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A graph-theoretic approach to Grothendieck-Teichmüller and Kashiwara-Vergne theories

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**A GRAPH-THEORETIC APPROACH TO
GROTHENDIECK-TEICHMÜLLER AND KASHIWARA-VERGNE
THEORIES**

THÈSE

présentée à la Faculté des Sciences de l'Université de Genève
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par
Matteo Felder
de
Lugano (TI)

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Thèse de Monsieur Matteo FELDER

intitulée :

**«A Graph-Theoretic Approach to
Grothendieck-Teichmüller and Kashiwara-Vergne Theories»**

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Genève, le 19 septembre 2019

Thèse - 5392 -

Le Doyen

Résumé

Cette thèse est consacrée à l'étude de divers aspects de l'algèbre de Lie de Grothendieck-Teichmüller \mathfrak{grt} . Introduite par Drinfeld à partir des idées de Grothendieck, \mathfrak{grt} joue un rôle important dans une variété de domaines en mathématiques. En particulier, nous traitons des sujets suivants.

- Nous considérons ses liens avec le problème de Kashiwara-Vergne (KV) qui a ses origines en théorie de Lie. Ce dernier concerne certaines propriétés de la série de Baker-Campbell-Hausdorff. L'algèbre de Lie de Kashiwara-Vergne a été introduite par Alekseev et Torossian et encode les symétries de l'ensemble des solutions du problème de KV. Il a été démontré que \mathfrak{grt} est une sous-algèbre de Lie de l'algèbre de Lie de KV, et il est conjecturé qu'elles sont isomorphes.

Dans l'**article A**, nous construisons une séquence infinie de sous-algèbres de Lie de l'algèbre de Lie de KV dont l'intersection est \mathfrak{grt} . Cela donne un moyen d'interpoler entre ces deux algèbres de Lie. Les sous-algèbres de Lie sont définies à l'aide de techniques de diagrammes issues des travaux de Willwacher sur l'interprétation de \mathfrak{grt} en tant que cohomologie en degré zéro du complexe de graphes de Kontsevich.

- La conjecture de Deligne, Drinfeld et Ihara affirme que l'algèbre de Lie de Grothendieck-Teichmüller est isomorphe (jusqu'à complétion) à l'algèbre de Lie libre avec générateurs $\sigma_3, \sigma_5, \sigma_7, \dots$ - un pour chaque degré impair. Selon un important théorème de Brown, cette algèbre de Lie libre forme une sous-algèbre de Lie de \mathfrak{grt} . De plus, \mathfrak{grt} admet une filtration combinatoire, appelée filtration de profondeur. Il est bien connu que les crochets des générateurs conjecturaux se trouvent en profondeur deux, et que modulo profondeur trois, ils satisfont un ensemble d'équations intéressantes qui se rapportent en outre à la théorie des formes modulaires.

Dans l'**article B**, nous donnons une interprétation graphique de ces relations. On espère que cette approche se généralise à toute profondeur, ce qui nous permettrait d'étudier les relations entre crochets itérés des générateurs au niveau du gradué associé à la filtration par la profondeur.

- Le théorème de Duflo est un résultat classique de la théorie de Lie. Il indique que pour une algèbre de Lie de dimension finie \mathfrak{g} , alors qu'il n'existe qu'un isomorphisme d'espaces vectoriels entre l'algèbre symétrique $S\mathfrak{g}$ et l'algèbre enveloppante universelle $U\mathfrak{g}$, il existe un isomorphisme explicite d'algèbres de l'espace des éléments invariants $(S\mathfrak{g})^{\mathfrak{g}}$ au centre $Z(U\mathfrak{g})$ de l'algèbre enveloppante universelle. Il se généralise en termes de cohomologie d'algèbre de Lie à valeurs dans $S\mathfrak{g}$ et $U\mathfrak{g}$, où le morphisme induit est un isomorphisme d'algèbres sur la cohomologie, mais pas au niveau des complexes de Chevalley-Eilenberg correspondants .

Dans l'**article C**, nous donnons une preuve alternative du résultat de Alm qui affirme que le morphisme induit entre les complexes de Chevaly-Eilenberg ne s'étend pas de manière universelle à un isomorphisme d'algèbre A_∞ . La preuve utilise une variante du complexe de graphes de Kontsevich et se base sur un calcul de graphes élémentaire.

Riassunto

Questa tesi è dedicata allo studio di vari aspetti dell'algebra di Lie di Grothendieck-Teichmüller \mathfrak{grt} . Introdotta da Drinfeld, basandosi su idee di Grothendieck, \mathfrak{grt} vanta una spettacolare gamma di applicazioni che coprono diversi campi della matematica. In particolare, affrontiamo i temi seguenti.

- Consideriamo la sua connessione al problema di Kashiwara-Vergne (KV) le cui origini sono nella teoria di Lie. Quest'ultimo ispeziona alcune proprietà della serie di Baker-Campbell-Hausdorff. L'algebra di Lie di Kashiwara-Vergne è stata introdotta da Alekseev e Torossian e codifica le simmetrie dell'insieme di soluzioni del problema di KV. È stato dimostrato che contiene \mathfrak{grt} come sottoalgebra di Lie ed è stato ipotizzato che sia ad essa isomorfa.

Nell'**articolo A**, costruiamo una sequenza infinita di sottoalgebra di Lie dell'algebra di Lie di KV la cui intersezione è \mathfrak{grt} . Ciò consente di interpolare tra queste due algebra di Lie. Le sottoalgebra di Lie sono definite usando tecniche diagrammatiche che hanno origine nel lavoro di Willwacher sull'interpretazione di \mathfrak{grt} come la coomologia in grado zero del complesso di grafi di Kontsevich.

- La congettura di Deligne, Drinfeld e Ihara afferma che l'algebra di Lie di Grothendieck-Teichmüller è isomorfa (fino al completamento) all'algebra di Lie libera con generatori $\sigma_3, \sigma_5, \sigma_7, \dots$ - uno per ogni grado dispari. Secondo l'importante teorema di Brown, questa particolare algebra di Lie libera forma una sottoalgebra di Lie di \mathfrak{grt} . Inoltre, \mathfrak{grt} ammette una filtrazione combinatoria, chiamata filtrazione di profondità. È ben noto che i bracket dei congetturati generatori si trovano in profondità due, e modulo profondità tre, soddisfano una serie di equazioni interessanti che sono legate inoltre alla teoria delle forme modulari.

Nell'**articolo B**, diamo un'interpretazione grafica di queste relazioni. La speranza è che questo approccio possa essere generalizzato a una profondità maggiore, permettendoci di studiare le relazioni tra bracket iterati dei generatori congetturali al livello del graduato associato alla filtrazione di profondità.

- Il teorema di Duflo è un risultato classico della teoria di Lie. Afferma che per un'algebra di Lie finita dimensionale \mathfrak{g} , mentre esistono solo isomorfismi di spazi vettoriali dall'algebra simmetrica $S\mathfrak{g}$ all'algebra universale involuante $U\mathfrak{g}$, esiste un isomorfismo esplicito di algebra dallo spazio degli elementi invarianti $(S\mathfrak{g})^{\mathfrak{g}}$ al centro $Z(U\mathfrak{g})$ della sua algebra universale involuante. Si generalizza in termini di coomologia dell'algebra di Lie con valori in $S\mathfrak{g}$ e $U\mathfrak{g}$, in cui il morfismo indotto è un isomorfismo di algebra sulla coomologia, ma non a livello dei corrispondenti complessi di Chevalley-Eilenberg.

Nell'**articolo C**, diamo una prova alternativa del risultato di Alm che afferma che il morfismo indotto tra i complessi Chevally-Eilenberg non si estende in modo universale a un A_∞ -isomorfismo. La dimostrazione fa uso di una variante del complesso di grafi di Kontsevich e si basa su un elementare calcolo di grafi.

Summary

This thesis studies various aspects of the Grothendieck-Teichmüller Lie algebra \mathfrak{grt} . Introduced by Drinfeld, building on ideas of Grothendieck, \mathfrak{grt} boasts a spectacular range of applications spanning a variety of fields in mathematics. In particular, we address the following topics.

- We consider its connections to the Kashiwara-Vergne (KV) problem arising in Lie theory. It inspects certain properties of the Baker-Campbell-Hausdorff series. The Kashiwara-Vergne Lie algebra was introduced by Alekseev and Torossian and captures the symmetries of the set of solutions of the KV problem. It has been shown to contain \mathfrak{grt} as a Lie subalgebra and, backed by strong numerical experiments, they are conjectured to be isomorphic.

In **Paper A**, we construct an infinite nested sequence of Lie subalgebras of the KV Lie algebra whose intersection is \mathfrak{grt} . This gives a way to interpolate between these two Lie algebras. The Lie subalgebras are defined using diagrammatic techniques which originate from Willwacher's work on the interpretation of \mathfrak{grt} as the zeroth cohomology of Kontsevich's graph complex.

- The Deligne-Drinfeld-Ihara conjecture states the Grothendieck-Teichmüller Lie algebra is isomorphic (up to completion) to the free Lie algebra with generators $\sigma_3, \sigma_5, \sigma_7, \dots$, one for every odd degree. By Brown's important theorem, this particular free Lie algebra forms a Lie subalgebra of \mathfrak{grt} . Moreover, \mathfrak{grt} allows a combinatorial filtration, called the depth filtration. The Lie brackets of the conjectural generators lie in depth two, and modulo depth three, they are known to satisfy an interesting set of equations which furthermore relate to the theory of modular forms.

In **Paper B**, we give two graph theoretic interpretation of these relations. The hope is that this approach might generalize to higher depth, allowing us to study relations among iterated brackets of the conjectural generators on the level of the associated graded with respect to the depth filtration.

- Duflo's theorem is a classical result from Lie theory. It states that for a finite dimensional Lie algebra \mathfrak{g} , while there is only an isomorphism of vector spaces from its symmetric algebra $S\mathfrak{g}$ to its universal enveloping algebra $U\mathfrak{g}$, there exists an explicit isomorphism of algebras from the space of invariant elements $(S\mathfrak{g})^{\mathfrak{g}}$ to the center $Z(U\mathfrak{g})$ of its universal enveloping algebra. It generalizes to a statement in terms of Lie algebra cohomology with values in $S\mathfrak{g}$ and $U\mathfrak{g}$, where the induced morphism is an isomorphism of algebras on cohomology, but not on the level of its corresponding Chevalley-Eilenberg complexes.

In **Paper C**, we give an alternative proof of a result by Alm which states that the induced morphism between the Chevalley-Eilenberg complexes does not extend in a universal way to

an A_∞ -isomorphism. The proof makes use of a variant of Kontsevich's graph complex and elementary graphical calculus.

Ai miei genitori

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Chapter 1

Introduction

This thesis studies various aspects of a theory whose origins trace back to Grothendieck and Drinfeld. We start by recalling some historical context and background material in order to motivate the questions addressed in the subsequent chapters.

1.1 Origins of the Grothendieck-Teichmüller groups

The profinite¹ Grothendieck-Teichmüller group $\widehat{\text{GT}}$ was introduced by Drinfeld [17], inspired by the works of Grothendieck [26]. Despite having a wide range of applications in various seemingly unrelated fields of mathematics, it still remains a somewhat mysterious object. Its origins lie in Grothendieck's plan to study the absolute Galois group of \mathbb{Q} , $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Here $\overline{\mathbb{Q}}$ is the algebraic closure of \mathbb{Q} . There exists no explicit description for

$$\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) = \{\phi \mid \phi \text{ is a field automorphism of } \overline{\mathbb{Q}}\}$$

and so far, the only explicitly known non-trivial element is the complex conjugation. Grothendieck proposed to study the absolute Galois group through its action on objects of simpler nature. Roughly, this amounts to finding an appropriate object, say O , on which $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts and whose automorphism group $\text{Aut}(O)$ is easier to understand. Ideally, the induced morphism $\text{Gal}(\overline{\mathbb{Q}}, \mathbb{Q}) \rightarrow \text{Aut}(O)$ provides a way to extract some information on the absolute Galois group of \mathbb{Q} (and in the best of cases it is an isomorphism). More concretely, Grothendieck suggested that one should consider the (outer) action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on (an appropriate version of) the fundamental groups of the moduli spaces of closed genus g curves defined over \mathbb{Q} with n marked points, $\mathcal{M}_{g,n}$. In particular, in the case of $\mathcal{M}_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, this procedure produces a morphism

$$\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Out}(\widehat{\pi}_1(\mathcal{M}_{0,4})) = \text{Out}(\widehat{\text{Free}}(x, y))$$

where $\widehat{\pi}_1(\mathcal{M}_{0,4}) \cong \widehat{\text{Free}}(x, y)$ is the profinite completion of the free group on two generators and $\text{Out}(-)$ denotes the group of outer automorphisms. Belyi's theorem provides the following useful fact.

Theorem (Belyi [12]). The morphism above is injective.

¹A topological group is profinite if it is homeomorphic to the inverse limit of an inverse system of finite groups endowed with the discrete topology.

Moreover, Grothendieck suggested that the image of the absolute Galois group under the above injection should be contained in a subgroup of the outer automorphism group characterized by simple equations. Famously, however, he did so without writing explicit formulas. These were discovered by Drinfeld [17] and the subgroup cut out by those equations is the profinite Grothendieck-Teichmüller group $\widehat{\text{GT}}$. He proves the following theorem.

Theorem (Drinfeld [17]). The outer action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\widehat{\text{Free}}(x, y)$ factors through $\widehat{\text{GT}}$, i.e. there is an injective morphism $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \widehat{\text{GT}}$ making the diagram

$$\begin{array}{ccc} & \widehat{\text{GT}} & \\ \swarrow & & \searrow \\ \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) & \xrightarrow{\quad} & \text{Out}(\widehat{\text{Free}}(x, y)) \end{array}$$

commute.

An exciting conjecture is then the following.

Conjecture. The morphism $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \widehat{\text{GT}}$ is an isomorphism.

Notice that Grothendieck’s plan was to let $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ act on the (profinite completion of the) “Teichmüller tower”, that is, the collection, indexed by g and n , of all fundamental groups of $\mathcal{M}_{g,n}$ together with certain natural connecting homomorphisms between them. Drinfeld suggests that $\widehat{\text{GT}}$ is the group of automorphisms of the subtower (i.e. the automorphisms are compatible with the natural homomorphisms between the fundamental groups) corresponding to the moduli of genus zero curves $\mathcal{M}_{0,n}$. Furthermore, he conjectures it to be the automorphism group of the whole Teichmüller tower. Thus, in summary, the profinite version of the Grothendieck-Teichmüller group contains the absolute Galois group of \mathbb{Q} as a subgroup compatible with the respective actions on a whole “tower” of algebro-geometric objects.

Remark that along with the profinite version, Drinfeld gave an analogous definition of the Grothendieck-Teichmüller group in the pronilpotent context. Although further away from Grothendieck’s dream of demystifying $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the pronilpotent group GT and its graded version GRT boast a spectacular range of applications to which we shall turn shortly. For concreteness, let us present the standard definition of GRT which is the easiest among the many different versions of the Grothendieck-Teichmüller groups. For this, let \mathbb{K} be a field of characteristic zero and consider the set of equations parametrised by $\mu \in \mathbb{K}$ for a group-like element $\Phi \in \mathbb{K}\langle\langle x, y \rangle\rangle$ of the completed free associative algebra in generators x and y ,

$$\begin{aligned} \Phi(t_{12}, t_{23} + t_{24})\Phi(t_{13} + t_{23}, t_{34}) &= \Phi(t_{23}, t_{34})\Phi(t_{12} + t_{13}, t_{24} + t_{34})\Phi(t_{12}, t_{23}) \\ 1 &= e^{\frac{\mu}{2}(-x-y)}\Phi(x, y)e^{\frac{\mu}{2}x}\Phi(y, -x - y)e^{\frac{\mu}{2}y}\Phi(-x - y, x) \\ 1 &= \Phi(x, y)\Phi(y, x) \end{aligned}$$

Note that the first equation takes values in the Drinfeld-Kohno Lie algebra \mathfrak{t}_4 . For $n \geq 2$, \mathfrak{t}_n is the Lie algebra with generators $t_{ij} = t_{ji}$ ($1 \leq i \neq j \leq n$) subject to the relations $[t_{ij}, t_{kl}] = 0$ and $[t_{ij}, t_{ik} + t_{kj}] = 0$ whenever i, j, k, l are all distinct. Pairs (μ, Φ) for $\mu \in \mathbb{K}^\times$ and Φ a group-like solution to these equations are called Drinfeld associators. The set of group-like solutions for $\mu = 0$ may be endowed with a certain group structure defining the graded Grothendieck-Teichmüller

group GRT_1 . We set $\text{GRT} := \mathbb{K}^\times \times \text{GRT}_1$ where the action of the multiplicative group rescales the variables. Concerning GT , let us only remark that it is a subset of the pronipotent completion of the free group on two generators, satisfying a similar set of equations, but with the (pronipotent completion of the) pure braid group on four strands taking the role of the Drinfeld-Kohno Lie algebra. Drinfeld's important theorem states that such solutions exist.

Theorem (Drinfeld [17]). The set of Drinfeld associators is non-empty.

In fact, he even proves the existence of a Drinfeld associator with rational coefficients. It remains however a hard open problem to construct such a rational associator explicitly. By Furusho's [23] remarkable result, any group-like solution Φ of the first equation automatically satisfies the other two, for μ depending only the coefficient in front of xy in Φ . Interestingly, despite being widely studied, there are only few explicit constructions describing solutions to the system of equations above. Drinfeld's ([17], [36]) Knizhik-Zamolodchikov associator Φ_{KZ} , defined borrowing methods from conformal field theory, had been the only known construction for a long time, before Alekseev and Torossian ([4], relying on Ševera and Willwacher's work [43]) introduced a second associator using techniques from deformation quantization. A more recent approach due to Rossi and Willwacher [40] then yielded a whole one-parameter family interpolating between the two constructions. Additionally, the set of Drinfeld associators is a $\text{GT} - \text{GRT}$ -torsor [17]. That is, both GT and GRT act freely and transitively on the set of Drinfeld associators. Let us emphasize this with the following diagram.

$$\text{GT} \circlearrowleft \{\text{Drinfeld associators}\} \circlearrowright \text{GRT}$$

In particular, this implies that $\text{GT} \cong \text{GRT}$, any Drinfeld associator determining such an isomorphism. This torsor structure is highlighted further in the work of Bar-Natan [11] in which he lays the foundations for an operadic interpretation of GT and GRT . Concretely, GT is the group of automorphisms of the operad of parenthesized braids $\widehat{\text{PaB}}$, GRT is the group of automorphism of the operad of parenthesized chord diagrams $\widehat{\text{PaCD}}$ and Drinfeld associators may be identified with isomorphisms $\widehat{\text{PaB}} \rightarrow \widehat{\text{PaCD}}$. The operadic viewpoint is thoroughly explained in Fresse's book ([22], see also Merkulov's recent survey [38]).

In the pronipotent setting, understanding the group GRT_1 reduces to the study of its (infinite-dimensional) graded Lie algebra \mathfrak{grt}_1 . It is a linear subspace of the (completion of the) free Lie algebra in two generators $\text{Lie}(x, y)$. It is, however, not a Lie subalgebra, since it is equipped with a different Lie bracket. Similarly to its group, its standard definition is rather technical, involving a system of equations in $\text{Lie}(x, y)$ which we refrain from giving at this point. Instead, let us recall the Deligne-Drinfeld-Ihara conjecture which might elucidate at least some of the structure of \mathfrak{grt}_1 .

Conjecture (Deligne-Drinfeld-Ihara). The Grothendieck-Teichmüller Lie algebra \mathfrak{grt}_1 is isomorphic (up to completion) to the free Lie algebra

$$\text{Lie}(\sigma_3, \sigma_5, \sigma_7, \dots)$$

generated by elements σ_{2j+1} of odd degree.

Brown's important result yields the inclusion [13]

$$\text{Lie}(\sigma_3, \sigma_5, \sigma_7, \dots) \hookrightarrow \mathfrak{grt}_1.$$

Remark also that the Lie algebra corresponding to $\text{GRT} = \mathbb{K}^\times \times \text{GRT}_1$ may be written as the semi-direct product $\mathfrak{grt} := \mathfrak{t}_2 \ltimes \mathfrak{grt}_1$, since \mathfrak{t}_2 is one-dimensional.

1.2 Number theory

It is worth noting that Drinfeld's motivation for introducing associators and the Grothendieck-Teichmüller groups lay in the theory of quantum groups in which they play a key role in the study of representations of quasitriangular quasi-Hopf algebras ([16], [17], [18]). Surprisingly, and as already noted by Drinfeld [18], the Φ_{KZ} associator arising from these works has close ties to number theory. More precisely, they are related to multiple zeta values (MZV's), that is, the numbers (where $n_1 \geq 2$, $n_i \geq 1$ for $2 \leq i \leq k$)

$$\zeta(n_1, \dots, n_k) := \sum_{j_1 > j_2 > \dots > j_k \geq 1} \frac{1}{j_1^{n_1} \dots j_k^{n_k}}.$$

These were already studied by Euler [21] in 1775. For instance, he computed the values of $\zeta(2n)$ for $n \in \{1, 2, 3, 4, 5, 6\}$ using a method which works for general n . The result is

$$\zeta(2n) = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!}$$

where $B_{2n} \in \mathbb{Q}$ is the $2n$ -th Bernoulli number. For odd values, no such closed expression is known, and they are the subject of many long standing conjectures in number theory. Their generalization, the MZV's satisfy two types of combinatorial relations, known as the (regularized) double shuffle relations. Roughly, one set of relations is obtained from reordering of the summation indices when considering products of MZV's. For example,

$$\zeta(2)\zeta(5) = \sum_{k,l} \frac{1}{k^2} \frac{1}{l^5} = \left(\sum_{k>l} + \sum_{l>k} + \sum_{k=l} \right) \frac{1}{k^2} \frac{1}{l^5} = \zeta(2, 5) + \zeta(5, 2) + \zeta(7).$$

A less obvious second set comes from rewriting MZV's in terms of iterated integrals - a technique that is attributed to Kontsevich. It is a hard open conjecture that all the (regularized) double shuffle relations exhaust all algebraic relations over \mathbb{Q} satisfied by the MZV's. The link to associators is established by the fact that all MZV's appear (modulo factors of $2\pi i$) as the coefficients of the Knizhik-Zamolodchikov associator Φ_{KZ} . For instance, the coefficient of the associative word

$$x^{n_1-1} y x^{n_2-1} y \dots x^{n_k-1} y$$

is

$$\frac{\pm \zeta(n_1, n_2, \dots, n_k)}{(2\pi i)^{n_1 + \dots + n_k}}.$$

Moreover, by a result of Furusho [24], every Drinfeld associator produces a solution to the (regularized) double shuffle relations. Conjecturally, the set of Drinfeld associators generates all solutions. Together with the previous conjecture, it would infer that algebra of (regularized) MZV's is isomorphic to the algebra of functions on the set of Drinfeld associators (over \mathbb{Q}).

1.3 Knot invariants

Further applications of Drinfeld associators are found in the theory of knot invariants. Recall that a knot (in \mathbb{R}^3) is an embedding $S^1 \rightarrow \mathbb{R}^3$. A knot invariant is a function on the set of knots invariant

under deforming the knot by isotopy. This notion may be generalized to singular knots, i.e. maps $S^1 \rightarrow \mathbb{R}^3$ with a finite number of self-intersections. In particular, any knot invariant extends to an invariant of singular knots by means of resolving the self-crossings via Vassiliev's skein relation. A finite type invariant (or Vassiliev invariant) of type $m \in \mathbb{N}$ is an invariant of singular knots which vanishes on all knots with at least $m + 1$ self-crossings. In [31] Kontsevich defined a universal finite type invariant for knots in \mathbb{R}^3 , now known as the Kontsevich integral. It is universal in the sense that all finite type invariants may be recovered from Kontsevich's. It is thus at least as powerful as the set of all finite type invariants, and other knot invariants coming from the theory of quantum groups (e.g. the Jones polynomial and its various generalizations) [31]. It is conjectured that the Kontsevich integral is complete, that is, it distinguishes all non-isotopic knots. Combinatorial constructions of Kontsevich's invariant, which to each knot assigns an element in a certain algebra of chord diagrams, have been given by Le and Murakami [37] and independently by Bar-Natan [10]. In [37], the authors show that every Drinfeld associator gives rise to an invariant of framed oriented links, called the universal Kontsevich-Vassiliev invariant, from which the Kontsevich integral may be recovered. Remarkably, the result is independent of the choice of associator [37]. In particular, this independence is the main ingredient in the proof that the coefficients of the Kontsevich integral are rational ([37], as foreseen by Kontsevich in [31]).

1.4 Duflo's isomorphism and the Kashiwara-Vergne problem

Let us recall a classical result from Lie theory. Let \mathfrak{g} be a finite dimensional Lie algebra. Consider its symmetric algebra

$$S(\mathfrak{g}) = T\mathfrak{g}/(x \otimes y - y \otimes x)$$

and its universal enveloping algebra

$$U\mathfrak{g} = T\mathfrak{g}/([x, y] - x \otimes y + y \otimes x).$$

The adjoint action of \mathfrak{g} on itself extends by derivation to a Lie algebra action on both $S\mathfrak{g}$ and $U\mathfrak{g}$. By the famous Poincaré-Birkhoff-Witt theorem, the symmetrization map

$$\text{Sym} : S\mathfrak{g} \rightarrow U\mathfrak{g}, \quad v_1 \dots v_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} v_{\sigma(1)} \dots v_{\sigma(n)}$$

defines an isomorphism of vector spaces. However, it does not define an isomorphism of algebras, unless \mathfrak{g} is abelian, the product on $S\mathfrak{g}$ being commutative, while the one $U\mathfrak{g}$ is not. Nevertheless, it is compatible with the \mathfrak{g} -action which allows us to study the induced map

$$\text{Sym} : (S\mathfrak{g})^{\mathfrak{g}} \rightarrow (U\mathfrak{g})^{\mathfrak{g}} = Z(U\mathfrak{g})$$

on \mathfrak{g} -invariant elements. Notice that in the case of the universal enveloping algebra, the algebra of \mathfrak{g} -invariant elements correspond to its center $Z(U\mathfrak{g})$. Despite having commutative algebras on both sides, the induced map is still not a morphism of algebras. This defect may be cured by precomposing with a certain automorphism of $S\mathfrak{g}$, the so-called Duflo element J , whose definition involves the power series $\sinh(x/2)/(x/2)$.

Theorem (Duflo [19]). The composition $\text{Duf} := \text{Sym} \circ J$ restricts to an isomorphism of algebras $(S\mathfrak{g})^{\mathfrak{g}} \rightarrow Z(U\mathfrak{g})$.

The result generalizes those of Dixmier [15] (in the case of nilpotent Lie algebras), Duflo [20] (for solvable Lie algebras) and Harish-Chandra [27] (in the case of semi-simple case). The proof relies on Kirillov’s orbit method in representation theory of Lie groups [30]. The Kashiwara-Vergne conjecture was put forward by Kashiwara and Vergne [29] with the aim to give an algebraic proof of Duflo’s theorem. Roughly, their idea was to find an automorphism F (not necessarily of algebras) of $S\mathfrak{g} \otimes S\mathfrak{g}$, which is the identity on invariants, and such that

$$m_{U\mathfrak{g}}(\text{Duf} \otimes \text{Duf})(x \otimes y) = \text{Duf} \circ m_{S\mathfrak{g}}(F(x \otimes y))$$

where $m_{U\mathfrak{g}}$ is the product on $U\mathfrak{g}$ and $m_{S\mathfrak{g}}$ the one on $S\mathfrak{g}$. An equivalent (and universal) way to state the KV problem is to find an automorphism of (degree completion of the) free Lie algebra in two variables, say x and y , which maps their sum

$$x + y \mapsto \text{bch}(x, y) = x + y + \frac{1}{2}[x, y] + \dots$$

to the Baker-Campbell-Hausdorff series, and additionally, satisfies two further technical equations which we choose to omit. All three equations allow a universal formulation, i.e. one that depends only on Lie brackets, and not on the specific choice of the Lie algebra. Solutions to the set of (non-universal) KV equations were found for solvable Lie algebras by Kashiwara and Vergne [29], by Rouvière [41] in the case of $\mathfrak{sl}(2, \mathbb{R})$, for quadratic Lie algebras by Vergne in [46], and also by Alekseev and Torossian [3]. The first universal solution of the KV conjecture was obtained by Alekseev and Meinrenken [2] building on results of Torossian [45] and Kontsevich’s deformation quantization. A second extraordinary proof was later formulated by Alekseev and Torossian [5] with the main ingredient being the existence of Drinfeld associators. In particular, they show how by starting with any Drinfeld associator one can construct a solution to the KV equations. Additionally, they construct the same type of torsor structure found in the associator setting, i.e. there exist groups KV and KRV acting freely and transitively on the set of solutions to the KV problem:

$$\text{KV} \circ \{\text{Solutions to the KV equations}\} \circ \text{KRV}$$

It comes with an injective homomorphism $\text{GRT} \rightarrow \text{KRV}$ which in turn is compatible with the respective actions on Drinfeld associators and solutions to the KV problem. Let \mathfrak{krv} denote the Lie algebra of KRV. Alekseev and Torossian state the following conjecture.

Conjecture (Alekseev-Torossian [5]). The Kashiwara-Vergne Lie algebra \mathfrak{krv} is isomorphic to the Grothendieck-Teichmüller Lie algebra $\mathfrak{krt} = \mathfrak{t}_2 \rtimes \mathfrak{gtrt}_1$.

Evidence for this conjecture to hold was provided by numerical experiments due to Albert, Harinck and Torossian [1] who checked that the two graded Lie algebras coincide up to degree 16. If true, this would in particular imply that all solutions to the KV problem come from Drinfeld associators.

1.5 Kontsevich’s graph complex

Typically, elements of a graph complex are linear combinations or formal series of (isomorphism classes of) some type of diagrams endowed with orientation data. Such diagrams, usually given by a collection of edges and vertices, come in a variety of flavours. For instance, the edges might be

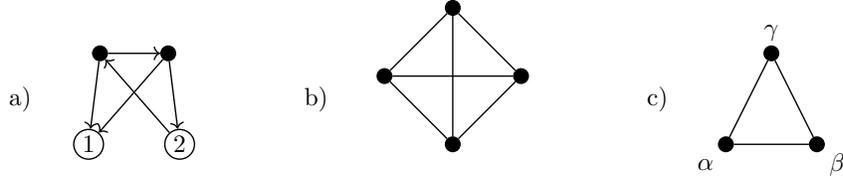
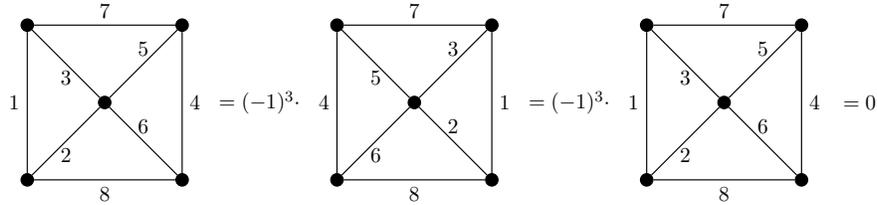


Figure 1.1: Different types of graphs.

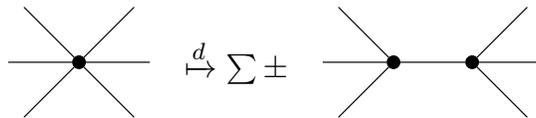
directed or undirected, while the vertices might be of different colours, labeled, decorated or simply indistinguishable.

In each case, one defines a combinatorial way to assign a degree to each diagram and, in addition, a differential, i.e. a linear combinatorial operation of degree one which, if applied twice, gives zero. Surprisingly, many important problems in mathematics may be encoded using graph complexes and, more precisely, reduce to a computation of the cohomology of an appropriate graph complex.

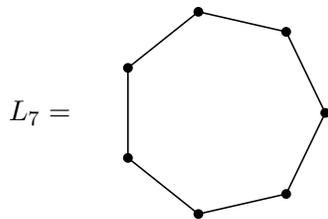
Several graph complexes were introduced by Kontsevich in his works on deformation quantization ([32], [33], [34], [35]). One version in particular, usually referred to as Kontsevich's graph complex, GC allows a simple combinatorial description. Its elements are given by linear combinations of connected undirected graphs with indistinguishable vertices of valence at least two, as depicted in b) of Figure 1.1. In this case, the orientation data is given by specifying a linear order on the set of edges, and identifying graphs with different orderings up to sign (given by the permutation mapping one ordering to the other). In particular, graphs with odd symmetries, such as the one below, are zero.



The differential d is described by the combinatorial operation which splits a vertex into two vertices connected by an edge, and summing over all possible ways of reconnecting the edges previously adjacent to the splitted vertex.



It is a hard open problem to compute the graph cohomology $H(\text{GC}) = \ker(d)/\text{im}(d)$, but partial and interesting results exist. Let us recall a few. For instance, notice that the differential preserves the number of loops, thus the complex decomposes into a direct product of subcomplexes, one for each loop order. In loop order one, the only possible elements are given by the loops L_k with k vertices. For instance, for $k = 7$,



By symmetry, however, only those with $k \equiv 1 \pmod{4}$ are non-zero. They yield the only non-trivial classes in loop order one. Furthermore, it is not hard to see that the differential does not produce any bivalent vertices (or rather they always appear twice, but with opposite signs and thus cancel), therefore we may consider the subcomplex $\text{GC}^{\geq 3}$ spanned by graphs whose vertices are at least trivalent. A result of Willwacher [47] (sketched by Kontsevich ([32],[33])) shows that in loop orders ≥ 2 , the inclusion

$$\text{GC}^{\geq 3} \hookrightarrow \text{GC}$$

is a quasi-isomorphism (that is, it induces an isomorphism in cohomology), i.e. including loop order one, we have

$$H(\text{GC}) = H(\text{GC}^{\geq 3}) \oplus \bigoplus_{\substack{k \geq 1 \\ k \equiv 1 \pmod{4}}} \mathbb{K}L_k.$$

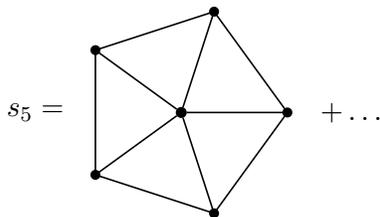
Note that GC naturally carries the structure of a differential graded Lie algebra. The Lie bracket is defined through certain combinatorial insertion operations. A major breakthrough was obtained by Willwacher [47] by establishing an isomorphism of Lie algebras between the zeroth cohomology of Kontsevich's graph complex and the Grothendieck-Teichmüller Lie algebra.

Theorem (Willwacher [47]). There is an isomorphism of Lie algebras

$$H^0(\text{GC}) \cong \mathfrak{gtr}_1.$$

Furthermore, $H^{-1}(\text{GC}) = \mathbb{K}$ and $H^{<-1}(\text{GC}) = 0$.

Notice in particular that the conjectural generators σ_{2j+1} of odd degree appear as the cohomology classes represented by linear combinations of graphs s_{2j+1} with “leading term” a wheel graph with $2j + 1$ spokes. For instance, in the case of σ_5 , we have



Additionally, let us emphasize that Willwacher provides a beautiful explicit combinatorial algorithm to construct an element of \mathfrak{gtr}_1 from a degree zero graph cocycle (i.e. an element representing a cohomology class in $H^0(\text{GC})$).

1.6 Content of the thesis

The chapters of this thesis consist of the following three articles by the author:

- A** Internally connected graphs and the Kashiwara-Vergne Lie algebra, *Lett. Math. Phys.*, 108 (6), 1407–144, (2018)
- B** Filtrations on graph complexes and the Grothendieck-Teichmüller Lie algebra in depth two, *Selecta Math. (N.S.)*, 24, 2063–2092, (2018)
- C** On a homotopy version of the Duflo isomorphism, *Lett. Math. Phys.* <https://doi.org/10.1007/s11005-019-01223-6> (2019)

1.6.1 Paper A

The results of Paper A follow from the realization due to Alekseev, Ševera, Torossian and Willwacher that not only \mathfrak{grt}_1 can be expressed in terms of diagrams, but also the Kashiwara-Vergne Lie algebra \mathfrak{kv} . In fact, while \mathfrak{grt}_1 may be expressed as the set of solutions of a certain set of diagrammatic equations, one for each loop order, arising from Willwacher’s algorithm, \mathfrak{kv} is encoded by those graphs solving the equation in loop order one. The main theorem of Paper A may be reformulated as follows.

Theorem. There exists a nested sequence

$$\mathfrak{grt} = \mathfrak{t}_2 \times \mathfrak{grt}_1 \subset \dots \subset \mathfrak{kv}^{(k+1)} \subset \mathfrak{kv}^{(k)} \subset \dots \subset \mathfrak{kv}^{(1)} = \mathfrak{kv}.$$

of Lie subalgebras of \mathfrak{kv} whose intersection is \mathfrak{grt} .

Naturally, $\mathfrak{kv}^{(k)}$ consists of diagrams satisfying the same type of equations up to loop order k . The proof of the fact that it forms a Lie algebra is non-trivial and fairly involved, relying on techniques from the theory of graph complexes. The result tries to set the framework for a systematic approach to the conjecture that $\mathfrak{grt} \cong \mathfrak{kv}$. If true, the isomorphism would in particular imply that any solution to the defining graphical equation in loop order one may be extended naturally and uniquely to a linear combination of diagrams verifying an infinite set of equations. In some sense, while the numerical experiments of Albert and Torossian [1] seem to confirm Alekseev and Torossian’s hypothesis, the equivalence looks less likely to be true from the point of view of graphical calculus.

1.6.2 Paper B

Paper B studies the structure of the conjectural generators of \mathfrak{grt}_1 . Observe that while \mathfrak{grt}_1 is graded (by the grading on $\text{Lie}(x, y)$, i.e. the total number of x and y of any Lie monomial), it may also be equipped with a natural combinatorial filtration $\mathcal{F}^p \mathfrak{grt}_1$ given by the number of y in any Lie word. In particular, it is a well-known fact that all Lie brackets $\{\sigma_{2i+1}, \sigma_{2j+1}\}$ lie in $\mathcal{F}^2 \mathfrak{grt}_1$. Here the curly brackets denote the Lie bracket in \mathfrak{grt}_1 . While conjecturally such brackets should not satisfy any linear relations, they do modulo expressions containing at least three y . For instance,

$$\{\sigma_3, \sigma_9\} - 3\{\sigma_5, \sigma_7\} = 0 \quad \text{mod } \mathcal{F}^3 \mathfrak{grt}_1.$$

These type of relations were first studied by Ihara and Takao [28], independently by Goncharov [25] and ultimately by Schneps [42] yielding deep connections to the theory of modular forms. A result of Ihara and Takao [28] shows that modulo expressions with at least three y , the brackets $\{\sigma_{2i+1}, \sigma_{2j+1}\}$ actually generate the whole quotient

$$\mathcal{F}^2 \mathfrak{grt}_1 / \mathcal{F}^3 \mathfrak{grt}_1.$$

Paper B gives two graph theoretic interpretations of this quotient. The first establishes a link to the two-loop part of the cohomology of a certain “hairy graph complex” C , used by Arone and Turchin [9] to study algebraic properties of so-called “long knots” coming from topology. In particular, the elements of interest reduce to certain “hairy” θ -shaped graphs as depicted in Figure 1.2.

