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Local supersymmetry of the Chern–Simons theory and finiteness *

Claudio Lucchesi

Max-Planck-Institut für Physik, Werner-Heisenberg-Institut, Föhringer Ring 6, Postfach 40 12 12, D-8000 Munich 40, Germany

Olivier Piguet

Département de Physique Théorique, 24, quai Ernest Ansermet, CH-1211 Geneva 4, Switzerland

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We prove the existence, for the Chern–Simons theory in curved space-time, of a renormalizable local supersymmetry and we use it to derive perturbative finiteness at all orders.

1. Introduction

The Chern–Simons theory [1–3] (for a review see ref. [4]) in $\mathbb{R}^3$ space-time formulated in the Landau gauge is known [5,6] to be invariant under a set of supersymmetry transformations whose generators form a Lorentz three-vector. Together with the generator of BRS transformations the supersymmetry generators span a Wess–Zumino type algebra which closes on space-time translations. This supersymmetric structure was essential in the proof of the finiteness of the theory as given in ref. [7].

We consider here – in perturbation theory – the Chern–Simons theory defined on an arbitrary space-time three-manifold. One result presented in this paper is the derivation of a local version of the supersymmetry mentioned above *. Furthermore, we investigate the quantized theory. We prove that this local supersymmetry is free of anomalies and show UV finiteness at all orders.

Obtaining a local version of the supersymmetry is not as straightforward as one could imagine by considering the topological nature of the Chern–Simons theory **.

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* See [8] for an earlier presentation of the classical results and ref. [9] for a brief account.

** Steps in this direction have been reported in ref. [10]. An interesting connection between supersymmetry, differential geometric structure and power-counting finiteness may be found in ref. [11].

Indeed, in the perturbative lagrangian approach we use, one has to fix the gauge by adding to the action a gauge-fixing term. In the Landau gauge – our gauge choice – the gauge fixing term depends explicitly on the metric tensor. And choosing the Landau gauge is essential: already in $\mathbb{R}^3$ space-time, the existence of the supersymmetric structure mentioned above is intimately dependent on this specific gauge choice *.

The paper is organized as follows. We start by recalling known facts about the Chern–Simons theory in $\mathbb{R}^3$ space-time and its off-shell supersymmetry algebra (sect. 2).

In sect. 3, we formulate the local theory by following the requirement of invariance under the diffeomorphisms. Our next task then is to control the metric dependence introduced by the gauge fixing. We see that the "physical" content of the theory is metric independent [2,13–17]. This is done by noticing that the metric plays the role of a gauge parameter. Its nonphysical character is made manifest through the recourse to the technique of extended BRS symmetry [18], which amounts to letting the metric transform under the BRS transformations into a Grassmann parameter. The extended BRS symmetry allows one to prove that the metric-dependent part of the action is non-physical, the latter being given by a BRS variation [18]. This is obvious at the classical level. For the quantum theory, the technique applies as well since – as we prove in sect. 7 – the extended BRS symmetry is renormalizable.

In sect. 4 we deal with the local version of the supersymmetry algebra. Superdiffeomorphisms turn out to yield the local version of the Lorentz three-vector generator. The gauge-fixed action is in general not invariant under these superdiffeomorphisms, i.e. the corresponding Ward identity is broken by a "hard" term. We are, however, able to control this breaking by coupling it to a set of external fields which transform nontrivially. The broken Ward identity can then be reformulated as an unbroken one, the breaking now being contained in the structure of the Ward operator. Doing so, we manage to establish a local supersymmetric structure for the Chern–Simons theory. Furthermore, this local supersymmetry is formulated in terms of unbroken Ward identities, a formalism that is very convenient for performing the perturbative quantization (sect. 7).

The diffeomorphisms and the superdiffeomorphisms, together with the (rigid) BRS transformations, are shown to form a closed algebra.

We prove in sect. 5 that the classical action we consider is the most general one obeying the constraints of BRS and diffeomorphism invariance, together with that of (broken) invariance under superdiffeomorphisms. There is at this stage only one free parameter. But the theory obeys an antighost equation of the same type as the one given in ref. [12] for gauge theories in the Landau gauge. It follows that the

* The fact that our results rely on the use of the Landau gauge is not surprising in view of the remarkable finiteness properties peculiar to this gauge choice [12].

parameter is fixed. This means that the classical theory is absolutely stable: no deformations are allowed.

In sect. 6 we give the conditions which the superdiffeomorphisms must obey in order to yield an exact symmetry: the vector fields along which they act must be covariantly constant *. These rigid supersymmetries reduce to the known ones [5,6] in the limit where the space-time is $\mathbb{R}^3$.

The quantum theory (sect. 7) is shown to be completely UV finite: no counterterms are allowed. Since the classical theory is stable under deformations, it is sufficient for proving perturbative finiteness to show that the constraints defining the classical theory can be maintained to all orders. In particular the local supersymmetry of the Chern–Simons theory is shown to be free of anomalies.

One remark must be added. Our strategy relies on two steps. The first step consists in considering the construction of a general ultraviolet-subtracted perturbation expansion. In the second step we apply the constraints defining the theory in order to characterize the possible free counterterms – and to show eventually their absence. We have thus to rely on a well-defined renormalization scheme. Our model is defined on an arbitrary space-time manifold, in which case such a scheme does not exist in general, for the time being. A consistent renormalization scheme can be defined in the restricted case of a manifold that is topologically equivalent to a flat manifold and which admits an asymptotically flat metric. Therefore, our results regarding the quantum theory hold rigorously only in this restricted case. Of course, all the classical results (sects. 2–6) are free of such a restriction.

2. Chern–Simons theory in $\mathbb{R}^3$ space-time

In $\mathbb{R}^3$ space-time and in the Landau gauge the classical action of Chern–Simons theory writes:

$$\Sigma\big(A_\mu, B, c, \bar{c}, \gamma^\mu, \tau\big) = \operatorname{Tr}\int d^3x\, \Big(-\epsilon^{\mu\nu\rho}\big(\tfrac{1}{2}A_\mu\partial_\nu A_\rho + \tfrac{1}{3}\lambda A_\mu A_\nu A_\rho\big)$$

$$-s\big(\partial^\nu\bar{c}A_\nu\big) - \gamma^\mu D_\mu c + \lambda\tau c^2\Big), \tag{2.1}$$

where $D_\mu c = \partial_\mu c + \lambda[A_\mu, c]$ is the covariant derivative and the coupling constant λ is related to the parameter k in the notation of ref. [2] by $\lambda^2 = 2\pi/k$. The gauge group is assumed to be simple. All fields ϕ belong to its adjoint representation and

* This is quite analogous to the case of global supersymmetries in curved 4-dimensional space-time – e.g., anti-de Sitter – generated by Killing spinor fields [19–21].

we use the matrix notation $\phi = \phi^a \lambda_a$. γ^μ and τ are external fields coupled to the nonlinear variations of the fields A_μ and c under the BRS transformation s:

$$sA_\mu = -\mathrm{D}_\mu c, \qquad s\bar{c} = B,$$
$$sc = \lambda c^2, \qquad sB = 0. \tag{2.2}$$

