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**FACULTÉ DE PSYCHOLOGIE
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Section des sciences de l'éducation

Sous la co-direction de Sylvie Coppé et Jean-Luc Dorier

**THE MATHEMATICAL EXPLORATION WITHIN THE
INTERNATIONAL BACCALAUREATE: INSTITUTIONAL
ANALYSIS AND CASE STUDIES OF PRACTICES OF TWO
MATHEMATICS TEACHERS AND THEIR STUDENTS IN
GENEVA**

THESE

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par

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**The Mathematical exploration within the International Baccalaureate:
institutional analysis and case studies of practices of two mathematics
teachers and their students in Geneva**

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Intitulée : « The Mathematical exploration within the International Baccalaureate: institutional analysis and case studies of practices of two mathematics teachers and their students in Geneva »

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ABBREVIATIONS

ATD:	anthropological theory of the didactic
CU:	cognitive universe
GDC:	graphic display calculator
HL:	higher level
IB:	International Baccalaureate
IBE:	inquiry-based education
IBL:	inquiry-based learning
IBME:	inquiry-based mathematics education
IBO:	International Baccalaureate Organization
IBSE:	inquiry-based science education
Me-Mi:	media-milieu
ME:	mathematical exploration
MMU:	media-milieu unit
Q-A:	question-answer
QAU:	question-answer unit
SL:	standard level
SRP:	study and research path
TDS:	theory of didactic situations
TOK:	theory of knowledge

INTRODUCTION AND PLAN

Passionate about mathematics, despite being myself a product of a very traditional school system where teaching and learning of mathematics were based on the unshakable foundations of hypothetico-deductive reasoning of the highly elevated sanctuary devoted to the gods of mathematics. I was lucky, I enjoyed and admired this rigorous beauty, but as a mathematics teacher, I quickly realized that only a minority of my students shared the same enthusiasm and the same convictions. The idealism of a young and passionate teacher has taken a slap until I was confronted with something called “inquiry-based learning” when I started a new job at a secondary school that offered the International Baccalaureate (IB) Diploma Program in Denmark. I was indeed part of the many teachers whose “first experience with explicit inquiry objectives” (Chichekian & Shore, 2014, p. 77) was provided by the IB. After a couple of years of professional experience as an IB mathematics teacher, I realized that it was not easy to implement inquiry-based activities into day-to-day teaching. It was time-consuming, there was pressure of the final external exams, the students were not used to it and there was a lack of resources. On the other hand, I also experienced many ‘eurekas’ when my students discovered a new property, understood a concept or solved a problem. In addition, one of the requirements of the IB was to supervise the students conducting a small investigation project into mathematics on a topic of their choice, referred to as Mathematical Exploration (ME). This experience raised my interest in mathematics education and research in the field of inquiry-based education. So naturally, when I had the opportunity to start doctoral research, I did not hesitate. The research team in mathematics education at the University of Geneva (DiMaGe¹) benefited already from taking part in an important European project PRIMAS devoted to promoting inquiry-based mathematics and science education and was thus ready and willing to transform my naïve ideas about inquiry into a doctoral research project. It was thus quite natural that my PhD research took me back to the IB where my intuitions for inquiry-based education were born and the ME became the heart of this research.

More precisely, the objective of our² PhD thesis is to observe how the ME contributes to implementing inquiry in mathematics and since it obviously can’t be done in an isolated manner, we need to place it in a more general context of the IB education. This thesis is therefore organized into three major parts: from an institutional study of the larger context through educational practices in the classroom to the implementation of the ME.

¹ Didactique des mathématiques Genève (DiMaGe)

² From now on, I will use the plural to talk about the work made in the doctorate.

Part A presents the context of the study, situates inquiry within the trends and traditions of mathematics education research, provides theoretical elements allowing to formulate the research questions and design the general methodology of the research.

At the very beginning of our research, we realized that the IB is generally not very well known outside of this institution. Therefore in Part A, Chapter 1, shall provide the context of this study by introducing the institutional context and, in particular, describing the IB curriculum and its assessment. As the ME is an object linked to inquiry objectives, Chapter 2 provides a review of literature to situate the emergence of inquiry-based education and its implementation within the research in mathematics education. Chapter 3 presents the theoretical framework of the Anthropological theory of the didactic (ATD) developed by Yves Chevallard. This theory provides tools to conduct an institutional study, to study the dissemination of knowledge through institutions and finally to model inquiry-based education. Considering the object of our research, the ATD was the best candidate to cover our research and enabled us to precise our research questions, lay out the general architecture of our theses and construct the methodology.

Part B is devoted to the institutional study of the IB. In order to understand the educational philosophy that forged the IB. It was indeed important to place it in its historical, ideological and sociological context; therefore we are proposing an institutional analysis in Chapter 1. In Chapter 2 we provide the results of a teacher survey that we conducted to identify the IB mathematics teachers' convictions concerning the IB and the implementation of inquiry-based education as well as their declared practices. To complete the big picture of the IB we provide an analysis of the final exams in Chapter 3.

In **Part C** we present a case study analyzing the teachers' and students' practices by comparing and contrasting the reality with the institutional expectations. This part is organized around five chapters. In Chapter 1 we focus on the ordinary classes and analyze practices of two observed IB mathematics teachers when attempting to put in place problem solving or investigation activities. In Chapter 2 we describe the context of the observation, explain the data treatment and introduce the framework that we developed to analyze the ME. Finally, we provide an insight into the institutional infrastructure of implementing the ME and describe the two mathematics practices when launching the ME. Chapters 3 and 4 are dedicated to detailed analysis of the work of two students. To conclude these two case studies, we provide a comparative conclusion of the two observed MEs in Chapter 5.

Finally, we propose a general conclusion in which we highlight the contributions of this PhD thesis at the theoretical level, point out the contributions and limitations of the institution in respect to the implementation of inquiry and present some perspectives.

PART A

RESEARCH CONTEXT, THEORETICAL FRAMEWORK AND GENERAL METHODOLOGY

In Part A, chapter 1 is devoted to a description of the institutional context of the IB, including the IB curriculum, the forms of assessment and the mathematics classes to situate the object of our research. Our initial questioning led us to encounter inquiry-based education (IBE) as an approach to teaching and learning and recognize the ME as an object linked to inquiry specific objectives. It is thus quite natural that chapter 2 is devoted to documenting its emergence within the existing educational research through a literature review in this field. Chapter 3 presents the general framework of our research. The Anthropological Theory of the Didactic (ATD) presented itself as a first choice. Indeed, the vast theoretical tools offered by the ATD theoretical framework enabled us to formulate our research questions and design an adapted methodology to answer them. At the end of this part, we present the research questions and introduce the general methodology that we devised to question the place of inquiry in the IB.

Chapter I

THE INSTITUTIONAL CONTEXT

The International Baccalaureate is a non-profit educational foundation established in Geneva in 1968 and currently offers a continuum of international education for students aged from 3 to 19 years old through 4 programs: Primary Years Program (PYP), Middle Years Program (MYP), Diploma Program (DP) and Career-Related Program (CP).

The IB Diploma Program offers a challenging, holistic, internationally oriented, and balanced education for students aged from 16 to 19. It was the first program designed by the IB, it covers the last two years of secondary education, and the diploma provides access to universities worldwide.

First, we look at the IB curriculum and the assessment in general and we finish this chapter with zooming on the mathematics syllabus and describe the Mathematical Exploration (ME).

1 The IB Curriculum

Students study and are examined in six subjects that they choose from each of the following groups:

Group 1: Studies in language and literature

- Language A: literature, available in 55 languages
- Language A: language and literature, available in 17 languages.
- Literature and performance: available in English, by special request in Spanish and French.

Group 2: Language acquisition

- Modern languages:
 - Language ab initio courses for beginners (students who have little or no previous experience of learning the language they have chosen).
 - Language B courses are intended for students who have had some previous experience of learning the language.
- Classical languages:
 - Latin or Classical Greek coursework provides opportunities for students to study the language, literature and culture of ancient Rome or Greece.

Group 3: Individuals and societies

- business management, economics, geography, global politics, history, information technology in a global society, philosophy, psychology, social and cultural anthropology, world religions

Group 4: Sciences

- biology, computer science, chemistry, design technology, physics, sports exercise and health science

Group 5: Mathematics (curriculum until 2019)

- Mathematical Studies (SL)
- Mathematics Standard Level (SL)
- Mathematics Higher Level (HL)
- Further Mathematics (HL)

Group 5: Mathematics (curriculum from 2020)

- Mathematics: analysis and approaches SL
- Mathematics: analysis and approaches HL
- Mathematics: applications and interpretation SL
- Mathematics: applications and interpretation HL

Group 6: The arts,

- dance, music, film, theater, visual arts

A subject choice from Group 6 can be replaced by an additional subject from groups 1 to 5.

In order to provide depth of learning, some degree of specialization is encouraged in the IB diploma by students choosing to study a minimum of 3 and a maximum of 4 subjects at higher level (HL) with the allocation of 240 hours of teaching. The remaining 3 respectively 2 courses are studied at a standard level (SL) with 150 teaching hours and thus contribute to a balanced and broad general education.³

In order to assure a holistic approach to education, the study of the academic subjects is supplemented by three core requirements:

- 1) Theory of knowledge (TOK)
- 2) The extended essay (EE)
- 3) Creativity, Action, Service (CAS)

³ On a basis of 30 weeks per year 240 hours represent roughly 4 hours per week over 2 years and 150 is 2.5 hours over 2 years. In terms of 45-minute periods this represents 5,5 periods per week for an HL course and 3,5 periods per week for an SL course.

The TOK course gives opportunities for discussion, reflection and instruction about the nature of human knowledge and develops interdisciplinary understanding. Schools are required to provide at least 100 teaching hours⁴ for the TOK course over the two-year program. Through the extended essay students conduct academic research of individual interest and develop research and writing skills needed for university education. The CAS program pushes the educational experience beyond the school grounds and encourages students to be involved in artistic, sports and community service activities.

Mathematics has a special position as discipline within the IB curriculum because it consists of a group in itself and it is thus mandatory for all students. Within this group, the IB offers four courses⁵ at different levels to meet the needs and interests:

- 1) Mathematics SL
- 2) Mathematics HL
- 3) Mathematical Studies SL
- 4) Further Mathematics HL

Mathematics SL and mathematics HL courses prepare students for university entry to pursue academic studies in scientific disciplines, respectively in mathematics. Mathematical Studies SL is a course offering more applied mathematics for students pursuing literary studies or humanities. The Further mathematics HL course, on the contrary, is a course offered to gifted students using Mathematics HL course as prerequisite.

2 Assessment

The IB mission statement mentions that “the organization works with schools, governments and international organizations to develop challenging programs of international education and rigorous assessment” (IBO, 2009b, p. 3). The main objective of the Diploma Program assessment is that “it should support curricular goals and encourage appropriate student learning”(IBO, 2009a, p. 10). Assessments are available in English, French or Spanish.

Each subject is graded on a scale from 1 to 7, there are also 3 possible bonus points for CAS, TOK and the EE which adds up to a maximum of 45 points. The minimum score needed to obtain the diploma is 24 under certain conditions. Certain universities nevertheless require higher scores to guarantee access and might not take the bonus points into account.

Most formal assessment is external in the form of examinations at the end of the two years or work completed during the course and sent to an external examiner. In each subject, some work is assessed internally, it is marked by the teacher and then moderated by an

⁴ On a basis of 30 weeks per year 100 hours represent roughly 1,5 hour or two 45-minute periods per week over 2 years.

⁵ These subject were in place until 2019

external moderator and is referred to as internal assessment. An internal assessment exists in each subject and contributes to about 20% of the final grade. It intends to evaluate students' performances and skills that are impossible to assess through written examinations or tests and gets students involved in investigations by conducting mini research projects or designing science experiments.

The Mathematical Studies course internal assessment requirement is a Math Project. For the Mathematics SL and HL courses, the internal assessment is the Mathematical Exploration (ME). The ME is "a piece of written work that involves investigating an area of mathematics" (IBO, 2012, p. 43). It is thus an individual, stand-alone research activity conducted on a topic chosen by a student and supervised by the mathematics teacher. The math guide recommends dedicating 10 hours of in-class time and 10 hours of personal work to the exploration. Its concrete implementation is left to the discretion of each school but is usually conducted from April to June in the first year or from October to January in the second year of the Diploma Program. Through this brief description, we have shown that the IB institution, while maintaining classical objectives and methods, is trying to introduce more innovative work, promoting problem-solving and taking it partially into account in the assessment.

Chapter II

LITERATURE REVIEW

In this chapter, we provide an overview of literature referring to inquiry. We trace it back to its philosophical foundations and compare and contrast the contributions of two important thinkers and educators: Dewey and Bachelard. Then we look at this concept that was applied to education, in particular to teaching and learning of science and mathematics and finally we consider the obstacles and benefits of its implementation.

1 The emergence of inquiry

The emergence of inquiry-based approaches is generally attributed to the American philosopher and educator John Dewey (1859–1952). However, the educational philosophy inspiring inquiry-based education (IBE) can be tracked back to the 17th century with the writings (*Didactica magna* or *Orbis pictus*) of Comenius (1592–1670), born in Moravia (Crown of Bohemia) and who according to Maaß & Artigue (2013, p. 781) complained that many people left school without permanent education pointing at the irrelevant content and the inefficacy of learning by reading and drills. Therefore, the idea of students having an active role in their learning to stimulate their curiosity and interest, is not new. Educators and philosophers such as Rousseau (1712–1778) and his followers Pestalozzi (1746–1827), Fröbel (1782–1852) and Herbart (1776–1841) or von Humboldt (1767–1835) “expressed and contributed to a shift in epistemology from seeing knowledge given as faith to knowledge based on thinking, reflection experimentation and science” (Artigue & Blomhøj, 2013, p. 798). Maaß and Artigue (2013, p. 781) also link the sources of IBE to the strong Hungarian tradition of problem-solving, especially to the writings of Pólya and to the work of psychologists such as Piaget, Vygotsky and Bruner. On top of that, when looking at the roots of inquiry, Artigue and Blomhøj (2013, p. 798) emphasize the importance of comparing and contrasting Dewey’s vision with the epistemology of Bachelard. We will now highlight, compare and contrast the work and contributions of the two most important thinkers: Dewey and Bachelard and provide an overview of definitions of inquiry-based approaches in mathematics education research.

1.1 Inquiry for Dewey and Bachelard

John Dewey, as a pragmatic philosopher, devoted his work to “overcoming the dualisms that afflict the field of education along with the rest of the modern world—the dualisms of thought and action, research and practice, science and common sense, the academy and everyday life” (Schön, 1992, p. 121). It is in his theory of inquiry that Dewey’s fight against

epistemological dualism culminates. Dewey (1938, pp. 104–105) defines inquiry as the controlled or directed transformation of an indeterminate situation into a determinate situation with the idea of converting the elements of the original situation into a unified whole. For Dewey, this indeterminate situation is the starting point for investigation. It is the situation itself which must question, arouse curiosity, it may even be conflicting or ambiguous. According to Dewey (1938, p. 108), the first indispensable step in the investigation will lead to the problematization of the situation, which otherwise risks losing its relevance. The next step in identifying the problem and finding a solution is observation, which could be a source of ideas or suggestions and lead to a relevant idea that would present a possible solution. He points out that one should beware of accepting ideas too quickly to avoid hasty and unfounded conclusions, as the sudden solution is usually the result of much prior research. Schön (1992, p. 122) concludes that the investigator is thus involved and takes ownership of the situation instead of being a mere external spectator. For Dewey (1938), we learn by doing (inquiry) and by thinking about what we do (reflection), which lead him to developing the concept of reflective inquiry and thus attempting to overcome the dualism of thought and action.

We can also observe attempts to deal with the problem of dualism by another important thinker: Bachelard, who contributed to the epistemology of science. According to Fabry (2021) “Bachelard’s philosophy is rationalism, but it is applied, and it is also materialism, but it is technical and instructed” (p. 4). Rationalism, according to Bachelard, puts forward the theory, emphasizing theoretical coherence and rigor. From this point of view, experiments are conducted in accordance with theoretical hypotheses and experimental knowledge has only value through theoretical interpretation. The place of experimentation is slightly different in empiricism: experimental knowledge can indeed reveal new facts, lead to discovering new theories or put existing theories to test. In philosophy, according to Bachelard as depicted by Fabry (2021) “these attitudes towards theory and experimentation have crystallized in two antithetic views on knowledge and science” (p. 5). In physicists’ practices, however, “rationalism and empiricism are two professional attitudes” and according to Bachelard “physics as a discipline lies at their intersection” (Fabry, 2021, p. 5). He claims that “if one of these two terms is missing, we can still do experiments and we can still do mathematics, but we cannot participate in the scientific activity of contemporary physical science” (Bachelard, 1953, p. 234, in (Fabry, 2021, p. 5). This led Bachelard to proceed to a certain conciliation between the two antagonisms of rationalism and empiricism and apply expressions such as *applied rationalism* to combine these two attitudes. According to Fabry (2021), Bachelard applied this philosophical approach to physics, “where mathematics must, by definition, be applied to the knowledge of physical reality” (p. 6). On the other hand, Bachelard also opposed the pure rationalism and accused it of misunderstanding “the way the mind constitutes knowledge because it underestimated the way our reasoning is shaped by experience” (Fabry, 2021, p. 6). To overcome this philosophical problem of dualism between theory and experiment,

Bachelard always referred to rationalism as applied and materialism as instructed “as the legitimate way of conceiving the relation between theory and experiment, giving them an equal epistemological weight and stressing their deep solidarity” (Fabry, 2021, p. 10). To resume Bachelard’s engagement to solve the two tensions, we borrowed the words of (Fabry, 2021):

Since *The New Scientific Spirit* (Bachelard, 2020; Eng. tr. 1984), his presentation of a dialectic between theory and experiment can, indeed, be regarded as a way of combining two requirements which seemed incompatible. Bachelard granted that mathematical reasoning does have the power to “get ahead of observation and predict laws which refine and extend experimental laws” (Bachelard, 1973: 168), but he denied that it could do so independently from a perpetual dialogue with experiment. On the other hand, he granted that reality could exceed our rational constructions, but denied that this should be regarded as a fundamental limitation of scientific knowledge and rather claimed that this limit manifests itself in the form of concrete technical difficulties or experimental objections, which can only lead to a refinement of our theoretical knowledge. (p. 10)

For both, Dewey and Bachelard, the continuous effort to overcome the dualism between theory and experimentation is a crucial aspect of acquiring knowledge. However, their epistemological approaches to tackle the problem of this existing dualism are not the same. According to Fabre (2005), for Dewey and Bachelard the idea of problematization is in the center of inquiry, respectively research and both situate it into a semantically rich environment of the complexity of a real-life problem or scientific research. Dewey (1938) claims that a problematic situation is never completely undetermined and the capacity to formulate and test hypotheses belongs to knowledge of questioning often confused with the art of answering. For Bachelard, according to (Fabre, 2005), the most important scientific competence is the capacity to pose problems; indeed problems do not appear on their own. Both, Bachelard and Dewey, insist on the importance of the construction of problems before even considering solving them. They both criticize empiricism taking observation for foundation of knowledge which implies a refusal of immediate access to knowledge as depicted by Fabre (2005). For Dewey, all knowledge is the result of an inquiry and for Bachelard:

Avant tout, il faut savoir poser des problèmes. Et quoi qu’on dise, dans la vie scientifique, les problèmes ne se posent pas d’eux-mêmes. [...] Pour un esprit scientifique, toute connaissance est une réponse à une question. S’il n’y a pas eu de question, il ne peut y avoir de connaissance scientifique. Rien ne va de soi. Rien n’est donné. Tout est construit.⁶
(Bachelard, 1967, p. 17)

According to Fabre (2005), the main difference between Dewey and Bachelard consist in their perception of the nature of inquiry. Indeed, if Dewey positions the nature of scientific,

⁶ Our translation of the French original: First of all, one must know how to pose problems. And whatever one may say, in scientific life, problems do not arise by themselves. [...] For a scientific mind, all knowledge is an answer to a question. If there is no question, there can be no scientific knowledge. Nothing is self-evident. Nothing is given. Everything is constructed.

professional or current life inquiries at the same level, Bachelard on the contrary, distinguishes between common sense and scientific thinking. This divergent position results in two different visions of the inquiry process. For Dewey, it is a generic process regardless the nature of the inquiry and a progressively built process based on making connections between specific methods general concepts (Artigue et Blomhøj, 2013). From the point of view of educational practice, this difference positions Dewey on the side of project-based pedagogy aiming to develop “inquiry habits of mind considered as generic” (p. 800). On the other hand, Bachelard’s approach aiming at the construction of clearly specified concepts “leads to the careful organization of students’ experience and inquiries to allow them to face the limitation of common sense, overcome the epistemological obstacles inherent in the progression of scientific knowledge regarding specific areas and concepts, and progressively structure and connect their knowledge, incorporating local constructs into more regional perspectives. (Artigue & Blomhøj, 2013, p. 800)

After these epistemological reflections, we continue our review by examining how these philosophical concepts were brought into educational practice.

2 From philosophical concepts to educational practices

Already Dewey in 1916, as a former science teacher, pointed out that “there was too much emphasis on facts without enough emphasis on science for thinking and an attitude of the mind” (Barrow, 2006, p. 266). Barrow (2006) depicts several features in Dewey’s approach: “the student is actively involved”, “the teacher has role as facilitator and guide”, the studied problems “must be related to students’ experiences and within their intellectual ability” (p. 266). The launching of Sputnik I in 1957 was a trigger element leading to question and to rethink the teaching of sciences in the USA. The National Science Foundation (NSF) funded the development of an innovative science curriculum and consequently the professional development for its implementation “with an emphasis on ‘thinking like a scientist’ (DeBoer,1991)” (Barrow, 2006, p. 266). Another thinker, professor of education and natural sciences at the University of Chicago, Joseph Schwab⁷ “believed that students should view science as a series of conceptual structures that should be continually revised when new information or evidence is discovered” (Barrow, 2006, p. 266) and recommended to use inquiry to teach science. As a result of the momentum of the Sputnik effect combined with the influence of prominent thinkers, inquiry became a central topic in the policy documents aiming to reform the K-12 science curriculum. Barrow (2006) provides an overview of the various policy documents that succeeded one another within the Project 2061⁸. The first one issued by the American Association for the Advancement of Science

⁷ <https://education.stateuniversity.com/pages/2401/Schwab-Joseph-1909-1988.html>

⁸ <https://www.aaas.org/programs/project-2061>

(AAAS) in 1989, entitled *Science for All Americans* focused on defining scientific literacy. It even contained a chapter *Habits of the Mind* where:

inquiry was considered as a science content topic using the following recommendations: start with questions about nature, engage students actively, concentrate on the collection and use of evidence, provide historical perspective, insist on clear expression, use a team approach, do not separate knowledge from finding out, and deemphasize the memorization of technical vocabulary. (Barrow, 2006, p. 267)

The next document, *Benchmarks for Scientific Literacy* from 1993, organized the topics to be taught by grades while a more recent document *Atlas of Scientific Literacy from 2001* proposed their interpretation of inquiry and how inquiry can be treated as a content.

Finally, the release of the new National Science Education Standards (NSES) in 1996, whose prominent feature was a focus on inquiry, was another important step in concretizing and implementing inquiry into science education “in reaction to the observed failure of traditional forms of instruction” (Artigue & Blomhøj, 2013, p. 800). For these authors, the introduction of the term inquiry-based education (IBE) “implies that it is possible and meaningful to have it as a dominant feature of an educational program and that the associated form of learning has certain specific qualities” (p. 797).

In Europe, the IBE ideas have generated a strong political will in response to the declining interest of young people in scientific fields, as suggested by the Rocard report (Rocard et al., 2007). They point out the fact that despite the consensus of the education community concerning the benefits of IBE, the current science teaching does not follow these methods. Rocard et al. (2007) remind that the deductive approach also referred to as *top-down transmission*, where the “the teacher presents the concepts, their logical—deductive implications and gives examples of applications” (p. 9) is traditionally used at schools. They are opposing it with the inductive approach which “gives more space to observation, experimentation and the teacher-guided construction by the child of his/her own knowledge” (p. 9).

In recent years, several projects (Fibonacci, PRIMAS, S-Team) and working groups have stated in their reports that it is necessary to “rethink the teaching of science through inquiry” (Coquidé, Fortin, & Rumelhard, 2009, p. 51). Matheron (2010) reminds two points characterizing mathematics education today: “a crisis manifested by a loss of social visibility of its meaning” and “an institutional desire for teaching that engages in authentic scientific activity” (p. 14). These alarming findings resulted in the EU launching and financing several projects⁹ with the objective “to support the further development and implementation of IBE in mathematics (IBME) and science (IBSE)” (Artigue & Blomhøj, 2013, p. 797).

⁹ For more information about the various EU projects concerning IBE see <http://www.scientix.eu>

2.1 Inquiry-based science education (IBSE)

When referring to inquiry, several generic definitions are in the circuit. In the NSES inquiry is defined as:

... a multifaceted activity that involves making observations; posing questions; examining books and other sources of information to see what is already known; planning investigations; reviewing what is already known in light of experimental evidence; using tools to gather, analyze, and interpret data; proposing answers, explanations and predictions; and communicating the results. Inquiry requires identification of assumptions, use of critical and logical thinking, and consideration of alternative explanations and scientific inquiry refers to the diverse ways in which scientists study the natural world and propose explanations based on the evidence derived from their work. (National Research Council, 1996, p. 23)

The Rocard report refers to inquiry as to “the intentional process of diagnosing problems, critiquing experiments, and distinguishing alternatives, planning investigations, researching conjectures, searching for information, constructing models, debating with peers, and forming coherent arguments” (Rocard et al., 2007, p. 9). What these definitions have all in common is that they place the students in the center of the activity and invite them “to work in ways similar to how mathematicians and scientists work” (Artigue & Blomhøj, 2013, p. 797).

In the NSES (National Research Council, 2000) inquiry is considered as both science content and as a way to learn science and is thus treated “as both a learning goal and as a teaching method” (p. 18). The National Research Council (NRC) put in evidence that students whose teachers used the new curriculum materials,

were spending large amounts of time in inquiry-based activities. They were making observations, manipulating materials, and conducting laboratory investigations. As a result, they were developing cognitive abilities, such as critical thinking and reasoning, as well as learning science content (Bredderman, 1982; Shymansky et al., 1983). (National Research Council, 2000, p. 18)

The NSES (National Research Council, 2000) also depict that learning through inquiry requires to develop abilities necessary to conduct a scientific inquiry and. The expected abilities are designed for all grades from primary to upper secondary classes at a growing level of complexity. Below we provide the description of the fundamental abilities to do scientific inquiry for secondary students (Grades 9–12) as formulated in the NSES:

- Identify questions and concepts that guide scientific investigations.
- Design and conduct scientific investigations.
- Use technology and mathematics to improve investigations and communications.
- Formulate and revise scientific explanations and models using logic and evidence.
- Recognize and analyze alternative explanations and models.
- Communicate and defend a scientific argument. (National Research Council, 2000, p. 19)

Another feature of learning through inquiry is according to the NSES the development of fundamental understanding about scientific inquiry. All grade 9–12 students should thus understand that:

- Scientists usually inquire about how physical, living, or designed systems function.
- Scientists conduct investigations for a wide variety of reasons.
- Scientists rely on technology to enhance the gathering and manipulation of data.
- Mathematics is essential in scientific inquiry.
- Scientific explanations must adhere to criteria such as a proposed explanation must be logically consistent; it must abide by the rules of evidence; it must be open to question a possible modification; and it must be based on historical and current scientific knowledge.
- Results of scientific inquiry—new knowledge and methods—emerge from different types of investigations and public communication among scientists. (National Research Council, 2000, p. 20)

Having considered inquiry from the side of learning outcomes raises necessarily a question of how teaching through inquiry should look like and led the conceivers of the NSES (National Research Council, 2000) to the formulation of science teaching standards. Below we provide examples of some of these standards that are directly linked to inquiry:

- Teachers of science plan an inquiry-based science program for their students.
- Encourage and model the skills of scientific inquiry, as well as the curiosity, openness to new ideas and data, and skepticism that characterize science.
- Structure the time available so that students are able to engage in extended investigations.
- Create a setting for student work that is flexible and supportive of science inquiry.
- Teachers of science develop communities of science learners that reflect the intellectual rigor of scientific inquiry and the attitudes and social values conducive to science learning.
- Structure and facilitate ongoing formal and informal discussion based on a shared understanding of the rules of scientific discourse.
- Model and emphasize the skills, attitudes, and values of scientific inquiry. (National Research Council, 2000, pp. 22–23)

Since the NSES standards are very broad and general, the NRC decided to focus more on inquiry in the classroom and proposes to “distinguish inquiry-based teaching and learning from inquiry in general sense and from inquiry as practiced by scientists.” (National Research Council, 2000, p. 24) This distinction led to identifying and formulating five essential features of classroom inquiry:

1. Learners are engaged by scientifically oriented questions.
2. Learners give priority to **evidence**, which allows them to develop and evaluate explanations that address scientifically oriented questions.

3. Learners formulate **explanations** from evidence to address scientifically oriented questions.
4. Learners evaluate their explanations in light of alternative explanations; particularly those reflecting scientific understanding
5. Learners communicate and justify their proposed explanations. (National Research Council, 2000, p. 25)

According to NRC (2000), “these essential features introduce students to many important aspects of science while helping them develop a clearer and deeper knowledge of some particular science concepts and processes” (p. 27). The NRC oversees the many facets and forms that inquiry can take; from a very guided or partial to open or full inquiry. The variation in the amount of guidance and structure in the teacher’s instruction (Appendix 1) determine the “degree” of inquiry. The inquiry is considered to be full, when each of the above five essential elements of classroom inquiry is present. The NRC (2000) also claims that:

Guided inquiry can best focus learning on the development of particular science concepts. More open inquiry will afford the best opportunities for cognitive development and scientific reasoning. Students should have opportunities to participate in all types of inquiries in the course of their science learning. (p. 30)

As far as the process of inquiry is concerned Artigue and Blomhøj (2013) referring to Dewey, emphasize two features: the importance of the formulation of questions and the dialectic between inductive and deductive phases. In this sense, a new approach stems from the problem-solving tradition: problem posing (Cai et al., 2013; Kilpatrick, 1987; Silver & Cai, 1996). This approach emphasizes the importance of the construction of questions to be answered later. In addition, Silver and Cai (1996) claim that the ability to pose problems improves the capacity to solve them. Nonetheless, researchers have not attached as much importance to knowing how to pose problems as to knowing how to solve them (Liljedahl, Santos-Trigo, Malaspina, & Bruder, 2016, p. 32).

Artigue and Blomhøj (2013) also make it clear that “the process of scientific inquiry process cannot be reduced to a linear process” but “involves several cycles with revision of previous steps and more or less complex interactions between its different components” (p.800). As an example, this non-linearity can be observed on the model describing inquiry process provided in the Fibonacci project¹⁰ as shown in Figure 1.

¹⁰ <http://www.fibonacci-project.eu/>

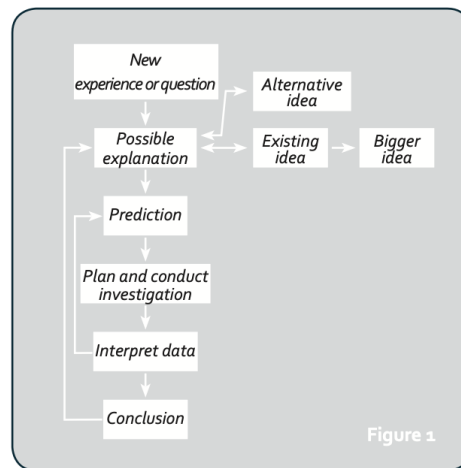


Figure 1: Inquiry process in the Fibonacci project (Artigue, Dillon, Harlen, & Léna, 2012, p. 7)

In the Fibonacci project (Artigue et al., 2012), learning through scientific inquiry refers to “the process of building understanding through collecting evidence to test possible explanations and the ideas behind them in a scientific manner” (p. 7). For Artigue and Blomhøj (2013), this model of inquiry process resonates with Bachelard’s ideas, and “the importance given to the progressive elaboration of ‘big ideas’ can be interpreted as a way of expressing the movement from local to regional levels in the progression of scientific knowledge” (p. 801). As inquiry is considered a particularly important way of learning and teaching involved in science education, Artigue and Blomhøj (2013) raise the questions of the appropriateness of inquiry-based instruction and its limits as well as the minimal requirements that educational IBSE practices should obey to. Keeping this overview of the role and place of inquiry in experimental sciences, let us examine now if and how this concept could be imported and applied within the long tradition of problem-solving approaches in mathematics teaching and learning.

2.2 Inquiry-based mathematics education (IBME)

As already mentioned above, IBE initially emerged from experimental sciences and was thus naturally first implemented into science education. The Rocard report (2007) claims that “in mathematics teaching, the educators’ community often refers to ‘Problem-Based Learning’ rather than to IBSE” because in many cases, experimental approaches turn out to be more difficult. There have, nevertheless, been successful attempts to conceptualize and define IBME (Artigue, 2012; Artigue et al., 2012; Artigue & Blomhøj, 2013; Maaß & Artigue, 2013; Bosch & Winsløw, 2016; Dorier & Maaß, 2020).

Since Euclid’s *Elements*, we associate mathematics with hypothetical-deductive reasoning, but we know that the Greek mathematicians did not find all the properties in the way these were written up. There has, indeed, been a reconstruction due to the passage to writing and the pre-eminence of synthesis over analysis. The mathematician Jean-Paul Allouche (2007, in Hersant & Orange-Ravachol, 2012, p. 1381) indicates that there are various types of

experiments in mathematics, which provide ‘a pool of conjectures’ but, in scientific publications, mathematicians convey their results as straight forwardly and accurately as possible. However, this does not mean that they have not made trials, taken blind paths or formulated conjectures, some of which may have turned out to be false and have thus been rejected. It is true that the methods of experimentation and validation are not identical in science and mathematics, but seeking to draw a demarcation line between them, or even attempting to oppose them, is not satisfactory, as Hersant and Orange-Ravachol (2012) explain. Using the examples of experimental approaches (establishing an approximation of the value of π or the formula for the area of the circle), Matheron (2010, pp. 18–19) shows that experimentation finds its place in the investigative approach in mathematics. Perrin (2007) goes even further, for him “mathematics is also an experimental science and a science of observation” (p. 10). The method of systematic investigation that he proposes “includes several phases, which may eventually be repeated: experimentation, observation of the experiment, formulation of conjectures, attempt of proof, counter-experiment (possible production of counter-examples), formulation of new conjectures, new attempt of proof, etc.”¹¹ (Perrin, 2007, p. 10). Gardes (2013) considers two principles of the experimental character of mathematics:

Le premier est l’existence d’un mode empirique de constitution de certains objets mathématiques. Le recours à cette dimension expérimentale entraîne alors le second principe: la mise en œuvre d’une démarche expérimentale qui se caractérise par des allers et retours entre les objets (naturalisés et/ou en cours de naturalisation) par des confrontations, des vérifications et des argumentations. (pp. 34–35)¹²

The experimentation according to Gardes (2013) ensures the articulation between the empirical and theoretical components of the experimental method. Chanudet (2019) concludes:

La démarche de type expérimental, comme caractérisée ci-dessus, est par ailleurs proche de la démarche scientifique telle que définie dans le cadre du problème ouvert. Les deux démarches se retrouvent sur les phases d’établissement de conjecture et de tentative de preuve mais se distinguent quant à la place accordée aux essais/à l’expérience et à l’aspect dynamique de la démarche. La caractérisation de la démarche scientifique met l’accent sur la présence des phases d’essais, de conjecture, de preuve, tandis que celle associée à la

¹¹ Our translation from the French original: La méthode d’investigation systématique qu’il propose « comprend plusieurs étapes, à répéter éventuellement : expérience, observation de l’expérience, formulation des conjectures, tentative de preuve, contre-expérience (production éventuelle de contre-exemple), formulation de nouvelles conjectures, nouvelle tentative de preuve, etc. » (Perrin, 2007, p. 10)

¹² Our translation of the French original: “The first is the existence of an empirical mode of constitution of certain mathematical objects. The recourse to this experimental dimension then leads to the second principle: the implementation of an experimental approach that is characterized by pendling back and forth between the objects (naturalized and/or in the process of naturalization) through confrontations, verifications and argumentations” (Gardes, 2013, pp. 34–35)

démarche de type expérimental souligne la dialectique entre expérimentation, formulation et validation. (p. 37)¹³

Artigue et al. (2012) also claim that developing appropriate education strategies when implementing IBME “must acknowledge the experimental dimension of mathematics and the new opportunities that digital technologies offer in support” (p. 8). Finally, Dias (2008) while clearly assigning mathematical activity to problem-solving, recognizes that it is fully appropriate to associate it with an experimental approach. This way “induction and abduction regain a preponderant place without forsaking the deduction necessary at the time of the establishment of the proof which represents the specificity of mathematics compared to the other experimental sciences” (p. 37)¹⁴.

If inquiry-based education (IBE) is quite naturally linked to experimental sciences, it does not benefit from the same tradition when applied to mathematics. The relatively recent migration of IBE into mathematics as compared to science, requires, according to Artigue and Blomhøj (2013), to be conceptualized from the point of view of “existing theories and forms of practice in mathematics education” (p. 798). Since “for Dewey, all knowledge results from an inquiry” (Fabre, 2005, p. 58), there is, however, no epistemological reason to prevent the development and implementation of IBME in mathematics education without obviously putting aside its long tradition of problem-solving. In the French tradition of the didactics of mathematics, there is rather a variety of terminology to designate objects more or less close to the investigation approaches: the open problem and the situation-problem (Arsac, Germain, & Mante, 1991) or the situation-research (Duchet & Mainguené, 2003). Georget (2009) brings them together under the generic term of research and peer review activity and introduces five potentials¹⁵ to better characterize them. Artigue and Blomhøj (2013) also link IBME to problem-solving tradition in mathematics but not only. They mention several theoretical frameworks and traditions that have the potential to contribute to the conceptualization of IBME: theory of didactic situations (TDS) from Guy Brousseau, realistic mathematics education developed by Hans Freudenthal, modeling perspective and the anthropological theory of the didactic (ATD) initiated by Yves Chevallard. The ATD framework provides tools for both: to model and

¹³ Our translation of the French original: The experimental approach, as characterized above, is also close to the scientific approach as defined in the case of the open problem. The two approaches are similar in the phases of establishing a conjecture and in attempting to prove it, but they differ in the role given to trials/experiments and in the dynamic aspect of the approach. The characterization of the scientific approach emphasizes the presence of the phases of testing, conjecture, and proof, while that associated with the experimental type approach emphasizes the dialectic between experimentation, formulation and validation. (Chanudet, 2019, p. 37)

¹⁴ Our translation from the French original: Ainsi induction et abduction retrouvent une place prépondérante sans pour autant délaisser la déduction nécessaire au moment de l'établissement de la preuve qui représente la spécificité des mathématiques par rapport aux autres sciences de l'empirie. (Dias, 2008, p. 37)

¹⁵ This concerns the potentials of research, debate, resistance and dynamic resistance and the didactic potential.

analyze inquiry (Bosch & Winsløw, 2016) and to implement inquiry via the study and research activities (SRA) and the study and research paths (SRP) (Chevallard, 2009a; Winsløw, Matheron, & Mercier, 2013). In the ATD (Chevallard, 2009a), inquiry is implemented via the development of study and research activities (SRA) or study and research paths (SRP). These can be carried out in secondary schools (Parra, Otero, & Fanaro, 2013; Parra & Otero, 2018; Gazzola, Otero, & Llanos, 2019) or designed and conducted over a one-semester university course in mechanical engineering (Florensa, Bosch, Gascón, & Mata, 2016).

In the Fibonacci project (Artigue et al., 2012), IBE is considered appropriate not only for teaching and learning of science but also finds its place in teaching and learning of mathematics. They point out the similarities and depict some differences between IBSE and IBME. The main point of convergence is that IBME “refers to an education which does not present mathematics to pupils and students as a ready-built structure to appropriate” (Artigue et al., 2012, p. 8). They also emphasize the experimental dimension of mathematics but remind that in mathematics, experimentation “is not limited to what is usually called ‘real world’” (p. 8), because diverse mathematical objects such as numbers or geometrical forms constitute a proper field for experimentation on mathematics. In addition, the development of digital technologies enhances exploration of patterns and other forms of intra-mathematical investigations. It is thus necessary to keep in mind that IBME doesn’t just “rely on situations arising from real world phenomena, even if the consideration of these is, of course, very important, but uses the diversity of contexts which can nurture investigative practices in mathematics” (Artigue et al., 2012, p. 8). An important feature specific for mathematics depicted by Artigue et al. (2012) is its cumulative dimension. It is thus important that in IBME, students are invited to make connections instead of just dealing with isolated problems, regardless of their challenge, because

mathematical tools developed for solving particular problems need to build on each other to become methods and techniques which can be productively used for solving classes of problems, eventually leading to new mathematical ideas and even theories, and new fields of application. (p. 9)

Another crucial aspect of inherent of mathematics is that mathematical objects are not directly accessible and are generally accessed through semiotic systems such as graphs, tables, figures or symbols. That is why IBME “must be sensitive to this semiotic dimension of mathematical learning and to the progressive development of associated competencies, without forgetting the evolution in semiotic potential and needs resulting from technological advances” (Artigue et al., 2012, p. 9). Last but not least, when considering IBME one needs to take into account the impact of modern digital technologies as these ensure immediate access to information and as a result, “the ‘milieux’¹⁶ in which students

¹⁶ This concept from the theory of didactic situations is developed in detail in Chapter III

can interact in investigative practices are potentially much richer than those usually used for developing investigative practices in mathematics” (Artigue et al., 2012, p. 9).

In our work, we retain the definition of IBME from PRIMAS published in the encyclopedia of mathematics education as it represents and summarizes the essential aspects of inquiry-based mathematics education:

Inquiry-based mathematics education (IBME) refers to a student-centered paradigm of teaching mathematics and science, in which students are invited to work in ways similar to how mathematicians and scientists work. This means they have to observe phenomena, ask questions, look for mathematical and scientific ways of how to answer these questions (like carrying out experiments, systematically controlling variables, drawing diagrams, calculating, looking for patterns and relationships, and making conjectures and generalizations), interpret and evaluate their solutions, and communicate and discuss their solutions effectively. (Dorier & Maaß, 2020, p. 2014)

In the next part, we examine the possibilities and eventual obstacles in respect to the implementation of inquiry into day-to-day teaching.

3 Implementation of inquiry into day-to-day teaching

Since 2009, the EU has funded several important international projects (PRIMAS, Fibonacci) whose objective was to promote and disseminate inquiry-based learning. In the previous section, we mostly focused on the learning part of IBME, i.e., what happens on the side of the learner. It is, however, also important to flip the coin and examine the teaching side of IBME even though it cannot be completely separated from learning. Dorier and Maaß (2020) point out that “in spite of research evidence and political pressure, IBME remains quite marginal in day-to-day mathematics teaching and often limited to softer versions compared to more ambitious experiments”, which “raises the issue of the role to be given to IBME in teachers’ training and professional development courses” (p. 302). Furthermore, in order to obtain this change “it seems essential (yet not sufficient) that teachers have a chance to experience this type of teaching personally in their own mathematical or professional training” (Dorier & Maaß, 2020). Maaß and Artigue (2013) point to the fact that the role of the teacher in IBME differs significantly from traditional transmissive teaching approaches. It requires put in place several changes such as fostering “students’ construction of knowledge through inquiry, exploring, and finding their own paths to solution” (p. 782), supporting collaborative work and orienting students towards interesting questions and problems and ensuring their learning potential. Another important feature of inquiry-based teaching is guidance, it is indeed important that “students are not left alone in their discovery but are guided by the teacher who supports them in learning to work independently” (p.782). Bell et al. (Bell, Smetana, & Binns, 2005) point out that “not all inquiry activities are equal” (p. 31) and refer to a four-level model of

inquiry instruction (Appendix 2) illustrating “how inquiry-based activities can range from highly teacher directed to highly student oriented” (p. 31). The discrepancies in teachers thinking to be doing inquiry-based teaching and actually doing meaningful inquiry-based activities led Marshall et al. (Marshall, Horton, & White, 2009) to develop a protocol to guide and improve the quality of inquiry-based instruction considering five specific aspects (time usage, instruction, discourse, assessment and curriculum); these are further characterized by specific indicators. The indicators concerning the aspect of instruction are provided in (Appendix 3). In their tool they also distinguish 4 levels of inquiry: pre-inquiry, developing inquiry, proficient inquiry and exemplary inquiry). Ideally, teachers should be aspiring to reach levels 3 and 4 of inquiry-based instruction which remains, nevertheless, quite challenging endeavor.

Artigue and Blomhøj (2013) describe more specifically the conditions under which inquiry can develop in the classroom. There must be a challenging element of the unknown in the initial situation. However, this part of the unknown must be able to be approached by what is already known in order to be able to generate ideas or hypotheses, which according to these authors “constitute a major didactic challenge when implemented in educational practice” (p. 799). Referring to Dewey’s contribution, Artigue and Blomhøj (2013) point out that the concept of *reflective inquiry* could serve as a basis for pedagogical practice. Hiebert et al. (1996) also suggest revisiting Dewey’s concept of reflective inquiry and propose an alternative approach when considering problems and problem-solving in mathematics. According to them, the main difficulty in teaching and learning mathematics through problem-solving resides in the distinction between acquiring knowledge and applying it, which they judge inappropriate for education and claim that even problem-based courses do not resolve this somewhat artificially created dualism. Reflective inquiry has got three fundamental features:

- Identification of a problem or problematization
- Studying a problem through an active engagement
- Reaching conclusions after problems are (at least partially) resolved

To implement reflective inquiry into educational practices Hiebert et al. (1996) propose to consider two aspects of understanding:

- 1) a functional understanding (pointing to social organization and the culture of a classroom) and based on Dewey’s vision of knowing which “is not the act of outside spectator but of a participator” (Dewey, 1929, p. 196)
- 2) a structural understanding (pointing to the content knowledge, i.e., what will students retain after the activity)

On the one hand, Hiebert et al. (1996) put forward that “reflective inquiry and problematizing depends more on the student and on the culture of the class than on the

task” (p. 16), without obviously neglecting the content of the task whose careful selection falls under the responsibility of the teacher. Indeed,

to select appropriate tasks, the teacher must draw on two resources: knowledge of the subject to select tasks that encourage students to wrestle with key ideas and knowledge of students’ thinking to select tasks that link with students’ experience and for which students can see the relevance of the ideas and skills they already possess. (Hiebert et al., 1996, p. 16)

Nevertheless, “the culture of the classroom will determine how tasks are treated by the students” (Hiebert et al., 1996, p. 16). This leads them to a promising conclusion that teachers can finally find and use a greater variety of tasks to help students problematize mathematics. In order to approach acquiring of content knowledge, on the other hand, Hiebert et al. (1996), build on three ideas:

(1) Dewey’s (1929) idea that knowledge is the fruit of activity that resolves problematic situations, (2) Brownell’s (1946) observation that understanding is better viewed as a by-product of activity than as a direct target of instruction, and (3) Davis’s (1992) more recent formulation of this idea as the residue that gets left behind when students solve problems. (Hiebert et al., 1996, p. 17)

These authors (Hiebert et al., 1996) conclude that “when students develop methods for constructing new procedures, they are integrating their conceptual knowledge with their procedural skill” (p. 17), because “students who treat the development of procedures as problematic must rely on their conceptual understanding to drive their procedural advances” (p. 17).

According to Dorier and García (2013), “IBMSE is a complex a multifaceted notion” (p. 838) whose essential ingredients fall into five areas situated around the teacher guidance, valued outcomes, classroom culture, type of questions and students’ actions as shown in Figure 2.

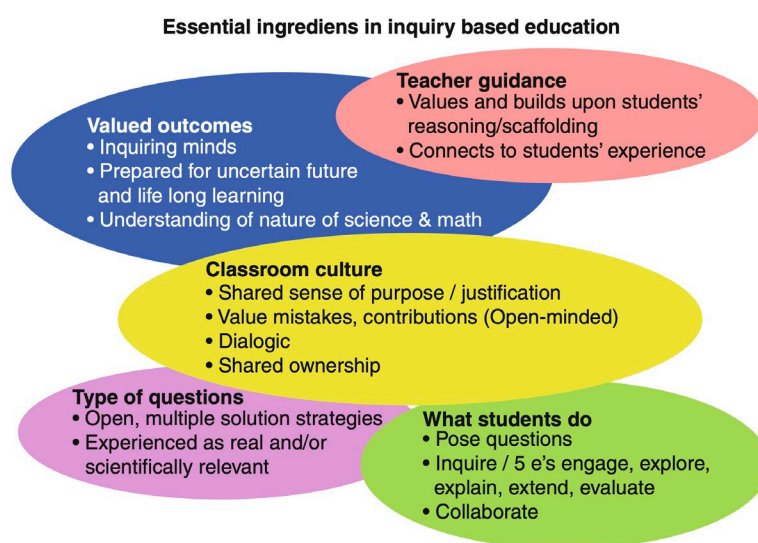


Figure 2: Essential ingredients in inquiry-based education (Dorier & García, 2013, p. 838)

This shows that the implementation of inquiry into daily teaching requires a completely different infrastructure than typical transmissive approaches to teaching and learning. From the institutional point of view, Dorier and García (2013) conducted within the PRIMAS project a large-scale study in the context of 12 countries and “point to the major types of conditions and constraints in the various institutional levels of each country that could favor or hinder a large-scale implementation of IBMSE” (p. 848). In this study, they have identified five areas (1) national policies, (2) didactic resources, (3) national assessment, (4) pre-service teachers’ training, and (5) in-service training and professional development) and discuss what further action is required and is possible to support the implementation of IBMSE. We do not provide a detailed review of this study but rather pinpoint some of the findings. To foster the implementation of IBMSE, it is vital to develop appropriate material and resources that support IBMSE and it is more crucial that these resources come with didactic comments specifying how it could be used. Under certain conditions, inquiry-specific assessment could also play an important role in fostering the implementation of IBMSE. For Dorier and García (2013), “teachers are the actors and vectors of changes and consequently play a central role” (p. 838) therefore pre-service and in-service training could also offer opportunities for implementation of IBMSE. This study shows, however, that the needs of primary and secondary school teachers are not the same. Future primary school teachers “must be offered sufficient support in order to overcome their general disinclination towards mathematics and sciences and to deepen their knowledge in these sciences in order to be able to use IBMSE in class flexibly” (Dorier & García, 2013, p. 848). For secondary school teachers, generally confident in their subject knowledge, the question to tackle is to convince them about the benefits of IBMSE. Bruder and Prescott (2013) also emphasize the importance of the teacher, in particular in relation to a careful choice of tasks:

As with any teaching, the choice of tasks and the actions of the teacher have an important effect on the student learning. Teachers must choose the task for the students carefully, ensuring that the tasks are challenging but not demotivating because they are too difficult. The Guided Inquiry strategy where the teacher has a supportive role in the students’ learning is the most effective in several aspects of student learning, with tasks that extend the students’ problem-solving. (p. 18)

Implementing inquiry in everyday teaching and learning presents a certain amount of challenge and it seems important to us to consider and examine more in detail what might be detected as possible obstacles when attempting to implement inquiry in day-to-day teaching.

3.1 Obstacles in implementation of inquiry

Barrow (2006) provides a more general overview of some other possible obstacles:

Welch et al. (1981) recognized reasons that teachers do not use inquiry and identified limited teacher preparation, including management; lack of time, limited available materials; lack

of support; emphasis only on content and difficult to teach. Subsequently, Eltinge and Roberts (1993) identified three reasons for avoiding inquiry (state documents emphasizing content, easier to access content, and textbooks' emphasis of science as a body of knowledge). (p. 267)

Even though these refer to implementation of inquiry-based instruction for experimental sciences but we can see that they could easily apply to mathematics, too.

The main obstacle hindering the implementation of inquiry is according to Chevallard (2015b) due to the current prevailing paradigm of visiting works where each piece of knowledge “is approached as a monument that stands on its own, that students are expected to admire and enjoy, even when they know next to nothing about its *raison d'être*, now or in the past” (p. 175). Chevallard (2015b) claims that as a result of this ‘visiting monuments’ approach, most people fall into the trap of *retrocognition*, which refers to a “cognitive attitude that leads one to refer preferentially and almost exclusively to knowledge *already known* to one”, in other words “if you don’t know in advance the answer to the question that faces you, then you’d better renounce all pretension to arrive at a sensible answer (p. 178). Research with the ATD framework (Chevallard & Matheron, 2002; Matheron, 2010) also points to the problem of a missing infrastructure when reacting to the appearance of interdisciplinary projects¹⁷ in French secondary schools. Matheron (2010) claims that without changing certain conditions of ordinary teaching,

les activités se présentant comme d’investigation, issues de bricolages plus ou moins heureux, risquent fort de rester « le plus souvent et fondamentalement au service de l’illustration d’un contenu conceptuel » (Coquidé, Fortin, Rumelhard, 2009), et non de sa production par les élèves au sein d’un processus effectif qui les engagerait dans la recherche. (Matheron, 2010, p. 35)¹⁸

Unfortunately, as pointed out by Dorier and García (2013), “even explicitly IBMSE or problem-solving orientated documents can be used by teachers in an inadequate manner, leading to very poor practice in reality” (p. 845). Bruder and Prescott (2013) report that “some studies showed that IBL needs a high level of content knowledge and strategies for problem-solving as a precondition for success” on the one hand, and “often there are problems for the students in posing mathematical questions or making hypothesis” (p. 820).

Dorier and García (2013) depict that some obstacles appear when national policies are concerned. Indeed, in several countries, a series of successive reforms led to a rejection of changes by teachers and sometimes even parents and to the reappearance of ideas claiming

¹⁷ In French TPE: Travaux pratiques encadrés

¹⁸ Our translation of the French original: [activities] that present themselves as investigative, resulting from more or less successful tinkering, are likely to remain "most often and fundamentally at the service of the illustration of a conceptual content" (Coquidé, Fortin, Rumelhard, 2009), and not of its production by the students within an effective process that would engage them in research. (Matheron, 2010, p. 35)

a return traditional educational practices which is also an important factor hindering the implementation of IBMSE.

Other two obstacles depicted by Chichekian and Shore (2014) concern directly the International Baccalaureate (IB). On the one hand, IB teachers often reported that “they felt rushed and overwhelmed due to the amount of content to cover in time for the IB examinations (Hertberg-Davis & Callahan, 2008)” (Chichekian & Shore, 2014, p. 82). On the other hand, “teachers were observed to teach to the test (Hertberg-Davis & Callahan, 2008; Hertberg-Davis et al., 2006)” often because “the validation of their own teaching quality and effectiveness was attached to students’ examination scores (Mayer, 2008)” (Chichekian & Shore, 2014, p. 85). For Dorier and García (2013), content-oriented assessment “puts added pressure on teachers to cover the program; thus, it is a hindering factor for IBSME” and is indeed seen “as time-consuming for little efficiency in terms of assessment results” (p. 848). It might, however, “be seen as a key lever to make the situation evolve”, especially making “large-scale assessment more compatible with IBMSE would undoubtedly turn the situation around, changing assessment from a strong restriction into a powerful driving force” (p. 848). Due to the obvious difficulty organizing assessment compatible with IBMSE and the lack of research in this matter (Dorier & García, 2013), assessment acts currently rather as a break than an accelerator with regards to the implementation of IBMSE. Bruder and Prescott (2013) also point to the problem of creating appropriate assessment for IBL.

Despite the above-mentioned difficulties that IBE promoters must tackle to ensure its successful implementation, there are already studies pointing to some benefits procured by IBE.

3.2 The benefits and critiques of IBE

Dorier and García (2013) claim that “nowadays there is a large consensus about the benefits of IBMSE (Hiebert et al. 1996; Feldon et al. 2010; King et al. 2008; Minner et al. 2010)” (p. 838) among the scientific community. This claim is also confirmed by Chichekian and Shore (2014) who consider that “inquiry is an effective approach to teaching and learning, when done well” (p. 76). Their review of literature specifies that:

Potential outcomes of inquiry have been shown to encourage the development of self-regulation and metacognitive strategies (e.g., organizing information, goal setting, monitoring understanding, self-evaluating), to enhance critical thinking and problem-solving skills, to promote curiosity and confidence, and ultimately to foster the motivation for autonomous learning (Llewellyn, 2002, 2005; Saunders-Stewart, Gyles, & Shore, 2012). (Chichekian & Shore, 2014, p. 76)

Bruder and Prescott (2013) provide a large overview of empirical studies and research evidence on the benefits of IBE and conclude that “most IBL projects showed some positive gains for the students—sometimes the gains were in mathematics and science content, but

more often the gains concerned the processes” (p. 817). In addition, the recognized effects contribute to motivation, enhance understanding of mathematics, affect the beliefs about mathematics and their relevance for life and society. They also consider the effects of IBL in respect to the form of inquiry distinguished as structured, guided and open. These correspond to levels 2, 3, respectively 4 of the 4-level inquiry model (Appendix 2). To briefly remind the differences; in structured inquiry the teacher provides the problem as well as the method, in guided inquiry the teacher gives the problem or question and provides material, but the choice of strategies and methods is left upon the students, in open inquiry it is the students’ responsibility to find a problem or a question and decide about the methods and strategies to tackle it. According to Bruder and Prescott (2013), the most important gains in terms of content and process were identified within guided inquiry projects. Even though such a generalization requires some prudence, “there is a growing body of research showing that students learn more deeply from strongly guided learning...” (p. 817).

On the other hand, critical voices were also raised against inquiry-based approaches to teaching and learning science and mathematics. Most of these criticizing studies concern, according to Bruder and Prescott (2013), discovery learning and Kirscher et al. (Kirschner, Sweller, & Clark, 2006) criticize open inquiry and conclude that minimally guided instructional approaches are ineffective and inefficient, putting thus constructivist, discovery, problem-based, experiential and inquiry-based forms of instruction into the same bag. This criticism has resulted in a strong response with the objective to dismantle their arguments (Hmelo-Silver, Duncan, & Chinn, 2007). These authors show that IBL and PBL do not fall into the category of minimal guidance instruction and provide an overview of evidence-based research supporting the benefits of scaffolded inquiry and problem-solving. They also point out that the question of “Does it work?” is not a relevant one. For them,

the more important questions to ask are the ones such as under what circumstances do these guided inquiry approaches work, what are the kinds of outcomes for which they are effective, what kinds of valued practices do they promote, and what kinds of support and scaffolding are needed for different populations and learning goals. (Hmelo-Silver et al., 2007, p. 105)

Anyway, they invite us to leave this dichotomy in thinking and arguing in terms of for and against that might be useful in defending an ideology but neither lead to very productive exchange nor contribute to advancing of the cause.

3.3 New forms of study in secondary schools’ curricula

Finally, it is also important to recognize a certain institutional will to create conditions to promote teaching that combines open research and the intersection of several disciplines. In this part, we propose to look at some of the projects implemented by school systems in

France, Switzerland and Denmark and in the IB that are required for high school graduation.

In 2000, the French Ministry of Education introduced the interdisciplinary project called TPE¹⁹, a new compulsory course for all students in all general high school series, based on an interdisciplinary approach and aiming "to develop students' capacities for autonomy and initiative in researching and exploiting documents, with a view to producing a written and oral summary" ("Travaux Personnels Encadrés," n.d.). This work is carried out in groups of a maximum of four students under the direction of their teacher. The Official Bulletin of the French Ministry of Education (2005, p. 2175) specifies the assessment methods and distinguishes three important components: the student's personal approach and engagement, the response to the problem and the oral presentation of the project. This system seems to respond to the desire to make students active and to evaluate skills that do not easily lend themselves to traditional evaluation through standardized exams. However, as Chevallard and Matheron (2002) point out, the implementation of the TPEs is not self-evident, because this new device disrupts the established habits and represents "an authentic break with the ordinary teaching process" (p. 148). These same authors also raise the difficulties linked to the elaboration of a real problematic situation around the chosen subject, the danger being that the oral presentations become simple lectures.

In Switzerland, the students in secondary schools are introduced to research work is through a project called 'Travail de maturité'. The regulation on the recognition of Gymnasium Matura Certificates mentions in article 10 that "each student must complete, alone or in teams, an independent assignment of a certain importance. This work will be the result of a written text or commentary and an oral presentation" (RRM, 1995)²⁰. The implementation may, however, vary according to the directives of the cantons and even from one institution to another in the same canton. The proposal to introduce personal written work into the curriculum of gymnasium students dates back to 1969. According to the EVAMAR report (Secrétariat d'Etat à l'éducation et à la recherche (SER), 2011), the initial project required students to do "independent work in two or three subjects" (p. 283) that would be evaluated in the form of an interview. However, the assessment of this work was not mandatory until the 1990s.

Among the Scandinavian countries, we can give the example of the specialized study project of the STX program in Danish gymnasiums, whose objective is also to introduce students to research work and which leads to the preparation of a written report.

¹⁹ <http://eduscol.education.fr/cid47789/definition-et-themes-nationaux-des-tpe.html#lien1>

²⁰ https://os.zhdk.cloud.switch.ch/tind-tmp-edudoc/c19479cd-22d5-4f47-bf25-114d5155d390?response-content-disposition=attachment%3B%20filename%3DUTE-8%27%27VO_MAR_f.pdf&response-content-type=application%2Fpdf&AWSAccessKeyId=1e2525c929074b90b98e19eb95e6256c&Expires=1668603143&Signature=WsW4GbVFbDHW22imY1wHAnOl9%2FY%3D

The common features of these schemes are that it is work carried out independently under the direction of the teacher, compulsory and evaluated, generally interdisciplinary, which aims to introduce the students to scientific methods.

In the IB the students must carry out some sort of research project in each subject and the one in mathematics is called a Mathematical Exploration (ME). The main difference with the objects cited above is that the ME is not an interdisciplinary project but must be carried out individually, on a topic of choice in mathematics and that is also why we decided to make it the object of our research.

In this literature review, we have followed the concept of inquiry from its emergence, through its philosophical roots to the implementation into science and mathematics education. We saw that questions about learning with empirical rather than deductive methods were asked a long time ago and are coming back to the stage, for example, in the Rocard report, which was commissioned to reflect on the growing lack of interest for sciences and mathematics in Europe. The teachers' practices are being challenged and need to evolve, but the progress is slow because of the many constraints and obstacles hindering the implementation of IBE. The national curricula introduced new forms of study as compulsory part of secondary education with the aim to introduce students to some research activity and interdisciplinary approaches. This being said, we set on to explore the place devoted to inquiry within the IB, to examine whether the Mathematical Exploration carries any IBME features and hope to contribute to a better comprehension of IBME.

Chapter III

THEORETICAL FRAMEWORK, RESEARCH QUESTIONS AND METHODOLOGY

In this chapter, we present the theoretical framework of the ATD and adapt some of its concepts to model the ME as a potential inquiry process. Afterwards, we formulate the research questions and provide an overview of the general methodology used in this research.

1 Introduction

The Anthropological Theory of the Didactic (ATD) is undeniably linked with the name of Yves Chevallard, a French mathematician and didactician of mathematics. The first developments of this framework date back to the '80s with questions concerning the didactic transposition (Chevallard, 1985). Since then this research framework has known a constant evolution and has pursued its “explicit aim to experimentally model the knowledge and know-how that is at the core of teaching and learning processes” (Bosch, Chevallard, García, & Monaghan, 2019, p. xii). From the ATD perspective, the objective of didactics is to explain and understand the mechanisms that are behind the diffusion of knowledge within institutions and among individuals (Bosch, 2018). Artigue and Blomhøj (2013) highlight that the ATD provides a theoretical framework that is particularly well adapted to approach the inquiry-based education (IBE). Certain ATD concepts and tools are thus going to be essential for our work, and we develop them more in detail in the following parts of this chapter. We focus, on the one hand, on the tools that enable us to conduct an institutional analysis, in particular the *didactic codeterminacy scale* and, on the other hand, on the concepts that allow us to describe any activity such as a *praxeology* and to model and capture the dynamics of an inquiry process such as the *Herbartian scheme* and the bipolar tensions referred to as *dialectics*. To complete our toolbox we are also going to revisit the concept of the *milieu* and reflect on the *student's cognitive universe*.

2 Analyzing an institution in the ATD

According to Bosch and Gascon (2002), one cannot correctly interpret teaching practices without studying the institution in which the teachers practice their profession. To consider the viability of inquiry-based activities in a classroom, it is essential to look at what is happening in the higher echelons of the didactic codeterminacy scale (Figure 3) because “what can be done at such a level [...] depends on the constraints and conditions created

by the higher echelon” (Chevallard, 2005, p. 241). The short glossary of the ATD (Bosch et al., 2019, pp. xx–xxi) defines a *condition* as anything pretending to have influence over some system and thus over the values of a set of variables that govern this system. A *constraint* is any condition that cannot be modified by the actors of a given institutional position. Often, for any condition, there exists at least one position from which this condition can be modified.

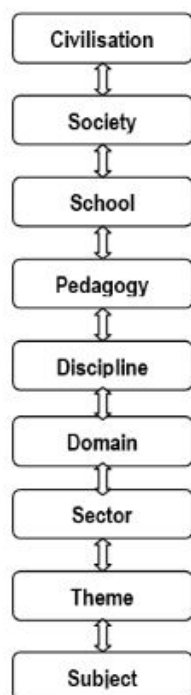


Figure 3: The didactic codeterminacy scale for the paradigm of visiting works

The higher levels of the scale refer to the conditions and constraints to how teaching and learning processes are organized in general. These include for instance “the way teaching and learning processes are conceived and managed in *societies* [...]” or even among *civilizations* (Bosch, 2018, p. 4003). The level of *school* concerns the infrastructure put in place by educational institutions such as organizations of teachers and students, courses and modules, physical and virtual spaces, time schedules, final examinations, etc. The different instructional formats (interactive lectures, participative tutorials, etc.) and teaching traditions (cooperative learning, discovery learning, etc.) defined independently of any particular content are identified at the level of *pedagogy*. Bosch highlights that the fact the responsibility of applying these formats to specific content is left out to teacher “is not always a trivial affair...” (2018, p. 4003). The lower levels of the scale correspond to what is specific to the content. A set of selected pieces of knowledge are organized in a curriculum and determine specific knowledge organizations that according to their size can be found at the lower levels of the didactic codeterminacy scale (discipline, sector, domains, theme and subject).

As part of the European project PRIMAS, Dorier and García (2013) used the didactic codeterminacy scale to categorize the conditions and constraints regarding the integration of education based on inquiry. To analyze the context in the 12 partner countries, they considered the higher levels of the scale (society, school, pedagogy and discipline), in which they define the following indicators:

1. Levels of society: Specific role of mathematics and sciences in society, tradition or recent changes in education relevant regarding IBMSE.
2. Level of school (global organization): Differentiation between primary, lower and upper secondary education, pre-service and in-service teachers' training structures, etc.
3. Level of pedagogy: Law of education, general statement on pedagogy, tradition in education (transmissive or constructivist tradition, place of the learner...), type and role of national assessments, etc.
4. Level of discipline: Links between mathematics and sciences in the curricula, integrated science or separate subjects, etc., place of mathematics and sciences in the curricula (number of hours), competence of teachers in mathematics and sciences (profile of teachers), type of curricula in mathematics and sciences (signs of IBMSE?), type of resources for teachers in mathematics and sciences (textbooks, web, etc.). Are mathematics and science teachers using IBMSE? Why? If it is a requisite in the curriculum, even in the textbooks, why not? (Dorier & García, 2013, pp. 839–840).

We refer to this work as a basis for describing the IB context and identifying the role and the place of IBE at the institutional level. The ATD researchers also reacted to the appearance of new scholar forms (personal projects, interdisciplinary project, etc.) and enlarged the theory's existing and introduced some new tools to address this phenomenon. In the following parts, we depict the tools and notions from the ATD that enable us to model inquiry-based education in general and apply them to specifically model the ME.

3 Modeling inquiry in ATD

After considering the tools enabling us to conduct an institutional analysis, in this part we use the tools and concepts from the ATD, we build a model for the Mathematical exploration. In order to do so, we visit the concept of praxeology, consider the dynamics of an inquiry process, zoom on the concept of the milieu from the point of view of two major theoretical frameworks and describe the cognitive universe of the student.

3.1 The concept of a praxeology

The fundamental principle of the ATD is that every human activity can be described in terms of *praxeology*. A praxeology consists of the know-how part called *praxis* and the *logos*, the organized discourse about the praxis (Chevallard, 2002b). The praxis block is composed of a certain *type of tasks*, accomplished using a certain *technique*, and justified by a *technology*, which is based on a *theory*, the two latter ones forming the logos block. Therefore, in the

ATD, the transmission of knowledge can be modeled through the notion of praxeology and its dissemination takes mainly place through *didactic systems*. A didactic system is created when a group of individuals Y (teachers/lecturers) and another group of individuals X (pupils/students) gather in order to study or work with a given piece of knowledge or praxeology. This is symbolically represented as a tern $S(X, Y, \wp)$.

3.2 Towards a change in didactic paradigms

Chevallard (2005, 2015b) recalls that the dominant paradigm of school study is that of visiting works in which the teacher's role consists in making students visit the designated mathematical works to be taught. He calls this phenomenon the *monumentalization* of the teaching of mathematics, that is to say that one studies mathematical objects without having access to their meaning, nor their *raisons d'être*. In opposition and willing to encourage a procognitive attitude, "which inclines one to behave as if knowledge was essentially still to conquer" (Chevallard, 2015b, p. 179), he defines a new didactic paradigm: *the paradigm of questioning the world* (Chevallard, 2009a, 2015b).

What the new didactic paradigm aims to create is a new cognitive ethos in which, when any question Q arises, x^{21} will consider it, and, as often as possible, will study it in order to arrive at a valuable answer A, in many cases with a little help from some y. In other words, x is supposed not to systematically balk at situations involving problems that he/she never came across or never solved (Chevallard, 2015b, p. 178).

Bosch (2018) mentions that the potential shift from the paradigm of visiting works to the paradigm of questioning the world would require "important changes in the organization of study processes at the different levels of the scale of didactic codeterminacy" (p. 4019). Indeed, in this change the didactic systems are no longer formed around a given praxeology, but rather around a question which often raises the need to search for answers in other disciplines. This led to a slight adaptation of the didactic codeterminacy scale, leading to placing the levels above the didactic system vertically above the levels under the discipline organized horizontally as shown on Figure 4.

²¹ x in ATD stands for a student or a learner in general

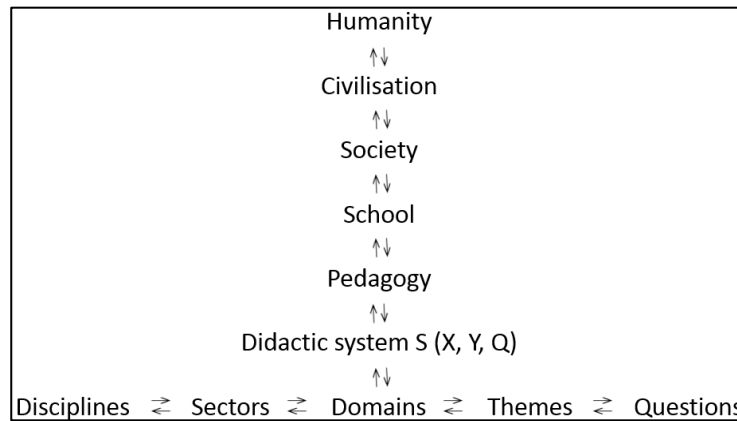


Figure 4: The didactic codeterminacy scale for the paradigm of questioning the world

In the paradigm of questioning the world (Chevallard, 2009b), a didactic system (a group of students X and a group of teachers Y) is formed in order to investigate a question Q and provide an answer $A^\heartsuit$. The superscript symbol ($\heartsuit$) means that $A^\heartsuit$ is the official answer to Q produced and agreed on by the didactic system satisfying a certain number of constraints (Chevallard, 2008; Bosch, 2018). We can see a change from “traditional teaching” because the didactic system $S(X, Y, Q)$ is formed around a question instead of around a given praxeology, and the inquiry allows them to work on several praxeologies often issued from different disciplines. In other words, in the ATD, to engage in an inquiry, means to engage in a study and research paths (SRP) (Chevallard, 2009a). In order to find the answer $A^\heartsuit$, the didactic system has to build a learning environment—*didactic milieu* M , which consists of *ready-made* answers $A^\diamond$ validated by the institution and other works W (theories, experiments, etc.) which are analysis tools for validating or rejecting partial responses. Following the type of inquiry these could be eventually completed by some derived questions Q and data collection D . The Herbartian scheme was proposed to model the transmission of knowledge (Chevallard, 2008) and applied to different learning contents (Bosch & Winsløw, 2016). Figure 5 represents a condensed way to model inquiry-based activities.

$$S(X, Y, Q) \rightarrow M = \{A_i^\diamond, W_j, Q_k, D_l\} \rightarrow A^\heartsuit$$

Figure 5: The Herbartian scheme for an inquiry process

A study process through inquiry, as suggested by Chevallard (2002a) can be modeled as an intertwining of five-step cycles: *observation* of the answers $A^\diamond$, *experimental or theoretical analysis* of the answers $A^\diamond$, *evaluation* of the answers $A^\diamond$, *development* of an answer $A^\heartsuit$, *defense and illustration* of the produced answer $A^\heartsuit$. The study of a question Q and formulation of an answer $A^\heartsuit$ is therefore a result of a rather dynamic process (represented by the two arrows in the above-mentioned scheme) which according to Chevallard (2002a) should be carried out in a workspace structured by bipolar tensions, the so-called dialectics.

3.3 The dynamics of the inquiry process

3.3.1 The role of dialectics in an inquiry process

The construct of dialectics as an important condition for an inquiry process was introduced by Chevallard (2014) as a reaction to the appearance of guided personal projects²² in French secondary schools. At the same occasion, he adds that in traditional school systems these remain largely blocked. In the ATD “dialectics are key gestures to achieve a teaching by SRP” (Parra & Otero, 2018, p. 242) and currently nine dialectics were identified during an SRP. In the following part, we propose a compilation of several works that provide definitions, characteristics (Chevallard, 2002a, 2008; Gazzola et al., 2019) and indicators (Parra & Otero, 2018) of the nine dialectics.

D1 Research and study (R-S) or Questions and answers (Q-A) dialectic

This dialectic refers to asking questions and elaborating answers continuously, that is, to study and research the knowledge that might be appropriate to answer Q. The research extends to a broad context while the study focuses on specific knowledge. Three types of indicators help to identify this dialectic: a search (on the Internet, in the library, consulting teachers or other experts, etc.), a study of available answers A_i or the works W_j and a formulation of derived questions Q_k .

D2 Media and milieu (Me-Mi) dialectic

The information from media (including the teacher) is admitted into the milieu M after being tested, while the answers to the partial questions are developed. In opposition to traditional teaching, in an SRP the teacher is not a privileged source of information and therefore answers coming from the teacher should be objected to the same evaluation as from any other media. Media refers to any system containing information (textbooks, websites, video tutorials, experts, etc.) and the milieu is a system that is supposed to be devoid of intention with respect to the answer it contributes to. This dialectic appears when the information from a media is questioned (“why”—or “how”—questions are asked) or different answers are studied.

D3 Individual and collective (I-C) dialectic

The study of a question Q often requires giving individual responsibilities and assigning tasks which will then be presented to the whole group during the process leading to a possible collective answer. Some examples indicating the presence of this dialectic are: a group decision is taken by the students, an individual uses a production of the group, the teacher and the students decide what to study, etc.

D4 Skydiver and truffle-seeker (S-Ts) dialectic

²² In French: Travaux personnels encadrés

This dialectic puts in contrast two types of research during an SRP: one comparable to a skydiver who can explore large areas from above and the other one to a truffle seeker who inspects thoroughly the soil to search for hidden truffles. Both of these are necessary: look through large areas of knowledge until identifying the little piece that will help to solve the problem. It is identified when the following is observed: the search on the Internet is wide, the students rule out topics that aren't useful or when the search starts narrowing and focuses on some more specific information.

D5 Subject and out-of-subject (S-Os) dialectic

The elaboration of the answer may sometimes lead away from the original question or research or go beyond the discipline to return later to the original problem. This dialectic appears when knowledge from other domains of mathematics or even from other disciplines is mobilized.

D6 Black boxes and clear boxes (Bb-Cb) dialectic

This dialectic invites to give primacy to the relevant knowledge. It is obviously not evident to identify the relevant knowledge, and this requires constant juggling between the potential knowledge in the black boxes and the available knowledge in the clear boxes. In other words, the inquirer has to determine the appropriate level of gray (or the depth of study) at which a particular piece of knowledge will be used or applied. This dialectic is identified when, at some point, a partial study of some topics appears.

D7 Reading and writing (R-W) dialectics

This dialectic comes into play after identifying the relevant knowledge which needs to be processed, analyzed and interpreted. This can take the form of highlighting, note taking, reformulation, creating a glossary, putting bits and pieces of $A^\bullet$ together until the final production.

D8 Praxeological analysis-synthesis and didactic analysis-synthesis (PAS-DAS) dialectic

This dialectic appears when new knowledge is at stake when analyzing the different answers $A^\bullet$ available in the works and deciding what answers to study (PAS) and how (DAS) to study them in order to build $A^\heartsuit$.

D9 Dissemination and reception (D-R) dialectic

This dialectic concerns the communication and justification of the final answer. The obtained answers need to be explained and eventually questioned in order to be accepted by the rest of the community. It appears when different study groups share their answers and these are accepted or rejected by the community.

Even though these nine dialectics constitute important didactic gestures during an SRP, they do not necessarily appear at the same moment of a study process or at the same frequency. Gazzola et al. (2019) depict that “the changes demanded by teaching through SRP

are substantial and require a long and continuous process” and conclude that the difficulties to carry out these gestures encountered when implementing an SRP are due to the prevailing paradigm of visiting works. Nevertheless, the dialectics provide a useful tool when modeling and analyzing teaching based on inquiry and we are going to see how they affect the didactic functions of chronogenesis, mesogenesis and topogenesis (Chevallard, 1985, 1992) during the inquiry process.

3.3.2 Topogenesis, mesogenesis and chronogenesis of an inquiry process

The topogenesis is the process that determines the place and role of the teacher and students in the classroom. However, with the expansion of the students' topos in IBE, the teacher's role is expected to evolve too. Indeed, Chevallard (2011, pp. 15–16) defines the new role of the teacher as that of the director of studies, who is invited to give up certain gestures linked to the paradigm of visiting works. The teacher should not behave like a distributor of knowledge, who provides ready-made answers, but rather help students to seek various means of validation. The mesogenesis is the process by which the milieu is created, developed and enriched. In traditional teaching, the teacher chooses the media and controls the milieu, whereas according to Wozniak (2015, p. 159) in an approach based on inquiry, it is the student who is in charge of constructing the milieu (with the help of the teacher) by mobilizing a diversity of didactic praxeologies (documentary research, experimentation, observation, validation of conjectures, group work, etc.). The chronogenesis refers to the creation of a didactic time during which knowledge is disseminated and acquired. This condition seems to come up against overloaded curricula, yet for an inquiry to unfold, time is needed. Wozniak (2015, p. 160) has identified some didactic practices that allow the investigation to progress in sufficient and reasonable time without reducing the student's topos:

- Making judicious choices of students' productions to guide the research.
- Use time outside the classroom to carry out documentary research.
- Use maieutic to suggest certain paths of reflection.

Wozniak (2015, p. 161) concludes that the implementation of IBE is closely linked to the above-mentioned conditions and depends on the presence of the Me-Mi dialectic. Bosch (2018) puts forward that the milieu evolves and that the dynamics of an inquiry process “is captured in terms of some dialectics that describe the production, validation and dissemination of A♥” (p. 4008). The dialectics taken into account by Bosch (2018) are the Q-A-answer dialectic, the Me-Mi dialectic and the I-C dialectic. The Q-A dialectic helps to unfold and describe the different paths followed during the inquiry process (including dead ends and abandoned attempts). When approaching the question Q at the beginning of the inquiry, one starts by searching for available answers A♦ and has to study them. This study usually generates new derived questions that help to move on with the inquiry. According

to Bosch (2018) the Q-A dialectics “provides visible proof of the progress of the inquiry and contributes to what is called the chronogenesis of the process” (p. 4008). In order to depict and visualize the chronogenesis the ATD was enriched by a new tool, the so-called question-answer map (Bosch & Winsløw, 2016; Florensa et al., 2016; Winsløw et al., 2013). It takes the form of a tree (Figure 6) and takes into account the derived question raised during the SRP and the partial answers obtained when elaborating the final answer $A^\heartsuit$.

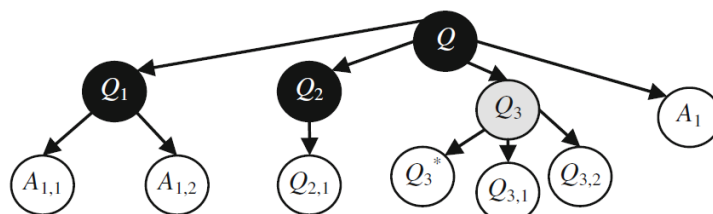


Figure 6: Example of a question-answer map (Winsløw et al., 2013, p. 271)

During an exploratory study of an implementation of a SRP, Parra et al. (2013) noticed that in addition to the Q-A dialectic, the S-Os dialectic also influenced the chronogenesis and they considerably expanded the time.

Very often an inquiry process requires a different distribution of the responsibilities within the didactic system and the topogenesis of the inquiry can be therefore approached through the I-C dialectic (Bosch, 2018). In an inquiry process, the inquirers X act in cooperation with the supervisor Y and therefore the roles and responsibilities are therefore distributed differently than in an instruction based on transmission. Parra et al. (2013) identified two other dialectics provoking radical changes in students' roles that affect the topogenesis of an inquiry process: the S-Ts dialectic and the D-R dialectic.

The Me-Mi dialectic defined as the “continuous interaction between available (partial) answers given by the media and their testing through the interaction with an a-didactic milieu” (Kidron et al., 2014, p. 158) plays according to Bosch (2018) an essential role in the mesogenesis of the inquiry. As far as the Me-Mi dialectic is concerned, Chevallard (2008) states that the validation process is carried out thanks to this dialectic because:

L'existence d'une dialectique vigoureuse (et rigoureuse) entre médias et milieux est une condition cruciale pour qu'un processus d'étude et de recherche ne se réduise pas au recopiage acritique d'éléments de réponse épars dans les institutions de la société. (Chevallard, 2008, p. 345)²³

The role of the milieu in the inquiry process is to confront and test the validity of any information from the media, in other words, any new $A^\heartsuit$ has to be deconstructed and reconstructed in order to be admitted in the milieu and turned into “verified” knowledge. Parra et al. (2013) observed that the mesogenesis was also influenced by the black boxes and

²³ Our translation: The existence of a vigorous (and rigorous) dialectic between media and milieu is a crucial condition for a process of study and research not to be reduced to the uncritical copying of scattered elements of response in the institutions of the society.

clear boxes and reading and writing dialectics. A brief summary of the relation between the didactic functions and the dialectics as proposed by Parra et al. (2013) is dressed in the following table (Table 1):

Chronogenesis	Topogenesis	Mesogenesis
Q-A dialectic	I-C dialectic	Me-Mi dialectic
S-Os dialectic	S-Ts dialectic	Bb-Cb dialectic
	D-R dialectic	R-W dialectic

Table 1: The relation between the didactic functions and the dialectics

The only dialectic that was not attributed to any of the didactic functions (chrono-, topo- or mesogenesis) was the PAS-DAS dialectic (Parra et al., 2013) even though it was identified within the same SRP (Parra & Otero, 2018). The latter exploratory work also revealed that certain dialectics are interrelated, for example the PAS-DAS dialectic appeared together with the Bb-Cb dialectic which is justified by the fact that “the realization of the analysis to a synthesis requires to determine a level of gray useful to the study of the works” (Parra & Otero, 2018, p. 251). Another surprising result was that the Me-Mi dialectic was identified less frequently despite the fact that it is considered a key gesture of an SRP. Parra and Otero (2018) explain that this result was obtained “due to the disjoint classification of the dialectics since the indicators of one of the dialectics also could be an indicator of the other” (p. 252). This possible interrelation makes us consider a different classification of the dialectics where we distinguish two major dialectics (Q-A, and Me-Mi dialectics) linked with chronogenesis and mesogenesis as suggested by Bosch (2018) and see if certain of the remaining ones could act as subdialectics of one of these major dialectics. In other words, we could use the subdialectics in order to better describe the main dialectic as the inquiry unfolds. Since the Q-A dialectic also related to as R-S dialectic (Gazzola et al., 2019) involves the formulation of derived questions and the search for answers, the S-Os (where to search) and S-Ts (how to search) dialectics seem to be good candidates for subdialectics to characterize the ways to search. In this sense the S-Os dialectics can be either extra-mathematical that is searching and importing information from other disciplines is necessary in or intra-mathematical when occasional trips to different theories are needed in order to move on with the inquiry. The S-Ts dialectic enables a toggle switch between general information and specific information. An example of the S-Ts dialectic is: reading Wikipedia to get the gist of the topic, retrieve some potentially relevant information and follow more thoroughly within this area. The Me-Mi dialectic ensures that the information from the media is processed and admitted to enrich the milieu and participates therefore on the mesogenesis. Parra et al. (2013) identified that the Bb-Cb and R-W dialectics influenced the mesogenesis of an SRP. The Bb-Cb dialectic enables to determine the depths of a study and at which level of gray the answers from the media will be admitted to the milieu. The R-W dialectic allows to keep track of the information in the milieu. As we have seen, the ATD uses certain dialectics to approach the mesogenesis, in other words, in how

the milieu *M* is built by the didactic system and how it evolves during the inquiry process. Yet, it seems important for us to step aside for a while from the ATD to look at the notion of the milieu from another perspective as it appears and is used in another major theoretical framework in didactics of mathematics, namely the theory of didactic situations (TDS). This allows us to come back to the notion of the milieu in ATD and propose an enriched vision of this concept, for the use in our work.

3.4 The notion of the *milieu*

In this part, we revisit the concept of the milieu as it appears in two major French theories in mathematics education.

3.4.1 The *milieu* in TDS

The notion of the *milieu* is central for the TDS and “is attached to the vision of learning as an adaptation process and to the ambition of optimizing such a process” (Kidron et al., 2014, p. 155). According to Piaget, “the student learns by adapting to a milieu that is a factor of contradictions, difficulties, disequilibria, much like human society does” (Brousseau, 1986, p. 48) but as highlighted by Brousseau “a milieu without didactic intentions is clearly insufficient to induce in the pupil all the cultural knowledge that one wishes he would acquire” (p. 49). Perrin-Glorian (1999, p. 285) and Kidron et al. (2014, p. 155) summarize that the milieu represents an antagonist system of the subject which should be a source of contradictions, but should also offer the possibilities of action and feedback that enable an emergence of new strategies and eventually allow the construction of new knowledge. Salin (2002, p. 114) adds that the milieu should ensure autonomous work of the student and should lead to building proficiency in the targeted mathematical knowledge (institutional knowledge) and not only to acquiring some local situational knowledge²⁴ or even only a particular know-how. The milieu is organized by the teacher who decides what material, artifacts and tools to include according to the social organization of the situation, but in TDS, the teacher is not part of it. It is important to note, as depicted by Kidron et al. (2014), “that for interacting with the milieu, learners always need to mobilize some of their already constructed knowledge” and that “some but not all authors include this knowledge into the milieu” (p. 155). We consider previous knowledge being part of the milieu if it is mastered, and the student is able to activate and mobilize it when necessary. Even though the concept of milieu has been present in the TDS from the origins, it often raised interest of researchers and was the object of several discussions and questioning ever since (Brousseau, 1990; Margolinas, 1995; Perrin-Glorian,

²⁴ in French, Brousseau distinguishes between “connaissance” and “savoir” and that following Margolinas (see ICMI AMOR) we have chosen to render this nuance by speaking of institutional knowledge (savoir) and situational knowledge (connaissance). Margolinas C., in collaboration with Bessot, A. (2020). Module 2: The dual aspect of knowledge. ICMI AMOR Project - Guy Brousseau’s Units. <https://www.mathunion.org/icmi/awards/amor/guy-brousseau-unit>

1999; Salin, 2002). Kidron and al. (2014) bring forward that one of the essential developments of this concept “has been its vertical and nested structuring” (p. 156). The nested structure model of the milieu was initiated by Brousseau (1990) and further developed and continued by Margolinas (1992, 1995). Even though we are not going to develop this concept further in detail, we need to briefly enter this vertical structure, just as Kidron et al. (2014, p. 156) did, in order to better illustrate our approach later in this chapter. We are specifically interested in two specific layers (S_{-1} and S_0) of the vertical structure (see Figure 7) because they represent the didactic and a-didactic situations which corresponds to the type of milieu we want to analyze. In this structure, the milieu M_{-1} at the level S_{-1} corresponds to the a-didactic milieu as described above and the level S_0 containing the milieu M_0 is the place where “the knowledge developed through a-didactic interaction with the milieu is made explicit, partially decontextualized, and connected to the institutional forms of knowledge aimed at” (Kidron et al., 2014, p. 156). In this model, the whole a-didactic situation S_{-1} is thus the milieu of the didactic situation S_0 .

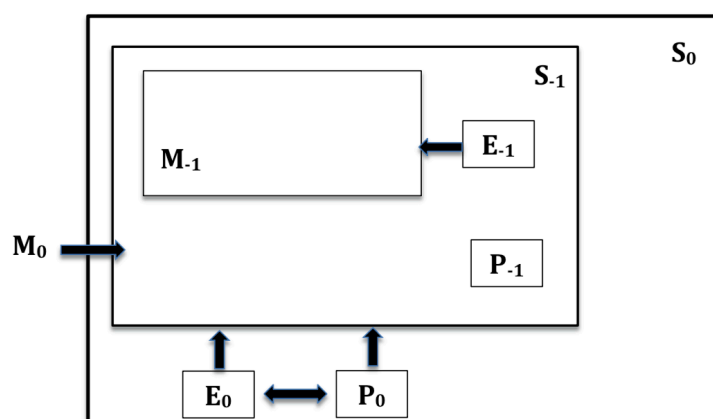


Figure 7: Simplified scheme of nested milieus (Kidron et al., 2014, p. 156)

According to Comitì, Grenier & Margolinas (1995) this nested structure model brought to light the possible bifurcations of the milieu which results in the fact that the didactic milieu M_0 of some students within the same didactic situation S_0 does not necessarily correspond to the one assumed by the teacher. In addition, as highlighted by Amade-Escot and Venturini (2009, p. 26), the nested structure model of the milieu allows us to grasp “that certain objects, instead of belonging to the objective milieu, become learning issues in their turn”. Perrin-Glorian (1999, p. 294) emphasizes that for the a-didactic milieu to play its role, it must contain pieces of institutional knowledge or at least situational knowledge mastered by the student. She raises therefore the problematic of a personal relation of the students towards these institutional objects.

Si le rapport personnel de certains élèves n'est pas idoine au rapport institutionnel, ces objets de savoir placés dans le milieu de la situation adidactique risquent de ne pas pouvoir apporter les rétroactions attendues aux actions de l'élève. Pour ces élèves la situation adidactique ne permettra pas l'apprentissage prévu : la dévolution ne peut se faire convenablement du fait de l'absence dans les connaissances disponibles de l'élève d'éléments supposés dans le milieu.

C'est aussi le cas quand le maître ne peut pas maîtriser complètement le milieu du fait des potentialités énormes de ce milieu, par exemple en présence d'un logiciel. Il peut alors y avoir présence dans le milieu d'objets non prévus puisqu'on introduit dans le milieu matériel, avec l'artefact, des savoirs auxquels les élèves n'ont pas des rapports stables, et qui vont leur amener des rétroactions externes. Dans ces cas, le milieu avec lequel interagit l'élève n'est pas le milieu construit par l'enseignant (Perrin-Glorian, 1999, p. 294)²⁵.