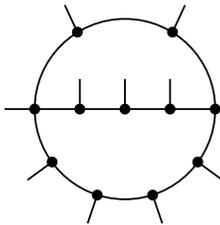


Figure 1.2: A “hairy” θ -graph.

We show that in degree one, the cohomology of this complex is generated by θ -graphs $\theta_{2i,2j}$ for which exactly two of the three strands making up the theta shape are decorated by $2i$ and $2j$ “hairs”, respectively. The first main result of Paper B is the following theorem.

Theorem. The map $\mathcal{F}^2 \mathfrak{grt}_1 / \mathcal{F}^3 \mathfrak{grt}_1 \rightarrow H^1(C)$ which on generators is given by

$$\{\sigma_{2i+1}, \sigma_{2j+1}\} \text{ mod } \mathcal{F}^3 \mathfrak{grt}_1 \mapsto \theta_{2i,2j},$$

defines an isomorphism of vector spaces.

In particular, there is an explicit and simple equivalence between the linear relations satisfied by the Lie brackets $\{\sigma_{2i+1}, \sigma_{2j+1}\}$ modulo expressions containing at least three y and the relations among the cohomology classes represented by the graphs $\theta_{2i,2j}$. The proof relies on a classification result by Schneps [42], and the fact that the two-loop part of the “hairy graph complex” may be expressed in terms of certain symmetric polynomials in three variables [14].

In a second step, we try to describe the quotient through the isomorphism $H^0(\mathfrak{GC}) \cong \mathfrak{grt}_1$. More precisely, we equip the graph complex \mathfrak{GC} with a combinatorially defined filtration inducing a filtration on $H^0(\mathfrak{GC})$. In particular, the filtration is chosen such that the class corresponding to σ_{2i+1} is represented modulo $\mathcal{F}^2 \mathfrak{GC}$ by the aforementioned wheel graph with $2i + 1$ spokes. Additionally, the filtration is compatible with the combinatorial Lie algebra structure on \mathfrak{GC} . Thus, considering brackets of wheel graphs, we obtain the second result of Paper B which states that, via Willwacher’s isomorphism the degree two part of the respective associated graded complexes agree.

Theorem. Willwacher’s map induces an isomorphism

$$\mathcal{F}^2 H^0(\mathfrak{GC}) / \mathcal{F}^3 H^0(\mathfrak{GC}) \xrightarrow{\cong} \mathcal{F}^2 \mathfrak{grt}_1 / \mathcal{F}^3 \mathfrak{grt}_1.$$

The proof consists in showing that on the induced map factors through an isomorphism

$$\mathcal{F}^2 H^0(\mathrm{GC})/\mathcal{F}^3 H^0(\mathrm{GC}) \cong H^1(C)$$

which, together with our first result, implies the statement. There are a variety of hopes and dreams arising from this result. For instance, it is unclear whether the chosen filtration on GC yields an isomorphism on the entire associated graded. Ideally, the quotient

$$\mathcal{F}^p H^0(\mathrm{GC})/\mathcal{F}^{p+1} H^0(\mathrm{GC})$$

would again relate to the p -loop part of the hairy graph complex which, hopefully, might be again encoded by some quotient of an algebra of polynomials. This would reduce the study of the quotients $\mathcal{F}^p \mathrm{grt}_1/\mathcal{F}^{p+1} \mathrm{grt}_1$ and the relations among iterated brackets of the σ_{2i+1} to polynomial identities.

1.6.3 Paper C

Paper C addresses a question related to the Duflo isomorphism. Let us sketch the setup first. Let (A, d_A, m_A) and (B, d_B, m_B) be differential graded associative algebras, and $f : (A, d_A) \rightarrow (B, d_B)$ be an isomorphism of complexes, which is only compatible with the algebra structure on the level of cohomology, i.e. the induced map

$$H(f) : H(A, d_A) \rightarrow H(B, d_B)$$

is an isomorphism of algebras, while f is not. Within this setting, it is natural to ask whether there exists a bilinear map $f_2 : A^{\otimes 2} \rightarrow B$ which measures the failure of f to be an algebra morphism before passing to cohomology. That is, f_2 should satisfy

$$f \circ m_A = m_B \circ (f \otimes f) + d_B \circ f_2 + f_2 \circ (d_A \otimes 1 + 1 \otimes d_A).$$

Typically, one requires f_2 to verify further compatibility conditions, again up to a homotopy $f_3 : A^{\otimes 3} \rightarrow B$. Explicitly, the natural relation is

$$f_2 \circ (m_A \otimes 1 - 1 \otimes m_A) + f_3 \circ (d_A \otimes 1 \otimes 1 + 1 \otimes d_A \otimes 1 + 1 \otimes 1 \otimes d_A) = m_B \circ (f \otimes f_2 - f_2 \otimes f) + d \circ f_3.$$

More generally, the collection of maps $\{f_n : A^{\otimes n} \rightarrow B\}_{n \geq 1}$ (setting $f_1 = f$) obtained by iterating this procedure is called an A_∞ -morphism between the associative algebras A and B . It is a natural question to ask whether the map f may be extended to an A_∞ -morphism. Note that since $f_1 = f$ is an isomorphism of vector spaces, the corresponding A_∞ -morphism is in fact called A_∞ -isomorphism.

Let us apply the setting above to the following version of Duflo's isomorphism. For this, let \mathfrak{g} be a finite dimensional Lie algebra and consider its Chevalley-Eilenberg complexes with values in its symmetric and universal enveloping algebra, respectively, i.e.

$$C(\mathfrak{g}, S\mathfrak{g}) = \wedge \mathfrak{g}^* \otimes S\mathfrak{g} \quad \text{and} \quad C(\mathfrak{g}, U\mathfrak{g}) = \wedge \mathfrak{g}^* \otimes U\mathfrak{g}.$$

The Duflo map induces an isomorphism of complexes, but not of algebras. However, by the works of Pevzner and Torossian [39], and Shoikhet [44], it induces an isomorphism

$$H(\mathfrak{g}, S\mathfrak{g}) \rightarrow H(\mathfrak{g}, U\mathfrak{g})$$

of algebras on the level of Lie algebra cohomology. The main result of Paper C may be rephrased as follows.

Theorem. The Duflo map $\text{Id} \otimes \text{Duf} : C(\mathfrak{g}, S\mathfrak{g}) \rightarrow C(\mathfrak{g}, U\mathfrak{g})$ does not extend to a universal (i.e. independent of the specific choice of the Lie algebra \mathfrak{g}) A_∞ -isomorphism.

This result was previously shown by Alm ([6], [7], [8]) using various techniques from the theory of graph complexes and a deep result by Willwacher [47]. Paper C provides a self-contained, elementary and (hopefully) more accessible proof of this result using a variant of Kontsevich's graph complex.

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Chapter 2

Internally connected graphs and the Kashiwara-Vergne Lie algebra

Abstract

It is conjectured that the Kashiwara-Vergne Lie algebra $\widehat{\mathfrak{kv}}_2$ is isomorphic to the direct sum of the Grothendieck-Teichmüller Lie algebra \mathfrak{grt}_1 and a one-dimensional Lie algebra. In this paper, we use the graph complex of internally connected graphs to define a nested sequence of Lie subalgebras of $\widehat{\mathfrak{kv}}_2$ whose intersection is \mathfrak{grt}_1 , thus giving a way to interpolate between these two Lie algebras.

Introduction

The Kashiwara-Vergne Lie algebra $\widehat{\mathfrak{kv}}_2$ was introduced by A. Alekseev and C. Torossian in [1]. It describes the symmetries of the Kashiwara-Vergne problem [5] in Lie theory. It has been shown in [1] to contain the Grothendieck-Teichmüller Lie algebra \mathfrak{grt}_1 as a Lie subalgebra. Conjecturally though,

$$\widehat{\mathfrak{kv}}_2 \cong \mathfrak{grt}_1 \oplus \mathbb{K}t =: \mathfrak{grt}$$

where t is a generator of degree 1. The aim of this work is to define a nested sequence of Lie subalgebras of $\widehat{\mathfrak{kv}}_2$ whose intersection is \mathfrak{grt} . This infinite family therefore interpolates between these two Lie algebras. Our hope is that this construction will provide the framework to a more systematic approach to tackle the conjecture. The technical tool used for this construction is the operad of internally connected graphs ICG introduced by P. Ševera and T. Willwacher in [13]. Elements of $\text{ICG}(n)$ are linear combinations of (isomorphism classes of) graphs with n “external” and an arbitrary number of “internal” vertices satisfying some connectivity condition. On these spaces, there are (among others) two natural operations. One is given by splitting internal (external) vertices into two internal (an external and an internal) vertices connected by an edge. The other splits external vertices into two external vertices. In both cases, we sum over all ways of reconnecting the “loose” edges (see Figure 2.1). While the former defines a differential d on $\text{ICG}(n)$, the latter, denoted by δ , maps $\text{ICG}(n)$ to $\text{ICG}(n+1)$ and is therefore of a more simplicial nature.

The central character throughout this story will be the equation

$$dX = \delta Y \tag{2.1}$$

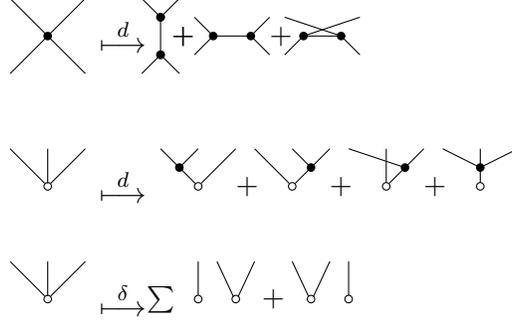


Figure 2.1: A schematic description of the operators d and δ . Black vertices represent “internal”, white vertices “external” vertices. For simplicity, we omit all signs.

where solutions X and Y should lie in $\text{ICG}(n)$ and $\text{ICG}(n-1)$, respectively. Note that $\text{ICG}(n)$ is filtered by the number of the internal loops (i.e. loops that do not contain any external vertices). While the simplicial differential δ preserves this number, the differential d might increase it.

Let us now trace the connection to A. Alekseev and C. Torossian’s work. It is given by the identification of internally trivalent trees in $\text{ICG}(n)$ modulo some relation with the Lie algebra of special derivations \mathfrak{sdet}_n of the free Lie algebra in n variables. This construction first appeared in some form in V. Drinfeld’s famous paper [4]. Also, one-loop graphs in $\text{ICG}(n)$ modulo some relations may be identified with a subspace of the graded vector space \mathfrak{tw}_n of cyclic words in n letters. Both $\widehat{\mathfrak{tw}}_2$ and \mathfrak{gt}_1 are Lie subalgebras of \mathfrak{sdet}_2 , meaning that their description as graphs should be in terms of (equivalence classes of) trees. For this, let x be the internally trivalent tree part of $X \in \text{ICG}(2)$ which solves equation (2.1) for some $Y \in \text{ICG}(1)$ only up to internal loop order 1, i.e.

$$dX = \delta Y \quad \text{mod 2 internal loops.}$$

Then the one-loop part of this equation (which only involves the tree part x of X on the left hand side) can be viewed as an identity in the space of cyclic words in two letters. In fact, it encodes exactly the defining relation of the Kashiwara-Vergne Lie algebra, where the differential d takes the role of the “divergence” map, $\text{div} : \mathfrak{sdet}_2 \rightarrow \mathfrak{tw}_2$, and δ corresponds to A. Alekseev and C. Torossian’s simplicial operator $\mathfrak{tw}_1 \rightarrow \mathfrak{tw}_2$. We may therefore identify $\widehat{\mathfrak{tw}}_2$ with (equivalence classes of) internally trivalent trees which correspond to the tree part of a solution to equation (2.1) up to loop order 2. On the other hand, the Grothendieck-Teichmüller Lie algebra is related to graph complexes through T. Willwacher’s result [16]

$$\mathfrak{gt}_1 \cong H^0(\text{GC}_2)$$

where GC_2 is a version of M. Kontsevich’s graph complex. Surprisingly, the algorithm describing the isomorphism $H^0(\text{GC}_2) \rightarrow \mathfrak{gt}_1$, produces first a pair $(X, Y) \in \text{ICG}(2) \times \text{ICG}(1)$ which satisfies

$$dX = \delta Y \quad \text{for any loop order,}$$

for which the tree part of X eventually represents the desired \mathfrak{gt}_1 -element. Thus, it appears as if the Lie algebras \mathfrak{gt}_1 and $\widehat{\mathfrak{tw}}_2$ live on opposite ends of a chain described in terms of solutions to equation (2.1) up to a certain loop order. More precisely, we consider solutions to the equation

$$dX = \delta Y \quad \text{mod } k + 1 \text{ internal loops.}$$

and set $\widehat{\mathfrak{kv}}_2^{(k)}$ to consist of the tree part of such X . Then, to summarize, our main result may be rephrased as follows.

Theorem 1. There exists a family of subspaces $\{\widehat{\mathfrak{kv}}_2^{(k)}\}_{k \in \mathbb{N}}$ of \mathfrak{sderv}_2 satisfying:

1. For all $k \geq 1$, $\widehat{\mathfrak{kv}}_2^{(k)}$ is a Lie subalgebra of \mathfrak{sderv}_2
2. They define an infinite nested sequence between $\widehat{\mathfrak{kv}}_2$ and \mathfrak{grt}_1 , that is,

$$\mathfrak{grt}_1 \subset \cdots \subset \widehat{\mathfrak{kv}}_2^{(k+1)} \subset \widehat{\mathfrak{kv}}_2^{(k)} \subset \cdots \subset \widehat{\mathfrak{kv}}_2^{(1)} = \widehat{\mathfrak{kv}}_2.$$

3. Their intersection is $\bigcap_{k \geq 1} \widehat{\mathfrak{kv}}_2^{(k)} \cong \mathfrak{grt}_1 \oplus \mathbb{K}t$.

The proofs of (1) and (3) are non-trivial and require several results from the theory of graph complexes. Additionally, we recall a similar construction which stems from the work of P. Ševera and T. Willwacher [13] for the kernel of the divergence map, $\ker(\text{div} : \mathfrak{sderv}_n \rightarrow \mathfrak{tv}_n) =: \mathfrak{kv}_n$ (which is also referred to as the Kashiwara-Vergne Lie algebra). More precisely, we show that there exists a nested sequence of Lie subalgebras $\{\mathfrak{kv}_n^{(k)}\}_{k \geq 1}$ of \mathfrak{sderv}_n extending the notion of the Lie algebra \mathfrak{kv}_n . In this instance, the intersection of these Lie subalgebras is the Drinfeld-Kohno Lie algebra \mathfrak{t}_n .

Figure 2.2: A pair $(X, Y) \in \text{ICG}(2) \times \text{ICG}(1)$ solving equation (2.1) in loop order 1.

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2.1 Preliminaries: Results from homotopy theory

In this first section, we recall some well-known facts from homotopy theory. Throughout the text, we work over a field \mathbb{K} of characteristic zero.

Definition 1. Let f and g be chain maps between two chain complexes (V, d_V) and (W, d_W) . A *homotopy* between f and g is a map $h : V \rightarrow W$ of degree -1 such that

$$f - g = d_W h + h d_V$$

We say f is homotopic to g .

Definition 2. A *homotopy retract* consists of the following data:

- two chain complexes (W, d_W) and (V, d_V) ,
- chain maps

$$\begin{aligned} i &: (W, d_W) \xrightarrow{\sim} (V, d_V) \\ p &: (V, d_V) \longrightarrow (W, d_W) \end{aligned}$$

where i is a quasi-isomorphism,

- a homotopy h between Id and ip .

Sometimes, it is more convenient to say (W, d_W) is a homotopy retract of (V, d_V) .

Proposition 1. Let (V, d) denote a differential graded vector space. If $\pi : V \rightarrow V$ is a projection ($\pi^2 = \pi$) and $h : V \rightarrow V$ is a map of degree -1 such that $\text{Id} - \pi = dh + hd$ (i.e. h is a homotopy between Id and π), then $\pi(V) \hookrightarrow V$ is a quasi-isomorphism.

Proof. Denote the inclusion map by $i : \pi(V) \hookrightarrow V$. Notice that

$$\begin{aligned} \pi i &= \text{Id}|_{\pi(V)} \\ i\pi &= \pi. \end{aligned}$$

Moreover, since π is homotopic to the identity Id_V , the induced maps on cohomology coincide, i.e. $\pi^* = \text{Id}_V^* = \text{Id}_{H(V)}$. Also

$$\begin{aligned} i^* &: H(\pi(V)) \rightarrow H(V) \\ \pi^* &: H(V) \rightarrow H(\pi(V)) \end{aligned}$$

satisfy $i^*\pi^* = \pi^* = \text{Id}_{H(V)}$, $\pi^*i^* = \text{Id}_{H(\pi(V))}$. Thus i^* is an isomorphism. \square

Proposition 2. Let (V, d) be as above. There exist graded subspaces $H, U, U' \subset V$ such that $d(H) = 0$, $H \cong H(V, d)$, d restricted to U is an isomorphism onto U' , i.e. $d : U \xrightarrow{\sim} U'$ and V decomposes as $V = H \oplus U \oplus U'$.

Proof. Let $Z := \{v \in V \mid dv = 0\}$ be the subset of closed elements. Let $U \subset V$ be some complement of Z in V , so that $V = Z \oplus U$. Define $U' := dU \subset Z$ and let $H \subset Z$ be some complement of U' in Z , so that $Z = H \oplus U'$. Then $V = H \oplus U \oplus U'$. By construction, $dH = dU' = 0$ and $d|_U : U \rightarrow U'$ is surjective. Since $U \cap Z = \{0\}$, it is also injective. Clearly, $H \cong H(V, d)$ as graded vector spaces. \square

Corollary 1. Let (V, d) be as above. Then there exist a projection π and a homotopy h between Id and π (i.e. $\text{Id} - \pi = dh + hd$) satisfying

$$d\pi = \pi d = 0 \text{ and } h^2 = \pi h = h\pi = 0. \quad (2.2)$$

For every such π and h , we have $\pi(V) \cong H(V, d)$ as graded vector spaces.

Proof. We have $V = H \oplus U \oplus U'$, with $d|_U : U \rightarrow U'$ an isomorphism, and $dH = dU' = 0$. Let π be the projection onto H and $h : U' \rightarrow U$ be an inverse for $d|_U$, i.e. $dh|_{U'} = \text{Id}$. Extend h to H and U by 0. Note that this way, $h : V \rightarrow V$ is a right inverse to $d : V \rightarrow V$. All requested relations are now easily checked.

Given such π and h , the relation $d\pi = 0$ implies $\pi(V) \subset \ker(d)$. Let

$$W := \ker(\pi) \cap \ker(d) = \{v \in V \mid dv = \pi v = 0\}.$$

Then

$$\ker(d) = \pi(V) \oplus W.$$

We claim that $W = \text{im}(d)$. Let $v = du \in \text{im}(d)$. Then, $dv = d^2u = 0 = d\pi(v)$, i.e. $v \in W$. On the other hand, if $w \in W$, then $(\text{Id} - \pi)(w) = w = (dh + hd)(w) = dhw \in \text{im}(d)$. Now $\ker(d) = \pi(V) \oplus \text{im}(d)$ implies $\pi(V) \cong H(V, d)$. \square

Suppose (V, d) is a complex and $\{V_n\}_{n \in \mathbb{Z}}$ a family of subsets of V such that $V \cong \prod_{n \in \mathbb{Z}} V_n$ as graded vector spaces. Assume that the differential decomposes as $d = d_0 + d_1 + d_2 + \dots$ with $d_j : V_n \rightarrow V_{n+j}$ for all n . Note that V is bigraded. The degree within the complex will be denoted by a superscript. Moreover, suppose that the V_n are bounded in this degree, that is for every degree j , there is an $\tilde{n}(j)$ such that $V_n^j = 0$ for all $n < \tilde{n}(j)$. With this setting, we have a bounded above, complete and descending filtration $\mathcal{F}^p V := \prod_{n \geq p} V_n$. Note that as complexes, the completed associated graded complex $\hat{g}rV$ with differential d_0 is isomorphic to (V, d_0) , i.e. $(V, d_0) \cong (\hat{g}rV, d_0)$.

Proposition 3. Suppose (V, d) is a complex as above. Let $\pi_0 : V \rightarrow V$ be a projection (i.e. $\pi_0^2 = \pi_0$) and h_0 be a homotopy between Id and π_0 for d_0 (i.e. $\text{Id} - \pi_0 = d_0 h_0 + h_0 d_0$) such that

$$\begin{aligned} d_0 \pi_0 &= \pi_0 h_0 = 0 \\ h_0^2 &= \pi_0 h_0 = h_0 \pi_0 = 0. \end{aligned}$$

Then

$$h := h_0 - h_0 d' h_0 + h_0 d' h_0 d' h_0 - \dots = h_0 \cdot \frac{1}{1 + d' h_0} = \frac{1}{1 + h_0 d'} \cdot h_0$$

and

$$\pi := \text{Id} - (dh + hd)$$

where $d' = d - d_0$ satisfy

- (i) $\pi^2 = \pi$
- (ii) $d\pi = \pi d$
- (iii) $h^2 = 0$
- (iv) $h\pi = \pi h = 0$
- (v) $\text{Id} - \pi = dh + hd$

Proof. By definition, we have $\text{Id} - \pi = dh + hd$ and since $h_0^2 = 0$, it clearly follows that $h^2 = 0$. Moreover,

$$d\pi = d(\text{Id} - dh - hd) = d - dh d = (\text{Id} - dh - hd)d = \pi d.$$

Using $h_0 d_0 h_0 = h_0(\text{Id} - \pi_0 - h_0 d_0) = h_0$, a cumbersome computation shows $hdh = h$. Hence,

$$\pi h = (\text{Id} - dh - hd)h = h - hdh = 0 = h(\text{id} - dh - hd) = h\pi,$$

and as $(dh + hd)^2 = dh d h + h d h d = dh + hd$, we find $(\text{Id} - \pi)^2 = \text{Id} - \pi \Leftrightarrow \pi^2 = \pi$. \square

Corollary 2. Let (V, d) and π be as a in the proposition above. Then $(\pi(V), d)$ is a quasi-isomorphic subcomplex of (V, d) . Moreover, as a graded vector space, $\pi(V)$ is isomorphic to $H^\bullet(V, d_0) \cong H^\bullet(\hat{g}rV, d_0)$.

Proof. That $(\pi(V), d) \hookrightarrow (V, d)$ is a quasi-isomorphism follows directly from Proposition 1. From Corollary 1, we get that $\pi_0(V) \cong H^\bullet(V, d_0)$ as graded vector spaces. To prove that $H^\bullet(V, d_0) \cong \pi(V)$ as graded vector spaces, we show that

$$\pi_0|_{\pi(V)} : \pi(V) \xrightarrow{\cong} \pi_0(V) : \pi|_{\pi_0(V)}$$

are mutual inverses. Note that as $\pi_0 h = h \pi_0 = 0$, $\pi_0 \pi \pi_0 = \pi_0(\text{Id} - dh - hd)\pi_0 = \pi_0^2 = \pi_0$ and therefore $\pi_0 \pi|_{\pi_0(V)} = \text{Id}_{\pi_0(V)}$. The other direction is more technical. First of all, note that $h_0 h = h h_0 = 0$ (as $h_0^2 = 0$) and $d_0 h_0 d_0 = (\text{Id} - \pi_0 - h_0 d_0)d_0 = d_0$. Also, a somewhat tedious, but elementary calculation shows

$$\begin{aligned} h_0 dh &= h_0 d_0 h - h + h_0 \\ hdh_0 &= h d_0 h_0 - h + h_0. \end{aligned}$$

Using these identities, a lengthy algebraic manipulation produces the desired result, $\pi \pi_0 \pi = \pi$. \square

Lemma 1. Let G be a finite group acting on a chain complex (V, d) (i.e. the action commutes with the differential). Then there exists a projection π and a homotopy h between Id and π which satisfy the equations (2.2) as in Corollary 1 and commute with the action of G . Moreover, for every such π and h , we have $\pi(V) \cong H(V, d)$ as graded G -vector spaces.

Proof. We need to adapt the proof of Corollary 1 slightly. We have $V = H \oplus U \oplus U'$, with $d|_U : U \rightarrow U'$ an isomorphism, and $dH = dU' = 0$. Let π be the projection onto H . This is a G -equivariant map. Let $h_0 : V \rightarrow V$ be any right inverse to d , i.e. $dh_0 = \text{Id}$. To construct an G -equivariant map out of h_0 , define

$$h := \frac{1}{|G|} \sum_{g \in G} g h_0 g^{-1}.$$

This is still a right inverse to d as

$$dh = \frac{1}{|G|} \sum_{g \in G} g d h_0 g^{-1} = \frac{1}{|G|} \sum_{g \in G} g \text{Id} g^{-1} = \frac{|G|}{|G|} \text{Id} = \text{Id}$$

and it is G -equivariant. Let $k \in G$, then

$$k.h = \frac{1}{|G|} \sum_{g \in G} kgh_0g^{-1} = \frac{1}{|G|} \sum_{g' := kg \in G} g'h_0(k^{-1}g')^{-1} = \frac{1}{|G|} \sum_{g' \in G} g'h_0g'^{-1}k = hk.$$

To show that $\pi(V) \cong H(V, d)$ as G -vector spaces, we need to find a G -equivariant right inverse $i : H \rightarrow V$ to $\pi : V \rightarrow H \cong H(V, d)$. For this, let i_0 be any right inverse to π (which exists as π is surjective). By the same averaging trick as above, we define

$$i := \frac{1}{|G|} \sum_{g \in G} gi_0g^{-1}.$$

That $\pi i = \text{Id}$ and i is G -invariant is checked in exactly the same way as for h . The map i can then be extended to $U \oplus U'$ by 0, thus giving a G -equivariant inverse to π and $\pi(V) \cong H(V, d)$ as graded G -vector spaces. \square

The following homotopy transfer theorem for L_∞ -algebras can be found in chapter 10.3. of J.-L. Loday and B. Vallette's book [11]. Another good introductory survey is B. Vallette's text [14].

Theorem 2. ([11], Theorem 10.3.5) Let (W, d_W) be a homotopy retract of (V, d_V) with maps $i : W \xrightarrow{\sim} V$, $p : V \rightarrow W$ and homotopy $h : V \rightarrow V$. Moreover, let $\{l_n : V^\otimes \rightarrow V\}_{n \geq 2}$ be an L_∞ -structure on V . This L_∞ -structure can be transferred to an L_∞ -structure on W such that i extends to an L_∞ -quasi-isomorphism. The transferred structure $\{m_n : W^\otimes \rightarrow W\}_{n \geq 2}$ is given by

$$m_n := \sum_{t \in RT_n} \pm pt(l, h)i^{\otimes n} \quad (2.3)$$

where the sum runs over rooted trees t with n leaves and where the notation $t(l, h)$ stands for the n -multilinear operation on V defined by the composition scheme t with vertices labeled by the l_k and internal edges labeled by h .

Remark 1. ([11], Theorems 10.3.11. and 10.3.15) Both maps i and p may be extended to L_∞ -morphisms $\tilde{i} = (i, i_2, i_3, \dots)$ and $\tilde{p} = (p, p_2, p_3, \dots)$ between the L_∞ -algebras $(V, d_V, \{l_n\}_{n \geq 2})$ and $(W, d_W, \{m_n\}_{n \geq 2})$. The higher arity maps $\{i_n\}_{n \geq 2}$ and $\{p_n\}_{n \geq 2}$ are constructed using composition schemes involving only i, p, h and $\{l_n\}_{n \geq 2}$. For instance, from [14]

$$i_n := \sum_{t \in RT_n} \pm ht(l, h)i^{\otimes n}$$

where the notation is as in equation (2.3).

2.2 The L_∞ -algebra of internally connected graphs

We follow P. Severa and T. Willwacher's work [13]. In their paper, we learn that the tools to define the L_∞ -algebra of internally connected graphs are based on M. Kontsevich's graph complex which can be found in [9] and [10]. Fix $n \geq 1$.

Definition 3. An *admissible graph* is an unoriented graph Γ with labeled vertices $1, 2, \dots, n$ (called external), possibly other vertices (unlabeled and called internal) satisfying the following properties:

1. There is a linear order on the set of edges.
2. Γ has no double edges, nor simple loops (edges connecting a vertex with itself).
3. Every internal vertex is at least trivalent.
4. Every internal vertex can be connected by a path with an external vertex.

Let $\mathbf{graphs}(n)$ be the vector space spanned by finite linear combinations of admissible graphs with n external vertices, modulo the relation $\Gamma^\sigma = (-1)^{|\sigma|} \Gamma$, where Γ^σ differs from Γ by a permutation σ on the order of edges. Define the degree by

$$\deg \Gamma = \#\text{edges} - 2\#\text{internal vertices}$$

and let the differential be given by vertex splitting. More precisely, an external vertex splits into an external and an internal vertex connected by an edge, and we sum over all possible ways of reconnecting the “loose” edges to the two newly created vertices, while only keeping admissible graphs. Similarly, an internal vertex splits into two internal vertices, before summing over all ways of reconnecting the edges previously connected to the splitted vertex.

Definition 4. A graph in $\mathbf{graphs}(n)$ which is connected after we cut off all external vertices is called *internally connected*. Denote by $\mathbf{ICG}(n)$ the space spanned by internally connected graphs modulo sign relations obtained from the order of edges. Define the grading on $\mathbf{ICG}(n)$ to be

$$\deg \Gamma = 1 - \#\text{edges} + 2\#\text{internal vertices}.$$

Set the differential on $\mathbf{ICG}(n)$ to be given by vertex splitting.

Since any graph in $\mathbf{graphs}(n)$ may be written as the disjoint union of its internally connected components (after identifying the external vertices), the internally connected graphs freely generate $\mathbf{graphs}(n)$ as a coalgebra. We therefore have an isomorphism of cocommutative coalgebras

$$\mathbf{graphs}(n) \cong S(\mathbf{ICG}(n)[1]).$$

By definition, the differential on $\mathbf{graphs}(n)$ defines the following L_∞ -structure on the graded vector space $\mathbf{ICG}(n)$. The k -ary bracket $[\Gamma_1, \dots, \Gamma_k]$ is given by gluing the Γ_i 's at the corresponding external vertices, applying the differential in $\mathbf{graphs}(n)$, and keeping only the graphs that are internally connected (we thus necessarily split only external vertices, and only in ways that connect all Γ_i 's together).

Finally, note that both \mathbf{graphs} and \mathbf{ICG} form operads in the category of cochain complexes. The operadic composition in \mathbf{graphs} (and also in \mathbf{ICG}) is given by insertion. That is, for $\Gamma_1 \in \mathbf{graphs}(r)$, $\Gamma_2 \in \mathbf{graphs}(s)$,

$$\Gamma_1 \circ_j \Gamma_2$$

is constructed by replacing the j th external vertex by Γ_2 , summing over all possible ways of reconnecting the “loose” edges (which were previously adjacent to vertex j) to vertices of Γ_2 , and keeping only admissible graphs (in the case of \mathbf{ICG} , we only keep the internally connected ones).

2.2.1 A natural filtration on $\text{ICG}(n)$

On $\text{ICG}(n)$, there is a natural descending filtration given by the number of internal loops (loops that do not contain any external vertices). For $p \in \mathbb{N}_0$, we denote by $\mathcal{F}^p := \mathcal{F}^p \text{ICG}(n)$ the subspace of $\text{ICG}(n)$ having *at least* p internal loops. Clearly,

$$\dots \subset \mathcal{F}^{p+1} \subset \mathcal{F}^p \subset \dots \subset \mathcal{F}^0 = \text{ICG}(n)$$

The completed associated graded with respect to this filtration is

$$\hat{\text{gr}}\text{ICG}(n) = \prod_{p \geq 0} \mathcal{F}^p / \mathcal{F}^{p+1}.$$

Remark that the p -th piece of the associated graded, $\mathcal{F}^p / \mathcal{F}^{p+1}$, is the space of graphs having *exactly* p internal loops. Also note that the differential d on $\text{ICG}(n)$ can be decomposed into a sum $d = d_0 + d_1 + d_2 + \dots$ where by applying d_i the vertex splitting produces i new internal loops. Note that thus d_0 splits internal vertices only. All others components come from splitting external vertices.

Remark 2. Occasionally, we drop the word internal. It should be noted that by loops we always mean internal loops.

Proposition 4. There exists a projection $\pi : \text{ICG}(n) \rightarrow \text{ICG}(n)$ and a homotopy h between Id and π which satisfy the equations (i) to (v) as in Proposition 3 and are such that $(\text{im}(\pi), d)$ is a quasi-isomorphic subcomplex of $(\text{ICG}(n), d)$. Moreover, $\text{im}(\pi) \cong H^\bullet(\hat{\text{gr}}\text{ICG}(n), d_0)$ as graded vector spaces.

Proof. Denote by $\pi_0 : \text{ICG}(n) \rightarrow \text{ICG}(n)$ the projection onto $H^\bullet(\text{ICG}(n), d_0) \cong H^\bullet(\hat{\text{gr}}\text{ICG}(n), d_0)$, and by h_0 a homotopy between Id and π_0 for d_0 . These exist by Lemma 1. Then Proposition 3 ensures the existence of π and the rest of the statement is an immediate consequence of Corollary 2. \square

Proposition 5. On $H^\bullet(\hat{\text{gr}}\text{ICG}(n), d_0)$, one can define a differential ∇ in such a way that the complex $(H^\bullet(\hat{\text{gr}}\text{ICG}(n), d_0), \nabla)$ is quasi-isomorphic to $(\text{ICG}(n), d)$.

Proof. Denote the isomorphism of graded vector spaces relating $H^\bullet(\hat{\text{gr}}\text{ICG}(n), d_0)$ to $\text{im}(\pi)$ by Φ ,

$$\Phi : H^\bullet(\hat{\text{gr}}\text{ICG}(n), d_0) \xrightarrow{\cong} \text{im}(\pi)$$

To turn this into an isomorphism of chain complexes, we define a differential on $H^\bullet(\hat{\text{gr}}\text{ICG}(n), d_0)$ by

$$\nabla := \Phi^{-1} \circ d \circ \Phi$$

Defined this way, $\nabla^2 = 0$, Φ commutes with the differentials and

$$(H^\bullet(\hat{\text{gr}}\text{ICG}(n), d_0), \nabla) \stackrel{\Phi}{\cong} (\text{im}(\pi), d)$$

as chain complexes. Since $(\text{im}(\pi), d) \xrightarrow{\text{incl}} (\text{ICG}(n), d)$ is a quasi-isomorphism, $(H^\bullet(\hat{\text{gr}}\text{ICG}(n), d_0), \nabla)$ is quasi-isomorphic to $(\text{ICG}(n), d)$ as well. \square

Remark 3. Note that the differential ∇ splits as $\nabla = \nabla_1 + \nabla_2 + \dots$ where applying ∇_i creates i new internal loops.

Remark 4. Denote the compositions by

$$\begin{aligned} i &: H^\bullet(\hat{\text{gr}}\text{ICG}(n), d_0) \cong \text{im}(\pi) \xrightarrow{\text{incl}} \text{ICG}(n) \\ p &: \text{ICG}(n) \xrightarrow{\pi} \text{im}(\pi) \cong H^\bullet(\hat{\text{gr}}\text{ICG}(n), d_0). \end{aligned}$$

The compositions pi and ip are

$$\begin{aligned} pi &= \Phi^{-1} \circ \pi \circ \text{incl} \circ \Phi = \text{Id} \text{ as } \pi|_{\text{im}(\pi)} = \text{Id} \\ ip &= \text{incl} \circ \Phi \circ \Phi^{-1} \circ \pi = \text{incl} \circ \pi. \end{aligned}$$

Note that i and p are chain maps, that is they satisfy

$$\begin{aligned} i\nabla &= di \\ \nabla p &= pd. \end{aligned}$$

Moreover, Proposition 3 ensures that there is a homotopy h between Id and ip , i.e.

$$\text{Id} - ip = dh + hd$$

In the setting above, $(H^\bullet(\hat{\text{gr}}\text{ICG}(n), d_0), \nabla)$ together with the chain maps i, p and the homotopy h form a homotopy retract of $(\text{ICG}(n), d)$.

Applying the homotopy transfer theorem 2, we readily obtain the following result.

Proposition 6. The L_∞ -structure on $\text{ICG}(n)$ may be transferred to an L_∞ -structure on $H^\bullet(\hat{\text{gr}}\text{ICG}(n), d_0)$ such that the map $i : H^\bullet(\hat{\text{gr}}\text{ICG}(n), d_0) \hookrightarrow \text{ICG}(n)$ may be extended to an L_∞ -quasi-isomorphism.

2.2.2 ICG and $H^\bullet(\hat{\text{gr}}\text{ICG}, d_0)$ as cosimplicial objects

The family of L_∞ -algebras $\{\text{ICG}(n)\}_{n \geq 1}$ together with the strict L_∞ -morphisms $\{\delta_j : \text{ICG}(n) \rightarrow \text{ICG}(n+1)\}_{j=0}^{n+1}$ and $\{s_j : \text{ICG}(n) \rightarrow \text{ICG}(n-1)\}_{j=1}^n$ for all $n \geq 0$ given by

- δ_0 (and δ_{n+1}): add an additional external vertex labeled by 1 ($n+1$) and raise the labels of the other external vertices by one (leave the labels invariant).
- δ_j for $j \neq 0, n+1$: split the j th vertex into two (rename them by j and $j+1$) and sum over all ways of reconnecting the “tangling” loose edges. The labels of the external vertices which were greater than j are all raised by one.
- s_j : delete the j th external vertex and all edges connected to it. All labels of external vertices greater than j get lowered by one.

form a cosimplicial object in the category of L_∞ -algebras. Operadically, for $\Gamma \in \text{ICG}(n)$, $\delta_j(\Gamma) = \Gamma \circ_j (\circ \circ)$. For all n , we define a *cosimplicial differential* $\delta : \text{ICG}(n) \rightarrow \text{ICG}(n+1)$ by

$$\delta := \sum_{j=0}^{n+1} (-1)^j \delta_j.$$

Proposition 7. On $\{H^\bullet(\widehat{\text{grICG}}(n), d_0)\}_{n \geq 1}$ we may define L_∞ -morphisms $\{\delta'_j : H^\bullet(\widehat{\text{grICG}}(n), d_0) \rightarrow H^\bullet(\widehat{\text{grICG}}(n+1), d_0)\}_{j=0}^{n+1}$ and $\{s'_j : H^\bullet(\widehat{\text{grICG}}(n), d_0) \rightarrow H^\bullet(\widehat{\text{grICG}}(n-1), d_0)\}_{j=1}^n$ for all $n \geq 0$ which turn $\{(H^\bullet(\widehat{\text{grICG}}(n), d_0))\}_{n \geq 1}$ into a cosimplicial object in the category of L_∞ -algebras.

Lemma 2. For all n , there exists a homotopy h_n on $(\text{ICG}(n), d)$ between Id and ip that commutes with the cosimplicial maps $\{s_j\}_{j=1}^n$ and $\{\delta_j\}_{j=0}^{n+1}$, i.e.

$$\begin{aligned} h_{n+1}\delta_j &= \delta_j h_n \\ s_j h_n &= h_{n-1} s_j. \end{aligned}$$

Let us prove Proposition 7 using Lemma 2.

