cocycle $[b] \in B^k \cap {}^I K_p / {}^I K_{p+1}$. Then, since b is a cocycle, $db = 0$. Hence $Db = \delta b$ and as the map i is just an inclusion map it follows that the inverse image of δb under i is δb. Hence

$$ k_1([b]) \quad = \quad [\delta b], $$

and

$$ d_1([b]) = j_1 \circ k_1([b]) = j_1([\delta b]) = [\delta b] $$

It follows that

$$ {}^I E_2 = H({}^I E_1, {}^I d_1) = H_\delta H_d(K), $$

and thus the first two pages are indeed as in Theorem 3.0.2.

Remark 3.3.1. *Even though both the spectral sequences ${}^I E_r$ and ${}^{II} E_r$ converge to $H_D^*(K)$ one should note that they do not converge to the same associated graded complex.*

Chapter 4

The boundary model for $H_c^*(\mathrm{Isom}^+(\mathbb{H}^n);\mathbb{R})$

4.1 Spectral sequence for $\mathrm{Isom}^+(\mathbb{H}^n)$

Let H be a locally compact second countable group. If H acts on a measure space X we define $H_m^*(H \curvearrowright X;\mathbb{R})$ to be the cohomology of the cocomplex $(C^*(X;\mathbb{R})^H, \delta)$, with δ the standard homogeneous coboundary operator. Fix a basepoint $x \in X$. Given a cocycle $\alpha \in C^p(X;\mathbb{R})^H$ we obtain a cocycle $\alpha_x \in C(H^{p+1};\mathbb{R})^H$ by

$$\alpha_x(h_0, \dots, h_p) := \alpha(h_0 \cdot x, \dots, h_p \cdot x).$$

The class of α_x does not depend on the chosen basepoint x. Indeed, given another $y \in X$ define

$$\beta(h_0, \dots, h_{p-1}) := \sum_{i=0}^{p-1} (-1)^i \alpha(h_0 \cdot x, \dots, h_i \cdot x, h_i \cdot y, \dots, h_{p-1} \cdot y).$$

Then a straightforward calculation gives

$$\begin{aligned}(\alpha_y - \alpha_x - \delta\beta)(h_0, \dots, h_p) &= \sum_{j=0}^{p} (-1)^j \delta\alpha(h_0 \cdot x, \dots, h_i \cdot x, h_i \cdot y, \dots, h_p \cdot y)\\ &= 0.\end{aligned}$$

Thus the cochain map $\alpha \mapsto \alpha_x$ defines a map

$$\iota_X : H_m^*(H \curvearrowright X;\mathbb{R}) \to H_m^*(H;\mathbb{R}).$$

We say that the measurable cohomology of H (with trivial $\mathbb{R}$-coefficients) is *measurably realized on* X if this map is an isomorphism. Since measurable and continuous cohomology coincide for trivial $\mathbb{R}$-coefficients we will then also say that the continuous cohomology of H is measurably realized on X. Let $\mathrm{Isom}^+(\mathbb{H}^n)$ be the group of orientation preserving isometries of real hyperbolic n-space $\mathbb{H}^n$. In this chapter we prove that

Theorem 4.1.1. *Let $n \geq 2$. The continuous cohomology of* $\mathrm{Isom}^+(\mathbb{H}^n)$ *is measurably realized on the boundary $\partial\mathbb{H}^n$ of real hyperbolic space (with the natural action). Furthermore, all cocycles have essentially bounded representatives in this boundary resolution.*

Let $G := \mathrm{Isom}^+(\mathbb{H}^n)$ and let

$$C^p := C((\partial\mathbb{H}^n)^{p+1};\mathbb{R})$$

be the Polish Abelian G-module endowed with the standard diagonal G-action. To prove Theorem 4.1.1 we will apply the spectral sequence constructed in the previous chapter to the double complex

$$K^{p,q} := (C(G^{q+1};C^p)^G, d, \delta),$$

where by definition $C(G^{q+1};C^p) = 0$ if $p < 0$ or $q < 0$. The vertical differential operator

$$d : C(G^{q+1};C^p)^G \to C(G^{q+2};C^p)^G$$

is given by the standard homogeneous coboundary operator, i.e. for $f \in C(G^{q+1};C^p)^G$ and $g_0,\dots,g_{q+1} \in G$

$$df(g_0,\dots,g_{q+1}) = \sum_{i=0}^{q+1}(-1)^i f(g_0,\dots,\hat{g}_i,\dots,g_{q+1}),$$

and the horizontal differential operator

$$\delta : C(G^{q+1};C^p)^G \to C(G^{q+1};C^{p+1})^G$$

is induced by the standard homogeneous coboundary operator

$$\delta : C((\partial\mathbb{H}^n)^{p+1};\mathbb{R}) \to C((\partial\mathbb{H}^n)^{p+2};\mathbb{R}),$$

given by

$$\delta\alpha(x_0,\dots,x_{p+1}) = \sum_{j=0}^{p+1}(-1)^j \alpha(x_0,\dots,\hat{x}_j,\dots,x_{p+1}),$$

for $\alpha \in C((\partial\mathbb{H}^n)^{p+1};\mathbb{R})$ and $x_0,\dots,x_{p+1} \in \mathbb{H}^n$. These two differential operators commute so as before we have a commutative diagram

$$\begin{array}{ccc}
C(G^{q+2},C^p)^G & \xrightarrow{\delta} & C(G^{q+2},C^{p+1})^G \\
{\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow \\
C(G^{q+1},C^p)^G & \xrightarrow{\delta} & C(G^{q+1},C^{p+1})^G.
\end{array}$$

In [Blo00, Section 7.4] Bloch considers the same spectral sequence but only for the case $n = 3$. For $p > 2$ we have ${}^{I}E_1^{p,q} = H_m^*(\mathrm{SO}(n-2);C^{p-3})$, with $\mathrm{SO}(n-2)$ the stabilizer of 3 points in the boundary of hyperbolic space.

In the case of $n = 3$ this group is trivial and thus ${}^{I}E_1^{p,q}$ automatically vanishes for $q > 0$. Because of this, in Bloch's proof the fact that the spectral sequence degenerates at the second page already follows from looking at the first page. One would expect that $H_m^*(K; A)$ vanishes if K is compact no matter what the coefficients A are. However, if the coefficients A are not locally convex there is no way to integrate over them so it is not possible to construct a coboundary in the usual way. Since $C((\partial\mathbb{H}^n)^{p-2}; \mathbb{R})$ is a (non-locally convex) F-space (see Remark 2.2.14) it is not clear (how to prove) that $H_m^*(\mathrm{SO}(n-2); C^{p-3})$ vanishes for $n > 3$. Here we will instead prove that if a cocycle $[\alpha] \in {}^{I}E_1^{p,q}$ survives to the second page of the spectral sequence, i.e. if $\delta\alpha = d\lambda$, then it is cohomologous in ${}^{I}E_2^{p,q}$ to a coboundary in ${}^{I}E_1^{p,q}$. As in the proof in [AusMoo13] of the isomorphism between measurable and continuous cohomology in the case of Fréchet coefficients (see [AusMoo13, Theorem A] or Theorem 2.5 in this text), we would like to show by dimension-shifting induction that each cocycle has a representative that is locally totally bounded. In fact, by a double induction argument, we will show that this is the case for a cocycle in ${}^{I}E_2^{p,q}$. The first step of this dimension-shifting induction argument is Proposition 4.2.4 that implies in particular that all cocycles in $(C^p)^G$ have essentially bounded representatives, which is the second part of Theorem 4.1.1.

4.2 Proof of Theorem 4.1.1

By Theorem 3.0.2 we have ${}^{II}E_1 = H(K, \delta)$. If $\alpha \in C((\partial\mathbb{H}^n)^{p+1}; \mathbb{R})^G$ is a cocycle then

$$\delta\alpha(y, x_0, \ldots, x_p) = 0,$$

for almost all $(y, x_0, \ldots, x_p)$. By the Fubini theorem, there exists a $y \in \partial\mathbb{H}^n$ such that $\delta\alpha(y, x_0, \ldots, x_p) = 0$ for almost all $(x_0, \ldots, x_p)$. Now define

$$\beta(x_0, \ldots, x_{p-1}) := \alpha(y, x_0, \ldots, x_{p-1}).$$

Then

$$\alpha(x_0, \ldots, x_p) = \sum_j (-1)^j \alpha(y, x_0, \ldots, \hat{x}_j, \ldots, x_p),$$

for almost all $(x_0, \ldots, x_p)$. Thus (C^*, δ) is an acyclic cocomplex and therefore

$${}^{II}E_1^{p,q} = \begin{cases} C(G^{q+1}; \mathbb{R})^G, & \text{if } p = 0; \\ 0, & \text{otherwise,} \end{cases}$$

which implies that the second page of ${}^{II}E_r^{p,q}$ is

$${}^{II}E_2^{p,q} = \begin{cases} H_m^q(G; \mathbb{R}), & \text{if } p = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Hence the spectral sequence degenerates at the second page and we obtain that ${}^{II}E_r^{p,q}$ converges to $H_m^*(G;\mathbb{R})$ which is isomorphic to the continuous cohomology $H_c^*(G;\mathbb{R})$ of G. On the other hand, we establish

Proposition 4.2.1. *The spectral sequence ${}^{I}E_r^{p,q}$ converges to the cohomology group $H_m^p(G \curvearrowright \partial\mathbb{H}^n;\mathbb{R})$*

Proof. By Theorem 3.0.2, the first page of ${}^{I}E_r^{p,q}$ is

$$\begin{aligned} {}^{I}E_1^{p,q} &= H^q(C(G^*, C^p), d) \\ &= H_m^q(G; C^p). \end{aligned}$$

In the next section we will calculate that this first page is as in Figure 4.1 below.

$$\begin{tikzcd} q & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & H^2_m(G,C^3) \arrow[r] & H^2_m(G,C^4) \arrow[r] & \cdots \\ \mathbb{R} \arrow[r, "d_1"] & \mathbb{R} \arrow[r] & 0 \arrow[r] & H^1_m(G,C^3) \arrow[r] & H^1_m(G,C^4) \arrow[r] & \cdots \\ \mathbb{R} \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R} \arrow[r] & (C^3)^G \arrow[r] & (C^4)^G \arrow[r] & \cdots \; p \end{tikzcd}$$

Figure 4.1 – First page of ${}^{I}E_r^{p,q}$

We finish the proof of Proposition 4.2.1 using the two following propositions which we will prove in the next two sections.

Proposition 4.2.2. *The map $d_1 : {}^{I}E_1^{0,1} \to {}^{I}E_1^{1,1}$ is an isomorphism.*

Proposition 4.2.3. ${}^{I}E_2^{p,q} = 0$ *for $p > 2$ and $q > 0$.*

It then follows that the spectral sequence degenerates at the second page, that is

$${}^{I}E_2^{p,q} = \begin{cases} H_m^p(G \curvearrowright \partial\mathbb{H}^n;\mathbb{R}), & \text{if } q = 0; \\ 0, & \text{otherwise,} \end{cases}$$

which proves Proposition 4.2.1. ∎

Hence $H_c^p(G;\mathbb{R})$ is isomorphic to $H_m^p(G \curvearrowright \partial\mathbb{H}^n;\mathbb{R})$ and thus measurably realized on the boundary. This proves the first part of Theorem 4.1.1. For the second part, note that a locally totally bounded cocycle $(\partial\mathbb{H}^n)^{p+1} \to \mathbb{R}$ is essentially bounded. Indeed, since $(\partial\mathbb{H}^n)^{p+1}$ is itself compact, a cocycle that sends compact subsets of $(\partial\mathbb{H}^n)^{p+1}$ to precompact subsets of $\mathbb{R}$ is bounded a.e. Thus Proposition 4.2.4 below finishes the proof of Theorem 4.1.1.

Proposition 4.2.4. *Let A be a Polish Abelian G-module. Any a.e. G-invariant cocycle $\alpha : (\partial\mathbb{H}^n)^{p+1} \to A$ has a locally totally bounded representative. That is, there exists a G-invariant cochain $\sigma : (\partial\mathbb{H}^n)^p \to A$ such that $\kappa = \alpha + \delta\sigma$ is a locally totally bounded cocycle.*

Proof. This follows by a dimension-shifting argument. For $p = 0$ cocycles correspond to constants in A and are thus bounded. Therefore the proposition holds in degree $p = 0$ for all Polish Abelian G-modules A. Suppose now that $p > 0$ and define $Q : C((\partial\mathbb{H}^n)^{p+1}; A)^G \to C((\partial\mathbb{H}^n)^p; C(G; A))^G$ by

$$Q\alpha(x_0, \ldots, x_{p-1})(g) = (-1)^p \alpha(x_0, \ldots, x_{p-1}, gG_\infty).$$

Then

$$\begin{aligned}
\delta(Q\alpha)(x_0, \ldots, x_p)(g) &= \sum_{i=0}^{p} (-1)^i Q\alpha(x_0, \ldots, \widehat{x_i}, \ldots, x_p)(g) \\
&= \sum_{i=0}^{p} (-1)^i (-1)^p \alpha(x_0, \ldots, \widehat{x_i}, \ldots, x_p, gG_\infty) \\
&= (-1)^p \delta\alpha(x_0, \ldots, x_p, gG_\infty) \\
&\quad -(-1)^{p+1}(-1)^p \alpha(x_0, \ldots, x_p) \\
&= \alpha(x_0, \ldots, x_p),
\end{aligned}$$

and thus $\delta(Q\alpha)(x_0, \ldots, x_p)$ is independent of g. Let $\iota(A)$ be the image of the natural embedding of A into $C(G; A)$ as the closed submodule of constant maps which we identify with A. The image $\overline{Q\alpha}$ of $Q\alpha$ in the module $C((\partial\mathbb{H}^n)^p; C(G; A)/\iota(A))^G$ defines a cocycle. By the induction hypothesis it has a representative that is locally totally bounded, i.e.

$$\overline{Q\alpha} = \overline{\beta} + \delta\overline{\gamma},$$

with $\overline{\beta} : (\partial\mathbb{H}^n)^p \to C(G; A)/\iota(A)$ a G-invariant locally totally bounded cocycle and $\overline{\gamma} : (\partial\mathbb{H}^n)^{p-1} \to C(G; A)/\iota(A)$ a G-invariant cochain. By Lemma 2.3.2 there exist G-invariant measurable lifts β, γ of $\overline{\beta}, \overline{\gamma}$ such that β is still locally totally bounded (but no longer a cocycle). Hence

$$Q\alpha = \beta + \delta\gamma + \sigma,$$

with $\sigma : (\partial\mathbb{H}^n)^p \to A$ a G-invariant measurable cochain. We have $\delta Q\alpha = \alpha$ and thus

$$\alpha = \delta\beta + \delta\sigma.$$

It follows that $\delta\beta$ is a cocycle that takes its values in $\iota(A) \subset C(G;A)$ and therefore can be seen as a cocycle $\delta\beta : (\partial\mathbb{H}^n)^{p+1} \to A$. Hence $\kappa := \delta\beta$ is a representative of α. Furthermore, for any compact $L \subset \partial\mathbb{H}^n$:

$$\kappa(L^{p+1}) \subset \kappa(L^p) - \kappa(L^p) + \cdots + (-1)^p\kappa(L^p),$$

and thus $\kappa(L^{p+1})$ is precompact as a subset of $A \subset C(G;A)$ and κ is a locally totally bounded representative of α. □

4.3 Computation of ${}^IE_1^{p,q}$

Let $G_\infty = (\mathbb{R}_{>0} \times \mathrm{SO}(n-1)) \ltimes \mathbb{R}^{n-1}, G_{\infty,0} = \mathbb{R}_{>0} \times \mathrm{SO}(n-1)$ and $G_{\infty,0,1} = \mathrm{SO}(n-2)$ be the stabilizers of respectively $\{\infty\}, \{\infty, 0\}$ and $\{\infty, 0, 1\}$, where these are viewed as points in the upper half space model of $\mathbb{H}^n$. Then

Lemma 4.3.1. *The first page of the spectral sequence ${}^IE^{p,q}$ is as follows:*

$$\begin{aligned} {}^IE_1^{0,q} &\cong H_m^q(G_\infty;\mathbb{R}), \\ {}^IE_1^{1,q} &\cong H_m^q(G_{\infty,0};\mathbb{R}), \\ {}^IE_1^{2,q} &\cong H_m^q(G_{\infty,0,1};\mathbb{R}), \ \textit{and} \\ {}^IE_1^{p,q} &\cong H_m^q(G_{\infty,0,1}, C^{p-3}), \ \textit{for } p > 2. \end{aligned}$$

Proof. We will prove below that

$$C^0 = \mathrm{Ind}_{G_\infty}^G(\mathbb{R}), \tag{4.1}$$
$$C^1 = \mathrm{Ind}_{G_{\infty,0}}^G(\mathbb{R}), \tag{4.2}$$
$$C^2 = \mathrm{Ind}_{G_{\infty,0,1}}^G(\mathbb{R}), \text{ and} \tag{4.3}$$
$$C^p = \mathrm{Ind}_{G_{\infty,0,1}}^G(C^{p-3}) \text{ for } p > 2. \tag{4.4}$$

Combining this with the Eckmann-Shapiro Lemma the Lemma immediately follows. For the proof of 4.1 note that there is an isomorphism $G/G_\infty \cong \partial\mathbb{H}^n$ given by

$$gG_\infty \mapsto g \cdot \infty.$$

Indeed this map is

1. well-defined: If $g_1, g_2 \in G$ are such that $g_1G_\infty = g_2G_\infty$, then $g_1 = g_2g$ with $g \in G_\infty$. Thus $g_1 \cdot \infty = g_2g \cdot \infty = g_2 \cdot \infty$;
2. injective: If $g_1, g_2 \in G$ are such that $g_1 \cdot \infty = g_2 \cdot \infty$ then $g_1^{-1}g_2 \cdot \infty = \infty$. Hence $g_1^{-1}g_2 \in G_\infty$ and thus $g_1G_\infty = g_1(g_1^{-1}g_2)G_\infty = g_2G_\infty$;
3. surjective: Since G acts transitively on $\partial\mathbb{H}^n$ the map is clearly surjective.

