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Pseudo-telepathy: input cardinality and Bell-type inequalities

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Abstract

Pseudo-telepathy is the most recent form of rejection of locality. Many of its properties have already been discovered: for instance, the minimal entanglement, as well as the minimal cardinality of the output sets, have been characterized. This paper contains two main results. First, we prove that no bipartite pseudo-telepathy game exists, in which one of the partners receives only two questions; as a corollary, we show that the minimal “input cardinality”, that is, the minimal number of questions required in a bipartite pseudo-telepathy game, is 3×3 . Second, we study the Bell-type inequality derived from the pseudo-telepathy game known as the Magic Square game: we demonstrate that it is a tight inequality for 3 inputs and 4 outputs on each side and discuss its weak resistance to noise.

1 Introduction

Implicitly present in the works of Heywood and Redhead [1] and of Greenberger, Horne, Zeilinger and Mermin [2, 3], the notion of *pseudo-telepathy* was first explicitly introduced by Brassard, Cleve and Tapp [4]. In this work, the authors used a game which was possible to win classically only if the players exchanged information, while quantum players sharing entanglement could dispense with communication altogether. They used their result to show a lower bound on the number of bits of communication classical players need to exchange in order to simulate quantum measurements on many maximally entangled pairs of qubits.

Since no communication whatsoever is required in the quantum strategy to the game, we can see that their game is actually a proof against locality in the same sense as the result of Bell [5]. Furthermore, it was shown recently that pseudo-telepathy is a stronger rejection of the locality assumption than the traditional Bell theorems, or even the Hardy [6] theorem [7, 8]. We will give a complete definition of pseudo-telepathy in Section 2, for a survey of known results see [9].

Much work has been done in order to characterize pseudo-telepathy. A question of interest is “What kind of correlations are required in order to give rise to a pseudo-telepathy game?” The minimal entanglement necessary in order to have a pseudo-telepathy game was resolved in [7] where the authors have shown that either a pair of entangled qutrits or three entangled qubits are needed. In [10], the minimal cardinality of the outputs sets was uncovered: in the bipartite scenario, at least one player must have more than two possible outputs, while in the multipartite scenario two outputs per player are enough. In Section 3 we provide the answer to the last question along these lines that has remained open: how many possible questions must one incorporate in a game for it to be a pseudo-telepathy game.

In Section 4, we consider pseudo-telepathy in the context of the characterization of non-local correlations. In the last years, the understanding of non-locality has been significantly improved using a geometric view of probability distributions: local distributions form a closed convex set with a finite number of extremal points (a “polytope”), whose facets define tight Bell-type inequalities [11, 12]. It is interesting to ask whether pseudo-telepathy games are associated to tight Bell inequalities: while a general proof or a counter-example are still missing, the answer is positive for the three-partite Mermin-GHZ game [3] as proved in [13] and for the bipartite Magic Square game [14] as we prove here. We show also that the inequality derived from the Magic Square game is less resistant to noise than another inequality, which is not associated to a pseudo-telepathy game: this is not surprising, since pseudo-telepathy is stronger a refutation of locality.

2 Definitions and mathematical tools

In this Section, we present formal definitions and the mathematical tools that are required to demonstrate the results of Sections 3 and 4.

2.1 Pseudo-telepathy

In this paper, we choose the following notations:

Alice’s input: $x \in \mathcal{X} = \{0, 1, \dots, m_A - 1\}$;

Alice’s output: $a \in \mathcal{A} = \{0, 1, \dots, n_A - 1\}$;

Bob's input: $y \in \mathcal{Y} = \{0, 1, \dots, m_B - 1\}$;
 Bob's output: $b \in \mathcal{B} = \{0, 1, \dots, n_B - 1\}$.

Definition 1 (Bipartite Game). A bipartite game $G = (\mathcal{I}, \mathcal{O}, W)$ is a set of inputs $\mathcal{I} = \mathcal{X} \times \mathcal{Y}$, a set of outputs $\mathcal{O} = \mathcal{A} \times \mathcal{B}$ and a relation $W \subseteq \mathcal{I} \times \mathcal{O}$ that inputs and outputs should satisfy.

Definition 2 (Winning Strategy). A winning strategy for a bipartite game $G = (\mathcal{I}, \mathcal{O}, W)$ is a strategy according to which for every $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, Alice and Bob output a and b respectively such that $(x, y, a, b) \in W$.

Definition 3 (Pseudo-telepathy). We say that a bipartite game G exhibits pseudo-telepathy if bipartite measurements of an entangled quantum state can yield a winning strategy, whereas no classical strategy that does not involve communication is a winning strategy.

The extension to the multipartite case is trivial, but we consider only bipartite scenarios in this paper. The generalization of Definition 3 can be translated into a set of multipartite measurements on an entangled state where any local classical model which is to attempt to only produce outputs that are not forbidden by quantum mechanics will fail.