Proof of Proposition 7. Lemma 2 tells us that we have a family of homotopies $\{h_n\}_{n \geq 1}$ between Id and ip that commute with the cosimplicial maps. To simplify notation, we shall omit the index n for the homotopy. By Remark 1 the maps i, p may be extended to L_∞ -morphisms \tilde{i}, \tilde{p} . Note that $\tilde{i}\tilde{p}$ contains only compositions of the maps $h, \{l_n\}_{n \geq 2}$ and the composition ip , all of which commute with the cosimplicial maps. As L_∞ -maps they thus satisfy

$$\begin{aligned} \delta_j \tilde{i}\tilde{p} &= \tilde{i}\tilde{p}\delta_j \\ s_j \tilde{i}\tilde{p} &= \tilde{i}\tilde{p}s_j. \end{aligned}$$

Possible candidates for the cosimplicial maps on $H^\bullet(\widehat{\text{grICG}}(n), d_0)$ are

$$\begin{aligned} \delta'_j &:= \tilde{p} \circ \delta_j \circ \tilde{i} \\ s'_j &:= \tilde{p} \circ s_j \circ \tilde{i}, \end{aligned}$$

where the composition is composition as L_∞ -maps. We need to check whether they satisfy the cosimplicial relations, i.e. for $i < j$

$$\begin{aligned} \delta'_j \delta'_i &= \tilde{p} \circ \delta_j \circ \tilde{i} \circ \tilde{p} \circ \delta_i \circ \tilde{i} = \tilde{p} \circ \tilde{i} \circ \tilde{p} \circ \delta_j \circ \delta_i \circ \tilde{i} = \tilde{p} \circ \tilde{i} \circ \tilde{p} \circ \delta_i \circ \delta_{j-1} \circ \tilde{i} \\ &= \tilde{p} \circ \delta_i \circ \tilde{i} \circ \tilde{p} \circ \delta_{j-1} \circ \tilde{i} = \delta'_i \delta'_{j-1}. \end{aligned}$$

Analogously, for $i \leq j$, $s'_j s'_i = s'_i s'_{j+1}$. The relations

$$s'_j \delta'_i = \begin{cases} \delta'_i s'_{j-1} & \text{if } i < j \\ \text{Id} & \text{if } i = j \text{ or } i = j + 1 \\ \delta'_{i-1} s'_j & \text{if } i > j + 1 \end{cases}$$

follow from a similar easy computation. □

Proof of Lemma 2. Fix $n, k \in \mathbb{N}$. Consider the space $\text{ICG}^{1-\text{val}}(k)$ of internally connected graphs with k univalent external vertices. There is an obvious S_k -action which permutes the labels of the k external vertices. This action extends to the direct sum

$$J_{k,n} := \bigoplus_{\substack{k_1 \geq 0, \dots, k_n \geq 0 \\ \sum k_i = k}} \text{ICG}^{1-\text{val}}(k).$$

By Lemma 1, on the chain complex $(J_{k,n}, d_0)$ there exists a projection π_0 and a homotopy h_0 between Id and π_0 which commute with this S_k -action. In particular, π_0 and h_0 restrict to

$$I_{k,n} := \bigoplus_{\substack{k_1 \geq 0, \dots, k_n \geq 0 \\ \sum k_i = k}} (\text{ICG}^{1-\text{val}}(k))^{S_{k_1} \times \dots \times S_{k_n}}$$

Here, the action of $S_{k_1} \times \dots \times S_{k_n} \subset S_k$ is obviously the induced one. We take a partition $\{k_1, \dots, k_n\}$ of the k external edges and each S_{k_i} will act only on the k_i part by permutation. To see that π_0 and h_0 restrict to this space, let $\Gamma \in I_{k,n}$ and $\sigma \in S_{k_1} \times \dots \times S_{k_n}$. Then,

$$\begin{aligned} \sigma.h_0(\Gamma) &= h_0(\sigma.\Gamma) = h_0(\Gamma) \Rightarrow h_0(\Gamma) \in I_{k,n} \\ \sigma.\pi_0(\Gamma) &= \pi_0(\sigma.\Gamma) = \pi_0(\Gamma) \Rightarrow \pi_0(\Gamma) \in I_{k,n}. \end{aligned}$$

In particular, this means that π_0 and h_0 preserve each S_{k_i} -invariant part. Denote by $\text{ICG}(n)(k)$ the space of graphs with n external vertices and k edges connecting internal and external vertices. There is an isomorphism of chain complexes

$$\text{Sym} : (\text{ICG}(n)(k), d_0) \longrightarrow (I_{k,n}, d_0)$$

Abbreviate the group $S_{k_1} \times \dots \times S_{k_n} =: G(k_1, \dots, k_n)$. The map is given by

$$\text{Sym}(\Gamma) := \frac{1}{k_1! \dots k_n!} \sum_{\sigma \in G(k_1, \dots, k_n)} \sigma.\tilde{\Gamma}$$

where $\tilde{\Gamma}$ is obtained by assigning an external vertex to each edge connecting an internal vertex to an external one. An external vertex i is thus sent to k_i univalent external vertices, labeled by following the order of the k_i incoming edges (for the symmetrization, the order in which the k_i external vertices are labeled is actually irrelevant). Note that, because d_0 splits only internal vertices, Sym is indeed an isomorphism of chain complexes, i.e.

$$\text{Sym}(d_0\Gamma) = d_0\text{Sym}(\Gamma).$$

For $j \in \{1, \dots, n\}$, the cosimplicial maps $\delta_j : \text{ICG}(n)(k) \rightarrow \text{ICG}(n+1)(k)$ are given by splitting the j -th external vertex and summing over all ways of reconnecting the ‘‘tangling loose’’ edges. On $I_{k,n}$, the corresponding operations are given by maps $\tilde{\delta}_j$ satisfying

$$\tilde{\delta}_j \text{Sym}(\Gamma) = \text{Sym}(\delta_j \Gamma)$$

for $\Gamma \in \text{ICG}(n)(k)$. Explicitly, the right hand side is given by

$$\text{Sym}(\delta_j \Gamma) = \sum_{l=0}^{k_j} \sum_{\tau \in \text{Unsh}(l, k_j-l)} \frac{1}{k_1! \dots l! (k_j-l)! \dots k_n!} \sum_{\sigma \in G(k_1, \dots, k_{j-1}, l, k_j-l, k_{j+1}, \dots, k_n)} \sigma.\tau.\tilde{\Gamma}$$

With this formula at hand, it is easy to see that π_0 and h_0 commute with $\tilde{\delta}_j$ on $I_{k,n}$. For this, let

$\Gamma' \in I_{k,n}$. Then there exists a $\Gamma \in \text{ICG}(n)(k)$ satisfying $\Gamma' = \text{Sym}(\Gamma)$ and

$$\begin{aligned}
h_0 \tilde{\delta}_j(\Gamma') &= h_0 \tilde{\delta}_j(\text{Sym}(\Gamma)) = h_0 \text{Sym}(\delta_j \Gamma) \\
&= \sum_{l=0}^{k_j} \sum_{\tau \in \text{Unsh}(l, k_j-l)} \frac{1}{k_1! \dots l!(k_j-l)! \dots k_n!} \sum_{\sigma \in G(k_1, \dots, k_{j-1}, l, k_j-l, k_{j+1}, \dots, k_n)} h_0 \sigma \cdot \tau \tilde{\Gamma} \\
&= \sum_{l=0}^{k_j} \sum_{\tau \in \text{Unsh}(l, k_j-l)} \frac{1}{k_1! \dots l!(k_j-l)! \dots k_n!} \sum_{\sigma \in G(k_1, \dots, k_{j-1}, l, k_j-l, k_{j+1}, \dots, k_n)} \sigma \cdot \tau \cdot h_0 \tilde{\Gamma} \\
&= \tilde{\delta}_j h_0(\text{Sym}(\Gamma)) = \tilde{\delta}_j h_0(\Gamma').
\end{aligned}$$

The proof that π_0 commutes with $\tilde{\delta}_j$ is analogous. Next, define a projection π and a homotopy h on $(\text{ICG}(n)(k), d_0)$ via

$$\begin{aligned}
\pi &:= \text{Sym}^{-1} \pi_0 \text{Sym} \\
h &:= \text{Sym}^{-1} h_0 \text{Sym}.
\end{aligned}$$

Because Sym is a chain map, h is a homotopy between Id and π with respect to the differential d_0 . Moreover, π and h commute with the cosimplicial maps δ_j . For $\Gamma \in \text{ICG}(n)(k)$,

$$\begin{aligned}
h_0 \tilde{\delta}_j \text{Sym}(\Gamma) &= h_0 \text{Sym}(\delta_j \Gamma) = \text{Sym}(h \delta_j \Gamma) \\
&= \tilde{\delta}_j h_0 \text{Sym}(\Gamma) = \tilde{\delta}_j \text{Sym}(h \Gamma) = \text{Sym}(\delta_j h \Gamma)
\end{aligned}$$

Using the fact that Sym is an isomorphism, we find,

$$\delta_j h = h \delta_j.$$

Analogously, one can show

$$\pi \delta_j = \delta_j \pi.$$

Remark that because h_0 and π_0 preserve the S_{k_i} -invariant parts of some $\text{Sym}(\Gamma) \in (\text{ICG}^{1-\text{val}}(k))^{S_{k_1} \times \dots \times S_{k_n}}$, h and π will preserve the k_i edges connected to the i th external vertex of Γ , for all i (as in, after applying h or π the images of these k_i edges will be connected to the image of the external vertex i). Also note that π and h correspond to the chain complex $(\text{ICG}(n), d_0)$ (note that $\text{ICG}(n)$ is the direct product over $k \geq 1$ of all $\text{ICG}(n)(k)$), and *not* to $(\text{ICG}(n), d)$. However, by Proposition 3, we can extend these two maps to $(\text{ICG}(n), d)$. Call them H and P . These extensions are constructed using only maps which commute with the δ_j . Therefore the extended projection and homotopy will still commute with the cosimplicial maps.

Note that π and h preserve the k_j edges connecting internal to external vertices. Thus for $j = 0$,

$$h \delta_0 \Gamma = h \left(\begin{smallmatrix} \circ & & \\ & \Gamma & \\ \circ & & \end{smallmatrix} \right) = \begin{smallmatrix} \circ & & \\ & h \Gamma & \\ \circ & & \end{smallmatrix} = \delta_0 h \Gamma$$

and analogously for π . Therefore $h \delta_0 = \delta_0 h$, $\pi \delta_0 = \delta_0 \pi$. Similarly, this holds also for $j = n + 1$.

The s_j maps are given by simply forgetting the j th external vertex and all edges connected to it. Again, as the homotopy h and the projection preserve the edges connected to external vertices,

$$\begin{aligned}
h s_j &= s_j h \\
\pi s_j &= s_j \pi
\end{aligned}$$

for all $j \in \{1, \dots, n\}$. Also, by construction, the extended homotopy H and projection P commute with the maps s_j .

As in Remark 4, denote by i and p the compositions

$$\begin{aligned} i : H^\bullet(\widehat{\text{gr}}\text{ICG}(n), d_0) &\cong \text{im}(P) \xrightarrow{\text{incl}} \text{ICG}(n) \\ p : \text{ICG}(n) &\xrightarrow{P} \text{im}(P) \cong H^\bullet(\widehat{\text{gr}}\text{ICG}(n), d_0). \end{aligned}$$

The extended homotopy H is a homotopy between Id and ip satisfying the properties requested in Lemma 2. \square

2.3 The Kashiwara-Vergne Lie algebra

2.3.1 A spectral sequence leading to the Kashiwara-Vergne Lie algebra

Consider the spectral sequence obtained through the filtration by internal loops. We find that the first page $E_1^{\bullet, \bullet}$ is exactly the aforementioned cohomology of the associated graded complex with respect to the differential d_0 , that is,

$$E_1^{p,q} = H^{p+q}(\widehat{\text{gr}}\text{ICG}(n)^p, d_0).$$

P. Severa and T. Willwacher explain in [13] that $H^0(\widehat{\text{gr}}\text{ICG}(n)^0, d_0)$, which consists of internally trivalent trees in $\text{ICG}(n)$ modulo the IHX relation, can be identified (as a Lie algebra) with the Lie algebra of special derivations (for an introduction see [1]). In formulas,

$$E_1^{0,0} = H^0(\widehat{\text{gr}}\text{ICG}(n)^0, d_0) \cong \mathfrak{sd}\mathfrak{er}_n.$$

We give the isomorphism in the appendix. This result already appeared in some form in V. Drinfeld's paper [4]. The Lie bracket on $E_1^{0,0}$ is given by identifying external vertices, summing over all ways of splitting external vertices without creating new internal loops and then keeping only internally connected, internally trivalent trees. This is justified in the following remark.

Remark 5. Denote by $m_2 : E_1^{\bullet, \bullet} \otimes E_1^{\bullet, \bullet} \rightarrow E_1^{\bullet, \bullet}$ the arity two component of the L_∞ -structure on the cohomology of the associated graded. In terms of the structure on $\text{ICG}(n)$, it is given by

$$m_2 = p \circ [-, -] \circ i^{\otimes 2}.$$

Denote by $[-, -]_{Ih}$ the projection to $E_1^{0,0}$ of the image of m_2 restricted to $E_1^{0,0} \otimes E_1^{0,0}$. As it correspond to the bracket on $\mathfrak{sd}\mathfrak{er}_n$ (which is sometimes named after Y. Ihara) we shall refer to it as the Ihara bracket. It is thus a map

$$[-, -]_{Ih} : E_1^{0,0} \otimes E_1^{0,0} \rightarrow E_1^{0,0}.$$

For $\overline{x_0}, \overline{y_0} \in E_1^{0,0}$, $[\overline{x_0}, \overline{y_0}]_{Ih} = \overline{[x_0, y_0]_0}$. Here $[-, -]_0$ is the term of the bracket $[-, -]$ on $\text{ICG}(n)$ that does not create any new loops. To see this, first note,

$$\begin{aligned} i(\overline{x_0}) &= x_0 + x_1 + \dots \\ i(\overline{y_0}) &= y_0 + y_1 + \dots \end{aligned}$$

Then

$$[i(\overline{x_0}), i(\overline{y_0})] = [x_0, y_0]_0 + [x_1, y_0]_0 + [x_0, y_1]_0 + \dots$$

Applying the projection p we obtain $m_2(\overline{x_0} \otimes \overline{y_0}) = \overline{[i(\overline{x_0}), i(\overline{y_0})]} \in E_1^{\bullet, \bullet}$. The only term in $E_1^{0,0}$ is $\overline{[x_0, y_0]_0}$, which therefore equals $\overline{[x_0, \overline{y_0}]_{Ih}}$ by definition. Since $[-, -]_0$ is a Lie bracket on $\text{ICG}(n)$, $[-, -]_{Ih}$ defines a Lie bracket on $E_1^{0,0}$.

Furthermore, also from [13], we know that the internally trivalent one-loop part of $\text{ICG}(n)$ modulo IHX (given by $H^1(\widehat{\text{grICG}}(n)^1, d_0)$) is isomorphic to cyclic words in n letters, denoted by \mathfrak{tr}_n in [1], modulo the relation

$$\text{tr}(w) = -(-1)^{\text{length}(w)} \text{tr}(\tilde{w})$$

where \tilde{w} corresponds to the word w but read backwards. We will denote the space of cyclic words modulo this relation by $\mathfrak{tr}_n^{(1)}$, i.e.

$$E_1^{1,0} = H^1(\widehat{\text{grICG}}(n)^1, d_0) \cong \mathfrak{tr}_n^{(1)}.$$

Moreover, Ševera and Willwacher show in ([13] Proposition 5.) that there is an injective map $\mathfrak{tr}_n^{(1)} \hookrightarrow \mathfrak{tr}_n$ (see the appendix) making the diagram

$$\begin{array}{ccc} E_1^{0,0} & \xrightarrow{\cong} & \mathfrak{sdet}_n \\ \downarrow \nabla_1 & & \downarrow \text{div} \\ E_1^{1,0} & \hookrightarrow & \mathfrak{tr}_n. \end{array}$$

commute. Here, $\text{div} : \mathfrak{sdet}_n \rightarrow \mathfrak{tr}_n$ is the “divergence map” defined by A. Alekseev and C. Torossian in [1]. In particular, $E_2^{0,0} = \ker \nabla_1 \cong \ker \text{div}$ is a Lie algebra, as shown in [1].

Definition 5. The *Kashiwara-Vergne Lie algebra* is

$$\mathfrak{kv}_n := \ker \text{div} = \{x \in \mathfrak{sdet}_n \mid \text{div}(x) = 0 \in \mathfrak{tr}_n\}.$$

Since $\mathfrak{kv}_n \cong \ker \nabla_1 = \{\bar{x} \in E_1^{0,0} \mid \nabla_1(\bar{x}) = 0 \in E_1^{1,0}\}$, all information determining \mathfrak{kv}_n is given by an equation involving internally trivalent trees and internally trivalent one-loop graphs. In what follows, we extend this notion to higher loop orders. Note that for $r \in \mathbb{N}$ (see for instance [15])

$$E_r^{0,0} = \frac{\{x = x_0 + x_1 + x_2 + \dots \in \text{ICG}(n) \mid \begin{array}{l} x_i \text{ graph with exactly } i \text{ loops,} \\ \deg(x)=0, dx=0 \pmod r \text{ loops} \end{array}\}}{Q}$$

where

$$\begin{aligned} Q := & \{dy \mid y \in \text{ICG}(n), \deg(y) = -1\} \\ & + \{x = x_1 + x_2 + \dots \in \mathcal{F}^1 \text{ICG}(n) \mid \begin{array}{l} x_i \text{ graph with exactly } i \text{ loops,} \\ \deg(x)=0, dx=0 \pmod r \text{ loops} \end{array}\}. \end{aligned}$$

Lemma 3. The map

$$\begin{aligned} i_r : E_r^{0,0} & \rightarrow E_1^{0,0} \cong \mathfrak{sdet}_n \\ \bar{x} = \overline{x_0 + x_1 + \dots} & \mapsto \bar{x_0} \end{aligned}$$

is injective.

Proof. Let $x = x_0 + x_1 + \dots \in \text{ICG}(n)$, $\deg(x) = 0$, $dx = 0 \pmod r$ loops and assume $i_r(\bar{x}) = \bar{x}_0 = \bar{0}$, that is $x_0 = d_0 y_0$ for some tree y_0 of degree -1 . Set $\tilde{x} := x - dy_0$. It satisfies $d\tilde{x} = dx - 0 = 0 \pmod r$ loops and $\tilde{x} = x_0 + x_1 + \dots - d_0 y_0 \in \mathcal{F}^1 \text{ICG}(n)$ (all elements have at least one internal loops). Therefore $x = x - dy_0 + dy_0 = \tilde{x} + dy_0$ and $\bar{x} = \bar{0} \in E_r^{0,0}$. \square

Definition 6. We set

$$\mathfrak{trv}_n^{(k)} := i_{k+1}(E_{k+1}^{0,0}).$$

More explicitly, $\mathfrak{trv}_n^{(k)}$ consists of classes $\bar{x}_0 \in E_1^{0,0}$ for which there are graphs $x_1, x_2, \dots \in \text{ICG}(n)$ (where x_i has exactly i loops) of degree zero such that $x = x_0 + x_1 + \dots \in \text{ICG}(n)$ satisfies $dx = 0 \pmod k + 1$ loops.

Lemma 4. The new definition extends our previous notion of the Kashiwara-Vergne Lie algebra in the sense that $\mathfrak{trv}_n^{(1)} \cong \mathfrak{trv}_n$.

Proof. Note that $\mathfrak{trv}_n^{(1)}$ consists of $\bar{x}_0 \in E_1^{0,0}$ which may be extended to a degree zero element $x = x_0 + x_1 + x_2 + \dots \in \text{ICG}(n)$ satisfying $dx = 0 \pmod 2$ internal loops. This equation is equivalent to $d_0 x_0 = 0$ (which is satisfied by the definition of x_0) and $d_1 x_0 + d_0 x_1 = 0 \in \text{ICG}(n)$. To prove the statement, let $\bar{x}_0 \in E_1^{0,0}$. Then

$$\begin{aligned} \nabla \bar{x}_0 &= \nabla_1 \bar{x}_0 + \nabla_2 \bar{x}_0 + \dots = \nabla p i \bar{x}_0 = p d i \bar{x}_0 = p d(x_0 + x_1 + x_2 + \dots) \\ &= p(d_1 x_0 + d_0 x_1 + \dots) = \overline{d_1 x_0 + d_0 x_1 + \dots} \end{aligned}$$

This is an equation in $E_1^{\bullet, \bullet}$. Consider its $E_1^{1,0}$ component. It is given by

$$\nabla_1 \bar{x}_0 = \overline{d_1 x_0 + d_0 x_1} = \overline{d_1 x_0}.$$

Therefore, $\ker \text{div} \cong \ker \nabla_1 \cong \mathfrak{trv}_n^{(1)}$. \square

We obtain a sequence of inclusions

$$\dots \subset \mathfrak{trv}_n^{(k)} \subset \mathfrak{trv}_n^{(k-1)} \subset \dots \subset \mathfrak{trv}_n^{(2)} \subset \mathfrak{trv}_n \subset \mathfrak{sdet}_n.$$

Proposition 8. ([13]) The subspaces $\mathfrak{trv}_n^{(k)}$ are Lie subalgebras of \mathfrak{sdet}_n for all $k \geq 1$.

Proof. The Ihara bracket of $\bar{x}_0, \bar{y}_0 \in E_1^{0,0}$ coincides with $\overline{[x_0, y_0]_0}$, where $[-, -]_0$ is the component of the bracket on $\text{ICG}(n)$ which does not produce any new loops. To prove the claim, let $\bar{x}_0, \bar{y}_0 \in \mathfrak{trv}_n^{(k)}$. Denote their extensions by $x = x_0 + x_1 + \dots$ and $y = y_0 + y_1 + \dots$. We claim that $[x, y]$ is a suitable extension of the bracket $[\bar{x}_0, \bar{y}_0]_{Ih}$. Indeed,

$$[x, y] = [x_0, y_0]_0 + [x_1, y_0]_0 + [x_0, y_1]_0 + \dots,$$

where $[x_1, y_0]_0 + [x_0, y_1]_0$ are already graphs of loop order 1, and

$$d[x, y] = [dx, y] + [x, dy] = 0 + 0 \pmod k + 1 \text{ loops.}$$

\square

Definition 7. The *Drinfeld-Kohno Lie algebra* \mathfrak{t}_n is generated by elements $t^{i,j} = t^{j,i}$, where $1 \leq i, j \leq n$ and relations

$$\begin{aligned} [t^{i,j}, t^{k,l}] &= 0 \text{ if } \#\{i, j, k, l\} = 4, \\ [t^{i,j} + t^{i,k}, t^{j,k}] &= 0 \text{ for } \#\{i, j, k\} = 3. \end{aligned}$$

Remark 6. As shown in [13], the aforementioned spectral sequence converges to the Drinfeld-Kohno Lie algebra, more precisely $\mathfrak{t}_n \cong E_\infty^{0,0}$. A generator $t^{i,j}$ is mapped to the equivalence class represented by the graph with no internal vertices and one edge connecting the external vertices i and j . In particular, this implies that

$$\bigcap_{k \geq 1} \mathfrak{krv}_n^{(k)} \cong \mathfrak{t}_n.$$

Remark 7. Most of the material presented in this section already appeared in some form in P. Severa and T. Willwacher's paper [13]. Our aim was to give an explicit description of the Lie algebras $\mathfrak{krv}_n^{(k)}$ to which they hinted at in their work. Moreover, the techniques developed here will be useful in the next section.

2.3.2 The extended Kashiwara-Vergne Lie algebra

For $n = 2$, A. Alekseev and C. Torossian defined in [1] the following extension of \mathfrak{krv}_2 ,

$$\widehat{\mathfrak{krv}}_2 := \{x \in \mathfrak{sdcr}_2 \mid \text{div}(x) = \text{tr}(f(u) - f(u+v) + f(v)) \text{ for some } f(u) = \sum_{k=2}^{\infty} f_k u^k\}.$$

They show that this is a Lie subalgebra of \mathfrak{sdcr}_2 . In fact, $[\widehat{\mathfrak{krv}}_2, \widehat{\mathfrak{krv}}_2] \subset \mathfrak{krv}_2$. Moreover, they prove that for $x \in \widehat{\mathfrak{krv}}_2$ the corresponding power series f is odd, i.e. $f_k = 0$ for k even. In particular this implies that $\text{tr}(f)$ corresponds to some linear combination of internally trivalent one-loop graphs under the injective map $E_1^{1,0} \cong \mathfrak{tr}_1^{(1)} \hookrightarrow \mathfrak{tr}_1$. On the level of graphs, it is not difficult to see that the map $\delta_{AT} : \text{tr}(f) \mapsto \text{tr}(f(u) - f(u+v) + f(v))$ corresponds to applying the cosimplicial differential $\delta' := p \circ \delta \circ i : \mathfrak{tr}_1^{(1)} \rightarrow \mathfrak{tr}_2^{(1)}$ to the graph associated to $\text{tr}(f)$. Including the vertex splitting differential d and ∇ , the global picture is encoded in the following commutative diagram.

$$\begin{array}{ccccc} \text{ICG}(2) & \xrightleftharpoons[i]{p} & E_1^{0,0} & \xrightarrow{\cong} & \mathfrak{sdcr}_2 \\ \downarrow d_0 + d_1 & & \downarrow \nabla_1 & & \downarrow \text{div} \\ \text{ICG}(2) & \xrightleftharpoons[i]{p} & E_1^{1,0} \cong \mathfrak{tr}_2^{(1)} & \hookrightarrow & \mathfrak{tr}_2 \\ \uparrow \delta & & \uparrow \delta' & & \uparrow \delta_{AT} \\ \text{ICG}(1) & \xrightleftharpoons[i]{p} & E_1^{1,0} \cong \mathfrak{tr}_1^{(1)} & \hookrightarrow & \mathfrak{tr}_1 \end{array}$$

The diagram implies the following equalities.

$$\begin{aligned}
\widehat{\mathfrak{trv}}_2 &= \{x \in \mathfrak{sdet}_2 \mid \text{div}(x) = \text{tr}(f(u) - f(u+v) + f(v)) \text{ for some } f(u) = \sum_{k=2}^{\infty} f_k u^k\} \\
&= \{\bar{x} \in E_1^{0,0} \cong \mathfrak{sdet}_2 \mid \nabla_1(\bar{x}) = \delta'(f) \text{ for some } f \in \mathfrak{tt}_1^{(1)}\} \\
&= \{x \in \mathfrak{sdet}_2 \mid \exists X \in \text{ICG}(2) : \text{deg}(X) = 0, X = x + x_1 + \dots \\
&\quad \text{and } d_1 x + d_0 x_1 = \delta Y \text{ for some } Y \in \text{ICG}(1)\} \\
&= \{x \in \mathfrak{sdet}_2 \mid \exists X \in \text{ICG}(2) : \text{deg}(X) = 0, X = x + x_1 + \dots \\
&\quad \text{and } dX = \delta Y \pmod{2 \text{ internal loops}} \text{ for some } Y \in \text{ICG}(1)\}.
\end{aligned}$$

As an extension of $\widehat{\mathfrak{trv}}_2$ we suggest,

$$\widehat{\mathfrak{trv}}_2^{(k)} := \{x \in \mathfrak{sdet}_2 \mid \exists X \in \text{ICG}(2) : dX = \delta Y \pmod{k+1 \text{ internal loops}} \text{ for some } Y \in \text{ICG}(1) \text{ and } \text{deg}(X)=0, [X]=x\}$$

By $[X] = x$ we mean that the tree part of X is x (for some choice of representative of the class of $x \in \mathfrak{sdet}_2$, by abuse of notation), i.e. X may be decomposed as

$$X = x + x_1 + x_2 + x_3 + \dots$$

with x_i having i internal loops. The equation $dX = \delta Y \pmod{k+1 \text{ internal loops}}$ means that the equation holds up to loop order $k+1$, i.e. we discard all graphs having more than k internal loops appearing on either side of the equation. Note that $\widehat{\mathfrak{trv}}_2 = \widehat{\mathfrak{trv}}_2^{(1)}$. Again, there is a filtration

$$\dots \subset \widehat{\mathfrak{trv}}_2^{(k)} \subset \widehat{\mathfrak{trv}}_2^{(k-1)} \subset \dots \subset \widehat{\mathfrak{trv}}_2^{(2)} \subset \widehat{\mathfrak{trv}}_2^{(1)} \subset \mathfrak{sdet}_2.$$

Our main result is

Theorem 3. For all $k \geq 1$, $\widehat{\mathfrak{trv}}_2^{(k)}$ is a Lie subalgebra of \mathfrak{sdet}_2 .

For the proof we need a few additional tools and results from the theory of graph complexes.

2.3.3 The graph complex GC_2

The graph complex GC_2 is a variant of M. Kontsevich's graph complex ([6],[7],[8]). We follow T. Willwacher's paper [16].

Definition 8. Let Γ be an undirected graph with N labeled vertices and k edges satisfying the following properties:

1. All vertices have valence at least three.
2. There is a linear order on the set of edges.
3. Γ has no simple loops.

We denote by $\text{Gra}_2(N, k)$ the graded vector space spanned by isomorphism classes of connected graphs satisfying the conditions above, modulo the relation $\Gamma \cong (-1)^{|\sigma|} \Gamma^\sigma$, where Γ^σ differs from Γ just by a permutation $\sigma \in S_k$ on the order of the edges. The degree of such a graph Γ is given by

$$\text{deg}_{\text{Gra}_2} \Gamma = -k.$$

Set,

$$\mathbf{Gra}_2(N) := \bigoplus_{k \geq 0} \mathbf{Gra}_2(N, k).$$

The collection $\{\mathbf{Gra}_2(N)\}_{N \geq 1}$ naturally defines an operad \mathbf{Gra}_2 in the category of graded vector spaces. For $\Gamma \in \mathbf{Gra}_2(N)$, the S_N -action permutes the labels of the vertices. For $r, s \geq 1$, $\Gamma_1 \in \mathbf{Gra}_2(r)$ and $\Gamma_2 \in \mathbf{Gra}_2(s)$, the operadic composition $\Gamma_1 \circ_j \Gamma_2 \in \mathbf{Gra}_2(r + s - 1)$ is given by inserting the graph Γ_2 at vertex j of Γ_1 and summing over all ways of reconnecting the edges incident to vertex j in Γ_1 to vertices of Γ_2 . As in the case of ICG, we ask that the order on the set of edges of $\Gamma_1 \circ_j \Gamma_2$ is such that all edges of Γ_1 come before those of Γ_2 while the respective orderings are left unaltered. Next, define,

$$\mathbf{GC}_2 := \prod_{N \geq 1} (\mathbf{Gra}_2(N)[2 - 2N])^{S_N}.$$

The space \mathbf{GC}_2 carries the structure of a differential graded Lie algebra. The degree of a graph $\Gamma \in \mathbf{GC}_2$ with k edges and N vertices is

$$\deg \Gamma = -2 - k + 2N.$$

For the Lie bracket, consider the operadic pre-Lie product on \mathbf{Gra}_2 ,

$$\Gamma_1 \circ \Gamma_2 = \sum_{j=1}^r \Gamma_1 \circ_j \Gamma_2.$$

Using this, the Lie bracket on \mathbf{GC}_2 is defined on homogeneous elements via,

$$[\Gamma_1, \Gamma_2] := \Gamma_1 \circ \Gamma_2 - (-1)^{\deg \Gamma_1 \cdot \deg \Gamma_2} \Gamma_2 \circ \Gamma_1.$$

The differential d is given by vertex splitting, where again we ask that the newly created edge is placed last in the ordering of the edges.

Remark 8. More generally, one defines \mathbf{Gra}_n for any n by setting the degree of each edge to be $1 - n$. Thus, a graph $\Gamma \in \mathbf{Gra}_n(N, k)$ has degree $\deg_{\mathbf{Gra}_n} \Gamma = (1 - n)k$. Also, the equivalence relation given by the ordering on the set of edges becomes $\Gamma \cong (-1)^{|\sigma|(n-1)} \Gamma^\sigma$. Thus, when n is odd, permuting the order of the edges does not produce any signs. However, in the n odd case, we additionally ask that the edges are directed. For $\Gamma \in \mathbf{Gra}_n(N, k)$, there is then a natural S_2^k -action given by flipping the directions of the edges. In this case, we identify a graph with an edge direction flipped with minus the original graph. Moreover, one then defines,

$$\mathbf{GC}_n := \begin{cases} \prod_{N \geq 1} (\mathbf{Gra}_n(N)[n(1 - N)])^{S_N} & n \text{ even,} \\ \prod_{N \geq 1} (\mathbf{Gra}_n(N) \otimes \text{sgn}_N[n(1 - N)])^{S_N} & n \text{ odd.} \end{cases}$$

Here, sgn_N denotes the one-dimensional representation of S_N . We will only be interested in the $n = 2$ case. For more details, we refer to ([16], Section 3.).

Remark 9. There is a map

$$\begin{aligned} (-)_1 : \mathbf{GC}_2 &\rightarrow \mathbf{graphs}(1) \\ \gamma &\mapsto \gamma_1 \end{aligned} \tag{2.4}$$

given by marking vertex 1 as “external”. For $\Gamma_1 \in \mathbf{graphs}(1)$ and $\Gamma_r \in \mathbf{graphs}(r)$, $r \in \mathbb{N}$, let

$$\Gamma_1 \cdot \Gamma_r := \Gamma_1 \circ_1 \Gamma_r - (-1)^{\deg \Gamma_1 \cdot \deg \Gamma_r} \sum_{j=1}^r \Gamma_r \circ_j \Gamma_1.$$

be an action of $\mathbf{graphs}(1)$ on $\mathbf{graphs}(r)$.

Lemma 5. The action defined above satisfies the identity

$$\gamma \cdot (\gamma' \cdot \Gamma) - (-1)^{\deg \gamma \cdot \deg \gamma'} \gamma' \cdot (\gamma \cdot \Gamma) = (\gamma \cdot \gamma' - (-1)^{\deg \gamma \cdot \deg \gamma'} \gamma' \cdot \gamma) \cdot \Gamma \quad (2.5)$$

for all $\gamma, \gamma' \in \mathbf{graphs}(1)$ and $\Gamma \in \mathbf{graphs}(r)$, $r \in \mathbb{N}$.

Remark 10. Note that for any operad in the category of cochain complexes \mathcal{P} , $\mathcal{P}(1)$ together with the operadic composition forms a graded algebra. Moreover, $\mathcal{P}(1)$ acts on \mathcal{P} via

$$a \cdot b := a \circ_1 b - (-1)^{\deg a \cdot \deg b} \sum_{j=1}^r b \circ_j a$$

for any $r \in \mathbb{N}$. The identity in Lemma 5 holds also in this case. Its proof is a simple computation and we refer to ([3], Section 6.1.).

Let $r \in \mathbb{N}$. Following [16], we define an action of \mathbf{GC}_2 on $\mathbf{graphs}(r)$ by

$$\gamma \bullet \Gamma := \gamma_1 \cdot \Gamma + \sum_v \Gamma \circ_v \gamma = \Gamma_1 \circ_1 \Gamma_r - (-1)^{\deg \Gamma_1 \cdot \deg \Gamma_r} \sum_{j=1}^r \Gamma_r \circ_j \Gamma_1 + \sum_v \Gamma \circ_v \gamma,$$

for $\gamma \in \mathbf{GC}_2$ and $\Gamma \in \mathbf{graphs}(r)$. The composition $\Gamma \circ_v \gamma$ is constructed by “inserting” γ into the internal vertex v in Γ and summing over all ways of reconnecting edges incident to v to vertices of γ . This action is compatible with the differentials on \mathbf{graphs} and \mathbf{GC}_2 , i.e.

$$d(\gamma \bullet \Gamma) = (d\gamma_1) \cdot \Gamma + \gamma_1 \cdot (d\Gamma) + \sum_v (d\Gamma) \circ_v \gamma + \sum_v \Gamma \circ_v (d\gamma).$$

Remark 11. Denote by \mathbf{GC}_2^{1-vi} the subcomplex of (\mathbf{GC}_2, d) spanned by 1-vertex irreducible graphs (that is graphs which remain connected after deletion of any of its vertex). As shown in [2], the subcomplex \mathbf{GC}_2^{1-vi} is quasi-isomorphic to \mathbf{GC}_2 . Also, note that the map $(-)_1$ restricted to \mathbf{GC}_2^{1-vi} maps to internally connected graphs $\mathbf{ICG}(1)$.

Remark 12. Denote by (C, d) the subcomplex of $(\mathbf{ICG}(1), d)$ spanned by graphs having only one edge incident to the unique external vertex. It follows from ([16], Proposition 6.13.) that

$$H^0(\mathbf{GC}_2, d) \cong H^2(C, d).$$

On the level of the corresponding cochain complexes, the map inducing this isomorphism has the simple combinatorial form [16]

$$F : \mathbf{GC}_2^{1-vi} \rightarrow C$$

$$\Gamma \mapsto \left(\begin{array}{c} \circ \\ \text{---} \\ \circ \\ \text{---} \\ \circ \\ \text{---} \\ \circ \end{array} \right) \circ_2 \Gamma.$$

It preserves the number of loops and thus if we denote by $H^2(C, d)^{(l)}$ and $H^0(\mathrm{GC}_2, d)^{(l)}$ the l -loop parts, we still have an isomorphism

$$H^0(\mathrm{GC}_2, d)^{(l)} \cong H^2(C, d)^{(l)} \quad (2.6)$$

for all $l \geq 1$. In particular, we have the following.

Lemma 6. For $l \geq 1$, given $Z \in \mathrm{ICG}(1)$ satisfying

$$Z \bmod l+1 \text{ loops} \in C, \deg(Z) = 2, dZ = 0 \bmod l+1 \text{ loops},$$

there exist a $Z' \in C$ and a (1-vertex irreducible) $\Gamma \in H^0(\mathrm{GC}_2)$ such that $Z + dZ' = \begin{pmatrix} \circ & -\circ \\ 1 & 2 \end{pmatrix} \circ_2 \Gamma \bmod l+1 \text{ loops}$.

Proof. The conditions on Z imply that it represents a cohomology class in $\bigoplus_{k=1}^l H^2(C, d)^{(k)}$. This class corresponds to the class of some $\Gamma \in \mathrm{GC}_2$ of degree 0 in $\bigoplus_{k=0}^l H^0(\mathrm{GC}_2, d)^{(k)} \subset H^0(\mathrm{GC}_2, d)$ under the isomorphism which sends Γ to $\begin{pmatrix} \circ & -\circ \\ 1 & 2 \end{pmatrix} \circ_2 \Gamma$. Therefore, there must be some $Z' \in C$ such that $Z + dZ' = \begin{pmatrix} \circ & -\circ \\ 1 & 2 \end{pmatrix} \circ_2 \Gamma \bmod l+1 \text{ loops}$. By Remark 11, we may assume that Γ is 1-vertex irreducible. \square

Lemma 7. It is true that $\ker(\delta : \mathrm{ICG}(1) \rightarrow \mathrm{ICG}(2)) = C$.