This is one of the reasons why Perrin-Glorian (1999) distinguishes a posteriori between the situation proposed by the teacher and the situation encountered by the student and also proposes to consider the milieu from the students' point of view. She defines as cognitive milieu the institutional and situational pieces of knowledge that the student must be able to mobilize by himself, in other words, towards which the student developed a stable personal relation in order to be able to interact with the milieu of the situation organized by the teacher. The teacher can only organize a milieu for an a-didactic situation a priori based on the objects available in the institutional milieu, but it may happen that for some students, certain objects intended by the teacher will be absent, while other objects not necessarily foreseen by the teacher may be present. From the student's point of view, Perrin-Glorian (1999) distinguishes in the cognitive milieu a *potential milieu* which refers to the milieu from the ATD, on the one hand, and an *activated milieu*, on the other hand. The potential milieu contains from the teacher's point of view the objects that should a priori be available for the student while the activated milieu reflects a posteriori what was really available for the student. For a generic student, since the potential milieu coincides with the institutional milieu the cognitive milieu (comprised in the institutional milieu) will always be activated. In reality, the activated milieu will be unique for each individual and may in some cases be insufficient. This allows to explain a posteriori why certain students succeed while others may stumble across the same situation.

The concept of an activated milieu allows to distinguish between the generic/ideal student and individuals whose personal relation to certain objects is not stable and consistent with the institutional relation. In what follows, we are going to see if and how we could import, adapt and operationalize the concept of the activated milieu within the ATD in order to better describe and understand the process of mesogenesis. Before that, it will be, however,

²⁵ Our translation: "If the personal relation of some students to the institutional milieu is not adequate, these objects of institutional knowledge placed in the a-didactic situation may not be able to provide the expected feedback to the student's actions. For these students, the a-didactic situation will not allow the planned learning: devolution cannot take place properly due to the absence in the student's available situational knowledge of assumed elements in the milieu. It is also the case when the teacher cannot completely master the milieu because of the enormous potentialities of this environment, e.g. in the case of adding a software. In this case, there may be the presence of unexpected objects in the milieu, since the artefact introduces knowledge into the material milieu that the students do not have a stable relation with, and which will bring them external feedback. In these cases, the milieu with which the student interacts is not the teacher-built milieu."

necessary to briefly review how the concept of the milieu appeared and evolved within the ATD.

3.4.2 The *milieu* in ATD

Chevallard (1992, pp. 88–89) imports the notion of the milieu from Brousseau’s TDS (1986) but considers it always as institution-related. Thus, the *institutional milieu* from the ATD perspective is defined as a set of objects characterized through a stable institutional relation, which means that any object in the institution appears to be self-evident, transparent and non-problematic for the subjects of the institution at a given time. For Chevallard (1992, pp. 94–95) the existence of an institutional milieu is a necessary condition for a didactic system to function. This means that at any moment a minimum of institutional objects with a stable institutional relation must exist. On the other hand, a didactic system acts upon the milieu and makes it evolve: “some of the elements of the milieu will be destabilized and will momentarily cease to belong to the milieu, before becoming stable again [...]” (p. 95). This process of a constant evolution of the milieu is referred to as mesogenesis in the ATD as already mentioned earlier. Kidron et al. (2014) claim that:

the *didactic milieu* of the Herbartian formula can include an *a-didactic milieu* in the sense given by the TDS, that is a system of objects acting as a fragment of ‘nature’ for Q, able to produce objective feedback about its possible answers without any didactic intention towards X.

And for Chevallard (2008) the milieu here is also to be regarded as “any system that can be considered as devoid of intention in the answer it can give, explicitly or implicitly, to a given question” (p. 344). This is close to the a-didactic milieu from the TDS. In addition, in an SRP or more generally during an inquiry process “the production and organization of an appropriate a-didactic milieu for Q is an essential aspect of the study process carried out by both X and Y”, but “finding and creating an appropriate experimental milieu [...] can be one of the most challenging issues to tackle” (Kidron et al., 2014, p. 158), which is obviously not the case in TDS because the milieu is organized by the teacher.

Several authors (Perrin-Glorian, 1999; Amade-Escot & Venturini, 2009; Kidron et al., 2014) tackled the question of articulation of different theoretical frameworks around the notion of the milieu. Perrin-Glorian (1999, p. 288) recognizes that an institutional milieu, as described in the ATD, containing objects towards which the institutional relation is stable, has the characteristics to play the role of an a-didactic milieu in sense of TDS. She emphasizes that it is nevertheless important to realize that the milieu in the sense of ATD is not defined in respect to a situation but in respect to an institution and its temporality but that these two notions are compatible and could be articulated. From his part, Chevallard (2003a) also distinguishes between the institutional and personal relation to institutional objects. If a person becomes a subject of an institution, he is also subject to institutional relations which will affect his personal relation to the existing objects. In other

words, a person x will tend to resemble the institutional relation, unless x turns out to be, in this respect, a bad subject of the institution. This opens the possibility to import the concept of activated milieu into the ATD.

In our modeling of inquiry, we try to combine all these elements and notions around the milieu, in order to better understand the student's work in the ME. First of all, we agree that the didactic (institutional) milieu of the Herbartian scheme corresponds to the concept of the potential milieu as developed by Perrin-Glorian (1999) when situated in the paradigm of visiting works where its organization is the responsibility of the teacher. However, when applying these concepts to the paradigm of questioning the world, in particular to teaching and learning through an SRP, a new parameter enters the scene: the access to media. Supposing that the access to media includes the access to the Internet, the didactic system has a quasi-infinite pool of options to organize an inquiry milieu which will extend the potential milieu beyond the institutional milieu. But the real capacity of the didactic system to organize an appropriate didactic milieu, create an a-didactic milieu and produce an answer to the question will strongly depend on the activated milieu.

In order to adapt and operationalize the concept of the activated milieu (Perrin-Glorian, 1999) within the ATD we finally need to consider the relation to institutional knowledge from the ATD point of view (Chevallard, 2003b) which we will develop in the following part.

3.5 The relation to institutional knowledge

The ATD also provides theoretical tools (Chevallard, 2003b, 2015a) to explain the cognitive dynamics but it is first necessary to introduce some ATD-specific terms and notation and define the key notions. We will start by defining the notion of a relation to an object. For an individual x and an object o this notion refers to all the ways in which x can relate to o and is symbolically represented as $R(x, o)$. There is a variety of ways that x can relate to o , the relation $R(x, o)$ combines thus what x knows (or thinks to know) about o , what he can say about it, how he can use or misuse it..., in other words, the system $R(x, o)$ specifies how x knows o . It is thus quite obvious that the personal relation to the same object is not identical for all individuals. In the course of time, the system of personal relations of x evolves: objects that did not exist for him begin to exist; others cease to exist; for others finally, the personal relation of x changes. To take into account this evolution, the ATD introduces the notion of a *person* which represents the individual x and his personal relations $R(x, o)$ at a given moment in the life of x . In this evolution, the invariant is the *individual*; what changes is the *person*. This brings us to the definition of the *cognitive universe* (CU) of x as a set of objects towards which a person has developed a personal relation (Chevallard, 2003b).

On the other hand, Chevallard (2003b) defines an *institution* I as a social structure which “allows—and imposes—to its subjects, that is to say to the persons x who come to occupy the different positions p offered in I , to put in place their own ways of doing and thinking” (p. 2)²⁶. Given an object o , an institution I , and a position p in I , we call the institutional relation to o in position p , and we note $RI(p, o)$, the relation to the object o which should be, ideally, that of the subjects of I in position p . The ATD explains the constitution and evolution of the CU of an individual x through a system of *subjections*²⁷ to a multitude of *institutions*. By becoming a subject of I in position p , an individual x , who is always already a person endowed with a certain CU, becomes subject to the institutional relations $RI(p, o)$ which will reshape his personal relations. If an object o exists for the subjects of I in position p , the personal relation of x to o , $R(x, o)$, will tend to resemble the institutional relation $RI(p, o)$. Generally speaking, our “personal” relations are thus the fruit of our past and present institutional subjections. Ideally, the personal relations are in adequacy with the institutional relations which, in reality, is not always the case. The CU is a complex system that includes also objects that the individual knows “badly”, meaning that certain $R(x, o)$ are not consistent with $RI(p, o)$ despite a given institutional subjection.

In the previous part (3.4 The notion of the *milieu*) we considered the concept of the milieu from different perspectives which we link now to the concept of the CU. Indeed, the TSD (Perrin-Glorian, 1999) raised the problematic that the situation prepared by the teacher a priori is not the one encountered by the student because of the discrepancy between the activated and potential cognitive milieu. On the other hand, the ATD approaches this discrepancy by foreseeing a nonconformity between personal and institutional relations towards an object revealing thus a great complexity of the student’s CU.

In the light of these theoretical reflections, we can formulate a hypothesis that the advancement of inquiry and the construction of the $A^\heartsuit$ will be closely linked to the capacity of the didactic system to build an appropriate inquiry milieu and organize an experimental milieu and that the CU of the didactic system will determine this capacity. Because of the access to media (which is not the case in TSD) we can expect that new personal relations towards some objects will be created and this makes us wonder to what extent they will be in adequacy with the institutional relations. In the next part, we are going to apply these theoretical tools to model the mathematical exploration.

3.6 Modeling the ME in the ATD

Based on these theoretical elements we propose to slightly modify the Herbartian scheme to include the organization of an a-didactic milieu and clarify the terms. In the Herbartian

²⁶ In French: « qui permet – et impose – à ses *sujets*, c’est-à-dire aux personnes x qui viennent y occuper les différentes *positions* p offertes dans I , la mise en jeu de *manières de faire et de penser propres* »

²⁷ We refer to the French term “assujettissement”

scheme (Figure 8), we denote M_{INQ} the didactic milieu and refer to it as to the inquiry milieu (INQ-milieu). We denote M_{EXP} the a-didactic milieu in sense of TDS as and refer to it as to the experimental milieu (EXP-milieu).

$$S(X, Y, Q) \rightarrow M_{\text{INQ}} = [\{A_i^*, W_j, Q_k, D_l\} \rightarrow \{M_{\text{EXP}}\}] \rightarrow A^*$$

Figure 8: A modified Herbartian scheme to model inquiry-based activities

In the ME, the didactic system $S(X, Y, Q)$ is composed of a student x ($X=\{x\}$), a teacher y and eventually a tutor (TUT) or another expert (EXP) ($Y= \{y, (\text{TUT}),(\text{EXP})\}$) gathered around a question Q chosen by x and approved by y . The answers A^* or W can be mobilized from the student's CU (previous knowledge or experiences), received from the teacher, another expert or a tutor (human help) or can be found in the media (Internet, textbook, etc.). Even though in the ATD the “human help” (including the teacher, an expert or a peer), is generally considered as a media, we consider it as a separate source of potential A^* because of the ability of adaptation to the needs of the didactic system. We suggest therefore splitting the media into two categories (static and dynamic). The static media are the classical media such as textbooks, websites or videos and the specificity of the dynamic media (tutor, teacher, peer, etc.) consists in their ability to adapt to the needs of the inquirer. On top of that the student's CU is not only another source of answers or works but will probably also have an effect on the capacity of the didactic system to search and process the answers from the media, to organize the inquiry milieu and to create an experimental milieu and interpret its feedback.

4 Research questions

The initial questions that motivated our research concerned our encounter with inquiry specific objectives within the IB, the implementation of inquiry-based activities into teaching of mathematics and the particular institutional requirement represented by the ME. The theoretical framework of the ATD provides an appropriate model to conduct our research and enables us to formulate the research questions (RQ) concerning the study of the institution, the activity of a teacher and finally the ME.

RQs concerning the institution

From the ATD point of view, we cannot study the dissemination of knowledge and the teaching practices without taking into account the institution in which teachers practice their profession. It is clear then that in the first instance we must question the place of inquiry in this institution which led us to formulate the first group of research questions.

- What are the conditions and constraints at each level of the didactic codeterminacy scale that foster or hinder the implementation of inquiry-based activities within the IB?
- To what extent can the IB be considered as an institution promoting inquiry?

- Is the IB able to (and if so, how and to what extent) assess inquiry-specific objectives; investigation tasks and problem-solving?

RQs concerning the teachers

The second group of research questions the teachers' convictions and declared practices concerning the inquiry and its implementation in teaching.

- What are the teachers' attitudes, values and beliefs about inquiry?
- What are the major constraints perceived as obstacles in implementing inquiry by teachers?
- To what extent, given the constraints, could the teachers be good subjects of their institution?
- What personal praxeologies are developed by the teachers when implementing the ME in reaction to the institutional conditions?

RQ concerning ordinary classes

- Are (and if so, how and to what extent) inquiry-based activities and problem-solving tasks integrated into teaching in ordinary mathematics classes?
- How does the institutional object of the ME live in relation to the ordinary classes?
- To what extent does it influence the day-to-day teaching practices, or live in a completely isolated way?

RQ concerning the ME

The particularity of the ME is that part of it is elaborated in class but an important amount of the students' work on the ME takes place outside of the classroom. We thus have to tackle the question of accessing the private work of the student which explains the rather methodological nature of the first research question in this group. In addition, we wish to characterize the students' practices concerning inquiry through an exploration of an existing institutional object that was a priori recognized as a potential vector of inquiry-specific objectives. This resulted in formulating research questions closely related to the treatment and analysis of the collected experimental data.

- How does the student's cognitive universe effect the activated milieu?
- What is the role of media in the ME? What type of media affects most the ME?
- How do the students obtain and process institutional answers?
- To what extent could the media ensure a creation/evolution of personal relations that are in line with the institutional relations towards mathematical objects?
- What are the learning outcomes of the students when engaging in the ME?

In order to answer the above-mentioned RQ we have designed a general methodology following a funnel-like plan descending the levels of the didactic codeterminacy scale.

5 Overview of the general methodology

Our theoretical framework enabled us to formulate research questions concerning the place dedicated to inquiry-based education in the IB as an institution and the Mathematical Exploration. That is why we organized our research into two major parts: an institutional study (Part B) and case study involving two IB mathematics teachers and two of their students (Part C).

As already mentioned, according to Bosch and Gascon (2002), one cannot correctly interpret teaching practices without studying the institution in which the teacher practices her profession. From a methodological point of view, Dorier and Garcia (2013) provide the sources to be investigated to document this analysis (curriculum, legal texts, textbooks, interviews with teachers, students, parents, press articles, etc.).

Our research begins with an in-depth analysis of the institutional context of the IB based on different texts such as the official documentation of the institution (the annual bulletins, curriculum and subject guides and other accompanying texts), some historical sources and texts issued from research in international education order to understand the social and ideological reasons behind the creation of this institution. When using certain “historical” sources, it is important to keep in mind their strong ideological background because these are often written by the actors of IB and provide more of a commemorative celebration rather than a critical historical narrative as depicted by Dugonjic-Rodwin (2014). According to the same author, they can nevertheless be precious sources of information to access the ideology that shaped the institution and collect certain data that are inaccessible otherwise. To complete our vision of the institution, we built an online questionnaire to address the IB teachers’ attitudes, values and beliefs concerning inquiry and mathematics teaching and to collect their declared teaching practices when implementing the ME. This questionnaire was conceived and adopted using the methodology deployed in two major European project (PRIMAS and mascil) about inquiry-based education (Maaß & Engeln, 2013, 2014). We finalize the institutional study at the level of discipline of the didactic codeterminacy scale. We analyze the type of tasks used in the items of the final exam to determine whether and how the objectives related to investigation and problem-solving are addressed in the assessment. This analysis uses the methodological and theoretical tools based on the work of Robert & Rogalski (2002) and Roditi & Salles (2015) leading to a classification of items based on “an analysis of the wording of the item whose purpose is to determine the nature of the mathematical knowledge required to answer the question posed in the item”²⁸

²⁸ In French: une analyse de l'énoncé visant à déterminer la nature de la mise en fonctionnement des connaissances mathématiques nécessaires pour répondre à la question posée dans l'item

(p. 238). In addition to see how the IB addresses assessment of inquiry-specific objectives, this analysis will allow us to compare and contrast with what is actually done in the ordinary classes.

The second part of this research is designed as a case study and brings us to observation of the work done in the ordinary classes, on the one hand, and during the ME, on the other hand. Because of the diversity and complexity of the data that we wanted to collect, we had to make certain choices to assure the feasibility of this research. The first choice that we were confronted with, was related to the mathematics courses we were going to observe. We have opted for the Mathematics Standard Level (SL) course because it is taken by the majority of the students and avoids introducing a bias, by having extremely weak or extremely talented students, which could present itself in Mathematical Studies, respectively Mathematics Higher Level (HL) course. The International school of Geneva has got two campuses in Geneva and we were advised on the one where they were more open towards implementing inquiry. Finally, we found two volunteer mathematics teachers that we followed for two teaching sequences each from October to April. For the ordinary classes, we placed a camera at the back of the classroom and equipped the teacher with a clip-on microphone.

In order to conduct the analysis, we decided to characterize the topogenesis, chronogenesis and mesogenesis of the ME. To study in detail the ME component of the IB mathematics program we focused on three aspects:

- 1) Institutional and teachers' practices when implementing the ME
- 2) Students' practices when elaborating the ME
- 3) Students' experiences of the ME

For the collective part of the implementation of the ME, we used the same methodology to collect the data as for the ordinary classes: a camera at the back of the classroom and a clip-on microphone on the teacher. For collecting data concerning the ME, we decided to follow six volunteer students, three from each class. Let us specify that there is an important methodological difficulty to have access to observables, because the only information that we can normally obtain comes from the final copy that the student gives to see to the teacher. We needed to figure out how to access the private work of the students (Coppé, 1998) which comprises students' actions, reflections, trials conducted out of the sight of the teacher. Coppé (1993, 1998) put in place a methodology that enabled her to gain access to the private work by collecting scrap papers with all traces and conducting explicatory interviewing technique developed by Vermersch (1994). She also emphasized that observing students closely during their work despite a rigorous discretion of the researcher results in alteration of the students' activity. Based on these elements, we asked the students to keep a logbook with all traces, we equipped them with an action camera asked them to record their work at home and they shared with us their ME folder on google docs. The in-class work was obtained via a camera in the classroom as well as the exchanges with the

teacher who was equipped with a clip-on microphone. We expect to be able to transcribe and synthesize the obtained data identifying each element from the Herbartian scheme. We propose a list of possible ingredients that we can think of a priori (Appendix 11) comprised between each question and produced answer and revealing thus the dynamics of the chronogenesis. To follow the evolution of the milieu, we will label each media as relevant or irrelevant according to its potential to contribute to the exploration and activated or non-activated based on the interaction of the student with the media. We will consider that a media was activated when we find observable evidence such as reading, taking notes, highlighting, watching a video, etc. The content of the media will be labeled as admitted, rejected or potential according to whether it was admitted or not in M_I based on how the student processed the information found in the media. Each answer, work or data will be coded as correct or incorrect, and relevant or irrelevant from the point of view of the researcher and according to its capacity to contribute to a given exploration at a given moment. The mesogenesis will thus be captured by zooming on the interactions of the student with each media and we will attempt to characterize these interactions from the point of view of the evolution of the milieu. Finally, the finest zoom will enable us to characterize each “ingredient” of the Herbartian scheme and identify if certain elements tend to appear more frequently or whether certain type sources are privileged by the student, etc. To complete our vision of the ME we will conduct two interviews to obtain information concerning the mathematical profile (pre-ME) of the observed students and their experience with the exploration (post-ME). Finally, we are also interested in how the students in both classes experienced the ME and we therefore decided to obtain this data via a questionnaire addressed to the students after they handed in their final paper. This questionnaire was adapted from a research project on problem-solving²⁹ piloted by Jean-Luc Dorier and Sylvie Coppé. We provide more detailed explanations of the data processing and analysis at the beginning of each concerned chapter. We can now move on to Part B and present the institutional analysis of the IB.

²⁹ FNS - Subside n° 100019_173105/1 - from 31.08.2017 to 31.01.2022

PART B

INSTITUTIONAL STUDY

In this part we are going to approach the IB as an institution taking place in the historical and social context that contributed to its foundation and analyze it using the scale of didactic codeterminacy to confirm our intuitions about it being an institution promoting inquiry in its midst. The above-mentioned scale also enables us to identify the system of conditions and constraints that shaped their educational philosophy and contributed to the creation and development of the IB curriculum. We will progressively descend the levels of the didactic codeterminacy scale until the level of the discipline. At the level of discipline, mathematics in our case, we will analyze the results of the teachers' survey to identify the attitudes of the teachers towards their institution, their personal convictions about mathematics teaching and inquiry-based education. The last element that completes this institutional analysis and closes this part is an analysis of the final external exam to see whether it includes inquiry-based tasks and problems.

Chapter I

THE INSTITUTIONAL ANALYSIS

The IB represents a particular institutional context that emphasizes the development of skills such as critical reflection or research skills through teaching that reflects pedagogical principles based on inquiry (IB, 2015). Thus, from its conception, the IB has expressed doubts about the effectiveness of encyclopedic teaching (Service d'examens des écoles internationales (ISES), 1967). In addition, it is one of the few institutions that requires a summative assessment of inquiry-specific nature in mathematics, compulsory for all students and contributing to 20% of the final grade. The internal assessment is "a piece of written work that involves investigating an area of mathematics" (IBO, 2012, p. 43). Its main objective is to evaluate students' performances and skills that are impossible to assess through written examinations or tests. On top of that, it "should, as far as possible, be woven into normal classroom teaching and not be a separate activity conducted after a course has been taught" (IBO, 2012, p. 43). These signs detected in the official IB documentation seem to indicate the presence of inquiry-based education and motivated our interest in pursuing our research in this institution. Dorier and Garcia (2013) highlight that "conditions and constraints are determinant factors in order to understand how a didactic system works" (p. 839). Using Chevallard's (2004) didactic codeterminacy scale, we will conduct an analysis at the levels of society, school, pedagogy and discipline. At each level we will study relevant sources and attempt to identify the conditions and constraints that might foster or hinder the implementation of inquiry-based education within the IB. In order to better understand this institutional context, the reasons that motivated its current orientations and positions, it is also necessary to look into the genesis of this institution.

1 The Level of Society

The first level that we are going to look at is the level of society. This is the place where the general debate about education and schooling takes place. Dorier and Garcia (2013) state that this "includes the institutional organization of the educational system in a country or a region, and the most general level of the curriculum..." (p. 839). Even though the IB doesn't depend on any national system, it took advantage of the presence of experts delegated by different government authorities who participated in the elaboration of syllabuses. These, as highlights Renaud (1974, p. 8), did not come as much to represent their governments but rather express their personal views and promote ideas, which they would like to see adopted. In the following parts, we attempt to depict the sense of the

general debate about education in the 1960s and identify the conditions that lead to the creation of the international baccalaureate.

1.1 The IB as a response to a social need

The origins of the International Baccalaureate Diploma Program (IBDP) are closely linked to the International School of Geneva, which, with its foundation in 1924, can rightly claim the title of “the oldest of modern international schools” (Peterson, 2003, p. 15). Already in 1925 the need for an internationally recognized secondary leaving examination was expressed, however, the context was not ready for such an endeavor. On the one hand, the education of expatriates’ children was looked after by the national governments and, on the other hand, there were few genuine international schools. The idea of an internationally recognized university entrance diploma had been raised three more times since, however, without any apparent success (Hill, 2002). Meyer (1968, p. 17) reports that in the 1960s about two and a half million people (half of them Americans) resided temporarily (for 1 to 3 years) far from their homes. This temporary expatriation concerned no longer military positions for American armed forces, which by the way disposed of a good network of schools following the American curriculum. Indeed, a new phenomenon occurred in the after-war society: more than half of the transients left their homes to accomplish a diplomatic mission or to engage in a private enterprise and these expatriates were not necessarily Americans. According to Meyer (1968) the lack of adequate educational facilities “hinder[ed] the recruitment of able and experienced people for international work in most areas of the globe” (p.18). The society faced a new problem: the education of foreigners. The existing and newly created international schools needed help in dealing with common difficulties. One of the biggest problems that these schools had to cope with, when their students grew older, was their preparation to enter universities, since university acceptance became more and more competitive and university entrance exams were based on the national programs. For example, in the International School of Geneva at that time, students in advanced physics class, were “divided into four small groups, one following the syllabus for the Swiss ‘maturité fédérale’, one that of the English GCE A-level, one that of the French ‘baccalauréat’, and a fourth preparing for the *American College Board Advanced Placement*” (Peterson, 2003, p. 16-17). As a response to help deal with these common difficulties, *Ecolint*, as the International School of Geneva is familiarly called, founded and housed the *International Schools Association* (ISA) in 1951. In 1960, a problem of a strong national bias was observed by Bob Leach, the head of history at Ecolint, when discussing the Second World War in his history class with 20 different nationalities present. These conditions raised a need for a new way of teaching history — “[...] one that invited critical inquiry of historical sources and statements” (Saxton & Hill, 2014, p. 43).

Progressively, the idea of establishing an international baccalaureate became more pertinent and realistic but there was still a long walk to do. Dugonjic-Rodwin (2014, pp.

139-140) states that the implementation of such a large-scale project requires more than a group of highly mobilized individuals. For her, the case of the IB reveals that this process requires a strong social capital: favorable circumstances and supportive institutions. The circumstances were favorable, there was a strong social demand for an international baccalaureate and the group of highly motivated Ecolint's teachers led by Desmond Cole-Baker³⁰, to give it a start. In 1962 the name *International Baccalaureate* (IB) was first mentioned at a small conference organized for social studies teachers thanks to a minor grant that Leach received from UNESCO, the first experimental contemporary history syllabus was created during the General Assembly of ISA in Paris in 1964. Nonetheless, Ecolint did not have any government subsidies for curriculum development and a lot of credit needs to be given to the enthusiasm and motivation of a group of its teachers who were willing to walk the extra mile for their cause. However, without proper funding and support the IB dream was hard to bring to reality. The next important step forward was the creation of the *International Schools Examination Syndicate* (ISES) in 1964, an organization that would work on the IB project, and the first major grant of \$75,000, obtained from the *Twentieth Century Fund* (TCF) in 1965, enabled a more structured planning and development. This grant was designed to support a research project whose objective was to examine the conditions of creating a common curriculum for international schools and the possibility of examinations for university entrance (Peterson, 2003). Martin Meyer was designated by TCF to conduct this research and write up a report about the feasibility of such a project. He was personally convinced that for the expatriates' community, who accepted to leave their homeland, the education of their children constituted a 'social problem' and therefore this deserves the attention of the society. (Meyer, 1968, p. 9) As part of this research he visited different international schools, universities and ministries of education or similar authorities, to try to identify the possible audience for the International Baccalaureate being developed.

1.1.1 The Construction Period (1964–1969)

During the years 1965–1966, the ISES organized many work conferences and international committees around different subjects or subject groups with the participation of a total of 170 participants out of which 60% were secondary education agents and 40% of university education, governments and international organization representatives (I.S.E.S., 1967, p. 33). The project seemed to have been buried under the struggles and frustration of unfruitful discussions, meetings and personal crises when according to Meyer (1968, p. 225) Oxford took over at the beginning of the negotiations for an important grant from the Ford Foundation in 1966. Alec Peterson, director of the Department of Education at Oxford

³⁰ appointed director of the English-language section in 1961

University, agreed to become the first Director General of the IBO³¹, and brought expertise and new vision to the project. From Peterson's (2003) point of view, the Oxford's decision to step in was "not so much of a calculated takeover as of a co-operative rescue mission." (p. 23).

Peterson (2003, p. 24) identified five elements that were necessary to show the feasibility of an international baccalaureate:

- 1) The need of a unified international curriculum and an examination system that actually worked on a small scale and had the potential of expansion.
- 2) A considerable number of universities would recognize these examinations as sufficient for entry to the university.
- 3) A certain number of schools willing to commit to the new curriculum.
- 4) It was necessary to find parents who would be willing to take the risk of their children not being accepted to university because of this new examination.
- 5) The last but not least was to provide sufficient funds in order to support schools and students in case the project would not succeed.

In order to achieve these goals, the 'Geneva group' needed to internationalize and therefore three important figures joined the Council: Jean Capelle, rector of the University of Nancy, Edmée Hatinguais, founder and the first directress of the *Centre International d'Etudes Pédagogiques de Sèvres* (CIEP, France) and Helmut Becker, director of the *Max Planck Institute*, Germany. Their common characteristic is that they were all considered as 'reformers' in their respective countries and the reason that put these three together "was the opportunity to try out in practice some of the reform proposals which were making such slow progress in (our) respective countries." (Peterson, 2003, p. 25). The two consultants Frank Bowles and Ralph Tyler designated by the Ford Foundation were also supportive of the view that "it was much easier to design one (curriculum) from scratch and then let it demonstrate its value." (Peterson, 2003, p. 25). The Council also had representatives from the academic sphere such as the rectors of the University of Stockholm and of the Moroccan Universities or an inspector in the Polish Ministry of Education. The IBO got an organizational structure consisting of a counsel that meets annually, an Executive Committee of five members and later a Board of Chief Examiners.

The veritable breakthrough can be attributed to the first *Sèvres Conference* in February 1967 which "established the purposes, structure, regulations, and pattern of studies for the IB." (Fox, 1985, p. 57)

The first Sèvres Conference agreed on the following measures:

³¹ The International Schools Examination Syndicate became officially the International Baccalaureate Office based in Geneva in 1968

- 1) The experimental period would last six years and includes cooperation with a certain number of selected schools and universities.³²
- 2) The number of candidates would be limited to 500 per year in order to allow an in-depth study of the procedures and the results.
- 3) A research group at a large university would be created.
- 4) Another conference was envisaged before the end of the experimental period in order to evaluate the experiment by experts and to determine whether the project would become permanent.

The *International Schools Examination Syndicate* was transformed and officially registered as the *International Baccalaureate Office* (IBO) in 1968. With the support of institutions like TCF, Oxford University, CIEP, Max-Planck Institute and others, the IBO managed to put together a strong social capital, achieved its ‘construction period (1964–1969)’ and headed to its six-year ‘experimental period (1970–1976)’.

1.1.2 The “Experimental Period” (1970–1976)

The first official general guide to the *International Baccalaureate Diploma Program* was released in 1970 and contained the syllabus and information on assessment of all the subjects that were offered and the first 29 students took the new IB examinations leading to an official IB diploma or certification (Peterson, 2003, p. 67). By that time, the IBO managed to negotiate a provisional recognition of the IB diploma by universities of most major countries as equivalent to the national high school diplomas. Renaud (1974, p. 15) reports that the number of schools went from 11 to 27 by 1974 and the number of candidates went over a thousand which largely exceeded the limit set by the first Sèvres Conference. Yet, this did not take into account the schools on the waiting list and state schools that were awaiting the IBO’s official status before enrolling. Table 2 shows the gradual growth of the number of candidates taking the IB examinations during the experimental period (Renaud, 1974, p. 16):

³² See the list of schools and universities in appendices

Sessions	Candidates entered	Nationalities	Diploma candidates	Diplomas awarded	Certificate candidates	% of passes (Dipl. and Cert.)
1970	312	49	29	20	283	69.4
1971	601	50	76	54	525	71.7
1972	631	57	151	96	480	74.9
1973	840	65	311	236	529	75.9
1974 (approx.)	1050	72	400		650	

Table 2: Number of candidates taking the IB examinations during the experimental period

The French Government found the project interesting and thus financed the second IB conference at Sèvres, which took place in April 1974. The 64 attendees came from different horizons representing 21 different nationalities: thirty of them were IBO representatives and the rest consisted of experts in curriculum development and examinations suggested by Unesco. The objective of this conference was to review the present state of the IB experiment in order to determine its viability, its long-term extension and consider its future developments and procedures. The conference contained both, discussions in plenary sessions and meetings in three thematic commissions were set up to study: the general policy and administration (Commission A), syllabuses (Commission B) and methods of assessment (Commission C). The conference did not question the transition into a long-term operation and recommended that the IBO continue its offer and spread around the globe. Neither it took any formal decisions regarding the future of the organization but rather provided advice and recommendations and left their realization up to the IBO. A report on the Second Sèvres Conference was made in the Annual Bulletin form November 1974 (Office du Baccalauréat International, 1974).

In this context, due to the diversity of national contexts, the IB couldn't define very precise conditions for setting up lessons, and must have therefore taken into account general dimensions (policy, syllabus) and focused on the question of assessment because it determines its viability at the international level, since it must guarantee university entrance in all countries.

1.2 The conditions and constraints at the society level

Saxton and Hill (2014) identify three reasons that brought the IB diploma program to existence: pragmatic, idealistic and pedagogical. We situate the first two (pragmatic and idealistic) at the level of society as they create a series of conditions and constraints at the lower level. From the pragmatic point of view, there was a need for a program that would

enable students from different backgrounds to study together and for a diploma that would guarantee university entrance worldwide. Thus, it can be said that the IB diploma program was created as a response to the needs of a changing society, and in particular to meet the needs of the growing group of expatriates concerning the education of their children. If the parents, as mentions Meyer (1968, p. 37), were open to almost any sort of primary education, this was no longer the case for upper secondary years because of the terminal examination that played an important role in university acceptance. It was precisely university acceptance conditions that created an important constraint, and strongly conditioned the elaboration of the curriculum of the IB diploma as will be shown at the level of school. Nonetheless, the parents also wanted an education that would reflect the values and ideals of the many organizations they worked for: fostering intercultural understanding, considering global issues from an international perspective and to promoting world peace (Hill, 2012).

Keeping these identified elements in mind, we will step down to the level of school to analyze how the IBO translated these expectations into educational aims and a viable curriculum sanctioned through a series of examinations.

2 The Level of School

Different countries presented different approaches to general education. On the one hand, there was the very encyclopedic approach of the German *Abitur* or the French *Baccalauréat* which considered general education as transmission of information over a wide range of subjects. On another hand, the upper-secondary system in Britain promoted a premature specialization through the system of the A-level examinations, while in the USA, general education was left up to colleges. Peterson (2003) insists on the fact that general education is not to be mistaken for acquisition of general knowledge and opposes the premature specialization of students preparing for the English A-levels. His views on general education are expressed in the Oxford Department of Education report on *Arts and Science in the Sixth Form*. Even though he did not manage to achieve the reform of the English Sixth Form while in Oxford, it is clear that his proposals deeply influenced the conception of the IB diploma program. In this report, Peterson (2003) proposes to extend the number of specialized subjects from two to three or four distributed over both liberal arts (humanities) and sciences, a fifth block suggesting religious and physical education and arts, and a course that would focus on the methodology of the subjects. Another reformer whose name is closely connected with the International Baccalaureate is Jean Capelle, the rector of the University of Nancy (France) and a former pedagogical director at the French ministry of education. He fought his battle on the field of the French Baccalaureate where he criticized encyclopedism and examination based on recalling of memorized facts and suggested to reduce the number of subjects to six in order to maintain diversity and sufficient depth. His main argument was that

one prepares better the acquisition of a general education by study in depth at secondary level of half a dozen subjects sufficiently diverse in their nature (which avoids specialization) than skimming over a dozen different fields of knowledge solely in order to be able to recall something about each of them on the day of examination. (Peterson, 2003, p. 43)

Thus, the reformers around IB brought quite a different vision of what general education might be:

We sought not to ensure that as ‘generally educated men and women’ our students should have acquired a wide range of knowledge, but they should have developed, as far as they were able, their power and desire to engage in a wide range of what Montaigne called ‘ways of thinking’ (Peterson, 2003, p. 41).

Also, the amount of content knowledge in each discipline is such that an encyclopedic approach appears to be insufficient. The general IB educational philosophy is expressed in the first General Guide to the International Baccalaureate in 1970:

The weight of available information in each discipline is such that an encyclopedic approach to education is not only outdated but inappropriate: learning how to learn has now become the prime function of school education. The challenge to educators arises in the attempt to provide youth with an education that is broad enough to enhance the awareness of a ‘common humanity’ and social responsibility, and at the same time specific enough to ensure the acquisition of those skills (both disciplinary and interdisciplinary) that are essential tools for higher education or employment in a competitive world. (IBO, 1970, pp. 21-22, in Fox, 1985, p. 58)

The IBO was about to put together a great puzzle with the aim to translate these principles into a curriculum and an examination that would “encourage the teaching of ‘minds well formed’ rather than ‘minds well stuffed’ ” (Peterson, 2003, p. 43). In this sense, I.S.E.S (1967, p. 8) insists on the importance of developing students’ intellectual abilities rather than feeding them with encyclopedic facts, providing methodology and allowing a certain specialization in order to prepare them for future university studies. This was clearly expressed in the 1967 guide:

1. to develop the student’s intellectual skills rather than requiring superficial and ephemeral memorization of encyclopedic knowledge: form rather than inform.
2. to instill in him the working methods that will be essential to pursue higher education.
3. to provide him, within a well-defined field, with sufficient training to enable him to specialize.
4. to familiarize him, at least in broad terms, with the problems of the contemporary world from an international perspective. (I.S.E.S, 1967, p. 8)

We can also notice that the last point reflects the ideological expectation of the parents identified at the level of society.

Nevertheless, emphasizing the active involvement of students does not exclude the need for memorization of certain facts and the development of critical thinking skills presupposes

solid foundations and acquisition of content knowledge. The Second Sèvres Conference also emphasized the innovatory aspect in the educational strategy worded by Jean Capelle:

It should be noted that IBO's field of activity is not limited as the title might lead one to suppose, to organizing a terminal examination of upper secondary studies; if the specific function of the Office is evaluation, its objective is education... it has started boldly with the renewal of programs and methods, both in curriculum and the evaluation of results.”
(Renaud, 1974, p. 30)

We will now look more in detail at how the IBO dealt with the general organization of the program and how these choices were reflected in the syllabus and assessment requirements.

2.1 The General Structure

The new approach to general education consisted in finding a compromise between a premature specialization and encyclopedism. The key was to allow a certain degree of selection within a balanced offer of scientific and literary disciplines.

Thus, in experimental sciences the important thing is not to study simultaneous programs in physics, chemistry and biology but to acquire, through one of these sciences, a sense of the experimental method. (Renaud, 1974, p. 34)

The general structure for the last two years of secondary education, introduced in the 1967 syllabus (Appendix 4), can be summarized in three points:

- 1) Students must choose six disciplines equitably distributed among the humanities and sciences (group 1: first language; group 2: human sciences—philosophy, history, geography, economics; group 3: experimental sciences, group 4: language 2; group 5: mathematics, group 6 is as varied as language 3, ancient language, applied mathematics, art or another discipline from group 2 or 3);
- 2) Students must study 3 or 4 of the 6 subjects chosen at higher level (HL), the others at standard³³ level (SL).
- 3) All students take a course called “Theory of Knowledge”, which “requires students to reflect critically and reflect on the cognitive process rather than learn a set of specific knowledge” (IBO, 2013a, p. 8). This course must represent a minimum of 100 hours of instruction.

Another particularity of the IB programs is the elaboration of the *Extended essay*, a piece of personal research work of choice that the candidates have to complete in a subject of choice in order to obtain the diploma. The 1972 guide (OBI, 1972) states that it was mandatory in higher-level language A, history and geography and optional for any other subjects. This resulted in overloading certain students who had to write up to three extended essays depending on the choice of their higher-level subjects. The Second Sèvres Conference acknowledged the importance and the educational value of such work but suggested to

³³ The term ‘standard level’ was introduced in 1998 replacing the term ‘subsidiary level’

reduce it to one extended essay in any subject of choice, which is reflected in the 1977 guide. (OBI, 1977)

Since educating the *whole man* was in the heart of the IB educational philosophy, already the 1967 syllabus emphasized that the core academic activity should not completely fill up the weekly schedule of an IB student. Indeed, schools should allow room for ethical, religious, artistic and physical education. The 1974 program became more specific about the extracurricular activities: every candidate must show involvement in some kind of artistic activity for the equivalent of one afternoon per week. (OBI, 1972) These were given a mandatory character in 1989 (Hill, 2007) and constitute the CAS (creativity, action and service) requirement in order to obtain the IB diploma.

2.2 Assessment

As one of the main reasons of creating the IB diploma program was to grant access to post-secondary education, the question of examination came up inevitably. Various methods of assessment ranged from sitting a series of terminal examinations spread over several days, which determine students' future through examination sprinkled over several periods to awarding diplomas based on continuous assessment of each subject. Renaud (1974) reports that the IBO chose "a definite but prudent innovation" (p. 43). The traditional examination approach recognized by university admissions was supplemented by introducing diverse means and methods of assessment and giving a certain weight to internal assessment in each subject that was examined. In foreign languages, the use of a tape recorder for oral exams was introduced, where the recordings were done at the school and sent to external examiners for marking. IB Candidates were internally assessed by the school, and their work was then sent to an external examiner for moderation. At the time of the Second Sèvres Conference, the following subjects were assessed internally: theory of knowledge, some artistic activities, schools' special programs and practical work in experimental sciences. As far as mathematics is concerned, the internal assessment was first introduced in 1977 to be presented at the 1979 examinations within the mathematical studies. This course was designed for students who encountered difficulties in mathematics and who generally continued their academic career in humanities. As of today all subjects have components that are object to internal assessment contributing to about 20% of the final grade.

2.3 Pre-service and in-service teacher training

The diversity of locations hosting the IB program and their geographical spread make the issue of teacher training necessarily multifaceted. In general, teachers have received initial training in their home country and have applied to a school offering the IB diploma program that may or may not be based in their home country. In any case, it is part of a voluntary process that is likely to be a sign of adherence to IB values. In addition, teachers must attend regular training workshops and there is a large amount of online material

available as well as a discussion forum. The collaborative dimension at the international level is also highlighted.

2.4 Conditions and constraints at the school level

At the level of school, we have examined the elements leading to the elaboration of the curriculum. First, a compromise between the very broad encyclopedic approach and too narrow premature specialization needed to be found. The six disciplines model guaranteeing a balance between sciences and humanities proved its worth during the experimental period and remained unchanged until today. Dugonjic-Rodwin (2014, p. 139) states that even though the IB is not materially dependent on any single nation state, its genesis and structure depend on the recognition of the diploma by their institutions or ministries. For example, the Commission B (in charge of the syllabuses) argued that an average student, summing up the requirements of all six subjects, would quickly reach the limits of his abilities. A proposition was raised to reduce the general scheme to two higher level and three standard level subjects. But this would be objected by universities and the report concludes that “it seems difficult to lower the requirements for a higher-level subject for a degree that benefits from many equivalence agreements and that has often been recognized as having a better preparatory value for higher education than traditional degrees” (OBI, 1974, p. 25). The Second Sèvres Conference concluded that it was necessary to maintain the current scheme and suggested to reduce the ambitious content in certain disciplines, in particular in mathematics. Some experts pointed out that “universities attach less importance to the student’s intellectual maturity than to the subjects listed on the diploma” (OBI, 1974, p. 26). The main constraint—university admissions—leaves therefore very little room for leeway and every change in the curriculum and assessment is object to new negotiations with university admission offices. Despite these constraints, the IB still managed to slip in a certain amount of less traditional assessment techniques, even though traditional exams (valued by universities) remain the main assessment tool.

3 The Level of Pedagogy

The 20th century brought new and alternative views on education where educating the *whole man* became important and the IB made original pedagogical choices within the constraints identified above. Renaud (1974, p. 36) highlights that as far as the teaching is concerned, emphasis needs to be given on the development of critical thinking skills and reasoning rather than to the volume of knowledge. In this sense, the Theory of Knowledge (TOK) course is a singular element of the system, which is the heart of the core curriculum. According to the official guide (IBO, 2013a), this course provides students with opportunities to reflect on the nature of knowledge. It is intended to equip students with epistemological tools and encourage them to reflect on the origins and reliability of the different sources of knowledge. The guide resumes that “TOK is a course about critical

thinking and inquiring into the process of knowing, rather than about learning a specific body of knowledge” (IBO, 2013a, p. 8). In addition, in order to encourage personal and independent work and to develop research skills, students must prepare an extended essay of around 4000 words on a subject of their choice. Finally, the Creativity, Action and Service (CAS) program, which is the third component of the core curriculum, completes the “education of the whole man” promoted by IB since its origins (OBI, 1973, p. 24). As depicted by Gérard Renaud:

The responsibility of educators is no longer just to prepare good mathematicians, good biologists or good historians. The mission of schools is to prepare young people—decision makers of tomorrow—to live in a complex multicultural society undergoing a rapid process of change and opening up a new world. Of course, the cognitive component of an educational system is fundamental for the acquisition of intellectual and professional skills. Even more important is the acquisition of attitudes in the learning process in a context of cultural exchanges. (Renaud, 2012, p. 8)

The general curriculum structure of the IB diploma program remained pretty much the same since its introduction. Nevertheless, a certain evolution cannot be left unnoticed (Figure 7). Whereas the earlier models (the 1993 and 2004 model) reflected essentially the academic part of the program, in the latter ones, the appearance of the IB learner profile (the 2008 model), the international mindedness and approaches to teaching and learning (the 2017 model) seem to reflect the willingness to emphasize its holistic approach to education.

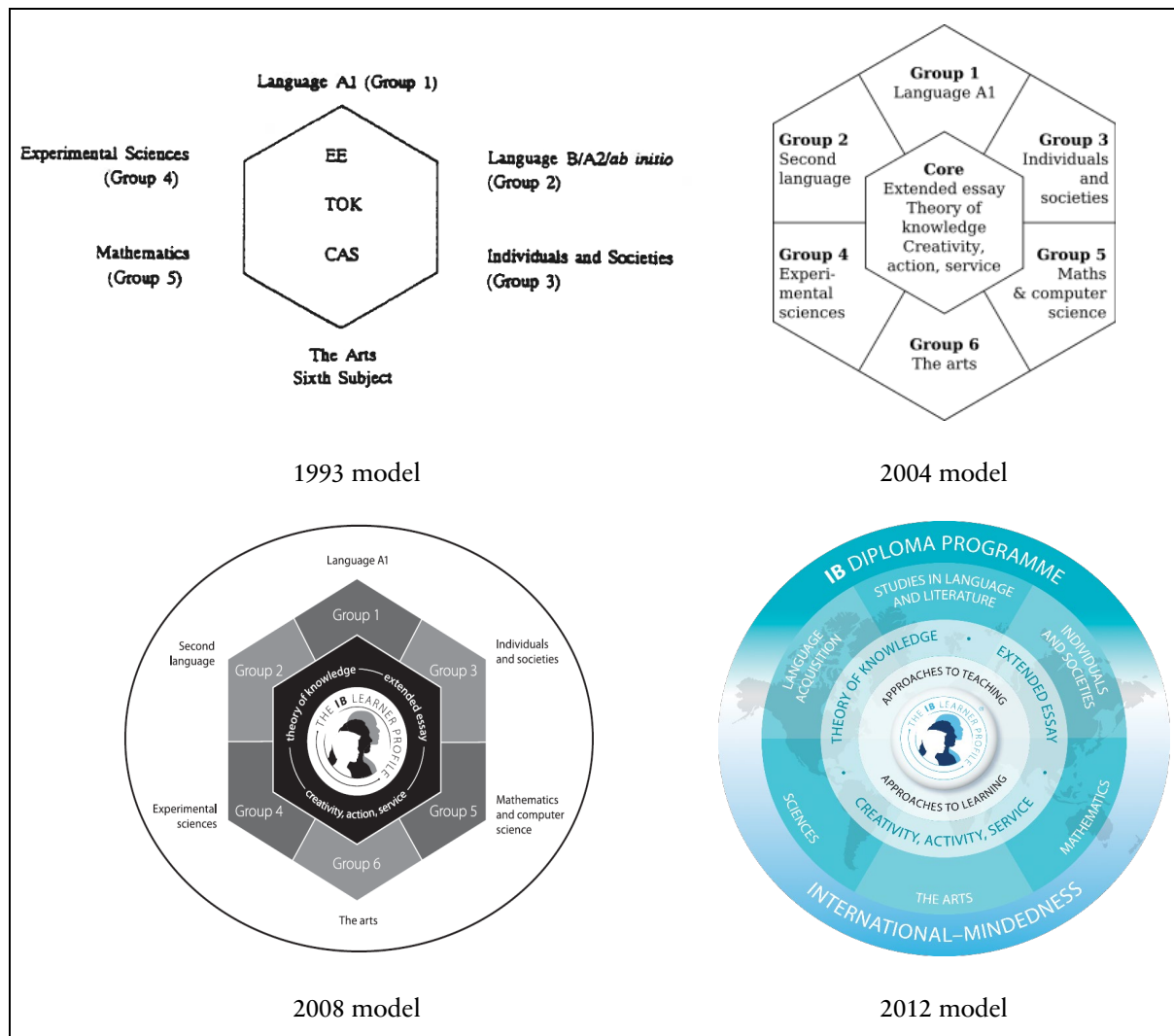


Figure 9: Evolution of the Diploma program model

The IB learner profile, first introduced as the IB student profile within the primary years program (PYP) and it underpins all programs since 2006. With the official publication of the IB learner profile, The IB “places the learner firmly at the heart of IB programs and focuses attention on the processes and the outcomes of learning” (IBO, 2008, p. 2). To emphasize its importance the IB decided to insert it into each major publication (IBO, 2008, p. 4). Thus, all subject guides published since 2007 contain the IB learner profile. We interpret this initiative as a strong will of the IB to emphasize a student-centered approach to education, which Dorier and Garcia (2013) classify as an essential ingredient in inquiry-based education. The IB learner profile (Appendix 5) consists of 10 attributes and descriptors that “define the type of learner the IB hopes to develop through its programs” (IBO, 2008, p. 1). IB learners strive to be inquirers, knowledgeable, thinkers, communicators, principled, open-minded, caring, risk takers, balanced and reflective (IBO, 2013b, p. 1). Among the 10 attributes, we identify the first three as inquiry indicators. These are described as follows:

INQUIRERES

We nurture our curiosity, developing skills for inquiry and research. We know how to learn independently and with others. We learn with enthusiasm and sustain our love of learning throughout life.

KNOWLEDGEABLE

We develop and use conceptual understanding, exploring knowledge across a range of disciplines. We engage with issues and ideas that have local and global significance.

THINKERS

We use critical and creative thinking skills to analyze and take responsible action on complex problems. We exercise initiative in making reasoned, ethical decisions. (IBO, 2013b, p. 1)

Continuing its efforts to make the IB educational philosophy more explicit, the IB launched in 2011 a new reflection focusing on ways of learning and teaching. The documentation made available to the IB community in 2015, entitled ‘Approaches to teaching and learning’, is intended to guide teachers by highlighting the principles of teaching and learning as part of the Diploma.

In the approaches to learning (ATL), the IB identifies five categories of skills: thinking skills, communication skills, social skills, self-management skills and research skills. They specify that “the term ‘skill’ is [...] used in a broad sense in the DP to encompass cognitive, metacognitive and affective skills” (IBO, 2015, p. 3). In the IB educational philosophy, “developing thinking skills is a key feature of the constructivist approach” and one of the attributes of the IB learner is being a ‘thinker’. Within this category “particular focus is placed on skills such as metacognition, reflection and critical thinking” (IBO, 2015, p. 3) and we can say that the TOK course plays an important role as it provides students with opportunities to reflect on the nature of knowledge. The IB places research skills “at the heart of inquiry-based pedagogy” and the elaboration of the Extended Essay (EE) provides students with opportunities to conduct a personal research project and thus contributes to develop these skills. (IBO, 2015, p. 12) The second part of this document describes six pedagogical principles that reflect the constructivist and student-centered approach that the IB has adopted from its beginnings. They claim that:

Teaching in IB programs is:

1. based on **inquiry**
2. focused on conceptual understanding
3. developed in local and global **contexts**
4. focused on effective teamwork and **collaboration**
5. **differentiated** to meet the needs of all learners
6. informed by **assessment** (formative and summative). (IBO, 2015, p. 14)

We would like to point out that the first two principles go in line with the first two attributes (inquirers and thinkers) of the learner profile. The IB does not prescribe any particular form or model of inquiry but rather emphasizes that teachers should:

[...] focus on making sure their students are engaging in inquiry, on finding their own information and constructing their own understandings, as often as possible in their classrooms. In an inquiry-based approach, learning is self-directed, “because it is driven by students’ own decisions about appropriate ways in which an issue or scenario might be approached [...]” (IBO, 2015, p. 15).

Nevertheless, this guide considers experiential learning and problem-based learning as “two particularly well-known inquiry-based approaches” (IBO, 2015, p. 15) and provides a more detailed description referencing research in this domain. The teacher is a key element in a successful implementation of teaching based on inquiry because it requires “a shift in teaching style from teachers whose primary role is to supply answers to teachers whose primary role is to promote questions” and also “a shift in some of the responsibility for learning from teachers to students” (IBO, 2015, p. 17). The IB is aware of the difficulties in implementing these principles and overcoming the retrocognitive habits (Chevallard, 2015b, p. 178) resulting from teaching based on transmission, encourages the teachers to persevere and concludes returning to its origins:

The implementation of process-oriented, skills-based teaching can be challenging for both teachers and students. The teacher’s role becomes more facilitative and the student’s role more inquiring. Many students, especially those comfortable with, or habituated by, transmission teaching will find it difficult to adjust to a classroom scenario where they are expected to do the learning for themselves rather than simply being told what to learn. These approaches to teaching and learning do, however, have the potential to develop “minds well formed rather than minds well stuffed” (Alec Peterson, 2003: 43), an aspiration at the heart of an IB education. (IB, 2015, p. 39)

3.1 Conditions and constraints identified at the pedagogy level

We perceive a clear tendency within the IB to affirm the pedagogical principles underlying its educational philosophy. However, Renaud pointed out that at the IB diploma program level “the constraints of the forthcoming examination [...] impose serious limitations on the implementation of the curriculum which would really correspond to the philosophy of an international system of education” (Renaud, 2012, p. 9). Providing a meta-analysis of research studies focusing on the IB’s principal pedagogical approach—inquiry-based teaching and learning, Chichekian and Shore (2014) claim that “a possible barrier in sustaining inquiry as the pedagogical approach in the IB programs has been the increasing focus on “teaching to the test” (p. 85). On the other hand, some of the measures (TOK, internal assessments, the extended essay, etc.) put in place by the IB aim probably for counterbalancing the weight of the final exams. Peterson (2003, p. 34) pointed out already at the time of designing the curriculum that the only ‘power’ to impose anything in the curriculum that the IBO had was the examinations.

4 The Level of Discipline

4.1 The development of the mathematics syllabus

It is interesting to note that the conception of the mathematics syllabus was not a straightforward matter. One could expect that the difficulties would be encountered mostly within the social sciences and it should be easier to come up with an agreement on the syllabus of a supposedly neutral subject such as mathematics. To his great surprise Meyer (1968) realized that even in mathematics “the suggested programs were substantially different in emphasis and in content” (p. 4). Especially at the beginnings of the IB, the mathematics program was too ambitious to be covered within the last years of secondary education not only as far as the volume was concerned but also content-wise. The report from the Second Sèvres Conference (OBI, 1974) mentions that it impinged on the first-year university programs. At the same time, the syllabus conceivers were aware of the different needs, abilities and interests of the student body and developed therefore mathematics courses that would suit these different needs. Obviously, such a challenging mathematics program would have caused a certain number of failures and resulted in not awarding the diploma to some more humanities-oriented students. The IB introduced therefore in 1972 a new course Mathematics Subsidiary Level B with the objective to offer a basic mathematics course to students who encountered difficulties in mathematics and who generally continue their academic career in humanities. This raised the issue of whether mathematics should be maintained as mandatory discipline. One of the schools suggested making mathematics optional arguing that doing a “lighter version” destined for the literary students or nothing was pretty much the same. This proposition was largely rejected by the participants of the Second Sèvres Conference claiming that “in the world where the student is supposed to live, it is inconceivable that he does not have a minimal training in the sense of ‘mathematical studies’ ” (OBI, 1974, p. 26)³⁴. They (OBI, 1974, p. 26) referred to the situation in France where mathematics was no longer required for certain literary sections of the French baccalaureate and qualified it as a disastrous. We can understand this reaction when placed in a larger context: the one of the launching of Sputnik by the Soviets in 1957 and its impact on promoting scientific education in the western world in the sixties (Cha, 2015) and the New mathematics reform as a reaction to it. Mathematics therefore remains mandatory and is offered at three levels and a special advanced course that can be chosen in group 6. In order to avoid confusion between Mathematics Subsidiary Level A and Level B courses, a new course Mathematical Studies Subsidiary Level was introduced in 1977 integrating the recommendations of the 2nd Sèvres Conference from April 1974. The big novelty includes the elaboration of a personal project as internal assessment that

³⁴ Our translation of the French original: “dans le monde où l’étudiant est appelé à vivre, il est inconcevable qu’il n’ait pas une formation minimale dans le sens des ‘études mathématiques’.”

weighed 20% of the final grade. The written report of the project should comprise at least 2000 but no more than 4000 words. Subsidiary and higher-level mathematics, however, are assessed externally only. Nevertheless, these students had the choice to write their extended essay in mathematics, but it was not mandatory. The Figure 10 provides a comprehensible overview of the courses offered from 1970 to 1998 showing also the introduction and evolution of the internal assessment in mathematics.

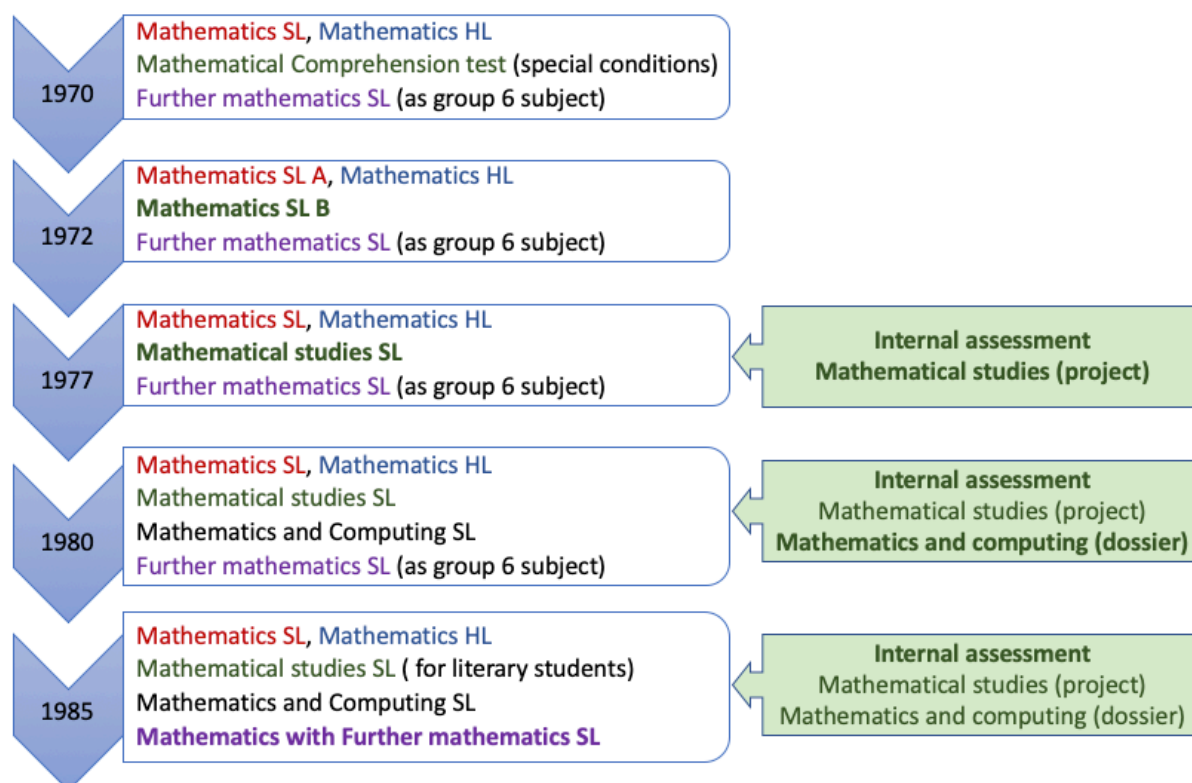


Figure 10: Overview of mathematics courses offered from 1970 to 1985

This situation remained practically unchanged until the curriculum review in 1999 with first examinations in 2001 when an internal assessment was introduced for the standard and higher-level courses in the form of portfolios (IBO, 2001). The overview of the mathematics courses offered from 1999 until 2019 is shown on Figure 11.

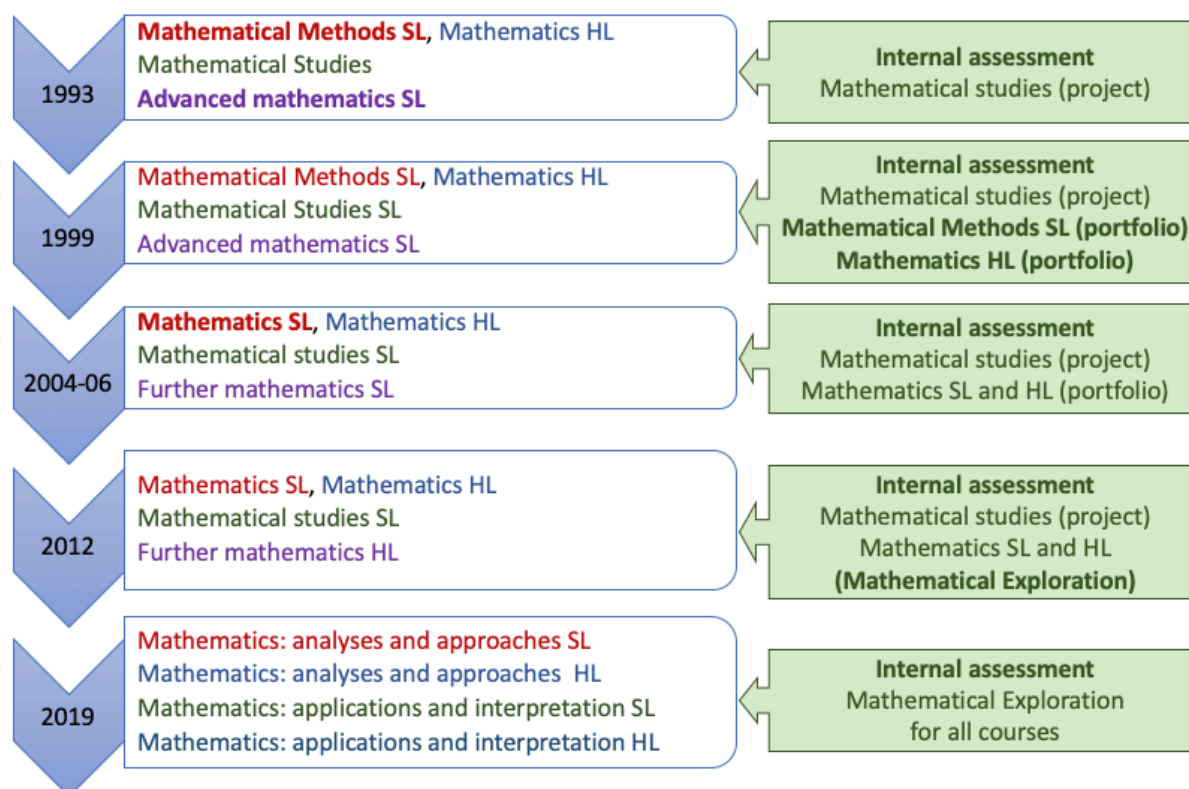


Figure 11: Overview of mathematics courses offered from 1993 to 2019

The Portfolio model for the internal assessment was replaced in 2012 (examination 2014) with the current model of Mathematical Exploration which is the object of our research. In addition, the 2012 guide is the first guide where inquiry-specific approaches are made explicit for all courses, always respecting the different needs, interests and abilities of the students in order to fulfill university admissions requirement or career choices of the students. Nonetheless, in each subject guide problem-solving is central in teaching and learning mathematics.

Problem-solving is central to learning mathematics and involves the acquisition of mathematical skills and concepts in a wide range of situations, including non-routine, open-ended and real-world problems. (IBO, 2012, p. 9)

On top of that the mathematics SL and HL guides mention that “the processes of **mathematical inquiry, mathematical modeling and applications** and **the use of technology** should be introduced appropriately” (IBO, 2012, p. 11).

The IB learner profile encourages learning by experimentation, questioning and discovery. In the IB classroom, students should generally learn mathematics by being active participants in learning activities rather than recipients of instruction. Teachers should therefore provide students with opportunities to learn through mathematical inquiry. (IBO, 2012, p. 11)

The guides also provide models (Figure 12) of how a mathematical inquiry and mathematical modeling processes should be approached.

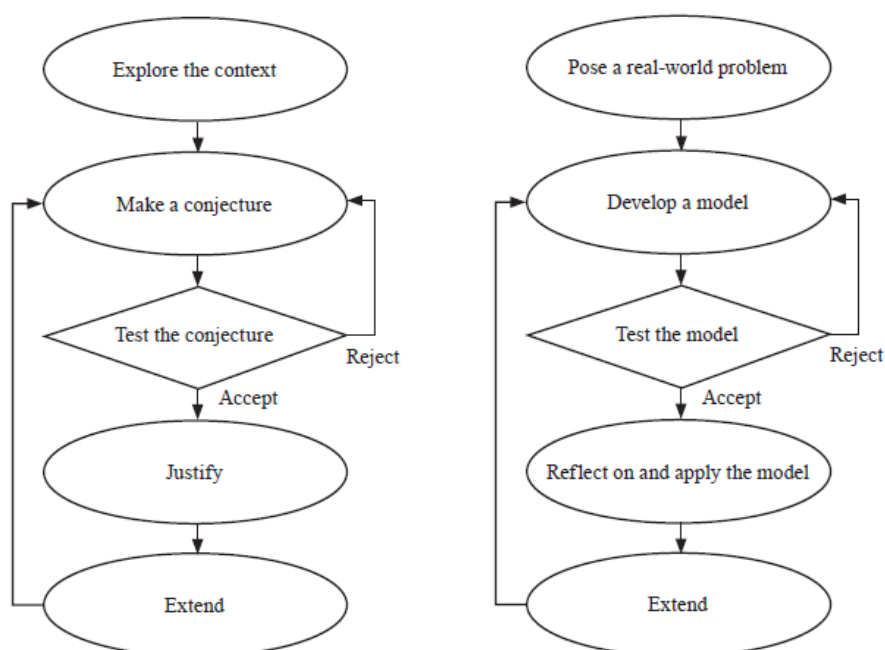


Figure 12: Inquiry and modeling processes model (IBO, 2012, pp. 11–12)

The latest curriculum changes in 2019 (first exams in 2021) was the most radical since it brought novelty not only in organization and but also in the content of the mathematics courses. It removes the existing model with four level-based (Mathematical Studies SL, Mathematics SL, Mathematics HL and Further Mathematics HL) and replaces it with a completely new approach. The Further Mathematics HL course was discontinued because of the lack of candidates taking it. From 2019 the IB is offering two new mathematics subjects different in nature, each at SL and HL in order to better address the two facets of mathematics (pure and applied) and thus better respond to the students' needs and abilities. The Mathematics: analyses and approaches course focuses more on the construction of mathematical arguments and development of strong skills in mathematical thinking, whereas the Mathematics: applications and interpretation course through a more applied approach “emphasizes the meaning of mathematics in context” (IBO, 2019, p. 14). The guide also states that 25 hours should be dedicated to a non-content-driven, “investigative, problem-solving and modeling skills development leading to an individual exploration” (IBO, 2019, p. 25). In order to help teachers to invest these 30 hours, the IB published teacher support material (TSM)³⁵ for each subject. This document contains information and guidance on implementing the approaches to teaching and learning specifically into the mathematics courses and incorporating the IB Learner profile (Appendix 6). An important part of this document is dedicated to supporting classroom activities and

³⁵Mathematics: analysis and approaches TSM

https://christosnikolaidis.com/_files/200000489-c9cd6cac66/MAA%20TSM%202019.pdf

Mathematics: applications and interpretation TSM

https://christosnikolaidis.com/_files/200000491-336a6345f5/MAI%20TSM%202019.pdf

proposes resources aiming to develop mathematical thinking skills. Making problem-solving and inquiry approaches specific is also reflected in assessment. The 2019 guides provide approximate percentage weight of how the individual objectives will be assessed in the new courses. Table 3 illustrates this distribution for the Mathematics Analysis and Approaches course and shows that problem-solving and inquiry approaches will be assessed not only through the internal assessment (Exploration) but in some way also within the external examination. We can thus see that the IB in order to reinforce the approaches to teaching by imposing it on examination being aware that “teachers who do not use these approaches are likely to disadvantage their students, both in the learning experience, and in the examinations” (IBO, 2010, p. 1).

Assessment objectives	Paper 1 %	Paper 2 %	Paper 3 % HL only	Exploration %
Knowledge and understanding	20-30	15-25	10-20	5-15
Problem solving	20-30	15-25	20-30	5-20
Communication and interpretation	20-30	15-25	15-25	15-25
Technology	0	25-35	10-30	10-20
Reasoning	5-15	5-10	10-20	5-25
Inquiry approaches	10-20	5-10	15-30	25-35

Table 3: Assessment objectives for Mathematics Analysis and Approaches course (IBO, 2019, p. 24)

To conclude we would like to point out that there is a clear tendency within the mathematics syllabuses starting in 1999 to make specific inquiry-related objectives more explicit (at least at the documentation level). Since the only ‘power’ the IB has to ensure the implementation of such objectives is through assessment, requiring an internal assessment in all mathematics courses should guarantee that these objectives were addressed in classes. Let us look at the different types of internal assessment and its role in the mathematics programs from 1999 until its current form—the Mathematical Exploration.

4.2 The Internal Assessment in Mathematics

4.2.1 The Project

An internal assessment in mathematics was introduced for the first time within the Mathematical Studies SL in 1977 and it contributed to 20% of the final grade. Naturally, the question that comes up is why a mathematics project was introduced first for the weaker students. We can make the hypothesis that the weaker students needed to experience mathematics in a less abstract way and to meet one of the aims the 1972 guide stated, that is “to encourage those lacking confidence in their knowledge, competencies and experience

of mathematics³⁶ (OBI, 1977, p. 185). The 2004 guide is more specific about the purposes of the project:

- develop students' personal insight into the nature of mathematics and to develop their ability to ask their own questions about mathematics
- encourage students to initiate and sustain a piece of work in mathematics
- enable students to acquire confidence in developing strategies for dealing with new situations and problems
- provide opportunities for students to develop individual skills and techniques, and to allow students with varying abilities, interests and experiences to achieve a sense of personal satisfaction in studying mathematics
- enable students to experience mathematics as an integrated organic discipline rather than fragmented and compartmentalized skills and knowledge
- enable students to see connections and applications of mathematics to other areas of interest
- provide opportunities for students to show, with confidence, what they know and what they can do. (IBO, 2004, p. 35)

The 2004 guide also mentions that the internal assessment is more appropriate to assess certain objectives without the constraints of written examinations. These are:

- organize and present information and data in tabular, graphical and/or diagrammatic forms
- know and use appropriate notation and terminology
- recognize patterns and structures in a variety of situations and make generalizations.
- recognize and demonstrate an understanding of the practical applications of mathematics
- use appropriate technological devices as mathematical tools
- demonstrate an understanding of and the appropriate use of mathematical modeling. (IBO, 2004, p. 34)

The same guide stipulates that these objectives imply the use of certain teaching and learning strategies such as:

- Students need to be provided with opportunities, as an integral part of the course, to experiment, to explore, to generate hypotheses and to ask questions.
- Students should be provided with experience of taking initiatives to develop their own mathematics in the classroom.
- Students should be encouraged to become active learners of mathematics, inside and outside the classroom.
- Topics in the syllabus should be organized to allow an active inquiry approach.

³⁶ Our translation

- Students should be encouraged to identify and reflect on the variety of mathematical processes that may be used to solve problems.
- Teachers should aim to offer advice about the choice of topics and titles, bearing in mind that the most valuable projects are those that reflect the student's interests and enthusiasms. (IBO, 2004, p. 35)

Based on the above-mentioned elements we can conclude that this kind of activity was specifically designed for the needs of the weaker students with little appreciation for and little interest in mathematics. In other words, it was probably believed that the motivation of students taking mathematics at standard or higher level was sufficiently self-driven and thus such an activity was not necessary. We would also like to point out that in 1979 about 30% of the candidates took mathematics at higher level and about 10% did Mathematical studies whereas in 2019 only 15% presented HL mathematics and 36% Mathematical studies. This little statistical trip only confirms that the IB population was not preserved from the growing decline of young people's interest in mathematics and science as highlighted in the Rocard's report. (2007, p. 5) This leads us to think that this situation contributed to the fact that the IB curriculum developers introduced an internal assessment in Mathematical methods SL and Mathematics HL in 1999 in the form of a Portfolio.

4.2.2 The Portfolio

The argument that the IB puts forward to explain the purpose of the portfolio is "to provide students with opportunities to be rewarded for mathematics carried out under ordinary conditions, that is, without the time limitations and stress associated with written examinations" (IBO, 2001, p. 35, 2006, p. 38). The portfolio activities should provide opportunities for good mathematical writing, thoughtful reflection, they should help increase understanding of mathematical concepts and processes, and last but certainly not least, they should be a source of stimulation and reward. When introduced in 1999, the portfolio consisted of three types of tasks: mathematical investigation, extended closed problem-solving and modeling. From 2005, the extended closed problem-solving task was dropped, and the portfolio required two pieces of student work, one based on investigation and the other on mathematical modeling. (Appendix 7) According to the Mathematics SL guide (IBO, 2006) "the mathematical investigation is intended to highlight that the idea of investigation is fundamental to the study of mathematics" (p. 40). On the other hand, the aim of the modeling tasks was to solve "a practical problem set in a real-world context" (IBO, 2001, p. 36). Through these two types of tasks, the IB hoped to "provide students with the opportunity to experience for themselves the beauty, power and usefulness of mathematics" (IBO, 2006, p. 39). The fact that "the discovery aspect of investigation work deepens understanding and provides intrinsic motivation" (IBO, 2006, p. 40) confirms our hypothesis that the internal assessment in SL and HL mathematics courses was introduced because of the decrease of interest in science and mathematics. Generally, before each exam session, the IBO published a certain number of these tasks that the teachers could use for

the internal assessment, but they also encouraged teachers to develop and use their own tasks complying with the IB requirements (IBO, 2006). The Portfolio was maintained as an internal assessment form until 2012 when it was replaced by a new model called Mathematical Exploration (ME).

4.2.3 The Mathematical Exploration (ME)

The mathematical exploration is an individual, stand-alone research activity conducted on a topic chosen by each student and supervised by their mathematics teacher. The IB expects that exploration will:

- develop students' personal insight into the nature of mathematics and to develop their ability to ask their own questions about mathematics
- provide opportunities for students to complete a piece of mathematical work over an extended period of time
- enable students to experience the satisfaction of applying mathematical processes independently
- provide students with the opportunity to experience for themselves the beauty, power and usefulness of mathematics
- encourage students, where appropriate, to discover, use and appreciate the power of technology as a mathematical tool
- enable students to develop the qualities of patience and persistence, and to reflect on the significance of their work
- provide opportunities for students to show, with confidence, how they have developed mathematically. (IBO, 2012, p. 46)

Compared to the Portfolio, the exploration is more open, encourages each individual student to explore an area of mathematics based on their interests and reinforces the approaches to teaching promoting inquiry. The 2012 curriculum recommends dedicating 10 hours of in-class time and 10 hours of personal work to the exploration. Students can choose to explore any area of mathematics and their activity should involve modeling, investigation or application of mathematics commensurate with the level of the course. The internal assessment is criterion-based and each exploration is marked against the following five criteria: communication (4 points), mathematical presentation (3 points), personal engagement (4 points), reflection (3 points) and use of mathematics (6 points). The maximum mark is therefore 20 points. The descriptors for the first four criteria are identical for both mathematics SL and HL, the use of mathematics criterion has different descriptors for the two subjects because more mathematical sophistication is expected from HL mathematics student. To assist teachers, the IB developed a website³⁷ that contains general guidance instruction and examples of assessed student explorations. The explorations are assessed internally by the subject teacher and a random sample from each school is

³⁷ https://ibpublishing.ibo.org/server2/rest/app/tsm.xml?doc=d_5_matsl_tsm_1205_1_e&part=1&chapter=1

generated and sent to the IB for an external moderation process. The purpose of the moderation process is to ensure that objectivity of the assessment. The IB publishes subject reports after each exam session general appreciation of the quality of submitted work, the performance of candidates on each criterion and recommendations for teaching. The May 2017 subject report expresses satisfaction confirming that the mathematical exploration is meeting its objectives because “candidates are collecting their own data, researching independently, conducting experiments, running simulations—all of which represents the true joy of learning” (IBO, 2017, p. 1). On the other hand, the report states that “some schools obviously coached their candidates to follow a particular format, sometimes producing near identical modeling style explorations” and adds that “schools are strongly discouraged from this approach” (IBO, 2017, p. 2). This suggests that not all schools and not all teachers comply with the IB philosophy and do not conduct the internal assessment in a way to meet its purpose.

4.3 Conditions and constraints at the level of discipline

At the level of discipline, we have noticed that several choices were motivated and guided by the conditions created at the higher levels of the didactic codeterminacy scale. For example, the fact that mathematics has always been a mandatory discipline in the IB is closely linked to the “sputnik effect” and reinforced by some of the participants of the Second Sèvres Conference, who found unacceptable that certain literary section of the French baccalaureate no longer required mathematics. We have also observed changes in the conception of mathematics as a discipline. The SL and HL syllabuses from the seventies emphasize the abstract concepts and structures which is probably the result of the New mathematics reform in Europe. However, the mathematical studies course offered a more applied approach and included an internal assessment in the form of a personal project. This leads us to the conclusion that the form of a more applied mathematical activity was reserved to weaker students with little interests in mathematics. This conception changed though in the late nineties with the introduction of an internal assessment in the form of a Portfolio requiring completing tasks based on mathematical investigation and modeling in all mandatory mathematics subjects. This change was probably motivated by the observed decline of interest for science and mathematics (Rocard et al., 2007) and the newly published National Science Education Standards in the USA in 1996 providing a decent definition of inquiry-based education (National Research Council, 1996, p. 23). Indeed, from the late nineties until present, we have identified at the level of pedagogy a strong will to make inquiry-specific objectives more explicit, which has also been reflected at the level of discipline. In 2012 the Portfolios are replaced by the Mathematical Exploration and in 2019 a totally new approach to teaching and learning mathematics has been introduced. In order to reinforce these choices and help their implementation in the classrooms, the IB

proceeded through four means: explicit documentation, teacher training workshop, resources and assessment. In addition, the 2019 syllabus claims to have revised and reduced content and has reserved 30 hours to the development of investigative, problem-solving and modeling skills as a response to the time constraints related with the implementation of inquiry-based activities.

4.3.1 The conditions and constraints of the ME

The institutional conditions and constraints that govern the application of the ME within the IB create a sort of paradoxical situation concerning the topogenesis and the didactic contract.

The first major constraint of the ME is that in the IB one teacher has to manage about 20 different explorations. This means that there are about twenty different didactic systems, where $X=x$ represents an individual student, $Y=y$ the classroom teacher and Q is the topic to be explored chosen by the student. This also means that for the teacher it is quite difficult, if not impossible, to conduct a detailed investigation of each question and in this sense the teacher does not have a ready-made answer as the exploration unfolds for him at about the same pace as for the student. What differentiates him from the students are his experience and mathematical background. When following the official institutional recommendation, 10 hours of in class time is spent on the exploration. In a class of 20 students, the teacher can dedicate realistically about 20 minutes³⁸ to each individual student. Since the students are supposed to spend about 10 hours working on their exploration outside of the classroom, this leads to quite a paradoxical situation concerning the topogenesis. The ratio of teacher-student topos in this case is quite disproportional (1:49) which puts the students into quite an unusual position, especially when considering a typical mathematics class spending the rest of the school year under a classical didactic contract of the paradigm of visiting works. This results in the fact that the students will most likely spend more time searching for answers in the media rather than actually expect ready-made answers from the teacher. The teacher is thus no longer a unique source of information and answers coming from the teacher should be object to the same evaluation as from any other media. Indeed, the teacher being part of the didactic system is an inquirer himself with a richer cognitive universe but is not necessarily in possession of all ready-made answers to the inquiry. Nevertheless, we can suppose that in a class governed by a classical didactic contract, the teacher will remain a privileged source of information.

³⁸ This amount was calculated by removing 200 minutes from the 600 available minutes that are spent collectively and dividing the remaining 400 minutes among the student of a class of 20.

5 The Institutional Analysis: Conclusion

In this chapter, we have attempted to provide an insight into the IB institution using the didactic codeterminacy scale and taking into account the historical development of this institution. At the level of society, we have identified the main conditions that lead to the creation of an international baccalaureate necessary to meet the needs of expatriates' community. We showed how the existing conceptions of general secondary education (the early specialization vs. broad encyclopedic approach) influenced the creation of the IB curriculum and lead to the six subject group general scheme. The IB was also lucky to have attracted experts whose efforts to conduct reforms in their national systems were making slow progress. (Peterson, 2003, p. 25) This led to a certain number of innovations that we identified at the level of pedagogy which was, however, limited by the university admissions requirements. Thus, the need that conditioned the creation of the international baccalaureate became a major constraint in implementing innovative pedagogical approaches in the IB curriculum. The IB needed therefore to navigate between the deep waters of pedagogical innovation and the shores imposed by university admissions for the acceptance of the diploma. The compromise resulted in assessment dedicating 80% to standardized external exams, 20% of internal assessment and a core requirement (extended essay, theory of knowledge and CAS) to ensure the education of the whole man. At the level of pedagogy, we have noticed a clear tendency to better define and document the IB educational philosophy drawn at the same time from its own history and responding to the 21st century skills. In mathematics, these changes started in the late nineties and inquiry-specific objectives were stated explicitly in the 2012 guides. The IB authorities are probably aware that the implementation of these pedagogical principles is not an easy affair and put therefore in place several means to reinforce them. The key is the teachers and the 2019 syllabus tried to take into account the time constraint by reducing the content and reserving 30 hours to inquiry-based activities. In addition, resources have been developed to support teachers in implementing these types of activities.

This analysis shows the existence of favorable conditions for inquiry-based education at each level of the didactic codeterminacy scale, in particular the fact that 20% of the final grade is based on an inquiry-based activity and mandatory regardless of the level of mathematics chosen by the student. In this sense, we consider that the IB provides better conditions for further research compared to the conditions in national systems. We are going to pursue our work by analyzing the results of a survey conducted with IB mathematics teachers in order to identify their perceptions of this system, their beliefs about mathematics teaching, their views of the ME and compare and contrast them with the institutional expectations.

Chapter II

TEACHER SURVEY

From the institutional perspective of the IB, teachers are the vectors for the implementation of its educational principles in the classroom and the mathematics SL guide (IBO, 2012) emphasizes the importance of the teachers' role in the implementation of the ME. They must ensure that the topic chosen by the students remains within their reach, accompany the students throughout the process with appropriate advice, direct the students to research paths or suggest sources of information. This also shows the teacher's responsibility with respect to devolution. In fact, it is intended that teachers may provide formative feedback on a draft of the exploration and the same guide states that "this advice should be in terms of how the work could be improved [...]" (IBO, 2012, p. 43). Therefore, we developed a questionnaire to ask about IB teachers' conceptions and practices regarding the implementation of ID and problem-solving in mathematics. The following are the broad categories addressed by this questionnaire:

1. General information
2. Previous experience with IBL
3. Teachers' perception of the institution
4. Attitudes, beliefs, and convictions about mathematics teaching
5. Attitudes, beliefs and convictions about IBL
6. Implementation difficulties
7. Beliefs and convictions about the ME
8. Practices regarding the ME
9. Day-today teaching practices
10. Estimated amount of teaching time that is/should be dedicated to IBL.

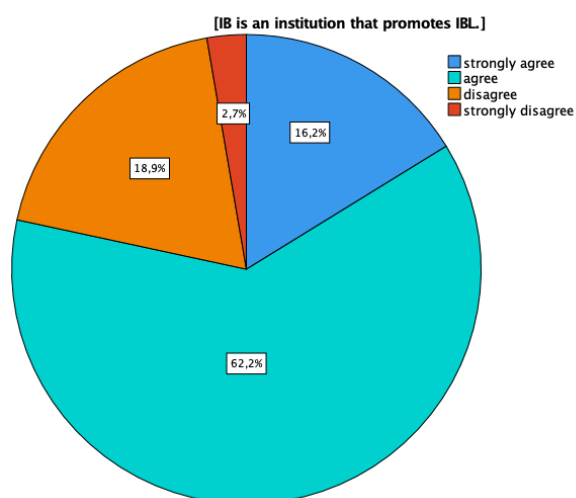
We collected 74 complete responses from IB teachers working in international schools in 20 different countries. This sample is composed of 52.7% men and 47.3% women and includes 25 nationalities, of which approximately half are native English speakers. Of these 74 teachers, 46% work in an international school in their home country and 54% are expatriates. In terms of the distribution of schools by type, our sample also reflects a certain diversity (public school 28%, private non-profit school 49% and private for-profit school 23%). In addition, 10 schools of all types offer residential boarding. We can therefore attest to have a sample that sufficiently reflects the diversity of the reality of the international context. We also collected information on training and professional experience. Thus, 89% of the teachers have, in addition to their academic training in their academic discipline, completed some form of teacher training.

Most of the items were statements that teachers were asked to respond to on a Likert scale (1 = strongly disagree, 2 = disagree, 3 = agree and 4 = strongly agree). The Table 4 serves as an example of the items proposed in this questionnaire.

	Strongly disagree	Disagree	Agree	Strongly agree
1 It is too difficult to cover all content through inquiry.				
2 It is difficult to sustain inquiry because of the exam pressure.				
3 I am afraid that basic skills will not be mastered.				
4 I am afraid that students get lost and frustrated.				
5 I am afraid that discipline issues might appear (the students will be "off-task").				
6 I don't feel well prepared to teach by inquiry.				
7 I would need to change my teaching style.				
8 It takes too much time to develop meaningful inquiry activities.				
9 I don't have enough IBL resources.				
10 The available instructional material is often of poor quality.				
11 Computer and other learning technologies are not available at my school.				
12 I am not equipped with the necessary mathematics knowledge.				
13 I find connecting mathematics to real-world context rather difficult.				
14 The mathematics needed to do inquiry is often inaccessible to students.				
15 Other:				

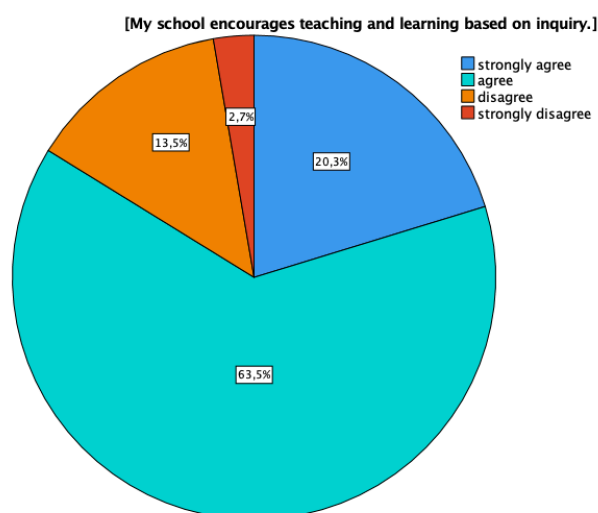
Table 4: Example of the questionnaire items in the "Implementation difficulties" category

We have collected information on teachers' perceptions of the IB's educational philosophy in order to compare it with the institutional discourse. Concerning the item: "The IB is an institution that promotes IBL" almost 80% of the teachers from the survey agree or strongly agree, as shown in Graphics 1.



Graphics 1: Distribution of responses on teachers' perception of the IB

As can be seen on Graphics 2, the majority of schools (almost 85%) in which the participating teachers work adhere to the IB educational philosophy and encourage IBL. Obviously, this is based on a subjective perception of the respondents.



Graphics 2: Distribution of the answers about the school attitude towards IBL

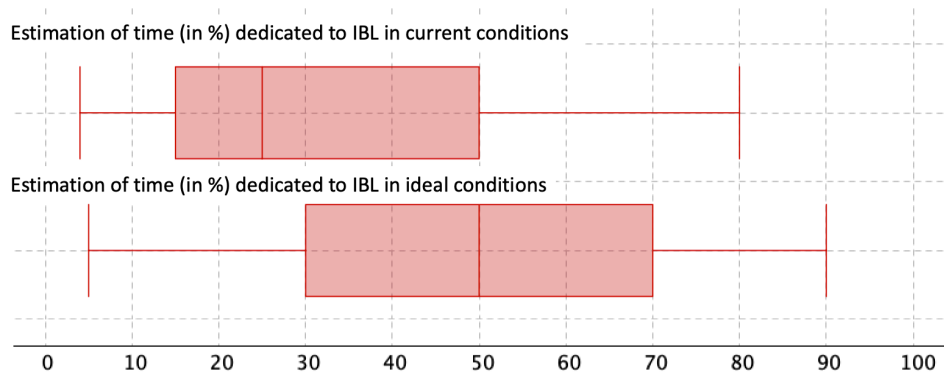
We also judged it important to ask teachers about their own experiences with IBL as former students. Of all the teachers who completed the questionnaire (74), only 27 (36.5%) had experienced IBL as students. Those who had experienced IBL as a student then responded to three items:

1. This experience has helped me better understand mathematics.
2. It was interesting but not very useful.
3. It was a waste of time.

Of these 27 teachers, only 3 think that this experience did not help them understand mathematics better, about a third find the IBL process interesting but not very useful, and none checked off “it was a waste of time.” On the other hand, 83% of those who did not experience IBL as a student think it could have contributed to a better understanding of mathematics for them.

Regarding their experience with IBL as a teacher, 77% of all teachers said they had used it before joining the IB. Overall, the participating teachers expressed a rather positive attitude towards IBL, with only 3 teachers stating that they were not interested in IBL.

We also asked teachers to estimate the amount of time dedicated to IBL in their classrooms in the current conditions and the amount of time they would like to dedicate to IBL if they had more leeway. We can see on the second whisker box (Graphics 3) a shift to the right that shows the teachers’ willingness to implement more IBL in their teaching.



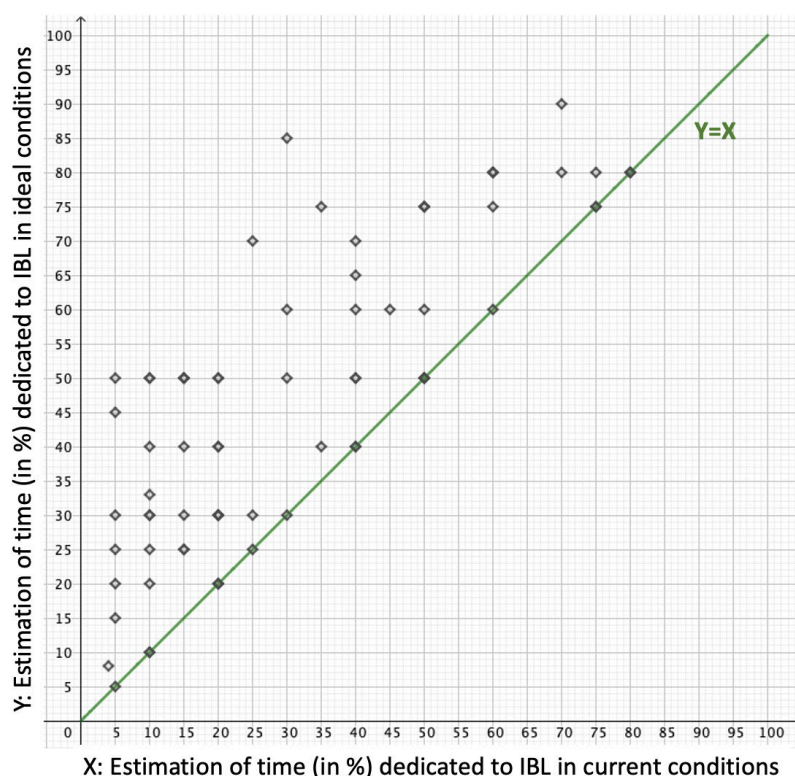
Graphics 3: Box and whisker plot to illustrate answers concerning time dedicated to IBL

Table 5 gives the details of the descriptive statistics of the same data. In both cases the mean is higher than the median, which means that it is influenced by some higher values. Therefore, the median gives a better idea about the central behavior of the data set. The standard deviation is relatively high which shows a large variability among teachers but remains almost invariant between the two sets of data which indicates a certain consistency of the range of the data. With the increased quartiles and interquartile range, we can conclude that overall, the majority of teachers would like to use IBL more in their practices.

n=74	Declared (%)	Ideal (%)	Augmentation (%)
Average	31,9	47,6	15,8
Standard deviation	22,4	22,2	13,9
Min	4	5	0
Q1	15	30	0
Median	25	50	15
Q3	50	70	25
Max	80	90	55

Table 5: Descriptive statistics concerning the time dedicated to IBL

The scatterplot (Graphics 4) below represents the covariation of the declared time dedicated to IBL versus the ideal time that teachers would like to dedicate to it. The points on the $Y=X$ line represent the 10 teachers (13.5%) who do not wish to change the time dedicated to ID in their practices, in other words, they are happy with what they do. The third column of Table 5 provides detailed descriptive statistics concerning this increase. These results are rather encouraging and show that the conditions for implementing IBL are generally met at both the institutional level and among the teachers (at least in the sample we were able to survey). In Graphics 3, we can also observe a wide range of declared times, which shows that there are significant differences between teachers and between the declared practices: actual and desired. We will try to explain this phenomenon by taking into account the constraints expressed by these same teachers.