Hence $C^0 = C(\partial\mathbb{H}^n;\mathbb{R}) \cong C(G/G_\infty;\mathbb{R}) \cong \mathrm{Ind}_{G_\infty}^G(\mathbb{R})$, where the last isomorphism follows from Proposition 2.4.4. Similarly, $G/G_{\infty,0}$ embeds in $\partial\mathbb{H}^n \times \partial\mathbb{H}^n$ via the map defined by

$$gG_{\infty,0} \hookrightarrow (g\cdot 0, g\cdot\infty)$$

whose image is the conull set $\{(x,y) \in \partial\mathbb{H}^n \times \partial\mathbb{H}^n \mid x \neq y\}$ in $\partial\mathbb{H}^n \times \partial\mathbb{H}^n$. This embedding therefore induces an isomorphism $C(\partial\mathbb{H}^n\times\partial\mathbb{H}^n;\mathbb{R}) \cong C(G/G_{\infty,0};\mathbb{R})$ and combined with Proposition 2.4.4 this implies 4.2.
Also, since G acts 3-transitively on $\partial\mathbb{H}^n$, $G/G_{\infty,0,1}$ embeds into $(\partial\mathbb{H}^n)^3$ as a conull set $\{(x_1,x_2,x_3) \in (\partial\mathbb{H}^n)^3 \mid x_i \neq x_j \text{ for } i\neq j\}$. Hence $C((\partial\mathbb{H}^n)^3;\mathbb{R}) \cong C(G/G_{\infty,0,1};\mathbb{R}) \cong \mathrm{Ind}_{G_{\infty,0,1}}^G(\mathbb{R})$.
For 4.4, applying Proposition 2.2.12, i.e. the Fubini theorem, gives

$$\begin{aligned} C^p &= C((\partial\mathbb{H}^n)^{p+1};\mathbb{R}) \cong C((\partial\mathbb{H}^n)^3, C((\partial\mathbb{H}^n)^{p-2};\mathbb{R})) \\ &= C((\partial\mathbb{H}^n)^3, C^{p-3}) = \mathrm{Ind}_{\mathrm{Stab}(\infty,0,1)}^G(C^{p-3}). \end{aligned}$$

□

4.3.1 Computation of ${}^IE_1^{0,q}$

We will prove

$${}^IE_1^{0,q} = H_m^q(G_\infty;\mathbb{R}) = \begin{cases} \mathbb{R}, & \text{if } q = 0,1; \\ 0, & \text{otherwise.} \end{cases}$$

Since for $\mathbb{R}$-coefficients measurable and continuous cohomology coincide it follows from Lemma 4.3.1 that ${}^IE_1^{0,q} \cong H_c^q(G_\infty;\mathbb{R})$. By the van Est isomorphism [BorWal00, Chapter IX, Corollary 5.6], for any connected Lie group G we have that $H_c^*(G;\mathbb{R}) \cong H^*(\Omega^*(G/K)^G)$, where K is the maximal compact subgroup of G and where $\Omega^q(G/K)^G$ denotes the set of G-invariant real differential q-forms on G/K. Furthermore, $\Omega^q(G/K)^G \cong \mathrm{Alt}^q(T_0(G/K))^K$. Note that the van Est isomorphism used here is more general than the one that is stated in Section 1.2 since G is no longer assumed to be a semisimple Lie group and therefore there are no geodesic simplices.
The maximal compact subgroup of $G_\infty = (\mathbb{R}_{>0} \times \mathrm{SO}(n-1)) \ltimes \mathbb{R}^{n-1}$ is $\mathrm{SO}(n-1)$. Thus $H_c^q(G_\infty;\mathbb{R})$ can be computed using the complex of multilinear alternating $\mathrm{SO}(n-1)$-invariant maps $(\mathbb{R}\times\mathbb{R}^{n-1})^q \to \mathbb{R}$, where $\mathrm{SO}(n-1)$ acts on the $\mathbb{R}^{n-1}$ factor. Let $T : \mathbb{R}\times\mathbb{R}^{n-1} \to \mathbb{R}$ be the projection onto the first factor and let $\det : (\mathbb{R}\times\mathbb{R}^{n-1})^{n-1} \to \mathbb{R}$ be the determinant defined on the second factor. We will prove below that up to scalar multiplication, the only nonzero alternating forms are constant maps in degree 0, the form T in degree 1, the determinant det in degree $n-1$ and $T\wedge\det$ in degree n. A straightforward calculation then gives $d(\det) = (1-n)\cdot T\wedge\det$ from which the result follows.

Lemma 4.3.2.

$$\mathrm{Alt}^q(\mathbb{R}\times\mathbb{R}^{n-1})^{\mathrm{SO}(n-1)} \cong \begin{cases} \mathbb{R}, & \textit{if } q=0,1,n-1,n \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Since the maximal compact subgroup $K=\mathrm{SO}(n-1)$ only acts on $\mathbb{R}^{n-1}$ a form in $\mathrm{Alt}^q(\mathbb{R}\times\mathbb{R}^{n-1})^K$ is a wedge product of a $\omega_1\in\mathrm{Alt}^{q_1}(\mathbb{R})$ and a $\omega_2\in\mathrm{Alt}^{q_2}(\mathbb{R}^{n-1})^K$ such that $q_1+q_2=q$. By Lemma 4.3.4 it follows that

$$\mathrm{Alt}^q(\mathbb{R}\times\mathbb{R}^{n-1})^K \cong \begin{cases} \mathbb{R}, & \text{if } q=0,1,n-1,n \\ 0, & \text{otherwise.} \end{cases}$$

□

Hence, up to scalar multiple, there is one nonzero $\omega\in\mathrm{Alt}^q(\mathbb{R}\times\mathbb{R}^{n-1})^K$ for $q=0,1,n-1,n$ which is given by:

- For $q=0$: $\mathrm{Alt}^0(\mathbb{R}\times\mathbb{R}^{n-1})^K$ is generated by a constant map $\{*\}\to\mathbb{R}$.
- For $q=1$: A generator of $\mathrm{Alt}^1(\mathbb{R}\times\mathbb{R}^{n-1})^K$ is given by $T:\mathbb{R}\times\mathbb{R}^{n-1}\to\mathbb{R}$ which is defined by
$$T:(t,v)\mapsto t$$
- For $q=n-1$: A nonzero form in $\mathrm{Alt}^n(\mathbb{R}^{n-1})^K$ is given by the determinant on the second factor, we denote the corresponding form in $\mathrm{Alt}^{n-1}(\mathbb{R}\times\mathbb{R}^{n-1})^K$ again by det, i.e.
$$\det:((t_1,v_1),(t_2,v_2),\dots,(t_{n-1},v_{n-1}))\mapsto\det(v_1,v_2,\dots,v_{n-1})$$
- For $q=n$: $\mathrm{Alt}^n(\mathbb{R}\times\mathbb{R}^{n-1})^K$ is generated by $T\wedge\det$ which is given by
$$\begin{aligned}&T\wedge\det((t_1,v_1),\dots,(t_n,v_n))\\ &=\frac{1}{(n-1)!}\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)t_{\sigma(1)}\cdot\det(v_{\sigma(2)},\dots,v_{\sigma(n)})\end{aligned}$$

Lemma 4.3.3. *Let* $\det\in\Omega^{n-1}(G_\infty/K)^{G_\infty}$ *be the form defined above. Then* $d(\det)=(1-n)T\wedge\det$.

Proof. For $\omega\in\Omega^{n-1}(G_\infty/K)$ and $X_0,\dots,X_{n-1}\in T(G_\infty/K)$ there is the following formula for $d\omega$ (see for example [Bre93] V.2)

$$\begin{aligned}d\omega(X_0,\dots,X_{n-1}) =&\sum_{i=0}^{n-1}(-1)^iX_i(\omega(X_0,\dots,\hat{X}_i,\dots,X_{n-1}))\\ &+\sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_{n-1}),\end{aligned}$$

We identify G_∞/K with $\mathbb{R}_{>0}\times\mathbb{R}^{n-1}$ by the map

$$(\lambda, v) \mapsto (\lambda, v)K = \{(\lambda k, kv) \mid k \in K\}$$

So in particular K is identified with $(1,0)\in\mathbb{R}_{>0}\times\mathbb{R}^{n-1}$. Let $X_0,\dots,X_{n-1}\in T(G_\infty/K)$ be the constant vector fields defined by $(X_0)_p=(1,0)$ and $(X_i)_p=(0,e_i)$ for all $p\in G_\infty/K$ and $i=1,\dots,n-1$. Then

$$\begin{aligned}
&d(\det)_K((X_0)_K,(X_1)_K,\dots,(X_{n-1})_K)\\
&d(\det)_{(1,0)}\left((1,0),(0,e_1),\dots,(0,e_{n-1})\right)\\
&=\frac{d}{dt}\Big|_{t=0}\det_{(1,0)+t(1,0)}\left((0,e_1),\dots,(e_{n-1},0)\right)\\
&+\sum_{i=1}^{n-1}\frac{d}{dt}\Big|_{t=0}\det_{(1,0)+t(0,e_i)}\Big((1,0),(0,e_1)\dots,\widehat{(0,e_i)},\dots,(0,e_{n-1})\Big)\\
&=\frac{d}{dt}\Big|_{t=0}\det_{(1+t,0)}\left((0,e_1),\dots,(0,e_{n-1})\right),
\end{aligned}$$

where the last equality follows from $\det\left((1,0),\cdot,\dots,\cdot\right)=0$
Since det is G_∞-invariant, i.e. invariant under taking the pullback by the action of G_∞ on G_∞/K, we have

$$\det_{(1+t,0)}\big((0,e_1),\dots,(0,e_{n-1})\big)=\det_{g\cdot(1+t,0)}\big(g\cdot(0,e_1),\dots,g\cdot(0,e_{n-1})\big),$$

for all $g\in G_\infty$. Let $g=(\frac{1}{1+t}I_{n-1},0)\in G_\infty$. Then

$$g\cdot(1+t,0)=(\frac{1}{1+t}I_{n-1}\cdot(1+t),0)=(1,0),$$

and

$$g\cdot(0,e_i)=(0,\frac{1}{1+t}I_{n-1}\cdot e_i)=(0,\frac{1}{1+t}e_i).$$

Hence

$$\begin{aligned}
\det_{(1+t,0)}\left((0,e_1),\dots,(0,e_{n-1})\right)=&\det_{g\cdot(1+t,0)}\left(g\cdot(0,e_1),\dots,g\cdot(0,e_{n-1})\right)\\
=&\det_{(1,0)}\left((0,\frac{1}{1+t}e_1),\dots,(0,\frac{1}{1+t}e_{n-1})\right)\\
=&\det\left(\frac{1}{1+t}e_1,\dots,\frac{1}{1+t}e_{n-1}\right)\\
=&\frac{1}{(1+t)^{n-1}}\det(e_1,\dots,e_{n-1})\\
=&\frac{1}{(1+t)^{n-1}}
\end{aligned}$$

It follows that

$$\left.\frac{d}{dt}\right|_{t=0} \det_{0+t(1,0)}((0,e_1),\ldots,(0,e_{n-1})) \quad = \quad \left.\frac{d}{dt}\right|_{t=0} \frac{1}{(1+t)^{n-1}} = 1-n$$

Since furthermore $T \wedge \det((1,0),(0,e_1),\ldots,(0,e_{n-1})) = 1$ the result follows. ☐

Lemma 4.3.4.

$$\mathrm{Alt}^q(\mathbb{R}^{n-1})^{\mathrm{SO}(n-1)} = \begin{cases} \mathbb{R}, & \textit{if } q = 0, n-1; \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. As before let $K = \mathrm{SO}(n-1)$. The alternating 0-forms are the constant maps and in top dimension $q = n-1$ there is (up to a constant) also a unique alternating K-invariant form. This form is given by the determinant and we will denote it by $\det \in \mathrm{Alt}^{n-1}(\mathbb{R}^{n-1})$. Furthermore, for $0 < q < n-1$ an alternating form $\omega \in \mathrm{Alt}^q(\mathbb{R}^{n-1})^K$ has to satisfy

$$\omega(kv_1,\ldots,kv_q) = \omega(v_1,\ldots,v_q)$$

for all $k \in K$ and all $v_i \in \mathbb{R}^{n-1}$. If $\omega \neq 0$ then there exists a set $\{w_1,\ldots,w_q\}$ of orthonormal vectors such that $\omega(w_1,\ldots,w_q) \neq 0$. Since $q < n-1$ there is a $k \in K$ such that $kw_1 = -w_1$ and $kw_j = w_j$ for $j = 2,\ldots,q$. Then

$$\begin{aligned} \omega(w_1,\ldots,w_q) &= \omega(kw_1,\ldots,kw_q) \\ &= \omega(-w_1,\ldots,w_q) \\ &= -\omega(w_1,\ldots,w_q), \end{aligned}$$

where the first equality follows from the K-invariance of ω and the last equality follows from its multi-linearity. So $\omega(w_1,\ldots,w_q) = 0$ therefore $\omega = 0$. It follows that $\mathrm{Alt}^q(\mathbb{R}^{n-1})^{\mathrm{SO}(n-1)} = 0$. ☐

4.3.2 Computation of ${}^IE_1^{1,q}$ and ${}^IE_1^{2,q}$.

The maximal compact subgroup of $\mathbb{R}_{>0} \times \mathrm{SO}(n-1)$ is $\mathrm{SO}(n-1)$ and thus

$$\begin{aligned} {}^IE_1^{1,q} &\cong H^q(\mathbb{R}_{>0} \times \mathrm{SO}(n-1);\mathbb{R}) \\ &\cong H^q\left(\Omega(\mathbb{R}_{>0})^{\mathrm{SO}(n-1)}\right) \cong \begin{cases} \mathbb{R}, & \text{if } q = 0,1; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Furthermore, since $\mathrm{SO}(n-2)$ is compact

$$\begin{aligned} E_1^{2,q} &\cong H^q(\mathrm{SO}(n-2);\mathbb{R}) \\ &\cong \begin{cases} \mathbb{R}, & \text{if q=0;} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

4.4 $d_1 : {}^I E_1^{0,1} \to {}^I E_1^{1,1}$ is an isomorphism

Recall that $G_\infty = (\mathbb{R}_{>0} \times \mathrm{SO}(n-1)) \ltimes \mathbb{R}^{n-1}$ and $G_{\infty,0} = \mathbb{R}_{>0} \times \mathrm{SO}(n-1)$. Let

$$j^* : H^1_m(G; C(G/G_\infty; \mathbb{R})) \to H^1_m(G; C(G/G_{\infty,0}; \mathbb{R})) \tag{4.5}$$

be the map induced by the natural surjection $G/G_{0,\infty} \twoheadrightarrow G/G_\infty$. By abuse of notation we will denote the map from $H^1_m(G; C(G/G_\infty; \mathbb{R}))$ to $H^1_m(G; C(G/G_{\infty,0}; \mathbb{R}))$ that is induced by the differential $d_1 : H^1_m(G; C^0) \to H^1_m(G; C^1)$ also by d_1. We will prove

Proposition 4.4.1. *The map*

$$d_1 : H^1_m(G; C(G/G_\infty; \mathbb{R})) \to H^1_m(G; C(G/G_{\infty,0}; \mathbb{R}))$$

is equal to $2j^*$.

