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Neutron-Proton Scattering and the Meson Theory of Nuclear Forces

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The angular dependence of the scattering of 14-Mev neutrons on protons is calculated with an interaction derived from Schwinger's mixed meson theory. The scattering shows a slight predominance for the backwards direction in the center of mass system. This result is in contradiction with the experimental results. The conclusion is drawn that the charge symmetrical theories cannot give the correct angular dependence of the scattering cross section.

1. INTRODUCTION

THE recent experiments by Amaldi and coworkers[1] on neutron-proton scattering indicate that for a neutron energy of 14 Mev the angular dependence of the scattered neutrons shows a marked preference for the scattering in the forward direction. It has already been pointed out by Wick[2] that an investigation of the angular distribution of the neutron-proton scattering will furnish a crucial test on the type of exchange forces. For the 14-Mev neutrons the angular dependence is mainly determined by the interaction in P states. In the so-called symmetrical meson theory, which makes use of both charged and neutral mesons in a symmetrical way, the potential is usually positive for P states, thus leading to a repulsive force. The symmetrical theory is the only existing satisfactory meson theory of nuclear forces which takes account of the existence of charged mesons in cosmic rays and the charge independence of nuclear forces.

A theoretical investigation of the angular dependence in the neutron-proton scattering has recently been published by Hulthén.[3] He uses for his calculations the interaction scheme proposed by Møller and Rosenfeld.[4] These authors use two different kinds of mesons, of spin zero and one, respectively, symmetrical with respect to the charge. The advantages of doing this are stated in the paper quoted above[4] and will not be repeated here. Møller and Rosenfeld chose the coupling constants of the two mesons with the nucleons equal, and as a consequence of this the tensor force vanishes. This is at variance with the known fact of the electric quadrupole moment of the deuteron. Attempts to explain this quadrupole moment with the non-static part of the forces have so far not met with success.[5] The result of Hulthén's calculations with this theory does not reproduce the experimentally observed angular dependence, the scattering being more backwards in this theory.

[1] Amaldi *et al.*, Naturwiss. **30**, 582 (1942); these results were recently confirmed by F. C. Champion and C. F. Powell, Proc. Roy. Soc. **183**, 64 (1944).
[2] G. Wick, Zeits. f. Physik **84**, 799 (1933).
[3] L. Hulthén, Arkiv. f. Mat. Astronom. Fys. **29**, No. 33 (1943); also B. Ferreti, Nuovo Cimento **XXI**, No. 1, 25 (1943).
[4] C. Møller and Rosenfeld, Kgl. Danske Vid. Sels. Math.-Fys. Medd. **XVII**, No. 8 (1940).
[5] C. Møller, Danske Vid. Sels. Math.-Fys. Medd. **XVIII**, No. 7 (1941).

The present investigation deals with the calculation of the angular dependence of the scattering with a modification of the Møller-Rosenfeld theory proposed by Schwinger.[6] In this theory we still have the two kinds of mesons, but the vector meson (spin one) is assumed to have a rest mass somewhat larger than that of the scalar meson (spin zero). This modification leads to a tensor force which has the correct sign in the static approximation (nucleons at rest). The question which we shall investigate in this paper is whether the tensor force modifies the scattering of 14-Mev electrons in such a way as to give the correct angular dependence.

A similar problem was treated recently by Rarita and Schwinger.[7] They use, however, square well potentials with range and depth so adjusted as to give correct values of deuteron binding energy and quadrupole moment. Their method for the calculation of the scattering is very convenient for the states of lowest value of the orbital angular momentum quantum number, but it becomes increasingly cumbersome for the higher states. We shall develop, therefore, the scattering theory with a tensor force in a more general way. It is to be expected that the result will not depend very much on the actual shape of the potential function. This expectation is borne out by the result of our calculation which differs very little from Rarita and Schwinger's result. In view of the importance of the conclusions which can be drawn from this result, it seemed to us worthwhile to carry out the numerical calculation for the meson potential of Schwinger's mixed theory.

