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## From fractal groups to fractal sets

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# From Fractal Groups to Fractal Sets

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# 1 Introduction

The idea of self-similarity is one of the most fundamental in the modern mathematics. The notion of “renormalization group”, which plays an essential role in quantum field theory, statistical physics and dynamical systems, is related to it. The notions of fractal and multi-fractal, playing an important role in singular geometry, measure theory and holomorphic dynamics, are also related. Self-similarity also appears in the theory of  $C^*$ -algebras (for example in the representation theory of the Cuntz algebras) and in many other branches of mathematics. Starting from 1980 the idea of self-similarity entered algebra and began to exert great influence on asymptotic and geometric group theory.

The aim of this paper is to present a survey of ideas, notions and results that are connected to self-similarity of groups, semigroups and their actions; and to relate them to the above-mentioned classical objects. Besides that, our aim is to exhibit new connections of groups and semigroups with fractals objects, in particular with Julia sets.

Let us review shortly some historical aspects of our research and list its main subjects.

## 1.1 Burnside groups

The second named author has constructed in 1980 [Gri80] two Burnside groups which (especially the first one) played a decisive role in the development of the idea of a self-similar group. Originally the Grigorchuk group (let us denote it by  $\mathbf{G}$ ) was defined as a transformation group of the segment  $[0, 1]$  without dyadically rational points. The generators were defined in a simple way as permutations of subintervals. One of the main properties of this action is the fact that if we restrict to an arbitrary dyadic subinterval  $I = [(k-1)/2^n, k/2^n]$  the action of its stabilizer  $\mathbf{G}_I = \text{st}_{\mathbf{G}}(I)$ , then the restriction will coincide with the action of the group  $\mathbf{G}$  on  $[0, 1]$  (after the natural identification of  $I$  with the whole interval  $[0, 1]$ ).

Another fundamental property is the fact that the action of the group  $\mathbf{G}$  is *contracting*, i.e., that the canonical homomorphism  $\phi_I : \mathbf{G}_I \rightarrow \mathbf{G}$  contracts the length of the group elements by a constant  $\lambda > 1$ . Finally, the third fundamental property is the *branching* nature of the action. This means that, up to finite-index inclusions, the stabilizers of the partition into the dyadic subintervals of the  $n$ th level are direct products of  $2^n$  isomorphic groups and that the lattice of subnormal groups has a tree-like structure.

Between 1983 and 1985 the second named author [Gri85b, Gri85a], and independently N. Gupta and S. Sidki [GS84], constructed other examples of similar groups and established other their important properties. It became clear that these examples are related to some big classes of groups. Namely, they belong to the class of *finitely automatic groups* (this was noted for the first time by Ju. I. Merzlyakov [Mer83]), and to the class of *branch groups*. The first class was defined in the early 1960's [Hoř63], while the definition of the class of branch groups was given in 1997 by R. Grigorchuk in his talk at the St-Andrews conference in Bath (see [Gri00]). The methods used in [GS83a, Gri80] were new to the theory of automata groups, and heralded the application of these groups to many new problems in group theory — see for instance [BG00a].

One of the distinctive features of the branch groups is their actions on spherically homogeneous rooted trees. Such trees appear naturally in the study of unimodal transformations [BORT96], in particular in problems of renormalization and in the study of dynamical systems in a neighborhood of Lyapunov-stable attractors [BS95]. A regular rooted tree is an example of a geometric object most closely related with the notion of self-similarity (every rooted subtree is naturally isomorphic to the original tree). The actions of the groups from [Gri80, Gri85a, Gri85c, GS83a, GS83b, BSV99] on the respective regular trees have self-similarity properties similar to the self-similarity property of the action of the group  $\mathbf{G}$  on the interval  $[0, 1]$ . These examples can be formalized in the general notion of a self-similar set and a self-similar action.

In Section 3 we discuss in detail the notion of a self-similar set and then define the notion of a self-similar action of a group and of an inverse semigroup. (The fact that self-similar inverse semigroups appear naturally in connection with self-similar sets, in particular, with Penrose tilings, was noted by V. Nekrashevych [Nekd].) One of the main sources of examples of self-similar group actions are the actions generated by finite automata. Groups of finite automata and their actions are discussed in Section 4.

## 1.2 Growth

The second named author noted in 1983 that the growth of the group  $\mathbf{G}$ , (which was initially introduced as a Burnside group, i.e., an infinite finitely generated torsion group), is intermediate between polynomial and exponential (and thus provides an answer to Milnor's question [Mil68b]). This observation was developed in different directions [Gri85a]. Besides, it was recently proved that many such groups, and in particular the Gupta-Sidki groups, also have intermediate growth [Bar02]. *In fine*, intermediate growth of these groups follows from strong contraction properties and branch structure. It became clear that the groups of such a type can be related with problems of fractal geometry, in particular with the problems of computation of the Hausdorff dimension. Later this conjecture was confirmed in different ways.

Most problems of computation of Hausdorff dimension (and dimensions of other type) reduce to the problem of finding the degree of polynomial growth or the base of exponential growth of

some formal language over a finite alphabet. Section 8 is devoted to questions of growth of formal languages, groups, semigroups, graphs and finite automata. In the same section the notion of amenability, which also plays a role in the theory of fractals, is considered.

Besides playing an important role in the study of growth of finitely generated groups, self-similar groups also appear in the study of the Hausdorff dimension of profinite groups. For instance, the profinite completion of the group  $\mathbf{G}$  has Hausdorff dimension  $5/8$  [GHZ00, Gri00].

Another relation of self-similar groups to fractals was found accidentally while studying the spectra of the non-commutative dynamical systems generated by the actions of self-similar groups (like the group  $\mathbf{G}$  mentioned above, or the Gupta-Sidki group) [BG00b, BG01]. First, it turned out that one has to use multi-dimensional rational mappings for the solution of the spectral problem and to study their invariant subsets (similar to the Julia sets). Secondly, the spectra turn out to be the Julia sets of polynomial mappings of the interval. Finally, amenability of the objects of polynomial growth (in particular groups and graphs) imply coincidence of the spectra of the above-mentioned dynamical systems with the spectra of the discrete Laplace operator on the Schreier graphs of the self-similar groups (where the Schreier graphs are defined with respect to the stabilizer of an end of the tree).

### 1.3 Schreier graphs

The Schreier graphs themselves are also interesting objects of investigation. They have polynomial growth in the case when the group action is contracting (though often the degree of the growth is non-integral). It was discovered that the Schreier graphs (defined with respect to the stabilizers of different points of the tree boundary) behave similarly to quasi-periodic tilings (like the Penrose tilings). Often there exist uncountably many non-isomorphic Schreier graphs in the bundle, while they all are locally isomorphic. Many other analogies can be found, in particular, the inflation and adjacency rules of the Penrose tilings have their counterparts in the Schreier graphs of self-similar groups. The finite Schreier graphs (defined with respect to the stabilizers of the tree vertices) of self-similar actions are often substitution graphs (here again an analogy with substitution dynamical systems and L-systems appears).

Their limit spaces (for example, in the sense of M. Gromov) often have a fractal nature. Moreover, recently V. Nekrashevych has introduced a notion of an iterated monodromy group (i.m.g.) of a branched covering and proved that the i.m.g. of a postcritically finite rational mapping is contracting (and thus is generated by a finite automaton) with the limit space homeomorphic to the Julia set of the mapping. Probably in the future the iterated monodromy groups and their Schreier graphs will play an important role in the holomorphic dynamics and the methods of the asymptotic and geometric group theory will be actively used in this part of mathematics. The basic definitions, facts and some examples of iterated monodromy groups are described in Section 5.

Schreier graphs of self-similar groups have very interesting spectral properties, as was discovered in [BG00b] and [GŻuk01]. For instance, in [BG00b] the first examples of regular graphs with a Cantor spectrum are constructed, while in [GŻuk01] the first example of a group with discrete spectral measure is given (which solves a question of Atiyah, see [GLSZ00]).

Some new examples of computations of the spectra and an example from [BG00b] concerning the Gupta-Fabrikovsky group is considered in Section 12.

As we have noted above, the Schreier graphs of contracting self-similar groups converge to nice fractal topological spaces. This was formalized by V. Nekrashevych [Nekc] in the notion of the limit space of a self-similar action. The limit space has a rich self-similarity structure with which self-replicating tiling systems and limit solenoids are related. The limit solenoid can be defined also for self-similar inverse semigroups. The obtained constructions agree with the well known notions of a self-affine tiling of the Euclidean plane (which correspond in our situation to self-similar actions of commutative groups). In the case of the iterated monodromy group of a rational mapping of the complex sphere the limit space is homeomorphic to the Julia set of the mapping.

## 1.4 Virtual endomorphisms and $L$ -presentations

An important role in the study of self-similar groups is played by *virtual endomorphisms*, i.e., endomorphisms defined on a subgroup of finite index. Every self-similar action of a group defines an associated virtual endomorphism of the group. Together with some simple additional data the virtual endomorphism determines uniquely the action. In this way the self-similar actions can be interpreted as abstract numeration systems on the groups with the virtual endomorphism playing the role of the base. Such numeration systems are natural generalizations of the usual numeration systems on the group  $\mathbb{Z}$ . Self-similar actions are also associated with the well-known numeration systems on the free abelian groups  $\mathbb{Z}^n$  (see [Vin00]). The respective tilings of Euclidean space can also be interpreted in the terms of self-similar actions and generalized to non-commutative groups.

I. G. Lysenok has obtained in 1985 a finite  $L$ -presentation of the Grigorchuk group  $\mathbf{G}$ , i.e. a presentation in which the defining relations are obtained from a finite set of relations using iterated application of a substitution  $f$  over the alphabet of generators (equivalently,  $f$  can be viewed as an endomorphism of the respective free group). Such presentations were obtained independently by S. Sidki for the Gupta-Sidki group [Sid87], and later L. Bartholdi gave a universal method of constructing  $L$ -presentations of branch groups using the virtual endomorphisms [Bar01a]. The  $L$ -presentations are convenient to construct embeddings of a group into a finitely presented group (using only one HNN-extension). If the  $L$ -presentation is defined using a usual endomorphism, then this embedding preserves amenability of the group. This was used in [Gri98] to construct a finitely presented amenable but not elementary amenable group. On one hand,  $L$ -presentations are similar to  $L$ -systems, which are well known in formal language theory. On the other hand, they have analogies with substitution dynamical systems [Que87]. The substitution dynamical systems have a direct relation with fractals.

## 1.5 Boundaries

One of the sources of fractal sets are the various boundaries of groups. There exist a great variety of different notions of boundary of a group connected with different compactifications: Baily-Borel compactification, Freudenthal boundary (the space of the ends) [Fre31, Fre42] boundary (see [Woe00]), Poisson-Furstenberg boundary [Fur71], Gromov boundary [Gro87], Higson-Roe corona [DF97], Floyd compactification [Kar01, Kar02, Flo84], etc. There exists a rather general method to construct a boundary of a finitely generated group based on the use of the metrics (or uniform structures) satisfying the condition  $d(gx, gy) \rightarrow 0$  when  $g$  tends to infinity (see, for instance, [Flo80] where a partial case of such metrics is considered). As it was noted probably for the first time by A. S. Mishchenko, the respective boundaries play an important role in the topics related to such famous conjectures as the Novikov conjecture or the Baum-Connes conjecture [DF97].

An important class of metric spaces and finitely generated groups are the Gromov-hyperbolic groups [Gro87]. They possess a natural boundary (the Gromov boundary) which is one of the most well studied boundaries. It is known, that the boundary of a hyperbolic group can be homeomorphic to the Sierpinski carpet, the Menger curve and to other sets of a fractal nature. The action of a hyperbolic space on its boundary is an example of a finitely presented dynamical system (the notion belongs also to M. Gromov, see [Gro87] and [CP93]). The boundary is a semi-Markovian space, and has a rich self-similarity structure. Many Kleinian groups are word hyperbolic and their limit sets are often homeomorphic to their Gromov boundaries.

A natural generalization of the boundary of hyperbolic groups are the Dynkin sets of  $\mathcal{D}$ -groups, defined by H. Furstenberg [Fur67]. The hyperbolic groups and their finitely generated subgroups, which are not virtually cyclic, belong to the class of  $\mathcal{D}$ -groups.

As the third named author noted recently, a naturally-defined hyperbolic graph can be associated with every contracting self-similar group action; and its boundary is homeomorphic to its limit space. A short survey of notions and facts about hyperbolic spaces, groups and their boundaries is presented in Sections 10, 11.

The classical notion of self-similarity (of topological spaces) and the notion of a self-similar

action can be interpreted from a common point of view using the notion of a Hilbert bimodule over a  $C^*$ -algebra. Some  $C^*$ -algebras (for instance the Cuntz-Pimsner algebras) associated with such self-similarity bimodules were studied in [Neka]; however the study of some other algebras is only at its beginning [GŻuk]. This is the topic of Section 13.

Finally, a word on “fractals”. We use the term “fractal group” only in the title, and not in the text. The reason is that, just as there is no unique generally accepted definition of a fractal set, there is no definition of a fractal group. However, in some papers, including the works of the authors, different variants of a definition corresponding to the notion of recurrent action (see Definition 6.2) where proposed. Roughly speaking, fractal groups are the groups acting self-similarly on a self-similar set and such that their geometry, analysis and dynamics are related in some way with fractal objects. We hope that numerous examples given in this paper will convince the reader that we are on the right path toward the notion of a fractal group.

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## 2 Preliminary definitions

### 2.1 Spaces of words

For a finite set  $X$  (an *alphabet*) we denote by  $X^* = \{x_1x_2\dots x_n | x_i \in X, n \geq 0\}$  the set of all finite words over the alphabet  $X$ , including the empty word  $\emptyset$ . We have  $X^* = \cup_{n \geq 0} X^n$  (we put  $X^0 = \{\emptyset\}$ ). If  $v = x_1x_2\dots x_n \in X^n$  then  $n$  is the length of the word  $v$  and is written  $|v|$ . The product (concatenation)  $v_1v_2$  of two words  $v_1, v_2 \in X^*$  is naturally defined.

By  $X^\omega$  we denote the set of all infinite unilateral sequences (words) of the form  $x_1x_2\dots$ ,  $x_i \in X$ . If  $v \in X^*$  and  $w \in X^\omega$ , then the product  $vw \in X^\omega$  is also naturally defined.

The set  $X^\omega$  is equipped with the topology of direct product of the discrete finite sets  $X$ . The basis of open sets in this topology is the collection of all *cylindrical sets*

$$a_1a_2\dots a_nX^\omega = \{x_1x_2\dots \in X^\omega | x_i = a_i, 1 \leq i \leq n\}$$

where  $a_1a_2\dots a_n$  runs through  $X^*$ . The space  $X^\omega$  is totally disconnected and homeomorphic to the Cantor set.

In a similar way we can introduce a topology on the set  $X^\omega \cup X^*$  taking a basis of open sets  $\{vX^* \cup vX^\omega : v \in X^*\}$ , where  $vX^* \cup vX^\omega$  is the set of all words (finite and infinite) beginning with  $v$ .

The topological space  $X^\omega \cup X^*$  is compact, the set  $X^\omega$  is closed in it and the set  $X^*$  is a dense subset of isolated points.

The *shift* on the space  $X^\omega$  is the map  $s : X^\omega \rightarrow X^\omega$ , that deletes the first letter of the word:

$$s(x_1x_2\dots) = x_2x_3\dots$$

The space  $X^\omega$  is also called the *full one-sided shift (space)*.

**Definition 2.1.** A subset  $\mathcal{F} \subseteq X^\omega$  is called a *subshift (space)* if it is closed and invariant under the shift  $s$ , i.e., if  $s(\mathcal{F}) \subseteq \mathcal{F}$ .

**Definition 2.2.** A subset  $\mathcal{F} \subseteq X^\omega$  is a *subshift (space) of finite type* if there exists a number  $m \in \mathbb{N}$  and a subset  $A \subset X^m$  of *admissible words*, such that a word  $w \in X^\omega$  belongs to  $\mathcal{F}$  if and every its subword of length  $m$  belongs to  $A$ . If  $m = 2$  then the subshift  $\mathcal{F}$  is also called a *topological Markov chain*.

It is easy to prove that every subshift of finite type is a subshift space.

## 2.2 Graphs

We will use two versions of the notion of a graph. The first is the most general one (directed graphs with loops and multiple edges). It will be used to construct Moore diagrams of automata (Subsection 4.1), structural graphs of iterated function systems (Definition 3.1), Schreier graphs of groups (Definition 7.1). All the other graphs, which will appear in our paper, are defined using a more classical notion of a (simplicial) graph, i.e., a non-directed graph without loops and multiple edges.

**Definition 2.3.** A *graph*  $\Gamma$  is defined by a set of *vertices*  $V(\Gamma)$ , a set of *edges (arrows)*  $E(\Gamma)$  and maps  $\alpha, \omega : E(\Gamma) \rightarrow V(\Gamma)$ . Here  $\alpha(e)$  is the *beginning* (or *source*) of the edge  $e$  and  $\omega(e)$  is its *end* (or *range*).

Two vertices  $v_1, v_2$  are *adjacent* if there exists an edge  $e$  such that  $v_1 = \alpha(e)$  and  $v_2 = \omega(e)$  or  $v_2 = \alpha(e)$  and  $v_1 = \omega(e)$ . Then we say that the edge  $e$  *connects* the vertices  $v_1$  and  $v_2$ .

The *(edge-)labeled graph* is a graph together with a map  $l : E(\Gamma) \rightarrow S$ , which assigns a *label*  $l(e) \in S$  to every edge of the graph. Here  $S$  is a given *label set*.

A *morphism* of graphs  $f : \Gamma_1 \rightarrow \Gamma_2$ , is a pair of maps  $f_v : V(\Gamma_1) \rightarrow V(\Gamma_2), f_e : E(\Gamma_1) \rightarrow E(\Gamma_2)$  such that

$$\begin{aligned}\alpha(f_e(e)) &= f_v(\alpha(e)) \\ \omega(f_e(e)) &= f_v(\omega(e))\end{aligned}$$

for all  $e \in E(\Gamma_1)$ . A *morphism of labeled graphs* is a morphism of graphs preserving the labels of the edges.

A *path* in a graph  $\Gamma$  is a sequence of edges  $e_1 e_2 \dots e_n$ , with  $\omega(e_i) = \alpha(e_{i+1})$  for every  $1 \leq i \leq n - 1$ . The vertex  $\alpha(e_1)$  is called the *beginning of the path*, and the vertex  $\omega(e_n)$  is its *end*. The number  $n$  is called the *length* of the path. In the similar way define *infinite to the right* paths  $e_1 e_2 \dots$ , *infinite to the left* paths  $\dots e_2 e_1$  and the *bi-infinite* paths  $\dots e_{-1} e_0 e_1 e_2 \dots$ .

**Definition 2.4.** A *simplicial graph*  $\Gamma$  is defined by its set of vertices  $V(\Gamma)$  and a set of edges  $E(\Gamma)$ , where every edge is a set  $\{v_1, v_2\}$  of two different vertices  $v_1, v_2 \in V(\Gamma)$ .

Thus, a simplicial graph is not directed and does not have loops or multiple edges.

If  $\{v_1, v_2\} \in E$  then we say that the vertices  $v_1$  and  $v_2$  are *adjacent*, and that the edge  $e = \{v_1, v_2\}$  *connects* the vertices.

A *morphism* of simplicial graphs  $f : \Gamma_1 \rightarrow \Gamma_2$  is a map  $V(\Gamma_1) \rightarrow V(\Gamma_2)$  which preserves the adjacency of the vertices.

The degree  $\deg v$  of a vertex  $v$  of a simplicial graph is the number of edges to which it belongs.

If  $\Gamma$  is a graph, then its *associated simplicial graph* is the simplicial graph with the same set of vertices, which contains an edge  $e = \{v_1, v_2\}$  if and only if the vertices  $v_1$  and  $v_2$  were adjacent in the original graph and  $v_1 \neq v_2$ .

A *path* in a simplicial graph  $\Gamma$  is a sequence of vertices  $v_1 v_2 \dots v_{n+1}$ , with  $\{v_i, v_{i+1}\} \in E(\Gamma)$  for every  $1 \leq i \leq n$ . The vertex  $v_1$  is called the *beginning of the path*, and the vertex  $v_{n+1}$  is its *end*. The number  $n$  is called the *length* of the path.

A *geodesic path*, connecting the vertices  $v_1$  and  $v_2$  is a path of minimal length, whose beginning and end are  $v_1$  and  $v_2$  respectively.

The length of a geodesic path connecting the vertices  $v_1$  and  $v_2$  is called their *distance* and is written  $d(v_1, v_2)$ . We define  $d(v, v) = 0$ . If the graph  $\Gamma$  is connected, then the distance  $d(v_1, v_2)$  is defined for every pair of vertices  $v_1, v_2 \in V(\Gamma_1)$  and is called the *natural* (or *combinatorial*) metric on the graph.

The distance between two vertices of a non-simplicial graph is defined as the distance between the vertices in the associated simplicial graph.

For a graph  $\Gamma$ , a vertex  $v \in V(\Gamma)$ , and  $r \in \mathbb{N}$ , the *ball*  $B(v, r)$  of *radius*  $r$  with *center* at the point  $v$  is the set of the vertices  $\{u \in V : d(v, u) \leq r\}$ .

A graph  $\Gamma$  is *locally finite* if for every vertex  $v$  the ball  $B(v, 1)$  is finite. If the graph is locally finite, then every ball  $B(v, r)$  is finite.

### 3 Self-similar sets and (semi)group actions

#### 3.1 Self-similar sets

**Definition 3.1.** A *graph-directed iterated function system* with *structural graph*  $\Gamma = \langle V, E, \alpha, \omega \rangle$  is a finite collection of sets  $\{F_v\}_{v \in V}$  together with a collection of injective maps  $\{\phi_e : F_{\alpha(e)} \rightarrow F_{\omega(e)}\}_{e \in E}$ , such that for every  $v \in V$

$$F_v = \bigcup_{\omega(e)=v} \phi_e(F_{\alpha(e)}). \quad (1)$$

If the structural graph contains only one vertex  $v$ , then we say that we have an *iterated function system* on the set  $F = F_v$ .

If the sets  $F_v$  are subsets of a common set  $F$  and  $F = \bigcup_{v \in V} F_v$ , then the graph-directed iterated function system will also be called a *self-similarity structure* on  $F$ . A set with a self-similarity structure on it is called *self-similar*.

A set with a fixed self-similarity structure is called *self-similar*.

Note that the structural graph of an iterated function system on a single set is the graph with a single vertex and  $|E|$  loops.

**Definition 3.2.** A *topological graph-directed iterated function system* is a system  $(\{F_v\}, \{\phi_e\})$  for which the sets  $F_v$  are compact Hausdorff topological spaces and the injections  $\phi_e$  are continuous.

In the case of the classical notion of a graph-directed iterated function system, the sets  $F_v$  are usually subsets of the Euclidean space  $\mathbb{R}^n$  and the maps  $\phi_e$  are contractions.

For more on the notion of (graph-directed) iterated function systems and self-similarity structures see the papers [Hut81, Hat85, Kig93, Ban89a, Ban89b] and the book [Fal97].

Graph-directed iterated function systems can be also viewed as particular examples of *topological graphs* (see [Dea99]).

**The full one-sided shift**  $X^\omega$  is one of the most important examples of the spaces with a standard self-similarity structure. The respective iterated function system on  $X^\omega$  has the structural graph with one vertex and  $|X|$  loops and is equal to the collection of the maps  $\{\mathbb{T}_x\}_{x \in X}$ , where

$$\mathbb{T}_x(w) = xw.$$

The image of  $X^\omega$  under the map  $\mathbb{T}_x$  is the cylindrical set  $xX^\omega$ , so that  $X^\omega = \bigcup_{x \in X} \mathbb{T}_x(X^\omega)$ .

**The shifts of finite type.** Let  $\mathcal{F} \subseteq X^\omega$  be a topological Markov chain. Let  $\mathcal{F}_x = \mathcal{F} \cap xX^\omega$  be the set of all the sequences  $x_1x_2 \dots \in \mathcal{F}$  such that  $x_1 = x$ . For every admissible word  $xy \in X^2$  we define  $\mathbb{T}_{xy} : \mathcal{F}_y \rightarrow \mathcal{F}_x$  by the formula

$$\mathbb{T}_{xy}(w) = xw.$$

It is easy to check that we get in this way a graph-directed iterated function system. The set of vertices of its structural graph is identified with  $X$ ; two vertices  $x, y \in X$  are connected with an arrow starting in  $x$  and ending in  $y$  if and only if the word  $xy$  is admissible.

The above two examples are in some sense universal, since they are used to encode all the other self-similar sets.

**Definition 3.3.** Let  $(\{F_v\}_{v \in V}, \{\phi_e\}_{e \in E})$  be a graph-directed iterated function system. Let  $p \in F_v$  be an arbitrary point. We define a *code* of the point  $p$  as an infinite sequence  $e_1e_2 \dots \in E^\omega$  such that for every  $k \in \mathbb{N}$  we have

$$p \in \phi_{e_1} \left( \phi_{e_2} \left( \dots \phi_{e_{k-1}} \left( \phi_{e_k} \left( F_{\alpha(e_k)} \right) \dots \right) \right) \right).$$

In general, a point can have different codes, since the sets  $F_i$  can overlap. Also, different points can have the same codes.

It follows directly from the definition that for every code  $e_1e_2\dots$  of a point we have  $\alpha(e_k) = \omega(e_{k+1})$  for all  $k \geq 1$ , i.e., the sequence  $\dots e_2e_1$  is a (left-)infinite path in the structural graph.

On the other hand, in the case of a topological iterated function system, for every infinite path  $\dots e_2e_1$  in the associated graph we have a decreasing sequence of compact sets

$$\phi_{e_1}(F_{\alpha(e_1)}) \supseteq \phi_{e_1}(\phi_{e_2}(F_{\alpha(e_2)})) \supseteq \phi_{e_1}(\phi_{e_2}(\phi_{e_3}(F_{\alpha(e_3)}))) \supseteq \dots$$

and every point in the intersection of these sets will have the code  $e_1e_2\dots$ ; therefore the set of all possible codes is a topological Markov chain in  $E^\omega$  (which is defined by the set of admissible words  $\{xy : \alpha(x) = \omega(y)\}$ ).

**Markov partitions.** If the sets  $F_v$  form a covering of a set  $F$ , and if all the maps  $\phi^{-1}$  (which are defined on subsets of the sets  $F_v$ ) are restrictions of a single map  $f : F \rightarrow F$ , then the collection  $\{\phi_e(F_{\alpha(e)})\}_{e \in E}$  is called the *Markov partition* of the dynamical system  $(F, f)$  (though, usually more restrictions are imposed on the sets  $F_v$ ). In this case the described encoding of the points of  $F$  is the classical tool of the Symbolic Dynamics. For the notion of a Markov partition see the survey [Adl98], the book [Kit98] and the bibliography in them.

Note that the maps  $T_x$  defining the iterated function system on  $X^\omega$  are the inverses of the shift  $s$ , so the sets  $sX^\omega$  form a Markov partition of the dynamical system  $(X^\omega, s)$ .

### Other examples of self-similar sets.

**1. The Cantor middle-third set.** The full shift  $X^\omega$  for  $|X| = 2$  can be naturally interpreted as the classical Cantor set. This is the set  $C$  obtained from the segment  $[0, 1]$  by successive removing from the segments their middle thirds (see Figure 1).



Figure 1: The Cantor set

It follows that  $C$  is the set of the numbers  $x \in [0, 1]$  which have only the digits 0 and 2 in their triadic expansion. In other words

$$C = \left\{ \sum_{n=1}^{\infty} \frac{d_n}{3^n} : d_1d_2d_3\dots \in \{0, 2\}^\omega \right\}.$$

Moreover, the map  $\Phi : d_1d_2d_3\dots \mapsto \sum_{n=1}^{\infty} d_n3^{-n}$  is a homeomorphism  $\{0, 2\}^\omega \rightarrow C$ .

The standard iterated function system on  $\{0, 2\}^\omega$  can be identified via the map  $\Phi$  with the following natural iterated function system on  $C$ : this system is the collection of two maps  $\phi_0 : x \mapsto x/3$  and  $\phi_1 : x \mapsto x/3 + 2/3$ .

It is easy to see now that the triadic numeration system gives exactly the standard encoding of the Cantor set with respect to the described iterated function system.

**2. The segment  $[0, 1]$ .** Let  $\phi_0(x) = x/2$  and  $\phi_1(x) = x/2 + 1/2$  be two maps of the interval  $[0, 1]$  to itself. In this way we obtain an iterated function system on the segment  $[0, 1]$ . The code of a point  $x \in [0, 1]$  in this case will be the sequence of digits in its dyadic expansion. Note that here the dyadic expansion is not uniquely defined. For instance, the point  $1/2$  has two different codes:  $100000\dots$  and  $011111\dots$

**3. The Sierpiński gasket and Sierpiński carpet.** The Sierpiński gasket (Figure 2 (a)) is constructed from the triangle with vertices  $(0, 0)$ ,  $(1, 0)$  and  $(1/2, \sqrt{3}/2)$  by successive deletion of the central triangles (see [Fal85]).

It is a self-similar set with the iterated function system consisting of three affine transformations  $\phi_1(\vec{x}) = \vec{x}/2$ ,  $\phi_2(\vec{x}) = \vec{x}/2 + (1/2, 0)$ ,  $\phi_3(\vec{x}) = \vec{x}/2 + (1/4, \sqrt{3}/4)$ .

The Sierpiński carpet (Figure 2 (b)) is constructed in a similar way starting from the square  $[0, 1] \times [0, 1]$  by deletion of the central squares. It is also a self-similar set, for the iterated function system

$$\begin{aligned} \phi_1(\vec{x}) &= \vec{x}/3, & \phi_2(\vec{x}) &= \vec{x}/3 + (1/3, 0), \\ \phi_3(\vec{x}) &= \vec{x}/3 + (2/3, 0), & \phi_4(\vec{x}) &= \vec{x}/3 + (0, 1/3), \\ \phi_5(\vec{x}) &= \vec{x}/3 + (0, 2/3), & \phi_6(\vec{x}) &= \vec{x}/3 + (1/3, 2/3), \\ \phi_7(\vec{x}) &= \vec{x}/3 + (2/3, 2/3), & \phi_7(\vec{x}) &= \vec{x}/3 + (2/3, 1/3). \end{aligned}$$

It is easy to see that the Sierpiński gasket (resp. the Sierpiński carpet) is uniquely determined by the condition (1) for the corresponding set of affine transformations. See the paper [Hut81], where properties of contracting iterated function systems are investigated in a general setup.

**4. The Apollonian net.** The Apollonian net (see [Man82]) is a subset of the Riemann sphere, constructed in the following way. Take four pairwise tangent circles  $A_1, A_2, A_3, A_4$  (see Figure 2 (c)). We will get four curvilinear triangles with the vertices in the tangency points of the circles and the sides coinciding with the respective arcs of the circles. Let us denote the obtained triangles  $T_1, T_2, T_3, T_4$ .

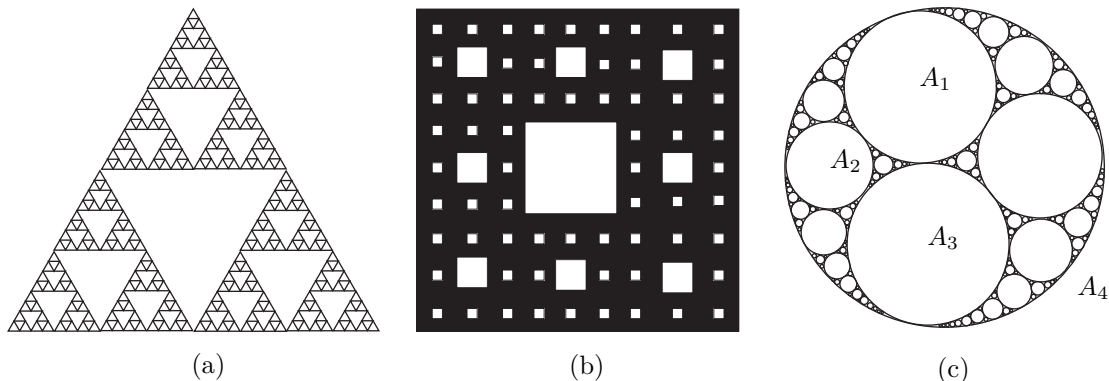


Figure 2: Self-similar sets

Let us remove from the sphere the open disks bounded by the circles  $A_i$ . Denote the obtained closed set by  $\mathcal{P}_1$ . It is the union of the triangles  $T_i$ ,  $i = 1, \dots, 4$ . At the next stage we inscribe into every triangle  $T_i$  a maximal circle (this can be done in a unique way), and remove the open disks bounded by these circles. The obtained set  $\mathcal{P}_2$  will be a union of 12 triangles. Then again, we inscribe a circle into each of these 12 triangles and remove from  $\mathcal{P}_2$  the open disks bounded by these circles. We denote the obtained union of 36 triangles by  $\mathcal{P}_3$ . We continue in a similar fashion. The *Apollonian net* is the intersection  $\mathcal{P} = \bigcap_{i=1}^{\infty} \mathcal{P}_i$ . See Figure 2 (c) for a picture of this fractal set.