This proves Proposition 4.2.2 since it implies in particular that d_1 is an isomorphism. Indeed, in degree 1

$$H^1_m(G_\infty; \mathbb{R}) \cong \mathrm{Hom}(G_\infty; \mathbb{R}) \cong \mathbb{R}$$

and a generator is given by the homomorphism $f_1 : G_\infty \to \mathbb{R}$ defined by

$$f_1(kA + v) = \ln(k),$$

where $kA + v \in G_\infty = (\mathbb{R}_{>0} \times \mathrm{SO}(n-1)) \ltimes \mathbb{R}^{n-1}$. Furthermore,

$$H^1_m(G_{\infty,0}; \mathbb{R}) \cong \mathrm{Hom}(G_{\infty,0}; \mathbb{R}) \cong \mathbb{R},$$

and a generator is given by the homomorphism $f_2 : G_{\infty,0} \to \mathbb{R}$ defined by

$$f_2(kA) = \ln(k), \tag{4.6}$$

for $kA \in G_{\infty,0} = \mathbb{R}_{>0} \times \mathrm{SO}(n-1)$. Under the Eckmann-Shapiro Lemma the map j^* from equation 4.5 corresponds to the map $i^* : H^1_m(G_\infty; \mathbb{R}) \to H^1_m(G_{\infty,0}; \mathbb{R})$ induced by the natural inclusion $i : G_{\infty,0} \hookrightarrow G_\infty$. This is an isomorphism as it sends f_1 to f_2. Let $J \in \mathrm{Isom}^+(\mathbb{H}^n)$ be a rotation by π centered on a point on the geodesic between 0 and ∞ so that $J(0) = \infty$, $J(\infty) = 0$ and $J^{-1} = J$. For an explicit formula of such a rotation see below. Let

$$J^* : H^1_m(G; C(G/G_{\infty,0}; \mathbb{R})) \to H^1_m(G; C(G/G_{\infty,0}; \mathbb{R})) \tag{4.7}$$

be the isomorphism defined on cochains by

$$J^*(\alpha)(g_0, g_1)(gG_{\infty,0}) = \alpha(g_0, g_1)(gJG_{\infty,0}),$$

for $\alpha \in C(G^2; C(G/G_{\infty,0};\mathbb{R}))^G$ and $g_0, g_1, g \in G$. Let

$$\psi_\infty : C(G^2; C(G/G_\infty;\mathbb{R}))^G \to C(G^2; C(\partial\mathbb{H}^n;\mathbb{R}))^G$$

be the isomorphism defined by

$$\psi_\infty(\beta)(g_0, g_1)(x) = \beta(g_0, g_1)(gG_\infty),$$

for $\beta \in C(G^2; C(G/G_\infty;\mathbb{R}))^G$, $g_0, g_1 \in G$ and $x \in \partial\mathbb{H}^n$ and with $g \in G$ such that $g \cdot \infty = x$. Furthermore, let

$$\psi_{\infty,0} : C(G^2; C(\partial\mathbb{H}^n \times \partial\mathbb{H}^n;\mathbb{R}))^G \to C(G^2; C(G/G_{\infty,0};\mathbb{R}))^G$$

be the isomorphism defined by

$$\psi_{\infty,0}(\alpha)(g_0, g_1)(gG_{\infty,0}) = \alpha(g_0, g_1)(g \cdot 0, g \cdot \infty),$$

for $\alpha \in C(G^2; C(\partial\mathbb{H}^n \times \partial\mathbb{H}^n;\mathbb{R}))^G$ and $g_0, g_1, g \in G$. We have the commutative diagram

$$\begin{tikzcd}
C(G^2; C(G/G_\infty;\mathbb{R}))^G \arrow[r, "d_1"] \arrow[d, "\psi_\infty"] & C(G^2; C(G/G_{\infty,0};\mathbb{R}))^G \\
C(G^2; C^0)^G \arrow[r, "\delta"] & C(G^2; C^1)^G, \arrow[u, "\psi_{\infty,0}"']
\end{tikzcd}$$

and furthermore, with j^* and J^* the maps defined in equation 4.5 and equation 4.7 respectively, we obtain

Lemma 4.4.2. *$d_1 = j^* - J^* \circ j^*$.*

Proof. By definition the differential operator $d_1 : H^1_m(G; C^0) \to H^1_m(G; C^1)$ is induced by $\delta : C^0 \to C^1$, i.e. for $[\sigma] \in H^1_m(G; C^0)$, $g \in G$ and $x_0, x_1 \in \partial\mathbb{H}^n$

$$d_1[\sigma] = [\delta \circ \sigma],$$

where

$$(\delta \circ \sigma)(g_0, g_1)(x_0, x_1) = \sigma(g_0, g_1)(x_1) - \sigma(g_0, g_1)(x_0).$$

For $\sigma \in C(G^2; C(G/G_\infty;\mathbb{R}_{>0}))$ and $g_0, g_1, g \in G$ we have

$$\begin{aligned}
\psi_{\infty,0} \circ \delta \circ \psi_\infty(\sigma)(g_0, g_1)(gG_{\infty,0}) &= \delta \circ \psi_\infty(\sigma)(g_0, g_1)(g \cdot 0, g \cdot \infty) \\
&= \psi_\infty(\sigma)(g_0, g_1)(g \cdot \infty) \\
&\quad - \psi_\infty(g_0, g_1)(g \cdot 0) \\
&= \sigma(g_0, g_1)(gG_\infty) - \sigma(g_0, g_1)(g \cdot JG_\infty) \\
&= j^*(\sigma)(g_0, g_1)(gG_{\infty,0}) \\
&\quad - J^* \circ j^*(\sigma)(g_0, g_1)(gG_{\infty,0}),
\end{aligned}$$

and it thus follows that $d_1 = j^* - J^* \circ j^*$. ☐

Since continuous cohomology and measurable cohomology coincide for $\mathbb{R}$-coefficients we can and will from now on work with continuous cochains. For such cochains an isomorphism $\varphi : C_c(G^2_{\infty,0};\mathbb{R})^{G_{\infty,0}} \to C_c(G_{\infty,0};\mathbb{R})$ between the homogeneous and inhomogeneous resolution is given by

$$\begin{aligned} \varphi(\beta)(g) &= \beta(e,g), \text{ with inverse} \\ \varphi^{-1}(\sigma)(g_0,g_1) &= \sigma(g_0^{-1}g_1). \end{aligned}$$

Lemma 4.4.3. *J^* acts as -1 on $H^1_m(G_{\infty,0};\mathbb{R})$.*

Proof. Let $s : G/G_{\infty,0} \to G$ be a Borel section such that $s(G_{\infty,0}) = e$. Let $\alpha \in \operatorname{Hom}(G_{\infty,0};\mathbb{R})$ be a cocycle, $h_1 \in G_{\infty,0}$, and $g \in G$. Let ϕ and ψ be the maps defined in Example 2.4.5. Then

$$\begin{aligned} &\varphi \circ \phi \circ J^* \circ \psi \circ \varphi^{-1}(\alpha)(h_1) \\ &= \phi \circ J^* \circ \psi \circ \varphi^{-1}(\alpha)(e,h_1) \\ &= J^* \circ \psi \circ \varphi^{-1}(\alpha)(s(gG_{\infty,0}), s(gG_{\infty,0})h_1)(gG_{\infty,0}) \\ &= \psi \circ \varphi^{-1}(\alpha)(s(gG_{\infty,0}), s(gG_{\infty,0})h_1)(gJG_{\infty,0}) \\ &= \varphi^{-1}(\lambda(s(gG_{\infty,0}), gJG_{\infty,0}), \lambda(s(gG_{\infty,0})h_1, gJG_{\infty,0})) \\ &= \alpha(\lambda(s(gG_{\infty,0}), gJG_{\infty,0})^{-1} \cdot \lambda(s(gG_{\infty,0})h_1, gJG_{\infty,0})) \\ &= \alpha(s[s(gG_{\infty,0})^{-1}gJG_{\infty,0}]^{-1} \cdot h_1 \cdot s[(s(gG_{\infty,0})h_1)^{-1}gJG_{\infty,0}]) \\ &= \alpha(s(JG_{\infty,0})^{-1} \cdot h_1 \cdot s(JG_{\infty,0})) \\ &= \alpha(h_2^{-1}J^{-1}h_1Jh_2) \\ &= \alpha(Jh_1J), \end{aligned}$$

where $h_2 = Js(JG_{\infty,0}) \in G_{\infty,0}$ and we use that $s(hJG_{\infty,0}) = s(JG_{\infty,0})$ for all $h \in G_{\infty,0}$. Thus J^* acts by conjugation on $H^1_m(G_{\infty,0};\mathbb{R}_{>0})$. In the upper half space model a possible formula for J is

$$J : (x_1,\ldots,x_n) \mapsto \frac{1}{|x|^2}(x_1,\ldots,x_{n-2},-x_{n-1},x_n) = \frac{1}{|x|^2}r(x),$$

for $x = (x_1,\ldots,x_n) \in \overline{\mathbb{H}^n} \setminus \{\infty\}$ and where r is the reflection in the hyperplane orthogonal to the $(n-1)$th coordinate axis. Let $g \in G_{\infty,0}$, say $g = kA$ with $k > 0$ and $A \in \mathrm{SO}(n-1)$ and let $x \in \overline{\mathbb{H}^n} \setminus \{\infty\}$. Then

$$\begin{aligned} JgJ(x) &= J \cdot \frac{k}{|x|^2} A \cdot r(x) \\ &= \frac{1}{\left|\frac{k}{|x|^2} \cdot |A \cdot r(x)|\right|^2} \cdot \frac{k}{|x|^2} rAr(x) \\ &= \frac{1}{\left|\frac{k}{|x|^2} \cdot |x|\right|^2} \cdot \frac{k}{|x|^2} rAr(x) \\ &= \frac{1}{k} rAr(x), \end{aligned}$$

Hence for the generator f_2 defined above in equation 4.6 we obtain that $f_2(JgJ) = -f_2(g)$. □

Together Lemma 4.4.2 and Lemma 4.4.3 imply Proposition 4.4.1.

4.5 Vanishing of ${}^IE_2^{p,q}$ for $p > 2$ and $q > 0$

In this section we give the proof of Proposition 4.2.3. For a Polish Abelian G-module A let $K^{p,q}(A) = C(G^{q+1}, C((\partial\mathbb{H}^n)^{p+1}; A))^G$.

Proposition 4.5.1. *Let $[[\alpha]_d]_\delta \in H_\delta H_d(K^{p,q}(A))$ with A an Abelian G-module. Then there exists a locally totally bounded representative κ of $[[\alpha]_d]_\delta$. That is, there exist $\sigma : G^{q+1} \times (\partial\mathbb{H}^n)^p \to A$ and $\lambda : G^q \times (\partial\mathbb{H}^n)^{p+1} \to A$ such that*

$$\kappa = \alpha + \delta\sigma + d\lambda : G^{q+1} \times (\partial\mathbb{H}^n)^{p+1} \to A$$

is locally totally bounded.

Proof. We will prove Proposition 4.5.1 by induction on q while allowing the module A to vary, thereby proving it for all Abelian G-modules A. Suppose $q = 0$. Then $d\alpha = 0$ implies that the cocycle $\alpha : G \to C((\partial\mathbb{H}^n)^{p+1}; \mathbb{R})$ is a constant function into $C((\partial\mathbb{H}^n)^{p+1}; A)^G$. We identify α with this element of $C((\partial\mathbb{H}^n)^{p+1}; A)^G$ and it thus follows from Proposition 4.2.4 that Proposition 4.5.1 holds in degree $q = 0$ for all Polish Abelian G-modules A.
Suppose now that $q > 0$ and that for $q' < q$ the proposition is true for all Polish Abelian G-modules A. Let $\alpha \in C(G^{q+1}; C((\partial\mathbb{H}^n)^{p+1}; A))^G$ be such that $d\alpha = 0$ and $\delta\alpha = d\gamma$, where $\gamma : G^q \times (\partial\mathbb{H}^n)^{p+1} \to A$. Define the function $Q\alpha$ in $C(G^q; C((\partial\mathbb{H}^n)^{p+1}; C(G; A)))^G$ by

$$Q\alpha(g_0, \dots, g_{q-1})(x_0, \dots, x_p)(g) := (-1)^{q+1}\alpha(g_0, \dots, g_{q-1}, g)(x_0, \dots, x_p).$$

Then

$$\begin{aligned}
d(Q\alpha)(g_0, \dots, g_q)(x_0, \dots, &x_p)(g) \\
&= \sum_{i=0}^{q} (-1)^i Q\alpha(g_0, \dots, \hat{g}_i, \dots, g_q)(x_0, \dots, x_p)(g) \\
&= \sum_{i=0}^{q} (-1)^{i+q+1}\alpha(g_0, \dots, \hat{g}_i, \dots, g_q, g)(x_0, \dots, x_p) \\
&= (-1)^{q+1} d\alpha(g_0, \dots, g_q, g)(x_0, \dots, x_p) \\
&\quad - (-1)^{2q+1}\alpha(g_0, \dots, g_q)(x_0, \dots, x_p) \\
&= \alpha(g_0, \dots, g_q)(x_0, \dots, x_p),
\end{aligned}$$

and we see that $d(Q\alpha)$ takes its values in $\iota(A)$ and therefore the image $\overline{Q\alpha}$ of $Q\alpha$ in $C(G^q; C((\partial\mathbb{H}^n)^{p+1}; C(G;A)/\iota(A)))^G$ is a cocycle with respect to the coboundary operator d. Furthermore,

$$\begin{aligned}
&\delta(Q\alpha)(g_0,\dots,g_{q-1})(x_0,\dots,x_{p+1})(g)\\
&\qquad\qquad\qquad=(-1)^{q+1}\delta\alpha(g_0,\dots,g_{q-1},g)(x_0,\dots,x_{p+1})\\
&\qquad\qquad\qquad=(-1)^{q}d\gamma(g_0,\dots,g_{q-1},g)(x_0,\dots,x_{p+1})\\
&\qquad\qquad\qquad=(-1)^{q}\gamma(g_0,\dots,g_{q-1})(x_0,\dots,x_{p+1})\\
&\qquad\qquad\qquad\quad\ (-1)^{q}dQ\gamma(g_0,\dots,g_{q-1})(x_0,\dots,x_{p+1})(g).
\end{aligned}$$

Hence $\overline{Q\alpha} \in C(G^q; C((\partial\mathbb{H}^n)^{p+1}; C(G;A)/\iota(A)))^G$ represents a cohomology class in $H_\delta H_d(K^{p,q-1}(C(G;A)/\iota(A)))$ and so by the induction hypothesis

$$\overline{Q\alpha} = \bar\beta + \delta\bar\mu + d\bar\nu,$$

where $\bar\beta$ is a locally totally bounded cocycle. Then, by Lemma 2.3.2, there exist G-invariant measurable lifts β, μ and ν of these maps such that

$$Q\alpha = \beta + \delta\mu + d\nu + \eta,$$

with $\eta : G^q \times (\partial\mathbb{H}^n)^{p+1} \to \iota(A)$ and such that β is still locally totally bounded. We obtain

$$\alpha = d\beta + d\delta\mu + d\eta.$$

Note that since the left-hand side takes values in $\iota(A)$ the right-hand side does as well. Hence, since $d\beta$ and $d\eta$ both take values in $\iota(A)$, $d\delta\mu$ can be identified with a coboundary in $C(G^{q+1}; C((\partial\mathbb{H}^n)^{p+1}; A))^G$. It follows that $\kappa := d\beta$ is a locally totally bounded representative of the class of α in $H_\delta H_d(K^{p,q}(A))$. $\square$

Let $K_c^{p,q}(A)$ be the G-module $C_c(G^{q+1}; C((\partial\mathbb{H}^n)^{p+1}; A))^G$ where A is from now on a Fréchet G-module and let $K_c^{p,q} = K_c^{p,q}(\mathbb{R})$. Then

Proposition 4.5.2. $H_\delta H_d(K_c^{p,q}(A)) \cong H_\delta H_d(K^{p,q}(A))$.