For the numerical evaluation of the scattering cross section and the angular dependence, we use the values of the coupling constants and the masses of the two mesons which have been recently determined from the deuteron problem and the scattering of slow neutrons on protons.[8] These constants have been calculated from the theory in the so-called weak coupling approximation, where the force between nucleons is obtained as a first approximation of a development in rising powers of the coupling constant. Recently it has been doubted whether this is a sufficiently good approximation.[9] This doubt is justified if we use the extended source for the heavy particle in the interaction operator. In that case the condition for validity of the weak coupling theory is

$$(f\kappa)^2 \ll (a\kappa)^3,$$

where f is the coupling constant of the dimension of a length; κ is the reciprocal Compton wavelength of the meson; and a is the size of the source. Now from our determination of the coupling constant we find $(f\kappa)^2 \sim 0.05$, while $a\kappa \sim 0.1$, because the range of the nuclear forces should be determined by the mass and not by the size of the source. The perturbation treatment of the weak coupling theory is, therefore, hardly justified.

The situation is, however, quite different if we use the point source model together with the λ limiting process.[10] The condition for the validity of the perturbation treatment is then[11]

$$f\kappa \ll 1, \quad r \gg f$$

and we are well within the region of the weak coupling case.

2. THE NUCLEAR FORCES IN THE MIXED SYMMETRICAL THEORY

We consider two nuclear particles situated at positions $\mathbf{y}_1$ and $\mathbf{y}_2$ respectively. A pseudoscalar meson field ψ_α of mass κ and a vector field $\psi_{\alpha\nu}$ of mass μ will interact with these particles. The index α distinguishes the three components in isotopic spin space and ν is the vector index. The fields ψ_α and $\psi_{\alpha\nu}$ are Hermitian operators.

The Hamiltonian for the field plus the interaction term is then,

$$H = H_f + H_{\text{int}},$$

with,

$$H_f = \frac{1}{2}\sum_\alpha \int \{\pi_\alpha^2 + (\nabla\psi_\alpha)^2 + \kappa^2\psi_\alpha^2\} d^3x + \frac{1}{2}\sum_\alpha \int \left\{\boldsymbol{\pi}_\alpha^2 + \frac{1}{\mu^2}(\nabla\cdot\boldsymbol{\pi}_\alpha)^2 + (\nabla\times\boldsymbol{\psi}_\alpha)^2 + \mu^2\boldsymbol{\psi}_\alpha^2\right\} d^3x,$$

[6] J. Schwinger, Phys. Rev. **61**, 387 (1942).

[7] W. Rarita and J. Schwinger, Phys. Rev. **59**, 556 (1941), also C. Kittel and G. Breit, Phys. Rev. **56**, 744 (1939).

[8] J. M. Jauch and Ning Hu, Phys. Rev. **65**, 289 (1944).

[9] G. Wentzel, Helv. Phys. Acta **13**, 269 (1940); J. R. Oppenheimer and J. Schwinger, Phys. Rev. **60**, 150 (1940).

[10] P. A. M. Dirac, Inst. H. Poincaré Ann. **9**, 13 (1939); W. Pauli, Rev. Mod. Phys. **15**, 175 (1943).

[11] W. Pauli, Phys. Rev. **64**, 332 (1943).

$$H_{\text{int}} = (4\pi)^{\frac{1}{2}} f \sum_{\alpha, r} \tau_\alpha^{(r)} (\boldsymbol{\sigma}^{(r)} \cdot \nabla) \psi_\alpha(\mathbf{y}_r) + 4\pi f \sum_{\alpha, r} \tau_\alpha^{(r)} (\boldsymbol{\sigma}^{(r)} \cdot [\nabla \times \boldsymbol{\psi}_\alpha(\mathbf{y}_r)])$$

where $\boldsymbol{\sigma}$ and τ_α are the operators of spin and isotopic spin.

The first-order term of the interaction between two nuclear particles situated at positions $\mathbf{y}_1$ and $\mathbf{y}_2$ is then,[12]

$$V_{\text{int}} = \tfrac{1}{3} f^2 T \{ \Lambda J + SK \},$$

with,

$$T = \sum_\alpha \tau_\alpha^{(1)} \tau_\alpha^{(2)} \quad S = 3\Sigma - \Lambda,$$

$$\Lambda = \boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)},$$

$$\Sigma = \frac{1}{r^2} (\boldsymbol{\sigma}^{(1)} \cdot \mathbf{y})(\boldsymbol{\sigma}^{(2)} \cdot \mathbf{y}) \quad \mathbf{y} = \mathbf{y}_1 - \mathbf{y}_2,$$

$$J = \kappa^2 \frac{e^{-\kappa r}}{r} + 2\mu^2 \frac{e^{-\mu r}}{r}.$$

$$K = \frac{1}{r^3} \{ (3 + 3\kappa r + \kappa^2 r^2) e^{-\kappa r} - (3 + 3\mu r + \mu^2 r^2) e^{-\mu r} \}.$$

The Schrödinger equation for the scattering problem is

$$-\frac{1}{M} \nabla^2 \psi + V_{\text{int}} \psi = \epsilon \psi, \tag{1}$$

where M is the mass of one nuclear particle[13] and ϵ is the energy in the center of mass system. The latter is half the energy in the laboratory system. The singularity of the function $K(r)$ near the origin is of the form $1/r$. The solutions of the two-body problem are, therefore, well behaved and form a complete orthonormal system.