Proof. An easy graphical calculation shows that $C \subset \ker \delta$. For the other inclusion, let $f \in \ker \delta$, and let the external vertex be of valence k . Then, $\delta_0 f + \delta_2 f = \delta_1 f$. Define a linear map $\Delta : \mathrm{ICG}(2) \rightarrow \mathrm{ICG}(1)$ given by simply merging the two external vertices into one (and keeping all incident edges). Applying this map to our equation yields, $2^k f = 2f$. This implies $k = 1$, and thus $f \in C$. \square

Lemma 8. Fix $k \geq 1$. Let $x \in \widehat{\mathrm{fv}}_2^{(k)}$. By definition, there exists an $X \in \mathrm{ICG}(2)$ such that $[X] = x$ and $dX = \delta Y \bmod k+1 \text{ internal loops}$ for some $Y \in \mathrm{ICG}(1)$. Denote by $(-)_1 : \mathrm{GC}_2 \rightarrow \mathrm{graphs}(1)$ the map defined in equation (2.4). It is given by marking vertex 1 as ‘‘external’’. In this setting, there exist an $X' \in \mathrm{ICG}(2)$ and a $\Gamma \in \mathrm{GC}_2$ (1-vertex irreducible, of degree 0 and satisfying $d\Gamma = 0$) such that

$$\begin{aligned} [X'] &= x \\ dX' &= \delta(\Gamma)_1 \bmod k+1 \text{ internal loops}. \end{aligned}$$

Proof. It follows from Lemma 6 that there is a $Y' \in \mathrm{ICG}(1)$ satisfying $dX = \delta Y' \bmod k+1 \text{ internal loops}$ and a 1-vertex irreducible $\Gamma \in \mathrm{GC}_2$ such that $d\Gamma = 0$ and $dY' = \begin{pmatrix} \circ & -\circ \\ 1 & 2 \end{pmatrix} \circ_2 \Gamma \bmod k+1 \text{ internal loops}$. To see this, note that the equation $dX = \delta Y' \bmod k+1 \text{ loops}$ implies in particular via

$$0 = d^2 X = d\delta Y' = -\delta dY' \bmod k+1 \text{ internal loops}$$

that $dY' \bmod k+1 \text{ loops} =: Z$ is in $\ker(\delta)$. Lemma 7 implies $Z \bmod k+1 \text{ loops} = Z \in C$. Moreover, we have $\deg(Z) = 2$ and $dZ = 0 \bmod k+1 \text{ loops}$. By Lemma 6 there exists a $\Gamma \in \mathrm{GC}_2$ of degree 0 such that $d\Gamma = 0$ and a $Z' \in C$ such that $Z + dZ' = \begin{pmatrix} \circ & -\circ \\ 1 & 2 \end{pmatrix} \circ_2 \Gamma \bmod k+1 \text{ loops}$. Set

$$Y' := Y + Z'.$$

It satisfies $\delta Y' = \delta Y + \delta Z' = \delta Y = dX \pmod{k+1}$ loops as $\delta Z' = 0$. Also, $dY' = dY + dZ' = Z + dZ' = (\circ_1 - \circ_2) \circ_2 \Gamma \pmod{k+1}$ loops $= F(\Gamma) \pmod{k+1}$ loops.

Next, note that ([12], [16] section 6.4.)

$$F(\Gamma) = d(\Gamma)_1 - (d\Gamma)_1. \quad (2.7)$$

As $d\Gamma = 0$, we have $F(\Gamma) = d(\Gamma)_1$, and since modulo $k+1$ loops, $F(\Gamma) = dY'$, we obtain

$$d(Y' - (\Gamma)_1) = 0 \pmod{k+1} \text{ loops.}$$

In [13], it is proven that $\mathfrak{t}_n \cong H(\text{ICG}(n), d)$ holds for all $n \in \mathbb{N}$. The isomorphism is given by mapping generators $t^{i,j}$ to graphs with no internal vertex and one edge connecting the external vertices i and j . In particular, this implies $H^k(\text{ICG}(n)) = 0$ for $k \neq 0$. Therefore, as $H^1(\text{ICG}(1)) = 0$ and $Y' - (\Gamma)_1 \in \ker(d) = \text{im}(d)$, there is a $W \in \text{ICG}(1)$ of degree 0 such that

$$Y' - (\Gamma)_1 = dW \pmod{k+1} \text{ loops.}$$

Because of degree reasons, W will not have a tree part. At this point, set

$$X' := X + \delta W.$$

It does indeed satisfy the required relations. As W does not contribute to the tree part, clearly $[X'] = x$. Moreover, everything modulo $k+1$ loops,

$$dX' = dX + d\delta W = \delta Y' + d\delta W = \delta(\Gamma)_1 + \delta dW + d\delta W = \delta(\Gamma)_1$$

as $\delta dW = -d\delta W$. □

Remark 13. The condition for Γ to be 1-vertex irreducible ensures that $(\Gamma)_1$ is internally connected.

Remark 14. For $\Gamma \in \text{graphs}(1)$,

$$\Gamma \cdot (\circ \ \circ) = -\delta\Gamma.$$

Additionally, for $\gamma \in \text{GC}_2$

$$\sum_v (\Gamma \circ_v \gamma) \cdot (\circ \ \circ) = \sum_v (\Gamma \cdot (\circ \ \circ)) \circ_v \gamma$$

where the sum runs over internal vertices of Γ .

Proof of Theorem 3. Fix $k \geq 1$. Let $x_1, x_2 \in \widehat{\mathfrak{t}\mathfrak{v}}_2^{(k)}$. By Lemma 8, there exist $X_1, X_2 \in \text{ICG}(2)$ and $\Gamma_1, \Gamma_2 \in \text{GC}_2$ (1-vertex irreducible, of degree 0 and satisfying $d\Gamma_1 = d\Gamma_2 = 0$) such that for $i = 1, 2$

(i) $[X_i] = x_i$

(ii) $dX_i = \delta(\Gamma_i)_1 \pmod{k+1}$ internal loops.

We need to find an $X \in \text{ICG}(2)$ which extends the bracket $[x_1, x_2]_{Ih}$ and a $Y \in \text{ICG}(1)$ such that $dX = \delta Y \pmod{k+1}$ internal loops. As an extension of $[x_1, x_2]_{Ih}$ we suggest the element

$$X := \Gamma_1 \bullet X_2 - \Gamma_2 \bullet X_1 + d(X_1 \wedge X_2) \pmod{k+1} \text{ internal loops} \in \text{graphs}(2). \quad (2.8)$$

The notation $X_1 \wedge X_2$ means that we identify the corresponding external vertices. The edges of the new graph are ordered by preserving their order in X_1 and X_2 and by $e_1 < e_2$ whenever e_1 is an edge of X_1 and e_2 is an edge of X_2 . Remark that a priori X might not be internally connected. It is a linear combination of graphs containing at most k loops. The higher loop part is set to zero. There are several things to check.

- (i) $[X] = [x_1, x_2]_{Ih}$: The tree part of X comes only from $d(X_1 \wedge X_2)$ as $\Gamma_1 \bullet X_2$ and $\Gamma_2 \bullet X_1$ both contain loops. Moreover, this tree part exactly coincides with the bracket $[x_1, x_2]_{Ih}$ which is given by gluing x_1 and x_2 (the tree parts of X_1 and X_2) at the corresponding external vertices, applying the differential and only keeping the loop-free internally connected graphs.
- (ii) $dX = \delta Y \pmod{k+1}$ internal loops: The differential is compatible with the action of GC_2 on $\text{graphs}(2)$. Therefore, everything modulo $k+1$ internal loops,

$$\begin{aligned} dX &= \underbrace{(d\Gamma_1)}_{=0} \bullet X_2 + \Gamma_1 \bullet \underbrace{(dX_2)}_{=\delta(\Gamma_2)_1} - \underbrace{(d\Gamma_2)}_{=0} \bullet X_1 - \Gamma_2 \bullet \underbrace{(dX_1)}_{=\delta(\Gamma_1)_1} \\ &= \Gamma_1 \bullet (\delta(\Gamma_2)_1) - \Gamma_2 \bullet (\delta(\Gamma_1)_1) \\ &= -\Gamma_1 \bullet ((\Gamma_2)_1 \cdot (\circ \circ)) + \Gamma_2 \bullet ((\Gamma_1)_1 \cdot (\circ \circ)) \\ &= -(\Gamma_1)_1 \cdot ((\Gamma_2)_1 \cdot (\circ \circ)) + (\Gamma_2)_1 \cdot ((\Gamma_1)_1 \cdot (\circ \circ)) \\ &\quad - \sum_v ((\Gamma_2)_1 \cdot (\circ \circ)) \circ_v \Gamma_1 + \sum_{v'} ((\Gamma_1)_1 \cdot (\circ \circ)) \circ_{v'} \Gamma_2. \end{aligned}$$

Remark 14 above, together with Lemma 5 enable us to write this as

$$\begin{aligned} &= -(\Gamma_1)_1 \cdot (\Gamma_2)_1 \cdot (\circ \circ) + ((\Gamma_2)_1 \cdot (\Gamma_1)_1) \cdot (\circ \circ) \\ &\quad - \sum_v ((\Gamma_2)_1 \circ_v \Gamma_1) \cdot (\circ \circ) + \sum_{v'} ((\Gamma_1)_1 \circ_{v'} \Gamma_2) \cdot (\circ \circ) \\ &= \underbrace{(\Gamma_2 \bullet (\Gamma_1)_1 - \Gamma_1 \bullet (\Gamma_2)_1)}_{=: -Y} \cdot (\circ \circ) \\ &= \delta Y. \end{aligned}$$

- (iii) $X \in \text{ICG}(2)$: Denote by k_i the number of edges of X_i . Remark that the signs in the wedge product \wedge behave as follows,

$$X_1 \wedge X_2 = (-1)^{k_1 k_2} X_2 \wedge X_1.$$

As $0 = \deg(X_i) = 1 - k_i + 2\#\text{internal vertices}$, we have that $k_i = 2\#\text{internal vertices} + 1$ is odd. Therefore,

$$X_1 \wedge X_2 = -X_2 \wedge X_1.$$

We find that the non-internally connected part of $\Gamma_1 \bullet X_2$ is

$$-(\delta(\Gamma_1)_1) \wedge X_2.$$

To see this, consider,

$$\Gamma_1 \bullet X_2 = (\Gamma_1)_1 \circ_1 X_2 - \sum_{j=1}^2 X_2 \circ_j (\Gamma_1)_1 + \sum_v X_2 \circ_v \Gamma_1.$$

The last sum will consist of internally connected graphs since we insert $\Gamma_1 \in \mathbf{GC}_2$ into the internal vertices of X_2 . When X_2 is inserted in the unique external vertex of $(\Gamma_1)_1$, the non-internally connected terms will arise when the edges of $(\Gamma_1)_1$ which were previously connected to the external vertex are distributed on the two external vertices. This corresponds to the expression $\delta_1(\Gamma_1)_1 \wedge X_2$. On the other hand, when $(\Gamma_1)_1$ is inserted in the first external vertex of X_2 we find the non-internally connected graphs by connecting all edges of X_2 previously connected to external vertex 1 to the unique external vertex of $(\Gamma_1)_1$. This yields $X_2 \wedge \delta_0(\Gamma_1)_1$. Similarly, we obtain $X_2 \wedge \delta_2(\Gamma_1)_1$ when considering the second external vertex of X_2 . Moreover, since Γ_1 is of degree zero in \mathbf{GC}_2 , all of Γ_1 , $(\Gamma_1)_1$ and $\delta_i(\Gamma_1)_1$ will have an even number of edges, and thus $X_2 \wedge \delta_i(\Gamma_1)_1 = \delta_i(\Gamma_1)_1 \wedge X_2$. These three terms together give the claim above. For a more schematic explanation, see Figures 2.3 and 2.4.

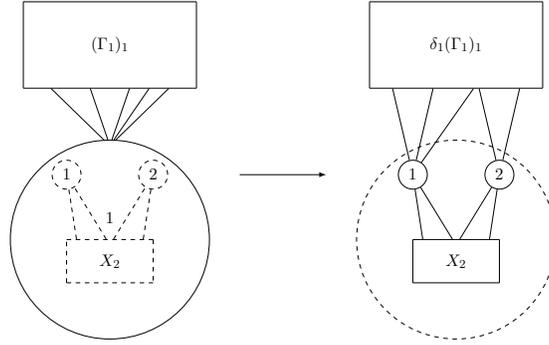


Figure 2.3: The non-internally connected part of $(\Gamma_1)_1 \circ_1 X_2$ is given by $\delta_1(\Gamma_1)_1 \wedge X_2$.

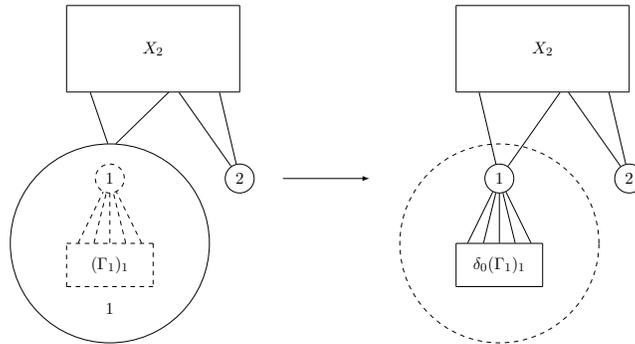


Figure 2.4: The non-internally connected part of $X_2 \circ_1 (\Gamma_1)_1$ is given by $X_2 \wedge \delta_0(\Gamma_1)_1$.

The non-internally connected part of $d(X_1 \wedge X_2)$ is

$$(dX_1) \wedge X_2 - X_1 \wedge (dX_2).$$

As $dX_i = \delta(\Gamma_i)_1 \pmod{k+1}$ internal loops the non-internally connected part of

$$X := \Gamma_1 \bullet X_2 - \Gamma_2 \bullet X_1 + d(X_1 \wedge X_2) \pmod{k+1} \text{ internal loops}$$

vanishes, i.e.

$$\underbrace{-(\delta(\Gamma_1)_1) \wedge X_2}_{\text{from } \Gamma_1 \bullet X_2} + \underbrace{(\delta(\Gamma_2)_1) \wedge X_1}_{\text{from } \Gamma_2 \bullet X_1} + \underbrace{(dX_1) \wedge X_2 - (-1)^{k_1(k_2+1)}(dX_2) \wedge X_1}_{\text{from } d(X_1 \wedge X_2)} = 0 \quad (2.9)$$

(iv) $Y \in \text{ICG}(1)$: The only non-internally connected part of

$$\Gamma_1 \bullet (\Gamma_2)_1 = (\Gamma_1)_1 \cdot (\Gamma_2)_1 + \underbrace{\sum_v (\Gamma_2)_1 \circ_v \Gamma_1}_{\in \text{ICG}(1)}$$

is given by $(\Gamma_1)_1 \wedge (\Gamma_2)_1 + (\Gamma_2)_1 \wedge (\Gamma_1)_1$. Therefore, in Y , the only non-internally connected part will be $(\Gamma_1)_1 \wedge (\Gamma_2)_1 + (\Gamma_2)_1 \wedge (\Gamma_1)_1 - ((\Gamma_2)_1 \wedge (\Gamma_1)_1 + (\Gamma_1)_1 \wedge (\Gamma_2)_1) = 0$.

Hence, the conditions for $[x_1, x_2]_{Ih} \in \widehat{\mathfrak{trv}}_2^{(k)}$ are satisfied. □

Definition 9. The *Grothendieck-Teichmüller Lie algebra* \mathfrak{grt}_1 is spanned by elements $(0, \psi) \in \mathfrak{tder}_2$, that satisfy the following relations:

$$\begin{aligned} \psi(x, y) &= -\psi(y, x) \\ \psi(x, y) + \psi(y, z) + \psi(z, x) &= 0 \text{ for } x + y + z = 0 \\ \psi(t^{1,2}, t^{2,3} + t^{2,4}) + \psi(t^{1,3} + t^{2,3}, t^{3,4}) &= \psi(t^{2,3}, t^{3,4}) + \psi(t^{1,2} + t^{1,3}, t^{2,4} + t^{3,4}) + \psi(t^{1,2}, t^{2,3}) \end{aligned}$$

where the last equation takes values in the Lie algebra \mathfrak{t}_4 .

Theorem 4. The Lie algebra \mathfrak{grt}_1 is contained in all of the $\widehat{\mathfrak{trv}}_2^{(k)}$.

Proof. In [16] it was proven that $H^0(\text{GC}_2) \cong \mathfrak{grt}_1$. The map

$$\begin{aligned} H^0(\text{GC}_2) &\rightarrow \mathfrak{grt}_1 \\ \gamma &\mapsto \phi_\gamma \end{aligned}$$

is given by the following algorithm [16].

1. Let γ be a closed element in GC_2 . We may assume it to be 1-vertex irreducible. Denote by $\gamma_1 \in \mathbf{graphs}(1)$ the linear combination of graphs obtained by marking the vertex 1 as “external” in each graph appearing in γ . As γ is 1-vertex irreducible, $\gamma_1 \in \text{ICG}(1)$.
2. Apply δ to γ_1 , i.e. split the external vertex into two vertices, and sum over all ways to reconnect the loose edges so that both vertices are hit by at least one edge. Call this linear combination $\gamma'_2 \in \text{ICG}(2)$.
3. It turns out that γ'_2 is the coboundary of some element $\gamma_2 \in \text{ICG}(2)$. We choose γ_2 to be symmetric under interchange of the external vertices 1 and 2.

4. Forget the non-internal trivalent tree part of γ_2 to obtain $T_2 \in \mathfrak{sdtr}_2$.
5. For each tree t appearing in T_2 construct a Lie word in formal variables X and Y as follows. For each edge incident to vertex 1, cut it and make it the “root” edge. The resulting tree is a binary tree with leafs labelled by 1 or 2. It can be seen as a Lie tree, and one gets a Lie word $\phi_1(X, Y)$ by replacing each 1 by X and 2 by Y . Set $\phi(X, Y) = \phi_1(X, Y) - \phi_1(Y, X)$. Summing over all such Lie words one gets a linear combination $\phi_\gamma(X, Y)$ of Lie words corresponding to γ . It is an element of \mathfrak{grt}_1 .

The algorithm and the fact that this map is an isomorphism imply that given $\phi_\gamma \in \mathfrak{grt}_1$, there exists a unique internal trivalent tree $T_2 \in \mathfrak{sdtr}_2$ which may be extended to $\gamma_2 \in \text{ICG}(2)$ satisfying that there is a $\gamma_1 \in \text{ICG}(1)$ with $d\gamma_2 = \delta\gamma_1$. This is exactly the required relation for T_2 to be in $\widehat{\mathfrak{trv}}_2^{(k)}$ for all $k \geq 1$. Hence, $\mathfrak{grt}_1 \subset \widehat{\mathfrak{trv}}_2^{(k)}$ for all $k \geq 1$. \square

Theorem 5. The intersection of all $\widehat{\mathfrak{trv}}_2^{(k)}$ is $\mathfrak{grt}_1 \oplus \mathfrak{t}_2$, i.e. in formulas

$$\mathfrak{grt}_1 \oplus \mathfrak{t}_2 \cong \bigcap_{k \geq 1} \widehat{\mathfrak{trv}}_2^{(k)} =: \widehat{\mathfrak{trv}}_2^{(\infty)}.$$

We will need two rather technical lemmas.

Lemma 9. For each $x \in \widehat{\mathfrak{trv}}_2^{(\infty)}$, there exists a pair $(X, Y) \in \text{ICG}(2) \times \text{ICG}(1)$ with $\deg(X) = 0$ such that the tree part of X is x and $dX = \delta Y$.

Proof. We define the following auxiliary grading on $\bigoplus_{r \geq 1} \text{ICG}(r)$. It is given by connecting the subsequent external vertices by an edge, and then counting the number of not necessarily internal loops in our graph. A brief graphical calculation shows that this degree is preserved by both δ and d . Let now $x \in \widehat{\mathfrak{trv}}_2^{(\infty)}$, and denote by x^M its (auxiliary) degree M component. Since $x \in \widehat{\mathfrak{trv}}_2^{(\infty)}$, in particular $x \in \widehat{\mathfrak{trv}}_2^{(M)}$, and there is a pair (X^M, Y^M) of degree M extending x^M such that $dX^M = \delta Y^M \pmod{M+1}$ internal loops. But then (X^M, Y^M) is an extension for x^M which satisfies $dX^M = \delta Y^M$ to infinite loop order, since the number of internal loops is bounded by the degree M . Applying this construction to each homogeneous component of x gives a pair $(X = \sum_M X^M, Y = \sum_M Y^M)$ satisfying all the required properties. \square

Lemma 10. Let (X, Y) be a pair corresponding to $x \in \widehat{\mathfrak{trv}}_2^{(\infty)}$. The map

$$\begin{aligned} B : \widehat{\mathfrak{trv}}_2^{(\infty)} &\rightarrow H^2(C, d) \\ (X, Y) &\mapsto dY. \end{aligned}$$

is well-defined. Here, (C, d) is the complex defined in Remark 12,

Proof. We define a map by

$$\begin{aligned} \widehat{\mathfrak{trv}}_2^{(\infty)} &\xrightarrow{E} H^0(\text{ICG}(3), d) \cong \mathfrak{t}_3 \\ (X, Y) &\mapsto \delta X. \end{aligned}$$

To show that E is well-defined, first note that δX is of degree 0 and that $d\delta X = -\delta dX = -\delta^2 Y = 0$, that is indeed $\delta X \in H^0(\text{ICG}(3), d)$. Let (X_1, Y_1) and (X_2, Y_2) be two extensions of x . The difference $X := X_1 - X_2$ has no tree part. Therefore, $E(X, Y) = \delta X = 0$ because elements of \mathfrak{t}_3 consists only of trees (and δX contains none). Thus, $E(X_1, Y_1) = \delta X_1 = \delta X_2 = E(X_2, Y_2)$ and E is well-defined.

To prove the same for B , note that since dY is obviously closed, of degree 2 (as $\deg(Y) = 1$) and satisfies $0 = d^2 X = d\delta Y = -\delta dY$, i.e. $dY \in \ker \delta = C$, the target space is indeed $H^2(C, d)$. Again, let (X_1, Y_1) and (X_2, Y_2) be two extensions of x and consider $Y := Y_1 - Y_2$. It follows from ([16], Proposition 6.13.), that the inclusion $(C, d) \hookrightarrow (\bigoplus_{r \geq 1} \text{ICG}(r)[1], d + \delta)$ is a quasi-isomorphism, in particular,

$$H^2(C, d) \cong H^3(\bigoplus_{r \geq 1} \text{ICG}(r), d + \delta). \quad (2.10)$$

The degree in the total complex for some $\Gamma \in \text{ICG}(r)$ is $\deg_{Tot} := \deg(\Gamma) + r$ (where $\deg(\Gamma)$ is the degree in $\text{ICG}(r)$). In the total complex dY is cohomologous to δX via

$$dY = \delta X - (d + \delta)(X - Y).$$

Therefore, since $\delta X = 0$, we have $dY = 0 \in H^3(\bigoplus_{r \geq 1} \text{ICG}(r), d + \delta)$. But the isomorphism (2.10) implies that therefore $dY = 0$ already in $H^2(C, d)$. This yields the result, as now $dY_1 = dY_2$, that is, B is well-defined. \square

Remark 15. As $H^0(\text{GC}_2) \xrightarrow{F} H^2(C, d)$, $dY = F(\gamma)$ for some $\gamma \in H^0(\text{GC}_2)$. Also, by equation (2.7), $F(\gamma) = d\gamma_1$, where γ_1 is obtained by marking vertex 1 as “external” (see equation (2.4)). Therefore, $d(\gamma_1 - Y) = 0$, and since $H^1(\text{ICG}(1), d) = 0$, $\gamma_1 - Y = d\alpha$ for some $\alpha \in \text{ICG}(1)$ of degree 0. We shall use this relation in the proof below.

Proof of Theorem 5. The algorithm in the proof of Theorem 4 provides us with a map

$$A : H^0(\text{GC}_2) \rightarrow \widehat{\mathfrak{ftv}}_2^{(\infty)}.$$

Let $\gamma \in H^0(\text{GC}_2)$, and denote by ϕ_γ the corresponding \mathfrak{grt}_1 element. Keeping the notation from the algorithm, the assignment $\gamma \mapsto \phi_\gamma$ produces a pair (γ_2, γ_1) satisfying $d\gamma_2 = \delta\gamma_1$ and thus the tree part of γ_2 , denoted T_2 , will lie in $\widehat{\mathfrak{ftv}}_2^{(\infty)}$. Abusing notation, set $A(\gamma) := (\gamma_2, \gamma_1)$. Consider the composition

$$H^0(\text{GC}_2) \xrightarrow{A} \widehat{\mathfrak{ftv}}_2^{(\infty)} \xrightarrow{B} H^2(C, d) \xrightarrow{F^{-1}} H^0(\text{GC}_2).$$

It equals the identity as

$$F^{-1} \circ B \circ A(\gamma) = F^{-1} \circ B(\gamma_2, \gamma_1) = F^{-1}(d\gamma_1) = \gamma,$$

implying that B is surjective.

We now determine the kernel of B . For this, let (X, Y) be a pair corresponding to $x \in \widehat{\mathfrak{ftv}}_2^{(\infty)}$ with $B(X, Y) = dY = 0 \in H^2(C, d)$. Then, $F(\gamma) = dY = 0$ and since F is an isomorphism $\gamma = 0 \in H^0(\text{GC}_2)$, i.e. $\gamma = d\tilde{\gamma}$ for some $\tilde{\gamma} \in \text{GC}_2$ of degree -1 . Remark that (by equation (2.7))

$$F(\tilde{\gamma}) = d\tilde{\gamma}_1 - (d\tilde{\gamma})_1 = d\tilde{\gamma}_1 - \gamma_1. \quad (2.11)$$

Define $\hat{\gamma} := \gamma_1 + F(\tilde{\gamma}) \in \text{ICG}(1)$. It satisfies,

$$\delta\hat{\gamma} = \delta\gamma_1 + \delta F(\tilde{\gamma}) = \delta\gamma_1$$

as $F(\tilde{\gamma}) \in C = \ker \delta$. Also, it follows directly from equation (2.11) that $\hat{\gamma} = d\tilde{\gamma}_1$. Finally, set

$$X' := X + \delta(\tilde{\gamma}_1 - \alpha) \in \text{ICG}(2),$$

where $\alpha \in \text{ICG}(1)$ is as in Remark 15. The degree of X' is 0 and it satisfies,

$$\begin{aligned} dX' &= dX + d\delta(\tilde{\gamma}_1 - \alpha) = dX - \delta d\tilde{\gamma}_1 + \delta d\alpha \\ &= dX - \delta\hat{\gamma} + \delta(\gamma_1 - Y) \\ &= dX - \delta\gamma_1 + \delta\gamma_1 - \delta Y = dX - \delta Y = 0. \end{aligned}$$

Hence, $X' \in H^0(\text{ICG}(2), d) \cong \mathfrak{t}_2$, i.e. $X' = \lambda \cdot (\circ\text{---}\circ_1\text{---}\circ_2)$ for some $\lambda \in \mathbb{K}$. But then,

$$X = X' - \delta(\tilde{\gamma}_1 - \alpha) = \lambda \cdot (\circ\text{---}\circ_1\text{---}\circ_2) - \delta(\tilde{\gamma}_1 - \alpha).$$

However, $\delta(\tilde{\gamma}_1 - \alpha)$ does not contribute to the tree part x of X , which therefore is of the form $\lambda \cdot (\circ\text{---}\circ_1\text{---}\circ_2)$. This implies $x \in \mathfrak{t}_2$ and $\ker B \subset \mathfrak{t}_2$. In fact, $\ker B = \mathfrak{t}_2$. The other inclusion is clear.

Since $t^{1,2}$ satisfies $d(t^{1,2}) = 0$, a pair corresponding to $t^{1,2}$ in $\widehat{\mathfrak{f}\mathfrak{v}}_2^{(\infty)}$ is $(t^{1,2}, 0)$, which lies in $\ker B$. And since B is well-defined, any pair corresponding to $t^{1,2}$ will lie in $\ker B$.

Thus, we eventually have

$$\widehat{\mathfrak{f}\mathfrak{v}}_2^{(\infty)} / \mathfrak{t}_2 \xrightarrow{\cong} H^0(\text{GC}_2) \cong \mathfrak{gr}\mathfrak{t}_1$$

and $\widehat{\mathfrak{f}\mathfrak{v}}_2^{(\infty)} \cong \mathfrak{gr}\mathfrak{t}_1 \oplus \mathfrak{t}_2$. □

Since it is conjectured that $\widehat{\mathfrak{f}\mathfrak{v}}_2 \cong \mathfrak{t}_2 \oplus \mathfrak{gr}\mathfrak{t}_1$, we expect all $\widehat{\mathfrak{f}\mathfrak{v}}_2^{(k)}$ to coincide.

Conjecture 1. For all $k \geq 1$

$$\widehat{\mathfrak{f}\mathfrak{v}}_2^{(k)} = \widehat{\mathfrak{f}\mathfrak{v}}_2^{(k+1)}.$$

2.4 The spaces $\mathfrak{t}\mathfrak{r}_n$, $\mathfrak{s}\mathfrak{d}\mathfrak{e}\mathfrak{r}_n$, $\mathfrak{t}\mathfrak{d}\mathfrak{e}\mathfrak{r}_n$

We follow [1]. Fix $n \geq 1$. Let \mathfrak{lie}_n denote the completed free Lie algebra over \mathbb{K} on n variables x_1, \dots, x_n and let $\text{Ass}_n = U(\mathfrak{lie}_n)$ be the completed free associative algebra in n generators. The graded vector space of *cyclic words* in n variables $\mathfrak{t}\mathfrak{r}_n$ is defined as

$$\mathfrak{t}\mathfrak{r}_n := \text{Ass}_n^+ / \langle (ab - ba), a, b \in \text{Ass}_n \rangle$$

where Ass_n^+ is the augmentation ideal of Ass_n . The Lie algebra $\mathfrak{t}\mathfrak{d}\mathfrak{e}\mathfrak{r}_n$ of *tangential derivations* on \mathfrak{lie}_n is defined as follows. A derivation u on \mathfrak{lie}_n is tangential if there exist $a_1, \dots, a_n \in \mathfrak{lie}_n$ such that $u(x_i) = [x_i, a_i]$ for all $i = 1, \dots, n$. The action of u on the generators completely determine the derivation. For $u = (a_1, \dots, a_n)$ and $v = (b_1, \dots, b_n)$ elements of $\mathfrak{t}\mathfrak{d}\mathfrak{e}\mathfrak{r}_n$, the Lie bracket is the

tangential derivation $[u, v] = (c_1, \dots, c_n)$, where $c_k = u(b_k) - v(a_k) + [a_k, b_k]$ for all $k = 1, \dots, n$. The Lie algebra of *special derivations* $\mathfrak{sd}\mathfrak{er}_n$ is

$$\mathfrak{sd}\mathfrak{er}_n := \{u \in \mathfrak{td}\mathfrak{er}_n \mid u(\sum_{i=1}^n x_i) = 0\}.$$

It is a Lie subalgebra of $\mathfrak{td}\mathfrak{er}_n$. For every $a \in \text{Ass}_n$, we have a unique decomposition

$$a = a_0 + \sum_{k=1}^n (\partial_k a) x_k,$$

where $a_0 \in \mathbb{K}$ and $(\partial_k a) \in \text{Ass}_n$. The *divergence map*

$$\begin{aligned} \text{div} : \mathfrak{td}\mathfrak{er}_n &\rightarrow \mathfrak{tr}_n \\ u = (a_1, \dots, a_n) &\mapsto \sum_{k=1}^n \text{tr}(x_k(\partial_k a_k)) \end{aligned}$$

is a cocycle for $\mathfrak{td}\mathfrak{er}_n$ ([1], Proposition 3.6.).

The following algorithm describes the isomorphism between $H^0(\hat{\text{gr}}\text{ICG}(n)^0, d_0)$, i.e. internally trivalent trees in $\text{ICG}(n)$ modulo IHX, and $\mathfrak{sd}\mathfrak{er}_n$. Let Γ be a tree representing an element of $H^0(\hat{\text{gr}}\text{ICG}(n)^0, d_0)$. Pick an edge incident to the external vertex 1, cut it and make it the “root” edge. The resulting tree is a binary tree with leafs labeled by $1, \dots, n$. Repeat this procedure for every edge incident to vertex 1, and take the sum of the trees obtained in this way. We want to interpret these binary trees as Lie words. The sign convention for this is as follows. The edges of the tree should be ordered such that its “root” edge comes first, then all edges of its left subtree, and then all edges of its right subtree. For each subtree, apply this convention recursively. The resulting linear combination of Lie words (these can be read off the trees by following the ordering of the edges) in the variables x_1, \dots, x_n corresponds to the first component a_1 of a special derivation $a = (a_1, \dots, a_n) \in \mathfrak{sd}\mathfrak{er}_n$. The i -th component a_i is obtained by applying the same procedure to the i -th external vertex.

We now give the map $H^1(\hat{\text{gr}}\text{ICG}(n)^1, d_0) \hookrightarrow \mathfrak{tr}_n$ as described in [13]. Let $\bar{\Gamma} \in H^1(\hat{\text{gr}}\text{ICG}(n)^1, d_0)$. We may assume that the representative Γ is such that the loop passes through all internal vertices. Order the edges as in Figure 2.6. In this case, we map

$$\bar{\Gamma} \mapsto \text{tr}(x_{m_1} \cdots x_{m_k}) - (-1)^k \text{tr}(x_{m_k} \cdots x_{m_1}).$$

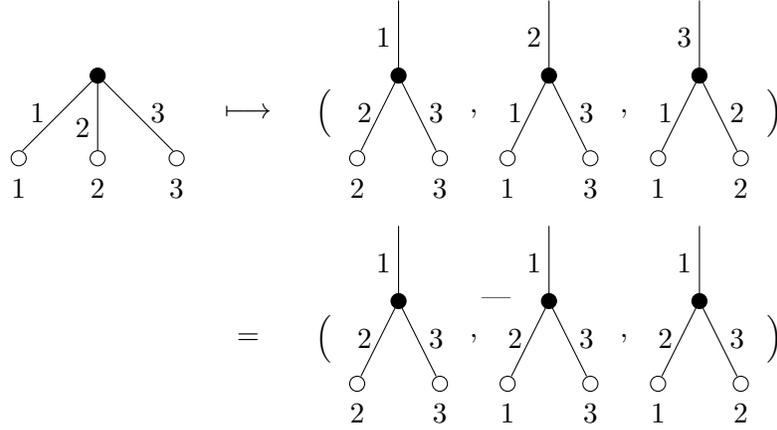


Figure 2.5: An example of the isomorphism $H^0(\widehat{\text{gr}}\text{ICG}(3)^0, d_0) \rightarrow \mathfrak{sd}\mathfrak{tr}_3$. The triple on the right corresponds to the element $([x_2, x_3], -[x_1, x_3], [x_1, x_2])$.

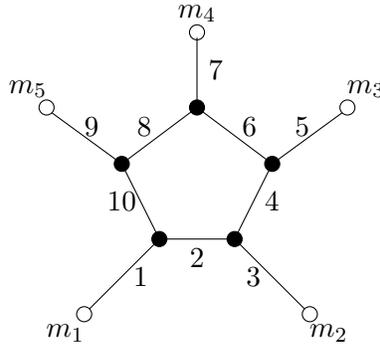


Figure 2.6: This graph will be sent to $\text{tr}(x_{m_1}x_{m_2}x_{m_3}x_{m_4}x_{m_5}) - (-1)^5\text{tr}(x_{m_5}x_{m_4}x_{m_3}x_{m_2}x_{m_1})$ under the injective map $H^1(\widehat{\text{gr}}\text{ICG}(n)^1, d_0) \rightarrow \mathfrak{tr}_n$.

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Chapter 3

Filtrations on graph complexes and the Grothendieck-Teichmüller Lie algebra in depth two

Abstract

We establish an isomorphism between the Grothendieck - Teichmüller Lie algebra \mathfrak{grt}_1 in depth two modulo higher depth and the cohomology of the two-loop part of the graph complex of internally connected graphs $\mathrm{ICG}(1)$. In particular, we recover all linear relations satisfied by the brackets of the conjectural generators σ_{2k+1} modulo depth three by considering relations among two-loop graphs.

The Grothendieck-Teichmüller Lie algebra is related to the zeroth cohomology of M. Kontsevich's graph complex GC via T. Willwacher's isomorphism. We define a descending filtration on $H^0(\mathrm{GC})$ and show that the degree two components of the corresponding associated graded vector spaces are isomorphic under T. Willwacher's map.

Introduction

The Grothendieck-Teichmüller Lie algebra \mathfrak{grt}_1 was introduced by V. Drinfeld [6]. Although being widely studied it still remains a somewhat mysterious object. It is spanned by series of Lie words in two variables x and y satisfying an antisymmetry, a hexagon and a pentagon equation. Its Lie bracket is given by,

$$\{\psi_1, \psi_2\} := D_{\psi_1}\psi_2 - D_{\psi_2}\psi_1 + [\psi_1, \psi_2],$$

where $[\psi_1, \psi_2]$ denotes the bracket on the completed free Lie algebra in x and y , and D_ψ is the unique derivation extending $x \mapsto 0$, $y \mapsto [y, \psi(x, y)]$. Moreover, \mathfrak{grt}_1 is graded by weight (i.e. the total number of x and y of any Lie word) and filtered by depth (i.e. the minimal number of y in any Lie word appearing in a series). Also, it is well-known that \mathfrak{grt}_1 contains, for $k \geq 1$, elements σ_{2k+1} of odd weight whose coefficient in front of the Lie word $\mathrm{ad}_x^{2k}(y)$ is non-vanishing (in particular, it can be normalized to be equal to one). An important result by F. Brown states that these elements freely generate a Lie subalgebra of \mathfrak{grt}_1 [3]. Conjecturally however, the free Lie algebra generated by the σ_{2k+1} should coincide with \mathfrak{grt}_1 .

Any Lie monomial appearing in the bracket of two such conjectural generators $\{\sigma_{2i+1}, \sigma_{2j+1}\}$ contains at least two y . Modulo elements of depth three and higher, these brackets satisfy a set of

linear relations of homogeneous weight. For instance in weight 12,

$$\{\sigma_3, \sigma_9\} = 3\{\sigma_5, \sigma_7\} \pmod{\mathcal{F}^3 \mathfrak{grt}_1}.$$

Linear relations of this sort were first studied by Y. Ihara and N. Takao [9], and independently by A. Goncharov [8]. In addition, Y. Ihara and N. Takao showed that modulo depth three, the brackets $\{\sigma_{2i+1}, \sigma_{2j+1}\}$ actually generate the whole quotient,

$$\mathcal{F}^2 \mathfrak{grt}_1 / \mathcal{F}^3 \mathfrak{grt}_1.$$

Later, L. Schneps fully classified such linear identities by relating them to restricted even period polynomials associated to cusp forms on $\mathrm{SL}_2(\mathbb{Z})$ [15]. The aim of this text is twofold. First, we add one more interpretation of the space spanned by the brackets $\{\sigma_{2i+1}, \sigma_{2j+1}\}$ modulo $\mathcal{F}^3 \mathfrak{grt}_1$, namely via the cohomology of a certain graph complex.

For this, consider the complex (C, d_0) ,

$$0 \rightarrow C_0 \xrightarrow{d_0} C_1 \xrightarrow{d_0} C_2 \rightarrow 0$$

where the graphs in C_i are as depicted in Figure 3.1. The differential d_0 sums over the three possible ways of splitting the 4-valent vertices, thus attaching an additional “hair” to each strand making up the theta shape of the graph. The cohomology of this complex was computed in [4], with the aim of determining the cohomology of the two-loop part of a larger complex introduced by G. Arone and V. Turchin to study the rational homotopy type of long embeddings [2]. In degree one the cohomology can be identified with a quotient of the algebra of even polynomials in three variables. Moreover, we find that graphs $\theta_{2i,2j}$ for which exactly two of the strands are decorated by $2i$ and $2j$ “hairs”, respectively, form a generating set for $H^1(C, d_0)$. Our first main result then states the following.