Let us denote now by  $\mathcal{T}_i$  the part of the Apollonian net bounded by the triangle  $T_i$ . The sets  $\mathcal{T}_i$  are called *Apollonian gaskets*. It is easy to prove that the Apollonian gasket is homeomorphic to the Sierpinski gasket, but as metric spaces they are very different. For instance, the Hausdorff dimension of the Sierpinski gasket is equal to  $\log 3 / \log 2$ , while the exact value of the Hausdorff dimension  $h$  of the Apollonian gasket is not known. At the moment very precise estimates of  $h$

exists, and it can also be computed with arbitrary precision (see [Boy73a, Boy73b] and [McM98]). C. T. McMullen has shown that  $h \approx 1.305688$ .

The Apollonian gasket can be obtained by the same procedure of removing inscribed disks but starting from a curvilinear triangle formed by three tangent circles. It is a self-similar set with self-similarity structure defined by the rational transformations  $f_i, i = 1, 2, 3$ , which map the triangle onto its three subtriangles. For instance, if we take the triangle with the vertices  $1, \exp(2\pi i/3), \exp(-2\pi i/3)$  then the iterated function system will be  $f_1(z) = \frac{(\sqrt{3}-1)z+1}{-z+\sqrt{3}+1}$ ,  $f_2(z) = \exp(2\pi i/3)f_1(z)$  and  $f_3(z) = \exp(-2\pi i/3)f_1(z)$  (see [MU98]).

### 3.2 Direct limits and self-replicating tilings

Let  $(\{F_v\}_{v \in V}, \{\phi_e\}_{e \in E})$  be a graph-directed iterated function system.

Let  $e_1 e_2 \dots$  be a path in the structural graph of the iterated function system. The *leaf* defined by the path  $e_1 \dots$  is the direct limit of the sequence

$$F_{\alpha(e_1)} \xrightarrow{\phi_{e_1}} F_{\alpha(e_2)} \xrightarrow{\phi_{e_2}} \dots \quad (2)$$

We identify in a natural way the leaf defined by the path  $e_1 e_2 e_3 \dots$  with the leaf defined by the path  $f_1 f_2 f_3 \dots$  if the paths are *cofinal*, i.e., if  $e_i = f_i$  for all  $i$  big enough. For a fixed path  $e_1 e_2 e_3 \dots$ , the *central tile* defined by the path is the image of  $F_{\alpha(e_1)}$  in the direct limit (2). The central tiles defined by other paths, cofinal with  $e_1 e_2 e_3 \dots$  are called *tiles* of the leaf.

It follows from Definition 3.1 that a leaf is the union of its tiles. A *tiled leaf* is a leaf together with its decomposition into the union of its tiles.

**Fibonacci tilings.** Let us construct an iterated function system with two sets  $F_1$  and  $F_2$ , where  $F_1$  is the segment  $[0, 1]$ , and  $F_2$  is the segment  $[0, \tau]$ , where  $\tau = (1 + \sqrt{5})/2$  is the *golden mean*.

The functions are  $\phi_{21} : F_2 \rightarrow F_1$ ,  $\phi_{12} : F_1 \rightarrow F_2$  and  $\phi_{22} : F_2 \rightarrow F_2$ , where  $\phi_{21}(x) = x/\tau$ ,  $\phi_{12}(x) = x/\tau$  and  $\phi_{22}(x) = (x+1)/\tau$ . So that  $\phi_{21}(F_2) = F_1$  and  $\phi_{12}(F_1) = [0, 1/\tau]$ ,  $\phi_{22}(F_2) = [1/\tau, \tau]$ , since  $1 + \tau = \tau^2$ . In this way we obtain the *Fibonacci iterated function system*.

Its leaves are homeomorphic to the real lines tiled by segments of length 1 and  $\tau$ . For every such leaf we get a bi-infinite sequence of symbols 1 and  $\tau$ , denoting the lengths of the respective tiles. The set of all such sequences is a closed subset of the space of all bi-infinite sequences over the alphabet  $\{1, \tau\}$  and is an example of a *substitution dynamical system*. It can also be viewed as a one-dimensional analogue of the Penrose tiling. See for example the paper [dB81], where such tilings are studied and are constructed using the *projection method*. (The Fibonacci tiling appears at the very end of the article.)

**Penrose tilings.** The Penrose tiling is the most famous example of an aperiodic tiling of the plane. A tiling is *aperiodic* if it is not preserved by a non-trivial translation.

See the interesting description of the way in which it was invented by R. Penrose in [Pen84], the first article on these tilings by M. Gardner in [Gar77] and its detailed analysis in the book [GS87].

The tiles of the Penrose tiling are the triangles shown on Figure 3. The angles of the triangle (a) are equal to  $\frac{\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{5}$  and the angles of the triangle (b) are equal to  $\frac{3\pi}{5}, \frac{\pi}{5}, \frac{\pi}{5}$ . These triangles are cut from a regular pentagon by the diagonals issued from a common vertex, so that the ratio between the lengths of the shorter and the longer sides of the triangles is equal to  $\tau = \frac{1+\sqrt{5}}{2}$ .

A tiling of the plane by the triangles is a Penrose tiling if it satisfies the *matching rules*, which require that the common vertices of two tiles are marked by the same color (black or white) and that the arrows on the adjacent sides of the triangles point at the same direction. The colors of the vertices and the arrows on the sides are shown on Figure 3.

It follows from the matching rules that if we mark the sides of the tiles by the letters  $S, L$  and  $M$  in the way it is done on Figure 3, then in every Penrose tiling the common sides of any two adjacent tiles will be marked by the same letter.

An example of a patch of a Penrose tiling is shown on Figure 4.

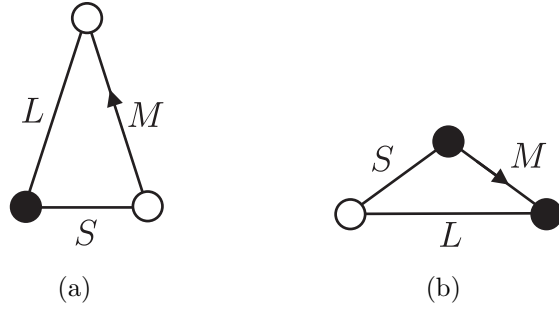


Figure 3: Tiles and matching rules

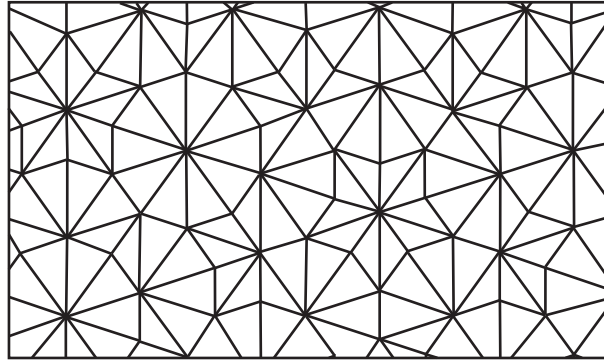


Figure 4: Penrose tiling

It follows from the matching rules that in any Penrose tiling one can group the tiles into blocks of two or three tiles, shown on Figure 5 (their mirror images are also allowed).

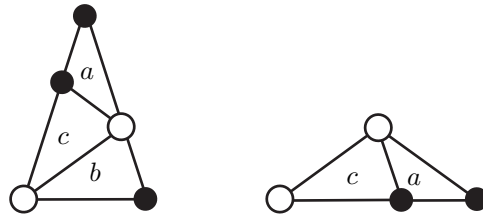


Figure 5: Inflation

The grouping into blocks is unique [GS87] and it is easy to see that the blocks are similar to the original tiles, with similarity coefficient  $\tau$ . Moreover, the grouping agrees with the matching rules, so that the blocks also form a Penrose tiling. This tiling is called the *inflation* of the original one.

It follows from existence and uniqueness of the inflation that every Penrose tiling is aperiodic. It also follows from inflation that any two Penrose tilings are locally isomorphic. See the book [GS87] for the proofs of these and other properties of the Penrose tilings.

The inflation rule defines obviously a graph-directed iterated function system with two sets  $F_1$  and  $F_2$ , equal to the triangles and five similarities, as shown on Figure 6. This iterated function system is called the *Penrose iterated function system*.

The symmetry group of a Penrose tiling is either trivial, or is of order two (then the only

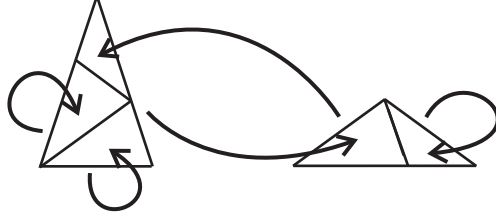


Figure 6: Penrose iterated function system

non-trivial symmetry of the tiling is a symmetry with respect to a line), or is the dihedral group  $D_5$  (then the only symmetries of the tiling are rotations around a point on the angles  $k \cdot \frac{2\pi}{5}$  and symmetries with respect to five lines passing through the rotation point). More detailed analysis of the symmetric situations will be made later.

The tiled leaves of the direct limit of the Penrose iterated function system will be either a Penrose tiling of the plane with trivial symmetry group, or the fundamental domain (a half-plane, or an 36-degree angle) of a Penrose tiling with a non-trivial symmetry group.

### 3.3 Self-similar groups

The aim of this section is to define *self-similar group actions*, which are analogous to self-similar spaces. The analogy will become more clear in Section 13, where both notions will be treated within a common algebraic approach.

**Definition 3.4.** Suppose a collection  $\{\phi_e\}_{e \in E}$  of functions  $F \rightarrow F$  defines an iterated function system on the space  $F$ . An action of a group  $G$  on  $F$  is then said to be *self-similar* if for every  $e \in E$  and for every element  $g \in G$  there exists  $f \in E$  and  $h \in G$  such that

$$(\phi_e(p))^g = \phi_f(p^h) \quad \text{for all } p \in F.$$

The most favorable case is when the images of the set  $F$  under the maps  $\phi_e$  do not intersect. We also assume that different points of  $F$  have different codes. The space  $F$  is then homeomorphic to  $X^\omega$  for the alphabet  $X = E$ , and the functions  $\phi_e$  become equal to the maps  $T_x : w \mapsto xw$  on  $X^\omega$ .

We therefore reformulate Definition 3.4 for the space  $F = X^\omega$  in the following way:

**Definition 3.4'.** An action of a group  $G$  on the space  $X^\omega$  is *self-similar* if for every  $g \in G$  and every  $x \in X$  there exist  $h \in G$  and  $y \in X$  such that

$$(xw)^g = y(w^h) \tag{3}$$

for every  $w \in X^\omega$ .

Applying Equation (3) several times we see that for every finite word  $v \in X^*$  and every  $g \in G$  there exist  $h \in G$  and a word  $u \in X^*$  such that  $|u| = |v|$  and

$$(vw)^g = u(w^h)$$

for all  $w \in X^\omega$ . Hence we get a naturally-defined associated action of  $G$  on  $X^*$ , for which the word  $u$  is the image of  $v$  under the action of  $g$ . If  $G$  acts faithfully then the group element  $h$  is defined uniquely. It is called then the *restriction* of  $g$  in the word  $v$  and is written  $h = g|_v$ .

It follows directly from the definitions that for any  $v, v_1, v_2 \in X^*$  and  $g, g_1, g_2 \in G$  we have

$$g|_{v_1v_2} = (g|_{v_1})|_{v_2} \quad (g_1g_2)|_v = (g_1|_v)(g_2|_{vg_1}). \tag{4}$$

**Definition 3.5.** A self-similar action is *level-transitive* for every  $n \geq 0$  the associated action on  $X^*$  is transitive on the set  $X^n$  of the words of length  $n$ .

All self-similar actions in this paper are assumed to be level-transitive.

### 3.4 Examples of self-similar group actions

**The adding machine.** Let  $a$  be the transformation of the space  $\{0, 1\}^\omega$  defined by the following recursion formulæ:

$$\begin{aligned}(0w)^a &= 1w \\ (1w)^a &= 0w^a,\end{aligned}$$

where  $w$  is an arbitrary infinite word over the alphabet  $\{0, 1\}$ .

These formulæ can be interpreted naturally as the rule of adding 1 to a dyadic integer. More precisely, identifying  $\{0, 1\}^\omega$  with the dyadic integers  $\mathbb{Z}_2$  via  $\Phi : x_1x_2\dots \mapsto \sum_{i \geq 1} x_i 2^{i-1}$ , we have

$$\Phi(w^a) = \Phi(w) + 1.$$

The term “adding machine” originates from this interpretation of the transformation  $a$ .

The transformation  $a$  generates an infinite cyclic group of transformations of the space  $\{0, 1\}^\omega$ . Thus we get an action of the group  $\mathbb{Z}$ , which will be also called *adding machine action*. It is easy to see that the adding machine action is self-similar.

**The dihedral group.** Let  $a$  and  $b$  be the transformations of the space  $X^\omega = \{0, 1\}^\omega$ , defined by the rules

$$\begin{aligned}(0w)^a &= 1w & (0w)^b &= 0w^a \\ (1w)^a &= 0w & (1w)^b &= 1w^b,\end{aligned}$$

where  $w \in X^\omega$  is arbitrary.

The group generated by the transformations  $a$  and  $b$  is isomorphic to the infinite dihedral group  $\mathbb{D}_\infty$ , and thus we get a self-similar action of this group on  $X^\omega$ .

**The Grigorchuk group.** The Grigorchuk group (see [Gri80] for the original definition) is the transformation group of the space  $\{0, 1\}^\omega$  generated by the transformations  $a, b, c, d$ , which are defined by the rules:

$$\begin{aligned}(0w)^a &= 1w & (1w)^a &= 0w \\ (0w)^b &= 0w^a & (1w)^b &= 1w^c \\ (0w)^c &= 0w^a & (1w)^c &= 1w^d \\ (0w)^d &= 0w & (1w)^d &= 1w^b.\end{aligned}$$

The Grigorchuk group is an infinite finitely generated torsion group. This fact relates it to the General Burnside Problem. It is also the first example of a group of intermediate growth [Gri83] (about growth see Section 8). For more on the Grigorchuk group see [Gri00] and the last chapter of [Har00].

**Other examples.** The “*lamplighter group*”  $(\mathbb{Z}/2\mathbb{Z})^\mathbb{Z} \rtimes \mathbb{Z}$ , where  $\mathbb{Z}$  acts on  $(\mathbb{Z}/2\mathbb{Z})^\mathbb{Z}$  by the shift is isomorphic to the transformation group of the space  $\{0, 1\}^\omega$  generated by the transformations

$$\begin{aligned}(0w)^a &= 1w^b & (0w)^b &= 0w^b \\ (1w)^a &= 0w^a & (1w)^b &= 1w^a.\end{aligned}$$

To see this, identify  $\{0, 1\}^\omega$  with  $(\mathbb{Z}/2\mathbb{Z})[[t]]$  via  $\Phi : x_1x_2\dots \mapsto \sum_{i \geq 1} x_i t^{i-1}$ . We then have (see [GNS00])

$$\Phi(w^{b^{-1}a}) = \Phi(w) + 1, \quad \Phi(w^b) = (1+t)\Phi(w).$$

This action was used in [GZuk01] to compute the spectrum of the lamplighter group.

The self-similar representations of *abelian* and *affine groups* were studied in the papers [BS98] and [NS01].

### 3.5 Self-similar inverse semigroups

A *partial permutation* of a set  $M$  is a bijection  $\pi : \text{Dom } \pi \rightarrow \text{Ran } \pi$  between a subset  $\text{Dom } \pi \subseteq M$  (the *domain* of the permutation) and a subset  $\text{Ran } \pi$  (the *range* of the permutation). We admit also the *empty permutation*  $0$  with empty domain and range.

Product of two partial permutations  $\pi_1, \pi_2$  is the partial permutation  $\pi_1\pi_2$  with the domain

$$\text{Dom } \pi_1\pi_2 = \{x \in \text{Dom } \pi_1 : x^{\pi_1} \in \text{Dom } \pi_2\}$$

defined on it by the condition  $x^{\pi_1\pi_2} = (x^{\pi_1})^{\pi_2}$ . In particular,  $\pi \cdot 0 = 0 \cdot \pi = 0$  for every partial permutation.

Every partial permutation has its *inverse*  $\pi^{-1} : \text{Ran } \pi \rightarrow \text{Dom } \pi$ . Note that the product  $\pi^{-1}\pi$  is not the identity but an *idempotent*. It is the identical permutation with the domain coinciding with the range of the permutation  $\pi$ .

An *inverse semigroup* acting on a set  $M$  is a set of partial permutations of the set  $M$  which is closed respectively to the composition and the inversion.

An example of an inverse semigroup is the *symmetric inverse semigroup*  $IS(M)$  which is the semigroup of all partial permutations of the set  $M$ .

If  $\pi$  and  $\rho$  are two partial permutations such that  $\text{Dom } \pi \cap \text{Dom } \rho = \emptyset$  and  $\text{Ran } \pi \cap \text{Ran } \rho = \emptyset$  then the *sum* of the permutations  $\pi$  and  $\rho$  is permutation  $\pi + \rho$  with domain  $\text{Dom } \pi \cup \text{Dom } \rho$ , defined by the condition

$$x^{\pi+\rho} = \begin{cases} x^\pi & \text{if } x \in \text{Dom } \pi \\ x^\rho & \text{if } x \in \text{Dom } \rho. \end{cases}$$

Just as in the case of group actions, we consider only actions of inverse semigroups on topological Markov chains (with standard iterated function systems on them).

**Definition 3.6.** Let  $\mathcal{F} \subseteq X^\omega$  be a topological Markov chain.

An inverse semigroup  $G$  acting on  $\mathcal{F}$  is *self-similar* if for every  $g \in G$  and  $x \in X$  there exist  $y_1, y_2, \dots, y_k \in X$  and  $h_1, h_2, \dots, h_k \in G$  such that the sets  $\text{Dom } h_i$  are disjoint,  $\cup_{i=1}^k y_i \text{Dom } h_i = xX^\omega \cap \text{Dom } g$ , and for every  $w \in X^\omega$  we have

$$(xw)^g = y_i(w)^{h_i},$$

where  $i$  is such that  $w \in \text{Dom } h_i$ .

It follows from the definition that the sets  $y_i \text{Dom } h_i$  are also disjoint.

Let us denote by  $\mathbb{T}_x$  the partial permutation of the space  $\mathcal{F}$  which acts by the rule  $\mathbb{T}_x(w) = xw$ . The domain of  $\mathbb{T}_x$  is the set of those elements  $w \in \mathcal{F}$ , for which  $xw$  belongs to  $\mathcal{F}$ . The range of  $\mathbb{T}_x$  is then the set  $xX^\omega \cap \mathcal{F}$ .

The previous definition can be reformulated in the following way:

**Definition 3.6'.** An inverse semigroup  $G$  acting on the space  $X^\omega$  is *self-similar* if for every  $g \in G$  and  $x \in X$  there exist  $y_1, y_2, \dots, y_k \in X$  and  $h_1, h_2, \dots, h_k \in G$  such that

$$\mathbb{T}_x g = h_1 \mathbb{T}_{y_1} + h_2 \mathbb{T}_{y_2} + \dots + h_k \mathbb{T}_{y_k}. \quad (5)$$

### 3.6 Examples of self-similar inverse semigroups.

**Fibonacci transformations.** Let  $\mathcal{F}$  be the set of all infinite words over the alphabet  $X = \{0, 1\}$  which do not contain a subword 11. The space  $\mathcal{F}$  is a shift of finite type called the *Fibonacci shift* (see [LM95, Kit98]).

Let us define two partial homeomorphisms  $a, b$  of the space  $\mathcal{F}$  having domains  $0X^\omega \cap \mathcal{F}$  and  $1X^\omega \cap \mathcal{F}$  respectively, by the inductive formula

$$\begin{cases} (00w)^a & = 10w \\ (01w)^a & = 0(1w)^b \\ (1w)^b & = 0(w^a). \end{cases}$$

It follows from the inductive definition that the transformation  $a$  acts by the rule

$$a : (01)^n 00w \mapsto (00)^n 10w,$$

where  $n \geq 0$  and  $w \in \mathcal{F}$  is an arbitrary word. Additionally, (in some sense, for  $n = \infty$ ) we have  $(010101\dots)^a = 000000\dots$ .

The transformation  $b$  acts by the rule

$$b : 1(01)^n 00w \mapsto 0(00)^n 10w,$$

where  $n \geq 0$  and  $w \in \mathcal{F}$ . In the limit case we have  $(101010\dots)^b = 000000\dots$ .

It follows from the formulæ above that  $\text{Ran } a$  is equal to the set of all the words from  $\mathcal{F}$  with even or infinite number of leading zeros and  $\text{Ran } b$  is the set of the words with odd or infinite number of leading zeros. These two sets intersect in the point  $000\dots$ . Thus the sum  $a + b$  (which is defined as the map  $\mathcal{F} \rightarrow \mathcal{F}$  equal to  $a$  on  $0X^\omega \cap \mathcal{F}$  and equal to  $b$  on  $1X^\omega \cap \mathcal{F}$ ) is not invertible.

The transformations  $a$  and  $b$  generate a self-similar inverse semigroup.

The partial homeomorphisms  $a$  and  $b$  are associated to a special numeration system (called the *Fibonacci system*) on the integers, in a similar way like the binary adding machine is associated to the usual dyadic numeration system.

Let  $u_1 = 1, u_2 = 2, u_3 = 3, u_4 = 5, u_5 = 8, \dots$  be the Fibonacci sequence defined by the recursion  $u_n = u_{n-1} + u_{n-2}$ . Then (see for instance [Knu69]) every positive integer  $m$  can be uniquely presented as a finite sum

$$a_1 u_1 + a_2 u_2 + \dots + a_n u_n,$$

where  $a_i \in \{0, 1\}$  and  $a_i = 1$  for  $i < n$  implies that  $a_{i+1} = 0$ .

If we put into correspondence to the word  $a_1 a_2 \dots \in \mathcal{F}$  a formal infinite sum

$$a_1 u_1 + a_2 u_2 + \dots,$$

then the map  $a + b$  can be interpreted as adding of 1 to this formal expression. The recurrent definitions of  $a$  and  $b$  come from the ‘‘carrying’’ rules for addition in the Fibonacci system, which in turn come from the relations  $u_1 + 1 = u_2$  and  $u_i + u_{i+1} = u_{i+2}$ .

A relation between the Fibonacci transformations and the Fibonacci tiling, introduced before, will be clarified in Subsection 9.6.

**A group and an inverse semigroup associated to the Penrose tilings.** Let us label the tiles of the Penrose tiling by letters  $a, b, c$  according to their participation in the inflation process as it is shown in Figure 5. Note that every obtuse-angled triangle is labeled by  $a$  and the acute-angled triangles are labeled either by  $b$  or by  $c$  depending on their role in the inflation.

Let  $u$  be a tile of a Penrose tiling of the plane. Let  $x_1 \in \{a, b, c\}$  be its label. After inflation the tile  $u$  becomes a part of a tile  $s(u)$  in the new inflated Penrose tiling. Let  $x_2$  be the label of the tile  $s(u)$ . In general, let  $x_n$  be the label of the tile to which  $u$  belongs in the Penrose tiling obtained from the original one by  $n - 1$  successive inflations. The sequence  $x_1 x_2 \dots \in \{a, b, c\}^\omega$  is called the *code* of the tile  $u$ .

A sequence  $x_1 x_2 \dots \in \{a, b, c\}^\omega$  is the code of a tile in a Penrose tiling if and only if it does not contain a subsequence  $x_i x_{i+1}$  such that  $x_i = b, x_{i+1} = a$ . Let  $\mathcal{P} \subset \{a, b, c\}^\omega$  be the set of such sequences.

The shift  $s : \mathcal{P} \rightarrow \mathcal{P}$  encodes the inflation, more precisely, it maps the code of a tile  $u$  to the code of the tile  $s(u)$ , to which  $u$  belongs after the inflation.

Two sequences  $u, v \in \mathcal{P}$  are codes of tiles belonging to a common tiling if and only if they are *cofinal*, i.e., if for some  $n \in \mathbb{N}$  the sequences  $s^n(u)$  and  $s^n(v)$  are equal. Since every cofinality class is countable, there exist uncountably many non-isomorphic Penrose tilings.

Two tiles  $u$  and  $v$  have the same codes if and only if they belong to a common tiling and there exists a symmetry of the tiling which carries  $u$  to  $v$ .

Therefore, we shall identify a tile with its code, keeping in mind the non-uniqueness described in the previous paragraph. Then  $\mathcal{P}$  becomes a union of the sets of the tiles of all Penrose tilings and the set of cofinality classes on  $\mathcal{P}$  becomes the set of all Penrose tilings.

Let  $L, M, S : \mathcal{P} \rightarrow \mathcal{P}$  be the maps which carry a tile to the neighbor, adjacent to the side labeled by the letter  $L, M$  or  $S$ , respectively. We follow the labeling rules shown on Figure 3.

It follows from the matching rules that the maps  $L, M$  and  $S$  are involutions, since common sides of tiles are labeled by the same letters. Therefore, the maps are invertible, and they give us an action of the free product  $F = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$  of three cyclic groups of order two on the set  $\mathcal{P}$ . Then the orbits of the action of  $F$  on  $\mathcal{P}$  are in one-to-one correspondence with the Penrose tilings.

The fact that the action of  $F$  on  $\mathcal{P}$  is continuous can be easily deduced from the properties of the Penrose tilings, but this also directly follows from the formulæ defining  $L, M$  and  $S$ , written below:

**Theorem 3.1.** *The transformations  $L, M, S$  act on  $\mathcal{P}$  according to the following rules:*

$$\begin{array}{ll}
(aw)^S = cw & (aaw)^L = b \cdot (aw)^S \\
(bw)^S = b \cdot (w)^M & (abw)^L = a \cdot (bw)^M \\
(cw)^S = aw & (acw)^L = a \cdot (cw)^M \\
& (bbw)^L = b \cdot (bw)^S \\
(aw)^M = a \cdot (w)^L & (bcw)^L = a \cdot (cw)^S \\
(bw)^M = cw & (cw)^L = c \cdot (w)^L \\
(caw)^M = c \cdot (aw)^M & \\
(cbw)^M = bbw & \\
(ccw)^M = bcw & 
\end{array}$$

Let us now define partial homeomorphisms  $S_x, M_x, L_x$ , respectively, equal to the restriction of  $S, M$  or  $L$  onto the set  $x\mathcal{P} \cap \mathcal{P}$  when  $x$  traverses  $\{a, b, c\}$ . Let  $G$  be the inverse semigroup generated by the obtained partial homeomorphisms. Then the homeomorphisms  $S, M$  and  $L$  are equal to the sums  $S_a + S_b + S_c, M_a + M_b + M_c$  and  $L_a + L_b + L_c$  respectively. By  $\mathbb{T}_x, x \in \{a, b, c\}$  we denote, as usual, the partially defined transformations  $\mathbb{T}_x : w \mapsto xw$ . Let  $P_x = \mathbb{T}_x^{-1}\mathbb{T}_x$  be the projection onto the cylindrical set  $xX^\omega \cap \mathcal{P}$ . We have  $1 = P_a + P_b + P_c$ .

In this notation the formulæ from Theorem 3.1 can be rewritten as:

$$\begin{array}{ll}
\mathbb{T}_a S_a = \mathbb{T}_c & \mathbb{T}_a M_a = L\mathbb{T}_a \\
\mathbb{T}_b S_b = (M_b + M_c)\mathbb{T}_b & \mathbb{T}_b M_b = \mathbb{T}_c \\
\mathbb{T}_c S_c = \mathbb{T}_a & \mathbb{T}_c M_c = M_a\mathbb{T}_c + (P_b + P_c)\mathbb{T}_b \\
\mathbb{T}_a L_a = S_a\mathbb{T}_b + (M_b + M_c)\mathbb{T}_a & \\
\mathbb{T}_b L_b = S_b\mathbb{T}_b + S_c\mathbb{T}_a & \\
\mathbb{T}_c L_c = L\mathbb{T}_c. & 
\end{array}$$

These are the only non-zero products of a generator with a transformation of the form  $\mathbb{T}_x$ .

**Corollary 3.2.** *The inverse semigroup  $G = \langle L_x, M_x, S_x, P_x : x = a, b, c \rangle$  is self-similar.*

Suppose that we have a Penrose tiling with non-trivial symmetry. Let  $u$  be a tile adjacent to the symmetry axes. Then either  $u^M = u$ , or  $u^L = u$ , or  $u^S = u$ . A careful analysis of the formulæ shows that in the first case  $u$  has to be of the form  $v_0 c^{n_1} v_1 c^{n_2} v_2 c^{n_3} v_3 \dots$ , where  $v_0$  is equal either to  $ca$  or to  $a$ , the  $v_i$  for  $i \geq 1$  belong to the set  $\{aca, bbca\}$ , and  $n_i \geq 1$  arbitrary integers. In the second case  $u$  is of the form  $c^{n_1} v_1 c^{n_2} v_2 c^{n_3} v_3 \dots$  and in the third case it has the form  $bbcac^{n_1} v_1 c^{n_2} v_2 c^{n_3} v_3 \dots$ , with the same conditions on the words  $v_i$ .

Thus the set of the tiles in the Penrose tilings with a non-trivial symmetry is uncountable, although it is nowhere dense.

All these tilings have a symmetry group of order two (i.e., they have only one non-trivial symmetry), except for a unique tiling with symmetry group  $D_5$ . It is the tiling containing the tile with the code  $cacaca \dots$ . This code is fixed by  $M$  and by  $L$ . See the monograph [GS87] for more details about this exceptional tiling.

### 3.7 Semigroups and groups of self-similarities

The self-similarities also generate interesting (semi)group actions.

If  $\langle \{F_v\}_{v \in V}, \{\phi_e\}_{e \in E} \rangle$  is a self-similarity structure on a set  $F$ , then the functions  $\phi_e$  generate a semigroup of *self-similarities*, or an inverse semigroup of self-similarities, if we take also the inverses of the maps  $\phi_e$ .

**Example.** Recall that the middle-thirds Cantor set  $C$  from page 9 is self-similar with respect to the self-similarity structure  $\{\phi_0(x) = \frac{1}{3}x, \phi_1(x) = \frac{1}{3}x + \frac{2}{3}\}$ . The semigroup  $G$  generated by the transformations  $\phi_0$  and  $\phi_1$  acts on the Cantor set  $C$ . It is easy to see that the semigroup  $G$  is free.

More generally, if we have an iterated function system  $\langle F, \{\phi_i\} \rangle$  such that the sets  $\phi_i(F) \subset F$  have disjoint subsets, then the semigroup generated by the maps  $\phi_i$  will be free.

As another class of examples, consider the actions which locally coincide with the self-similarities. More precisely: let  $H_0$  be the inverse semigroup of self-similarities. Let  $H$  be the semigroup consisting of all sums of the elements of the semigroup  $H_0$ . We say that a group  $G$  acts on  $F$  by *piecewise self-similarities*, if it is a subgroup of the pseudogroup  $H$ .

**The Thompson groups.** The maximal group acting by piecewise self-similarities on the full shift space  $X^\omega$  is called the *Higman-Thompson group*.

The elements of the Higman-Thompson group  $V_d$  are the transformations of the set  $X^\omega$  (for  $|X| = d$ ) defined by *tables* of the form

$$\begin{pmatrix} v_1 & v_2 & \dots & v_m \\ u_1 & u_2 & \dots & u_m \end{pmatrix},$$

where  $m \in \mathbb{N}$  and  $v_i, u_i \in X^*$  are finite words such that for every infinite word  $w \in X^\omega$  exactly one word  $v_i$  and exactly one word  $u_j$  is a beginning of  $w$ . The transformation  $\tau$  defined by the above table acts on the infinite words by the rule

$$(v_i w)^\tau = u_i w,$$

where  $w$  is an arbitrary infinite word over the alphabet  $X$ .

Denote by  $V'_d = [V_d, V_d]$  the commutant of the group  $V_d$ . The following theorem holds (see [Hig74, CFP96, Tho80]).

**Theorem 3.3.** *For every  $d > 1$  the Higman-Thompson group  $V_d$  is finitely presented. For even  $d$  the groups  $V_d$  and  $V'_d$  coincide. For odd  $d$  the group  $V'_d$  is of index 2 in  $V_d$ .*

*The group  $V'_d$  is the only non-trivial normal subgroup of  $V_d$  and is simple.*

The group  $V_2$  and its analogs were constructed by R. Thompson in 1965. These groups were used in [MT73] to construct an example of a finitely presented group with unsolvable word problem and in [Tho80] for embeddings of groups into finitely presented simple groups.

The maximal group of piecewise self-similarities of the segment  $[0, 1]$  (with respect to the iterated function system  $\{x \mapsto x/2, x \mapsto x/2 + 1/2\}$ ) is also called the Thompson group, and is denoted  $F$ . It is the group of the piecewise linear continuous transformations of the segment  $[0, 1]$ , which are differentiable in every point, except for a finite number of points of the form  $m/2^n$ , ( $n \geq 1, 0 \leq m \leq 2^n$ ) and such that the derivative in all the points where it exists is an integral power of 2.