Proof. By Proposition 4.5.1 any cohomology class $[[\alpha]_d]_\delta \in H_\delta H_d(K^{p,q}(A))$ has a locally totally bounded representative κ. Then, as in the proof of Theorem 2.5, such a cocycle is effaced by the inclusion $A \hookrightarrow C_c(G;A)$ and the result follows by Buchsbaum's criterion. More precisely, there exists an $\eta : G^p \times (\partial\mathbb{H}^n)^{q+1} \to C_c(G;A)$ s.t. $d\eta = \kappa$ where κ is viewed as a map $G^{p+1} \times (\partial\mathbb{H}^n)^{q+1} \to C_c(G;A)$ taking values in $\iota(A) \subset C_c(G;A)$. For example, we can define η by

$$\begin{aligned}
&\eta(g_0,\dots,g_{p-1})(x_0,\dots,x_q)(g) :=\\
&(-1)^p \int_G \kappa(g_0,\dots,g_{p-1},gh)(x_0,\dots,x_q)\xi(h)d\mu_G(h),
\end{aligned}$$

where $g_0,\dots,g_{p-1},g\in G$, $x_0,\dots,x_q\in\partial\mathbb{H}^n$ and $\xi: G\to\mathbb{R}_{>0}$ is a compactly-supported continuous function with $\int_G \xi d\mu_G = 1$. Since furthermore

$$H_\delta H_d(K^{p,0}(A)) = H_\delta H_d(K_c^{p,0}(A)) = H_\delta((C((\partial\mathbb{H}^n)^{p+1};A))^G)$$

and there exist long exact sequences in the case of Fréchet modules for both, Buchsbaum's criterion applies. □

Let $K=\mathrm{SO}(n-2)$ and let $s: G/K\to G$ be a locally totally bounded Borel section such that $s(K)=e$. By the Eckmann-Shapiro Lemma we have the isomorphism $H_m^q(G;C(G/K;C^{p-3}))\cong H_m^q(K;C^{p-3})$. From Proposition 4.5.2 it follows that if we restrict to cocycles in $H_\delta H_d(K^{p,q})$ we can assume them to be continuous in G^{q+1}. As discussed in Remark 2.4.5 we then get an explicit map $\phi: C_c(G^{q+1};C(G/K;C^{p-3}))^G\to C_c(K^{q+1};C^{p-3})^K$ that induces the isomorphism

$$H_\delta H_d(C(G^{q+1};C(G/K;C^{p-3}))^G)\cong H_\delta H_d(C(K^{q+1};C^{p-3})^K).$$

Proof of Proposition 4.2.3. Let $[[\alpha]_d]_\delta\in H_\delta H_d(K_c^{p,q}) = E_2^{p,q}$. We will show that in this case α is cohomologous in $H_\delta H_d(K_c^{p,q})$ to a coboundary in $H_d(K^{p,q})$. By Proposition 4.5.1, α has a locally totally bounded representative

$$\beta: G^{q+1}\times(\partial\mathbb{H}^n)^{p+1}\to\mathbb{R}.$$

Then $\phi(\beta): K^{q+1}\times(\partial\mathbb{H}^n)^{p-2}\to\mathbb{R}$ is also a locally totally bounded cocyle and furthermore we have $\phi(\beta)=d\eta$, where $\eta: K^q\times(\partial\mathbb{H}^n)^{p-2}\to\mathbb{R}$ is defined by

$$\begin{aligned}&\eta(k_0,\dots,k_{q-1})(x_0,\dots,x_{p-3}) :=\\ &(-1)^q\int_K \phi(\beta)(k_0,\dots,k_{q-1},k)(x_0,\dots,x_{p-3})d\mu_K(k).\end{aligned}$$

It follows that ${}^IE_2^{p,q}=0$ for $p>2$ and $q>0$. □

Chapter 5

Injectivity and stability results

5.1 Injectivity of the comparison map

An immediate consequence of Theorem 4.1.1 is

Corollary 5.1.1. *The comparison map from continuous bounded cohomology to continuous cohomology for real hyperbolic space $\mathbb{H}^n$ is injective in degree 3, i.e.*

$$c : H^3_{c,b}(\mathrm{Isom}^+(\mathbb{H}^n);\mathbb{R}) \hookrightarrow H^3_c(\mathrm{Isom}^+(\mathbb{H}^n);\mathbb{R}).$$

Proof. Let $G := \mathrm{Isom}^+(\mathbb{H}^n)$. By Theorem 4.1.1 we have

$$H^3_c(G;\mathbb{R}) = \frac{\ker(\delta : C((\partial\mathbb{H}^n)^4;\mathbb{R})^G \to C((\partial\mathbb{H}^n)^5;\mathbb{R})^G)}{\mathrm{im}(\delta : C((\partial\mathbb{H}^n)^3;\mathbb{R})^G \to C((\partial\mathbb{H}^n)^4;\mathbb{R})^G)}.$$

Furthermore, the continuous bounded cohomology of G can also be calculated with maps that are defined on the boundary of hyperbolic space. That is,

$$H^3_{c,b}(\mathrm{Isom}^+(\mathbb{H}^n);\mathbb{R}) = \frac{\ker(\delta : L^\infty((\partial\mathbb{H}^n)^4;\mathbb{R})^G \to L^\infty((\partial\mathbb{H}^n)^5;\mathbb{R})^G)}{\mathrm{im}(\delta : L^\infty((\partial\mathbb{H}^n)^3;\mathbb{R})^G \to L^\infty((\partial\mathbb{H}^n)^4;\mathbb{R})^G)},$$

where $L^\infty((\partial\mathbb{H}^n)^p;A) \subset C((\partial\mathbb{H}^n)^p;A)$ consists of essentially bounded measurable function classes. By 3-transitivity of the action of G on the boundary of hyperbolic space it then follows that there are no coboundaries in degree 3. Hence $H^3_c(G;\mathbb{R})$ and $H^3_{c,b}(G;\mathbb{R})$ are equal to the corresponding spaces of cocycles and it follows that the comparison map is injective. □

Injectivity in degree 3 for $\mathrm{Isom}^+(\mathbb{H}^n)$ also follows from a simpler argument which only uses some basic properties of hyperbolic space and the injectivity in degree 3 for $n = 3$. Denote by $\mathbb{R}_\epsilon$ the $\mathrm{Isom}(\mathbb{H}^n)$-module $\mathbb{R}$ with $\mathrm{Isom}(\mathbb{H}^n)$-action given by the homomorphism

$$\epsilon : \mathrm{Isom}(\mathbb{H}^n) \to \mathrm{Isom}(\mathbb{H}^n)/\mathrm{Isom}^+(\mathbb{H}^n) \cong \{1,-1\}.$$

Furthermore, let

$$p : C_{c(,b)}((\mathbb{H}^n)^{k+1};\mathbb{R})^{\mathrm{Isom}^+(\mathbb{H}^n)} \to C_{c(,b)}((\mathbb{H}^n)^{k+1};\mathbb{R})^{\mathrm{Isom}(\mathbb{H}^n)}$$

and

$$\bar{p}: C_{c(,b)}((\mathbb{H}^n)^{k+1};\mathbb{R})^{\mathrm{Isom}^+(\mathbb{H}^n)} \to C_{c(,b)}((\mathbb{H}^n)^{k+1};\mathbb{R}_\epsilon)^{\mathrm{Isom}(\mathbb{H}^n)}$$

be the maps defined for $x_0,\dots,x_k \in \mathbb{H}^n$ and $\beta \in C_{c(,b)}((\mathbb{H}^n)^{k+1};\mathbb{R})^{\mathrm{Isom}^+(\mathbb{H}^n)}$ by

$$\begin{aligned} p(\beta)(x_0,\dots,x_k) &= \frac{1}{2}[\beta(x_0,\dots,x_k)+\beta(\tau x_0,\dots,\tau x_k)], \\ \bar{p}(\beta)(x_0,\dots,x_k) &= \frac{1}{2}[\beta(x_0,\dots,x_k)-\beta(\tau x_0,\dots,\tau x_k)], \end{aligned}$$

where $\tau \in \mathrm{Isom}(\mathbb{H}^n)\setminus\mathrm{Isom}^+(\mathbb{H}^n)$ is any orientation reversing symmetry. Then the cochain map $(p,\bar{p})$ induces an isometric isomorphism

$$H^*_{c(,b)}(\mathrm{Isom}^+(\mathbb{H}^n);\mathbb{R}) \cong H^*_{c(,b)}(\mathrm{Isom}(\mathbb{H}^n);\mathbb{R}) \oplus H^*_{c(,b)}(\mathrm{Isom}(\mathbb{H}^n);\mathbb{R}_\epsilon).$$

Also, by Bloch's result, $H^3_{c(,b)}(\mathrm{Isom}^+(\mathbb{H}^3);\mathbb{R})$ is generated by the volume cocycle which is equivariant, i.e. in $H^3_{c(,b)}(\mathrm{Isom}(\mathbb{H}^3),\mathbb{R}_\epsilon)$, and therefore $H^3_{c(,b)}(\mathrm{Isom}(\mathbb{H}^3);\mathbb{R}) = 0$. On the other hand, for $n>3$ we have that $H^3_{c(,b)}(\mathrm{Isom}(\mathbb{H}^n),\mathbb{R}_\epsilon)=0$ (see [BucBurIoz13]). Thus it follows that

$$H^3_{c(,b)}(\mathrm{Isom}^+(\mathbb{H}^n);\mathbb{R}) = H^3_{c(,b)}(\mathrm{Isom}(\mathbb{H}^n);\mathbb{R}).$$

Lemma 5.1.2. *Let $n>3$. Then $H^3_{c,b}(\mathit{Isom}(\mathbb{H}^n);\mathbb{R})=0$.*

Proof. Let $[\beta] \in H^3_{c,b}(\mathrm{Isom}(\mathbb{H}^n);\mathbb{R})$, so $\beta: (\mathbb{H}^n)^4 \to \mathbb{R}$ is an $\mathrm{Isom}(\mathbb{H}^n)$-invariant cocycle. Let $i: \mathbb{H}^3 \hookrightarrow \mathbb{H}^n$ be the natural embedding. We will identify the image $i(\mathbb{H}^3) \subset \mathbb{H}^n$ with $\mathbb{H}^3$. Now $\beta|_{\mathbb{H}^3} \in C_{c,b}(\mathbb{H}^3;\mathbb{R})^{\mathrm{Isom}(\mathbb{H}^3)}$ is a cocycle (an isometry of $\mathbb{H}^3$ can be extended to an isometry of $\mathbb{H}^n$.) It follows by assumption that there is an $\alpha \in C_{c,b}((\mathbb{H}^3)^3;\mathbb{R})^{\mathrm{Isom}(\mathbb{H}^3)}$ such that $\delta\alpha = \beta|_{\mathbb{H}^3}$.
Let $x_0,x_1,x_2 \in \mathbb{H}^n$. These points lie in a 2-dimensional hyperplane which we denote by $H(x_0,x_1,x_2)$. Since $\mathrm{Isom}(\mathbb{H}^n)$ acts transitively on such hyperplanes there always exists a $g \in \mathrm{Isom}(\mathbb{H}^n)$ such that $g(H(x_0,x_1,x_2)) \subset \mathbb{H}^3$. Define $\bar{\alpha}: (\mathbb{H}^n)^3 \to \mathbb{R}$ by

$$\bar{\alpha}(x_0,x_1,x_2) = \alpha(gx_0,gx_1,gx_2),$$

where $g \in \mathrm{Isom}(\mathbb{H}^n)$ such that $g(H(x_0,x_1,x_2)) \subset \mathbb{H}^3$. This is well defined, $\bar{\alpha} \in C_{c,b}((\mathbb{H}^n)^3;\mathbb{R})^{\mathrm{Isom}(\mathbb{H}^n)}$ and $\delta\bar{\alpha}=\beta$:

- $\bar{\alpha}$ is well defined: Suppose that $G; g' \in \mathrm{Isom}(\mathbb{H}^n)$ are both such that $g(H(x_0,x_1,x_2)) \subset \mathbb{H}^3$ and $g'(H(x_0,x_1,x_2)) \subset \mathbb{H}^3$. Then $g'g^{-1}|_{\mathbb{H}^3} \in \mathrm{Isom}(\mathbb{H}^3)$ and hence

$$\begin{aligned} \alpha(gx_0,gx_1,gx_2) &= \alpha(g'g^{-1}gx_0, g1g^{-1}gx_1, g'g^{-1}gx_2) \\ &= \alpha(g'x_0,g'x_1,g'x_2) \end{aligned}$$

- $\bar{\alpha} \in C_{c,b}((\mathbb{H}^n)^3;\mathbb{R})^{\mathrm{Isom}(\mathbb{H}^n)}$: The only thing that needs to be checked is that $\bar{\alpha}$ is $\mathrm{Isom}(\mathbb{H}^n)$-invariant.
  Let $x_0, x_1, x_2 \in \mathbb{H}^n$ and $g \in \mathrm{Isom}(\mathbb{H}^n)$ such that $gH(x_0,x_1,x_2) \subset \mathbb{H}^3$ so that $\bar{\alpha}(x_0,x_1,x_2) = \alpha(gx_0,gx_1,gx_2)$. Let g' be another element of $\mathrm{Isom}(\mathbb{H}^n)$. Then $gg'^{-1}(H(g'x_0,g'x_1,g'x_2)) \subset \mathbb{H}^3$ and thus

$$\begin{aligned} \bar{\alpha}(g'x_0,g'x_1,g'x_2) &= \alpha(gg'^{-1}g'x_0, gg'^{-1}x_1, gg'^{-1}g'x_2) \\ &= \alpha(gx_0,gx_1,gx_2) \\ &= \bar{\alpha}(x_0,x_1,x_2) \end{aligned}$$

- $\delta\bar{\alpha} = \beta$: Let $x_0,x_1,x_2,x_3 \in \mathbb{H}^n$ and $g \in \mathrm{Isom}(\mathbb{H}^n)$ such that

$$g(H(x_0,x_1,x_2,x_3)) \subset \mathbb{H}^3.$$

  Then

$$\begin{aligned} \beta(x_0,x_1,x_2,x_3) &= \beta(gx_0,gx_1,gx_2,gx_3) \\ &= \beta|_{i(\mathbb{H}^3)}(gx_0,gx_1,gx_2,gx_3) \\ &= \delta\alpha(gx_0,gx_1,gx_2,gx_3) \\ &= \delta\bar{\alpha}(x_0,x_1,x_2,x_3) \end{aligned}$$

It follows that $H^3_{c,b}(\mathrm{Isom}(\mathbb{H}^n);\mathbb{R}) = 0$. □

5.2 Stability results

The proof of Lemma 5.1.2 only uses the following two facts:

1. An isometry of $\mathbb{H}^n$ can be extended to an isometry of $\mathbb{H}^{n+1}$.
2. If $x_0,\dots,x_k \in \mathbb{H}^{n+1}$ where $k+1 \leq n$ then there exists a $g \in \mathrm{Isom}(\mathbb{H}^{n+1})$ such that $gH(x_0,\dots,x_k) \subset \mathbb{H}^n$ where $H(x_0,\dots,x_k)$ denotes the linear subspace spanned by the points $x_0,\dots,x_k$.

These also hold in complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$. Hence we immediately obtain the following lemma.