The BRS invariance of the theory is implemented at the functional level by the Slavnov identity:

$$\mathscr{S}(\Sigma) = \mathrm{Tr}\int \mathrm{d}^3x \left(\frac{\delta\Sigma}{\delta\gamma^\mu}\frac{\delta\Sigma}{\delta A_\mu} + \frac{\delta\Sigma}{\delta\tau}\frac{\delta\Sigma}{\delta c} + B\frac{\delta\Sigma}{\delta\bar{c}} \right) = 0. \tag{2.3}$$

The associated linearized Slavnov operator is

$$\mathscr{S}_\Sigma = \mathrm{Tr}\int \mathrm{d}^3x \left(\frac{\delta\Sigma}{\delta\gamma^\mu}\frac{\delta}{\delta A_\mu} + \frac{\delta\Sigma}{\delta A_\mu}\frac{\delta}{\delta\gamma^\mu} + \frac{\delta\Sigma}{\delta\tau}\frac{\delta}{\delta c} + \frac{\delta\Sigma}{\delta c}\frac{\delta}{\delta\tau} + B\frac{\delta}{\delta\bar{c}} \right). \tag{2.4}$$

Note that it acts on the fields A_μ, c, $\bar{c}$ and B like the operator s in (2.2); on the external fields γ^μ and τ its action, which we shall also denote by s, is given by

$$s\gamma^\mu \equiv \mathscr{S}_\Sigma\gamma^\mu = \frac{\delta\Sigma}{\delta A_\mu}, \qquad s\tau \equiv \mathscr{S}_\Sigma\tau = \frac{\delta\Sigma}{\delta c}. \tag{2.5}$$

The Chern–Simons model in $\mathbb{R}^3$ space-time has been shown to be invariant under three further (linear) supersymmetry transformations whose generators form a Lorentz three-vector [5]. The corresponding Ward identities are

$$\mathscr{V}_\mu\Sigma = \Delta_\mu^{\mathrm{cl}} = \mathrm{Tr}\int \mathrm{d}^3x \left(-\gamma^\nu\partial_\mu A_\nu - \epsilon_{\mu\nu\rho}\gamma^\nu\partial^\rho B + \tau\partial_\mu c \right). \tag{2.6}$$

The r.h.s. is a classical breaking which, being linear in the quantum fields, can be controlled in higher orders. The Ward operator $\mathscr{V}_\mu$ in (2.6) has the form

$$\mathscr{V}_\mu = \mathrm{Tr}\int \mathrm{d}^3x \left(-\epsilon_{\mu\nu\rho}(\gamma^\rho + \partial^\rho\bar{c})\frac{\delta}{\delta A_\nu} - A_\mu\frac{\delta}{\delta c} + \partial_\mu\bar{c}\frac{\delta}{\delta B} - \delta_\mu^\nu\tau\frac{\delta}{\delta\gamma^\nu} \right), \tag{2.7}$$

it acts on the fields as

$$\mathscr{V}_\mu c = -A_\mu, \qquad \mathscr{V}_\mu B = \partial_\mu\bar{c},$$
$$\mathscr{V}_\mu A_\nu = -\epsilon_{\mu\nu\rho}(\gamma^\rho + \partial^\rho\bar{c}), \qquad \mathscr{V}_\mu\bar{c} = 0,$$
$$\mathscr{V}_\mu\gamma^\nu = -\delta_\mu^\nu\tau, \qquad \mathscr{V}_\mu\tau = 0. \tag{2.8}$$

For any functional of the fields $\Gamma = \Gamma(A_\mu, B, c, \bar{c}, \gamma^\mu, \tau)$ the Slavnov operator $\mathscr{S}$, its linearized version $\mathscr{S}_\Gamma$ and the three Ward operators $\mathscr{V}_\mu$ form a nonlinear algebra:

$$\mathscr{S}_\Gamma \mathscr{S}(\Gamma) = 0,$$

$$\{\mathscr{V}_\mu, \mathscr{V}_\nu\}\Gamma = 0,$$

$$\mathscr{V}_\mu \mathscr{S}(\Gamma) + \mathscr{S}_\Gamma\left(\mathscr{V}_\mu \Gamma - \Delta_\mu^{\mathrm{cl}}\right) = \mathscr{P}_\mu \Gamma. \tag{2.9}$$

This algebra closes on the Ward operator of translations $\mathscr{P}_\mu$:

$$\mathscr{P}_\mu = \mathrm{Tr}\int \mathrm{d}^3x \sum_{\text{all fields } \phi} \partial_\mu \phi \frac{\delta}{\delta\phi}. \tag{2.10}$$

In terms of the linear Ward operators $\mathscr{S}_\Sigma$ and $\mathscr{V}_\mu$ only, one has – if Σ is a functional satisfying eqs. (2.3) and (2.6) – the following algebra:

$$\mathscr{S}_\Sigma^2 = 0, \qquad \{\mathscr{V}_\mu, \mathscr{V}_\nu\} = 0, \qquad \{\mathscr{V}_\mu, \mathscr{S}_\Sigma\} = \mathscr{P}_\mu, \tag{2.11}$$

which has the form of an off-shell $N = 1$ supersymmetry algebra. Note that the number of bosonic (Faddeev–Popov even) degrees of freedom A_μ, B, τ equals the number of fermionic (FP odd) ones c, $\bar{c}$, γ^μ.

3. Local theory and metric dependence

In order to define the theory on a three-manifold $\mathscr{M}$, let us impose the invariance of the classical action under diffeomorphisms, i.e. local translation symmetry. Under diffeomorphisms, the gauge field A_μ is a covariant vector, the Lagrange multiplier B as well as the ghosts and antighosts c and $\bar{c}$ behave like scalars, γ^μ is a contravariant vector density and τ a scalar density. The corresponding weights are given in table 1 of sect. 5.

The action of the local theory on $\mathscr{M}$ consists of three pieces:

$$\Sigma = \Sigma_{\mathrm{C.S.}} + \Sigma_{\mathrm{gf}} + \Sigma_{\mathrm{ext}}. \tag{3.1}$$

In terms of the 1-form $A = A_\mu \, \mathrm{d}x^\mu$, the pure Chern–Simons action $\Sigma_{\mathrm{C.S.}}$ writes *:

$$\Sigma_{\mathrm{C.S.}}(A) = -\tfrac{1}{2}\mathrm{Tr}\int_{\mathscr{M}}\left(A \, \mathrm{d}A + \tfrac{2}{3}\lambda A^3\right). \tag{3.2}$$

* The wedge product symbol shall be omitted.