The group  $F$  is a finitely presented group. Its commutant is simple and  $F/F'$  is isomorphic to the group  $\mathbb{Z}^2$ . The group  $F$  does not contain a free subgroup and has no group laws, i.e., no identities of the form  $w(x_1, x_2, \dots, x_n) = 1$ , which are true for any substitution of the group elements into the variables  $x_1, x_2, \dots, x_n$ . Here  $w(x_1, x_2, \dots, x_n)$  is a nonempty freely reduced word in the alphabet  $x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_n, x_n^{-1}$ .

See the survey [CFP96] for more properties of the Thompson groups.

**The group associated with the Apollonian net.** We use here the notations from the definition of the Apollonian net in Subsection 3.1, page 10.

Let  $C_i$ ,  $i = 1, \dots, 4$  denote the circle circumscribed around the triangle  $T_i$ . The circle  $C_i$  is orthogonal to the circles forming the triangle  $T_i$  (in other words, it is orthogonal to the sides of the triangle  $T_i$ ).

Let  $\gamma_i$  be the inversion of the Riemann sphere with respect to the circle  $C_i$ . It is easy to deduce from the definition of the Apollonian net that it is invariant under each of the transformations  $\gamma_i$ , so it is invariant under the action of the group  $\Gamma$  generated by the set  $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ .

It is also easy to see that  $\gamma_i$  maps the gasket  $\mathcal{T}_i$  onto the union of the other three gaskets  $\mathcal{T}_j$ ,  $j \neq i$ . On the other hand, the union of the sets  $\mathcal{T}_j$  for  $j \neq i$  is mapped by  $\gamma_i$  onto  $\mathcal{T}_i$  and each of the sets  $\mathcal{T}_j$  is mapped onto one of the three smaller triangles forming  $\mathcal{T}_i$ . Thus the inversions  $\gamma_i$  restricted onto the respective triangles  $\mathcal{T}_i$  form a self-similarity structure on the Apollonian gasket  $\mathcal{P}$ .

Therefore, we can define the encoding of the points of the net using this self-similarity structure. Let  $X = \{1, 2, 3, 4\}$  be our alphabet. Recall that the encoding is defined in such a way that if  $x$  is a point of the Apollonian net  $\mathcal{P}$  then the first letter of the code is the index  $i_1$  such that  $x \in \mathcal{T}_{i_1}$ . The next letters of the code are defined by the condition that if the word  $i_1 v \in X^*$  is the code of the point  $x$  then the word  $v$  must be the code of the point  $\gamma_{i_1}(x)$ . Then the set of the points whose code starts with a given word of length  $n$  is one of the  $4 \cdot 3^{n-1}$  triangles constructed on the  $n$ th stage of the definition of the Apollonian net. It then follows that there exists not more than one point of the net having a given code  $w \in X^\omega$ . It also follows from the construction that a word  $w \in X^\omega$  is a code of a point of  $\mathcal{P}$  if and only if it has no equal consecutive letters.

We immediately conclude that the action of the generators  $\gamma_i$  of the group  $\Gamma$  is the following:

$$\begin{aligned} (iw)^{\gamma_i} &= w; \\ (w)^{\gamma_i} &= iw \quad \text{if the first letter of } w \text{ is not } i. \end{aligned}$$

It then easily follows that the group  $\Gamma$  is isomorphic to the free product of four groups of order 2; the set of finite-length codes is a principal  $\Gamma$ -space through the natural identification  $i_1 i_2 \dots i_n \leftrightarrow \gamma_{i_1} \gamma_{i_2} \dots \gamma_{i_n}$ .

The above formulæ also imply that the action of the group  $\Gamma$  is minimal, i.e. that the closure of every  $\Gamma$  orbit is the whole Apollonian net.

## 4 Automata groups and actions on rooted trees

### 4.1 Automata

Equation (3) in Subsection 3.3 can be interpreted as a work of the machine, which being in a state  $g$  and receiving as input a letter  $x$ , goes into state  $h$  and outputs the letter  $y$ . Such a machine is formalized by the following definition:

**Definition 4.1.** Let  $X$  be an alphabet. An *automaton* (*transducer*)  $\mathcal{A}$  over an alphabet  $X$  is a triple  $\langle Q, \lambda, \pi \rangle$ , where

1.  $Q$  is a set (*the set of the internal states* of the automaton  $\mathcal{A}$ );
2.  $\lambda : Q \times X \rightarrow X$  is a map, called *the output function* of the automaton;
3.  $\pi : Q \times X \rightarrow Q$  is a map, called *the transition function* of the automaton.

An automaton is *finite* if the set  $Q$  is finite.

A subset  $S \subseteq Q$  is a *subautomaton* of  $\mathcal{A}$  if for all  $s \in S, x \in X$  the state  $\pi(s, x)$  belongs to  $S$ .

For a general theory of automata see [Eil74].

It is convenient to define automata by their diagram (their *Moore diagram*). Such a diagram is a directed labeled graph with vertices identified with the states of the automaton. For every state

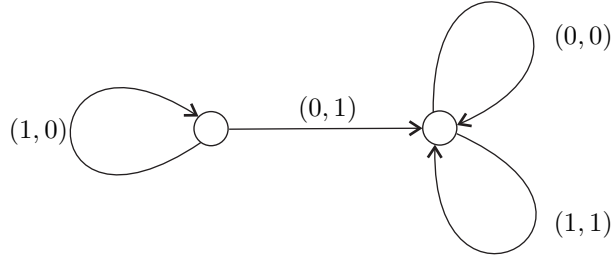


Figure 7: A Moore diagram

$q$  and letter  $x \in X$  the diagram has an arrow from  $q$  to  $\pi(q, x)$  labeled by the pair  $(x, \lambda(q, x))$ . An example of Moore diagram is given on Figure 7.

We interpret the automaton as a machine, which being in a state  $q$  and reading on the input tape a letter  $x$ , goes to the state  $\pi(q, x)$ , types on the output tape the letter  $\lambda(q, x)$ , then moves both tapes to the next position and proceeds further.

In this way, we get a natural action of the automaton on the words. Namely, we extend the functions  $\lambda$  and  $\pi$  to  $Q \times X^*$  in the following way:

$$\pi(q, \emptyset) = q \quad \pi(q, xv) = \pi(\pi(q, x), v), \quad (6)$$

$$\lambda(q, \emptyset) = \emptyset \quad \lambda(q, xv) = \lambda(q, x)\lambda(\pi(q, x), v). \quad (7)$$

Equations (7) also define uniquely a map  $\lambda : Q \times X^\omega \rightarrow X^\omega$ .

Therefore every automaton with a fixed initial state  $q$  (such automata are called *initial*) defines a transformation  $\lambda(q, \cdot)$  on the sets of finite and infinite words  $X^*$  and  $X^\omega$ . *In fine*, we get a transformation of the set  $X^* \cup X^\omega$  which will be written  $\mathcal{A}_q$ . The image of a word  $x_1x_2 \dots$  under the map  $\mathcal{A}_q$  can be easily found using the Moore diagram of the automaton. One must find a directed path starting in state  $q$  with consecutive labels  $(x_1, y_1), (x_2, y_2), \dots$ . Such a path will be unique and the word  $(x_1x_2 \dots)^{\mathcal{A}_q}$  will be equal to  $y_1y_2 \dots$ .

**Definition 4.2.** A transformation  $g : X^\omega \rightarrow X^\omega$  (or  $g : X^* \rightarrow X^*$ ) is *finite-state* if it is defined by a finite automaton.

The product  $\mathcal{A}_q \cdot \mathcal{B}_p$  of two transformations defined by automata  $\mathcal{A} = \langle Q_1, \lambda_1, \pi_1 \rangle$  and  $\mathcal{B} = \langle Q_2, \lambda_2, \pi_2 \rangle$ , with respective initial states  $q \in Q_1$  and  $p \in Q_2$ , is defined as the *composition*  $\mathcal{A} * \mathcal{B}$  of the automata with initial state  $(q, p)$ . The composition  $\mathcal{A} * \mathcal{B}$  is defined as the automaton  $\langle Q_1 \times Q_2, \lambda, \pi \rangle$ , where

$$\begin{aligned} \lambda((s, r), x) &= \lambda_2(r, \lambda_1(s, x)); \\ \pi((s, r), x) &= (\pi_1(s, x), \pi_2(r, \lambda_1(s, x))). \end{aligned}$$

Thus the product of two finite-state transformations is again a finite-state transformation.

An automaton is *invertible* if each of its states defines an invertible transformation of the set  $X^\omega$  (or equivalently of the set  $X^*$ ). This condition holds if and only if for every state  $q$  the transformation  $\lambda(q, \cdot) : X \rightarrow X$  (i.e., the transformation  $\mathcal{A}_q$  restricted onto the set  $X^1 \subset X^*$ ) is invertible (see [Eil74, Sus98, GNS00]).

If the automaton  $\mathcal{A} = \langle Q, \lambda, \pi \rangle$  is invertible, then the inverse transformation to the transformation  $\mathcal{A}_q$  is defined as the *inverse automaton*  $\mathcal{A}^* = \langle Q, \lambda^*, \pi^* \rangle$  with initial state  $q$ , where

$$\begin{aligned} \lambda^*(s, x) &= y \\ \pi^*(s, x) &= \pi(s, y); \end{aligned}$$

here  $y \in X$  is such that  $\lambda(s, y) = x$  (such a  $y$  exists and is uniquely defined, since the automaton  $\mathcal{A}$  is invertible).

It also follows from these formulæ that if an invertible transformation is finite-state, then its inverse is also finite-state.

For more facts on automatic transformations see [GNS00, Sid98, Sus98].

## 4.2 The complete automaton of a self-similar action

Suppose that the group  $G$  acts by a self-similar action on the space  $X^\omega$ . Then this action defines an automaton over the alphabet  $X$  with the set of states  $G$  with the output and the transition functions  $\lambda$  and  $\pi$  defined in such a way that

$$g(xw) = \lambda(g, x)w^{\pi(g, x)}$$

for all  $w \in X^\omega$ .

The obtained automaton  $\mathcal{A}$  has the property that the transformation  $\mathcal{A}_g$  coincides with the action of the element  $g$ . The automaton  $\mathcal{A}$  is called the (*complete*) *automaton* of the self-similar action.

Therefore the notion of a self-similar action of a group  $G$  on the space  $X^\omega$  can be defined in terms of the automata theory in the following way:

**Definition 3.4''.** An action of a group  $G$  on the space  $X^\omega$  is *self-similar* if there exists an automaton  $\mathcal{A} = \langle G, \lambda, \pi \rangle$  such that

$$w^{\mathcal{A}_g} = w^g$$

for all  $w \in X^\omega$ .

We have the following obvious characterization of the automata which are associated with the self-similar actions:

**Proposition 4.1.** *Let  $G$  be a group. An automaton  $\mathcal{A} = \langle G, \lambda, \pi \rangle$  is associated with a self-similar action of the group  $G$  on the space  $X^\omega$  if and only if  $\mathcal{A}_1$  is the identical transformation (here 1 is the identity of the group  $G$ ) and for all  $g_1, g_2 \in G$  we have*

$$\mathcal{A}_{g_1} \mathcal{A}_{g_2} = \mathcal{A}_{g_1 g_2}.$$

## 4.3 Groups generated by automata

The complete automaton of an action is infinite for infinite groups, so it is not very convenient to define a group by its complete automaton. A better way is to define the group *generated* by an automaton in the following way.

Let  $\mathcal{A} = \langle Q, \lambda, \pi \rangle$  be an invertible automaton. Then every transformation  $\mathcal{A}_q$  defined by an initial state  $q \in Q$  will be invertible. The group (or the group action) generated by the transformations  $\mathcal{A}_q$  is called the *group generated by the automaton  $\mathcal{A}$*  and is denoted  $G(\mathcal{A})$ .

**Proposition 4.2.** *An action of a group on the set  $X^\omega$  is self-similar if and only if it is generated by an automaton.*

The groups of the form  $G(\mathcal{A})$ , where  $\mathcal{A}$  is a finite automaton, are the most interesting; we give here several examples. These groups were introduced in Subsection 3.4. More examples will be given in Subsection 5.2.

**The adding machine** action of the cyclic group is the action generated by the automaton shown on Figure 7. Namely, the state, corresponding to the left vertex of the Moore diagram defines the adding machine transformation  $a$ . The state corresponding to the right vertex defines the trivial transformation. Thus the group defined by the automaton is the adding machine action of the group  $\mathbb{Z}$ .

**The dihedral group** is generated by the automaton shown on Figure 8.

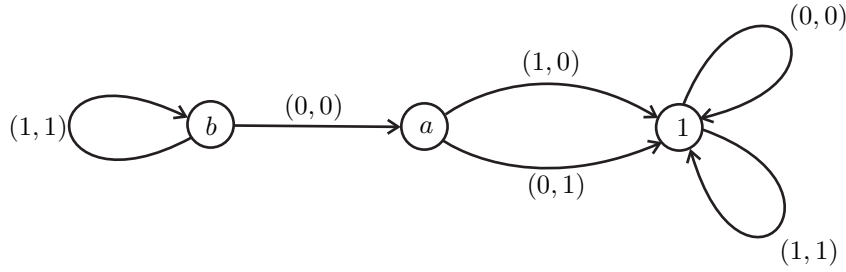


Figure 8: An automaton generating  $\mathbb{D}_\infty$

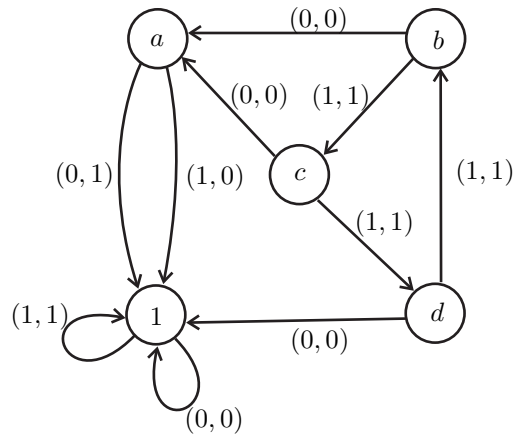


Figure 9: The automaton generating the Grigorchuk group

**The Grigorchuk group** is defined by the automaton shown on Figure 9.

#### 4.4 Trees

A *tree* is a connected simplicial graph without cycles.

We consider only *locally finite* trees, i.e., trees in which every vertex has a finite degree.

A rooted tree is a tree with a fixed vertex called its *root*. An isomorphism of rooted trees  $f : T_1 \rightarrow T_2$  is an isomorphism of the trees which maps the root of the tree  $T_1$  to the root of the tree  $T_2$ .

The vertices of a rooted tree are naturally partitioned into *levels*. If the distance between a vertex  $v$  and the root is equal to  $n$  then we say that the vertex belongs to the  $n$ th level. In particular, the 0th level contains only the root.

A rooted tree is *spherically homogeneous* (or *isotropic*) if all the vertices belonging to the same level have the same degree.

A spherically homogeneous tree is uniquely defined (up to isomorphism) by its *spherical index*. This is the sequence  $(m_0, m_1, \dots)$ , where  $m_0$  is the degree of the root of the tree and  $m_n + 1$  is the degree of the vertices of the  $n$ th level.

The regular  $n$ -ary tree is the spherically homogeneous tree with the spherical index  $(n, n, n, \dots)$ .

**Example.** The set  $X^*$  of finite words over the alphabet  $X$  is a vertex set of a naturally-defined rooted tree. Namely, the root of this tree is the empty word  $\emptyset$ , and two words are connected by an edge if and only if they are of the form  $v$  and  $vx$  for some  $x \in X$  and  $v \in X^*$ . We denote this rooted tree by  $T(X)$ .

The rooted tree  $T(X)$  is spherically homogeneous with spherical index  $(d, d, d, \dots)$ , where  $d = |X|$ . Any regular  $d$ -ary rooted tree is isomorphic to the tree  $T(X)$ .

The  $n$ th level of the tree  $T(X)$  coincides with the set  $X^n$ .

An *end* of a rooted tree  $T$  is an infinite sequence of pairwise different vertices (an infinite simple path)  $v_0, v_1, v_2, \dots$  such that  $v_0$  is the root of the tree and for every  $i$ , the vertices  $v_i$  and  $v_{i+1}$  are adjacent. The vertex  $v_n$  will then belong to the  $n$ th level of the tree.

The *boundary*  $\partial T$  of the tree  $T$  is the set of all of its ends. For every vertex  $v$  denote by  $\partial T_v$  the set of all the ends passing through  $v$ . The sets  $\partial T_v$  form a basis of neighborhoods for the natural topology on  $\partial T$ . In this topology the space  $\partial T$  is totally discontinuous and compact.

Every automorphism of a rooted tree  $T$  acts naturally on its boundary and it directly follows from the definitions that it acts on  $\partial T$  by homeomorphisms.

In the case of the tree  $T(X)$ , every end has the form  $(\emptyset, x_1, x_1x_2, \dots)$ , and can be identified with the infinite word  $x_1x_2\dots \in X^\omega$ . Thus the boundary  $\partial T(X)$  is naturally identified with the space  $X^\omega$ . It is easy to see that this identification agrees with the topologies on these sets, since  $\partial T(X)_v$  is identified with the cylindrical set  $vX^\omega$ .

## 4.5 Action on rooted trees

It follows from the definition of the action of an initial automaton  $\mathcal{A}_q$  on finite words that for any  $v \in X^*$ ,  $x \in X$  we have  $(vx)^{\mathcal{A}_q} = v^{\mathcal{A}_q}y$  for some  $y \in X$ . Thus the adjacent vertices of the tree  $T(X)$  are mapped onto adjacent vertices. Therefore, if the transformation  $\mathcal{A}_q$  is invertible, then  $\mathcal{A}_q$  defines an automorphism of the rooted tree  $T(X)$ .

More generally, the following holds:

**Proposition 4.3.** *A bijection  $f : X^* \rightarrow X^*$  is defined by an automaton if and only if it induces an automorphism of the rooted tree  $T(X)$ .*

*The action of an automorphism of the rooted tree  $T(X)$  on the boundary  $\partial T(X) = X^\omega$  coincides with the action of the respective initial automaton on the space  $X^\omega$ .*

Let us denote by  $\Sigma(X)$  the symmetric group of permutations of the set  $X$ . Every permutation  $\alpha \in \Sigma(X)$  can be extended to an automorphism of the whole tree  $T(X)$  in a standard way:

$$(xw)^\alpha = x^\alpha w,$$

where  $w \in X^*$  is arbitrary. This extension defines a canonical embedding of the symmetric group  $\Sigma(X)$  into the automorphism group  $\text{Aut}T(X)$  of the tree  $T(X)$ .

Suppose  $g$  is an automorphism of the tree  $T(X)$ . It fixes the root  $\emptyset$  and it permutes the first level  $X^1$  of the tree. Let  $\alpha_\emptyset \in \Sigma(X)$  be the permutation of the set  $X^1$  induced by  $g$ . Then the automorphism  $g\alpha_\emptyset^{-1} \in \text{Aut}T(X)$  fixes the points of the first level (here  $\alpha_\emptyset$  is identified with its canonical extension).

**Definition 4.3.** The set of all automorphisms which fix pointwise the first level is a subgroup called the *first level stabilizer* and is written  $\text{St}_1$ . It is a normal subgroup of index  $n!$  in the group  $\text{Aut}T(X)$ .

In general, the  *$n$ th level stabilizer*  $\text{St}_n$  is the subgroup of those elements of the  $\text{Aut}T(X)$ , which fix all the elements of the  $n$ th level  $X^n$  of the tree  $T(X)$ . The  $n$ th level stabilizer is also a normal subgroup of finite index and the quotient  $\text{Aut}T(X)/\text{St}_n$  is isomorphic to the automorphism group of the finite rooted subtree of  $T(X)$  consisting of the first  $n + 1$  levels.

Let  $G$  be a subgroup of  $\text{Aut}T(X)$ . We say  $G$  is *recurrent* if  $G \cap \text{St}_n$  is a subgroup of  $G^X$ , where the latter group is a direct product of  $|X|$  copies of  $G$  acting independently on the level-1 subtrees of  $T(X)$ .

The first level stabilizer is isomorphic to the direct product of  $d = |X|$  copies of the group  $\text{Aut}T(X)$ . Namely, every automorphism  $g \in \text{St}_1$  acts independently on the  $d$  subtrees  $T_x$ ,  $x \in X$ , where  $T_x$  is the subtree rooted at  $x$ , with the set of vertices  $xX^*$ . Every subtree  $T_x$  is isomorphic to the whole tree  $T(X)$ , with the isomorphism given by restriction of the shift  $s$  to the set  $xX^*$ .

Let us consider the restriction of the automorphism  $g$  to the subtree  $T_x$  and conjugate it with the isomorphism  $s : T_x \rightarrow T(X)$  to get an automorphism of the whole tree  $T(X)$ . Let us denote the obtained automorphism by  $g|_x$ . It is easy to see that we have

$$(xw)^g = x(w)^{g|_x}$$

for all  $w \in X^*$ . Thus this notion of restriction agrees with the one introduced before for self-similar groups.

**Notation.** Set  $X = \{x_1, x_2, \dots, x_d\}$ . Write  $g|_{x_i} = g_i$ . We have a map  $\Psi : g \mapsto (g_1, g_2, \dots, g_d) : \text{St}_1 \rightarrow \text{Aut}T(X)^d$ . It is easy to see that the map  $\Psi$  is an isomorphism.

Identifying the first level stabilizer with the direct product  $\text{Aut}T(X)^d$  we write  $g = (g_1, g_2, \dots, g_d)$ . In general, every element of the group  $\text{Aut}T(X)$  can be written as a product  $(g_1, g_2, \dots, g_d)\alpha_\emptyset$  of an element of the stabilizer and an element of the symmetric group  $\Sigma(X)$ .

Therefore, the group  $\text{Aut}T(X)$  is isomorphic to a semi-direct product  $\text{Aut}T(X)^X \rtimes \Sigma(X)$  with the natural action of  $\Sigma(X)$  on the multiples of the direct product  $\text{Aut}T(X)^X$ .

**Definition 4.4.** Let  $H$  be a group and let  $G$  be a group acting on a set  $M$ . A (*permutational wreath product*)  $H \wr G$  is the semi-direct product  $H^M \rtimes G$ , where  $G$  acts on  $H^M$  by the permutations of the direct multiples coming from its action on  $M$ .

Permutational wreath products are also called *non-standard wreath products*. The standard wreath product  $H \wr G$  is the permutational wreath product respectively to the regular action on  $G$  on itself by right multiplication.

Hence, the automorphism group  $\text{Aut}T(X)$  is isomorphic to the permutational wreath product  $\text{Aut}T(X) \wr \Sigma(X)$ .

We can proceed further in this manner. Denote by  $\alpha_x$  the permutation of the points of the first level, defined by the automorphism  $g|_x$ . Then  $g|_x = (g|_{xx_1}, g|_{xx_2}, \dots, g|_{xx_d})\alpha_x$ . In general, denote by  $\alpha_v \in \Sigma(X)$  the permutation of the first level of the tree, defined by the restriction  $g|_v$ . In this way we get a labeling of the tree  $T(X)$ , where each vertex  $v$  is labeled by a permutation  $\alpha_v$  from  $\Sigma(X)$ . This labeled tree is called the *portrait* of the automorphism  $g$ .

The portrait defines the automorphism uniquely. The action of an automorphism  $g$  is computed via its portrait using the formula

$$(a_1 a_2 \dots)^g = (a_1)^{\alpha_\emptyset} (a_2)^{\alpha_{a_1}} (a_3)^{\alpha_{a_1 a_2}} (a_4)^{\alpha_{a_1 a_2 a_3}} \dots$$

Let us recall here the definition of branch and weakly branch groups; for more details see [Gri00].

**Definition 4.5.** Let  $G$  be a level-transitive automorphism group of the rooted tree  $T(X)$ . The *rigid stabilizer*  $\text{Rist}_G(v)$  of a vertex  $v \in X^*$  in the group  $G$  is the set of all elements of the group  $G$  which fix all the vertices of  $T(X)$ , except perhaps the vertices of the form  $vu$ ,  $u \in X^*$ . The  *$n$ th level rigid stabilizer*  $\text{Rist}_G(n)$  of the group  $G$  is the subgroup generated by the rigid stabilizers  $\text{Rist}_G(v)$  of all the vertices of the  $n$ th level (i.e., all the vertices  $v \in X^n$ ).

The group  $G$  is *weakly branch* if none of its rigid stabilizers  $\text{Rist}_G(v)$  is trivial. It is *branch* if  $\text{Rist}_G(n)$  has finite index in  $G$  for all  $n \in \mathbb{N}$ . The group is *tough* if it is not weakly branch.

Furthermore, we say  $G$  is *regular (weakly) branch* if there is a subgroup  $K < G$  such that  $K^X < K$  as a geometric embedding induced by restriction to the level-1 subtrees in  $T(X)$ , and  $K$  has finite index in  $G$  (respectively, is non-trivial).

If the group is weakly branch, then all of its rigid stabilizers are infinite. If it is tough, then for all  $n$  big enough the  $n$ th level rigid stabilizer is trivial.

## 5 Iterated monodromy groups

### 5.1 Definitions and main properties

Here we present a class of examples of self-similar group actions, which are associated with topological dynamical systems. These examples show the close relation between the classical self-similar

fractals, like the Julia sets of polynomials and the self-similar group actions. We will show later how the Julia set of a rational function can be reconstructed from its iterated monodromy group (Subsection 9.3).

Let  $M_1$  and  $M_2$  be arcwise connected and locally arcwise connected topological spaces. A continuous map  $f : M_1 \rightarrow M_2$  is a *local homeomorphism* at the point  $p \in M_2$  if there exists a neighborhood  $U$  of the point  $p$  such that  $f^{-1}(U)$  is a disjoint union of sets  $U_i$  such that  $f : U_i \rightarrow U$  is a homeomorphism.

We have the following classical result (see [Mas91]):

**Lemma 5.1.** *Let  $f : M_1 \rightarrow M_2$  be a local homeomorphism in every point of an arcwise connected open set  $U \subset M_2$ . Let  $\gamma$  be a continuous path in  $U$  starting in a point  $p \in U$ . Then for every  $x \in M_1$  such that  $f(x) = p$  there exists a unique path  $\gamma'$  in  $M_1$  starting in  $x$  such that  $f(\gamma') = \gamma$ .*

Suppose  $M$  is an arcwise connected and locally arcwise connected topological space and let  $f : M \rightarrow M$  be a branched  $d$ -fold self-covering. In other words, there exists a subset  $R \subset M$  (called the *set of branching points*) such that for every point  $x \in M \setminus R$  the map  $f$  is a local homeomorphism in  $x$ .

By  $f^n$  we denote the  $n$ th iterate of the mapping  $f$ . Let  $P = \overline{\bigcup_{n=0}^{\infty} f^n(R)}$  be the closure of the union of the forward orbits of the branching points. The set  $P$  is called the *set of postcritical points*. Then every preimage of a point from  $M \setminus P$  also belongs to  $M \setminus P$ , and  $f$  is a local homeomorphism at every point of  $M \setminus P$ .

An important example of a branched covering is the branched covering of the Riemann sphere  $M = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  defined by a rational function  $f$ . The map  $f$  is a  $d$ -fold branched covering, for  $d$  equal to the degree of the rational function, i.e.,  $d = \max(\deg p, \deg q)$ , where  $p, q \in \mathbb{C}[z]$  are such that  $p/q = f$  is a reduced fraction. The set of branching points is in this case the set of *critical values* of the rational function  $f$ , i.e., the values of the function at the critical points.

We impose throughout this section the condition that the  $d$ -fold branched covering  $f : M \rightarrow M$  is such that the space  $M \setminus P$  is arcwise connected. If furthermore  $P$  is finite, we say  $f$  is *postcritically finite*. See the paper [DH93] and Appendix B of the book [McM95] for an interesting Thurston's criterion for a postcritically finite branched covering of a sphere to be defined by a rational function.

Let  $t \in M \setminus P$  be an arbitrary point. Then it has  $d^n$  preimages under  $f^n$  for every  $n \in \mathbb{N}$ . Denote by  $T$  the formal disjoint union of the sets  $f^{-n}(t)$  for  $n \geq 0$ , (where  $f^{-0}(t) = \{t\}$  and  $f^{-n}(t)$  denotes the preimage of the point  $t$  under the map  $f^n$ ). Formally the set  $T$  can be defined as

$$T = \bigcup_{n=0}^{\infty} f^{-n}(t) \times \{n\},$$

so that its element  $(z, n)$  is the point  $z$  seen as an element of the set  $f^{-n}(t)$ .

The set  $T$  is a vertex set of a naturally-defined  $d$ -regular rooted tree with the root  $(t, 0)$  in which a vertex  $(z, n) \in T$  is connected with the vertex  $(f(z), n - 1)$ . Let us call the tree  $T$  the *preimages tree* of the point  $t$ .

Let now  $\gamma$  be a loop in  $M \setminus P$  based at  $t$ , i.e., a path starting and ending at  $t$ . Then, by Lemma 5.1, for every element  $v = (z, n)$  of the preimage tree  $T$  there exists a single path  $\gamma_v$ , which starts in  $z$  and is such that  $f^n(\gamma_v) = \gamma$ . Denote by  $z^\gamma$  the end of the path  $\gamma_v$ . Then we obviously have  $f^n(z^\gamma) = t$ , so the element  $v^\gamma = (z^\gamma, n)$  also belongs to the preimage tree.

We have the following proposition:

**Proposition 5.2.** *The map  $v \mapsto v^\gamma$  is an automorphism of the preimage tree, which depends only on the homotopy class of  $\gamma$  in  $M \setminus P$ . The set of all such automorphisms is a group, which is a quotient of the fundamental group of the space  $M \setminus P$ . Up to an isomorphism, this group does not depend on the choice of the base point  $t$ .*

**Definition 5.1.** The group from Proposition 5.2 is called the *iterated monodromy group (i.m.g.)* of the map  $f$ , and is written  $IMG(f)$ .

The term *iterated monodromy group* comes from the fact that the quotient of this group by the stabilizer of the  $n$ th level is the monodromy group of the mapping  $f^n$ .

The preimage tree  $T$  is  $d$ -regular, so that it can be identified with the tree  $T(X)$  for an alphabet  $X$  of cardinality  $d$ . There is no canonical identification, but we will define a class of natural identifications using paths in the space  $M \setminus P$ .

Let us take an alphabet  $X$  with  $d$  letters and a bijection  $\Lambda : X \rightarrow f^{-1}(t)$ . For every point  $x \in f^{-1}(t)$  choose a path  $\ell_x$  in  $M \setminus P$ , starting at  $t$  and ending at  $\Lambda(x)$ .

Define a map  $\Lambda : T(X) \rightarrow T$  inductively by the rules:

1.  $\Lambda(\emptyset) = (t, 0)$ ,
2. for every  $n \geq 1$ ,  $v \in X^n$  and  $x \in X$  the point  $\Lambda(xv)$  is  $(z, n + 1)$ , where  $z$  is the end of the path  $\gamma$  which starts at  $\Lambda(v)$  and is such that  $f^n(\gamma) = \ell_x$ .

**Proposition 5.3.** *The constructed map  $\Lambda : T(X) \rightarrow T$  is an isomorphism of the rooted trees.*

**Definition 5.2.** The *standard action* of the i.m.g. on the tree  $T(X)$  is obtained from its action on the preimage tree  $T$  conjugating it by the isomorphism  $\Lambda : T(X) \rightarrow T$ .

**Proposition 5.4.** *The standard action of an iterated monodromy group is self-similar. More precisely, if  $\gamma$  is a loop based at  $t$  and  $x \in X$  is a letter, then, respectively to the standard action, for every  $v \in X^*$  we have*

$$(xv)^\gamma = y \left( v^{\ell_x \gamma_x \ell_y^{-1}} \right), \quad (8)$$

where  $\gamma_x$  is the preimage of  $\gamma$  starting at  $\Lambda(x)$  and  $y$  is such that  $\Lambda(y)$  is the end of  $\gamma_x$  (i.e.,  $x^\gamma = y$ ).

Note, that  $\ell_x \gamma_x \ell_y^{-1}$  is a loop based at  $t$  (see Figure 10).

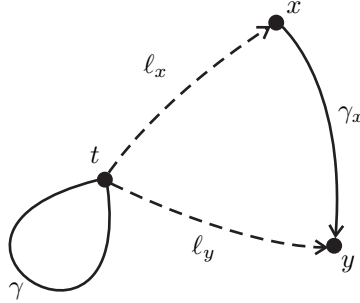


Figure 10: The recurrent formula

In many cases the action of the iterated monodromy group is generated by a finite automaton. This is the case, for instance, when the map  $f$  is expanding with respect to some Riemann metric on  $M$ , or for postcritically finite rational mappings of the complex sphere, as follows from Theorem 9.7.

## 5.2 Examples of iterated monodromy groups

We give in this subsection some examples of iterated monodromy groups. Most of these groups are described for the first time here, and we will only mention their most elementary properties — a more detailed study will be given in [Bar] and [Nekb].

In this section we will use very often the notation from Subsection 4.5 coming from the decomposition of the group  $\text{Aut}T(X)$  into the semidirect product  $\text{Aut}T(X)^d \rtimes \Sigma(X)$ .

**The adding machine.** Let  $\mathbb{T}$  be the circle  $\{z \in \mathbb{C} : |z| = 1\}$ . The map  $f : z \mapsto z^2$  is a two-fold self-covering of the circle  $\mathbb{T}$ . Let us compute the iterated monodromy group of the covering  $f : \mathbb{T} \rightarrow \mathbb{T}$ .