Graphics 4: Declared time versus ideal time

We sought to investigate which constraints reported by teachers concerned them and prevented them from implementing IBL in a sustainable manner. We identified six categories (see Table 6) that could potentially pose difficulties for implementing IBL in the classroom. Table 6 provides an overview of these categories, the items and the descriptive statistics of the responses for which we solicited the teachers' opinions.

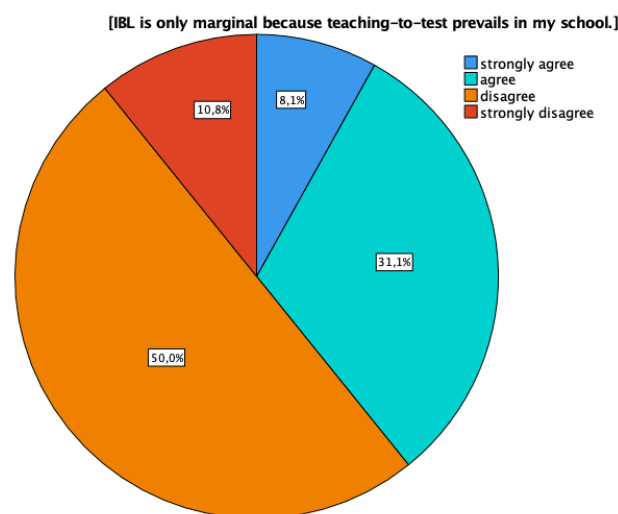
Category of difficulties concerning:	Item	Descriptive statistics				
		Mean	SD ³⁹	Q1	Median	Q3
the lack of time	It is too difficult to cover all content through inquiry.	3.28	0.65	3	3	4
	It takes too much time to develop meaningful inquiry activities.	2.74	0.79	2	3	3
the pressure of the exam	IBL is only marginal because teaching-to-test prevails in my school.	2.36	0.78	2	2	3
	It is difficult to sustain inquiry because of the exam pressure.	3.26	0.70	3	3	4
the lack of resources	I don't have enough IBL resources.	2.84	0.77	2	3	3
	The available instructional material is often of poor quality.	2.65	0.74	2	3	3

³⁹ Standard deviation

	Computer and other learning technologies are not available at my school.	1.61	0.80	1	1	2
the teacher	I don't feel well prepared to teach by inquiry.	2.18	0.84	2	2	3
	I would need to change my teaching style.	2.14	0.78	2	2	3
	I am not equipped with the necessary mathematics knowledge.	1.38	0.56	1	1	2
the students	I am afraid that basic skills will not be mastered.	2.64	0.71	2	3	3
	I am afraid that students get lost and frustrated.	2.53	0.68	2	2.5	3
	I am afraid that discipline issues might appear (the students will be "off task").	2.07	0.86	1	2	3
the mathematics at stake	I find connecting mathematics to real-world context rather difficult.	1.81	0.77	1	2	2
	The mathematics needed to do inquiry is often inaccessible to students.	2.15	0.83	2	2	3

Table 6: Difficulties concerning the implementation of IBL

Let us now examine closely the difficulties reported by the teachers. Graphics 5 shows the details of the gathered responses concerning the following item: "IBL is only marginal because teaching-to-test prevails in my school."



Graphics 5: Distribution of answers concerning teaching-to-the test attitude

Nearly 40% of the participating teachers reported that “teaching to the test” prevails at their school, which shows that the pressure to pass the standardized exam that schools may exert on teachers is not negligible. In addition, 88% (49% agree and 39% strongly agree) endorsed the statement, “It is difficult to sustain inquiry because of the exam pressure.” Finally, 92% of teachers find it too difficult to cover all curriculum content through IBL. Almost 70% of teachers report that they do not have enough resources that would facilitate the implementation of IBL in a sustainable way. While about two thirds felt that preparing IBL-based activities was too time-consuming and that existing resources were of poor quality.

Table 7 provides the overview of responses regarding potential teacher-related difficulties. About one third of the teachers do not feel well prepared to teach through inquiry and the same proportion fear having to change their teaching style. On the contrary, the majority of teachers are confident about their mathematical knowledge.

	I don't feel well prepared to teach by inquiry.		I would need to change my teaching style.		I am not equipped with the necessary mathematics knowledge.	
N=74	Frequency	%	Frequency	%	Frequency	%
Strongly disagree	16	21.6	15	20.3	49	66.2
Disagree	34	45.9	37	50	22	29.7
Agree	19	25.7	19	27.7	3	4.1
Strongly agree	5	6.8	3	4.1	0	0

Table 7: Difficulties concerning the teacher

Table 8 summarizes the responses regarding student-related difficulties. As can be seen, teachers are quite divided on this subject. As for the mastery of basic skills almost 60% express concern that these will not be mastered by the students. Half of the teachers fear that students will be lost and frustrated if they engage them in IBL. On the other hand, only one third of the respondents fear discipline problems.

	I am afraid that basic skills will not be mastered.		I am afraid that students get lost and frustrated.		I am afraid that discipline issues might appear (the students will be “off task”).	
N=74	Frequency	%	Frequency	%	Frequency	%
Strongly disagree	3	4.1	3	4.1	21	28.4
Disagree	28	37.8	34	45.9	31	41.9
Agree	36	48.6	32	43.2	18	24.3
Strongly agree	7	9.5	5	6.8	4	5.4

Table 8: Difficulties concerning the students

Responses to questions in the last category of potential difficulties related to the implementation of IBL are shown in Table 9. Finally, unlike our predictions, only slightly more than 20% of teachers believe that it is difficult to find mathematical applications

related to the real-world context, and this percentage rises to about 35% regarding the belief that IBL requires mathematics that is inaccessible to students.

	I find connecting mathematics to real-world context rather difficult.		The mathematics needed to do inquiry is often inaccessible to students.	
N=74	Frequency	%	Frequency	%
Strongly disagree	30	40.5	18	24.3
Disagree	28	37.8	30	40.5
Agree	16	21.6	23	31.1
Strongly agree	0	0	3	4.1

Table 9: Difficulties concerning the mathematics at stake

We also gave the teachers the opportunity to freely formulate the difficulties they encounter and 19 of them gave a response. Practically all of them say that the biggest obstacle to the implementation of IBL is the lack of time and the pressure of practicing for the final exam, which leaves little space to propose IBL-based activities in regular classes (outside the mathematical exploration device), which confirms our previous analyses. However, we did find some comments that can be interpreted in terms of didactic contract or conceptions of mathematics.

- The most severe problem when teaching mathematics is that quite a number of students have a negative preconception of maths; therefore inquiry-based activities sometimes frustrate them at the very beginning.
- Student expectations of what mathematics and mathematics teaching should look like.
- Also, I have found recently that students want everything on demand, they are less willing to engage in IBL activities than they were 5–10 years ago.

The teachers' responses quoted above suggest that students' conceptions of mathematics and course routines influence their expectations of instruction and may also be a barrier to IBL implementation. However, it is important to keep in mind that these are teachers' statements and not students' conceptions of mathematics.

Our analysis of the official documentation shows that the IB has integrated IBL into its educational philosophy and is trying to create the favorable conditions for its implementation over time. Teachers, too, have an overall positive and open attitude and declare their willingness to offer inquiry-based education. However, even in an institution that offers IBL a privileged place both in its philosophy and in its institutional conditions, there are constraints that hinder the implementation of IBL in classrooms outside of the space that is strictly reserved for it. Two of these constraints stand out clearly from the analysis of the responses to our survey: the pressure of the exam and the lack of time. Thus, the teachers state that it is difficult to implement IBL into the regular classes and that the only possibility remains the Mathematical Exploration. In a recent curriculum review, IB

authorities tried to address time-related difficulties by increasing the number of hours dedicated to IBL-based activities and reducing the content in the curriculum. Further analysis is, however, needed to see what impact this may have.

After examining the IB teachers' attitudes towards IBL revealing that one of the main obstacles hindering the implementation of IBL is the final exam, in the following chapter, we analyze in detail whether and to what extent the IB includes inquiry-specific objectives and problem-solving in the various types of assessment.

Chapter III

ANALYSIS OF ASSESSMENT IN THE IB

1 General context of the IB assessment

As mentioned in the institutional analysis, one of the main reasons for creation of the IB Diploma Program was to provide access to higher education, and as a result the question of assessment has become one of the important constraints that have shaped curriculum design. Among various assessment methods ranging from taking a series of standardized exams spread over several days to issuing diplomas based on a continuous assessment of each subject, Renaud (1974) reports that the IB “a definite but prudent innovation” (p. 43). It was thus decided that an external assessment in line with the traditional approach to examinations recognized by university admissions would be complemented by an internal assessment in each subject. Thus, in mathematics, external standardized exams form 80% of the assessment and 20% are dedicated to an internal assessment. The IB states in the Mathematics SL Guide to guarantees the following skills for the diploma holders:

1. **Knowledge and understanding:** recall, select and use their knowledge of mathematical facts, concepts and techniques in a variety of familiar and unfamiliar contexts.
2. **Problem-solving:** recall, select and use their knowledge of mathematical skills, results and models in both real and abstract contexts to solve problems.
3. **Communication and interpretation:** transform common realistic contexts into mathematics; comment on the context; sketch or draw mathematical diagrams, graphs or constructions both on paper and using technology; record methods, solutions and conclusions using standardized notation.
4. **Technology:** use technology, accurately, appropriately and efficiently both to explore new ideas and to solve problems.
5. **Reasoning:** construct mathematical arguments through use of precise statements, logical deduction and inference, and by the manipulation of mathematical expressions.
6. **Inquiry approaches:** investigate unfamiliar situations, both abstract and real world, involving organizing and analyzing information, making conjectures, drawing conclusions and testing their validity. (IBO, 2012, p. 9)

In the following two sections, we analyze what place is dedicated to the evaluation of problem-solving and the investigation approaches and how the above objectives are included in the internal, respectively external assessment.

2 The internal assessment

As already mentioned above, the internal assessment constitutes 20% of the final grade and is entirely dedicated to evaluation of inquiry-specific objectives involving an investigation in a field of mathematics. This shows the IB's willingness to assess primarily two of these global objectives: problem-solving and inquiry approaches. The exploration is evaluated out of 20 points according to 5 criteria that aim to assess not only the knowledge and understanding of a strict mathematical content but also take into account the wider skills needed to carry out this exploration. The synthetic description of these criteria linked to the global evaluation objectives is given in Table 10.

Criterium	This criterium assesses:	Corresponding objective
A: Communication <i>4 points</i>	organization and coherence of the exploration	3. Communication and interpretation
B: Mathematical presentation <i>3 points</i>	the student's ability to use appropriate mathematical language, define terminology, and use various forms of mathematical representation	3. Communication and interpretation
C: Personal Engagement <i>4 points</i>	the student's ability to engage in and take ownership of the exploration (independent and/or original thinking, addressing issues of personal interest, etc.)	6. Inquiry approaches
D: Reflection <i>3 points</i>	the student's ability to review, analyze, and evaluate their exploration	5. Reasoning
E: Use of mathematics <i>6 points</i>	the student's ability to mobilize and use mathematical tools in the exploration	1. Knowledge and understanding 2. Problem-solving

Table 10: Assessment criteria of the ME

These criteria are aligned with all of the overall objectives except for the fourth one (Technology) which is not directly assessed within the ME, although students are strongly encouraged to use technology in their explorations.

The primary function of this grid is to provide a unique tool for the certificative evaluation of the ME. It is made available to teachers who must familiarize themselves with it. To ensure that these criteria are interpreted and applied consistently by all teachers, the institution offers training in the form of workshops and provides teachers with corrected explorations commented on by expert examiners. During these training sessions, delivered by a senior examiner, the participants become familiar with the grid and the criteria, apply it to correct some proposed explorations and can exchange or compare their decisions with other teachers and the expert. Thus, the institution assumes the teachers' training in the use of the grid. To ensure the objectivity of the internal evaluation, the work of IB students is

evaluated internally by the school and a sample is then sent to an external examiner for moderation. This also encourages teachers in the same school to work closely together, sometimes even double marking students' work. The IB encourages the teachers to make the rubric available to the students, ensuring that the students understand how the criteria will be applied and what they mean. The teacher can also share "sample" explorations made available by the IB with the students. These sample explorations (elaborated by real students) have obtained different grades which helps to demonstrate the application of the criteria on concrete examples. One of the activities proposed to help students become familiar with the rubric is to let them use the rubric themselves to grade the examples of the "model" explorations. This represents in a certain way peer evaluation, aiming at the appropriation of the evaluation criteria, and perhaps even allowing them to have a sort of a "meta" view of their own exploration. Although this assessment is primarily certificate-based, the process of exploration includes formative interventions by the teacher. The mathematics guide states that:

The teacher is expected to give appropriate guidance at all stages of the exploration by, for example, directing students into more productive routes of inquiry, making suggestions for suitable sources of information, and providing advice on the content and clarity of the exploration in the writing-up stage. Teachers are responsible for indicating to students the existence of errors but should not explicitly correct these errors. It must be emphasized that students are expected to consult the teacher throughout the process. (IBO, 2012, p. 46)

This grid allows us to better understand the formative function of the ME, which we illustrate on the following example. If we examine closely the criterion E: Use of mathematics, we can see that it has, with its 6 points, the greatest weight in this grid. The detailed descriptors for this criterion are provided in Table 10. However, it can be seen that it represents only 30% of the total weight dedicated to the evaluation of the students' ability to mobilize and use mathematical tools. The mathematics guide (IBO, 2012) states that "the mathematics explored should either be part of the syllabus, or at a similar level or beyond" (p. 49) and specifies that no more than 2 of the 6 points may be awarded, "if the level of the mathematics is not commensurate with the level of the course" (p. 49).

Achievement level	Descriptor
0	The exploration does not reach the standard described by the descriptors below.
1	Some relevant mathematics is used.
2	Some relevant mathematics is used. Limited understanding is demonstrated.
3	Relevant mathematics commensurate with the level of the course is used. Limited understanding is demonstrated.
4	Relevant mathematics commensurate with the level of the course is used. The mathematics explored is partially correct. Some knowledge and understanding are demonstrated.
5	Relevant mathematics commensurate with the level of the course is used. The mathematics explored is mostly correct. Good knowledge and understanding are demonstrated.
6	Relevant mathematics commensurate with the level of the course is used. The mathematics explored is correct. Thorough knowledge and understanding are demonstrated.

Table 11: Detailed descriptors of Criterion E: Use of mathematics (IBO, 2012, p. 49)

Therefore, the students, regardless of their mathematical abilities in *stricto sensu*, are encouraged to engage in mathematical activity during their exploration, as the level and the use of mathematics alone are not a decisive criterion for success. If a weak student explores mathematics adjusted to his/her level (and scores fewer than 3 points on this criterion), he/she has the opportunity to demonstrate the skills assessed on the other four criteria and not only obtain a passing grade for the ME but also develop other skills and retain a positive mathematical experience as a bonus. In this sense, the teacher's guidance plays a crucial role in exploiting the full potential of the ME. The mathematics guide (IBO, 2012) emphasizes the importance of the role of the teacher not only in ensuring that the students are aware of the requirements and assessment criteria for this type of work, but also in encouraging students to initiate discussions for guidance and to seek help when necessary.

In conclusion, the ME is shaped by many conditions and constraints that weigh on the institution and represents the compromise that the IB has put in place to reflect its educational philosophy. The rubric developed by the IB is a tool with a dual purpose: to foster a rich mathematical activity, on the one hand, and to ensure that the certificative assessment is as objective as possible, on the other hand. Conceived this way, the ME allows students to experience and to a certain extent to engage in genuine mathematical activity without an obligation to achieve results. As for the teachers, they can adjust the choice of subjects and the level of the mathematics resulting from it to the real abilities of the student without compromising the obtention of the diploma. The ME also allows us to see that the evaluation of problem-solving and more broadly of inquiry approaches is a complex process

that is not just about ticking right or wrong on an exam paper. We now analyze the external assessment, which remains, with the weight of 80%, the main criterion for awarding the IB diploma.

3 The external assessment

The external assessment in the form of standardized examinations consists of two 90-minute papers (1 and 2), further split into two 45-minute sections (A and B). Students are not allowed to use a calculator in Paper 1 while a graphical display calculator is indispensable for Paper 2. In each paper, Section A consists of 7 short-answer questions where a small number of resolution steps is required. The objective of this section is to test the knowledge and application of techniques throughout the syllabus. In addition, the questions respect an ascending order of difficulty and the last two generally go beyond the scope of the classic types of tasks by requiring a thorough reflection on the choice of mathematical tools to apply. In sections B of each paper, as indicated in the SL Mathematics Guide, students must answer 3 questions that require answers developed with sustained reasoning. These questions are also proposed in an ascending order of difficulty and are dedicated, according to the same guide, to the evaluation of problem-solving.

3.1 Theoretical background and methodological notes

To determine more precisely the place of problem-solving in the IB's external assessment. To do this, we have established objective criteria that allow us to characterize and categorize the items offered in the IB exams, based on the work of Robert and Rogalski (2002) and Roditi and Salles (2015) synthesized in Table 12. The classification of Roditi and Salles (2015) inspired by the work of Robert and Rogalski (2002) is based on an analysis of the item text to determine the nature of the application of the mathematical knowledge necessary to find a solution.

Level 1: DA direct application	At Level 1, the student performs a routine task and directly obtains the expected result through the implementation of a procedure, often unique, that is indicated or suggested by the text of the item. These tasks are considered as acquired and automated for the students.
Level 2: AA application with adaptation	Items at Level 2 require the student to some kind of adaptation or transformation of the text of the item, the data or the question) before applying his knowledge. Some initiative and responsibility are postponed to the student.
Level 3: AI application with intermediaries	Solving a task at Level 3 requires that the student, introduces autonomously one or more intermediaries not suggested in the text of the item. The initiative to introduce intermediary steps as well as the responsibility to carry out the task is totally left upon the student.

Table 12: Categorization of types of problems

As shown in Table 12, these authors distinguish three levels of application of knowledge. Items at level 1 or direct application (DA) represent tasks that require the execution of a known technique and are intended to verify isolated knowledge. These are according to Robert and Rogalski (2002) simple applications (involving immediate conceptualization) and isolated applications (involving one piece of situational knowledge at a time). For Roditi and Salles (2015) it is a direct application of a procedure. Items at the second level or applications with adaptation (AA) require the adaptation of the mathematical tool (mentioned directly or indirectly in the text of the item) to the proposed situation. According to Roditi and Salles (2015) this adaptation can take different forms: carry out a change of frame or register, relate different mathematical objects, isolate a relevant figure from a complex figure, etc.). They then speak of an application of a procedure with adaptation of the text of the item. The third level or applications with intermediates (AI) concerns tasks whose resolution requires the student to mobilize situational knowledge and/or techniques and introduces intermediate steps that are not given by the text of the item. These applications are characterized by Robert and Rogalski (2002) as non-simple (text of the tasks without indications requiring several steps) and non-isolated (requiring to put together several pieces of situational knowledge and/or interpret the results). Robert and Rogalski (2002) conclude that the success of this type of item indicates “that this knowledge is available to students” (p. 12). In addition, “levels of students’ current knowledge” must also be taken into account (p. 12), because the same exercise can lead to different applications. It is also important to point out that the analysis of Roditi and Salles (2015) also showed a “relatively large dispersion of the difficulty of the items” (p. 251) within each level of knowledge implementation and conclude that this criterion is not

sufficient to determine the difficulty of an item. However, they also show that the three levels of knowledge implementation, which represent “an increasing demand for mathematical activity, also correspond to an increasing difficulty for students” (p. 252).

To conduct our analysis, from each text of an item over the period 2014–2017, we determined the level of knowledge implementation (DA, AA, and AI) of each item. In addition, as the IB ranks the questions in ascending order of difficulty in each test, this allowed us to assign a level of difficulty to each question according to the position in the test (easy E, intermediate I and challenging C). In addition, we also had access to the examiners’ report for each item, which allowed us to know the overall level of success of each item: overall success (S), some difficulty (D), overall failure (F). This set of information allowed us to conduct a cross analysis of these items.

3.2 Analysis and results

To illustrate this classification, we start by giving three examples of different test items (Example 1, Example 2, Example 3) situated within the Sequences and Series topic.

In an arithmetic sequence, the first term is 2 and the second term is 5.

- | | |
|--|-----|
| (a) Find the common difference. | [2] |
| (b) Find the eighth term. | [2] |
| (c) Find the sum of the first eight terms of the sequence. | [2] |

Example 1: Example of a direct application task (DA), Easy (E) and Overall success (S)

In this first example, the text of the item asks to perform isolated techniques for which the main knowledge (in an arithmetic sequence...) is given, limiting the student’s initiative it to applying the formulas and carrying out the calculations.

The sums of the terms of a sequence follow the pattern

$$S_1 = 1 + k, S_2 = 5 + 3k, S_3 = 12 + 7k, S_4 = 22 + 15k, \dots, \text{ where } k \in \mathbb{Z}.$$

- | | |
|---|-----|
| (a) Given that $u_1 = 1 + k$, find u_2 , u_3 and u_4 . | [4] |
| (b) Find a general expression for u_n . | [4] |

Example 2: Example of an application with adaptation task (AA), Challenging (C) and Overall failure (F)

This second text of the item indirectly situates the task in the context of sequences and series and the process to follow is only partially indicated in question (a). However, to find the general expression for part b) students must recognize from the first generated terms that, on the one hand, they form an arithmetic sequence and a geometric sequence, on the other hand.

[Maximum mark: 7]

Ramiro walks to work each morning. During the first minute he walks 80 metres. In each subsequent minute he walks 90 % of the distance walked during the previous minute. The distance between his house and work is 660 metres. Ramiro leaves his house at 08:00 and has to be at work by 08:15.

Explain why he will not be at work on time.

Example 3: Example of an application with intermediaries task (AI), Challenging (C) and Overall failure (

Finally, in the third example, nothing in the text of the item suggests the use of a geometric series, the initiative of the choice of mathematical tools is thus left upon the student as well as the overall modeling of the task.

There are also items requiring an investigation approach where the students are asked to search for patterns, make conjectures and find a general formula (see Example 4) or to recognize an arithmetic or geometric sequence or series based on a pattern (see Example 5).

The following figures consist of rows and columns of squares. The figures form a continuing pattern.

Figure 1 has two rows and one column. Figure 2 has three rows and two columns.

Figure 1

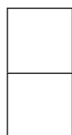


Figure 2

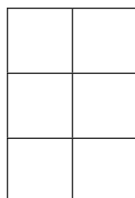


Figure 3

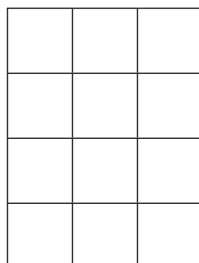


Figure 4

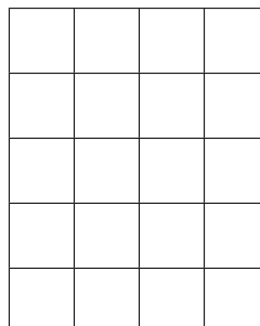


Figure 5 has p rows and q columns.

(a) Write down the value of

(i) p ;

(ii) q .

[2]

Each small square has an area of 1 cm^2 . Let A_n be the total area of Figure n . The following table gives the first five values of A_n .

n	1	2	3	4	5
$A_n (\text{cm}^2)$	2	6	12	20	k

(b) Find the value of k .

[2]

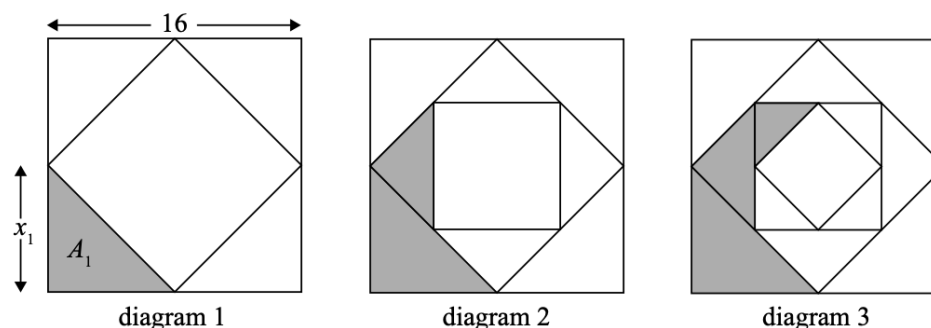
(c) Find an expression for A_n in terms of n .

[2]

Example 4: An investigation task leading to a generalization (AA, I, S)

10. [Maximum mark: 15]

The sides of a square are 16 cm in length. The midpoints of the sides of this square are joined to form a new square and four triangles (diagram 1). The process is repeated twice, as shown in diagrams 2 and 3.



Let x_n denote the length of one of the equal sides of each new triangle.
Let A_n denote the area of each new triangle.

- (a) The following table gives the values of x_n and A_n , for $1 \leq n \leq 3$. **Copy** and complete the table. (Do **not** write on this page.) [4]

n	1	2	3
x_n	8		4
A_n	32	16	

- (b) The process described above is repeated. Find A_6 . [4]

- (c) Consider an initial square of side length k cm. The process described above is repeated indefinitely. The total area of the shaded regions is k cm². Find the value of k . [7]

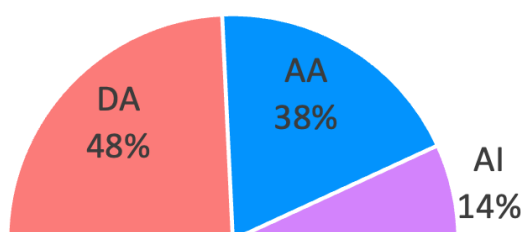
Example 5: An investigation task leading to a generalization (AI, C, F)

This categorization of items allowed us to map the different types of items we identified in the standardized exams to the overall goals set out at the beginning of this section. The items of the first level of knowledge application (DA) mainly reflect the first objective (Knowledge and understanding), they are generally overall succeeded, even if some may pose difficulties related in particular to the complexity of the technique to be mobilized. Among items classified AA or AI, the majority of those proposed by designers require hypothetical-deductive reasoning and/or logical implication and aim to evaluate objectives 2 (Problem-solving) and 5 (Reasoning). In addition, items that require the use of investigative approaches aim to assess overall objectives 2 (Problem-solving) and 6 (Investigative Approach) and are classified as AA or AI. This is, however, not sufficient to say that all AA-rated items are representative of problem-solving, as the student is often very guided in the solving process through a series of proposed sub-questions and ultimately only performs isolated DA tasks with a few reasoning steps to make connections. AI items, on the other hand, generally require a greater initiative from the student. This shows that

we are getting closer to problem-solving. Objective 4 (Technology) is assessed through the items in Paper 2 which are designed in a way that their effective resolution requires an appropriate use of a graphic display calculator.

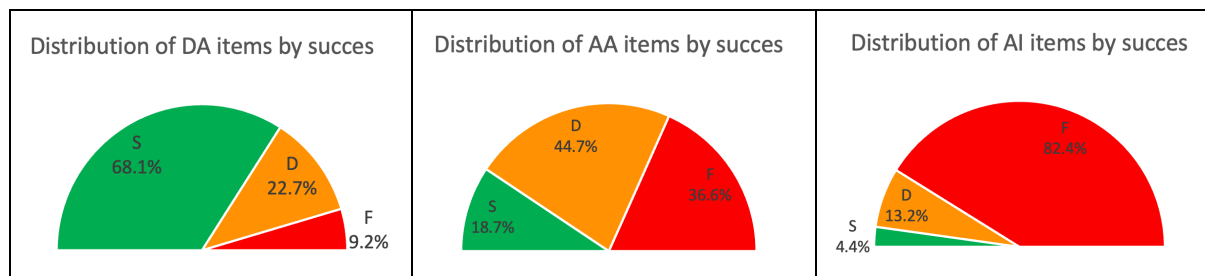
The Graphics 6 shows the distribution of items according to the level of implementation of knowledge. We can see that about half of the exam is dedicated to DA tasks and the other half to AA and AI task requiring a certain degree of initiative of the student.

Distribution of items according to application of knowledge



Graphics 6: Distribution of items according to the application of knowledge

If we cross the type of items with the level of success, we notice a strong correlation: the more the initiative is left to the student, the less succeeded the item is. This trend is represented on Graphics 1.



Graphics 1—Distribution of the items by success

We have observed on the AA items that the execution of the simple tasks requested by the sub-questions is generally well succeeded, but the examiners' reports emphasize that the main difficulty encountered by the students is often due to the lack of reasoning enabling to make links between the different sub-tasks. This same phenomenon has been observed in Demonty et al. (2015) regarding the success of complex tasks in the sense of Rey et al. (2005). Indeed, these authors found that in this case, dividing a task into simple subtasks is not really beneficial for succeeding in a complex task.

Thus, we can see that the IB, in addition to the ME, also attempts to evaluate “problem-solving” tasks in the external part of the assessment. Even though graduation is not conditioned by the success of AI items, these items are nevertheless used to assign higher grades according to students' abilities to tackle problems. University admissions also seem to take this phenomenon into account because in order to be admitted to certain prestigious

branches⁴⁰, it is often requested in addition to having followed the courses at higher level, to have obtained a sufficient grade that does not allow to bypass the items of type AI. In the next section, we move to examine the place and the role of the internal assessment, more specifically its objectives and criteria.

4 Conclusion

The IB as an institution has found ways to integrate problem-solving into assessment through mechanisms that are in line with its educational philosophy. It is the internal assessment system, which is entirely dedicated to the evaluation of problem-solving and investigation processes. We have shown that this one is not strictly limited to mathematical results but broadens the field to other skills related to the investigation approaches. In this system, the final written work of the pupils is essentially the subject of a certificative evaluation. However, the process of elaboration gives rise to interventions by the teacher and interactions with the pupils, which are more related to formative assessment. It is in these interactions that students may learn to develop skills for research, questioning, critical thinking, writing, etc.

As far as the external assessment, accounting for 80%, is concerned, we find standardized examinations, and thus an assessment more oriented towards knowledge and techniques. However, our analysis shows that the problem-solving dimension is not totally absent from this more traditional part of the IB assessment. In fact, if problem-solving is not sufficiently preponderant in the overall IB mathematics assessment to be a discriminating factor of success, it nevertheless seems to carry enough weight to certainly discriminate the best students and thus access to the most prestigious studies. We can infer from the various constraints on the system and the nature of the external evaluation that problem-solving will have a fairly modest place in day-to-day teaching.

In the next part, we confront these findings from the institutional analysis with the day-to-day classroom reality through two case studies focusing on the teaching practices of two IB mathematics teachers (teaching in ordinary classes) and the work of their students (the elaboration of the ME).

⁴⁰ For example, at the “École polytechnique fédérale de Lausanne” (EPFL) to be admitted, the student must have followed Mathematics HL course and obtained a minimum of 38 points out of the 42 possible, which requires students to have 6 or 7 (the maximum) in any examined subject.

PART C

THE LEVEL OF THE DISCIPLINE: CLASSROOM WORK AND EXPLORATION WORK

In order to compare and contrast our findings at the institutional level, the declared practices of teachers collected via the online survey with the reality in the classrooms we designed a case study and followed two mathematics teachers giving the Mathematics Standard Level (SL) course and their students at the International School of Geneva from October to June during the school year 2018/2019. As already mentioned, the IB offers three mathematics courses adapted to the needs and level of the students. We chose to observe the Mathematics SL course which represents a regular secondary school mathematics course comparable to standard national curricula requirements and taken by the majority of IB students. This choice was done in order to avoid bias due to either very weak or very talented mathematics students. On the one hand, we were interested in whether and eventually how inquiry-based activities are used in day-to-day teaching and, on the other hand, we wanted to observe the implementation of the Mathematical Exploration as a means of giving the students an opportunity to conduct a substantial inquiry-based work over a longer period.

Chapter I

OBSERVATION OF ORDINARY MATHEMATICS SL CLASSES

This chapter is dedicated to the observation of the ordinary classes in which we identify and examine what in the teachers' regular practices mirrors most IBME and problem-solving features. We start by providing the general context of the study, followed by a description of the theoretical tools that we use to analyze the proposed activities. We examine then three activities proposed by each teacher and finish by a comparative conclusion of the practices of the two teachers.

1 Generalities and context of the study

The Mathematics SL syllabus (IBO, 2012) suggests to dedicate 150 hours to cover the content over two years. This is equivalent to 3–4 45-minute periods per week depending on the number of school weeks. The content is organized into six main topics and the guide suggests the number of hours that should be dedicated to each topic:

- 1) Algebra (9 hours equivalent to 12 periods)
- 2) Functions and equations (24 hours equivalent to 32 periods)
- 3) Circular functions and trigonometry (16 hours equivalent to 21 periods)
- 4) Vectors (16 hours equivalent to 21 periods)
- 5) Statistics and probability (35 hours equivalent to 47 periods)
- 6) Calculus (40 hours equivalent to 53 periods)

The Diploma Program is designed for the last two years of secondary education and a considerable amount of the content is thus considered as prior learning (see Appendix 8). After presenting the content topics and the syllabus, we move to describe the context of our observations.

We observed two volunteer mathematics teachers (we call them David and Eliot) for a period from October 2018 to June 2019. Both of them were experienced IB mathematics teachers with over 20 years of teaching experience. When questioned about implementing inquiry and investigation activities into daily teaching they both agreed that inquiry approaches are beneficial to students but because of the loaded syllabus and the pressure of the final exams, they both claimed that it is too time-consuming and therefore difficult to implement on a regular basis.

The timetable for mathematics was organized into 1 double period and 2 single periods per week. It actually turned out impossible to observe the same teaching sequences of the two

teachers, we thus had to adapt our methodology because the two Mathematics SL courses were taught in parallel time slots. We therefore decided to observe each teacher for alternating teaching sequences which enabled us to obtain video recordings of the following topics:

- David: Quadratic functions and equations, Differential calculus Part 2, Vectors, Infinite series.
- Eliot: Exponentials and Logarithms, Differential calculus Part 1, Statistics, Sequences and Series, Binomial expansion.

All classrooms were equipped with giant tactile screen (Figure 13), the students had the TI nspire graphic display calculator (GDC) and the teachers had the corresponding software to project the simulation of the GDC screen. In addition, most of the student had their personal laptop in class.

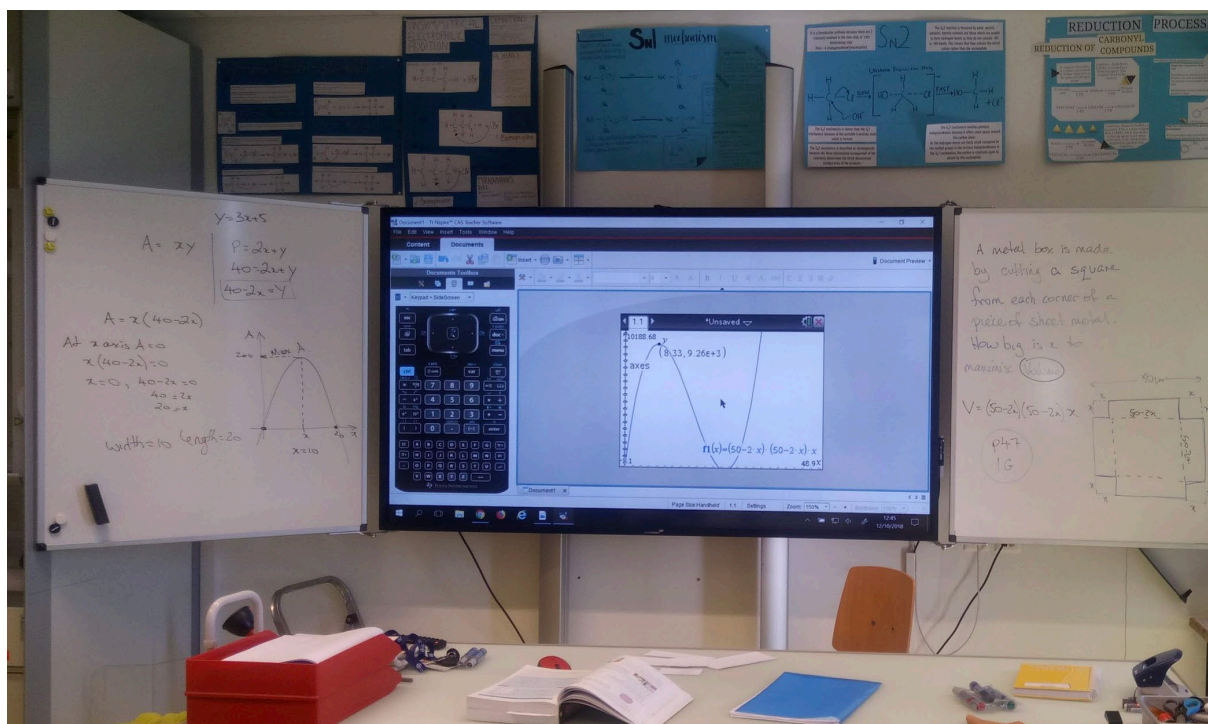


Figure 13: TI nspire GDC software projected on a tactile screen

The textbook used for this class was the *Mathematics for the international student* (Haese, Haese, Haese, Mäenpää, & Humphries, 2013) issued by an Australian editor⁴¹ specializing in mathematics textbook for various anglophone curricula.

In this part, our objective is neither to provide a detailed analysis of David's and Eliot's teaching practices nor to conduct a detailed praxeological analysis of the proposed types of tasks, techniques and technologies but rather to identify and to zoom on examples of forms of IBME or problem-solving activities that these teachers were able to implement in the regular classes despite the given constraints. This raises a methodological question of how

⁴¹ <https://www.haesemathematics.com/>

to identify IBME-like activities within the ordinary classes. Before getting to the examples of the teachers' practices, we therefore briefly describe the criteria that enabled us to choose them and we briefly describe the theoretical tools that we used to conduct our analysis and interpret the results.

2 Analysis according to three axes

In order to choose the examples to analyze, explain and interpret some of the observed phenomena in the activities and problems we approach it according to three criteria:

- 1) We consider the type of the problem or activity
- 2) We examine the format of the work and the allocated time
- 3) We look at the students' topos

In the following briefly present and characterize the concept of potentials to characterize research and peer review activities developed by Georget (2009) and operationalized by Chanudet (2019).

2.1 Activity potentials

As already mentioned in the literature review, in his PhD thesis, Georget (2009) presented an overview of all types of existing activities whose objective is to bring IBME-like activity characterizing a research activity of mathematicians. Georget (2009) split these into two categories: (1) activities of research and peer review, whose objective is to provide the students with opportunities of mathematical investigation and research in a way similar to the work of professional mathematicians and (2) activities oriented to introducing a new notion or technique. The first group of activities allows, according to Georget, to develop what Chevallard (1985, p. 50) refers to as *paramathematical notions* that are indispensable in the work of a mathematician. On the other hand, for Georget (2009), the situations-problem (Arsac, Germain, & Mante, 1991) and a-didactic situations are examples of the second category of activities. He then characterized these activities in terms of five potentials pointing out that while these are relatively independent, they are not completely hermetic:

The *research potential* of a problem ensures that different ways can be found and various strategies exist in order to solve the problem and engages thus the students in a search for different elements and requires them to think about and chose an appropriate strategy. The problem presents itself as a new problem for the students, in other words, the students shouldn't be able to solve the problem in one step or by simply applying a technique that they know in advance. In addition, the research potential contributes to raising students' interest, motivation and curiosity.

The *resistance* and *dynamic resistance potential* ensure that the problem is not too easy and that it presents a certain resistance to the students' attempts to solve it. Georget also specifies the dynamic aspect of the resistance as the capacity of the resistance to evolve with the advancement of the research as new elements appear during the research. This dynamic aspect of the resistance creates conditions for the teacher to provide some hints for students with difficulties without actually solving the problem for them.

The *debate potential* fosters discussion and debate of mathematical nature concerning the strategies to approach the problem, the obtained results and their validity.

The *didactic potential* concerns pieces of knowledge emerging from problem-solving or research. Georget's distinction of two categories of activities leads naturally to distinguishing two different natures of knowledge emerging from problem-solving: (1) paramathematical notions and methodological elements and (2) mathematical concepts or techniques.

An approach to characterize problems through the use of these potentials allows to overcome according to Chanudet (2019) a "dichotomous and rather sterile vision in terms of belonging or not to a certain category of problems" (p. 28) and leads to consider placing a cursor on a continuum between 0 and 1 to characterize a problem or an activity. In this sense, the above-mentioned potentials could be used to analyze a problem or a situation a priori and the same tool could complete the a posteriori analysis to see how much of the potential was actually realized when the same problem or activity is carried out in class. This way we could better understand and diagnose the nature of difficulties related to implementing IBME in mathematics classes.

2.2 Registers of semiotic representation

Mathematical objects, abstract by nature, are not directly accessible by perception and need thus to be represented. On the one hand, while it is necessary, in order to study a mathematical object, to approach it through its various semiotic representations, on the other hand, Duval (1993) shows that it is the ability to translate from one register of semiotic representation to another that plays a central role in the learning of mathematics. He also points out that the semiotic representations are an essential feature of cognitive activity concerning especially: (1) the development of mental representations, (2) the accomplishment of various cognitive functions (communication, treatment...) and (3) the production of situational knowledge. He concludes that

On ne peut pas faire comme si les représentations sémiotiques étaient simplement subordonnées aux représentations mentales, puisque le développement des secondes dépend d'une intériorisation des premières et que seules les représentations sémiotiques permettent

de remplir certaines fonctions cognitives essentielles, comme celle de traitement⁴². (Duval, 1993, p. 39)

Duval names *semiosis* the process of the production of a semiotic representation and calls *noesis* the conceptual understanding of an object. His main thesis is that one cannot exist without the other and vice versa. For him, the underestimation of this fact. This reveals a cognitive paradox of mathematical thinking and explains some of the difficulties of learning mathematics because “there is in fact no noesis without semiosis; we insist on teaching mathematics as if the semiosis was a negligible operation compared to noesis”⁴³ (Duval, 1993, p. 40). Moreover, Duval claims that it is essential, in a mathematical activity, to be able to either mobilize several registers of semiotic representation or privilege the choice of one register over another. For him, “this use of several registers even seems to be a necessary condition so that mathematical objects are not confused with their representations and that they can also be recognized in each of their representations”⁴⁴ (Duval, 1993, p. 40). The above-mentioned tools are particularly useful to describe or interpret difficulties that teachers and students may encounter with the concept of a function. This mathematical object, to be studied and conceptualized correctly, needs to be presented in several registers of representation. Since two of the proposed investigation activities deal with functions and the fact the students have a permanent possibility to use a GDC, the registers of semiotic representation seem to provide an appropriate tool to interpret some of the observed phenomena.

In the next two parts, we examine three activities proposed by David and Eliot in which we identified a priori an inquiry potential according to the three above-mentioned axes (type of problem/activity, time allocation and format of work and the students’ topos). As already mentioned above we focused on identifying activities that carry some IBME features, we thus removed all lecture-like classes, drills and practice exercises. The examples presented below were chosen from what we identified to best represent IBME-like activities from the two teachers’ practices.

⁴² Our translation of the French original: One cannot pretend that semiotic representations are simply subordinate to mental representations, since the development of the latter depends on the internalization of the former, and that only semiotic representations can fulfill certain essential cognitive functions, such as processing.

⁴³ Our translation from the French original: Il n’y pas de noésis sans sémiosis alors qu’on veut enseigner les mathématiques comme si la sémiosis était une opération négligeable par rapport à la noésis

⁴⁴ Our translation from the French original: Ce recours à plusieurs registres semble même une condition nécessaire pour que les objets mathématiques ne soient pas confondus avec leurs représentations et qu’il puissent être aussi reconnus dans chacune de leurs représentations

3 Examples of IBME in David's practices

In David's practices, we zoom on three different situations through which we describe his conception and implementation of problem-solving and IBME. In the first part dedicated to problem-solving on quadratic optimization, we also briefly analyze the organization and the proposition of problems in the textbook. We also would like to say that we did not have a video recording of the first session on optimization and that the session was reconstructed based on the notes and photos of the whiteboard from our observation.

3.1 Problem-solving session on optimization

Even though according to the official IB documentation, the quadratic function topic is considered as prior learning topic, students generally join the IB diploma program from different backgrounds and the schools have to react to the differences. To ensure that the student has the same starting point, the mathematics department at the school where our observation took place decided to start the school year with quadratic functions and equations: revision for some, novelty for others. As first example we propose to analyze a double period dedicated to problem-solving on optimization in the end of the teaching sequence on quadratic functions and equations at the beginning of the school year. On this example, we show how the research potential is influenced and can even be minimized by the way a problem is implemented.

The double period was organized into three parts:

- 1) Problem of maximizing the area of a rectangular field:
A farmer has 40m of fencing to enclose a rectangular field. He uses a barn for one side. What dimensions maximize the area of the field?
- 2) Problem of maximizing the volume of a metal box:
A metal box is made by cutting a square from each corner a piece of sheet metal. How big is x^{45} to maximize volume?
- 3) Practice exercises

3.1.1 Problem of maximizing the area of a rectangular field

At the moment of posing the first problem, the institutional milieu contained the necessary praxeological equipment concerning quadratic functions and equations but the students have never encountered such problem which presented as an optimization problem on the whiteboard as shown on Figure 14. We can see that the teacher presented the problem in

⁴⁵ x is the length of the side of the square to be cut out, the problem was formulated like this on the whiteboard

two registers of semiotic representation: the register of natural language (on the left) and a figural representation (on the right).

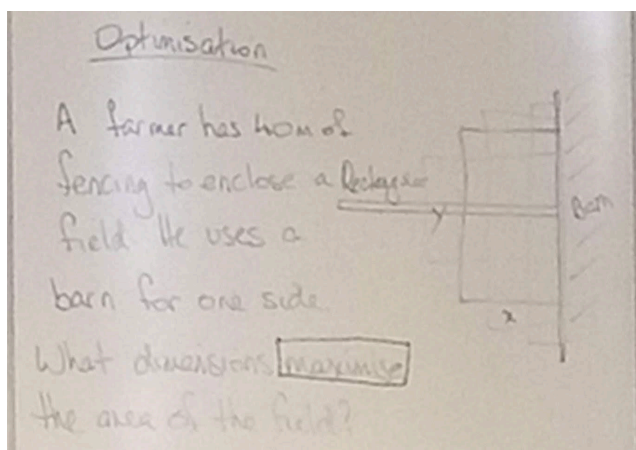


Figure 14: Rectangular field problem

David started by posing a question to the whole class:

T: Do you know this problem?

A student suggested making a square but this suggestion was not really taken into account. It looked as if David asked questions but wasn't really interested in the students' answers and went directly on with the "expert" solution. He started drawing various rectangles as can be seen on Figure 14 to suggest the variability of the area and then he wrote the formula of the area of a rectangle on the whiteboard $A = x \cdot y$ and said:

T: The problem is that we have two unknowns. Is there anything that allows me to get rid of an unknown?

A student proposed to use the formula of a perimeter of a rectangle and without counting the side taken by the barn. The teacher wrote it on the whiteboard and solved it for y as shown on Figure 15:

The image shows a whiteboard with handwritten equations. On the left side, it says $A = xy$. On the right side, it says $P = 2x + y$, $40 = 2x + y$, and $40 - 2x = y$.

Figure 15: Applying the formula of a perimeter of a rectangle

The same student continued and suggested to substitute y in the area formula with the y found in the perimeter formula to obtain: $A = x(40 - 2x)$. The next step was to recognize a quadratic function in this formula and find the vertex.

David then asked:

T: What graph do I get?

S: Parabola.

David continued asking questions without really giving the students enough time to think about the answers:

T: What kind? Opening upwards or downwards? Where does it intersect the x-axis?

Still in the algebraic register, David that continued to solve the problem on the whiteboard as shown in Figure 16.

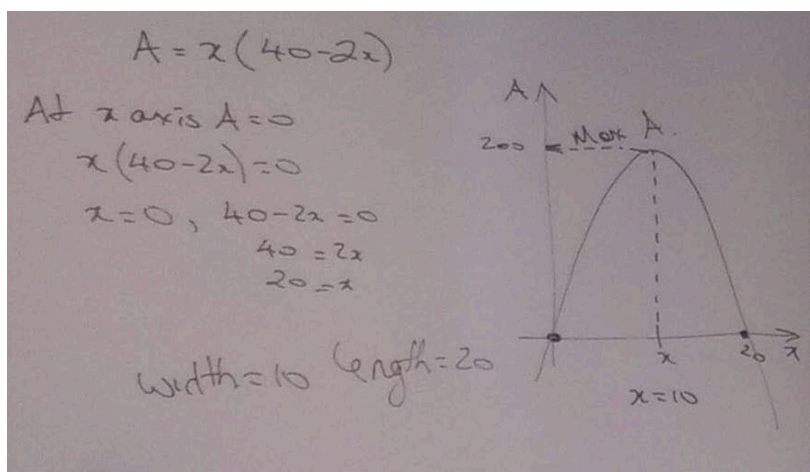


Figure 16: Algebraic solution of the optimization problem

To illustrate the algebraic solution and to determine the vertex, David used the graphic register of semiotic representation. Finally, he interpreted the results in terms of width and length in the light of the initial problem.

It took about 13 minutes for the teacher to “present” and solve this problem and it is legitimate to ask what the students actually took out of it and how this problem could have been implemented differently. Besides 2–3 students trying to keep up with the teacher’s pace and attempt to answer his questions with technically no time to think, the remaining students were just passively listening and taking notes.

Let us now proceed to an analysis of the implementation of this problem. This problem presents a priori a good potential of resistance and dynamic resistance because it is easily accessible by trial and successive adjustment strategy and any student could just calculate the area of the field by picking various dimensions of the field. This strategy was, however, discouraged by the teacher at the very beginning of the problem when the suggestion of a student to try a square wasn’t given a chance to be explored. Let us illustrate how this problem could have been approached and how its remaining potentials could have been explored if the suggestion of the student had been taken into account. Let’s imagine that the students would have calculated the area of the field assuming it is a square. This would definitely have raised a discussion (debate potential) about how to verify that this is the maximum area and still without using any algebra, they could calculate the area for different dimensions and find out that for some of them it’s greater and for others it’s smaller which would have revealed the variability of the data. Bringing the students’ calculations to the graphic register of semiotic representation and graphing them as a scatter

plot would have enabled the teacher to explore the didactic potential of this problem because parabolic shape would have ‘appeared’ on its own. What actually “killed” all potentials of this problem was that the teacher’s reduced vision of this problem, which we could probably extrapolate on his vision of teaching and learning mathematics. Most likely, David saw this problem as an application problem that comes at the end of the chapter which was also reinforced by the textbook as we show later in this chapter. His objective was actually to illustrate the utility of a quadratic function in solving optimization pseudo-problems and show the expert solution so that the students can practice and apply it to similar “problems” or rather exercises for the rest of the lesson. Imposing the expert solution brought this problem too quickly to the algebraic register of semiotic representation. Indeed, the way it was implemented privileged the algebraic register of the expert solution and the graphic register was used only to provide an illustration of the algebraic reality. In addition, the reduced students’ topos, the fact that the teacher did not give the students enough time to think and did not really expect or use the answers that did not fit the expert solution resulted probably to a disengagement of many students in this class because they knew anyway that a solution would come sooner or later. This example shows that the teacher’s vision of a problem and the way it is implemented can actually drastically reduce the inquiry potentials of an activity.

3.1.2 Problem of maximizing the volume of a metal box

After this David proposed another optimization problem whose text is below:

A metal box is made by cutting a square from each corner a piece of sheet metal. How big is x to maximize volume?

As the previous problem, this one was also presented in the register of natural language and figural register as shown in Figure 17. The “ x ” represents the length of the side of the square that needs to be cut out, this was not cleared out in the register of natural language but given in the figural register.

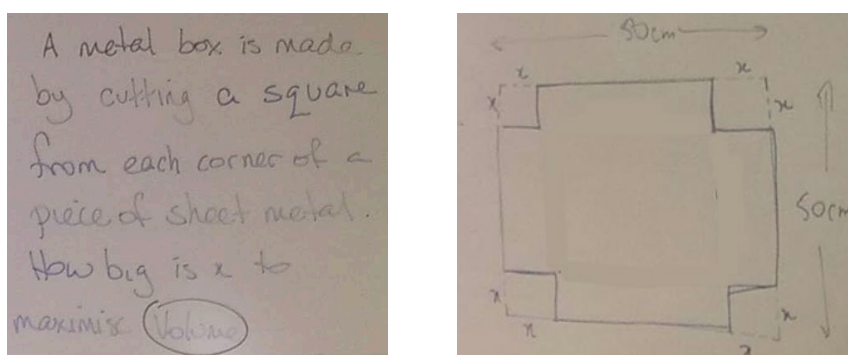


Figure 17: Problem of the metal box

David proceeded in a similar way as before by asking questions, completing the figure and basically solving the problem for the students. Below is an extract of how this problem was implemented:

T: What do we need an expression for? What are we trying to maximize?

S: Volume

T: How long is the bottom of the box?

Without really waiting for an answer, David continued to annotate the figure on the whiteboard (Figure 18).

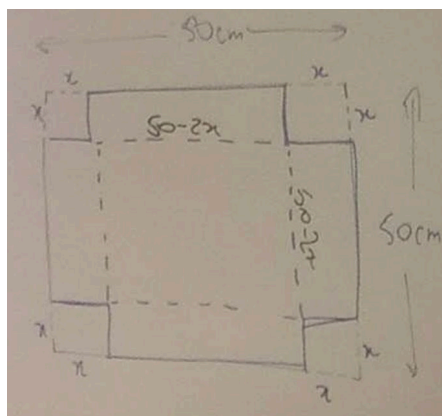


Figure 18: Annotated figure of the metal box

And finally, he wrote the expression for the volume of the box on the whiteboard:

$$V = (50 - 2x)(50 - 2x)x$$

And said:

T: It is not gonna be a quadratic. We can't do it algebraically but... Any ideas grabbing technology?

S: Graphing?

Getting to this stage of the problem took about 7 minutes which shows that David's main objective was not problem-solving, which was just a pretext to show the students how to use the GDC to solve it because the algebraic method was not available. David projected the GDC screen and show them how to type in the function, set the window to fit the curve on the calculator screen and how to find the maximum of the function. This included the following type of tasks and sub-tasks:

- Determine the maximum/minimum of a function using a GDC.
 - Type the functional expression into the input line.
 - Set correctly the window for the calculator screen.
 - Determine the lower respectively upper band of the interval including the maximum or minimum.

The final calculator screen is shown in Figure 19.

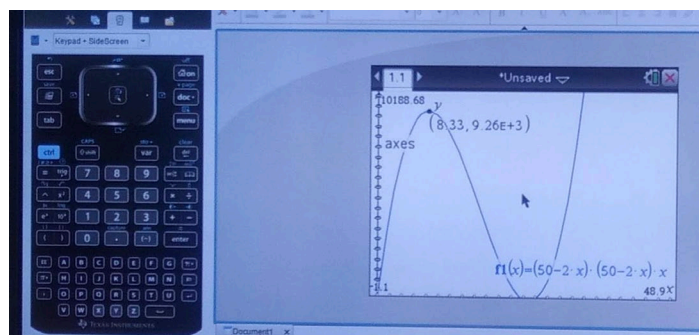


Figure 19: Solving the problem of the metal box with technology

This second example also reveals in a certain way of the reduced vision of problem-solving as a pretext to teach certain types of tasks and consequent sub-tasks, which was in this case meant to enhance an effective use of the GDC. The research potential of this problem was also reduced by bringing it too early to the algebraic register and the graphic register was, as a consequence, explored in a minimal way. The construction of the graphic register was finally reduced to a list of instructions of a series of buttons to press. Obviously, we do not underestimate the importance and the need to build an appropriate praxeological and instrumental equipment, we just want to point out that the “all in one” solutions do not necessarily work out.

3.1.3 Practice exercises in the textbook proposed by David

The rest of the double period was dedicated to practicing and applying these newly acquired techniques to similar problems from the textbook. The students were working individually, they could check their answers with the results provided at the back of the textbook or eventually call the teacher to their desk. There are altogether eight practice exercises designed in growing difficulty which we analyze below. This section of the textbook starts by a worked example showing the techniques necessary to determine the maximum respectively minimum of a quadratic function by using the formula of the x-coordinate of the vertex: $x = -\frac{b}{2a}$ (see Figure 20).

Example 27		Self Tutor	
Find the maximum or minimum value of the following quadratic functions, and the corresponding value of x :			
a	$y = x^2 + x - 3$	b	$y = 3 + 3x - 2x^2$
a	$y = x^2 + x - 3$ has $a = 1$, $b = 1$, and $c = -3$. Since $a > 0$, the shape is 	b	$y = -2x^2 + 3x + 3$ has $a = -2$, $b = 3$, and $c = 3$. Since $a < 0$, the shape is 
The minimum value occurs when $x = \frac{-b}{2a} = -\frac{1}{2}$ and $y = (-\frac{1}{2})^2 + (-\frac{1}{2}) - 3$ $= -3\frac{1}{4}$ So, the minimum value of y is $-3\frac{1}{4}$, occurring when $x = -\frac{1}{2}$.		The maximum value occurs when $x = \frac{-b}{2a} = \frac{-3}{-4} = \frac{3}{4}$ and $y = -2(\frac{3}{4})^2 + 3(\frac{3}{4}) + 3$ $= 4\frac{1}{8}$ So, the maximum value of y is $4\frac{1}{8}$, occurring when $x = \frac{3}{4}$.	

Figure 20: A worked example showing a technique to determine the coordinates of the vertex of a quadratic function (Haese et al., 2013, p. 47)⁴⁶

In Exercise 1 as shown in Figure 21, the authors propose to practice a direct application of the above technique.

- 1 Find the maximum or minimum values of the following quadratic functions, and the corresponding values of x :
- | | | | | | |
|---|--------------------|---|--------------------|---|---------------------|
| a | $y = x^2 - 2x$ | b | $y = 7 - 2x - x^2$ | c | $y = 8 + 2x - 3x^2$ |
| d | $y = 2x^2 + x - 1$ | e | $y = 4x^2 - x + 5$ | f | $y = 7x - 2x^2$ |

Figure 21: Exercise proposing a direct application of a technique

And the same technique is to be applied in Exercise 2 in which a pseudo real-life “cover” is proposed (see Figure 22).

- 2 The profit in manufacturing x refrigerators per day, is given by the profit relation $P = -3x^2 + 240x - 800$ dollars.
- How many refrigerators should be made each day to maximise the total profit?
 - What is the maximum profit?

Figure 22: Exercise 2 providing a pseudo real-life context

From exercise 3 on, the textbook proposes various optimization problems all besides one presented in two registers of semiotic representation: the register of natural language and the figural register. In Exercises 3 to 6, the authors propose a variety of problems to maximize the area of a rectangular field for a given perimeter. In Exercise 3, two sub-questions (Figure 23) are included in the form of “Show that...” which basically gives out the correct answer and provides a strong suggestion of the mathematical tools that need to be mobilized as illustrated in. This reduces a potential problem-solving activity to “show me how to do it, give me the right tool and I’ll do it” type of activity.

⁴⁶ Used with permission by the Copyright Act under the exception for “any fair dealing for the purposes of private study, research, criticism or review” (Haese, Haese, Haese, Mäenpää, & Humphries, 2013).

3 A rectangular plot is enclosed by 200 m of fencing and has an area of A square metres. Show that:

- a** $A = 100x - x^2$ where x m is the length of one of its sides
- b** the area is maximised if the rectangle is a square.



Figure 23: Example of a problem with sub-questions

Exercise 4 is the only one presented only in the register of natural language and does not contain any sub-questions (see Figure 24). It also slightly differs from the previous one because only 3 sides need to be considered as far as the perimeter is concerned but the same (apart from the numerical values) as the one proposed by David.

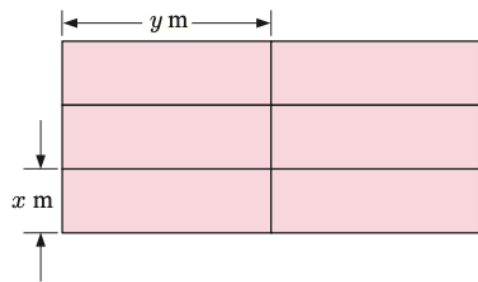
4 Three sides of a rectangular paddock are to be fenced, the fourth side being an existing straight water drain. If 1000 m of fencing is available, what dimensions should be used for the paddock so that it encloses the maximum possible area?

Figure 24: Optimization problem without any sub questions.

This same format is used for the following two exercises (5 and 6) with some more complex algebra (see Figure 25).

5 1800 m of fencing is available to fence six identical pens as shown in the diagram.

- a** Explain why $9x + 8y = 1800$.
- b** Show that the area of each pen is given by $A = -\frac{9}{8}x^2 + 225x$ m².
- c** If the area enclosed is to be maximised, what are the dimensions of each pen?



6 500 m of fencing is available to make 4 rectangular pens of identical shape. Find the dimensions that maximise the area of each pen if the plan is:

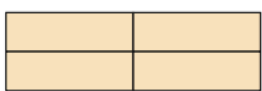
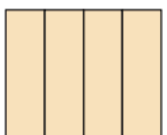
- a** 
- b** 

Figure 25: Examples of quadratic optimization exercises

Finally, the remaining two exercises (7 and 8) shown in Figure 26 could have presented a better problem-solving potential but this was reduced by adding the sub-questions in both cases. On top of that, Exercise 7 is deployed off all connection to some meaningful modeling situation where actually calculating the maximum difference of two functions would make sense and would have provided a *raison d'être* to this technique.

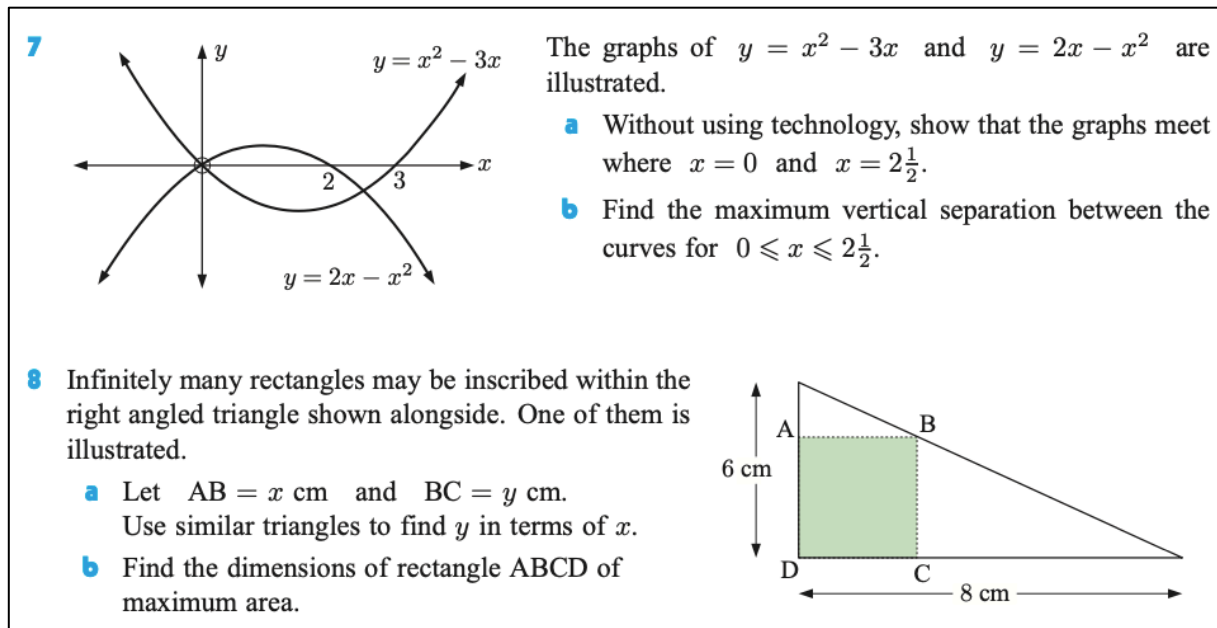


Figure 26: More complex problems deployed of their problem-solving potential

The above examples reveal something about how the conceivers of the textbook consider problem-solving. Moreover, the textbook provides advice and gives a general problem-solving. In the method proposed in the textbook (Figure 27), we can recognize the influence of the well-known Pólya's linear model (1971) in four stages:

- 1) Understand the problem
- 2) Make a plan
- 3) Carry out the plan
- 4) Look back at your work.

This model based on a linear approach to problem-solving has been, however, shown as insufficient and Schoenfeld (1985) suggests that problem-solving processes are rather cyclical.

F
PROBLEM SOLVING WITH QUADRATICS

Some real world problems can be solved using a quadratic equation. We are generally only interested in any **real solutions** which result.

Any answer we obtain must be checked to see if it is reasonable. For example:

- if we are finding a length then it must be positive and we reject any negative solutions
- if we are finding 'how many people are present' then clearly the answer must be an integer.

We employ the following general problem solving method:

Step 1: If the information is given in words, translate it into algebra using a variable such as x for the unknown. Write down the resulting equation. Be sure to define what the variable x represents, and include units if appropriate.

Step 2: Solve the equation by a suitable method.

Step 3: Examine the solutions carefully to see if they are acceptable.

Step 4: Give your answer in a sentence.

Figure 27: A general problem-solving method provided in the textbook (Haese et al., 2013, p. 44)

In addition, as proposed in Step 1, the expert solution using algebra is privileged from the very beginning leaving thus no space to any other strategy such as experimentation or trial and successive adjustments approaches. This is probably also reinforced by the fact that the sequence on problem-solving is proposed at the end of the chapter on quadratic function. The way this sequence of exercises is built does not foster genuine problem-solving activity, but rather the objective seems to train the students to:

- mobilize the right mathematical tool,
- recognize a particular type of task,
- apply a correct technique.

Another feature we would like to bring to your attention is the way the problems are formulated especially in the presence of the sub-questions, we recognize the influence of the final exam. As already shown in the analysis of the external assessment in the institutional analysis (see p.100), many items are accompanied by complementary sub-questions in the form of “Show that...” or “Use the following ... to...” which actually turns a problem into a sequence of simple tasks to execute. In this sense, the textbook does not provide meaningful resources to do inquiry and thus the teachers in order to implement IBME need to look for resources elsewhere or adapt the ones from the textbook.

3.2 Two examples from the calculus sequence in David’ class

3.2.1 Making connections: an example of a reflective activity on differentiation

This lesson started with a quick check of the practice exercises on quotient rule differentiation technique.

David then projected the introduction activity entitled as a *Thinking question*, proposing a graph of a matching activity as shown in Figure 28 and said:

01:54⁴⁷ T: Here is something for you to think about. Let’s see if you can figure out which is the answer. Multiple choice, it is taken from an exam paper...

⁴⁷ The time is given in the following format: “mm:ss”

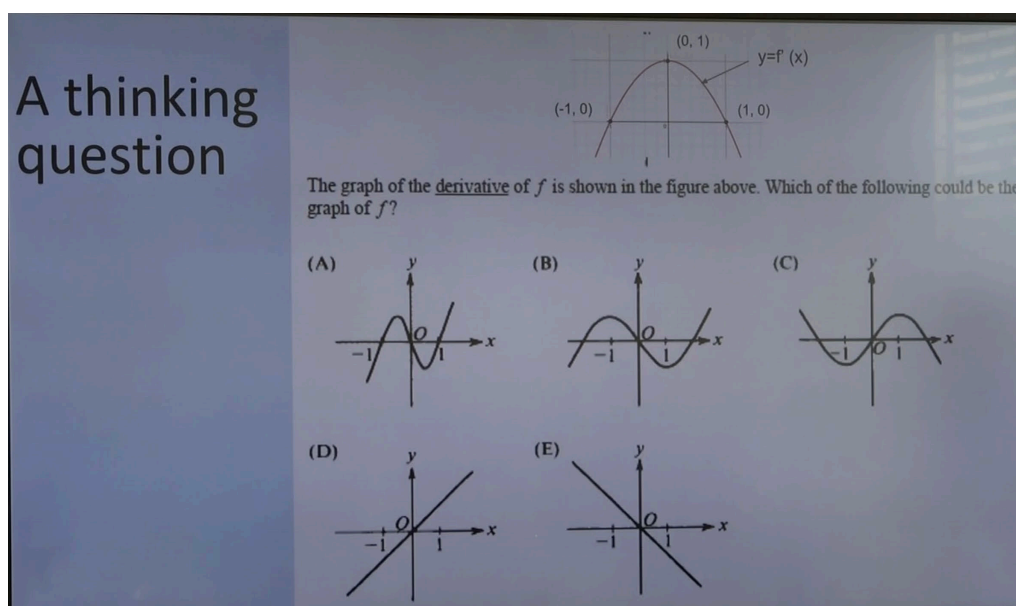


Figure 28: Thinking question for derivatives

David read the question out loud and wrote the hints on the whiteboard Figure 29.

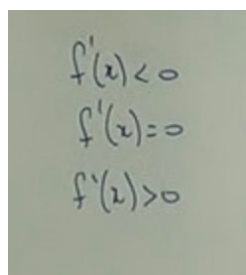


Figure 29: Hints proposed by the teacher to tackle the question

And commented:

02:33—02:47 T: You should know what happens when the derivative is less than, equal to or greater than zero and that should help you identify which graph is the original function.

02:47—03:18 Silence (31 seconds)

After this 30-second silence, David asked:

03:19—03:23 T: Any ideas? Anyone could explain how we could figure this out?

03:23—03:27 Silence (4 seconds)

03:27—03:35 T: It is an IB paper question... Paper 1, no calculator, can't use the calculator anyway.

03:35—03:38 Silence (3 seconds)

Nobody from the class reacted, despite the fact that the allusion to the exam was used and expected to act as a motivator.

From this short extract, we can see that the students actually knew that another hint or another answer would come sooner or later and were just waiting for it.

David returned to the whiteboard, pointed to the three derivative conditions (Figure 29) and said:

03:38—03:47 T: Who can tell us what happens when the derivative is less than zero, what is happening to the curve, the original curve?

03:47—03:49 Silence (2 seconds)

03:49—03:52 T: When the derivative is negative, what's happening to the original curve?

03:52—03:55 Silence (3 seconds)

03:55 T: It's going down!

After this David basically solved the problem for the students and explained step-by-step the connections between the graph of the derivative of the function and the original function. Pointing to the part where the derivative was negative (to the left of -1 of the graph) he said the original function must be going down and stated by elimination that the only two possibilities are the answers C or E. Then he asked:

04:30—04:33 T: What happens when the derivative is zero?

04:33—04:34 S: It's flat

04:35—04:50 T: Yes, it's flat, horizontal gradient... This one works (A), this one works (B), this one works (C) but not these two (D and E) because they don't have anywhere where the gradient is flat.

This would have actually been enough to find the answer but David continued to make the connection between the sign of the derivative and the graph of the original function and illustrated it on the answer C. Finally, he concluded:

05:26—06:03 T: Do you see the connection between the derivative of a function and the original function? We don't care about the shape of the derivative function; we just care about its sign. Ok? It's negative, the curve goes down, it's positive the curve is going up, it's negative the original curve is going down again. Alright? It's just the sign of the derivative... You've got to keep that in mind because there is a question like that on the exam paper remember?

The students were actually just passive receivers as their topos was quasi inexistent and the way this activity was implemented was more or less close to some form of lecturing with several allusions to the final exam used to motivate the students to remember this stuff. This example shows definitely the teacher's impact on the implementation of this activity and by simply augmenting the students' topos it would already push the curser closer to the inquiry tale of an imaginary scale with traditional and inquiry-based approaches on the opposite ends. Yet, we could also question the inquiry potential of the activity itself. As an exam item, we would classify it as AI (application with intermediaries) because several pieces of knowledge have to be mobilized and their choice as well as the strategy are left to the student's initiative unless they were trained before on similar types of tasks. Seeing this question makes us actually wonder in what situation, we would actually need to mobilize the connection between the derivative and the original function. The objective here is not

to propose such a situation but rather to point out that this *thinking question* the way it was presented was deprived of its *raison d'être*. While we recognize that such an endeavor would be rather complicated in a standardized examination, the mathematics classroom could and should be the place to bring up and to tackle such a situation. It certainly wouldn't be doable in 6 minutes but we believe it would lead to doing some more meaningful mathematics.

3.2.2 Discovering the derivative of $f(x) = e^x$

The last example that we analyze is an activity proposed by David to have the students discover the derivative of the exponential function with base e . This activity was carried out in about 14 minutes (from 06:04—20:00). It started with David's introduction of the activity while projecting the slide shown on Figure 30:

06:04—06:33 T: We have sort of moved around e in the natural log but we haven't talked about e and what's so special about e . We are gonna talk about it now, so what I want you to do, I want you on your calculator to graph the function $y=e^x$.

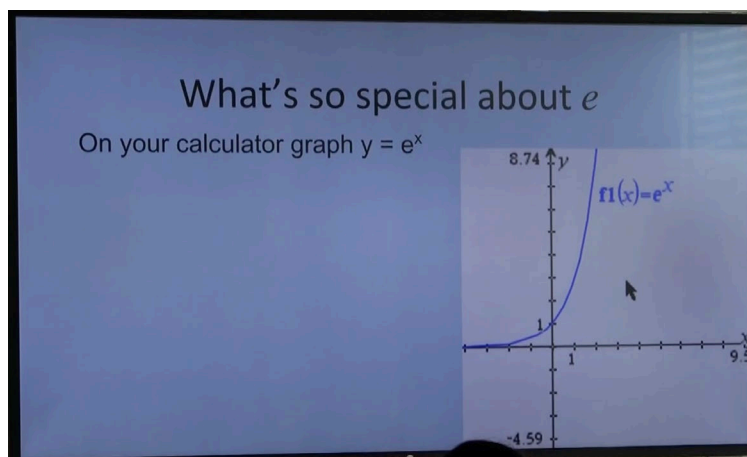


Figure 30: Introduction slide for discovering the derivative of $y=e^x$

So far, we can say that the lesson started with a question which is the first sign revealing a potential inquiry feature. Posed this way, the question has a priori a good research potential because it could be a source of many new derived questions. It also has a good didactic potential leading to the construction of new knowledge about this mysterious function with the base of e . For the resistance potential, it might turn out to be too difficult to start because nothing indicates that the derivative should be explored.

After this short introduction, the students sorted out their calculators to type in the function. David was walking around to help those who did not know how to do it. After about a minute, he returned to his computer and showed how to do it on the giant board. After about another minute, he asked the students to add a point on the curve providing a step-by-step instruction on the next slide (Figure 31) and showing it several times on the screen and helping a student who encountered difficulties doing it.

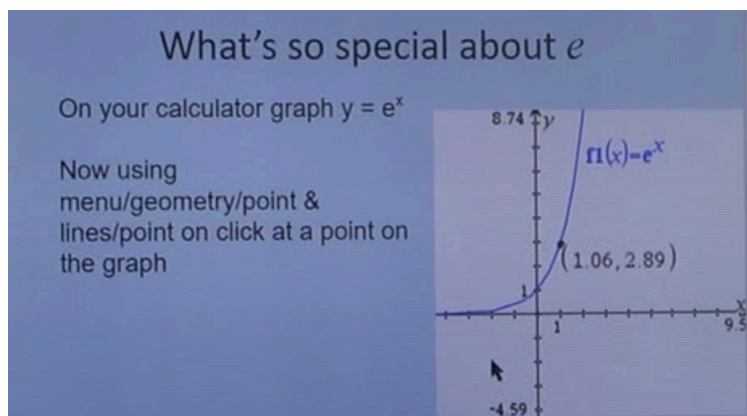


Figure 31: GDC instruction to place a point on the curve

In this part, David wanted to use the graphic register but the students seemed overwhelmed by its construction on the GDC.

As David wanted to move on, another student asked for help:

10:37—10:43 T: Now what we are gonna do... (a student raised hand) You didn't get that?

10:44—10:45 S: I don't even know how to graph...

10:46—10:47 T: You don't even know how to graph?

10:47—10:49 S: I can't even figure out what I am doing wrong...

After this David moved towards the student and showed her how to do it on the GDC which took about 40 seconds, returned to his computer and said to the whole class:

11:30—11:32 T: If you can't do it, just watch up here...

11:35—12:28 What we are gonna do now, we're gonna graph a tangent to that. Because remember the derivative represents the gradient of a tangent at any point. So what we are going to do is we are going to the same menus (doing it on the screed) but this time we are going to put a tangent there and we just go and click on the same place and there is the tangent.

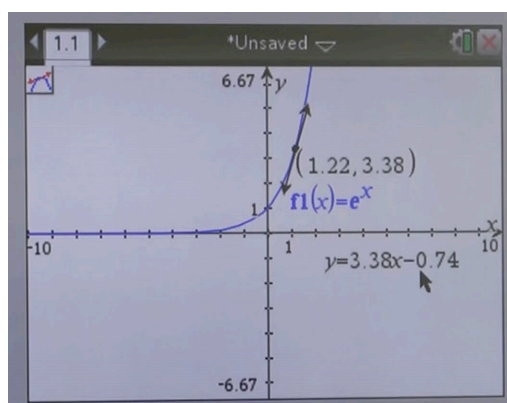


Figure 32: The GDC screen with the tangent

After this David asked again if the students managed to do that and showed that they could grab the tangent and move it alongside the curve. Up to now the main activity was the preparation of the “experimental milieu” on the students’ GDCs but as we show later, the

time spent on this took up half of the time of the activity (7 minutes) to finally not to be really used by the students because the teacher remained in the center of attention.

He continued:

13:23—13:45 T: You should be able to notice a connection between the point and the equation of the tangent as you move it. There is a connection between the point and the gradient of the tangent. Which of these numbers (pointing at the equation of the tangent) is the gradient of the tangent? Is it the 3.38 or the -0.74?

13:46—13:47 S: The 3.38...

13:47—13:54 T: The 3.38 is the gradient of the tangent (continues to move the tangent). What's the connection between that and the point?

13:54—14:00 Silence (6 seconds)

14:00—14:11 T: It's the same. The y value is always the same as the gradient of the tangent. Ok? (Continues to move the tangent). Every time.

The teacher moved then to the whiteboard and wrote $y = f'(x)$ saying

14:12—14:41 T: The y-value at the point equals the gradient of the tangent at that point. Ok? How do we figure out the y-value at the point? Given the x how do we figure out the y-value?

14:41—14:46 Silence (5 seconds)

14:46—15:04 T: If I gave the equation of the curve, which we know (writing the equation $y=e^x$ on the whiteboard), and I told you the x-value at the point $x=1.44$, how do you figure out what y equals?

15:04—15:05 S: Substitute

15:05—15:24 T: Substitute (writing $y=e^{1.44}$ on the whiteboard). Ok? If I told you the derivative (writing $\frac{dy}{dx}$ on the whiteboard), how would you figure out the derivative of the tangent at the point?

15:24—15:28 Silence (4 seconds)

15:28—15:33 T: How would you figure out the gradient of the tangent at a point if I give you the derivative of the function, how do you figure it out?

15:33—15:40 Silence (7 seconds)

15:33—15:54 T: Come on, here is a function, we did it yesterday, here is the function (writing $y = 3x^2$ on the whiteboard), here is the derivative (writing $\frac{dy}{dx} = 6x$), how do you figure out the gradient of the tangent where x equals 1.44?

15:54—15:55 S: Substitute

15:55—17:02 T: Substitute, yeah? So you would do here (pointing to the exponential function) the same thing. Ok? And the gradient of the tangent is exactly the same as the y-value (writing $y = e^{1.44}$) It is exactly the same thing. So the derivative function (writing $\frac{dy}{dx} = e^x$) is exactly the same thing as the original function, because the gradient of the tangent is the same as the y-value (pointing to the coordinates of the point on the GDC screen), they are the same thing. Yeah? It's nice if you could see and make that connection, it's not super important, but it's nice if you can make that connection... And what's so special about e is

that the original function is exactly the same as the derivative function. The y-value at any point is the same as the gradient of the tangent at this point. Alright?

After this long institutionalization monologue, David continued to explore and point to some particular points on the function such as $x=0$ and $x=1$ which took about two more minutes. Finally, he summarized that the point of the exercise is to remember that if $y = e^x$, $\frac{dy}{dx}$ remains the same, it doesn't change, that's what's special about e .

We could question if this were the best way to approach e and the exponential function with a base of e and its derivative. We are convinced that even a brief a historical and epistemological analysis of these mathematical objects would have affected the way it was approached. One could also argue whether this activity carries any features of inquiry and we admit that it is rather a disguised lecture than genuine inquiry but it also reveals that David had certainly good intentions when preparing this activity and was probably even convinced about proposing reasonable problem-solving and inquiry-based activities in general. This actually shows that in the ordinary classes, there is some space even if it's limited to include inquiry and problem-solving activities. The fact that David attempted to do some problem-solving and implement investigation activities regardless its format and efficiency, creates actually favorable conditions for implementation of IBME in mathematics SL classes. Research projects like PRIMAS (Dorier & García, 2013) showed that teachers are the key to successful dissemination and implementation of IBME and if the teachers resist to change, this creates a serious obstacle hindering the implementation of IMBE. Even with the best resources and professional development ever, if the teachers resist, the expected change does not come. It's in this sense that we refer to favorable conditions that could provide a good starting point for the development of collaborative research with the objective to conceive good quality resources including didactic comments and put in place efficient professional development.

4 Examples of IBME in Eliot's practices

In this part, we provide three examples of Eliot's attempts to implement investigation and problem-solving activities in the regular mathematics classes.

4.1 An example of an investigation on Exponential functions

This first example illustrates the implementation of an investigation activity of the graphs of exponential functions proposed by the textbook (see Appendix 9). In this activity, the students are expected to explore the effect of different parameters on the shape of the exponential function. The main register of representation for this activity is graphic. When considering this activity a priori from the perspective of the potentials, the too many subtasks significantly reduced the research and the resistance potentials. The students do

not need to come up with a strategy but simply follow the steps given in the textbook. Eliot started the lesson by giving a general description of the proposed activity:

01:19 –02:26 T: All right, if you look at page 95, it's an investigation, you are going to do on a GDC, because it's good for you to practice how to use your graphic calculator. So, first of all, you need to do a new graph page for each of the five different question. So for example you put in a new graph page It says "Explore the family of curves in the form $y=b^x$. Ok? So you gonna do those four [...] (see Figure 33).

- 1** Explore the family of curves of the form $y = b^x$ where $b > 0$.
For example, consider $y = 2^x$, $y = 3^x$, $y = 10^x$, and $y = (1.3)^x$.
- a** What effect does changing b have on the shape of the graph?
 - b** What is the y -intercept of each graph?
 - c** What is the horizontal asymptote of each graph?

Figure 33: First task of the investigation

Now you might need to change the window a little bit but you can actually see what's going on just with the one that comes up. So you put in those four functions and it says "What effect does changing b have on the shape of the graph?" So I'd like you to answer these five questions. You can see immediately what effect the b has, so describe that in your own words, try to make it as mathematical as possible. Then it says: "What's the y -intercept of each graph?" That should be obvious. And that it says: "What's the horizontal asymptote of each graph?"

02:27—02:54 T: Do you all know what horizontal asymptote means? Does your silence mean you don't know or you do know? Does anybody not know what horizontal asymptote means? You are not sure... I can show you... It's... horizontal means that way (making a horizontal gesture with his hand), it means it's a line that gets closer and closer and closer to flat but never actually gets flat exactly. You have heard of it before probably. Yeah, ok. Alright?

The introduction of this activity can be divided in two parts. First a general description of the activity illustrated on the first question (Figure 33) and making sure that the students understand what to do. This led to a brief explanation of what a horizontal asymptote is because a student made a sign that he was not so sure what it was. After this Eliot returned to the main activity and gave out more methodological instructions on how to carry out the exploration as shown on the transcript below.

02:55—04:11 T: So that's what you are going to do for all five questions, you haven't got to draw the graphs, you can just say for each one what you are investigating, for example if you are doing the $y=b^x$ and you would say what effect the b has and then you answer the questions about that one. Alright? Ok to do that? I think if you do it reasonably quickly, you should get through all of them, we'll see how well you do. If you are stuck on any of them, just tell me, don't draw the graphs each time, but each time, open a new graph page. Then on your calculator you will have five different graphics pages. At the end of the lesson, you can actually save those on your calculator, so you can start using your calculator for notes. So at the end, save the document as Exponential investigation...

Then Eliot found out that several students did not have their calculator, so he went out to get extra calculators and came back after about three minutes.

The students were all set and started working individually, in pairs or in groups of three. Eliot let them choose the modality. After about four and a half minutes, Eliot asked whether they figured out the answers of the first one and a group of three girls said yes.

Short conversation Group 1:

11:31—11:33 T: How did you describe it?

11:33—11:37 S: The b? Slope?

11:37—11:40 T: Slope is good... it makes it broader or higher. Yeah great.

Eliot walked around the class to verify the progression of the students, checking on their partial results and the way they communicated their findings. The students were actually actively involved in the activity and progressed at their own pace. He did not give out ready-made answers but rather made a good use of maieutic. After about 12 minutes, Eliot organized a summary of the first question and had a student formulate the answer. Then he summarized question 2 (see Figure 34) see for them saying that it was easy.

- 2** Explore the family of curves of the form $y = 2^x + d$ where d is a constant. For example, consider $y = 2^x$, $y = 2^x + 1$, and $y = 2^x - 2$.
- a** What effect does changing d have on the position of the graph?
 - b** What effect does changing d have on the shape of the graph?
 - c** What is the horizontal asymptote of each graph?
 - d** What is the horizontal asymptote of $y = 2^x + d$?
 - e** To graph $y = 2^x + d$ from $y = 2^x$ what transformation is used?

Figure 34: Question 2 of the investigation on graphs of exponential functions

He then told them to continue on question 3, not to worry about the time since they would get some extra time at the beginning of the next lesson if needed. The students continued working and Eliot continued having some exchanges with the small groups/pairs/individual students. In what follows we provide an example of a small conversation with two students to illustrate some of Elliot's activity from the topogenesis and the chronogenesis aspects, on the one hand, and to show how some of the students' findings were destabilizing for the teacher, on the other hand.

Short conversation Group 2 (Part 1):

T: How did you describe number 3 (see Figure 35), what's a nice mathematical way of describing it?

- 3** Explore the family of curves of the form $y = 2^{x-c}$. For example, consider $y = 2^x$, $y = 2^{x-1}$, $y = 2^{x+2}$, and $y = 2^{x-3}$.
- a** What effect does changing c have on the position of the graph?
 - b** What effect does changing c have on the shape of the graph?
 - c** What is the horizontal asymptote of each graph?
 - d** To graph $y = 2^{x-c}$ from $y = 2^x$ what transformation is used?

Figure 35: Question 3 of the investigation on graphs of exponential functions

S1: Horizontal translation.

T: Horizontal translation. What's a bit weird about it though? If you've got $x-c$, does move backward or forwards?

S2: Forwards.

T: Do you remember, it was the same with quadratics, $x-b$ moved actually forward by b . That is the one thing about the horizontal translation, so that's one thing to bear in mind. Does it change the shape?

S2: No.

T: How about part 4 then? (see Figure 36). What's the effect of the b^x and b^{-x} , what does that do?

- 4** Explore the relationship between $y = b^x$ and $y = b^{-x}$ where $b > 0$.
For example, consider $y = 2^x$ and $y = 2^{-x}$.
- a** What is the y -intercept of each graph?
 - b** What is the horizontal asymptote of each graph?
 - c** What transformation moves $y = 2^x$ to $y = 2^{-x}$?

Figure 36: Question 4 of the investigation on graphs of exponential functions

S1: Changing the (making a flipping gesture with his hand)

S2: In the y -axis...

T: The y -axis or the x -axis

S2: The y -axis

T: Are you sure?

S2: Yeah.

T: Because you went like that (making a vertical flip gesture)... Does it change the horizontal asymptote?

S2: No.

In this example we can see that Eliot respected quite well the students' topoi, let them come up with their answers first and then institutionalized the findings of the group to advance the didactic time. The second example follows still Eliot's exchange with the same group of students. This time we illustrate how the students' observations raised a new question, managed even to destabilize the teacher for a moment and could have potentially led to:

Short conversation Group 2 (Part 2):

T: Did you do part 5? (see Figure 37)

- 5** Explore the family of curves of the form $y = a \times 2^x$ where a is a constant.
- a** Consider functions where $a > 0$, such as $y = 2^x$, $y = 3 \times 2^x$, and $y = \frac{1}{2} \times 2^x$. Comment on the effect on the graph.
 - b** Consider functions where $a < 0$, such as $y = -2^x$, $y = -3 \times 2^x$, and $y = -\frac{1}{2} \times 2^x$. Comment on the effect on the graph.
 - c** What is the horizontal asymptote of each graph? Explain your answer.

Figure 37: Question 5 of the investigation on graphs of exponential functions

S2: Yes

T: What does it do?

S1: It changes the y-intercept.

T: What else does it do?

S1: Changing the y-intercept... I don't know

T: Yeah? It's maybe more difficult to see... It's actually making it steeper, because if we look at where this goes, that's where it's steeper (pointing to their calculator). So it takes each of the y-values and makes it 3 times bigger or on third the size of it.

S2: It looks the same, what if I just moved it?

T: Oh that's interesting...

S1: I think that if we moved this one over there, would it match the other shape?

T: So if you move this on down onto there, do you think that would match up?

S2: Yeah, I think so.

T: Let me tell you what... (manipulating the graph on the calculator) Let me move it a bit...

S1 and S2: It worked.

T: Hmm... It's interesting. I'll tell you what... (The teacher looked surprised and intrigued)... You will have to believe me, it's actually steeper than that one.

In order to illustrate what was at stake in this small exchange between the students and the teacher, we need to look at the mathematics behind this observation. The three curves that were the object of the above conversation are shown in Figure 38. When looking at them, we can understand the teacher's point, that the curve goes steeper because for $a > 1$, each y-value of $f(x) = a \cdot 2^x$ is a times bigger than $f(x) = 2^x$. The students, however, were talking about the shape of the graph that seemed to be the same and they were even able to translate it on the calculator in a way that it superimposed on the graph of $f(x) = 2^x$. The teacher tried to zoom on the two curves in order to prove them wrong but gave up because the GDC screen was too small and the class was going to end soon. So what is the truth?

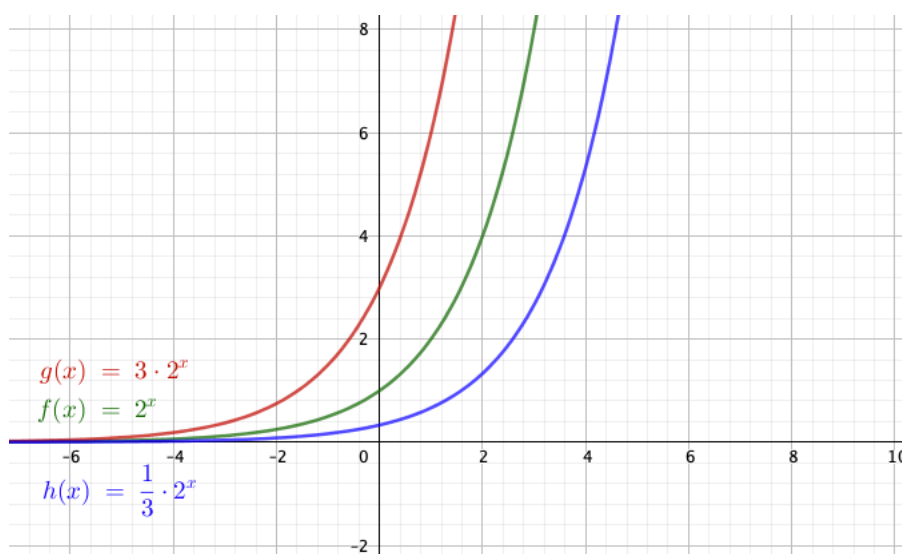


Figure 38: Exploring $f(x) = a \cdot 2^x$

The students were actually right about the shape of these curves. In fact, if we translate $f(x) = a \cdot b^x$, for $a > 1$, by the vector $\left(\frac{\ln(a)}{\ln(b)}, 0\right)$, we obtain $f(x) = b^x$. We leave the proof of this transformation up to the curiosity of the reader. In the video, we could see that the teacher was intrigued for a moment by the observation of this group of students but in the given circumstances quickly returned to his truth: when considering the gradients of $f(x) = b^x$ and $g(x) = a \cdot b^x$ at a given point, $f(x) = b^x$ is steeper at the point, we can even add the gradient of the tangent at the given point is a times greater. This small discussion and the observation of this group could have been the starting point of a new investigation and maybe even a new discovery for the teacher if he let the curiosity of the students go or if he knew that there was a good potential for a new exploration and investigation of this curious phenomenon. Of course, it would also have been difficult to prove this intuition because neither vectors nor exponential equations were available for the students; nevertheless, some of it could have actually been taken care of by technology and the proof could have been saved for a bit later where it could have provided a *raison d'être* for exponential equations. This short extract shows that when the students' topos is big enough, they are actually able to come up with new findings and new questions. As for the teacher, he was surprised for a moment but in the given circumstances (the approaching end of the lesson and other groups raising hands) did not grasp the created momentum to push this investigation further.