We further introduce a dual 2-form $\tilde{\gamma}$ and a dual 3-form $\tilde{\tau}$ associated to the external fields γ^μ and τ:

$$\tilde{\gamma} = \tfrac{1}{2}\epsilon_{\mu\nu\rho}\gamma^\rho \,\mathrm{d}x^\mu \,\mathrm{d}x^\nu, \qquad \tilde{\tau} = \tfrac{1}{6}\epsilon_{\mu\nu\rho}\tau \,\mathrm{d}x^\mu \,\mathrm{d}x^\nu \,\mathrm{d}x^\rho, \tag{3.3}$$

where $\epsilon_{\mu\nu\rho}$ is the antisymmetric tensor density defined by $\epsilon_{012} = 1$. The contribution Σ_{ext} from the external fields γ^μ and τ to the action (3.1) hence reads

$$\Sigma_{\text{ext}}(A, c, \tilde{\gamma}, \tilde{\tau}) = \mathrm{Tr}\int_{\mathcal{M}} (\tilde{\gamma}\mathrm{D}c + \lambda\tilde{\tau}c^2). \tag{3.4}$$

d denotes the exterior derivative and $\mathrm{D}c = \mathrm{d}c + \lambda\{A, c\}$ is the covariant one *; the former is nilpotent and anticommutes with s,

$$\mathrm{d}^2 = 0, \quad \{s, \mathrm{d}\} = 0. \tag{3.5}$$

Note that the pure Chern–Simons and external field contributions to the action (3.1) are both of topological nature. The remaining piece of the action Σ_{gf} is the gauge-fixing for the (covariant) Landau gauge; it is the only nontopological contribution to Σ:

$$\Sigma_{\text{gf}}(A, B, c, \bar{c}, g_{\mu\nu}) = -\mathrm{Tr}\int \mathrm{d}^3x \, s\left(\sqrt{g}\, g^{\mu\nu}\partial_\mu \bar{c} A_\nu\right), \tag{3.6}$$

i.e. it depends explicitly on the metric **, considered here as an external field. In the limit of $\mathbb{R}^3$ space-time, $g_{\mu\nu}$ is just the minkowskian or the euclidean metric and one recovers from (3.1) the action (2.1).

In the general case, the metric plays the role of a gauge parameter and will be treated as such. Accordingly, we extend the BRS transformation [18] to transform $g_{\mu\nu}$ in order to make explicit the nonphysical character of the metric. The BRS transformation (2.2), which writes in terms of differential forms:

$$sA = \mathrm{D}c, \qquad s\bar{c} = B, \tag{3.7}$$

$$sc = \lambda c^2, \qquad sB = 0. \tag{3.8}$$

* Our convention for the exterior derivative d on any form ω is: $\mathrm{d}\omega = \mathrm{d}x^\mu\partial_\mu\omega$. The grading is defined as the sum of the ghost number and the form degree; hence the anticommutator in $\mathrm{D}c$.

** In our convention $g^{\mu\nu}$ is the inverse of the metric tensor and $g = \det(g_{\mu\nu})$, the latter beeing a scalar density of weight 2. Moreover, the Levi-Civita density $\epsilon^{\mu\nu\rho}$ (defined by $\epsilon^{012} = 1$) has weight 1 and the scalar volume element density d^3x has weight -1. The relation between $\epsilon^{\mu\nu\rho}$ and the tensor density $\epsilon_{\mu\nu\rho}$ defined above (see (3.3)) is given by $\epsilon_{\mu\nu\rho} = g_{\mu\alpha}g_{\nu\beta}g_{\rho\gamma}g^{-1}\epsilon^{\alpha\beta\gamma}$, the weight of $\epsilon_{\mu\nu\rho}$ is -1. Hence, the three pieces of the action (3.1) are invariant under diffeomorphisms, provided the weights of the fields are those in table 1 of sect. 5.

hence acts now also on $g_{\mu\nu}$ as

$$s g_{\mu\nu} = \hat{g}_{\mu\nu}, \qquad s\hat{g}_{\mu\nu} = 0. \tag{3.9}$$

As a consequence, the gauge-fixing action (3.6) contains a $\hat{g}_{\mu\nu}$-dependent term:

$$-\operatorname{Tr}\int \mathrm{d}^3x\, s\big(\sqrt{g}\, g^{\mu\nu}\big)\partial_\mu \bar{c} A_\nu . \tag{3.10}$$

Since the metric now transforms under BRS, the Slavnov identity has to be extended [18] accordingly *

$$\mathscr{S}(\Sigma) = \operatorname{Tr}\int_{\mathscr{M}}\left(\frac{\partial\Sigma}{\delta\tilde{\gamma}}\frac{\delta\Sigma}{\delta A} + \frac{\delta\Sigma}{\delta\tilde{\tau}}\frac{\delta\Sigma}{\delta c} + B\frac{\delta\Sigma}{\delta\bar{c}}\right) + \int \mathrm{d}^3x\, \hat{g}_{\mu\nu}\frac{\delta\Sigma}{\delta g_{\mu\nu}} = 0. \tag{3.11}$$

This last identity allows one to control the dependence of the classical theory on the metric $g_{\mu\nu}$. For this being feasible also at the quantum level, one has to show that the extended Slavnov identity (3.11) can be renormalized, i.e., it is not plagued by anomalies – a result we prove in sect. 7.

Let us come back to the classical action Σ; it is moreover characterized by the gauge condition

$$\frac{\delta\Sigma}{\delta B} = \partial_\mu\big(\sqrt{g}\, g^{\mu\nu} A_\nu\big), \tag{3.12}$$

and by the ghost equation of motion

$$\left(\frac{\delta}{\delta\bar{c}} + \partial_\mu\left(\sqrt{g}\, g^{\mu\nu}\frac{\delta}{\delta\gamma^\nu}\right)\right)\Sigma = -\partial_\mu\big(s\big(\sqrt{g}\, g^{\mu\nu}\big)A_\nu\big), \tag{3.13}$$

which follows from the Slavnov identity and the gauge condition.

The action Σ can be split into a B-independent part $\bar{\Sigma}$ (solution of the homogeneous gauge condition) and a rest

$$\Sigma\big(A, B, c, \bar{c}, \tilde{\gamma}, \tilde{\tau}, g_{\mu\nu}, \hat{g}_{\mu\nu}\big) = \bar{\Sigma}\big(A, c, \bar{c}, \tilde{\gamma}, \tilde{\tau}, g_{\mu\nu}, \hat{g}_{\mu\nu}\big)$$
$$-\operatorname{Tr}\int \mathrm{d}^3x\, \big(\sqrt{g}\, g^{\mu\nu}\partial_\mu B A_\nu + s\big(\sqrt{g}\, g^{\mu\nu}\big)\partial_\mu\bar{c}A_\nu\big). \tag{3.14}$$

Moreover, $\bar{\Sigma}$ obeys the homogeneous ghost equation; it therefore depends on γ^μ

* The functional derivatives with respect to differential forms are to be understood in the following sense: $\delta S/\delta f = X$, if $S = \int_{\mathscr{M}} fX$, where f is a generic differential form.

and $\bar{c}$ only through the combination $\Omega^\mu = \gamma^\mu + \sqrt{g}\, g^{\mu\sigma}\partial_\sigma \bar{c}$, which reads as a 2-form:

$$\tilde{\Omega} = \tilde{\gamma} + \tfrac{1}{2}\epsilon_{\mu\nu\rho}\sqrt{g}\, g^{\mu\sigma}\partial_\sigma \bar{c}\ \mathrm{d}x^\nu\ \mathrm{d}x^\rho. \tag{3.15}$$

A linearized Slavnov operator $\mathcal{S}_\Sigma$ can be defined similarly to eq. (2.4), but now with an additional metric-varying term:

$$\mathcal{S}_\Sigma = \mathrm{Tr}\int_{\mathcal{M}}\left(\frac{\delta\Sigma}{\delta\tilde{\gamma}}\frac{\delta}{\delta A} + \frac{\delta\Sigma}{\delta A}\frac{\delta}{\delta\tilde{\gamma}} + \frac{\delta\Sigma}{\delta\tilde{\tau}}\frac{\delta}{\delta c} + \frac{\delta\Sigma}{\delta c}\frac{\delta}{\delta\tilde{\tau}} + B\frac{\delta}{\delta\bar{c}}\right) + \int \mathrm{d}^3x\, \hat{g}_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}. \tag{3.16}$$