Let us chose the base-point  $t$  equal to 1. It has two preimages: itself and  $-1$ . So we can take  $\ell_1$  equal to the trivial path in the point 1 and  $\ell_{-1}$  equal to the upper semicircle starting at 1 and ending in  $-1$ .

The fundamental group of the circle is the cyclic group generated by the loop  $\tau$  which starts in 1 and goes around the circle once in the positive direction. So it is sufficient to compute the action of the element  $\tau$  on the tree  $T(X)$ . The path  $\tau$  has two preimages. One is the upper semicircle  $\tau_1$  starting in 1 and ending in  $-1$  (i.e., equal to the path  $\ell_{-1}$  defined above), another is the lower semicircle  $\tau_{-1}$  starting in  $-1$  and ending in 1. Thus  $\tau$  acts on the first level of the tree  $T(X)$  by the transposition. So using Equation (8) we get, in notation of Subsection 4.5,

$$\tau = (\ell_1 \tau_1 \ell_{-1}^{-1}, \ell_{-1} \tau_{-1} \ell_1^{-1}) \sigma = (1, \tau) \sigma,$$

since the path  $\ell_1 \tau_1 \ell_{-1}^{-1}$  is trivial and  $\ell_{-1} \tau_{-1} \ell_1^{-1}$  is equal to  $\tau$ . Therefore,  $\tau$  acts on the tree  $T(X)$  as the adding machine.

**The torus.** The above example can be obviously generalized to tori in the following way:

Let  $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$  be the  $n$ -dimensional torus. Let  $A$  be a  $n \times n$ -matrix with integral entries and with determinant equal to  $d > 1$ . Then the linear map  $A$  on the space  $\mathbb{R}^n$  induces an  $d$ -fold (non-branched) self-covering of the torus  $\mathbb{T}^n$ . Since the fundamental group of the torus is the free abelian group  $\mathbb{Z}^n$ , the i.m.g. of such coverings are abelian. More precisely, the following proposition holds.

**Proposition 5.5.** *Let  $A$  be the matrix defining the covering  $f : \mathbb{T}^n \rightarrow \mathbb{T}^n$ . Then the iterated monodromy group of  $f$  is the quotient  $\mathbb{Z}^n / H$ , where*

$$H = \bigcap_{n \geq 1} A^n (\mathbb{Z}^n).$$

The group  $H$  is trivial if and only if no eigenvalue of the matrix  $A^{-1}$  is an algebraic integer (see [NS01] Proposition 4.1 and [BJ99] Proposition 10.1).

The corresponding actions of  $\mathbb{Z}^n$  on rooted trees are studied in Subsection 6.4.

**Chebyshev polynomials and example of S. Lattès.** Consider the Chebyshev polynomials  $T_d(z) = \cos(d \arccos z)$ , satisfying the recursion

$$T_0(z) = 1, \quad T_1(z) = z, \quad T_d(z) = 2zT_{d-1} - T_{d-2}.$$

Then  $T_d$  is an even or odd polynomial of degree  $d$ .

**Proposition 5.6.** *The group  $IMG(T_d)$  is infinite dihedral for all  $d \geq 2$ .*

*Proof.* We have the following commutative diagramm:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{dz} & \mathbb{C} \\ \cos z \downarrow & & \downarrow \cos z \\ \mathbb{C} & \xrightarrow{T_d(z)} & \mathbb{C} \end{array} \quad (9)$$

Let  $t \in (-1, 1)$  be a basepoint not belonging to the postcritical set of  $T_d$ . From the diagram (9) follows that

$$T_d^{-n}(t) = \left\{ \cos \left( \pm \frac{\alpha + 2\pi k}{d^n} \right) : k \in \mathbb{Z} \right\} = \left\{ \cos \left( \frac{\alpha + 2\pi k}{d^n} \right) : k = 0, 1, \dots, d^n - 1 \right\},$$

where  $\alpha = \arccos t$ .

Let  $\gamma$  be an arbitrary loop at  $t$  not passing through the branching points of the functions  $T_d(z)$  and  $\cos z$ . The function  $\cos z$  is even and  $2\pi$ -periodic, thus the preimages of  $\gamma$  under  $\cos z$  are either paths starting at  $\alpha + 2\pi k$  and ending at  $\alpha + 2\pi(k+l)$  and paths starting at  $-\alpha - 2\pi k$  and ending at  $-\alpha - 2\pi(k+l)$ , where  $l$  is fixed and  $k \in \mathbb{Z}$ , or paths starting at  $\alpha + 2\pi k$  and ending at  $-\alpha + 2\pi(k+l)$  and paths starting at  $-\alpha - 2\pi k$  and ending at  $\alpha - 2\pi(k+l)$ , where  $l$  is fixed and  $k \in \mathbb{Z}$ .

This implies that the preimages of  $\gamma$  under  $T_d^n$  either start at  $\cos\left(\frac{\alpha+2\pi k}{d^n}\right) = \cos\left(\frac{-\alpha-2\pi k}{d^n}\right)$  and end at  $\cos\left(\frac{\alpha+2\pi(k+l)}{d^n}\right) = \cos\left(\frac{-\alpha-2\pi(k+l)}{d^n}\right)$ , or they start at  $\cos\left(\frac{\alpha+2\pi k}{d^n}\right) = \cos\left(\frac{-\alpha-2\pi k}{d^n}\right)$  and end at  $\cos\left(\frac{\alpha-2\pi(k+l)}{d^n}\right) = \cos\left(\frac{-\alpha+2\pi(k+l)}{d^n}\right)$ , where  $l$  is fixed and  $k \in \mathbb{Z}$ .

Therefore, the iterated monodromy group  $\text{IMG}(\cdot T_d)$  is isomorphic to the group of affine functions  $\{z + l, -z + l : l \in \mathbb{Z}\}$  under composition. This group is isomorphic to the infinite dihedral group  $\mathbb{D}_\infty$ .  $\square$

An explicit computation shows that the associated standard action of  $\mathbb{D}_\infty$  on the tree  $T(X)$ , for  $d$  odd, is generated by two involutions  $a$  and  $b$ , where

$$a = (a, 1, 1, \dots, 1)\sigma_1, \quad b = (1, 1, \dots, 1, b)\sigma_2,$$

where  $\sigma_1$  is the permutation  $(2, 3)(4, 5) \dots (d-1, d)$  and  $\sigma_2 = (1, 2)(3, 4) \dots (d-2, d-1)$  and for  $d$  even by

$$a = (1, 1, \dots, 1)\sigma_1, \quad b = (a, 1, \dots, 1, b)\sigma_2,$$

where  $\sigma_1 = (1, 2)(3, 4) \dots (d-1, d)$  and  $\sigma_2 = (2, 3)(4, 5) \dots (d-2, d-1)$ . Here the alphabet  $X$  is  $\{1, 2, \dots, d\}$ .

Also related are the following examples of S. Lattès [Lat18]. Let  $\Lambda$  be a lattice in  $\mathbb{C}$ , and let  $\alpha$  be a multiplier of  $\Lambda$ , i.e., some  $\alpha \in \mathbb{C}$  such that  $\alpha\Lambda \subset \Lambda$ . Then  $\mathbb{C}/\Lambda$  is a torus, and the affine function  $\alpha \cdot z$  induces an  $|\alpha|^2$ -fold self-covering of  $\mathbb{C}/\Lambda$ . Note that the iterated monodromy group of this self-covering is isomorphic to  $\Lambda$ , as follows from Proposition 5.5.

The Weierstrass elliptic function

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left[ \frac{1}{(z + \omega)^2} - \frac{1}{\omega^2} \right]$$

is  $\Lambda$ -periodic and even (see for example [Lan87]), so that it we get a well-defined 2-fold covering  $\wp : \mathbb{C}/\Lambda \rightarrow \widehat{\mathbb{C}}$ , branched at the four points  $\frac{1}{2}\Lambda/\Lambda$ . Then the function  $f(z) = \wp(\alpha\wp^{-1}(z))$  is rational of degree  $|\alpha|^2$ . The dynamics of such maps were first studied by S. Lattès [Lat18].

Let  $\Lambda$  be a lattice in  $\mathbb{C}$ , and let  $\alpha$  be a multiplier of  $\Lambda$ , i.e. some  $\alpha \in \mathbb{C}$  such that  $\alpha\Lambda \subset \Lambda$ . Then  $\mathbb{C}/\Lambda$  is a torus, and  $\wp : \mathbb{C}/\Lambda \rightarrow \widehat{\mathbb{C}}$  is a degree-2 covering, branched at the four points  $\frac{1}{2}\Lambda/\Lambda$ . Consider  $f(z) = \wp(\alpha\wp^{-1}(z))$ . Its Julia set is the entire  $\widehat{\mathbb{C}}$ . Then  $f$  is a rational map of degree  $|\alpha|^2$ , and  $\text{IMG}(f)$  is a 4-generated group.

For instance, for  $\alpha = 2$  the function  $f$  is

$$f(z) = \frac{z^4 + \frac{g_2}{2}z^2 + 2g_3z + \frac{g_2^2}{16}}{4z^3 - g_2z - g_3},$$

where  $g_2 = 60s_4$  and  $g_3 = 140s_6$  for  $s_m = \sum_{\omega \in \Lambda, \omega \neq 0} \omega^{-m}$ , see [Bea91] p. 74.

There exists a lattice  $\Lambda$  with given values of  $g_2$  and  $g_3$  if and only if  $g_2^3 - 27g_3^2 \neq 0$  (see [Lan87], p. 39). In particular, there exists a lattice  $\Lambda$  such that  $g_3 = 0$  and  $g_2 = 4$ , so that

$$f(z) = \frac{(z^2 + 1)^2}{4z(z^2 - 1)}. \tag{10}$$

The following proposition is proved in the similar way as Proposition 5.6.

**Proposition 5.7.** *Let  $\Lambda$  be a lattice in  $\mathbb{C}$  and let  $\alpha \in \mathbb{C}$  be such that  $\alpha\Lambda \subset \Lambda$  and  $|\alpha| \neq 1$ . Let a rational function  $f \in \mathbb{C}(z)$  be such that  $\wp(\alpha z) = f(\wp(z))$ . Then the iterated monodromy group  $\text{IMG}(f)$  is isomorphic to the semi-direct product  $\mathbb{Z}^2 \rtimes (\mathbb{Z}/2\mathbb{Z})$ , where  $\mathbb{Z}/2\mathbb{Z}$  acts on  $\mathbb{Z}^2$  by the automorphism  $(a, b) \mapsto (-a, -b)$ .*

$z^2 - 1$ . The critical points of the polynomial  $z^2 - 1$  are  $\infty$  and  $0$  and the postcritical set is  $P = \{0, -1, \infty\}$ .

Choose as a basepoint the fixed point of the polynomial  $t = \frac{1-\sqrt{5}}{2}$ . It has two preimages: itself, and  $-t$ . Choose the path  $\ell_0$  to be trivial path at  $t$  and  $\ell_1$  to be the path, connecting  $t$  with  $-t$  above the real axis, as on the lower part of Figure 11 (the point  $t$  is marked there by a star). Let  $a$  and  $b$  be the generating elements of  $\text{IMG}(z^2 - 1)$ , defined by the small loops going in the positive direction around the points  $-1$  and  $0$ , respectively and connected to the basepoint by straight segments. The loops  $a$  and  $b$  are shown on the upper part of the figure.

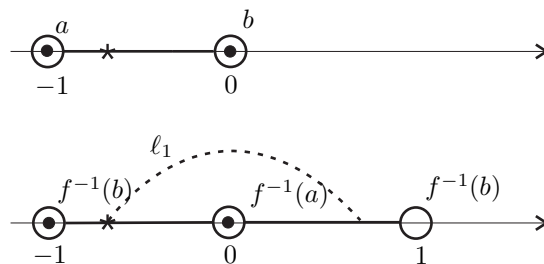


Figure 11: Computation of the group  $\text{IMG}(z^2 - 1)$

The preimages of the loops  $a$  and  $b$  are shown on the lower part of Figure 11. It follows that

$$a = (b, 1)\sigma, \quad b = (a, 1),$$

so that the group  $\text{IMG}(z^2 - 1)$  is generated by the automaton with the Moore diagram shown on Figure 12.

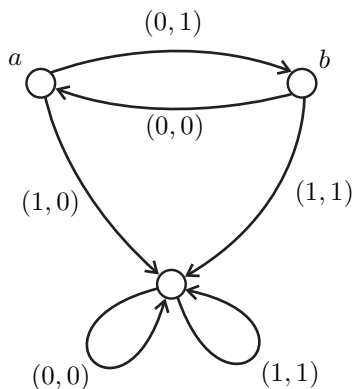


Figure 12: The automaton generating the group  $\text{IMG}(z^2 - 1)$ .

The following properties of the group  $\text{IMG}(z^2 - 1)$  were proved by R. Grigorchuk, A. Żuk and L. Bartholdi.

**Theorem 5.8.** *The group  $\text{IMG}(z^2 - 1)$*

1. *is torsion free;*

2. has exponential growth (actually, the semigroup generated by  $a$  and  $b$  is free);
3. is just non-solvable, i.e., is not solvable, but all of its proper quotients are solvable;
4. has a presentation

$$\text{IMG}(z^2 - 1) = \langle a, b \mid \left[ \left[ a^{2^k}, b^{2^k} \right], b^{2^k} \right] = \left[ \left[ b^{2^k}, a^{2^{k+1}} \right], a^{2^{k+1}} \right] = 1, k = 1, 2, \dots \rangle.$$

5. has solvable word and conjugacy problems;
6. has no free non-abelian subgroups of rank 2.

The properties of the group  $\text{IMG}(z^2 - 1)$  are similar to the properties of the branch groups (see [Gri00]) and of the just non-solvable group of A. Brunner, S. Sidki and A. Vieira in [BSV99].

**More examples.** We present here the iterated monodromy groups of quadratic rational maps with size of postcritical set at most 3, arranged in a table. The algebraic structure of most of them is not yet well understood.

$f(z)$	standard action	comments
$z^2$	$\tau = (1, \tau)\sigma$	The adding machine (see above)
$z^{-2}$	$\mu = (1, \mu^{-1})\sigma$	A conjugate of the adding machine. The group, generated by the transformation $\mu$ and the adding machine $\tau$ was studied by A. Brunner, S. Sidki and A. C. Vieira [BSV99].
$z^2 - 1$	$a = (b, 1)\sigma,$ $b = (a, 1)$	see Theorem 5.8 above
$\frac{z^2-1}{z^2}$	$a = (1, b),$ $b = (a^{-1}, 1)\sigma$	
$z^2 - 2$	$a = \sigma,$ $b = (a, b)$	Isomorphic to $\mathbb{D}_\infty$ , since $z^2 - 2$ is conjugate to the Chebyshev polynomial $T_2$ , (see above).
$z^2 + c,$ with $c \in \mathbb{R}$ such that $c^3 + 2c^2 + c + 1 = 0$	$a = (1, b)\sigma,$ $b = (1, c),$ $c = (a, 1)$	
$z^2 + c,$ with $c^3 + 2c^2 + c + 1 = 0,$ $c \notin \mathbb{R}$	$a = (1, b)\sigma,$ $b = (c, 1),$ $c = (a, 1)$	The action is not conjugate to the previous one, though it is not known if the groups are isomorphic. Their closures in $\text{Aut}T(X)$ coincide.
$\frac{z^2-2}{z^2}$	$a = (b, a),$ $b = (b^{-1}, a^{-1})\sigma$	The group is isomorphic to $\mathbb{Z}^2 \rtimes (\mathbb{Z}/4)$ , where $\mathbb{Z}/4\mathbb{Z}$ acts by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ .
$\frac{z^2-\phi^2}{z^2},$ with $\phi = \frac{1+\sqrt{5}}{2}$	$a = (b, 1),$ $b = (1, c),$ $c = (a^{-1}, b^{-1})\sigma$	
$\frac{z^2-\phi^2}{z^2},$ with $\phi = \frac{1-\sqrt{5}}{2}$	$a = (1, b),$ $b = (1, c),$ $c = (a^{-1}, 1)\sigma$	
$\frac{z^2-1}{z^2+1}$	$a = (1, b)\sigma,$ $b = (a, a^{-1})$	
$\frac{z^2-1}{z^2-\omega},$ with $\omega^3 = 1, \omega \neq 1$	$a = (1, b)\sigma,$ $b = (c, 1),$ $c = (c^{-1}b^{-1}, a^{-1})\sigma$	
$z^2 + i$	$a = \sigma,$ $b = (a, c),$ $c = (b, 1)$	This group has intermediate growth

**Questions.**

1. When is the iterated monodromy group of a rational function torsion free?
2. When are these groups just non-solvable?
3. What do their presentations look like?
4. Do they all have solvable conjugacy problem?
5. Can any be non-amenable? or contain a free subgroup of rank  $\geq 2$ ?
6. Which rational functions have iterated monodromy groups of exponential growth?

**Galois groups.** The following construction is due to R. Pink (private communication).

Suppose the branched covering  $f : M \rightarrow M$  is a polynomial mapping of the complex sphere  $M = \mathbb{C}$ . Define the polynomials  $F_n(x) = f^n(x) - t$  over the field  $\mathbb{C}(t)$ . Let  $\Omega_n$  be the decomposition field of  $F_n(x)$ . Then  $\Omega_n \subseteq \Omega_{n+1}$ ; write  $\Omega = \cup_{n \geq 0} \Omega_n$ . Then the Galois group of the extension  $\mathbb{C}(t) \subset \Omega$  is isomorphic to the closure of the group  $IMG(f)$  in  $\text{Aut}T$ , where  $T$ , the disjoint union of the roots of  $F_n(x)$ , has the natural structure of a  $\deg(f)$ -regular tree.

## 6 Virtual endomorphisms

### 6.1 Definitions

**Definition 6.1.** A *virtual endomorphism*  $\phi : G \dashrightarrow G$  is a homomorphism from a subgroup of finite index  $\text{Dom } \phi \leq G$  into  $G$ .

The product  $\phi = \phi_1 \phi_2$  of two virtual endomorphisms is again a virtual morphism with domain

$$\text{Dom } \phi = \{g \in G : g \in \text{Dom } \phi_2, \phi_2(g) \in \text{Dom } \phi_1\} = \phi_2^{-1}(\text{Dom } \phi_1).$$

Let us fix a faithful self-similar action of a group  $G$  on the space  $X^\omega$ . For every  $x \in X$  we denote by  $G_x$  the stabilizer of the one-letter word  $x$  in the associated action of the group  $G$  on  $X^*$ . Define a map  $\phi_x : G_x \rightarrow G$  by the formula

$$\phi_x(g) = g|_x.$$

It follows from Equation (4) that  $\phi_x$  is a homomorphism from  $G_x$  into  $G$ . The group  $G_x$  is a subgroup of finite index in  $G$ . The index is equal to the cardinality of  $X$ , since we assume that the action is level-transitive.

Thus, given a self-similar action of a group  $G$  for any  $x \in X$  we have a virtual endomorphism  $\phi_x : G \dashrightarrow G$ . We call this endomorphism the endomorphism *associated with the self-similar action*.

For example, for the adding machine action, the associated virtual endomorphism is the map  $\mathbb{Z} \dashrightarrow \mathbb{Z} : n \mapsto n/2$  with domain equal to the set of even numbers.

Let again  $\phi : G \dashrightarrow G$  be any virtual endomorphism. Choose a right transversal  $T = \{g_0 = 1, g_1, g_2, \dots, g_{d-1}\}$  to the subgroup  $\text{Dom } \phi$  (where  $d = [G : \text{Dom } \phi]$ ), i.e., elements of  $G$  such that  $G$  is a disjoint union of the sets  $\text{Dom } \phi \cdot g_i$ . Choose also a sequence  $C = (h_0 = 1, h_1, h_2, \dots, h_{d-1}) \in G^d$  of arbitrary elements of the group. Define an automaton over the alphabet  $X = \{x_0, x_1, \dots, x_{d-1}\}$ , with the set of states equal to  $G$ , by the equations

$$\lambda(g, x_i) = x_j, \quad \pi(g, x_i) = h_i^{-1} \phi(g_i g g_j^{-1}) h_j, \quad (11)$$

where  $j$  is such that  $g_i g g_j^{-1} \in \text{Dom } \phi$ .

The obtained automaton  $\mathcal{A}(\phi, T, C)$  will be called the *automaton defined by the virtual endomorphism  $\phi$ , the coset transversal  $T$  and the sequence  $C$* .

Using Proposition 4.1, one can prove the following assertion:

**Proposition 6.1.** *The automaton  $\mathcal{A}(\phi, T, C)$  defines a self-similar action of the group  $G$  with the associated virtual endomorphism  $\phi$ .*

A set  $K \subseteq G$  is *invariant* under a virtual endomorphism  $\phi$  if  $K \subseteq \text{Dom } \phi$  and  $\phi(K) \subseteq K$ .

The kernel of a self-similar action can be determined using the following description (see [Nek00, NS01]):

**Proposition 6.2.** *The kernel of a self-similar action of a group  $G$  is the maximal normal  $\phi$ -invariant subgroup of  $G$ , where  $\phi$  is the virtual endomorphism, associated with the action. It is equal to the group*

$$\bigcap_{n \geq 1} \bigcap_{g \in G} g^{-1} \cdot \text{Dom } \phi^n \cdot g.$$

The above construction is universal, namely:

**Proposition 6.3.** *Any self-similar action is defined by the associated virtual endomorphism  $\phi_{x_0}$ , a coset transversal  $T = \{g_y : y \in X\}$  and the sequence  $C = \{h_y : y \in X\}$ , where  $x_0^{g_y} = y$  and  $g_y|_{x_0} = h_y$ .*

## 6.2 Recurrent actions and abstract numeration systems

Let  $G$  be a group with a self-similar action over the alphabet  $X = \{x_0, x_1, \dots, x_{d-1}\}$ .

**Definition 6.2.** The self-similar action is *recurrent* if the associated virtual endomorphism  $\phi_x$  is onto, i.e., if  $\phi_x(\text{Dom } \phi_x) = G$ .

It follows from Proposition 6.3 and Equation (11) that every recurrent action has a *digit set* in the sense of the following definition.

**Definition 6.3.** A *digit set* for the self-similar action is a set  $T = \{r_0 = 1, r_1, \dots, r_{d-1}\} \subset G$  such that for every  $g \in G$  we have  $x_0^{r_i} = x_i$  and  $r_i|_{x_0} = 1$ .

The self-similar action defined by a virtual endomorphism  $\phi$  and a digit set  $T$  is the self-similar action defined by  $\phi$ , the coset transversal  $T$  and the sequence  $C = (1, 1, \dots, 1)$ .

Suppose now that the action of the group  $G$  is recurrent and let  $T = \{r_0 = 1, r_1, \dots, r_{d-1}\}$  be its digit set. Then the action of the group  $G$  can be interpreted in the following way. Let  $w = x_{i_1} x_{i_2} x_{i_3} \dots \in X^\omega$  be an infinite sequence. We put it in correspondence with a formal expression

$$w = \dots r_{i_4})^{\phi^{-1}} r_{i_3})^{\phi^{-1}} r_{i_2})^{\phi^{-1}} r_{i_1},$$

where  $(\cdot)^{\phi^{-1}}$  is another notation for  $\phi^{-1}(\cdot)$ . In order to know the image of the sequence  $x_{i_1} x_{i_2} x_{i_3} \dots$  under the action of an element  $g \in G$ , we multiply the expression from the right by  $g$  and then reduce it to a similar form. There exists a unique index  $j_1$  such that  $r_{i_1} g = \tilde{g} r_{j_1}$  for  $\tilde{g} \in \text{Dom } \phi$ , so it is natural to write

$$\begin{aligned} wg &= \dots r_{i_3})^{\phi^{-1}} r_{i_2})^{\phi^{-1}} r_{i_1} g \\ &= \dots r_{i_3})^{\phi^{-1}} r_{i_2})^{\phi^{-1}} \tilde{g} r_{j_1} \\ &= \dots r_{i_3})^{\phi^{-1}} r_{i_2} \phi(\tilde{g})^{\phi^{-1}} r_{j_1}. \end{aligned}$$

Next, we determine the index  $j_2$  such that  $r_{i_2} \phi(\tilde{g}) = \tilde{\tilde{g}} r_{j_2}$  for  $\tilde{\tilde{g}} \in \text{Dom } \phi$  and proceed further. It follows from formula (11) that in this way we will get correctly all the indices  $j_1, j_2, \dots$  of the image  $x_{j_1} x_{j_2} \dots = (x_{i_1} x_{i_2} \dots)^g$ . In the general (non-recurrent) case the formulæ are slightly more complicated.

This interpretation can be viewed as some sort of “ $\phi$ -adic” numeration system on the space  $X^\omega$ . In particular, in the case of the binary adding machine we will get the usual dyadic numeration system if we chose  $r_0 = 0, r_1 = 1$  (recall that the virtual endomorphism in this case is  $\phi(n) = n/2$ ). So the elements of the coset transversal  $\{r_0, r_1, \dots, r_{d-1}\}$  play the role of the digits and  $\phi^{-1}$  is the “base” of the numeration system.

### 6.3 Contracting actions

**Definition 6.4.** A self-similar action of a group  $G$  is *contracting* if there exists a finite set  $\mathcal{N} \subset G$  such that for every  $g \in G$  there exists  $k \in \mathbb{N}$  such that  $g|_v \in \mathcal{N}$  for every word  $v \in X^*$  of length  $\geq k$ . The minimal set  $\mathcal{N}$  with this property is called the *nucleus* of the self-similar action.

Obviously, every contracting action is finite state.

**The adding machine.** Let  $a$  be the adding machine transformation of the space  $\{0, 1\}^\omega$  introduced in Subsection 3.4. Then for even  $n$  we have

$$a^n|_0 = a^{\frac{n}{2}}, \quad a^n|_1 = a^{\frac{n}{2}},$$

and for odd  $n$  we have

$$a^n|_0 = a^{\frac{n-1}{2}}, \quad a^n|_1 = a^{\frac{n+1}{2}}.$$

It easily follows from this that the adding machine action of the group  $\mathbb{Z}$  is contracting with nucleus  $\{-1, 0, 1\}$ .

**The Grigorchuk group** is contracting with the nucleus equal to the standard set of generators  $\{1, a, b, c, d\}$ . In the original paper [Gri80], where the Grigorchuk group was defined, the contraction of the group was used to prove that each of its elements has finite order. Also many other properties of the Grigorchuk group and its analogs are proved using the contraction properties, since it allows to prove statements by induction on the length of group elements.

**The iterated monodromy groups.** A rational function is said to be *sub-hyperbolic* (see [Mil99]) if there exists an orbifold metric on a neighborhood of its Julia set such that the rational function is expanding with respect to this metric. We have the following theorem (for a proof see [Mil99]).

**Theorem 6.4.** *A rational function is sub-hyperbolic if and only if every orbit of a critical point is either finite or converges to an attracting finite cycle.*

Thus, in particular, every postcritically finite rational function is sub-hyperbolic.

If  $f \in \mathbb{C}(x)$  is a sub-hyperbolic rational function, then the group  $IMG(f)$ , with respect to any natural action, is contracting (see Theorem 9.7).

**Other examples** include the Gupta-Sidki group [GS83a], the Fabrykowski-Gupta group [FG91], the group from the paper [BSV99] and many others [BG01].

**A non-contracting action** is, for example, the action of the lamplighter group described in Subsection 3.4. See also the paper [Dah01] for an example of a weakly branch non-contracting group.

It follows from the definition that the restrictions of the elements of the nucleus also belong to the nucleus. Thus, the nucleus is a subautomaton of the complete automaton of the action. For instance, the diagram of the nucleus of the adding machine is shown on Figure 13.

For the Grigorchuk group the nucleus is the automaton defining the generators (see Figure 9, page 22).

An equivalent definition of contracting action uses the contraction of the length of the group elements and is based on the following proposition.

**Proposition 6.5.** *Let  $G$  be a finitely generated group with a self-similar action. Let  $|g|$  denote the word-length of  $g \in G$  with respect to some fixed generating set of  $G$ . Then the number*

$$\rho = \limsup_{n \rightarrow \infty} \max_{v \in X^n} \sqrt[n]{\limsup_{|g| \rightarrow \infty} \frac{|g|_v}{|g|}}$$

*is finite and does not depend on the choice of the generating system of the group.*

*The number  $\rho$  is less than 1 if and only if the action of the group  $G$  is contracting.*

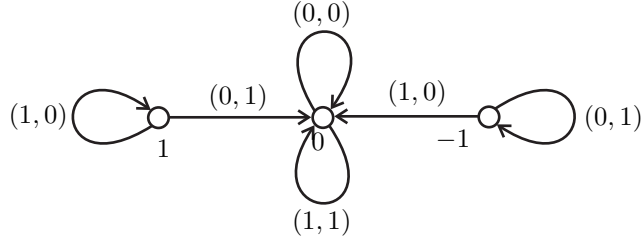


Figure 13: The nucleus of the adding machine action

## 6.4 Abelian self-similar groups

Here we show, following [NS01], some of the properties of self-similar actions of the free abelian group  $\mathbb{Z}^n$ . We will use the additive notation in this subsection.

Let  $\phi = \phi_x : \mathbb{Z}^n \dashrightarrow \mathbb{Z}^n$  be the virtual endomorphism associated to a self-similar action over an alphabet of  $d$  letters.

Let  $G_1 = \text{Dom } \phi$  be its domain. The map  $\phi$  extends uniquely to a linear map  $\phi : \mathbb{Q}^n \rightarrow \mathbb{Q}^n$ . Since  $\phi$  maps a subgroup of index  $d$  of  $\mathbb{Z}^n$  to  $\mathbb{Z}^n$ , its matrix  $A$  in a basis of  $\mathbb{Z}^n$  has the form

$$\begin{pmatrix} \frac{a_{11}}{k_1} & \frac{a_{12}}{k_2} & \cdots & \frac{a_{1n}}{k_n} \\ \frac{a_{21}}{k_1} & \frac{a_{22}}{k_2} & \cdots & \frac{a_{2n}}{k_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n1}}{k_1} & \frac{a_{n2}}{k_2} & \cdots & \frac{a_{nn}}{k_n} \end{pmatrix},$$

where  $a_{ij}, k_j$  are integers and  $d = k_1 k_2 \cdots k_n$ . In particular,  $d \cdot \det A \in \mathbb{Z}$  and the action is recurrent if and only if  $\det A = \pm d^{-1}$ .

Every  $\phi$ -invariant subgroup must be trivial; this is equivalent to the condition that the characteristic polynomial of the matrix  $A$  is not divisible by a monic polynomial with integral coefficients (see [NS01] Proposition 4.1). In particular, the matrix  $A$  is non-degenerate.

If the action is recurrent then the matrix  $A^{-1}$  has integral entries. Any such action coincides with a natural action of  $\mathbb{Z}^n$  as the i.m.g. of the self-covering of the torus  $\mathbb{T}^n$  defined by the matrix  $A^{-1}$ .

The following theorem is proved in [NS01] for the case  $d = 2$ . The proof in the general case is similar.

**Theorem 6.6.** *A self-similar action of the group  $\mathbb{Z}^n$  is finite-state if and only if the associated virtual endomorphism  $\phi$  is a contraction, i.e., if its matrix  $A$  has spectral radius less than one.*

Let us fix a recurrent self-similar action of the group  $\mathbb{Z}^n$  over an alphabet  $X = \{0, 1, \dots, d-1\}$ , with associated virtual endomorphism  $\phi$ , and let  $R = \{r_0 = 0, r_1, \dots, r_{d-1}\}$  be a digit set of the action. Consider the closure  $\hat{\mathbb{Z}}^n$  of the group  $\mathbb{Z}^n$  in  $\text{Aut}T(X)$ . Then  $\hat{\mathbb{Z}}^n$  is an abelian compact topological group. We have the following properties (see [NS01]):

**Theorem 6.7.** *For every sequence  $w = i_0 i_1 \dots \in X^\omega$  the series*

$$\Phi(w) = r_{i_0} + \phi^{-1}(r_{i_1}) + \phi^{-2}(r_{i_2}) + \cdots + \phi^{-k}(r_{i_k}) + \cdots$$

*converges in  $\hat{\mathbb{Z}}^n$ .*

*The map  $\Phi : X^\omega \rightarrow \hat{\mathbb{Z}}^n$  is a continuous bijection, which agrees with the action of the group  $\mathbb{Z}^n$  on its closure, i.e.*

$$\Phi(w^g) = \Phi(w) + g$$

*for all  $g \in \mathbb{Z}^n, w \in X^\omega$ .*

*The group  $\hat{\mathbb{Z}}^n$  is isomorphic to the profinite completion of the group  $\mathbb{Z}^n$  with respect to the series of subgroups of finite index*

$$\text{Dom } \phi \geq \text{Dom } \phi^2 \geq \text{Dom } \phi^3 \geq \dots$$

So, recurrent self-similar actions of abelian groups give us “ $\phi$ -adic” numeration systems on these groups and define naturally their profinite  $\phi$ -adic completions  $\hat{\mathbb{Z}}_\phi^n$ . For more on such numeration systems see [Vin00, Gil82].