Theorem. The map

$$\begin{aligned} \mathcal{F}^2 \mathfrak{grt}_1 / \mathcal{F}^3 \mathfrak{grt}_1 &\rightarrow H^1(C, d_0) \\ \{\sigma_{2i+1}, \sigma_{2j+1}\} \pmod{\mathcal{F}^3 \mathfrak{grt}_1} &\mapsto \theta_{2i,2j} \end{aligned}$$

is an isomorphism of vector spaces.

In particular, there is a canonical equivalence between the linear relations satisfied by the brackets $\{\sigma_{2i+1}, \sigma_{2j+1}\}$ modulo $\mathcal{F}^3 \mathfrak{grt}_1$ and the cohomology classes represented by the graphs $\theta_{2i,2j}$. The proof of this theorem follows from one of the classification results of L. Schneps [15].

In a second part, we show that the equivalence between the cohomology of two-loop graphs and the Grothendieck-Teichmüller Lie algebra in depth two modulo higher depth is no coincidence. For instance, \mathfrak{grt}_1 is related to graphs via T. Willwacher’s important result [17] which states that there is an isomorphism of Lie algebras,

$$H^0(\mathrm{GC}) \cong \mathfrak{grt}_1,$$

where GC denotes a variant of Kontsevich’s graph complex. Our aim is to define a filtration on $H^0(\mathrm{GC})$ which is compatible with the depth filtration on \mathfrak{grt}_1 under T. Willwacher’s isomorphism. Our second main result states that this can be done in depth two modulo higher depth.

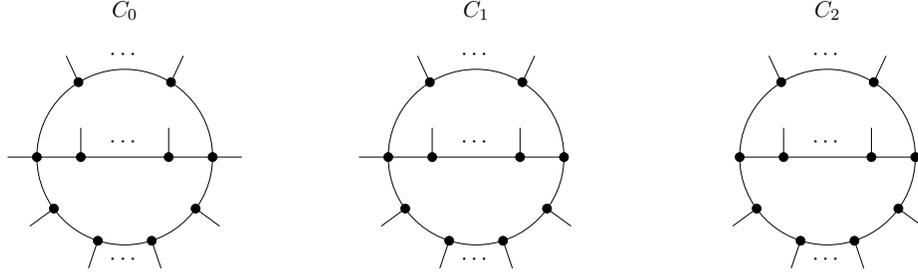


Figure 3.1: Graphs in C_0, C_1, C_2 .

Theorem. T. Willwacher’s map induces an isomorphism,

$$\mathcal{F}^2 H^0(\text{GC})/\mathcal{F}^3 H^0(\text{GC}) \xrightarrow{\cong} \mathcal{F}^2 \text{grt}_1/\mathcal{F}^3 \text{grt}_1.$$

In the proof of this theorem we show that on the degree two part of the associated graded, the map induced by T. Willwacher’s isomorphism factors through an isomorphism,

$$\mathcal{F}^2 H^0(\text{GC})/\mathcal{F}^3 H^0(\text{GC}) \xrightarrow{\cong} H^1(C, d_0).$$

Together with our first main result, this implies the statement.

The hope at this point is that the reasoning above can be generalized. More precisely, we expect that the filtration on $H^0(\text{GC})$ is defined in such a way that we have an isomorphism on the entire associated graded. Additionally, the degree p part of the associated graded could then ideally be identified with the cohomology of the p -loop part of a graph complex, similar to the one of which C is a subcomplex, i.e

$$\mathcal{F}^p H^0(\text{GC})/\mathcal{F}^{p+1} H^0(\text{GC}) \cong \text{“cohomology of } p\text{-loop graphs”}.$$

This would in turn allow us to study the Grothendieck-Teichmüller Lie algebra’s relations in higher depth by computing the cohomology of the space of higher loop order graphs. Possibly, such a cohomology could again be expressed in terms of some quotient of an algebra of polynomials, thus simplifying some of the relations found in the associated graded with respect to the depth filtration on grt_1 . For now, however, these questions remain open.

Acknowledgements

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3.1 The Grothendieck-Teichmüller Lie algebra

3.1.1 Definitions

We recall the definition of the Grothendieck-Teichmüller Lie algebra \mathfrak{grt}_1 . It was first introduced by V. Drinfeld [6].

Definition 10. The *Drinfeld-Kohno Lie algebra* \mathfrak{t}_n is generated by elements $t^{i,j} = t^{j,i}$, where $1 \leq i \neq j \leq n$, which satisfy the relations,

$$\begin{aligned} [t^{i,j}, t^{k,l}] &= 0 \text{ if } \#\{i, j, k, l\} = 4, \\ [t^{i,j} + t^{i,k}, t^{j,k}] &= 0 \text{ if } \#\{i, j, k\} = 3. \end{aligned}$$

Definition 11. The *Grothendieck-Teichmüller Lie algebra* \mathfrak{grt}_1 is spanned by elements $\psi \in \widehat{\mathfrak{lie}}_2$ (the completed free Lie algebra in two generators x, y) satisfying the following relations:

$$\begin{aligned} \psi(x, y) &= -\psi(y, x), \\ \psi(x, y) + \psi(y, z) + \psi(z, x) &= 0 \text{ for } x + y + z = 0, \\ \psi(t^{1,2}, t^{2,3} + t^{2,4}) + \psi(t^{1,3} + t^{2,3}, t^{3,4}) \\ &= \psi(t^{2,3}, t^{3,4}) + \psi(t^{1,2} + t^{1,3}, t^{2,4} + t^{3,4}) + \psi(t^{1,2}, t^{2,3}), \end{aligned}$$

where the last equation takes values in the Lie algebra \mathfrak{t}_4 . The bracket on \mathfrak{grt}_1 is given by

$$\{\psi_1, \psi_2\} := D_{\psi_1}\psi_2 - D_{\psi_2}\psi_1 + [\psi_1, \psi_2],$$

where D_ψ is the unique derivation extending $x \mapsto 0, y \mapsto [y, \psi(x, y)]$.

Definition 12. Let $a(x, y) \in \widehat{\mathfrak{lie}}_2$. The *depth* $\text{depth}(a)$ of a is the minimal number of y 's contained in any of its Lie monomials. Let b be a Lie word in x, y . The *weight* $\text{wt}(b)$ is the total number of x and y in b .

Define the following descending filtration on \mathfrak{grt}_1 by

$$\mathcal{F}^p \mathfrak{grt}_1 := \{\psi \in \mathfrak{grt}_1 \mid \text{depth}(\psi) \geq p\}$$

for $p \geq 1$. Note that $\mathcal{F}^1 \mathfrak{grt}_1 = \mathfrak{grt}_1$. It is not hard to see that the filtration is compatible with the Lie algebra structure, i.e. for all $p, q \geq 1$,

$$\{\mathcal{F}^p \mathfrak{grt}_1, \mathcal{F}^q \mathfrak{grt}_1\} \subset \mathcal{F}^{p+q} \mathfrak{grt}_1.$$

Remark 16. Note that the weight defines a grading on \mathfrak{grt}_1 . For $k \geq 2$, we denote by $[\mathfrak{grt}_1]_k$ the subspace of \mathfrak{grt}_1 spanned by homogeneous elements of weight k .

Remark 17. It is well-known that \mathfrak{grt}_1 contains, for $k \geq 1$, homogeneous elements σ_{2k+1} of weight $2k+1$ for which the coefficient in front of the Lie word $\text{ad}_x^{2k}(y)$ is non-vanishing. As it is a common convention, we shall assume this coefficient to be 1. It has been shown by F. Brown that the Lie algebra freely generated by these σ_{2k+1} forms a Lie subalgebra of \mathfrak{grt}_1 [3]. In fact, it is conjectured that these elements freely generate \mathfrak{grt}_1 as a Lie algebra.

Let us recall one more well-known result of which we give a proof for convenience.

Lemma 11. ([9], Theorem III-II-1) In depth one modulo higher depth there are no elements of even weight, that is, if $\text{ad}_x^l(y) \in \mathcal{F}^1 \mathfrak{grt}_1 / \mathcal{F}^2 \mathfrak{grt}_1$, then $l = 2k$ for some $k \geq 1$.

Proof. Assume $\psi(x, y) = \text{ad}_x^l(y) + R \in \mathfrak{grt}_1$ with $R \in \mathcal{F}^2 \mathfrak{grt}_1$. The second defining relation can be rewritten as,

$$\psi(x, y) = \psi(-x - y, y) - \psi(-x - y, x). \quad (3.1)$$

We would like to pick out the linear terms in y . For this, recall first that Theorem 4.1 in [1] implies in particular that $[x, \psi(-x - y, x)] + [y, \psi(-x - y, y)] = 0$. Since $[y, \psi(-x - y, y)] \in \mathcal{F}^2 \mathfrak{grt}_1$, $[x, \psi(-x - y, x)] \in \mathcal{F}^2 \mathfrak{grt}_1$ and hence also $\psi(-x - y, x) \in \mathcal{F}^2 \mathfrak{grt}_1$, i.e. $\psi(-x - y, x)$ has no linear term in y . Therefore, modulo $\mathcal{F}^2 \mathfrak{grt}_1$, equation (3.1) reads, $\text{ad}_x^l(y) = (-1)^l \text{ad}_x^l(y)$ and l is even. \square

3.1.2 Relations among the conjectural generators

Next, we recall some useful results on the brackets of the conjectural generators. Consider the composition,

$$\bigoplus_{i, j \geq 1} \mathbb{K} \cdot \{\sigma_{2i+1}, \sigma_{2j+1}\} \xrightarrow{i} \mathcal{F}^2 \mathfrak{grt}_1 \xrightarrow{\pi} \mathcal{F}^2 \mathfrak{grt}_1 / \mathcal{F}^3 \mathfrak{grt}_1.$$

Y. Ihara and N. Takao proved the following.

Theorem 6 ([9], Corollary to Theorem III-II-1.). In depth two modulo depth three the brackets of the conjectural generators span the whole of \mathfrak{grt}_1 . That is, the composition,

$$\begin{aligned} \pi \circ i : \bigoplus_{i, j \geq 1} \mathbb{K} \cdot \{\sigma_{2i+1}, \sigma_{2j+1}\} &\rightarrow \mathcal{F}^2 \mathfrak{grt}_1 / \mathcal{F}^3 \mathfrak{grt}_1 \\ \{\sigma_{2i+1}, \sigma_{2j+1}\} &\mapsto \{\sigma_{2i+1}, \sigma_{2j+1}\} \pmod{\mathcal{F}^3 \mathfrak{grt}_1} \end{aligned}$$

is surjective.

Remark 18. A. Goncharov [8], and independently Y. Ihara and N. Takao [9] discovered that for $k \geq 4$ even, there are $\lfloor \frac{k-4}{4} \rfloor - \lfloor \frac{k-2}{6} \rfloor$ linear relations modulo $\mathcal{F}^3 \mathfrak{grt}_1$ among the brackets $\{\sigma_{2i+1}, \sigma_{k-1-2i}\}$, $1 \leq i \leq \frac{k-4}{2}$. That is, relations of the form,

$$\sum_{i=1}^{\frac{k-4}{2}} a_i \{\sigma_{2i+1}, \sigma_{k-1-2i}\} = 0 \pmod{\mathcal{F}^3 \mathfrak{grt}_1}.$$

In weight k and depth 2 modulo higher depth, and taking symmetry into account, we find that there are $\lfloor \frac{k-4}{4} \rfloor$ different brackets. The result above now implies that the dimension of the vector space spanned by the brackets of weight k and depth 2 modulo elements of higher depth is $\lfloor \frac{k-2}{6} \rfloor$. Together with Theorem 6, we obtain that the dimension of the quotient space $[\mathcal{F}^2 \mathfrak{grt}_1 / \mathcal{F}^3 \mathfrak{grt}_1]_k$ of \mathfrak{grt}_1 -elements of weight k and depth 2 modulo higher depth is also of $\lfloor \frac{k-2}{6} \rfloor$.

The linear relations mentioned above were later fully classified by L. Schneps in [15] and we recall her result below. Consider the ring $\mathbb{K}\langle x, y \rangle$ of polynomials in two non-commutative variables x and y . Denote by $\mathbb{K}^d\langle x, y \rangle$ the vector space of $\mathbb{K}\langle x, y \rangle$ generated by monomials containing exactly d y 's. There is a canonical S_{d+1} -action on monomials in $\mathbb{K}^d\langle x, y \rangle$ given by,

$$\sigma.(x^{i_1} y x^{i_2} y \dots x^{i_d} y x^{i_{d+1}}) = x^{i_{\sigma(1)}} y x^{i_{\sigma(2)}} y \dots x^{i_{\sigma(d)}} y x^{i_{\sigma(d+1)}},$$

for $\sigma \in S_{d+1}$. With this notation, L. Schneps' theorem states the following.

Theorem 7 ([15], Theorem 4.1. and Corollary 4.2.). Let

$$F = \sum_{i=1}^{\frac{k-4}{2}} a_i \text{ad}_x^{2i}(y) \text{ad}_x^{k-2-2i}(y) \in \mathbb{K}^2 \langle x, y \rangle$$

where $\text{ad}_x(y) = xy - yx$. Then F satisfies

$$F + (13).F = 0 \text{ and } F + (123).F + (132).F = 0$$

if and only if

$$\sum_{i=1}^{\frac{k-4}{2}} a_i \{\sigma_{2i+1}, \sigma_{k-1-2i}\} = 0 \pmod{\mathcal{F}^3 \text{grr}_1}. \quad (3.2)$$

Remark 19. It will be useful to note the following easy technical subtlety. The polynomial expansion of F as in the theorem above is given by,

$$F = \sum_{i=1}^{\frac{k-4}{2}} a_i \sum_{u=0}^{2i} \sum_{v=0}^{k-2-2i} (-1)^{u+v} \binom{2i}{u} \binom{k-2-2i}{v} x^u y x^{2i-u+v} y x^{k-2-2i-v}.$$

Since we are mainly interested in the S_3 -action, the relevant information is contained in the exponents of x . These may be encoded in the expression,

$$\begin{aligned} G &= \sum_{i=1}^{\frac{k-4}{2}} a_i \sum_{u=0}^{2i} \sum_{v=0}^{k-2-2i} (-1)^{u+v} \binom{2i}{u} \binom{k-2-2i}{v} \alpha^u \beta^{2i-u+v} \gamma^{k-2-2i-v} \\ &= \sum_{i=1}^{\frac{k-4}{2}} a_i (\alpha - \beta)^{2i} (\beta - \gamma)^{k-2-2i}. \end{aligned}$$

The S_3 -action on $\mathbb{K}^2 \langle x, y \rangle$ is compatible with the S_3 -action on $\mathbb{K}[\alpha, \beta, \gamma]$ which for $\sigma \in S_3$ is given by,

$$\sigma.(\alpha^{k_1} \beta^{k_2} \gamma^{k_3}) = \alpha^{k_{\sigma(1)}} \beta^{k_{\sigma(2)}} \gamma^{k_{\sigma(3)}}$$

on monomials. Clearly, keeping this identification in mind, L. Schneps' result reads as follows.

$$G = \sum_{i=1}^{\frac{k-4}{2}} a_i (\alpha - \beta)^{2i} (\beta - \gamma)^{k-2-2i}$$

satisfies $G + (13).G = 0$ and $G + (123).G + (132).G = 0$ if and only if equation (3.2) holds.

3.2 Internally connected graphs

3.2.1 Definitions

The graph complex of internally connected graphs was introduced by P. Ševera and T. Willwacher [16] based on the works of M. Kontsevich (see for instance [13], [14]). Fix $n \geq 1$.

Definition 13. An *admissible graph* is an unoriented graph Γ with labeled vertices $1, 2, \dots, n$ (called external), possibly other vertices (unlabeled and called internal) satisfying the following properties:

1. There is a linear order on the set of edges.
2. Γ has no double edges, nor simple loops (edges connecting a vertex with itself).
3. Every internal vertex is at least trivalent.
4. Every internal vertex can be connected by a path with an external vertex.

Let $\mathbf{graphs}(n)$ be the vector space spanned by admissible graphs with n external vertices, modulo the relation $\Gamma^\sigma = (-1)^{|\sigma|}\Gamma$, where Γ^σ differs from Γ by a permutation σ on the order of edges. Here $|\sigma|$ denotes the parity of the permutation σ .

Definition 14. A graph in $\mathbf{graphs}(n)$ which is connected after we delete all external vertices is called *internally connected*. Denote by $\mathbf{ICG}(n)$ the space spanned by internally connected graphs modulo sign relations obtained from the order of edges. Define the grading on $\mathbf{ICG}(n)$ to be,

$$\deg \Gamma = 1 - \#\text{edges} + 2\#\text{internal vertices}.$$

Set the differential d on $\mathbf{ICG}(n)$ (on $\mathbf{graphs}(n)$, respectively) to be given by vertex splitting. More precisely, an external vertex splits into an external and an internal vertex connected by an edge, and we sum over all possible ways of reconnecting the “loose” edges to the two newly created vertices. Similarly, an internal vertex splits into two internal vertices, before summing over all ways of reconnecting the edges previously connected to the splitted vertex. In both cases, we only keep graphs that are still internally connected (admissible, respectively). As a convention, we set the newly created edge to come last in the new order of the edges. In this way, we have $d^2 = 0$.

Remark 20. The collections $\{\mathbf{graphs}(n)\}_{n \geq 1}$ and $\{\mathbf{ICG}(n)\}_{n \geq 1}$ both form operads in the category of cochain complexes. The symmetric action permutes the labels of the external vertices. The operadic composition in \mathbf{graphs} (and also in \mathbf{ICG}) is given by insertion. That is, for $\Gamma_1 \in \mathbf{graphs}(r)$, $\Gamma_2 \in \mathbf{graphs}(s)$,

$$\Gamma_1 \circ_j \Gamma_2 \in \mathbf{graphs}(r + s - 1)$$

is constructed by replacing the j th external vertex by Γ_2 , summing over all possible ways of reconnecting the “loose” edges (which were previously adjacent to vertex j) to vertices of Γ_2 , and keeping only admissible graphs (in the case of \mathbf{ICG} , we only keep the internally connected ones). The order on the set of edges of the new graphs is simply given by letting the edges of Γ_1 come before those of Γ_2 while leaving the respective ordering unchanged.

Remark 21. Since any graph in $\mathbf{graphs}(n)$ may be written as the disjoint union of its internally connected components (after identifying the external vertices), the internally connected graphs freely generate $\mathbf{graphs}(n)$ as a coalgebra. For a suitable choice of grading on $\mathbf{graphs}(n)$, we therefore have an isomorphism of cocommutative coalgebras

$$\mathbf{graphs}(n) \cong S(\mathbf{ICG}(n)[1]).$$

By definition, the differential on $\mathbf{graphs}(n)$ defines an L_∞ -structure on the graded vector space $\mathbf{ICG}(n)$.

Definition 15. Let $\Gamma \in \text{ICG}(n)$. An *internal loop* of Γ is a loop in Γ which does not pass through any external vertex.

The space $\text{ICG}(n)$ is filtered by the number of internal loops. That is, we may define the following descending filtration on $\text{ICG}(n)$,

$$\mathcal{F}^p \text{ICG}(n) := \{\Gamma \in \text{ICG}(n) \mid \Gamma \text{ has at least } p \text{ internal loops}\}.$$

Consider the spectral sequence corresponding to this filtration. On its first page, we find the cohomology of the associated graded complex,

$$E_1^{p,q}(n) = H^{p+q}(\mathcal{F}^p \text{ICG}(n) / \mathcal{F}^{p+1} \text{ICG}(n), d_0)$$

where d_0 is the part of the of the differential d on $\text{ICG}(n)$ which does not create any internal loops (i.e. it splits internal vertices only).

Remark 22. The first page of this spectral sequence turns out to be very useful. It is related to the works of A. Alekseev and C. Torossian on the Kashiwara-Vergne conjecture [1]. For instance,

$$\bigoplus_{q \in \mathbb{Z}} E_1^{0,q}(n) = E_1^{0,0}(n) = H^0(\text{ICG}(n) / \mathcal{F}^1 \text{ICG}(n), d_0)$$

consists of (internally) 3-valent trees modulo the IHX relation and forms a Lie algebra. It corresponds bijectively (as a Lie algebra) to the Lie algebra of special derivations $\mathfrak{sd}\mathfrak{er}_n$ (for an introduction, see [1]). Moreover,

$$\bigoplus_{q \in \mathbb{Z}} E_1^{1,q}(n) = E_1^{1,0}(n) = H^1(\mathcal{F}^1 \text{ICG}(n) / \mathcal{F}^2 \text{ICG}(n), d_0)$$

describes the space of one-loop graphs, again modulo IHX. These correspond to a quotient of the space of cyclic words \mathfrak{tt}_n in n variables. These equivalences are made precise in P. Ševera and T. Willwacher's paper [16]. These two spaces together with the induced differential on the first page yield a definition of the Kashiwara-Vergne Lie algebra in terms of graphs ([1],[16]). Furthermore, using some additional simplicial structure on $\text{ICG}(n)$, we constructed a nested sequence of Lie subalgebras of $E_1^{0,0}(2)$ interpolating between the Kashiwara-Vergne Lie algebra and the Grothendieck-Teichmüller Lie algebra [7]. In this text we describe an application of the two-loop part, that is,

$$\bigoplus_{q \in \mathbb{Z}} E_1^{2,q}(n) = \bigoplus_{q \in \mathbb{Z}} H^{2+q}(\mathcal{F}^2 \text{ICG}(n) / \mathcal{F}^3 \text{ICG}(n), d_0).$$

3.2.2 Cohomology of the two-loop part

In [2], G. Arone and V. Turchin defined a graph complex $(\mathcal{E}_\pi^{m,N}, \partial)$ ($m, N \in \mathbb{N}$) which provides some insight on the rational homotopy type of the spaces of long embeddings, i.e. a certain kind of embeddings $\mathbb{R}^m \hookrightarrow \mathbb{R}^N$. While the cohomology of the tree- and one-loop part of this complex was computed in [2], the cohomology of the two-loop part was established in [5]. In this section, we recall the results on the two-loop part from which we will later deduce our main theorem.

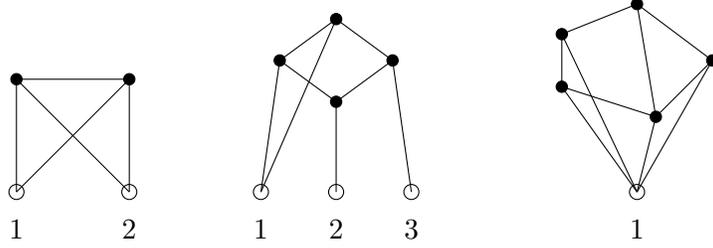


Figure 3.2: A tree in $\text{ICG}(2)$, a one-loop graph in $\text{ICG}(3)$ and a two-loop graph in $\text{ICG}(1)$.

Remark 23. It is clear from the definition given in [5] that, for $m = 0$, $N = 2$, the graph complex $(\mathcal{E}_\pi^{0,2}, \partial)$ corresponds (up to some degree issues) to the complex $(\text{ICG}(1), d_0)$. The only difference between the two definitions is that while in $\mathcal{E}_\pi^{0,2}$ the degree of a graph is given by $\#\text{edges} - 2\#\text{internal vertices}$, in $\text{ICG}(1)$, it will be $1 - \#\text{edges} + 2\#\text{internal vertices}$. Note however, that the way one draws graphs in the respective complexes is different, but the correspondence is obvious (see Figure 3.3).

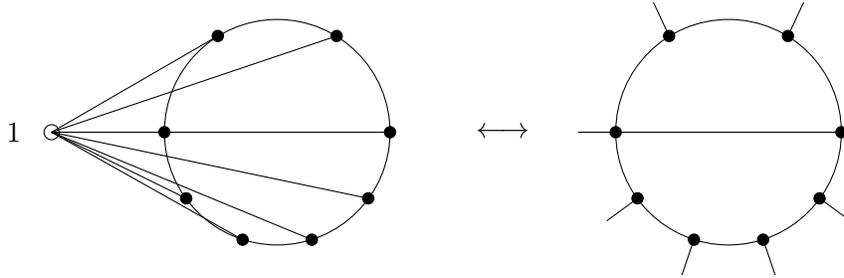


Figure 3.3: The correspondence between graphs in $\text{ICG}(1)$ and $\mathcal{E}_\pi^{0,2}$.

Remark 24. In [5] it is shown that the cohomology of the two-loop part of the complex $(\text{ICG}(1), d_0) \simeq (\mathcal{E}_\pi^{0,2}, \partial)$ can be calculated by the quasi-isomorphic subcomplex (C, d_0) given by

$$0 \rightarrow C_0 \rightarrow C_1 \rightarrow C_2 \rightarrow 0$$

where the graphs in C_i are as depicted in Figure 3.4. In our notation, the cohomology of this complex is $E_1^{2,\bullet}(1) = H^{2+\bullet}(\mathcal{F}^2\text{ICG}(1)/\mathcal{F}^3\text{ICG}(1), d_0)$.

Definition 16. On a two-loop graph as in Figure 3.3 on the left, there is exactly one 4-valent vertex and one 3-valent vertex which is not connected to the external vertex. These two vertices may be joined by three paths on the graph which pass through internal vertices only and any two of which do not share an edge. We refer to these paths as the three *strands*. They correspond to the dashed parts of the θ -graph depicted in Figure 3.5. The edges connecting an internal vertex to the external one are called *hairs*. Similarly, when the graph lies in C_0 (in C_2) and after we identify it with a graph in $\text{ICG}(1)$, it will have exactly two 4-valent vertices (two vertices which are not connected to the external vertex, respectively), and one should modify the definition accordingly.

Definition 17. By definition, a hair h connects the external vertex to an internal one. Call this internal vertex v . We say that the hair h is *attached to* or *on* a strand, if the path defining the strand passes through its corresponding internal vertex v .

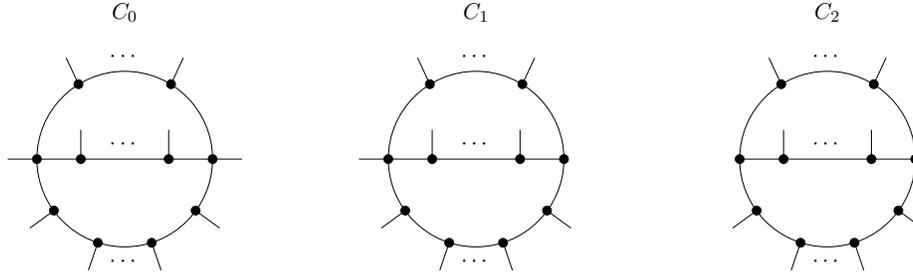


Figure 3.4: Graphs in C_0, C_1, C_2 .

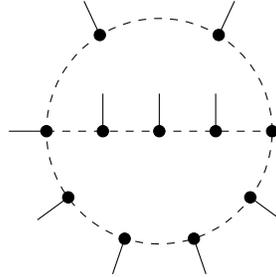


Figure 3.5: We refer to the dashed parts of such a θ -graph as the three strands. The black edges are called hairs.

Remark 25. Following [5], we encode such θ -graphs by the space of coinvariants of certain polynomials under a symmetric action. To fix the overall sign, we set a graph in C_0 with edges ordered as in Figure 3.6 to correspond to $x^{k_1}y^{k_2}z^{k_3}$. Here k_1 is the number of hairs attached to the strand whose edges come first in the order of the edges, k_2 is the number of hairs on the strand whose edges come second in the ordering (i.e. they will start at $k_1 + 2$), and k_3 is the number of hairs on the strand whose edges come last in the ordering (i.e. they will start at $2k_1 + 2k_2 + 3$). The action of the symmetric group is described below.

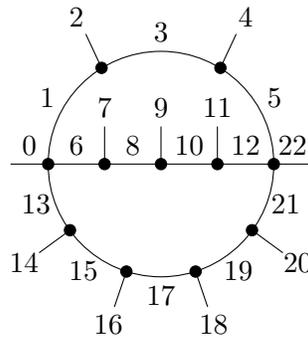


Figure 3.6: A θ -graph with this ordering on the set of edges corresponds to the polynomial $x^2y^3z^4$.

By abuse of notation, we identify C_1 and C_2 with the same type of polynomials. The different symmetry relations on graphs imply the following sign relations on our spaces of polynomials.

Lemma 12 ([5], Lemma 4.2.). 1. Let $x^{k_1}y^{k_2}z^{k_3} \in C_0$. The symmetry with respect to the vertical line produces the sign

$$(-1)^{k_1+k_2+k_3+1}.$$

Therefore, graphs in C_0 with an even number of hairs on the strands are zero.

2. Let $x^{k_1}y^{k_2}z^{k_3} \in C_2$. The symmetry with respect to the vertical line produces the sign

$$(-1)^{k_1+k_2+k_3}.$$

Therefore, graphs in C_2 with an odd number of hairs are zero.

3. Let $x^{k_1}y^{k_2}z^{k_3} \in C_i$, $i = 0, 1, 2$. The S_3 -action which permutes the three strands acts by sign, i.e. for $\sigma \in S_3$,

$$x^{k_1}y^{k_2}z^{k_3} \text{ is identified with } (-1)^{|\sigma|}x^{k_{\sigma(1)}}y^{k_{\sigma(2)}}z^{k_{\sigma(3)}}$$

where $|\sigma|$ denotes the parity of the permutation σ .

Proof. The proof is an exercise in graphical calculus. It can be found in [5]. □

Definition 18. Let $\mathbb{K}[x, y, z]$ be the algebra of polynomials in the variables x, y, z . We say that a polynomial $p \in \mathbb{K}[x, y, z]$ is even (odd) if all of its monomials are of even (odd) degree. We denote by $\mathbb{K}[x, y, z]^{\text{even}}$ ($\mathbb{K}[x, y, z]^{\text{odd}}$) the space of even (odd) polynomials. Given a polynomial $p \in \mathbb{K}[x, y, z]$, we denote by $[p]_{\text{even}}$ ($[p]_{\text{odd}}$) its even (odd) part.

Remark 26. Let the action of S_3 on $\mathbb{K}[x, y, z]$ be given on monomials by

$$\sigma.(x^{k_1}y^{k_2}z^{k_3}) := (-1)^{|\sigma|}x^{k_{\sigma(1)}}y^{k_{\sigma(2)}}z^{k_{\sigma(3)}}.$$

Using Lemma 12 we can identify the components of the graph complex (C, d_0) with the following spaces of coinvariants with respect to this action of the symmetric group.

$$\begin{aligned} C_0 &= (\mathbb{K}[x, y, z]^{\text{odd}})_{S_3} \\ C_1 &= \mathbb{K}[x, y, z]_{S_3} \\ C_2 &= (\mathbb{K}[x, y, z]_{>0}^{\text{even}})_{S_3} \end{aligned}$$

where $\mathbb{K}[x, y, z]_{>0}^{\text{even}}$ denotes the space of even polynomials in which all monomials have strictly positive degree.

Lemma 13 ([5], Lemma 5.6.). The action of the differential d_0 translates under the above identifications to

$$\begin{aligned} d_0 : C_0 &\longrightarrow C_1 \\ p(x, y, z) &\longmapsto 2(x + y + z)p(x, y, z) \\ d_0 : C_1 &\longrightarrow C_2 \\ p(x, y, z) &\longmapsto \frac{1}{2}(x + y + z)(p(x, y, z) - p(-x, -y, -z)) = [(x + y + z)p(x, y, z)]_{\text{even}} \end{aligned}$$

Proof. The differential $d_0 : C_0 \rightarrow C_1$ splits each of the two 4-valent vertices in three ways, and every time it adds an additional edge to one of the three strands. In the space of polynomials, this corresponds to multiplying with $x + y + z$. The same operations is performed by $d_0 : C_1 \rightarrow C_2$ on the unique 4-valent vertex. In this case, by symmetry, all graphs with an odd number of hairs will equal zero. \square

Lemma 14. The cohomology of (C, d_0) is

$$\begin{aligned} E_1^{2,-2}(1) &= H^0(C, d_0) = 0 \\ E_1^{2,-1}(1) &= H^1(C, d_0) = \left(\frac{\mathbb{K}[x, y, z]^{\text{even}}}{(x + y + z) \cap \mathbb{K}[x, y, z]^{\text{even}}} \right)_{S_3} \\ E_1^{2,0}(1) &= H^2(C, d_0) = \left(\frac{\mathbb{K}[x, y, z]_{>0}^{\text{even}}}{\text{im}(d_0 : C_1 \rightarrow C_2)} \right)_{S_3} \end{aligned}$$

where $(x + y + z)$ denotes the ideal in $\mathbb{K}[x, y, z]$ generated by $x + y + z$.

Proof. The differential $d_0 : C_0 \rightarrow C_1$ is injective. Thus $H^0(C, d_0) = 0$. Next, let $p \in C_1$ be such that $d_0 p = \frac{1}{2}[(x + y + z)p]_{\text{even}} = 0$. This is equivalent to $[p]_{\text{odd}} = 0$, and therefore $p \in \mathbb{K}[x, y, z]^{\text{even}}$. The image of $d_0 : C_0 \rightarrow C_1$ is the ideal generated by $(x + y + z)$ which by symmetry we need to intersect with the algebra of even polynomials, and we obtain the result for $H^1(C, d_0)$. Since $d_0 : C_2 \rightarrow 0$, the formula for $H^2(C, d_0)$ follows by definition. \square

Definition 19. Let $\gamma \in \text{ICG}(1)$. The number of edges adjacent to the unique external vertex in $\text{ICG}(1)$ is called the *weight* of γ . Denote by $[H^i(C, d_0)]_k = [E_1^{2,1-i}(1)]_k$ ($i = 1, 2$) the space of graphs of weight k in $H^i(C, d_0) = E_1^{2,1-i}(1)$.

Remark 27. In [5], the authors work with invariants rather than with coinvariants to describe the cohomology. In particular, they give a generating set for $H^1(C, d_0)$ and $H^2(C, d_0)$ in terms of anti-symmetric polynomials. This enables them to compute the dimensions of the homogeneous weight components of these spaces (see Theorem 8 below). For our purposes, working with coinvariants seems to be more suitable. In fact, the generating set for $H^1(C, d_0)$ that we determine in Section 3.2.3 contains fewer elements than the one given in [5].

Theorem 8 ([5], Theorem 6.2.). The dimensions of the aforementioned spaces are given by

$$\begin{aligned} \dim([H^1(C, d_0)]_k) &= \dim([E_1^{2,-1}(1)]_k) = \begin{cases} 0, & \text{for } k \text{ even} \\ \lfloor \frac{k}{6} \rfloor, & \text{for } k \text{ odd} \end{cases} \\ \dim([H^2(C, d_0)]_k) &= \dim([E_1^{2,0}(1)]_k) = \begin{cases} \lfloor \frac{k}{6} \rfloor, & \text{for } k \text{ even} \\ 0, & \text{for } k \text{ odd.} \end{cases} \end{aligned}$$

Remark 28. As vector spaces $E_1^{2,-1}(1)$ and $\mathcal{F}^2 \text{grt}_1 / \mathcal{F}^3 \text{grt}_1$ decompose into their respective weight components. That is,

$$\begin{aligned} E_1^{2,-1}(1) &= \bigoplus_{k \geq 6} [E_1^{2,-1}(1)]_{k+1} \\ \mathcal{F}^2 \text{grt}_1 / \mathcal{F}^3 \text{grt}_1 &= \bigoplus_{k \geq 8} [\mathcal{F}^2 \text{grt}_1 / \mathcal{F}^3 \text{grt}_1]_k. \end{aligned}$$

Corollary 3. There is an isomorphism of vector spaces,

$$\mathcal{F}^2 \mathbf{grt}_1 / \mathcal{F}^3 \mathbf{grt}_1 \cong E_1^{2,-1}(1).$$

Proof. From Remark 18 and Theorem 8, we obtain for all even $k \geq 8$,

$$\dim([E_1^{2,-1}(1)]_{k-1}) = \lfloor \frac{k-1}{6} \rfloor = \lfloor \frac{k-2}{6} \rfloor = \dim([\mathcal{F}^2 \mathbf{grt}_1 / \mathcal{F}^3 \mathbf{grt}_1]_k)$$

which implies $[\mathcal{F}^2 \mathbf{grt}_1 / \mathcal{F}^3 \mathbf{grt}_1]_k \cong [E_1^{2,-1}(1)]_{k-1}$ and the result. \square

Still, there remains a choice on the isomorphism. Next, we determine a generating set for $E_1^{2,-1}(1)$ which will enable us to describe a particularly easy isomorphism.