The action of $\mathcal{S}_\Sigma$ (which we keep denoting by s) on the individual fields and on the metric is given by (3.7)–(3.9) and

$$s\tilde{\gamma} \equiv \mathcal{S}_\Sigma\tilde{\gamma} = \frac{\delta\Sigma}{\delta A}, \qquad s\tilde{\tau} \equiv \mathcal{S}_\Sigma\tilde{\tau} = \frac{\delta\Sigma}{\delta c}, \qquad s\tilde{\Omega} \equiv \mathcal{S}_\Sigma\tilde{\Omega} = \frac{\delta\bar{\Sigma}}{\delta\ [illegible]}. \tag{3.17}$$

4. Local supersymmetry

What is the local version of the superalgebra $\{\mathcal{S}_\Sigma, \mathcal{V}_\mu, \mathcal{P}_\mu\}$ * in eq. (2.11) – the supersymmetry algebra in $\mathbb{R}^3$ space-time – for the case where the Chern–Simons theory is defined on an arbitrary space-time manifold $\mathcal{M}$?

Local translations are just diffeomorphisms (see sect. 3). At the functional level the invariance of the classical action (3.1) under these is expressed by the Ward identity

$$\mathcal{W}^{\mathrm{D}}_{(\epsilon)}\Sigma = \mathrm{Tr}\int_{\mathcal{M}}\sum_f \mathcal{L}_\epsilon f\frac{\delta\Sigma}{\delta f} + \int \mathrm{d}^3x\left(\mathcal{L}_\epsilon g_{\mu\nu}\frac{\delta\Sigma}{\delta g_{\mu\nu}} + \mathcal{L}_\epsilon \hat{g}_{\mu\nu}\frac{\delta\Sigma}{\delta\hat{g}_{\mu\nu}}\right) = 0, \tag{4.1}$$

where the summation runs over $f = A, B, c, \bar{c}, \tilde{\gamma}, \tilde{\tau}$. The infinitesimal transformation is given by the Lie derivative $\mathcal{L}_\epsilon = i_\epsilon\, \mathrm{d} + \mathrm{d}i_\epsilon$ along the vector field ϵ; i_ϵ is the inner product, $i_\epsilon\, \mathrm{d}x^\mu = \epsilon^\mu$.

The local version of the supersymmetry generators $\mathcal{V}_\mu$ in (2.11) turns out to be given by superdiffeomorphisms. These lead, together with the BRS transforma-

* In this minimal localization scheme, the BRS transformation s remains rigid.

tions and the diffeomorphisms, to a closed algebra (see below). The superdiffeomorphisms act infinitesimally as *:

$$\begin{aligned}
&\delta^{S}_{(\xi)}c = -i_\xi A, && \delta^{S}_{(\xi)}B = \mathscr{L}_\xi \bar{c},\\
&\delta^{S}_{(\xi)}A = i_\xi \tilde{\Omega}, && \delta^{S}_{(\xi)}\bar{c} = 0,\\
&\delta^{S}_{(\xi)}\tilde{\Omega} = \delta^{S}_{(\xi)}\tilde{\gamma} = -i_\xi \tilde{\tau}, && \delta^{S}_{(\xi)}\hat{g}_{\mu\nu} = \mathscr{L}_\xi g_{\mu\nu},\\
&\delta^{S}_{(\xi)}\tilde{\tau} = 0, && \delta^{S}_{(\xi)}g_{\mu\nu} = 0,
\end{aligned} \tag{4.2}$$

where ξ is an odd ** vector field. The Ward operator associated to the superdiffeomorphisms writes:

$$\mathscr{W}^{S}_{(\xi)} = \int_{\mathscr{M}} \sum_{\phi} \delta^{S}_{(\xi)}\phi \frac{\delta}{\delta\phi}. \tag{4.3}$$

The summation runs over all fields (forms), the metric, etc; traces are understood. The application of this operator (4.3) to the action (3.1) yields:

$$\mathscr{W}^{S}_{(\xi)}\Sigma = \Delta^{\text{cl}}_{(\xi)} + \int \mathrm{d}^3x\, g_{\mu\rho}\xi^\rho s\Xi^\mu, \tag{4.4}$$

where:

$$\Delta^{\text{cl}}_{(\xi)} = \mathrm{Tr}\int_{\mathscr{M}}\left(-\tilde{\gamma}\mathscr{L}_\xi A + \tilde{\tau}\mathscr{L}_\xi c\right) - \mathrm{Tr}\int \mathrm{d}^3x\ \xi^\mu \epsilon_{\mu\nu\rho}\gamma^\nu s\left(\sqrt{g}\, g^{\rho\sigma}\partial_\sigma \bar{c}\right), \tag{4.5}$$

$$s\Xi^\mu = -\mathrm{Tr}\ \epsilon^{\mu\nu\rho}\partial_\nu\bar{c}\partial_\rho B, \qquad \text{with} \qquad \Xi^\mu = \tfrac{1}{2}\mathrm{Tr}\ \epsilon^{\mu\nu\rho}\partial_\nu\bar{c}\partial_\rho\bar{c}. \tag{4.6}$$

The invariance under superdiffeomorphisms is seen to be broken. Whereas the first term in the right hand side of (4.4) is nothing else than a generalization of the "classical" breaking of the supersymmetry Ward identity (2.6), the second term is a genuine, "hard" (nonlinear in the quantum fields) breaking. One checks that it disappears in particular in the limit of constant metric and constant ξ since Ξ^μ as given by (4.6) is a total derivative.

* With the change of variable (3.15) it is apparent that the field representation of superdiffeomorphisms reduces to three irreducible representations $\{c,\ A,\ \tilde{\Omega},\ \tilde{\tau}\}$, $\{B,\ \bar{c}\}$ and $\{\hat{g}_{\mu\nu},\ g_{\mu\nu}\}$. In component language, the transformations involving i_ξ read: $\delta^{S}_{(\xi)}c = -\xi^\mu A_\mu$, $\delta^{S}_{(\xi)}A_\mu = \epsilon_{\mu\nu\rho}\xi^\nu\Omega^\rho$, $\delta^{S}_{(\xi)}\Omega^\mu = \delta^{S}_{(\xi)}\gamma^\mu = -\xi^\mu\tau$.

** ξ being odd, the inner derivative i_ξ is a derivation, instead of an antiderivation as i_ϵ. The Lie derivative $\mathscr{L}_\xi = i_\xi\,\mathrm{d} - \mathrm{d}i_\xi$ is an antiderivation.