## 7 Schreier graphs

### 7.1 Definitions

Let  $G$  be a group generated by a finite set  $S$ . We suppose  $1 \notin S$  and  $S = S^{-1}$ . Suppose that  $G$  acts faithfully on a set  $M$ .

Let us define the *labeled Schreier graph*  $\Gamma(G, S, M)$  of the group  $G$  acting on  $M$ . It is a labeled graph with set of vertices  $M$  and set of edges  $M \times S$ . The label of every edge  $(x, s)$  is  $s$ . We set  $\alpha(x, s) = x$  and  $\omega(x, s) = x^s$ .

It is obvious that the labeled Schreier graph uniquely defines the action of the generating elements on the set  $M$ , so it also determines uniquely the group  $G$ .

Sometimes we will consider just *Schreier graphs*, i.e., the graphs defined in the same way but without the labeling.

A *simplicial Schreier graph* is the simplicial graph associated to a Schreier graph.

Obviously, the orbits of  $G$  are exactly the vertex sets of the connected components of the Schreier graph  $\Gamma(G, S, M)$ .

If  $x \in M$ , then by  $\Gamma(G, S, x)$  we denote the Schreier graph of the action of  $G$  on the  $G$ -orbit of  $x$ . Such Schreier graphs are called *orbit Schreier graphs*.

If the group  $G$  acts transitively on the set  $M$ , then the Schreier graph  $\Gamma(G, S, M)$  can be interpreted in a more classical way:

**Definition 7.1.** Let  $G = \langle S \rangle$  be a group with distinguished generating set  $S$ , and let  $H < G$  be a subgroup. The corresponding *Schreier graph* is the graph whose vertices are the right cosets  $H \backslash G = \{Hg : g \in G\}$  and whose set of edges is  $(H \backslash G) \times S$ , with maps  $\alpha(Hg, s) = Hg$  and  $\omega(Hg, s) = Hgs$ .

If  $G$  acts transitively on the set  $M$ , then the Schreier graph  $\Gamma(G, S, M)$  is isomorphic to the Schreier graph corresponding to the stabilizer  $\text{St}(m)$ , for any point  $m \in M$ .

In the special case of  $G$  acting on itself by right-multiplication, the Schreier graph  $\Gamma(G, S, G)$  is called the *Cayley graph* of  $G$ ; it is the Schreier graph corresponding to the trivial subgroup.

### 7.2 Schreier graphs of groups acting on rooted trees

Suppose  $G$  acts on the rooted tree  $T(X)$  by automorphisms (this holds, for instance, if  $G$  is a self-similar group).

Then the levels  $X^n$  are invariant under the action of  $G$ . Let us denote by  $\Gamma_n(G, S)$  the Schreier graph of the action of  $G$  on the  $n$ th level. Then the Schreier graph  $\Gamma(G, S, X^*)$  of the action on  $X^*$  is the disjoint union of the graphs  $\Gamma_n(G, S)$ .

For every  $n \geq 0$ , let  $\pi_n : \Gamma_{n+1}(G, S) \rightarrow \Gamma_n(G, S)$  be the map, defined on the vertex sets, given by  $\pi_n(x_1 \dots x_n x_{n+1}) = x_1 \dots x_n$ . Then, since  $G$  acts by automorphisms of the rooted tree, the maps  $\pi_n$  induce surjective morphisms between the labeled graphs. In this way we get an inverse spectrum of finite labeled graphs

$$\Gamma_0(G, S) \leftarrow \Gamma_1(G, S) \leftarrow \Gamma_2(G, S) \leftarrow \dots \quad (12)$$

We therefore get the following simple description of the graph  $\Gamma(G, S, X^\omega)$ :

**Proposition 7.1.** *The labeled Schreier graph  $\Gamma(G, S, X^\omega)$  is the inverse limit of the sequence (12).*

The graphs which are isomorphic to an inverse limit of finite graphs are called *profinite graphs* (see [RZ00]).

On the other side, it is possible to interpret the orbit Schreier graphs on  $X^\omega$  as limits of the finite Schreier graphs  $\Gamma_n(G, S)$ . The following proposition holds (see [GŽuk97] for applications).

**Proposition 7.2.** *Let  $G$  be a finitely generated group acting on the tree  $T(X)$  by automorphisms. Let  $v = x_1x_2\dots \in X^\omega$  be a point on the boundary. Then the pointed orbit Schreier graph  $(\Gamma(G, S, v), v)$  is isomorphic to the limit of the pointed Schreier graphs  $(\Gamma_n(G, S), x_1x_2\dots x_n)$  with respect to the local topology on the space of pointed graphs.*

Here the local topology on the space of pointed graphs is defined by the metric

$$d((\Gamma_1, v_1), (\Gamma_2, v_2)) = 2^{-R},$$

where  $R$  is maximal among such that the ball  $B(v_1, R)$  in  $\Gamma_1$  and the ball  $B(v_2, R)$  in  $\Gamma_2$  are isomorphic (respectively to an isomorphism mapping  $v_1$  to  $v_2$ ).

**Definition 7.2.** A *graph contraction* from a graph  $(V, E)$  to a graph  $(V', E')$  is a pair  $f = (f_V, f_E)$  of maps  $f_V : V \rightarrow V'$  and  $f_E : E \rightarrow E' \cup \{\emptyset\}$  such that for edges  $e \in E$  with  $f_E(e) \neq \emptyset$ ,

$$\alpha(f_E(e)) = f_V(\alpha(e)), \quad \omega(f_E(e)) = f_V(\omega(e));$$

and  $(f_E)|_{f_E^{-1}(E')}$  is a bijection from  $f_E^{-1}(E')$  onto  $E'$ .

In essence, some edges may be deleted (by sending them to  $\emptyset$ ); all other edges map bijectively onto the edges of  $(V', E')$ .

A graph  $\Gamma = (V, E)$  is *self-similar* if  $V$  is infinite, and there is a graph contraction  $f : \Gamma \rightarrow \Gamma$  and a finite set  $N \subset V$  such that  $\bigcup_{n \geq 0} f^{-n}(N) = V$ .

**Proposition 7.3.** *Let  $G$  be a self-similar group generated by a finite set  $S$  such that for every  $g \in S$  and  $x \in X$  the restriction  $g|_x$  also belongs to  $S$ . Then for every  $n \in \mathbb{N}$  the shift  $\mathfrak{s} : xv \mapsto v : X^n \rightarrow X^{n-1}$  can be extended to a contraction of the Schreier graph  $\Gamma_n(G, S)$  onto the Schreier graph  $\Gamma_{n-1}(G, S)$ .*

### 7.3 Examples of Schreier graphs of self-similar group actions

**The Grigorchuk group.** The Schreier graphs of the action of the Grigorchuk group on the orbit of the point  $111\dots$ , with its self-similar nature, was described in [BG00b].

The Schreier graph  $\Gamma_1(\mathbf{G}, S)$  of the action of the Grigorchuk group is shown on left-hand side part of Figure 14. In order to obtain the Schreier graph  $\Gamma_n(\mathbf{G}, S)$ , one has to replace in the graph  $\Gamma_{n-1}(\mathbf{G}, S)$  simultaneously all the labels  $b$  by the labels  $d$ , the labels  $c$  by  $b$ , the labels  $d$  by  $c$  and all the edges labeled by  $a$  by the graph depicted on the right part of Figure 14 (the ends of the original edge correspond to the marked vertices of the graph). See, for example, the Schreier graph  $\Gamma_3$  in Figure 15. (We do not indicate the orientation of the arrows on the figures, since the generators are involutions.)

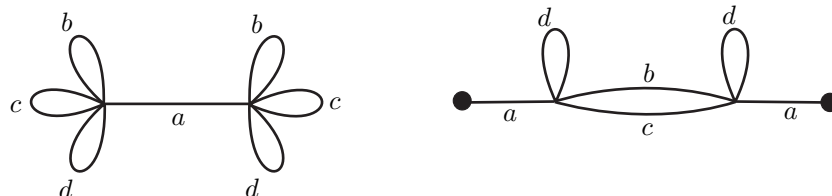


Figure 14: Substitution rule for the Schreier graphs of the Grigorchuk group

A component of the Schreier graph of the action of the Grigorchuk group on the space  $X^\omega$  is either the infinite line shown in Figure 16 (a), or the infinite ray shown in Figure 16 (b) (this last case occurs only for the orbit of the point  $1111\dots$ ).

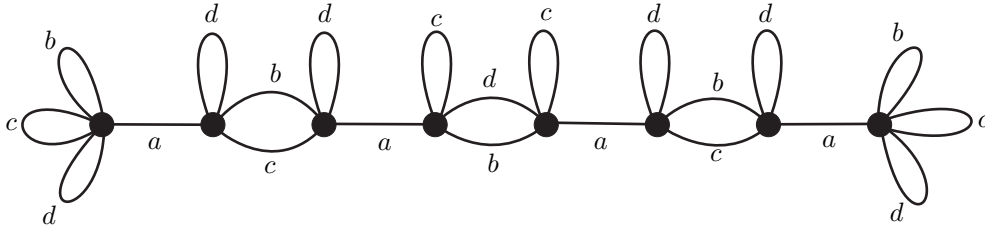


Figure 15: A Schreier graph of the Grigorchuk group

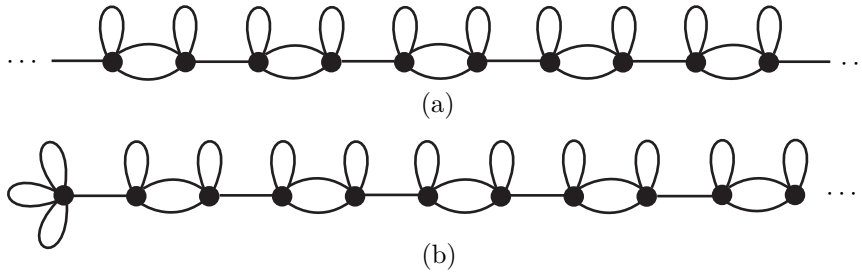


Figure 16: The orbit Schreier graphs of the Grigorchuk group on the space  $X^\omega$

**IMG( $x^2 - 1$ ).** The iterated monodromy group of the polynomial  $x^2 - 1$  is generated by two generators  $a = (b, 1)\sigma$  and  $b = (a, 1)$ , where  $\sigma$  is the transposition.

Some of the (simplicial) Schreier graphs of the group  $\text{IMG}(z^2 - 1)$  acting on the finite levels are shown on Figure 17. Compare with the Julia set in Figure 20, page 49.

The Schreier graphs of the groups  $\text{IMG}(z^2 - 1)$  are unions of  $2^n$ -gons.

**The Fabrykowski-Gupta group.** The Schreier graph of this group, introduced in Subsection 9.3, is planar and is a union of triangles. The finite Schreier graph  $\Gamma_6(G, S)$  is given in Figure 18. As can be seen, the limit space and the Schreier graph have a similar aspect.

**Penrose tilings.** If we take the group  $F$  generated by the transformations  $L$ ,  $M$  and  $S$  defined by the formulæ from Theorem 3.1, then it will act on the space  $\mathcal{P}$ , with Schreier graphs isomorphic to the dual graphs of the Penrose tilings (i.e., to the graphs whose vertices are tiles of the tiling, with two vertices connected by an edge if and only if the respective tiles have a common side), except for the Penrose tilings having non-trivial symmetry. In that case the corresponding Schreier graph will be isomorphic to the adjacency graph of the fundamental domain of the symmetry group of the tiling, with loops at the vertices bounding the domain.

## 8 Growth and languages

In its most general form, the problem we deal with here is the association to a geometric or combinatorial object of a numeric invariant, the *degree* or *rate* of growth, or of a string of numeric invariants, the *growth power series*. We sketch in this section the main notions of growth, and present them in a unified way.

The geometric objects described in this paper are of two natures: some are compact ( $X^\omega$ , or the closure of  $G$  in  $\text{Aut}T(X)$ ), while some are discrete ( $G$ , its Cayley graph, Schreier graphs, etc.)

Some other, more algebraic notions of growth or dimension may also be integrated to this picture. To name the main ones, growth of monoids and automata (that are intimately connected to growth of groups); cogrowth of groups (related to spectral properties of groups — see Section 12);

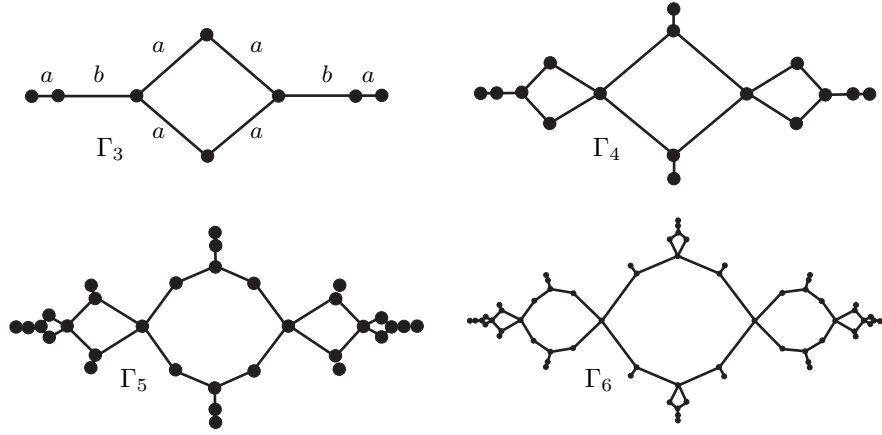


Figure 17: The Schreier graphs  $\Gamma_n(\text{IMG}(z^2 - 1), \{a^{\pm 1}, b^{\pm 1}\})$  of the  $n$ th level action, for  $3 \leq n \leq 6$

subgroup growth [Lub95]; growth of number of irreducible representations [PT96]; growth of planar algebras [Jon01]; growth of the lower central series [Gri89, BG00a, Pet99], etc.

## 8.1 Compact spaces

Let  $K$  be a compact metric space. Its *Hausdorff dimension* (see [Fal97, Mot01]) is defined as follows: for  $\beta > 0$ , the  $\beta$ -volume of  $K$  is

$$H^\beta(K) = \lim_{\epsilon \searrow 0} \inf_{\substack{\text{covers } \{\mathcal{U}_i\} \text{ of } K \text{ with} \\ \text{diameter at most } \epsilon}} \sum \text{diam}(\mathcal{U}_i)^\beta.$$

Clearly  $H^\beta(K)$  is a decreasing function of  $\beta$ . The *Hausdorff dimension*  $\dim_H(K)$  of  $K$  is defined as the unique value in  $[0, \infty]$  such that  $H^\beta(K) = \infty$  if  $0 < \beta < \dim_H(K)$  and  $H^\beta(K) = 0$  if  $\beta > \dim_H(K)$ .

A connected, but easier-to-grasp notion, is that of *box dimension*. It is defined, when it exists, as

$$\dim_\square(K) = - \lim_{\epsilon \searrow 0} \frac{\ln(\text{number of } \epsilon\text{-balls needed to cover } K)}{\ln \epsilon}.$$

If  $\dim_\square(K)$  exists, then  $\dim_H(K)$  exists too and takes the same value.

For arbitrary topological spaces  $F$ , the following notion, which does not refer to any metric, has been introduced: the *topological dimension*, also called (*Lebesgue*) *covering dimension*  $\dim_T(F)$  of  $F$  is the minimal  $n \in \mathbb{N}$  such that any open cover of  $F$  admits an open refinement of order  $n + 1$ , i.e. such that no point of  $F$  is covered by more than  $n + 1$  open sets.

## 8.2 Discrete spaces

Let  $\Gamma$  be a connected, locally finite graph, viewed as a discrete metric space by assigning length 1 to each edge. Choose a base vertex  $v \in V$ . Then the growth of  $\Gamma$  at  $v$  is the integer-valued function  $\gamma_{\Gamma,v} : n \mapsto |B(v, n)|$  measuring the volume growth of balls at  $v$ .

We introduce a preorder on positive-real-valued functions: say  $\gamma \lesssim \delta$  if there is an  $N \in \mathbb{N}$  such that  $\gamma(n) \leq \delta(n + N)$  for all  $n \in \mathbb{N}$ ; and say  $\gamma \sim \delta$  if  $\gamma \lesssim \delta$  and  $\delta \lesssim \gamma$ .

Clearly  $\gamma_{\Gamma,v}(n) \leq \gamma_{\Gamma,w}(n + d(v, w))$ , so the  $\sim$ -equivalence class of  $\gamma_{\Gamma,v}$  does not depend on  $v$ ; we call it the *growth of  $\Gamma$* , written  $\gamma_\Gamma$ .

Note that if  $\Gamma$  has degree bounded by a constant  $D$ , then  $\gamma_\Gamma \lesssim D^n$ . The graph  $\Gamma$  has *polynomial growth* if  $\gamma_\Gamma \lesssim Kn^d$  for some  $K, d \in \mathbb{R}$ ; the infimal such  $d$  is called the *degree* of  $\Gamma$ . The graph has

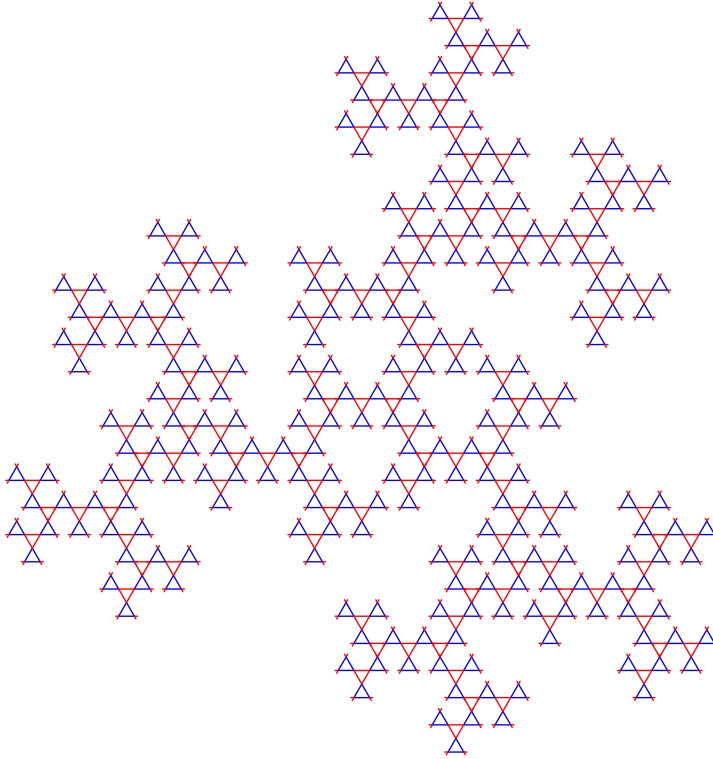


Figure 18: The Schreier graph  $\Gamma_6(G, S)$  of the Fabrykowski-Gupta group

*exponential growth* if  $\gamma_\Gamma \gtrsim b^n$  for some  $b > 1$ ; the supremum of such  $b$ 's is called the *growth rate* of  $\Gamma$ . In all other cases,  $\Gamma$  has *intermediate growth*.

The (polynomial) degree of growth is an exact analogue of the box dimension defined above. Indeed, given a graph  $\Gamma$  and a vertex  $v$ , consider the metric spaces  $K_n = \frac{1}{n}B(v, n)$ , namely the balls of radius  $n$  with the metric scaled down by a factor of  $n$ . Then each  $K_n$  is compact (of diameter 1). Assume  $\Gamma$  has growth degree  $d$ . Take the limit  $K$  of a convergent subsequence (in the Gromov-Hausdorff metric [Gro81]) of  $(K_n)_{n \geq 1}$ . Then  $\dim_\square(K) = d$ .

Conversely, let  $K$  be a compact space of box dimension  $d$ , with a fixed point  $*$ . For  $\epsilon = 1/n$  cover  $K$  by a minimal number of  $\epsilon$ -balls, and consider the graph  $\Gamma_n$ , with vertex set the set of balls, and edges connecting adjacent balls. Take the limit  $\Gamma$  of a convergent subsequence of  $(\Gamma_n)_{n \geq 0}$  (in the local topology), with each  $\Gamma_n$  based at the ball containing  $*$ . Then  $\Gamma$  is a graph of growth degree  $d$ .

We shall see in Subsection 8.9 examples of Schreier graphs of polynomial growth, with associated compact spaces of finite box dimension.

### 8.3 Amenability

**Definition 8.1.** Let  $G$  act on a set  $X$ . The action is *amenable* (in the sense of von Neumann [vN29]) if there exists a finitely additive measure  $\mu$  on  $X$ , invariant under the action of  $G$ , with  $\mu(X) = 1$ .

We then say a group is amenable if its left- (or right-) multiplication action on itself is amenable.

Amenability can be tested using the following criterion, due to Følner for the regular action [Føl57] (see also [CSGH99] and the literature cited there):

**Theorem 8.1.** *Assume the group  $G$  acts on a discrete set  $X$ . Then the action is amenable if and*

only for every if for every  $\lambda > 0$  and every  $g \in G$  there exists a finite subset  $F \subset X$  such that  $|F \Delta gF| < \lambda|F|$ , where  $\Delta$  denotes symmetric difference and  $|\cdot|$  cardinality.

**Corollary 8.2.** *Let  $G$  act on a discrete space  $X$ , and assume the growth of  $X$  is subexponential. Then the action is amenable.*

Many other characterizations of amenability were discovered — see the reference [CSGH99]. The following are equivalent:

1. The discrete  $G$ -space  $X$  is non-amenable;
2.  $X$  admits a *paradoxical decomposition*, i.e. a partition  $X = X_1 \cup X_2$  with  $X_1 \cong X \cong X_2$ , the  $\cong$  sign indicating there is a piecewise-translational bijection between the spaces;
3. There exists a piecewise-translational map  $X \rightarrow X$  with cardinality-2 fibers;
4. For any generating set  $S$  of  $G$ , the simple random walk on the graph of the action of  $G$  on  $X$  has spectral radius strictly less than 1.

## 8.4 Languages

Most of the properties of growth of discrete spaces can be expressed in terms of growth of languages; given a rooted locally finite graph  $(\Gamma, v)$ , this is done by labeling the edges of  $\Gamma$  by some alphabet  $X$ , and identifying each vertex  $w$  with some shortest path from  $v$  to  $w$ , usually the lexicographically minimal such path. In case  $\Gamma$  is a Cayley graph of a group  $G$ , whose edges are then naturally labelled by generators, such a choice of paths is a *geodesic normal form*.

Let then  $X$  be an alphabet. A *language* is a subset  $\mathcal{L}$  of  $X^*$ . Its *growth* is the function  $\gamma_{\mathcal{L}} : n \mapsto |\{w \in \mathcal{L} : |w| \leq n\}|$ . Polynomial, intermediate and exponential growth are defined as in Subsection 8.2. The language’s *growth series* is the formal power series

$$\Phi_{\mathcal{L}}(t) = \sum_{w \in \mathcal{L}} t^{|w|} = \sum_{n \geq 0} \gamma_{\mathcal{L}}(n)t^n.$$

As an excellent reference on more general power series consult [SS78].

In addition to their asymptotic behaviour, languages are classified by their *complexity* in the Chomsky hierarchy (see for instance [HU79] as a good reference). The simplest languages in that hierarchy are the *regular languages*, and it is the smallest class of languages containing  $\{x\}$  for all  $x \in X$ , and closed under the operations of union, intersection, concatenation, and iteration; this last one is defined as  $\mathcal{L}^* = \{w^n : w \in \mathcal{L}, n \geq 0\}$ , and actually intersection is not necessary; if one replaces union by “+” and concatenation by “.”, one may write  $\mathcal{L}$  as a word over the alphabet  $X \cup \{(\cdot), +, \cdot, *\}$ , called a *regular expression*.

Let  $N$  be a set disjoint from  $X$ ; its elements are called *nonterminals*. A *grammar* is a collection of rules of the form  $v \rightarrow w$  for some  $v, w \in (X \cup N)^*$ ; we write  $v \models w$  if  $v = \alpha v_0 \beta$ ,  $w = \alpha w_0 \beta$ , and  $v \rightarrow w$  for some  $\alpha \in X^*$  and  $\beta, v_0, w_0 \in (X \cup N)^*$ . Given an initial  $n \in N$ , the grammar *produces* the language  $\{w \in X^* : n \models \dots \models w\}$ . A language is regular if and only if it is produced by a grammar with rules of the form  $N \rightarrow X^*$  and  $N \rightarrow X^*N$ .

Equivalently, a language  $\mathcal{L}$  is regular if and only if there exists a finite state automaton  $\mathcal{A}$  (see Subsection 4.1), an “initial” state  $q \in Q$  and a set  $F \subset Q$  of “accepting” states, such that  $\mathcal{L}$  is the set of words in  $X^*$  for which  $\mathcal{A}$ , if started in state  $q$ , ends in  $F$ . (The output of the automaton is discarded.)

A language is *context-free* if it is produced by a grammar with rules of the form  $N \rightarrow (X \cup N)^*$ ; or equivalently if it is recognized by a *push-down automaton*, i.e., a finite-state device that has access to the top of a linear stack.

A language is *indexed* if it is recognized by a finite-state device that has access to a tree-shaped stack; or equivalently if it is produced by an *indexed grammar*.

A language is *context-sensitive* if it is produced by a grammar, with no restrictions on the rules.

In addition to this hierarchy “regular  $\subset$  context-free  $\subset$  indexed  $\subset$  context-sensitive”, a language can be *unambiguously* in one of these classes if there exists such a grammar producing it, and such that for each  $w \in \mathcal{L}$  the derivation  $n \models \cdots \models w$  is unique. “Unambiguously regular” is equivalent to “regular”, but this does not hold for the other classes.

It turns out that the formal power series captures essential properties of  $\mathcal{L}$ . We sum up the main facts:

- Proposition 8.3.** *1. If the language  $\mathcal{L}$  is regular, then  $\Phi_{\mathcal{L}}$  is a rational function.*  
*2. If  $\mathcal{L}$  is unambiguously context-free, then  $\Phi_{\mathcal{L}}$  is an algebraic function.*  
*3. If  $\mathcal{L}$  is ambiguously context-free, then  $\Phi_{\mathcal{L}}$  may be transcendental, but its growth is either polynomial or exponential.*  
*4. If  $\mathcal{L}$  is indexed, then  $\mathcal{L}$  may have intermediate growth.*  
*5. If  $\mathcal{L}$  has intermediate growth, then  $\Phi_{\mathcal{L}}$  is transcendental.*

*Proof.* For the first point: If  $\mathcal{L}$  is regular, then it may be expressed unambiguously as a regular expression. In that expression, replace each variable by  $t$  and  $e^*$  by  $(1 - e)^{-1}$  for any expression  $e$ ; the result is a rational function in  $t$  expressing the growth series of  $\mathcal{L}$ .

The second point, due to N. Chomsky and M.-P. Schützenberger [CS63], is proved along similar lines.

The third point follows, in its first part, from [Fla87] and, in its second part, from the independent work of [BG99] and [Inc01].

The fourth point follows from an example supplied in [GM99].

The fifth point follows from a result by P. Fatou [Fat06, page 368]. □

## 8.5 Groups

In this section we sum up results on growth of groups, i.e. the growth of  $G$  seen as a metric space for the word metric. As a reference, see [Har00, Chapters VI–VIII] and [GH97].

Fix a symmetric monoid-generating set  $S$  for  $G$ . It induces a *word metric*

$$d(g, h) = \min\{n \in \mathbb{N} \mid g = s_1 \dots s_n h, \text{ with } s_i \in S\},$$

on  $G$ , turning it into a (discrete) metric space. This metric is the natural metric on the Cayley graph of  $G$  introduced in Subsection 7.1. If  $S$  is finite, then  $G$  is a locally compact discrete space, and hence has a growth function  $\gamma_{G,S}(n) = |B(1, n)|$ , and associated power series  $\Phi_{G,S} = \sum_{g \in G} t^{d(1,g)}$ . Often we write  $|g| = d(1, g)$ , and inversely  $d(g, h) = |gh^{-1}|$ . Note that we need  $S$  to be symmetric for  $d$  to be a distance, but that last fact is not in itself necessary.

An even stronger algebraic invariant of  $(G, S)$  is the *complete growth series*, introduced by F. Liardet [Lia96]. It is

$$\tilde{\Phi}_{G,S} = \sum_{g \in G} gt^{|g|} \in \mathbb{Z}G[[t]],$$

i.e., a power series over the group algebra. A power series in  $\mathbb{Z}G[[t]]$  is rational if it lies in the closure of the polynomials ring  $\mathbb{Z}G[t]$  under the operations of addition, subtraction, multiplication and *quasi-inversion*  $f^* = \sum_{n \geq 0} f^n$  of a series with no constant term.

There are many good reasons to consider properties of the growth series of  $G$ ; it is connected to the Hilbert-Poincaré series of the classifying space  $K(G, 1)$ , and in particular one often (but not always) has  $\Phi_G(1) = 1/\chi(G)$ , the latter being the Euler characteristic [FP87, Smy84, Gri95]. Other results are listed in the following

- Theorem 8.4.** *1. If  $G$  either has a finite-index abelian subgroup, or is hyperbolic in the sense of Gromov, or is a Coxeter group, then its growth series is rational.*  
*2. There are soluble groups with algebraic, but not rational growth series.*

3. *There are nilpotent groups  $G$  that have rational growth for some generating sets and transcendental growth for others.*

*Proof.* The first point is due to M. Benson [Ben83] (“virtually abelian”), M. Gromov [GH90, Chapitre 9] (“hyperbolic”), and N. Bourbaki [Bou68, “exercice 26”] (“Coxeter”). Actually in all these cases the complete growth series is also rational. F. Liardet extended the virtually abelian case to the complete growth series, R. Grigorchuk and T. Smirnova-Nagnibeda computed explicitly the series for orientable surface groups [GN97], and L. Paris (unpublished) extended the result to Coxeter groups.

The second point is due to W. Parry [Par92]; explicitly, he computes the growth of some wreath products, and in particular of the wreath product  $F_2 \wr \mathbb{Z}/2\mathbb{Z}$  of a free group of rank 2 with a cyclic group of order 2.

The last point is due to M. Shapiro [Sha89] (“rational”) and M. Stoll [Sto96]; for generalizations to many central extensions see [Sha94].

The first author (unpublished) showed that although  $\Phi$  is rational for some nilpotent groups, the complete growth series  $\tilde{\Phi}$  is always transcendental. He also obtained examples of soluble groups for which  $\Phi$  is rational and  $\tilde{\Phi}$  is algebraic but non-rational.  $\square$

The growth function  $\gamma_{G,S}$  depends heavily on the choice of  $S$ . Write  $\gamma \lesssim \delta$  for two functions  $\gamma, \delta : \mathbb{N} \rightarrow \mathbb{N}$  if there exists a constant  $C \in \mathbb{N}$  with  $\gamma(n) \leq \delta(Cn)$  for all  $n \in \mathbb{N}$ , and  $\gamma \sim \delta$  if  $\gamma \lesssim \delta \lesssim \gamma$ . (Note that this equivalence relation is coarser than the one introduced in Subsection 8.2.) Then the  $\sim$ -equivalence class of  $\gamma_{G,S}$  depends only on  $G$ .

- Theorem 8.5.** *1. A group has polynomial growth if and only if it has a nilpotent finite-index subgroup; this implies that the growth degree is an integer.*
- 2. A soluble or linear group has either polynomial or exponential growth.*
- 3. There exist uncountable chains and antichains of groups of intermediate growth.*

*Proof.* The first point is due to Y. Guivarc’h and independently H. Bass [Gui70, Bas72] (“if”) and M. Gromov [Gro81] (“only if”).

The second point is due to J. Milnor and J. Wolf [Mil68a, Wol68] (“soluble”) and J. Tits [Tit72] (“linear”).

The last point is due to the second author [Gri83, Gri91], who constructed the first example of a group  $G$  of intermediate growth. See [Bar98, Bar01c] for the best-known estimates

$$e^{n^{0.5157}} \lesssim \gamma_G(n) \lesssim e^{n^{0.7675}}.$$

$\square$

The following notions have also been introduced (see [GH97, Boz80]): a group  $G$  has *uniformly exponential growth* if there is a  $b > 1$  such that  $\gamma_{G,S}(n) \geq b^n$  for all  $S$ ; the main point is that  $b$  does not depend on  $S$ . Soluble [Osi01], non-elementary Gromov-hyperbolic [Kou98], one-relator [GH01], and most amalgamated products and  $HNN$  extensions [BH00] are known to have uniformly exponential growth; no example is known of a group with exponential growth, but not uniformly so.

The group  $G$  is *growth tight* if  $\sqrt[n]{\gamma_{G/N,S/N}(n)} < \sqrt[n]{\gamma_{G,S}(n)}$  for any infinite normal subgroup  $N \triangleleft G$ . Hyperbolic groups have uniformly exponential growth, and are growth tight [AL01].

It is unknown whether all groups of exponential growth have uniformly exponential growth; however, there are examples of groups that do not reach the infimum of their growth rates [Sam00].

Suppose  $G$  is residually a  $p$ -group, and let  $\{G_n\}$  be its lower  $p$ -central series. It is known [BG00a] that if the rank of  $G_n/G_{n+1}$  increases exponentially in  $n$ , then  $G$  has uniformly exponential growth.