Finally, Eliot did not come back to this discussion, he gave some more time to the students to finish this investigation and then institutionalized the findings from the investigation about the graphs of exponential functions and enlarged it on functions in general as shown on the screenshot of the whiteboard (Figure 39).

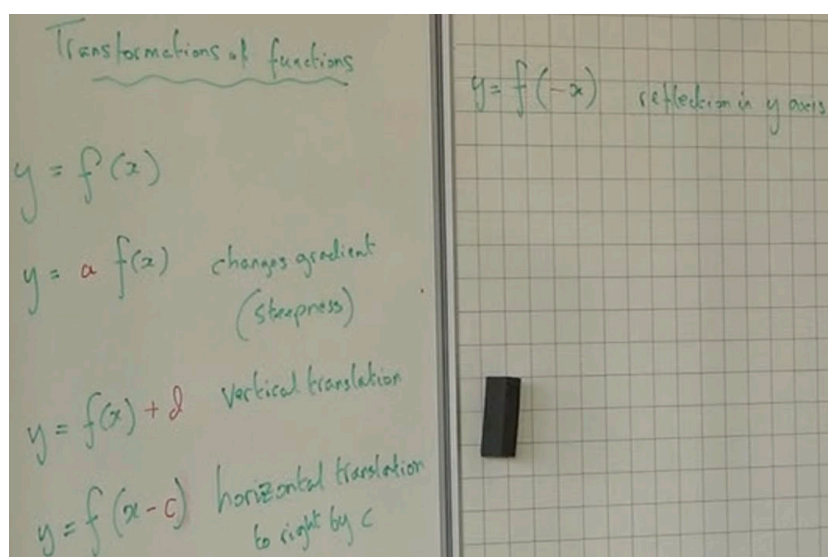


Figure 39: Institutionalization of transformation of functions

Nevertheless, we identified in this investigation several interesting IBME features. From the teacher's perspective, he managed to provide enough time for the students and enlarged

their topos. At the same time, he walked around the class, verified the students' progression, did not hesitate to engage in discussions with the students and provide some guidance if necessary. His behavior could be compared to the one of a senior fellow supervising a bunch of juniors. From the point of view of the resource, we are a little bit more critical. The proposed investigation was too much guided and split into isolated sub-tasks which significantly diminished the research potential and turned it into a sort of disguised ostension. A thought-provoking moment emerged from an observation of a group of students and a discussion with their teacher, which in our opinion has not been foreseen by the authors of the textbook and contributed to the research potential. The teacher, however, did not manage to grasp and build on the appeared research potential, because he simply did not know about it either. In this sense, this observation confirms a need for developing and providing good quality resources with didactic comments nourished by research in mathematics education.

4.2 Application of exponential functions to interest rates

Just after introducing the exponential functions and transformations of their graphs, Eliot proposed some applications of exponential function on “modeling” exponential growth and decay. He started by giving the problem shown in Figure 40. This problem was meant to be solved using technology because the students hadn't done logarithmic equations.

T: The next thing, we're going to look at, has to do with exponential growth and decay. That is used a lot, it's used in analyzing populations, it's used particularly in economics, because it's used with interest rates. So to get you to start off with that, we have about 25 minutes left so we could do some practical applications of exponential functions, in which exponential growth and decay are the most obvious ones. We are going to look at interest rates, how they work...

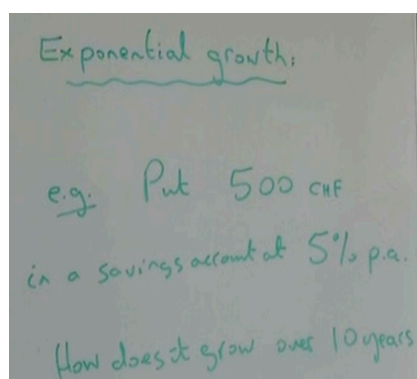


Figure 40: A problem on exponential growth

Right from the beginning we can see that he planned to give the students enough time to do this problem.

T: You'd be extremely lucky if you found a bank that does that. At the moment, the maximum interest rate is about a quarter of a percent, which is one twentieth of that. But in

the old days, you used to be able to get a 5% percent interest rate... Things have changed, however, inflation rates and interest rates have been low for years now. So, how are we gonna figure out how it grows? We're gonna do it over 10 years...

T: So this is an exponential function, but how? Because what you are doing is each year you are multiplying by a fixed amount, which is increasing it by 5%. So can anybody tell me what you would need to multiply by to increase something by 5%?

S1: 1.05

T: Exactly alright, so you have a multiplier 1.05 (writing on the whiteboard, see Figure 41), which corresponds to one plus five percent.

$$\text{multiplier } 1.05 = 1 + \frac{5}{100}$$

Figure 41: The modeling part guided by the teacher—multiplier

T: One really common mistake is to say you multiply by 5%. If you put your money in a bank, you'd come back a year later and you'd only have 5% of it, you'd be rather fed up. So you're not multiplying by 5% but by 1 plus 5%. That is what you said to get 1.05.

T: So that's what you are going to do... (writing the equation of the exponential function on the whiteboard, see Figure 42). You are going to have the amount 500 and you're gonna multiply by 1.05 each year, so you're gonna do it to the power of n . And we're gonna have some function of n where we do it per year. Ok?

$$f(n) = 500 (1.05)^n$$

Figure 42: The modeling part guided by the teacher—equation of the function

T: So here are some questions for you to answer (writing on the whiteboard, see Figure 43). You can draw the graph. Now, I would like you to draw that using your calculators but then do it in your notebooks, label it, put the correct axes on it, put the correct amounts on it... Nice, easy question, how much after six years?

Exponential growth,
e.g. Put 500 CHF
in a savings account at 5% p.a.
How does it grow over 10 years.

i) Draw the graph
ii) How much after 6 years.
iii) How long for it to double?

Figure 43: Adding questions to the problem

T: And then one question that you could think about but you haven't actually got enough maths to do but you could try to work it out... How long would it take to double? How many years you would need to leave it before it doubles? I'll see if you can figure out the way of doing that.

Eliot then summarized once more what he wanted the students to do and set them off. This first part which we could characterize as a mixture of giving instruction and ensuring a good devolution of the problem took about four minutes.

Eliot let the students work and figure out how to graph it on the GDC for about eight minutes. Three students came up to his desk saying that it didn't work. He did not actually show them how to do it but rather had them think why they did not see anything on the screen. After this he turned to the whole class:

T: Now, you probably drew it and thought why does it look weird and then you started thinking about the scale hopefully. It's not a very interesting graph, by the way, it's not gonna excite you particularly...

S1: Is it normal that it doesn't show up at all?

T: (turning to the student) Because of the scale...

S1: How do I change the scale?

T: Did you know how to use your calculator last year?

S1: In my old school we did not use it...

T: Ah ok. If you don't know what to do, what do press?

S1: Menu?

T: Yes. It's a good thing to remember that one. What do you think you might do?

S1: View?

T: That would be sensible but it isn't. What else?

S1: Trace?

T: That would also be sensible but "trace" is where you put something on it and you move it along, but you can't see anything yet. Window... Window Settings... What's the x gonna be between?

S1: Zero and ten?

T: Yeah

S1: And then y?

T: You can start it a zero or at 500... How big do you think it's gonna get? If it's 5% a year, is it gonna get huge?

S1: No.

T: About how much? How big do you think it might go up to?

S1: Something like 900 or 1000?

T: Ok, so you can try 1000 and adjust it if needed...

Eliot went around the classroom and helped 3 more groups trying to make them grasp the sense of the problem, which took other 12 minutes. Eliot turned again to the whole class:

T: You should by now have a graph that looks something like this one and not the horrible ones you started with. Sorry to call them horrible but they were horrible. Because you need to adjust the axes to make sense so should get something like this one (see Figure 44)

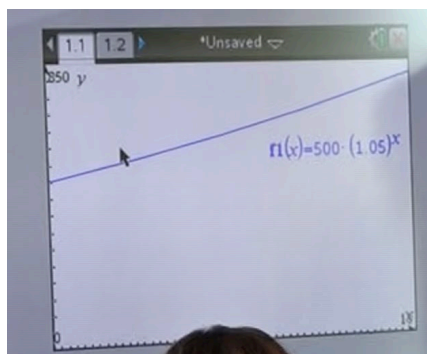


Figure 44: The correctly adjusted window

T: Why does it look like this? Because you gonna go from zero, because you starting it year zero up to ten. You could have started this at zero or at 500 (pointing to the y-axis). It's not a very interesting graph, why not? Because your money is not growing very fast. Ok? Five percent isn't gonna make a huge difference. So see if any of you by next week can actually answer the third question. How long does it take to double? There are at least three ways to do it, see if you can figure that out.

What Eliot thought to be an easy task turned out to occupy the students for almost all the time reserved to this problem.

Here we have an example of the intertwining process of semiosis and noesis showing that one cannot exist without the other. Indeed, in order to best represent this problem, the graphics register was needed and we saw that it was not an easy issue for the students to tackle. In order to correctly set the window on their calculator some conceptual understanding of the exponential was required, at the same time, the graphic register was supposed to enhance the understanding of the concept of a function and find the solution for this problem because the algebraic register had not been available. It is thus important to realize that giving out technology to the students must not be reduced to providing a list of buttons to press.

4.3 What is the formula for the sum of the first n terms of the arithmetic sequence?

In this third example, we examine the implementation of an activity building up progressively the conceptual understanding from a particular case to generalization of a formula. To introduce the sum of the first n terms of the arithmetic sequence, Eliot used the classical story and problem, attributed the school years of Gauss. His objective was that by the end of the lesson, the students would have found the formula for the sum of the first n terms of an arithmetic sequence. The implementation of this activity took about 14 minutes and can be split into three parts:

- 1) Part 1: Introduction of the problem of the sum of the first 100 integers (about 5 minutes)
- 2) Part 2: Preparative work for the general case (about 4 minutes)
- 3) Part 3: Work on the general case institutionalization (about 5 minutes)

4.3.1 Part 1: The sum of the first 100 integers

T: There is a classic famous problem that I've just written up on the board. It is believed to have been given to a class where one of the greatest mathematicians Gauss was in. His math teacher thought he was chatting too much so he said: "Add up the first 100 integers". And about 5 minutes later Gauss walk up to him and gave him the answer. And he said: "How did you do it?". So let's see if you can figure it out. I'll give you one clue in a minute... If you think you know, find the answer. I'll give you one clue. Write it down beneath, but this time, write it backward (writing on the whiteboard, see Figure 45).

What is the sum of first 100 integers?

$$S_n = 1 + 2 + 3 + 4 + \dots + 99 + 100$$

$$S_n = 100 + 99 + 98 + 97 + \dots + 2 + 1$$

Figure 45: The hint

This time, the students were not actually given time to figure it out on their own and Eliot gave away a hint that provided actually a key to the solution (see Figure 45).

S1: (a student suggesting something incomprehensible) 5500

T: You are so close but not right.

S2: 5050?

T: So you were careful, he wasn't. Now add them together (waiting for about 5 seconds). So the first one is $100 + 1$, which is? 101. The next one is $99 + 2$, which is? 101. The next one is $98 + 3$, which is?

S3: 101

T: So if you add them together you get two lots of (writing on the whiteboard $2S_n = 101 + 101 + \dots + 101 + 101$). Agree? How many of those are there?

S4: Aren't there a hundred?

T: There are a hundred terms there. So we know that 2 lots of (writing on the whiteboard $2S_n = 100(101)$). Which is 10100. So if two lots of the sum is 10100, the sum is equal to? (writing on the whiteboard $S_n = 5050$) That's 5050 and not 5500, right (turning to S2).

S1: I didn't do it like that.

T: Did you do the average in the middle?

S1: I did $1 + 2 + 3 + \dots + 10 = 55$. And then up to 20 and then up to 30, which gave 100 times $55 = 5500$

T: I am trying to think where you went wrong but. I can't think of it at a moment...

Let us stop at the suggestion of the student S1 for a while. We agree that it is not obvious for the teacher to pick it up on the go and that he wanted to move on with the lesson. But there was actually an explorable path in the student's idea that we propose in Figure 46. Maybe if Eliot had the student come up to the whiteboard, he could have better explained his idea and might even have spotted the error.

$$\begin{array}{rcl}
 1 + 2 + 3 + \dots + 9 + 10 & = & 55 \\
 11 + 12 + 13 + \dots + 19 & = & 20 = (1 + 2 + \dots + 9 + 10) + 100 \\
 21 + 22 + \dots + 29 + 30 & = & 200 = (1 + 2 + \dots + 9 + 10) + 200 \\
 \vdots & & \vdots \\
 91 + 92 + \dots + 99 + 100 & = & 900 = (1 + 2 + 3 + \dots + 9 + 10) + 900
 \end{array}$$

$$\begin{array}{r}
 10 \cdot 55 + (100 + 200 + \dots + 900) \\
 10 \cdot 55 + 100(1 + 2 + \dots + 9) \\
 10 \cdot 55 + 100 \cdot 45 \\
 550 + 4500 \\
 \hline
 5050
 \end{array}$$

Figure 46: Another way of finding the sum of the first 100 integers.

His idea was actually not so bad and could have contributed to the ongoing inquiry, include the rest of the class, raise a discussion, etc. These important moments are, however, difficult to be picked up on the go by the teacher. In this case, even though Eliot stopped for a while, listened to the students' suggestion, he wasn't able to identify the student's error right away and did not recognize a potential way to approach the problem. This shows that students actually come up with ideas when given the time. These ideas are not always fully operational and need some fine-tuning. That is why they also need to be given space to express their ideas and to share them with the whole class, which could lead to new discussions on what to agree on or what needs to be improved.

4.3.2 Part 2: Preparative work for the general case

Eliot then moved on to show the students another way to approach this problem because his objective was actually to have the students find the formula for the sum of an arithmetic series.

T: Another way to do this, ok if you take the first term and the last term, you get a 101, if you take the next one plus 99, you get a 101, if you take the third one plus the third from the end, you get 101. So when you are doing the pairs and you are starting off at a hundred, how many pairs you're gonna end up with?

S: Fifty

T: And each one is worth 101, so therefore the sum must be 5050

S5: This one is better.

S6: No, the first one makes more sense. Like when you see the first, oh like that makes sense.

T: Yeah. But the thing is...why does it work that way? Because it is balanced, it's symmetrical. So you get these two ideas, if you reverse it and add two things that are symmetrical, you're gonna get a rectangle. Or if it's symmetrical, the first plus the last is the same as the... (making symmetrical gestures) alright?

Having explained this, Eliot continued to prepare the “tools” needed to attain the objective of this activity: finding the formula of the arithmetic series.

T: That's how I want you to work out what's the sum of arithmetic series. Now I want you to come up with a formula yourselves (writing the title, “Sum of arithmetic series on the board”). Now how are you gonna do it? What is an arithmetic series? It's a sum of an arithmetic sequence... So if we write it, just to start you off, with the notation we've used before, what's the first term?

S: u_1

T: (writing on the whiteboard $S_n = u_1 +$) What's the next term? If it's arithmetic, how would you get to the next term of the arithmetic sequence?

Ss: (Silence)

T: Do you remember? You wrote it down last week... The first term is u_1 , what's the next term?

S: u_2

T: (smiling) Ok, we could write $u_1 + u_2 + u_3 + \dots$ but what is it defined by, the first term and ...?

S: (silence)

T: So what's the next term?

S: (silence)

T: How would you get to the next term?

S: You add something...

T: You add something, what's it called?

S: d

T: So the next term in the series is?

S: $u_1 + d$

T: (writing on the whiteboard $S_n = u_1 + (u_1 + d)$) I am going to put it in brackets. I hope this not too painful. What's the next term?

S: (several students speaking) $u_1 + d + d$

T: (writing on the whiteboard $S_n = u_1 + (u_1 + d) + (u_1 + 2d)$) What's the next one?

S: $u_1 + 3d$

T: Ok, all the way up to the... last term. What's the last term? It's n , what was the last term of the arithmetic sequence? It's the n -th term. We did this last week, no (smiling, yet desperate expression on his face)... How many d 's do you need to add to get to the last term?

S: (two students raising hands, one of them designated by the teacher) u_1 plus brackets, n minus one time d .

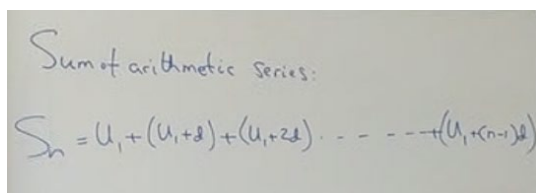
T: Perfect (writing on the whiteboard $S_n = u_1 + (u_1 + d) + (u_1 + 2d) + \dots + (u_1 + (n - 1)d$). Remember that? Right? Why? Because to get to the 100th term, how many differences do you add? 99. So now, using Gauss's method, you could use either of them, work out what S_n is equal to. Try it. Try figuring out what is the formula for the sum of the n terms.

This part actually shows, helps us understand and even explain why Eliot rushed so much through the introductory problem. In fact, his objective was not that much problem-solving but rather the formula for the sum of the arithmetic series. And the sum of the first 100 integers provided a neat method leading to this formula.

4.3.3 Part 3: Work on the general case and institutionalization

Eliot let the students work on their own. After about one and a half minute, a student raised his hand and asked:

S: (looking at the whiteboard and talking to the teacher, see Figure 47) Is it two terms of u_1 plus eeuuum, d times n minus 1, times eehhh, n divided by 2?



Sum of arithmetic series:

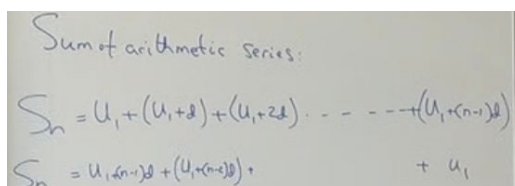
$$S_n = u_1 + (u_1 + d) + (u_1 + 2d) + \dots + (u_1 + (n-1)d)$$

Figure 47: The preparative work provided by the teacher

T: (making a thumb up gesture) That's the formula...

We can see that this student made the connection and came up quite quickly with the generalization. Eliot waited for about one more minute and gave the procedure to follow:

T: Ok. So what you're gonna do for the Gauss's method is: Write it down underneath in the opposite order, then add the two together and please do the algebra carefully, see what you get and figure out what the formula for the sum is. (moving to the whiteboard and writing, see Figure 48)



Sum of arithmetic series:

$$S_n = u_1 + (u_1 + d) + (u_1 + 2d) + \dots + (u_1 + (n-1)d)$$

$$S_n = (u_1 + (n-1)d) + (u_1 + (n-2)d) + \dots + u_1$$

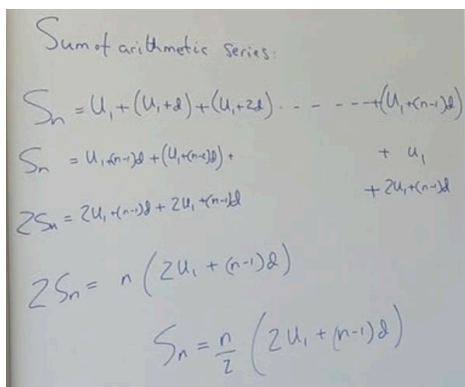
Figure 48: Writing the hint

After about another minute, he continued:

T: Alright. (speaking while writing on the whiteboard) So you add the two together, you get two lots of S_n . If you take this first term, it's just u_1 (pointing to it on the whiteboard) plus $u_1 + (n - 1)d$, so you get (writing on the whiteboard) $2u_1 + (n - 1)d$. if you take the next one, you've got... (doing the algebra term by term). What's this last one here? It's u_1 plus u_1 which is $2u_1 + (n-1)d$. So what are all those terms? There are all $2u_1 + (n - 1)d$. How many of them have we got?

S: n

T So therefore we've got $2S_n = n(2u_1 + (n - 1)d)$, which gives when divided by two gives (writing the expression on the whiteboard, see Figure 49)



Sum of arithmetic series:

$$S_n = u_1 + (u_1 + d) + (u_1 + 2d) + \dots + (u_1 + (n-1)d)$$

$$S_n = u_1 + (n-1)d + (u_1 + (n-1)d) + \dots + u_1 + 2u_1 + (n-1)d$$

$$2S_n = 2u_1 + (n-1)d + 2u_1 + (n-1)d$$

$$2S_n = n(2u_1 + (n-1)d)$$

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

Figure 49: The formula for the sum of arithmetic series

Even though, out of the three analyzed activities proposed by Eliot, this was the most teacher controlled one and the students were given relatively little time to work on their own, the overall impression is rather positive. The research potential of the sum of the first 100 integers was certainly “killed” by giving out the first hint about reversing the order of the two sums too early. Despite reducing the students’ topos drastically, key-hint given out too early and little time, this activity carried some IBME features: a student (S1) tackled the “Gauss’s problem” differently, the class was engaged in the activity or another student who came up with the generalization.

Based on our observation of Eliot’s practices, we can say that Eliot generally tried to listen to the students, to take their suggestions into consideration and to build on what came from the students. This attitude contributed to keeping the students engaged in the activities and actually do things on their own. In the next part, we propose a comparative conclusion of David’s and Eliot’s practices, attempt to formulate some generalities concerning the conditions and constraints fostering or hindering the implementation of IBME within the ordinary classes.

5 Conclusion of David’s and Eliot’s practices

A lot of the teaching and learning time in the regular classes was just teaching-to-the-test and the types of tasks proposed as practice questions in class and for homework as well as the exercises in the textbook were just the same as the items on the standardized exams. The

end-of-chapter tests were literally generated via a question bank from past exam questions. An exam-like atmosphere was very palpable already in the first year of the diploma program, even though the final exam was not scheduled until the end of the second year. Another generality that we observed was an abundant use of technology including GDC and the corresponding software to project the calculator screen. This was, on the one hand, facilitated by the material equipment of the school and reinforced by the fact that one of the two exam papers required the use of a GDC, on the other hand.

Even though most of the in-class time and especially practicing was dedicated to the exam preparation, we were still able to identify attempts to implement some investigation and problem-solving activities into the regular classes. Both teachers either made students raise questions or tried to elicit answers from students by making use of maieutic which is a didactic praxeology suggested by Wozniak (2015) to allow the advancement of the investigation in a reasonable time without reducing too much the student's topos. Unfortunately, in David's practices, we often saw that when he proposed some reflective situations to introduce a new topic or problems that required to mobilize and combine several pieces of knowledge, he did not give the students enough time to actually think about it or solve it on their own. On top of that, when some students came up with a suggestion or proposed a possible path to explore, David did not take it into account unless it contributed to the expert solution he was proposing. This resulted in the fact that most of the students were often disengaged from the problem-solving or inquiry activity and let the teacher come up with the solution and before they copied it down. David certainly created a very teacher-controlled learning environment even in the space dedicated to what could have potentially been IBME or problem-solving. This is also how Sophie, one of the students from this class, described what they usually do in the regular mathematics class:

Most of the time when we are in class, we are kind of looking at the formulas that we are learning and then we apply that to different things and things that we haven't seen before and we have to adopt the ideas, that's the main thing that we do in class.

On the contrary, Eliot seems more of a risk taker. In the activities we analyzed, he attempted introducing new topics through investigations, proposed some interesting problems and made a better use of maieutic which, within the constraints, created conditions that contributed to extend the students' topos. Eliot shared definitely more topos with the students, provided more time to think and tried to build on the students' suggestions and inputs. This led to interesting discussions with the students, most of the students were actively engaged in the proposed activities, some raised some interesting new questions, which weren't, however, always further explored. It is therefore important that the most interesting ideas are shared with the rest of the students which would lead to more confrontations, exchanges and discussions. This would actually bring the classroom work closer to how scientists or mathematicians work: as a community. Nevertheless, this comparison enables us to make the following conclusion. While both teachers work in the

same conditions, benefit from the same material and technological equipment and have to deal with the same constraints due to the amount of content to cover for the exam, in Eliot's class, the students were generally more engaged in the proposed activities. Indeed, Eliot, in addition to provide the extra time, tried to build on the students' answers and suggestions, which turns the class into a community of learners responsible for their learning instead of just being passive receivers of what the teacher says.

Another obstacle for a genuine inquiry-based activity to deploy was the rather poor quality of the resources for the introductory investigations proposed in the textbook. The "too much guidance" approach led to splitting the activity into many isolated subtasks which turned a potential inquiry into disguised ostension as described by Berthelot and Salin (1992): "Instead of showing the student what is to be seen, the teacher hides it behind a fiction: that it is making the student believe that it's himself who discovers it on the spatial objects subjected to his observation or to his action" (p. 80)⁴⁸. Unfortunately, most of the "investigations" proposed by the textbook, the way they are designed to generate and reinforced disguised ostension. Despite this fact, we were still able to observe some differences related to the personal praxeologies of the two teachers. In Eliot's class, the students' topos was greater and the students were generally given some time to think and carry out the activity on their own which led, even with a poor activity, to a greater engagement of the students in class, to interesting discussions with the teacher and even raised some new questions. In the other hand, in David's class, the reduced students' topos contributed to a disengagement of the class in the activity and resulted in students following the teacher's instructions and executing isolated instructions even with a problem or an activity that presented good inquiry potentials a priori.

Based on the observed evidence we identified several obstacles of various nature in implementing IBME and problem-solving within the IB mathematics SL classes. From what we have observed, we can conclude that the main constraint hindering the implementation of IBME in the Mathematics SL classes is of institutional nature: the pressure of the final exam and the amount of content to cover in a relatively short period (2 years). Despite this main institutional constraint, we were, however, able to identify in the practices of the two observed teachers some attempts to implement IBME and problem-solving into the regular classes. Paradoxically, one of the main constraints, i.e., the fact that Paper 2 of the final exams requires to use a GDC efficiently, created favorable conditions as far as the material equipment and access to technology are concerned. Our analysis also reveals that some of the obstacles preventing the deployment of genuine inquiry were created by the way the problems or activities were implemented. The reasons behind point

⁴⁸ Our translation from the French original: « Au lieu de montrer à l'élève ce qui est à voir, le maître le dissimule derrière une fiction: celle que c'est l'élève lui-même qui le découvre sur les objets spatiaux soumis à son observation ou à son action » (p. 80).

either to the way the resource was conceived or to the teachers' personal praxeologies put in place.

Even though important changes in the way standardized exams are conceived or a significant reduction of the content knowledge is most likely not scheduled on the IB agenda, our analysis shows that there is space for implementation of IBME in the IB mathematics SL classes. To improve the quality of the already existing IBME, individual schools via their mathematics departments could actually put in place relatively effective measures such as targeted continuous teacher training or support their participation in collaborative research oriented to conception and testing of appropriate resources as part of their teachers' professional development.

After taking the "temperature" of inquiry in the regular classes, we now move on to present the two case studies of our observation of the ME.

Chapter II

OBSERVATION OF THE MATHEMATICAL EXPLORATION

We have finally arrived at the heart of our research situated at the level of discipline. The institutional analysis allowed us to identify elements characteristic for inquiry-based education and the ME provided an existing and observable object reflecting inquiry-specific objectives as pointed out in Chapter I in Part B (see p. 83). As a reminder, the ME is an individual, stand-alone research activity conducted on a topic chosen by each student and supervised by their mathematics teacher. The syllabus reserves 10 hours which corresponds to about 13 periods of in-class time and suggests that the students spent about the same amount of time working on their explorations at home.

In this chapter, we are first going to describe the contextual circumstances which we faced during the data collection. Then we will describe the data treatment and the construction of an analysis framework that we developed from the theoretical framework. We will describe the institutional and the teachers' practices in which the ME is implemented and then conduct an analysis of two MEs. Finally, we will analyze the students' experiences of the ME via a questionnaire addressed to the students from the two observed classes.

1 Context of the observation

The observed MEs took place from March 20 to June 24, 2019, towards the end of the second semester of the first year of the IB diploma program due to the school agenda needed to fit different internal assessments. This is indeed an important institutional constraint as it affected and limited the covered topics of the math curriculum and the students had therefore less mathematics tools available to conduct the MEs. During the exploration assignment, the institutional milieu of the observed classes had available knowledge on functions (quadratic, trigonometric, exponential and logarithmic functions) and their transformations as well as the basics of differential calculus, including the derivative of some known functions and their applications for optimization, and descriptive statistics (Appendix 10). The implementation of the ME followed the timeline (Figure 50) provided by the teachers during the introduction session.

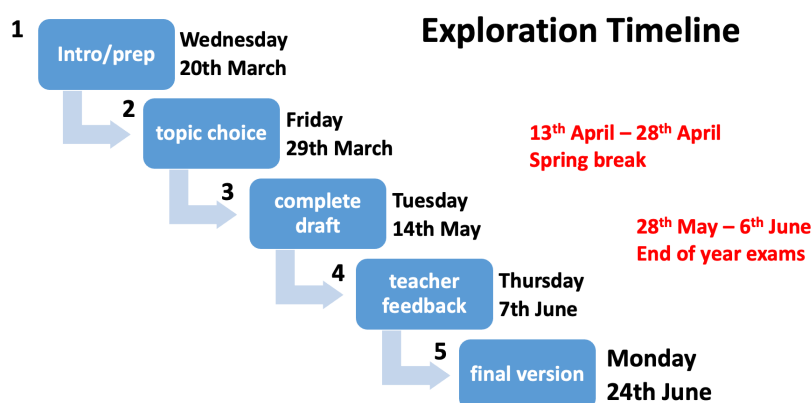


Figure 50: Timeline for the exploration provided by the teacher

For this period of the ME, the original plan was to follow six volunteer students, however, only four volunteers were found (two per classroom) probably due to the complexity of the protocol to respect. Unfortunately, in Eliot's class one student had to stop participating in the research because of health issues and the other student did not respect the protocol and produced unexploitable data. Therefore, we finally managed to collect exploitable data of only two volunteer students both from David's class. Another difficulty that we encountered were the gaps in the data because the students did not always think of turning on the action camera and this caused some gaps in the chronology.

We are now proceeding to the presentation of the data treatment and construction of an analysis framework for the two case studies. On the one hand, we used the Herbartian scheme to synthesize the transcribed videos and, on the other hand, we proceeded inductively from what we found in our experimental data.

2 The case studies data treatment

The designed methodology enabled us to obtain reasonable data (notes in a logbook, recordings of the computer screen, Google docs), so that for two students' ME we could reconstruct the chronology of the exploration, identify the accessed media and the research carried out by the students and transcribe the data and list each action of the student and categorized each media and its content that appeared in the recordings based on the elements from the Herbartian scheme (Appendix 11). Obviously, the questions were reconstructed by the researcher based on the visible data, particularly on the Google search input line. This enabled us to create a chronological list of all events in Excel (Appendix 13) in respect to the order of appearance of the different identified elements that we will call *elementary units* or *E-units*). The E-unit is the smallest unit in our analysis and corresponds to the "ingredients" of the Herbartian scheme. It is important to note that the chronology is not expressed in terms of unit of time but in chronological order in terms of the appearance of the different elements from the Herbartian scheme (E-unit = 1). The reason behind is quite simple and rather pragmatic, since the collected data came from different

sources (video, logbook, the teacher's annotations, shared Google docs, interviews), it was impossible to obtain a consistent time unit. The advantage of using the elements from the Herbartian scheme as a base unit is that we can still express the chronology of the exploration, if not its pace in time units while combining data from different sources.

With the appearance of questions, the chronology was quite naturally split into larger sections to provide a global view of the ME. We will call a *QA-unit* a sequence of all E-units following a question until a new question is identified. We numbered the questions and labeled each E-unit accordingly, an example of QA-unit 31 is given in Table 13. The label WS 31.1, for example, refers to the first website from QA-unit 31. When the same media appeared several times, we kept it with its original label indicating when it appeared for the first time. For example, WS 28.1 was first identified in QA-unit 28 and was used again in QA-unit 31.

Chronolo	E-unit	Description
258	Q 31	What is a catenary curve?
259	GS 31.1	Catenary curve
260	R 31.1	The catenary is a plane curve whose shape corresponds to a hanging homogeneous flexible chain supported at its ends and sagging under the force of gravity.
261	A♦ 31.1	The catenary is a plane curve whose shape corresponds to a hanging homogeneous flexible chain supported at its ends and sagging under the force of gravity.
262	WS 31.1	Math24-equation of catenary – too complicated
263	A♦ 31.2	a physics model of a catenary—inaccessible
264	GD 1.1	Math sites
265	L 31.1	The link of WS 31.1 is copied to GD 1.1
266	WS 28.1	How to calculate half parabolic curve?
267	A♦ 31.3	Gives a step-by-step technique to graph half a parabola by only taking the x-values less (greater then) the x-coordinate of the vertex

Table 13: An example of QA-unit 31

The Herbartian scheme provided a good model to categorize the different “ingredients” of inquiry, synthesize the experimental data and prepare them for analysis.

2.1 Construction of the analysis framework

We want to capture the dynamics of an inquiry process and identify the interactions of the student with the media, understand how the information from the media enriches the inquiry milieu M_I and contributes to the production of the $A^\heartsuit$, and finally if and how an a-didactic milieu M_A is organized to test or validate the $A^\heartsuit$. Our objective is to observe different phenomena, identify any regularities and study different aspects of the dynamics of an inquiry process to describe it and eventually attempt some generalizations. These reflections led us to the conception of an analysis framework with three levels (question-answer, media-milieu and elementary,) as summarized in the scheme on Figure 51.

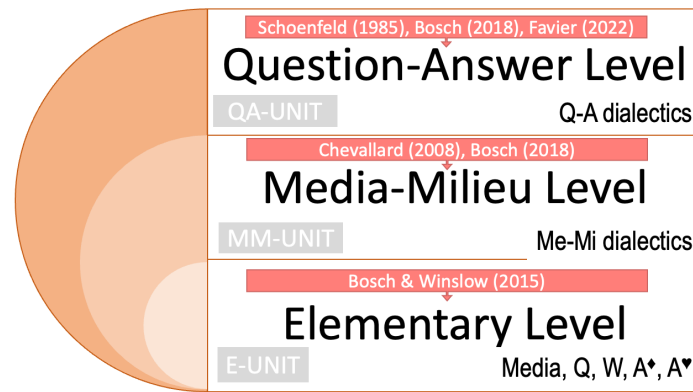


Figure 51: Analysis framework scheme

In what follows, we present in detail each of these three levels and show how they enable us to analyze the chronogenesis and the mesogenesis of an inquiry process.

2.1.1 Question-Answer Level

The QA-level appeared quite naturally because as Bosch (2018) claims, the chronogenesis of an inquiry process can be captured through the question-answer dialectic. As already mentioned earlier the base unit at this level is a QA-unit (QAU): each new question marking the beginning of a new QAU. In order to characterize the QA-units based on the main activity of the inquirer, we adapted Schoenfeld's (1985, pp. 297–301) model enriched by Favier (2022) developed to analyze problem-solving. To depict the main activity of the student we used Schoenfeld's definition of an episode:

A period of time during which an individual or a problem-solving group is engaged in one large task [...] or a closely related body of tasks in the service of the same goal [...]
(Schoenfeld, 1985, p. 292).

Compared to Schoenfeld's model, it is important to note that we removed the episodes of reading and transition because these were not applicable to modeling inquiry with access to media, we split planning-implementation into two separate episodes and finally, we introduced an episode of study.

In relation to the student's main activity and the pursued objective and based on the above definition, we identified and characterized the following episodes for inquiry processes:

- **Exploration:** this episode is characterized by a rather unstructured search for information, preceded by a rather general question.
- **Study:** this episode begins when a media catches the inquirer's attention and the latter stops and studies the content of this media. New information may be admitted into the milieu M_{INQ} .
- **Planning:** during this episode, the inquirer sets up a structured plan for proceeding; this phase may be implicit.
- **Implementation:** the inquirer carries out a plan or a procedure.

- Analysis: the inquirer tries to better understand the problem and analyzes it in light of new information.
- Verification: refers to episodes when the inquirer verifies responses obtained from the media or his or her own generated responses.
- Regulation: this episode considers interventions initiated by the teacher, tutor or another expert in the field.

On top of that Schoenfeld (1985) suggests to mark new-information points when “a previously unnoticed piece of information is obtained or recognized” (p. 299). Based on this idea we created a mark that we called particular event.

- Particular event: important media, adjusting search and new information.

We then characterized each episode according to the themes it treated: mathematical (these categories depend obviously on the main topic of the ME) or non-mathematical such as methodology and writing report and its productivity (productive, non-productive, ill-productive and irrelevant). We refer to a productive episode when the research and work carried out in respect to the given question helped either to advance the inquiry or contributed to the production of A♥. A non-productive episode occurs when, after a search, a relevant media is activated but no immediate action is taken or answer is produced. An ill-productive episode is identified when, after a search, incorrect information is retrieved and/or an incorrect answer is produced. An irrelevant episode corresponds to moments when an irrelevant media is activated, irrelevant information is retained and/or an irrelevant path is pursued. It seemed important to us to consider separately the irrelevant episodes, because they seem to have a much less harmful effect on the inquiry than ill-productive episodes but are still less neutral than non-productive episodes.

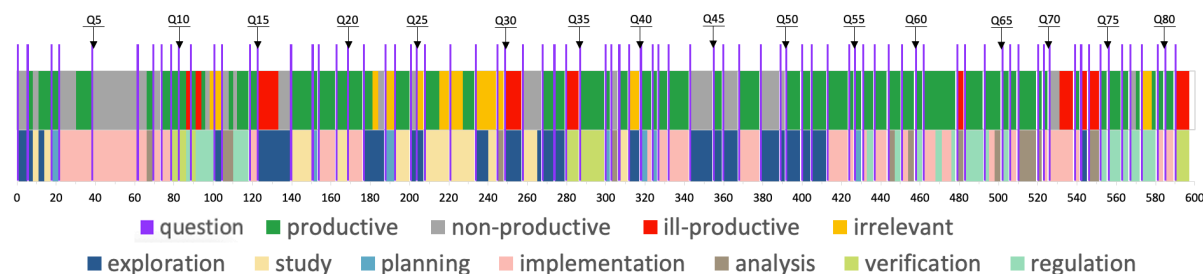
When coding the chronology into episodes, we sometimes identified several different episodes within the same QA-unit. In order to keep track of the episodes in relation to the QA-units and to the overall chronology, we labeled them as shown in Table 14.

Chronology	E-unit	Description	Theme	Type of episode	Productivity	Ep_label
245 Q 29		How am I going to validate the quadratic model?	3: choice of model	question	question	Q29-Ep48
246 Std 29.1		Student's reflection	3: choice of model	analysis	irrelevant	Q29-Ep48
247 A♥ 29.1		logbook: I will assume it to be a parabola, find the best fit curve and then find the half of it and try	3: choice of model	analysis	irrelevant	Q29-Ep48
248 A♥ 29.2		logbook: if it fits in perfectly, then it is a parabola.	3: choice of model	analysis	irrelevant	Q29-Ep48
249 Q 30		Does the suspension cable of the san francisco golden gate bridge form a parabola?	3: choice of model	question	question	Q30-Ep49
250 GS 30.1		Does the suspension cable of the san francisco golden gate bridge form a parabola	3: choice of model	exploration	ill-productive	Q30-Ep49
251 WS 15.1		Math central: online help type of website	3: choice of model	exploration	ill-productive	Q30-Ep49
252 R 30.1		Excerpt from WS 15.1	3: choice of model	exploration	ill-productive	Q30-Ep49
253 A♥ 30.1		The suspension is a catenary	3: choice of model	exploration	ill-productive	Q30-Ep49
254 R 30.2		Are suspension bridges parabolas?	3: choice of model	exploration	ill-productive	Q30-Ep49
255 A♥ 30.2		suspension bridges are rather parabolas because forces of compression and tension are acting	3: choice of model	exploration	ill-productive	Q30-Ep49
256 GD 1.1		Math sites	3: choice of model	exploration	ill-productive	Q30-Ep49
257 L 30.1		R 31.1 is copied to GD 1.1 Math sites	3: choice of model	exploration	ill-productive	Q30-Ep49
258 Q 31		What is a catenary curve?	4: catenary model	question	question	Q31-Ep50
259 GS 31.1		Catenary curve	4: catenary model	study	non-productive	Q31-Ep50
260 R 31.1		The catenary is a plane curve whose shape corresponds to a hanging homogeneous flexible ch	4: catenary model	study	non-productive	Q31-Ep50
261 A♥ 31.1		The catenary is a plane curve whose shape corresponds to a hanging homogeneous flexible ch	4: catenary model	study	non-productive	Q31-Ep50
262 WS 31.1		Math24-equation of catenary – too complicated	4: catenary model	study	non-productive	Q31-Ep50
263 A♥ 31.2		a physics model of a catenary – inaccessible	4: catenary model	study	non-productive	Q31-Ep50
264 GD 1.1		Math sites	4: catenary model	study	non-productive	Q31-Ep50
265 L 31.1		The link of WS 31.1 is copied to GD 1.1	4: catenary model	study	non-productive	Q31-Ep50
266 WS 28.1		How to calculate half parabolic curve?	2: quadratic model	exploration	non-productive	Q31-Ep51
267 A♥ 31.3		Gives a step-by-step technique to graph half a parabola by only taking the x-values less (greate	2: quadratic model	exploration	non-productive	Q31-Ep51

Table 14: Extract of coding in episodes in Excel.

This coding allowed us to create different graphical representations of the exploration chronology on the same scale. This allowed us to superpose Productivity and Type of Episode as shown on (Graphics 7).

Exploration chronology



Graphics 7: The chronology of the ME (Productivity and Type of Episode)

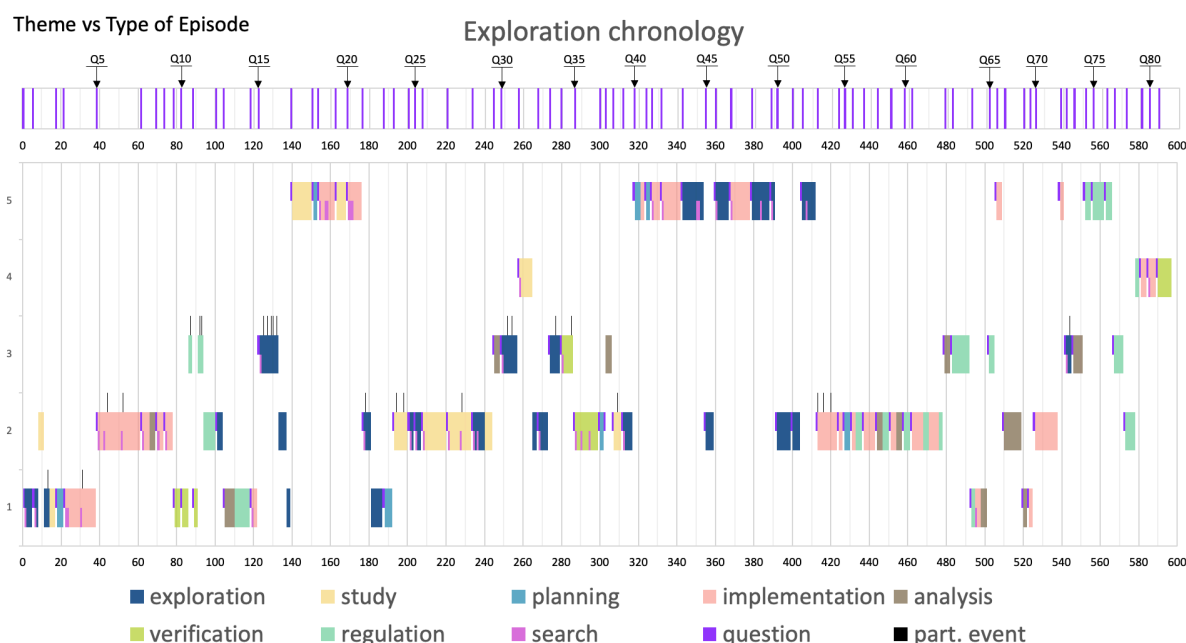
We were also able to produce a split view of the chronology according to theme showing either the productivity (Graphics 8) or the type of episode (Graphics 9). Each horizontal line of the graphics corresponds to a theme. Note that the size of each episode is expressed in terms of the number of E-units and not in terms of time length.

Theme vs Productivity

Exploration chronology



Graphics 8: The Chronology of the ME—Productivity vs. Theme

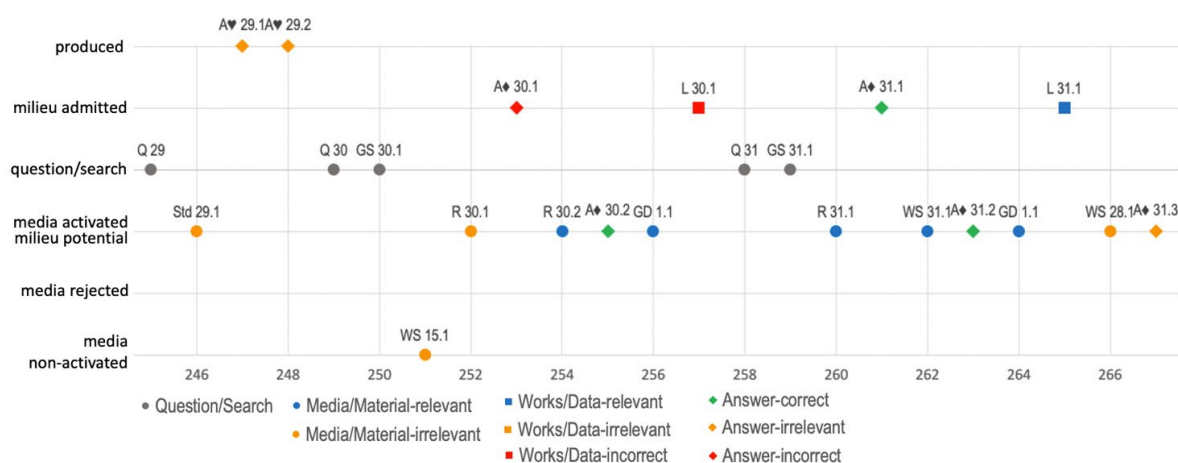


Graphics 9: The Chronology of the ME—Type of Episode vs. Theme

These representations provide a sort of cognitive map of the whole ME and allow to visualize its chronogenesis.

2.1.2 Media-Milieu Level

To proceed to the analysis at the MM-level, we graphed all questions and searches on the main horizontal axis, all admitted and produced e-units are above it and all potential or rejected e-units are placed below. We obtained thus a visual graphical representation of the evolution of the milieu during the inquiry process with an excerpt shown on Graphics 10. In addition, a color code and a shape code (provided in the legend) enable us to have an instant information about each E-unit.



Graphics 10: Excerpt from the chronology of the exploration

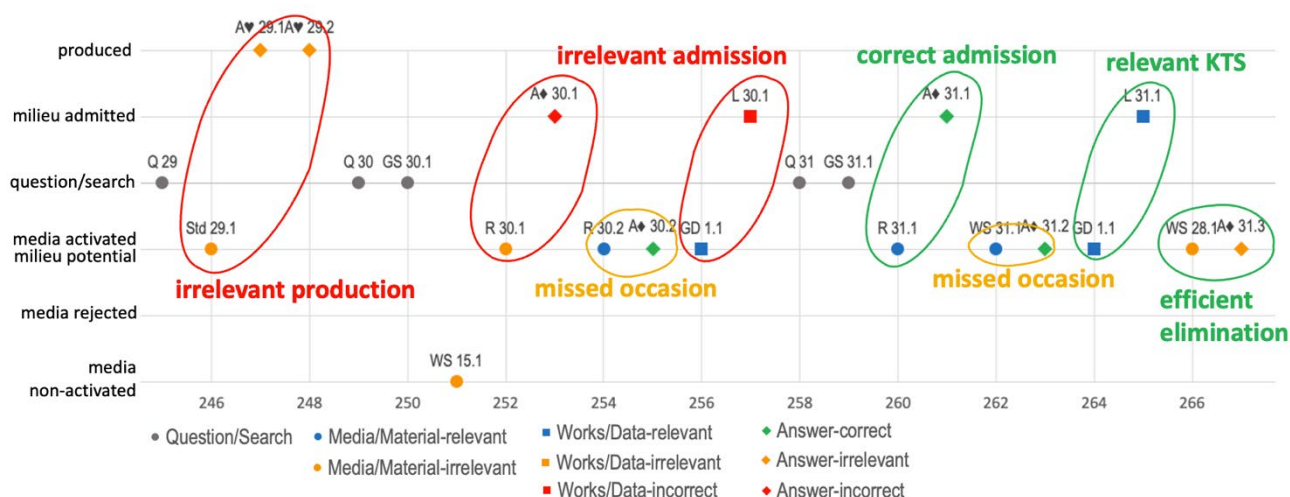
This graphical representation provides thus the chronology of the exploration and enables us to make visible the interactions with media and capture the moments when different

elements were admitted to the milieu M_I and eventually some answers were produced. While observing these interactions, we noticed some recurring patterns of certain clusters of E-units which led us to consider a new level of analysis: media-milieu level (MM-level).

At the MM-level, the basic unit that we will call *MM-unit* (MMU) is composed of one or more E-units and depicts the interaction of the inquirer with a media. The beginning of an MMU is marked when a new media occurs and contains generally institutional answers, data or works, and eventually produced answers. We proceeded empirically and identified five major categories of MMUs according to the way the information was processed and to the quality of the contained information:

- 1) Non-activation: a media appeared in the recording but there was no visible evidence of activation. If the media was irrelevant, we coded it an “efficient media judgment”, if on the contrary the media could be potentially relevant for the exploration, we considered it as a “missed occasion”. This is by the way the only case when an MMU is composed of a media alone and has thus a length of one.
- 2) Non-admission: a media is activated and information is processed because at first sight it seems potentially pertinent for the exploration but is not retained. We identified two types of MMUs: a “missed occasion” when the media is relevant and an “efficient elimination” when the media is irrelevant for the exploration.
- 3) Admission: a media is successfully activated, new information is processed (reading, watching a video, note taking, etc.) and thus new elements ($A^\star$, W, Dt) are admitted into M_D .
Admission MMUs can be correct, incorrect or irrelevant.
- 4) Production/reflection: a partial answer is produced by the student either after a reflection or an interaction with a media
- 5) Keeping track of sources (KTS): this type of MMU occurs when a link of an activated media was copied and kept as a source.

An example of coding into MMUs is provided on Graphics 11.



Graphics 11: Example of coding into MMUs

In order to understand the role and the impact of the different media on the exploration, we described the effect of each MMU on the produced final answer. We proceeded again inductively based on the empirical data and identified the following effects of the MMUs on the exploration:

- 1) answer found later
- 2) different solution found
- 3) advancing inquiry
- 4) contributing to A♥
- 5) following an irrelevant path
- 6) negative effect on A♥
- 7) continuing search
- 8) unexplored path
- 9) no visible effect

Table 15 provides an example of coding at the MM-level in Excel.

Chronology	E-unit	Description	MM-unit name	MM-unit characteristics	Effect on exploration	MMunit#	S_label
246	Std 29.1	Student's reflection	production	irrelevant reflection	5_following an irrelevant path	76	Q29
251	WS 15.1	Math central: online help type of website	non-activation	efficient media judgment	7_continuing search	77	Q30
252	R 30.1	Excerpt from WS 15.1	admission	incorrect admission	6_negative effect on A♥	78	Q30
254	R 30.2	Are suspension bridges parabolas?	non-admission	missed occasion	6_negative effect on A♥	79	Q30
256	GD 1.1	Math sites	KTS	incorrect source	6_negative effect on A♥	80	Q30
260	R 31.1	The catenary is a plane curve whose shape co	non-admission	correct admission	8_unexplored path	81	Q31
262	WS 31.1	Math24-equation of catenary – too complicated	non-admission	missed occasion	8_unexplored path	82	Q31
264	GD 1.1	Math sites	KTS	relevant source	9_no visible effect	83	Q31
266	WS 28.1	How to calculate half parabolic curve?	non-admission	efficient elimination	7_continuing search	84	Q31

Table 15: Extract of coding in MM-units in Excel

At the MM-level, we expect to be able to describe the evolution of the milieu and characterize the conditions in which a Me-Mi dialectic can exist.

2.1.3 Elementary Level

The E-level provides detailed characteristics of each E-unit concerning the statuses of activation, relevance and correctness, on the one hand, and a categorization of each E-unit according to its type and function in the inquiry.

Chronology	E-unit	Description	Status 1	Status 2	Type 1	Type 2	Type 3
245 Q 29		How am I going to validate the quadratic model?			action	question	question
246 Std 29.1		Student's reflection	irrelevant	activated	media	human	student
247 A♥ 29.1		logbook: I will assume it to be a parabola, find the best fit curve and	irrelevant	produced	answers	produced answer	student's answer
248 A♥ 29.2		logbook: if it fits in perfectly, then it is a parabola.	irrelevant	produced	answers	produced answer	student's answer
249 Q 30		Does the suspension cable of the san francisco golden gate bridge form a parabola?			action	question	question
250 GS 30.1		Does the suspension cable of the san francisco golden gate bridge form a parabola			action	internet search	google search
251 WS 15.1		Math central: online help type of website	irrelevant	non-activated	media	website	website
252 R 30.1		Excerpt from WS 15.1	irrelevant	activated	media	result	result
253 A♦ 30.1		The suspension is a catenary	incorrect	admitted	answers	retrieved information	institutional answer
254 R 30.2		Are suspension bridges parabolas?	relevant	activated	media	result	result
255 A♦ 30.2		suspension bridges are rather parabolas because forces of compress correct	correct	potential	answers	retrieved information	institutional answer
256 GD 1.1		Math sites	relevant	activated	material	software	google doc
257 L 30.1		R 31.1 is copied to GD 1.1 Math sites	incorrect	admitted	works	keeping track of sources	copied link
258 Q 31		What is a catenary curve?			action	question	question
259 GS 31.1		Catenary curve			action	internet search	google search
260 R 31.1		The catenary is a plane curve whose shape corresponds to a hanging	relevant	activated	media	result	result
261 A♦ 31.1		The catenary is a plane curve whose shape corresponds to a hanging	correct	admitted	answers	retrieved information	institutional answer
262 WS 31.1		Math24-equation of catenary – too complicated	relevant	activated	media	website	website
263 A♦ 31.2		a physics model of a catenary – inaccessible	correct	potential	answers	retrieved information	institutional answer
264 GD 1.1		Math sites	relevant	activated	material	software	google doc
265 L 31.1		The link of WS 31.1 is copied to GD 1.1	relevant	admitted	works	keeping track of sources	copied link
266 WS 28.1		How to calculate half parabolic curve?	irrelevant	activated	media	website	website
267 A♦ 31.3		Gives a step-by-step technique to graph half a parabola by only takin	irrelevant	activated	answers	retrieved information	institutional answer

Table 16: Extract of coding the E-units in Excel

On the data organized this way, we can thus easily run some descriptive statistics and use pivot tables to see what media was accessed and activated or abandoned, what institutional answers were retrieved and finally capture moments when answers were produced.

Combining the 3-levels of this framework enables us to choose an appropriate level of zoom navigating between a global view and a detailed one to capture the dynamics of the chronogenesis and the mesogenesis of an inquiry process in terms of Q-A and Me-Mi dialectics of the ME.

In our description of the implementation of the ME now proceed by giving some information on the way the ME has been launched in the two observed classes and the general institutional framework.

3 Launching the ME and institutional context

The ME was introduced during a 30-minute session using a PowerPoint presentation for the two classes in collaboration with the two Mathematics SL teachers (David and Eliot) and was essentially led by David. Before the presentation, David emphasized the three topics covered in their ordinary classes (Functions, Differential Calculus and Statistics) which should, according to him, provide the students with sufficient mathematics tools to conduct the ME. During this session David went over the institutional expectations and specified the assessment criteria. He reminded the importance of citing sources and warned the students of the consequences of plagiarism. He described what the complete draft

should look like according to the planning (Figure 50) we can see that the students had about 10 days to choose the topic of their ME. Within this time, they were expected to start thinking about their interests, do some initial research and approach their teacher for the topic approval. The students were provided with a link to a list of about 200 possible exploration topics (Appendix 12) organized according to the mathematics it addresses.

To guide the choice of the topic David proposed to keep in mind three important questions:

- 1) Does the topic involve math at a suitable level for an SL exploration?
- 2) Is the topic narrow enough so that it can be treated sufficiently in a 6–12-page report?
- 3) Does the topic lend itself to demonstrating personal engagement (criterion C)? That is, can you envision some way that you could apply something of your own—your own viewpoint, your own examples, your own models (conceptual or physical), your own question & ideas, etc.?

Another important part of this introduction session was devoted to the elaboration of the first draft. According to the planning, the students had about 6 weeks to complete it once their topic approved. This was the only written feedback from the teacher and the draft should therefore present a rather complete version of the exploration. Here is David's view on how the first draft should look like:

A first draft is not a sketch, it's not a bunch of bullet points, it's not a thought bubble, a first draft is a finished product that's got everything in it, it's got the statement of what you gonna do, it's got how you gonna do it, it's got what you need, it's got your results, it's got the mathematics in it, it's got a bibliography, it's got appendices, it's got all in it and we just look at it and give you ideas on how to tweak it and make it better. (David, March 20, 2019, Introduction session)

After this introductory session, both teachers supervised the explorations in their respective classes. It turned out impossible to follow entirely two different teachers in parallel time slots, but since we finally only obtained the collaboration of two volunteer students in David's class, we thus naturally concentrated in obtaining more data on David's practices and only partial elements on Eliot. Despite this constraint, we were still able to get some data through exchanges with the two teachers and to compare their practices.

Both of them used some of the collective in-class time to address some generalities such as the assessment criteria, bibliography and the choice of the topic. David was convinced that modeling explorations are the most accessible for the student taking into account the covered topics and oriented his class towards using functions to model logos, bridges or repeating phenomena such as tides. Indeed, for David, modeling explorations were the most accessible ones for the students in respect to their level and the covered topics as he explained. Even though, institutionally, they could still decide not to do a modeling ME, practically as a result of the didactic contract in this case, their free choice remained highly theoretical. In addition, David's vision of mathematical modeling consisted of finding a mathematical function whose graph would fit a given set of data or superimpose on a curve

on a photo or a logo, determining the equation of this function analytically and reflecting on the results. This clearly affected the choice of topics in his class, because 15 out of 17 explorations were on modeling whereas Eliot did not suggest any specific type of exploration and there was more variety in the chosen topics in his class. For more detail the lists of the topics for the two classes are in Appendix 14. Each student benefited from a 10–15-minute interview with their teacher concerning the choice of their topic. Most of the in-class work took place until the first draft deadline and in both classes the students were working individually, but could ask for the teacher's assistance when needed.

We come now to the analysis of the MEs of the two students: Rachel and Sophie. In order to set up the landscape, we first present the question each of them chose to tackle and one possible a priori investigation. Then, we give some information on the profile of each student and finally we describe the overall chronogenesis and mesogenesis of each ME. Let us start with Rachel's ME.

Chapter III

RACHEL'S ME: THE SHAPE OF THE SUSPENSION CABLE OF THE GOLDEN GATE BRIDGE

Rachel chose to investigate the Golden Gate Bridge in San Francisco (Figure 52). She decided to analyze the shape of the curve formed by the bridge's suspension cable and find out if it forms a parabola or not.

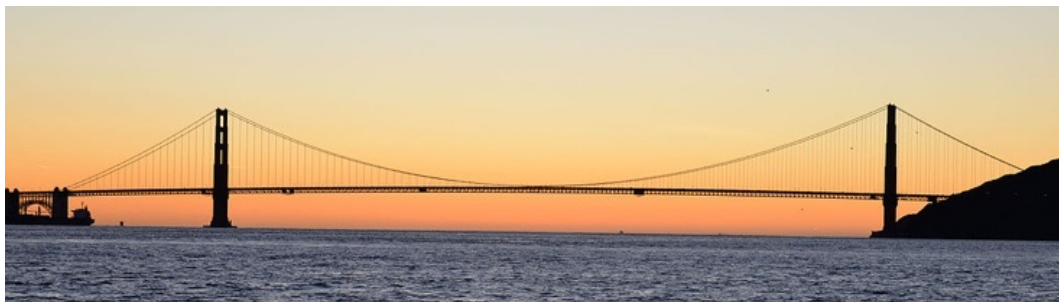


Figure 52: The photo of the San Francisco Bridge used by Rachel⁴⁹

Before tackling the analysis of Rachel's work, we first propose a possible exploration of the shape of the Golden Gate Bridge in order to help understanding the Rachel's work.

1 An “expert” investigation of the Golden Gate Bridge suspension cable

At first glance, two curves that have a similar shape could be potential candidates for modeling: a parabola and a catenary. A parabola is the graphical representation of a quadratic function, and a catenary can be modeled by a hyperbolic cosine and is obtained by suspending a cable or a chain between two points. The suspension cables of bridges are largely believed to be catenaries, but the answer is far from being straightforward. Indeed, if a cable is freely suspended between two points, the curve thus formed is a catenary, but this is not the case for the suspension cable of the San Francisco Bridge. In general, when the horizontal deck of the bridge is suspended on vertical rods attached to the main suspension cable, different forces act on the suspension cable, and it will take the shape of a parabola. We are thus going to provide an analytical solution justifying a quadratic model for a suspension bridge cable supporting a uniform load.

⁴⁹ https://en.wikipedia.org/wiki/Golden_Gate_Bridge#/media/File:Golden_Gate_Bridge_Dec_15_2015_by_D_Ramey_Logan.jpg

1.1 Analytical justification of the quadratic model

The key element of the resolution lies in the vertical rods that connect the deck (support of the roadway) to the supporting cables on which the bridge is suspended (Figure 53).

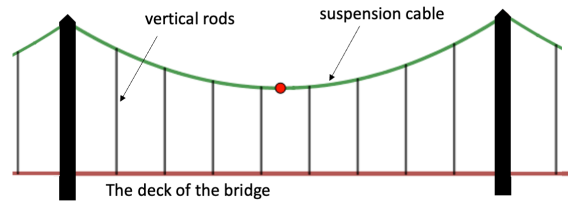


Figure 53: A simplified model of a suspension bridge

Even though the modeling of the suspension cable of a suspension bridge is much more complex and we will only consider the analytical solution for the simplest case where the cable supports a constant and uniformly distributed weight horizontally. We also assume the towers to be rigid and the cable to be flexible (can be bent without resistance) and inextensible (does not stretch).

Let us take a point P on the arc AVA' representing the suspension cable of a suspension bridge. More precisely, as shown in Figure 54, the tension exerted by the arc $A'P$ on the rest of the cable has the direction of the tangent at P represented by the magnitude vector T.

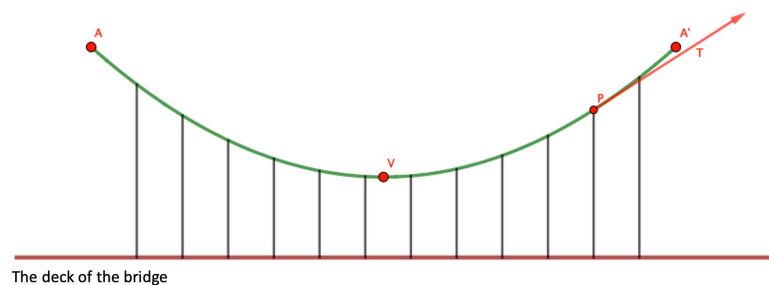


Figure 54: The tension on the suspension cable

Let us now consider any section (say the section from V to P) of the cable as shown in Figure 55. Since the traction on V from the section to the left of V is tangent to the cable, the pull is thus horizontal and directed to the left. This tension is a constant that we will call T_0 . The tension exerted on VP at P by the section PA' of the cable is exerted along the tangent at P. We will refer to the magnitude of this tension T and we will call θ the angle that the direction of the tension (or tangent) at P makes with the horizontal. There is another force acting on the VP section of the cable, namely the tension of the portion of the cable, load, and pavement that extends from O to P'.

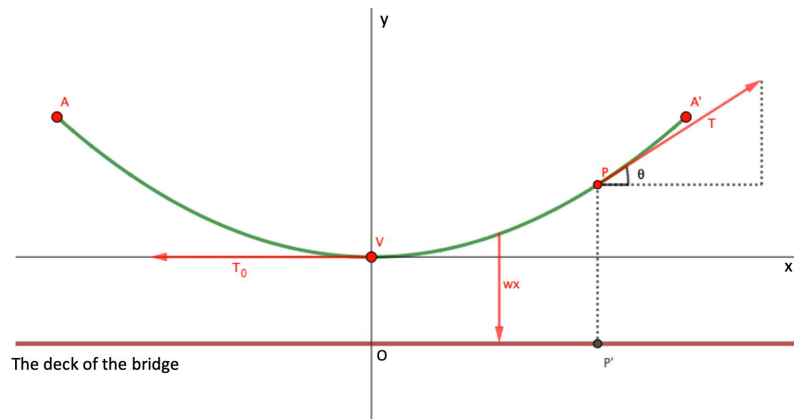
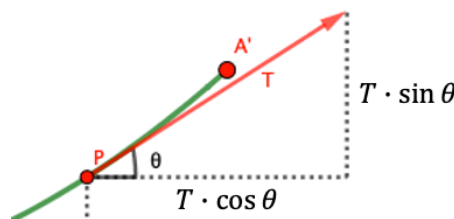


Figure 55: Distribution of the traction forces on the suspension cable of a suspension bridge

Let if w be the weight per horizontal meter. Since the total pull is assumed to be constant for each horizontal meter, the load carried by the arc VP is $w \cdot x$, where x is the abscissa of P . The pull of this load is actually distributed along VP , but all of the tractions are directed downward, therefore the total downward pull on VP is equal to $w \cdot x$.

Thus, there are three forces acting on VP : the horizontal pull T_0 to the left, the downward pull $w \cdot x$ of the total load, and the tangent pull T at P . Since the arc VP is in equilibrium, the three forces must somehow compensate for each other, because if there were a net force, the rope would bend under the action of that force. We can also apply the same reasoning to the vertical forces. The tension T is equivalent to a horizontal and a vertical force acting simultaneously, it is thus equivalent to the combined action of the horizontal component: $T \cdot \cos \theta$ and the vertical component: $T \cdot \sin \theta$ as shown in Figure 56.

Figure 56: Components of the tension T

Hence,

$$T \cdot \cos \theta = T_0 \quad (1) \quad \text{et} \quad T \cdot \sin \theta = wx \quad (2).$$

If we divide (2) par (1), we get:

$$\tan \theta = \frac{w}{T_0} x \quad (3)$$

Because $\tan \theta$ est the gradient of the tangent at P , we have got:

$$\frac{dy}{dx} = \frac{w}{T_0} x \quad (4)$$

Where $\frac{dy}{dx}$ is the derivative of the function representing the shape of the suspension cable. By solving this simple differential equation, we obtain:

$$y = \frac{w}{2T_0}x^2 + C \quad (5)$$

To determine the constant C , we place the vertex at $V=(0, 0)$. We have then: $y=0$ when $x=0$ and thus $C=0$. We obtain a particular solution for our differential equation:

$$y = \frac{w}{2T_0}x^2 \quad (6).$$

This rather simplified excursion into physics allowed us to show that under the above-mentioned circumstances a suspension cable can be modeled using a quadratic function. From now on, we just need to find another point and we can easily determine the equation of the parabola describing the shape of the suspension cable. The span between the two towers of the Golden Gate Bridge is 1280m and the height of the tower from the deck of the bridge is 152m. We place the vertex at $V=(0, 0)$ and $A=(-640, 152)$ at the top of the left tower and $A'=(640, 152)$ corresponding to the top of the right tower based on the dimensions of the bridge. Using GeoGebra, we can quickly determine the equation of the parabola (Figure 57) passing through these 3 points and we obtain:

$$y = 0.00037x^2 \quad (6).$$

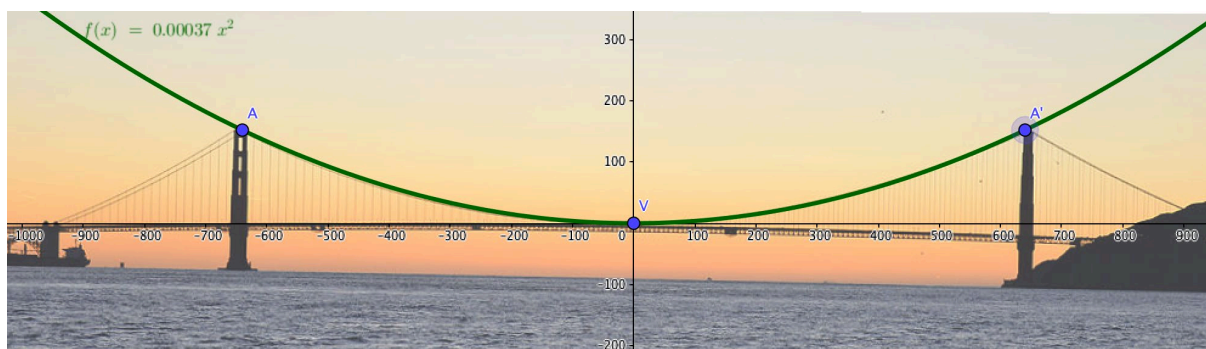


Figure 57: The parabola superposed on the Golden Gate Bridge suspension cable

1.2 Conclusion

This is rather a simplified model, but sufficient to demonstrate the complexity of conducting a modeling exploration for a SL mathematics student. It is important to state that solid knowledge on vector analysis and calculus and their applications in physics would be necessary to justify a quadratic model of a suspension bridge. We now describe Rachel's profile to better explain the difficulties that she encountered during her exploration.

2 Rachel's profile

Overall, Rachel is a rather scholar and hard-working student, coming from a culture where teaching and learning mathematics is associated with repetitive exercises and calculations.

She describes her level of mathematics as follows: *So basically, I don't know if I am good at it or is it just because I know the basics, I don't know if I am logical.* She herself prefers the calculative side of mathematics because: *it's not easy for me to visualize and apply math in the real world.* As for choosing the topic of her exploration, she says:

So first of all, I was thinking to do something in statistics only but then I was not able to get an exploration in statistics that has standard level, I mean that was standard level. So, I thought I would switch. I'll do modeling of the San Francisco bridge using functions and because I have never done functions, doing this might help me you know...

Thus, she thought that the exploration could help her better understand the concept of a function. Regarding the exploration itself, she found it stressful but also fun because:

[...] you can do it on your own and it's not gonna come on the exams so it's something apart from the curriculum and it is interesting, so you get to know other things, like it's not from the textbook. So, you don't have that pressure that you need to remember it for a long time. Like you can positively learn it. Like genuinely... because you are genuinely interested in that, it will actually get into your head.

The justification of the quadratic model requires knowledge on vectors and its application in physics and differential equations and modeling using a catenary curve would lead hyperbolic functions, which were not available in the institutional milieu and will necessarily require the use of media. On top of that, Rachel's particularity is that she missed the introductory courses on functions, including the quadratic function, so the cognitive milieu was rather potential that activated. The teacher suggested to approach this modeling problem was to try to superpose a curve on a photo of the bridge. This raises, by the way, questions of what means of validation the student could use in order to accept/reject the quadratic model or the catenary model and it seems that the only way was perception.

3 Q-A level analysis of Rachel's ME

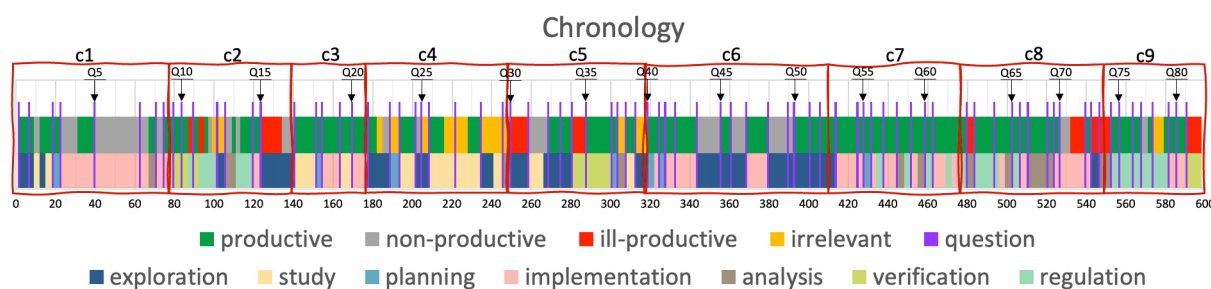
The generating question Q of Rachel's ME is: What form does the suspension cable of the Golden Gate Bridge take? The overall chronology (597 E-units) contains 81 QA-units (which corresponds to the number of questions) that were further refined into 113 episodes. Within this ME five main themes were identified:

1. Methodology (23 episodes)
2. Exploration of the quadratic model (49 episodes)
3. Choice of model (15 episodes)
4. Exploration of the catenary model (5 episodes)
5. Writing the report (21 episodes)

In the following parts we examine first the chronogenesis of the ME and then conduct an analysis of taking into account the productivity of the identified episodes.

3.1 The chronogenesis of Rachel's ME

We now further investigate the chronogenesis of the ME in respect to the productivity and the type of episode to characterize conditions in which inquiry processes unfold. In order to do so we split the whole chronology into 9 clusters in respect to the student's main activity and the similarity of the pursued objectives as shown on Graphics 12.



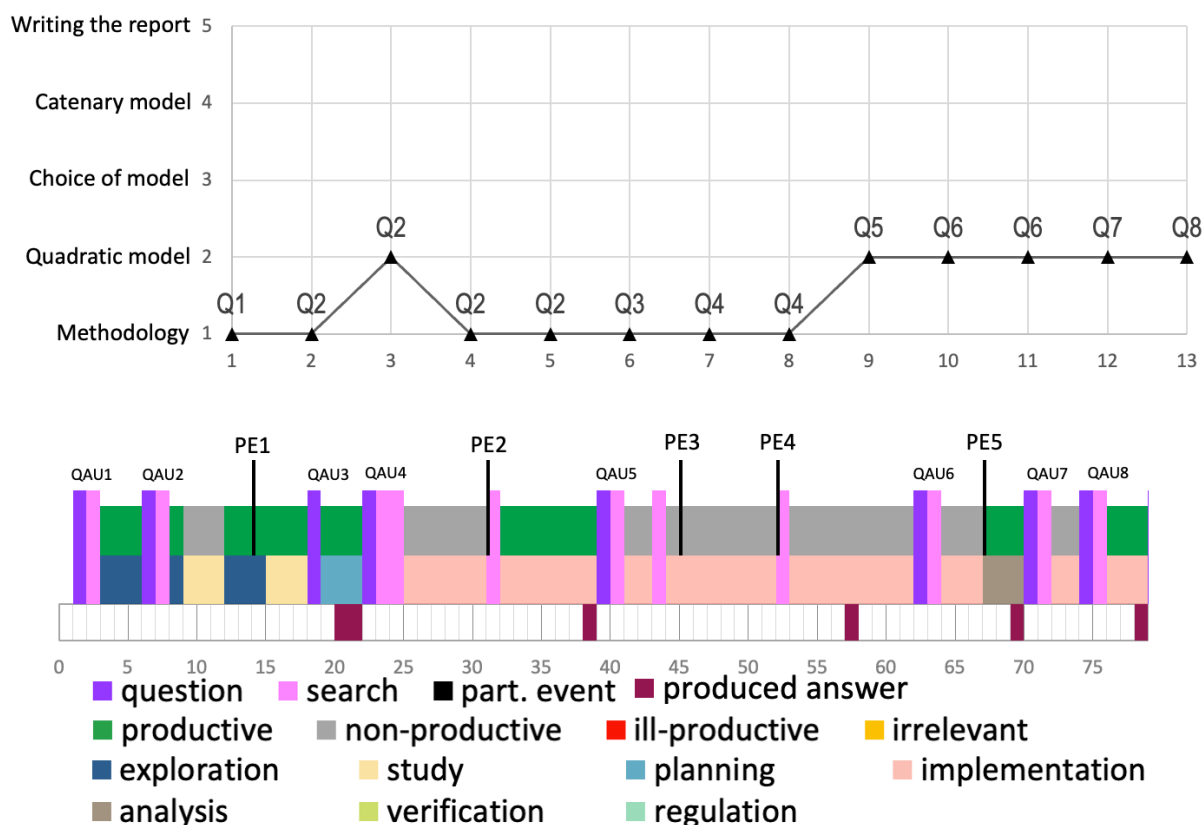
Graphics 12: Chronology of Rachel's ME split into clusters

It also seems important to have a word on the type of the graphical representations that we will be using in this part. For each cluster, we will provide its visual representation using two graphical representations as shown on Graphics 13 (example of Cluster 1). The top part of the graphics gives a detailed view of how the questions and episodes in the given cluster are spread around the identified themes. The vertical axis corresponds to a given theme (1: methodology, 2: quadratic model, 3: choice of the model, 4: catenary model and 5: writing the report) and the horizontal axis is in terms of episodes. For example, the point Q2 (3, 2) corresponds to QA-unit 2 and episode 3 in theme 2: quadratic model (QAU2_Ep3_Th2). This graphical representation enables us to see the spread of the chronology of the ME into the themes and to easily spot occasional short “trips” to a different theme. The bottom part of Graphics 13 provides the chronology of the given cluster including a new question and search, a particular event (PE) and a produced answer, productivity and type of episodes. The scale of the horizontal axis is in terms of E-units.

3.1.1 Cluster 1: Search for a method (QAU1—QAU8)

Cluster 1 is composed of 8 QA-units and 13 episodes spread in between two themes: 1: methodology and 2: quadratic model. This cluster can be characterized by research for a method (QAU1-2) containing episodes of exploration and study, which resulted in planning (QAU3) and a series of implementation episodes (QAU4-QAU8). In addition, we identified 5 particular events and 6 produced answers as shown on Graphics 13.

Cluster 1



Graphics 13: Cluster 1 of Rachel's ME

The QAU 1 was reconstructed a posteriori and contains a productive episode of exploration that resulted in creating a Google document (GD1.1 Math sites) used to copy links of media found during the exploration.

The QAU 2 started with a rather general question: “Q2—How to model a bridge using functions?” The exploration episode (QAU2_Ep2_Th1) resulted in a short trip into the quadratic model (non-productive study episode QAU2_Ep3_Th2). The activated media (V2.2) showed how to determine the quadratic function of a parabolic bridge using the intercept form of a quadratic function on a textbook type of exercise (Figure 58).

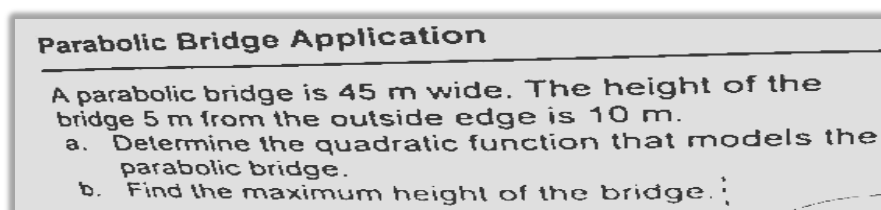


Figure 58: Extract from V2.2

We have the following hypothesis to explain its non-productivity. In V2.2 the problem was already well-formulated and proposed an application of a particular type of task. However, it appeared within a QAU devoted to methodology when the student's problem hadn't

been mathematized and therefore the student was not able to determine what information would be useful for her inquiry. In QAU 2, she returned to Theme 1 and continued with a productive exploration (QAU2_Ep4_Th1). The media (Doc 2.1) shown on Figure 59 was activated, raised Rachel's interest and resulted in a productive study episode (Q2_Ep5). The Doc 2.1 gave a protocol of how to proceed with modeling of a parabolic bridge.

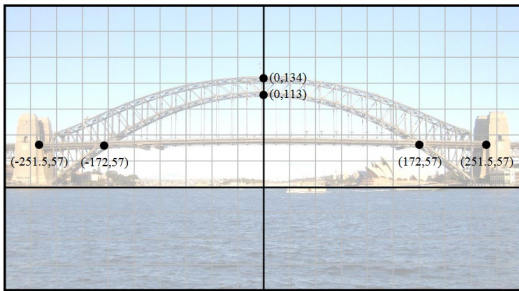
How Do Functions Shape the Sydney Harbour Bridge? We can use functions to accurately describe shapes. Consider the Sydney Harbour Bridge. The approximate coordinates of key points of the bridge have been identified:	
 Image Source: http://sydney-city.blogspot.jp/2010/06/sydney-harbour-bridge.html	
Steps to get you started: • Find a photograph of a suitable bridge. It should be taken perfectly from the side, not at an angle. • Print out the photograph. • By hand, draw x and y axes on top of your photograph. • Determine coordinates of key points of your bridge. You may have to research facts about your bridge to determine these, and then determine the scale of your photograph to calculate other points. • Use these coordinates to determine the equation of the parabola(s), and use the parabolas to determine coordinates of supports. Then recreate the bridge in Excel. • Create a report in Word explaining how you created your bridge (explain the mathematics, do <i>not</i> explain how to use Excel). Your report should <i>at least</i> recreate and explain everything you did by hand in the steps listed here.	

Figure 59: Extract of the document Doc 2.1⁵⁰

This media (marked as PE1 on Graphics 13) had indeed a significative effect on the course of the ME, probably because it was in adequacy with Q2 and accessible to the student who retrieved answers that contributed to advancing the inquiry. We have thus here a first example of Q-A dialectic identified in a series of exploration-study episodes, leading to a productive episode of planning (QUA3_Ep6_Th1) based on the retrieved answers and thus two produced correct answers. The QA dialectics can be characterized through an *exploration-study-planning* cycle which resulted in a series of implementation episodes.

The QAU4 (search for a suitable photo of the bridge) started first with a non-productive implementation (QAU4_Ep7_Th1). The gesture of “adjusting search” (marked as PE2 on Graphics 13) put an end to a series of non-productive searches in this episode and resulted

⁵⁰ <https://mrbertman.com/projects/bridge.pdf>

finally in finding a photo (Img4.7) of the Golden Gate Bridge (Figure 52) used the ME in QAU4_Ep8_Th1.

The QAUs 5–8 moved to Th2 and were devoted to the search for the dimensions of the bridge needed for the quadratic model. QAU5 was composed of a long non-productive implementation episode with two PEs. The PE3 stands for a missed key information. The search went through a figure of an architectural plan of the Golden Gate Bridge (Fgr 5.1) as shown on Figure 60. This could have been indeed a good alternative for Img 4.7 but was not retrieved. We can explain this by the fact that the student just found a photo of the Golden Gate Bridge and in this QA-unit was searching rather for the dimensions of the bridge and thus this picture wasn't activated.

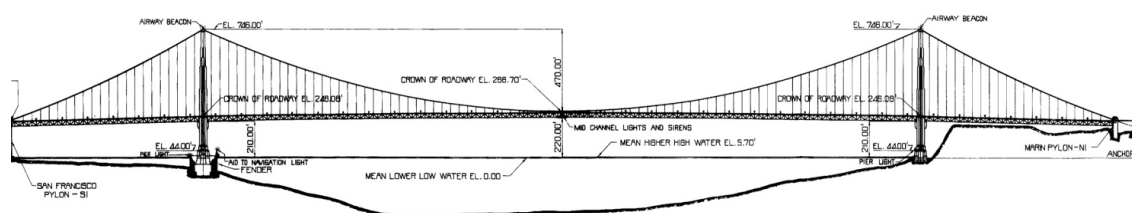


Figure 60: Architectural plan of the Golden Gate Bridge (Fgr5.1)

Despite the “adjusting search” gesture (PE4), the search remained non-productive, the difficulty consisted of not being able to use the appropriate key words in the search line. Several media containing the dimensions of the bridge were activated but no information was retrieved at this stage. This difficulty was overcome during QAU6 thanks to the E-unit Img 6.1 (marked as PE5 on Graphics 13) showing the tower of the bridge (Figure 61) and leading to an analysis episode (QAU6_Ep11) when the student suddenly realized what data to look for which led to finding the height of the towers (Dt 8.1) in QAU8.

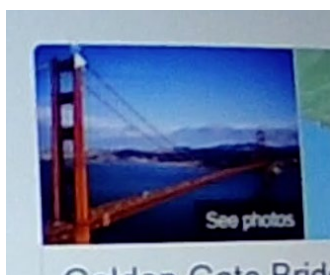


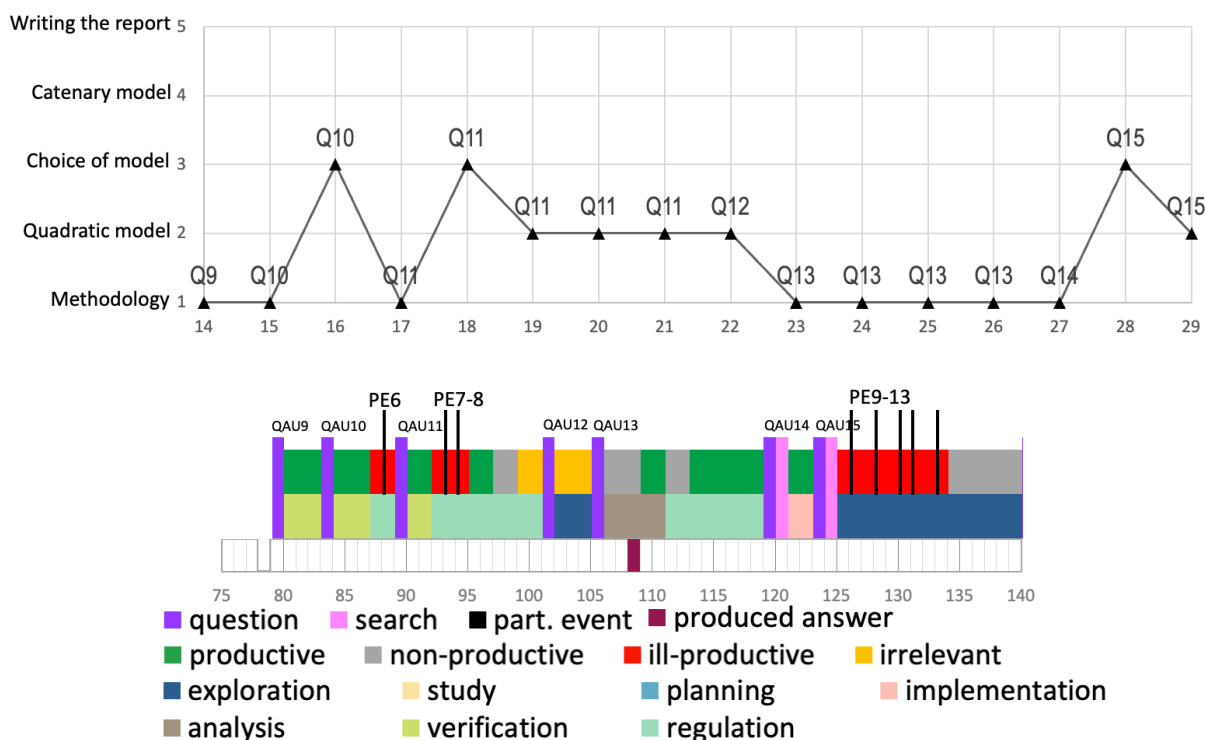
Figure 61: The tower of the bridge (Img6.1)

In this cluster, globally productive, we saw a search for methodology of the ME resulting in finding and activating an important media that shaped the direction of the ME, opened new paths and became a source of new questions. In other words, the inquiry advanced through a functioning Q-A dialectic. We also saw two moments when the student came across relevant media but didn't retrieve any information because this one wasn't within the scope of her current search or question. We will further analyze these particular moments and characterize the conditions in which they appear in terms of Me-Mi dialectic later in the mesogenesis section.

3.1.2 Cluster 2: Interview with the teacher (QAU9—QAU15)

Cluster 2 was mostly situated in the classroom and started with an exchange of the student with the researcher and the teacher (QAU9-14) and ended with Rachel's individual work (QAU15). The spread of the questions over the themes is shown on Graphics 14. Note that this is the first time in the ME that episodes concerning theme 3 (Choice of the model) appear. QAU10, 11 and 15 spread over several themes, which means that within the same question or search, the student came across media that were not necessarily linked to the question. We suppose that this might be an indicator a rather chaotic exploration without pursuing a fixed objective.

Cluster 2



Graphics 14: Cluster 2

In QAU 9, Rachel approached the researcher and the teacher to verify the suitability of the methodology proposed in Doc 2.1 and in QAU10 she raised a question concerning the skewness of the found Golden Gate Bridge photo. Indeed, it was not a perfect side view photo (as suggested in Doc 2.1) because it was taken from an angle and the picture was thus a bit skewed. We identified an episode of a productive regulation (QAU10_Ep15_Th1) pointing out the skewness of the picture and a suggestion to find the dimensions of the bridge. The discussion moved then the Theme 3 and was followed by an episode of ill-productive regulation (QAU10_Ep16_Th3) concerning the type of curve formed by a suspension cable of a bridge. Indeed, the following answer (*A*10.3 It is not a parabola, ...*

but it doesn't matter, you can try to find a parabola that fits, it is probably symmetrical right here...), marked as PE6 on Graphics 14, was provided by the teacher. This answer came in within a QAU dealing with the skewness of the picture and the question concerning the choice of a model was not raised by the student. On the contrary, as at this stage of the inquiry the student was still trying to specify the theme and the methodology of the exploration, the second part of this answer raised a new question (*Q11 Could my research question be: "is the Golden Gate Bridge symmetrical?"*). The teacher brought it back to modeling (*A*11.1 No you wanna model it*) and provided two answers concerning the choice of the model (*A*11.2 It's not a parabola, it's a catenary curve* and *A*11.3 You can try to model it using a parabola and your conclusion could be that it's not*) moving again away from the student's question. These answers (PE6-8 on Graphics 14) had a significant effect on the inquiry and we can say that Rachel literally shaped her ME based on these answers. Within QAU 11, we also noticed that the teacher injected a lot of ideas probably because he realized that he only had a limited time to dedicate to the student individually and tried to provide as much information as possible. In QAU 12, Rachel attempted to follow up on one of the teachers' ideas (*A*12.1 You can try to plot points on the side and match them to make a parabola or a series of parabolas, every 3 points could be a different parabola*) but this resulted in an abandoned irrelevant exploration because she did not really understand what the teacher meant. In QAU 13, she pointed out the difficulty to find a better picture. The answers provided by the teacher

- *A*13.1 You should try to find another picture*
- *A*13.2 You can use that one and make it part of your reflection*
- *A*13.6 You model it and the fact that it is taken from an angle could be part of your reflection*

seemed efficient and productive at the moment but as seen later in the ME, the student would have needed more guidance with how to deal with the skewness or with reflecting on the results.

The ME moved then to individual work. QAU 14 contains a productive implementation episode (QAU14_Ep27_Th2) where Rachel searched for the missing dimensions of the bridge resulting in finding the span between the towers (Dt 14.1). QAU 15 is dedicated to exploring the catenary curve and contains two episodes. The first one, an ill-productive exploration (QAU15_Ep28_Th3), was directly affected by the teacher's input from the previous QAUs. The answer A*15.1 (Table 17) coded as PE9 (Graphics 14) was taken for granted because it confirmed A*11.2 from the teacher. On the other hand, all answers (PE 10–13 on Graphics 14) from media suggesting a quadratic model were systematically rejected as show in Table 17 (E-units 127–133).