In order to keep this "hard" breaking under control [22], we couple the field polynomials Ξ^μ and $s\Xi^\mu$ to two external covariant vector fields * L_μ and M_μ, i.e. we add a term

$$\Sigma_{L,M} = \int d^3x \, \left(L_\mu \Xi^\mu - M_\mu s\Xi^\mu \right) \tag{4.7}$$

to the action. Invariance under the diffeomorphisms is guaranteed if $L = L_\mu \, dx^\mu$ and $M = M_\mu \, dx^\mu$ transform as 1-forms. In order to preserve BRS invariance and nilpotency, and in order to write the broken superdiffeomorphisms Ward identity (4.4) in the way of an unbroken one – up to the classical breaking (4.5) – we impose the transformation rules:

$$sM = L, \qquad \delta^{S}_{(\xi)} M = g_{\mu\rho}\xi^\rho dx^\mu, \tag{4.8}$$

$$sL = 0, \qquad \delta^{S}_{(\xi)} L = \mathcal{L}_\xi M. \tag{4.9}$$

The summations over all fields in the previously defined Ward and Slavnov operators *include from now on the new fields M and L.*

The Slavnov identity (3.11) and the Ward identity of diffeomorphisms (4.1) are otherwise unchanged, whereas the gauge condition (3.12) and the ghost equation (3.13) are modified into:

$$\frac{\delta\Sigma}{\delta B} = \partial_\mu\left(\sqrt{g}\, g^{\mu\nu} A_\nu\right) - \epsilon^{\mu\nu\rho}\partial_\mu\left(M_\nu \partial_\rho \bar{c}\right), \tag{4.10}$$

$$\left(\frac{\delta}{\delta\bar{c}} + \partial_\mu\left(\sqrt{g}\, g^{\mu\nu} \frac{\delta}{\delta\gamma^\nu} \right) \right)\Sigma = -\partial_\mu\left(s\left(\sqrt{g}\, g^{\mu\nu}\right) A_\nu \right) + \epsilon^{\mu\nu\rho}\partial_\mu\left(L_\nu \partial_\rho \bar{c} - M_\nu \partial_\rho B \right). \tag{4.11}$$

As a consequence of enforcing the transformations (4.8) (4.9) and of extending the summation in (4.3) to run also on $\phi = M,\ L$, the Ward identity of superdiffeomorphisms takes the simpler, functional form:

$$\mathcal{W}^{S}_{(\xi)} \Sigma = \Delta^{\text{cl}}_{(\xi)}. \tag{4.12}$$

This result can be interpreted as establishing a supersymmetric structure at the local level. The "hard" classical breaking is now contained in the structure of the Ward operator. Putting things in this way – at the price of introducing the external fields M and L – will reveal very convenient for the renormalization of the local supersymmetry, performed in sect. 7.

* The weights, dimensions and ghost numbers of L_μ and M_μ are shown in table 1 of sect. 5.

The algebra of the BRS transformations, of the diffeomorphisms and of the superdiffeomorphisms is given by:

$$
\begin{aligned}
&\mathcal{S}_\Gamma \mathcal{S}(\Gamma) = 0, && \mathcal{W}^{\rm S}_{(\xi)} \mathcal{S}(\Gamma) - \mathcal{S}_\Gamma\left(\mathcal{W}^{\rm S}_{(\xi)}\Gamma - \Delta^{\rm cl}_{(\xi)}\right) = \mathcal{W}^{\rm D}_{(\xi)}\Gamma, \\
&\mathcal{S}_\Gamma \mathcal{W}^{\rm D}_{(\epsilon)}\Gamma - \mathcal{W}^{\rm D}_{(\epsilon)}\mathcal{S}(\Gamma) = 0, && \left[\mathcal{W}^{\rm S}_{(\xi)}, \mathcal{W}^{\rm D}_{(\epsilon)}\right]\Gamma = -\mathcal{W}^{\rm S}_{([\xi,\epsilon])}\Gamma, \\
&\left[\mathcal{W}^{\rm D}_{(\epsilon)}, \mathcal{W}^{\rm D}_{(\epsilon')}\right]\Gamma = -\mathcal{W}^{\rm D}_{([\epsilon,\epsilon'])}\Gamma, && \left[\mathcal{W}^{\rm S}_{(\xi)}, \mathcal{W}^{\rm S}_{(\xi')}\right]\Gamma = 0, \qquad (4.13)
\end{aligned}
$$

where $[\epsilon, \epsilon']^\mu = \mathcal{L}_\epsilon \epsilon'^\mu = \epsilon^\lambda \partial_\lambda \epsilon'^\mu - \epsilon'^\lambda \partial_\lambda \epsilon^\mu$ is the Lie bracket of the vector fields ϵ and ϵ'. Eqs. (4.13) are valid for any functional $\Gamma = \Gamma(A, B, c, \bar{c}, \tilde{\gamma}, \tilde{\tau}, g_{\mu\nu}, \hat{g}_{\mu\nu}, M, L)$. If the functional Σ is a solution of the extended Slavnov identity, of the Ward identity of diffeomorphisms and of the Ward identity of superdiffeomorphisms, then the linear algebra is given by:

$$
\begin{aligned}
&\mathcal{S}^2_\Sigma = 0, && \left[\mathcal{W}^{\rm S}_{(\xi)}, \mathcal{S}_\Sigma\right] = \mathcal{W}^{\rm D}_{(\xi)}, \\
&\left[\mathcal{S}_\Sigma, \mathcal{W}^{\rm D}_{(\epsilon)}\right] = 0, && \left[\mathcal{W}^{\rm S}_{(\xi)}, \mathcal{W}^{\rm D}_{(\epsilon)}\right] = -\mathcal{W}^{\rm S}_{([\xi,\epsilon])}, \\
&\left[\mathcal{W}^{\rm D}_{(\epsilon)}, \mathcal{W}^{\rm D}_{(\epsilon')}\right] = -\mathcal{W}^{\rm D}_{([\epsilon,\epsilon'])}, && \left[\mathcal{W}^{\rm S}_{(\xi)}, \mathcal{W}^{\rm S}_{(\xi')}\right] = 0. \qquad (4.14)
\end{aligned}
$$

As expected from the case of $\mathbb{R}^3$ space-time, the superdiffeomorphisms commuted with the BRS transformations yield diffeomorphisms. But unlike the fact that the translations commute with supersymmetry in this same $\mathbb{R}^3$ situation, the diffeomorphisms and the superdiffeomorphisms do not commute in general.

5. Stability

In the two preceding sections we have constructed an action (see eqs. (3.1) and (4.7)) which we denote here by Σ:

$$
\Sigma = \Sigma_{\rm C.S.} + \Sigma_{\rm g.f.} + \Sigma_{\rm ext.} + \Sigma_{L,M} \qquad (5.1)
$$

and which is the solution of

(i) the gauge condition and of the ghost equation of motion (4.10) and (4.11),

(ii) the extended Slavnov identity

$$
\mathcal{S}(\Sigma) = \int_{\mathcal{M}} \left[\mathrm{Tr}\left(\frac{\delta\Sigma}{\delta\tilde{\gamma}} \frac{\delta\Sigma}{\delta A} + \frac{\delta\Sigma}{\delta\tilde{\tau}} \frac{\delta\Sigma}{\delta c} + B \frac{\delta\Sigma}{\delta\bar{c}} \right) + L \frac{\delta\Sigma}{\delta M} \right] + \int \mathrm{d}^3x\, \hat{g}_{\mu\nu} \frac{\delta\Sigma}{\delta g_{\mu\nu}} = 0, \qquad (5.2)
$$

(iii) the Ward identity of diffeomorphisms (4.1) with the summation running now also on L, M,
(iv) the Ward identity of superdiffeomorphisms (4.12).
(v) Moreover this action obeys the antighost equation * [12]

$$\int \mathrm{d}^3x \left(\frac{\delta \Sigma}{\delta c} + \lambda \left[\bar{c}, \frac{\delta \Sigma}{\delta B} \right] \right) = \int_{\mathscr{M}} \lambda(\{\tilde{\gamma}, A\} + \{\tilde{\tau}, c\}). \tag{5.3}$$

Note that the r.h.s. – in analogy to the r.h.s. of the Ward identity of superdiffeomorphisms (4.12) – consists of terms linear in the quantum fields A and c.