3.2.3 A generating family of θ -graphs

Remark 29. The algebra homomorphism $\phi : \mathbb{K}[x, y, z] \rightarrow \mathbb{K}[x, y]$ which on generators is defined via $x \mapsto x$, $y \mapsto y$, $z \mapsto -x - y$, induces an isomorphism,

$$\phi : \frac{\mathbb{K}[x, y, z]}{(x + y + z)} \rightarrow \mathbb{K}[x, y].$$

If we restrict to even polynomials, we still get an isomorphism,

$$\phi : \frac{\mathbb{K}[x, y, z]^{\text{even}}}{(x + y + z) \cap \mathbb{K}[x, y, z]^{\text{even}}} \rightarrow \mathbb{K}[x, y]^{\text{even}}.$$

This enables us to define an S_3 -action on $\mathbb{K}[x, y]$ via the formula,

$$\sigma_*(x^{k_1} y^{k_2}) := \phi(\sigma.(x^{k_1} y^{k_2}))$$

for $\sigma \in S_3$. It is indeed a group action, since for $\sigma, \tau \in S_3$, we have,

$$\begin{aligned} (\sigma\tau)_*(x^{k_1} y^{k_2}) &= \phi((\sigma\tau).(x^{k_1} y^{k_2})) = \phi(\sigma.(\tau.(x^{k_1} y^{k_2}))) \\ &= \sigma_*(\phi(\tau.(x^{k_1} y^{k_2}))) = \sigma_*(\tau_*(x^{k_1} y^{k_2})), \end{aligned}$$

and also

$$\text{Id}_*(x^{k_1} y^{k_2}) = \phi(\text{Id}.(x^{k_1} y^{k_2})) = \phi(x^{k_1} y^{k_2}) = x^{k_1} y^{k_2}.$$

Lemma 15. The algebra homomorphism ϕ is S_3 -equivariant, that is, for all $\sigma \in S_3$ and $p \in \mathbb{K}[x, y, z]^{\text{even}}$, we have

$$\sigma_* \phi(p) = \phi(\sigma.p).$$

Proof. First notice that for $f, g \in p[x, y, z]^{\text{even}}$ and $\sigma \in S_3$ we have, $\sigma.(fg) = (-1)^{|\sigma|}(\sigma.f) \cdot (\sigma.g)$. From this, we deduce $\sigma.(x^{k_1}) = (-1)^{|\sigma|(k_1-1)}(\sigma.x)^{k_1}$ and analogously for y and z . Moreover, it is

easy to check by direct computation that $\phi(\sigma.(-x-y)) = \phi(\sigma.z)$ for all $\sigma \in S_3$. Finally, equivariance follows from,

$$\begin{aligned}
\sigma_*\phi(x^{k_1}y^{k_2}z^{k_3}) &= \sigma_*(x^{k_1}y^{k_2}(-x-y)^{k_3}) = \sum_{j=0}^{k_3} (-1)^{k_3} \binom{k_3}{j} \sigma_*(x^{k_1+j}y^{k_2+k_3-j}) \\
&= \sum_{j=0}^{k_3} (-1)^{k_3} \binom{k_3}{j} \phi(\sigma.(x^{k_1+j}y^{k_2+k_3-j})) = \phi(\sigma.\sum_{j=0}^{k_3} (-1)^{k_3} \binom{k_3}{j} x^{k_1+j}y^{k_2+k_3-j}) \\
&= \phi(\sigma.(x^{k_1}y^{k_2}(-x-y)^{k_3})) = \phi(\sigma.(x^{k_1})\sigma.(y^{k_2})\sigma.((-x-y)^{k_3})) \\
&= \phi(\sigma.(x^{k_1})\sigma.(y^{k_2}))\phi(\sigma.((-x-y)^{k_3})) \\
&= (-1)^{|\sigma|(k_3-1)} \phi(\sigma.(x^{k_1})\sigma.(y^{k_2}))\phi(\sigma.(-x-y)^{k_3}) \\
&= (-1)^{|\sigma|(k_3-1)} \phi(\sigma.(x^{k_1})\sigma.(y^{k_2}))\phi(\sigma.z)^{k_3} \\
&= \phi(\sigma.(x^{k_1})\sigma.(y^{k_2}))\phi(\sigma.(z^{k_3})) = \phi(\sigma.(x^{k_1})\sigma.(y^{k_2})\sigma.(z^{k_3})) = \phi(\sigma.(x^{k_1}y^{k_2}z^{k_3})).
\end{aligned}$$

Thus, by linearity, $\sigma_*\phi(p) = \phi(\sigma.p)$ holds for all $p \in \mathbb{K}[x, y, z]^{\text{even}}$ and all $\sigma \in S_3$. \square

Since ϕ preserves the S_3 -action, we may take coinvariants on both sides to obtain an isomorphism

$$\phi : \left(\frac{\mathbb{K}[x, y, z]^{\text{even}}}{(x+y+z) \cap \mathbb{K}[x, y, z]^{\text{even}}} \right)_{S_3} \longrightarrow (\mathbb{K}[x, y]^{\text{even}})_{S_3}.$$

Next, consider the subalgebra A of $\mathbb{K}[x, y]^{\text{even}}$ generated by monomials $x^a y^b$ with $a \leq b$ even, i.e.

$$A := \text{span}(x^a y^b | 0 \leq a \leq b \text{ even})$$

We define recursively a linear map,

$$\psi : \mathbb{K}[x, y]^{\text{even}} \longrightarrow A$$

$$x^a y^b \mapsto \begin{cases} 0, & \text{if } a = 0 \text{ or } b = 0, \\ x^a y^b, & \text{if } a \leq b \text{ both even,} \\ \frac{-1}{a+1} \psi \left(x^{a+1} y^{b-1} + \sum_{\substack{j=1 \\ j \neq a}}^{a+1} \binom{a+1}{j} x^j y^{a+b-j} \right), & \text{if } a \leq b \text{ both odd,} \\ -\psi(x^b y^a) & \text{if } a \geq b. \end{cases}$$

Note that for $a = 1, b \geq 3$ odd, we have $\psi(x y^b) = -\psi(x^2 y^{b-1}) = -x^2 y^{b-1}$. Moreover, for $a \leq b$ both odd, $\psi(x^a y^b)$ may be reformulated as

$$\begin{aligned}
\psi(x^a y^b) &= \frac{-1}{a+1} \left(\psi(x^{a+1} y^{b-1}) + \sum_{\substack{j=2 \\ j \text{ even}}}^{a+1} \binom{a+1}{j} \psi(x^j y^{a+b-j}) \right) \\
&\quad + \sum_{\substack{j=1 \\ j \text{ odd}}}^{a-2} \binom{a+1}{j} \psi(x^j y^{a+b-j}).
\end{aligned}$$

The first term and the first sum are obviously well-defined as ψ acts (up to sign) as the identity on monomials with even exponents while the second sum is well-defined by induction on a (the exponent of x is always strictly smaller than a). Let us denote by B the subspace of A generated by the image of the relations on $\mathbb{K}[x, y]_{S_3}^{\text{even}}$ under ψ , that is,

$$B := \text{span}_{\mathbb{K}}(\psi(\sigma_*v) - \psi(v) | \sigma \in S_3, v \in \mathbb{K}[x, y]^{\text{even}}).$$

In this way, the map ψ induces a well-defined surjection,

$$\psi : \mathbb{K}[x, y]_{S_3}^{\text{even}} \longrightarrow A/B.$$

Lemma 16. Let $i : A/B \rightarrow \mathbb{K}[x, y]_{S_3}^{\text{even}}$, $x^a y^b \mapsto x^a y^b$, $2 \leq a \leq b$ even. We have $\psi \circ i = \text{Id}$ and $i \circ \psi = \text{Id}$. Thus, as vector spaces,

$$A/B \cong \mathbb{K}[x, y]_{S_3}^{\text{even}}.$$

Proof. Since ψ is the identity on $x^a y^b$, $2 \leq a \leq b$ even, clearly $\psi \circ i = \text{Id}$. For the other composition, we proceed by induction. For $a = 1$, $b \geq 3$ odd, we have $i\psi(xy^b) = -i(x^2 y^{b-1}) = -x^2 y^{b-1}$ which can easily be checked to equal xy^b in $\mathbb{K}[x, y]_{S_3}^{\text{even}}$. Moreover, whenever a and b are even, $i\psi(x^a y^b) = x^a y^b \in \mathbb{K}[x, y]_{S_3}^{\text{even}}$. Therefore, for $a \leq b$ odd,

$$\begin{aligned} i\psi(x^a y^b) &= \frac{-1}{a+1} \left(i\psi(x^{a+1} y^{b-1}) + \sum_{\substack{j=2 \\ j \text{ even}}}^{a+1} \binom{a+1}{j} i\psi(x^j y^{a+b-j}) \right) \\ &\quad + \sum_{\substack{j=1 \\ j \text{ odd}}}^{a-2} \binom{a+1}{j} i\psi(x^j y^{a+b-j}) \\ &= \frac{-1}{a+1} \left(x^{a+1} y^{b-1} + \sum_{\substack{j=2 \\ j \text{ even}}}^{a+1} \binom{a+1}{j} x^j y^{a+b-j} + \sum_{\substack{j=1 \\ j \text{ odd}}}^{a-2} \binom{a+1}{j} x^j y^{a+b-j} \right) \\ &= \frac{-1}{a+1} \left(x^{a+1} y^{b-1} + (-x-y)^{a+1} y^{b-1} - (a+1)x^a y^b \right) \\ &= \frac{-1}{a+1} \left(x^{a+1} y^{b-1} - (13)_*(x^{a+1} y^{b-1}) - (a+1)x^a y^b \right) = x^a y^b, \end{aligned}$$

in $\mathbb{K}[x, y]_{S_3}^{\text{even}}$. In the first line we used that by induction all compositions $i \circ \psi$ appearing on the right hand side equal Id . \square

Remark 30. Under the chain of identifications,

$$A/B \cong \mathbb{K}[x, y]_{S_3}^{\text{even}} \cong \left(\frac{\mathbb{K}[x, y, z]^{\text{even}}}{(x+y+z) \cap \mathbb{K}[x, y, z]^{\text{even}}} \right)_{S_3} \cong E_1^{2,-1}(1)$$

a basic element $x^{2i} y^{2j} \in A/B$ corresponds to the theta graph $\theta_{2i,2j} \in E_1^{2,-1}(1)$ that has only two of its strands decorated by $2i$ and $2j$ hairs, respectively (see Figure 3.7). In particular, we find that these graphs generate $E_1^{2,-1}(1)$.

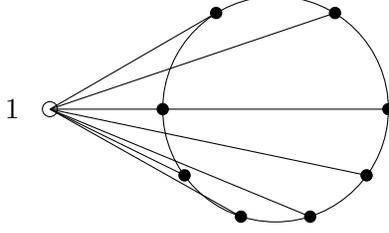


Figure 3.7: The graph $\theta_{2,4}$.

3.2.4 Main theorem and relations among θ -graphs

Our main result states the following.

Theorem 9. Let $\theta_{2i,2j} \in E_1^{2,-1}(1)$ denote the theta graph that has only two of its strands decorated by $2i$ and $2j$ hairs, respectively. The map

$$\begin{aligned} \Phi : \mathcal{F}^2 \mathbf{grt}_1 / \mathcal{F}^3 \mathbf{grt}_1 &\rightarrow E_1^{2,-1}(1) \\ \{\sigma_{2i+1}, \sigma_{2j+1}\} \text{ mod } \mathcal{F}^3 \mathbf{grt}_1 &\mapsto \theta_{2i,2j}. \end{aligned}$$

is an isomorphism of vector spaces.

Proof. We show that Φ is well-defined, injective and surjective. It is a map degree -1 with respect to the weight grading. It will be enough to check its properties on elements of homogeneous degree. Let $k \geq 8$ and

$$L := \sum_{i=1}^{\frac{k-4}{2}} a_i \{\sigma_{2i+1}, \sigma_{k-1-2i}\} = 0 \text{ mod } \mathcal{F}^3 \mathbf{grt}_1.$$

By Remark 19, this is equivalent to $G = \sum_{i=1}^{\frac{k-4}{2}} a_i (\alpha - \beta)^{2i} (\beta - \gamma)^{k-2-2i}$ satisfying $G + (13).G = 0$ and $G + (123).G + (132).G = 0$. Note that the first equation implies that $a_i = -a_{k/2-1-i}$, and thus we may write G as,

$$G = \sum_{i=1}^{\lfloor \frac{k-4}{2} \rfloor} a_i \left((\alpha - \beta)^{2i} (\beta - \gamma)^{k-2-2i} - (\alpha - \beta)^{k-2-2i} (\beta - \gamma)^{2i} \right).$$

Also, remark that $G \in \mathbb{K}[\alpha - \beta, \beta - \gamma]^{\text{even}}$ and that there is an isomorphism of algebras,

$$\frac{\mathbb{K}[x, y, z]^{\text{even}}}{(x + y + z) \cap \mathbb{K}[x, y, z]^{\text{even}}} \longrightarrow \mathbb{K}[\alpha - \beta, \beta - \gamma]^{\text{even}},$$

induced by the surjective map $\mathbb{K}[x, y, z]^{\text{even}} \rightarrow \mathbb{K}[\alpha - \beta, \beta - \gamma]^{\text{even}}$, $x \mapsto \alpha - \beta$, $y \mapsto \beta - \gamma$, $z \mapsto \gamma - \alpha$. Note that the isomorphism is not S_3 -equivariant. Consider the preimage of the second equation for G under this isomorphism. It is represented by,

$$\sum_{i=1}^{\lfloor \frac{k-4}{2} \rfloor} a_i \sum_{\sigma \in S_3} \sigma.x^{2i} y^{k-2-2i} = 0.$$

This is an equation of S_3 -invariant elements, and thus the linear combination on the left is zero also in the space of invariants,

$$\left(\frac{\mathbb{K}[x, y, z]^{\text{even}}}{(x + y + z) \cap \mathbb{K}[x, y, z]^{\text{even}}} \right)^{S_3},$$

which in turn is isomorphic to the space of coinvariants. Moreover, under the corresponding isomorphism, the above equation reads,

$$3! \sum_{i=1}^{\lfloor \frac{k-4}{2} \rfloor} a_i x^{2i} y^{k-2-2i} = 0 \in \left(\frac{\mathbb{K}[x, y, z]^{\text{even}}}{(x + y + z) \cap \mathbb{K}[x, y, z]^{\text{even}}} \right)_{S_3} \cong E_1^{2,-1}(1).$$

Hence, since $\theta_{2i, k-2-2i} = -\theta_{k-2-2i, 2i}$,

$$\begin{aligned} \Phi(L) &= \sum_{i=1}^{\frac{k-4}{2}} a_i \theta_{2i, k-2-2i} = \sum_{i=1}^{\lfloor \frac{k-4}{2} \rfloor} a_i (\theta_{2i, k-2-2i} - \theta_{k-2-2i, 2i}) \\ &= 2 \cdot \sum_{i=1}^{\lfloor \frac{k-4}{2} \rfloor} a_i \theta_{2i, k-2-2i} = 0. \end{aligned}$$

In fact, since we are only dealing with isomorphisms, the argument above may be traced backwards to imply the injectivity of Φ . Since the graphs $\theta_{2i, 2j}$ generate $E_1^{2,-1}(1)$, Φ is clearly surjective. \square

Remark 30 and Theorem 9 establish the following equivalence. For even $k \geq 8$,

$$\sum_{i=1}^{\lfloor \frac{k-4}{2} \rfloor} a_i x^{2i} y^{k-2-2i} = 0 \in A/B \Leftrightarrow \sum_{i=1}^{\lfloor \frac{k-4}{2} \rfloor} a_i \{\sigma_{2i+1}, \sigma_{k-1-2i}\} = 0 \pmod{\mathcal{F}^3 \mathfrak{grt}_1}.$$

On the left hand side, finding non-trivial coefficients a_i amounts to finding elements which are zero in $\mathbb{K}[x, y]_{S_3}^{\text{even}}$, but are sent to a non-trivial linear combination in A by ψ . Natural candidates which might satisfy this condition are expressions of the form $\phi(x^a y^b z^a) = x^a y^b (-x - y)^a$ where $2a + b = k$. Their image $\psi(x^a y^b (-x - y)^a)$ might be equal to zero already in A . If not, however, this will produce a relation in A/B . Here is a list for the first few relations obtained in this way.

$$\begin{aligned} k = 10 : 0 &= \psi(x^3 y^4 (-x - y)^3) = \frac{1}{2}(-3x^4 y^6 + x^2 y^8) \\ k = 14 : 0 &= \psi(x^4 y^6 (-x - y)^4) = \frac{1}{3}(11x^6 y^8 - 7x^4 y^{10} + 2x^2 y^{12}) \\ k = 16 : 0 &= \psi(x^5 y^6 (-x - y)^5) = \frac{1}{12}(26x^6 y^{10} - 25x^4 y^{12} + 8x^2 y^{14}) \\ k = 18 : 0 &= \psi(x^5 y^8 (-x - y)^5) = \frac{1}{2}(-13x^8 y^{10} + 14x^6 y^{12} - 10x^4 y^{14} + 3x^2 y^{16}) \\ k = 20 : 0 &= \psi(x^6 y^8 (-x - y)^6) = \frac{1}{10}(-85x^8 y^{12} + 136x^6 y^{14} - 105x^4 y^{16} + 32x^2 y^{18}) \end{aligned}$$

These yield linear relations for the corresponding two-loops graphs $\theta_{2i, 2j}$. Via the isomorphism Φ , we recover the relations from Remark 18 for the brackets $\{\sigma_{2i+1}, \sigma_{2j+1}\}$ in $\mathcal{F}^2 \mathfrak{grt}_1 / \mathcal{F}^3 \mathfrak{grt}_1$. The coefficients in the list below agree with L. Schneps' calculations [15]. All identities hold modulo $\mathcal{F}^3 \mathfrak{grt}_1$.

$$k = 12 : 0 = -3\{\sigma_5, \sigma_7\} + \{\sigma_3, \sigma_9\}$$

$$k = 16 : 0 = 11\{\sigma_7, \sigma_9\} - 7\{\sigma_5, \sigma_{11}\} + 2\{\sigma_3, \sigma_{13}\}$$

$$k = 18 : 0 = 26\{\sigma_7, \sigma_{11}\} - 25\{\sigma_5, \sigma_{13}\} + 8\{\sigma_3, \sigma_{15}\}$$

$$k = 20 : 0 = -13\{\sigma_9, \sigma_{11}\} + 14\{\sigma_7, \sigma_{13}\} - 10\{\sigma_5, \sigma_{15}\} + 3\{\sigma_3, \sigma_{17}\}$$

$$k = 22 : 0 = -85\{\sigma_9, \sigma_{13}\} + 136\{\sigma_7, \sigma_{15}\} - 105\{\sigma_5, \sigma_{17}\} + 32\{\sigma_3, \sigma_{19}\}$$

In weight $k = 24$, we expect, from Goncharov's [8], and Ihara and Takao's [9] formula, two independent linear relations for the brackets $\{\sigma_{2i+1}, \sigma_{24-1-2i}\}$ modulo $\mathcal{F}^3 \mathbf{grt}_1$. These can be found by calculating for instance $\psi(x^6 y^{10} (-x-y)^6)$ and $\psi(x^7 y^8 (-x-y)^7)$. The relations in $\mathcal{F}^2 \mathbf{grt}_1 / \mathcal{F}^3 \mathbf{grt}_1$ obtained in this way are,

$$\begin{aligned} 20\{\sigma_{11}, \sigma_{13}\} - 33\{\sigma_9, \sigma_{15}\} + 44\{\sigma_7, \sigma_{17}\} - 33\{\sigma_5, \sigma_{19}\} + 10\{\sigma_3, \sigma_{21}\} &= 0 \\ -672\{\sigma_{11}, \sigma_{13}\} + 915\{\sigma_9, \sigma_{15}\} - 1106\{\sigma_7, \sigma_{17}\} + 805\{\sigma_5, \sigma_{19}\} - 242\{\sigma_3, \sigma_{21}\} &= 0. \end{aligned}$$

3.3 Relations to M. Kontsevich's graph complex

We claim that the equivalence between \mathbf{grt}_1 in depth two modulo higher depth and the cohomology of the space of two-loop graphs is not a coincidence. For this, let us recall one further graph complex, GC.

3.3.1 Definitions

The graph complex GC is a variant of M. Kontsevich's graph complex ([10],[11],[12]). We follow T. Willwacher's paper [17].

Definition 20. Let Γ be an undirected graph with N labeled vertices and k edges satisfying the following properties:

1. All vertices have valence at least three.
2. There is a linear order on the set of edges.
3. Γ has no simple loops.

We denote by $\mathbf{Gra}_2(N, k)$ the graded vector space spanned by isomorphism classes of connected graphs satisfying the conditions above, modulo the relation $\Gamma \cong (-1)^{|\sigma|} \Gamma^\sigma$, where Γ^σ differs from Γ just by a permutation $\sigma \in S_k$ on the order of the edges. The degree of such a graph Γ is given by

$$\deg_{\mathbf{Gra}_2} \Gamma = -k.$$

Set,

$$\mathbf{Gra}_2(N) := \bigoplus_{k \geq 0} \mathbf{Gra}_2(N, k).$$

The collection $\{\text{Gra}_2(N)\}_{N \geq 1}$ naturally defines an operad Gra_2 in the category of graded vector spaces. For $\Gamma \in \text{Gra}_2(N)$, the S_N -action permutes the labels of the vertices. For $r, s \geq 1$, $\Gamma_1 \in \text{Gra}_2(r)$ and $\Gamma_2 \in \text{Gra}_2(s)$, the operadic composition $\Gamma_1 \circ_j \Gamma_2 \in \text{Gra}_2(r + s - 1)$ is given by inserting the graph Γ_2 at vertex j of Γ_1 and summing over all ways of reconnecting the edges incident to vertex j in Γ_1 to vertices of Γ_2 . As in the case of ICG, we ask that the order on the set of edges of $\Gamma_1 \circ \Gamma_2$ is such that all edges of Γ_1 come before those of Γ_2 while the respective orderings are left unaltered. Next, define,

$$\text{GC} := \bigoplus_{N \geq 1} (\text{Gra}_2(N)[2 - 2N])^{S_N}.$$

The space GC carries the structure of a differential graded Lie algebra. The degree of a graph $\Gamma \in \text{GC}$ with k edges and N vertices is

$$\deg \Gamma = -2 - k + 2N.$$

For the Lie bracket, consider the operadic pre-Lie product on Gra_2 ,

$$\Gamma_1 \circ \Gamma_2 = \sum_{j=1}^r \Gamma_1 \circ_j \Gamma_2.$$

Using this, the Lie bracket on GC is defined on homogeneous elements via,

$$[\Gamma_1, \Gamma_2] := \Gamma_1 \circ \Gamma_2 - (-1)^{\deg \Gamma_1 \cdot \deg \Gamma_2} \Gamma_2 \circ \Gamma_1.$$

The differential d is given by vertex splitting, where again we ask that the newly created edge is placed last in the ordering of the edges.

Let $\gamma \in \text{GC}$. For any vertex v of γ , we denote its valence by $\text{val}(v)$. Set,

$$\begin{aligned} m_\gamma &:= \max\{\text{val}(v) \mid v \text{ vertex of } \gamma\} \\ v_\gamma &:= \#\text{vertices of } \gamma. \end{aligned}$$

The number of vertices v_γ minus the ‘‘maximal valence’’ m_γ then defines the following descending filtration on GC .

$$\mathcal{F}^p \text{GC} := \{\gamma \in \text{GC} \mid v_\gamma - m_\gamma \geq p\}$$

where $p \geq 1$. Note that $\mathcal{F}^1 \text{GC} = \text{GC}$.

Lemma 17. The filtration is compatible with the differential graded Lie algebra structure on GC , i.e. for all $p, q \geq 1$,

$$\begin{aligned} d(\mathcal{F}^p \text{GC}) &\subset \mathcal{F}^p \text{GC} \\ [\mathcal{F}^p \text{GC}, \mathcal{F}^q \text{GC}] &\subset \mathcal{F}^{p+q} \text{GC} \end{aligned}$$

Thus, $\mathcal{F}^p H^0(\text{GC}) := \{\Gamma \in H^0(\text{GC}) \mid v_\Gamma - m_\Gamma \geq p\}$ defines a descending filtration on $H^0(\text{GC})$.

Proof. Let $\Gamma \in \mathcal{F}^p \text{GC}$. Note that the differential decreases the valence of a vertex at least by one while also creating one additional vertex (whose valence will be at most equal to the valence of the splitted vertex minus one). Therefore, $m_{d\Gamma} \leq m_\Gamma$ and $v_{d\Gamma} = v_\Gamma + 1$, and,

$$v_{d\Gamma} - m_{d\Gamma} \geq v_\Gamma + 1 - m_\Gamma \geq p + 1 \geq p.$$

Let $\Gamma_1 \in \mathcal{F}^{p_1}\text{GC}$ and $\Gamma_2 \in \mathcal{F}^{p_2}\text{GC}$. Note that the insertion operation $\Gamma_1 \circ \Gamma_2$ decreases the total number of vertices by 1. Hence, $v_{[\Gamma_1, \Gamma_2]} = v_{\Gamma_1} + v_{\Gamma_2} - 1$. On the other hand, assuming the vertices of maximal valence are labeled by 1, the expression,

$$\Gamma_1 \circ_1 \Gamma_2 - \Gamma_2 \circ_1 \Gamma_1,$$

produces two graphs with maximal valence $m_{\Gamma_1} + m_{\Gamma_2}$. One is obtained when the vertex of maximal valence in Γ_1 is replaced by Γ_2 and all “loose” edges are reattached to the vertex of maximal valence in Γ_2 . The other stems from an analogous construction, but with the roles of Γ_1 and Γ_2 exchanged. Luckily, due to symmetry, these two terms always cancel each other out. All other insertions yield $m_{[\Gamma_1, \Gamma_2]} \leq m_{\Gamma_1} + m_{\Gamma_2} - 1$, and thus

$$v_{[\Gamma_1, \Gamma_2]} - m_{[\Gamma_1, \Gamma_2]} \geq v_{\Gamma_1} + v_{\Gamma_2} - 1 - (m_{\Gamma_1} + m_{\Gamma_2} - 1) \geq p_1 + p_2,$$

from which the statement follows. \square

Remark 31. The proof of Lemma 17 shows that $d(\mathcal{F}^p\text{GC}) \subset \mathcal{F}^{p+1}\text{GC}$. Therefore the associated graded has trivial differential.

Remark 32. A graph in GC is called 1-vertex irreducible if it stays connected after deletion of any single vertex. Denote by GC^{1-vi} the subcomplex of (GC, d) spanned by 1-vertex irreducible graphs. As shown in [4], the subcomplex GC^{1-vi} is quasi-isomorphic to GC . We may thus assume that all cohomology classes we work with are represented by 1-vertex irreducible graphs.

3.3.2 Compatibility with T. Willwacher’s isomorphism

The graph complex GC is related to the Grothendieck-Teichmüller Lie algebra via its degree zero cohomology. More precisely, T. Willwacher’s important result states the following.

Theorem 10 ([17], Theorem 1.1.). There is an isomorphism of Lie algebras,

$$\omega : H^0(\text{GC}) \xrightarrow{\cong} \mathfrak{grt}_1.$$

The aim of this section is to prove the following statement.

Theorem 11. T. Willwacher’s isomorphism ω induces an isomorphism,

$$\bar{\omega} : \mathcal{F}^2 H^0(\text{GC}) / \mathcal{F}^3 H^0(\text{GC}) \longrightarrow \mathcal{F}^2 \mathfrak{grt}_1 / \mathcal{F}^3 \mathfrak{grt}_1.$$

As a consequence, we obtain identifications,

$$\mathcal{F}^2 H^0(\text{GC}) / \mathcal{F}^3 H^0(\text{GC}) \cong \mathcal{F}^2 \mathfrak{grt}_1 / \mathcal{F}^3 \mathfrak{grt}_1 \cong E_1^{2, -1}(1).$$

Conjecturally, ω should induce an isomorphism on the whole associated graded. Moreover, the hope is that the degree p part of the associated graded may be identified with $E_1^{p, -p+1}(1)$, that is, the cohomology of p -loop graphs in $\text{ICG}(1)$ of degree 1. This would yield the following isomorphisms,

$$\mathcal{F}^p H^0(\text{GC}) / \mathcal{F}^{p+1} H^0(\text{GC}) \cong E_1^{p, -p+1}(1) \cong \mathcal{F}^p \mathfrak{grt}_1 / \mathcal{F}^{p+1} \mathfrak{grt}_1,$$

which in turn would allow us to study the Grothendieck-Teichmüller Lie algebra’s relations in higher depth by computing the cohomology of the space of higher loop order graphs in $\text{ICG}(1)$. Notice



Figure 3.8: The wheel graphs w_3 and w_5 .

however that for now $E_1^{p,-p+1}(1)$ is unknown for $p \geq 3$ and that it is a hard combinatorial problem to compute it.

To start, recall from ([17], Proposition 9.1.) that all representatives of the cohomology classes s_{2k+1} in $H^0(\text{GC})$ corresponding to the conjectural generators σ_{2k+1} under the isomorphism ω have coefficient equal to 1 in front of the wheel graph w_{2k+1} with $2k + 1$ spokes (see Figure 3.8).

Lemma 18. Let $\gamma \in \text{GC}$ be a graph of degree zero satisfying $v_\gamma - m_\gamma = 1$. Then γ is a wheel graph. It follows that $s_{2k+1} = w_{2k+1} + R_{2k+1}$, where $R_{2k+1} \in \mathcal{F}^2 H^0(\text{GC})$.

Proof. Let $\gamma \in \text{GC}$ be a graph of degree zero with $v_\gamma = m_\gamma + 1$. This implies that there must be one vertex which is connected to all other vertices. As $\deg \gamma = -2 - \#\text{edges} + 2\#\text{vertices} = 0$, we have $\#\text{edges} = 2m_\gamma$ and since we already fixed a total of m_γ edges (attached to the vertex of maximal valence), we need to decide how to place the other m_γ edges. It turns out that in order to respect 1-vertex irreducibility and the condition that all vertices must be of valence at least three, the remaining edges must be placed as to form such a wheel graph. Since by ([17], Proposition 9.1.) w_{2k+1} corresponds to the $\text{ad}_x^{2k}(y)$ term in σ_{2k+1} under the isomorphism ω , s_{2k+1} cannot contain any wheel graph other than w_{2k+1} , and thus, $s_{2k+1} = w_{2k+1} + R_{2k+1}$, where $R_{2k+1} \in \mathcal{F}^2 H^0(\text{GC})$. \square

Consider the space of brackets of the elements s_{2k+1} in $H^0(\text{GC})$ corresponding to the conjectural generators σ_{2k+1} of grt_1 , that is,

$$\bigoplus_{i,j \geq 1} \mathbb{K} \cdot [s_{2i+1}, s_{2j+1}] \subset \mathcal{F}^2 H^0(\text{GC}) \subset H^0(\text{GC}).$$

Since the filtration on $H^0(\text{GC})$ is compatible with the Lie algebra structure, the space of brackets is a subspace of $\mathcal{F}^2 H^0(\text{GC})$. Moreover, also by compatibility, the bracket descends to a map,

$$\begin{aligned} (\mathcal{F}^1 H^0(\text{GC}) / \mathcal{F}^2 H^0(\text{GC}))^{\otimes 2} &\longrightarrow \mathcal{F}^2 H^0(\text{GC}) / \mathcal{F}^3 H^0(\text{GC}) \\ \bar{s}_{2i+1} \otimes \bar{s}_{2j+1} &\longmapsto [\bar{s}_{2i+1}, \bar{s}_{2j+1}] = [\bar{w}_{2i+1}, \bar{w}_{2j+1}] = \overline{[w_{2i+1}, w_{2j+1}]}. \end{aligned}$$

In this quotient, the element $[w_{2i+1}, w_{2j+1}]$ is represented by the difference of “bowtie”-graphs as in Figure 3.9.

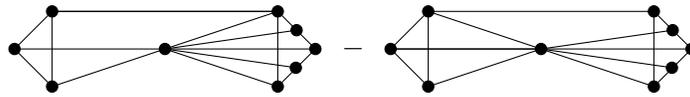


Figure 3.9: Modulo $\mathcal{F}^3 H^0(\text{GC})$, $[w_3, w_5]$ is given by a nonzero multiple of graphs as depicted above. Individually, we shall refer to such a graphs as “bowtie”-graphs.

It is obtained by first inserting w_{2j+1} into the highest valent vertex of w_{2i+1} and connecting all but one of the edges (to preserve 1-vertex irreducibility) to the highest valent vertex of w_{2j+1} .

The second graph is constructed by the same procedure with the roles of w_{2i+1} and w_{2j+1} reversed. One can check that all other graphs produced in $[w_{2i+1}, w_{2j+1}]$ lie in $\mathcal{F}^3 H^0(\text{GC})$. Next consider the composition of the inclusion with the quotient map,

$$\bigoplus_{i,j \geq 1} \mathbb{K} \cdot [s_{2i+1}, s_{2j+1}] \xrightarrow{i} \mathcal{F}^2 H^0(\text{GC}) \xrightarrow{\pi} \mathcal{F}^2 H^0(\text{GC}) / \mathcal{F}^3 H^0(\text{GC}).$$

It induces an injective map,

$$\bigoplus_{i,j \geq 1} \mathbb{K} \cdot [s_{2i+1}, s_{2j+1}] / \ker(\pi \circ i) \hookrightarrow \mathcal{F}^2 H^0(\text{GC}) / \mathcal{F}^3 H^0(\text{GC}).$$

For the next step, we need the following technical tool.

Proposition 9 ([17], Section 6.4.1.). Let

$$\begin{aligned} (-)_1 : \text{GC} &\rightarrow \text{graphs}(1) \\ \Gamma &\mapsto \Gamma_1, \end{aligned}$$

where Γ_1 is obtained by summing over all ways of marking one vertex of Γ as external. This map satisfies,

$$d\Gamma_1 - (d\Gamma)_1 = \left(\begin{array}{c} \circ \\ \text{---} \\ \circ \end{array} \right)_2 \circ_2 \Gamma,$$

where the right hand side means that we insert Γ in vertex number two and sum over all possible ways of reconnecting the edge attached to vertex one to vertices of Γ .

Lemma 19. The map $(-)_1$ induces a surjective map,

$$\mathcal{F}^2 H^0(\text{GC}) / \mathcal{F}^3 H^0(\text{GC}) \longrightarrow H^1(\mathcal{F}^2 \text{ICG}(1) / \mathcal{F}^3 \text{ICG}(1), d_0) = E_1^{2,-1}(1).$$

Proof. Let us first check that the map is well-defined. For this, consider $\Gamma \in \text{GC}$. By Remark 32, we may assume that it is 1-vertex irreducible. This ensures that Γ_1 lies in $\text{ICG}(1)$. Next, project Γ_1 onto its two-loop part via the quotient map,

$$\pi : \text{ICG}(1) \rightarrow \mathcal{F}^2 \text{ICG}(1) / \mathcal{F}^3 \text{ICG}(1).$$

Note that there are no two-loop graphs of weight one in $\text{ICG}(1)$ (i.e. with only one edge adjacent to the unique external vertex). Thus, by Proposition 9, the composition $\pi \circ (-)_1$ is a chain map, that is,

$$d\pi\Gamma_1 = \pi(d\Gamma)_1,$$

and it induces a map on the level of cohomology,

$$H^0(\text{GC}) \rightarrow H^1(\mathcal{F}^2 \text{ICG}(1) / \mathcal{F}^3 \text{ICG}(1), d_0) = E_1^{2,-1}(1). \quad (3.3)$$

Let now $\Gamma \in \mathcal{F}^2 H^0(\text{GC})$ be such that it represents a non-zero equivalence class in the quotient

$$\mathcal{F}^2 H^0(\text{GC}) / \mathcal{F}^3 H^0(\text{GC}).$$

If m denotes the vertex of maximal valence of Γ and N the number of vertices of Γ , we have $N - \text{val}(m) \geq 2$. In particular, this means that there is at least one vertex which is not directly

connected to m . Moreover, the part Γ_2 of Γ which represents the non-zero element in the quotient has exactly one such vertex. Call it w , and assume the graph we are considering has k edges. Using the elementary formula,

$$\sum_{v \text{ vertex of } \Gamma_2} \text{val}(v) = 2k,$$

and the fact that $k = 2N - 2$ (since Γ is of degree zero in GC), and that $\text{val}(m) = N - 2$, we find,

$$\sum_{v \text{ vertex of } \Gamma_2} \text{val}(v) = \text{val}(m) + \text{val}(w) + \sum_{v \neq w, m} \text{val}(v) = 2k = 4N - 4. \quad (3.4)$$

As all vertices are at least trivalent,

$$\sum_{v \neq w, m} \text{val}(v) \geq 3(N - 2).$$

If there is equality, that is all vertices other than m and w are trivalent, we find $\text{val}(w) = 4$. If the inequality is strict, first of all,

$$\text{val}(m) + \text{val}(w) + 3(N - 2) < 4N - 4,$$

and thus,

$$3 \leq \text{val}(w) < N + 2 - m = 4,$$

i.e. $\text{val}(w) = 3$. Equation (3.4) then reads,

$$\text{val}(m) + 3 + \sum_{v \neq w, m} \text{val}(v) = 4N - 4,$$

which implies that there is exactly one 4-valent vertex, and with the exception of m , the rest is trivalent. From this we deduce, that the only graphs which represent non-zero elements in the quotient above are of the following form. Either the non-maximal valent vertices make up two loops which do not share an edge and which are only connected through one edge, and all except one non-maximal valent vertices are connected to m (i.e. a “bowtie”-graph), or the non-maximal valent vertices make up two loops which share at least one edge and again all but one non-maximal valent vertices are connected to m . Examples of the two cases are depicted in Figure 3.10. Applying $(-)_1$ to Γ , we find that the only part which produces a two-loop graph in $\text{ICG}(1)$ is given by the part which is non-zero in the quotient $\mathcal{F}^2 H^0(\text{GC})/\mathcal{F}^3 H^0(\text{GC})$, and only when we mark m as external. The map from (3.3) therefore factors through the projection,

$$H^0(\text{GC}) \rightarrow \mathcal{F}^2 H^0(\text{GC})/\mathcal{F}^3 H^0(\text{GC}),$$

and thus induces a map

$$\mathcal{F}^2 H^0(\text{GC})/\mathcal{F}^3 H^0(\text{GC}) \longrightarrow H^1(\mathcal{F}^2 \text{ICG}(1)/\mathcal{F}^3 \text{ICG}(1), d_0) = E_1^{2,-1}(1).$$

It is surjective, since (for $i, j \geq 1$) all the $\theta_{2i, 2j}$ -graphs which generate $E_1^{2,-1}(1)$ lie in the image. They are obtained by marking the maximal valent vertex of a graph similar to the one on the right in Figure 3.10 as external. □

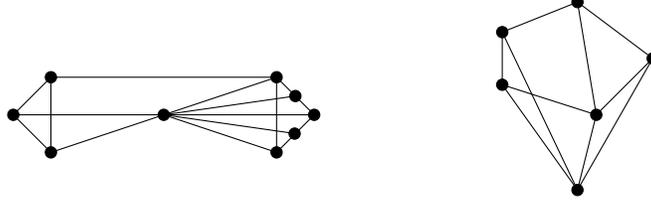


Figure 3.10: Graphs which represent a non-zero class in the quotient $\mathcal{F}^2 H^0(\text{GC})/\mathcal{F}^3 H^0(\text{GC})$.

Proposition 10. The map from Lemma 19 restricted to the quotient space of brackets modulo $\ker(\pi \circ i)$ is an isomorphism. It can be normalized to map generators to generators, i.e.

$$\Omega : \bigoplus_{i,j \geq 1} \mathbb{K} \cdot [s_{2i+1}, s_{2j+1}] / \ker(\pi \circ i) \rightarrow E_1^{2,-1}(1)$$

$$\overline{[s_{2i+1}, s_{2j+1}]} \mapsto \theta_{2i,2j}$$

Proof. We have already seen that in the quotient $\mathcal{F}^2 H^0(\text{GC})/\mathcal{F}^3 H^0(\text{GC})$ brackets $[s_{2i+1}, s_{2j+1}]$ are represented by the differences of “bowtie”-graphs. Applying the map from Lemma 19 to this class, we obtain a difference of two-loop graphs in $\text{ICG}(1)$ (see Figure 3.11). Denote it by $D_{2i+1,2j+1}$. It is cohomologous to a multiple of the theta graph $\theta_{2i,2j}$ (see Figure 3.7).

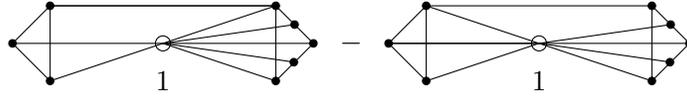


Figure 3.11: The vertex with highest valence is marked as external to obtain an element $D_{3,5} \in \text{ICG}(1)$.

To see this, denote by $E_{2i,2j} \in \text{ICG}(1)$ the “figure-8” two-loop graph of degree zero as depicted in Figure 3.12. The differential d_0 splits the vertex of valence five in 10 different ways. The four graphs which are still of “figure-8” type graphs are zero due to symmetry. By ordering the edges in a consistent way, we are left with,

$$d_0 E_{2i,2j} = D_{2i+1,2j+1} + 4 \cdot \theta_{2i,2j}.$$

Since the theta-graphs $\theta_{2i,2j}$ generate $E_1^{2,-1}(1)$, Ω is clearly surjective, and we can normalize it such that it satisfies $\Omega(\overline{[s_{2i+1}, s_{2j+1}]}) = \theta_{2i,2j}$.