Growth tightness can also be defined for languages: given a language  $\mathcal{L}$  and a word  $w \in \mathcal{L}$ , one defines  $\mathcal{L}_w = \{v \in \mathcal{L} : v \text{ does not contain } w\}$ ; then  $\mathcal{L}$  is *growth tight* if for all  $w \in \mathcal{L}$  the growth rate of  $\mathcal{L}_w$  is strictly less than that of  $\mathcal{L}$ . T. Ceccherini-Silberstein and W. Woess have shown that “ergodic” context-free languages (i.e., such that for any  $n_1, n_2 \in N$  we have  $n_1 \models \dots \models un_2v$  for some  $u, v \in (T \cup N)^*$ ) are growth tight.

## 8.6 $L$ -presentations

Let  $S$  be a finite alphabet, and let  $\Phi$  be a finite set of monoid endomorphisms:  $S^* \rightarrow S^*$ . In effect, each  $\phi \in \Phi$  is determined by the values  $\phi(s)$  for all  $s \in S$ . Let  $I \subset S^*$  be a finite set of initial words. A *DOL system* [RS80] is the closure of  $I$  under iterated application of  $\Phi$ . This formalism, named after A. Lindenmayer, was introduced to study growth of biological organisms.

The following definition was introduced in [Bar01a]:

**Definition 8.2.** A group  $G$  has a *finite  $L$ -presentation* if it can be presented as  $F_S/N$ , where  $F_S$  is the free group on  $S$  and  $N$  is the normal closure of a DOL system.

The class of groups with a finite  $L$ -presentation clearly contains finitely presented groups, and enjoys various closure properties, in particular that of being closed under wreath products by finite groups.

The main motivation in studying  $L$ -presentations is the following result, also taken from [Bar01a]:

**Theorem 8.6.** *Let  $G$  be a contracting regular branch group. Then  $G$  has a finite  $L$ -presentation, but is not finitely presented.*

This is a generalization to branch groups of an earlier result due to I. Lysionok [Lys85]:

**Theorem 8.7.** *The Grigorchuk group  $\mathbf{G}$  admits the following presentation:*

$$\mathbf{G} = \langle a, c, d \mid \sigma^i(a^2), \sigma^i(ad)^4, \sigma^i(adacac)^4 \ (i \geq 0) \rangle,$$

where  $\sigma : \{a, c, d\}^* \rightarrow \{a, c, d\}^*$  is defined by  $\sigma(a) = aca, \sigma(c) = cd, \sigma(d) = c$ .

Note incidentally that the Schreier graphs  $\Gamma_n(\mathbf{G}, S)$  of  $\mathbf{G}$  described in Subsection 7.3 are isomorphic to line segments of length  $2^n - 1$ ; the labeling on the edge is  $\sigma^n(a)$ . This is the tip of a connection between the lower central series of a branch group and its Schreier graph structure [Bar01b].

## 8.7 Semigroups and automata

The automata from Subsection 4.1 are not necessarily invertible; the most general setting in which growth questions can be studied is that of *semigroups*. This subsection discusses growth of semigroups, and in particular semigroups generated by automata.

Let  $T$  be a semigroup generated by a finite set  $S$ . Analogously to the group situation, the *growth function* of  $T$  is the function  $\gamma(n) = |\{t \in T \mid t = s_1 \dots s_n, s_i \in S\}|$ .

On the other hand, let  $\mathcal{A}$  be a finite automaton, and let  $\mathcal{A}^n$  be the  $n$ -fold composition  $\mathcal{A} * \dots * \mathcal{A}$  of  $\mathcal{A}$  defined in Subsection 4.1. The *growth function* of  $\mathcal{A}$  is the function  $\gamma(n) =$  minimal number of states of  $\mathcal{A}^n$ .

The following connection is clear:

**Proposition 8.8.** *The growth function of  $\mathcal{A}$  is the growth function of the semigroup generated by  $\{\mathcal{A}_q \mid q \in Q\}$ .*

Let  $T$  be a cancellative semigroup (i.e. a semigroup satisfying the axiom  $(xz = yz \vee zx = zy) \Rightarrow x = y$ ), and let  $G$  be its group of (left) quotients. The second author studied in [Gri90] the connections between amenability, and growth, of  $T$  and  $G$ . He showed in [Gri88] that a cancellative semigroup has polynomial growth if and only if its group of left quotients  $G$  is virtually nilpotent — and in that case that the growth degrees of  $G$  and  $T$  are the same. This generalizes Gromov’s statement in Theorem 8.5.

Essentially, semigroups can have any growth function at least quadratic and at most exponential; and it was long known that Milnor’s question (“Do all finitely generated groups have either polynomial or exponential growth?”) has a negative answer in the context of semigroups — see the example by V. Belyaev, N. Sesekin and V. Trofimov [BST77].

Later many other examples of semigroups of intermediate growth were discovered [Shn01]. Consult also the book [Okn98], and [LM01], where semigroups of intermediate growth are found even among  $2 \times 2$ -matrices.

## 8.8 Growth and dimension of self-similar groups

For subgroups  $G$  of  $\text{Aut}T(X)$ , we have the following extra results on growth. First,  $\text{Aut}T(X)$  is a compact metric space, for which  $\{\text{St}_n\}_{n \in \mathbb{N}}$  is a basis of neighbourhoods: define a metric on  $\text{Aut}T(X)$  by

$$d(g, h) = \max\{|\text{Aut}T(X)/\text{St}_n| : g^{-1}h \in \text{St}_n\}.$$

Then the *Hausdorff dimension* of  $G$  is the Hausdorff dimension of its closure  $\overline{G}$ , with the restricted metric, in  $\text{Aut}T(X)$ . Since in the given metric all balls are cosets of  $\text{St}_n$  for some  $n$ , we have the simpler definition [BS97]

$$\dim_H(G) = \lim_{n \rightarrow \infty} \frac{\log |(G\text{St}_n)/\text{St}_n|}{\log |\text{Aut}T(X)/\text{St}_n|}.$$

For instance, consider again the Grigorchuk group  $\mathbf{G}$ . It is known that  $\mathbf{G} \cap \text{St}_1$  has index 2 in  $\mathbf{G}$ , and that  $\mathbf{G} \cap \text{St}_1$  embeds in  $\mathbf{G} \times \mathbf{G}$  with index 8. A simple calculation gives  $|\mathbf{G}/(\mathbf{G} \cap \text{St}_3)| = 2^7$ , so by induction  $|\mathbf{G}/(\mathbf{G} \cap \text{St}_n)| = 2^{\frac{5}{8}2^n + 2}$  for  $n \geq 3$ . On the other hand,  $|\text{Aut}T(X)/\text{St}_n| = 2^{2^n - 1}$ . Therefore  $\dim_H(\mathbf{G}) = \frac{5}{8}$ . This generalizes as follows:

**Theorem 8.9 ([Bar]).** *Let  $G$  be a regular branch group. Then its Hausdorff dimension  $\dim_H(G)$  is a rational number in  $(0, 1)$ .*

All known groups of intermediate growth are subgroups of  $\text{Aut}T(X)$  for some finite  $X$ , and are described by the following result:

**Theorem 8.10 ([Bar02]).** *Let  $G < \text{Aut}T(X)$  be generated by an automaton  $X$ . Assume the states of  $T$  are partitioned as  $\{1\} \cup T_1 \cup T_2$ , where 1 is an inactive state that maps to itself; all states in  $T_1$  map to 1; and all states in  $T_2$  are inactive, and all their maps except one are to  $T_1 \cup \{1\}$ . Then  $G$  has subexponential growth.*

## 8.9 Schreier graphs of contracting actions and their growth

Let  $(G, X^\omega)$  be a self-similar action of a finitely generated group. We assume that the orbits of the action are infinite. This is the case for any level-transitive action of a group.

A graph  $\Gamma$  has polynomial growth if and only if the number

$$\alpha = \limsup_{r \rightarrow \infty} \frac{\log |B(v, r)|}{\log r}$$

is finite. The number  $\alpha$  is then called the *degree* of the growth.

**Proposition 8.11.** *The growth of every orbit Schreier graph of a contracting action  $(G, X^\omega)$  is polynomial.*

*The growth degree is not greater than  $-\frac{\log |X|}{\log \rho}$ , where  $\rho$  is the contraction coefficient of the action.*

### Examples.

1. The orbit Schreier graph of the Grigorchuk group have linear growth (i.e., the growth degree is equal to 1). This follows directly from their description. The contraction coefficient of the Grigorchuk group is equal to  $1/2$ .
2. The orbit Schreier graph of the iterated monodromy group  $\text{IMG}((z^2 - 1))$  has polynomial growth of degree 2. For example, the Schreier graph on the orbit of the point  $111\dots = 1^\infty$  is described as follows:

There is a  $b$ -labeled loop at  $1^\infty$ . Between  $0^{2n}1^\infty$  and  $0^{2n+1}1^\infty$  there is a  $2^{n+1}$ -gon labeled by  $a$ -edges, and between  $0^{2n-1}1^\infty$  and  $0^{2n}1^\infty$  there is a  $2^n$ -gon labeled by  $b$ -edges. The vertices on these polygons are labeled by strings in  $(00|10)^n(0|1)1^\infty$  and  $(00|01)^n1^\infty$  respectively.

Then at each of these new vertices on the polygons finite graphs are attached; if the vertex is labeled  $0^k 1 w 1^\infty$  for some  $w \in \{0, 1\}^*$ , then the attached graph has  $2^k$  vertices labeled by all words in  $(0|1)^k 1 w^\infty$ .

It therefore follows that the ball of radius  $2^n - 2$  at  $1^\infty$  contains only vertices with labels in  $(0|1)^{2^n - 2} 1^\infty$ , and contains all vertices with labels in  $(0|1)^{2^n - 4} 1^\infty$ . It follows that the cardinality of  $B(1^\infty, 2^n)$  is approximately  $2^{2^n}$ .

This graph is self-similar under the graph contraction  $f = (f_V, f_E)$  given by  $f_V = s : s_1 s_2 \dots \mapsto s_2 \dots$ , and

$$f_E : \begin{cases} (0w, a) \mapsto (w, b), \\ (0w, b) \mapsto (w, a), \\ (1w, a) \mapsto \bar{\partial}, \\ (1w, b) \mapsto \bar{\partial}. \end{cases}$$

It contracts distances by a factor of  $\sqrt{2}$ , while collapsing 2 points to 1;

## 9 Limit spaces of self-similar group actions

In this section we return from self-similar (semi)group actions to self-similar topological spaces, showing that a naturally-defined self-similar topological space is associated with every contracting self-similar action. This space can be defined in different ways: as a quotient of the Cantor set by an *asymptotic equivalence relation* (Definition 9.1), as a limit of finite Schreier graphs (Theorem 9.6) or as the boundary of a naturally-defined hyperbolic graph (Theorem 10.1).

### 9.1 The limit space $\mathcal{J}_G$

Let us fix a self-similar contracting action of a group  $G$  on the space  $X^\omega$ .

Denote by  $X^{-\omega}$  the space of all sequences infinite to the left:

$$X^{-\omega} = \{\dots x_3 x_2 x_1\}$$

equipped with the product topology.

**Definition 9.1.** Two elements  $\dots x_3 x_2 x_1, \dots y_3 y_2 y_1 \in X^{-\omega}$  are said to be *asymptotically equivalent* with respect to the action of the group  $G$ , if there exist a finite set  $K \subset G$  and a sequence  $g_k \in K, k \in \mathbb{N}$  such that

$$(x_k x_{k-1} \dots x_2 x_1)^{g_k} = y_k y_{k-1} \dots y_2 y_1$$

for every  $k \in \mathbb{N}$ .

It follows directly from the definition that the asymptotic equivalence is an equivalence relation.

**Proposition 9.1.** *Let  $\mathcal{N}$  be the nucleus of the action. Then two sequences  $\dots x_2 x_1, \dots y_2 y_1 \in X^{-\omega}$  are asymptotically equivalent if and only if there exists a sequence  $h_n \in \mathcal{N}, n \geq 0$  such that*

$$x_n^{h_n} = y_n, \quad \text{and} \quad h_n|_{x_n} = h_{n-1} \tag{13}$$

for all  $n \geq 1$ .

Proposition 9.1 can be reformulated in the following terms:

**Proposition 9.2.** *Let  $\Gamma$  be the Moore diagram of the nucleus  $\mathcal{N}$  of the action of the group  $G$ . Two sequences  $\dots x_2 x_1, \dots y_2 y_1 \in X^{-\omega}$  are asymptotically equivalent if and only if  $\Gamma$  has a path  $\dots e_2 e_1$  such that every edge  $e_i$  is labeled by the pair  $(x_i, y_i)$ .*

**Definition 9.2.** The *limit space* of the self-similar action (written  $\mathcal{J}_G$ ) is the quotient of the topological space  $X^{-\omega}$  by the asymptotic equivalence relation.

It follows from the definition of the asymptotic equivalence relation that  $\mathcal{J}_G$  is invariant under the shift map  $\mathfrak{s} : \dots x_3 x_2 x_1 \mapsto \dots x_4 x_3 x_2$ , and therefore the shift  $\mathfrak{s} : X^{-\omega} \rightarrow X^{-\omega}$  induces a surjective continuous map  $\mathfrak{s} : \mathcal{J}_G \rightarrow \mathcal{J}_G$  on the limit space. Every point  $\xi \in \mathcal{J}_G$  has at most  $|X|$  preimages under the map  $\mathfrak{s}$ .

**Definition 9.3.** The dynamical system  $(\mathcal{J}_G, \mathfrak{s})$  is called the *limit dynamical system* associated to the self-similar action.

**Example.** In the case of the adding machine action of  $\mathbb{Z}$ , one sees from the diagram of the nucleus given in Figure 13 that two sequences in  $X^{-\omega}$  are asymptotically equivalent if and only if they are equal or are of the form

$$\dots 0001x_m x_{m-1} \dots x_1, \text{ and } \dots 1110x_m x_{m-1} \dots x_1.$$

This is the usual identification of dyadic expansions of reals in  $[0, 1]$

$$0.x_1 x_2 \dots x_m 0111 \dots = 0.x_1 x_2 \dots x_m 1000 \dots$$

Consequently, the limit space  $\mathcal{J}_{\mathbb{Z}}$  is the circle that we obtain after identifying the endpoints of the unit segment (since the asymptotic equivalence relation glues the sequences  $\dots 000$  and  $\dots 111$ ). The map  $\mathfrak{s}$ , induced on the circle from the shift on the space  $X^{-\omega}$ , is the two-fold covering map  $\mathfrak{s}(x) = 2x \pmod{1}$ .

Proposition 9.1 implies now the following properties of the limit spaces (see [Nekc]).

**Theorem 9.3.** *Let  $\mathcal{N}$  be the nucleus of a contracting self-similar action of a group  $G$ . The asymptotic equivalence is a closed equivalence relation on  $X^{-\omega}$  and every point is equivalent to at most  $|\mathcal{N}|$  points.*

*The limit space  $\mathcal{J}_G$  is metrizable and has topological dimension  $\leq |\mathcal{N}| - 1$ .*

*If the group is  $G$  finitely generated then the limit space  $\mathcal{J}_G$  is connected.*

## 9.2 Self-similarity of the space $\mathcal{J}_G$

Here we construct a self-similarity structure on the space  $\mathcal{J}_G$  which will be defined by a Markov partition of the dynamical system  $(\mathcal{J}_G, \mathfrak{s})$

**Definition 9.4.** For a given finite word  $v \in X^*$ , define the *tile*  $\mathcal{T}_v$  to be the image of the set  $X^{-\omega} v = \{\dots x_2 x_1 v\}$  under the canonical map  $X^{-\omega} \rightarrow \mathcal{J}_G$ .

It follows from the definitions that  $\mathcal{T}_{\emptyset} = \mathcal{J}_G$  and that

$$\mathfrak{s}(\mathcal{T}_v) = \mathcal{T}_{v'},$$

where  $v'$  is the word obtained from the word  $v$  by deletion of its last letter. We also have

$$\mathcal{T}_v = \bigcup_{x \in X} \mathcal{T}_{xv}.$$

Consequently, for every fixed  $n$  the set of the tiles  $\{\mathcal{T}_v : v \in X^n\}$  is a Markov partition of the dynamical system  $(\mathcal{J}_G, \mathfrak{s})$ . Therefore, the tiles together with the restrictions of the maps inverse to the shift  $\mathfrak{s}$  define a self-similarity structure on the space  $\mathcal{J}_G$ .

Every tile is a compact set and any point of the limit space belongs to not more than  $|\mathcal{N}|$  tiles, where  $\mathcal{N}$  is the nucleus of the action.

**Proposition 9.4.** *The tiles  $\mathcal{T}_v, \mathcal{T}_u$ , for  $u, v \in X^n$ , intersect if and only if there exists  $h \in \mathcal{N}$  such that  $v^h = u$ .*

**Definition 9.5.** Denote by  $J_n(G)$  the simplicial graph whose vertices are the tiles  $\mathcal{T}_v$  for  $v \in X^n$ , with two vertices connected by an edge if and only if the respective tiles have a nonempty intersection.

Then Proposition 9.4 can be formulated in the following way:

**Corollary 9.5.** *The map  $v \rightarrow \mathcal{J}_v$  is an isomorphism between the simplicial Schreier graph  $\Gamma(\langle \mathcal{N} \rangle, \mathcal{N}, X^n)$  and the graph  $J_n(G)$ .*

If the action is recurrent, then the nucleus  $\mathcal{N}$  generates the group  $G$ , and thus the graphs  $J_n(G)$  are the Schreier graphs of the group  $G$ .

The following theorem shows that the limit space  $\mathcal{J}_G$  is a limit of the graphs  $J_n(G)$ :

**Theorem 9.6.** *A compact Hausdorff space  $\mathfrak{X}$  is homeomorphic to the limit space  $\mathcal{J}_G$  if and only if there exists a collection  $\mathfrak{T} = \{T_v : v \in X^*\}$  of closed subsets of  $\mathfrak{X}$  such that the following conditions hold:*

1.  $T_\emptyset = \mathfrak{X}$  and  $T_v = \cup_{x \in X} T_{xv}$  for every  $v \in X^*$ ;
2. For every word  $\dots x_3 x_2 x_1 \in X^{-\omega}$  the set  $\cap_{n=1}^{\infty} T_{x_n x_{n-1} \dots x_1}$  contains only one point;
3. The intersection  $T_v \cap T_u$  for  $u, v \in X^n$  is non-empty if and only if there exists an element  $s$  of the nucleus such that  $v^s = u$ .

In particular, it follows that if for some sequence of numbers  $R_n > 0$  the metric spaces  $(J_n(G), d(u, v)/R_n)$  converge in the Gromov-Hausdorff metric to a metric space  $\mathfrak{X}$ , then  $\mathfrak{X}$  is homeomorphic to the limit space  $\mathcal{J}_G$ .

If  $\mathfrak{X}$  is a metric space then condition (2) can be replaced by the condition:

$$\lim_{n \rightarrow \infty} \max_{v \in X^n} \text{diam}(T_v) = 0.$$

### 9.3 Examples of limit spaces

**The Grigorchuk group.** It follows from the Moore diagram of the nucleus of the Grigorchuk group (see Figure 9) that two sequences are asymptotically equivalent if and only if they are either equal or have the form

$$\xi = \dots 111101w \quad \zeta = \dots 111100w,$$

where  $w \in X^*$ .

Let us define a homeomorphism  $F$  of the space  $X^{-\omega}$ :

$$F(\dots x_3 x_2 x_1) = \dots y_3 y_2 y_1,$$

with  $y_i = (1 + x_1) + (1 + x_2) + \dots + (1 + x_i) \pmod 2$ .

Then two elements  $\xi, \zeta \in X^{-\omega}$  are asymptotically equivalent with respect to the action of the Grigorchuk group if and only if the  $F(\xi)$  and  $F(\zeta)$  are asymptotically equivalent with respect to the adding machine action, except in the case of the points  $\dots 1111$  and  $\dots 11110$ . These two points are not equivalent, while their  $F$ -images  $\dots 0000$  and  $\dots 1111$  are equivalent. Therefore the limit space of the Grigorchuk group is homeomorphic to the real segment  $[0, 1]$ .

The shift  $\mathfrak{s}$  on the space  $\mathcal{J}_G$  is the “tent” map  $\mathfrak{s} : x \mapsto 1 - |2x - 1|$ , folding the interval in two.

The fact that the limit space of the Grigorchuk group is the segment also follows from Theorem 9.6 and the description of the Schreier graphs of its action on the sets  $X^n$ .

**The Fabrykowski-Gupta group.** The limit space of this group, defined by J. Fabrykowski and N. D. Gupta [FG91], is the dendrite fractal shown on Figure 19. The picture is drawn using the description of Schreier graphs of the group (see [BG00b]) and Theorem 9.6. This dendrite fractal is also homeomorphic to the fractal described on Figure 8.2 of the book [Fal85].

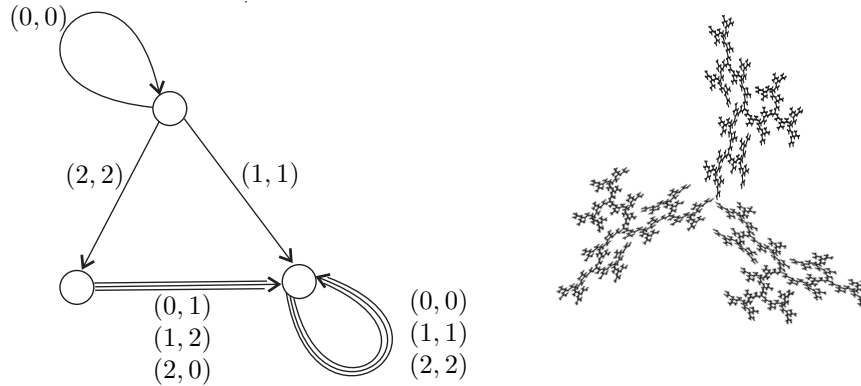


Figure 19: The Fabrykowski-Gupta group and its limit space

**The Sierpiński gasket.** Let the alphabet  $X$  be  $\{0, 1, 2\}$ . Define three transformations  $b_i$ ,  $i \in X$  of the space  $X^\omega$  as follows: put  $(iw)^{b_i} = i(w^{b_i})$  and  $(jw)^{b_i} = kw$  for all  $i, j, k$  such that  $\{i, j, k\} = \{0, 1, 2\}$ . In the standard notation:

$$b_0 = (b_0, 1, 1)\sigma_{12}, \quad b_1 = (1, b_1, 1)\sigma_{02}, \quad b_2 = (1, 1, b_2)\sigma_{01}.$$

One can show, using Theorem 9.6, that the limit space of the group  $\langle b_1, b_2, b \rangle$  is the Sierpinski gasket.

**The Julia sets of rational mappings.** The *Julia set* of a rational function  $f \in \mathbb{C}(z)$  can be defined as the closure of the union of its repelling cycles (see [Mil99, Lyu87]). If  $f$  is polynomial, its Julia set is the boundary of the attraction basin of  $\infty$ .

The proof of the next theorem will appear in [Nekb].

**Theorem 9.7.** *The iterated monodromy group of a sub-hyperbolic rational function (respectively to any natural action on a regular tree) is contracting; and its limit space is homeomorphic to the Julia set of the rational function.*

### Examples.

1. The Julia set of the polynomial  $z^2$  is the circle. Its iterated monodromy group is the adding machine action, which is contracting and with the limit set homeomorphic to the circle.
2. The Julia set of the polynomial  $z^2 - 2$  is the segment  $[-2, 2]$ . The iterated monodromy group is the dihedral group  $\mathbb{D}_\infty$  generated by the automaton in Figure 8. The limit space of this action is homeomorphic to the segment, and this is proved in the same way as for the Grigorchuk group (in fact the asymptotic equivalence relations in these two cases are the same).
3. The Julia set of the polynomial  $z^2 - 1$  is shown in Figure 20. Compare its shape with the pictures of the Schreier graphs of the iterated monodromy group of this polynomial given in Figure 17.
4. The Julia set of the polynomial  $z^2 + i$  is the dendrite shown in Figure 21. Its tree-like structure parallels the fact that the Schreier graphs of its iterated monodromy group are all trees. This holds in fact for all Misiurewicz polynomials, i.e., quadratic polynomials  $z^2 + c$  for which the critical point 0 is strictly pre-periodic. See the paper [Kam93], where the self-similarity of Julia sets of Misiurewicz polynomials is studied.

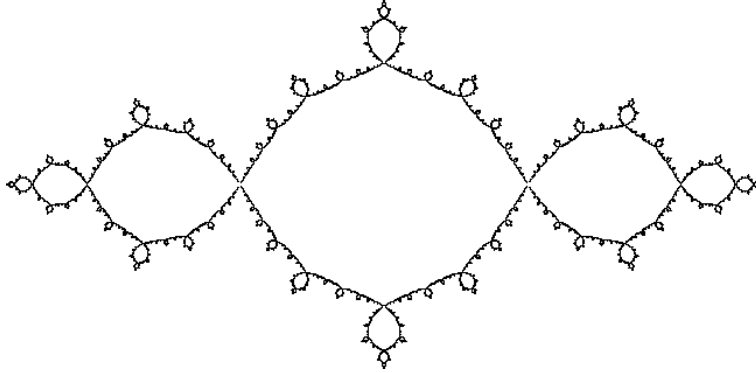


Figure 20: The Julia set of the polynomial  $z^2 - 1$ .

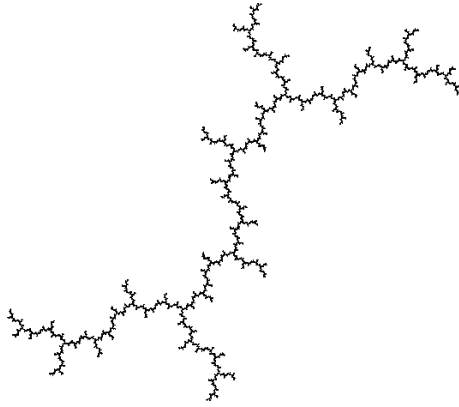


Figure 21: The Julia set of the polynomial  $z^2 + i$ .

5. The Julia sets of the polynomials  $z^2 + c$ , where  $c^3 + 2c^2 + c + 1 = 0$ , are shown in Figure 22. The left one, called the “airplane”, corresponds to the real root  $c$ , while the right one, called “Douady’s rabbit”, corresponds to the complex root with positive imaginary part. Their corresponding groups have almost the same recursion — see Subsection 5.2.

## 9.4 The solenoid $\mathcal{S}_G$

Let us fix a self-similar contracting action of a group  $G$  over the alphabet  $X$ . Denote by  $X^{\mathbb{Z}}$  the space of all two-sided infinite sequences over the alphabet  $X$  with the product topology. The elements of this space have the form

$$\xi = \dots x_{-3}x_{-2}x_{-1} \cdot x_0x_1x_2 \dots,$$

with  $x_i \in X$ , and where the dot marks the place between the  $(-1)$ -st and 0th coordinates. The sequence  $x_0x_1x_2 \dots$  is called the *integer part* of the sequence  $\xi$  and is written  $[\xi]$ .

The map

$$s : \dots x_{-3}x_{-2}x_{-1} \cdot x_0x_1x_2 \dots \mapsto \dots x_{-4}x_{-3}x_{-2} \cdot x_{-1}x_0x_1 \dots$$

is called the *shift*. It is a homeomorphism of the space  $X^{\mathbb{Z}}$ .

We say that two sequences  $\dots x_{-2}x_{-1} \cdot x_0x_1 \dots$  and  $\dots y_{-2}y_{-1} \cdot y_0y_1 \dots \in X^{\mathbb{Z}}$  are *asymptotically equivalent* if there exists a sequence  $\{g_k\}_{k=1}^{\infty}$  taking a finite number of different values in  $G$ , such that

$$(x_{-k}x_{-k+1}x_{-k+2} \dots)^{g_k} = y_{-k}y_{-k+1}y_{-k+2} \dots$$

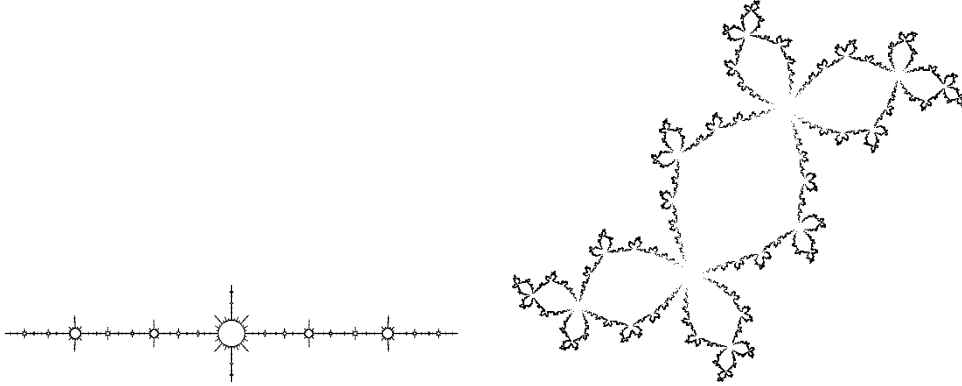


Figure 22: The Julia sets of the polynomials  $z^2 + c$ , with  $(c^2 + c)^2 + c = 0$ .

**Definition 9.6.** The *(limit) solenoid* of the contracting action of the group  $G$  is the quotient of the space  $X^{\mathbb{Z}}$  by the asymptotic equivalence relation. We denote it by  $\mathcal{S}_G$ .

Proposition 9.2 gives a more convenient description of the asymptotic equivalence relation, which is also true for the space  $X^{\mathbb{Z}}$  (the only difference is that the path must be infinite in both directions).

The next proposition shows that the solenoid is uniquely defined by the dynamical system  $(\mathcal{J}_G, \mathfrak{s})$ :

**Proposition 9.8.** *The space  $\mathcal{S}_G$  is homeomorphic to the inverse limit of the topological spaces*

$$\mathcal{J}_G \xleftarrow{\mathfrak{s}} \mathcal{J}_G \xleftarrow{\mathfrak{s}} \cdots .$$

**Definition 9.7.** For a given  $w \in X^\omega$ , the *tile*  $\mathcal{T}_w$  is the image of the set  $\{\xi \in X^{\mathbb{Z}} : [\xi] = w\}$  under the canonical quotient map  $X^{\mathbb{Z}} \rightarrow \mathcal{S}_G$ .

If  $O \subset X^\omega$  is a  $G$ -orbit, then the corresponding *leaf* is the image in  $\mathcal{S}_G$  of the set of all  $\xi \in X^{\mathbb{Z}}$  with  $[\xi] \in O$ .

It follows from the definition of the asymptotic equivalence that the integer parts of asymptotically equivalent elements of  $X^{\mathbb{Z}}$  belong to one  $G$ -orbit. Thus the solenoid is a disjoint union of leaves.

The shift  $\mathfrak{s} : X^{\mathbb{Z}} \rightarrow X^{\mathbb{Z}}$  induces a homeomorphism  $\mathfrak{s}$  on the space  $\mathcal{S}_G$ , which will be also called the *shift*. We have

$$\mathfrak{s}(\mathcal{T}_w) = \bigcup_{x \in X} \mathcal{T}_{xw},$$

so that every tile  $\mathcal{T}_w$  is homeomorphic to a union of  $|X|$  tiles.

The *type* of the tile  $\mathcal{T}_w$  is the set of elements of the nucleus  $\mathcal{N}$  which fix the word  $w$ .

**Proposition 9.9.** *If the tiles  $\mathcal{T}_{w_1}$  and  $\mathcal{T}_{w_2}$  have the same type, then the map  $w_2 : u \cdot w_1 \mapsto u \cdot w_2$ , for all  $u \in X^{-\omega}$ , induces a homeomorphism  $\mathcal{T}_{w_1} \rightarrow \mathcal{T}_{w_2}$ .*

Consequently, there exist not more than  $2^{|\mathcal{N}|}$  different tiles up to homeomorphism.

Therefore, every contracting action defines an iterated function system on the sets  $\mathcal{T}_w$ , with maps  $T_x : \mathcal{T}_w \mapsto \mathcal{T}_{xw}$ . In this system we identify the tiles of same type. Then we get a finite number of sets. The type of the tile  $\mathcal{T}_{xw}$  is uniquely defined by the type of the tile  $\mathcal{T}_w$  and the letter  $x$ , since  $xw$  is fixed under the action of  $h_0 \in \mathcal{N}$  if and only if  $x^{h_0} = x$  and  $h_0|_x = h_1$  for an element  $h_1 \in \mathcal{N}$  fixing  $w$ . We call this system the *tiling* iterated function system.

In fact, the term “tile” may be misleading, since in general the tiles  $\mathcal{T}_w$  may overlap. See the paper [Vin95] where this problem is discussed in the abelian case. The following theorem gives a criterion determining when the tiles intersect only on their boundaries.

**Theorem 9.10.** *If for every element  $g$  of the nucleus there exists a finite word  $v \in X^*$  such that  $g|_v = 1$ , then any two different tiles  $\mathcal{T}_w$  have disjoint interiors.*

*On the other hand, if for some element  $g$  of the nucleus all the restrictions  $g|_v$ ,  $v \in X^*$  are non-trivial, then there exists a tile  $\mathcal{T}_w$  covered by other tiles.*

We say that a contracting action satisfies the *open set condition* if every element of its nucleus has a trivial restriction. It is easy to see that this is equivalent to the condition that every element of the group has a trivial restriction.