We want now to study the stability of the theory, i.e. we look for the most general allowed deformation of the classical solution Σ of the constraints. This amounts to consider a perturbed action

$$\Sigma' = \Sigma + \Delta, \tag{5.4}$$

where Δ is an integrated local field polynomial of dimension zero ** and ghost number zero, and to require it to obey all the constraints listed above. The perturbation Δ is hence constrained by

$$\frac{\delta \Delta}{\delta B} = 0, \tag{5.5}$$

$$\left(\frac{\delta}{\delta \bar{c}} + \partial_\mu \left(\sqrt{g}\, g^{\mu\nu} \frac{\delta}{\delta \gamma^\nu} \right) \right) \Delta = 0, \tag{5.6}$$

$$\mathscr{S}_\Sigma \Delta = 0, \tag{5.7}$$

$$\mathscr{W}^{\mathrm{D}}_{(\epsilon)} \Delta = 0, \tag{5.8}$$

$$\mathscr{W}^{\mathrm{S}}_{(\xi)} \Delta = 0, \tag{5.9}$$

$$\int \mathrm{d}^3x \left(\frac{\delta \Delta}{\delta c} + \lambda \left[\bar{c}, \frac{\delta \Delta}{\delta B} \right] \right) = 0. \tag{5.10}$$

The first two equations imply that $\Delta = \Delta(A, c, \tilde{\Omega}, \tilde{\tau}, g_{\mu\nu}, \hat{g}_{\mu\nu}, M, L)$: Δ is independent of B and depends on $\bar{c}$ and $\tilde{\gamma}$ only through the combination $\tilde{\Omega}$ (3.15). The last equation is then reduced to

$$\int \mathrm{d}^3x \, \frac{\delta \Delta}{\delta c} = 0. \tag{5.11}$$

* The derivation of the antighost equation follows the lines of the proof presented in ref. [12] and will not be repeated here.
** The integration measure has dimension −3.

TABLE 1
Weights, dimensions and ghost numbers

Field	A_μ	B	c	$\bar{c}$	γ^μ	Ω^μ	τ	M_μ	L_μ	ϵ^μ	ξ^μ
Weight W	0	0	0	0	1	1	1	0	0	0	0
Dimension	1	1	0	1	2	2	3	−1	−1	−1	−1
Ghost number	0	0	1	−1	−1	−1	−2	1	2	1	2

The remaining equations (5.7)–(5.9) are conveniently condensed into a single cohomology problem:

$$\delta \Delta = 0, \tag{5.12}$$

where

$$\delta = \mathscr{S}_\Sigma + \mathscr{W}^{\mathrm{D}}_{(\epsilon)} + \mathscr{W}^{\mathrm{S}}_{(\xi)}. \tag{5.13}$$

The operator δ is made a coboundary operator

$$\delta^2 = 0 \tag{5.14}$$

by changing the statistics of the vector fields ϵ and ξ, which now become, respectively, odd and even, and by defining the action of δ on the latter to be

$$\delta\epsilon = \tfrac{1}{2}[\epsilon, \epsilon] - \xi, \qquad \delta\xi = [\epsilon, \xi], \tag{5.15}$$

where [,] is the graded Lie bracket. The ghost numbers attributed to ϵ and ξ are shown in table 1. Note that with these new conventions, both the Ward operators $\mathscr{W}^{\mathrm{D}}_{(\epsilon)}$ and $\mathscr{W}^{\mathrm{S}}_{(\xi)}$ carry ghost number one, just as the linearized Slavnov operator $\mathscr{S}_\Sigma$.

Noticing that δ-invariance (5.12) of the perturbation Δ involves its invariance under the diffeomorphisms (condition (5.8)), we restrict the search of solutions to the space of diffeomorphism invariant integrated local functionals.

In order to solve the cohomology problem (5.12), it is useful to split the operator δ [23, p. 202] as

$$\delta = \delta_0 + \delta_1 \tag{5.16}$$

where δ_0 is the part of δ which does not increase the homogeneity degree * in the fields, whereas δ_1 increases the degree by one. Both δ_0 and δ_1 are nilpotent:

$$\delta_0^2 = \{\delta_0, \delta_1\} = \delta_1^2 = 0. \tag{5.17}$$

The reason for performing this split is that the cohomology of δ, i.e. classes of solutions Δ of (5.12) which are equivalent modulo coboundaries $\delta\hat{\Delta}$, is isomorphic

* Degree 1 is attributed to all the fields of the theory, $g_{\mu\nu}$, $\hat{g}_{\mu\nu}$, ϵ and ξ included.

to a subspace of the cohomology of δ_0 (see ref. [23,24]). Hence, let us begin by solving the latter:

$$\delta_0 \Delta\Big(A,\, c,\, \tilde{\Omega},\, \tilde{\tau},\, g_{\mu\nu},\, \hat{g}_{\mu\nu},\, M,\, L\Big) = 0. \tag{5.18}$$

The action of the coboundary operator δ_0 on the fields comes from summing the homogeneity degree preserving parts of the BRS transformations, of the diffeomorphisms and of the superdiffeomorphisms. It reads

$$\begin{aligned} \delta_0 A &= \mathrm{d}c, & \delta_0 c &= 0, \\ \delta_0 \tilde{\Omega} &= -\mathrm{d}A, & \delta_0 \tilde{\tau} &= \mathrm{d}\tilde{\Omega}, \\ \delta_0 u &= v, & \delta_0 v &= 0, \end{aligned} \tag{5.19}$$

where (u, v) stands generically for the doublets $(g_{\mu\nu}, \hat{g}_{\mu\nu})$, (M, L) and $(\epsilon, -\xi)$.

We first remark that, according to a general result [23,25], the δ_0 cohomology classes are represented by local field functionals of homogeneity degree zero in the doublet fields u and v. In the present situation this implies independence from the fields M, L, ϵ, ξ, $g_{\mu\nu}$ and $\hat{g}_{\mu\nu}$. Note that although the dependence on the metric is in general not polynomial, there is no possible $g_{\mu\nu}$- or $\hat{g}_{\mu\nu}$-dependent contribution to Δ which is both invariant under the diffeomorphisms and homogeneous of degree zero * in $g_{\mu\nu}$ and $\hat{g}_{\mu\nu}$.