Lemma 5.2.1. *Let $k \leq n$ and suppose that $H^k_{c,b}(\mathrm{Isom}(\mathbb{H}^n_{(\mathbb{C})});\mathbb{R}) = 0$. Then*

$$H^k_{c,b}(\mathrm{Isom}(\mathbb{H}^{n+1}_{(\mathbb{C})});\mathbb{R}) = 0.$$

By a similar proof we furthermore obtain

Theorem 5.2.2. *If $k+1 \leq n$ then there exists an injection*

$$H^k_{c,b}(\mathrm{Isom}(\mathbb{H}^n_{(\mathbb{C})});\mathbb{R}) \hookrightarrow H^k_{c,b}(\mathrm{Isom}(\mathbb{H}^{n+1}_{(\mathbb{C})});\mathbb{R}).$$

Proof. On cochains define

$$j : C((\mathbb{H}^n_{(\mathbb{C})})^{k+1};\mathbb{R})^{\mathrm{Isom}(\mathbb{H}^n_{(\mathbb{C})})} \to C((\mathbb{H}^{n+1}_{(\mathbb{C})})^{k+1};\mathbb{R})^{\mathrm{Isom}(\mathbb{H}^{n+1}_{(\mathbb{C})})}$$

by

$$j(\beta)(x_0,\dots,x_k) := \beta(gx_0,\dots,gx_k),$$

where $g \in \mathrm{Isom}(\mathbb{H}^n_{(\mathbb{C})})$ is such that $g(H(x_0,\dots,x_k)) \subset i(\mathbb{H}^n_{(\mathbb{C})})$. Then

$$\delta(j(\beta))(x_0,\dots,x_{k+1}) = \delta\beta(g'x_0,\dots,g'x_{k+1}),$$

with $g' \in \mathrm{Isom}(\mathbb{H}^{n+1}_{(\mathbb{C})})$ such that $g'(H(x_0,\dots,x_{k+1})) \subset \mathbb{H}^n_{(\mathbb{C})}$. Note that such a g' exists since $k+1 \leq n$. It follows that if β is a cocycle then $j(\beta)$ is as well. Let furthermore

$$r : C((\mathbb{H}^{n+1}_{(\mathbb{C})})^{k+1};\mathbb{R})^{\mathrm{Isom}(\mathbb{H}^{n+1}_{(\mathbb{C})})} \to C((\mathbb{H}^n_{(\mathbb{C})})^{k+1};\mathbb{R})^{\mathrm{Isom}(\mathbb{H}^n_{(\mathbb{C})})}$$

be the map defined by restricting a k-cochain to $(\mathbb{H}^n_{(\mathbb{C})})^{k+1}$. Then for a cocycle β in degree k

$$\begin{aligned} r \circ j(\beta)(x_0,\dots,x_k) &= j(\beta)(x_0,\dots,x_k) \\ &= \beta(x_0,\dots,x_k), \end{aligned}$$

for all $x_0,\dots,x_k \in \mathbb{H}^n_{(\mathbb{C})}$. It follows that j induces an injective map on cohomology. □

Part II

Chapter 6

Complex hyperbolic geometry

In this chapter we give a brief introduction to complex hyperbolic geometry, mostly restricting to the complex hyperbolic plane. For more details on this subject we refer to [Gol99] and [Par10]. Denote by $\mathbb{C}^{n,1}$ the complex vector space $\mathbb{C}^{n+1}$ equipped with a Hermitian form of signature $(n,1)$. A standard choice for the Hermitian matrix defining the Hermitian form $\langle\cdot,\cdot\rangle$ is

$$J_1 := \begin{bmatrix} 1 & \cdots & 0 & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \\ 0 & \cdots & 0 & -1 \end{bmatrix}.$$

Since $\langle \mathbf{z},\mathbf{z}\rangle$ is real for all $\mathbf{z} \in \mathbb{C}^{n,1}$ we can define the following subsets of $\mathbb{C}^{n,1}$:

$$\begin{aligned} V_- &:= \left\{\mathbf{z} \in \mathbb{C}^{n,1} \middle| \langle \mathbf{z},\mathbf{z}\rangle < 0\right\}, \\ V_0 &:= \left\{\mathbf{z} \in \mathbb{C}^{n,1} \setminus \{0\} \middle| \langle \mathbf{z},\mathbf{z}\rangle = 0\right\}, \\ V_+ &:= \left\{\mathbf{z} \in \mathbb{C}^{n,1} \middle| \langle \mathbf{z},\mathbf{z}\rangle > 0\right\}. \end{aligned}$$

Denote by $\mathbb{P}$ the canonical projection of $\mathbb{C}^{n,1}\setminus\{0\}$ onto $\mathbb{C}P^n$. The *projective model* of complex hyperbolic n-space $\mathbb{H}^n_{\mathbb{C}}$ is then defined to be $\mathbb{P}(V_-)$ and the boundary, $\partial\mathbb{H}^n_{\mathbb{C}}$, is defined to be $\mathbb{P}(V_0)$. Thus

$$\mathbb{H}^n_{\mathbb{C}} = \left\{ z \in \mathbb{P}(\mathbb{C}^{n,1}\setminus\{0\}) \middle| \sum_{i=1}^{n} |\mathbf{z}_i|^2 - |\mathbf{z}_{n+1}|^2 < 0 \right\},$$

and

$$\partial\mathbb{H}^n_{\mathbb{C}} = \left\{ z \in \mathbb{P}(\mathbb{C}^{n,1}\setminus\{0\}) \middle| \sum_{i=1}^{n} |\mathbf{z}_i|^2 - |\mathbf{z}_{n+1}|^2 = 0 \right\},$$

where $\mathbf{z} \in \mathbb{C}^{n,1}$ can be any lift of $z \in \mathbb{P}(\mathbb{C}^{n,1}\setminus\{0\})$ since the sign of $\langle \mathbf{z},\mathbf{z}\rangle$ does not depend on the chosen lift. In general we take for $\mathbf{z}$ the standard lift

$$\begin{bmatrix} \mathbf{z}_1 \\ \vdots \\ \mathbf{z}_n \\ 1 \end{bmatrix}.$$

This $\mathbf{z}_{n+1} = 1$ section of V_- gives the *Ball model* of complex hyperbolic space. Concretely, denote by $\langle\langle\ ,\ \rangle\rangle$ the standard positive definite Hermitian inner product of $\mathbb{C}^n$, i.e. $\langle\langle z, w\rangle\rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n}$. Complex hyperbolic n-space $\mathbb{H}^n_{\mathbb{C}}$ can then be identified with the unit ball $\mathbb{B}^n = \{z \in \mathbb{C}^n \mid \langle\langle z, z\rangle\rangle < 1\}$ as follows: Let $A : \mathbb{C}^n \to \mathbb{P}(\mathbb{C}^{n,1})$ be defined by

$$A : z \mapsto \begin{bmatrix} z \\ 1 \end{bmatrix}.$$

A identifies $\mathbb{B}^n$ with $\mathbb{H}^n_{\mathbb{C}}$ and $\partial\mathbb{B}^n = S^{2n-1} \subset \mathbb{C}^n$ with $\partial\mathbb{H}^n_{\mathbb{C}}$.

Yet another model is the *Siegel domain model.* Consider

$$J_2 = \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \\ 0 & 1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & 0 \\ 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}.$$

This is again a Hermitian matrix and the Siegel domain $\mathfrak{h}^n$ is the section defined by $\mathbf{z}_{n+1} = 1$ for the corresponding Hermitian form $\langle\cdot,\cdot\rangle_2$. Hence

$$\mathfrak{h}^n = \left\{ z \in \mathbb{C}^n \mid 2\Re(z_1) + \sum_{i=2}^{n} |z_i|^2 < 0 \right\}.$$

and its boundary is

$$\mathcal{H} = \left\{ z \in \mathbb{C}^n \mid 2\Re(z_1) + \sum_{i=2}^{n} |z_i|^2 = 0 \right\}.$$

$\mathcal{H}$ is not the complete boundary of $\mathbb{H}^n_{\mathbb{C}}$, to compactify it we need to add a point at infinity, hence $\partial\mathbb{H}^n_{\mathbb{C}} = \mathcal{H} \cup \{\infty\}$. Note that in this model for $n = 1$ we obtain the left half-plane, contrary to the in real hyperbolic geometry customary upper half-plane. We can pass from the ball model to the Siegel domain via a Cayley transformation which is given by the matrix

$$C = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & \sqrt{2} & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & \sqrt{2} & 0 \\ 1 & 0 & \cdots & 0 & -1 \end{bmatrix},$$

which interchanges the first with the second Hermitian form, i.e. $J_2 = CJ_1C$.

The metric on $\mathbb{H}^n_{\mathbb{C}}$ is defined by

$$\cosh^2\left(\frac{1}{2}d(z, w)\right) = \frac{\langle\mathbf{z},\mathbf{w}\rangle\langle\mathbf{w},\mathbf{z}\rangle}{\langle\mathbf{z},\mathbf{z}\rangle\langle\mathbf{w},\mathbf{w}\rangle},$$

with $d(\cdot,\cdot)$ the distance function. Here $\langle\cdot,\cdot\rangle$ denotes either of the Hermitian forms defined above, depending on the model. For the ball model and the Siegel domain $\mathbf{z}$ and $\mathbf{w}$ can be the standard lifts of z, w but any other lifts will give the same result. On $\mathbb{B}^n$ this metric is the *Bergman metric*. With this scaling $\mathbb{H}^n_{\mathbb{C}}$ has holomorphic sectional curvature -1, with real sectional curvature pinched between -1 and $-1/4$.

From the metric it is clear that $\mathrm{PU}(n,1)$ acts isometrically on $\mathbb{H}^n_{\mathbb{C}}$ and thus $\mathrm{PU}(n,1)$ is a subgroup of $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{C}})$. Also, coordinate-wise complex conjugation $z \mapsto \overline{z}$ clearly leaves the metric invariant. In fact, the isometry group of $\mathbb{H}^n_{\mathbb{C}}$ is generated by $\mathrm{PU}(n,1)$ and complex conjugation. The holomorphic isometry group of $\mathbb{H}^n_{\mathbb{C}}$ is $\mathrm{PU}(n,1)$ while every anti-holomorphic isometry is given by complex conjugation followed by an element of $\mathrm{PU}(n,1)$ (for a proof see Theorem 3.5 in [Par10]).

6.1 Totally geodesic subspaces

There are two types of totally geodesic subspaces of real dimension 2 in $\mathbb{H}^2_{\mathbb{C}}$ [Gol99, Section 3.1.11]:

1. Complex lines, i.e. intersections of complex lines in $\mathbb{C}P^2$ with $\mathbb{H}^2_{\mathbb{C}}$. The boundary of a complex line in $\mathbb{H}^2_{\mathbb{C}}$ is called a $\mathbb{C}$-circle in $\partial\mathbb{H}^2_{\mathbb{C}}$.
2. Totally real Lagrangian planes, i.e. subspaces R of real dimension 2 such that $\langle\mathbf{v},\mathbf{w}\rangle \in \mathbb{R}$ for all $v, w \in R$. The boundary of a Lagrangian plane in $\mathbb{H}^2_{\mathbb{C}}$ is a $\mathbb{R}$-circle.

Together with (real) geodesics these are the only totally geodesic proper subspaces of $\mathbb{H}^2_{\mathbb{C}}$.

6.1.1 Complex lines

Let L be a complex line in $\mathbb{H}^2_{\mathbb{C}}$. For $\mathbf{v} \in \mathbb{C}^{2,1}$ denote by $\mathbf{v}^\perp$ its orthogonal complement, i.e.

$$\mathbf{v}^\perp := \{\mathbf{z} \in \mathbb{C}^{2,1} \mid \langle\mathbf{v},\mathbf{z}\rangle = 0\}.$$

There exists a unique $c \in \mathbb{P}(V_+)$ such that $L = \mathbb{P}(\mathbf{c}^\perp) \cap \mathbb{H}^2_{\mathbb{C}}$. The vector $\mathbf{c}$ is called a *polar vector* of L.

Definition 6.1.1. Let η be a unit complex number. The *complex reflection in L with reflection factor η* is the map $\rho^\eta_L : \mathbb{C}^{2,1} \to \mathbb{C}^{2,1}$ defined by

$$\mathbf{z} \mapsto \mathbf{z} + (\eta - 1)\frac{\langle\mathbf{z},\mathbf{c}\rangle}{\langle\mathbf{c},\mathbf{c}\rangle}\mathbf{c},$$

for $\mathbf{z} \in \mathbb{C}^{2,1}$. If $\eta = -1$ we simply call this map the *complex reflection in L*.

Example 6.1.2. *An example of a complex line is*

$$L = \mathbb{P}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^{\perp}\right) \cap \mathbb{H}^2_{\mathbb{C}} = \{(z_1, 0) \mid |z_1|^2 < 1\} \subset \mathbb{B}^2.$$

The complex reflection ρ_L^η is given by

$$\begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} z_1 \\ \eta z_2 \\ 1 \end{bmatrix},$$

and hence a matrix representative in $\mathrm{U}(2,1)$ *is*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Remark 6.1.3. *On complex lines in $\mathbb{B}^2$ the Bergman metric restricts to the Poincaré metric of constant curvature -1 [Gol99, Theorem 3.1.9].*

6.1.2 Totally real Lagrangian planes

Example 6.1.4. *The subspace consisting of all those points in $\mathbb{H}^2_{\mathbb{C}}$ with real coordinates is a totally real Lagrangian plane. It is an embedded copy of the real hyperbolic space $\mathbb{H}^2_{\mathbb{R}} = \{(p_1, p_2) \mid p_1, p_2 \in \mathbb{R}\} \subset \mathbb{B}^2$ on which the Bergman metric restricts to the Klein-Beltrami metric of curvature $-1/4$ [Gol99, Section 3.1.9].*

6.2 Heisenberg model of the boundary

The standard lift to $\mathbb{C}^{2,1}$ of a point in the boundary of the Siegel domain is

$$\begin{bmatrix} \frac{-|z|^2+it}{2} \\ z \\ 1 \end{bmatrix},$$

with $z \in \mathbb{C}$ and $t \in \mathbb{R}$. Hence we can identify $\partial\mathbb{H}^2_{\mathbb{C}}$ with $(\mathbb{C} \times \mathbb{R}) \cup \{\infty\}$. A standard lift for the point at infinity is

$$\hat{\infty} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

We will write 0 for the point $(0, 0)$ and $\hat{0}$ for its standard lift.

In this model the boundary is naturally endowed with the structure of a Heisenberg group on $\mathbb{C} \times \mathbb{R}$.

Definition 6.2.1. The *Heisenberg group* $\mathfrak{N}$ is $\mathbb{C} \times \mathbb{R}$ with multiplication defined by

$$(z,t) * (w,s) = (z+w, t+s+2\Im(z\overline{w})).$$

We denote by $T_{z,t}$ the *Heisenberg translation* which is the left translation by $(z,t) \in \mathfrak{N}$, i.e.

$$T_{z,t} : (w,s) \mapsto (w+z, s+t+2\Im(z\overline{w})),$$

and we identify $\mathfrak{N}$ with this group of Heisenberg translations of $\mathcal{H}$. A matrix in $\mathrm{U}(2,1)$ that represents $T_{z,t}$ is

$$\begin{bmatrix} 1 & -\overline{z} & \frac{1}{2}(-|z|^2+it) \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}.$$

Complex numbers $A \in \mathrm{U}(1)$ of norm 1 act by so-called *Heisenberg rotations about the vertical axis* on $\mathcal{H}$

$$A : (w,s) \mapsto (Aw, s).$$

Furthermore, nonzero complex numbers $\lambda \in \mathbb{C}^*$ act by *Heisenberg (complex) dilations about the origin*:

$$\lambda : (w,s) \mapsto (\lambda w, |\lambda|^2 s).$$

The group of isometries fixing the point ∞ is the *Heisenberg similarity group* $\mathbf{Sim}(\mathcal{H}) = (\mathbb{R}_+ \times \mathrm{U}(1)) \rtimes \mathfrak{N}$. Furthermore, the group of isometries that fix both ∞ and 0 is given by $\mathbf{Sim}_0(\mathcal{H}) = \mathbb{R}_+ \times \mathrm{U}(1)$.