Therefore, in the case of contracting actions satisfying the open set condition, we get tilings of the leaves in the usual sense. In the next subsection we show how the self-affine tilings of Euclidean space appear as tilings associated with self-similar actions of free abelian groups.

## 9.5 Self-affine tilings of Euclidean space

Let us fix a recurrent action  $(\mathbb{Z}^n, X^\omega)$  over the alphabet  $X = \{0, 1, \dots, d-1\}$ . Let  $R = \{r_0 = 0, r_1, \dots, r_{d-1}\}$  be a digit set for the action. We keep the notation of Subsection 6.4, and in particular write the group additively.

Suppose that the action is finite-state; then by Theorem 6.6 the associated virtual endomorphism yields a linear map  $\phi : \mathbb{Q}^n \rightarrow \mathbb{Q}^n$ , which is a contraction.

Then for all sequences  $w = i_1 i_2 \dots \in X^\omega$  the series

$$F(w) = \sum_{k=1}^{\infty} \phi^k(r_{i_k})$$

is convergent in  $\mathbb{R}^n$ . Let  $\mathcal{T}(\phi, R) = \{F(w) : w \in X^\omega\}$  be the set of their sums. We will call  $\mathcal{T}(\phi, R)$  the *set of fractions*, or the *tile*.

In the classical situation of the dyadic numeration system, given by  $\phi(n) = n/2$  and  $R = \{0, 1\}$ , the set of fractions  $\mathcal{T}(\phi, R)$  is the interval  $[0, 1]$ .

The set  $\mathcal{T}(\phi, R)$  is an attractor of the following affine iterated function system (see [Hut81]) on  $\mathbb{R}^n$ :

$$\{p_i(r) = \phi(r_i + r) \text{ for all } r \in \mathbb{R}^n, i \in X\}$$

which means that it is the unique fixed point of the transformation

$$P(C) = \bigcup_{i=0}^{d-1} p_i(C)$$

defined on the space of all non-empty compact subsets of  $\mathbb{R}^n$ . Moreover, for any nonempty compact set  $C \subset \mathbb{R}^n$ , the sequence  $P^n(C)$  converges in this space to  $\mathcal{T}(\phi, R)$  with respect to the Hausdorff metric. This can be used in practice to obtain approximations of these sets.

Sometimes, the set of fractions is just a rectangle; this is the case for instance with  $\phi = \begin{pmatrix} 0 & 1 \\ 1/2 & 0 \end{pmatrix}$  and  $R = \{(0, 0), (1, 0)\}$  (its set of fractions is then the square  $[0, 1] \times [0, 1]$ ), but often the set of fractions has an interesting fractal appearance.

One of the most famous examples is the region bounded by the “dragon curve”, corresponding to the case  $\phi = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$  and  $R = \{(0, 0), (1, 0)\}$ . An approximation of this set is shown in Figure 23. The associated numeration system can be interpreted as a numeration system of the Gaussian integers  $\mathbb{Z}[i]$  in base  $(1+i)$ , with the set of digits  $\{0, 1\}$ . See its discussion in [Knu69]. See also [Gil82, GPS95], where similar numeration systems of the complex numbers and their related set of fractions are considered.

Other examples of sets of fractions are shown on Figure 24.

See [Vin95, Vin00, Pra99] for examples of fraction sets and their properties. These sets have applications to wavelet theory, computer image processing, toral dynamical systems and other fields. See also the book [BJ99] for relations with the representation theory of the Cuntz  $C^*$ -algebra.

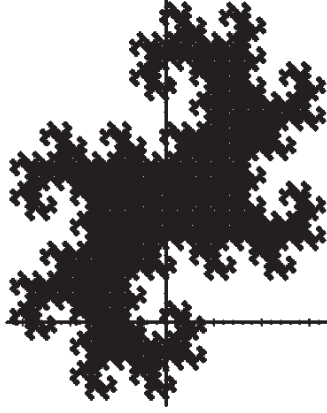


Figure 23: The set of fractions bounded by the dragon curve.

**Theorem 9.11.** *Two sequences  $(\dots x_2 x_1)$  and  $(\dots y_2 y_1) \in X^{-\omega}$  are asymptotically equivalent if and only if*

$$\sum_{k=1}^{\infty} \phi^k(r_{x_k}) - \sum_{k=1}^{\infty} \phi^k(r_{y_k}) \in \mathbb{Z}^n.$$

*Two sequences  $\xi = (\dots x_{-2} x_{-1} \cdot x_0 x_1 \dots)$  and  $\zeta = (\dots y_{-2} y_{-1} \cdot y_0 y_1 \dots) \in X^{\mathbb{Z}}$  are asymptotically equivalent if and only if*

$$\sum_{k=1}^{\infty} \phi^k(r_{x_{-k}}) - \sum_{k=1}^{\infty} \phi^k(r_{y_{-k}}) = \sum_{k=0}^{\infty} \phi^{-k}(r_{y_k}) - \sum_{k=0}^{\infty} \phi^{-k}(r_{x_k}),$$

where the left-hand side part is calculated in  $\mathbb{R}^n$ , while the right one is calculated in the closure  $\widehat{\mathbb{Z}}^n$  of  $\mathbb{Z}^n$ ; both differences must belong to  $\mathbb{Z}^n$ .

Let  $\mathcal{L}$  be a leaf of the solenoid  $\mathcal{S}_G$ . Then it decomposes into the union of its tiles, and thus can be equipped with the direct limit topology coming from this decomposition. More explicitly, a set  $A \subseteq \mathcal{L}$  is open in the direct limit topology if and only if for any finite union of tiles  $B$  the set  $A \cap B$  is open in the relative topology of  $B$ .

**Corollary 9.12.** *Let  $(\mathbb{Z}^n, X^\omega)$  be a self-similar recurrent finite-state action. Then*

1. *the limit space  $\mathcal{J}_{\mathbb{Z}^n}$  is homeomorphic to the torus  $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ ;*
2. *for every leaf  $\mathcal{L}$  (with its direct limit topology) of the solenoid  $\mathcal{S}_{\mathbb{Z}^n}$  there exists a homeomorphism  $\Phi : \mathcal{L} \rightarrow \mathbb{R}^n$  such that for every tile  $\mathcal{T}_w$  of  $\mathcal{L}$  we have  $\Phi(\mathcal{T}_w) = \mathcal{T}(\phi, R) + r(w)$  for some  $r(w) \in \mathbb{Z}^n$ .*

Essentially,  $r(w)$  is the base- $\phi$  evaluation of  $w$ .

It follows from the description we obtained of the limit space  $\mathcal{J}_{\mathbb{Z}^n}$  that the shift  $\mathfrak{s}$  on it coincides with the map on the torus  $(\mathbb{R}/\mathbb{Z})^n$  given by the linear transformation  $\phi^{-1}$ . This map is obviously a  $d$ -to-1 covering. The tiles, just as in the general case, define a Markov partition for this toral dynamical system.

Corollary 9.12 shows that the tiled leaves of contracting recurrent self-similar actions of abelian groups are the classical digit tilings of Euclidean space. For example, a part of the tiling by “dragons” is shown on Figure 25. The union of the two marked central tiles is similar to the original tile.

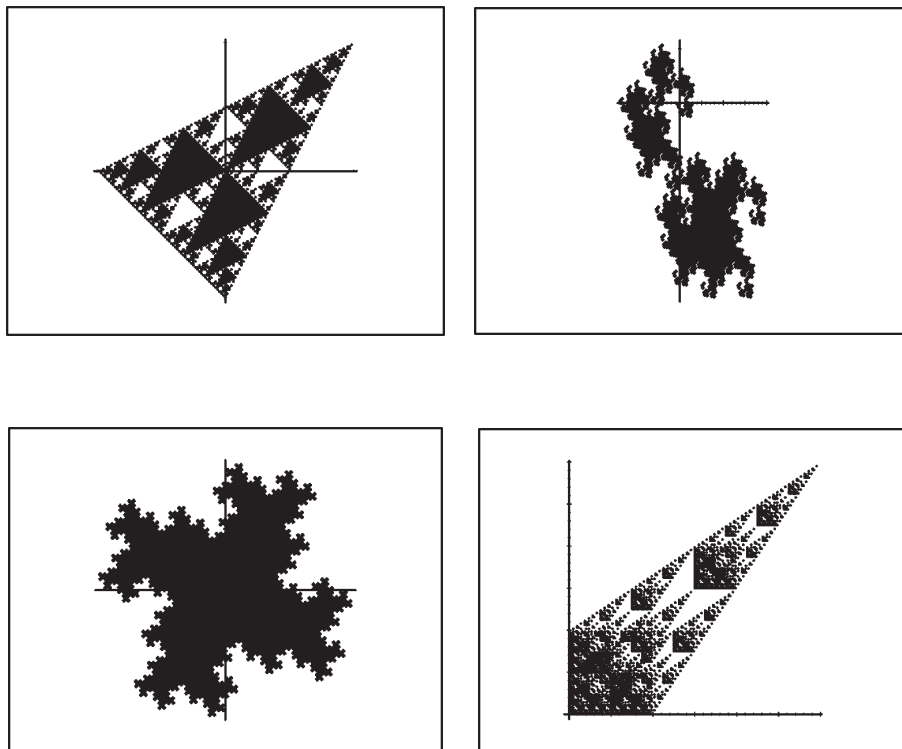


Figure 24: Tiles of abelian groups

## 9.6 Limit spaces of the inverse semigroups

**Definition 9.8.** A self-similar inverse semigroup  $G$  acting on a topological Markov chain  $\mathcal{F} \subseteq X^\omega$  is *contracting* if there exists a finite set  $\mathcal{N} \subset G$  such that for every  $g \in G$  and for every word  $w \in \mathcal{F}$  there exists a finite prefix  $v$  of  $w$  such that

$$\mathsf{T}_v g = h_1 \mathsf{T}_{u_1} + h_2 \mathsf{T}_{u_2} + \cdots + h_k \mathsf{T}_{u_k},$$

where  $h_i \in \mathcal{N}$  and  $u_i$  are words of length  $|v|$ , and  $\mathsf{T}_u$  denotes the partial permutation  $\mathsf{T}_u : x \mapsto ux$ .

It is easy to prove that, if  $G$  is a group, this definition agrees with Definition 6.4 of a contracting self-similar action of a group on a space  $X^\omega$ .

In the case of inverse semigroup actions we can not define the limit space  $\mathcal{J}_G$ , since we do not have a canonical action of the semigroup on the set of finite words. However, we can define the limit solenoid  $\mathcal{S}_G$  using the action on infinite sequences.

Let  $G$  be an inverse semigroup acting on a shift of finite type  $\mathcal{F} \subseteq X^\omega$ . Define  $\mathcal{F}^\mathbb{Z} \subseteq X^\mathbb{Z}$  to be the two sided shift space defined by the same set of admissible words as  $\mathcal{F}$ . More prosaically,  $\mathcal{F}$  is the set of all biinfinite sequences  $\dots x_{-2}x_{-1}.x_0x_1x_2\dots \in X^\mathbb{Z}$  such that for every  $n \in \mathbb{Z}$  the sequence  $x_n x_{n+1} x_{n+2} \dots$  belongs to  $\mathcal{F}$ .

We define the asymptotic equivalence relation on  $\mathcal{F}^\mathbb{Z}$  for the action of the semigroup  $G$  exactly in the same way as it is defined on the space  $X^\mathbb{Z}$  for group actions. The *limit solenoid*  $\mathcal{S}_G$  is then the quotient of the topological space  $\mathcal{F}^\mathbb{Z}$  by the asymptotic equivalence relation.

We also define the tiles of the solenoid in the same way as it is done for groups (Definition 9.7).

By the *type* of a tile  $\mathcal{T}_w$  in the case of an inverse semigroup we mean a pair  $(D, F)$  of subsets of the nucleus. The set  $D$  is the set of those elements  $g$  of the nucleus for which  $w$  is contained in the domain  $\text{Dom } g$ . The set  $F$  is, as in the case of the group actions, the set of the elements fixing the word  $w$ .

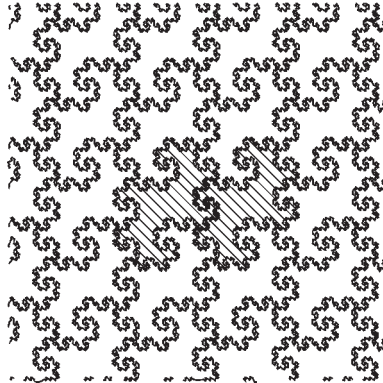


Figure 25: Plane tiling by dragon curves

Proposition 9.9 remains true for contracting actions of inverse semigroups. Therefore, for self-similar contracting inverse semigroups the tiling iterated function systems are also well defined.

**The Fibonacci transformations.** The semigroup generated by the Fibonacci transformations is contracting with contracting coefficient  $\tau^{-1}$ , where  $\tau = \frac{1+\sqrt{5}}{2}$ .

The corresponding iterated function system on the tiles is the *Fibonacci iterated function system* described among the Examples of Subsection 3.2.

**Penrose tilings.** The semigroup related to the Penrose tilings is also contracting, with contraction coefficient  $\tau^{-1}$ . The tiling iterated function system on the tiles corresponding to this semigroup is exactly the Penrose iterated functions system.

## 10 Hyperbolic spaces and groups

### 10.1 Definitions

**Definition 10.1.** A metric space  $(\mathfrak{X}, d)$  is  $\delta$ -hyperbolic (in the sense of M. Gromov) if for every  $x_0, x, y, z \in \mathfrak{X}$  the inequality

$$\langle x \cdot y \rangle_{x_0} \geq \min \{ \langle x \cdot z \rangle_{x_0}, \langle y \cdot z \rangle_{x_0} \} - \delta$$

holds, where

$$\langle x \cdot y \rangle_{x_0} = \frac{1}{2} (d(x_0, x) + d(x_0, y) - d(x, y))$$

denotes the *Gromov product* of the points  $x$  and  $y$  with respect to the base point  $x_0$ .

Examples of hyperbolic metric spaces are all bounded spaces (with  $\delta$  equal to the diameter of the space), trees (which are 0-hyperbolic) and the usual hyperbolic space  $\mathbb{H}^n$ , which is hyperbolic with  $\delta = \log 3$ .

**Definition 10.2.** A finitely generated group is *hyperbolic* if it is hyperbolic as a word metric space.

The definition is independent of the choice of the generating set with respect to which the word metric is defined. For the proof of this fact, and for the proof of other properties of hyperbolic groups, see [Gro87, CDP90, CP93, GH90].

Here is a short summary of examples and properties of hyperbolic groups:

1. Every finite group is hyperbolic.

2. Every finitely generated free group is hyperbolic.
3. If  $G_1$  is a subgroup of finite index of the group  $G$ , then  $G$  is hyperbolic if and only if  $G_1$  is hyperbolic.
4. A free product of two hyperbolic groups is hyperbolic.
5. The fundamental group of a compact Riemannian space of negative curvature is hyperbolic.
6. A hyperbolic group is finitely presented.
7. The word problem in a hyperbolic group is solvable in linear time.
8. A subgroup of a hyperbolic group either contains the free subgroup  $F_2$ , or is a finite extension of a cyclic group.
9. Hyperbolic groups have a rational growth series, and are either virtually cyclic or have uniformly exponential growth (see Subsection 8.5).

## 10.2 The boundary of a hyperbolic space

Let  $(\mathfrak{X}, d)$  be a hyperbolic space. We say that a sequence  $(x_n)_{n \geq 1}$  of points of the space *converges to infinity* if for a fixed  $x_0 \in \mathfrak{X}$

$$\langle x_n \cdot x_m \rangle_{x_0} \rightarrow +\infty \quad \text{when } n, m \rightarrow +\infty.$$

It is easy to prove that the definition does not depend on the choice of the point  $x_0$ .

Two sequences  $(x_n), (y_n)$  converging to infinity are said to be *equivalent* if  $\langle x_n \cdot y_n \rangle_{x_0} \rightarrow +\infty$ , when  $n \rightarrow +\infty$ .

This definition also does not depend on  $x_0$ . The quotient of the set of sequences converging to infinity by this equivalence relation is called the *boundary* of the hyperbolic space  $\mathfrak{X}$ , and is denoted  $\partial\mathfrak{X}$ .

If a sequence  $(x_n)$  converges to infinity, then its *limit* is the equivalence class  $a \in \partial\mathfrak{X}$  to which the sequence  $(x_n)$  belongs, and we say that  $(x_n)$  *converges to*  $a$ .

If  $a, b \in \partial\mathfrak{X}$  are two points of the boundary, then their *Gromov product* is defined as

$$\langle a \cdot b \rangle_{x_0} = \sup_{(x_n) \in a, (y_m) \in b} \liminf_{m, n \rightarrow \infty} \langle x_n \cdot y_m \rangle_{x_0}.$$

For every  $r > 0$  define

$$V_r = \{(a, b) \in \partial\mathfrak{X} \times \partial\mathfrak{X} : \langle a \cdot b \rangle_{x_0} \geq r\}.$$

Then the set  $\{V_r : r \geq 0\}$  is a fundamental neighborhood basis of a uniform structure on  $\partial\mathfrak{X}$  (see [Bou71] and [GH90] for the necessary definitions and proofs). We topologize the boundary  $\partial\mathfrak{X}$  by this uniform structure.

Another way to define the topology on the boundary is to introduce the *visual metric* on it.

Namely, let  $\mathfrak{X}$  be a geodesic hyperbolic metric space with a base-point  $x_0$ . Recall that a metric space  $\mathfrak{X}$  is said to be *geodesic* if any two points  $x, y \in \mathfrak{X}$  can be connected by a path defined by an isometric embedding of the real segment  $[0, d(x, y)]$  into the space.

Let  $a \in (0, 1)$  be a number close to 1. Then for every path  $\gamma : [0, t] \rightarrow \mathfrak{X}$  we define its  $a$ -length  $l_a(\gamma)$  as the integral

$$l_a(\gamma) = \int_{x \in \gamma[0, t]} a^{-d(x, x_0)} dx.$$

Then we define the  $a$ -distance between two points  $x, y \in \mathfrak{X}$  as the infimum of  $a$ -lengths of the continuous paths connecting  $x$  and  $y$ . There exists  $a_0 \in (0, 1)$  such that for all  $a \in (a_0, 1)$  the completion of the  $a$ -metric on  $\mathfrak{X}$  is  $\mathfrak{X} \cup \partial\mathfrak{X}$ . The restriction of the extended metric on  $\partial\mathfrak{X}$  is called *visual metric* on the boundary.

The boundary of a hyperbolic group has a rich self-similar structure (see Section 11 and the book [CP93]). Some of the classical fractals (for instance, the Sierpinski carpet) can also be realized as the boundary of a hyperbolic group (see [CP93, KK00] and their bibliography).

**Questions.** What topological spaces can be realized as the boundaries of hyperbolic groups? How can one compute the Hausdorff dimension of the boundary of a hyperbolic group respectively to the visual metric?

**Definition 10.3.** Suppose a group  $G$  acts by isometries on a hyperbolic space  $\mathfrak{X}$ . Then its *limit set* is the set of all the limits in  $\partial\mathfrak{X}$  of sequences of the form  $x_k = x_0^{g_k}, k \geq 1$ , where  $g_k \in G$  and  $x_0 \in \mathfrak{X}$  is a fixed point.

It is easy to prove that the limit set does not depend on the point  $x_0$ . An important case is the limit set of a Kleinian group acting on  $\mathbb{H}^n$ . It is a subset of  $S^{n-1} = \partial\mathbb{H}^n$ .

**Example.** Let  $\Gamma$  be the group generated by the four inversions  $\gamma_i$  defining the self-similarity structure of the Apollonian net (see “The group associated with the Apollonian gasket” in Subsection 3.7). Since the group of conformal automorphisms of the Riemannian sphere is isomorphic to the group of isometries of the hyperbolic space  $\mathbb{H}^3$ , the action of  $\Gamma$  on the sphere is extended in the standard way to an action by isometries on the space  $\mathbb{H}^3$  (see [EGM98]), where the sphere is identified with the boundary  $\partial\mathbb{H}^3$ . In the extended action the generators  $\gamma_i$  are reflections. For instance, in the Kleinian model of the space  $\mathbb{H}^3$ , the generators  $\gamma_i$  can be defined by the matrices

$$\begin{pmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{pmatrix}.$$

It is easy to see that the limit set of the obtained group is the Apollonian net  $\mathcal{P}$ .

Let  $\Gamma_1 < \Gamma$  be the subgroup of orientation-preserving transformations. It is of index 2 in  $\Gamma$  and is a free group of rank 3 generated by the elements  $\gamma_1\gamma_2, \gamma_1\gamma_3$  and  $\gamma_1\gamma_4$ . It is a subgroup of the Moebius group and its limit set coincides with the limit set of the group  $\Gamma$ , i.e., with the Apollonian net.

### 10.3 The limit space of a self-similar action as a hyperbolic boundary

**Definition 10.4.** Let  $(G, X^\omega)$  be a self-similar action of a finitely generated group. For a given finite generating system  $S$  of the group  $G$  we define the *self-similarity complex*  $\Sigma(G, S)$  as the 1-complex with set of vertices  $X^*$ , and with two vertices  $v_1, v_2 \in X^*$  belonging to a common edge if and only if either  $v_i = xv_j$  for some  $x \in X$  (the edges of the *first type*) or  $v_i^s = v_j$  for some  $s \in S$  (the edges of the *second type*); here  $\{i, j\} = \{1, 2\}$ .

The set of vertices of the self-similarity complex splits into the levels  $X^n, n \in \mathbb{N}$ . Every edge of the first type connects two vertices from neighboring levels, while every edge of the second type connects the vertices of the same level. The set of edges of the second type spans  $\Gamma(G, S, X^*)$ , the disjoint union of all the finite Schreier graphs  $\Gamma_n(G, S)$  of the group  $G$ .

A part of the self-similarity complex of the adding machine is shown in Figure 26.

If all the restrictions of the elements of the generating set  $S$  also belong to  $S$  (this is the case, for instance, when  $S$  is the nucleus  $\mathcal{N}$  of  $G$ ), then the self-similarity complex  $\Sigma(G, S)$  is an *augmented tree* in the sense of V. Kaimanovich (see [Kai02]).

**Theorem 10.1.** *If the action of a finitely generated group  $G$  is contracting, then the self-similarity complex  $\Sigma(G, S)$  is a Gromov-hyperbolic space.*

*The limit space  $\mathcal{J}_G$  is then homeomorphic to the hyperbolic boundary  $\partial\Sigma(G, S)$  of the self-similarity complex  $\Sigma(G, S)$ . Moreover, there exists a homeomorphism  $\mathcal{J}_G \rightarrow \partial\Sigma(G, S)$ , which makes the diagram*

$$\begin{array}{ccc} X^{-\omega} & & \\ \pi \downarrow & \searrow \ell & \\ \mathcal{J}_G & \longrightarrow & \partial\Sigma(G, S) \end{array}$$

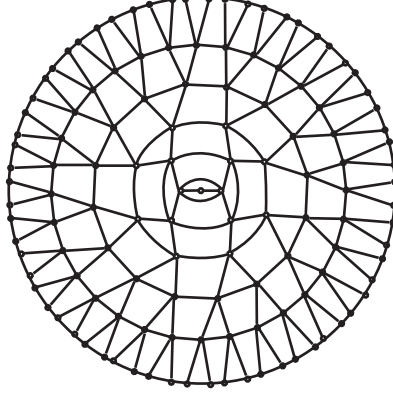


Figure 26: The self-similarity complex of the adding machine

commutative. Here  $\pi$  is the canonical projection and  $\ell$  carries every sequence  $\dots x_2 x_1 \in X^{-\omega}$  to its limit

$$\lim_{n \rightarrow \infty} x_n x_{n-1} \dots x_1 \in \partial \Sigma(G, S).$$

## 11 Finitely presented dynamical systems and semi-Markovian spaces

Let  $(\{F_v\}_{v \in V}, \{\phi_e\}_{e \in E})$  be a self-similarity structure on a compact space  $F = \cup_{v \in V} F_v$ .

Suppose that the self-similarity structure is such that for every infinite path  $\dots e_2 e_1$  in the structural graph of the iterated function system, the intersection

$$\bigcap_{k \geq 1} \phi_{e_1} (\phi_{e_2} (\dots \phi_{e_k} (F_{\alpha(e_k)})) \dots),$$

contains exactly one point of the set  $F$ .

Let  $\mathcal{F} \subset E^\omega$  be the set of all sequences  $e_1 e_2 \dots$  for which  $\dots e_2 e_1$  is a path in the structural graph of the self-similarity structure.

**Proposition 11.1.** *The map  $\Pi : \mathcal{F} \rightarrow F$  defined by the rule*

$$\{\Pi(e_1 e_2 \dots)\} = \bigcap_{k \geq 1} \phi_{e_1} (\phi_{e_2} (\dots \phi_{e_k} (F_{\alpha(e_k)})) \dots)$$

*is continuous.*

The space  $E^\omega \times E^\omega$  is naturally identified with the one-sided shift space  $(E \times E)^\omega$  via the homeomorphism

$$(e_1 e_2 \dots, f_1 f_2 \dots) \mapsto (e_1, f_1)(e_2, f_2) \dots$$

**Definition 11.1.** The self-similarity structure  $(\{F_v\}_{v \in V}, \{\phi_e\}_{e \in E})$  on the space  $F$  is *finitely presented* if the set

$$\{(w_1, w_2) : \Pi(w_1) = \Pi(w_2)\} \subseteq E^\omega \times E^\omega$$

is a subshift of finite type in  $(E \times E)^\omega$ .

A topological space  $F$  which has a finitely presented self-similarity structure is called *Markovian*.

If the maps  $\phi_e^{-1}$  are restrictions of a map  $f : F \rightarrow F$ , and the self-similarity structure is finitely presented, then the dynamical system  $(F, f)$  is also called *finitely presented* (see [CP93]).

The notion of a finitely presented dynamical system and a (semi-)Markovian space was formulated for the first time by M. Gromov (see [Gro87]).

We recall here the definitions of a shift of finite type and of a finitely presented dynamical system in the general case of a semigroup action, following [CP93]. The one-sided shift is then the case of the semigroup  $\mathbb{N}$ ; the bilateral shift is the case of the group  $\mathbb{Z}$ .

Let  $X$  be an alphabet and let  $G$  be a semigroup. The *shift space* is the Cartesian power  $X^G$ , i.e., the set of all functions  $G \rightarrow X$ . We put on the shift space the direct product topology. The semigroup  $G$  acts on the elements of the shift space  $X^G$  by the rule

$$\xi^g(h) = \xi(gh).$$

A subset  $C \subseteq X^G$  is called a *cylinder* if there exists a finite set  $F \subseteq G$  and a finite set  $A$  of functions  $F \rightarrow X$  such that  $\xi$  is an element of  $C$  if and only if the restriction  $\xi|_F$  belongs to  $A$ . Any cylinder is a closed and open subset of the space  $X^G$ .

In the same way as for the unilateral shift, we can identify the direct product  $X^G \times X^G$  with the full shift space  $(X \times X)^G$  over the alphabet  $X \times X$ . Namely, the pair  $(\xi, \zeta) \in X^G \times X^G$  is identified with the function  $h \mapsto (\xi(h), \zeta(h))$ .

**Definition 11.2.** A subset  $\mathcal{F} \subseteq X^G$  is a *subshift of finite type* if there exists a cylinder  $C \subset X^G$  such that

$$\mathcal{F} = \bigcap_{g \in G} C^{g^{-1}},$$

where  $C^{g^{-1}} = \{\xi : \xi^g \in C\}$ .

It is easy to see that every subshift of finite type is a closed  $G$ -invariant subset of  $X^G$ .

**Definition 11.3.** A dynamical system  $(\mathfrak{X}, G)$  (i.e., a topological space  $\mathfrak{X}$  with a continuous action of a semigroup  $G$ ) is a *system of finite type* if there exists a finite alphabet  $X$ , a subshift of finite type  $\mathcal{F} \subseteq X^G$  and a continuous, surjective and  $G$ -equivariant map  $\pi : \mathcal{F} \rightarrow \mathfrak{X}$ .

A dynamical system  $(\mathfrak{X}, G)$  is *finitely presented* if in addition the set  $\{(\xi, \zeta) \in \mathcal{F} \times \mathcal{F} : \pi(\xi) = \pi(\zeta)\}$  is a subshift of finite type in  $(X \times X)^G$ .

**Definition 11.4.** A dynamical system  $(\mathfrak{X}, G)$  is *expansive* if there exists an open set  $U \subset \mathfrak{X} \times \mathfrak{X}$  such that

$$\Delta = \bigcap_{g \in G} U^{g^{-1}},$$

where  $\Delta = \{(x, x) | x \in \mathfrak{X}\}$  is the diagonal in  $\mathfrak{X} \times \mathfrak{X}$ .

For the proof of the following theorem see [CP93] (see also [Fri87], where the case  $G = \mathbb{Z}$  is considered):

**Theorem 11.2.** *Let  $(\mathfrak{X}, G)$  be a dynamical system. Let  $\mathcal{F} \subseteq X^G$  be a subshift of finite type and  $\pi : \mathcal{F} \rightarrow \mathfrak{X}$  a continuous, surjective and  $G$ -equivariant map. Then the space  $\{(\xi, \zeta) \in \mathcal{F} \times \mathcal{F} : \pi(\xi) = \pi(\zeta)\}$  is a subshift of finite type if and only if the system  $(\mathfrak{X}, G)$  is expansive.*

**Corollary 11.3.** *A dynamical system  $(\mathfrak{X}, G)$  is finitely presented if and only if it is both expansive and of finite type.*

### Examples.

1. Any subshift of finite type is a finitely presented dynamical system.
2. Let  $G$  be a hyperbolic group. Then the dynamical system  $(\partial G, G)$  is finitely presented. See the book [CP93], where different finite presentations of this dynamical system are given.

**Definition 11.5.** A Hausdorff topological space  $\mathfrak{X}$  is *semi-Markovian* if there exists a finite alphabet  $X$ , a cylinder  $C \subseteq X^\omega$ , a subshift of finite type  $\mathcal{F} \subseteq X^\omega$  and a continuous surjection  $\Pi : C \cap \mathcal{F} \rightarrow \mathfrak{X}$  such that the set  $\{(w_1, w_2) : \Pi(w_1) = \Pi(w_2)\}$  is an intersection of a cylinder and a subshift of finite type in  $(X \times X)^\omega$ .

**Examples.**

1. Obviously the shifts of finite type and the Cantor space  $X^\omega$  are semi-Markovian spaces.
2. The real segment  $[0, 1]$  is a semi-Markovian space. The semi-Markovian presentation is defined, for instance, by the dyadic expansion of the reals.
3. In general, any finite simplicial complex is semi-Markovian.
4. The boundary of a torsion-free hyperbolic group is semi-Markovian.

For more details on these examples see [CP93].

## 12 Spectra of Schreier graphs and Hecke type operators

Let  $\Gamma$  be a graph, and consider the Hilbert space  $\mathcal{H} = \ell^2(\Gamma, \text{deg})$ , the complex space of finite-norm functions on  $V(\Gamma)$  determined by the scalar product

$$\langle f | g \rangle = \sum_{v \in V(\Gamma)} f(v) \overline{g(v)} \text{deg}(v).$$

The adjacency of  $\Gamma$  defines an operator  $M$  on  $\mathcal{H}$ , called its *Markov* or *transition* operator, by

$$M(f)(v) = \frac{1}{\text{deg}(v)} \sum_{\substack{e \in E(\Gamma) \\ \alpha(e)=v}} f(\omega(e)).$$

The *spectrum* of  $\Gamma$  is defined as the spectrum of  $M$ . If  $\Gamma$  has bounded degree, then  $M$  is a bounded operator, and hence the spectrum of  $\Gamma$  is compact.

The spectrum of  $\Gamma$  is intimately connected to random walk properties of  $\Gamma$ : the Green function is essentially the resolvent of  $M$ ; the graph  $\Gamma$  is amenable if and only if the spectral radius of  $M$  is 1, as was shown by Kesten [Kes59].

**Definition 12.1.** Let  $G$  be a group with fixed generating set  $S$ , and let  $\pi$  be a unitary representation of  $G$  on a Hilbert space  $\mathcal{H}$ . The associated *Hecke type operator* is

$$H_\pi = \frac{1}{|S|} \sum_{s \in S} \pi(s) \in \mathcal{B}(\mathcal{H}).$$

In the case when  $S$  is symmetric (meaning  $S = S^{-1}$ ),  $H$  is a self-adjoint operator. The *spectrum* of  $\pi$  is the spectrum of its Hecke type operator  $H_\pi$ . Note that it depends also on  $S$ , although this is not apparent in the notation.

In our situation, we have a group  $G$  acting on the boundary  $X^\omega$  of a  $d$ -regular tree  $T(X)$ , and hence  $G$  acts unitarily on the Hilbert space  $\mathcal{H} = L^2(X^\omega, \mu)$ , where  $\mu$  is the uniform measure. By restriction to the subsets  $xX^\omega$  for  $x \in X$ , we get an isomorphism  $\phi : \mathcal{H} \cong \mathcal{H}^X$  given by  $\phi(f)(x) = f\Gamma_x : w \mapsto f(xw)$ . We use this isomorphism to obtain finite approximations to the spectrum of the representation on  $\mathcal{H}$ .