We are thus left with solving (5.18) for an integrated polynomial $\Delta(A, c, \tilde{\Omega}, \tilde{\tau})$ of dimension zero and zero ghost number. Let us first write

$$\Delta = \int_{\mathcal{M}} f_3^0, \tag{5.20}$$

where f_p^q is a local field polynomial of form degree p and ghost number q. The cohomology condition (5.18) is equivalent to

$$\delta_0 f_3^0 = \mathrm{d} f_2^1. \tag{5.21}$$

Due to the nilpotency of δ_0, to its anticommutativity with d and to the triviality of the d-cohomology in the space of local field polynomials [25], this leads to a chain of so-called "descent equations" connecting f_3^0 to a local polynomial f_0^3 of ghost number 3 and form degree 0:

$$\begin{aligned} \delta_0 f_2^1 &= \mathrm{d} f_1^2, \\ \delta_0 f_1^2 &= \mathrm{d} f_0^3, \\ \delta_0 f_0^3 &= 0. \end{aligned} \tag{5.22}$$

* The factor $\sqrt{g}$ in the metric-dependent invariant integrals carries an homogeneity degree $\frac{3}{2}$.

The most general form for $f_0^3 = f_0^3(A, c, \tilde{\tau}, \tilde{\Omega})$ is given by $f_0^3 = x \operatorname{Tr} c^3$, where x is some coefficient. By going upwards through the descent equations (5.21)–(5.22), one gets the corresponding local solution in the sector of ghost number 0:

$$f_3^0 = -x \operatorname{Tr}\left(-\tfrac{1}{3}A^3 + \tilde{\Omega}\{A, c\} + \tilde{\tau}c^2\right) + \delta_0 l_3^{-1} + \mathrm{d}l_2^0; \tag{5.23}$$

the two last terms are trivial ones. The solution of the integrated δ_0 cohomology hence reads

$$\Delta = x\Delta_c + \delta_0\hat{\Delta}, \tag{5.24}$$

where $\hat{\Delta} = \int_{\mathcal{M}} l_3^{-1}$ and Δ_c is just proportional to the trilinear part of the classical action:

$$\Delta_c = -\operatorname{Tr}\int_{\mathcal{M}}\left(-\tfrac{1}{3}A^3 + \tilde{\Omega}\{A, c\} + \tilde{\tau}c^2\right) \tag{5.25}$$

It turns out that Δ_c is also invariant under the complete coboundary operator δ (5.13). Hence, the general solution of the cohomology problem (5.12) looks like (5.24), but with δ_0 replaced by δ. The two possible independent contributions to $\hat{\Delta}$ (invariant under diffeomorphisms, of ghost number -1 and dimension 0) and their δ-variations are

$$\begin{aligned} \hat{\Delta}_1 &= \operatorname{Tr}\int_{\mathcal{M}} \tilde{\tau}c, \quad \delta\hat{\Delta}_1 = \operatorname{Tr}\int_{\mathcal{M}}\left(\mathrm{d}\tilde{\Omega}\, c + \tilde{\tau} i_\xi A\right), \\ \hat{\Delta}_2 &= \operatorname{Tr}\int_{\mathcal{M}} \tilde{\Omega}A, \quad \delta\hat{\Delta}_2 = \operatorname{Tr}\int_{\mathcal{M}}\left(-\mathrm{d}A\, A - i_\xi\tilde{\tau}A + \tilde{\Omega}\, \mathrm{d}c + \tilde{\Omega} i_\xi \tilde{\Omega}\right). \end{aligned} \tag{5.26}$$

Both $\delta\hat{\Delta}_1$ and $\delta\hat{\Delta}_2$ depend on the vector field ξ and there exists no linear combination of these objects which would be ξ-independent. We must thus reject them, as we are looking for deformations of the action which of course cannot depend on the vector fields, the latter being nothing else than infinitesimal parameters of field transformations. Finally, the most general deformation of the classical action preserving the conditions (5.5)–(5.9) is given by $\Delta = x\Delta_c$. If acceptable, this would correspond to a possible continuous renormalization of the coupling constant λ – the free parameter multiplying the trilinear part of the action. However, the deformation Δ_c is forbidden by the antighost equation (constraint (5.10) or (5.11)) and must consequently be rejected as well.

In conclusion the action (5.1) is completely fixed by the requirements listed in the beginning of the present section: no deformations are allowed.

6. Rigid supersymmetry

We have seen in sect. 4 that the supersymmetry is in general broken by a hard term and we have given a method for controlling this breaking. As a by-product of our main result, we now investigate the class of manifolds on which an exact supersymmetry holds, i.e., one without hard breakings. In order to do this, let us rewrite the broken Ward identity of superdiffeomorphisms (4.12) in the following way:

$$\mathscr{V}^{\mathrm{S}}_{(\xi)}\Delta \equiv \int_{\mathscr{M}} \sum_{\phi = A,B,c,\bar{c},\tilde{\gamma},\tilde{\tau}} \delta^{\mathrm{S}}_{(\xi)}\phi \frac{\delta\Sigma}{\delta\phi} = \Delta^{\mathrm{cl}}_{(\xi)} + \int \mathrm{d}^3x \left(g_{\mu\rho}\xi^\rho s\Xi^\mu - \mathscr{L}_\xi g_{\mu\nu} \frac{\delta\Sigma}{\delta\hat{g}_{\mu\nu}} \right). \quad (6.1)$$

We have set the external fields M and L to zero, since we now deal with the explicitly broken theory. We define the Ward operator $\mathscr{V}^{\mathrm{S}}_{(\xi)}$ in such a way that when acting on Σ, it yields a r.h.s. which contains all the manifold-dependent information. Enforcing exact supersymmetry means requiring the two last terms in this r.h.s. (6.1) to vanish; that these must vanish separately can be seen by setting $\hat{g}_{\mu\nu}$ to zero. The resulting constraints are given by

$$\partial_\mu\xi_\nu - \partial_\nu\xi_\mu = 0, \qquad \mathscr{L}_\xi g_{\mu\nu} = 0. \quad (6.2)$$

The first condition comes from integrating by parts one of the derivatives contained in $s\Xi^\mu$ (4.6); the second one is equivalent to writing $\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = 0$, where ∇_μ denotes the covariant derivative with respect to the metric connection and $\xi_\mu = g_{\mu\rho}\xi^\rho$. Note that the second of conditions (6.2) is the Killing equation; as a consequence, the space-time must be symmetric. The number of independent Killing vectors, which is bounded by six – this bound is reached in the case of maximally symmetric spaces – is further reduced to a maximum of three by the first condition (6.2). This maximum number of three rigid supersymmetries is realized e.g., in $\mathbb{R}^3$, T^3.

The two conditions (6.2) together mean that the vector field ξ is covariantly constant,

$$\nabla_\mu\xi^\rho = 0. \quad (6.3)$$

This represents a natural generalization of the situation in $\mathbb{R}^3$ space-time: on manifolds where the condition (6.3) is solvable, its solutions (covariantly constant vector fields ξ) generate rigid supersymmetries. On such manifolds one has – assuming that Σ satisfies the Slavnov identity – the following rigid linear algebra:

$$\mathscr{S}^2_\Sigma = 0, \qquad \left[\mathscr{V}^{\mathrm{S}}_{(\xi)}, \mathscr{V}^{\mathrm{S}}_{(\xi')}\right] = 0, \qquad \left[\mathscr{S}_\Sigma, \mathscr{V}^{\mathrm{S}}_{(\xi)}\right] = \mathscr{P}_{(\xi)}. \quad (6.4)$$

This is the generalization of (2.11) we were looking for. We complete it by giving the commutation relations with translations:

$$\left[\mathscr{S}_\Sigma, \mathscr{P}_{(\epsilon)}\right] = 0, \quad \left[\mathscr{V}^S_{(\xi)}, \mathscr{P}_{(\epsilon)}\right] = \mathscr{V}^S_{([\xi,\epsilon])} = 0. \tag{6.5}$$

Here $\mathscr{P}_{(\epsilon)}$ is the Ward operator for translations, i.e., the Ward operator of eq. (4.1) restricted to diffeomorphisms generated by covariantly constant vector fields ϵ. Note that the last equality follows from the vanishing of the Lie bracket of two covariantly constant vector fields.