For the injectivity, assume that for $k \geq 8$, $A := \sum_{i=1}^{\lfloor \frac{k-4}{2} \rfloor} a_i [s_{2i+1}, s_{k-1-2i}] \in \ker(\Omega)$, i.e.

$$\Omega(A) = \sum_{i=1}^{\lfloor \frac{k-4}{2} \rfloor} a_i D_{2i+1,k-1-2i} = 0 \in E_1^{2,-1}(1).$$

This implies the existence of some $B \in \mathcal{F}^2 \text{ICG}(1)/\mathcal{F}^3 \text{ICG}(1)$ of degree 0 such that $\Omega(A) = d_0(B)$. Note that $\Omega(A)$ and B are both of weight k , and by degree reasons of all vertices (internal and external) the external vertex is the one of highest valence. Next, consider the graphs in GC obtained by, first, marking the unique external vertex of $\Omega(A)$ and of B as internal again, and then

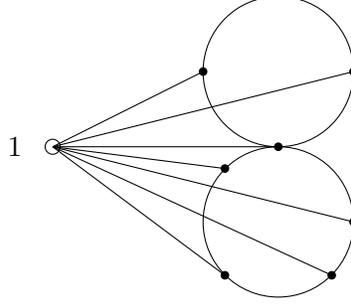


Figure 3.12: The graph $E_{2,4}$ which is mapped $D_{3,5} + 4 \cdot \theta_{2,4}$ by d_0 .

summing over all ways of labeling the vertices to make them indistinguishable. Call them A' and B' , respectively. Note that A' is a scalar multiple of A . Apply the differential of GC on B' . We remark that the part (denoted by Q_1 in the following) of the differential acting on the vertex that was previously external will lie in $\mathcal{F}^3 H^0(\text{GC})$, since it will invariably produce at least two vertices which are not directly connected to the vertex of maximal valence. The remaining part (denoted by Q_2) of the differential will only operate on vertices which were previously internal. It mimics the differential d_0 on $\text{ICG}(1)$, and thus produces a multiple of A' , hence a multiple of A . To summarize, the reasoning above implies that there is a $\lambda \in \mathbb{K}^\times$ such that $dB' = \lambda A + Q_1$, and thus,

$$A = \lambda^{-1} dB' \in \mathcal{F}^2 H^0(\text{GC}) / \mathcal{F}^3 H^0(\text{GC}),$$

Equivalently, $A = 0 \in \mathcal{F}^2 H^0(\text{GC}) / \mathcal{F}^3 H^0(\text{GC})$, and Ω is injective. \square

Corollary 4. The map $(-)_1$ from Lemma 19 is an isomorphism, and we may thus identify the following spaces,

$$\bigoplus_{i,j \geq 1} \mathbb{K} \cdot [s_{2i+1}, s_{2j+1}] / \ker(\pi \circ i) \cong \mathcal{F}^2 H^0(\text{GC}) / \mathcal{F}^3 H^0(\text{GC}) \cong E_1^{2,-1}(1).$$

Proof. The map Ω describes the composition,

$$\bigoplus_{i,j \geq 1} \mathbb{K} \cdot [s_{2i+1}, s_{2j+1}] / \ker(\pi \circ i) \hookrightarrow \mathcal{F}^2 H^0(\text{GC}) / \mathcal{F}^3 H^0(\text{GC}) \twoheadrightarrow E_1^{2,-1}(1).$$

Since it is an isomorphism, the statement easily follows. \square

Proof of Theorem 11. Clearly, the foregoing discussion implies,

$$\begin{aligned} \mathcal{F}^2 H^0(\text{GC}) / \mathcal{F}^3 H^0(\text{GC}) &\stackrel{\Omega}{\cong} E_1^{2,-1}(1) \stackrel{\Phi^{-1}}{\cong} \mathcal{F}^2 \mathfrak{grr}_1 / \mathcal{F}^3 \mathfrak{grr}_1 \\ \overline{[s_{2i+1}, s_{2j+1}]} &\mapsto \theta_{2i,2j} \mapsto \{\sigma_{2i+1}, \sigma_{2j+1}\} \pmod{\mathcal{F}^3 \mathfrak{grr}_1}, \end{aligned}$$

from which it follows that $\Phi^{-1} \circ \Omega = \omega$. \square

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Chapter 4

On a homotopy version of the Duflo isomorphism

Abstract

For a finite dimensional Lie algebra \mathfrak{g} , the Duflo map $S\mathfrak{g} \rightarrow U\mathfrak{g}$ defines an isomorphism of \mathfrak{g} -modules. On \mathfrak{g} -invariant elements it gives an isomorphism of algebras. Moreover, it induces an isomorphism of algebras on the level of Lie algebra cohomology $H(\mathfrak{g}, S\mathfrak{g}) \rightarrow H(\mathfrak{g}, U\mathfrak{g})$. However, as shown by J. Alm and S. Merkulov, it cannot be extended in a universal way to an A_∞ -isomorphism between the corresponding Chevalley-Eilenberg complexes. In this paper, we give an elementary and self-contained proof of this fact using a version of M. Kontsevich's graph complex.

Introduction

For a finite dimensional Lie algebra \mathfrak{g} , the Duflo map $\text{Duf} : S\mathfrak{g} \rightarrow U\mathfrak{g}$ is an isomorphism of \mathfrak{g} -modules. It is defined as the composition of the symmetrization map with the Duflo element, the formal power series on \mathfrak{g} ,

$$\det \left(\frac{e^{\text{ad}_x/2} - e^{-\text{ad}_x/2}}{\text{ad}_x} \right)^{1/2},$$

viewed as a differential operator of infinite order acting on $S\mathfrak{g}$. It is a non-trivial fact that when restricted to \mathfrak{g} -invariant elements, the map $\text{Duf} : S\mathfrak{g}^{\mathfrak{g}} \rightarrow U\mathfrak{g}^{\mathfrak{g}}$ is an isomorphism of algebras [9]. Moreover, it induces an isomorphism of algebras $H(\mathfrak{g}, S\mathfrak{g}) \rightarrow H(\mathfrak{g}, U\mathfrak{g})$ on the level of Lie algebra cohomology ([17], [14]). In particular, this implies that on chains $\text{Duf} : C(\mathfrak{g}, S\mathfrak{g}) \rightarrow C(\mathfrak{g}, U\mathfrak{g})$ respects the algebra structures up to homotopy. More precisely, there exists a map $\text{Duf}_2 : C(\mathfrak{g}, S\mathfrak{g})^{\otimes 2} \rightarrow C(\mathfrak{g}, U\mathfrak{g})$ which measures the failure of the Duflo map to be an algebra morphism, i.e. it satisfies,

$$\text{Duf}(m_{C(\mathfrak{g}, S\mathfrak{g})}) = m_{C(\mathfrak{g}, U\mathfrak{g})}(\text{Duf} \otimes \text{Duf}) + d(\text{Duf}_2) + \text{Duf}_2(d \otimes 1 + 1 \otimes d).$$

It is natural to require Duf_2 to satisfy further compatibility conditions up to homotopy. However, it turns out that this procedure may not be extended (in a universal way) to higher homotopies of arbitrary order. In other words, the Duflo map does not extend to an A_∞ -isomorphism. This was shown by J. Alm in [1],[2], and later again in collaboration with S. Merkulov [3]. They work with a variety of techniques from the theory of graph complexes, and use deep results by T. Willwacher

[18]. The purpose of this text is to present a self-contained, elementary and (hopefully) more accessible proof of this fact using yet another variant of M. Kontsevich’s graph complex.

Denote by \tilde{m}_{Duf} the product on $C(\mathfrak{g}, S\mathfrak{g})$ defined by pulling back the product on $C(\mathfrak{g}, S\mathfrak{g})$ via the Duflo map. J. Alm and S. Merkulov’s theorem ([2], Proposition 5.3.0.10) may then be reformulated as follows.

Theorem. There does not exist a universal (i.e. independent of the specific choice of the Lie algebra \mathfrak{g}) A_∞ -isomorphism,

$$f : (C(\mathfrak{g}, S\mathfrak{g}), d_{C(\mathfrak{g}, S\mathfrak{g})}, m_{C(\mathfrak{g}, S\mathfrak{g})}) \rightarrow (C(\mathfrak{g}, S\mathfrak{g}), d_{C(\mathfrak{g}, S\mathfrak{g})}, \tilde{m}_{\text{Duf}}),$$

whose first component f_1 is the identity.

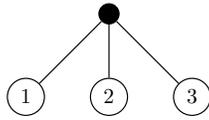
The strategy of our proof is the following. We begin by introducing a version of M. Kontsevich’s graph complex [12]. For $n \geq 1$, we denote by $\mathbf{dgraphs}(n)$ the graded vector space spanned by directed graphs having n “external” vertices (labeled, of arbitrary valence) and possibly some “internal” vertices (unlabeled, at most trivalent) modulo a set of relations encoding the Lie algebra structure of \mathfrak{g} . The product $\prod_{n \geq 1} \mathbf{dgraphs}(n)$ carries the structure of a graded Lie algebra. Next, we recall a variant of M. Kontsevich’s representation $B : \mathbf{dgraphs}(n) \rightarrow \text{Hom}(C(\mathfrak{g}, S\mathfrak{g})^{\otimes n}, C(\mathfrak{g}, S\mathfrak{g}))$. It turns out that both $m_{C(\mathfrak{g}, S\mathfrak{g})}$ and $\tilde{m}_{\text{Duf}} \in \text{Hom}(C(\mathfrak{g}, S\mathfrak{g})^{\otimes 2}, C(\mathfrak{g}, S\mathfrak{g}))$ may be described via this representation. In particular, \tilde{m}_{Duf} corresponds precisely to M. Kontsevich’s celebrated star product on $C^\infty(\mathfrak{g}^*)$, i.e. smooth functions on the dual Lie algebra \mathfrak{g}^* , viewed as a Poisson manifold ([12],[16]). Within this setting, we find that *universal* A_∞ -structures on $C(\mathfrak{g}, S\mathfrak{g})$ are in bijection with Maurer-Cartan elements of the graded Lie algebra $\prod_{n \geq 1} \mathbf{dgraphs}(n)$, that is, (linear combinations of) graphs α of degree one, satisfying the equation,

$$[\alpha, \alpha] = 0.$$

Moreover, A_∞ -isomorphic structures correspond to gauge equivalent Maurer-Cartan elements. This reduces the question of the existence of such an A_∞ -isomorphism to a combinatorial problem in graph theory. If α_0 and α_{Duf} are the Maurer-Cartan elements inducing the two products on $C(\mathfrak{g}, S\mathfrak{g})$, then for them to be gauge equivalent means that there is a $\xi \in \prod_{n \geq 1} \mathbf{dgraphs}(n)$ of degree zero such that,

$$\alpha_0 = \alpha_{\text{Duf}} + [\xi, \alpha_{\text{Duf}}] + \frac{1}{2!}[\xi, [\xi, \alpha_{\text{Duf}}]] + \cdots = e^{[\xi, -]} \alpha_{\text{Duf}}.$$

The graphs α_0 and α_{Duf} are given explicitly up to a certain order. We may thus define ξ inductively in such a way that it deforms α_{Duf} into α_0 . In this process, we find that the graph,



defines an obstruction class which cannot be forced to vanish by gauge transformations.

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4.1 Preliminaries

4.1.1 A_∞ -structures

The material presented in this section is well-known. We follow the textbook by J.-L. Loday and B. Vallette [13] and B. Keller's exposition [11]. We work over a field \mathbb{K} of characteristic zero. Let A be a graded vector space. The *suspension* sA of A is defined via the degree shift $(sA)_p = A_{p+1}$. Recall also the *suspension map* $s : A \rightarrow sA$, the canonical map of degree -1 which sends a to a . This sign convention corresponds to the one in [11], and *not* to the one in [13].

Definition 21. An A_∞ -structure on A is a collection of maps $m = \{m_n : A^{\otimes n} \rightarrow A\}_{n \geq 1}$ of degree $2 - n$ satisfying for $n \geq 1$,

$$\sum (-1)^{r+st} m_u(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0, \quad (4.1)$$

where the sum runs over all decompositions $n = r + s + t$ and $u := r + 1 + t$.

Definition 22. For (A, m) , (B, l) two A_∞ -algebras, a morphism of A_∞ -algebras $f : A \rightarrow B$ is a collection of maps $\{f_n : A^{\otimes n} \rightarrow B\}_{n \geq 1}$ of degree $1 - n$ satisfying for $n \geq 1$,

$$\sum (-1)^{r+st} f_u(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = \sum (-1)^s l_r(f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_r}), \quad (4.2)$$

where the first sum runs over all decompositions $n = r + s + t$ and $u := r + 1 + t$, and the second sum runs over all $1 \leq r \leq n$ and all decompositions $n = i_1 + \cdots + i_r$. Also,

$$s = (r-1)(i_1-1) + (r-2)(i_2-1) + \cdots + 2(i_{r-2}-1) + (i_{r-1}-1).$$

Given two A_∞ -morphisms $f : A \rightarrow B$ and $g : B \rightarrow C$, the n -th component of their composition is defined via the identity,

$$(f \circ g)_n = \sum (-1)^s f_r(g_{i_1} \otimes \cdots \otimes g_{i_r}),$$

where the summation runs over the same elements as in equation (4.2).

Remark 33. We adopt the notation as in the two previous definitions. Note that m_1 is of degree one and satisfies $m_1^2 = 0$, and therefore defines a differential on A . Moreover, as $f_1 m_1 = l_1 f_1$, f_1 defines a chain map between the complexes (A, m_1) and (B, l_1) .

Definition 23. A morphism of A_∞ -algebras $f = \{f_n\}_{n \geq 1}$ is called *isomorphism* (*quasi-isomorphism*) if f_1 is an isomorphism (quasi-isomorphism). An A_∞ -algebra (A, m) is called *strict* if $m_n = 0$ for all $n \geq 3$.

4.1.2 A_∞ -structures via the convolution Lie algebra

Consider the endomorphisms operad of A , $\text{End}_A = \{\text{Hom}(A^{\otimes n}, A)\}_{n \geq 1}$. For $f \in \text{Hom}(A^{\otimes n}, A)$ and $g \in \text{Hom}(A^{\otimes m}, A)$, the partial composition $f \circ_i g \in \text{Hom}(A^{\otimes n+m-1}, A)$ is simply,

$$(f \circ_i g)(a_1 \otimes \cdots \otimes a_{n+m-1}) := f(a_1 \otimes \cdots \otimes a_{i-1} \otimes g(a_i \otimes \cdots \otimes a_{i+m}) \otimes a_{i+m+1} \otimes \cdots \otimes a_{n+m-1}).$$

Next, consider the non-symmetric operad \mathbf{As} . It is one-dimensional in each arity, $\mathbf{As}(n) := \mathbb{K}\mu_n$, where the generators μ_n are of degree zero. The partial composition is given by $\mu_n \circ_i \mu_m = \mu_{n+m-1}$

for any i . The Koszul dual cooperad of \mathbf{As} , denoted $\mathbf{As}^{\dot{i}}(n)$, is also one-dimensional in each arity, but with generators μ_n^c of degree $1 - n$ ([13], Section 9.1.5). The *convolution Lie algebra* is,

$$\mathfrak{g}_{\mathbf{As}, A} := \text{Hom}(\mathbf{As}^{\dot{i}}, \text{End}_A) = \prod_{n \geq 1} \text{Hom}(\mathbf{As}^{\dot{i}}(n), \text{End}_A(n)).$$

We equip it with the pre-Lie product \star defined for $F, G \in \text{Hom}(\mathbf{As}^{\dot{i}}, \text{End}_A)$ by,

$$F \star G := \gamma_{(1)} \circ (F \circ_{(1)} G) \circ \Delta_{(1)},$$

where $\circ_{(1)}$ denotes the infinitesimal composite, $\gamma_{(1)} : \text{End}_A \circ_{(1)} \text{End}_A \rightarrow \text{End}_A$ the infinitesimal composition map and $\Delta_{(1)} : \mathbf{As}^{\dot{i}} \rightarrow \mathbf{As}^{\dot{i}} \circ_{(1)} \mathbf{As}^{\dot{i}}$ the infinitesimal decomposition map ([13], Section 6.4.2). We consider $\mathfrak{g}_{\mathbf{As}, A}$ as differential graded Lie algebra with zero differential. As graded vector spaces, we identify,

$$\mathfrak{g}_{\mathbf{As}, A} = \prod_{n \geq 1} \text{Hom}(\mathbf{As}^{\dot{i}}(n), \text{End}_A(n)) \cong \prod_{n \geq 1} \text{End}_A(n) \otimes \mathbf{As}^{\dot{i}*}(n) = \prod_{n \geq 1} \text{End}_A(n) \otimes \mathbb{K}\mu_n^{c*},$$

where μ_n^{c*} is dual to μ_n^c , and therefore of degree $n - 1$. The element $F \in \text{Hom}(\mathbf{As}^{\dot{i}}(n), \text{End}_A(n))$ which sends μ_n^c to $f \in \text{End}_A(n)$ will be mapped to $f \otimes \mu_n^{c*}$ by the bijection above. Next, recall that,

$$\prod_{n \geq 1} \text{End}_{sA}(n) = \prod_{n \geq 1} \text{Hom}((sA)^{\otimes n}, sA),$$

endowed with the operation,

$$\tilde{f} \star' \tilde{g} := \sum_{i=1}^n \tilde{f} \circ_i \tilde{g},$$

for $\tilde{f} \in \text{End}_{sA}(n)$ and $\tilde{g} \in \text{End}_{sA}(m)$ also defines a pre-Lie algebra ([13], Section, 5.9.15). Moreover, note that the commutative diagram,

$$\begin{array}{ccc} (sA)^{\otimes n} & \xrightarrow{\tilde{f}} & sA \\ s^{\otimes n} \uparrow & & \uparrow s \\ A^{\otimes n} & \xrightarrow{f} & A. \end{array}$$

defines a bijection between $\text{End}_{sA}(n)$ and $\text{End}_A(n)$. This correspondence yields the identification,

$$\prod_{n \geq 1} \text{End}_{sA}(n) = \prod_{n \geq 1} \text{Hom}((sA)^{\otimes n}, sA) \cong \prod_{n \geq 1} \text{End}_A(n) \otimes \mathbb{K}\mu_n^{c*},$$

by mapping $\tilde{f} \in \text{End}_{sA}(n)$ to $f \otimes \mu_n^{c*} \in \text{End}_A(n) \otimes \mathbb{K}\mu_n^{c*}$. We then have the following result.

Proposition 11. We endow the space $\prod_{n \geq 1} \text{End}_A(n) \otimes \mathbb{K}\mu_n^{c*}$ with the operation,

$$(f \otimes \mu_n^{c*}) \star'' (g \otimes \mu_m^{c*}) := \sum_{i=1}^n (-1)^{|g|(1-n)+(i+1)(m-1)} (f \circ_i g) \otimes \mu_{n+m-1}^{c*}.$$

Then \star'' defines a pre-Lie product and the isomorphisms of graded vector spaces defined above are isomorphisms of pre-Lie algebras, i.e.

$$(\mathfrak{g}_{\mathbf{As}, A}, \star) \cong \left(\prod_{n \geq 1} \text{End}_A(n) \otimes \mathbb{K}\mu_n^{c*}, \star'' \right) \cong \left(\prod_{n \geq 1} \text{End}_{sA}(n), \star' \right).$$

Proof. See [13], Proposition 10.1.16. \square

We recall one more identification. Let V be a graded vector space and $\overline{TV} = \prod_{n \geq 1} V^{\otimes n}$ the reduced (completed) tensor algebra. Equipped with the usual deconcatenation coproduct Δ defined by,

$$\Delta(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n-1} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n),$$

\overline{TV} describes a graded coalgebra. Let now W denote another graded vector space. A linear map $f : (\overline{TV}, \Delta_V) \rightarrow (\overline{TW}, \Delta_W)$ is called a *coalgebra morphism* if $\Delta_W \circ f = (f \otimes f) \circ \Delta_V$. A *coderivation* b of \overline{TV} is a linear map $b : \overline{TV} \rightarrow \overline{TV}$ such that $\Delta \circ b = (b \otimes 1 + 1 \otimes b) \circ \Delta$. The graded commutator of two coderivations is again a coderivation, and we denote by $\text{Coder}(\overline{TV})$ the Lie algebra of coderivations of \overline{TV} . It is a well-known result that there is a natural bijection between coderivations of \overline{TV} and collections of linear maps $\prod_{n \geq 1} \text{Hom}(V^{\otimes n}, V)$ (see, for instance, [13], Proposition 6.3.7.). More precisely, given a coderivation $b : \overline{TV} \rightarrow \overline{TV}$, its component mapping $V^{\otimes n}$ to $V^{\otimes u}$ is given by,

$$\sum_{\substack{r+s+t=n \\ u=r+1+t}} 1^{\otimes r} \otimes b_s \otimes 1^{\otimes t},$$

where $b_s = \text{proj}_V \circ b : V^{\otimes s} \rightarrow V$ is the projection of b on the space of cogenerators. Note also that the composition $a \circ b$ of two coderivations of \overline{TV} yields the following family of maps in $\prod_{n \geq 1} \text{Hom}(V^{\otimes n}, V)$,

$$(a \circ b)_n = \sum_{\substack{r+s+t=n \\ u=r+1+t}} a_u(1^{\otimes r} \otimes b_s \otimes 1^{\otimes t}) = \sum_{\substack{n+1=u+s \\ 1 \leq i \leq u}} a_u(1^{\otimes i-1} \otimes b_s \otimes 1^{\otimes u-i}) = \sum_{\substack{n+1=u+s \\ 1 \leq i \leq u}} a_u \circ_i b_s = (a \star' b)_n.$$

This gives an isomorphism of Lie algebras,

$$\text{Coder}(\overline{TV}) \cong \prod_{n \geq 1} \text{End}_V(n). \quad (4.3)$$

Definition 24. An element $\alpha \in \mathfrak{g}_{A_S, A}$ is called *Maurer-Cartan element* if it is of degree one and satisfies the *Maurer-Cartan equation*,

$$\alpha \star \alpha = 0.$$

We denote the set of Maurer-Cartan elements of $\mathfrak{g}_{A_S, A}$ by $\text{MC}(\mathfrak{g}_{A_S, A})$.

Remark 34. By Proposition 11, the data of a Maurer-Cartan element $\alpha \in \text{MC}(\mathfrak{g}_{A_S, A})$ is equivalent to a collection $b = (b_n)_{n \geq 1} \in \prod_{n \geq 1} \text{End}_{sA}(n)$ of degree one elements satisfying $b \star' b = 0$. Equivalently, b describes a degree one coderivation of \overline{TsA} which squares to zero. This in turn translates to,

$$(b \star' b)_n = (b \circ b)_n = \sum_{\substack{n=r+s+t \\ u=r+1+t}} b_u(1^{\otimes r} \otimes b_s \otimes 1^{\otimes t}) = 0$$

for all $n \geq 1$. Moreover, the commutative diagram above applied to any b_n gives a map $m_n := s^{-1}b_n s^{\otimes n} : A^{\otimes n} \rightarrow A$ of degree $|m_n| = -n + 1 - (-1) = 2 - n$, and the collection $(m_n)_{n \geq 1}$ satisfies precisely the set of equations (4.1), thus defining an A_∞ -structure on A . Denoting by $\text{Codiff}(\overline{TsA})$ the set of degree one coderivations squaring to zero (also called *codifferentials*), the reasoning above gives two bijections,

$$\text{MC}(\mathfrak{g}_{A_S, A}) \cong \text{Codiff}(\overline{TsA}) \cong A_\infty - \text{structures on } A.$$

Remark 35. Assume that $(\overline{T}sA, a)$ and $(\overline{T}sB, b)$ are differential coalgebras (i.e. coalgebras equipped with a codifferential). A similar reasoning as above implies that a differential coalgebra map $\varphi : \overline{T}sA \rightarrow \overline{T}sB$ (i.e. a coalgebra map that commutes with the codifferentials) of degree zero gives a family of degree zero maps $\{\varphi_n : (sA)^{\otimes n} \rightarrow sB\}_{n \geq 1}$ satisfying,

$$\sum \varphi_u(1^{\otimes r} \otimes a_s \otimes 1^{\otimes t}) = \sum b_r(\varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_r}), \quad (4.4)$$

where the summations run over the same elements as in equation (4.2). Setting $f_n := s^{-1}\varphi_n s^{\otimes n}$, $m_n := s^{-1}a_n s^{\otimes n}$ and $l_n := s^{-1}b_n s^{\otimes n}$, equation (4.4) becomes exactly equation (4.2), that is, the collection $f = \{f_n\}_{n \geq 1}$ describes an A_∞ -morphism $f : (A, \{m_n\}_{n \geq 1}) \rightarrow (B, \{l_n\}_{n \geq 1})$. This yields a bijection between differential coalgebra maps $(\overline{T}sA, a) \rightarrow (\overline{T}sB, b)$ of degree zero and A_∞ -morphisms $A \rightarrow B$, equipped with the corresponding A_∞ -structures.

Remark 36. The previous remark shows that the group of degree zero differential coalgebra automorphisms of $\overline{T}sA$, denoted $\text{Aut}^0(\overline{T}sA)$ acts on $\text{Codiff}(\overline{T}sA)$ by the adjoint action, that is, $\varphi.b := \varphi \circ b \circ \varphi^{-1}$ for $b \in \text{Codiff}(\overline{T}sA)$ and $\varphi \in \text{Aut}^0(\overline{T}sA)$. Moreover, both b and $\varphi.b$ give A_∞ -structures on A (say m and m_φ). The family $\{\varphi_n : (sA)^{\otimes n} \rightarrow sA\}_{n \geq 1}$ obtained by the automorphism φ induces an A_∞ -isomorphism $f : (A, m) \rightarrow (A, m_\varphi)$. We thus have a bijection,

$$\text{Codiff}(\overline{T}sA)/\text{Aut}^0(\overline{T}sA) \cong \{A_\infty - \text{structures on } A\}/\{A_\infty - \text{isomorphisms}\}.$$

Remark 37. We denote the Lie subalgebra of $\text{Coder}(\overline{T}sA)$ given by degree zero coderivations by $\text{Coder}^0(\overline{T}sA)$. It acts on degree one coderivations via the adjoint action. Furthermore, it corresponds to the Lie algebra of the automorphism group $\text{Aut}^0(\overline{T}sA)$ and therefore acts on the set of codifferentials via the formula,

$$c.b := e^{\text{ad}_c}(b) = e^c \circ b \circ e^{-c},$$

for $c \in \text{Coder}^0(\overline{T}sA)$, $b \in \text{Codiff}(\overline{T}sA)$ and $\text{ad}_c(-) = [c, -]$.

4.1.3 Gauge equivalences

A clear exposition of the material below can be found in W. M. Goldman and J. J. Millson's paper [10]. We follow the more concise Appendix B of [5]. The graded Lie algebra $\mathfrak{g}_{As,A}$ has a natural descending filtration given by,

$$\mathcal{F}^p \mathfrak{g}_{As,A} := \prod_{n \geq p+1} \text{Hom}(As^i(n), \text{End}_A(n)).$$

This filtration is complete and compatible with the Lie bracket, that is,

$$\mathfrak{g}_{As,A} = \varprojlim \mathfrak{g}_{As,A}/\mathcal{F}^p \mathfrak{g}_{As,A} \text{ and } [\mathcal{F}^p \mathfrak{g}_{As,A}, \mathcal{F}^q \mathfrak{g}_{As,A}] \subset \mathcal{F}^{p+q} \mathfrak{g}_{As,A}.$$

Moreover, $\mathcal{F}^0 \mathfrak{g}_{As,A} = \mathfrak{g}_{As,A}$. The degree zero elements $\mathfrak{g}_{As,A}^0$ form a Lie subalgebra of $\mathfrak{g}_{As,A}$ and the completeness condition ensures that $\mathfrak{g}_{As,A}^0$ is a pro-nilpotent Lie algebra. It may thus be exponentiated to the pro-unipotent group $\exp(\mathfrak{g}_{As,A}^0)$ which consists of the set $\mathfrak{g}_{As,A}^0$ equipped with the Baker-Campbell-Hausdorff product bch . It acts on the set of degree one elements $\mathfrak{g}_{As,A}^1$ via the formula,

$$\xi.\alpha := \exp(\text{ad}_\xi)\alpha = \alpha + \text{ad}_\xi(\alpha) + \frac{1}{2!}\text{ad}_\xi^2(\alpha) + \dots$$

for $\xi \in \exp(\mathfrak{g}_{\mathcal{A}s,A}^0)$ and $\alpha \in \mathfrak{g}_{\mathcal{A}s,A}^1$. Again, completeness allows us to make sense of the series above. Notice also that this is indeed a group action since $\exp(\text{ad}_{\xi_1})\exp(\text{ad}_{\xi_2}) = \exp(\text{ad}_{\text{bch}(\xi_1, \xi_2)})$. We then have the following well-known result.

Lemma 20. The action of $\exp(\mathfrak{g}_{\mathcal{A}s,A}^0)$ on $\mathfrak{g}_{\mathcal{A}s,A}^1$ preserves the set of Maurer-Cartan elements $\text{MC}(\mathfrak{g}_{\mathcal{A}s,A})$.

Proof. We refer to Section 1 of [10]. □

Definition 25. The action of $\exp(\mathfrak{g}_{\mathcal{A}s,A}^0)$ defines an equivalence relation on the set of Maurer-Cartan elements $\text{MC}(\mathfrak{g}_{\mathcal{A}s,A})$. We say that two Maurer-Cartan elements $\alpha_1, \alpha_2 \in \text{MC}(\mathfrak{g}_{\mathcal{A}s,A})$ are *gauge equivalent* if there is a $\xi \in \exp(\mathfrak{g}_{\mathcal{A}s,A}^0)$ such that $\alpha_2 = \xi.\alpha_1$.

Proposition 12. Gauge equivalent Maurer-Cartan elements of $\mathfrak{g}_{\mathcal{A}s,A}$ correspond bijectively to A_∞ -isomorphic A_∞ -structures on A . We thus have the following bijections,

$$\text{MC}(\mathfrak{g}_{\mathcal{A}s,A}) / \exp(\mathfrak{g}_{\mathcal{A}s,A}^0) \cong \text{Codiff}(\overline{T}sA) / \text{Aut}^0(\overline{T}sA) \cong \{A_\infty\text{-structures on } A\} / \{A_\infty\text{-isomorphisms}\}.$$

Proof. The identifications in Proposition 11 and the isomorphism (4.3) give an isomorphism of Lie algebras,

$$\mathfrak{g}_{\mathcal{A}s,A}^0 \cong \text{Coder}^0(\overline{T}sA),$$

which preserves the respective adjoint actions on $\mathfrak{g}_{\mathcal{A}s,A}^1 \cong \text{Coder}^1(\overline{T}sA)$. Moreover, this allows us to identify $\exp(\mathfrak{g}_{\mathcal{A}s,A}^0)$ with the group of automorphisms $\text{Aut}^0(\overline{T}sA)$ such that the respective actions on $\text{MC}(\mathfrak{g}_{\mathcal{A}s,A}) \cong \text{Codiff}(\overline{T}sA)$ coincide. Thus,

$$\text{MC}(\mathfrak{g}_{\mathcal{A}s,A}) / \exp(\mathfrak{g}_{\mathcal{A}s,A}^0) \cong \text{Codiff}(\overline{T}sA) / \text{Aut}^0(\overline{T}sA).$$

which, together with Remark 36, implies the statement. □

We finish this section by recalling a useful technical lemma.

Lemma 21. Let \mathfrak{a} be a graded Lie algebra. Assume that there is a second positive grading on \mathfrak{a} compatible with the Lie algebra structure, i.e.

$$\mathfrak{a} = \bigoplus_{i \geq 1} \mathfrak{a}^{(i)} \quad \text{and} \quad [\mathfrak{a}^{(i)}, \mathfrak{a}^{(j)}] \subset \mathfrak{a}^{(i+j)}.$$

Let $\alpha \in \text{MC}(\mathfrak{a})$ be a Maurer-Cartan element of \mathfrak{a} . Decompose α with respect to the second grading, that is,

$$\alpha = \alpha^{(1)} + \alpha^{(k)} + \alpha^{(k+1)} + \dots$$

where $\alpha^{(i)} \in \mathfrak{a}^{(i)}$ and $k \geq 2$. Then $\alpha^{(1)} \in \text{MC}(\mathfrak{a})$, the bracket $[\alpha^{(1)}, -]$ defines a differential on \mathfrak{a} and if the cohomology class $[\alpha^{(k)}] \neq 0 \in H^1(\mathfrak{a}, [\alpha^{(1)}, -])$, the Maurer-Cartan elements α and $\alpha^{(1)}$ are not gauge equivalent.

Proof. The low order expansion of the Maurer-Cartan equation for α reads,

$$0 = \frac{1}{2}[\alpha, \alpha] = \frac{1}{2}[\alpha^{(1)}, \alpha^{(1)}] + [\alpha^{(1)}, \alpha^{(k)}] + \dots$$

Since $[\alpha^{(1)}, \alpha^{(1)}]$ is the only contribution to $\alpha^{(2)}$, it must equal zero. Thus $\alpha^{(1)} \in \mathbf{MC}(\mathfrak{a})$. Together with the Jacobi identity this implies $[\alpha^{(1)}, [\alpha^{(1)}, -]] = 0$, i.e. bracketing with $\alpha^{(1)}$ defines a differential on \mathfrak{a} . Next, assume $[\alpha^{(k)}] \neq 0 \in H^1(\mathfrak{a}, [\alpha^{(1)}, -])$ and that α and $\alpha^{(1)}$ are gauge equivalent Maurer-Cartan elements, that is, there exists $b \in \mathfrak{a}$ of degree zero such that $\alpha = e^{\text{ad}(b)}\alpha^{(1)}$. Expanding this equation with respect to the second grading yields inductively,

$$[b^{(j)}, \alpha^{(1)}] = 0,$$

for $1 \leq j \leq k-2$, and $[b^{(k-1)}, \alpha^{(1)}] = \alpha^{(k)}$. But then $[\alpha^{(k)}] = 0 \in H^1(\mathfrak{a}, [\alpha^{(1)}, -])$, leading to a contradiction. Therefore, α and $\alpha^{(1)}$ cannot be gauge equivalent. \square

4.1.4 The Duflo isomorphism

We follow D. Calaque and C. A. Rossi's lecture series [6]. Let \mathfrak{g} be a finite dimensional Lie algebra. It acts on the symmetric algebra $S\mathfrak{g}$ and the universal enveloping algebra $U\mathfrak{g}$ by the adjoint action. Consider the formal power series on \mathfrak{g} given by the (modified) Duflo element,

$$J(x) := \det \left(\frac{e^{\text{ad}_x/2} - e^{-\text{ad}_x/2}}{\text{ad}_x} \right).$$

Moreover, recall the symmetrization map,

$$\text{Sym} : S\mathfrak{g} \rightarrow U\mathfrak{g}, \quad v_1 \cdots v_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} v_{\sigma(1)} \cdots v_{\sigma(n)}.$$

It is an isomorphism of filtered vector spaces, but not an algebra isomorphism (the product on $S\mathfrak{g}$ being commutative, while the one on $U\mathfrak{g}$ is not unless \mathfrak{g} is abelian). M. Duflo's theorem [9] states that the composition,

$$\text{Duf} := \text{Sym} \circ J^{1/2} : (S\mathfrak{g})^{\mathfrak{g}} \rightarrow (U\mathfrak{g})^{\mathfrak{g}}$$

defines an algebra isomorphism on \mathfrak{g} -invariant elements. Here, we identify $x \in \mathfrak{g}$ with the vector field $\frac{\partial}{\partial x^*}$. It acts by derivation on $S\mathfrak{g}$, which may be viewed as polynomial functions on \mathfrak{g}^* . In this way, we may view the formal power series $J^{1/2}(x)$ on \mathfrak{g} as an infinite-order differential operator $J^{1/2}$ on \mathfrak{g}^* .

By pulling back the product $m_{U\mathfrak{g}}$ on $U\mathfrak{g}$ to $S\mathfrak{g}$ via the Duflo isomorphism, we obtain a second associative product,

$$m_{\text{Duf}} := \text{Duf}^{-1} \circ m_{U\mathfrak{g}} \circ \text{Duf}^{\otimes 2} : S\mathfrak{g}^{\otimes 2} \rightarrow S\mathfrak{g}$$

on the symmetric algebra. On invariant elements m_{Duf} coincides with the usual commutative product m_0 .

Next, consider the Chevalley-Eilenberg complexes, $(C^\bullet(\mathfrak{g}, S\mathfrak{g}) = S\mathfrak{g} \otimes \bigwedge^\bullet \mathfrak{g}^*, d_{S\mathfrak{g} \otimes \bigwedge^\bullet \mathfrak{g}^*})$ and $(C^\bullet(\mathfrak{g}, U\mathfrak{g}) = U\mathfrak{g} \otimes \bigwedge^\bullet \mathfrak{g}^*, d_{U\mathfrak{g} \otimes \bigwedge^\bullet \mathfrak{g}^*})$ [7]. Results by B. Shoikhet [17] and M. Pevzner and C. Torossian [14] show that the map $\text{Duf} \otimes \text{Id} : S\mathfrak{g} \otimes \bigwedge^\bullet \mathfrak{g}^* \rightarrow U\mathfrak{g} \otimes \bigwedge^\bullet \mathfrak{g}^*$ induces an isomorphism of algebras on the level of cohomology. Moreover, this allows us to define two associative products on $S\mathfrak{g} \otimes \bigwedge^\bullet \mathfrak{g}^*$, namely,

$$\begin{aligned} \tilde{m}_0 : (S\mathfrak{g} \otimes \bigwedge^\bullet \mathfrak{g}^*)^{\otimes 2} &\cong (S\mathfrak{g})^{\otimes 2} \otimes (\bigwedge^\bullet \mathfrak{g}^*)^{\otimes 2} \xrightarrow{m_0 \otimes m_{\bigwedge^\bullet \mathfrak{g}^*}} S\mathfrak{g} \otimes \bigwedge^\bullet \mathfrak{g}^* \\ \tilde{m}_{\text{Duf}} : (S\mathfrak{g} \otimes \bigwedge^\bullet \mathfrak{g}^*)^{\otimes 2} &\cong (S\mathfrak{g})^{\otimes 2} \otimes (\bigwedge^\bullet \mathfrak{g}^*)^{\otimes 2} \xrightarrow{m_{\text{Duf}} \otimes m_{\bigwedge^\bullet \mathfrak{g}^*}} S\mathfrak{g} \otimes \bigwedge^\bullet \mathfrak{g}^* \end{aligned}$$

where $m_{\wedge \mathfrak{g}^*}$ denotes the usual graded anticommutative product on $\wedge \mathfrak{g}^*$.

A natural question at this point is whether the map $\text{Duf} \otimes \text{Id}$ may be extended to an A_∞ -isomorphism in a universal way (i.e. independent of the specific choice of the Lie algebra \mathfrak{g}). It has been answered in the negative by J. Alm ([1], Remark 4.0.1), ([2], Proposition 5.3.0.10), see also J. Alm and S. Merkulov's paper [3]). The aim of this text is to describe a self-contained and elementary proof of this fact. The non-existence of such an A_∞ -isomorphism is equivalent to the following statement.

Theorem 12. There does not exist a universal A_∞ -isomorphism of strict A_∞ -algebras,

$$f : (S\mathfrak{g} \otimes \wedge \mathfrak{g}^*, (d_{S\mathfrak{g} \otimes \wedge \mathfrak{g}^*}, \tilde{m}_0)) \rightarrow (S\mathfrak{g} \otimes \wedge \mathfrak{g}^*, (d_{S\mathfrak{g} \otimes \wedge \mathfrak{g}^*}, \tilde{m}_{\text{Duf}})),$$

whose first component f_1 is the identity.