3. THE MATRIX ELEMENTS OF THE TENSOR FORCE OPERATOR

The term in the interaction operator which contains the factor S is usually called the tensor force. Since the angular momentum operator $\mathbf{L}$ does not commute with S, it follows that the angular momentum is no longer an integral of motion. The total angular momentum

$$J = L + \tfrac{1}{2} (\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)}),$$

however, is still an integral, since it commutes with S. It is, therefore, still possible to classify the states according to their total angular momentum quantum number J, defined by the eigenvalues of the operator $\mathbf{J}^2$:

$$\mathbf{J}^2 = J(J+1).$$

In order to write down the eigenfunctions which belong to these eigenvalues J, we denote with $Y_l^m(\vartheta, \varphi)$ the normalized spherical harmonics and with u_1^m, u_0 the normalized spin functions for triplet and singlet states, respectively. The triplet and singlet functions which belong to a given value of J, normalized to unity, are then:

$$\begin{aligned}
W_{J, J-1}^M &= \frac{1}{[J(2J-1)]^{\frac{1}{2}}} \sum_{m+m'=M} C_{mm'}^J (J-1, 1) Y_{J-1}^m u_1^{m'}, \\
W_{J, J+1}^M &= \frac{1}{[(J+1)(2J+3)]^{\frac{1}{2}}} \sum_{m+m'=M} C_{mm'}^J (J+1, 1) Y_{J+1}^m u_1^{m'}, \\
W_{J, J}^M &= \frac{1}{2[J(J+1)]^{\frac{1}{2}}} \sum_{m+m'=M} C_{mm'}^J (J, 1) Y_J^m u_1^{m'}, \\
V_J^M &= Y_J^M u_0.
\end{aligned} \tag{2}$$

The coefficients $\overset{J}{C}_{mm'}(l, l')$ are the Clebsch-Gordon coefficients for the composition of two angular momenta l, l', to a resultant angular momentum J, which may assume the values $l+l'$, $l+l'-1$, $\cdots$, $|l-l'|$. They may be determined from group theoretical considerations alone. We do not give the explicit form of these coefficients.[14]

Since the operators Λ and Σ commute with the total angular momentum operator $\mathbf{J}$, these operators will have no matrix elements with respect

[12] For the derivation of this result cf. reference 4.

[13] We neglect the small difference between neutron and proton mass.

[14] Cf. B. L. van der Waerden, *Die gruppentheoretische Methode in der Quantenmechanik* (Springer, Berlin, 1932), p. 69 ff.

to states of different angular momentum. The four-dimensional submatrices of these operators in the space of the functions (W, V) belonging to a fixed value of J are easily evaluated by direct computation. A simplification is introduced by the fact that both the operators Λ and Σ commute with the permutation operator P, as well as with the total spin operator $\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}$. From these relations it follows immediately that the two functions $W_{J,J}$ and V are eigenfunctions of Λ and Σ. The other two functions, however, $W_{J,J-1}$ and $W_{J,J+1}$ are transformed into each other by the operator Σ. This is the characteristic property of the tensor force, which couples states of different orbital angular momentum, thus giving rise to the quadrupole moment in the deuteron ground state. The result of a straightforward calculation of these matrix elements may best be written in the matrix notation. If we order the states as indicated above [Eq. (2)], we find for these matrices

$$\Sigma=\begin{pmatrix} \frac{1}{2J+1} & \frac{2}{2J+1}[J(J+1)]^{\frac{1}{2}} & 0 & 0 \\ \frac{2}{2J+1}[J, J+1]^{\frac{1}{2}} & -\frac{1}{2J+1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\Lambda=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

The eigenvalues of Σ follow immediately from the relation $\Sigma^2=1$ and $tr\Sigma=0$. The first relation implies that the eigenvalues of Σ are all ± 1. From the second we find that the sum of these eigenvalues is zero and therefore both values occur as eigenvalues twice. The operator S for the tensor force is then in this notation

$$S=3\Sigma-\Lambda$$

$$=\begin{pmatrix} -2\frac{J-1}{2J+1} & \frac{6}{2J+1}[J(J+1)]^{\frac{1}{2}} & 0 & 0 \\ \frac{6}{2J+1}[J(J+1)]^{\frac{1}{2}} & -\frac{2(J+2)}{2J+1} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and its eigenvalues are 2, 2, -4, 0. This follows from the fact that $\Sigma\Lambda=\Lambda\Sigma$ and Λ and Σ may therefore be transformed simultaneously into the diagonal form.