7. Quantization and finiteness

Let us imagine a theory – defined by some set of constraints – that is stable under classical deformations, i.e. the number of parameters of the classical action is fixed. Showing further that these classical constraints also hold at the quantum level means that the theory is renormalizable: the parameters of the classical action may receive finite corrections from counterterms at all orders of perturbation theory, but the number of such parameters if fixed.

In the case we consider of the Chern–Simons theory on a three-dimensional manifold $\mathscr{M}$, the classical result is stronger: we showed in sect. 5 that the parameters of the classical theory are *completely fixed* by the classical constraints: no deformations are allowed. Showing that these constraints hold at all order then means that the quantum theory is not only renormalizable but *completely finite*: no counterterms are allowed and the parameters of the theory are the ones defined at the classical level.

Our proof of finiteness makes use of a cohomological argument that is valid for an arbitrary manifold $\mathscr{M}$. Nevertheless, one should keep in mind that the quantum theory has to be consistently defined and this shall restrict the class of three-manifolds on which our results *rigorously* holds. Indeed, the problem consists in defining a consistent renormalization scheme and this can be done, in the present state of the art, only on manifolds which are topologically equivalent to a flat manifold and which admit an asymptotically flat metric. The present treatment of the quantized theory hence applies only to the Chern–Simons theory defined on such manifolds. However, we stress that all the classical results derived above (sects. 2–6) are not affected by this restriction and hold for the general case of an arbitrary space-time three-manifold.

Let us now show that the classical constraints which ensured the stability of the classical action can be renormalized. These constraints are listed at the beginning of sect. 5. Our goal is to prove that they hold as well with the classical action Σ replaced by the quantum generating functional of vertex functions $\Gamma = \Sigma + O(\hbar)$. The following comments are in order.

(i) The gauge condition and the ghost equation of motion can easily be shown to hold at all orders, by using standard arguments [26].
(ii) The renormalizability of the antighost equation has been proved for the case of $\mathbb{R}^3$ space-time in ref. [12]. Generalizing the proof to make it hold in the context of the present approach is straightforward.
(iii) Renormalizing the extended Slavnov identity, the Ward identity of diffeomorphisms and the Ward identity of superdiffeomorphisms amounts to show that these are not plagued with anomalies.
(iv) The validity to all orders of the Ward identity of diffeomorphisms will be assumed in the following. Indeed, the absence of diffeomorphisms anomalies has been proved in ref. [27] for the class of manifolds we consider here. Therefore we shall be working in the space of diffeomorphism-invariant functionals.

The proof of absence of anomalies for the extended BRS symmetry and for the superdiffeomorphisms is conveniently carried out by using the coboundary operator $\delta = \mathcal{S}_\Sigma + \mathcal{W}^{\mathrm{D}}_{(\epsilon)} + \mathcal{W}^{\mathrm{S}}_{(\xi)}$ defined in (5.13).

In the cohomological approach, proving the absence of anomalies to all orders amounts to show that the cohomology is trivial in the space of classical integrated local field functionals of ghost number one and dimension zero. In other words, the general solution of the consistency condition

$$\delta \Delta = 0 \tag{7.1}$$

for the anomaly Δ must be the δ-variation of some integrated local field functional. This is the cohomology defined by eqs. (5.12) to (5.15), but with Δ carrying ghost number one.

The space of functionals in which we shall solve the cohomology is restricted by the gauge condition, the ghost equation and the antighost equation, i.e., the anomaly Δ has to obey the same conditions (5.5), (5.6) and (5.10) as the classical deformation discussed in sect. 5. Moreover, as previously stated, we need consider only diffeomorphism-invariant functionals.

By using the same arguments as in sect. 5, we reduce the cohomology problem (7.1) to the simpler one of solving the local cohomology of the degree-preserving operator δ_0 defined by (5.16). Writing $\Delta = \int_{\mathcal{M}} f_3^1$, we get the following descent equations:

$$\begin{aligned} \delta_0 f_3^1 &= \mathrm{d} f_2^2, \\ \delta_0 f_2^2 &= \mathrm{d} f_1^3, \\ \delta_0 f_1^3 &= \mathrm{d} f_0^4, \\ \delta_0 f_0^4 &= 0. \end{aligned} \tag{7.2}$$

These now descend up to a ghost number four zero-form local field polynomial f_0^4. The most general possible form for it is

$$f_0^4 = t_{[abcd]} c^a c^b c^c c^d, \tag{7.3}$$

which vanishes since there exists no totally antisymmetric invariant tensor of rank 4 in the adjoint representation. Hence, the sector with maximal ghost number being empty, one finds, after going upwards through the descent equations, the following local solution in the anomaly sector:

$$f_3^1 = \mathrm{d}l_2^1 + \delta_0 l_3^0, \tag{7.4}$$

where l_p^q are local field polynomials of ghost number q and form degree p. This shows the triviality of the δ_0 cohomology, hence that of the δ-cohomology (which is included in the former [23,24]).

The triviality of the δ-cohomology concludes our proof of the absence of anomalies. This, in turn, implies UV finiteness at all orders of perturbation theory, according to the discussion above.

8. Conclusions

A local realization is obtained for the supersymmetry of the Chern–Simons theory defined on an arbitrary three-dimensional manifold. In order to achieve this, we add to the "matter supermultiplets" $\{c, A_\mu, \Omega^\mu, \tau\}$ and $\{B, \bar{c}\}$ – out of which the gauge fixed classical action is constructed – the "metric supermultiplet" of external fields $\{g_{\mu\nu}, \hat{g}_{\mu\nu}, M_\mu, L_\mu\}$. The latter supermultiplet is constructed out of two requirements. The first is the ability to control the metric (in)dependence; it is made possible by introducing $\hat{g}_{\mu\nu}$ together with the corresponding new terms in the classical action. The second is the requirement of invariance of the action under the local supersymmetry. The latter breaks down at the classical level on a general space-time manifold. This local invariance can be restored by coupling the breaking to a set of external fields M, L. The total classical action exhibiting local supersymmetry is then the one constructed out of both the matter and the metric supermultiplets.

The set of constraints the theory is submitted to (Ward identities, equations of motion, etc) completely fixes the total classical action, forbidding any deformations of it.

The classical constraints can be renormalized to all orders of perturbation theory; in particular, the local supersymmetry is free of anomalies. This, together with the stability of the classical theory, ensures all order UV finiteness of the Chern–Simons theory, wherever a renormalized perturbation series can be defined.

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