Chronology	E-unit	Description	Status 1	Status 2
123	Q 15	Is the golden gate bridge suspension a parabola?		
124	GS 15.1	is the golden gate bridge suspension a parabola		
125	WS 15.1	Math central: online help type of website	irrelevant	activated
126	A♦ 15.1	The suspension cable is a catenary	incorrect	admitted
127	WS 15.2	Math beyond school: Suspension bridges	relevant	activated
128	A♦ 15.2	Explanation of how a catenary and a parabolic curve are formed on a bridge	correct	potential
129	R 15.1	provides an accessible explication on why a suspension bridge is a parabola	relevant	activated
130	A♦ 15.3	When the main cables are attached to the towers, the curve is a catenary	correct	potential
131	A♦ 15.4	The cable of a suspension bridge is a parabola, because the weight of the deck is equally distributed on the curve	correct	potential
132	WS 15.3	Catenary Wikipedia	relevant	activated
133	A♦ 15.5	Accessible explanation of how a catenary and a parabolic curve are formed on a bridge	correct	potential
134	Doc 15.1	PDF The main suspension cables between the towers of the Golden Gate bridge	relevant	activated
135	A♦ 15.6	Solved problem using a quadratic equation of the Golden Gate Bridge in feet	correct	admitted
136	GD 1.1	Math sites	relevant	activated
137	L 15.1	The link of Doc 15.1 is copied to GD 1.1 Math sites	relevant	admitted
138	WS 15.4	Suspension Cable Size vs. Tower Height	relevant	activated
139	A♦ 15.7	useful information about the GGB	correct	potential

Table 17: Detail of QA-unit 15

This is also an interesting phenomenon when considering the topogenesis of the ME. On the one hand, the “physical” topos of the teacher is rather limited by the institutional constraints, on the other hand, however, the answers provided by the teacher or an expert have a significant impact on the course of the inquiry. We have also noticed that the teacher injects a lot of information that is not necessarily linked to the asked questions which is not always very productive and can sometimes be even counterproductive.

The following episode, a non-productive exploration (QAU15_Ep29_Th2), brought the student to a media proposing a solved problem of the Golden Gate Bridge using a quadratic model. The link was copied to GD1.1 but was never used later. We have two hypotheses to explain this: first, the solved problem was modeled in feet and second, this media came in within a QAU dedicated to research about the catenary curve and affected by the teacher’s previous answer.

3.1.3 Cluster 3: Study of former ME’s (QAU16—QAU20)

Cluster 3 is composed of 5 QAUs and is fully dedicated to Theme 5: Writing the report as shown on Graphics 15. Since Rachel did not manage to resolve the question concerning the choice of model, she completely changed activity and moved to Theme 5. We think that the reason behind this sudden change of theme was the fact that she encountered very different and sometimes even contradictory information in her previous search on the shape of the suspension cable which she couldn’t resolve and thus decided to do something different.



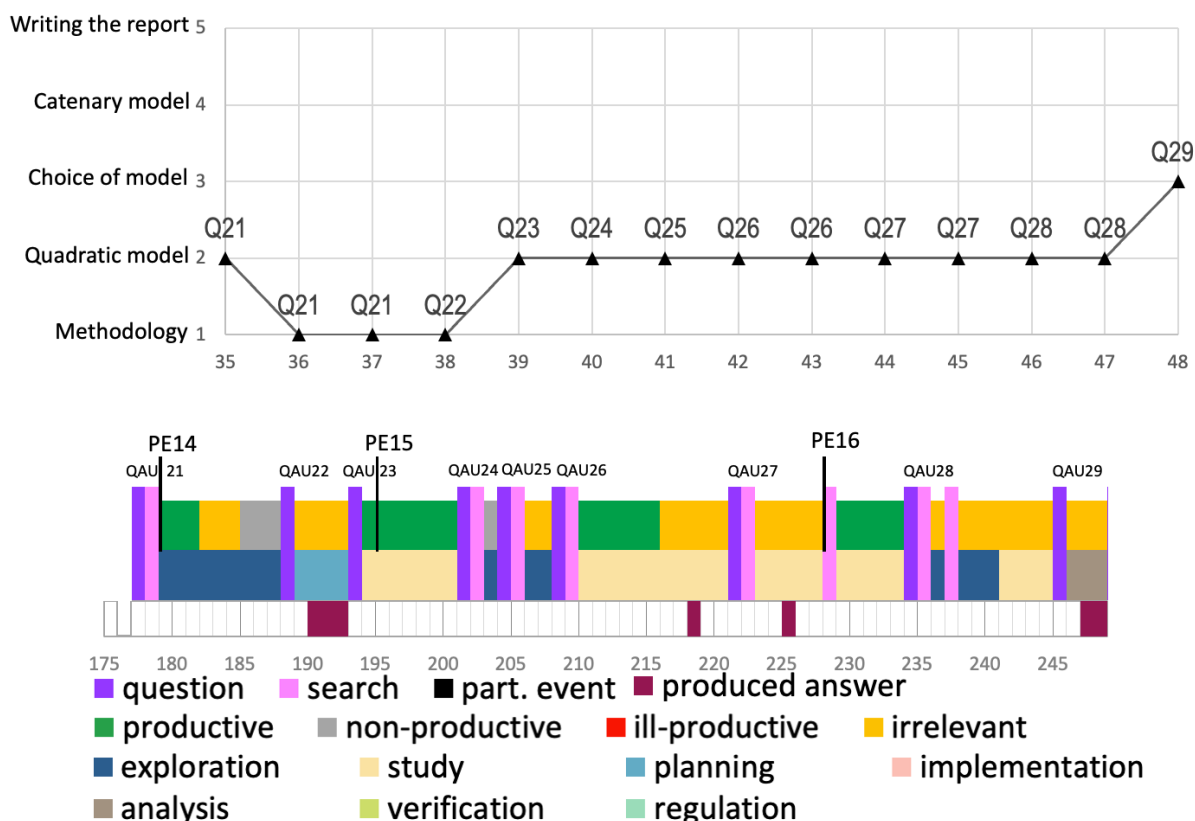
Graphics 15: Cluster 3

This cluster is overall productive and composed of study, planning and implementation episodes. In QAU16_Ep30, we have a productive study episode where Rachel studied the structure of different past MEs provided by the teacher. This resulted in a production of a plan (QAU17_Ep31) and its productive implementation (QAU18_Ep32) where she searched for factual information about the bridge to include in the introduction of the ME. QAU19_Ep33 is another study episode where she returned to the past explorations and focused on the assessment criteria. QAU20_Ep34 is again an implementation episode during which Rachel wrote the introduction of the ME.

3.1.4 Cluster 4: Exploring the quadratic model (QAU21—QAU29)

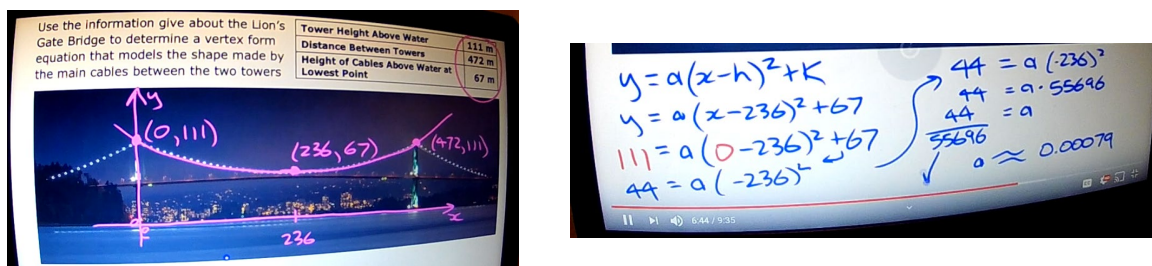
After the short excursion into theme 5 in cluster 3, cluster 4 is mostly devoted to theme 2 as seen on Graphics 16. Rachel returned to tackle the modeling of the shape of the suspension cable.

Cluster 4



Graphics 16: Cluster 4

QAU 21 was reconstructed based on the history of Google docs and links found in GD1.1 serving as document to keep track of links to media that retained the student's attention at the moment. GD1.1 contained three new links corresponding to three exploration episodes of in QAU 21. Rachel found a tutorial in the form of a video (V21.1 in QUA21_Ep35_Th2) that proposed a worked example using a quadratic model of a suspension cable of the Lion's Gate Bridge (Figure 62). As shown on the screenshots below, this media provided a perfect transferable know-how and she was able to extract the methodology of this document, personalize and apply it to her work later. Without continuing the exploration of this media at this moment, Rachel copied the link of this video to GD1.1. This video is marked as an important media (PE14) because of its effect on the ME.

Figure 62: Screenshot of the video V21.1⁵¹

The two following episodes correspond to a link to an irrelevant algebra website coded as an irrelevant exploration episode (QAU21_Ep36_Th1) and a link to a relevant document containing information on the Golden Gate bridge coded as non-productive exploration (QAU21_Ep37_Th1) because this source has never been used.

QAU 22 corresponds to an irrelevant planning episode (Table 18) and drew our attention because it also had a non-negligent effect on the ME bringing the student on an irrelevant path later on.

188	Q 22	How am I going proceed to find the curve that fits the suspension cable?
189	Std 22.1	Student's reflection: misconception of a pair function
190	A♥ 22.1	logbook: Model the curve of the suspension and find out if it is symmetrical or not
191	A♥ 22.3	logbook: Imperpose the half curve and check if it fits
192	A♥ 22.4	logbook: Then transform the equation to fit it on the other side

Table 18: Episode Q22-Ep28

Here we can see that the student made a plan to validate a curve by superposing it on the photo of the Golden Gate Bridge and attempted to use the concept of symmetry to do so. But due to her insufficient knowledge of the properties of quadratic functions, in particular the concept of a pair function, a potentially relevant plan resulted in following an irrelevant path.

After noting the plan in the logbook, Rachel returned to study V21.1, which resulted in a productive study episode (QAU23_Ep39_Th2) and provided key-answers that were important for the advancement of the ME and the work within the quadratic model. This video provided the praxis part (type of task and technique) of a praxeology.

QAUs 24 through 28 are an example of following an irrelevant path. Indeed, Rachel in order to implement the plan from QAU 22 tried to find the equation of half a curve, specifying it later as half a parabola. The result of this search was the video V26.2 Semicircles and half parabolas. This video was activated and led to an irrelevant study episode (QAU26_Ep43). Rachel watched the video several times, took notes in the logbook and copied its link into Gd1.1. The video itself made her encounter and study a new mathematical object (the square root function) and even though this particular praxeology was not directly applicable to the exploration, it nevertheless resulted in studying a new

⁵¹ <https://www.youtube.com/watch?v=CAu3LdK8xS8>

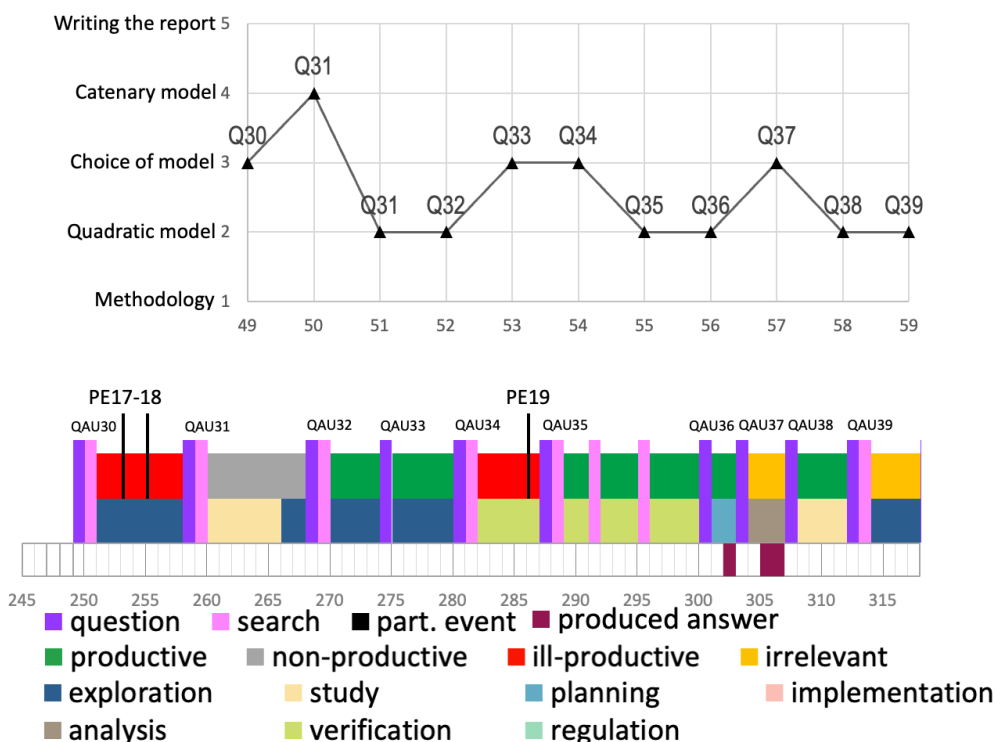
mathematical object. This sequence contained mostly irrelevant episodes (see Graphics 16 for more detail). We also identified two productive study episodes: QAU26_Ep42 and QAU27_Ep_45. Indeed, even though, the question and the search were focused on finding the equation of a half-parabola, Rachel activated two videos that contained again the praxis part of a praxeology in theme 2 (Quadratic model). The first video (V26.1) from Ep42 explained the graph of a quadratic function, its vertex and the axis of symmetry. We think that this video retained the student's attention because of the notion of symmetry, however, did not provide any information about how to restrain the domain in order to graph a half-parabola only. The second video (V27.1) showed how to find the equation of a quadratic function given the vertex and a point. A link to both videos was pasted to GD1.1 but none was used later. In QAU 28, Rachel pursued the search for the general formula of a half-parabola but did not retrieve any answers from WS 28.1 explaining how to restrain the domain to obtain the graph of a half-parabola.

After this irrelevant path, Rachel initiated a new reflection, which we coded as an irrelevant analysis episode (QAU29_Ep48_Th3), pursuing the idea of using symmetry and half-parabolas to validate or invalidate the quadratic model (*A♥29.1 I will assume it to be a parabola, find the best curve and then find the half of it and transform it. If it fits in perfectly, then it is a parabola*). At this stage of the ME, she had the necessary praxeological equipment (Doc2.1, Img4.7, V21.1 and V27.1) to be able to find the equation of a quadratic function using a point and vertex but is still missing the dimensions of the bridge to determine the coordinates of the vertex and the points corresponding to the top of the towers.

3.1.5 Cluster 5: (Re)Questioning the shape of the suspension cable (QAU30—QAU39)

After pursuing the irrelevant path about half-parabolas, in cluster 5, Rachel returned to questioning the shape of the suspension cable and her activity can be characterized as chaotic navigating between themes 2, 3 and 4 and represents a rather large variety of episodes as shown on Graphics 17.

Cluster 5



Graphics 17: Cluster 5

In QAU 30 (an ill-productive exploration), we can see again the influence of the teacher's answer concerning the shape of the suspension cable. Rachel came across two media (PE17-18), one stating that the suspension cable is a catenary and the other claiming that suspension bridges are rather parabolas and explaining why. This QAU ends by copying the answer and accepting that the suspension cable is a catenary.

QAU 31 contains a non-productive study episode (QAU31_Ep50_Th4) of two media concerning the catenary curve: its definition and physics model, which were rather inaccessible to the student. The link to this media was copied to GD1.1 but never used later. Then we have a non-productive exploration episode (QAU31_Ep51_Th2) coming across a website explaining how to graph a half-parabola by restraining the domain of the function. Again, we can see that this information came within a QAU exploring and studying the catenary curve and remained thus unexploited.

QAU32 and 33 consist each of a productive exploration episode. The question (Q32 *Do parabolas have always quadratic functions?*) reveals Rachel's gaps in the knowledge of quadratic functions and explains in a certain way the inaccessibility of the catenary model. In QAU 33, she kept on questioning the shape of the suspension cable and returned to two previously found videos from GD1.1 and watched again V21.1 proposing a quadratic model of the Lion's gate bridge.

In QAU34, we have an ill-productive verification episode (QAU34_Ep54) where she questioned whether the Lion's Gate Bridge formed a parabola. She activated WS 34.1 (a

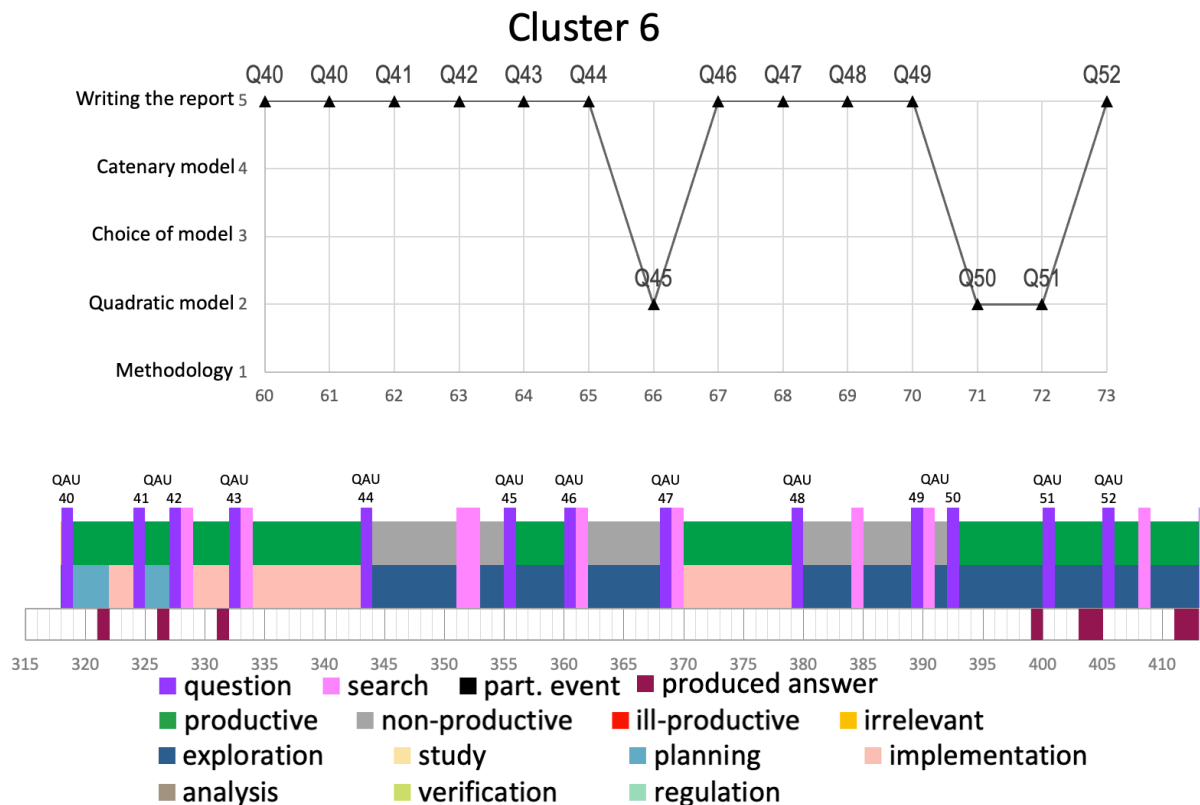
homework help blog) and retrieved the answer A*34.2 *The cables of most suspension bridges form a catenary* marked as PE19 on Graphics 17. This website also contained the general equation of a catenary in terms of e but this answer was not activated. It is interesting to note that WS 34.1 also contained the solution using the quadratic model of the Lion's Gate Bridge but was not activated either. In QAU35, Rachel continued to explore and to verify whether parabolas are always quadratic and came across several media providing basic information about quadratic functions and their graphs.

QAU 36 provides a plan (A*36.1 *logbook: I will assume the full curve to be parabolic and model it using a quadratic function*) probably because she has got the necessary praxeological equipment for, even though she had accepted that the shape of a suspension cable is a catenary. In QAU 37, Rachel analyzed the situation and proposed a reflection on how to validate or reject the model based on an erroneous application of symmetry. This plan seems to put in place the teacher's advice from QAU10 (A*10.3 *It is not a parabola, ... but it doesn't matter, you can try to find a parabola that fits, it is probably symmetrical right there*).

The QAU 38 is a productive study episode where the student watched V21.1 again and in the QAU 39, Rachel returned to WS 28.1 *How to calculate half parabolic curve?* and copied its link to GD1.1.

3.1.6 Cluster 6: Annotating a picture (QAU40—QAU52)

Cluster 6 is mostly dedicated to theme 5 with three occasional trips to theme 2 as shown on Graphics 18. The main objective of this cluster was to annotate the picture of the Golden Gate bridge and place it on the coordinate plane.



Graphics 18: Cluster 6

It is rather a long cluster because it took some time to Rachel before finding the right software. In QAU 40–43 and 47, she searched and finally found the missing dimensions of the bridge. In QAU 44–46, she explored the Desmos graphing software attempting to annotate the picture of the Golden Gate Bridge. QAU 45 brought her to explore the effects of the parameters a , b , c on the shape of a parabola. Being unsuccessful with Desmos, Rachel attempted to annotate the picture in Google Docs (QAU 48) and again in Desmos (QAU 49) without any success. In QAUs 50 and 51, she explored again the effects of the parameters a , b and c on the shape of the parabola. In QAU 52, she found the right software (OneNote) and produced an annotated picture of the Golden Gate Bridge (Figure 63).



Figure 63: The annotated picture of the Golden Gate Bridge

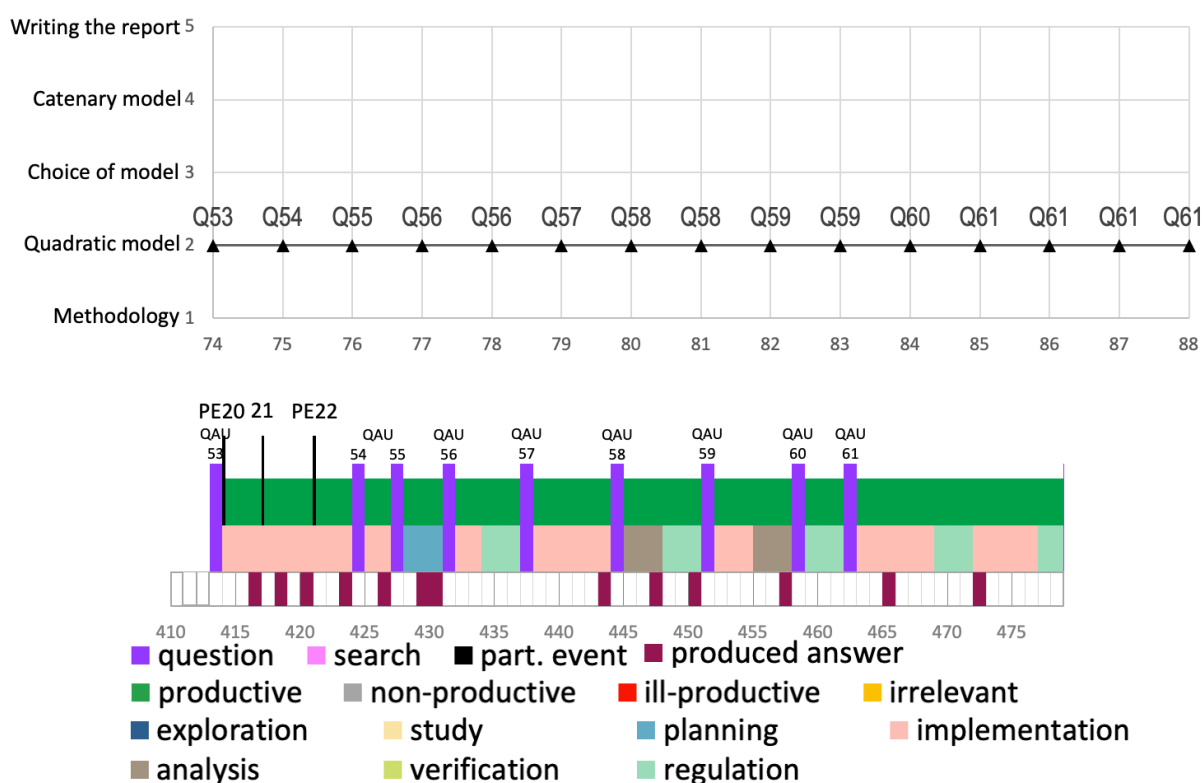
Even though the activity characterizing this cluster was essentially non-mathematical, the exploration of a graphing software enabled the student to encounter an experimental

dimension of mathematics when testing the effects of the parameters on the shape of the graph.

3.1.7 Cluster 7: Analytical solution in the quadratic model (QAU53—QAU 61)

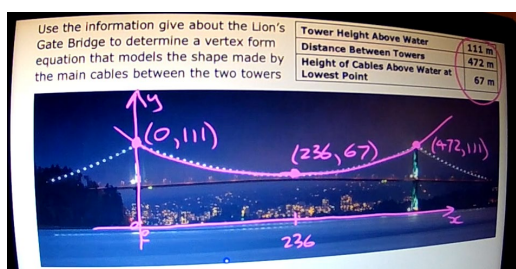
Cluster 7 is fully situated in theme 2 and provides examples of a continuous productive activity of the student and efficient regulations.

Cluster 7



Graphics 19: Cluster 7

In QAU 53, Rachel used a tutorial video (V21.1 found in QAU21_Ep35_Th2) that proposed a quadratic modeling of a suspension cable of the Lion's Gate Bridge (Figure 64). She was able to extract the methodology of this document, personalize and apply it to her work.



$y = a(x-h)^2 + k$

$y = a(x-236)^2 + 67$

$111 = a(0-236)^2 + 67$

$44 = a(-236)^2$

$44 = a \cdot 55696$

$44 = a$

$a \approx 0.00079$

Figure 64: Screenshot of the video V20.1⁵²

If we compare these documents with the productions found in the student's logbook (Figure 65) we can see that she succeeded in extracting relevant information from these two media, integrating it into M_I , adapting it and applying it to the problem of her exploration.

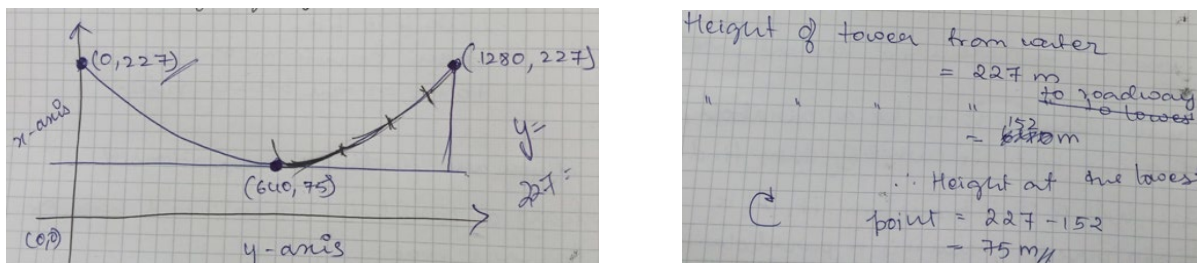


Figure 65: Quadratic model of the Golden Gate Bridge

$$\begin{aligned}
 y &= a(x-h)^2 + k \\
 y &= a(x-640)^2 + 75 \\
 \therefore y &= a(x-640)^2 + 75 \\
 \text{now, for the equation value for } a \text{ we take } (0, 227) \\
 227 &= a(0-640)^2 + 75 \\
 227 &= a(409600) + 75 \\
 152 &= a(409600) \\
 a &= 0.00036 \\
 \therefore \text{equation } \Rightarrow y &= a(x-640)^2 + 75 \\
 y &= 0.00036(x-640)^2 + 75
 \end{aligned}$$

Figure 66: Finding the equation of the quadratic function

This production, taken from the logbook, illustrates the question-answer and Me-Mi dialectics that led to the production of a correct answer and to building a situational knowledge related to quadratic functions.

The QAU 54 contains the vertex form of the quadratic equation of the suspension cable with a typing error in the a parameter and in the QAU 55, Rachel produced a plan on how to validate the found answer.

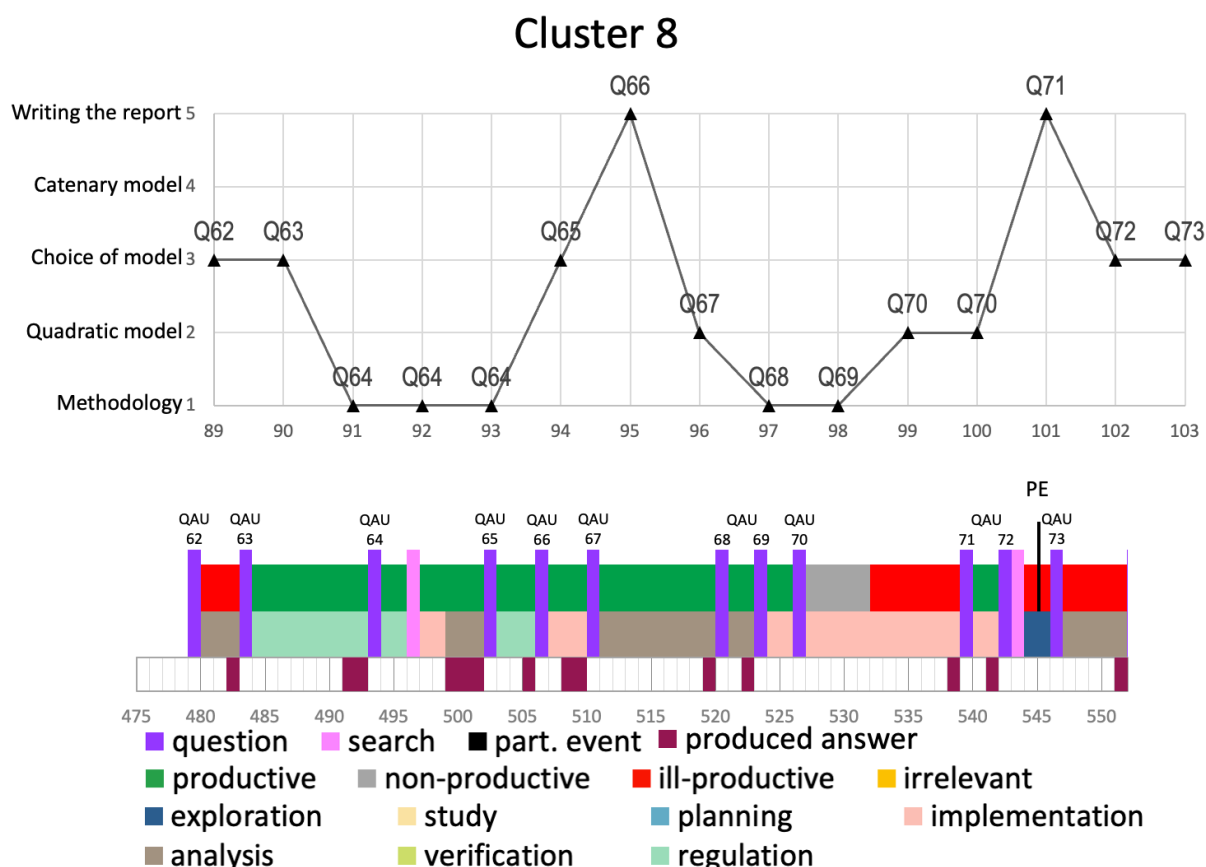
QAUs 56–61 took place in the classroom in the presence of the researcher and provide examples of several efficient regulations. First (QAU 56), Rachel received some help with placing the picture of the Golden Gate Bridge on the coordinate plane in Desmos. Thus, this created a functioning experimental milieu. Then, she graphed the found function with the typing error and obtained a graph of a narrow parabola that did not fit the suspension

⁵² <https://www.youtube.com/watch?v=CAu3LdK8xS8>

cable at all (QAU 57). This typing error was corrected in QAUs 58–61 and the student finally obtained a graph of a parabola superposing approximately the suspension cable.

3.1.8 Cluster 8: Rejection of the quadratic model (QAU62—QAU73)

Cluster 8 shows the last activity on the first draft and is mostly about deciding whether the shape of the suspension cable is a parabola or not and writing the report. We can also see that the chronology is zigzagging among all 5 themes (Graphics 20).



Graphics 20: Cluster 8

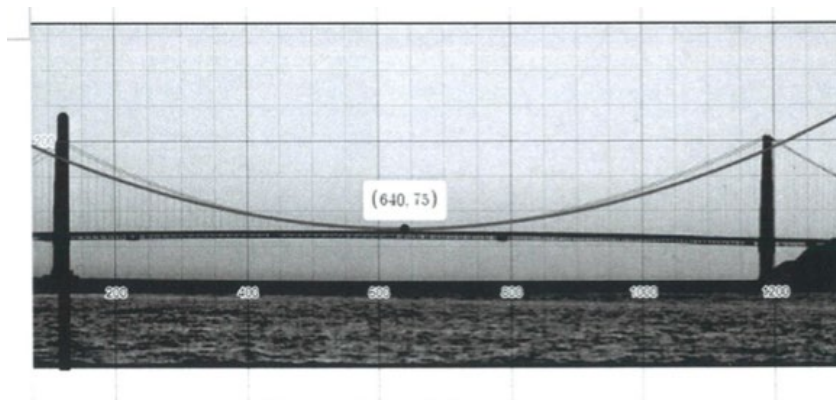
In QAU 62 (ill-productive analysis), Rachel decided to reject the quadratic model, because it did not fit perfectly on the picture. This led to an exchange with the researcher (QAUs 63–65) about the validation of the model. The skewness of the picture was questioned again and resulted in a new plan (*A♥64.3 I think I can deal with the skewness on my phone*). Rachel came to the conclusion that she cannot reject the quadratic model based on the idea of symmetry because the picture was skewed (*A♥65.1 So I couldn't like come to a conclusion because of the picture, so by doing that I would be stating that the bridge is not symmetrical but it is actually symmetrical*).

QAU 66 was dedicated to writing the introduction and typing the solution using the quadratic model.

In QAU 67, she explained why the parabola is slightly off on the right side pointing out the skewness of the picture.

QAUs 68 and 69 were dedicated to correcting the skewness of the picture which was partially corrected using a photo software with a help of a friend.

In the QAU 70, she superposed the parabola on the “unskewed” picture as shown in Figure 67.



The curve on the non-skewed picture

Figure 67: Parabola superimposed on the unskewed picture

However, the picture of the bridge was not inserted to scale in Desmos and resulted in providing incorrect feedback (W70.1 *The parabola is off at both towers and passes under the top of the towers*). Rachel interpreted it as follows:

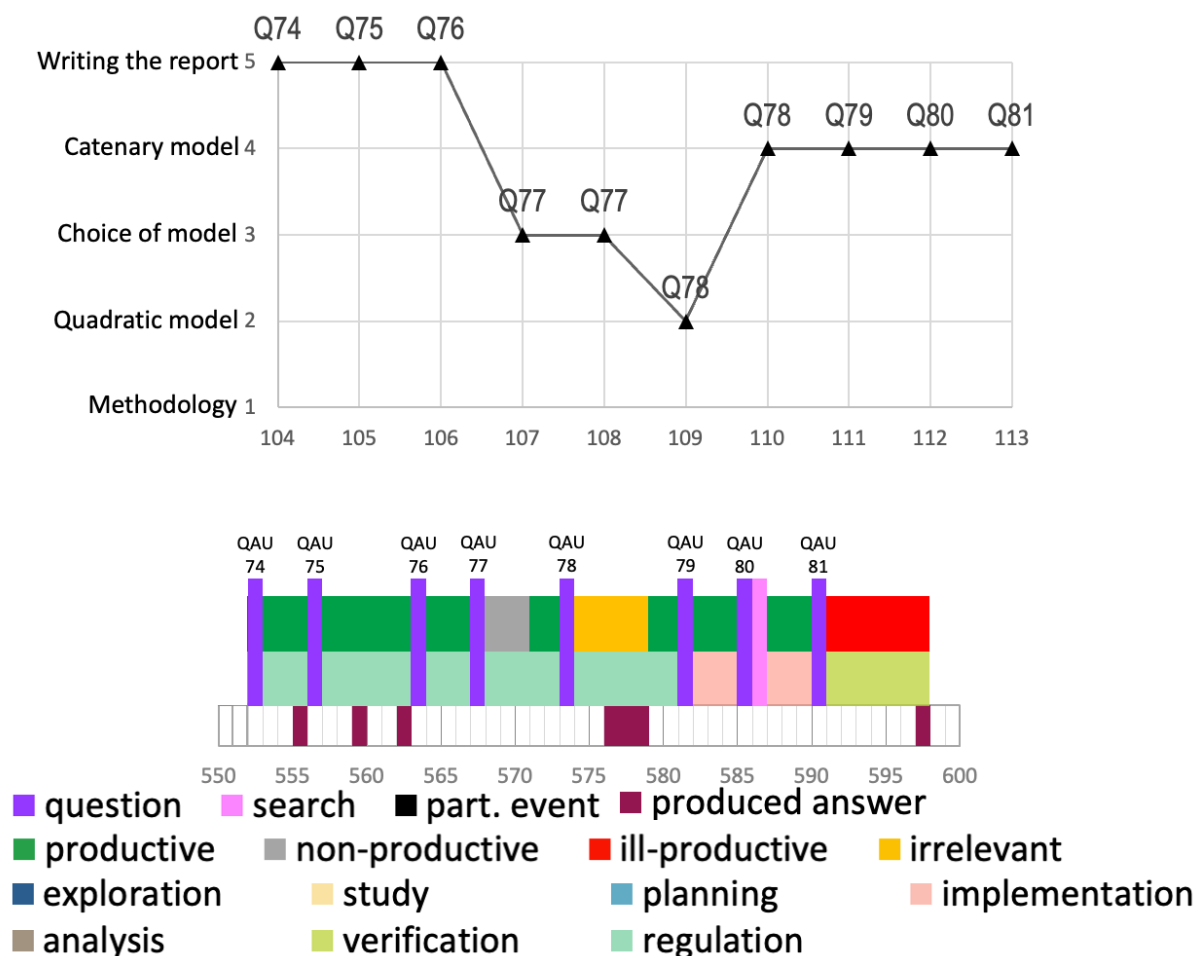
A♥70.1 As we can see, the parabolic curve does not go through the curve properly and this indicates that the suspension cables of the Golden Gate Bridge do not form a parabola between the two towers.

Rachel rejected definitively the quadratic model and conducted a short research on catenaries in QAUs 72 and 73 to justify her answer. The paradox in this situation is that she actually correctly interpreted the milieu's feedback putting together several pieces of wrong information.

3.1.9 Cluster 9: Implementing teacher's feedback (QAU74—QAU81)

Cluster 9 contains the implementation of the teacher's written feedback on the first draft.

Cluster 9



Graphics 21: Cluster 9

QAUs 74–76 provided feedback and its implementation concerning Communication, Mathematical representation and Personal engagement and are rather successful.

Then in QAU 78 we have an irrelevant implementation where the student just added some point on the curve as asked by the teacher.

In QAUs 79 and 80, she followed the teacher's feedback (*A*78.2 You could perhaps find the general form of a catenary curve and try that*) on Use of mathematics and conducted further research. She found an equation of the catenary with the correct a parameter on a website and copied it to Desmos.

In the QAU 81, Rachel validated the catenary model based on perception and an incorrect experimental milieu (the picture of the bridge wasn't placed correctly on the coordinate plane and the catenary curve was superposed on a very small picture) as shown on Figure 68 and concluded:

*A*81.1* As we can see the catenary function fits the bridge perfectly which indicates the suspension of the bridge from a catenary curve.

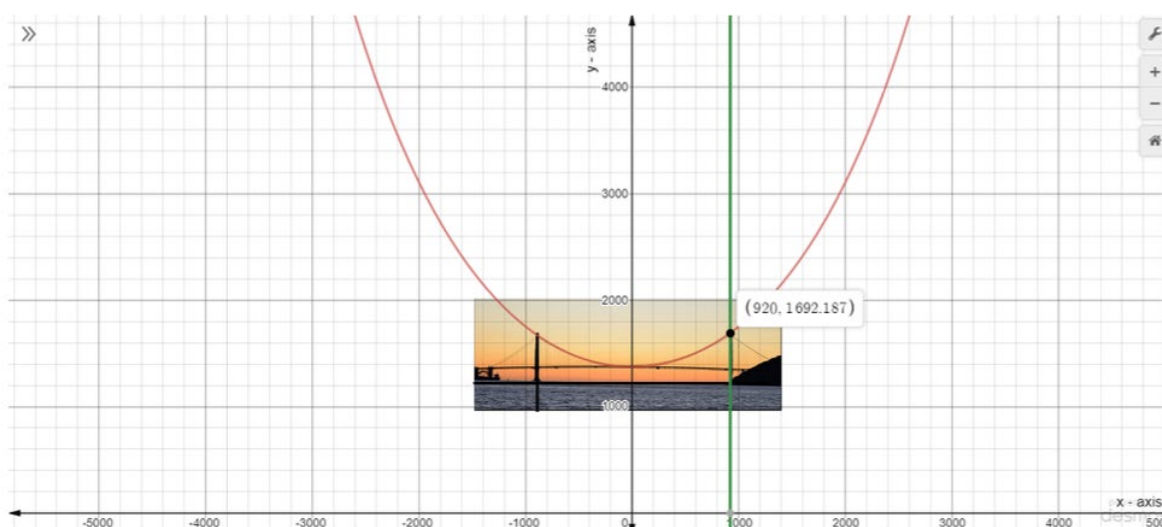


Figure 68: The validation of the catenary model based on perception

This conclusion seems very hasty but the work in cluster 9 was done over a short period of time because of the upcoming deadline and the student was simply running out of time.

3.1.10 Conclusion of the chronogenesis of Rachel's ME

The chronogenesis reveals the presence of a rich mathematical but also non-mathematical activity. The advancement of the inquiry can be characterized by cycles of exploration-study episodes resulting in cycles of (planning)-implementation episodes. These were generally productive when the media contained the praxis part of a praxeology needed to answer the question.

We have also noticed that the main difficulty for the student concerned the choice of model. In addition to the fact that it is a difficulty inherent to modeling, we identified three other phenomena that reinforced this difficulty. The first one is a series of unhappy teacher's regulations influenced by the institutional constraints and the nature of this particular ME. As already mentioned, even though the teacher's physical topos was relatively limited, the provided answers had an important impact on the ME and led to a systematic rejection of media opposing the teacher's information. Secondly, the difficulty with choosing the right model was probably also reinforced by the lack of study episodes. Despite the weight of the teacher's regulations, the student attempted to verify this information and came across media explaining the circumstances characteristic for each model (parabola or catenary). These were, however, too theoretical and provided rather the logos part of a praxeology. And thirdly, the validation of the model was compromised by the organization of a malfunctioning experimental milieu and the interpretation of its feedback. We will address this issue more in detail when analyzing the mesogenesis of the ME.

Another consequence of the limited teacher's physical topos is what we can call "machine gun" phenomenon concerning the teacher's interventions: the teacher provides as much

information as possible, yet, the answers are not always clearly related to the questions of the student. We have noticed that the student makes a limited use of these answers. The same happens when the student comes across some relevant media that is not in adequacy with her question.

We will now proceed to a more general analysis of the ME at the QA-level to confirm these intuitions from the chronogenesis.

3.2 Analysis in terms of QA-units

We can notice that the majority of the student's work concerns the theme 2: Quadratic model which represents 43% of the total number of episodes. We then have about the same proportion of episodes devoted to theme 1: methodology (20%) and theme 5: writing the report (19%), finally 23% to the theme 3: choice of the model and only 4% to the theme 4: catenary model. Indeed, the quadratic model is characterized by a rich activity and seems to represent a certain comfort zone for the student. We have then a fair number of episodes devoted to the methodology, the choice of a model and to writing the report but only very little activity in the catenary model. The number of episodes is decreasing as the distance from the student's comfort zone increases. We are going to further explore this phenomenon by crossing the productivity and the type of episodes versus the theme involved.

3.2.1 Analysis according to theme

In this part, we propose to examine the activity of the student analyzing the productivity of the episodes (Table 19) in respect to theme.

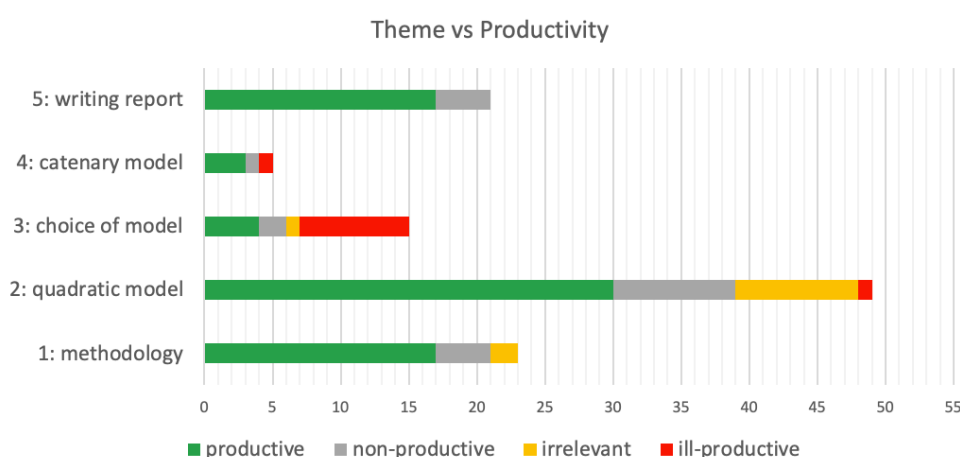
Theme	productive	non-productive	irrelevant	ill-productive	Total
1: methodology	17	4	2	0	23
2: quadratic model	30	9	9	1	49
3: choice of model	4	2	1	8	15
4: catenary model	3	1	0	1	5
5: writing report	17	4	0	0	21
Total	71	20	12	10	113

Table 19: Theme vs. Productivity

In the *Quadratic model*, over 60% of episodes are productive and only one is ill-productive. The irrelevant episodes represent an abandoned irrelevant path that the student took based on an irrelevant methodological choice. This confirms that the inquiry was advancing because the student was able to access most of the mathematical objects from the quadratic model. We can interpret this in terms of the student's cognitive universe which contained objects at a comparable level to the ones accessed by the student within the quadratic model.

When we move to *Methodology* and *Writing report*, there are mostly productive episodes and no ill-productive episodes. This can be explained through the fact that the expectations concerning the elaboration of the ME were understood and well assimilated by the student.

Even though, there is clearly less activity in the *Choice of model* than in the *Quadratic model*, we can still say that the student attempted to tackle the question of choosing a model. When we look at the distribution of ill-productive episodes over the whole exploration, out of the total of 10 ill-productive episodes, 8 are found in theme 3: *choice of model*. This confirms that the student encountered a lot of difficulties when determining the right model to use for the suspension cable. There is very little activity when dealing with objects from the catenary model which suggests that these objects were out of reach for the student. The Graphics 22 summarizes the distribution of episodes according to their productivity over each identified theme in the ME and thus provides a clear visualization of what we just described.



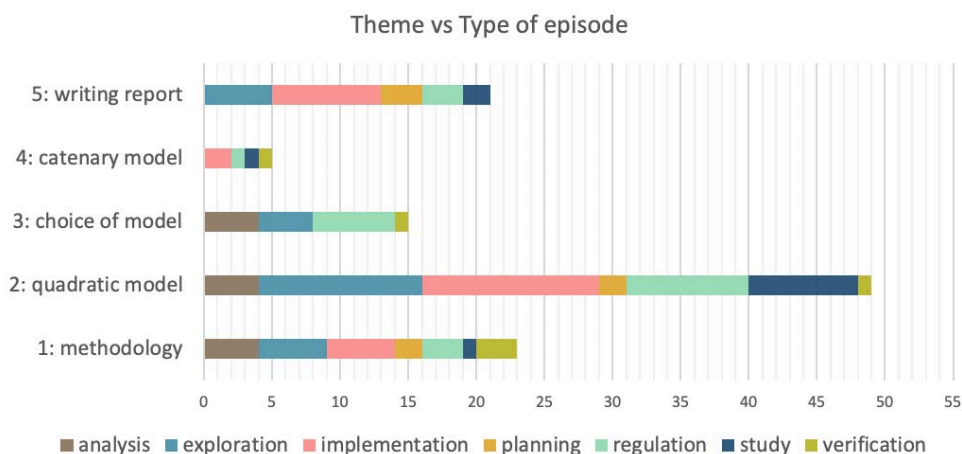
Graphics 22: Distribution of Productivity over Theme

We can interpret this phenomenon using the concept of the student's cognitive universe (CU) which is a set of objects towards which an individual has developed a stable personal relation. (Chevallard, 2003b). We think the student's CU plays an important role in how far a student will be able to conduct an inquiry. Some of the objects from the theme *Quadratic model* were already in the student's CU while others became part of it as a result of the work done on the ME. There is a clear tendency that the more the nature of the inquiry leads the student to deal with objects at a more complex level than the ones in her CU, the less activity there is and the proportion of ill-productive episodes increases.

We would also like to point out another interesting feature. When looking at the distribution of the types of episode over the different themes (Table 20 and Graphics 23) we noticed that most of the study episodes are in the *Quadratic model*, two in *Writing report*, one in *Methodology* one in the *Catenary model* and none in the *Choice of model*.

Theme	analysis	exploration	implementation	planning	regulation	study	verification	Total
1: methodology	4	5	5	2	3	1	3	23
2: quadratic model	4	12	13	2	9	8	1	49
3: choice of model	4	4	0	0	6	0	1	15
4: catenary model	0	0	2	0	1	1	1	5
5: writing report	0	5	8	3	3	2	0	21
Total	12	26	28	7	22	12	6	113

Table 20: Theme vs. Type of episode



Graphics 23: Distribution of Episodes over Theme

This reinforces the hypothesis that a certain proximity of the inquired questions with the objects in the student's CU contributes to a rich mathematical activity, creates favorable conditions for a Me-Mi dialectic to be deployed and leads eventually to construction of new situational knowledge.

3.3 Conclusion of the Q-A level analysis of Rachel's ME

The chronogenesis of the ME revealed a rich mathematical and non-mathematical activity. The inquiry advances through cycles of exploration-study episodes leading to cycles of planning-implementation episodes when the encountered objects are at a similar level of complexity as those contained in the student's CU. In this case, the productive episodes were mostly situated within Theme 2: Quadratic model. Rachel was able to activate relevant media, retrieve, process and apply a particular praxeology or reject an irrelevant one. It was not always a straightforward process and productive implementation episodes are often preceded by non-productive explorations. These episodes are nevertheless an essential part of an inquiry process and indirectly contribute to the advancement of the ME through either adjusting the keywords in the search or a better formulation of the question. We can see that she is generally assiduous in the research, raises a lot of questions and attempts to search for answers.

However, it happens that Rachel missed important information and relevant media was not activated. This happened in case of “zapping” between the different media and absence of study episodes, when the information in the media is “too far” from her CU or when erroneous or irrelevant answer is already activated in the M_I . In the last case, irrelevant or erroneous information was admitted into the inquiry milieu M_I . In this situation she rejected relevant media, while inclined to pursue irrelevant or erroneous paths. Pursuing irrelevant paths was time-consuming and generally, an external regulation was needed to put it to an end. The identified ill-productive episodes resulted from the presence of certain elements in the M_I , the difficulty to organize an appropriate experimental milieu and from an “insufficient” CU. In this case a targeted regulation of the teacher would have been necessary to change the course of this ME.

The ME leads inevitably the student beyond her cognitive universe to build new skills and acquire new knowledge. This learning seems to be closely linked to the “distance” from the student’s CU. Indeed, when Rachel encountered mathematical objects at a level comparable to the ones in her cognitive universe and when the media provides a suitable praxeology, and more specifically its praxis part, she was able to “manage” on her own with the media to which she had access and to conduct a relevant and fruitful inquiry. This conclusion is, of course, based on one case study so far but we have all reasons to think that these findings could be generalized. The QA-level analysis showed that the ME advances from question to question by progressively building and enriching an inquiry milieu. In the next part, we examine and analyze the mesogenesis of the ME by zooming on the findings from the chronogenesis and using the media-milieu level of our analysis framework.

4 M-M level analysis of Rachel’s ME

To conduct the M-M level analysis we start by considering the mesogenesis in general, then we examine the content and the evolution of the inquiry milieu M_I in terms of M-M units and finally we describe the conditions and the role of creation of the experimental milieu M_{EXP} .

4.1 The mesogenesis of Rachel’s ME

When approaching the mesogenesis of the ME, we need to consider two processes: the construction of the inquiry milieu and the organization of an appropriate experimental milieu M_{EXP} . As already mentioned, Chevallard (2008) claims that without a Me-Mi dialectic, a study and research path would be reduced to an uncritical copying of answers from the media. For Kidron and al. (2014), however, the organization of an appropriate experimental milieu M_{EXP} within the inquiry milieu M_I is not an easy affair.

The analysis of the chronogenesis enabled us to identify three patterns concerning the capacity of the student to activate a media, retrieve and process relevant information and enrich the inquiry milieu M_I :

- the media is in adequacy with the questioning at the moment it appears
- the media contains mathematical objects at a comparable level of complexity as those already contained in the student's cognitive universe
- the media in the form of videos and documents provide a type of task and a technique (the praxis part of a praxeology) that is directly applicable or easily transferable.

On the other hand, the experimental milieu M_{EXP} seems to play a crucial role in the validation of the produced answers. In the case of modeling explorations, where the mathematics or physics justification of the model is often inaccessible to secondary school students because of its level of complexity, students often make an “intelligent guess” from the functions available in their CU and test them by superposing on a picture or on the collected data based on visual perception. In our case study, this procedure was reinforced by the teacher's personal praxeology put in place in this class. The teacher was convinced that modeling explorations were the “easiest” and most accessible for the students given the institutional milieu at the moment of elaboration of the ME. On top of that, he was mainly interested by the “mathematics” part of modeling and reduced it to finding the curve of best fit analytically without considering modeling within larger context.

The mesogenesis of inquiry is a complex process and in the following analysis, in addition to describing the content of the milieu we also attempt to capture the dynamics of its evolution.

At the M-M level, we explore and analyze the interactions of the student with the media that led to the construction of the inquiry milieu M_I and to the organization of an experimental milieu M_{EXP} . To conduct this analysis, we focus on two entries: (1) the conditions of the activated milieu within the inquiry milieu and (2) the organization of an experimental milieu. For the inquiry milieu, we first present some generalities and describe its content in respect to the different characteristics of MMUs identified inductively in Rachel's ME, then we explain how we approach the activated milieu and analyze it. Finally, we attempt to depict the organization, evolution and the role of the experimental milieu M_{EXP} in the ME.

4.2 Content and evolution of the inquiry milieu

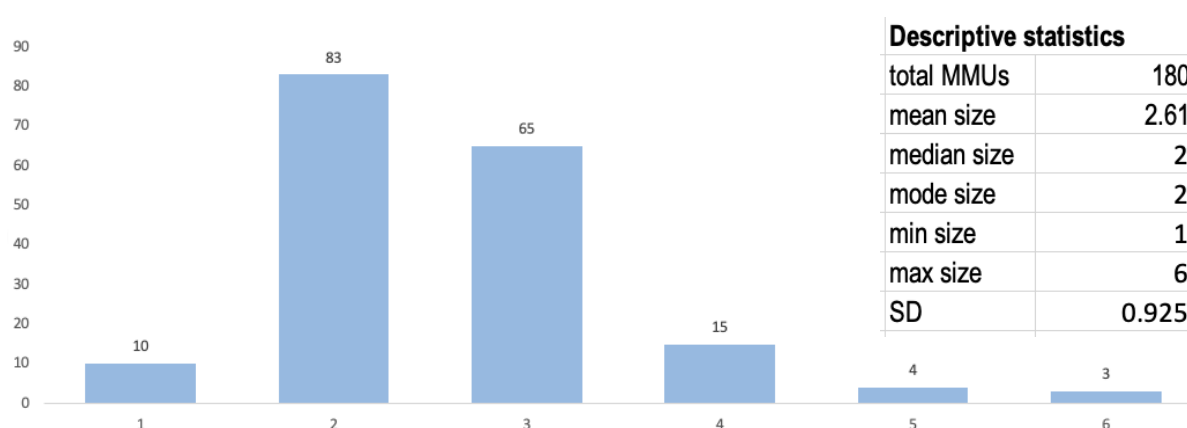
The first part of the M-M level analysis presents a global view on the experimental data in terms of MMUs in order to set up the general context before entering into a more detailed analysis. In this general approach, we provide the distribution of the MMUs according to

various criteria such as their size, type and subtype, effect on the ME, theme, and type of the processed information.

In this introductory note for the M-M level analysis, we would like to point out that in all graphics, we generally use a color code providing a visual representation of certain phenomena. Green and shades of green refer to something correct or desirable, red and shades of red are generally used for something incorrect or non-desirable and orange and shades of orange are applied for something irrelevant. Of course, when this is not possible, especially when several parameters overlap and we have to choose between two colors, we either use different colors or precise each exception from this general rule.

4.2.1 Distribution of MMUs according to size

We start with a general description of the identified media-milieu units (MMUs). As a reminder, an MMU is composed of one or more E-units and depicts the interaction of the inquirer with a media. The beginning of an MMU is marked when a new media occurs and contains generally institutional answers, data or works, and eventually produced answers. In Rachel's exploration, we identified 180 MMUs with an average size of 2.61 E-units; with over 80% MMUs of size 2 or 3. This means that in about 80% of the interactions with a media, she retrieved 1 or 2 answers, this could, however, also be an effect of the way the data was coded. A detail of the distribution of the size of the MM-units as well as some descriptive statistics are provided on Graphics 24. We first consider the MMUs at the two ends of the bar graph to see what type of MMUs have got size 1 respectively size 5 or 6, to determine whether these are rather outliers or there is a reason for their size. On the one hand, there are 10 MMUs of size 1 that correspond to efficient media judgment MMUs (rejected media or non-activated media). This justifies the size 1 because no answer is retrieved from the media.



Graphics 24: Distribution of the MM-units according to size

On the other hand, there are 7 MMUs of size 5 or 6. The first two MMUs of size 6 were identified in Cluster 1 (Q4-Ep7-MMU7 and Q5-Ep9-MMU13) and represent a long non-

productive, yet assiduous search for a suitable picture of the Golden Gate Bridge and for the dimensions of the bridge, which explains their size. At first sight, we could think of this long and non-productive search as of an inefficient use of time but it is simply an intrinsic and inevitable part of research especially when using the Internet. In addition, this long and assiduous, yet non-productive search shows that it was difficult if not impossible to find a perfect front view picture of the Golden Gate Bridge. On the other hand, this is also what differentiates inquiry from a ready-made textbook problem.

In cluster 4, one MMU of size 5 was an important contribution to A♥ within a study episode (Q23-Ep39-MMU60) of V21.1 in the quadratic model theme. Indeed, the study of this video provided the missing praxeology (Figure 69) and enabled Rachel to find the quadratic function analytically.

193	Q 23	How do I find the equation of the suspension cable of the Golden Gate bridge?
194	GD 1.1	Math sites
195	V 21.1	The Lion's Gate Bridge... Modelling the Shape Using a Quadratic Function
196	A♦ 23.1	shows how to draw the coordinate plane on a picture of a suspension bridge, y-axis on the left tower
197	A♦ 23.2	shows how to use the vertex form of the quadratic function
198	A♦ 23.3	shows how to determine the parameter a

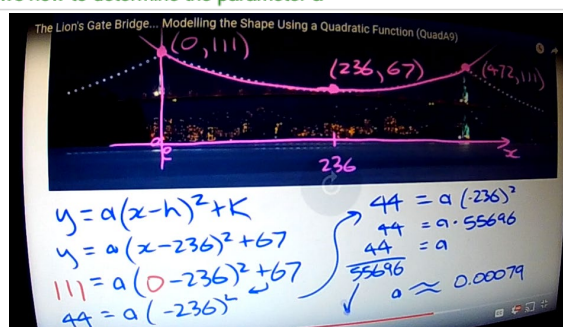


Figure 69: Detail of MMU 60 showing contained in the study episode of V21.1

Then we have got three MMUs that contributed to advancing the inquiry in cluster 8, all three leading to correct productions. The first one (Q64-Ep92-MMU155) corresponds to an analysis episode when dealing with the skewness of the picture. The following two MMUs (Q67-Ep96-MMU158 and Q70-Ep99-MMU162) correspond to creating an experimental milieu in Desmos graphing software. The last MMU (Q78-Ep109-MMU175) of size 6 from Cluster 9 corresponds to an irrelevant production Figure 70 when implementing a misinterpreted teachers' feedback. This can be interpreted as an effect of the didactic contract where Rachel just executed what the teacher asked for.

573	Q 78	What can I do to improve on the Criterion E – Use of mathematics?
574	TFB 78.1	Criterion E – Use of mathematics
575	A♦ 78.1	"Why not pick more points on the curve, you may even be able to find more dimensions"
576	A♥ 78.1	Explains how she estimated the height of the vertical cables from the picture.
577	A♥ 78.2	Image of the parabola superposing the suspension cable with the additional points (Fig5.1)
578	A♥ 78.3	Justifies her choice: "because information on the suspender cables was not very clear on the Internet"

Figure 70: Detail of an irrelevant production

Globally, these large size MMUs represent an assiduous and continuous activity either a search, a study or a production and even if some of these are non-productive at the moment, they are not only an inevitable and but also a necessary part of the maturation in every inquiry process.

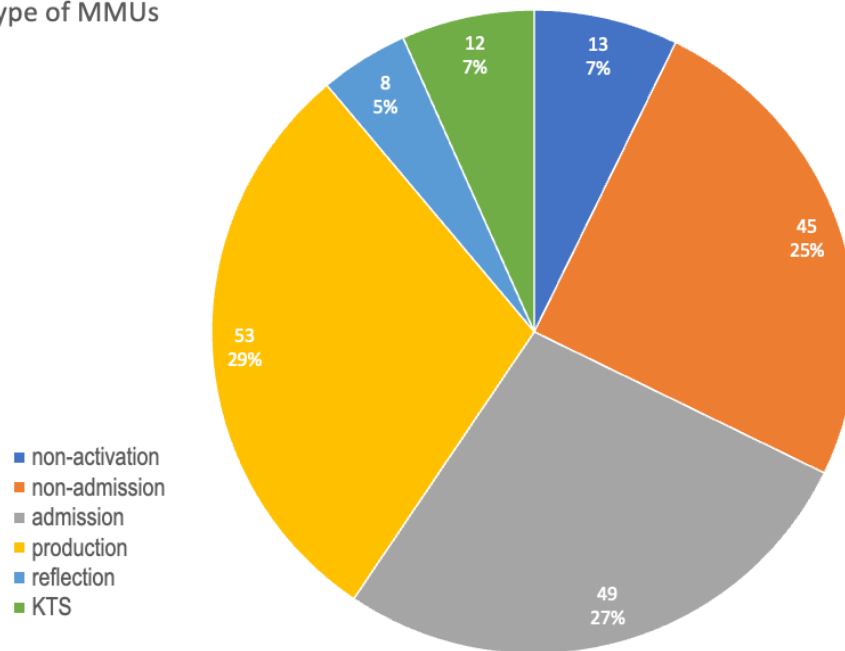
4.2.2 Types of MMUs and the effect on the ME

The MMUs are categorized based on the interaction of the student with the media into seven main types: non-activation, non-admission, admission, production, reflection and keeping track of sources (KTS). The main types are further refined into subtypes to specify the interaction with the media. From the point of view of the activated milieu, these subtypes of MMUs provide a priori indicators of the status of the activated milieu at the moment of the interaction of the student with the media. We consider two cases:

- *sufficient activated milieu*: efficient media judgment respectively elimination, relevant source, correct reflection/admission/production
- *insufficient activated milieu*: missed occasion, incorrect/irrelevant source incorrect/irrelevant reflection/admission/production)

As shown on Graphics 25, the three most represented types of MMUs in Rachel's ME are productions (29%), admissions (27%) and non-admissions (25%). The fact that more than half of the ME is composed of admission or non-admission MMUs witnesses of a rich research activity facilitated by the access to the Internet where information is available in a couple of clicks. On the other hand, reflection MMUs represent only 5% of Rachel's activity whereas production MMUs fill up 29% of the ME. This result goes in line with her profile of a scholar student and with her previous experience of mostly procedural aspects of mathematics.

Type of MMUs



Graphics 25: Distribution of the MMUs by type (Rachel's ME)

About a third (58) of the MMUs corresponds to non-activations or non-admissions. These MMUs contain the subtypes of efficient media judgment respectively efficient elimination and missed occasion as shown in Table 21. In terms of subtypes of MMUs, we noticed that about 47% are efficient media judgment and efficient elimination MMUs and over a half (53%) of non-activation and non-admission MMUs are missed occasions possibly interpretable in terms of insufficient activated milieu. This leads us to two conclusions: (1) the access to Internet facilitates exploration and research because it makes information more accessible in terms of comfort which significantly differs from a regular classroom situation and (2) the use of Internet does not necessarily facilitate the selection of information.

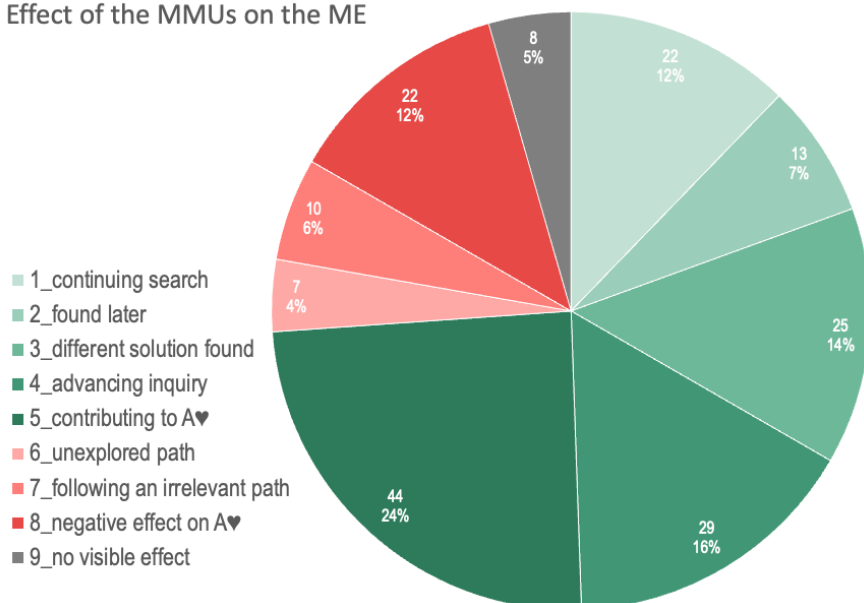
Type of MMU	# of MMUs	% of parent row
non-activation	13	7.2%
efficient media judgment	9	69.2%
missed occasion	4	30.8%
non-admission	45	25.0%
efficient elimination	18	40.0%
missed occasion	27	60.0%
reflection	8	4.4%
correct reflection	3	37.5%
irrelevant reflection	5	62.5%
admission	49	27.2%
correct admission	41	83.7%
incorrect admission	6	12.2%
irrelevant admission	2	4.1%
production	53	29.4%
correct production	40	75.5%
incorrect production	11	20.8%
irrelevant production	2	3.8%
KTS	12	6.7%
incorrect source	1	8.3%
irrelevant source	4	33.3%
relevant source	7	58.3%
Grand Total	180	100.0%

Table 21: Types and subtypes of MMUs (Rachel's ME)

Almost 84% of the admission MMUs about 75% of the production MMUs were correct admissions respectively production meaning that most of the time, Rachel was able to select and process information from media and produce correct answers. In addition, only three of these MMUs have got a negative effect on the ME and they concern the choice of the model. We also identified MMUs that we called Keeping track of sources (KTS), a procedure emphasized by the teacher, is another type of MMU that appears in the ME and there 4 irrelevant and 1 incorrect sources against 7 relevant sources were identified. We examine in detail the circumstances indicating a sufficient or an insufficient activated milieu later in this analysis. So far, we can say that the access to the Internet makes a lot of information available but the selection and judgment of the appropriateness of information from media, especially from the Internet, is not evident for Rachel and we have good reasons to think that in this case Rachel could be considered a generic student and her difficulty is actually a difficulty intrinsically related to research on the Internet.

Another way of looking at the content of the inquiry milieu is to consider the effect of the MMUs on the ME a posteriori, i.e., to examine the contribution of each MMU from the point of view of the final A[♥]. The pie chart below (Graphics 26) shows the distribution of the effect of the MMUs on the ME identified a posteriori from the point of view of their effect on the final answer according to 9 categories. The categories 1 to 5 (73% of the MMUs) correspond to a *desirable effect* on the exploration and categories 6 to 8 (22% of the MMUs) are considered to have an *undesirable effect* on the ME. As for the category 9 (5% of the MMUs), we were unable to determine their effect on the ME.

Effect of the MMUs on the ME

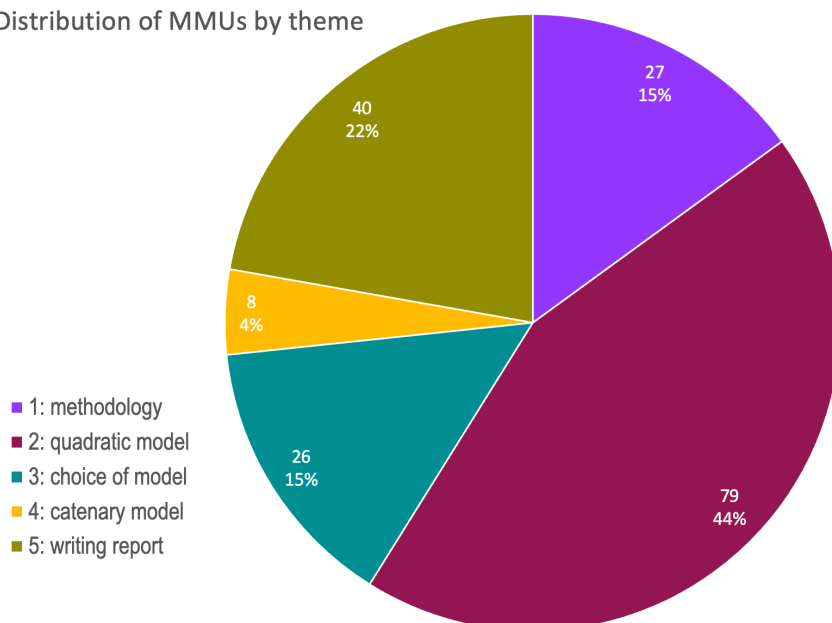


Graphics 26: Effect of the MMU on the ME (all themes)

Crossing these two criteria in a pivot table enables us to compare the a priori defined indicators of the activated milieu (sufficient/insufficient) at the moment of the interaction with the media and the final effect of these MMUs on the ME.

We can also look at the distribution of the MMUs in respect to the 5 themes identified in the chronogenesis (Graphics 27). As we can see, the distribution of the MMUs over the 5 themes is coherent with what was already shown in the QA-analysis, confirming the solidity of our coding.

Distribution of MMUs by theme



Graphics 27: The distribution of the MMUs by theme in Rachel's ME

Most (79, 44%) of the MMUs were in theme 2 (Quadratic model) and least number of MMUs (8, 4%) in theme 4 (Catenary model). In the next part, we analyze the MMUs from the point of view of processing information and its effect on the ME in respect to theme.

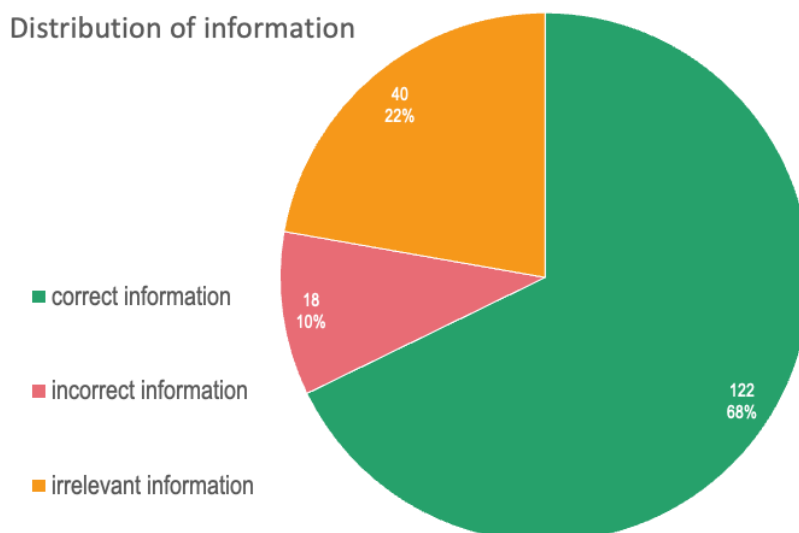
4.2.3 Processing of information

Another way of examining the content of the inquiry milieu M_I is to look at how the encountered information was processed. Specific subtypes of MMUs according to how the information was processed within three types of information (correct, incorrect and irrelevant) are shown in Table 22. These can also be considered from the point of view of the sufficient/insufficient activated milieu.

Correct information	Incorrect information	Irrelevant information
Sufficient activated milieu		
correct admission	efficient elimination	efficient elimination
correct production	efficient media judgement	efficient media judgement
correct reflection		
relevant source		
Insufficient activated milieu		
missed occasion	incorrect source	irrelevant admission
	incorrect admission	irrelevant production
	incorrect production	irrelevant reflection
		irrelevant source

Table 22: Types of MMUs in respect to correctness of information

The Graphics 28 shows the distribution of the MMUs based on the correctness in terms the encountered information all themes included.

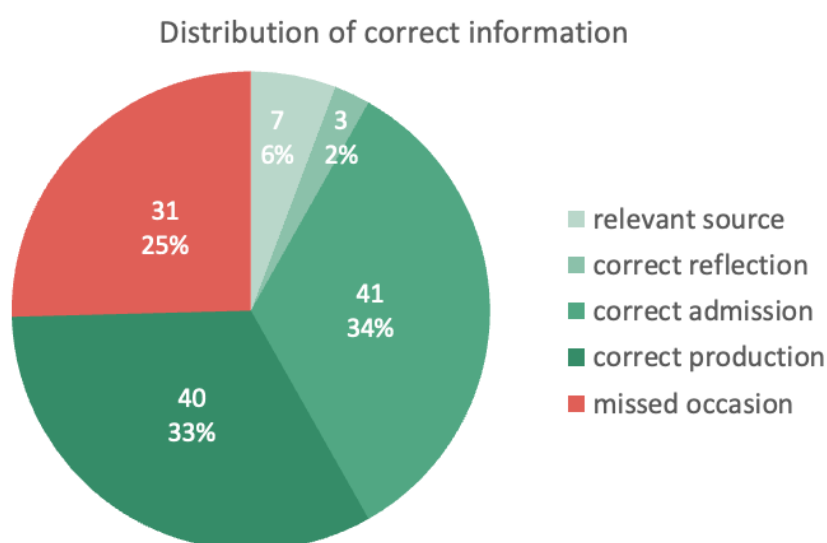


Graphics 28: Distribution of information in the inquiry milieu

We can see that 68% of the MMUs contained correct information against 10% of incorrect information and 22% of irrelevant information. This shows that with the access to the Internet, even though most of the encountered information was correct, with almost a third of it being incorrect or irrelevant, the student must not be able to select between what is relevant and irrelevant for the ME but also determine what is correct and incorrect which is again different from a situation in regular class where the information comes from the teacher and is supposed to be relevant and correct. In the following parts, we explore more in detail how Rachel was able to deal with this difficulty.

4.2.3.1 Processing of correct information

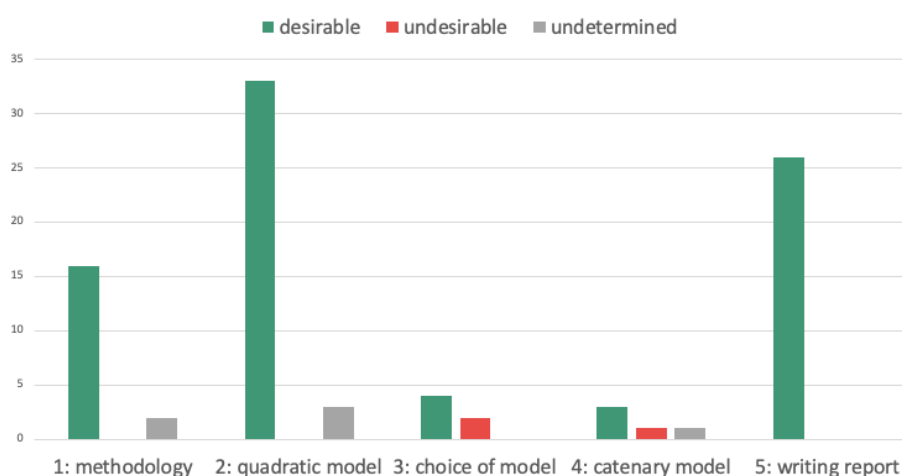
In this part we propose to closely examine the MMUs related to the processing of correct information. The detailed distribution is provided on Graphics 29.



Graphics 29: Distribution of correct information according to its processing

In the category processing correct information, we identified 5 subtypes of MMUs (relevant source, correct reflection/admission/production and missed occasion) and these are distributed as follows: for 75% of the correct MMUs the activated milieu was sufficient at the moment of the encounter and in 25% of the correct MMUs it was insufficient which led to missed occasions.

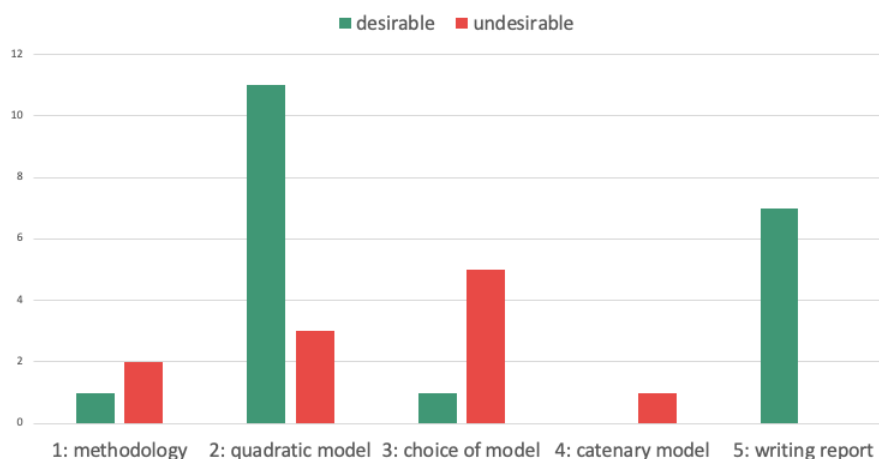
When examining closely the effect of the 75% of the relevant source, correct reflection/admission and production and MMUs with respect to theme, we can observe (see Graphics 30) that most of these MMUs with a desirable effect are situated within the themes 1 (Methodology), 2 (Quadratic model) and 5 (Writing report). In Themes 3 (Choice of model) and 4 (Catenary model), however, there are 3 MMUs with an undesirable effect on the ME which shows that even correct information can lead to a negative effect on the ME.



Graphics 30: Effect of the correct MMUs with a sufficient milieu in respect to theme

This confirms the difficulty the student encountered when dealing with objects further away from the CU. This makes us think that these elements might have been admitted to the milieu by chance, without a Me-Mi dialectic, ensuring the deconstruction and reconstruction of the retrieved answers and ultimately the advancement of the inquiry. Among these MMUs was the definition of a catenary curve, a comparison of the shape of a parabola and a catenary and finally a teacher's regulation asking to check the conclusions.

On the other hand, a quarter of MMUs with correct information were not recognized as potentially useful at the moment of the encounter. Since the mesogenesis is a dynamic process, this raises a question of whether and how (if applicable) these missed occasions affected the ME. Indeed, it might not necessarily mean that because the student was unable to process an information at a given moment, that this one is not definitely lost. As can be seen on Graphics 31, especially in themes 2 and 5, most of the missed occasions ended up having a desirable effect on the ME, which again is not the case in theme 1, 3 and 4. Under a desirable effect, we mean that the information was found later or a different solution was found. This shows again that that Rachel was definitely more at ease when dealing with objects closer to her CU.



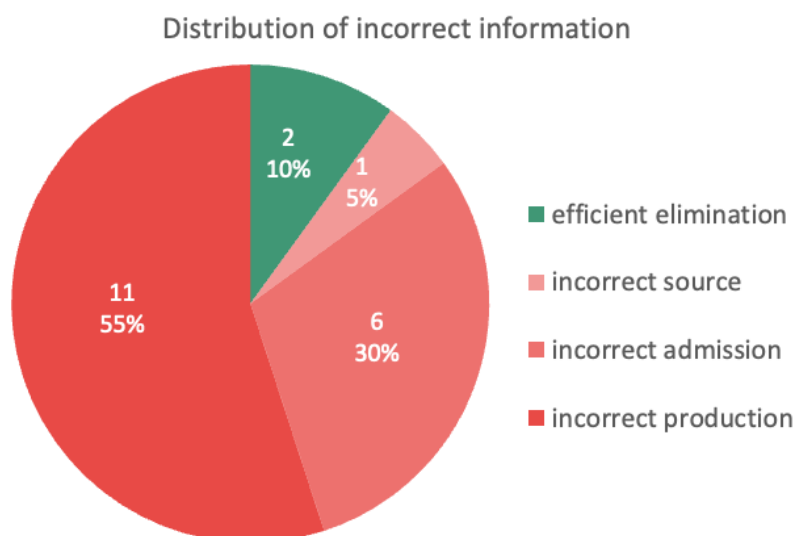
Graphics 31: Effect of the missed occasions on the ME in respect to theme

We further analyze the effect of the missed occasion on the ME later in this analysis (see p. 210).

We obtained a surprising result concerning relevant source MMUs: six out of seven sources had no visible effect on the ME, they were just copied and kept as a source but never activated again. The only relevant source that contributed to the production of A♥ is the V21.1 containing a worked example showing how to find the equation of a parabola using the dimension of the Lion's Gate Bridge which was indeed a key media that had a significant effect on the whole ME. This makes us think that it is because this media proposed a directly applicable method illustrated on a worked example which was actually close enough to the situation Rachel had to deal with. A detailed distribution of them MMUs by theme and effect on the ME can be found in Appendix 15.

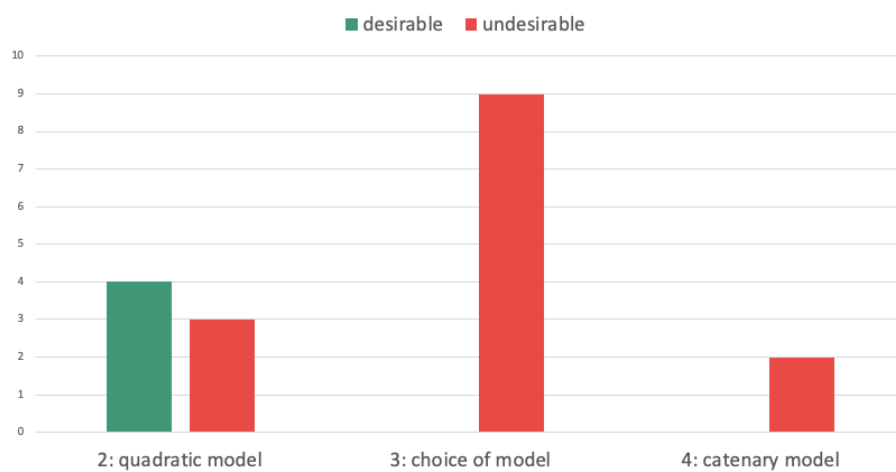
4.2.3.2 Processing of incorrect information

When dealing with incorrect information, the student can have a good judgment of the encountered media which leads to efficient media judgment or an efficient elimination or in case of an absent Me-Mi dialectic, this leads to keeping an incorrect source, incorrect admission or incorrect production. There is a total of 20 MMUs that contain incorrect information distributed over 4 subtypes as shown on Graphics 32. There are, however, only 2 MMUs identified as efficient eliminations, which clearly indicates that Rachel encountered difficulties when processing incorrect information. In addition, in one of these two MMUs, the successful elimination was rather the product of chance than a Me-Mi dialectic. In the other MMU, on the contrary, Rachel was able to eliminate an incorrect production based on the feedback of the experimental milieu.



Graphics 32: Distribution of incorrect information according to its processing

18 MMUs are identified as incorrect source/admission/production and we were interested what effect these MMUs have on the ME. As shown Graphics 33, these had mostly a negative effect on A^* as we could expect, however, in theme 2 (Quadratic model), there are 4 incorrect production MMUs for which Rachel was able to find an alternative solution or a correct answer was found later. This, however, did not happen in themes 3 or 4. This shows again that when the inquiry remains close enough to the student's CU, the student is able to deal more efficiently with incorrect information and even proceed to corrections and alternative solutions.

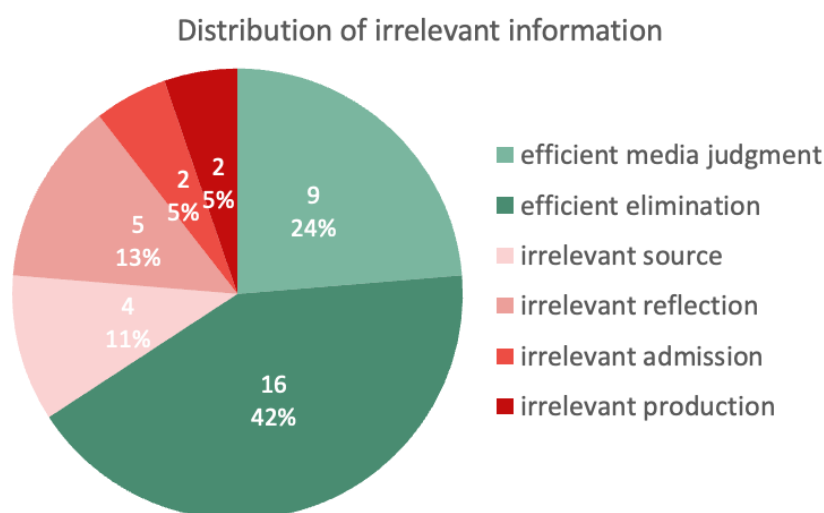


Graphics 33: Effect of incorrect source/admission/production MMUs on the ME

Globally, processing incorrect information is a source of difficulty and it is not easy for the student to select the right information when alone on the Internet. The Me-Mi dialectic seems to function when a certain proximity with the students' CU is ensured. A detailed distribution by theme and effect on the ME can be found in Appendix 16.

4.2.3.3 Processing of irrelevant information

When dealing with irrelevant information we can notice that about two thirds of the MMUs are identified as efficient media judgment and efficient elimination MMUs as shown on Graphics 34. This shows that it seems to be easier to eliminate irrelevant information than incorrect information.



Graphics 34: Distribution of irrelevant information according to its processing

There are altogether 25 eliminated MMUs and the reasons leading to an elimination of an answer or a work from media depend on the nature of the given answer or work. We identified that some eliminations were related to an efficient regulation, but in most cases the choice was made based on the name of a video which seemed right away unrelated to the topic of the exploration such as:

- V 2.1 Khan Academy: Modeling with composite functions
- V 2.3 Bridge modeling with Dynabridge Dynamo package
- V 2.4 Modeling with function combination

Or based on the milieu's feedback:

- SW-DGr 44.2 Explores the settings of Desmos
- W 44.2 Types 227 into the field height (of desmos) which stretches the Img 4.7 vertically
- W 44.3 Types 2 into the field height (of desmos) which shrinks the Img 4.7 vertically

After an efficient elimination and efficient media judgment MMUs, most of the time Rachel continued to explore the results of a Google search or eventually adjusted the keyword in the Google input bar to launch a new search because the encountered information did not provide a satisfactory answer. This is quite an expected behavior, which seems to be even reinforced by the facility to access information on the Internet.

On the other hand, we have 13 irrelevant MMUs that were somehow admitted to the inquiry milieu after the encounter, 2 of these had no visible effect on the ME. Actually, 10 out of the remaining 11 MMUs resulted in effect 7 following an irrelevant path. We can explain this by the presence of 5 irrelevant reflection MMUs. Indeed, the reflection MMUs come from the student's initiative who is probably convinced about their validity. The example of an irrelevant reflection MMU in Table 23 illustrates how a misconception of a mathematical concept put the student on an irrelevant path.

QAU22_Ep_38_MMU 59		
189	Std 22.1	Student's reflection: misconception of a pair function
190	A♥ 22.1	logbook: Model the curve of the suspension and find out if it is symmetrical or not
191	A♥ 22.3	logbook: Imperpose the half curve and check if it fits
192	A♥ 22.4	logbook: Then transform the equation to fit it on the other side

Table 23: Example of an irrelevant reflection MMU

Nevertheless, mathematical objects such as the square root function encountered on the irrelevant path (Table 24), were still studied and enriched the student's CU even though they did not have an immediate application in the ME.

QAU26_Ep_43_MMU 66		
216	V 26.2	Semicircles and Half Parabolas
217	A♦ 26.4	Explains how to graph a square root function
218	A♥ 26.1	logbook: If you are finding the equations of half a parabola, you will have a square root, the sign is very important.
QAU26_Ep_44_MMU 67		
219	V 26.2	Semicircles and Half Parabolas
220	A♦ 26.5	$y^2=x$ à we have a horizontal parabola
QAU27_Ep_44_MMU 68		
223	V 26.2	Semicircles and Half Parabolas
224	A♦ 27.1	explains how to graph $x=\sqrt{y}$ —horizontal parabola
225	A♥ 27.1	logbook: you will be modeling a vertical parabola

Table 24: Examples of MMUs from an irrelevant path

Still within the irrelevant MMUs with an undesirable effect, we identified only 1 MMU that had a negative effect on A♥. This happened when Rachel attempted to implement an unhappy teacher's regulation (Table 25) from feedback on the first draft.

Q78-Ep109-MMU175		
574	TFB 78.1	Criterion E—Use of mathematics
575	A♦ 78.1	“Why not pick more points on the curve, you may even be able to find more dimensions”
576	A♥ 78.1	Explains how she estimated the height of the vertical cables from the picture.
577	A♥ 78.2	Image of the parabola superposing the suspension cable with the additional points (Fig5.1)
578	A♥ 78.3	Justifies her choice: “because information on the suspender cables was not very clear on the Internet.”

Table 25: Example of an irrelevant MMU with a negative effect on A♥

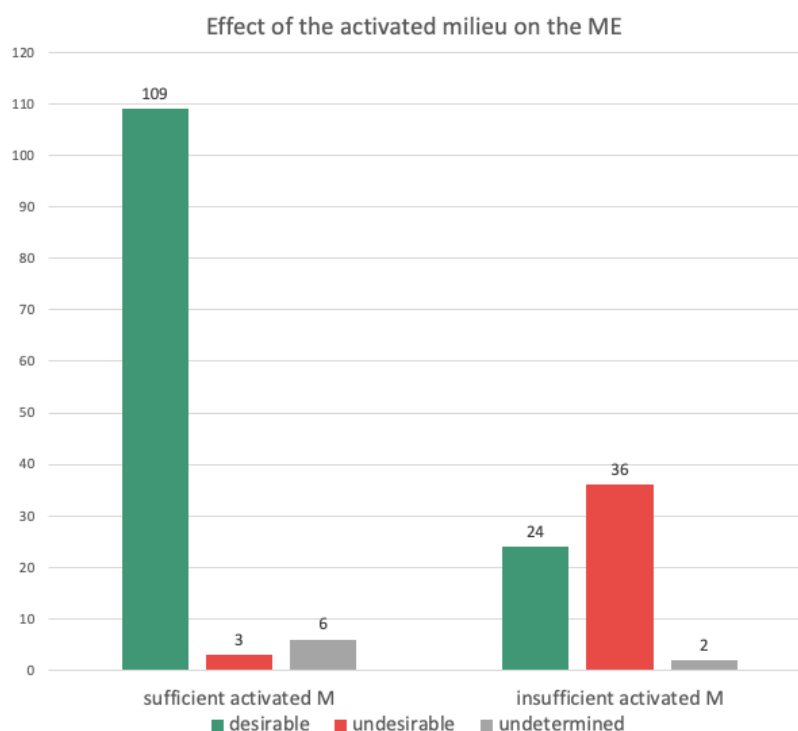
This situation provides another example of the effect of the didactic contract. Literally, without any understanding or making sense, Rachel just added some extra points on the curve. From the point of view of the chronogenesis, it is important to keep in mind that this happened towards the end of the ME and she was also running out of time.