The matrices given above degenerate if $J=0$ since for this case $W_{J,J-1}=W_{J,J}=0$, as may be seen from an inspection of the Gordon coefficients. We are then left with the two states $^1S_0$ and $^3P_0$ only.

4. THE RADIAL EQUATIONS

We are now in a position to derive the radial differential equations. For that purpose it is necessary to remember that the functions (W, V) form an orthonormal set of functions in the space of the spin functions and on the surface of the unit sphere. If there were no coupling of the states through the tensor force, we would obtain for each value of $J>0$ four independent radial equations. With the tensor force, however, we find instead two equations and a coupled system of two equations. The number of independent solutions is of course four again, for each value of $J>0$, because the system allows two linearly independent solutions which satisfy the boundary conditions of the problem, which we distinguish by an index + or $-$.

In order to write down the radial equations we need to know the value of the operator $T=\sum_\alpha \tau_\alpha^{(1)}\tau_\alpha^{(2)}$ for the different states. For the neutron-proton scattering problem we have a system that consists of a neutron and a proton. Such a system may be either in a triplet or a singlet state in isotopic spin space, for which T has the eigenvalues $+1$ and -3, respectively. The system as a whole, including space, spin, and isotopic spin degrees of freedom, must be in an antisymmetrical state according to the exclusion principle. Since the singlet states are symmetrical and the triplets antisymmetrical it follows that

$$T=\left\{\begin{matrix} -1+2(-1)^l & \text{for singlet} \\ -1-2(-1)^l & \text{for triplet} \end{matrix}\right\}\text{states in spin space.}$$

The complete solution may be written in the form $\frac{1}{r}\omega_{J,S}^{M}$, $\frac{1}{r}\chi_J^M$ with

$$\begin{aligned} \omega_{J,\pm}^{M}&=a_\pm(r)W_{J,J-1}^{M}+b_\pm(r)W_{J,J+1}^{M}, \\ \omega_{J,0}^{M}&=c(r)W_{J,J}^{M}, \\ \chi_J^M&=d(r)V_J^M. \end{aligned} \tag{3}$$

Inserting these expressions into the Schrödinger equation (1) and using the matrix form of S we find, by equating the coefficients of each W and V to zero, the following radial equations for the functions a, b, c and d.

$$a''-\frac{J(J-1)}{r^2}a+(1+2(-1)^{J-1})\frac{Mf^2}{3}\left[Ja+K\left(-\frac{2(J-1)}{2J+1}a+\frac{6}{2J+1}[J(J+1)]^{\frac{1}{2}}b\right)\right]+k^2a=0,$$

$$b''-\frac{(J+1)(J+2)}{r^2}b+(1+2(-1)^{J+1})\frac{Mf^2}{3}\left[Jb+K\left(\frac{6}{2J+1}[J(J+1)]^{\frac{1}{2}}a-\frac{2(J+2)}{2J+1}b\right)\right]+k^2b=0, \quad (4)$$

$$c''-\frac{J(J+1)}{r^2}c+(1+2(-1)^J)\frac{Mf^2}{3}[J+2K]c+k^2c=0,$$

$$d''-\frac{J(J+1)}{r^2}d-(1-2(-1)^J)Mf^2Jd+k^2d=0.$$

In these equations we have introduced the wave number $k'=(M\epsilon)^{\frac{1}{2}}$. Some special cases are of interest for our problem. We write them down explicitly:

(a) Triplet States

$^3S_1+^3D_1$: $$a''+Mf^2Ja+2\sqrt{2}Mf^2Kb+k^2a=0,$$
$$b''-\frac{6}{r^2}b+Mf^2(J-2K)b+2\sqrt{2}Mf^2Ka+k^2b=0,$$