This isomorphism also makes possible to interpret the self-similarity of the group actions in terms of operator algebras. We do this in Section 13.

Consider the constant functions  $\mathcal{H}_0 = \mathbb{C} \subset \mathcal{H}$ , and define subspaces  $\mathcal{H}_{n+1} = \phi^{-1}(\mathcal{H}_n^X)$  for all  $n \geq 0$ . All  $\mathcal{H}_n$  are closed  $G$ -invariant subspaces of  $\mathcal{H}$ , and  $\dim \mathcal{H}_n = d^n$ . We will compute the spectrum of  $\pi_n = \pi|_{\mathcal{H}_n}$ ; since  $\pi_n$  contains  $\pi_{n-1}$  as a subrepresentation, we write  $\pi_n \ominus \pi_{n-1}$  an orthogonal complement. The convergence of the spectrum of  $\pi_n$  towards the spectrum of  $\pi$  is given by the following result from [BG00b]:

**Theorem 12.1.**  $\pi$  is a reducible representation of infinite dimension whose irreducible components are precisely those of the  $\pi_n \ominus \pi_{n-1}$  (and thus are all finite-dimensional). Moreover

$$\text{spec}(\pi) = \overline{\bigcup_{n \geq 0} \text{spec}(\pi_n)}.$$

$H_\pi$  has a pure-point spectrum, and its spectral radius  $r(H_\pi) = s \in \mathbb{R}$  is an eigenvalue.

Consider also the following family of representations: let  $H$  be a subgroup of  $G$ , and let  $G$  act on  $\ell^2(G/H)$  by left translations. The resulting unitary representation  $\rho_{G/H}$  is called *quasi-regular*; its Hecke-type operator is the Markov operator of the Schreier graph of  $\Gamma(G, S, G/H)$ . If  $H = 1$ , this is the *regular* representation, and more generally if  $H$  is normal  $\rho_{G/H}$  is the regular representation of the quotient.

Let  $P_n$  be the stabilizer in  $X^*$  of a sequence  $w_n$  of length  $n$ . Then  $\rho_{G/P_n}$  is naturally isomorphic to  $\pi_n$ , since the cosets of  $P_n$  are in 1 – 1 correspondence with the orbit  $X^n$  of  $w$ . Consider also  $P$ , the stabilizer of an infinite sequence  $w \in X^\omega$ , and the quasi-regular representation  $\rho_{G/P}$ .

**Theorem 12.2.** The spectrum of  $\rho_{G/P}$  is contained in the spectrum of  $\pi$ , and if either  $G/P$  or  $P$  is amenable, these spectra coincide. If  $P$  is amenable, then they are contained in the spectrum of  $\rho_G$ .

The spectral radius of  $\rho_{G/P}$  is never an eigenvalue.

Note then that, under the amenability condition,  $\rho_{G/P}$  and  $\pi$  are two distinct representations with same spectrum.

Each  $\pi_n(g)$  is a  $d^n \times d^n$ -permutation matrix, giving  $g$ 's action on  $\mathcal{H}_n$ . By decomposition, each  $\pi_n(g)$  is a  $d \times d$ -matrix, whose entries are either 0 or  $\pi_{n-1}(g')$  for some  $g' \in G$ . In particular, if  $G$  is contracting, there is a finite generating set  $K$  such that for each  $k \in K$ , the matrix  $\pi_n(k)$  has entries of the form 0 or  $\pi_{n-1}(k')$  for some  $k' \in K$ .

The computation of the spectrum of  $\pi_n$  can be seen in a more general context, that of *Frobenius determinants*. Given a finite-dimensional representation  $\rho$  of a group  $G$ , let  $\{\lambda_g\}$  be a set of formal variables indexed by  $G$ , and define the Frobenius determinant as

$$\Phi_\rho = \det \left( \sum_{g \in G} \lambda_g \rho(g) \right).$$

It is known that the factorization of  $\Phi_\rho$  in prime components exactly parallels the decomposition of  $\rho$  in irreducible components.

Even though  $G$  is not assumed to be finite, it is clear that  $\Phi_\rho$  depends only on  $G/\ker \rho$ . In our case,  $\rho = \pi_n$  is a permutational representation, so  $G/\ker \pi_n$  is a finite group acting on  $X^n$ .

The spectrum of  $\rho$  is obtained by substituting  $\lambda_s = 1/|S|$  for all  $s \in S$ , preserving  $\lambda_1$ , and setting  $\lambda_g = 0$  for all other variables, in  $\Phi_\rho$ ; and then solving the resulting polynomial in  $\lambda_1$ . The following somewhat miraculous facts make it sometimes possible to compute this particular value of  $\Phi_\rho$ :

- If  $G$  is sufficiently contracting, it may be possible to consider only a finite subset  $V = \{\lambda_g\}_{g \in F}$  of variables, and to express  $\Phi_{\pi_n}(V)$  in the form  $R(\Phi_{\pi_{n-1}}(S(V)))$  for a rational function  $R \in \mathbb{C}(z)$  and  $S \in \mathbb{C}(V)$ ;
- Even though  $\Phi_\rho$  may not factor in many low-degree terms, it may happen that after imposing some extra conditions among the  $\lambda_g$ , that are both weak enough so that they allow  $\lambda_s = 1$  for  $s \in S$  and  $\lambda_1 = -|S|\lambda$  to be chosen, and strong enough so that a recursion still holds between  $\Phi_{\pi_n}$  and  $\Phi_{\pi_{n-1}}$ , the determinant  $\Phi_\rho$  does factor nicely.

These two properties hold for the Grigorchuk group, the Gupta-Sidki group, the Fabrykowski-Gupta group and the Bartholdi-Grigorchuk group [BG00b], and for the lamplighter group [GŻuk01], allowing a complete computation of the spectrum of  $\pi$ .

We make this process explicit for the group  $\text{IMG}(z^2 - 2)$ . It is the group, generated by the transformations  $a = (b, 1)\sigma, b = (a, 1)$  (see Subsection 5.2). Define the polynomial

$$\Phi_k(\lambda; \lambda_1, \lambda_2) = \det \left( \lambda + \lambda_1(\pi_k(a) + \pi_k(a^{-1})) + \lambda_2(\pi_k(b) + \pi_k(b^{-1})) \right).$$

Write  $\Lambda = \lambda + 2\lambda_2$ . Then we have

$$\begin{aligned} \Phi_{k+1}(\lambda; \lambda_1, \lambda_2) &= \det \begin{pmatrix} \lambda + \lambda_2(\pi_k(a) + \pi_k(a^{-1})) & \lambda_1(1 + \pi_k(b^{-1})) \\ \lambda_1(1 + \pi_k(b)) & \Lambda \end{pmatrix} \\ &= \Phi_k(\Lambda\lambda - 2\lambda_1^2; \Lambda\lambda_2, -\lambda_1^2). \end{aligned}$$

Consider the curve  $\Lambda$  in  $\mathbb{R}^3$ , and the polynomial transformation

$$P : (\lambda; \lambda_1, \lambda_2) \mapsto (\Lambda\lambda - 2\lambda_1^2; \Lambda\lambda_2, -\lambda_1^2)$$

on  $\mathbb{R}^{r+1}$ ; consider the intersection of  $P^{-k}(\Lambda)$  with  $\{\lambda_1 = \lambda_2 = -1/4\}$ . This is the spectrum of  $\pi_k$ . Therefore, the spectrum of  $\pi$  is the intersection of  $\bigcup_{k \geq 0} P^{-k}(\Lambda)$  with  $\{\lambda_1 = \lambda_2 = -1/4\}$ . This spectrum seems to be a Cantor set of null measure, although we have no complete proof of this fact; see Figure 27. The problem is that  $\Phi_k(\lambda; -1/4, -1/4)$  does not have any nice factorization properties (it has an irreducible factor of degree  $2^{k-2} + 1$ )

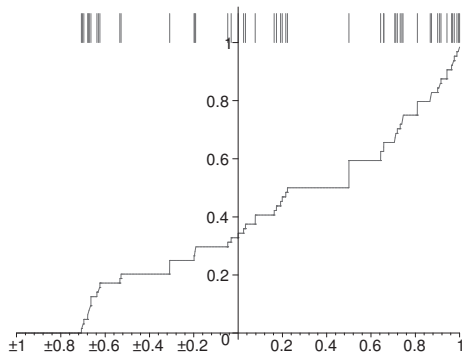


Figure 27: The spectrum and (counting) spectral measure of  $\text{IMG}(z^2 - 1)$

## 12.1 The Fabrykowski-Gupta group

The spectrum of the quasi-regular representation of this group, introduced in Subsection 9.3, was computed by the first two named authors in [BG00b]. In standard notation, it is the group

$$G = \langle a = (123), \quad s = (a, 1, s) \rangle.$$

We only quote the statements, whose proof appear in the above-mentioned paper. Denote by  $a_n$  and  $s_n$  the matrices of the action on  $\{0, 1, 2\}^n$ . We have

$$\begin{aligned} a_0 &= s_0 = (1), \\ a_n &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad s_n = \begin{pmatrix} a_{n-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s_{n-1} \end{pmatrix}. \end{aligned}$$

Let us define operators

$$A_n = a_n + a_n^{-1}, \quad S_n = s_n + s_n^{-1}, \quad Q_n(\lambda, \mu) = S_n + \lambda A_n - \mu;$$

then the combinatorial Laplacian of  $G$  on  $X^n$  is

$$\Delta_n = a_n + a_n^{-1} + s_n + s_n^{-1} = \begin{pmatrix} A_{n-1} & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & S_{n-1} \end{pmatrix}.$$

Define the polynomials

$$\begin{aligned} \alpha &= 2 - \mu + \lambda, & \beta &= 2 - \mu - \lambda, \\ \gamma &= \mu^2 - \lambda^2 - \mu - 2, & \delta &= \mu^2 - \lambda^2 - 2\mu - \lambda. \end{aligned}$$

**Lemma 12.3.**

$$\begin{aligned} \det Q_0(\lambda, \mu) &= 2 + 2\lambda - \mu = \alpha + \lambda, \\ \det Q_1(\lambda, \mu) &= (2 + 2\lambda - \mu)(2 - \lambda - \mu)^2 = (\alpha + \lambda)\beta^2, \\ \det Q_n(\lambda, \mu) &= (\alpha\beta\gamma^2)^{3^{n-2}} \det Q_{n-1} \left( \frac{\lambda^2\beta}{\alpha\gamma}, \mu + \frac{2\lambda^2\delta}{\alpha\gamma} \right) \quad (n \geq 2). \end{aligned}$$

Consider now the quadratic forms

$$H_\theta = \mu^2 - \lambda\mu - 2\lambda^2 - 2 - \mu + \theta\lambda,$$

and the function  $F : [-4, 5] \rightarrow [-4, 5]$  given by  $F(\theta) = 4 - 2\theta - \theta^2$ . Set  $X_2 = \{-1\}$ , and iteratively define  $X_n = F^{-1}(X_{n-1})$  for all  $n \geq 3$ . Note that  $|X_n| = 2^{n-2}$ .

**Lemma 12.4.** *We have for all  $n \geq 2$  the factorization*

$$\det Q_n(\lambda, \mu) = (2 + 2\lambda - \mu)(2 - \lambda - \mu)^{3^{n-1}+1} \prod_{\substack{2 \leq m \leq n \\ \theta \in X_m}} H_\theta^{3^{n-m}+1}.$$

Thus, according to the previous proposition, the spectrum of  $Q_n$  is a collection of two lines and  $2^{n-1} - 1$  hyperbolæ  $H_\theta$  with  $\theta \in X_2 \sqcup X_3 \sqcup \dots \sqcup X_n$ . The spectrum of  $\Delta_n$  is obtained by solving  $\det Q_n(1, \mu) = 0$ , as given in Figure 28 and the following theorem:

**Theorem 12.5.** *Let  $\pi_\pm : [-4, 5] \rightarrow [-2, 4]$  be defined by  $\pi_\pm(\theta) = 1 \pm \sqrt{5 - \theta}$ . Then*

$$\begin{aligned} \text{spec } \Delta_0 &= \{4\}; \\ \text{spec } \Delta_1 &= \{1, 4\}; \\ \text{spec } \Delta_n &= \{1, 4\} \cup \bigcup_{2 \leq m \leq n} \pi_\pm(X_m) \quad (n \geq 2). \end{aligned}$$

*The spectrum of  $\pi$  for the Fabrykowski-Gupta group  $G$  is the closure of the set (here  $\mu$  indicates the dimension of the eigenspace)*

$$\left\{ \begin{array}{l} 4 \quad (\mu = 0) \\ 1 \quad (\mu = \frac{1}{3}) \\ 1 \pm \sqrt{6} \quad (\mu = \frac{1}{9}) \\ 1 \pm \sqrt{6 \pm \sqrt{6}} \quad (\mu = \frac{1}{3^3}) \\ 1 \pm \sqrt{6 \pm \sqrt{6 \pm \sqrt{6}}} \quad (\mu = \frac{1}{3^4}) \\ \dots \end{array} \right\}.$$

*It is the union of a Cantor set of null Lebesgue measure that is symmetrical about 1, and a countable collection of isolated points supporting the empiric spectral measure, which has the values indicated as  $\mu$ .*

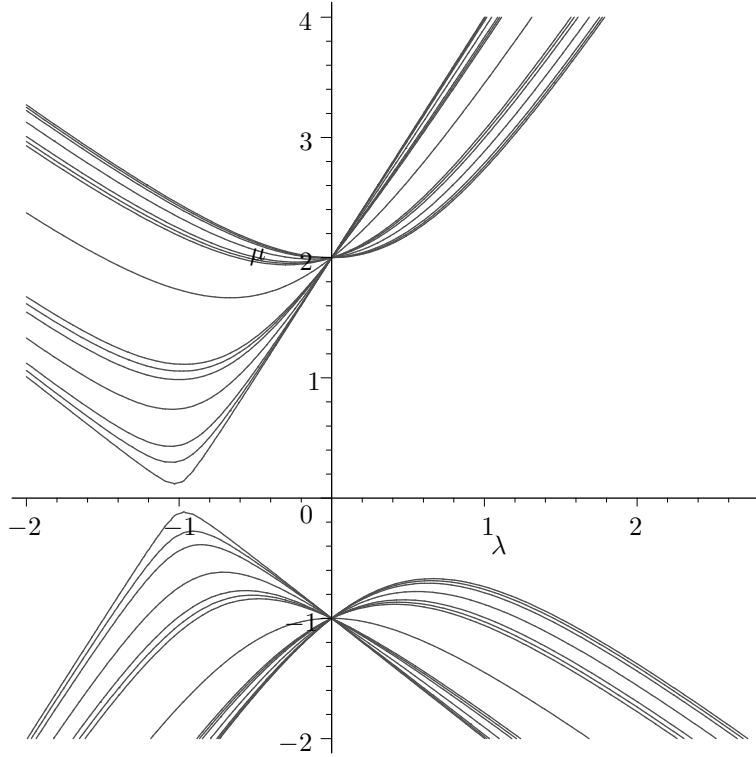


Figure 28: The zeroes of  $\det Q_n(\lambda, \mu)$ , giving the spectrum of the Fabrykowski-Gupta group  $G$

## 13 Self-similarity and $C^*$ -algebras

### 13.1 The self-similarity bimodule

The self-similarity of the sets, group actions and inverse semigroups can be treated from a common point of view through the notion of a  $C^*$ -bimodule or *correspondence*. For the notions of  $C^*$ -bimodules, correspondences and their applications see [Con94, Appendix A in Chapter II and Appendix B in Chapter V] and the papers [CJ85, JS97].

**Definition 13.1.** Let  $A$  be a  $*$ -algebra. An  $A$ -bimodule  $\Phi$  is a right  $A$ -module with an  $A$ -valued sesquilinear inner product and a  $*$ -homomorphism  $\phi : A \rightarrow \text{End}(\Phi)$ .

By an  $A$ -valued sesquilinear inner product on  $\Phi$  we mean a function  $\langle \cdot | \cdot \rangle$  from  $\Phi \times \Phi$  to  $A$  such that

1.  $\langle v | v_1 + v_2 \rangle = \langle v | v_1 \rangle + \langle v | v_2 \rangle$ ,
2.  $\langle v_1 | v_2 a \rangle = \langle v_1 | v_2 \rangle a$ ,
3.  $\langle v_1 | v_2 \rangle = (\langle v_2 | v_1 \rangle)^*$ ,
4.  $\langle v | v \rangle$  is a positive element of  $A$  and is equal to zero if and only if  $v = 0$ ,

where  $v, v_1, v_2 \in \Phi$  and  $a \in A$  are arbitrary.

For  $a \in A$  and  $v \in \Phi$  we will write  $av$  instead of  $\phi(a)(v)$ , so that the map  $\phi$  defines the left multiplication of the bimodule  $\Phi$ .

Let  $(\{F_v\}_{v \in V}, \{\phi_e\}_{e \in E})$  be a graph-directed iterated function system. Let  $A = \bigoplus_{v \in V} C(F_v)$  be the direct sum of the  $C^*$ -algebras of continuous  $\mathbb{C}$ -valued functions on the spaces  $F_v$ . The

algebra  $A$  can be also defined as the algebra of continuous functions on the disjoint union  $\tilde{F}$  of the spaces  $F_v$ .

Let  $\Phi_R$  be the right  $A$ -module which is the direct sum  $\bigoplus_{e \in E} e \otimes C(F_{\alpha(e)})$  (here  $e \otimes \cdot$  is only an index), with the  $A$ -valued inner product

$$\left\langle \sum_{e \in E} e \otimes f_e \middle| \sum_{e \in E} e \otimes h_e \right\rangle = \sum_{e \in E} \overline{f_e} h_e,$$

and the right multiplication defined by the rule

$$\left( \sum_{e \in E} e \otimes f_e \right) \cdot h = \sum_{e \in E} e \otimes (f_e \cdot h|_{F_{\alpha(e)}}).$$

We can also define the left multiplication by the formula

$$h \cdot \left( \sum_{e \in E} e \otimes f_e \right) = \sum_{e \in E} e \otimes (h \circ \phi_e \cdot f_e),$$

where  $h \circ \phi_e$  is the composition  $h \circ \phi_e(x) = h|_{F_{\omega(e)}}(\phi_e(x))$ .

In this way we get a well-defined *self-similarity bimodule*  $\Phi$  over the algebra  $A$  associated to the iterated function system.

The self-similarity of the group and of inverse semigroup actions can be also encoded in terms of bimodules.

Let us fix some self-similar action of a group  $G$  on the space  $X^\omega$  for the alphabet  $X = \{1, 2, \dots, d\}$ . Denote by  $\Phi_R$  the free right module over the group algebra  $\mathbb{C}G$  with the free basis equal to the alphabet  $X$ .

The algebra  $\mathbb{C}G$  is a  $*$ -algebra with the standard involution  $(\alpha \cdot g)^* = \overline{\alpha}g^{-1}$  (where  $\alpha \in \mathbb{C}$ ,  $g \in G$ ). On the module  $\Phi_R$  we define a  $\mathbb{C}G$ -valued sesquilinear form by the equality

$$\left\langle \sum_{x \in X} x \cdot a_x \middle| \sum_{y \in X} y \cdot b_y \right\rangle = \sum_{x \in X} a_x^* b_x,$$

where  $a_x, b_y \in \mathbb{C}G$ .

Now the definition of self-similarity gives us a structure of a left  $\mathbb{C}G$ -module on  $\Phi_R$ . Namely, for any element  $g \in G$  and any vector  $x \in X$  from the basis of  $\Phi_R$  we define

$$g \cdot x = y \cdot h, \tag{14}$$

where  $h$  and  $y$  are as in Equation (3) of Definition 3.4'. This multiplication is extended by linearity to the whole module  $\Phi_R$ . In this way we get a map from  $G$  to  $\text{End}(\Phi_R)$ . This map extends to a morphism of algebras  $\phi : \mathbb{C}G \rightarrow \text{End}(\Phi_R)$ . The algebra  $\text{End}(\Phi_R)$  is isomorphic to the algebra  $M_d(\mathbb{C}G)$  of  $d \times d$ -matrices over  $\mathbb{C}G$ .

The morphism  $\phi : \mathbb{C}G \rightarrow M_d(\mathbb{C}G)$  is called the *linear recursion* of the self-similar action. The obtained  $\mathbb{C}G$ -bimodule  $\Phi$  is called the *self-similarity bimodule* of the action.

For instance, in the case of the adding machine action, the self-similarity bimodule  $\Phi$  is 2-dimensional as a right  $\mathbb{C}\langle a \rangle$ -module and left multiplication by the generator  $a$  is the endomorphism of the right module defined by the matrix

$$\phi(a) = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}.$$

The linear recursions of self-similar actions were used by the first two named authors and A. Žuk to compute the spectra of random walks on the Cayley graphs and on the Schreier graphs of some self-similar groups (see [GŽuk01, BG00b]). See Section 12 for details.

Suppose that  $\rho$  is a unitary representation of the group  $G$  on a Hilbert space  $\mathcal{H}$ . Then we have a representation  $\rho_1 = \Phi \otimes_{\mathbb{C}G} \rho$  of the group  $G$  on the space

$$\Phi \otimes_{\mathbb{C}G} \mathcal{H} = \sum_{x \in X} x \otimes \mathcal{H},$$

where each  $x \otimes \mathcal{H}$  is a copy of  $\mathcal{H}$  with the natural isometry  $T_x : \mathcal{H} \rightarrow x \otimes \mathcal{H} : v \mapsto x \otimes v$ . Then the representation  $\rho_1$  acts on the space  $\Phi \otimes_{\mathbb{C}G} \mathcal{H}$  by the formula  $\rho_1(g)(x \otimes v) = y \otimes (\rho(h)(v))$ , where  $g \in G$ ,  $v \in \mathcal{H}$ , and  $h \in G$ ,  $y \in X$  are such that  $g \cdot x = y \cdot h$ .

A unitary representation  $\rho$  of the group  $G$  is said to be *self-similar* if  $\rho$  and  $\Phi \otimes \rho$  are unitary equivalent. Therefore, the representation  $\rho$  is self-similar if and only if there exist a decomposition of  $\mathcal{H}$  into a direct sum  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_d$  and isometries  $S_x : \mathcal{H} \rightarrow \mathcal{H}$  with range  $\mathcal{H}_x$ , such that

$$\rho(g)S_x = S_y\rho(h) \tag{15}$$

whenever  $g \cdot x = y \cdot h$ .

### Examples.

1. Since the group  $G$  acts on the space  $X^\omega$  by measure-preserving transformations, we get a natural unitary representation  $\rho$  of the group  $G$  on the space  $\mathcal{H} = L^2(X^\omega)$ . Since the set  $X^\omega$  is the disjoint union  $\cup_{x \in X} xX^\omega$ , the space  $\mathcal{H}$  is the direct sum of the spaces  $\mathcal{H}_x$  of functions with support in  $xX^\omega$ . We have a natural isometry  $S_x : \mathcal{H} \rightarrow \mathcal{H}_x \subset \mathcal{H}$  defined by the rule:

$$S_x(f)(w) = \begin{cases} 0 & \text{if } w \notin xX^\omega, \\ \sqrt{|X|} \cdot f(w') & \text{if } w = xw'. \end{cases}$$

It is checked directly that condition (15) holds, so the natural representation of  $G$  on the space  $L^2(X^\omega)$  is self-similar.

2. A set  $\mathcal{M} \subseteq X^\omega$  is *self-similar* if  $\mathcal{M} = \cup_{x \in X} x\mathcal{M}$  is a disjoint union.

Let  $\mathcal{M} \subseteq X^\omega$  be a countable self-similar  $G$ -invariant set. Set  $\mathcal{H} = \ell^2(\mathcal{M})$ . Then the space  $\mathcal{H}$  is also a direct sum  $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_d$ , where  $\mathcal{H}_x$  is the subspace spanned by the set  $x\mathcal{M}$ . The isometry  $S_x : \mathcal{H} \rightarrow \mathcal{H}_x$  is the linear extension of the map  $w \mapsto xw$ . Since the set  $\mathcal{M}$  is  $G$ -invariant, we have a natural permutational representation of the group  $G$  on the space  $\mathcal{H}$ . It is also easy to check that this representation satisfies condition (15), thus is self-similar.

Suppose that the representation  $\rho$  of the group  $G$  is self-similar and let  $S_x$  be the isometries for which (15) holds.

Let  $C_\rho^*(G)$  be the completion of the group algebra  $\mathbb{C}G$  with respect to the operator norm induced by the representation  $\rho$ . Let  $\Phi_\rho$  be the right  $C_\rho^*(G)$ -module with free basis  $X$ . Then formula (14) gives a well-defined  $C_\rho^*(G)$ -bimodule structure on  $\Phi_\rho$ .

Therefore, if the representation  $\rho$  is self-similar, then the  $\mathbb{C}G$ -bimodule  $\Phi$  can be extended to a  $C_\rho^*(G)$ -bimodule.

In general, we adopt the following definition:

**Definition 13.2.** A completion  $A$  of the algebra  $\mathbb{C}G$  respectively to some  $C^*$ -norm is said to be *self-similar* if the self-similarity bimodule  $\Phi$  extends to an  $A$ -bimodule.

We need the following auxiliary notion:

**Definition 13.3.** Let  $G$  be a countable group acting by homeomorphisms on the set  $X^\omega$ . A point  $w \in X^\omega$  is *generic* with respect to  $g \in G$  if either  $w^g \neq w$  or there exists a neighborhood  $U \ni w$  consisting of the points fixed under the action of  $g$ .

A point  $w \in X^\omega$  is  *$G$ -generic* if it is generic with respect to every element of  $G$ .

It is easy to see that for any countable group  $G$  acting by homeomorphisms on the set  $X^\omega$ , the set of all  $G$ -generic points is residual, i.e., is an intersection of a countable set of open dense subsets. In particular, it is non-empty.

For any point  $w \in X^\omega$  we denote by  $G(w)$  the  $G$ -orbit of  $w$ . Let  $\ell^2(G(w))$  be the Hilbert space of all square-summable functions  $G(w) \rightarrow \mathbb{C}$ . We have the permutation representation  $\pi_w$  of  $G$  on  $\ell^2(G(w))$ . Let  $\|\cdot\|_w$  be the operator norm on  $\mathbb{C}G$  defined by the representation  $\pi_w$ .

**Proposition 13.1.** *Let  $w_1, w_2 \in X^\omega$  and suppose that  $w_1$  is  $G$ -generic. Then for every  $a \in \mathbb{C}G$*

$$\|a\|_{w_1} \leq \|a\|_{w_2}.$$

By  $A_w$  we denote the completion of the algebra  $\mathbb{C}G$  with respect to the operator norm defined by the permutation representation of  $G$  on the orbit of the point  $w \in X^\omega$ . As an immediate corollary of Proposition 13.1, we get

**Theorem 13.2.** *Let  $w \in X^\omega$  be a  $G$ -generic point. Then for every  $u \in X^\omega$ , the algebra  $A_w$  is a quotient of the algebra  $A_u$ . If  $u$  is also generic, then the algebras  $A_w$  and  $A_u$  are isomorphic.*

Let us denote the algebra  $A_w$ , where  $w$  is a generic point, by  $\mathcal{A}_\Phi$ .

**Theorem 13.3.** *The algebra  $\mathcal{A}_\Phi$  is self-similar. If  $A$  is another self-similar completion of  $\mathbb{C}G$  then the identity map  $G \rightarrow G$  extends to a surjective homomorphism of  $C^*$ -algebras  $A \rightarrow \mathcal{A}_\Phi$ .*

## 13.2 Cuntz-Pimsner algebras

In [Pim97] M. Pimsner associates to every bimodule  $\Phi$  an algebra  $\mathcal{O}_\Phi$ , which generalizes the Cuntz algebra  $\mathcal{O}_d$  (see [Cun77]), and the Cuntz-Krieger algebra  $\mathcal{O}_A$  from [CK80]. In the case of the self-similarity bimodule the Cuntz-Pimsner algebra  $\mathcal{O}_\Phi$  can be defined in the following way.

**Definition 13.4.** Let  $\Phi$  be the self-similarity bimodule of the action of a group  $G$  over the alphabet  $X$ . The *Cuntz-Pimsner algebra*  $\mathcal{O}_\Phi$  is the universal  $C^*$ -algebra generated by the algebra  $\mathcal{A}_\Phi$  and  $|X|$  operators  $\{S_x : x \in X\}$  satisfying the relations

$$S_x^* S_x = 1, \quad S_x^* S_y = 0 \text{ if } x \neq y, \quad (16)$$

$$\sum_{x \in X} S_x S_x^* = 1, \quad (17)$$

$$a S_x = \sum_{y \in X} S_y a_{x,y}, \quad (18)$$

where  $a_{x,y} \in \mathcal{A}_\Phi$  are such that  $a \cdot x = \sum_{y \in X} y \cdot a_{x,y}$ , so  $a_{x,y} = \langle y | a \cdot x \rangle$ .

We will use the multi-index notation, so that for  $v = x_1 x_2 \dots x_n \in X^*$  the operator  $S_v$  is equal to  $S_{x_1} S_{x_2} \dots S_{x_n}$  and for the empty word  $\emptyset$  the operator  $S_\emptyset$  is equal to 1. Then, since  $g \cdot v = u \cdot h$ , we have  $g S_v = S_u h$  in the algebra  $\mathcal{O}_\Phi$ .

The *Cuntz algebra*  $\mathcal{O}_d$  is the universal algebra generated by  $d$  isometries  $S_1, S_2, \dots, S_d$  such that  $\sum_{i=1}^d S_i S_i^* = 1$ . This algebra is simple (see [Cun77, Dav96]) and thus any isometries satisfying such a relation generate an algebra isomorphic to  $\mathcal{O}_d$ . In particular, since the operators  $\{S_x : x \in X\}$  satisfy relations (16), (17), they generate a subalgebra of  $\mathcal{O}_\Phi$ , isomorphic to  $\mathcal{O}_d$  for  $d = |X|$ .

If  $\rho : \mathcal{O}_\Phi \rightarrow \mathcal{B}(\mathcal{H})$  is a representation of the algebra  $\mathcal{O}_\Phi$ , then its restriction onto the subalgebra generated by  $\mathcal{A}_\Phi$  is a self-similar representation of the algebra  $\mathcal{A}_\Phi$ , by (18). Conversely, if  $\rho$  is a self-similar representation, then by (15) we get a representation of the algebra  $\mathcal{O}_\Phi$ .

**Theorem 13.4.** *The algebra  $\mathcal{O}_\Phi$  is simple. Moreover, for any non-zero  $x \in \mathcal{O}_\Phi$  there exist  $p, q \in \mathcal{O}_\Phi$  such that  $pxq = 1$ .*

The Cuntz-Pimsner algebras of self-similarity bimodules of iterated function systems were studied by V. Deaconu (in the more general situation of a *continuous graph*) in [Dea99]. See also the work [Put96], where the *Ruelle algebras* of a *Smale space* are studied. The connection between the Ruelle algebras and the Cuntz-Pimsner algebras of topological graphs is described in [AR97].

### 13.3 The algebra $\mathcal{F}(\Phi)$

For  $k \in \mathbb{N}$  denote by  $\mathcal{F}_k$  the linear span in  $\mathcal{O}_\Phi$  over  $\mathbb{C}$  of the products  $S_v a S_u^*$ , with  $u, v \in X^k$  and  $a \in \mathcal{A}_\Phi$ .

The space  $\mathcal{F}_k$  is an algebra isomorphic to the algebra  $M_{d^k}(\mathcal{A}_\Phi)$  of  $d^k \times d^k$ -matrices over the algebra  $\mathcal{A}_\Phi$ .

By the linear recursion we have inclusions  $\mathcal{F}_k \subset \mathcal{F}_{k+1}$ . Denote by  $\mathcal{F}(\Phi)$  the closure of the union  $\cup_{k \geq 1} \mathcal{F}_k$ .

The algebra  $\mathcal{F}(\Phi)$  is isomorphic to the direct limit of the matrix algebras  $M_{d^n}(\mathcal{A}_\Phi)$  with respect to the embeddings defined by the linear recursion  $\phi$ .

**Theorem 13.5.** *Suppose the self-similar action of  $G$  is recurrent and that the point  $w_0 \in X^\omega$  and all its shifts  $s^n(w_0)$  are  $G$ -generic.*

*Let  $\pi_1$  be the permutation representation of  $G$  on the orbit of  $w_0$  and let  $\pi_2$  be the natural representation on  $\ell^2(G(w_0))$  of the algebra  $C(X^\omega)$  of continuous functions on  $X^\omega$ . Then the algebra  $\mathcal{F}(\Phi)$  is isomorphic to the  $C^*$ -algebra generated by  $\pi_1(G) \cup \pi_2(C(X^\omega)) \subset \mathcal{B}(\ell^2(G(w_0)))$ .*

**Example.** In the case of the adding machine action the algebra  $\mathcal{A}_\Phi$  is isomorphic to the algebra  $C(\mathbb{T})$  with the linear recursion  $C(\mathbb{T}) \rightarrow M_2(C(\mathbb{T}))$  coming from the double self-covering of the circle. Thus the algebra  $\mathcal{F}(\Phi)$  in this case is the Bunce-Dedden algebra. Then Theorem 13.5 in this case is the well known fact that the Bunce-Dedden algebra is the cross-product algebra of the odometer action on the Cantor space  $X^\omega$  (see [Dav96]).

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