Remark 6.2.2. *The Heisenberg group is 2-step nilpotent. Indeed,*

$$(z,t) * (w,s) * (-z,-t) * (-w,-s) = (0, 4\Im(\overline{z}w)).$$

This implies in particular that $\mathfrak{N}$ is amenable, and hence $\mathbf{Sim}(\mathcal{H})$ is as well.

Lemma 6.2.3. $\mathrm{PU}(2,1)$ *acts 2-transitively on the boundary $\partial\mathbb{H}^2_{\mathbb{C}}$.*

Proof. Let $p, q \in \partial\mathbb{H}^2_{\mathbb{C}}$. We will show that there is a $M \in \mathrm{PU}(2,1)$ such that $M \cdot \mathbf{p} = \hat{\infty}$ and $M \cdot \mathbf{q} = \hat{0}$. Suppose that we do not already have $p = \infty$ so that we can write $p = (z,t) \in \mathbb{C} \times \mathbb{R}$. Let furthermore ι denote an element of $\mathrm{PU}(2,1)$ that exchanges $\hat{\infty}$ and $\hat{0}$. For example we can take

$$\iota = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Then $\iota \circ T_{-z,-t}$ sends $\mathbf{p}$ to $\hat{\infty}$. Another Heisenberg translation will send the image of $\mathbf{q}$ under $\iota \circ T_{-z,-t}$ to $\hat{0}$ while fixing $\hat{\infty}$. $\square$

6.3 Cartan angular invariant

Let $\overline{p} = (p_1, p_2, p_3) \in (\mathbb{H}^n_{\mathbb{C}} \cup \partial\mathbb{H}^n_{\mathbb{C}})^{(3)}$ be a triple of distinct points with lifts $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3} \in \mathbb{C}^{n,1}$.
The *Hermitian triple product* is defined to be

$$\langle \mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3} \rangle = \langle \mathbf{p_1}, \mathbf{p_2} \rangle \langle \mathbf{p_2}, \mathbf{p_3} \rangle \langle \mathbf{p_3}, \mathbf{p_1} \rangle \in \mathbb{C},$$

and it has the following properties:

1. Replacing the $\mathbf{p_i}$ by $\xi_i \mathbf{p_i}$ with $\xi_i \in \mathbb{C}^*$ multiplies this complex number with the positive real number $|\xi_1\xi_2\xi_3|^2$.
2. It has negative real part: $\mathrm{Re}\langle \mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3} \rangle \leq 0$ for all triples of distinct points in the boundary.

The *Cartan Angular invariant* $\mathbb{A} : (\mathbb{H}^n_{\mathbb{C}} \cup \partial\mathbb{H}^n_{\mathbb{C}})^{(3)} \to \mathbb{R}$ is by definition

$$\mathbb{A}(\overline{p}) := \arg(-\langle \mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3} \rangle).$$

It follows from the above properties of the Hermitian triple product that $\mathbb{A}$ is well defined, i.e. it is independent of the chosen lifts, and

$$-\frac{\pi}{2} \leq \mathbb{A}(\overline{p}) \leq \frac{\pi}{2} \qquad \forall \overline{p} \in (\partial\mathbb{H}^n_{\mathbb{C}} \cup \mathbb{H}^n_{\mathbb{C}})^{(3)}.$$

We will extend it to all of $(\mathbb{H}^n_{\mathbb{C}} \cup \partial\mathbb{H}^n_{\mathbb{C}})^3$ by setting $\mathbb{A} \equiv 0$ for triples in which not all three points are distinct. The Cartan angular invariant $\mathbb{A}$ classifies triples in the boundary of complex hyperbolic space:

Proposition 6.3.1. *[Gol99, Theorem 7.1.1] Let (p_1, p_2, p_3) and (q_1, q_2, q_3) be triples of distinct points in the boundary of complex hyperbolic space. If $\mathbb{A}(p_1, p_2, p_3) = \mathbb{A}(q_1, q_2, q_3)$ then there exists a holomorphic automorphism $g \in \mathrm{PU}(n, 1)$ such that $gp_i = q_i$ for $i = 1, 2, 3$. This automorphism is unique unless (p_1, p_2, p_3) are contained in a $\mathbb{C}$-circle.*

Example 6.3.2. *In the Heisenberg model three points that are not contained in a same $\mathbb{C}$-circle can be represented by*

$$p_1 = \infty, p_2 = (0, 0), p_3 = (1, t),$$

with Cartan angular invariant $\mathbb{A}(p_1, p_2, p_3) = \arctan(t)$.

Example 6.3.3. *In the projective model three points that are not in the same $\mathbb{C}$-circle can be represented by*

$$p_1 = \begin{bmatrix} ie^{i\mathbb{A}} \\ 0 \\ 1 \end{bmatrix}, p_2 = \begin{bmatrix} -ie^{-i\mathbb{A}} \\ 0 \\ 1 \end{bmatrix}, p_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

with Cartan angular invariant $\mathbb{A}(p_1, p_2, p_3) = \mathbb{A}$.

In the following Lemma we list some useful and elementary properties of $\mathbb{A}$. Proofs can be found in Section 7.1 of [Gol99].

Lemma 6.3.4. *The Cartan angular invariant $\mathbb{A}$ has the following properties*

1. *$\mathbb{A}$ is alternating: If $\sigma \in \mathrm{Sym}(3)$ then*
$$\mathbb{A}(p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}) = \mathrm{sign}(\sigma)\mathbb{A}(p_1, p_2, p_3).$$
2. *If $g \in \mathrm{PU}(n,1)$ is a holomorphic automorphism then*
$$\mathbb{A}(gp_1, gp_2, gp_3) = \mathbb{A}(p_1, p_2, p_3),$$
and if g is an anti-holomorphic automorphism then
$$\mathbb{A}(gp_1, gp_2, gp_3) = -\mathbb{A}(p_1, p_2, p_3).$$
3. *p_1, p_2 and p_3 are in the same $\mathbb{C}$-circle $\iff \mathbb{A}(p_1, p_2, p_3) = \pm\frac{\pi}{2}$.*
4. *p_1, p_2 and p_3 are in the same $\mathbb{R}$-circle $\iff \mathbb{A}(p_1, p_2, p_3) = 0$.*

Now let Φ be the Kähler form on $\mathbb{H}^n_{\mathbb{C}}$ and let $(p_1, p_2, p_3) \in (\mathbb{H}^n_{\mathbb{C}} \cup \partial\mathbb{H}^n_{\mathbb{C}})^3$. Define
$$c_\Phi(p_1, p_2, p_3) = \int_{\Delta(p_1,p_2,p_3)} \Phi.$$
For a triple of points in $\mathbb{H}^n_{\mathbb{C}}$ when we fix a base point $x \in \mathbb{H}^n_{\mathbb{C}}$ this is equal to the image of Φ under the van Est isomorphism as defined before evaluated at the point (g_1, g_2, g_3) with g_i such that $x = g_i^{-1}p_i$. For a triple of points in the boundary one can take the limit $x \to \xi$ with $\xi \in \partial\mathbb{H}^n_{\mathbb{C}}$ and the cocycle c_Φ will still represent the same class. From now on we will consider c_Φ as a map $(\partial\mathbb{H}^n_{\mathbb{C}})^3 \to \mathbb{R}$. We clearly have

Lemma 6.3.5. *c_Φ is a cocycle, i.e.*
$$c_\Phi(p_1, p_2, p_3) - c_\Phi(p_1, p_2, p_4) + c_\Phi(p_1, p_3, p_4) - c_\Phi(p_2, p_3, p_4) = 0,$$
for all $(p_1, p_2, p_3, p_4) \in (\partial\mathbb{H}^n_{\mathbb{C}})^4$.

We will now show that c_Φ is proportional to the Cartan angular invariant.

Theorem 6.3.6. *$c_\Phi(\overline{p}) = 2\mathbb{A}(\overline{p})$ for all $\overline{p} \in (\partial\mathbb{H}^n_{\mathbb{C}})^3$.*

Proof. (Based on the proof of [Gol99, Theorem 7.1.11]). Let $\overline{p} = (p_1, p_2, p_3)$ be in $(\partial\mathbb{H}^n_{\mathbb{C}})^{(3)}$ and let $p_4 = \Pi_{12}(p_3)$, where $\Pi_{12} : \mathbb{H}^n_{\mathbb{C}} \cup \partial\mathbb{H}^n_{\mathbb{C}} \twoheadrightarrow L_{12}$ is the projection onto the complex line L_{12} spanned by p_1 and p_2. By Lemma 6.3.5 we have:
$$c_\Phi(p_1, p_2, p_3) - c_\Phi(p_1, p_2, p_4) = c_\Phi(p_1, p_3, p_4) - c_\Phi(p_2, p_3, p_4). \tag{6.1}$$
Claim: $c_\Phi(p_1, p_3, p_4) = c_\Phi(p_2, p_3, p_4) = 0$.

Proof of claim. We will work in the two-dimensional subspace spanned by p_1, p_2 and p_3 so that we can assume $n = 2$. If $p_3 \in L_{12}$ then $p_4 = p_3$ and thus clearly $c_\Phi(p_1, p_3, p_4) = c_\Phi(p_2, p_3, p_4) = 0$. Now suppose that $p_3 \notin L_{12}$. Then, up to the action of $\mathrm{PU}(2,1)$, we have

$$p_1 = \begin{bmatrix} \lambda_1 \\ 0 \\ 1 \end{bmatrix}, \; p_2 = \begin{bmatrix} \lambda_2 \\ 0 \\ 1 \end{bmatrix}, \; p_3 = \begin{bmatrix} 0 \\ \lambda_3 \\ 1 \end{bmatrix}, \text{ and } p_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Let $i = 1, 2$. Then $\langle p_i, p_3 \rangle = \langle p_i, p_4 \rangle = \langle p_3, p_4 \rangle = -1$. It follows that p_i, p_3 and p_4 span a totally real subspace. Since Φ vanishes on such a subspace this implies $c_\Phi(p_i, p_3, p_4) = 0$. $\square$

By equation 6.1 and the claim above

$$c_\Phi(p_1, p_2, p_3) = c_\Phi(p_1, p_2, p_4).$$

Because $\Phi|_{L_{12}}$ is the volume form on L_{12} (see Remark 6.1.3) and p_1, p_2, p_4 are all on this complex line, we conclude that $c_\Phi(\overline{p})$ is equal to the area of the triangle with vertices p_1, p_2 and $\Pi_{12}(p_3)$. Up to the action of $\mathrm{PU}(n,1)$ we can assume that p_1, p_2, p_3 are represented by

$$\hat{\infty} = \begin{bmatrix} 1 \\ 0' \\ 1 \end{bmatrix}, \qquad \hat{0} = \begin{bmatrix} -1 \\ 0' \\ 1 \end{bmatrix}, \text{ and } \qquad \mathbf{p} = \begin{bmatrix} z_n \\ z' \\ 1 \end{bmatrix},$$

in $\mathbb{P}(\mathbb{C}^{n,1})$, with $0', z' \in \mathbb{C}^{n-1}$. The projection Π_{12} onto the complex line L_{12} that contains p_1 and p_2 sends

$$\begin{bmatrix} z_n \\ z' \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} z_n \\ 0' \\ 1 \end{bmatrix}.$$

Let $w_n = \frac{z_n+1}{z_n-1}$. Under the Cayley transform C, the points $\hat{\infty}$, $\hat{0}$ and $\Pi_{12}(\mathbf{p})$ get send to $\hat{\infty}$, $\hat{0}$ and $\begin{bmatrix} w_n \\ 0' \\ 1 \end{bmatrix}$ and thus to $\infty, 0$ and w_n in the left half-plane $\mathfrak{h}$. It follows that

$$\begin{aligned} c_\Phi(\overline{p}) := \mathrm{Area}(\Delta(p_1, p_2, \Pi_{12}(p_3))) &= \mathrm{Area}(\Delta(\infty, 0, w_n)) \\ &= \pi - (-\pi + 2\arg(w_n)) \\ &= 2(\pi - \arg(w_n)). \end{aligned}$$

Furthermore,

$$\begin{aligned} w_n &= \frac{z_n + 1}{z_n - 1} \frac{-1 - \overline{z_n}}{-1 - \overline{z_n}} \\ &= \frac{-|z_n + 1|^2}{w} \\ &= \frac{c}{w}, \end{aligned}$$

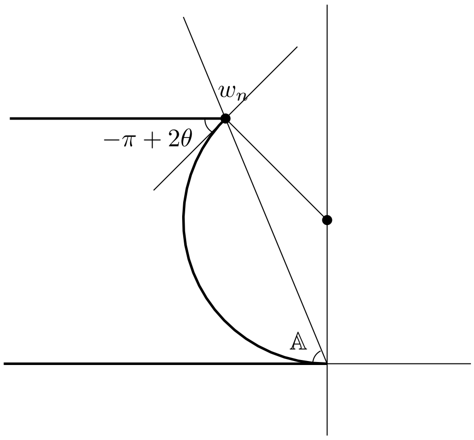


Figure 6.1 – Projection of the triangle $\Delta(p_1, p_2, p_3)$ onto L_{12}.

with $w := (1 + \overline{z_n})(1 - z_n)$ and $c \in \mathbb{R}_{<0}$. Hence $\arg(w_n) = \pi - \arg(w)$ and we obtain

$$\begin{aligned} c(\overline{p}) &= 2(\pi - \arg(w_n)) \\ &= 2\arg(w). \end{aligned}$$

On the other hand, $< \hat{\infty}, \hat{0}, \mathbf{p} >= 2(\overline{z_n} + 1)(z_n - 1) = -2w$ and hence

$$\begin{aligned} \mathbb{A}(\overline{p}) &= \arg(-\langle \hat{\infty}, \hat{0}, \mathbf{p} \rangle) \\ &= \arg(w). \end{aligned}$$

It follows that indeed $c(\overline{p}) = 2\mathbb{A}(\overline{p})$.

$\square$

Chapter 7

Complex hyperbolic surfaces

7.1 Simplicial volume

Let M be a closed manifold that is locally isometric to the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$. Recall that the simplicial volume $\|M\|$ of M is defined by

$$\|M\| = \inf \left\{ \sum |a_\sigma| \middle| \sum a_\sigma \sigma \text{ represents the real fundamental class } [M] \right\}.$$

By Proposition 1.4.2

$$\|M\| = \frac{\mathrm{Vol}(M)}{\|\omega\|_\infty},$$

where $\omega \in H^4_c(\mathrm{PU}(2,1);\mathbb{R})$ is the image under the van Est isomorphism of the volume form.