$^3P_0$: $$b''-\frac{2}{r^2}b-\frac{Mf^2}{3}[J-4K]b+k^2b=0,$$

$^3P_1$: $$c''-\frac{2}{r^2}c-\frac{Mf^2}{3}[J+2K]c+k^2c=0, \quad (5)$$

$^3P_2+^3F_2$: $$a''-\frac{2}{r^2}a-\frac{Mf^2}{3}\left[Ja+K\left(-\frac{2}{5}a+\frac{6\sqrt{6}}{5}b\right)\right]+k^2a=0,$$
$$b''-\frac{12}{r^2}b-\frac{Mf^2}{3}\left[Jb+K\left(\frac{6\sqrt{6}}{5}a-\frac{8}{5}b\right)\right]+k^2b=0.$$

(b) Singlet States

$^1S_0$: $$d''+Mf^2Jd+k^2d=0,$$

$^1P_1$: $$d''-\frac{2}{r^2}d-3Mf^2Jd+k^2d=0, \quad (6)$$

$^1D_2$: $$d''-\frac{6}{r^2}d+Mf^2Jd+k^2d=0.$$

5. THE SCATTERING PROBLEM

The asymptotic behavior of the solutions of these equations determines the scattering cross section. In order to determine the connection between this cross section and the phase shifts at infinity we write down first the asymptotic form of the stationary state wave function appropriate for the scattering problem.

$$\psi_1^m\sim e^{ikz}u_1^m+\sum_{m'}f_{mm'}(\vartheta)u_1^{m'}\frac{e^{ikr}}{r} \quad (7)$$

for triplet states and

$$\psi\sim e^{ikz}u_0+f(\theta)u_0\frac{e^{ikr}}{r} \quad (8)$$

for singlet states.

The differential scattering cross section is obtained by averaging the square of the amplitude of the scattered wave over the most general linear superposition of these four states.

$$\sigma(\theta)=\left|\sum_{m,m'}\lambda_1^m f_{mm'}u_1^{m'}+\lambda_0 f u_0\right|_{\text{Av}}=\tfrac{1}{4}\{\sum_{m,m'}|f_{mm'}|^2+|f|^2\}. \quad (9)$$

The next step is to find the connection of the phase shifts at infinity with the quantities f, $f_{mm'}$. Let the functions at infinity be denoted with

$$a_{J,\pm}=(2/\pi)^{\frac{1}{2}}\alpha_{J,\pm}\sin\left(kr-\frac{J-1}{2}\pi+\xi_{J,\pm}\right),$$

$$b_{J,\pm}=(2/\pi)^{\frac{1}{2}}\beta_{J,\pm}\sin\left(kr-\frac{J+1}{2}\pi+\eta_{J,\pm}\right),$$

$$c_J=(2/\pi)^{\frac{1}{2}}\sin\left(kr-\frac{J}{2}\pi+\zeta_J\right),$$

$$d_J=(2/\pi)^{\frac{1}{2}}\sin\left(kr-\frac{J}{2}\pi+\delta_J\right).$$

This notation is convenient because in the absence of any interaction we have $\xi=\eta=\zeta=\delta=0$. We distinguish the two linearly independent but otherwise arbitrary solutions for the system with the index + or −.

It is clear that, because of the linear independence of triplet and singlet spin functions, the amplitude f depends only on the phase shifts δ_J of the singlet states, while $f_{mm'}$ depends on the phases ξ_J, η_J, ζ_J of the triplet states. The connection between f and δ_J is given by the ordinary scattering theory of Faxén and Holtsmark and does not need to be derived here:[15]

$$f=\frac{1}{2ik}\sum_J (2J+1)^{iJ}(e^{2i\delta_J}-1)P_J(\cos\vartheta) \quad (10)$$

where $P_J(x)$ are the Legendre polynomials of order J.

In order to derive the connection between $f_{mm'}$ and the phase shifts ξ, η, ζ, we write the solution for triplet states as a linear combination of the solutions (3).

$$\psi_1{}^M=\frac{1}{kr}\sum_{J,S}{}^MC_{J,S}\overset{M}{\omega}_{J,S}$$
$$=e^{ikz}u_1{}^M+\text{scattered wave.} \quad (11)$$