4.1.5 Kontsevich's product

Let x_1, \dots, x_d be a basis of \mathfrak{g} and denote by $c_{ij}^k \in \mathbb{K}$ its structure constants. Set $A := S\mathfrak{g} \otimes \wedge \mathfrak{g}^*$. When equipped with the product \tilde{m}_0 we may identify A with the polynomial algebra $\mathbb{K}[x_1, \dots, x_d, p^1, \dots, p^d]$, where p^1, \dots, p^d are of degree 1 and describe the (degree shifted) dual basis. The x_i are set to have degree 0. To describe the product \tilde{m}_{Duf} induced by the Duflo isomorphism on A , recall that in [12], M. Kontsevich gave an explicit universal formula for the deformation quantization of any Poisson manifold. In particular, this can be applied to the dual Lie algebra \mathfrak{g}^* which defines a Poisson manifold with a linear Poisson structure. If we think of x_1, \dots, x_d as local coordinates on \mathfrak{g}^* , then for $f, g \in C^\infty(\mathfrak{g}^*)$, the Poisson bracket is given by,

$$\{f, g\} := c_{ij}^k x_k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

It extends the linear Poisson structure on \mathfrak{g}^* . We set $\pi_{ij} = c_{ij}^k x_k$ and

$$\pi := \pi_{ij} p^i p^j = c_{ij}^k x_k p^i p^j \in A.$$

The latter can be identified with the Poisson bivector field on \mathfrak{g}^* given by $\pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ in local coordinates.

Definition 26. A deformation quantization of \mathfrak{g}^* is a $\mathbb{K}[[\epsilon]]$ -linear, associative product m_π on $C^\infty(\mathfrak{g}^*)[[\epsilon]]$ such that for all $f, g \in C^\infty(\mathfrak{g}^*)$,

$$m_\pi(f \otimes g) = fg + \frac{\epsilon}{2} \{f, g\} + O(\epsilon^2).$$

Remark 38. We shall not recall M. Kontsevich's construction at this point. Note however that (by formally setting $\epsilon = 1$) it yields one further associative product m_π on the space of polynomial functions on \mathfrak{g}^* , i.e. on $S\mathfrak{g} = \mathbb{K}[x_1, \dots, x_d]$. This product corresponds precisely to the one induced by the Duflo isomorphism (see [12],[16]), that is,

$$m_\pi = m_{\text{Duf}}.$$

Moreover, if we define the product \tilde{m}_π on A in an analogous way as we did for \tilde{m}_{Duf} , we also have $\tilde{m}_\pi = \tilde{m}_{\text{Duf}}$.

4.2 A variant of M. Kontsevich's graph complex

We consider the following version of M. Kontsevich's graph complex ([12], Section 6.1).

Definition 27. An *admissible directed graph* is a directed graph Γ with labeled vertices $1, 2, \dots, n$ (called external), possibly other vertices (unlabeled and called internal) satisfying the following properties:

1. There is a linear order on the set of edges.
2. Γ has no double edges, nor simple loops (edges connecting a vertex with itself).
3. Every internal vertex is at most trivalent.
4. Every internal vertex has at most one incoming edge, and at most two outgoing edges.
5. Every internal vertex can be connected by a path with an external vertex.

Let $\mathbf{dgr}(n)$ be the vector space spanned by finite linear combinations of admissible directed graphs with n external vertices, modulo the relation $\Gamma^\sigma = (-1)^{|\sigma|}\Gamma$, where Γ^σ differs from Γ by a permutation σ on the order of edges. Here $|\sigma|$ denotes the parity of the permutation σ .

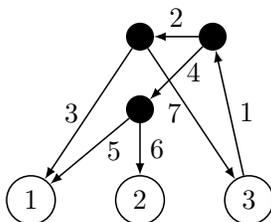


Figure 4.1: A graph in $\mathbf{dgr}(3)$.

- Definition 28.**
1. Let Γ be an admissible directed graph and fix any one of its internal vertices. Call it v . Consider the linear combination obtained by replacing v by two vertices connected by a directed edge e and summing over all possible ways of reconnecting the edges previously adjacent to v to the endpoints of e while creating only admissible directed graphs.
 2. Let Γ be a directed graph as in Definition 27 but with one four-valent internal vertex v with one incoming and three outgoing edges. Consider the linear combination obtained by replacing v by two vertices connected by a directed edge e and summing over all possible ways of reconnecting the edges previously adjacent to v to the endpoints of e while creating only admissible directed graphs (see Figure 4.2).

In both cases, the order of the set of edges of the new graphs is given by placing the newly added edge last. The *generalized IHX relations* are given by setting such linear combinations equal to zero.

Our main object of study will be the collection of quotients ($n \geq 1$),

$$\mathbf{dgraphs}(n) := \mathbf{dgr}(n) / \text{generalized IHX relations.}$$

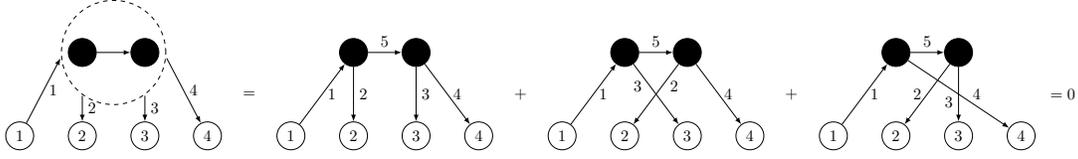


Figure 4.2: An example of the generalized IHX relations.

Remark 39. For each $n \geq 1$, $\mathbf{dgraphs}(n)$ defines a graded vector space. The degree of a graph $\Gamma \in \mathbf{dgraphs}(n)$ is given by

$$|\Gamma| := 2\#\text{internal vertices} - \#\text{edges}.$$

Furthermore, the collection $\{\mathbf{dgraphs}(n)\}_{n \geq 1}$ assembles to a non-symmetric operad $\mathbf{dgraphs}$ in the category of graded vector spaces. The operadic composition in $\mathbf{dgraphs}$ is given by insertion. That is, for $\Gamma_1 \in \mathbf{dgraphs}(r)$, $\Gamma_2 \in \mathbf{dgraphs}(s)$, $1 \leq j \leq r$,

$$\Gamma_1 \circ_j \Gamma_2 \in \mathbf{dgraphs}(r + s - 1)$$

is constructed by replacing the j -th external vertex by Γ_2 , summing over all possible ways of reconnecting the “loose” edges (which were previously adjacent to vertex j) to vertices of Γ_2 , and keeping only admissible directed graphs. The order on the set of edges of the new graphs is simply given by letting the edges of Γ_1 come before those of Γ_2 while leaving the respective orderings unchanged. Moreover, the product,

$$\Gamma_1 \circ \Gamma_2 := \sum_{j=1}^r \Gamma_1 \circ_j \Gamma_2,$$

defines a pre-Lie product ([13], Section 5.9.15). Its graded commutator thus yields a graded Lie algebra structure on $\mathbf{dgraphs}$.

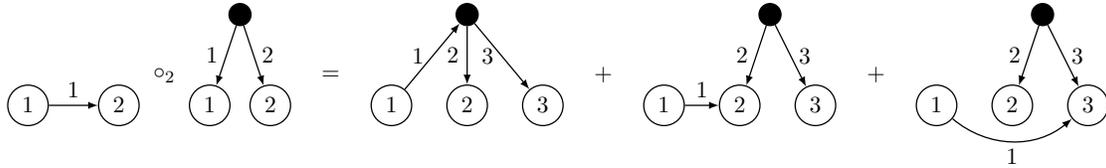


Figure 4.3: The operadic composition of two graphs in $\mathbf{dgraphs}(2)$.

We will need the following two subspaces of $\mathbf{dgraphs}(n)$.

Definition 29. Let $\mathbf{dgraphs}^{\text{uni}}(n)$ be the subspace of $\mathbf{dgraphs}(n)$ spanned by graphs for which all external vertices are univalent and denote by $\mathbf{dgraphs}(n, q)$ the subspace spanned by graphs with exactly q edges connected to the n external vertices.

Remark 40. J. Alm and S. Merkulov ([2],[3]) make use of a similar graph complex. They require, however, internal vertices to be *at least trivalent*. For us, it will be crucial that the internal vertices are allowed to be uni- and bivalent.

Remark 41. Undirected edges in our figures mean that we take the sum over all possible directions, i.e.

$$\bullet \text{---} \bullet = \bullet \text{---} \bullet + \bullet \text{---} \bullet .$$

4.3 M. Kontsevich's representation $\mathbf{dgraphs} \rightarrow \text{End}_A$

Next, we recall M. Kontsevich's construction ([12], Section 6.3) of a linear map $B : \mathbf{dgraphs} \rightarrow \text{End}_A$, $\Gamma \mapsto B_\Gamma$ following D. Calaque and C. A. Rossi's text [6]. For this, let $\Gamma \in \mathbf{dgr}(n)$ and assume it has m internal vertices. Label these m internal vertices by $\bar{1}, \dots, \bar{m}$ in an arbitrary way. Define the operator of degree one,

$$\tau := \sum_{l=1}^d \frac{\partial}{\partial p^l} \otimes \frac{\partial}{\partial x_l} : A^{\otimes 2} \rightarrow A^{\otimes 2}.$$

Moreover, for any finite index set I and any pair (i, j) of distinct elements in I , let $\tau_{ij} : A^{\otimes I} \rightarrow A^{\otimes I}$ be the operator acting as τ on the i -th and j -th factors and as the identity on all other factors of A . Using this data, we define for any $n \geq 1$,

$$B'_\Gamma : A^{m+n} \rightarrow A$$

$$B'_\Gamma(\gamma_1, \dots, \gamma_m, f_1, \dots, f_n) := \mu_{m+n} \prod_{(i,j) \in E(\Gamma)} \tau_{ij}(\gamma_{\bar{1}} \otimes \dots \otimes \gamma_{\bar{m}} \otimes f_1 \otimes \dots \otimes f_n)$$

where $\mu_{m+n} : A^{\otimes(m+n)} \rightarrow A$ is the iterated graded commutative product, $E(\Gamma)$ denotes the edge set of Γ , (i, j) describes the edge starting at vertex i and ending at j for $i, j \in I := \{1, \dots, n\} \cup \{\bar{1}, \dots, \bar{m}\}$ and the order of the product of the endomorphisms τ_{ij} is determined by the order on the set of edges (i.e. the automorphism corresponding to the first edge in the linear order is applied first).

Notice that since the automorphisms τ_{ij} are of degree one, any permutation σ in the order of their product produces a sign $(-1)^{|\sigma|}$, where $|\sigma|$ denotes the parity of the permutation. This is compatible with the equivalence relation on \mathbf{dgr} given by the ordering on the set of edges.

The map B'_Γ depends on the choice of labeling of the internal vertices. However, the map,

$$B_\Gamma : A^{\otimes n} \rightarrow A$$

$$B_\Gamma(f_1 \otimes \dots \otimes f_n) := B'_\Gamma(\pi, \dots, \pi, f_1, \dots, f_n)$$

is independent of the labeling we choose, and therefore yields a well-defined element of $\text{End}_A(n)$.

Proposition 13. The map $B : \mathbf{dgr}(n) \rightarrow \text{End}_A(n)$ factors through the projection $\mathbf{dgr}(n) \twoheadrightarrow \mathbf{dgraphs}(n)$.

Proof. The generalized IHX relations are all obtained by replacing one internal vertex v (which in this case is of valence less or equal to four) of some graph $\Gamma \in \mathbf{dgr}(n)$ by a directed edge connected by two internal vertices and summing over all possible ways of reconnecting the edges previously adjacent to v . If v is for instance of valence four with one incoming and three outgoing edges, this will produce a term of the following form within the large product defined by $B_\Gamma(f_1 \otimes \dots \otimes f_n)$ (for $f_1, \dots, f_n \in A$),

$$\frac{\partial^4}{\partial x_i \partial p^j \partial p^k \partial p^l} \left(\frac{\partial \pi}{\partial x_m} \frac{\partial \pi}{\partial p^m} \right),$$

where the term $\frac{\partial \pi}{\partial x_m} \frac{\partial \pi}{\partial p^m}$ corresponds to the newly inserted directed edge. If the valence of v is smaller the derivatives in front of this factor change accordingly. However, a short calculation shows that,

$$\frac{\partial \pi}{\partial x_m} \frac{\partial \pi}{\partial p^m} = 2\pi_{mi} \frac{\partial \pi_{jk}}{\partial x_m} p^i p^j p^k = \frac{2}{3} \left(\pi_{mi} \frac{\partial \pi_{jk}}{\partial x_m} + \pi_{mj} \frac{\partial \pi_{ki}}{\partial x_m} + \pi_{mk} \frac{\partial \pi_{ij}}{\partial x_m} \right) p^i p^j p^k = 0.$$

This is zero since the term in the bracket is equivalent to the Jacobi identity in terms of the structure constants. Hence, the generalized IHX relations are sent to zero under the map $B : \mathbf{dgr}(n) \rightarrow \text{End}_A(n)$ from which the statement follows. \square

Remark 42. Proposition 13 ensures that there is a well-defined map $\mathbf{dgraphs} \rightarrow \text{End}_A$. We denote this map by B as well. It follows from the product rule that this is a map of operads. More precisely, for $\Gamma_1 \in \mathbf{dgraphs}(n)$, $\Gamma_2 \in \mathbf{dgraphs}(m)$ and $f_1, \dots, f_{n+m-1} \in A$ we have,

$$\begin{aligned} B_{\Gamma_1 \circ_i \Gamma_2}(f_1 \otimes \dots \otimes f_{n+m-1}) &= B_{\Gamma_1}(f_1 \otimes \dots \otimes f_{i-1} \otimes B_{\Gamma_2}(f_i \otimes \dots \otimes f_{i+m-1}) \otimes f_{i+m} \otimes \dots \otimes f_{n+m-1}) \\ &= B_{\Gamma_1} \circ_i B_{\Gamma_2}(f_1 \otimes \dots \otimes f_{n+m-1}). \end{aligned}$$

Example 1. Consider $\Gamma_1 \in \mathbf{dgraphs}(2)$ and $\Gamma_2 \in \mathbf{dgraphs}(3)$ as in Figure 4.4. Then,

$$B_{\Gamma_1}(f_1 \otimes f_2) = \frac{\partial^2 \pi}{\partial p^{i_1} \partial p^{i_2}} \frac{\partial^3 \pi}{\partial x_{i_2} \partial p^{i_3} \partial p^{i_4}} \frac{\partial^2 f_1}{\partial x_{i_1} \partial x_{i_3}} \frac{\partial f_2}{\partial x_{i_4}} = c_{i_1, i_2}^j x_j c_{i_3, i_4}^{i_2} \frac{\partial^2 f_1}{\partial x_{i_1} \partial x_{i_3}} \frac{\partial f_2}{\partial x_{i_4}},$$

and,

$$B_{\Gamma_2}(f_1 \otimes f_2 \otimes f_3) = \frac{\partial^3 \pi}{\partial x_{i_1} \partial p^{i_2} \partial p^{i_3}} \frac{\partial f_1}{\partial p^{i_1}} \frac{\partial f_2}{\partial x_{i_2}} \frac{\partial f_3}{\partial x_{i_3}} = c_{i_2, i_3}^{i_1} \frac{\partial f_1}{\partial p^{i_1}} \frac{\partial f_2}{\partial x_{i_2}} \frac{\partial f_3}{\partial x_{i_3}}.$$

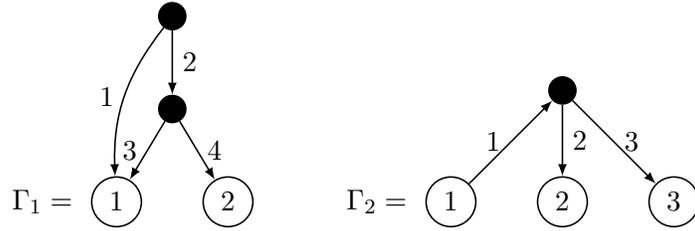


Figure 4.4: The graphs corresponding to the calculations given in Example 1.

Remark 43. For $n = 2$, when the graph $\Gamma \in \mathbf{dgraphs}(2)$ has no edges starting at any external vertex and f_1, f_2 lie in $\mathbb{K}[x_1, \dots, x_d] \cong \mathbf{Sg}$, the map B_Γ corresponds precisely to M. Kontsevich's bidifferential operator $B_{\Gamma, \pi}$ ([12], Section 2). Moreover, when $f_1, f_2 \in A$, they decompose as $f_i = a_i(x_1, \dots, x_d) b_i(p^1, \dots, p^d)$ ($i = 1, 2$), and if $\Gamma \in \mathbf{dgraphs}(2)$ has no edges starting at any external vertex, B_Γ satisfies,

$$B_\Gamma(f_1, f_2) = B_\Gamma(a_1, a_2) b_1 b_2 = B_{\Gamma, \pi}(a_1, a_2) b_1 b_2.$$

One can verify (see [12]) that up to order ϵ^2 (before setting $\epsilon = 1$), M. Kontsevich's product $\tilde{m}_\pi = \tilde{m}_{\text{Duf}}$ is given by,

$$\tilde{m}_\pi = \begin{array}{c} \circ \quad \circ \\ + \frac{1}{2} \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ 1 \quad 2 \end{array} - \frac{1}{12} \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \bullet \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ 3 \quad 4 \end{array} + \frac{1}{12} \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \bullet \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ 1 \quad 2 \\ 3 \quad 4 \end{array} + \frac{1}{8} \begin{array}{c} \bullet \quad \bullet \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ \circ \quad \circ \\ 1 \quad 2 \quad 3 \quad 4 \end{array} + \frac{1}{24} \begin{array}{c} \bullet \quad \bullet \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ \bullet \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ 1 \quad 4 \end{array} + \dots \end{array}$$

Example 2. The Chevalley-Eilenberg differential d_A on A is represented by the graph,

$$a_1 = \begin{array}{c} \bullet \\ | \\ \circ \end{array} = \begin{array}{c} \bullet \\ | \\ \circ \end{array} + \begin{array}{c} \bullet \\ | \\ \circ \end{array}.$$

For $f \in A$, we find,

$$d_A f = \frac{\partial \pi}{\partial p^l} \frac{\partial f}{\partial x_l} + \frac{\partial f}{\partial p^l} \frac{\partial \pi}{\partial x_l} = -2c_{il}^k x_k p^i \frac{\partial f}{\partial x_l} + \frac{\partial f}{\partial p^l} c_{ij}^l p^i p^j.$$

On basis elements it therefore acts as,

$$d_A x_m = -2c_{im}^k x_k p^i, \quad d_A p^m = c_{ij}^m p^i p^j.$$

Note that the usual convention is to define the Chevalley-Eilenberg differential as $-\frac{1}{2}d_A$.

Definition 30. A morphism $f : A^{\otimes n} \rightarrow A \in \text{End}_A(n)$ is called *universal* if there exists a graph $\Gamma \in \text{dgraphs}(n)$ such that $f = B(\Gamma) \in \text{End}_A(n)$. Any such morphism does not depend on the specific choice of the Lie algebra \mathfrak{g} .

4.4 Proof of Theorem 12

In Proposition 11 we have seen that as pre-Lie algebras,

$$\mathfrak{g}_{\text{As}, A} \cong \prod_{n \geq 1} \text{End}_{sA}(n).$$

By Definition 30, universal morphisms $f : (sA)^{\otimes n} \rightarrow sA$ are in bijection with homomorphisms $F : \text{As}^i(n) \rightarrow \text{End}_A(n)$ which factor through the map $B : \text{dgraphs}(n) \rightarrow \text{End}_A(n)$, i.e. homomorphisms for which there exists a morphism $\tilde{F} : \text{As}^i(n) \rightarrow \text{dgraphs}(n)$ making the diagram,

$$\begin{array}{ccc} \text{As}^i(n) & \xrightarrow{F} & \text{End}_A(n) \\ \tilde{F} \searrow & & \nearrow B \\ & \text{dgraphs}(n) & \end{array}$$

commute. Set,

$$\mathfrak{g}_{\text{As}, \text{dgraphs}} := \text{Hom}(\text{As}^i, \text{dgraphs}) = \prod_{n \geq 1} \text{Hom}(\text{As}^i(n), \text{dgraphs}(n)).$$

It forms a pre-Lie algebra, the product being given by the convolution product. A similiar identification as in Proposition 11 yields,

$$\mathfrak{g}_{\text{As,dgraphs}} \cong \prod_{n \geq 1} \text{dgraphs}(n) \otimes \mathbb{K} \mu_n^{c*} \cong \prod_{n \geq 1} s^{-n+1} \text{dgraphs}(n). \quad (4.5)$$

For $\Gamma_1 \in \text{dgraphs}(n)$ and $\Gamma_2 \in \text{dgraphs}(m)$, we set,

$$\Gamma_1 \star \Gamma_2 := \sum_{i=1}^n (-1)^{|\Gamma_2|(1-n)+(i+1)(m-1)} \Gamma_1 \circ_i \Gamma_2.$$

This defines a pre-Lie bracket on $\prod_{n \geq 1} s^{-n+1} \text{dgraphs}(n)$ which turns the bijections above into pre-Lie algebra isomorphisms.

Lemma 22. Let $\tilde{F}, \tilde{G} \in \mathfrak{g}_{\text{As,dgraphs}}$. Then, $(B \circ \tilde{F}) \star (B \circ \tilde{G}) = B \circ (\tilde{F} \star \tilde{G})$.

Proof. Since the respective pre-Lie products on $\mathfrak{g}_{\text{As},A}$ and $\mathfrak{g}_{\text{As,dgraphs}}$ are defined using the operadic composition, and B respects all such operations, the statement follows. \square

Lemma 23. There is a natural bijection,

$$\text{MC}(\mathfrak{g}_{\text{As,dgraphs}}) / \exp(\mathfrak{g}_{\text{As,dgraphs}}^0) \cong \{\text{universal } A_\infty\text{-structures on } A\} / \{\text{universal } A_\infty\text{-isomorphisms}\}.$$

Proof. A universal A_∞ -structure m on A corresponds to a Maurer-Cartan element $\alpha \in \text{MC}(\mathfrak{g}_{\text{As},A})$ for which there exists an $\tilde{\alpha} \in \mathfrak{g}_{\text{As,dgraphs}}$ such that $\alpha = B \circ \tilde{\alpha}$. By Lemma 22 we have,

$$0 = \alpha \star \alpha = (B \circ \tilde{\alpha}) \star (B \circ \tilde{\alpha}) = B \circ (\tilde{\alpha} \star \tilde{\alpha}),$$

which is equivalent to $\tilde{\alpha} \star \tilde{\alpha} = 0$ and $\tilde{\alpha} \in \text{MC}(\mathfrak{g}_{\text{As,dgraphs}})$. Moreover, universal A_∞ -isomorphisms correspond bijectively to elements of $\exp(\mathfrak{g}_{\text{As},A}^0)$ which factor through B . These may in turn be identified with elements of $\exp(\mathfrak{g}_{\text{As,dgraphs}}^0)$. Note that the discussion for $\mathfrak{g}_{\text{As},A}$ from Section 4.1.3 may also be applied to $\mathfrak{g}_{\text{As,dgraphs}}$ to define $\exp(\mathfrak{g}_{\text{As,dgraphs}}^0)$ and its action on $\text{MC}(\mathfrak{g}_{\text{As,dgraphs}})$. \square

Proposition 14. Consider the Maurer-Cartan element $\alpha_{\text{Duf}} \in \text{MC}(\mathfrak{g}_{\text{As,dgraphs}})$ corresponding to the universal A_∞ -structure $(d_A, \tilde{m}_{\text{Duf}}, 0, \dots)$. It is given by,

$$\alpha_{\text{Duf}} = \begin{array}{c} \bullet \\ | \\ \circ \end{array} + \begin{array}{c} \bullet \\ / \quad \backslash \\ \circ \quad \circ \end{array} + \begin{array}{c} \bullet \\ / \quad \backslash \\ \circ \quad \circ \end{array} + \frac{1}{2} \begin{array}{c} \bullet \\ / \quad \backslash \\ \circ \quad \circ \end{array} - \frac{1}{12} \begin{array}{c} \bullet \\ / \quad \backslash \\ \circ \quad \circ \end{array} + \frac{1}{12} \begin{array}{c} \bullet \\ / \quad \backslash \\ \circ \quad \circ \end{array} + \frac{1}{8} \begin{array}{c} \bullet \quad \bullet \\ / \quad \backslash \quad / \quad \backslash \\ \circ \quad \circ \quad \circ \quad \circ \end{array} + \frac{1}{24} \begin{array}{c} \bullet \quad \bullet \\ / \quad \backslash \\ \circ \quad \circ \end{array} + \dots$$

It is gauge equivalent to the following Maurer-Cartan element,

$$\alpha'_{\text{Duf}} = \begin{array}{c} \bullet \\ | \\ \circ \end{array} + \begin{array}{c} \bullet \\ / \quad \backslash \\ \circ \quad \circ \end{array} + \frac{1}{24} \begin{array}{c} \bullet \\ / \quad \backslash \\ \circ \quad \circ \end{array} + \dots$$

They are related via the gauge action of the element,

$$\xi = -\frac{1}{4}(\circ \rightarrow \circ - \circ \leftarrow \circ) - \frac{1}{16}(\begin{array}{c} \bullet \\ \swarrow \searrow \\ \circ \quad \circ \\ \uparrow \downarrow \\ \circ \end{array} - \begin{array}{c} \bullet \\ \swarrow \searrow \\ \circ \quad \circ \\ \downarrow \uparrow \\ \circ \end{array}) + \frac{1}{48}(\begin{array}{c} \bullet \\ \swarrow \searrow \\ \circ \quad \circ \\ \uparrow \downarrow \\ \circ \end{array} - \begin{array}{c} \bullet \\ \swarrow \searrow \\ \circ \quad \circ \\ \downarrow \uparrow \\ \circ \end{array}) + \dots$$

Proof. Rewrite α_{Duf} as

$$\alpha_{\text{Duf}} = a_1 + a_2 + \frac{1}{2}a_3 - \frac{1}{12}a_4 + \frac{1}{8}a_5 + \frac{1}{24}a_6 + \dots$$

Note that a_4 consists of the difference of two graphs, while the other a_i denote just one graph. Accordingly, we write,

$$\xi = -\frac{1}{4}\xi_1 - \frac{1}{16}\xi_2 + \frac{1}{48}\xi_3 + \dots,$$

where now ξ_i all correspond to the difference of two graphs as depicted in the proposition. It is easily verified that the terms contributing to $\alpha'_{\text{Duf}} = e^{\text{ad}\xi}\alpha_{\text{Duf}}$ up to a total number of four vertices are,

$$[\alpha'_{\text{Duf}}] = [\alpha_{\text{Duf}}] - \frac{1}{4}\text{ad}_{\xi_1}(a_1 + a_2 + \frac{1}{2}a_3) + \frac{1}{2!} \frac{1}{16}\text{ad}_{\xi_1}^2(a_1 + a_2) - \frac{1}{16}\text{ad}_{\xi_2}(a_1 + a_2) + \frac{1}{48}\text{ad}_{\xi_3}(a_1 + a_2), \quad (4.6)$$

where $[\alpha_{\text{Duf}}]$ ($[\alpha'_{\text{Duf}}]$) denotes the part of α_{Duf} (of $[\alpha'_{\text{Duf}}]$) having up to four vertices. To fix notation, set,

$$b = \begin{array}{c} \bullet \\ \swarrow \searrow \\ \circ \quad \circ \\ \uparrow \downarrow \\ \circ \end{array} = \begin{array}{c} \bullet \\ \swarrow \searrow \\ \circ \quad \circ \\ \downarrow \uparrow \\ \circ \end{array} + \begin{array}{c} \bullet \\ \swarrow \searrow \\ \circ \quad \circ \\ \uparrow \downarrow \\ \circ \end{array} + \begin{array}{c} \bullet \\ \swarrow \searrow \\ \circ \quad \circ \\ \downarrow \uparrow \\ \circ \end{array} =: b_1 + b_2 + b_3,$$

and

$$Q = -\begin{array}{c} \bullet \\ \swarrow \searrow \\ \circ \quad \circ \\ \uparrow \downarrow \\ \circ \end{array} - \begin{array}{c} \bullet \\ \swarrow \searrow \\ \circ \quad \circ \\ \downarrow \uparrow \\ \circ \end{array} + \begin{array}{c} \bullet \\ \swarrow \searrow \\ \circ \quad \circ \\ \uparrow \downarrow \\ \circ \end{array} - \begin{array}{c} \bullet \\ \swarrow \searrow \\ \circ \quad \circ \\ \downarrow \uparrow \\ \circ \end{array} - \begin{array}{c} \bullet \\ \swarrow \searrow \\ \circ \quad \circ \\ \uparrow \downarrow \\ \circ \end{array} + \begin{array}{c} \bullet \\ \swarrow \searrow \\ \circ \quad \circ \\ \downarrow \uparrow \\ \circ \end{array} \\ + \begin{array}{c} \bullet \\ \swarrow \searrow \\ \circ \quad \circ \\ \uparrow \downarrow \\ \circ \end{array} - \begin{array}{c} \bullet \\ \swarrow \searrow \\ \circ \quad \circ \\ \downarrow \uparrow \\ \circ \end{array} - \begin{array}{c} \bullet \\ \swarrow \searrow \\ \circ \quad \circ \\ \uparrow \downarrow \\ \circ \end{array} + \begin{array}{c} \bullet \\ \swarrow \searrow \\ \circ \quad \circ \\ \downarrow \uparrow \\ \circ \end{array}.$$

The graphical calculus allows us to compute the following identities. Note that some graphs cancel because of symmetries or the generalized IHX relations.

$$\begin{aligned} [\xi_1, a_1] &= 2a_3, \\ [\xi_1, a_2] &= 0, \\ [\xi_1, a_3] &= Q, \\ [\xi_1, [\xi_1, a_1]] &= 2[\xi_1, a_3] = 2Q, \\ [\xi_2, a_1] &= -a_4 + 2a_5, \\ [\xi_2, a_2] &= -[\xi_1, a_3] - b_1 - b_3 = -Q - b_1 - b_3, \\ [\xi_3, a_1] &= -2a_6 + a_4, \\ [\xi_3, a_2] &= -b_1 + 2b_2 - b_3. \end{aligned}$$

Inserting this back into equation (4.6) yields,

$$[\alpha'_{\text{Duf}}] = a_1 + a_2 + \frac{1}{24}(b_1 + b_2 + b_3) = a_1 + a_2 + \frac{1}{24}b.$$

□

Remark 44. The Maurer-Cartan element $\alpha_0 \in \text{MC}(\mathfrak{g}_{\text{As,dgraphs}})$ corresponding to the universal A_∞ -structure $(d_A, \tilde{m}_0, 0, \dots)$ is given by $\alpha_0 = a_1 + a_2$.

4.4.1 The cohomology of $H(\mathfrak{g}_{\text{As,dgraphs}}, \text{ad}_{a_2})$

Fix $n \geq 1$. Consider the polynomial coalgebra $P_n := \mathbb{K}[t_1, \dots, t_n]$. The variables t_i are of degree zero. It is equipped with the usual coproduct which on homogeneous elements is given by,

$$\Delta(t_{i_1} \cdots t_{i_k}) = 1 \otimes t_{i_1} \cdots t_{i_k} + \sum_{j=1}^{k-1} t_{i_1} \cdots t_{i_j} \otimes t_{i_{j+1}} \cdots t_{i_k} + t_{i_1} \cdots t_{i_k} \otimes 1.$$

Let $\Omega P_n = (Ts^{-1}P_n, d)$ be its cobar construction. Here, $d : (s^{-1}P_n)^{\otimes k} \rightarrow (s^{-1}P_n)^{\otimes k+1}$ is the degree one map explicitly given by the alternating sum,

$$d = \sum_{i=0}^{k+1} (-1)^i d_i,$$

where for $p \in (s^{-1}P_n)^{\otimes k}$, $d_0(p) = s^{-1} \otimes p$, $d_{k+1}(p) = p \otimes s^{-1}$ and for $p_i \in s^{-1}P_n$,

$$d_i(p_1 \otimes \cdots \otimes p_k) = p_1 \otimes \cdots \otimes \Delta(p_i) \otimes \cdots \otimes p_k.$$

On ΩP_n there is an \mathbb{N}_0^n -grading counting the number of t_1, \dots, t_n appearing in any monomial. The subspace of degree $(1, \dots, 1)$ elements (i.e. every variable occurs exactly once), denoted by $\Omega P_n^{(1, \dots, 1)}$, defines a subcomplex of ΩP_n . For instance, (up to suspension) $t_1 \otimes t_2 t_3$ lies in $\Omega P_3^{(1, 1, 1)}$, whereas $t_1 \otimes t_2 t_1$ does not. Note also that the S_n -action on $\Omega P_n^{(1, \dots, 1)}$ which permutes the variables is compatible with the differential.

Proposition 15. The cohomology of $(\Omega P_n^{(1, \dots, 1)}, d)$ is one-dimensional. More precisely, if we denote by,

$$\omega_n = \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{|\sigma|} s^{-1} t_{\sigma(1)} \otimes \cdots \otimes s^{-1} t_{\sigma(n)},$$

the totally antisymmetric element in $(s^{-1}P_n)^{\otimes n}$, then,

$$H(\Omega P_n^{(1, \dots, 1)}, d) = H^n(\Omega P_n^{(1, \dots, 1)}, d) = \mathbb{K} \cdot [\omega_n] \cong s^{-n} \mathbb{K}.$$

Proof. The cohomology of the complex $(\Omega P_n^{(1, \dots, 1)}, d)$ was computed for instance in ([8], Proposition 2.2) by V. Drinfeld, in ([4], Section 4.2) by D. Bar-Natan and in [15] by P. Ševera and T. Willwacher. \square

Lemma 24. Consider $\text{dgraphs}^{\text{uni}}(n)$ equipped with the S_n -action which permutes the labels of the external vertices. There is an isomorphism of complexes,

$$\left(s \text{dgraphs}^{\text{uni}}(n) \otimes_{S_n} \Omega P_n^{(1, \dots, 1)}, 1 \otimes d \right) \cong \left(\prod_{m \geq 1} s^{-m+1} \text{dgraphs}(m, n), \text{ad}_{a_2} \right).$$

A graphical interpretation of the isomorphism is given in Figure 4.5. In the following, we denote by $s^{-\bullet+1} \text{dgraphs}(\bullet, n) := \prod_{m \geq 1} s^{-m+1} \text{dgraphs}(m, n)$.

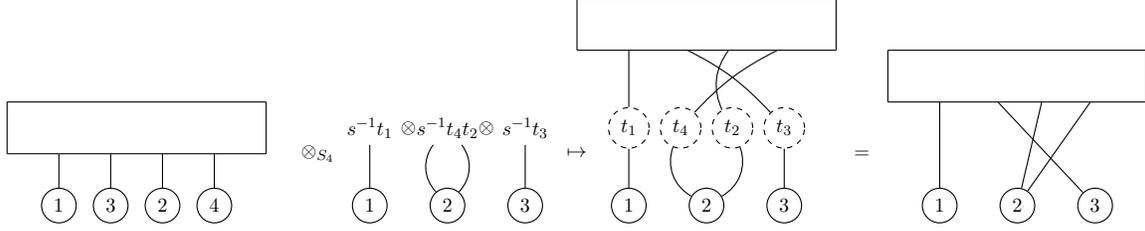


Figure 4.5: Schematic description of the isomorphism from Lemma 24. The element $\Gamma \otimes_{S_4} (s^{-1}t_1 \otimes s^{-1}t_4 t_2 \otimes s^{-1}t_3) \in \mathbf{dgraphs}^{\text{uni}}(4) \otimes_{S_4} \Omega P_4^{(1, \dots, 1)}$ is identified by this gluing procedure with a graph in $\mathbf{dgraphs}(3)$.

Proof. The identification as described in Figure 4.5 is bijective. Moreover, it is easy to check that the differentials $1 \otimes d$ and ad_{a_2} act in an equivalent way on their respective complexes. \square

Corollary 5. We have,

$$H(s \mathbf{dgraphs}^{\text{uni}}(n) \otimes_{S_n} \Omega P_n^{(1, \dots, 1)}, 1 \otimes d) = s \mathbf{dgraphs}^{\text{uni}}(n) \otimes_{S_n} \mathbb{K} \cdot [\omega_n].$$

Thus, $H(s^{-\bullet+1} \mathbf{dgraphs}(\bullet, n), \text{ad}_{a_2})$ is spanned by graphs with univalent external vertices which are totally antisymmetric with respect to the S_n -action permuting the external vertices. Moreover, by taking the direct product over all n , we obtain,

$$H(\mathfrak{g}_{\text{As, dgraphs}}, \text{ad}_{a_2}) \cong \prod_{n \geq 1} s \mathbf{dgraphs}^{\text{uni}}(n) \otimes_{S_n} \mathbb{K} \cdot [\omega].$$

Proof. The first part of the statement follows from the fact that taking coinvariants under finite group actions commutes with taking cohomology. The rest is a consequence of Lemma 24. \square

Lemma 25. The graph $b \in \mathbf{dgraphs}(3)$ represents a non-trivial class in $H^1(\mathfrak{g}_{\text{As, dgraphs}}, \text{ad}_{a_1+a_2})$ under the identification (4.5).

Proof. By Corollary 5, we know that b represents a non-trivial class in $H^1(\mathfrak{g}_{\text{As, dgraphs}}, \text{ad}_{a_2})$. It is also easily verified that $\text{ad}_{a_1}(b) = 0$. Moreover, b is cohomologous to the graph b' on the left in Figure 4.6. They satisfy $b = -b' + (\text{ad}_{a_1+a_2})(c)$, where c is the graph on the right in Figure 4.6. Since b' has no internal vertices it cannot be exact under $\text{ad}_{a_1+a_2}$, as all graphs in the image of the differential ad_{a_1} have at least one internal vertex. Thus, b is not exact and the statement follows. \square

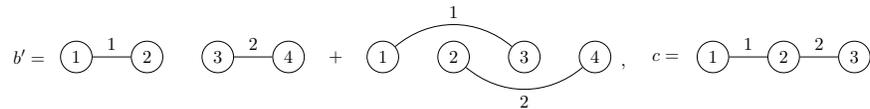


Figure 4.6: We have $b = -b' + [a_1 + a_2, c]$.

Corollary 6. The Maurer-Cartan elements α_{Duf} and α_0 are not gauge equivalent.

Proof. Define a second grading on $\mathfrak{g}_{As, dgraphs} \cong \prod_{n \geq 1} s^{-n+1} dgraphs(n)$ by

$$(\# \text{internal vertices} + \# \text{external vertices}) - 1.$$

The grading is compatible with the Lie algebra structure. The degree one part of α'_{Duf} , denoted by $\alpha'^{(1)}_{Duf}$, is given by $\alpha_0 = a_1 + a_2$. The degree three part $\alpha'^{(3)}_{Duf}$ equals $1/24 \cdot b$. Since $[b] \neq 0 \in H^1(\mathfrak{g}_{As, dgraphs}, \text{ad}_{\alpha_0})$, we may apply Lemma 21 to find that α_0 is not gauge equivalent to α'_{Duf} . The fact that α_{Duf} is gauge equivalent to α'_{Duf} establishes the result. \square

Proof of Theorem 12. The universal Maurer-Cartan elements α_{Duf} and α_0 correspond to the universal A_∞ -structures $(d_A, \tilde{m}_{Duf}, 0, \dots)$ and $(d_A, \tilde{m}_0, 0, \dots)$ on A . Since α_{Duf} is not gauge equivalent to α_0 , there is no universal A_∞ -isomorphism between these two structures, by Lemma 23. \square

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