Globally, however, the irrelevant information, besides being time-consuming, does not do much harm to the exploration, because irrelevant paths are often abandoned, or worst-case scenario the irrelevant productions are out of subject and unnecessary.

Considering the distribution of the MMUs according to different criteria enables us to see the content of the inquiry milieu and reveals certain generalities which confirms in some way our findings from the QA-level analysis. On its own it proves, however, insufficient to grasp the dynamics and to explain the evolution of the inquiry milieu. We therefore use pivot table analysis crossing sufficient or insufficient activated milieu with the effect on the ME as the main entry and target particular MMUs by adding a filter with another criterium such as theme or type of information. To analyze the effect of the MMUs on the ME we also place the targeted MMUs in the context of the chronology and consider their content.

4.2.4 Effect of the activated milieu on the ME

Based on the indicators of sufficient respectively insufficient milieu we proceed now to a cross analysis in respect to the effect (desirable, undesirable or undetermined) on the ME. Out of the total of 180 MMUs, for 118 MMUs the activated milieu was identified as sufficient at the moment of the interaction and for 62 MMUs the activated milieu was a priori insufficient. A priori we would expect that a sufficient activated milieu would lead to a desirable effect on the MU and an insufficient activated milieu would result in an undesirable effect on the ME which, as we can see on Graphics 35, is not always the case.



Graphics 35: Effect of the activated milieu on the ME

From the point of view of the effect of the activated milieu on the ME, we consider the distribution of the MMUs by subtype (Appendix 18) and by theme (Appendix 19).

To conduct a more detailed analysis we omit the MMUs resulting in an undetermined effect and focus on the following combinations:

- 1) Sufficient activated milieu MMUs with a desirable effect on the ME
- 2) Sufficient activated milieu MMUs with an undesirable effect on the ME
- 3) Insufficient activated milieu MMUs with a desirable effect on the ME
- 4) Insufficient activated milieu MMUs with an undesirable effect on the ME

We first consider the effects of the sufficient milieu on the ME by subtype of MMUs, filter by theme in a second step and complete by type of episode to better understand and interpret the results.

4.2.4.1 Effects of a sufficient activated milieu

A sufficient activated milieu was identified in about two thirds (118) of the total number (180) of MMUs and led, as expected, to over 90% (108 MMUs) of desirable effect on the ME. The detailed distribution of these MMUs is provided in Table 26.

MMU_effect_grouped	desirable				
# of MMUs	Effect on the ME				
Activated Milieu	1_continuing search	3_different solution found	4_advancing inquiry	5_contributing to A♥	Grand Total
☐ sufficient activated M	22	14	29	44	109
correct admission	0	3	14	21	38
correct production	0	7	12	21	40
correct reflection	0	0	2	1	3
efficient elimination	13	4	1	0	18
efficient media judgment	9	0	0	0	9
relevant source	0	0	0	1	1
Grand Total	22	14	29	44	109

Table 26: Sufficient activated milieu MMUs with a desirable effect on the ME by subtype of MMUs

As expected, most of the correct admission and production MMUs result in desirable effects (3_advancing inquiry and 4_contributing to A♥) and most of the efficient elimination and efficient media judgment MMUs lead to effect 1_continuing search. This witnesses of the presence of a functioning Me-Mi dialectic that is indispensable for the deployment of an inquiry process. In addition, there are 5 efficient elimination MMUs resulting in effect 3_different solution found (4 MMUs) respectively effect 4_advancing inquiry (1 MMU). All of them are linked to the experimental milieu either when exploring a new software (Effect 3) or when interpreting the experimental milieu feedback (Effect 4). We provide a detailed analysis of the experimental milieu later in this chapter (see p. 214 for more detail).

A complementary way of approaching the distribution of the MMUs and interpreting some of the results is adding some filters to the activated sufficient milieu versus effect on the ME cross analysis. A detailed table considering theme and subtypes of MMUs is provided in Appendix 20 and theme and type of episode in Appendix 21. In the following part, we focus on some of the MMUs that attracted our attention.

Out of the 109 MMUs from the sufficient activated milieu with a desirable effect on the ME only 10 (Table 27) were identified within the two themes that posed difficulties, i.e., 3: choice of model (7 MMUs) and 4: catenary model (3 MMUs).

# of MMUs	Effect on the ME _{TT}				
	desirable				desirable Total
Activated Milieu	1_continuing search	3_different solution found	4_advancing inquiry	5_contributing to A♥	
☐ sufficient activated M	22	14	29	44	109
⊕ 1: methodology	4	1	8	7	20
⊕ 2: quadratic model	12	6	12	16	46
☐ 3: choice of model	3	0	4	0	7
correct admission	0	0	2	0	2
correct production	0	0	2	0	2
efficient elimination	1	0	0	0	1
efficient media judgment	2	0	0	0	2
☐ 4: catenary model	0	0	1	2	3
correct admission	0	0	1	2	3
⊕ 5: writing report	3	7	4	19	33
Grand Total	22	14	29	44	109

Table 27: Sufficient activated milieu MMUs by theme and subtype

When considering the chronogenesis and the content, the 7 MMUs from theme 3 (choice of model) fall into 2 groups:

- Conflicting media (3 MMUs)
 - 2 efficient media judgment MMUs led to effect 1_continuing search
 - 1 correct admission resulted in effect 4_advancing inquiry
- Dealing with the skewed picture (4 MMUs)
 - 1 correct admission MMU resulted in effect 4_advancing inquiry
 - 1 efficient elimination MMU led to effect 1_continuing search
 - 2 correct production MMUs resulted in effect 4_advancing inquiry

Let us first examine and explain the effect of the 3 MMUs from what we called the conflicting media group. The first efficient media judgment (MMU77-WS15.1) appeared within an ill-productive exploration episode in QAU30 in cluster 5, when Rachel was trying to determine the shape of the suspension cable (Figure 71).

249	Q 30	Does the suspension cable of the san francisco golden gate bridge form a parabola?	question	question
250	GS 30.1	Does the suspension cable of the san francisco golden gate bridge form a parabola	exploration	ill-productive
251	WS-15.1	Math-central: online help type of website	exploration	ill-productive

Figure 71: Detail and context of QAU30-Ep49-MMU77 in Cluster 5

Despite this first efficient media judgment, a wrong information was accepted and a correct information rejected later in QAU30 as shown in Figure 72.

252	R 30.1	Excerpt from WS 15.1	exploration	ill-productive
253	A♦ 30.1	The suspension is a catenary	exploration	ill-productive
254	R 30.2	Are suspension bridges parabolas?	exploration	ill-productive
255	A♦ 30.2	suspension bridges are rather parabolas because forces of compression and tension are acting on it	exploration	ill-productive
256	GD 1.1	Math sites	exploration	ill-productive
257	L 30.1	R 31.1 is copied to GD 1.1 Math sites	exploration	ill-productive

Figure 72: Example of an effect of a parasitized milieu in cluster 5

At this moment the activated milieu had already been “parasitized” with the answers A♦10.3, A♦11.2 and A♦11.3 from the teacher from cluster 2. As a reminder, in these answers the teacher claimed that the suspension was a catenary. and resulted in systematic rejection of correct answers that would lead to the justification of the quadratic model.

Later in cluster 5, we have another efficient media judgment MMU87 and a correct admission MMU88 as shown in Figure 73. The correct admission is rather isolated within QAU33 and comes from V21.1 (The Lion’s Gate Bridge...) proposing a quadratic model for the suspension cable.

274	Q 33	What is the shape of a suspension cable?	question	question
275	GD 1.1	Math sites	exploration	productive
276	V-26.1	Quadratic Functions—Explained, Simplified and made easy	exploration	productive
277	GD 1.1	Math sites	exploration	productive
278	V 21.1	The Lion’s Gate Bridge... Modelling the Shape Using a Quadratic Function	exploration	productive
279	A♦ 33.1	Quadratic model of the Lion's Gate Bridge	exploration	productive

Figure 73: Detail and context of QAU33-Ep53-MMU87-88

We remind that the technique to determine the equation of a quadratic function using the vertex form from V21.1 based on a worked example was found in QAU 21 in cluster 4 but the equation of the quadratic function was produced analytically in QAU53 and 54 in cluster 7. Rachel was actually between two conflicting pieces of information: the teacher’s answers A♦10.3, A♦11.2 and A♦11.3 from cluster 2 claiming that the shape of the suspension cable is a catenary without a technique and answers from media in favor of the quadratic model with an accessible technique to find an analytical solution.

The second group of MMUs, concerning dealing with the skewed picture, is composed of 1 correct admission, 1 efficient elimination and 2 correct productions and appeared within regulation episodes in cluster 8 (QAU63-Ep90-MMU151, 152, 153 and QAU65-Ep94_MMU156). The content of QAU63 and QAU65 is shown in Figure 74.

483	Q 63	Can you reject the quadratic model based on this information?	question	question
484	Exp 63.1	Regulation	regulation	productive
485	W 63.1	(expert) Can you say that it's not a parabola?	regulation	productive
486	Std 63.1	Student reflection	regulation	productive
487	W 63.2	Explaining the idea of half parabolas from A♥ 29.2	regulation	productive
488	W 63.3	Explaining the idea of symmetry to invalidate the quadratic model	regulation	productive
489	Exp 63.2	Regulation: Pushing for reflection 2	regulation	productive
490	W 63.4	(expert) Based on the skewness of the picture, can you say this?	regulation	productive
491	A♥ 63.1	I might not be able to conclude because the picture is skewed	regulation	productive
492	A♥ 63.2	There were different data for the dimensions of the bridge and that is why it does not fit	regulation	productive
502	Q 65	What conclusion can you make?	question	question
503	Exp 65.1	Regulation: Dealing with the picture 2	regulation	productive
504	A♦ 65.1	(expert) What you can't say, it is not fitting because of the fact that it's not a parabola or is it not fitting because of the fact that it is skewed.	regulation	productive
505	A♥ 65.1	So I couldn't like come to a conclusion because of the picture, so by doing that I would be stating that the bridge is not symmetrical but it is actually symmetrical.	regulation	productive

Figure 74: Detail of QAU63-Ep90-MMUs151-153 and QAU65-Ep94-MMU156

Based on this information we can confirm that Rachel actually abandoned the irrelevant path and seems to have understood that the skewed picture would not help her to validate/reject the quadratic model. Since this regulation was efficient, the reason for the rejection of the quadratic model must be somewhere else. We show this in the part dedicated to the organization of the experimental milieu (see p. 214).

We remind that we are still examining the 10 MMUs having a desirable effect on the ME. In theme 4 (Catenary model), the 3 successive MMUs (Q78-Ep110-MMU176, Q79-Ep111-MMU177 and Q80-Ep112-MMU178) with a desirable effect are situated in cluster 9 and the detail of these MMUs is provided in Figure 75.

579	TFB 78.2	Criterion E – Use of mathematics	correct admission	4_advancing inquiry
580	A♦ 78.2	"You could perhaps find the general form of a catenary curve and try that"	correct admission	4_advancing inquiry
581	Q 79	What is the equation of the catenary for the Golden Gate Bridge?		
582	GD 1.1	Math sites	correct admission	5_contributing to A♥
583	WS 31.1	Math24-equation of catenary – too complicated	correct admission	5_contributing to A♥
584	A♦ 79.1	The equation of a catenary $y = a \cdot \cosh(x/a)$	correct admission	5_contributing to A♥
585	Q 80	How do I determine the "a" parameter for the Golden Gate Bridge?		
586	GS 80.1	unknown (equation of the catenary for the Golden Gate Bridge)		
587	WS 80.1	Math.stackexchange.com – questions: catenary-equation-for-golden-gate-bridge	correct admission	5_contributing to A♥
588	A♦ 80.1	For the Golden Gate Bridge, the parameter $a = 1371,98 \approx 1372$ (found numerically with Mathematica)	correct admission	5_contributing to A♥
589	W 80.1	The equation of the catenary function $y = 1372 \cdot \cosh(x/1372)$	correct admission	5_contributing to A♥

Figure 75: Detail of MMUs from theme 4 with a desirable effect on the ME in cluster 9

This happened after the teacher's feedback on the first draft (A♦78.2) and resulted in two implementation episodes when Rachel found the general equation of the catenary curve and copied the value of the parameter for the Golden Gate Bridge proposed on a website. In the post-exploration interview, when Rachel was questioned about how she found the value of a , she admitted that she just took it from the Internet because she did not know what to do with the “cosh” function which remained out of her reach. Even though, Rachel was able to find isolated information concerning the catenary curve on the Internet to follow up on the teacher's comment, further application of these answers was out of her reach. This illustrates a combination of an effect of the didactic contract and the absence of a media milieu dialectic.

The next considered combination is the MMUs from the sufficient activated milieu leading to an undesirable effect on the ME (Table 28).

MMU_effect_grouped	undesirable		
# of MMUs	Effect on the ME		
Activated Milieu	6_unexplored path	8_negative effect on A♥	Grand Total
sufficient activated M	1	2	3
correct admission	1	2	3
Grand Total	1	2	3

Table 28: Sufficient activated milieu MMUs with an undesirable effect on the ME

There are 3 correct admission MMUs leading to an undesirable effect (1 MMU with effect 6_unexplored path and 2 MMUs with effect 8_negative effect on A♥) whose detailed content is shown in Figure 76.

Chronolo	E-unit	Description	MM-unit_characterist	Effect on exploration
260	R 31.1	Definition of a catenary curve	correct admission	6_unexplored path
261	A♦ 31.1	The catenary is a plane curve whose shape corresponds to a hanging homogeneous flexible chain supported at its ends and sagging under the force of gravity.	correct admission	6_unexplored path
547	WS 73.1	Quora forum: the difference between a parabola a catenary	correct admission	8_negative effect on A♥
548	A♦ 73.1	Catenary curves look just like parabolas but the difference between them is that a goes to infinity much faster than the parabolic curve.	correct admission	8_negative effect on A♥
549	Img 73.1	Parabola and catenary on one graph	correct admission	8_negative effect on A♥
570	TFB 77.1	Criterion D – Reflection	correct admission	8_negative effect on A♥
571	A♦ 77.2	I would check this before stating it: “The suspension cables do not form a parabola”.	correct admission	8_negative effect on A♥
572	A♦ 77.3	I would check this before stating it: “We can clearly see that the curve between the golden gate bridge is not a parabola”	correct admission	8_negative effect on A♥

Figure 76: Detail of Q31-Ep50-MMU81-EU260-261, Q73-Ep103-MMU167-EU547-549 and Q77-Ep107-MMU174-EU570-572

The unexplored path (Q31-Ep50-MMU81-EU260-261) concerned the definition of the catenary curve in cluster 5 within theme 4 (Catenary model). In the context of the chronogenesis, this happened just after the unfruitful exploring of the irrelevant path about half parabolas. She returned thus to (re-question the shape of the suspension cable. This definition of a catenary on its own represents just an isolated piece of information and to contribute to the inquiry it would have needed to be combined with other answers. For

example, together with the fact that for a suspension bridge the deck is hanging on vertical cables attached to the main suspension cable it could have led to a new question and further exploration. The two remaining MMUs are from theme 3 (Choice of model) and are situated in cluster 8 respectively cluster 9. The reason for a negative effect on $A^\heartsuit$ of the first MMU (Q73-Ep103-MMU167-EU547-549) is a combination of several parameters. Rachel actually analyzed correctly the admitted information in this MMU but made a wrong conclusion due to a malfunctioning experimental milieu. We analyze the experimental milieu in detail later in this chapter (see p. 214). The second MMU (Q77-Ep107-MMU174-EU570-572) with a negative effect on $A^\heartsuit$ was a result of an inefficient teacher's feedback from the final draft. Rachel attempted to do something with this suggestion, but she would have probably needed more precise guidance than just "I would check this before stating it...". This part actually points to the difficulties of processing and further application of the information found on the Internet. Rachel found and retrieved potentially useful information yet, wasn't able to explore its full potential. In this case the inversed topoi clearly played in Rachel's disadvantage showing that she would have needed more guidance.

In the next part, we move to explore the effects of the insufficient activated milieu on the ME.

4.2.4.2 Effects of an insufficient activated milieu

In about one third (62 MMUs) of the total number of MMUs the activated milieu was identified as a priori insufficient, and we would expect an important proportion of an undesirable effect on the ME. However, almost 40% (24 MMUs) resulted in a desirable effect on the ME.

The distribution of MMUs having a desirable effect with an insufficient activated milieu is provided in Table 29.

MMU_effect_grouped	desirable		
# of MMUs	Effect on the ME	2_found later	3_different solution found
Activated Milieu			Grand Total
insufficient activated M	13	11	24
incorrect production	1	3	4
missed occasion	12	8	20
Grand Total	13	11	24

Table 29: Insufficient activated milieu MMUs with a desirable effect on the ME

Out of the 24 MMUs, 20 are missed occasions and 4 are incorrect productions. In 13 MMUs the information was found later and in 11 MMUs a different solution was found. Considering the total number of missed occasions (31 MMUs, see Table 21, p. 194 for detail), we can see that in 20 of them (Table 29) either an answer was found later (in 12 MMUs) suggesting that as the inquiry advanced, the inquiry milieu was enriched and

became sufficient or a different solution was found (in 8 MMUs) meaning that the student found a way around the insufficient activated milieu.

From the point of view of theme, 15 out of the 24 MMUs from the sufficient activated milieu with a desirable effect on the ME are situated in theme 2 (Quadratic model), 7 in theme 5 (Writing report), 1 MMU (missed occasion) in theme 1 (Methodology) and 1 MMU (missed occasion) in theme 3 (Choice of model). A detailed distribution of the MMUs filtered by theme/subtype and theme/episode are provided in Appendix 20 respectively in Appendix 21. We can thus conclude and attempt an extrapolation that in case of an insufficient milieu at the moment of the interaction with the media, an answer can be found later or a different solution is found when a certain proximity with the student's CU is guaranteed. In Rachel's exploration, the activity within the themes 2 and 5 seems to be the case.

Finally, the last combination left is the MMUs from the insufficient activated milieu with an undesirable effect on the ME as shown in Table 30.

MMU_effect_grouped	undesirable			
# of MMUs	Effect on the ME			
Activated Milieu	6_unexplored path	7_following an irrelevant path	8_negative effect on A♥	Grand Total
☒ insufficient activated M	5	10	21	36
incorrect admission	0	0	6	6
incorrect production	0	0	7	7
incorrect source	0	0	1	1
irrelevant admission	0	2	0	2
irrelevant production	0	0	1	1
irrelevant reflection	0	5	0	5
irrelevant source	0	3	0	3
missed occasion	5	0	6	11
Grand Total	5	10	21	36

Table 30: Insufficient activated milieu MMUs with an undesirable effect on the ME

As expected, all missed occasion MMUs result in effect 6_unexplored passed and irrelevant MMUs (except for one) lead to effect 7_following an irrelevant path (exploring half-parabolas in the second part of cluster 4 QAU 24–28 on page 172), which were already treated in the part on processing irrelevant information (see p. 201) earlier in this analysis. Finally, out of the 36 MMUs from the insufficient activated milieu, 21 MMUs resulted in effect 8_negative effect on A♥. In order to understand the reasons that led to the undesirable effect on the ME, we chose to examine some of these MMUs through the filter of theme as shown in Table 31 and zoom on those that seem to contribute to better characterize an inquiry process

MMU_effect_grouped	undesirable			
# of MMUs	Effect on the ME			
Activated Milieu	6_unexplored path	7_following an irrelevant path	8_negative effect on A♥	Grand Total
insufficient activated M	5	10	21	36
1: methodology	2	1	0	3
irrelevant reflection	0	1	0	1
missed occasion	2	0	0	2
2: quadratic model	1	7	6	14
incorrect production	0	0	3	3
irrelevant admission	0	2	0	2
irrelevant production	0	0	1	1
irrelevant reflection	0	2	0	2
irrelevant source	0	3	0	3
missed occasion	1	0	2	3
3: choice of model	1	2	13	16
incorrect admission	0	0	6	6
incorrect production	0	0	2	2
incorrect source	0	0	1	1
irrelevant reflection	0	2	0	2
missed occasion	1	0	4	5
4: catenary model	1	0	2	3
incorrect production	0	0	2	2
missed occasion	1	0	0	1
Grand Total	5	10	21	36

Table 31: Insufficient activated milieu MMUs by theme

We start by considering the nature of the MMUs in theme 2 (Quadratic model), because that is the closest to the student's CU and that is where we would expect the least undesirable effect on the ME. Since the 7 MMUs leading to effect 7_following an irrelevant path have already been treated earlier in this analysis, we would particularly like to zoom on 4 MMUs with effect 8_negative effect on A♥. Firstly, we consider 1 missed occasion MMU that drew our attention because it could have changed the course of the exploration and more precisely, it could have contributed to the organization of the experimental milieu. This MMU is situated in cluster 1 (QAU5-Ep9-MMU12-EU44-45) and corresponds to a picture of the architectural plan of the bridge which would have enabled the validation of the quadratic model by superimposition as shown on Figure 77. Rachel came across this picture just after having found the skewed picture of the Golden Gate Bridge and this potentially valuable source went unnoticed.

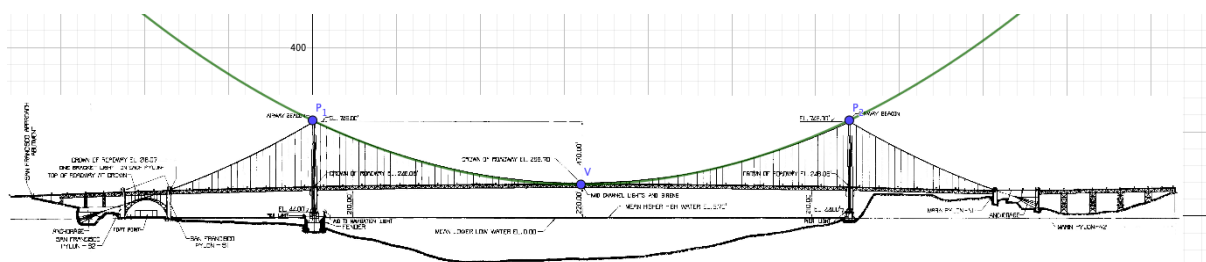


Figure 77: The quadratic function found by Rachel superimposed on the architectural plan of the Golden Gate bridge

The picture of the bridge that Rachel used found and used by the student was indeed not the best tool to use for the validation by superimposition. Here again more constructive regulations than just “You should find another picture” or “You could use this one and make it part of your reflection” would have been needed. This shows again a negative effect of the inversed topos and on top of that, the fact that the teacher is not in a position of an inquirer himself becomes a serious constraint in his ability to provide constructive feedback. Still in theme 2, there are 3 incorrect production MMUs resulting in effect 8_negative effect on A♥. The first one (QAU56-Ep77-MMU136-EU432-433) from cluster 5 corresponds to the very first attempt to build an experimental milieu and concerned respecting respectively non-respecting the scale when placing the photo of the bridge on the coordinate plane. This MMU was subject to a regulation which turned out to be sufficient at the moment, but Rachel was unable to reproduce and apply this idea later. The two remaining incorrect production MMUs (QAU70-Ep100-MMU163-164-EU532-538) situated in cluster 8 are linked to the construction of an experimental milieu and to interpretation of its feedback. In theme 2 (Quadratic model), we have illustrated the negative effect on A♥ on two types of MMUs (missed occasion and incorrect production) in which we identified an a priori insufficient activated milieu. In both situations, Rachel was unable to proceed with the inquiry on her own and would have needed specific guidance by the teacher. The teacher, however, in order to be able to provide efficient, targeted and constructive regulations, would need to adopt a posture of an inquirer himself. The difficulty resides, however, in the current institutional format of the ME. Indeed, with every student working on a different topic, it is close to impossible for the teacher to conduct all these inquiries by himself.

In theme 3 (Choice of model), we have 16 MMUs from the insufficient activated milieu with an undesirable effect out of which 13 led to effect 8_negative effect on A♥. A detail of these 13 MMUs is given in Table 32.

Chronology	Media	Cluster	Label	Episode	MMU	Effect on the ME	Reason
87	T 10.1	c2	Q10-Ep16-MMU25	regulation	incorrect admission	8_negative effect on A♥	key-answer in I-milieu
92	T 11.2	c2	Q11-Ep18-MMU27	regulation	incorrect admission	8_negative effect on A♥	key-answer in I-milieu
125	WS 15.1	c2	Q15-Ep28-MMU39	exploration	incorrect admission	8_negative effect on A♥	effect of key-answer in I-milieu
127	WS 15.2	c2	Q15-Ep28-MMU40	exploration	missed occasion	8_negative effect on A♥	effect of key-answer in I-milieu
129	R 15.1	c2	Q15-Ep28-MMU41	exploration	missed occasion	8_negative effect on A♥	effect of key-answer in I-milieu
132	WS 15.3	c2	Q15-Ep28-MMU42	exploration	missed occasion	8_negative effect on A♥	effect of key-answer in I-milieu
246	Std 29.1	c4	Q29-Ep48-MMU76	analysis	irrelevant reflection	7_following an irrelevant path	effect of key-answer in I-milieu
252	R 30.1	c5	Q30-Ep49-MMU78	exploration	incorrect admission	8_negative effect on A♥	effect of key-answer in I-milieu
254	R 30.2	c5	Q30-Ep49-MMU79	exploration	missed occasion	8_negative effect on A♥	effect of key-answer in I-milieu
256	GD 1.1	c5	Q30-Ep49-MMU80	exploration	incorrect source	8_negative effect on A♥	effect of key-answer in I-milieu
285	WS 34.1	c5	Q34-Ep54-MMU90	verification	incorrect admission	8_negative effect on A♥	effect of key-answer in I-milieu
304	Std 37.1	c5	Q37-Ep57-MMU97	analysis	irrelevant reflection	7_following an irrelevant path	effect of key-answer in I-milieu
480	Std 62.1	c8	Q62-Ep89-MMU150	analysis	incorrect production	8_negative effect on A♥	interpreting EXP-milieu feedback
544	N/A 72.1	c8	Q72-Ep102-MMU166	exploration	incorrect admission	8_negative effect on A♥	effect of key-answer in I-milieu
550	W 70.4	c8	Q73-Ep103-MMU168	analysis	incorrect production	8_negative effect on A♥	interpreting EXP-milieu feedback
568	TFB 77.1	c9	Q77-Ep107-MMU173	regulation	missed occasion	6_unexplored path	lack of time and understanding

Table 32: Detail of MMUs with an undesirable effect of the insufficient activated milieu from theme 3: Choice of model

When placing these MMUs into the context of the chronology of the exploration, we observe that the starting point of this series of undesirable effects was an unhappy teacher's regulation about the catenary curve (A♦10.3, A♦11.2 and A♦11.3) in cluster 2 as already pointed out in the chronogenesis. When a particular answer has an important effect on the ME, we refer to it as to a key answer in the inquiry milieu. In this case the key answer was incorrect and led to a series of missed occasions and incorrect admissions. This together with an interpretation of a malfunctioning experimental milieu's feedback resulted in the rejection of the quadratic model.

In theme 4 (Catenary model) with altogether 8 MMUs, there are 3 MMUs resulting in an undesirable effect, 1 missed occasion and 2 incorrect productions as shown in Table 33.

Chronology	Media	Cluster	Label	Episode	MMU	Effect on the ME	Reason
262	WS 31.1	c5	Q31-Ep50-MMU82	study	missed occasion	6_unexplored path	too far from CU
591	SW-DGr 81.1	c9	Q81-Ep113-MMU179	verification	incorrect production 8_negative effect on A♥		creating EXP-milieu
595	Std 81.1	c9	Q81-Ep113-MMU180	verification	incorrect production 8_negative effect on A♥		interpreting EXP-milieu feedback

Table 33: Detail of MMUs with an undesirable effect of the insufficient activated milieu from theme 4: Catenary model

We can confirm again that a missed occasion MMU led to effect 6_unexplored path because the content of the media was too far from the student's CU. Indeed, the answer contained in WS31.1 (Table 34) was a physics model of the catenary which was clearly inaccessible to Rachel.

Q31-Ep50-MMU82		
262	WS 31.1	Math24-equation of catenary—too complicated
263	A♦ 31.2	a physics model of a catenary—inaccessible

Table 34: Missed occasion MMU in Theme 4

The two incorrect productions with a negative effect on the A♥ were identified within a verification episode when creating an experimental milieu and interpreting its feedback.

In the next part, we move to explore the content of the material/experimental milieu and analyze in detail its role and effect on the ME.

4.3 Organization of a material/experimental milieu

As a material milieu, the student had a laptop, a graphing calculator as hardware and Google classroom, *Desmos graphing software* and, of course, an Internet connection available.

In respect to the material/experimental milieu, we identified four categories of MMUs:

- Exploring software (9 MMUs)
- Media acting as an a-didactic milieu (4 MMUs)
- Creating an experimental milieu for validation (18 MMUs)

- Interpreting an experimental milieu feedback (15 MMUs)

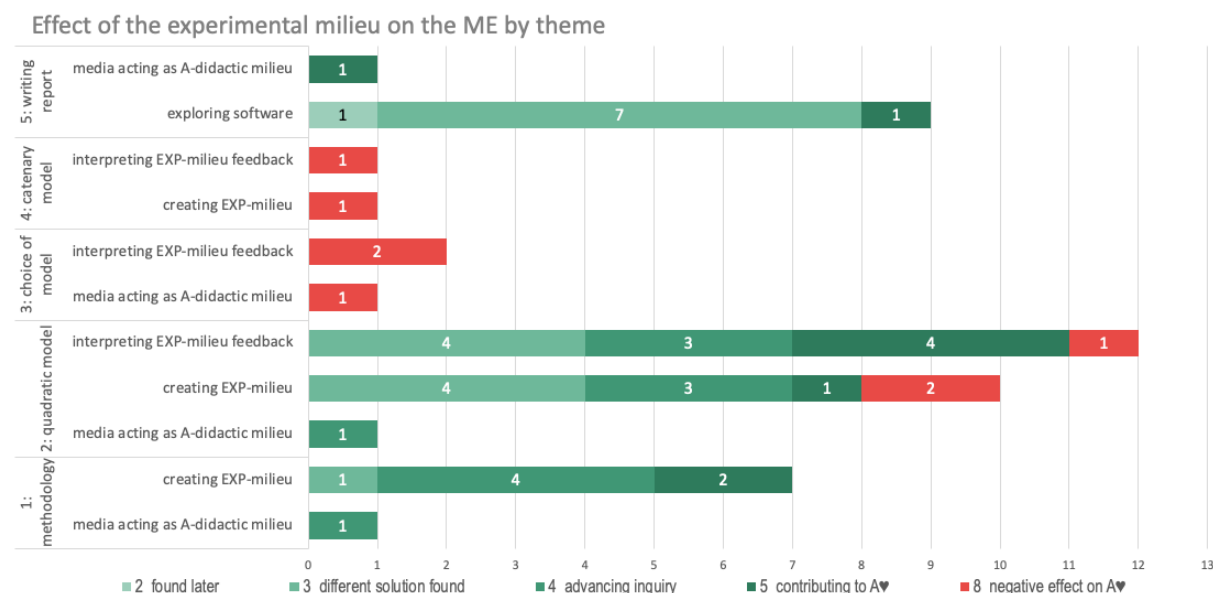
There is a total 46 MMUs in relation to the material/experimental milieu which corresponds to about 25% of all MMUs. Over 70% of them are directly related to the creation of an experimental milieu and interpreting its feedback. The detail of the distribution of these MMUs in respect to theme and effect on the ME is provided in Table 35.

# of MMUs	Effect on the M					
Experimental milieu	1: methodology	2: quadratic model	3: choice of model	4: catenary model	5: writing report	Grand Total
exploring software	0	0	0	0	9	9
media acting as A-didactic milieu	1	1	1	0	1	4
creating EXP-milieu	7	10	0	1	0	18
interpreting EXP-milieu feedback	0	12	2	1	0	15
Grand Total	8	23	3	2	10	46

Table 35: Content of the experimental milieu by effect and by theme

Again, most of the MMUs (23) are identified in theme 2 (Quadratic model) which represents 30% of the MMUs in theme 2. In addition, 22 of them have a direct impact on the organization of an experimental milieu and interpreting its feedback. On the other hand, all exploring software MMUs appeared in theme 5 (Writing report) and the 4 MMUs related to media acting an a-didactic milieu are in themes 1, 2, 3 and 5 each.

When considering the effect on the ME (Graphics 36), we noticed that in theme 3 and 4, none of the MMUs resulted in a desirable effect on the ME which indicates a clear difficulty when organizing an experimental milieu and interpreting its feedback within these themes.



Graphics 36: Effect of the experimental milieu on the ME by theme

In order to understand the reasons behind this undesirable effect we first examine in detail the characteristics of MMUs having a negative effect on A♥. As shown in Table 36, these MMUs are situated towards the end of the chronology, mostly in cluster 8 and 9. All MMUs except 1 are incorrect productions and appear within implementation, analysis or

verification episodes. They are evenly distributed over three themes: theme 2 (3MMUs), theme 3 (3 MMUs) and theme 4 (2 MMUs.)

Chronology	Media	Theme	Cluster	Label	Episode	MMU	Milieu
432	SW-DGr 56.1	2: quadratic model	c7	Q56-Ep77-MMU136	implementation	incorrect production	creating EXP-milieu
480	Std 62.1	3: choice of model	c8	Q62-Ep89-MMU150	analysis	incorrect production	interpreting EXP-milieu feedback
532	SW-DGr 70.2	2: quadratic model	c8	Q70-Ep100-MMU163	implementation	incorrect production	creating EXP-milieu
536	Std 70.1	2: quadratic model	c8	Q70-Ep100-MMU164	implementation	incorrect production	interpreting EXP-milieu feedback
547	WS 73.1	3: choice of model	c8	Q73-Ep103-MMU167	analysis	correct admission	media acting as A-didactic milieu
550	W 70.4	3: choice of model	c8	Q73-Ep103-MMU168	analysis	incorrect production	interpreting EXP-milieu feedback
591	SW-DGr 81.1	4: catenary model	c9	Q81-Ep113-MMU179	verification	incorrect production	creating EXP-milieu
595	Std 81.1	4: catenary model	c9	Q81-Ep113-MMU180	verification	incorrect production	interpreting EXP-milieu feedback

Table 36: Characteristics of MMUs related to the experimental milieu with a negative effect on A♥

Contentwise, these MMUs dealt with the superimposition of the curve on the photo of the Golden Gate Bridge and served to validate or reject a given model. The construction of an efficient experimental milieu and the interpretation of its feedback is indeed a challenging issue and rather inaccessible for Rachel. We could claim this because the experimental milieu in Desmos that included placing the photo correctly and in scale on the coordinate plane was first created within productive regulation episodes and used for superimposing the found parabola. Later, when Rachel was working alone, she did not respect the scale when placing the photo on the coordinate plane. An addition, when she considered that the “skewed” photo was “unskewed” perfectly and was providing reliable feedback. In combination with a correctly interpreted answer about the graph of a catenary curve (A♦ 73.1 Catenary curves look just like parabolas but the difference between them is that the catenary curve goes to infinity much faster than the parabolic curve) that was acting as an a-didactic milieu, the experimental milieu was sending incorrect feedback that finally led to rejecting the quadratic model and accepting the catenary model reinforced by the teacher’s input from the beginning of the exploration.

The organization of an appropriate experimental milieu plays an important role in the validation of answers. Since the teacher reduced the vision of modeling exploration to superimposing a curve on a picture and imposed it on the students, we would have expected more inputs and Rachel would have definitely needed more help since this was her main difficulty and resulted in validating and accepting an incorrect final answer.

4.4 Conclusion of M-M level analysis of Rachel’s ME

In the analysis at M-M level, we considered both the content and the evolution of the inquiry milieu based on the interactions of the student with the media, on the one hand, and the organization of an experimental milieu using the ingredients from the inquiry milieu, on the other hand.

The inquiry milieu evolves gradually as the ME advances in a double dialectical process: turning to media to search for answers and making connections between answers. First, the

didactic system needs to feed an inquiry milieu for the inquiry to unfold and the inquiry milieu is thus progressively enriched through different ingredients found and processed from media as shown in the Herbartian scheme. Globally, a sufficient inquiry milieu led to a desirable effect on the ME and in this case, the processed information and the encountered objects were at a level comparable to objects contained in Rachel's CU. Even though the proximity of the encountered objects with the student's CU ensures a certain operational ability to conduct an inquiry, i.e., to retrieve correct and relevant answers from the media, it does not seem to be a sufficient condition. We identified situations when an a priori sufficient activated milieu resulted in an undesirable effect on the ME. It is thus not enough just to be able to retrieve isolated answers from the media, another important condition is the ability to make connections between the ingredients of the inquiry milieu. From the point of view of research, this raises a new question: under what circumstances the students are able to make connections between the answers that they were able to retrieve from the media and add them to the inquiry milieu? Does this ability to make connections fall within the characteristics of the Me-Mi dialectic or is it a new dialectic that seems indispensable for an inquiry process?

Another interesting fact to consider is the influence of the access to the Internet on the evolution of the milieu. On the one hand, the Internet facilitates the exploration and enables more research, because a lot of information is accessible in a couple of clicks. This, on the other hand, adds a new challenge, which does not exist in a regular class where the main source of information is the teacher or an official textbook. In this context, in addition to the need of selecting potentially useful information and judging its relevance, the student must also be able to determine whether the information is correct or not. In case of Rachel, we saw that it is easier to reject irrelevant information than incorrect information which seems relevant for the ME. We identified, however, situations when Rachel pursued an irrelevant path based on her own irrelevant reflection. We can thus conclude that following an irrelevant path during inquiry can easily happen because it is not always evident to know in advance what correct information is also relevant for the given question. At the same time, irrelevant paths are generally harmless and abandoned.

We also saw that Rachel was able to retrieve certain answers later in the inquiry or find an alternative solution. This shows that the inquiry milieu is in constant evolution and sometimes several encounters with the same object are needed or an additional connection is made which leads to finding the right information.

We also identified moments when the inquiry milieu was "parasitized" by an incorrect information as an effect of the didactic contract and it was thus impossible for Rachel to process a contradictory information from another media even if this information was the correct one. We also identified a moment when a missed occasion could have changed the course of the whole exploration. This makes us think that in these moments, the student was finally very lonely to make decisions or to try to resolve the dilemmas, which is in a

direct contradiction with the fact how scientists work. Indeed, within a scientific community, they conduct research in teams where they can discuss their difficulties or confront their ideas.

Equally, we observed situations in which an answer or work retrieved from media acted as an a-didactic milieu, i.e., did not carry any didactic intention, yet contributed to advancing the inquiry. Rachel also explored various software and IT programs with a view to tackle a rather non-mathematical difficulty; annotating a picture in order to create appropriate figures for the final written report.

Another challenge was directly linked to the organization of an experimental milieu and to being able to interpret its feedback. We realized that despite what seemed an efficient and appropriate regulation when creating an experimental milieu, Rachel was not in a position to reproduce it when working alone, which led to an incorrect interpretation of its feedback. This proved even more complicated due to both the skewness of the initial picture of the bridge as well as its amended version. A combination of a malfunctioning experimental milieu, didactic contract and teacher effect, insufficient CU and finally lack of time towards the end of the ME resulted in an incorrect conclusion i.e., rejection of the quadratic model and acceptance of the catenary model in Rachel's case.

5 Summary of Rachel's ME

Rachel chose to explore the shape of the suspension cable of the Golden Gate Bridge. The generating question of her ME was "What form does the suspension cable of the Golden Gate Bridge take?" She is a scholar student who joined the IB from a rather traditional school system in which the procedural and calculational aspects of mathematics teaching are emphasized. Most likely, this explains the initial search for a procedure and her interest in a document providing a step-by-step protocol on how to model a bridge using functions.

We should also bear in mind that Rachel was not able to take part in lessons focusing on functions and more specifically, on quadratic function. Thus, the activated milieu did not correspond to the institutional milieu of the class. This fact together with the teacher orienting to students to do modeling explorations certainly influenced Rachel's choice of the topic. She decided thus to conduct a modeling exploration using functions in order to learn more about them as well as about transformations of their graphs. Considering the above circumstances, even for Rachel, the ME was beneficial, fulfilled its objective and resulted in an acquisition of new situational knowledge about quadratic functions. Rachel found the ME challenging but she globally enjoyed doing it.

However, she also encountered several difficulties; some of which she was able to overcome with the help of media or through efficient regulations, while others remained out of her reach. Generally, if

- a) the encountered objects were close to the ones within Rachel's CU,
- b) appeared within the scope of the question,
- c) the media provided a technique for a certain type of tasks explained using a worked example, the inquiry advanced and produced a desirable outcome.

If, however, these three conditions were not met, ill-productive episodes leading to a negative effect on the final answer occurred. Another phenomenon that we observed was the following: certain significantly incorrect or irrelevant answers, in particular, when provided by the teacher, had a tendency to remain in the activated milieu as though it were "parasitized" despite the fact that she encountered correct or relevant accessible information in the media or received targeted regulations.

The lack of the ability to create an appropriate experimental milieu was one of the major obstacles Rachel encountered when validating her responses. She definitely would have required more assistance in this area, which would have enabled her to sufficiently grasp the stakes behind the need of such milieu.

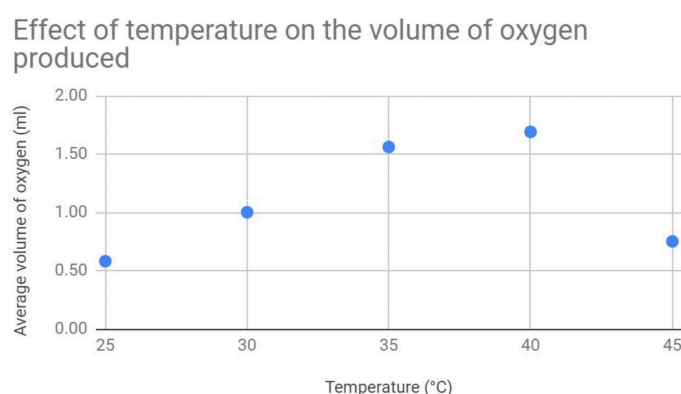
As a conclusion, Rachel's ME clearly illustrates that the reason behind her failure in validating the final correct model lied in her inability to create a functioning experimental milieu so essential to properly validate or reject conjectures. Beyond this, however, we could also see the effect of the teacher's reductive vision of mathematical modeling who, probably with a good intention to adapt the difficulty to secondary school students, prevented interesting questioning and reduced a potential genuine mathematical activity to finding analytically a function that fits a curve. In this sense, Rachel as a good subject of the institution fulfilled the contract, did some mathematics and even enriched her CU of a new praxeology.

In the next chapter, we zoom on our second case study and analyze Sophie's exploration and proceed afterwards to a comparative conclusion of the MEs of both students.

Chapter IV

SOPHIE’S EXPLORATION: THE OPTIMUM TEMPERATURE FOR PRODUCTION OF OXYGEN OF A CABOMBA PLANT

Sophie was keen on biology and chose to investigate the mathematics behind a biology experiment on production of oxygen. She wanted to determine the optimum temperature at which a Cabomba plant produces a maximum of oxygen. She conducted the experiment, collected her own data (Graphics 37), set on searching for the right function that would fit the data and enable her to find the optimum temperature.



Graphics 37: Data from Sophie’s ME—Temperature versus Production of oxygen in ml/20min

1 An “expert” investigation of the production of oxygen of the Cabomba plant

After an initial search on the rate of photosynthesis we found a website⁵³ explaining three limiting factors of photosynthesis and the shape of the corresponding graphs. We focused on the effects of temperature on the production of oxygen and the shape is shown on Figure 78.

⁵³ <https://www.elevise.co.uk/gab4b.html>

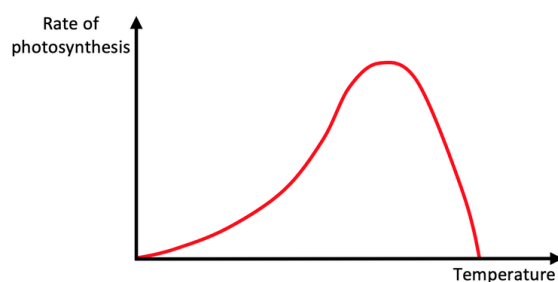


Figure 78: The effect of temperature on the rate of photosynthesis⁵⁴

According to the above-mentioned website, photosynthesis reactions are carried out by enzymes. When examining the shape of this curve, we observe that the rate of photosynthesis increases as the temperature rises. The rise in temperature provides more kinetic energy for the enzymes, leading therefore to more production of oxygen. This happens until the optimal temperature where the rate of photosynthesis is at its highest is reached. However, too high a temperature causes the enzymes to start denaturing, i.e., the bonds that make up the enzyme break. This results in a decreasing rate of photosynthesis. The denaturing temperature depends on the type of plant but is usually around 35–45°C.

After this short excursion into the world of plants, let's get back to mathematics. The shape of the graph suggests first an exponential growth until an inflection point is reached where the curve changes concavity and the rate of change starts slowing down. When the optimum temperature is attained, we observe a significant drop in the oxygen production. We decided to search for mathematical models for temperature as a limiting factor of photosynthesis. Using a Google image search, the circled image on Figure 79 retained our attention and brought us to a scientific article⁵⁵ comparing several models.

⁵⁴<https://www.elewise.co.uk/uploads/9/8/0/2/98021560/published/screenshot-2020-12-06-at-14-09-04.png?1637760522>

⁵⁵ <https://www.nature.com/articles/srep39930>

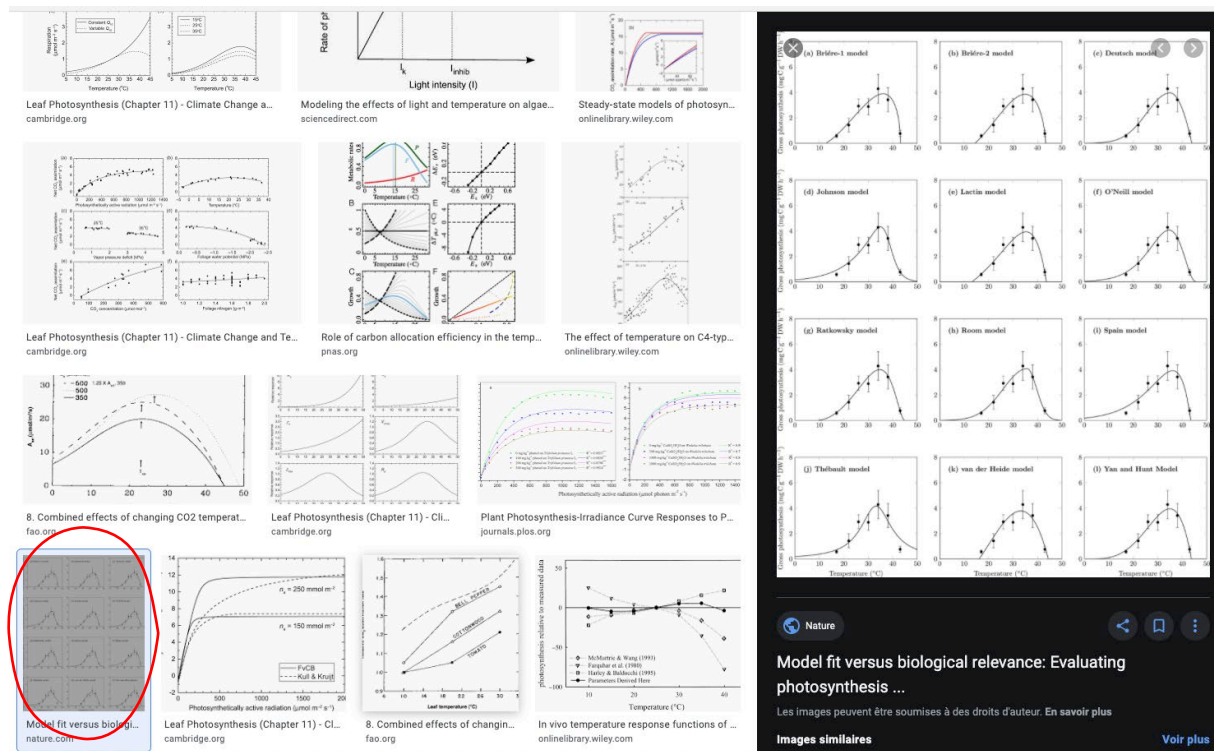


Figure 79: Search for mathematics models for temperature as limiting factor of photosynthesis

According to this article (Adams et al., 2017), “the Yan and Hunt model is the best of the 12 models considered here, when accounting for both the ease at which biologically meaningful parameters can be obtained, and achieving goodness of fit between model and data” (p. 5).

$$P(T) = P_{\max} \left(\frac{T_{\max} - T}{T_{\max} - T_{opt}} \right) \left(\frac{T}{T_{opt}} \right)^{T_{opt}/(T_{\max} - T_{opt})}$$

Figure 80: Yan and Hunt model

In this equation (Figure 80), $P(T)$ is the photosynthesis rate (in ml/minute) at temperature T (in °C), $P_{\max}$ is the maximum photosynthesis rate which occurs at the optimum temperature T_{opt} , and $T_{\max}$ is the temperature, after which the photosynthesis collapses, i.e., no oxygen is produced.

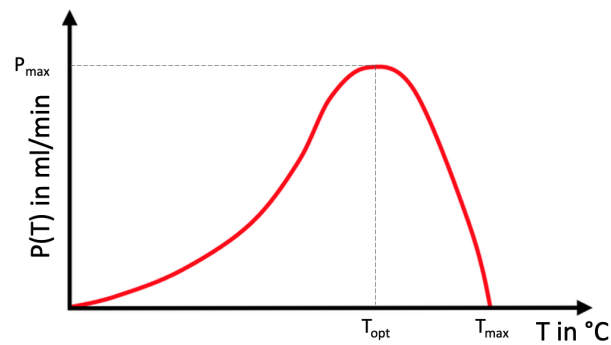


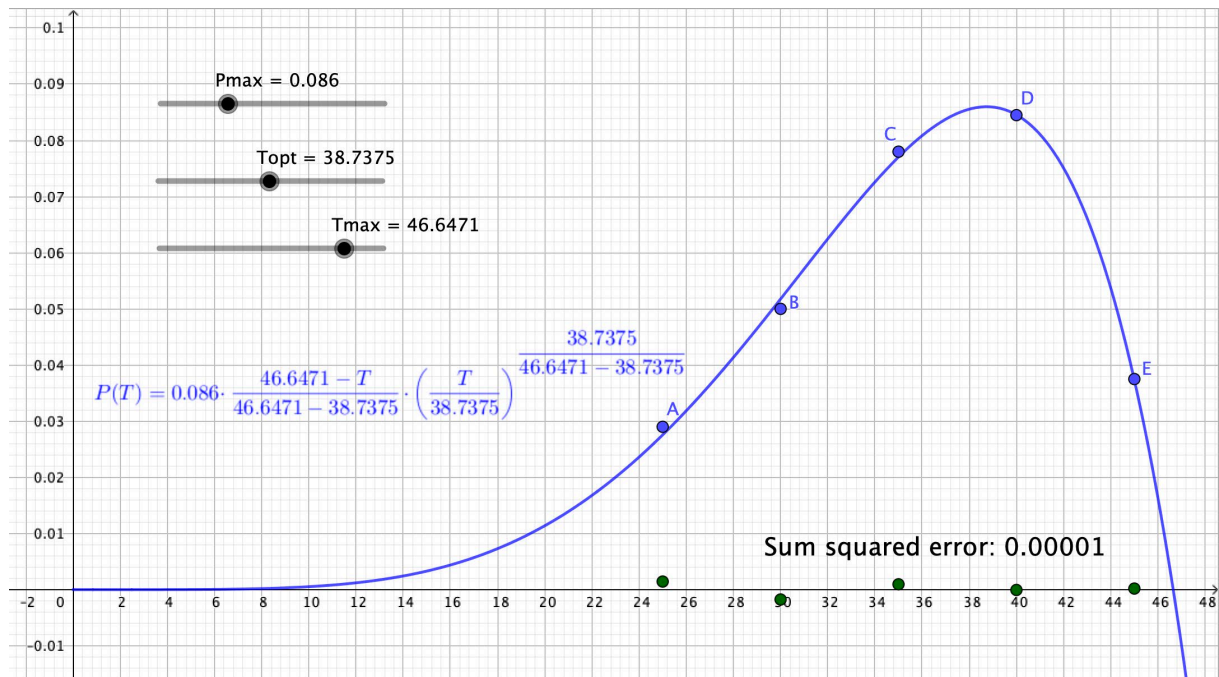
Figure 81: The parameters of the Yan and Hunt model

To find an approximate fit we use the data from Sophie's experiment (Table 37).

Temperature in $^{\circ}\text{C}$	Production of oxygen in ml/20min	Production of oxygen in ml/min
25	0.58	0.029
30	1	0.05
35	1.56	0.078
40	1.69	0.0845
45	0.75	0.0375

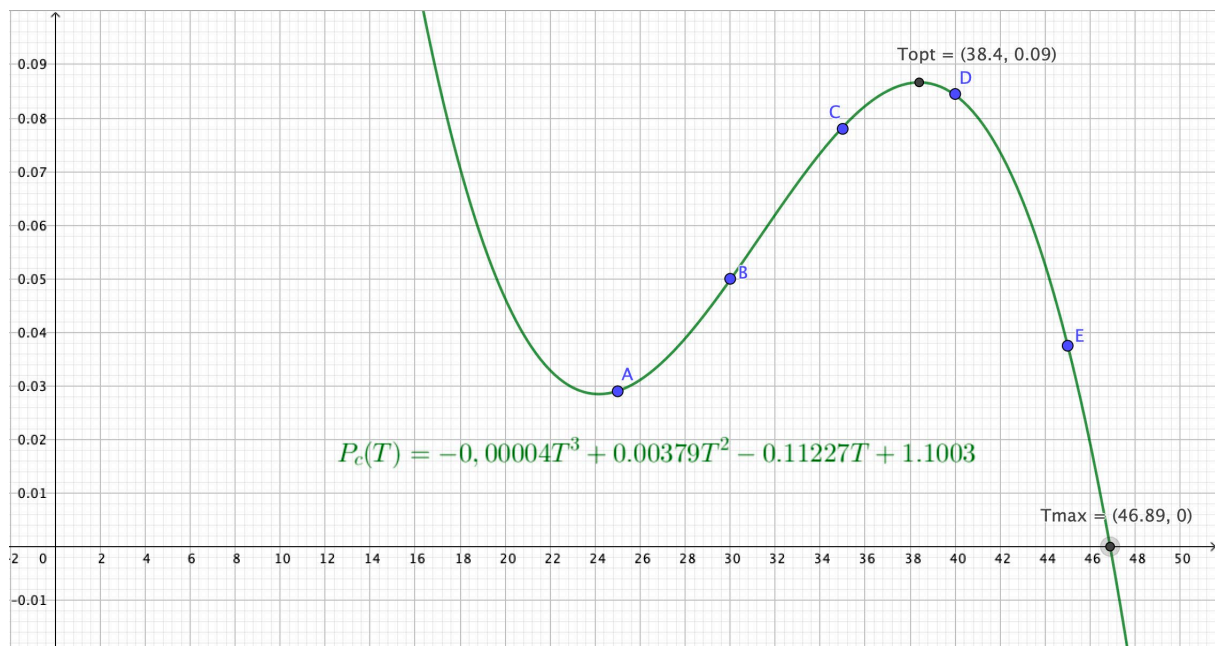
Table 37: Table of values Temperature versus Production of oxygen

An analytical solution using the equation of the Yan and Hunt model (Figure 80) would have led to quite an “ugly” system of three non-linear equations, we therefore used sliders in GeoGebra and obtained a decent fit (sum squared error of 0.00001) as shown on Graphics 38. According to this model, the optimum temperature at which the Cabomba plant produces a maximum of oxygen is approximately at 38.7°C .



Graphics 38: Yan and Hunt's model for fitting the data

Now, when considering the data between 24°C and 47°C, we could probably attempt a cubic polynomial fit (Graphics 39) to predict the optimum temperature.



Graphics 39: Cubic model for the Cabomba plant

For this cubic regression, we obtained a very good fit with a sum squared error of 1.75×10^{-7} . The optimum temperature predicted from the cubic model is at approximately 38.4°C which is quite close to the value from the Yan and Hunt model (38.7°C). At the optimum temperature 0.09 ml/min of oxygen are produced according to the cubic model and 0.86 ml/min for the Yan and Hunts model. And finally, for the maximum temperature is 46.9°C for the cubic model and 46.7°C for the Yan and Hunt

model. We can see that these values are quite close, thus we can conclude that the cubic model, also more accessible analytically, provides a good candidate for an approximation of the optimum temperature for production of oxygen of a Cabomba plant.

To find the cubic polynomial analytically we apply the technique that Sophie attempted to use in her ME; this way it can also serve to better comprehend Sophie's work. The cubic regression used in GeoGebra has got a sum squared error of 1.75×10^{-7} which means that we can consider applying the method of the finite differences to determine the coefficient and then solve a system of linear equations to find the remaining coefficients. When differentiating successively a cubic function, we obtain a constant third derivative:

$$f(x) = ax^3 + bx^2 + cx + d$$

$$f'(x) = 3ax^2 + 2bx$$

$$f''(x) = 6ax + 2b$$

$$f'''(x) = 6a$$

From this we can conclude that: $a = \frac{f'''(x)}{3!}$

Thus calculating the consecutive differences $\Delta = \frac{y_2 - y_1}{x_2 - x_1}$ from our data should lead to an approximately constant third difference and enable us to calculate the coefficient a .

x	y - ml/min	$\Delta 1$	$\Delta 2$	$\Delta 3$	$\Delta 4$	
25	0.029					average delta 3
30	0.05	0.0042				-0.000242
35	0.078	0.0056	0.00028			a=average delta3/3!
40	0.0845	0.0013	-0.0009	-0.00023		-0.0000403
45	0.0375	-0.009	-0.0021	-0.00026	-0.00000560	

Table 38: The method of finite differences applied to the “almost” cubic data

We can see in Table 38 that Δ_4 is almost equal to 0 and we can conclude that Δ_3 is almost constant. To calculate the coefficient $a \approx 0.0000403$, we took the average of the two Δ_3 values and divided by $3!$. Finally, to calculate the remaining coefficients b , c and d , we used the data points (25, 0.029), (35, 0.078) and (45, 0.0375), set a system of 3 linear equations and used Wolfram Alpha to solve it as shown on Figure 82.

Input interpretation	
	$-0.629688 + 625b + 25c + d = 0.029$
solve	$-1.72786 + 1225b + 35c + d = 0.078$
	$-3.67234 + 2025b + 45c + d = 0.0375$
Result	
$b = 0.003784$ and $c = -0.112323$ and $d = 1.10175$	

Figure 82: Solving the system of three linear equations using Wolfram Alpha

These values being quite close to the best fit cubic function obtained using a regression in GeoGebra, make that our investigation stops here. We could have solved the above system analytically but why to waste our energy if Wolfram Alpha did it in one click and no time. We now return to Sophie and her ME and start by describing her profile.

2 Sophie's profile

Sophie has pursued her scholarship as an international student and joined the IB diploma program from the IB middle years program. She describes herself alike that: *I am not the worst but I definitely take longer than other people... I can't just look at a problem and see the immediate solution.* For her, mathematics is *working out problems, any problem where you need to find a number or formula.* She would definitely enjoy mathematics more if it was more real-life connected and made sense as described below.

So often, maths I could apply to other subjects I really enjoy. So just looking at statistics for geography—I really like geography. So looking at GDP and HDI rankings and putting that into graphs and stuff like that I find that very interesting. And of course, what I am doing for my ME which is looking at the rate of photosynthesis. So things I can apply and things I can see in everyday life while I actually learn stuff about the world around me instead of kind of numbers on a paper that I don't understand anything from it.

Sophie also pointed out that the math curriculum did not really reflect what she likes about mathematics. She described the work in the ordinary classes as:

Most of the time when we are in class, we are kind of looking at the formulas that we are learning and then we apply that to different things and things that we haven't seen before and we have to adopt the ideas.

For her, the ME is *a better measure, perhaps, of how good you are at maths because you have to apply it and exams are gonna measure whatever you can do only on the exam as opposed to what you actually are gonna use.* She would like to have different types of internal assessment in form of exploration or projects because she thinks that one learns more when working on a project.

Sophie is keen on biology and that is what she wants to continue to study at the university. Without any doubt, she chose to explore the effect of temperature on the rate photosynthesis based on a former biology experiment. The quote below describes well her attitude towards the ME.

I think I will probably enjoy it much more than I have been enjoying the course because I am allowed to apply it to biology which I am really interested in. So hopefully that will kind of like give me, you know, the motivation to do maths...

In the next part, we analyze Sophie's ME in terms of the chronogenesis and mesogenesis.

3 The chronogenesis of Sophie's ME

The Q-A level analysis consists of two parts: the chronogenesis of Sophie's ME and an analysis in terms of QA-units taking into account the theme, productivity and type of episode.

3.1 Analysis in terms of clusters and content of Sophie's ME

The generating question of Sophie's ME is: "How does temperature affect the rate of production of oxygen of a Cabomba plant?". The whole ME consists of 422 E-units split into 59 QA-units and unfolds around the following 8 themes:

Theme 1: Methodology

Theme 2: Biology—rate of photosynthesis

Theme 3: Choice of model

Theme 4: Exploration of the quadratic model

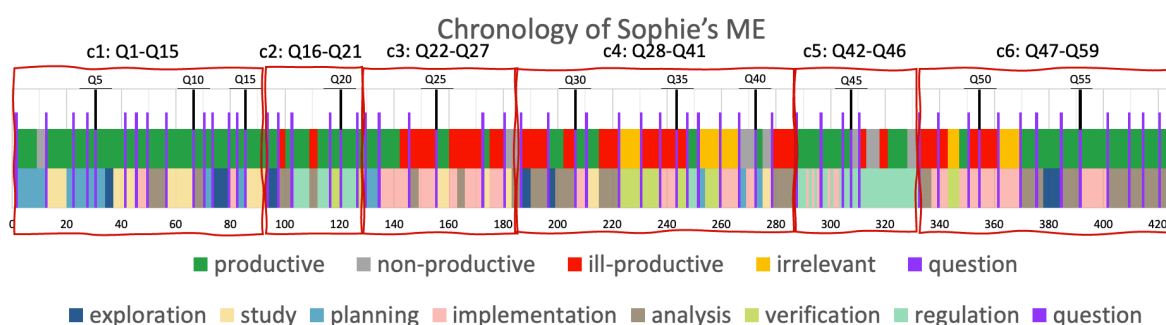
Theme 5: Exploration of the cubic model

Theme 6: Exploration of the quartic model

Theme 7: Calculus

Theme 8: Writing the report

To tackle the chronogenesis of the ME, we divided the chronology into 6 clusters as shown on Graphics 40.



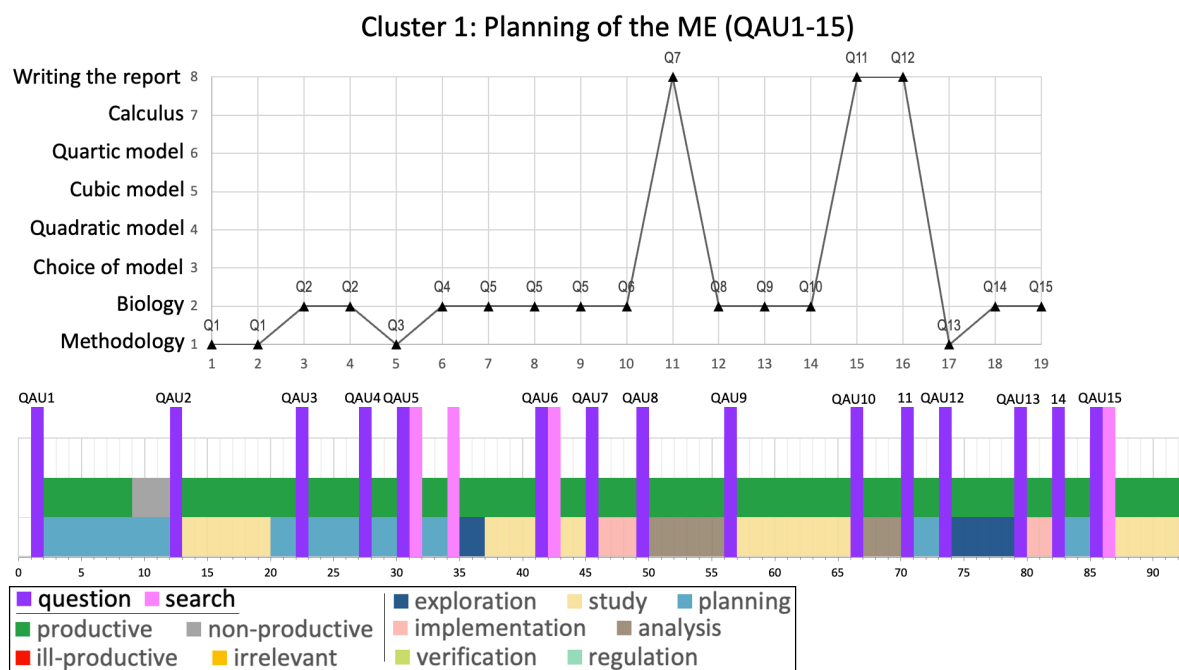
Graphics 40: Chronology of Sophie's ME split into clusters

In each cluster, we describe in detail Sophie's activity and attempt to depict the plausible reasons of her actions.

3.1.1 Cluster 1: Planning of the ME (QAU1—QAU15)

Cluster 1 contains 15 QAUs and 19 episodes which are mostly dedicated to planning and study and is overall productive. The activity unfolds around three themes, mostly in theme

2: Biology (12 episodes) with 4 episodes in theme 1 (Methodology) and 3 episodes in theme 8 (Writing the report) as can be seen on Graphics 41. For the type of episodes, this cluster contains mostly planning (8) and study (5) episodes and only 2 exploration and 2 implementation episodes. The lack of exploration episodes at the beginning of the ME could be explained by the fact that at this stage Sophie was set on the choice of topic based on a prior biology experiment.



Graphics 41: Cluster 1 of Sophie's ME

QAU 1 starts by filling out a document (GD1.1) provided by the teacher containing a rubric with various criteria (Appendix 22) to plan the ME. The rubric D (What research you will need to do or what data you will need to collect?) brought her to the following two questions: *Q2 What biology knowledge do I need?* and *Q3 What mathematics will I need?* In QAU2, Sophie studied the limiting factors of photosynthesis and their graphics from the biology textbook (Figure 83).

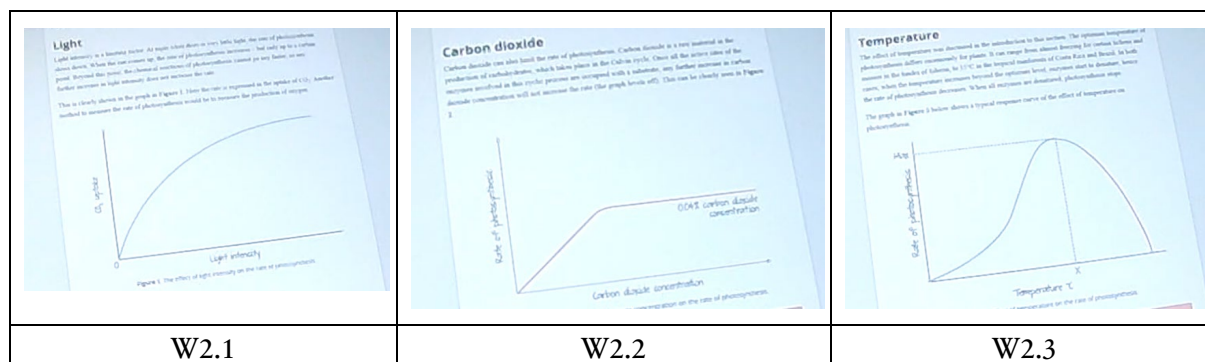


Figure 83: Graphics of limiting factors of photosynthesis

In QAU 3, she gave an outline of the mathematics she thought useful for her ME. She continued filling out GD1.1 and in QAU 4, specified more precisely the object of her ME

(A♥4.1 *During this task I shall be investigating the effect of the limiting factors (carbon dioxide and temperature) on the rate of photosynthesis*). In QAUs 5 and 6, Sophie set on searching for and studying information about setting up experiments to measure the rate of photosynthesis. In QAUs 7 and 8, she returned to filling out the rubrics C and E in GD1.1 (Appendix 22). This document enabled her to plan and specify and justify the theme of her ME. She thus narrowed down her investigation to two limiting factors (Carbon dioxide and Temperature) and retrieved the shape of their graphics. In QAU 9, Sophie pursued the study of the various ways to measure the rate of photosynthesis and concluded that the best option was to investigate the effect of temperature on the production of oxygen because she could use the school's lab gear and obtain reliable data. QAU 10 is thus dedicated to the planning of the experiment and analyzing of the conditions for its set-up. In QAU 11, Sophie created a new Google document (GD11.1) to keep track of the links to the visited websites and to take notes. QAU 12 is an example of an exploration episode leading to an exchange with her science teacher. She explored the purpose of finding the optimum temperature putting it into a larger context of climate change and global warming. Inspired by a website (WS12.1 A trillion trees), she wanted to create a map of where plants could be grown to maximize CO₂ absorption and oxygen production. This led to formulating the ME question (A♥13.1 *How can finding a plant's optimum temperature for photosynthesis help countries maximize their environmental sustainability?*), which showed; however, too ambitious and was reduced to its current form. QAU 14 contains a planning episode concerning the type of data that needs to be collected (A♥14.1) and QAU 15 contains a study episode of a protocol of an experiment designed to measure the effect of temperature on photosynthetic rate of a pondweed. At the end of this cluster, Sophie had a plan on how to conduct the biology experiment, to organize the data and knew the approximate shape of the curve to expect that she reproduced from the textbook (Figure 83—W2.3) to her logbook (Figure 84). We refer to this graphic as to works W2.4 Temperature vs. photosynthesis

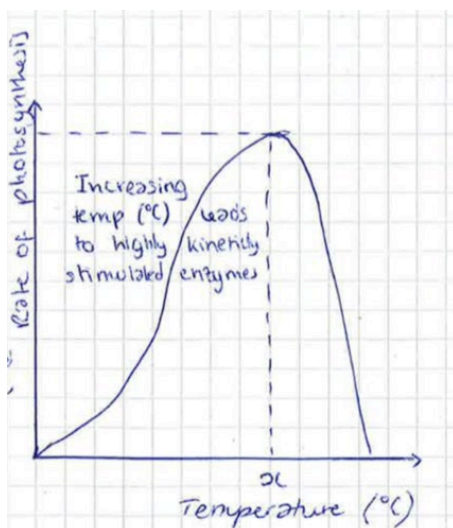


Figure 84: The reproduced shape of the graphics W2.3 describing the effect of temperature on the rate of photosynthesis

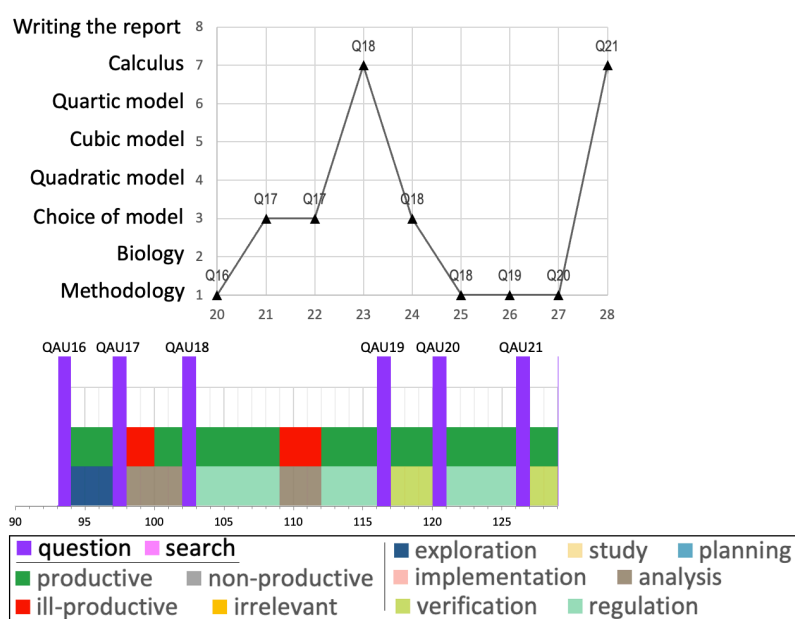
Cluster 1 is mostly dedicated to methodological questions and study of biology experiments that would enable Sophie to capture the rate of photosynthesis. A planning document (GD1.1) with various rubrics provided by the teacher guided the initial questioning. She was able to formulate the exploration question and determine the feasibility of the experiment in the school lab settings which oriented her choice towards exploring the effect of temperature on the rate of photosynthesis.

With this information in mind, Sophie approached her mathematics teacher to discuss the possible mathematics to do in her ME.

3.1.2 Cluster 2: Interview with the teacher (QAU16—QAU21)

Cluster 2 depicts the interactions between Sophie and her math teacher about how she could possibly explore the data from the biology experiment. It contains 9 episodes within 6 QAUs. The addressed themes concern methodology, choice of model and calculus. This cluster is overall productive; however, 2 ill-productive analysis episodes occur. The detail of theme, productivity and type of episode is provided in Graphics 42.

Cluster 2: Interview with the teacher (QAU16-21)



Graphics 42: Cluster 2 of Sophie's ME

In QAU16, Sophie asked how to treat the data, and the teacher showed how to organize it in a table of values and plot a scatter plot on a set of axes (A*16.1 and A*16.2) as show on Figure 85.

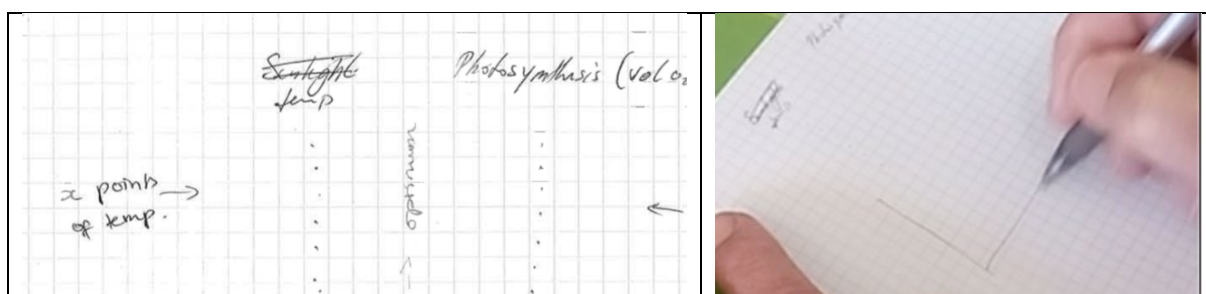


Figure 85: Detail of answers A*16.1 and A*16.2

This was followed by the first ill-productive analysis episode QAU17_Ep21) where Sophie interpreted the shape of the curve from W2.3 (Figure 84) as parabolic and added that there would be an optimum temperature (productive analysis episode QAU17_Ep22). In QAU18, the teacher relied on this information without verifying the shape of the curve and proposed some possible paths to explore the data:

- A*18.1 *You can look at the rate of change using calculus*
- A*18.2 *You can try the best some function fit*
- A*18.3 *You can try the best parabola fit*

After providing A*18.3, the teacher turned to Sophie and asked again whether the data looked like a parabola. She confirmed, which was marked as an ill-productive analysis episode (QAU18_Ep_24) and the teacher gave her a plan to follow:

- A*18.4 *So you get the parabola that fits*

- A*18.5 *You use the derivative to determine the maximum*
- A*18.6 *And that's enough, you model it and you use calculus to examine it.*

On QAU19, Sophie wanted to verify the feasibility of her big idea, i.e., to create a map where plants should be grown using the optimum temperature, which was approved by the teacher but never explored probably due to lack of time. In QAU20, the teacher resumed what to do and reminded her keep it simple and to focus on the mathematics. Figure 86 shows the extract of the teacher's explanations based on Sophie's interpretation of the shape of the data.

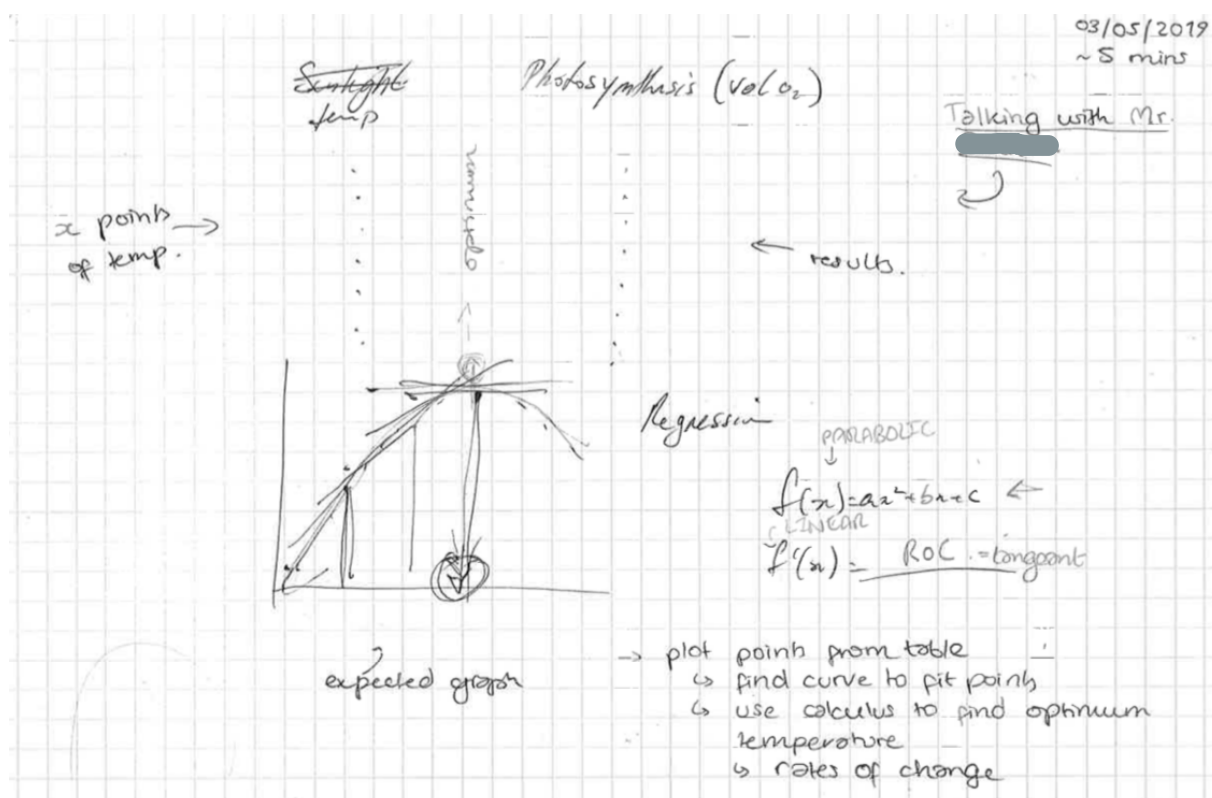


Figure 86: Logbook notes from the exchange with the teacher

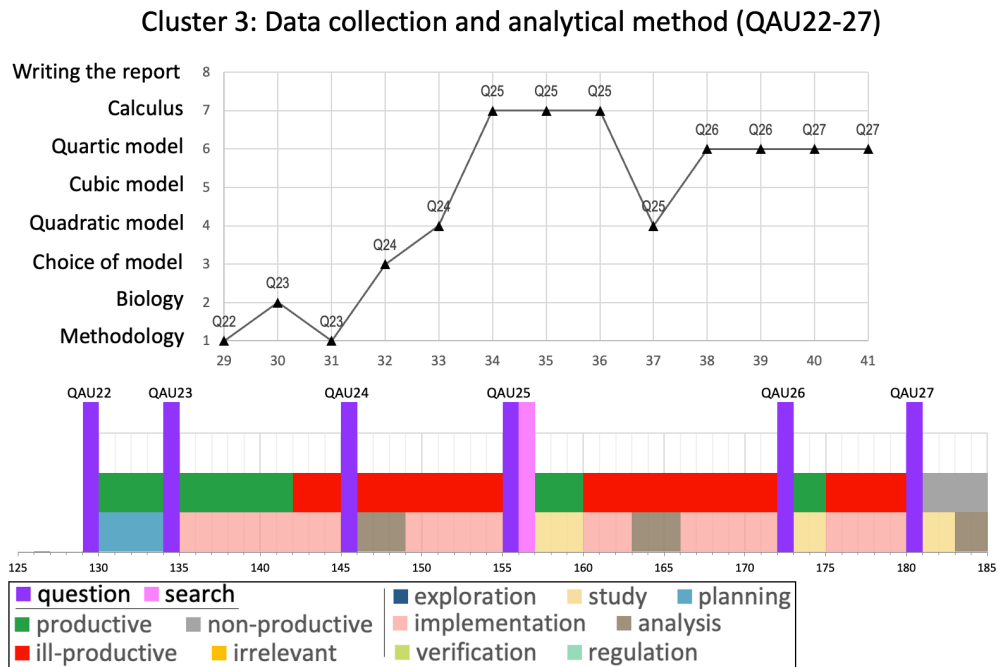
In QAU21, Sophie verified if she should use the derivative to find the optimum temperature and the teacher explained the optimum temperature would occur where the derivative is zero.

Apparently, she was all set to continue the ME but something went wrong and her conviction that the shape is a parabola led to other ill-productive episodes which we show in the next cluster.

3.1.3 Cluster 3: Data collection and analytical method (QAU22—QAU27)

Cluster 3 starts with a planning episode that resumes the exchange with the teacher (QAU22_Ep29)). It is, on the one hand, dedicated to data collection from the biology experiment (QAU23_Ep30: productive implementation) and its organization in a table of

values using Google sheets (QAU23_Ep31: ill-productive implementation). On the other hand, the prevailing ill-productive part of this cluster is situated in theme 1 (methodology) and consists of attempting to find a method to determine a polynomial equation analytically from data. The detail of theme, productivity and type of episode is provided in Graphics 43.



Graphics 43: Cluster 3 of Sophie's ME

After having written down a plan in QAU22, retrieved the data in QAU23_Ep30, the difficulties started to cumulate resulting in a series of ill-productive implementation and analysis episodes. In QAU23_Ep31, the ill-productive implementation episode corresponds to incorrect rounding when changing the units of the production of oxygen from ml/20min to ml/min. We can see in Figure 87 that $f(35)=1.56$ and $f(40)=1.69$ when expressed in ml/20min, however, $f(35)=f(40)=0.08$ when expressed in ml/min.

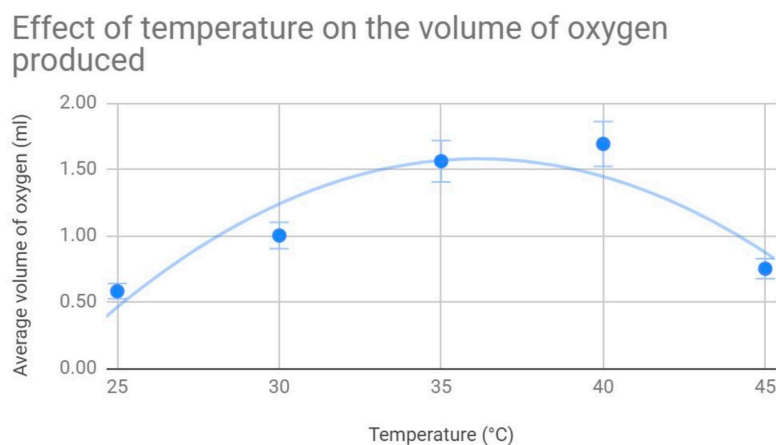
Temp. (°C)	Volume of oxygen gas produced during 20 minutes (ml)								
	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5	Trial 6	Trial 7	Trial 8	Average
25	0.50	0.5	0.50	0.50	1.00	0.60	0.50	0.50	0.58
30	1.00	0.90	1.00	1.50	1.00	0.80	1.00	0.80	1.00
35	2.00	1.00	1.00	2.00	2.00	1.00	2.00	1.50	1.56
40	2.00	1.50	2.00	1.00	1.50	2.00	2.00	1.50	1.69
45	0.50	1.00	1.00	0.50	1.00	1.00	0.50	0.50	0.75

I then calculated the rate of oxygen production by dividing the number of milliliters of oxygen produced by the number of minutes taken for the experiment to take place. This produced the following data:

Temp. (°C)	Rate of production of oxygen (ml/minute)								
	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5	Trial 6	Trial 7	Trial 8	Average
25	0.025	0.025	0.025	0.025	0.050	0.030	0.025	0.025	0.03
30	0.050	0.045	0.050	0.075	0.050	0.040	0.050	0.040	0.05
35	0.100	0.050	0.050	0.100	0.100	0.050	0.100	0.075	0.08
40	0.100	0.075	0.100	0.050	0.075	0.100	0.100	0.075	0.08
45	0.025	0.050	0.050	0.025	0.050	0.050	0.025	0.025	0.04

Figure 87: Organization of the data (excerpt from Sophie's ME)

QAU24 contains an ill-productive analysis followed by an ill-productive implementation episode. Sophie interpreted the shape of the graphics representing the temperature versus production of oxygen (W2.3) as parabolic and used a quadratic regression to find the best fit for the data as shown in Graphics 44. Even though the parabola did not follow the general trend of the data, Sophie pursued the exploration of the quadratic model and set on finding the equation of the parabola analytically.



Graphics 44: Quadratic regression

QAU25 starts with a productive study episode. In this episode, Sophie searched for a method to determine the coefficients of a polynomial and watched a video (QAU25-Ep34-V25.1) explaining how to apply the method of the finite differences to determine the degree of a polynomial from data and to calculate the value of the coefficient a . This method was demonstrated on the following example (see Figure 88).

Polynomial Equation From Data **Anil Kumar**

The following represent a set of points that lie on the graph of a function. Determine the equation of the polynomial function using an algebraic method.

(1, 0), (2, -12), (3, -10), (4, 18), (5, 84), (6, 200)

Figure 88: Excerpt from video V25.1⁵⁶—example

The video was entitled *Polynomial equation from data* what probably attracted Sophie's attention and provided only the praxis part of a praxeology without any technological or theoretical justification as shown in Figure 89.

(1, 0), (2, -12), (3, -10), (4, 18), (5, 84), (6, 200)

x	y	Δ^1	Δ^2	Δ^3
1	0			
2	-12	-12		
3	-10	2	14	
4	18	28	26	12
5	84	66	38	12
6	200	116	50	12

Cubic Function $\Delta^3 = 12$
 $a = \frac{\Delta^3}{3!} = \frac{12}{3 \times 2 \times 1} = 2$
 $f(x) = 2x^3 + bx^2 + cx + d$

Figure 89: Excerpt from video V25.1—technique

Sophie was able to reproduce this technique and apply it more or less correctly to her data (Figure 90) but committed a certain amount of calculation and inattention errors.

14/05/2020
20 min with video

x	y (average)	$\frac{y_2 - y_1}{x_2 - x_1}$	Δ_2	Δ_3	Δ_4
25	0.03				
30	0.05	-24.9			
35	0.08	-29.9	5		
40	0.08	-34.2	4.3	0.7	
45	0.04	-39.9	5.7	1.4	0.7

Cubic function $\leftarrow$
 difference = $a = \frac{\Delta_4}{4!} = \frac{0.7}{4 \times 3 \times 2 \times 1} = \frac{0.7}{24} = 0.029$
 $a = 0.0291666664 \rightarrow f(x) = 0.0292x^4 + bx^3 + cx^2 + dx + e$
 or maybe $0.0293x^4 + bx^3 + cx^2 + dx + e$

Figure 90: Application of the technique from V25.1

⁵⁶ <https://www.youtube.com/watch?v=YORfLpnppOU>

Below is a list of the accumulated errors when Sophie attempted to apply the method of the finite differences:

- 1) Sophie did not use the same units for the regression (ml/20min) and for the method of finite differences (ml/min)
- 2) The calculation of Δ_1 column is wrong because of missing parentheses:
 line 1: $\Delta_1 = \frac{y_2 - y_1}{x_2 - x_1} = 0.05 - 0.03 \div 30 - 25 \approx -24.9$
 line 2: $\Delta_1 = \frac{y_2 - y_1}{x_2 - x_1} = 0.08 - 0.05 \div 35 - 30 \approx -29.9$
 line 3: $\Delta_1 = \frac{y_2 - y_1}{x_2 - x_1} = 0.08 - 0.08 \div 40 - 35 \approx -34.2$
 line 4: $\Delta_1 = \frac{y_2 - y_1}{x_2 - x_1} = 0.04 - 0.08 \div 45 - 40 \approx -29.9$
- 3) Δ_2 , Δ_3 and Δ_4 were not divided by Δx
- 4) Δ_2 , Δ_3 and Δ_4 should have been all negative
- 5) Δ_4 is determined to be constant based on one entry.
- 6) Sophie wrote down Cubic function for a fourth “constant” difference—probably because the example in the video showed a cubic polynomial

Taking into account these “number crunching” errors, Sophie’s application and interpretation of the method of the finite differences were so far correct. Sophie was even able to adopt the formula used for calculating the coefficient a from degree 3 provided in the video to degree 4 using her constant Δ_4 value. Afterwards, however, she applied the calculated coefficient for Δ_4 universally (in a quadratic polynomial and a quartic polynomial) which suggests that a justification of this technique by a technology would have been needed.

This video (V25.1) shows then how to determine the remaining coefficients of the cubic polynomial using a system of 3 linear equations. In QAU26, Sophie watched this part of the video and set up a system of 3 linear equations (actually 4 would have been needed) using $f(25) = 0.03$, $f(30) = 0.05$ and $f(35) = 0.08$ with further calculation error due to misinterpretation of the result given in scientific notation on her calculator. The result $11,406 \cdot 3$ highlighted in yellow in Figure 91 comes from the scientific notation of $11\,406 \times 10^3$ which on a calculator screen looks like 11.406E3

$f(25) \hookrightarrow 11,406.3 + 15,625b + 625c + 25d + e = 0.03$
 $f(30) \hookrightarrow 23,652 + 27,000b + 900c + 30d + e = 0.05$
 $f(35) \hookrightarrow 43,818.3 + 42,875b + 1225c + 35d + e = 0.08$
 $\hookrightarrow$ would do simultaneous equations.

Figure 91: Setting up a system of 3 linear equations

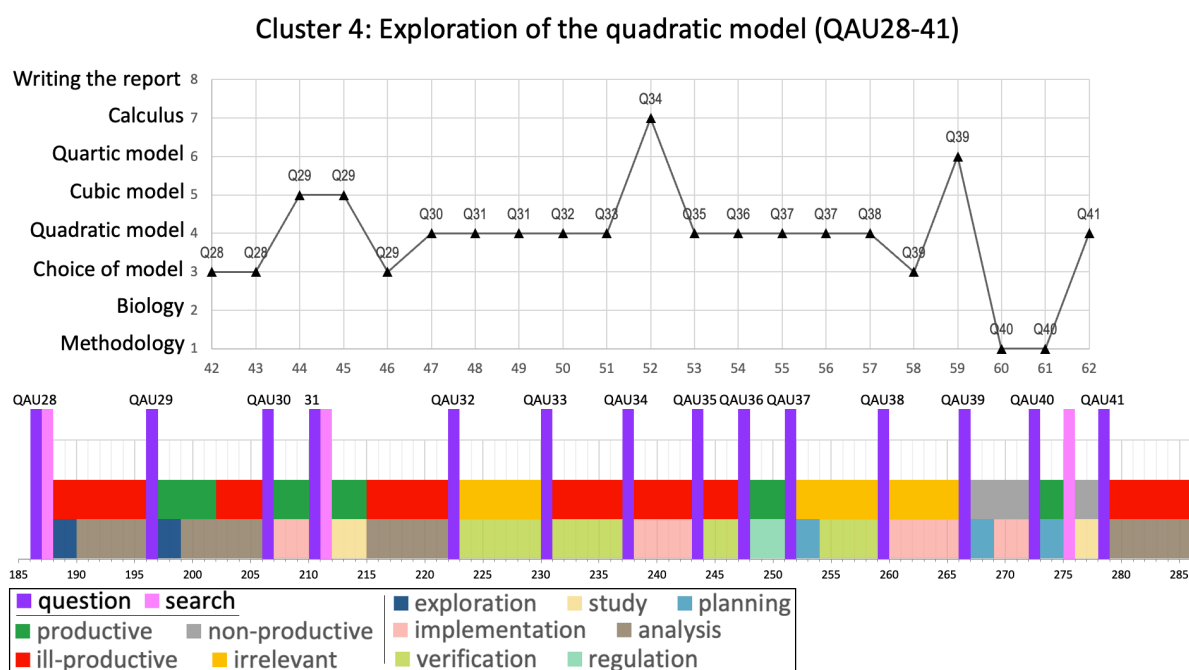
In QAU 27, Sophie then wrote down in the logbook *would do simultaneous equations* but never managed to learn how to solve them.