The plane wave part will be a superposition of solutions of the force-free equation for which we have

$$\overset{0M}{\omega}_{J,-}=(kr)^{\frac{1}{2}}J_{J-1/2}(kr)W_{J,J-1},$$
$$\overset{0M}{\omega}_{J,+}=(kr)^{\frac{1}{2}}J_{J+3/2}(kr)W_{J,J+1},$$
$$\overset{0M}{\omega}_{J,0}=(kr)^{\frac{1}{2}}J_{J+1/2}(kr)W_{J,J},$$
$$e^{ikz}u_1{}^M=\frac{1}{kr}\sum_{J,S}{}^MB_{J,S}\overset{0M}{\omega}_{J,S}.$$

The coefficients in this series may be evaluated in terms of the well known series[15]

$$e^{ikz}u_1{}^M=\sum A_JY_J{}^0(\vartheta,\varphi)\frac{1}{\sqrt{kr}}J_{J+1/2}(kr)u_1{}^M,$$
$$A_J=2\pi i^J(2J+1)^{\frac{1}{2}},$$

by identifying the coefficients of the linearly independent functions $Y_J{}^0(1/(ur)^{\frac{1}{2}})J_{J+\frac{1}{2}}(kr)u_1{}^M$. For each value of J and M we obtain then a set of three linear equations for the three unknown coefficients ${}^MB_{J+1,-}$, ${}^MB_{J-1,+}$, ${}^MB_{J,0}$ which may be expressed in terms of the A_J. The result of a straightforward calculation along these lines is the following:

$${}^{-1}B_{J,-}=\left[\frac{J+1}{2(2J-1)}\right]^{\frac{1}{2}}A_{J-1},$$
$${}^{-1}B_{J,+}=\left[\frac{J}{2(2J+3)}\right]^{\frac{1}{2}}A_{J+1},$$
$${}^{-1}B_{J,0}=A_J,$$
$${}^{+1}B_{J,-}=\left[\frac{J+1}{2(2J-1)}\right]^{\frac{1}{2}}A_{J-1},$$
$${}^{+1}B_{J,+}=\left[\frac{J}{2(2J+3)}\right]^{\frac{1}{2}}A_{J+1},$$
$${}^{+1}B_{J,0}=-A_J,$$
$${}^{0}B_{J,-}=\left[\frac{J}{2J-1}\right]^{\frac{1}{2}}A_{J-1},$$
$${}^{0}B_{J,+}=\left[\frac{J+1}{2J+3}\right]^{\frac{1}{2}}A_{J+1},$$
$${}^{0}B_{J,0}=0.$$

From the condition that the scattered wave contains only outgoing spherical waves, we obtain

$$\sum_{J,S}({}^MC_{J,S}\overset{M}{\omega}_{J,S}-{}^MB_{J,S}\overset{0M}{\omega}_{J,S})\sim\sum_{M'}f_{MM'}u_1{}^{M'}\frac{e^{ikr}}{r}. \quad (12)$$

In order to get the expression for ${}^MC_{J,S}$ into a convenient form, we write by suppressing the indices J and M

$$\begin{aligned}C_+\alpha_+e^{-i\xi_+}+C_-\alpha_-e^{-i\xi_-}&=C\alpha e^{-i\xi},\\ C_+\beta_+e^{-i\eta_+}+C_-\beta_-e^{-i\eta_-}&=C\beta e^{-i\eta},\\ C_+\alpha_+e^{i\xi_+}+C_-\alpha_-e^{i\xi_-}&=C\alpha e^{i\xi},\\ C_+\beta_+e^{i\eta_+}+C_-\beta_-e^{i\eta_-}&=C\beta e^{i\eta},\end{aligned} \quad (13)$$

or shorter

$$C_+\omega_++C_-\omega_-=C\omega\equiv C(aW_{J,J-1}+bW_{J,J+1}),$$

where a and b behave asymptotically like

$$a\sim(2/\pi)^{\frac{1}{2}}\alpha\sin\left(kr-\frac{J-1}{2}\pi+\xi\right),$$
$$b\sim(2/\pi)^{\frac{1}{2}}\beta\sin\left(kr-\frac{J+1}{2}\pi+\eta\right).$$

[15] Cf. N. F. Mott and H. S. W. Massey, *Theory of Atomic Collisions* (Oxford University Press, 1933).