Lemma 7.1.1. *Let $\omega \in H^4_c(\mathrm{PU}(2,1);\mathbb{R})$ be the image of the volume form in $\Omega^4(\mathbb{H}^2_{\mathbb{C}})^{\mathrm{PU}(2,1)}$ under the van Est isomorphism. Then $\omega = \frac{1}{2} \cdot c_\Phi \cup c_\Phi$.*

Proof. The volume form on $\mathbb{H}^2_{\mathbb{C}}$ with holomorphic sectional curvature -1 is equal to $\frac{1}{2}\Phi \wedge \Phi$ (see for example [Gol99, Chapter 3]). Since the van Est isomorphism is natural with respect to products it sends $\frac{1}{2}\Phi \wedge \Phi$ to $\frac{1}{2}c_\Phi \cup c_\Phi$. □

In the next section we prove

Theorem 7.1.2.

$$\frac{2}{9}\pi^2 \le \|[c_\Phi \cup c_\Phi]\|_\infty \le \pi^2$$

By Proposition 1.4.2 this implies

Corollary 7.1.3. *Let M be a closed oriented manifold which is locally isometric to $\mathbb{H}^2_{\mathbb{C}}$. Then*

$$\frac{2}{\pi^2}\mathrm{Vol}(M) \le \|M\| \le \frac{9}{\pi^2}\mathrm{Vol}(M).$$

We can also express this in terms of $\chi(M)$. Using Hirzeburch's proportionality principle, i.e. Proposition 1.4.3, we get

Lemma 7.1.4.

$$\mathrm{Vol}(M) = \frac{8}{3}\pi^2 \cdot \chi(M).$$

Proof. The compact dual of $\mathbb{H}^2_{\mathbb{C}}$ is the complex projective plane $\mathbb{C}P^2$. Therefore, by Proposition 1.4.3,

$$\frac{\mathrm{Vol}(M)}{\chi(M)} = \frac{\mathrm{Vol}(\mathbb{C}P^2)}{\chi(\mathbb{C}P^2)}.$$

As $\mathbb{C}P^2$ has zero homology groups in odd dimensions and one-dimensional homology groups in even dimensions $\chi(\mathbb{C}P^2) = 3$. Furthermore, the complex projective plane $\mathbb{C}P^2$ is a symplectic quotient $S^5(r)/S^1(r)$, with $S^n(r)$ the real n-sphere of radius r. Hence the volume of $\mathbb{C}P^2$ is

$$\mathrm{Vol}(\mathbb{C}P^2) = \mathrm{Vol}(S^5(r))/\mathrm{Vol}(S^1(r)) = \pi^3 r^5/2\pi r = \frac{1}{2}\pi^2 r^4,$$

To have holomorphic sectional curvature equal to 1 and therefore sectional curvature between 1/4 and 1 we have to set $r = 2$ and we thus obtain $\mathrm{Vol}(\mathbb{C}P^2) = 8\pi^2$. For a more elaborate description of the Fubini-Study metric on the complex projective space and its sectional curvature see e.g. the first pages of Chapter 6 in [Sak97]. □

We get the following lower and upper bound for the simplicial volume of M in terms of its Euler characteristic:

Corollary 7.1.5. *Let M be a closed oriented manifold which is locally isometric to $\mathbb{H}^2_{\mathbb{C}}$. Then*

$$\frac{16}{3}\chi(M) \leq \|M\| \leq 24\chi(M).$$

Let ξ be any $GL^+(4,\mathbb{R})$-bundle over M that admits a flat structure. Its Euler number $\chi(\xi)$ is by definition the pairing of the Euler class $\varepsilon_4(\xi) \in H^4(M;\mathbb{R})$ with the fundamental class $[M] \in H_4(M;\mathbb{R})$:

$$\chi(\xi) :=< \varepsilon_4(\xi), [M] > .$$

By Proposition 1.4.1

$$|\chi(\xi)| \leq \|\varepsilon_4(\xi)\|_\infty \cdot \|M\|,$$

and thus combining Lemma 1.4.7 with Corollary 7.1.5 we obtain the Milnor-Wood inequality

Corollary 7.1.6. *Let ξ be a flat $\mathrm{GL}^+(4,\mathbb{R})$-bundle over a closed complex hyperbolic surface M. Then*

$$|\chi(\xi)| \leq \frac{3}{2}\chi(M).$$

7.2 Proof of Theorem 7.1.2

Proof. (Theorem 7.1.2) For all triples $(x_0, x_1, x_2) \in (\partial \mathbb{H}^2_{\mathbb{C}})^3$ we have

$$|c_\Phi(x_0, x_1, x_2)| \leq \pi,$$

and we therefore obtain the trivial upper bound π^2 for $\|[c_\Phi \cup c_\Phi]\|_\infty$. In Proposition 7.2.6 below we obtain the lower bound $\frac{2}{9}\pi^2$. $\square$

Let X be a topological space. The alternation of a p-cochain $f : X^{p+1} \to \mathbb{R}$ is given by

$$\mathrm{Alt}(f)(x_0, \ldots, x_p) = \frac{1}{(p+1)!} \sum_{\sigma \in \mathrm{Sym}(p+1)} \mathrm{sign}(\sigma) f(x_{\sigma(0)}, \ldots, x_{\sigma(p)}).$$

Recall that for a p-cochain $f : X^{p+1} \to \mathbb{R}$ and a q-cochain $g : X^{q+1} \to \mathbb{R}$ the standard cup product $f \cup g$ is the $p+q$-cochain defined by

$$f \cup g(x_0, \ldots, x_{p+q}) = f(x_0, \ldots, x_p) g(x_p, \ldots, x_{p+q}).$$

Slightly abusing notation, we will still denote by $c_\Phi \cup c_\Phi$ the alternation of the standard cup product of c_Φ with itself. Thus $c_\Phi \cup c_\Phi(x_0, \ldots, x_4)$ is equal to

$$\frac{1}{120} \sum_{\sigma \in \mathrm{Sym}(5)} \mathrm{sign}(\sigma) c_\Phi(x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)}) \cdot c_\Phi(x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}).$$

Any $\sigma \in \mathrm{Sym}(5)$ can be uniquely written as $\tau^k \circ \alpha$, where $\alpha \in \mathrm{Sym}(5)$ maps 2 to 0, $\tau = (0\,1\,2\,3\,4)$ and k is an integer from 0 to 4. Then, exploiting the fact that c_Φ itself is already alternating, we get

$$\begin{aligned}
&c_\Phi \cup c_\Phi(x_0, \ldots, x_4) \\
&\qquad = \frac{1}{15} \sum_{\substack{\tau = (0\,1 \ldots 4)^k \\ k \in \{0,1,\ldots,4\}}} c_\Phi(x_{\tau(0)}, x_{\tau(1)}, x_{\tau(2)}) \cdot c_\Phi(x_{\tau(0)}, x_{\tau(3)}, x_{\tau(4)}) \\
&\qquad\quad - c_\Phi(x_{\tau(0)}, x_{\tau(1)}, x_{\tau(3)}) \cdot c_\Phi(x_{\tau(0)}, x_{\tau(2)}, x_{\tau(4)}) \\
&\qquad\quad + c_\Phi(x_{\tau(0)}, x_{\tau(1)}, x_{\tau(4)}) \cdot c_\Phi(x_{\tau(0)}, x_{\tau(2)}, x_{\tau(3)}).
\end{aligned}$$

We use the cocycle relation

$$\begin{aligned}
0 &= \delta c_\Phi(x_0, x_i, x_j, x_k) \\
&= c_\Phi(x_i, x_j, x_k) - c_\Phi(x_0, x_j, x_k) \\
&\quad + c_\Phi(x_0, x_i, x_k) - c_\Phi(x_0, x_i, x_j),
\end{aligned}$$

to rewrite the above sum such that c_Φ is always evaluated at x_0 and two other points. It then turns out that the five terms in the above sum are all the same and we therefore obtain that $c_\Phi \cup c_\Phi(x_0, \dots, x_4)$ is equal to

$$\frac{1}{3}\big[c_\Phi(x_0,x_1,x_2)\cdot c_\Phi(x_0,x_3,x_4) - c_\Phi(x_0,x_1,x_3)\cdot c_\Phi(x_0,x_2,x_4) + c_\Phi(x_0,x_1,x_4)\cdot c_\Phi(x_0,x_2,x_3)\big]. \tag{7.1}$$

Remark 7.2.1. *The natural map $Alt^* : H_c^p(X;\mathbb{R}) \to H_c^p(X;\mathbb{R})$ which sends a cocycle to its alternation is an isomorphism with inverse induced by the identity map. Since both maps do not increase norms at the cochain level, we have $\|[Alt(f)]\|_\infty = \|[f]\|_\infty$ for any $[f] \in H_c^p(X;\mathbb{R})$.*

7.2.1 Lower bound

Our strategy for finding a lower bound of $\|[c_\Phi \cup c_\Phi]\|_\infty$ is to find a set of 5-tuples $\overline{\mathbf{p}_i} \in (\partial\mathbb{H}^2_\mathbb{C})^5$ such that for all $\mathrm{PU}(2,1)$-invariant alternating cochains $b : (\partial\mathbb{H}^2_\mathbb{C})^4 \to \mathbb{R}$ we have

$$\sum \delta b(\overline{\mathbf{p}_i}) = 0.$$

If $b \in L^\infty((\partial\mathbb{H}^2_\mathbb{C})^4;\mathbb{R})$ then b is only defined a.e. and thus a pointwise equality as above would have no meaning. However, from Lemma 7.2.2 and Lemma 7.2.3 it will follow that we can instead consider $c_\Phi \cup c_\Phi : (\partial\mathbb{H}^2_\mathbb{C})^5 \to \mathbb{R}$ as a cocycle in $H_b^4(\mathrm{PU}(2,1)^\delta;\mathbb{R})$, with $\mathrm{PU}(2,1)^\delta$ the underlying discrete group of the topological group $\mathrm{PU}(2,1)$. We will denote this everywhere defined cocycle by $c_\Phi^\delta \cup c_\Phi^\delta$.

Lemma 7.2.2. *The bounded cohomology group $H_b^*(\mathrm{PU}(2,1)^\delta;\mathbb{R})$ is measurably realized on the boundary, i.e. by the resolution*

$$0 \to \ell^\infty(\partial\mathbb{H}^2_\mathbb{C};\mathbb{R})^G \to \ell^\infty((\partial\mathbb{H}^2_\mathbb{C})^2;\mathbb{R})^G \to \ell^\infty((\partial\mathbb{H}^2_\mathbb{C})^3;\mathbb{R})^G \to \dots$$

Proof. The minimal parabolic subgroup of $\mathrm{PU}(2,1)$ is the Heisenberg similarity group $\mathbf{Sim}(\mathcal{H}) = (\mathbb{R}_+ \times U(1)) \rtimes \mathfrak{N}$. This group is amenable as an abstract group and thus $H_b(\mathrm{PU}(2,1)^\delta;\mathbb{R})$ is given by the cohomology of the complex $(\ell^\infty((\mathrm{PU}(2,1)/\mathbf{Sim}(\mathcal{H}))^{*+1};\mathbb{R})^{\mathrm{PU}(2,1)}, \delta)$. □

Lemma 7.2.3. *Let $\Gamma = \mathrm{PU}(2,1;\mathbb{Z}[i])$. Then $\|[c_\Phi \cup c_\Phi]\|_\infty$ is equal to*

$$\inf\{\|c_\Phi^\delta \cup c_\Phi^\delta + \delta b\|_{\ell^\infty} \mid b : (\partial\mathbb{H}^2_\mathbb{C})^4 \to \mathbb{R}\text{ bounded and }\Gamma\text{−invariant}\}$$

Proof. We follow Section 6 of [BucMon12]. Note that for the proof presented there amenability of the minimal parabolic as an abstract group is not necessary. Let $G = \mathrm{PU}(2,1)$ and let P be its minimal parabolic subgroup. Denote by $\mathscr{L}^\infty((G/P)^{p+1};\mathbb{R})$ the Banach G-module of bounded measurable functions $(G/P)^{p+1} \to \mathbb{R}$ so that we have the natural quotient map

$$q : \mathscr{L}^\infty((G/P)^{p+1};\mathbb{R}) \twoheadrightarrow L^\infty((G/P)^{p+1};\mathbb{R}),$$

and the natural inclusion map

$$i : \mathscr{L}^\infty((G/P)^{p+1}; \mathbb{R}) \hookrightarrow \ell^\infty((G/P)^{p+1}; \mathbb{R}).$$

Then $q(c_\Phi \cup c_\Phi)$ is the function class of $c_\Phi \cup c_\Phi$ in $L^\infty((G/P)^5; \mathbb{R})^G$ (which by slight abuse of notation we also denote by $c_\Phi \cup c_\Phi$ in the rest of this text). On the other hand, $i(c_\Phi \cup c_\Phi) = c_\Phi^\delta \cup c_\Phi^\delta \in \ell^\infty((G/P)^{p+1}; \mathbb{R})^G$. The restriction maps $\mathrm{res}_{(c),b} : H^4_{(c,)b}(G; \mathbb{R}) \to H^4_b(\Gamma; \mathbb{R})$ send $[q(c_\Phi \cup c_\Phi)]$ and $[i(c_\Phi \cup c_\Phi)]$ to the same cohomology class in $H^4_b(\Gamma; \mathbb{R})$. It follows that

$$\|[\mathrm{res}_{c,b}(q(c_\Phi \cup c_\Phi))]\|_{\ell^\infty} = \|[\mathrm{res_b}(i(c_\Phi \cup c_\Phi))]\|_{\ell^\infty}.$$

Since restricting to a cocompact lattice preserves the seminorm in continuous bounded comology [Mon01, Proposition 8.6.2] we can conclude

$$\|[q(c_\Phi \cup c_\Phi)]\|_\infty = \|[\mathrm{res_b}(i(c_\Phi \cup c_\Phi))]\|_{\ell^\infty}.$$

Furthermore, note that the restriction map res_b is realized by the inclusion $\ell^\infty((G/P)^{p+1}; \mathbb{R})^G \hookrightarrow \ell^\infty((G/P)^{p+1}; \mathbb{R})^\Gamma$. This finishes the proof. □

Let

$$x_+ = \begin{bmatrix}1\\0\\1\end{bmatrix}, x_i = \begin{bmatrix}i\\0\\1\end{bmatrix}, y_+ = \begin{bmatrix}0\\1\\1\end{bmatrix}, y_i = \begin{bmatrix}0\\i\\1\end{bmatrix}, y_{-i} = \begin{bmatrix}0\\-i\\1\end{bmatrix}, v = \begin{bmatrix}\frac{1}{2}(1+i)\\\frac{1}{2}(1+i)\\1\end{bmatrix},$$

be points in the boundary of the complex hyperbolic plane in the projective model. We have

$$\begin{aligned}
&\mathbb{A}(x_+, x_i, y_+) = \mathbb{A}(x_+, x_i, y_i) = \mathbb{A}(x_+, x_i, y_{-i}) = \mathbb{A}(x_+, y_+, y_i) = \frac{\pi}{4},\\
&\mathbb{A}(x_+, y_+, y_{-i}) = \mathbb{A}(x_+, x_i, v) = -\frac{\pi}{4},\\
&\mathbb{A}(x_+, y_i, y_{-i}) = \mathbb{A}(x_+, y_+, v) = 0,\\
&\mathbb{A}(x_+, y_i, v) = -\frac{\pi}{2}.
\end{aligned}$$

Recall that equation 7.1 gives a convenient way for calculating the alternating cup product $c_\Phi \cup c_\Phi$ and furthermore that $c_\Phi = 2\mathbb{A}$. Therefore

$$\begin{aligned}
&c_\Phi^\delta \cup c_\Phi^\delta(x_+, x_i, y_+, y_i, y_{-i})\\
&\quad = \frac{1}{3}\Big[c_\Phi^\delta(x_+, x_i, y_+)c_\Phi^\delta(x_+, y_i, y_{-i}) - c_\Phi^\delta(x_+, x_i, y_i)c_\Phi^\delta(x_+, y_+, y_{-i})\\
&\qquad + c_\Phi^\delta(x_+, x_i, y_{-i})c_\Phi^\delta(x_+, y_+, y_i)\Big]\\
&\quad = \frac{1}{3} \cdot \left[\frac{\pi}{2} \cdot 0 - \frac{\pi}{2} \cdot \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} \cdot \frac{\pi}{2}\right]\\
&\quad = \frac{1}{6}\pi^2,
\end{aligned}$$

and

$$
\begin{aligned}
&c_\Phi^\delta \cup c_\Phi^\delta(x_+, x_i, y_+, y_i, v) \\
&\quad = \frac{1}{3}\Big[c_\Phi^\delta(x_+, x_i, y_+)c_\Phi^\delta(x_+, y_i, v) - c_\Phi^\delta(x_+, x_i, y_i)c_\Phi^\delta(x_+, y_+, v) \\
&\qquad + c_\Phi^\delta(x_+, x_i, v)\cdot c_\Phi^\delta(x_+, y_+, y_i)\Big] \\
&\quad = \frac{1}{3}\cdot\left[\frac{\pi}{2}\cdot(-\pi) - \frac{\pi}{2}\cdot 0 + \left(-\frac{\pi}{2}\right)\cdot\frac{\pi}{2}\right] \\
&\quad = -\frac{1}{4}\pi^2.
\end{aligned}
$$

Lemma 7.2.4. *Let $b : (\partial\mathbb{H}^2_{\mathbb{C}})^4 \to \mathbb{R}$ be an alternating Γ-invariant cochain. Then $\delta b(x_+, x_i, y_+, y_i, y_{-i}) = 2b(x_+, x_i, y_+, y_i)$.*

Proof. By definition

$$
\begin{aligned}
\delta b(x_+, x_i, y_+, y_i, y_{-i}) = {} & b(x_i, y_+, y_i, y_{-i}) - b(x_+, y_+, y_i, y_{-i}) \\
& + b(x_+, x_i, y_i, y_{-i}) - b(x_+, x_i, y_+, y_{-i}) \\
& + b(x_+, x_i, y_+, y_i).
\end{aligned}
$$