The ill-productive episodes in this cluster reveal the lack of basic mathematics skills and point out the consequences of applying a technique without the technological discourse of the given praxeology. The given video provided a technique to determine the degree of a polynomial from data. In addition to the calculation and inattention errors, this episode shows the need to understand the conditions in which a given technique can be applied to a given situation. Here the study of those condition was insufficient because the video did not clearly state the circumstances in which the given technique could be used. In Sophie's case, an adaptation of the technique of the finite differences was possible as we showed in the "expert" exploration but required an understanding of the technology justifying this technique.

Since Sophie got probably stuck with solving the above system of linear equations and as a consequence, she moved to explore other models, mostly the quadratic model. Her activity is analyzed in the next cluster.

3.1.4 Cluster 4: Exploration of the quadratic model (QAU28—QAU41)

Cluster 4 is mostly dedicated to exploration of the quadratic model with some occasional trips to cubic and quartic model, calculus and methodology as shown on Graphics 45. We also continue to see the prevalence of ill-productive and irrelevant episodes



Graphics 45: Cluster 4 of Sophie's ME

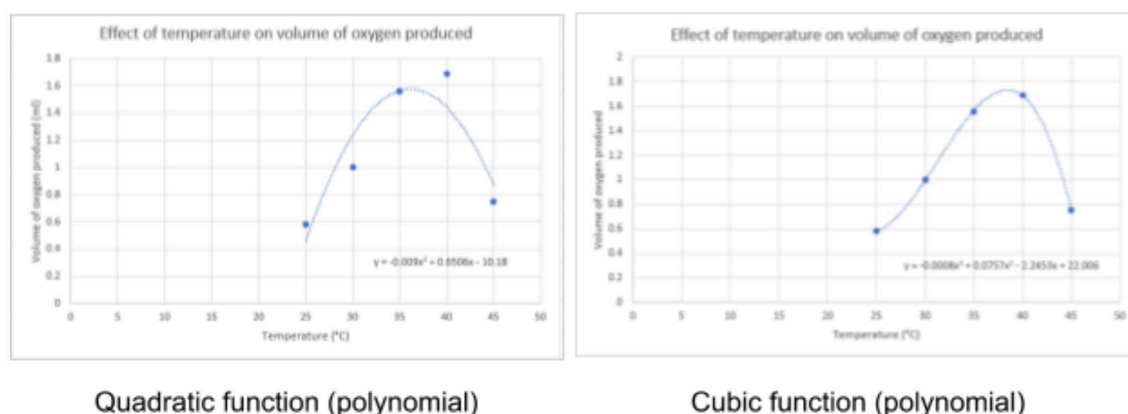
After the inability to solve the system of linear equations from the previous cluster, Sophie attempted to explore something more accessible: the quadratic model. She started by justifying the shape of her data to form a parabola. QAU28 is composed of an ill-productive exploration and an ill-productive analysis episode. Sophie searched for a new image

(W28.1) showing the effect of temperature on the rate of photosynthesis despite the fact that she already had W2.3 and W2.4. The three images are shown in Table 39.

W2.3	W2.4	W28.1

Table 39: Available works for Temperature versus rate of photosynthesis graphs

We want to point out here that she rejected the W2.3 and W2.4 and retained W28.1 because it was the one that resembled most the wanted parabolic shape. Based on this image, Sophie concluded the shape to be parabolic. However, in QAU29, she also ran a cubic regression on the data in Google sheets and obtained a decent fit within the range of her data points and provided the graphics of the quadratic and cubic regressions next to each other in her ME as shown in Graphics 46.



Graphics 46: Quadratic and Cubic regressions from Sophie's ME

Despite the good fit of the cubic polynomial, Sophie decided to continue with the quadratic function. We could explain this by the fact that, the quadratic function, its equation and the transformations of the graph were part of the institutional milieu and should be thus in her praxeological equipment. In addition, the teacher insisted on finding the equation analytically to score in Criterium E and as Sophie was probably unable to process the cubic function. She decided to investigate the quadratic model (QAU29_Ep46) and justified this choice by considering the data point (40, 1.69) as an anomaly and discarding it (*A ♥29.2 When discarding $f(40)$ as an anomaly, the graph becomes a parabola*). In QAU30, she was able to mobilize the general form of a quadratic equation as well as to set up a system of three linear equations using three different data points as shown in Figure 92. The solution of this system remained, however, out of her reach.

Figure 92: An attempt to solve a system of 3 linear equations

Vertex form of a quadratic function = $y = a(x-h)^2 + k$, $(h, k) = \text{vertex}$

↳ $y = a(x-35)^2 + 1.56$

↳ $y - 1.56 = a(x-35)^2 \rightarrow (x-35)(x-35)$
 $x^2 - 35x - 35x + 1225$

↳ $y - 1.56 = a(x^2 - 70x + 1225)$ ✓

↳ $y - 1.56 = \overset{-0.029}{a}x^2 - 70ax + 1225a$ ✓ } $+1.56$

↳ if $a = -0.029 \rightarrow p(x) = -0.029(x-35)^2 + 1.56$

This resulted in an ill-productive analysis episode and led to irrelevant and ill-productive episodes in QAU32 respectively QAU33. Indeed, Sophie wanted to make sure that her calculations were correct but in the mobilized verification mechanisms she used a circular definition and applied incorrectly a mathematical concept from calculus as shown in Figure 94. She actually set the vertex at (35, 1.69) to find the equation and then plugged it into the equation and interpreted the obtained 0 as the rate of change at the optimum temperature.

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$y - 1.56 = -0.029(x - 35)^2$
 $f(35) - 1.56 = -0.029(35)^2 - 70(0.029)(35) + 1225(-0.029) = 0$
 Checking if it works
 Correct - Should be no role of change, as optimum is reached!
 $f(35) - 1.56 = 0$
 $\hookrightarrow f(35) = 1.56$ ✓

Figure 94: Sophie's verification mechanism

QAU34 represents a short trip to calculus that resulted in an ill-productive implementation episode. She attempted to calculate the rate of change at each point but this path was later abandoned. We think that this episode occurred as an effect of the didactic contract because the teacher explicitly asked her to do some calculus. We also think that Sophie got confused by the terminology and interpreted the rate of photosynthesis as the rate of change in calculus. In QAU35, we have another ill-productive verification episode. Sophie calculated the first derivative of the found quadratic function and due to errors in signs, she obtained the optimum temperature at $t = -35^\circ\text{C}$. This error was corrected by a tutor in the logbook and the calculation were carried out “correctly” in the final version. Nevertheless, she was again “stuck” in a circular definition and applying the derivative to find the maximum could be possibly interpreted as an effect of the didactic contract. The verification attempt continued in QAU37 and this time it resulted in an irrelevant planning and verification episode. Sophie wanted to verify if the calculations were correct and superimposed the vertex form on the standard form and concluded that since both functions give the same graph, her calculations were correct. In QAU38 she calculated correctly the values of $f(25)$, $f(30)$, $f(40)$ and $f(45)$, stated that $f(35) = 0$ and interpreted these values as Δy without actually using the derivative.

In QAU39 she attempted to find a quartic polynomial and was able to set up a system of linear equations that would have enabled her to determine the missing coefficients as shown in Figure 95. We also point out that a fourth equation would have been needed to solve this system.

$$\begin{aligned}
 f(25) &= -11,406 + 15,625b + 625c + 25d + e = 0.03 \\
 f(30) &= -23,652 + 27,000b + 900c + 30d + e = 0.05 \\
 f(25) &= -43,818.3 + 42,875b + 1225c + 35d + e = 0.08
 \end{aligned}$$

Figure 95: Setting up a system of linear equations

This led to a non-productive study episode in QAU40 where she found and watched a video (V40.1) but did not retrieve the needed technique. An extract of V40.1 is shown in Figure 96.

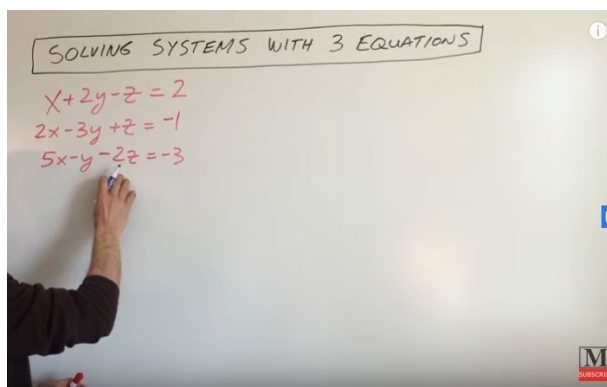
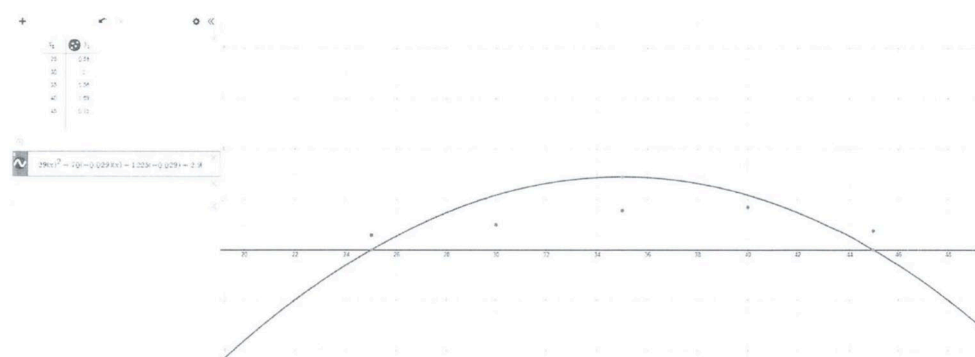


Figure 96: V40.1 providing a technique to solve a system of three linear equations⁵⁸

Sophie abandoned again the idea of solving a system of linear equations and returned to the quadratic function in QAU41 which gave another ill-productive analysis episode. Sophie actually realized that something was wrong with the quadratic function because this model predicted a negative production of oxygen at 25°C and 45°C. This was a correct observation but again, she was unable to find an appropriate solution. She actually “moved” the parabola up by changing the coefficient c in a way that that the x-intercept was at 25, respectively 45 (Figure 97) and concluded that the optimum temperature at which the Cabomba plant produces the maximum volume of oxygen is 35°C.

A better fit for the function used the c value: 2.9, instead of 1.56, as it crossed the correct x-intercepts. This parabola can be seen below:



This parabola, with the function: $y = -0.029(x)^2 - 70(-0.029)(x) + 1225(-0.029) + 2.9$ best represents the relationship between temperature and the volume of oxygen produced.

Figure 97: Conclusion of the first draft

For Sophie, the mathematics part turned out to be a source of important difficulty and she would have definitely needed more help to put in place the missing praxeological infrastructure. Nevertheless, the presence of the many analysis and verification episodes shows that Sophie had some good “inquiry” instincts but unfortunately their successful

⁵⁸ <https://www.youtube.com/watch?v=UcYbFN49uGc&feature=youtu.be>

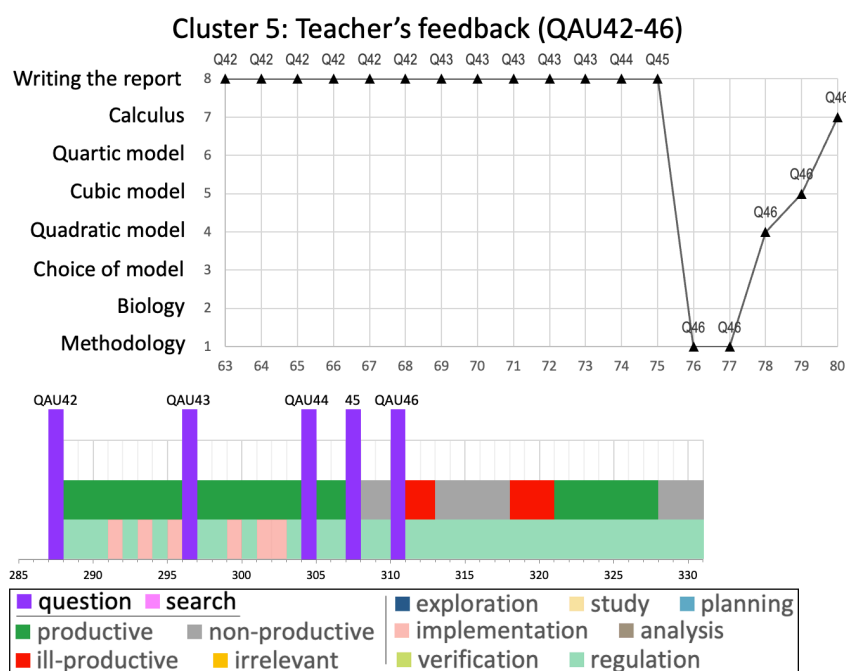
implementation was largely blocked by the missing mathematical knowledge. Sophie handed in the first draft with a very lame conclusion and in the following 2 clusters we analyze what feedback the teacher was able to provide and how she was able to implement it.

3.1.5 Cluster 5: Teacher's feedback (QAU42—QAU46)

In cluster 5 we examine the teacher's feedback according to the five assessment criteria which we analyzed in Chapter V of Part B (see p. 97). Each QAU corresponds to feedback on a given criterion as detailed below:

- QAU42: Feedback on Criterion A—Communication
- QAU43: Feedback on Criterion B—Mathematical Presentation
- QAU44: Feedback on Criterion C—Personal engagement
- QAU45: Feedback on Criterion D—Reflection
- QAU46: Feedback on Criterion E—Use of mathematics

In QAUs 42 and 43 we also included the implementation episodes because they were relatively short and the context was suitable for doing it this way. On the other hand, we analyze the implementation on the teacher's feedback on Criteria D and E in a separate cluster. We can see on Graphics 47 that globally, the feedback on the first three criteria (QAUs42-44) was productive and implemented correctly. However, as far as the Criteria D (QAU45) and E (QAU46) are concerned it does not seem very efficient because it led to several non-productive, respectively ill-productive episodes which we describe and analyze in detail below.



Graphics 47: Cluster 5 of Sophie's ME

As already mentioned, Sophie was able to correctly implement the teacher's feedback on Criterium A Communication and B Mathematical representation by rewriting the introduction paragraph and formulating clearly the objective of the ME (QAU42) and correcting various details concerning terminology and mathematical representation (QAU43) such as adding units, labeling axes, etc. In QAU 44, the teacher recognized a good personal engagement in the ME as *obviously something you are keen on*.

In QAU45, the feedback on Criterium D Reflection (*A45.1 Reflection could be deeper—link to aim*) was rather vague and difficult to implement. Nevertheless, to our surprise, Sophie was able to go deeper in her reflection despite this rather insufficient regulation. We get back to it in detail in cluster 6.

Finally, in QAU46 the teacher provides feedback on the use of mathematics. It starts with an ill-productive episode (QAU46-Ep76) based on the teacher's comment in *A*46.1 Your maths is correct*, which, as we saw in the previous clusters, was not really true. He probably meant that most of the calculations were correct because this comment was also linked to the lack of reflection. Nevertheless, it's formulation was quite unhappy and reinforced Sophie's conviction about using the mathematics correctly. The following comments led to a non-productive episode (QAU46-Ep77):

- *A*46.2 You could incorporate some more maths. Eg.: Rate of production of oxygen at a particular time*
- *A*46.3 I don't think this works (method of finite differences)*
- *A*46.4 Why don't you substitute in the points you have, solve each one for and then find the mean of all the a's you get?*

The teacher provides rather superficial feedback which is comprehensible because he is unable to treat the questions in depth. Sophie wasn't able to process and implement any of the above comments into her ME.

The next episode (QAU46-Ep78) is ill-productive and concerns theme 4 (Quadratic model). The teacher left again a rather vague comment concerning the vertex:

- *A*46.5 Where did the vertex come from? Did you read it from the graph?*

In addition, the part in which Sophie adjusted the x-intercepts of the parabola was marked as correct.

- *A*46.6 Adjusting the x-intercepts of the parabola (25, 0) and (45, 0) marked as correct.*

The only productive regulation episode (QAU46-Ep79) provides feedback on the cubic model and partially on the implementation of calculus:

- *A*46.7 Maybe it is cubic—looks good*
- *A*46.8 Maybe you should have modeled using a cubic.*
- *A*46.9 $y=ax^3+bx^2+cx+d$*

- A*46.10 *You could substitute all your x and y and create simultaneous equations to find a, b, c, d*
- A*46.11 *You could use calculus to figure out the maximum.*

Again, we examine in detail how Sophie handled the cubic model in cluster 6.

The last episode in QAU46 is non-productive and concerned theme 7 (Calculus). The teacher questioned the $\Delta y/\Delta x$ calculation (A*46.12 *Why did you find $\Delta y/\Delta x$?*) which was removed from the final version and suggested to find the rate of change at a point and check with her rate of change. We think that even the teacher interpreted the rate of photosynthesis as a rate of change.

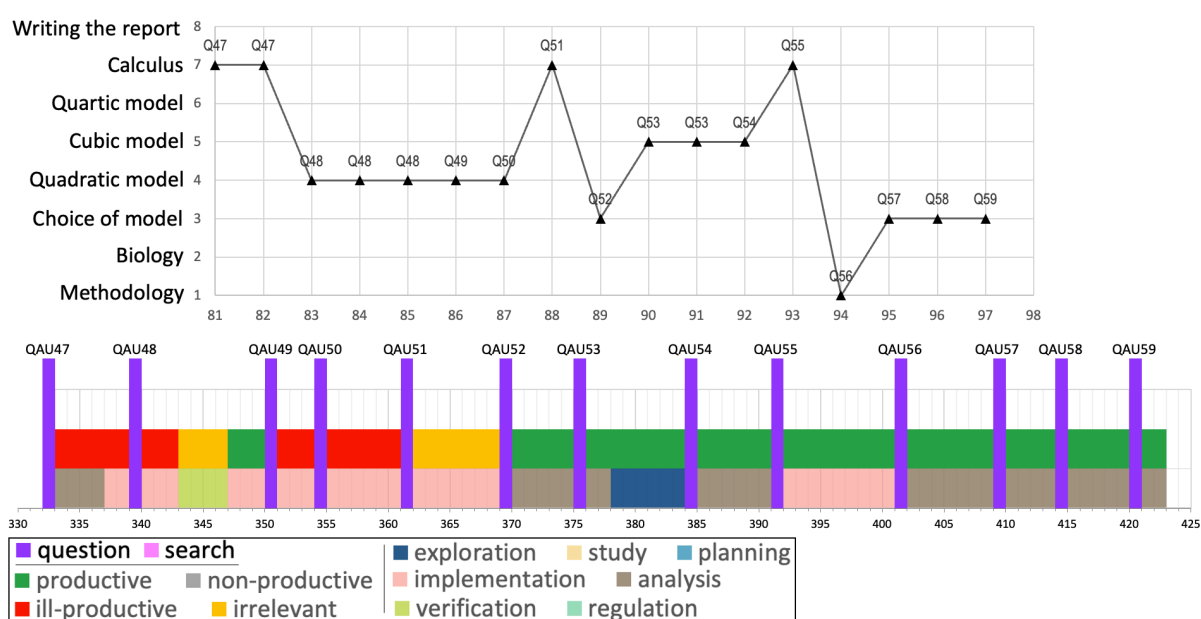
Globally, the teacher's comments were rather vague and sometimes confusing. At the same time, he did not investigate the question and had to rely on the student's information. We showed that sometimes an unhappy formulation of a comment could also lead to an ill-productive episode.

In examining the last cluster, we look in detail at how Sophie handled the teacher's feedback and in what way it contributed to the ME.

3.1.6 Cluster 6: Implementing teacher's feedback on Criteria D and E (QAU46—QAU59)

Cluster 6 starts with two episodes dedicated to methodology (QAU47) and continues with implementation of changes in the quadratic model (QAUs 48–50) with a last calculus touch (QAU51) and as we can observe in Graphics 48, the ill-productive and irrelevant episodes prevail. In the middle (QAUs 52–55) we have productive analysis, exploration and implementation episodes in theme 5 (Cubic model) the cluster finishes with productive analysis episodes in QAUs 56–59 with a final reflection on the choice of model. We also need to remind that the chronology of this cluster was reconstructed based on a comparison of the final version of the ME with the first draft annotated by the teacher and thus may not necessarily reflect the order of Sophie's actions. Nevertheless, we were able to obtain a logical succession of the E-units that serves our purpose.

Cluster 6: Implementing teacher's feedback (QAU46-59)



Graphics 48: Cluster 6 of Sophie's ME

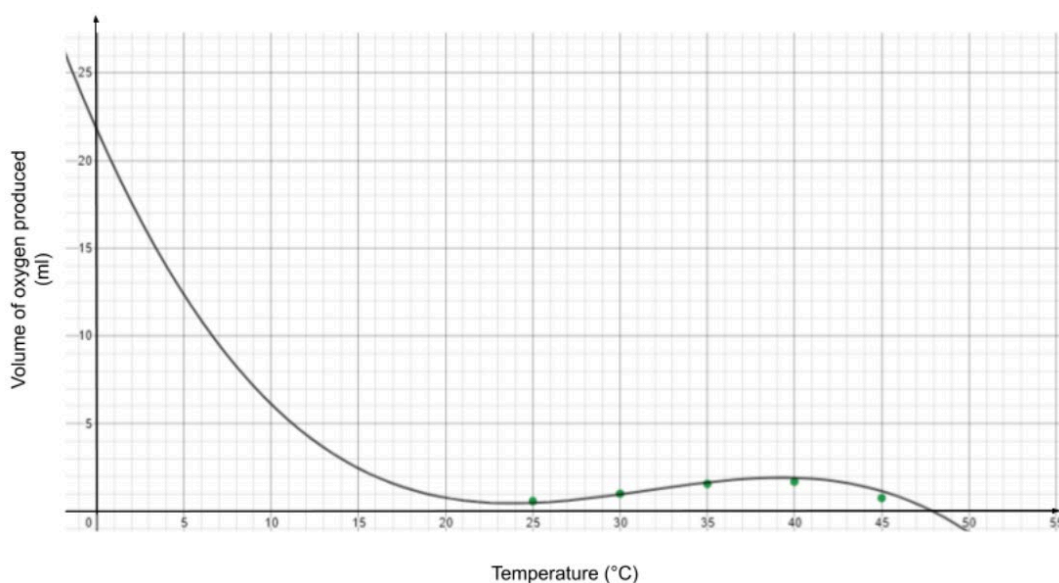
In QAU47, Sophie was unable to react to the teacher's comment (A♦46.3) and the suggestion (A♦46.4) to substitute all the data points and find the mean of all the values of a . She decided to keep the method of finite difference and the incorrect value of that came from this method. In QAU48 she decided to keep the vertex of the quadratic function and the verification by superposing the curves obtained from the vertex form and from the standard form. Finally, at the end of QAU48, Sophie correctly simplified the standard form of the quadratic function and removed the calculations and the table of what she called the rate of change. Even though this episode was coded as productive, as a result of the current correct calculations and choices, it is important to keep in mind that the quadratic function was still incorrect, using the coefficient a from the finite differences method and the vertex at (35, 1.56). In QAU49, Sophie just described that she just took the value of $t=35$ from her data for the vertex and used it in the vertex form of the quadratic function. She also kept the part of adjusting the x-intercepts of the parabola in QAU50 as the teacher marked it as correct on the first draft and used calculus to figure out the maximum as suggested by the teacher (A♦46.11). She correctly mobilized and determined the derivative of a quadratic function and calculated the maximum of this function. This, in the light of the work in the quadratic mode, however, only led to calculating the maximum chosen beforehand by Sophie from the collected data and thus did not correspond to predicting a potential optimum temperature based on a quadratic model. In this part, we have an example of the effect of the didactic contract (the teacher asked for calculus, so calculus there is...) and a sort of constant attachment to a technique even though that was marked as not operational by the teacher. Certainly, the teacher's feedback (A♦46.3 *I don't think this works (method of finite differences)*) was too vague and light and thus inefficient. In addition, Sophie was probably unable to process the teacher's proposition (A♦46.4 *Why don't you substitute in the*

points you have, solve each one for and then find the mean of all the a 's you get?) to find the coefficient a due to her rather limited praxeological equipment. Yet, it is quite intriguing why she kept this technique being aware of the fact that it did not work. A plausible interpretation is that Sophie started running out of time, did not know by what she could have possibly replaced her calculations and preferred to keep something wrong to nothing.

Surprisingly, in the next part of this cluster, Sophie was able to conduct some relevant reflection within the cubic model and the missing praxeological infrastructure was partially substituted by technology. Based on the teacher's comments (A♦46.7 and A♦46.8) suggesting to explore the cubic model, in QAU52, Sophie returned to the cubic curve from her first draft (W29.1) and wrote down:

- A♥52.1 *I will explore the cubic relationship since it was far closer to my actual data.*

In QAU53 Sophie (re)included the data point (40, 1.69) considered as outlier in the quadratic model, ran a cubic regression on the data in Desmos (Graphics 49) and retrieved the equation of a cubic function (W53.1), with a wrong coefficient d for some unknown reason.



Graphics 49: The cubic model in Desmos

She concluded that cubic function represented a good fit for her data (A♥53.1). In QAU54, Sophie carried out a relevant and productive reflection on the validity of the cubic model and correctly interpreted the results as shown in Figure 98.

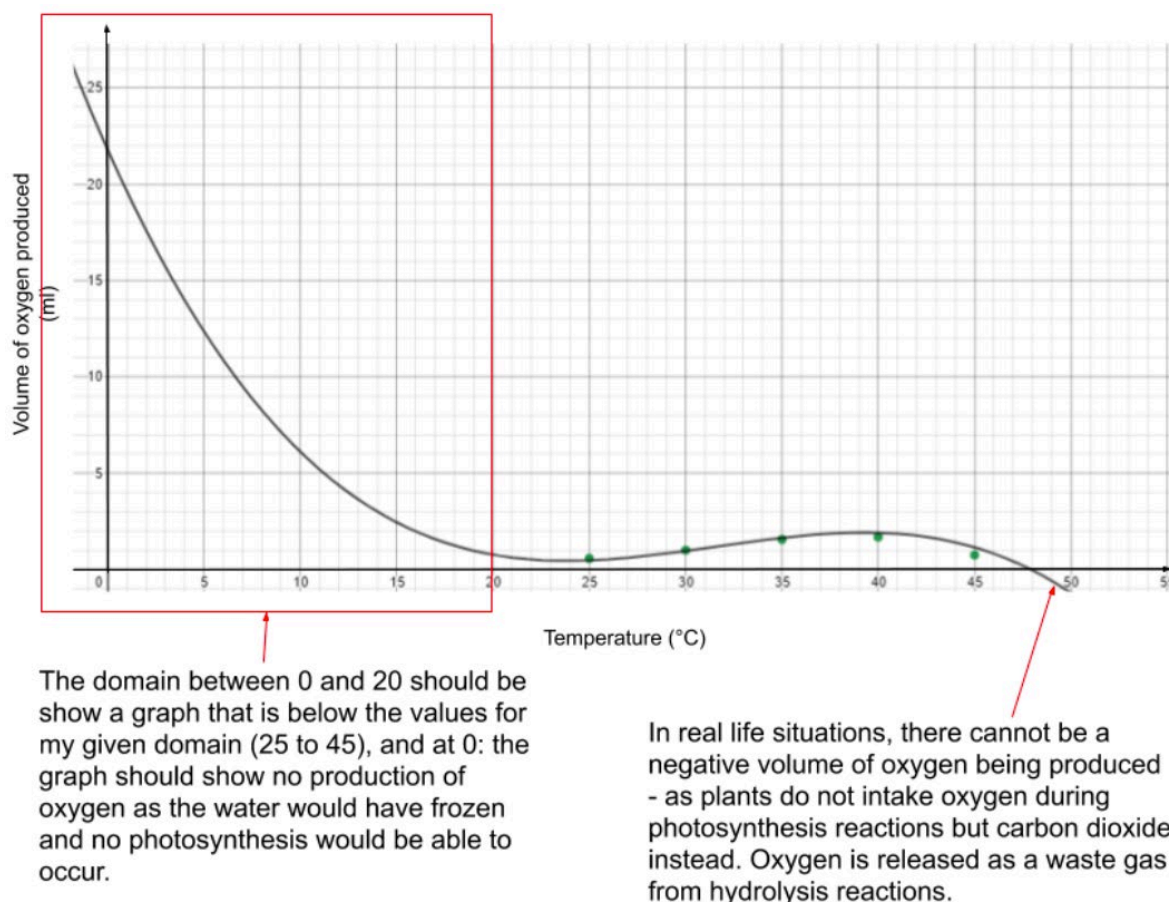


Figure 98: Reflection on the Cubic model (excerpt from Sophie's ME)

In QAU55, she used calculus to find the first derivative of the obtained cubic function and set it equal to zero. She was again unable to solve the obtained quadratic equation analytically and found the extremum using Desmos graphing software. In QAU57 she correctly reflected on the results from the quadratic model and concluded that the quadratic model wouldn't provide the optimum temperature and in QAU58, she reflected on the cubic model:

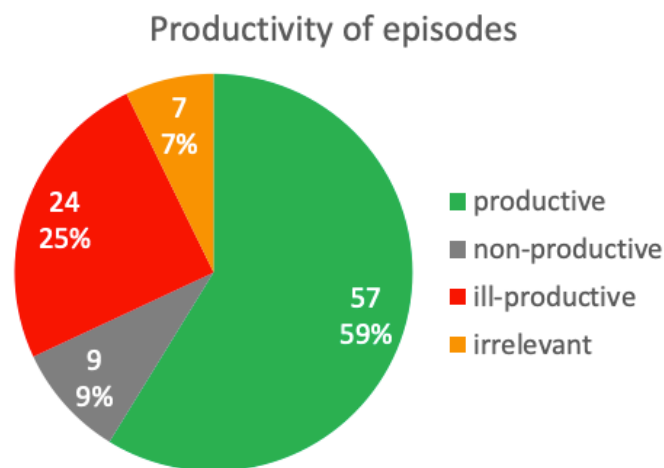
- A♥58.1 *The cubic function provides a better fit than a parabola*
- A♥58.2 *The cubic function does not work outside of the collected data*
- A♥58.3 *I predict that there would be an increasing temperature from 0°C until the maximum point, followed by a steep decrease due to the enzymes being denatured*

In the final conclusion (QAU59), Sophie suggested to use the cubic model to predict the optimum temperature in order to maximize the volume of the produced oxygen. This last note leaves an overall positive impression of the whole ME despite the demonstration of limited mathematics skills.

In the next part, we provide a global view on Sophie's and conduct an analysis in terms of the productivity of episodes according to the identified themes to examine any patterns concerning the productive respectively ill-productive episodes.

3.2 Analysis in terms of the productivity of episodes

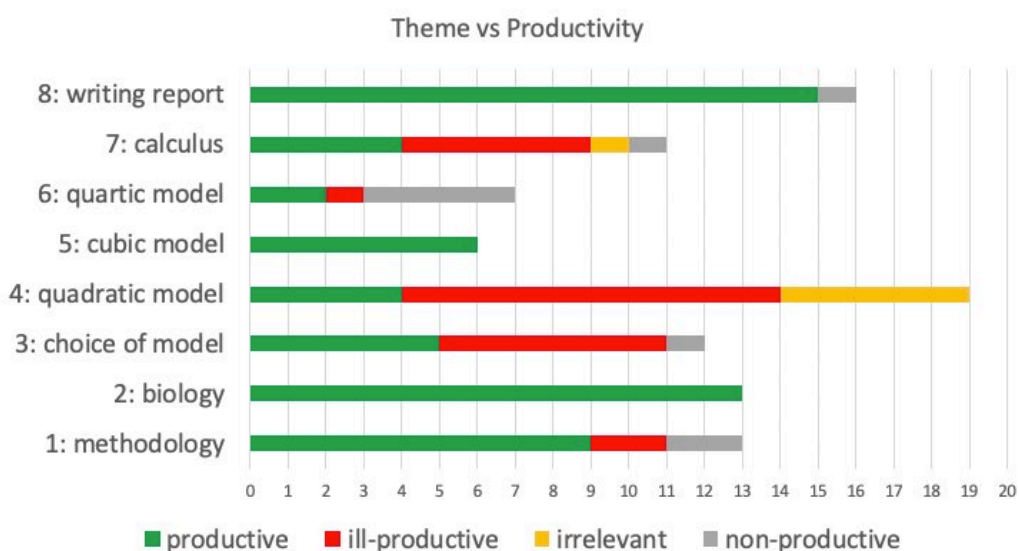
In Sophie's ME we have a total of 97 episodes distributed among 8 identified themes with almost 60% of productive episodes and about a quarter of ill-productive episodes as shown in Graphics 50. The relative under-representation of non-productive episodes is probably caused by the fact that Sophie delivered a less detailed and thus exploitable data compared to Rachel, which provided less access to unexplored or abandoned paths. We therefore focus on analyzing the productive and ill-productive episodes.



Graphics 50: Productivity of episodes

3.2.1 Theme versus productivity of episodes

Globally, we can observe that productive episodes appear in all themes and there aren't any ill-productive in theme 2, 5 and 8 as shown in Graphics 51 which provides a visual representation of where the productive, respectively ill-productive episodes are situated according to theme.



Graphics 51: Visual representation of episodes in respect to theme and productivity

Table 40 resumes the detailed distribution of episodes among the identified themes in respect to productivity.

Count of Productivity Column Labels					
Theme	productive	ill-productive	irrelevant	non-productive	Total
1: methodology	9	2	0	2	13
2: biology	13	0	0	0	13
3: choice of model	5	6	0	1	12
4: quadratic model	4	10	5	0	19
5: cubic model	6	0	0	0	6
6: quartic model	2	1	0	4	7
7: calculus	4	5	1	1	11
8: writing report	15	0	0	1	16
Total	58	24	6	9	97

Table 40: Theme versus productivity

We refer to the above-mentioned Graphics 51 and Table 40 in the analysis of the productive and ill-productive episodes below. We also remind that the analysis below needs to be read in the light of the chronogenesis of the ME.

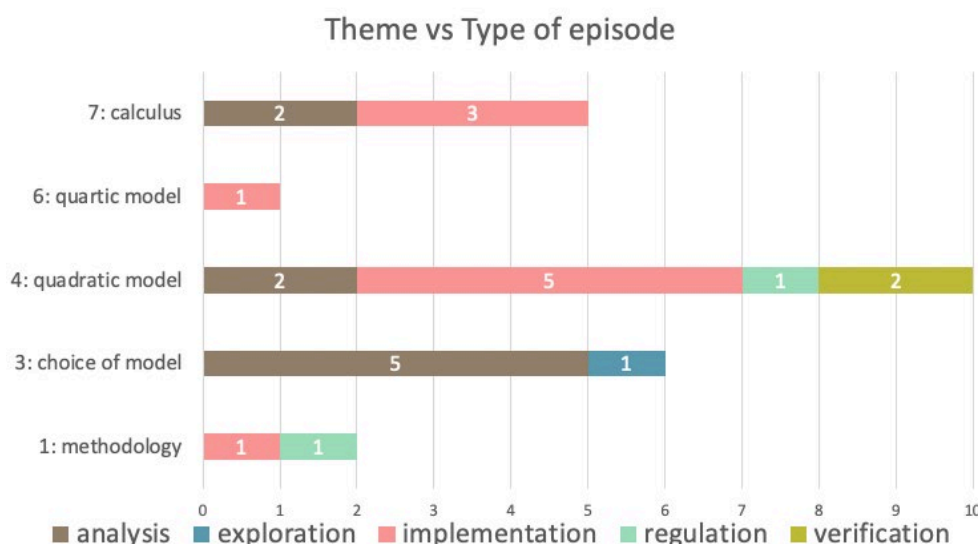
3.2.1.1 Productive episodes

As far as the 58 productive episodes are concerned, we observe that these are represented in each theme. Most of them are situated in theme 8 (15 out of 16 episodes), followed by theme 2 (13 out of 13 episodes) and theme 1 (9 out of 13 episodes) as shown in Table 40 and we interpret this in terms of a proximity to the student's CU and Sophie's profile. Indeed, Sophie had a good comprehension of the biology components and was thus able to conceive a comprehensive plan of the ME which led to collecting reasonable data. By the way, all episodes within the theme 2 (Biology) are productive. Thanks to her profile, despite the missing knowledge in mathematics, Sophie was able to conduct a coherent reflection, interpret results and write a satisfactory conclusion. On the other hand, we would definitely expect more productive episodes in theme 4 (only 4 out of 19) because these objects were

included in the institutional milieu which shows that for Sophie, the institutional milieu in this case (theme 4) was only potential. Another surprising finding is that there are only productive episodes in theme 5 (Cubic model): this happened when technology was used to supply for the missing techniques and in theme 3 (Choice of model) the productive episodes resulted from a correct interpretation of the results after the cubic model using technology had been introduced. Finally, we have 3 productive episodes in theme 7 (calculus) showing that Sophie mastered the application of the first derivative to determine an extreme of a polynomial and 1 productive study episode concerning the technique of finite differences.

3.2.1.2 Ill-productive episodes

Out of the total of the 24 ill-productive, 10 are in theme 4, 6 in theme 3 and 5 in theme 7 as shown in Table 40 revealing thus where Sophie's main difficulty resides. We can also consider the ill-productive episodes by their type as shown on Graphics 52.



Graphics 52: Distribution of ill-productive episodes

In theme 3, the prevailing ill-productive episodes are analysis episodes and the chronology as they appeared in the ME is shown in Table 41. The first three analysis episodes in cluster 2, respectively in cluster 3, came from the student's wrong appreciation of the shape of her data as a parabola.

ill-productive episodes: Theme 3		
Type of episode	Cluster	Label
analysis	cluster 2	Q17-Ep21
analysis	cluster 2	Q18-Ep24
analysis	cluster 3	Q24-Ep32
exploration	cluster 4	Q28-Ep42
analysis	cluster 4	Q28-Ep43
analysis	cluster 4	Q29-Ep46

Table 41: Chronology of ill-productive episodes in theme 3

The exploration and analysis episodes in cluster 4 reinforced this conviction and despite an exploration using showing a good cubic fit, Sophie chose to pursue the exploration of the quadratic model.

When looking at theme 4, we can observe that the prevailing ill-productive episodes are implementation episodes revealing the difficulties concerning the analytical solution. The two verification episodes were based on a circular definition and thus completely inefficient. As shown in the chronology of the ill-productive episode in Table 42, the regulation episode appeared in cluster 6 within the teacher's feedback on the first draft and marked the analytical work on the quadratic function as correct. This unhappy regulation resulted in the last three ill-productive episodes in cluster 6.

ill-productive episodes: Theme 4		
Type of episode	Cluster	Label
implementation	cluster 3	Q24-Ep33
implementation	cluster 3	Q25-Ep37
analysis	cluster 4	Q31-Ep49
verification	cluster 4	Q33-Ep51
verification	cluster 4	Q35-Ep53
analysis	cluster 4	Q41-Ep62
regulation	cluster 5	Q46-Ep78
implementation	cluster 6	Q48-Ep83
implementation	cluster 6	Q49-Ep86
implementation	cluster 6	Q50-Ep87

Table 42: Chronology of ill-productive episodes in theme 4

Finally, theme 7 (Calculus) contains 5 ill-productive episodes. The chronology of their appearance is shown Table 43.

ill-productive episodes: Theme 7		
Type of episode	Cluster	Label
implementation	cluster 3	Q25-Ep35
analysis	cluster 3	Q25-Ep36
implementation	cluster 4	Q34-Ep52
analysis	cluster 6	Q47-Ep81
implementation	cluster 6	Q47-Ep82

Table 43: Chronology of ill-productive episodes in theme 7

The two episodes in cluster 3 concern the wrong application of the method of finite of differences due to plenty of calculation errors, on the one hand, and missing logos of the studied praxeology, on the other hand. The implementation in cluster 4 concerned calculations of the rate of change as an effect of the didactic contract. Finally, the two episodes in cluster 6 are the result of inefficient regulation that Sophie was unable to react to.

3.3 Conclusion of the chronogenesis of Sophie's ME

The chronogenesis of Sophie's ME unfolded around three major parts:

- 1) A successful determination of the topic, exploration of the biology prerequisites and organization of an experiment to collect data
- 2) A rather laborious search and implementation of an analytical solution
- 3) A relevant reflection and interpretation on the result obtained by an effective use of technology

This reveals something about the strengths and the weaknesses of Sophie's abilities and skills. The ME was motivated by Sophie's interest in biology and this led to a successful choice of topic, detailed planning of an experiment resulting in a data collection to be explored mathematically. And that's where we encountered Sophie's major weakness. Unfortunately, most of the mathematics (apart from some isolated derivatives) wasn't applied correctly and the use of media either hasn't contributed to acquiring a new situational knowledge (solving a system of 3 linear equations) or led to studying and applying a technique without a corresponding technology. Even though we observed that Sophie was able to reproduce the technique of finite differences more or less correctly, the technologico-theoretical block of this praxeology would have probably provided Sophie with tools to evaluate the relevance of this technique to her situation. We also noticed that this incorrect production resisted the teacher's regulations. Regardless of the quality of the teacher's comments, we remind that the reason of this resistance could be probably explained by the fact that Sophie did not have another technique available and removing the work with the *technique of the finite differences* would have actually removed all the analytical work produced in the quadratic model. This is also due to an effect of the didactic contract as the teacher emphasized the need of an analytical solution to score points on Criterion E Use of mathematics. Another possible reason for using the *technique of the finite differences* is that the teacher also explained that using and applying mathematics from prior learning topics only would score a maximum of 2 points on Criterion E and reminded that quadratic equations and functions are considered prior learning.

When attempting to justify the choice of the quadratic model, we also observed a phenomenon of preferring to keep information from an unreliable source (an image close to a parabolic shape) that would confirm her personal conviction (about the data corresponding to a parabolic shape) and rejecting relevant information (from the biology textbook and all images from the Google s showing the same tendency). We think that this happened because Sophie considered that the mathematics in the quadratic model was probably more accessible for her and in some way, all means were good to justify this choice.

Even though, her ME seemed to have been seriously compromised by the lack of her praxeological equipment in mathematics, which was made visible on the production of the analytical solution in the quadratic model, Sophie surprised us once the maths was taken care of by technology. She was indeed able to conduct a relevant reflection and interpret correctly the results obtained using technology. We could conclude that her limited maths skills almost ruined her ME but she was nevertheless able to turn to her advantage her capacity of reflection and analysis through an effective use of technology.

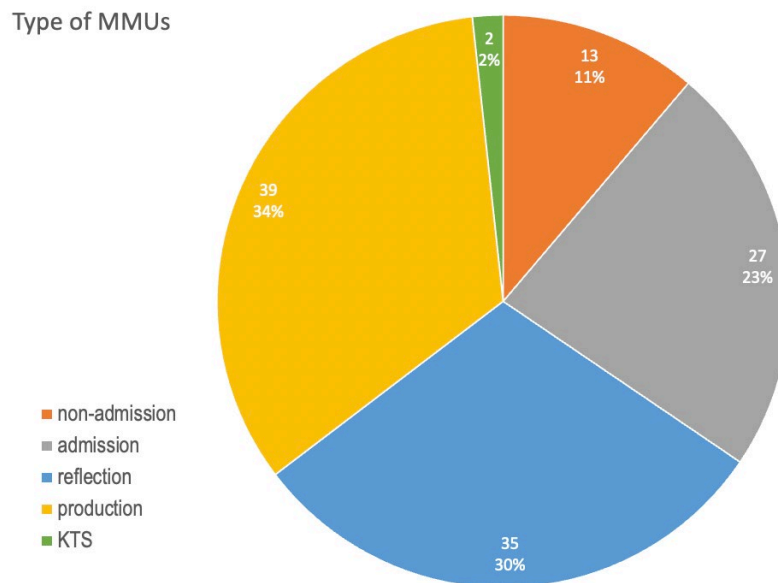
In the next part, we analyze of the content and the evolution of the milieu and its effects on the ME.

4 The mesogenesis of Sophie's ME

The chronogenesis revealed a series of difficulties concerning in particular Sophie's inability to provide an analytical solution of a possible model emphasized by the teacher. Later, however, when the analytical solution was taken care of by technology, she was able to come up with relevant reflection and correct interpretation of the results. In this part we analyze the content of the milieu and attempt to explain and interpret the finding from the chronogenesis by zooming on them from the point of view of the evolution of the milieu. We start by characterizing the content of the milieu in terms of the type of MMUs which characterizes the type of interaction Sophie had with the media including her own reflections. We consider the effect of the identified MMUs on the ME in the light of the activated milieu and finally we consider the effect of the experimental milieu on the ME.

4.1 The content of the inquiry milieu and its effect on the ME

In Sophie's ME, we identified a total of 116 MMUs whose distribution according to the type of MMU is provided in Graphics 53. We haven't identified any non-activation MMUs and even the non-admission and KTS (keeping track of sources) MMUs occupy a relatively small portion (13%) of the milieu. This could be explained by the fact that in Sophie's case we obtained less data concerning the research on the Internet and that is why we also have less access to the non-activated and abandoned part of the ME.



Graphics 53: Distribution of the MMUs by type (Sophie's ME)

For the three remaining types of MMUs, we observe that these are represented in approximately the same proportion: admission (23%), reflection (30%) and production (34%) which shows that Sophie dedicated more space to constructing answers from the available information than searching for new information on the Internet. This interpretation could be nonetheless biased by the fact that Sophie did not always turn on the action camera and some part of her search might not have been recorded. Table 44 shows the detail of the distribution of the MMUs according to subtype of MMUs.

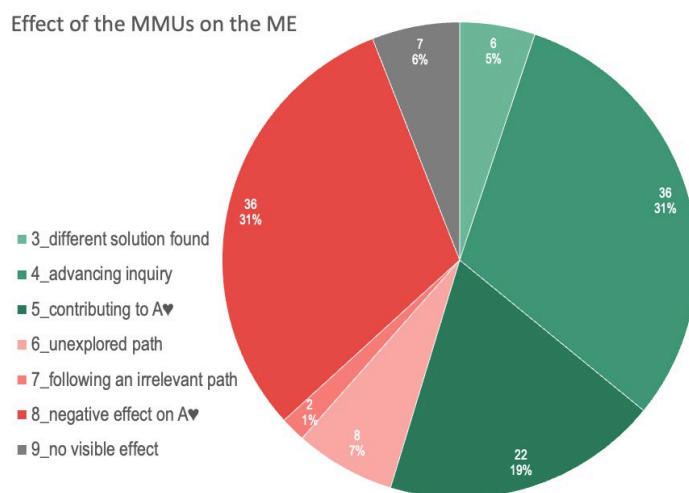
Type of MMU	# of MMUs	% of parent row
non-admission	13	11.2%
efficient elimination	6	46.2%
missed occasion	7	53.8%
admission	27	23.3%
correct admission	24	88.9%
incorrect admission	3	11.1%
reflection	35	30.2%
correct reflection	21	60.0%
irrelevant reflection	4	11.4%
incorrect reflection	10	28.6%
production	39	33.6%
correct production	20	51.3%
incorrect production	12	30.8%
irrelevant production	7	17.9%
KTS	2	1.7%
relevant source	2	100.0%
Grand Total	116	100.0%

Table 44: Types and subtypes of MMUs (Sophie's ME)

When examining the identified subtypes of MMUs from Table 44, we can see that within the reflection MMUs almost 40% are irrelevant (11.4%) or incorrect (28.6%) reflections whereas almost 90% of the admissions are correct. Moreover, when looking at the production MMUs, about half of them is correct, 30% are incorrect and something less than a fifth are irrelevant. This suggests a correlation between the incorrect reflection MMUs and incorrect production MMUs and shows that Sophie was able to mobilize the correct

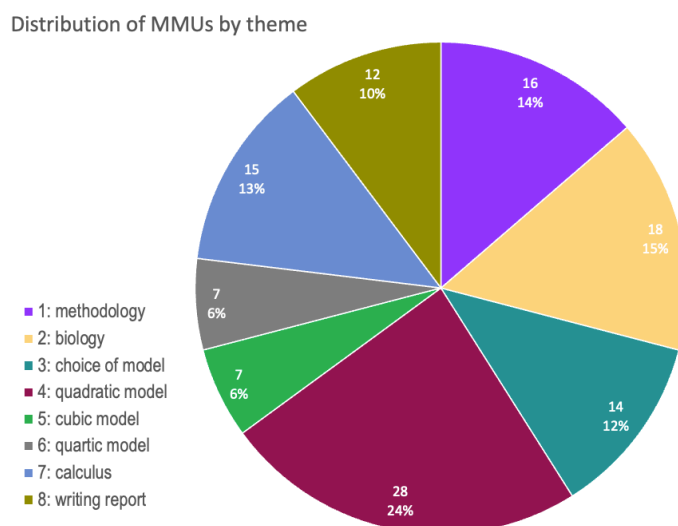
tools but her main difficulty resided in making connections, using and applying the information correctly.

When looking at the effect of the MMUs on the ME as shown in Graphics 54, we can see that 31% of the MMUs had a negative effect on the A♥ which raises a question of where these MMUs are and how and why they resulted in a negative effect on the ME.



Graphics 54: Effect of the MMUs on the ME

In order to tackle these questions later in this analysis, we also consider and refer to the distribution of the MMUs by theme that we provide now on Graphics 55.

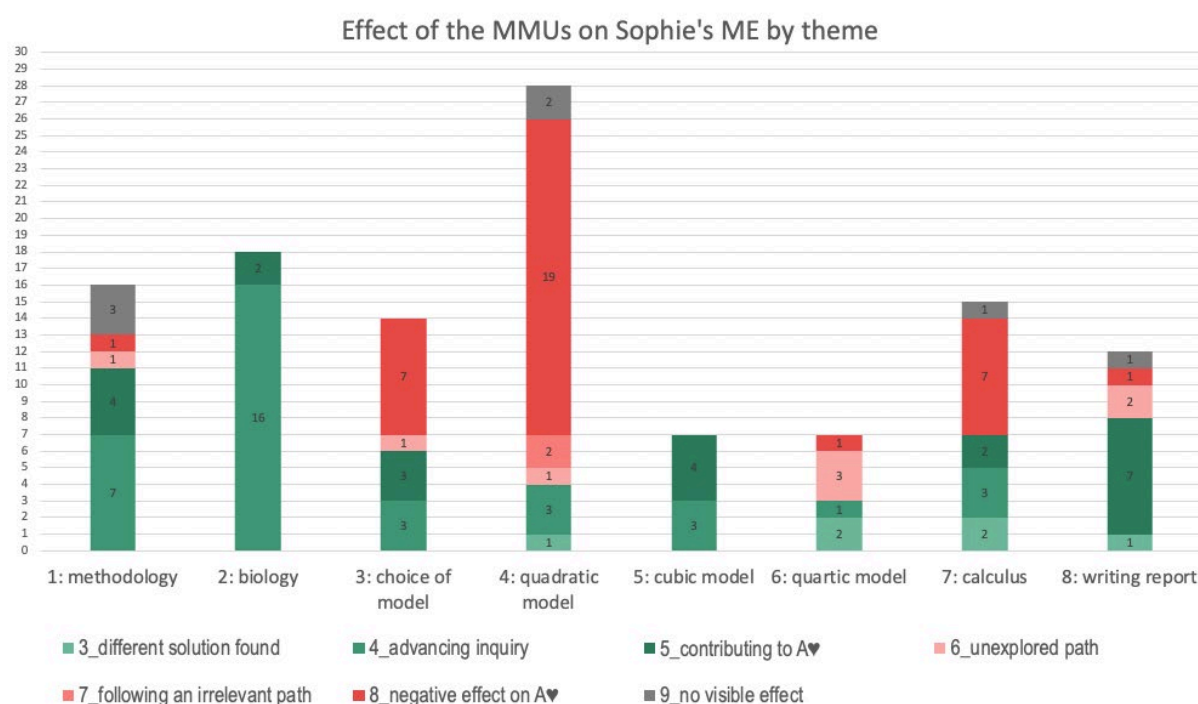


Graphics 55: The distribution of the MMUs by theme in Sophie's ME

We can see a variety of different themes represented in Sophie's ME with about 60% of the MMUs concerning directly a mathematical theme (3, 4, 5, 6 and 7), about a quarter dedicated to methodology and writing and in 15% of the MMUs Sophie explores the rate of photosynthesis in biology.

To illustrate our analysis, we combine the figures from Graphics 54 showing the global distribution of the effect on the ME, Graphics 55 showing the global distribution of MMUs

by theme and provide a visual representation on Graphics 56 combining thus the effect of MMUs by theme. This enables us to identify how the effect on the ME is distributed over the identified themes.



Graphics 56: Effect of the MMUs on Sophie's ME by theme

The largest portion (24%) of MMUs i.e., 28 MMUs, as shown in Graphics 55, was identified in theme 4 (quadratic model). Out of these 28 MMUs, 19 resulted in a negative effect on A♥ (see Graphics 56). In addition, these 19 MMUs identified in theme 4 (quadratic model) represent more than a half of all MMUs having a negative effect on A♥. This is followed by 7 MMUs with negative effect on A♥ in each; theme 3 (choice of model) and theme 7 (calculus). These three themes combined contain over 90% of the MMUs having a negative effect on A♥. This shows that most of Sophie's difficulties resided in these 3 themes and were thus related to mathematics. While we would quite naturally expect difficulties in the choice of model or calculus, it is rather surprising to find out that Sophie encountered most of the difficulty in the quadratic model. The applications of a quadratic function should have represented to most accessible mathematics in the ME and as Sophie revealed in the post-exploration interview, it was even strongly suggested to her to do the "maths" on the quadratic function because it would be the most doable for her. This "someone", whose identity Sophie did not want to reveal, evaluated her difficulties concerning using and applying mathematics. This confirms actually the fact that her mathematics skills were quite limited.

On the other hand, we can observe that the MMUs in themes 1 (methodology), 2 (biology), 5 (cubic model) and 8 (writing report) had mostly a desirable effect on the ME. This points to another unexpected fact: the work within the cubic model only contains MMU's having

a desirable effect on the ME. Indeed, the cubic function has not been taught prior to the ME and that is why we would expect some difficulty. Sophie, however, did not attempt to find the equation analytically. The equation and the graph were found through cubic regression using technology and most of the MMUs consisted thus of the interpretation of the results obtained when the cubic model was applied.

Finally, for the 7 MMUs related to work in Theme 6 (quartic model) in 2 MMUs a different solution was found and 3 MMUs led to an unexplored path which explains the relatively small amount of production in this theme.

In order to explain these surprising findings described above, we need to consider the evolution of the milieu from the point of view of the chronogenesis of the ME and we focus on the effects of the MMUs on the ME by theme in the next part.

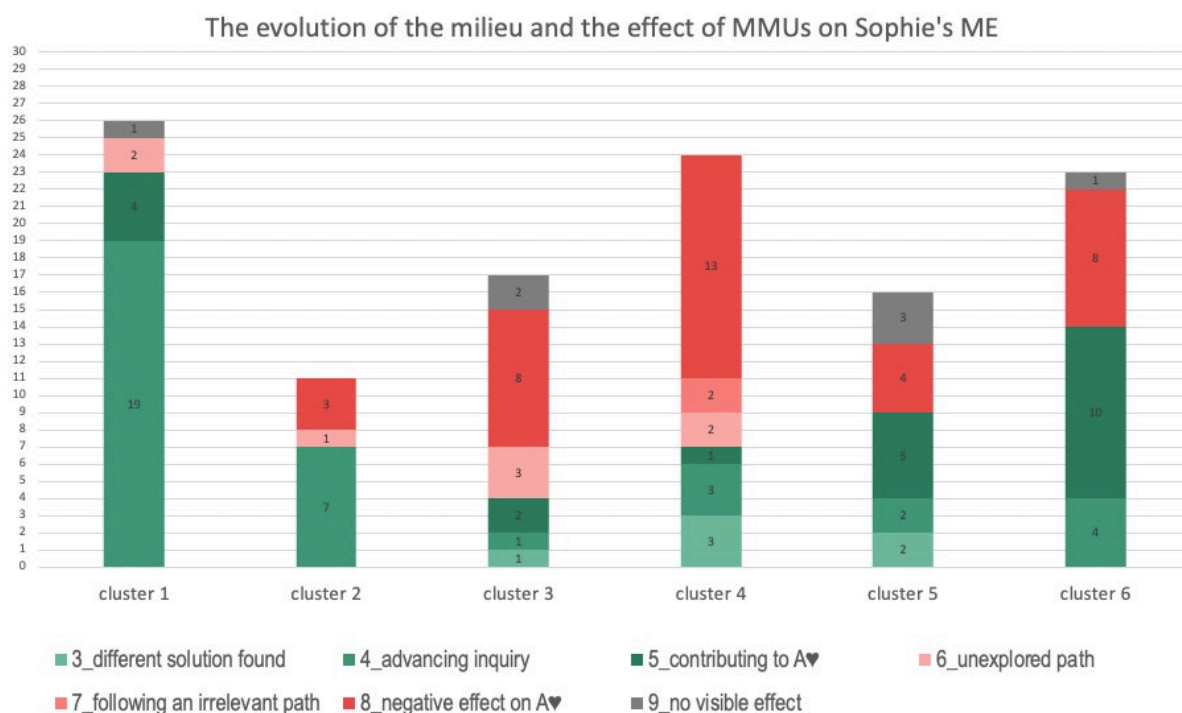
4.1 The evolution of the inquiry milieu

Considering the milieu from the point of view of the chronogenesis enables us to observe how the inquiry milieu evolved and locate the MMUs with a specific effect on the ME. Below is a quick reminder of the clusters identified in the chronogenesis:

- 1) Cluster 1: Planning of the ME (see p. 227)
- 2) Cluster 2: Interview with the teacher (see p. 230)
- 3) Cluster 3: Data collection and analytical method (see p. 232)
- 4) Cluster 4: Exploration of the quadratic model (see p. 237)
- 5) Cluster 5: Teacher's feedback and implementing criteria A, B and C (see p. 242)
- 6) Cluster 6: Implementing teacher's feedback on criteria D and E⁵⁹ (see p. 244)

As we can see on Graphics 57, the first MMUs with a negative effect on A♥ started appearing in cluster 2 during the exchange with the teacher. Almost 60% of the MMUs with a negative effect on A♥ is situated in clusters 3 and 4 where most of the analytical work took place suggesting the nature of the difficulties that Sophie encountered. We can also observe a growing tendency of the MMUs with a negative effect on A♥ from cluster 2 to cluster 4. In Cluster 5 (teacher's feedback on the first draft) as well as in cluster 6 (implementing teacher's feedback), we have a prevalence of MMUs that either contributed to advancing the inquiry or to building A♥. In both of these clusters, there are, however, MMUs that resulted in a negative effect on A♥ despite and even because of the teacher's intervention showing clearly that he does not have all the answers unless he would take time to adopt a posture of an inquirer himself.

⁵⁹ As reminder the assessment criteria are: A Communication, B Mathematical presentation, C Personal engagement, D Reflection and E Use of mathematics

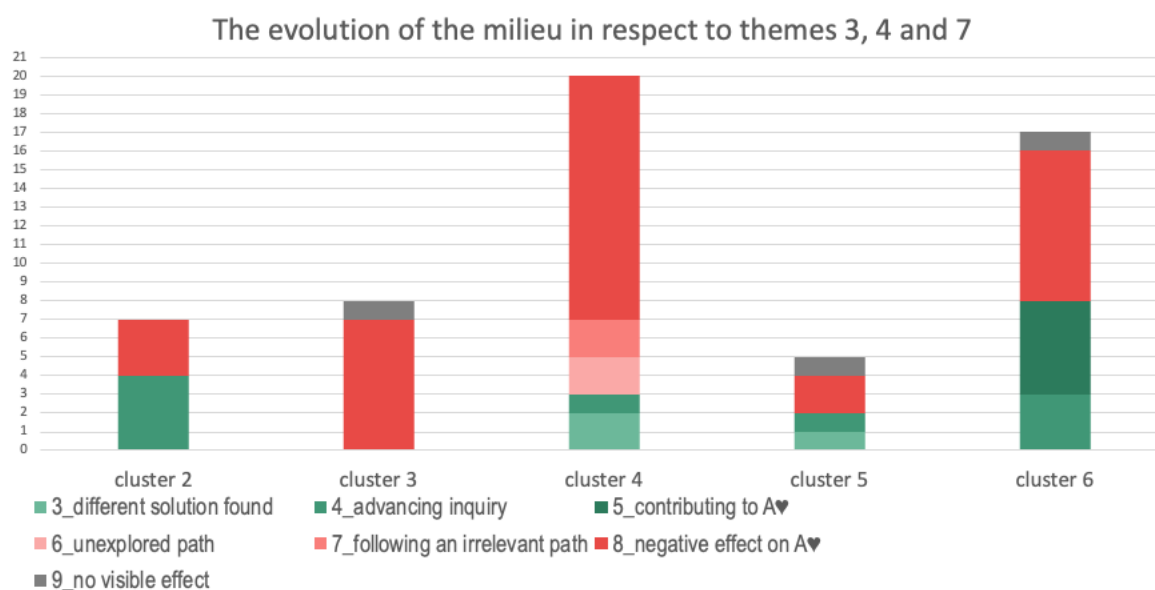


Graphics 57: The evolution of the milieu and the effect on Sophie's ME by theme.

In the next part, we examine the evolution of the milieu in respect to themes with the prevailing undesirable effect on the ME, on the one hand, and in respect to themes with a desirable effect, on the other hand.

4.1.1 Focus on the MMUs with negative effect on A♥

In this part, we examine the evolution of the milieu concerning the 3 themes in which most of the MMUs with a negative effect on A♥ were identified. The objective is to understand and explain why over 90% of the MMUs with a negative effect on the ME were identified within the 3 themes: choice of model, quadratic model and calculus. The Graphics 58 provides a visual representation of the MMUs related to the 3 themes mentioned above. There is a total of 57 MMUs of which 33 resulted in a negative effect on A♥ against 14 that contributed to advancing inquiry or to building A♥.

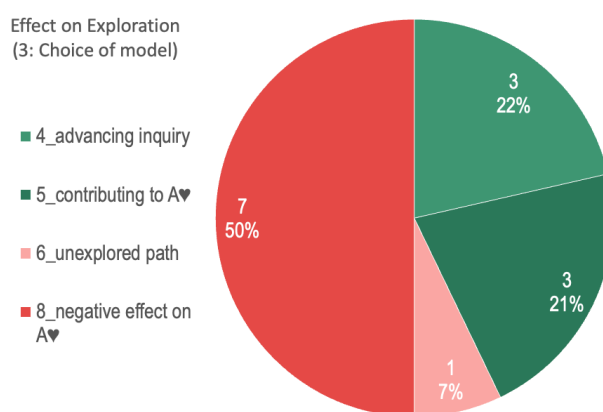


Graphics 58: The evolution of the milieu within themes 3, 4 and 7.

Let us now look in detail at the MMUs related to theme 3—choice of model.

4.1.1.1 The evolution of the milieu in respect to theme 3

There are altogether 14 MMUs in theme 3: as already mentioned above 7 of these MMUs resulted in a negative effect on A♥, 1 in an unexplored path and 3 MMUs contributed to advancing inquiry respectively to building A♥. Their distribution is shown in Graphics 59.



Graphics 59: Effect of the MMUs on Sophie's ME in the 3 (choice of model)

When we examined each MMU in this theme from the point of view to which model it contributed, we discovered that the 7 MMUs having a negative effect on A♥ are actually related to the theme 4 (quadratic model) as shown in Table 45.

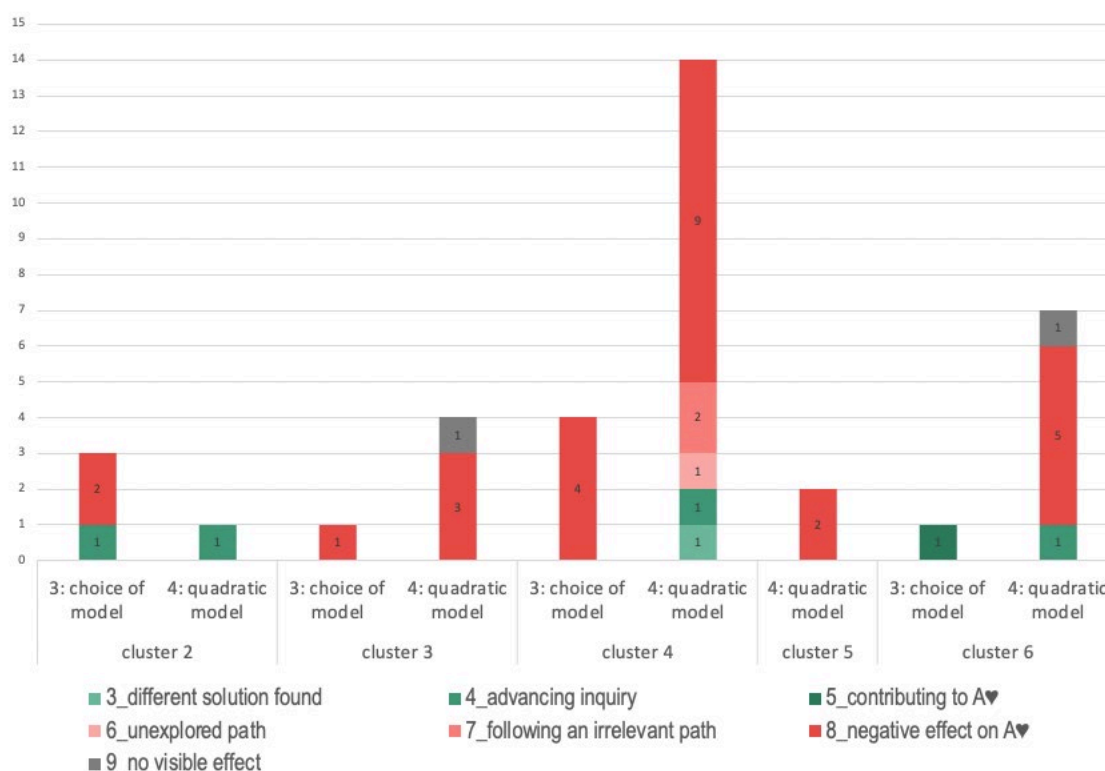
Theme		3: choice of model
Row Labels		Count of Effect on exploration
4_advancing inquiry		3
cubic		1
general		1
quadratic		1
5_contributing to A♥		3
cubic		2
quadratic		1
6_unexplored path		1
quartic		1
8_negative effect on A♥		7
quadratic		7
Grand Total		14

Table 45: Effect of the MMUs in them 3 and their relation to other models

This finding made us include the MMUs from the choice of the model when analyzing the evolution of the milieu in respect to the other themes in the next parts.

4.1.1.2 The evolution of the milieu in respect to theme 4

The Graphics 60 captures the evolution of the milieu in respect to theme 4 (quadratic model) including the MMUs from theme 3 (choice of model) related to the choice of the quadratic model.



Graphics 60: The evolution of the milieu in respect to theme 4

We can see on Graphics 60 that the first MMUs with a negative effect on A♥ appeared already in cluster 2 during an exchange with the teacher where Sophie said that the data

has got a parabolic shape despite the feedback provided by two works (W2.3 and W2.4 in Figure 99) from the research in cluster 1. The teacher did not see these two works and did not verify whether the data followed a parabolic trend.

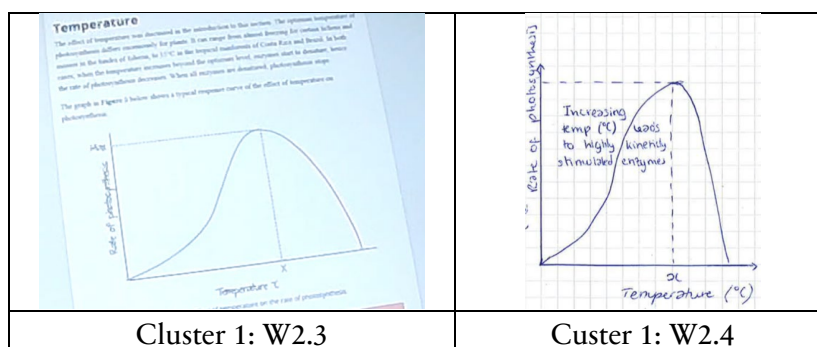


Figure 99: Works providing the shape of the rate of photosynthesis in respect to temperature

In cluster 3, Sophie stated again that the expected relationship was a parabola and ran a quadratic regression on the data. Despite the feedback of the experimental milieu that she was able to create (Figure 100) she pursued the work in the quadratic model because as revealed in the final interview, she was told so.

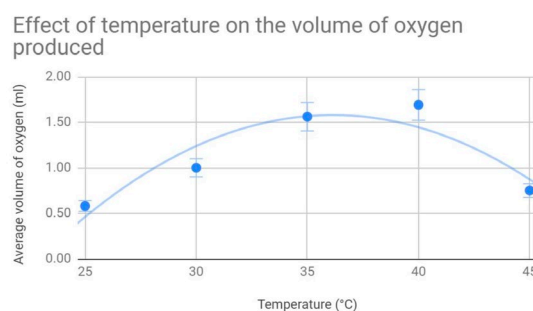


Figure 100: The experimental milieu feedback

She must not have been very convinced about her own conclusion and went on the Internet to search for something that could confirm the choice of the quadratic model and found the work W28.1 (see Figure 101) and used it to justify the quadratic model because it looked more like a parabola while two other works from the textbook and her notes (W2.3 and W2.4 see Figure 99) were already present in the milieu. In spite of their presence in the inquiry milieu, providing quite a visible “this is not a parabola” feedback, Sophie searched on purpose for something that looked like a parabola to justify her choice and used W28.1 in the report.

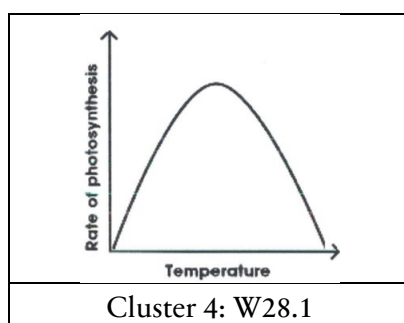


Figure 101: Incorrect admission MMU

As already shown in detail in the chronogenesis (Clusters 3 and 4), most of the work done analytically was incorrect or irrelevant and resulted mostly in a negative effect on the ME.

In Clusters 5 and 6, we would like to show another interesting finding. In cluster 5, the teacher made a general comment stating: “Your maths is correct but reflection could go deeper,” which explains why Sophie kept the incorrect productions in the final report. The evolution of the milieu shows how this happened. It is important to realize the influence of the teacher (David) on the exploration. In his conception, the mathematics that count and allow to score on the criterion E must be done analytically (by hand) and preferably in the algebraic register of representation. In cluster 5, there are 2 MMUs (see Table 46) from the teacher’s feedback resulting in a negative effect on A♥.

Q46-Ep77-MMU89		
315	TFB 46.3	Quadratic model (Criterion E—Use of mathematics)
316	A♦ 46.3	I don't think this works (method of finite differences)
317	A♦ 46.4	Why don't you substitute in the points you have, solve each one for and then find the mean of all the a's you get?
Q46-Ep78-MMU90		
318	TFB 46.4	Quadratic model (Criterion E—Use of mathematics)
319	A♦ 46.5	Where did the vertex (35, 1.56) come from? Did you read it from the graph?
320	A♦ 46.6	Adjusting the x-intercepts of the parabola (25, 0) and (45, 0) marked as correct

Table 46: MMUs from the teacher’s feedback on the first draft

The MMU89 provided two answers from the teacher: (1) stating vaguely that the method of the finite differences does not work and (2) a suggestion to calculate “a” differently. In the MMU90, we also have two answers (1) another comment that wasn’t very useful and (2) an incorrect production was marked as correct. When we look globally at Sophie’s ME the only “mathematics” acceptable for David that Sophie was able to do was the method of the finite differences and the equation of the quadratic function. Thus, on the one hand, we have the teacher’s comment stating that the mathematics was correct and, on the other hand, the only analytical method that Sophie was able to come up with was questioned. In addition, the alternative method suggesting finding the mean of all the “a” coefficients was

inaccessible without further guidance because it would have required to set up and solve a system of linear equations.

This shows the influence that the teacher has on the ME despite his reduced topos on the evolution of the milieu and finally on the production of the final answer. At the same time the implementation conditions of the ME, especially the fact that he has to follow about twenty individual explorations, make it impossible for him to provide pertinent comments and feedback together with the fact that the teacher's inputs seem to remain in the milieu, create thus an important constraint in the evolution of the milieu.

4.1.1.3 The evolution of the milieu in respect to theme 7

The last theme with a significant amount of MMUs with a negative effect on $A^\heartsuit$ is theme 7 (calculus).



Graphics 61: The evolution of the milieu in respect to theme 7 (calculus)

As shown on Graphics 61, the first MMU with a negative effect on $A^\heartsuit$ in cluster 2 during the exchange with the teacher: “You can look at the rate of change using calculus.” This reveals a wrong understanding and interpretation of the terminology used to describe a phenomenon in biology: rate of photosynthesis. In biology, this term refers to the volume of the produced oxygen. By the math teacher, it was, however, interpreted as and related to the mathematical concept of the rate of change in calculus. This explains the irrelevant teacher's inputs concerning the rate of change and the exploration of the concept of the rate of change by the student. Since the teacher insisted on using calculus, Sophie calculated the rate of change.

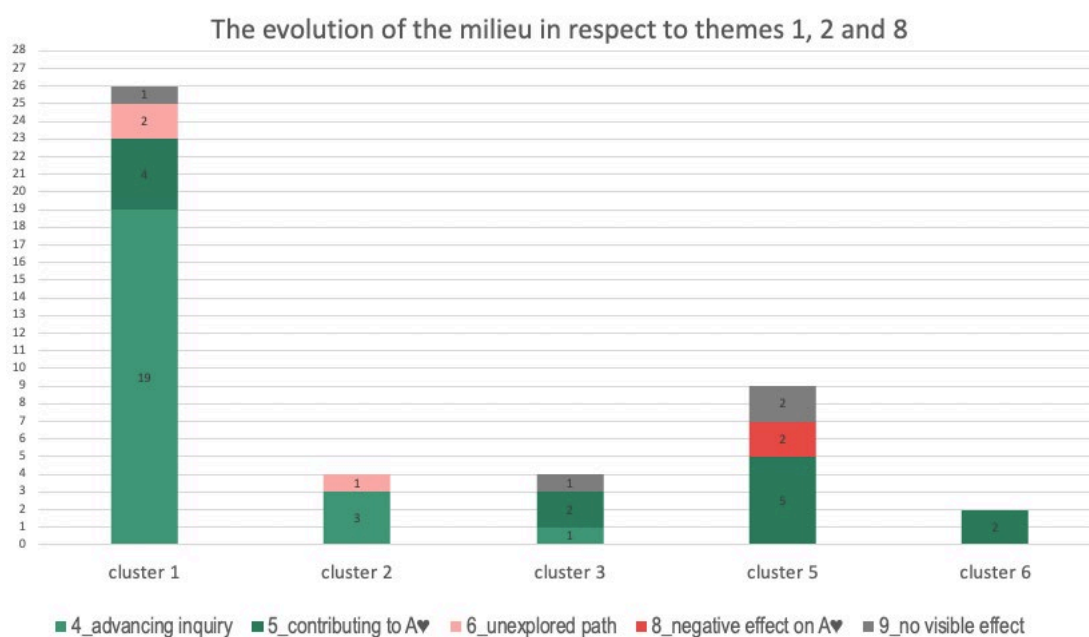
In cluster 3, the 3 MMUs with a negative effect on $A^\heartsuit$ are all related to the method of finite differences. Even though Sophie was able to use this method with her data as shown in the chronogenesis, she did not have enough conceptual understanding of calculus to be able to determine the conditions in which the method of the finite differences could be used and applied correctly as we showed in the “expert” investigation at the beginning of this chapter.

Besides the application of this particular method, as far as the theme 7 (calculus) is concerned, Sophie was able to find the derivatives of the quadratic and cubic functions analytically and demonstrate understanding of their applications to determine the maximum of a function.

The analysis of the MMUs with a negative effect on $A^\heartsuit$ in respect to the three themes (3, 4 and 7) shows that Sophie's main difficulty resided in actually doing the mathematics analytically.

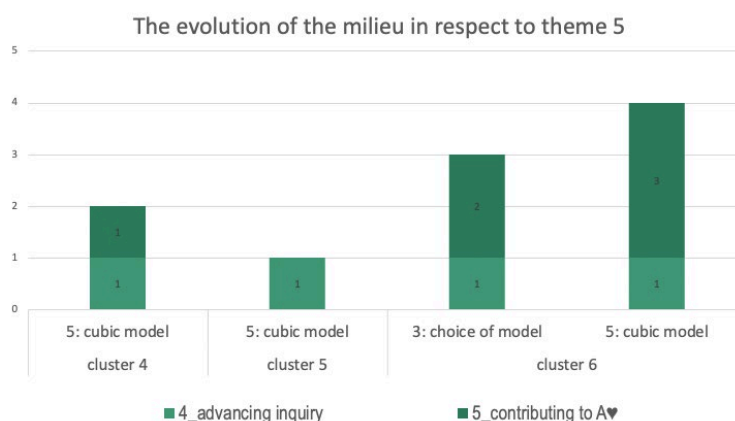
4.1.2 Zoom on the desirable effect on the ME

Globally, the work concerning themes 1 (methodology), 2 (biology) and 8 (writing) resulted mostly in a desirable effect (see Graphics 62) on the ME showing that Sophie was at ease with the planning and the topic choice as well as the writing of the report and that her difficulty as shown in the previous part resided in the use of mathematics. By the way, the only 2 MMUs with a negative effect on $A^\heartsuit$ are related to the teacher's feedback.



Graphics 62: The evolution of the milieu within themes 1, 2 and 8

One of these teacher's comments was: "Your math is correct but reflection could go deeper" and as we saw in the previous parts, Sophie kept all the incorrect productions concerning themes 3, 4 and 7. On the other hand, she was able to conduct a relevant reflection in cluster 6, which leads to another surprising finding: there aren't any MMUs resulting in a negative effect on $A^\heartsuit$ in theme 5 (cubic model) as shown on Graphics 63. We also included the MMUs for theme 3 (choice of model) related to theme 5 (cubic model).



Graphics 63: The evolution of the milieu in respect to theme 5 (cubic model)

None of the work related to theme 5 (cubic model) was actually done analytically. Sophie ran a cubic regression which provided quite a good fit within the given data and could thus be used to predict the optimum temperature. After the teacher's feedback on the first draft in cluster 5 encouraging the exploration of the cubic model, Sophie conducted a meaningful reflection to interpret the results and she was able to find the first derivative of the cubic function (see Figure 102).

Despite these issues, the function is certainly the best fit within my given domain. Therefore I worked out the derivative for the function to find the optimum point using the same process as before:

$$\begin{aligned} \text{If } f(t) &= -0.0008t^3 + 0.0757t^2 - 2.2453t \\ f'(t) &= -3(0.0008)t^2 + 2(0.0757)t - 2.2453 \\ f'(t) &= -0.0024t^2 - 0.1514t - 2.2453 \end{aligned}$$

Figure 102: Finding the first derivative of a cubic function

Sophie was then able to set the derivative equal to zero but again couldn't solve the related quadratic equation. As we can see in Figure 103 was only used to verify

So then, making the derivative equal to 0, to find where the rate of reaction is 0 and therefore where the optimum point is, we find this:

$$\begin{aligned} 0 &= -0.0024t^2 - 0.1514t - 2.2453 \\ -2.2453 &= 0.0024t^2 - 0.1514t \\ -2.2453 &= 0.0024(39.245)^2 - 0.1514(39.245) \\ t &= 39.245 \end{aligned}$$

Figure 103: Dealing with a quadratic equation

The optimum temperature must have been found using technology and Sophie, apparently unable to solve the quadratic equation, did something that made it look like she solved it.

In the last 2 MMUs in cluster 6, Sophie reflected on the choice of the cubic model and interpreted it as the best from the ones she explored and provided a satisfactory answer to her initial question without any “by hand” mathematics. In the next part, we put all the pieces of the puzzle together to see the bigger picture.

4.2 Conclusion of the mesogenesis of Sophie’s ME

The analysis of the mesogenesis of Sophie’s ME enabled us to show the composition of the inquiry milieu, understand how the milieu evolved during the ME and how certain ingredients of the inquiry milieu affected the construction of the final answer. As we have shown above, Sophie’s main difficulty resided in “doing the maths by hand”, but when the inquiry milieu was enriched by the cubic function and its graph obtained via technology, she was able to conduct a meaningful reflection and interpret the results correctly.

We could also wonder why Sophie kept all the incorrect productions once she justified the cubic model. In order to explain certain choices that Sophie made, it is important to remind David’s conception and vision of the ME and representation of what doing mathematics means. For him, the ME must contain mathematics done analytically, i.e., a function needs to be found algebraically, an extreme needs to be determined using the derivative and solving the equation algebraically, etc. In Sophie’s case, this personal praxeology of the teacher putting the analytical work in the center of the ME had a negative effect on the evolution of the inquiry milieu and finally on the construction of the final answer.

We can conclude that despite the relatively large topos left to the students, the influence of the teacher on the milieu is still quite important. The fact that the teacher is not an inquirer himself in the ME together with his reduced topos make it difficult to either provide meaningful, relevant and targeted regulations or confirm the answers brought up by the students. For example, during the interview in cluster 2, we pointed out that David relied on Sophie’s inputs without verifying their validity and the regulations based on this information influenced the course of the exploration. Indeed, by the effect of the didactic contract the teacher’s regulations benefit from a privileged place in the milieu and can even override other information in the milieu or its feedback. These have thus an important impact on the evolution of the inquiry milieu and as a result, on the construction of $A^\heartsuit$.

The way Sophie conducted her exploration reflects her profile of someone used to question phenomena and curious about domains she’s passionate about. Her final reflection and interpretation of the results demonstrate some inquiry habits of her mind. She was just not interested in the “number and letter crunching” part of mathematics. And honestly, who would be excited about solving a system of linear equations with coefficients using real-life data? This raises a question of what doing and applying mathematics means and to what extent technology supplying the missing techniques should be recognized as applying

mathematics. We saw that in Sophie's case, the technology provided useful works for the milieu and largely contributed to the construction of the A♥.

5 Summary of Sophie' ME

Sophie's profile played an important role in how she conceived and conducted her ME. She has always been passionate about biology, which influenced her choice of the topic of the ME. That can explain why she dedicated an important portion of her ME to constructing the question, the planning of the experiment and designing the experiment from the biology point of view and wanted it to make sense. This can also explain the important portion of reflection MMUs and the fact that the MMUs concerning methodology, biology and writing the report contained mostly MMUs with a desirable effect on the ME. On the other hand, Sophie's profile came to direct confrontation of the teacher's vision and expectations of the ME. Sophie wanted to explore the biology and apply it to environmental issues but her teacher oriented the work on doing mathematics and that is where she encountered most of the difficulties.

The expectation of demonstrating the ability to mobilize some mathematical tools and apply them led Sophie to search for a praxeology that would enable her to satisfy the teacher's expectations, even though this was not the part of the ME she was most interested in. She found the method of the finite differences, but she was not able to apply it on her data without a series of calculation and inattention errors. This resulted in a fourth constant difference (instead of only three). Sophie was, however, able to correctly interpret this result and recognize a quartic polynomial even though the video used an example of a cubic polynomial. Here, Sophie showed some understanding of this technique but would have definitely needed help with the justification of the application on her data, which was not possible because of the teacher's reduced topos. And when the teacher commented on this technique on the first draft ("I don't think this works"), it was just too late. Indeed, the whole analytical solution in Sophie's ME was based on this technique and if she had removed it, all the "mathematics" expected by the teacher (and the institution) would have been gone. After determining the degree of the polynomial, Sophie set up a system of 3 linear equations with 4 unknown coefficients of the quartic polynomial and was unable to find an appropriate technique to deal with this situation. Here again, a targeted regulation would have been needed. Sophie then used technology to run a quadratic and a cubic regression obtained the two curves on her data. This experimental milieu was sending out feedback that the cubic regression provided a good fit. What happened, however, since Sophie had to do the mathematics analytically, she turned to exploring the quadratic model because she could handle the techniques that enabled her to determine the quadratic function analytically. Despite the works in the milieu sending out clear feedback that the curve did not look like a parabola, Sophie went on the Internet to search for another "work" that resembled more a parabola to justify the pursuit of her work in the quadratic model.

Fortunately, there was a useful teacher's regulation on the first draft stating that the cubic regression looked good and engaged Sophie in the exploration of the cubic model. However, Sophie faced the same difficulties but this time she used technology to explore the cubic model. This actually took care of the missing praxeological equipment and enabled Sophie to conduct a meaningful reflection and interpret the results.

For David, Sophie's math teacher, doing the mathematics by hand seemed to have been more important than mobilizing the right mathematical concepts, asking questions. We obviously do not neglect the importance of building appropriate praxeological equipment when teaching and learning mathematics yet, it is also important, especially when engaging students in inquiry, to consider the added value of technology to supply the missing techniques. At least as shown on Sophie's case, the technology played an important role and enabled her to deepen her reflection and finally provide a reasonable answer to her initial question. We could also ask what would have happened if Sophie had been supervised by a biology teacher in collaboration with her math teacher. Most likely this would have obliged the math teacher to enlarge his vision of modeling and would have even brought some of the models used in biology as we showed in the "expert" investigation at the beginning of this chapter. This would have even further justified the use of technology because the models used by the biologists are more complex and often inaccessible with the mathematics toolbox of a secondary school student. This raises another question: To what extent the "applied mathematics" explorations can be approached from purely mathematics perspective? In the next chapter, we provide a comparative conclusion of the two MEs that we've analyzed.

Chapter V

COMPARATIVE CONCLUSION OF RACHEL'S AND SOPHIE'S ME.

When considering the two MEs that we have analyzed, we can observe some similarities related to the common institutional milieu and influenced by the teacher's personal praxeologies. Yet, we have also observed several differences in the way these MEs were conducted resulting from the students' profiles. If we were to depict one dominant feature of each student, we would refer to Rachel's assiduity and Sophie's creativity. We consider the similarities and differences between the two MEs resulting from the students' profiles according to two axes: (1) how the students responded to the teacher's constraints and (2) how the media was used in the elaboration of the ME.

1 The students' response to their teacher's constraint

The choice of the topic of the ME for Sophie and Rachel was certainly influenced by the teacher's vision of the ME as described at the end of Chapter II of Part C (see p. 156). As a matter of fact, both explorations are related to applications of function, yet each student approached the ME in an original way. This shows that the constraint imposed by the teacher was interpreted differently by the two students due to their different profiles. It is thus quite likely that the student's profile also contributes to the fact that the mathematics are done and learned in different ways.

1.1 The inquiry potential of the ME

Even though the way Rachel and Sophie searched and conducted their MEs diverged because of their different profiles, in both cases at some point the milieu created favorable conditions for a potential inquiry. Both students, Rachel and Sophie, as well as most of the other students in this class, conducted explorations related to mathematical modeling. Here we can see the influence of the institutional constraint on the teacher which led to reducing the choices of the students. Despite the reduced vision of mathematical modeling definitely limiting the inquiry potential of the ME, we were able to identify in both MEs some favorable conditions related to the milieu.

Rachel as a more scholar student accepted and adhered to the teacher's vision and in her ME, she searched for a curve that would perfectly superimpose the suspension cable of the Golden Gate Bridge. Her questions and search in the media were oriented rather pragmatically to fulfill her aim: find the function whose curve superimposes on the photo

of the bridge. She found and followed a protocol that enabled her to know how to proceed, which reduced the inquiry into a list of instructions to follow taking the suspension cable for a parabola without any justification. This together with the teacher's regulation stating that a suspension cable was a catenary curve raised a new question and created conditions for a new search: "Is the suspension cable a parabola?" Rachel while conducting the search on the Internet encountered answers stating that a suspension cable could be a parabola or a catenary. These contradictory answers in the inquiry milieu created favorable conditions for a potential inquiry into the conditions in which a suspension cable forms a parabola while in other cases this would be a catenary. As we showed in the previous chapter, this inquiry potential has not been exploited as the teacher's answer overruled the answers from the media.

In Sophie's case, the influence of the teacher was even more visible. Compared to Rachel, already her initial questioning, motivated by climate change issues, had a good inquiry potential. Indeed, she wanted to explore where to grow certain aquatic plants, knowing from biology that the rate of photosynthesis varies in respect to temperature. In this case the mathematics would be at the service of this question because it provides tools to find the optimum temperature to maximize the rate of photosynthesis. Sophie wasn't that much interested in the doing the mathematics, but rather considered it as an ally that provided answers to pursue her inquiry. The teacher, however, by imposing his vision of modeling, reduced the inquiry potential of Sophie's initial question and transformed it into a protocol to follow: find the equation of the function that fits best the data and do the derivative to determine the optimum temperature.

These two examples show that the inquiry milieu of the ME can create conditions for the inquiry potential to appear and that in our case study this inquiry potential was reduced by the teacher's personal praxeologies as a result of his conception of mathematics and his vision of mathematical modeling. There seems to be, however, an objective constraint created by the institutional conditions of the ME that hinders the inquiry potential. Let us demonstrate it on the following example.

1.2 The student's difficulties as result of an institutional condition

Influenced by the teacher, both students then attempted to provide analytical solutions for the best fit function and justifications of their choice. They both encountered difficulties but these were not of the same nature. Rachel's main difficulty resided in the organization of an appropriate experimental milieu whereas Sophie's resulted from her missing praxeological equipment. Regardless the nature of their difficulties, both students would have needed more guidance. This appears to be, however, a complicated issue because of the institutional conditions concerning the implementation of the ME, and more

specifically because of the fact that the MEs are conducted individually. We showed through the analysis of the mesogenesis four characteristics concerning the teacher's inputs:

- 1) these are not necessarily always relevant and may even be incorrect;
- 2) these are often very general and do not correspond to the student's specific needs;
- 3) as the teacher discovers the student's exploration on the go, he often relies on the information brought by the student. So, when a plausible information appears, even if this turns out to be incorrect or irrelevant later, the teacher may accept or confirm it.

Indeed, this creates a vicious circle: the students' crying need for a teacher's guidance and the difficulty of the teacher to provide efficient help resulting from his reduced topos. Moreover, the students tend to accept the teacher's inputs without further questioning benefiting thus from a sort of privileged status. Students are generally told to verify the reliability of the sources when searching on the Internet. As a result of the prevailing teaching and learning paradigm, the status of the teacher provides, however, a sort of guarantee that his answers are relevant and correct. This is generally true in the context of ordinary classes when a teacher prepares a lesson in advance and anticipates possible questions or difficulties. This, however, no longer works when a teacher has to supervise a research or an investigation project which puts the teacher in a different role: the one of a senior fellow or more experienced partner whose ideas can contribute to the inquiry but need to be subject to a similar process of verification as those from the Internet. The conditions of the ME change thus drastically the implicit didactic contract, which, in David's case, was even more reinforced by the topogenèse in the ordinary classes. This generates a supplementary difficulty for the students who need to realize that they cannot have the same expectations on their teacher as in the ordinary classes.

2 The choice of the media and their potential to enrich the milieu

In the analyzed MEs and also from the interviews with the two students and the answers collected from the other students in the same class, the privileged type of media was tutorial videos explaining on a worked example how to apply a technique. These videos explaining an application of a technique do not generally provide the technological justification or the conditions in which the technique could be applied i.e., do not provide the logos of the given praxeology. In both, Rachel's and Sophie's ME, we identified such videos.