Taking suitable linear combinations of the Eqs. (13), we have

$$C_+\alpha_+ \sin \xi_+ + C_-\alpha_- \sin \xi_- = C\alpha \sin \xi,$$
$$C_+\alpha_+ \cos \xi_+ + C_-\alpha_- \cos \xi_- = C\alpha \cos \xi,$$
$$C_+\beta_+ \sin \eta_+ + C_-\beta_- \sin \eta_- = C\beta \sin \eta,$$
$$C_+\beta_+ \cos \eta_+ + C_-\beta_- \cos \eta_- = C\beta \cos \eta.$$

From this we get

$$\mathrm{tg}\xi = \frac{\alpha_+ \sin \xi_+ + \lambda\alpha_- \sin \xi_-}{\alpha_+ \cos \xi_+ + \lambda\alpha_- \cos \xi_-}, \quad \lambda = \frac{C_-}{C_+} \quad (14)$$
$$\mathrm{tg}\eta = \frac{\beta_+ \sin \eta_+ + \lambda\beta_- \sin \eta_-}{\beta_+ \cos \eta_+ + \lambda\beta_- \cos \eta_-}.$$

These formulae enable us to calculate ξ and η for each value of J, M in terms of ξ_+, ξ_-, η_+, η_-, α_+, α_-, β_+, β_- and $\lambda = C_-/C_+$. The phases and amplitudes are determined from the integration of the systems of Eqs. (4). The only quantity which is not yet determined is λ. In order to find λ we must make use of the boundary condition (7), which requires that the scattered wave contains only outgoing waves. If we put the coefficients of the incoming waves in (2) equal to zero, we obtain for each value of J and M three equations of the form

$$\alpha C e^{-i\xi} = B_-,$$
$$\beta C e^{-i\eta} = B_+, \quad (15)$$
$$C_0 e^{-i\zeta} = B_0.$$

The first two of these equations are only compatible if

$$\sigma = \frac{B_-}{B_+} = \frac{\alpha}{\beta} e^{i(\eta-\xi)} = \frac{\alpha_+ e^{-i\xi_+} + \lambda\alpha_- e^{-i\xi_-}}{\beta_+ e^{-i\eta_+} + \lambda\beta_- e^{-i\eta_-}}.$$

Solving this equation for λ we obtain

$$\lambda = -\frac{\alpha_+ e^{-i\xi_+} - \sigma\beta_+ e^{-i\eta_+}}{\alpha_- e^{-i\xi_-} - \sigma\beta_- e^{-i\eta_-}}. \quad (16)$$

For each value of J and M we can therefore determine λ from (16) and then ξ and η with the help of (14). Inserting the values of (15) for C and C_0 in the expression (12) we find for the coefficients of the outgoing spherical waves

$$^M R_{J,-} = (2/\pi)^{\frac{1}{2}} (e^{2i\xi_J{}^M} - 1) e^{-i(J-1/2)\pi}\, {}^M B_{J,-},$$
$$^M R_{J,+} = (2/\pi)^{\frac{1}{2}} (e^{2i\eta_J{}^M} - 1) e^{-i(J+1/2)\pi}\, {}^M B_{J,+},$$
$$^M R_{J,0} = (2/\pi)^{\frac{1}{2}} (e^{2i\zeta_J} - 1) e^{-i(J/2)\pi}\, {}^M B_{J,0}$$

Finally we obtain for the scattering matrix $f_{MM'}$ from (7) and (2)

$$f_{MM'} = \frac{1}{2ik} \sum_J \left\{ {}^M R_{J,-} \frac{1}{[J_1(2J-1)]^{\frac{1}{2}}} C^J_{M-M',M}(J-1,1) Y_{J-1}^{M-M'} + \frac{1}{[(J+1)(2J+3)]^{\frac{1}{2}}} {}^M R_{J,+} C^J_{M-M',M'}(J+1,1) Y_{J+1}^{M-M'} + \frac{1}{[(J+1)J]^{\frac{1}{2}}} {}^M R_{J,0} C^J_{M-M',M'}(J,1) Y_J^{M-M'} \right\}. \quad (17)$$

Inserting this expression in the formula (9) gives the desired connection between the scattering cross section and the phase shifts.