Denote by L_x the complex line that contains x_+ and x_i and by L_y the complex line that contains y_+, y_i and y_{-i}. The reflection in L_x, represented by the matrix

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
$$

exchanges y_i and y_{-i} while fixing x_+ and x_i. Thus, as b is alternating,

$$
b(x_+, x_i, y_i, y_{-i}) = 0. \tag{7.2}
$$

The reflection in L_y with reflection factor $-i$, represented by the matrix

$$
\begin{bmatrix} -i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
$$

sends x_i to x_+ while fixing y_+, y_i and y_{-i}. It follows that

$$
b(x_i, y_+, y_i, y_{-i}) = b(x_+, y_+, y_i, y_{-i}). \tag{7.3}
$$

Lastly, the reflection in L_x with reflection factor i, represented by the matrix

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 1 \end{bmatrix},
$$

maps $y_+ \mapsto y_i$ and $y_{-i} \mapsto y_+$ while fixing x_+ and x_i. It follows that $b(x_+, x_i, y_+, y_{-i}) = b(x_+, x_i, y_i, y_+)$ and hence, since b is alternating,

$$b(x_+, x_i, y_+, y_{-i}) = -b(x_+, x_i, y_+, y_i). \tag{7.4}$$

Combining equations 7.2,7.3 and 7.4 gives

$$\delta b(x_+, x_i, y_+, y_i, y_{-i}) = 2b(x_+, x_i, y_+, y_i).$$

□

Lemma 7.2.5. *Let $b : (\partial \mathbb{H}^2_{\mathbb{C}})^4 \to \mathbb{R}$ be an alternating Γ-invariant cochain. Then $\delta b(x_+, x_i, y_+, y_i, v) = b(x_+, x_i, y_+, y_i)$.*

Proof. The isomorphism represented by the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

exchanges x_+ with y_+, and x_i with y_i while fixing v. Combined with the fact that b is alternating this gives

$$b(x_i, y_+, y_i, v) = b(x_+, x_i, y_i, v), \text{ and } b(x_+, y_+, y_i, v) = b(x_+, x_i, y_+, v).$$

Thus

$$\delta b(x_+, x_i, y_+, y_i, v) = 2b(x_+, x_i, y_i, v) - 2b(x_+, x_i, y_+, v) + b(x_+, x_i, y_+, y_i).$$

Furthermore, the isomorphism represented by the matrix

$$\begin{bmatrix} -1+i & 0 & 1 \\ 0 & -i & 0 \\ i & 0 & 1-i \end{bmatrix}$$

sends the 4-tuple (x_+, x_i, y_i, v) to (x_i, x_+, v, y_+) and thus

$$b(x_+, x_i, y_i, v) = b(x_+, x_i, y_+, v).$$

It follows that $\delta b(x_+, x_i, y_+, y_i, v) = b(x_+, x_i, y_+, y_i)$ □

Proposition 7.2.6. $\|[c_\Phi \cup c_\Phi]\|_\infty \geq \frac{2}{9}\pi^2$.

Proof. Note that since $c^\delta_\Phi \cup c^\delta_\Phi$ is alternating we can restrict to alternating cochains to compute $\inf \|c^\delta_\Phi \cup c^\delta_\Phi + \delta b\|_{\ell^\infty}$. By Lemma 7.2.4 and Lemma 7.2.5

$$\delta b(x_+, x_i, y_+, y_i, y_{-i}) - 2\delta b(x_+, x_i, y_+, y_i, v) = 0,$$

for all alternating cochains $b \in \ell^\infty((\partial\mathbb{H}^2_{\mathbb{C}})^4;\mathbb{R})^\Gamma$. Let $\mathbf{p}_1 = (x_+, x_i, y_+, y_i, y_{-i})$ and $\mathbf{p}_2 = (x_+, x_i, y_+, y_i, v)$. Then

$$\begin{aligned}\|c^\delta_\Phi \cup c^\delta_\Phi + \delta b\|_{\ell^\infty} &\geq \frac{1}{3}\Big((c^\delta_\Phi \cup c^\delta_\Phi + \delta b)(\mathbf{p}_1) - 2(c^\delta_\Phi \cup c^\delta_\Phi + \delta b)(\mathbf{p}_2)\Big)\\ &= \frac{1}{3}\Big(c^\delta_\Phi \cup c^\delta_\Phi(\mathbf{p}_1) - 2c^\delta_\Phi \cup c^\delta_\Phi(\mathbf{p}_2)\Big)\\ &= \frac{2}{9}\pi^2,\end{aligned}$$

for all alternating cochains $b \in \ell^\infty((\partial\mathbb{H}^2_{\mathbb{C}})^4;\mathbb{R})^\Gamma$ and it therefore follows from Lemma 7.2.3 that

$$\|[c_\Phi \cup c_\Phi]\|_\infty \geq \frac{2}{9}\pi^2.$$

□

Remark 7.2.7. *Note that*

$$\mathbb{A}(x_+, x_i, y_+) = \mathbb{A}(x_+, x_i, y_i) = \mathbb{A}(x_+, y_+, y_i) = \mathbb{A}(x_i, y_+, y_i) = \frac{\pi}{4},$$

and thus the 4*-tuple* (x_+, x_i, y_+, y_i) *is a regular special symmetric tetrahedron as defined in [Fal08]. Coordinates in the Heisenberg model for such a tetrahedron (with* $\mathbb{A} = \pi/4$*) are for example* $\infty, 0, (1,1)$ *and* $(i,1)$*.*

Remark 7.2.8. *The* 8 *vectors*

$$\begin{bmatrix}\pm 1\\0\\1\end{bmatrix}, \begin{bmatrix}\pm i\\0\\1\end{bmatrix}, \begin{bmatrix}0\\\pm 1\\1\end{bmatrix}, \begin{bmatrix}0\\\pm i\\1\end{bmatrix},$$

correspond to the eight vertices of a regular octahedron in $\mathbb{R}^4$ *with edge length* $\sqrt{2}$*. The* 5*-tuple* $(x_+, x_i, y_+, y_i, y_{-i})$ *corresponds to one of the simplices in the minimal triangulation of this octahedron. In fact,* $c^\delta_\Phi \cup c^\delta_\Phi$ *takes the value* $\pm\pi^2/6$ *on all the simplices of this triangulation. Furthermore the eight vertices of the form*

$$\begin{bmatrix}\frac{1}{2}(\pm 1 \pm i)\\\frac{1}{2}(\pm 1 \pm i)\\1\end{bmatrix},$$

with an even number of plus signs also correspond to a regular octahedron in $\mathbb{R}^4$ *with edge length* $\sqrt{2}$*. Together the* 16 *vertices correspond to a regular cube in* $\mathbb{R}^4$ *with edge length* 1*. The* 5*-tuple* (x_+, x_i, y_+, y_i, v) *is one of the simplices in the minimal triangulation of this cube found in [Ma76]. It corresponds to one of the eight corners that are "sliced off" in this construction and in fact* $c^\delta_\Phi \cup c^\delta_\Phi$ *is equal to* $\pm\pi^2/4$ *on all these eight simplices while on the remaining eight simplices in the triangulation it is again equal to* $\pm\pi^2/6$*.*

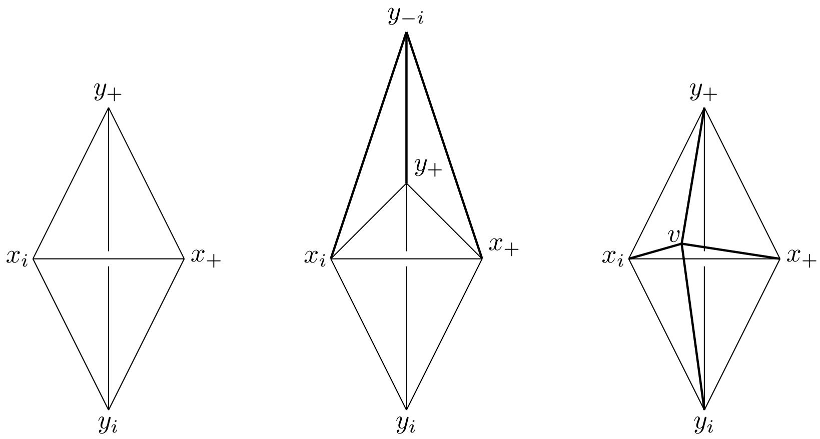


Figure 7.1 – The regular special symmetric tetrahedron (x_+, x_i, y_+, y_i) and the two types of simplices in the minimal triangulation of a regular cube.

Remark 7.2.9. *Let* $\Lambda = \mathrm{PU}(2,1;\mathbb{Z}[\omega])$ *be the Eisenstein-Picard modular group which is by definition the subgroup of* $\mathrm{PU}(2,1)$ *with entries in the ring* $\mathbb{Z}[\omega]$ *where* ω *is a cube root of unity. Let* Λ_∞ *be the stabilizer of* ∞ *in* Λ *and let* $\Lambda_T < \Lambda_\infty$ *be its torsion-free subgroup. A* 5*-tuple that realizes the lower bound* $2\pi^2/9$ *is given by the following points in the Heisenberg space:*

$$(0, -\sqrt{3}),\ (-\omega, 0),\ (1, 0),\ (0, \sqrt{3}),\ (0, 2\sqrt{3}).$$

These are the vertices p_0, p_1, p_2, p_3 *and* p_8 *of the fundamental domain of* Λ_T *described in [Gen10].*

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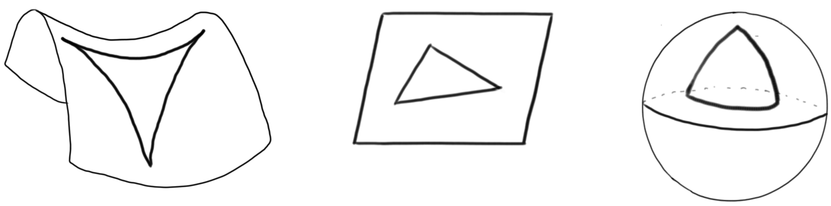
Samenvatting

Een van de bekendste stellingen in de wiskunde is de stelling die zegt dat de som van de hoeken van een driehoek 180 graden is. Dit is alleen waar in het platte vlak. Als we drie punten op een bol nemen en een driehoek vormen met als zijdes de kortste verbindingen tussen deze punten dan is de som van de hoeken van deze driehoek groter dan 180 graden. Een bol is een voorbeeld van een oppervlak met een positieve kromming. Op zo'n oppervlak is de som van de hoeken van een driehoek altijd groter dan 180 graden. Is er sprake van negatieve kromming dan is deze som kleiner dan 180 graden.



Negatieve kromming Geen kromming Positieve kromming

Een andere manier om de kromming van een oppervlak te beschrijven is als volgt. Een oppervlak met een positieve kromming buigt in ieder punt in twee richtingen dezelfde kant op terwijl een oppervlak met een negatieve kromming in ieder punt in twee richtingen verschillende kanten opbuigt. Een oppervlak zonder kromming is in minstens één richting plat. In het algemeen geldt voor de som van de hoeken van een driehoek:

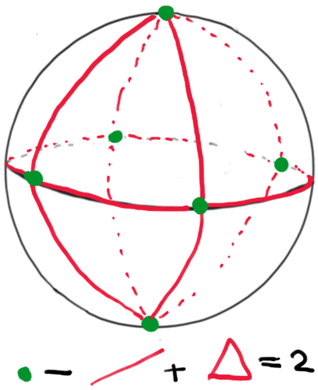
$$\sum \text{hoeken } \Delta = 180 \text{ graden} + \text{totale ingesloten kromming.}$$

Een generalisatie van deze som is de Euler karakteristiek. Als we een oppervlak opdelen in driehoeken dan is de Euler karakteristiek gelijk aan het aantal hoekpunten min het aantal zijdes plus het aantal driehoeken van deze zogenaamde triangulatie. Voor de triangulatie van de bol als in Figuur 1 zien we dat Euler karakteristiek 2 is. Het blijkt dat de Euler karakteristiek niet van de gekozen triangulatie afhangt, voor een bol is deze dus altijd gelijk aan 2.

De stelling van Gauss-Bonnet uit de 19^e eeuw zegt dat voor een gesloten compact oppervlak S (zoals bijvoorbeeld de bol) de integraal over de krommingsfunctie K gelijk is aan 2π keer de Euler karakteristiek van dit oppervlak:

$$\int_S K dA = 2\pi \cdot \chi(S)$$

Figuur 1 – Euler karakteristiek van een bol



De Euler karakteristiek is een topologische invariant, dat wil zeggen dat deze niet verandert als we het oppervlak vervormen of uitrekken. De kromming verandert wel bij zo'n vervorming. De stelling van Gauss-Bonnet zegt dus dat alhoewel vervorming de *meetkunde* en de kromming *lokaal* verandert de integraal over deze kromming een *globale* of *topologische* invariant is die gelijk blijft.

Een oppervlak is een twee-dimensionaal object dat we bekijken in de drie-dimensionale ruimte. Als wiskundigen beperken we ons natuurlijk niet tot de drie-dimensionale ruimte, maar willen we graag alles generaliseren naar willekeurig hoge dimensies. Zo'n hoger dimensionaal "oppervlak" noemen we een variëteit. Een recentere topologische invariant van een variëteit is het simpliciaal volume dat is geïntroduceerd door de wiskundige Gromov in de jaren 80. Voor variëteiten met een negatieve kromming is deze invariant ook proportioneel aan de Euler karakteristiek, het is alleen over het algemeen niet bekend met welke constante. In het tweede deel van mijn proefschrift geef ik schattingen van het simpliciaal volume van complexe hyperbolische oppervlakken. Dit zijn 4-dimensionale variëteiten met een (niet-constante) negatieve kromming. Deze nieuwe onder- en bovengrenzen hebben nog verdere consequenties voor de meetkunde van deze variëteiten. Ik gebruik hiervoor technieken uit de theorie van continue begrensde cohomologie, een vrij recente topologische theorie die onder andere gebruikt kan worden om topologische invarianten uit te rekenen. Over deze theorie is zelf ook nog veel onbekend, in het eerste deel van dit proefschrift bewijs ik iets nieuws over deze theorie en de meer klassieke continue cohomologie.

Hieronder beschrijf ik wat ik precies in dit proefschrift doe. Dit is een bijna letterlijke vertaling van de Franse samenvatting.

Wat bewijs ik in dit proefschrift?

In het eerste deel laat ik zien dat de continue cohomologie van de isometriegroep van de reële hyperbolische ruimte uitgerekend kan worden door middel van functies die gedefinieerd zijn op de rand van de hyperbolische ruimte. Bloch heeft dit eerder al bewezen voor de 3-dimensionale hyperbolische ruimte. De generalisatie is niet automatisch. Terwijl in dimensie 3 de stabilisateur van 3 punten triviaal is, is voor dimensie groter dan 3 de stabilisateur niet langer triviaal maar een compacte group. Het is niet duidelijk dat de cohomologie van zo'n groep met coëfficienten in Fréchet ruimtes die niet lokaal convex zijn triviaal is. Dit resultaat kan een eerste stap zijn naar een bewijs van het vermoeden van Dupont en Monod dat de natuurlijke vergelijkingsafbeelding tussen de continue begrensde cohomologie en de continue cohomologie een isomorfisme is voor alle samenhangende halfenkelvoudige Lie-groepen met een eindig-dimensionaal centrum. De surjectiviteit van deze afbeelding is al bewezen voor veel van deze groepen. De injectiviteit is echter alleen maar bewezen voor een aantal speciale gevallen in lage dimensie. Een direct gevolg van mijn resultaat is de injectiviteit van de vergelijkingsafbeelding in graad 3 voor reële hyperbolische groepen.

In het tweede deel vind ik nieuwe onder- en bovengrenzen voor de Gromov norm van de cohomologieklasse in top dimensie van de isometriegroep van het complexe hyperbolische vlak $\mathbb{H}^2_{\mathbb{C}}$. Een direct gevolg zijn de volgende onder- en bovengrenzen voor het simpliciaal volume $\|M\|$ van een gesloten complex hyperbolisch oppervlak M:

$$\frac{2}{\pi^2}\mathrm{Vol}(M) \leq \|M\| \leq \frac{9}{\pi^2}\mathrm{Vol}(M).$$