In Rachel's case, the video showing how to find the quadratic function of the Lion's Gate Bridge was perfectly transferable to the problem of the Golden Gate Bridge. Rachel was able to learn a new technique and apply it to her problem correctly. This resulted in producing a tool (the equation of a quadratic function) that contributed to the construction of an experimental milieu. In Sophie's case, we also identified a video which showed how

to identify the degree of a polynomial and the coefficient of the term with highest degree from data using the method of finite differences and determine the remaining coefficients of a cubic polynomial using a system of linear equations. To a certain extent, Sophie was also able to extract and apply the technique of finite differences to her data but was blocked by the adaptation of the technique from a system of 3 unknowns to a system of 4 unknowns. The difference between these two videos resided in the need of adaptation of the provided techniques to the situation in the MEs. This process was somewhat easier for Rachel because the technique was directly adaptable from the worked example to the Rachel's problem whereas in Sophie's case, the adaptation of the provided technique was more complex. On top of that, Sophie's difficulties were reinforced by the many calculation and inattention errors when applied to real-life data. Nevertheless, none of the two videos provided a reasonable justification for the given techniques. For the Lion's Gate Bridge, a quadratic model was applied for the suspension in the video without any justification of why a suspension cable of a bridge formed a parabola. In the video proposing the technique of the finite differences, applying it to polynomial data (cubic in the provided example) was a condition *sine qua non* for the example to work out.

Globally, Sophie explored different paths, attempted to study and apply a rather sophisticated technique which could have worked out for her, but the missing logos led to a wrong application. She was blocked by a system of linear equations, unable to learn and apply the technique from the video, probably because it required a more complex adaptation. In Sophie's case, the missing techniques were partially taken care of by the use of technology which, together with the inquiry habits of mind, led to an abundance of reflection and analysis in her ME and contributed to the construction of A♥ despite the lack of praxeological equipment.

Rachel was able to apply a technique to find the vertex form of a quadratic function by analogy because the worked example in the media was close enough to her situation. On the contrary, Sophie encountered more difficulties when trying to apply the method of the finite differences and we attribute this difficulty to the fact that the worked example in the media was not directly applicable to her situation and a thorough analysis of the situation would have been needed in order to determine the conditions in which the described technique could be used.

Based on these two quite different ME's we can say the following factors affect its elaboration and thus reduce or enhance its inquiry potential:

- 1) the inquiry habits of mind of the student;
- 2) the missing techniques which could be built partially with the use of media or substituted via technology.

Moreover, the content of media also plays an important role in the ME and in the construction of the inquiry milieu. When carrying out an analytical solution an appropriate

technique is necessary, the correct choice of a technique requires a good analysis of the situation and an ability to justify the choice of the technique. In case of tutorials or instructional videos, often a technique is explained on a worked example but generally, these videos do not provide any logos. In case of a direct application of the technique it somewhat works as shown on Rachel's ME but with an increasing complexity of the need of an adaptation of the given technique, as shown on Sophie's ME, a good understanding of the logos is crucial. We thus advance a hypothesis that if a more complex adaptation is required, a good understanding of the logos is indispensable to adapt and apply a given technique correctly. This seems to be an important condition of conducting a mathematical inquiry and in fine, a skill that should be taught and developed prior to the MEs. Simply raising the awareness of the students of this fact inherent to how mathematical knowledge works, could probably help the students seek more specific help and guidance from the teacher, on the one hand, and enable the teacher to provide more targeted regulations, on the other hand. In order to confirm this hypothesis, more MEs, however, would need to be analyzed.

We also observed that both students have used technology in their MEs. In Rachel's case, technology was used to organize an experimental milieu, which was proved to be a challenge to tackle. In Sophie's ME, technology was used in a certain way as a media, because it provided some of the mathematical objects that Sophie needed to conduct her ME but was unable to find analytically. In both MEs the access to technology contributed to the advancement of the inquiry. However, the supervisor of the ME should pay particular attention to the construction of an appropriate experimental milieu.

3 The students' view of the ME

Last, but not least, both students claimed when interviewed after they handed in the final report that they globally enjoyed working on their MEs and also that they found it challenging. When asked what they found most challenging about the ME, the way they described their difficulties actually goes in line with their profile and their conception of mathematics. Rachel, when asked what she found the most difficult said:

I think it was the beginning part, I think once I understood, what exactly I have to do, it was the equation, getting the proper and the right dimensions. But then once I understood what I have to do...

This confirms her need to follow a protocol and in a certain way her assiduity and determination to find a technique and apply it correctly. Sophie couldn't get past the system of linear equations with four unknowns and that is what she found most difficult in the ME. We could think that if Rachel was facing the system of linear equations with four unknowns, she would have probably persevered and found a way to solve it analytically. We could imagine that Sophie would have probably explored different paths if she had

been doing the Golden Gate Bridge. This makes us think that the student profile, past experiences with inquiry also affects the inquiry potential of the ME.

Sophie brought up several times that doing the ME and applying to biology “made it make more sense...” and helped her better understand the derivatives and the maximum of a function. She concluded that:

I would actually like to do more of this with different topics since this really reinforced what I knew and helped me learn new things. And I feel that if I did that with vectors or if I did that with... I don't know what else, you know different themes, I think that would really help.

For Rachel, too, the ME fulfilled her expectations as she said:

My aim was to look through a lot more on functions and the thing is I didn't know like what a negative a would do, you know, equation like I didn't know what negative a would like to flip the parabola. I didn't know which one would stretch and which one would [incomprehensible] and stuff like that. And I learned all of that now, when I was doing stuff.

Even though the inquiry potential of the ME has not been exploited to the full, both students took something out of the experience of conducting the MEs and they both claim that it helped them to better understand some of the encountered mathematics which turned their experience of doing mathematics into a positive one.

GENERAL CONCLUSION

The general objective of this PhD thesis was to determine whether and to what extent the IB as an institution creates favorable conditions for implementation of inquiry-based mathematics education. More specifically, whether and how the Mathematical Exploration in the form of an internal assessment represents an opportunity to experience a genuine research activity. We also wanted to better understand how the teachers perceive IBE and how they navigate within the leeway left by the institutional constraints. Finally, an important part of our research also concerned the work of the students, their interactions with the media and the role of the media in the ME. The conclusion is organized around three entries. First, we highlight some general contributions regarding primarily the methodological design and the enrichment of the theoretical framework. Then we consider the contributions concerning the comprehension of the IB as an institution, in particular the conditions and constraints identified at the different levels of the scale of didactic codeterminacy and their impact on the implementation of IBME. And finally, we formulate some perspectives, suggesting possible modifications at the levels of school and discipline that could help increase the inquiry potential of the ME and propose some ideas to optimize the leeway left for the teachers within the institutional conditions and constraints.

1 Theoretical contributions

In order to tackle our research questions concerning the chronogenesis and mesogenesis of the ME, we first needed to find a way to access the private work of a student, which is not an easy issue to tackle. The tools such as an action camera, a logbook and a shared document on Google drive that we put in place proved reliable and enabled us to access the private work of the students. Though, the complexity of the designed protocol makes it difficult to get volunteer students and, for the students who actually volunteer to collaborate in the research, it is not a straightforward task to fully respect the requirements of the protocol. This can result in inconsistent data due to the fact that the students can forget to turn on the camera or actually decide what to record and what not to record. It is also important to keep in mind that the students have a thousand other things to do and regular follow-up on the progression and recordings is necessary. Nonetheless, the designed protocol provides an operational tool to access the private work of the student and thus obtain unique data that is not accessible otherwise.

One of the main theoretical contributions of this PhD thesis is the adaptation of the ATD tools and concepts resulting in the development of a promising analysis framework enabling to process data coming from different sources and to capture and analyze the dynamic aspects of the chronogenesis and mesogenesis of an inquiry process. This analysis

framework has the potential to be used and applied universally to conduct a posteriori analysis of inquiry-based activities with access to media. The three-level analysis framework enables to examine the chronogenesis and mesogenesis of the ME at various levels of detail and access what would otherwise remain inaccessible. At the Q-A level, we were able to describe the student's activity based on the pursued objective thanks to the adaptation of Schoenfeld's protocol to analyze problem solving. The main adaptation consisted of creating a new episode that we called "study" reflecting thus the possibility to access the media. At the M-M level, we were able to depict and describe the nature of the interactions of the student with the media and capture the evolution of the milieu. The E-level provided the finest lens to zoom on each singular ingredient of the Herbartian scheme.

There is, obviously, still room for improvement and fine-tuning of the developed analysis framework, a need to further test its robustness and perhaps adjust certain descriptors but already in the current state, we can confirm its applicability to conduct a posteriori analysis of inquiry-based activities, particularly to capture the dynamics of the chronogenesis and mesogenesis of an inquiry process.

One of the limitations of this work is that we were only able to obtain explorable data of two students' MEs. Nevertheless, this presumed limitation allowed us to conduct an in-depth analysis of the work of each student, consider in detail each student's gesture, each interaction with the media and contributed both to the development of an analysis framework and to a better understanding of the media-milieu dialectic within an inquiry process.

2 Contributions concerning the comprehension of the IB as an institution

We address this issue in respect to the conditions and constraints concerning the implementation of IBME in day-to-day teaching and the role of the ME as a potential vector of inquiry in the IB mathematics classes. We organized it into two sections following the scale of didactic codeterminacy: the upper levels including the levels of school and pedagogy and the level of discipline.

2.1 At the levels of school and pedagogy

In the institutional analysis, we showed at the levels of school and pedagogy, that from its creation, the IB inscribed the desire for innovation and implementation of constructivist approaches to learning in their educational philosophy. Indeed, doubts about the efficiency of an encyclopedic approach to education were expressed and learning how to learn was the new mission of school. The creation of the IB curriculum was thus motivated by these principles, yet, they had to do so within the leeway left by the main constraint: university

acceptance. Alongside with the standardized external examination, the IB found ways to counterbalance the weight of the final exams and create conditions for their educational philosophy to be reflected in the classroom. From the curriculum point of view, they found a compromise in the midst of the different approaches to general education ensuring a balanced offer between sciences and humanities. The requirement to study 3 or 4 subjects on a higher level allows a certain specialization and the integration of the Theory of knowledge (TOK) course allows the students to reflect on and grasp the nature of knowledge. The Creativity and Service (CAS) requirement completes this vision of holistic education. On the side of assessment, this resulted in creating diverse forms of internal assessment based on the elaboration of various research projects contributing to about 20% of the final grade. We can conclude that the IB through the levers of the curriculum and assessment attempted to create favorable conditions to ensure the implementation of IBE. In mathematics, this translated and settled into the introduction of the ME as an individual mathematical research project supervised by the teacher with the objective to provide the students with opportunities to conduct an investigation into a field of mathematics. Indeed, it is one of the rare institutions that assesses the completion of such an individual exploration specifically in mathematics thus creating room for the implementation of IBME. At the same time, the IB created a system of assessment without specifying what the inquiry-specific objectives are, resulting in a paradox of evaluating something that has not been taught. We would also like to point to a possible risk related to the way the ME may be interpreted and finally, do a disservice to the implementation of inquiry. Instead of being the culmination of IBME in the form of assessment where everyday teaching and learning tend to be more and more inquiry-based, it exists in an isolated manner, next to the regular classes thus giving the actors of the IB a sort of a clean conscience that a relatively large portion (20%) of the program is dedicated to inquiry. This is even reinforced by the volume of the content knowledge to cover and the pressure of the final exams. Despite the fact that the institution created the ME and dedicated 20% of the assessment to foster inquiry in the classroom, our findings show that it is not enough for several reasons that we develop at the level of discipline.

In the next part, we move to the level of discipline to interpret these conditions and name some limiting factors of the inquiry potential of the ME in the light of the results of our two case studies.

2.2 At the level of discipline

Institutionally, 20% of the final grade results from the assessment of the ME but in reality, the portion dedicated to inquiry represents less than 20% of the classroom activity. The weight of the remaining 80%, certainly due to the pressure of the final exam and the volume of the content to be covered result in a rather traditional approach to teaching and learning. The observation of the practices of two mathematics teachers, who both expressed an

overall positive attitude towards inquiry, confirms the difficulty to implement IBME in the daily practice. Indeed, most of the day-to-day mathematics classes were dedicated to covering content knowledge and the preparation of the final exam. Both teachers recognized the need of conceptual understanding in mathematics and put in place some form of a ‘dialogued lecture’, nonetheless, most of the teaching remained quite upfront. Even though we were able to identify some inquiry potential within the analyzed activities proposed by both teachers, their implementation often resulted in a disguised ostension due to the lack of time or resources. This shows that the 20% dedicated to the ME are not enough to influence the day-to-day teaching and learning despite the fact that most of the teachers from our survey expressed a positive attitude towards inquiry. At least the two case studies haven’t provided any evidence of possible effects of the ME on the ordinary classes. Indeed, the ME lives its parallel life next to the ordinary classes. For about 4 to 5 weeks, the regular mathematics classes are dedicated to the elaboration of the MEs: students working individually, advancing at their own pace and the teacher walking around, answering individual questions. This results in a radical change in terms of chronogenesis, topogenesis and mesogenesis in comparison to the ordinary classes, yet it does not suffice for the teacher and the students to adapt to it. Most of the time, the students are used to listen to the teacher explaining the theory, showing the different techniques and constructing the tools, which they then apply to textbook exercises, practice on exam items or pseudo-problems. In contrast, the 4–5 weeks dedicated to the elaboration of the ME, inverting the topos and affecting the construction of the milieu, create a completely different learning environment. For a rather short period of time, the students find themselves in studying conditions that they are not used to, resulting in Rachel’s case in a very scholar approach to the ME. We observed that the teacher’s regulations had a tendency to remain in the milieu and override other information from the media as illustrated by Rachel opting for a catenary curve rather than a parabola. Another example of Rachel’s difficulty was related to the creation of an experimental milieu and to the inability to deal with the skewed picture of the bridge. The created paradox results in situations that we showed throughout the two analyzed MEs in particular (1) the students taking the teacher’s answers for granted, (2) the inability of the teacher to provide relevant and targeted regulations when needed, (3) a certain solitude of students when facing a difficulty or (4) the difficulties of students to process information from the media. Without any doubt, the ME creates a niche for inquiry to be deployed but the drastic change of teaching and learning paradigm thus created, prevents the inquiry potential of the ME to be optimally exploited. Indeed, the students and the teacher return constantly to their old habits because there is simply not enough time for the change of the paradigm to be operational. In a traditional setting, teachers prepare the problems or the lesson and know what they are doing or where it should lead. This creates conditions in which they are able to provide correct and relevant answers and ensure targeted regulations when needed. With the ME, however, as the teachers discover the explorations on the go, it is simply impossible to provide absolute

answers the same way as in the ordinary classes. It is crucial for both the teachers and the students to become aware of this fact because it affects the degree to which students' expectations for a clear answer may be met. With the reduced topos and the little time that the teachers have at their disposal to see each student, neither they nor the students can expect to function the same way as in the ordinary classes. In other words, the teachers have to realize that they can't give out the same affirmative answers as they would when they are checking an assignment that they have designed themselves. If they do so, though, we can expect a negative effect on the construction and evolution of the milieu. The students also need to understand that in the paradigm created by the ME, the teachers' answers do not have the same validity as in regular classes and have to be considered the same way as any other media. This raises a question of how to make aware the teachers and the students of the need of a change in their respective roles and how to make this change operational and effective.

Another limiting factor is that the type and the depth of the inquiry are conditioned by the level of the students. Institutionally, it is required that the applied mathematics is at least at the level of the course, the students can use more complex mathematics or mathematics that is not in the program provided that they show enough conceptual understanding of the applied tools. This is reflected in the choice of the topic and the teachers need to find ways to tackle this problem as it is their responsibility to make sure that the mathematics related to the topic chosen by the student is within the student's reach. This reveals a certain degree of dependence towards the teacher because the students don't have the means to evaluate the level of difficulty which the inquiry within the chosen topic would require. In David's case, this was ensured by guiding the students towards MEs exploring applications of functions, but we saw in the two observed explorations that this wasn't enough to ensure a certain proximity with the student's cognitive universe. Moreover, determining the suitability of the topic and the accessibility and appropriateness of the mathematics is a complex task affected by multiple factors such as the intersection between the activated and institutional milieu or the student's cognitive universe. Undeniably, the reduced topos and the limited time make it difficult for the teacher to guide and discern the students' choices and we can see that in David's case it's not necessarily a personal choice but rather a reaction to the institutional constraints and a solution to deal with this phenomenon created by the reduced topos.

We would also like to point to the question of decontextualization of knowledge and even of institutionalization. The ME is, first of all, defined as internal assessment aiming to evaluate inquiry-specific objectives. However, in order to assess a particular piece of knowledge or a competence, it must first be taught and institutionalized. Unfortunately, within the given conditions, this is often reduced to ensuring that the students are aware of the assessment criteria. We observed that during the introductory session mostly formal aspects of the ME were addressed. An important part of the teacher's discourse was

dedicated to plagiarism. He also emphasized the need of using correct terminology and notation and the personal engagement was mostly interpreted as personal interest. The students are told what they need to include their explorations but never taught how to conduct a mathematical inquiry. It seems that research skills, problem posing skills and paramathematical notions are considered as self-evident and are taken for granted as though there was nothing to learn, nothing to teach. Ultimately, the students are assessed on something that they have never really been taught. This raises a need to clearly identify the learning outcomes of the ME and the essential transversal skills necessary to conduct a mathematical inquiry, which are far from being obvious.

Finally, the last finding contributes to a better understanding of media management. First of all, we need to consider the nature of information contained in the media. The first feature to take into account is that the information from the media is not pre-processed for the student as is the case in class when the teacher prepared the lesson in advance or in a textbook when the destination of a particular piece of knowledge is clearly defined. On the one hand, research projects create a good occasion for the students to learn how to better deal with the selection of information outside of a school textbook. On the other hand, and especially when the Internet is involved in the search, in addition to the relevance, the students have to be able to judge the correctness of the retrieved information. We showed that under some circumstances, provided by a functional media-milieu dialectic, the access to media contributes to the advancement of the inquiry and enrichment of the milieu because the students were able to activate correct and relevant answers and reject irrelevant and incorrect information. Yet, we also observed moments when the milieu was saturated by incorrect or irrelevant information which prevented a good functioning of a media-milieu dialectic because the milieu was sending out irrelevant or incorrect feedback. This shows that the media cannot completely replace the teacher and a targeted teacher's intervention is necessary to adjust the saturated milieu because the students don't have the means to find an appropriate remedy by themselves. We can thus raise new questions of whether it is possible to identify some criteria that would help the students to make good choices, how the teacher could help the students evaluate the relevance of a given media and how the teacher could recognize a malfunctioning milieu and provide efficient regulations. Another issue that needs to be addressed is the solitude that the students experience when they are blocked and unable to process information from the media or when the teacher's regulations are inefficient.

Finally, we can see that even in an institution such as the IB, IBME experiences difficulty in finding its place in the mathematics classroom. And this despite the fact that the IB represents a certain social uniformity attracting students for the upper levels of the society, where we can expect parents keen on their children receiving good education. The IB students' population represents such a cultural diversity that a high degree of open-mindedness is an essential condition to function which should also contribute to cultivate

an inquiry attitude. Neither the conditions that shaped the creation of the IB, the development of the curriculum and the fact that 20% of the final grade is dedicated to inquiry-specific objectives, nor the teachers generally convinced about the benefits of IBE approaches provide enough weight to tip the scales. We could thus legitimately ask what else would need to be done to overcome this systemic resistance and to foster the implementation of IBME.

In the next part we propose, on the one hand, to formulate some modifications which, if done at the level of school, would have a significant effect at the level of discipline. On the other hand, we also present ideas, reflections and suggestions to modify the teachers' practices to better exploit the inquiry potential of the ME within the current conditions.

3 Perspectives

We identified some favorable conditions for the implementation of IBME and problem-solving in mathematics SL classes within the IB Diploma Program. Indeed, the ME provides a niche for inquiry, yet its potential can't be fully exploited due to the institutional constraints as well as to the fact that it is hermetically closed and lives on its own next to the ordinary classes. In this part, we revisit the levels of didactic codeterminacy scale and propose certain modifications resulting from our research.

At the level of school, the IB could slightly modify the format of the ME in order to improve its inquiry potential. The IB attempts to provide conditions for investigation or research activities but at the same time the students work individually. This is, of course, understandable since the ME is an official individual assessment. In research, however, scientists usually work in labs or research teams as a community and here the students are left a bit on their own because the teacher, who has to supervise these explorations, cannot simply be everywhere. In this sense, the current format of the ME reduces the opportunity for the students to experience genuine research activity. Moreover, the students experience a certain number of difficulties when searching and processing the information from the Internet alone, which shows that the media cannot replace human interaction with the supervisor or peers. Changing one parameter of the ME such as letting the students work in pairs or eventually groups of three would create conditions for discussion among peers, reduce the number of explorations to supervise for the teacher and thus increase the availability of the teacher. Conscious about the difficulty that this change could cause with regard to the assessment, we are firmly convinced that the benefits would outweigh the drawbacks. In particular, given the fact that in the science domain, the institution already created conditions in order to assess group work project as opposed to individual work in mathematics.

In the meantime, the IB proceeded to important curriculum changes in mathematics. These changes concern, *inter alia*, a reduction of the content knowledge and a creation of a

Toolkit dedicating 30 hours to the development of investigative, problem-solving and modeling skills leading to an individual exploration. This Toolkit is accompanied by a 60-page document highlighting some theoretical and conceptual principles of the approaches to teaching and learning based on inquiry and providing support for classroom activities fostering the development of the above-mentioned skills. This shows the institution's desire to take into account and react to the constraints that appeared at the level of discipline and provide better conditions for inquiry to be deployed in mathematics. Obviously, a detailed analysis of this document and further observations of its implementation are necessary to measure its impact on the implementation of IBME in the IB mathematics courses. Based on the findings resulting from our research, we would like to formulate some tips and suggestions for the teachers to improve the inquiry potential of the ME.

In our research we identified an important fact: if the 20% dedicated to the ME remain isolated in respect to what is done in the rest of mathematics classes, it will have very little to no effect on the ordinary classes. As a matter of fact, the ME represents the visible part of the “inquiry” iceberg and, to be effective, it must be seen as an end product of teaching and learning based on inquiry intended for assessment, rather than the only space to do inquiry. The day-to-day teaching and learning prior to the ME must entail a larger portion of inquiry-based activities leading to a general attitude of inquiry in order to create conditions for the students and the teachers to discover and adapt to their new roles resulting from the inversed topos. A visible change of paradigm in the ordinary classes and thus in the teachers' practices is indispensable, otherwise, there is a risk that even the newly attributed 30 hours of the Toolkit will continue their enlarged, yet parallel existence next to the ordinary classes. Although convinced about the benefits of IBME, we do not neglect the fact that teaching and learning of mathematics must necessarily include moments dedicated to building particular praxeologies and drills to master certain techniques.

Our research and findings also allow to formulate a couple of suggestions that would help to better prepare the students for the elaboration of the ME. It is crucial that the students are taught the transversal skills that seem to be often taken for granted and that the learning outcomes are identified. Prior to the MEs, it is important to teach the students how to use mathematical knowledge to conduct a mathematical investigation, how to deal with the information in the media, how to determine whether a particular technique described in a video can be applied to another situation, etc. Nowadays, an important part of the search for information is done on the Internet. It is thus crucial to understand that mathematical knowledge present in the information retrieved from the Internet is not directly accessible and pre-processed as is the case of mathematical knowledge brought to the classroom by the teacher. The a posteriori analysis of the two case studies pointed out several difficulties that the students experienced because they were evaluated on something they haven't really been taught. Without being exhaustive, we can mention their difficulties to select relevant information from the media, organize an appropriate experimental milieu or to be aware

of what they can actually expect from the teacher. It is thus essential for the students to experience a mathematical exploration before the internal assessment. This could be done collectively with the whole class on some mini-explorations of the already existing topics and the teacher could illustrate various skills that are necessary to conduct an investigation. If we take the example of the shape of the suspension cable of a bridge, we could let the students search for a while on the Internet and have them come back and discuss on the contradictory information pointing to a parabola or a catenary curve. Or we could also have them watch a video providing a particular technique on a worked example and discuss the conditions in which it would be applicable to another situation. A special attention needs to be given to the organization of an experimental milieu making sure that it provides relevant feedback. We could use the skewed picture of the Golden Gate Bridge to discuss the conditions and the limitations of superimposing as way to justification of the fit of a curve or question the need to keep the picture to scale. The teachers should really be aware of the fact that their regulations do not always have a desirable effect on the inquiry and that even a well-meant advice might have negative consequences on the answer because the student was unable to interpret or apply it correctly.

Another element that could increase the inquiry potential of the ME is the adaptation of the level of mathematics to the students' abilities. Even though for the majority of the class it should be possible to explore topics at the level of the course, for some weaker students, it could be beneficial to allow them to explore and use mathematics adapted to their abilities. This would, of course, lead to a lower score on the criterion related to the use of mathematics but there are chances that it would create conditions for the weak students to conduct a meaningful inquiry at their level.

The mathematics teachers could also use the potential of the TOK course which could contribute to a better understanding of the nature of mathematical knowledge. Maybe a closer collaboration between the TOK and the mathematics teachers would allow to treat some of the "meta" questions concerning mathematical knowledge and link it to the ME.

These findings reveal the need to rethink the pre-service and in-service teacher training; to include opportunities for the teachers to experience inquiry for themselves and make sure that they grasp the fundamental conditions when implementing IBME. Despite the fact that the IB has no influence on the pre-service training, they could do so through their system of workshops dedicated to the internal assessment. Indeed, as research shows, inquiry-based approaches are beneficial for the student when done well, which justifies the need to revisit pre-service and in-service teacher training.

We can also see a possible perspective and application of the results of this research in designing collaborative research to help optimize the space for implementation of IBME created within the institutional constraints and improve the quality of inquiry-based activities and problem-solving in day-to-day teaching. This becomes even more interesting in the context of the recent mathematics curriculum change claiming to have reduced the

content knowledge and created a block of 30 hours dedicated to the development of specific problem-solving and inquiry skills.

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- Amade-Escot, C., & Venturini, P. (2009). Le milieu didactique: D'une étude empirique en contexte difficile à une réflexion sur le concept. *Éducation et didactique*, 3(vol 3-n°1), 7–43. <https://doi.org/10.4000/educationdidactique.407>
- Arsac, G., Germain, G., & Mante, M. (1991). *Problème ouvert et situation-problème*. Lyon: IREM de Lyon.
- Artigue, M. (2012). What is Inquiry-Based Mathematics Education (IBME)? In S. Borda Carulla (Ed.), *Inquiry in Mathematics Education*. Fibonacci Project. Retrieved from http://fibonacci.uni-bayreuth.de/index.php?eID=tx_nawsecuredl&u=0&file=fileadmin/Dokumente/startingpackage/background/inquiry_in_mathematics_education.pdf&t=1490347811&hash=29992a1fa42c6fabba99e429f95e4327
- Artigue, M., & Blomhøj, M. (2013). Conceptualizing inquiry-based education in mathematics. *ZDM*, 45(6), 797–810. <https://doi.org/10.1007/s11858-013-0506-6>
- Artigue, M., Dillon, J., Harlen, W., & Léna, P. (2012). *Learning through inquiry*. Fibonacci Project. Retrieved from http://fibonacci.uni-bayreuth.de/index.php?eID=tx_nawsecuredl&u=0&file=fileadmin/Dokumente/startingpackage/background/learning_through_inquiry.pdf&t=1490347811&hash=64d27cf95545a320b79f2febf5d88a60
- Bachelard, G. (1967). *La formation de l'esprit scientifique. Contribution à une psychanalyse de la connaissance objective*. (5e édition). Paris: Librairie philosophique J. Vrin.
- Barrow, L. H. (2006). A Brief History of Inquiry: From Dewey to Standards. *Journal of Science Teacher Education*, 17(3), 265–278. <https://doi.org/10.1007/s10972-006-9008-5>
- Bell, R. L., Smetana, L., & Binns, I. (2005). Simplifying inquiry instruction. *The Science Teacher*, 30–33.
- Berthelot, R., & Salin, M. H. (1992). *L'enseignement de l'espace et de la géométrie dans la scolarité obligatoire* (Phdthesis, Université Sciences et Technologies - Bordeaux I). Université Sciences et Technologies - Bordeaux I. Retrieved from <https://tel.archives-ouvertes.fr/tel-00414065>

- Bosch, M. (2018). Study and research paths: A model for inquiry. In *Proceedings of the International Congress of Mathematicians (ICM 2018)* (Vols. 1–0, pp. 4015–4035). WORLD SCIENTIFIC. https://doi.org/10.1142/9789813272880_0210
- Bosch, M., Chevallard, Y., García, F., & Monaghan, J. (Eds.). (2019). *Working with the Anthropological Theory of the Didactic in Mathematics Education: A comprehensive casebook*. London: Routledge. <https://doi.org/10.4324/9780429198168>
- Bosch, M., & Gascon, J. (2002). Organiser l'étude. 2. Théories & empiries. In J.-L. Dorier, M. Artaud, M. Artigue, R. Berthelot, & F. Ruhal (Eds.), *Actes de la 11e école d'été de didactique des mathématiques* (pp. 23–40). La Pensée Sauvage.
- Bosch, M., & Winsløw, C. (2016). Linking problem solving and learning contents: The challenge of self-sustained study and research processes. *Recherches En Didactique Des Mathématiques*, 35(3), 357–401.
- Brousseau, G. (1986). Fondements et méthodes de la didactique des mathématiques. *Recherches En Didactique Des Mathématiques*, 7(2), 33–115.
- Brousseau, G. (1990). Le contrat didactique: Le milieu. *Recherches En Didactique Des Mathématiques*, 9(9.3), 309–336.
- Bruder, R., & Prescott, A. (2013). Research evidence on the benefits of IBL. *ZDM*, 45(6), 811–822. <https://doi.org/10.1007/s11858-013-0542-2>
- Cai, J., Moyer, J. C., Wang, N., Hwang, S., Bikai, N., & Garber, T. (2013). Mathematical problem posing as a measure of curricular effect on students' learning. *Educational Studies in Mathematics*, 83, 57–69.
- Cha, H. (2015). *Soviet Launch of Sputnik: Sputnik-Inspired Educational Reform and Changes in Private Returns in America*. Retrieved from https://tigerprints.clemson.edu/all_dissertations/1550
- Chanudet, M. (2019). *Etude des pratiques évaluatives des enseignants dans le cadre d'un enseignement centré sur la résolution de problèmes en mathématiques* (Thèse de doctorat en Sciences de l'Éducation). Université de Genève.
- Chevallard, Y. (1985). *La transposition didactique: Du savoir savant au savoir enseigné* /. La Pensée sauvage,.
- Chevallard, Y. (1992). Concepts fondamentaux de la didactique: Perspectives apportées par une approche anthropologique. *Recherches En Didactique Des Mathématiques*, 12(1), 73–112.
- Chevallard, Y. (2002a). Les TPE comme un problème didactique. In T. Assude & B. Grugeon (Eds.), *Actes du séminaire national de didactique des mathématiques. Année 2001* (pp. 177–188). Paris: IREM de Paris 7 et ARDM. Retrieved from <http://www.irem.univ-paris-diderot.fr/up/publications/AAR02005.pdf>

- Chevallard, Y. (2002b). Organiser l'étude. 1. Structure & fonction. In J.-L. Dorier, M. Artaud, M. Artigue, R. Berthelot, & F. Ruhal (Eds.), *Actes de la 11e école d'été de didactique des mathématiques* (pp. 3–22). La Pensée Sauvage.
- Chevallard, Y. (2003a). Approche anthropologique du rapport au savoir et didactique des mathématiques. In *Rapport au savoir et didactique* (Editions Fabert, pp. 81–104). Retrieved from http://yves.chevallard.free.fr/spip/spip/article.php3?id_article=42
- Chevallard, Y. (2003b). Approche anthropologique du rapport au savoir et didactique des mathématiques. In S. Maury & M. Caillot (Eds.), *Rapport au savoir et didactiques* (pp. 81–104). Paris: Fabert. Retrieved from http://yves.chevallard.free.fr/spip/spip/article.php3?id_article=42
- Chevallard, Y. (2004). La place des mathématiques vivantes dans l'éducation secondaire: Transposition didactique des mathématiques et nouvelle épistémologie scolaire. In C. Ducourtioux & P.-L. Hennequin, *La place des mathématiques vivantes dans l'enseignement secondaire*. Paris: APMEP. Retrieved from http://yves.chevallard.free.fr/spip/spip/article.php3?id_article=48
- Chevallard, Y. (2005). La place des mathématiques vivantes dans l'éducation secondaire: Transposition didactique des mathématiques et nouvelle épistémologie scolaire. *La Place Des Mathématiques Vivantes Dans l'enseignement Secondaire*. Presented at the 3e Université d'été Animath, (Saint-Flour, 22-27 août 2004), Paris. Paris: APMEP. Retrieved from http://yves.chevallard.free.fr/spip/spip/IMG/pdf/La_place_des_mathematiques_vivantes_au_secondaire.pdf
- Chevallard, Y. (2008). Un concept en émergence: La dialectique des média et des milieux. In G. Gueudet & Y. Matheron (Eds.), *Actes du Séminaire national de didactique des mathématiques*. Paris: IREM.
- Chevallard, Y. (2009a, April). *La notion de PER: problèmes et avancées*. Communication presented at the IUFM, Toulouse. Toulouse. Retrieved from http://yves.chevallard.free.fr/spip/spip/IMG/pdf/La_notion_de_PER___problemes_et_avancees.pdf
- Chevallard, Y. (2009b, May). *Remarques sur la notion d'infrastructure didactique et sur le rôle des PER*. Communication presented at the Journées Ampère, Lyon. Lyon. Retrieved from http://yves.chevallard.free.fr/spip/spip/IMG/pdf/Infrastructure_didactique_PER.pdf
- Chevallard, Y. (2011, February). *Journal du séminaire TAD/IDD. Théorie Anthropologique du Didactique & Ingénierie Didactique du Développement*. UMR-ADEF. Retrieved from <http://yves.chevallard.free.fr/spip/spip/IMG/pdf/journal-tad-idd-2010-2011-3.pdf>

- Chevallard, Y. (2015a). Pour une approche anthropologique du rapport au savoir. *Dialogue*, (155). Retrieved from https://www.gfen.asso.fr/images/documents/publications/dialogue/dial155_enligne_anthropo_rap_savoir_chevallard.pdf
- Chevallard, Y. (2015b). Teaching Mathematics in Tomorrow's Society: A Case for an Oncoming Counter Paradigm. In S. J. Cho (Ed.), *The Proceedings of the 12th International Congress on Mathematical Education* (pp. 173–187). Springer International Publishing.
- Chevallard, Y., & Matheron, Y. (2002). Travaux Personnels Encadrés: Un cadre d'analyse didactique pour un changement majeur dans l'enseignement au lycée. In M. Bridenne (Ed.), *Nouveaux dispositifs d'enseignement en mathématiques dans les collèges et les lycées*. (pp. 141–150). Dijon: IREM de Dijon.
- Chichekian, T., & Shore, B. M. (2014). The International Baccalaureate: Contributing to the Use of Inquiry in Higher Education Teaching and Learning. In *Innovations in Higher Education Teaching and Learning: Vol. 1. Inquiry-based Learning for Faculty and Institutional Development: A Conceptual and Practical Resource for Educators* (Vol. 1, pp. 73–97). Emerald Group Publishing Limited. Retrieved from <http://www.emeraldinsight.com/doi/full/10.1108/S2055-364120140000001006>
- Comiti, C., Grenier, D., & Margolinas, C. (1995). *Niveaux de connaissances en jeu lors d'interactions en situation de classe et modélisation de phénomènes didactiques*. 92–113.
- Coppé, S. (1993). *Processus de vérification en mathématiques chez les élèves de première scientifique en situation de devoir surveillé*. Thèse de l'Université Claude Bernard. Lyon I.
- Coppé, S. (1998). Composantes Privées et Publiques du Travail de l'Élève en Situation de Devoir Surveillé de Mathématiques. *Educational Studies in Mathematics*, 35(2), 129–151. <https://doi.org/10.1023/A:1003086628291>
- Coquidé, M., Fortin, C., & Rumelhard, G. (2009). L'investigation: Fondements et démarches, intérêt et limites. *Aster*, (49), 51–78.
- Demonty, I., Fagnant, A., & Dupont, V. (2015). Analyse d'un outil d'évaluation en mathématiques: Entre une logique de compétences et une logique de contenu. *Mesure et évaluation en éducation*, 38(2), 1–29. <https://doi.org/10.7202/1036761ar>
- Dewey, J. (1929). *The Quest For Certainty*. Minton Balch And Company. Retrieved from <http://archive.org/details/questforcertaint032529mbp>
- Dewey, J. (1938). *Logic: The theory of inquiry*. New York: Henry Holt and company.

- Dias, T. (2008). *La dimension expérimentale des mathématiques: Étude exploratoire dans des situations d'enseignement et de formation au sein de l'enseignement spécialisé* (Thèse de Doctorat). Université Claude Bernard Lyon 1, Lyon.
- Dorier, J.-L., & García, F. J. (2013). Challenges and opportunities for the implementation of inquiry-based learning in day-to-day teaching. *ZDM*, 45(6), 837–849. <https://doi.org/10.1007/s11858-013-0512-8>
- Dorier, J.-L., & Maaß, K. (2020). Inquiry-Based Mathematics Education. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 384–388). Cham: Springer International Publishing. https://doi.org/10.1007/978-3-030-15789-0_176
- Dugonjic-Rodwin, L. S. (2014). *Les IB Schools, une internationale élitiste: émergence d'un espace mondial d'enseignement secondaire au XXe siècle*. (Thèse de doctorat, Université de Genève). Université de Genève, Genève. Retrieved from <http://archive-ouverte.unige.ch/unige:43349>
- Duval, R. (1993). Registres de représentation sémiotique et fonctionnement cognitif de la pensée. *Annales de Didactique et de Science Cognitives*, 5, 37–65.
- Fabre, M. (2005). Deux sources de l'épistémologie des problèmes: Dewey et Bachelard. *Les Sciences de l'éducation - Pour l'Ère Nouvelle*, Vol. 38, 53–67.
- Fabry, L. (2021). A dialogical philosophy: Bachelard's Introduction to "Le Rationalisme appliqué." *Philosophical Inquiries*, 9(1), 211. <https://doi.org/10.4454/philing.v9i1.362>
- Favier, S. (2022). *Étude des processus de résolution de problèmes par essais et ajustements en classe de mathématiques à Genève* (University of Geneva). University of Geneva. <https://doi.org/10.13097/archive-ouverte/unige:159466>
- Florensa, I., Bosch, M., Gascón, J., & Mata, M. (2016). SRP design in an Elasticity course: The role of mathematic modelling. *First Conference of International Network for Didactic Research in University Mathematics*. Montpellier, France. Retrieved from <https://hal.archives-ouvertes.fr/hal-01337877>
- Fox, E. (1985). International Schools and the International Baccalaureate. *Harvard Educational Review*, 55(1), 53–69. <https://doi.org/10.17763/haer.55.1.cl26455642782440>
- Gardes, M.-L. (2013). *Étude de processus de recherche de chercheurs, élèves et étudiants, engagés dans la recherche d'un problème non résolu en théorie des nombres* (Mathématiques générales). Université Claude Bernard - Lyon I.
- Gazzola, M. P., Otero, M. R., & Llanos, V. C. (2019). THE CHARACTERISTICS DIDACTIC GESTURES OF A STUDY AND RESEARCH PATH INVOLVING MATHEMATICS AND PHYSICS AT SECONDARY SCHOOL. *European Journal of Education Studies*, 0(0). <https://doi.org/10.46827/ejes.v0i0.2704>

- Haese, R. C., Haese, S., Haese, M., Mäenpää, M., & Humphries, M. (2013). *Mathematics for the international student: Mathematics SL, for use with IB diploma programme* (Third edition. reprinted with minor corrections). Marleston, Australia: Haese & Harris Publications.
- Hersant, M., & Orange-Ravachol, D. (2012). La démarche d'investigation, les mathématiques et les SVT : des problèmes de démarcation aux raisons d'une union. In J.-L. Dorier & S. Coutat (Eds.), *Enseignement des mathématiques et contrat social: Enjeux et défis pour le 21e siècle: Actes du colloque EMF 2012*. Université de Genève, Genève, Suisse.
- Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Human, P., Murray, H., ... Wearne, D. (1996). Problem Solving as a Basis for Reform in Curriculum and Instruction: The Case of Mathematics. *Educational Researcher*, 25(4), 12–21. <https://doi.org/10.3102/0013189X025004012>
- Hill, I. (2002). The history of international education: An international baccalaureate perspective. In M. Hayden, J. Thompson, & G. Walker, *International Education in Practice: Dimensions for national & international schools* (Kogan Page, pp. 18–29). London.
- Hill, I. (2007). International Education as Developed by the International Baccalaureate Organization. In *The SAGE Handbook of Research in International Education* (pp. 25–37). London: SAGE Publications Ltd. <https://doi.org/10.4135/9781848607866>
- Hill, I. (2012). An international model of world-class education: The International Baccalaureate. *PROSPECTS*, 42(3), 341–359. <https://doi.org/10.1007/s11125-012-9243-9>
- Hmelo-Silver, C. E., Duncan, R. G., & Chinn, C. A. (2007). Scaffolding and Achievement in Problem-Based and Inquiry Learning: A Response to Kirschner, Sweller, and. *Educational Psychologist*, 42(2), 99–107. <https://doi.org/10.1080/00461520701263368>
- IB. (2015). *Approaches to teaching and learning in the Diploma Programme*. International Baccalaureate.
- IBO. (2001). *Mathematical Methods Standard Level* (Second edition). Grand-Saconnex: International Baccalaureate Organization.
- IBO. (2004). *Diploma Programme Mathematical studies SL guide*. Cardiff: International Baccalaureate Organization.
- IBO. (2006). *Diploma Programme Mathematics SL guide*. Cardiff: International Baccalaureate Organization.
- IBO. (2009a). *The Diploma Programme: A basis for practice*. International Baccalaureate Organization.

- IBO. (2009b). *The Diploma Programme: From principles into practice*. International Baccalaureate Organization.
- IBO. (2010). *Mathematics HL and SL Curriculum Review Report*. International Baccalaureate Organization.
- IBO. (2012). *Diploma Programme Mathematics SL guide*. Cardiff: International Baccalaureate Organization.
- IBO. (2013a). *Diploma programme: Theory of knowledge guide*. Cardiff: IBO.
- IBO. (2013b). *IB learner profile*. Cardiff: International Baccalaureate Organization.
- IBO. (2017). *May 2017 subject report Mathematics SL - Time Zone 1*. International Baccalaureate Organization. Retrieved from [https://www.ibdocuments.com/IB%20SUBJECT%20REPORTS/Group%205%20-%20Mathematics/Mathematics%20SL/ENGLISH/May%202017%20Subject%20Report%20\(TZ1\).pdf](https://www.ibdocuments.com/IB%20SUBJECT%20REPORTS/Group%205%20-%20Mathematics/Mathematics%20SL/ENGLISH/May%202017%20Subject%20Report%20(TZ1).pdf)
- IBO. (2019). *Mathematics: Analysis and approaches guide*. Cardiff: International Baccalaureate Organization.
- International Baccalaureate. (2015). *Approaches to teaching and learning in the Diploma Programme*. Cardiff: IB.
- International Baccalaureate Organization. (2008). *IB learner profile booklet*. Cardiff.
- I.S.E.S. (1967). *Le Baccalauréat International*. Genève: Service d'examens des écoles internationales.
- Kidron, I., Artigue, M., Bosch, M., Dreyfus, T., & Haspekian, M. (2014). Context, Milieu, and Media-Milieus Dialectic: A Case Study on Networking of AiC, TDS, and ATD. In A. Bikner-Ahsbabs & S. Prediger (Eds.), *Networking of Theories as a Research Practice in Mathematics Education* (pp. 153–177). Cham: Springer International Publishing. https://doi.org/10.1007/978-3-319-05389-9_10
- Kilpatrick, J. (1987). Problem formulating: Where do good problem come from? In Alan H. Schoenfeld (Ed.), *Mathematical problem solving* (Academic Press, pp. 123–148). New York.
- Kirschner, P. A., Sweller, J., & Clark, R. E. (2006). Why Minimal Guidance During Instruction Does Not Work: An Analysis of the Failure of Constructivist, Discovery, Problem-Based, Experiential, and Inquiry-Based Teaching. *Educational Psychologist*, 41(2), 75–86. https://doi.org/10.1207/s15326985ep4102_1
- Liljedahl, P., Santos-Trigo, M., Malaspina, U., & Bruder, R. (2016). *Problem Solving in Mathematics Education* (Springer Open; G. Kaiser, Ed.).

- Maaß, K., & Artigue, M. (2013). Implementation of inquiry-based learning in day-to-day teaching: A synthesis. *ZDM*, 45(6), 779–795. <https://doi.org/10.1007/s11858-013-0528-0>
- Maaß, K., & Engeln, K. (2013). *Promoting inquiry-based learning in mathematics and science education across Europe* (No. Deliverable N° 9.3, FP7 Grant Agreement n° 244380). Freiburg: University of Education Freiburg. Retrieved from University of Education Freiburg website: https://primas-project.eu/wp-content/uploads/sites/323/2017/11/PRIMAS_D-9.3_IBL-Implementation-survey-report.pdf
- Maaß, K., & Engeln, K. (2014). *Report on the large-scale survey about inquiry based learning and teaching in the European partner countries* (Project Deliverable No. Deliverable N° 10.2, FP7 Grant Agreement n° 320693). Freiburg: University of Education Freiburg. Retrieved from University of Education Freiburg website: https://mascil-project.ph-freiburg.de/images/pdf/reports/D10.2_Report_on_the_large-scale_survey_about_inquiry_based_learning_and_teaching_in_the_European_partner_countries.pdf
- Margolinas, C. (1992). Éléments pour l'analyse du rôle du maître: Les phases de conclusion. *Recherches En Didactique Des Mathématiques*, 12(1), 113–158.
- Margolinas, C. (1995). La structuration du milieu et ses apports dans l'analyse a posteriori des situations. *Les Débats de Didactique Des Mathématiques*, 89–102. Grenoble: La pensée sauvage.
- Marshall, J., Horton, R., & White, C. (2009). EQUIPPing Teachers: A protocol to guide and improve inquiry-based instruction. *The Science Teacher*, 76(4), 46–53.
- Matheron, Y. (2010). Démarche d'investigation" et Parcours d'Etude et de Recherche en mathématiques: Entre injonctions institutionnelles et étude raisonnée des conditions et contraintes de viabilité au sein du système éducatif. *Actes Du 17ème Colloque de La CORFEM. 17-18 Juin 2010*, 14–36. Caen: Université de Basse-Normandie.
- Meyer, M. (1968). *Diploma Internationa school and university entrance*. New York: The Twentieth Century Fund.
- Ministère de l'éducation national, de l'enseignement supérieur et de la recherche. (2005). *Le B.O. n°41*. Retrieved from <http://www.education.gouv.fr/bo/BoAnnexes/2005/41/41.pdf>
- National Research Council. (1996). *National Science Education Standards*. <https://doi.org/10.17226/4962>

- National Research Council. (2000). *Inquiry and the National Science Education Standards: A Guide for Teaching and Learning*. <https://doi.org/10.17226/9596>
- OBI. (1972). *Guide général du Baccalauréat International*. Genève: Office du Baccalauréat International.
- OBI. (1977). *Guide général du Baccalauréat International*. Genève: Office du Baccalauréat International.
- Office du Baccalauréat International. (1973). *Le Baccalauréat International*. Genève: OBI.
- Office du Baccalauréat International. (1974). *Bulletin Annuel No.10*. Genève: Office du Baccalauréat International.
- Parra, V., & Otero, M. R. (2018). Study and Research path: Indicators of the development of the dialectics. *Pré-Actes CITAD6*, 241–253. Autrans. Retrieved from https://citad6.sciencesconf.org/data/pages/Pre_proceedings_citad_7.pdf
- Parra, V., Otero, M. R., & Fanaro, M. de los A. (2013). *Comment fonctionnent les dialectiques dans l'implémentation d'un parcours d'étude et de recherche autour d'un système économique?* Presented at the IVe congrès international sur la TAD, Toulouse. Toulouse.
- Perrin, D. (2007). L'expérimentation en mathématiques. *Petit x*, (73), 6–34.
- Perrin-Glorian, M.-J. (1999). Problèmes d'articulation de cadres théoriques: L'exemple du concept de milieu. *Problèmes d'articulation de Cadres Théoriques: L'exemple Du Concept de Milieu*, 19(3), 279–321.
- Peterson, A. D. C. (2003). *Schools Across Frontiers: The Story of the International Baccalaureate and the United World Colleges*. Open Court Publishing.
- Pólya, G. (1971). *How to solve it: A new aspect of mathematical method* (2d ed). Princeton, N.J: Princeton University Press.
- Renaud, G. (1974). *Experimental period of the International baccalaureate: Objectives and results*. The Unesco Press.
- Renaud, G. (2012). The International School association (ISA): Historical and philosophical background. In P. L. Jonietz & D. Harris (Eds.), *World Yearbook of Education 1991: International Schools and International Education* (pp. 6–14). Routledge.
- Rey, B., Caffieaux, C., Defrance, A., & Marcoux, G. (2005). L'articulation entre savoirs et compétences dans l'enseignement secondaire. *Informations pédagogiques - RESTODE*, (57), 3–14.
- Robert, A., & Rogalski, J. (2002). Comment peuvent varier les activités mathématiques des élèves sur des exercices? Le double travail de l'enseignement sur les énoncés et sur la gestion en classe. *IREM*, 60, 6–25.

- Rocard, M., Csermely, P., Jorde, D., Lenzen, D., Walberg-Henriksson, H., & Hemmo, V. (2007). *Science Education Now: A Renewed Pedagogy for the Future of Europe*. Bruxelles: Commission européenne. Retrieved from Commission européenne website: <http://ec.europa.eu>.
- Roditi, E., & Salles, F. (2015). Nouvelles analyses de l'enquête PISA 2012 en mathématiques. *Education et Formations*, 86–87, 235–257.
- Salin, M.-H. (2002). Repères sur l'évolution du concept de milieu en théorie des situations. In J.-L. Dorier, M. Artaud, M. Artigue, R. Berthelot, & R. Floris (Eds.), *Actes de la 11e École d'été de didactique des mathématiques, Corps (Isère), du 21 au 30 août 2001* (pp. 111–124). Grenoble: la Pensée sauvage.
- Saxton, S. E., & Hill, I. (2014). The International Baccalaureate (IB) programme: An international gateway to higher education and beyond. *Higher Learning Research Communications*, 4(3), 42–52. <https://doi.org/10.18870/hlrc.v4i3.123>
- Schoenfeld, A. H. (1985). *Mathematical problem solving*. New York: Academic Press Inc.
- Schön, D. A. (1992). The Theory of Inquiry: Dewey's Legacy to Education. *Curriculum Inquiry*, 22(2), 119–139. <https://doi.org/10.2307/1180029>
- Secrétariat d'Etat à l'éducation et à la recherche (SER). (2011). *Évaluation de la réforme de la maturité 1995 (EVAMAR) Rapport final de la phase II* (p. 424). Berne. Retrieved from https://www.sbfi.admin.ch/dam/sbfi/fr/dokumente/evamar_evaluationdermaturitaetsreform1995phaseii.pdf.download.pdf/evaluation_de_lareformedelamaturite1995evamarrapportfinaldelapha.pdf
- Service d'examens des écoles internationales. (1967). *Le Baccalauréat International*. Genève: I.S.E.S.
- Silver, E. A., & Cai, J. (1996). An analysis of arithmetic problem posing by middle school students. *Journal for Research in Mathematics Education*, 27(5), 521–539.
- Travaux personnels encadrés. (n.d.). Retrieved May 24, 2017, from Éduscol website: <http://eduscol.education.fr/cid47789/definition-et-themes-nationaux-des-tpe.html#lien1>
- Vermersch, P. (1994). *L'entretien d'explicitation* (ESF). Paris.
- Winsløw, C., Matheron, Y., & Mercier, A. (2013). Study and research courses as an epistemological model for didactics. *Educational Studies in Mathematics*, 83(2), 267–284. <https://doi.org/10.1007/s10649-012-9453-3>
- Wozniak, F. (2015). La démarche d'investigation depuis la théorie anthropologique du didactique: Les parcours d'étude et de recherche. *Recherches En Éducation*, (21), 152–165.

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APPENDICES

Appendix 1: Essential features of classroom inquiry and their variations

Essential Feature	Variations			
1. Learner engages in scientifically oriented questions	Learner poses a question	Learner selects among questions, poses new questions	Learner sharpens or clarifies question provided by teacher, materials, or other source	Learner engages in question provided by teacher, materials, or other source
2. Learner gives priority to evidence in responding to questions	Learner determines what constitutes evidence and collects it	Learner directed to collect certain data	Learner given data and asked to analyze	Learner given data and told how to analyze
3. Learner formulate explanations from evidence	Learner formulates explanation after summarizing evidence	Learner guided in process of formulating explanations from evidence	Learner given possible ways to use evidence to formulate explanation	Learner provided with evidence and how to use evidence to formulate explanation
4. Learner connects explanations to scientific knowledge	Learner independently examines other resources and forms the links to explanations	Learner directed toward areas and sources of scientific knowledge	Learner given possible connections	
5. Learner communicates and justifies explanations	Learner forms reasonable and logical argument to communicate explanations	Learner coached in development of communication	Learner provided broad guidelines to use sharpen communication	Learner given steps and procedures for communication
More ————— Amount of Learner Self-Direction ————— Less Less ————— Amount of Direction from Teacher or Material ————— More				

(National Research Council, 2000, p. 29)

Appendix 2: The four-level inquiry model

Inquiry level	The information given to the student		
	Question?	Procedure?	Solution?
1 Confirmation inquiry <i>Students confirm a principle through an activity when the results are known in advance</i>	✓	✓	✓
2 Structured inquiry <i>Students investigate a teacher-presented question through a prescribed procedure</i>	✓	✓	
3 Guided inquiry <i>Students investigate a teacher-presented question using student designed/selected procedures</i>	✓		
4 Open inquiry <i>Students investigate questions that are student formulated through student designed/selected procedures</i>			

Appendix 3: Instruction Factors associated with inquiry-based instruction (extract from the EQUIP protocol)

Construct measured	Preinquiry (Level 1)	Developing Inquiry (Level 2)	Proficient Inquiry (Level 3)	Exemplary Inquiry (Level 4)
Instructional Strategies	Teacher predominantly lectured to cover content.	Teacher frequently lectured and/or used demonstrations to explain content. Activities were verification only .	Teacher occasionally lectured , but students were engaged in activities that helped develop conceptual understanding.	Teacher occasionally lectured, but students were engaged in investigations that promoted strong conceptual understanding .
Order of Instruction	Teacher explained concepts. Students either did not explore concepts or did so only after explanation.	Teacher asked students to explore concept before receiving explanation . Teacher explained.	Teacher asked students to explore before explanation . Teacher and students explained .	Teacher asked students to explore concept before explanation occurred. Though perhaps prompted by the teacher, students provided the explanation .
Teacher Role	Teacher was center of lesson; rarely acted as facilitator.	Teacher was center of lesson; occasionally acted as facilitator .	Teacher frequently acted as facilitator.	Teacher consistently and effectively acted as facilitator.
Student Role	Students were consistently passive as learners (taking notes, practicing on their own).	Students were active to a small extent as learners (highly engaged for very brief moments or to a small extent throughout lesson).	Students were active as learners (involved in discussions, investigations, or activities, but not consistently and clearly focused).	Students were consistently and effectively active as learners (highly engaged at multiple points during lesson and clearly focused on the task).
Knowledge Acquisition	Student learning focused solely on mastery of facts, information, and rote processes.	Student learning focused on mastery of facts and process skills without much focus on understanding of content.	Student learning required application of concepts and process skills in new situations.	Student learning required depth of understanding to be demonstrated relating to content and process skills.

Appendix 4: 1967 syllabus

1. Schéma Général de l'Examen

Les épreuves du Baccalauréat International porteront sur six disciplines. Trois d'entre elles – au choix du candidat et en fonction de sa future orientation – devront être présentées au Degré Supérieur* :

1. Langue A (langue d'enseignement) incluant une étude de Littérature Mondiale en traduction.
2. Etude de l'Homme. L'une des options suivantes: a. Philosophie, b. Histoire, c. Géographie, d. Economie. A ces différentes options s'ajoutera un cours commun sur la Théorie de la Connaissance.
3. Une science expérimentale. L'une des options suivantes: a. Physique, b. Chimie, c. Biologie.
4. Langue B (première langue étrangère).
5. Mathématiques,
6. L'une des options suivantes:
 - Langue C (seconde langue étrangère)
 - Langue classique I
 - Langue classique II
 - Etude de l'Homme, option II
 - Science expérimentale II
 - Science expérimentale III
 - Mathématiques appliquées
 - Géographie physique
 - Beaux-Arts (Arts plastiques ou Musique)

D'autres options sont actuellement à l'étude: Histoire et Géographie – Anthropologie – Sciences Physiques – Etudes scientifiques.

(Service d'examens des écoles internationales, 1967, p. 13)

Appendix 5: IB Learner Profile



IB learner profile

The aim of all IB programmes is to develop internationally minded people who, recognizing their common humanity and shared guardianship of the planet, help to create a better and more peaceful world.

As IB learners we strive to be:

INQUIRERS We nurture our curiosity, developing skills for inquiry and research. We know how to learn independently and with others. We learn with enthusiasm and sustain our love of learning throughout life. KNOWLEDGEABLE We develop and use conceptual understanding, exploring knowledge across a range of disciplines. We engage with issues and ideas that have local and global significance. THINKERS We use critical and creative thinking skills to analyse and take responsible action on complex problems. We exercise initiative in making reasoned, ethical decisions. COMMUNICATORS We express ourselves confidently and creatively in more than one language and in many ways. We collaborate effectively, listening carefully to the perspectives of other individuals and groups. PRINCIPLED We act with integrity and honesty, with a strong sense of fairness and justice, and with respect for the dignity and rights of people everywhere. We take responsibility for our actions and their consequences.	OPEN-MINDED We critically appreciate our own cultures and personal histories, as well as the values and traditions of others. We seek and evaluate a range of points of view, and we are willing to grow from the experience. CARING We show empathy, compassion and respect. We have a commitment to service, and we act to make a positive difference in the lives of others and in the world around us. RISK-TAKERS We approach uncertainty with forethought and determination; we work independently and cooperatively to explore new ideas and innovative strategies. We are resourceful and resilient in the face of challenges and change. BALANCED We understand the importance of balancing different aspects of our lives—intellectual, physical, and emotional—to achieve well-being for ourselves and others. We recognize our interdependence with other people and with the world in which we live. REFLECTIVE We thoughtfully consider the world and our own ideas and experience. We work to understand our strengths and weaknesses in order to support our learning and personal development.
--	---

The IB learner profile represents 10 attributes valued by IB World Schools. We believe these attributes, and others like them, can help individuals and groups become responsible members of local, national and global communities.

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(IBO, 2013b, p. 1)

Appendix 6: Learner profile attributes linked to mathematics aims

Link to mathematics aims	Learner profile attributes
Develop a curiosity and enjoyment of mathematics, and appreciate its elegance and power	Inquirer
Develop an understanding of the concepts, principles and nature of mathematics	Knowledgeable
Communicate mathematics clearly, concisely and confidently in a variety of contexts	Communicator
Develop logical and creative thinking, and patience and persistence in problem solving to instil confidence in using mathematics	Thinker, balanced
Employ and refine their powers of abstraction and generalization	Reflective
Take action to apply and transfer skills to alternative situations, to other areas of knowledge and to future developments in their local and global communities	Open-minded, risk-takers
Appreciate how developments in technology and mathematics influence each other	Knowledgeable, reflective
Appreciate the moral, social and ethical questions arising from the work of mathematicians and the applications of mathematics	Principled, caring
Appreciate the universality of mathematics and its multicultural, international and historical perspectives	Open-minded
Appreciate the contribution of mathematics to other disciplines, and as a particular “area of knowledge” in the TOK course	Knowledgeable, balanced
Develop the ability to reflect critically upon their own work and the work of others	Reflective, communicator
Independently and collaboratively extend their understanding of mathematics	Inquirer

Appendix 7: Examples of Portfolio tasks

Type 1: Mathematical investigation

MATRIX BINOMIALS

SL TYPE I

Let $X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Calculate $X^2, X^3, X^4; Y^2, Y^3, Y^4$

By considering integer powers of X and Y , find expressions for $X^n, Y^n, (X+Y)^n$.

Let $A = aX$ and $B = bY$, where a and b are constants.

Use different values of a and b to calculate $A^2, A^3, A^4; B^2, B^3, B^4$

By considering integer powers of A and B , find expressions for $A^n, B^n, (A+B)^n$.

Now consider $M = \begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}$.

Show that $M = A + B$, and that $M^2 = A^2 + B^2$.

Hence, find the general statement that expresses M^n in terms of aX and bY .

Test the validity of your general statement by using different values of a, b , and n .

Discuss the scope and/or limitations of your general statement.

Use an algebraic method to explain how you arrived at your general statement.

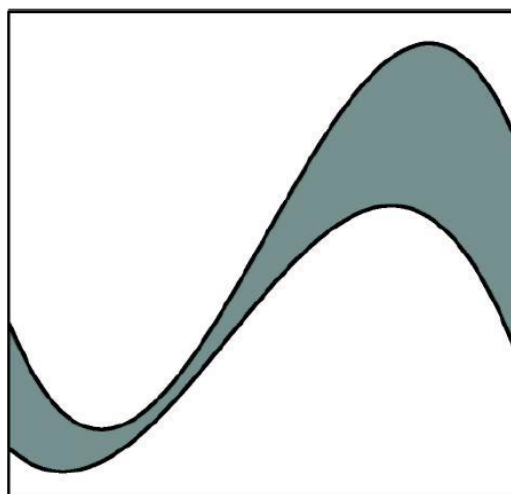
Type 2: Mathematical modeling

LOGAN'S LOGO

SL TYPE II

Note to teachers: The size of the square is not critical until it is measured. Variations may result when copies of the task are made. Students should measure the diagram as it is presented. It will be very helpful to moderators if you include a copy of the task with any work selected for the sample.

Logan has designed the logo below.



The diagram shows a square which is divided into three regions by two curves. The logo is the shaded region between the two curves. Logan wishes to develop mathematical functions that model these curves.

Using an appropriate set of axes, identify and record a number of data points on the curves which will allow you to develop model functions for them. Define all variables used and state any parameters clearly.

Using technology, plot these two sets of data points on a graph. What type of functions model the behaviour of the data? Explain why you chose these functions.

Find functions that represent the upper and lower curves forming the logo. Discuss any limitations.

Logan wishes to print T-shirts with the logo on the back. She must double the dimensions of the logo for this purpose. Describe how your functions must be modified.

Logan also wishes to print business cards. A standard business card is 9 cm by 5 cm. How must your functions be modified so that the logo extends from one end of the card to the other? Use technology to show the results.

What fraction of the area of the card does the logo occupy? Why might this be an important aspect of a business card?

Appendix 8: Prior learning topics

Topic	Content
Number	Routine use of addition, subtraction, multiplication and division, using integers, decimals and fractions, including order of operations. Simple positive exponents. Simplification of expressions involving roots (surds or radicals). Prime numbers and factors, including greatest common divisors and least common multiples. Simple applications of ratio, percentage and proportion, linked to similarity. Definition and elementary treatment of absolute value (modulus), a. Rounding, decimal approximations and significant figures, including appreciation of errors. Expression of numbers in standard form (scientific notation), that is, $a \times 10^k$, $1 \leq a < 10$, $k \in \mathbb{Z}$.
Sets and Numbers	Concept and notation of sets, elements, universal (reference) set, empty (null) set, complement, subset, equality of sets, disjoint sets. Operations on sets: union and intersection. Commutative, associative and distributive properties. Venn diagrams. Number systems: natural numbers; integers, $\mathbb{Z}$; rationals, $\mathbb{Q}$, and irrationals; real numbers, $\mathbb{R}$. Intervals on the real number line using set notation and using inequalities. Expressing the solution set of a linear inequality on the number line and in set notation. Mappings of the elements of one set to another. Illustration by means of sets of ordered pairs, tables, diagrams and graphs.
Algebra	Manipulation of simple algebraic expressions involving factorization and expansion, including quadratic expressions. Rearrangement, evaluation and combination of simple formulae. Examples from other subject areas, particularly the sciences, should be included. The linear function and its graph, gradient and y-intercept. Addition and subtraction of algebraic fractions. The properties of order relations: $<$, $\leq$, $>$, $\geq$. Solution of equations and inequalities in one variable, including cases with rational coefficients. Solution of simultaneous equations in two variables.
Trigonometry	Angle measurement in degrees. Compass directions and three figure bearings. Right-angle trigonometry. Simple applications for solving triangles. Pythagoras' theorem and its converse.

Topic	Content
Geometry	Simple geometric transformations: translation, reflection, rotation, enlargement. Congruence and similarity, including the concept of scale factor of an enlargement. The circle, its center and radius, area and circumference. The terms “arc”, “sector”, “chord”, “tangent” and “segment”. Perimeter and area of plane figures. Properties of triangles and quadrilaterals, including parallelograms, rhombuses, rectangles, squares, kites and trapeziums (trapezoids); compound shapes. Volumes of prisms, pyramids, spheres, cylinders and cones.
Coordinate geometry	Elementary geometry of the plane, including the concepts of dimension for point, line, plane and space. The equation of a line in the form: $y=mx+c$. Parallel and perpendicular lines, including $m_1 = m_2$ and $m_1m_2 = -1$. Geometry of simple plane figures. The Cartesian plane: ordered pairs (x, y), origin, axes. Mid-point of a line segment and distance between two points in the Cartesian plane and in three dimensions.
Statistics and probability	Descriptive statistics: collection of raw data; display of data in pictorial and diagrammatic forms, including pie charts, pictograms, stem and leaf diagrams, bar graphs and line graphs. Obtaining simple statistics from discrete and continuous data, including mean, median, mode, quartiles, range, interquartile range. Calculating probabilities of simple events.

(IBO, 2012, pp. 15–16)

Appendix 9: Investigation of graphs of exponential functions

INVESTIGATION 1

GRAPHS OF EXPONENTIAL FUNCTIONS

In this investigation we examine the graphs of various families of exponential functions.

Click on the icon to run the **dynamic graphing package**, or else you could use your **graphics calculator**.



What to do:

- 1 Explore the family of curves of the form $y = b^x$ where $b > 0$.
For example, consider $y = 2^x$, $y = 3^x$, $y = 10^x$, and $y = (1.3)^x$.
 - a What effect does changing b have on the shape of the graph?
 - b What is the y -intercept of each graph?
 - c What is the horizontal asymptote of each graph?
- 2 Explore the family of curves of the form $y = 2^x + d$ where d is a constant.
For example, consider $y = 2^x$, $y = 2^x + 1$, and $y = 2^x - 2$.
 - a What effect does changing d have on the position of the graph?
 - b What effect does changing d have on the shape of the graph?
 - c What is the horizontal asymptote of each graph?
 - d What is the horizontal asymptote of $y = 2^x + d$?
 - e To graph $y = 2^x + d$ from $y = 2^x$ what transformation is used?
- 3 Explore the family of curves of the form $y = 2^{x-c}$.
For example, consider $y = 2^x$, $y = 2^{x-1}$, $y = 2^{x+2}$, and $y = 2^{x-3}$.
 - a What effect does changing c have on the position of the graph?
 - b What effect does changing c have on the shape of the graph?
 - c What is the horizontal asymptote of each graph?
 - d To graph $y = 2^{x-c}$ from $y = 2^x$ what transformation is used?
- 4 Explore the relationship between $y = b^x$ and $y = b^{-x}$ where $b > 0$.
For example, consider $y = 2^x$ and $y = 2^{-x}$.
 - a What is the y -intercept of each graph?
 - b What is the horizontal asymptote of each graph?
 - c What transformation moves $y = 2^x$ to $y = 2^{-x}$?
- 5 Explore the family of curves of the form $y = a \times 2^x$ where a is a constant.
 - a Consider functions where $a > 0$, such as $y = 2^x$, $y = 3 \times 2^x$, and $y = \frac{1}{2} \times 2^x$.
Comment on the effect on the graph.
 - b Consider functions where $a < 0$, such as $y = -2^x$, $y = -3 \times 2^x$, and $y = -\frac{1}{2} \times 2^x$.
Comment on the effect on the graph.
 - c What is the horizontal asymptote of each graph? Explain your answer.

Appendix 10: Topics of the Mathematics SL syllabus covered before the ME

Topic 2—Functions and equations

	Content	Further guidance
2.1	Concept of function $f : x \mapsto f(x)$. Domain, range; image (value). Composite functions. Identity function. Inverse function f^{-1}. Not required: domain restriction.	<i>Example:</i> for $x \mapsto \sqrt{2-x}$, domain is $x \leq 2$, range is $y \geq 0$. A graph is helpful in visualizing the range. $(f \circ g)(x) = f(g(x))$. $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$. On examination papers, students will only be asked to find the inverse of a <i>one-to-one</i> function.
2.2	The graph of a function; its equation $y = f(x)$. Function graphing skills. Investigation of key features of graphs, such as maximum and minimum values, intercepts, horizontal and vertical asymptotes, symmetry, and consideration of domain and range. Use of technology to graph a variety of functions, including ones not specifically mentioned. The graph of $y = f^{-1}(x)$ as the reflection in the line $y = x$ of the graph of $y = f(x)$.	Note the difference in the command terms “draw” and “sketch”. An analytic approach is also expected for simple functions, including all those listed under topic 2. Link to 6.3, local maximum and minimum points.
2.3	Transformations of graphs. Translations: $y = f(x) + b$; $y = f(x - a)$. Reflections (in both axes): $y = -f(x)$; $y = f(-x)$. Vertical stretch with scale factor p: $y = pf(x)$. Stretch in the x-direction with scale factor $\frac{1}{q}$: $y = f(qx)$. Composite transformations.	Technology should be used to investigate these transformations. Translation by the vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ denotes horizontal shift of 3 units to the right, and vertical shift of 2 down. <i>Example:</i> $y = x^2$ used to obtain $y = 3x^2 + 2$ by a stretch of scale factor 3 in the y-direction followed by a translation of $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

2.4	The quadratic function $x \mapsto ax^2 + bx + c$: its graph, y-intercept $(0, c)$. Axis of symmetry. The form $x \mapsto a(x - p)(x - q)$, x-intercepts $(p, 0)$ and $(q, 0)$. The form $x \mapsto a(x - h)^2 + k$, vertex (h, k).	Candidates are expected to be able to change from one form to another. Links to 2.3, transformations; 2.7, quadratic equations.
	Content	Further guidance
2.5	The reciprocal function $x \mapsto \frac{1}{x}$, $x \neq 0$: its graph and self-inverse nature. The rational function $x \mapsto \frac{ax + b}{cx + d}$ and its graph. Vertical and horizontal asymptotes.	Examples: $h(x) = \frac{4}{3x - 2}$, $x \neq \frac{2}{3}$; $y = \frac{x + 7}{2x - 5}$, $x \neq \frac{5}{2}$. Diagrams should include all asymptotes and intercepts.
2.6	Exponential functions and their graphs: $x \mapsto a^x$, $a > 0$, $x \mapsto e^x$. Logarithmic functions and their graphs: $x \mapsto \log_a x$, $x > 0$, $x \mapsto \ln x$, $x > 0$. Relationships between these functions: $a^x = e^{x \ln a}$; $\log_a a^x = x$; $a^{\log_a x} = x$, $x > 0$.	Links to 1.1, geometric sequences; 1.2, laws of exponents and logarithms; 2.1, inverse functions; 2.2, graphs of inverses; and 6.1, limits.
2.7	Solving equations, both graphically and analytically. Use of technology to solve a variety of equations, including those where there is no appropriate analytic approach. Solving $ax^2 + bx + c = 0$, $a \neq 0$. The quadratic formula. The discriminant $\Delta = b^2 - 4ac$ and the nature of the roots, that is, two distinct real roots, two equal real roots, no real roots. Solving exponential equations.	Solutions may be referred to as roots of equations or zeros of functions. Links to 2.2, function graphing skills; and 2.3–2.6, equations involving specific functions. Examples: $e^x = \sin x$, $x^4 + 5x - 6 = 0$. Example: Find k given that the equation $3kx^2 + 2x + k = 0$ has two equal real roots. Examples: $2^{x-1} = 10$, $\left(\frac{1}{3}\right)^x = 9^{x+1}$. Link to 1.2, exponents and logarithms.
2.8	Applications of graphing skills and solving equations that relate to real-life situations.	Link to 1.1, geometric series.

Topic 3—Circular functions and trigonometry

	Content	Further guidance
3.4	The circular functions $\sin x$, $\cos x$ and $\tan x$: their domains and ranges; amplitude, their periodic nature; and their graphs. Composite functions of the form $f(x) = a \sin(b(x+c)) + d$. Transformations. Applications.	<i>Examples:</i> $f(x) = \tan\left(x - \frac{\pi}{4}\right), f(x) = 2\cos(3(x-4)) + 1.$ <i>Example:</i> $y = \sin x$ used to obtain $y = 3\sin 2x$ by a stretch of scale factor 3 in the y-direction and a stretch of scale factor $\frac{1}{2}$ in the x-direction. Link to 2.3, transformation of graphs. Examples include height of tide, motion of a Ferris wheel.

Topic 5—Statistics and probability

	Content	Further guidance
5.1	Concepts of population, sample, random sample, discrete and continuous data. Presentation of data: frequency distributions (tables); frequency histograms with equal class intervals; box-and-whisker plots; outliers. Grouped data: use of mid-interval values for calculations; interval width; upper and lower interval boundaries; modal class. Not required: frequency density histograms.	Continuous and discrete data. Outlier is defined as more than $1.5 \times \text{IQR}$ from the nearest quartile. Technology may be used to produce histograms and box-and-whisker plots.

	Content	Further guidance
5.2	Statistical measures and their interpretations. Central tendency: mean, median, mode. Quartiles, percentiles. Dispersion: range, interquartile range, variance, standard deviation. Effect of constant changes to the original data. Applications.	On examination papers, data will be treated as the population. Calculation of mean using formula and technology. Students should use mid-interval values to estimate the mean of grouped data. Calculation of standard deviation/variance using only technology. Link to 2.3, transformations. <i>Examples:</i> If 5 is subtracted from all the data items, then the mean is decreased by 5, but the standard deviation is unchanged. If all the data items are doubled, the median is doubled, but the variance is increased by a factor of 4.
5.3	Cumulative frequency; cumulative frequency graphs; use to find median, quartiles, percentiles.	Values of the median and quartiles produced by technology may be different from those obtained from a cumulative frequency graph.

Topic 6—Calculus

	Content	Further guidance
6.1	Informal ideas of limit and convergence. Limit notation. Definition of derivative from first principles as $f'(x) = \lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x)}{h} \right)$.	<i>Example:</i> 0.3, 0.33, 0.333, ... converges to $\frac{1}{3}$. Technology should be used to explore ideas of limits, numerically and graphically. <i>Example:</i> $\lim_{x \rightarrow \infty} \left(\frac{2x+3}{x-1} \right)$ Links to 1.1, infinite geometric series; 2.5–2.7, rational and exponential functions, and asymptotes. Use of this definition for derivatives of simple polynomial functions only. Technology could be used to illustrate other derivatives. Link to 1.3, binomial theorem. Use of both forms of notation, $\frac{dy}{dx}$ and $f'(x)$, for the first derivative.

	Content	Further guidance
	Derivative interpreted as gradient function and as rate of change. Tangents and normals, and their equations. Not required: analytic methods of calculating limits.	Identifying intervals on which functions are increasing or decreasing. Use of both analytic approaches and technology. Technology can be used to explore graphs and their derivatives.
6.2	Derivative of x^n ($n \in \mathbb{Q}$), $\sin x$, $\cos x$, $\tan x$, e^x and $\ln x$. Differentiation of a sum and a real multiple of these functions. The chain rule for composite functions. The product and quotient rules. The second derivative. Extension to higher derivatives.	Link to 2.1, composition of functions. Technology may be used to investigate the chain rule. Use of both forms of notation, $\frac{d^2y}{dx^2}$ and $f''(x)$. $\frac{d^n y}{dx^n}$ and $f^{(n)}(x)$.
6.3	Local maximum and minimum points. Testing for maximum or minimum. Points of inflexion with zero and non-zero gradients. Graphical behaviour of functions, including the relationship between the graphs of f, f' and f''. Optimization. Applications. Not required: points of inflexion where $f''(x)$ is not defined: for example, $y = x^{1/3}$ at $(0,0)$.	Using change of sign of the first derivative and using sign of the second derivative. Use of the terms “concave-up” for $f''(x) > 0$, and “concave-down” for $f''(x) < 0$. At a point of inflexion, $f''(x) = 0$ and changes sign (concavity change). $f''(x) = 0$ is not a sufficient condition for a point of inflexion: for example, $y = x^4$ at $(0,0)$. Both “global” (for large x) and “local” behaviour. Technology can display the graph of a derivative without explicitly finding an expression for the derivative. Use of the first or second derivative test to justify maximum and/or minimum values. Examples include profit, area, volume. Link to 2.2, graphing functions.

(IBO, 2012, pp. 19–35)

Appendix 11: List of the elements from the Herbartian scheme

	Action	Media	Material	Elements in M _I	
Q question	GS google search	V video	SW-DGr graphing software (Desmos)	A [•] institutional answer	A [•] produced answer
	GSI image search	Doc document	SW-GD google docs software	Img works-image	
	YTS YouTube search	WS website	SW-ON OneNote software	Fgr works-figure	
		R google result	C calculator	L works-link	
		iR image result	SW-LR LightRoom software	DGr works-graphics	
		PPT power point	SW-GG GeoGebra software	W various works	
		TB textbook		GD works-google doc	
		T teacher		DT various data	
		Exp expert			
		TUT tutor			
		Std Student's CU			

Appendix 12: List of possible exploration topics

Your topic must be approved by your teacher by the given deadline

200 Exploration Ideas/topics

Algebra & Number Theory

Modular arithmetic	Euler's identity: $e^{i\pi} + 1 = 0$	Goldbach's conjecture
Chinese remainder theorem	Probabilistic number theory	Fermat's last theorem
Applications of complex numbers	Natural logarithms + complex numbers	Continued fractions
Diophantine equations	Twin primes problem	Hypercomplex numbers
General solution of a cubic equation	Diophantine application: Cole numbers	Applications of logarithms
Odd perfect numbers	Polar equations	Euclidean algorithm for GCF
Patterns in Pascal's triangle	Palindrome numbers	Finding prime numbers
Factorable integers of the form $ak + b$	Random numbers	Algebraic congruences
Pythagorean triples	Inequalities & Fibonacci numbers	Mersenne primes
Combinatorics – art of counting	Magic squares & cubes	Boolean algebra
Loci and complex numbers	Roots of unity	Matrices & Cramer's rule
Divisibility tests	Complex numbers & transformations	Egyptian fractions
Graphical representation of roots of complex numbers		

Calculus/Analysis & Functions

Mean Value theorem	Torricelli's trumpet (Gabriel's horn)	Integrating to infinity
Applications of power series	Newton's law of cooling	Hyperbolic functions
Fundamental theorem of calculus	Brachistochrone (min.time) problem	The harmonic series
Second order differential equations	L'Hopital's rule and evaluating limits	Torus – solid of revolution

Probability & Probability Distributions

Normal distribution and natural phenomena	The Monty Hall problem
Monte Carlo simulations	Random walks
Insurance and calculating risks	Poisson distribution and queues
Determination of π by probability	Lotteries

Statistics & Modelling

Logistic function & constrained growth	Modelling growth of tumours	Traffic flow
Modelling epidemics/spread of a virus	Correlation coefficients	Hypothesis testing
Modelling the shape of a bird's egg	Central limit theorem	Modelling radioactive decay
Modelling growth of computer power	Least squares regression	Regression to the mean
Modelling change in record performances for a sport		

Numerical Analysis

Methods for solving differential eqns	Linear programming	Fixed point iteration
Methods of approximating π	Applications of iteration	Newton's method
Estimating size of large crowds	Generating the number e	Descartes' rule of signs

Logic & Sets

Codes and ciphers	Set theory and different 'size' infinities	
Mathematical induction (strong)	Proof by contradiction	Proving a number is irrational

Appendix 13: Extract from the Chronology of the Exploration

Chronolo	E-unit	Description
245	Q 29	How am I going to validate the quadratic model?
246	Std 29.1	Student's reflection
247	A♥ 29.1	logbook: I will assume it to be a parabola, find the best fit curve and then find the half of it and transform it.
248	A♥ 29.2	logbook: If it fits in perfectly, then it is a parabola.
249	Q 30	Does the suspension cable of the san francisco golden gate bridge form a parabola?
250	GS 30.1	Does the suspension cable of the san francisco golden gate bridge form a parabola
251	WS 15.1	Math-central: online help type of website
252	R 30.1	Excerpt from WS 15.1
253	A♦ 30.1	The suspension is a catenary
254	R 30.2	Are suspension bridges parabolas?
255	A♦ 30.2	suspension bridges are rather parabolas because forces of compression and tension are acting on it
256	GD 1.1	Math sites
257	L 30.1	R 31.1 is copied to GD 1.1 Math sites
258	Q 31	What is a catenary curve?
259	GS 31.1	Catenary curve
260	R 31.1	The catenary is a plane curve whose shape corresponds to a hanging homogeneous flexible chain supported at its ends and sagging under the force of gravity.
261	A♦ 31.1	The catenary is a plane curve whose shape corresponds to a hanging homogeneous flexible chain supported at its ends and sagging under the force of gravity.
262	WS 31.1	Math24-equation of catenary – too complicated
263	A♦ 31.2	a physics model of a catenary—inaccessible
264	GD 1.1	Math sites
265	L 31.1	The link of WS 31.1 is copied to GD 1.1
266	WS 28.1	How to calculate half parabolic curve?
267	A♦ 31.3	Gives a step-by-step technique to graph half a parabola by only taking the x-values less (greater then) the x-coordinate of the vertex

As far as the color code is concerned, blue stands for relevant, orange for irrelevant, green for correct and red for incorrect. A strikethrough text means that the given E-unit was either non-activated, potential or rejected from the milieu.

Appendix 14: List of exploration topics in the observed classes

David's class	Eliot's class
Functions	Modeling the decrease in the Sea ice extent of the arctic sea
Exponential curves	Fibonacci sequence
Sinusoidal curves—modeling	Bees vs. pesticide use—statistics
Trig—graphing musical intervals	Modeling tumour growth
Functions and modeling	Calculus
Modeling the spread of infectious diseases	Monty Hall problem
Theoretical investigation into the distribution of forces in a roof truss	Statistical analyses of Wellbeing survey
The effect of temperature on the rate of oxygen production	Traffic flow
The effect of new/full moons on the height of tides	Modeling the spread of diseases
Calculate the sun path using sundial	Bank mortgages and profit
Modeling of the Golden Gate bridge	Correlation and linear modeling in climate change
Investigating how the phase of the moon affects the tides	How does a GPS work?
Trigonometric functions modeling—predator-prey relationship	How probability affects ...?
Graphing the sine alignment of historical sites around the world	Probability and potential outcomes
Application of interval estimation in statistics	Radiocarbon dating—modeling the decay of c-14
Modeling a logo: Giogio Arman	The golden ratio
Modeling the subway logo	SIR model for epidemics

Appendix 15: Processing of correct information—Theme vs. Effect/Subtype of MMUs

MMU_info_grouped	correct information ☐					
Correct information MMUs	Theme ☐					
Effect on ME/subType MMUs ☐	1: methodology	2: quadratic model	3: choice of model	4: catenary model	5: writing report	Grand Total
☐ desirable	17	44	5	3	33	102
☐ 2_found later	1	8	0	0	3	12
missed occasion	1	8	0	0	3	12
☐ 3_different solution found	1	9	1	0	7	18
correct admission	0	2	0	0	1	3
correct production	1	4	0	0	2	7
missed occasion	0	3	1	0	4	8
☐ 4_advancing inquiry	8	11	4	1	4	28
correct admission	5	5	2	1	1	14
correct production	3	4	2	0	3	12
correct reflection	0	2	0	0	0	2
☐ 5_contributing to A♥	7	16	0	2	19	44
correct admission	5	4	0	2	10	21
correct production	2	10	0	0	9	21
correct reflection	0	1	0	0	0	1
relevant source	0	1	0	0	0	1
☐ undesirable	2	3	7	2	0	14
☐ 6_unexplored path	2	1	1	2	0	6
correct admission	0	0	0	1	0	1
missed occasion	2	1	1	1	0	5
☐ 8_negative effect on A♥	0	2	6	0	0	8
correct admission	0	0	2	0	0	2
missed occasion	0	2	4	0	0	6
☐ undetermined	2	3	0	1	0	6
☐ 9_no visible effect	2	3	0	1	0	6
relevant source	2	3	0	1	0	6
Grand Total	21	50	12	6	33	122

Appendix 16: Processing of incorrect information—Theme vs. Effect/Subtype of MMUs

MMU_info_grouped	incorrect information ☐					
Irrelevant MMUs	Theme ☐					
Effect on ME/subType of MMU ☐	2: quadratic model	3: choice of model	4: catenary model	5: writing report	Grand Total	
☐ 1_continuing search	0	0	0	1	1	
efficient elimination	0	0	0	1	1	
☐ 2_found later	1	0	0	0	1	
incorrect production	1	0	0	0	1	
☐ 3_different solution found	3	0	0	0	3	
incorrect production	3	0	0	0	3	
☐ 4_advancing inquiry	1	0	0	0	1	
efficient elimination	1	0	0	0	1	
☐ 8_negative effect on A♥	3	9	2	0	14	
incorrect admission	0	6	0	0	6	
incorrect production	3	2	2	0	7	
incorrect source	0	1	0	0	1	
Grand Total	8	9	2	1	20	

Appendix 17: Processing of irrelevant information—Theme vs. Effect/Subtype of MMUs

Irrelevant MMUs	Theme 				Grand
Effect on ME/subType of MMU 	1: methodology	2: quadratic model	3: choice of model	5: writing report	Total
 1_continuing search	4	12	3	2	21
efficient elimination	2	8	1	1	12
efficient media judgment	2	4	2	1	9
 3_different solution found	0	0	0	4	4
efficient elimination	0	0	0	4	4
 7_following an irrelevant path	1	7	2	0	10
irrelevant admission	0	2	0	0	2
irrelevant reflection	1	2	2	0	5
irrelevant source	0	3	0	0	3
 8_negative effect on A♥	0	1	0	0	1
irrelevant production	0	1	0	0	1
 9_no visible effect	1	1	0	0	2
irrelevant production	0	1	0	0	1
irrelevant source	1	0	0	0	1
Grand Total	6	21	5	6	38

Appendix 18: Effect of the activated milieu on the ME by subtype

# of MMUs	Effect on the ME ↕			
	+ desirable	+ undesirable	+ undetermined	Grand Total
Activated Milieu ▼				
☐ sufficient activated M	109	3	6	118
correct admission	38	3	0	41
correct production	40	0	0	40
correct reflection	3	0	0	3
efficient elimination	18	0	0	18
efficient media judgment	9	0	0	9
relevant source	1	0	6	7
☐ insufficient activated M	24	36	2	62
incorrect admission	0	6	0	6
incorrect production	4	7	0	11
incorrect source	0	1	0	1
irrelevant admission	0	2	0	2
irrelevant production	0	1	1	2
irrelevant reflection	0	5	0	5
irrelevant source	0	3	1	4
missed occasion	20	11	0	31
Grand Total	133	39	8	180

Appendix 19: Effect of the activated milieu on the ME by theme

# of MMUs	Effect on the ME ↕			
	+ desirable	+ undesirable	+ undetermined	Grand Total
Activated Milieu ▼				
☐ sufficient activated M	109	3	6	118
1: methodology	20	0	2	22
2: quadratic model	46	0	3	49
3: choice of model	7	2	0	9
4: catenary model	3	1	1	5
5: writing report	33	0	0	33
☐ insufficient activated M	24	36	2	62
1: methodology	1	3	1	5
2: quadratic model	15	14	1	30
3: choice of model	1	16	0	17
4: catenary model	0	3	0	3
5: writing report	7	0	0	7
Grand Total	133	39	8	180

Appendix 20: Desirable effect of the activated milieu on the ME by theme and by subtype of MMU

MMU_effect_grouped	desirable					
# of MMUs	Effect on the ME ₋₁					
Activated Milieu	1_continuing search	2_found later	3_different solution found	4_advancing inquiry	5_contributing to A♥	Grand Total
sufficient activated M	22	0	14	29	44	109
1: methodology	4	0	1	8	7	20
correct admission	0	0	0	5	5	10
correct production	0	0	1	3	2	6
efficient elimination	2	0	0	0	0	2
efficient media judgment	2	0	0	0	0	2
2: quadratic model	12	0	6	12	16	46
correct admission	0	0	2	5	4	11
correct production	0	0	4	4	10	18
correct reflection	0	0	0	2	1	3
efficient elimination	8	0	0	1	0	9
efficient media judgment	4	0	0	0	0	4
relevant source	0	0	0	0	1	1
3: choice of model	3	0	0	4	0	7
correct admission	0	0	0	2	0	2
correct production	0	0	0	2	0	2
efficient elimination	1	0	0	0	0	1
efficient media judgment	2	0	0	0	0	2
4: catenary model	0	0	0	1	2	3
correct admission	0	0	0	1	2	3
5: writing report	3	0	7	4	19	33
correct admission	0	0	1	1	10	12
correct production	0	0	2	3	9	14
efficient elimination	2	0	4	0	0	6
efficient media judgment	1	0	0	0	0	1
insufficient activated M	0	13	11	0	0	24
1: methodology	0	1	0	0	0	1
missed occasion	0	1	0	0	0	1
2: quadratic model	0	9	6	0	0	15
incorrect production	0	1	3	0	0	4
missed occasion	0	8	3	0	0	11
3: choice of model	0	0	1	0	0	1
missed occasion	0	0	1	0	0	1
5: writing report	0	3	4	0	0	7
missed occasion	0	3	4	0	0	7
Grand Total	22	13	25	29	44	133

Appendix 21: Desirable effect of the activated milieu on the ME by theme and by type of episode

MMU_effect_grouped	desirable					
# of MMUs	Effect on the ME					
Activated Milieu	1_continuing search	2_found later	3_different solution found	4_advancing inquiry	5_contributing to A♥	Grand Total
sufficient activated M	22	0	14	29	44	109
1: methodology	4	0	1	8	7	20
analysis	0	0	1	2	0	3
exploration	2	0	0	1	0	3
implementation	2	0	0	1	4	7
planning	0	0	0	1	0	1
regulation	0	0	0	1	2	3
verification	0	0	0	2	1	3
2: quadratic model	12	0	6	12	16	46
analysis	0	0	0	4	1	5
exploration	5	0	6	1	1	13
implementation	2	0	0	2	6	10
planning	0	0	0	0	2	2
regulation	0	0	0	2	3	5
study	3	0	0	1	3	7
verification	2	0	0	2	0	4
3: choice of model	3	0	0	4	0	7
exploration	2	0	0	1	0	3
regulation	1	0	0	3	0	4
4: catenary model	0	0	0	1	2	3
implementation	0	0	0	0	2	2
regulation	0	0	0	1	0	1
5: writing report	3	0	7	4	19	33
exploration	1	0	6	0	1	8
implementation	2	0	1	1	8	12
planning	0	0	0	2	1	3
regulation	0	0	0	1	3	4
study	0	0	0	0	6	6
insufficient activated M	0	13	11	0	0	24
1: methodology	0	1	0	0	0	1
exploration	0	1	0	0	0	1
2: quadratic model	0	9	6	0	0	15
exploration	0	0	1	0	0	1
implementation	0	7	3	0	0	10
regulation	0	0	1	0	0	1
study	0	1	1	0	0	2
verification	0	1	0	0	0	1
3: choice of model	0	0	1	0	0	1
verification	0	0	1	0	0	1
5: writing report	0	3	4	0	0	7
exploration	0	1	4	0	0	5
implementation	0	2	0	0	0	2
Grand Total	22	13	25	29	44	133

Appendix 22: Sophie's ME—GD1.1 A rubric with various criteria to plan the exploration

- A. Areas of interest
- B. Topic
- C. Reason for choice
- D. What research you will need to do or what data you will need to collect?
- E. Mathematical processes to be used in the analyses
 - NON SI math SL math
- F. Statement of task
- G. Overall plan, include timeline
- H. Bibliography