6. NUMERICAL EVALUATION

For the numerical computation of the phase shifts we must integrate the Eqs. (5) and (6) for triplet and singlet states. The following procedure was adopted: the equations were first made dimensionless by introducing the new variable $x = \kappa r$. The functions $j(x) = \lambda J$, $k(x) = \lambda K$, with $\lambda = (M/\kappa)(\kappa f)^2 = 0.532$ were then[8] calculated and tabulated. The energy E was assumed to be 14 Mev which gives for $k/\kappa = 0.6165$. The differential equation was then replaced by a difference equation with the differences in x chosen sufficiently small so as to keep the error in the final result of the order of five percent at most. It was found that a difference in x of $\Delta x = x_u - x_{u-1} = 0.2$ was sufficient for this accuracy. This was tested by carrying out sample integrations with the difference $\Delta x = 0.1$. The deviation from the result for $\Delta x = 0.2$ was taken as an approximate value of the error. This test was used both for an ordinary equation and a system. The integration was then carried out from $x = 0$ to $x = 4$. At $x = 4$ the potential functions are very small, less than $\frac{1}{10}$ percent of the value at $x = 1$. We have therefore neglected the potential functions for values of $x > 4$. The solutions for the region $4 < x < \infty$ are then the known functions of the force-free case. These functions are then pieced together at the point $x=4$ with the solutions obtained from the numerical integration of the differential equation in such a way that the function itself, as well as its derivative, is continuous. For the two systems which must be integrated, we obtained two linearly independent solutions a_+, b_+ and a_-, b_- by assuming the two different boundary conditions: $a_+(0.2)=1$, $b_+(0.2)=0$ and $a_-(0.2)=0$, $b_-(0.2)=1$, in addition to $a\pm(0)=b\pm(0)=0$.

The result of this calculation of the phase shifts is summarized in the following table:

$^3\mathbf{S}_1+{}^3\mathbf{D}_1$:

$$\xi_{1+}=+1.829$$
$$\eta_{1+}=\ 0.0068 \quad \beta_+/\alpha_+=18.96,$$
$$\xi_{1-}=\ 1.764$$
$$\eta_{1-}=\ 0.0073 \quad \beta_-/\alpha_-=22.84,$$

$^3\mathbf{P}_2+{}^3\mathbf{F}_2$:

$$\xi_{2+}=-0.0930$$
$$\eta_{2+}=+0.0655 \quad \beta_+/\alpha_+=6.95,$$
$$\xi_{2-}=-0.588$$
$$\eta_{2-}=+0.0641 \quad \beta_-/\alpha_-=118,$$

$^3\mathbf{P}_1$: $\zeta_1=-0.501$ $\quad$ $^1\mathbf{S}_0$: $\delta_0=+1.516$,
$^3\mathbf{P}_0$: $\eta_0=-0.0293$ $\quad$ $^1\mathbf{P}_1$: $\delta_1=-0.156$,
$^1\mathbf{D}_2$: $\delta_2=+0.00453$.

The next step is to determine the correct linear combination of the two solutions of the systems. For this we apply formula (16). Formula (14) will give us then the values of $\xi^M{}_J$ and $\eta^M{}_J$. Since λ is in general a complex number we have also for ξ and η in general complex numbers. The result is as follows:

$^3\mathbf{S}_1+{}^3\mathbf{D}_1$:

$$\xi_1\pm=1.679-0.0009i \qquad \xi_1{}^0=1.679-0.0018i,$$
$$\eta_1\pm=0.00267+0.0257i \qquad \eta_1{}^0=0.00583+0.0124i,$$
$$\xi_2\pm=-0.0618+0.00035i \qquad \xi_2{}^0=-0.0602+0.0005i,$$
$$\eta_2\pm=0.0521+0.0012i \qquad \eta_2{}^0=0.0554-0.0009i.$$

These quantities we must insert in formulae (17) and (9) to obtain the scattering cross section. We calculated only the ratio R of the cross section at $\vartheta=\pi$ and $\vartheta=\pi/2$. The result is:

$$R=\frac{\sigma(\pi)}{\sigma(\pi/2)}=1.1.$$

7. CONCLUSION

From the result of our calculation it can be seen that the symmetrical mixed theory gives an angular distribution in the center of mass system which is in contradiction with the experimental result which is $R=0.52\pm0.03$ for 14-Mev neutrons. Our result differs very little from the result of Schwinger and Rarita ($R=1.16$, $E=15.3$ Mev) which indicates that the preferential backwards scattering is virtually independent of the shape of the potential curve. This gives a great deal of generality to the conclusion which we want to draw from these calculations. *This conclusion is that the charge symmetrical meson theories lead to the wrong angular dependence for high energy neutron-proton scattering.*

We suggest therefore that, in the light of this result, the so-called meson pair theory of nuclear forces should be re-examined. They are known to give the same exchange operators in the nuclear force as the neutral theory.[16] But they lead to creation and absorption of charged mesons, which thus may be connected with the penetrating particles of the cosmic rays.

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