2.2 Polytope and facets inequalities

For the purposes of Section 4, let's recall the geometric formalism for probability distributions [11, 12]. As mentioned, we consider the situation in which Alice receives the input $x \in \{0, 1, \dots, m_A - 1\}$ and has to output $a \in \{0, 1, \dots, n_A - 1\}$; similarly, Bob receives the input $y \in \{0, 1, \dots, m_B - 1\}$ and has to output $b \in \{0, 1, \dots, n_B - 1\}$. Any probability distribution can be represented by a vector of dimension $m_A m_B n_A n_B$ whose entries are just the numbers $\Pr[a, b|x, y]$. In the space of probability distributions, we focus on the set of local strategies $\Omega_{m_A m_B n_A n_B}$ (written Ω_{mn} when $m_A = m_B = m$ and $n_A = n_B = n$). This set is a polytope, that is, a convex set with a finite number of extremal points. The extremal points are deterministic local strategies, that is, points of the form

$$\Pr[a, b|x, y] = \Pr[a|x] \Pr[b|y] = \delta_{a, a(x)} \delta_{b, b(y)} \quad (1)$$

for all possible functions $x \mapsto a(x)$, $y \mapsto b(y)$, and where $\delta_{i,j}$ is the Kronecker delta. There are clearly $n_A^{m_A} n_B^{m_B}$ deterministic strategies. Note that the local polytope $\Omega_{m_A m_B n_A n_B}$ is embedded in a subspace of the probability space, because local strategies automatically satisfy the so-called no-signalling condition. It can be shown that the dimension of the space of no-signalling probability distributions is $d = m_A m_B (n_A - 1)(n_B - 1) + m_A (n_A - 1) + m_B (n_B - 1)$.

A polytope embedded in a d -dimensional space is completely described by giving the list of its extremal points (vertices) or equivalently, of the $(d-1)$ -dimensional hypersurfaces that

bound it (facets). On the one hand, as we have just seen, it is very easy to list the vertices. On the other hand, the characterization in terms of facets is more useful, because it allows to decide whether a probability point is inside the polytope, and is therefore local; or outside it, and is then non-local; in other words, each facet represents a tight Bell-type inequality. The task of finding the facets given the list of vertices is computationally hard. In Section 4, we show how a particular pseudo-telepathy game provides a natural inequality that is in fact a facet for the corresponding local polytope.

3 Input Cardinality

In this Section, we want to prove the following

Theorem 1. *In pseudo-telepathy, the set of possible question $\mathcal{X} \times \mathcal{Y}$ cannot be of cardinality $2 \times n$.*

Proof. For definiteness, we suppose that it is Alice who receives only two questions, i.e. $\mathcal{X} = \{0, 1\}$. We prove the statement by *reductio ad absurdum*. Consider deterministic strategies: the values $a^{(0)}$ and $a^{(1)}$ that Alice is going to output upon receiving $x = 0$ or $x = 1$ are fixed. Suppose now that for a given pair $(a^{(0)}, a^{(1)})$ it so happens, that for all $y \in \mathcal{Y}$ there exist a $b^{(y)}$ such that $(0, y, a^{(0)}, b^{(y)}) \in W$ and $(1, y, a^{(1)}, b^{(y)}) \in W$: then there is clearly a classical winning strategy, and the game cannot be a pseudo-telepathy game. The rest of the proof consist in proving that exactly this situation is generated by the most general bipartite quantum strategy with two inputs on Alice side.

In a quantum strategy, Alice and Bob share a quantum state and obtain the outcomes by performing measurements on the state. Since we are not going to fix the dimension of the Hilbert space of Alice and Bob, we can assume without loss of generality that the measurements are projective — in other words: a strategy using positive-operator-valued measurements on states of dimension d can always be written as a strategy using projective measurement on states of suitable dimension $d' > d$. Let's suppose at first that Alice and Bob share a pure state $|\Psi\rangle_{AB}$. Upon receiving input x , Alice performs the measurement defined by the projectors $\{P_a^{(x)}\}$; similarly, upon receiving input y , Bob performs the measurement defined by the projectors $\{P_b^{(y)}\}$. We suppose that this is a winning quantum strategy. Now: if $x = 0$ and Alice obtains the outcome $a^{(0)} = a$, she prepares on Bob's side the state $|\varphi_{a^{(0)}}\rangle_B \simeq P_a^{(0)} \otimes \mathbb{1} |\Psi\rangle_{AB}$; if $x = 1$ and Alice obtains the outcome $a^{(1)} = a'$, she prepares on Bob's side the state $|\varphi_{a^{(1)}}\rangle_B \simeq P_{a'}^{(1)} \otimes \mathbb{1} |\Psi\rangle_{AB}$. Since each projective measurement is a resolution of the identity, there must exist $(a^{(0)}, a^{(1)})$ such that $\langle \varphi_{a^{(0)}} | \varphi_{a^{(1)}} \rangle \neq 0$. In turn, given these $(a^{(0)}, a^{(1)})$, for each y there must be at least an element $P_{b^{(y)}}^{(y)}$ in each of Bob's measurements such that both $\langle \varphi_{a^{(0)}} | P_{b^{(y)}}^{(y)} | \varphi_{a^{(0)}} \rangle$ and $\langle \varphi_{a^{(1)}} | P_{b^{(y)}}^{(y)} | \varphi_{a^{(1)}} \rangle$ are not zero. Therefore a subset of the outcomes of the quantum strategy defines the deterministic strategy $\{a^{(0)}, a^{(1)}; b^{(y)}\}$.

But if the quantum strategy is winning, any subset of outcomes must be winning as well: therefore, the assumption of a winning quantum strategy implies the existence of a winning classical strategy as well, contradicting the definition of pseudo-telepathy game.

Finally, the assumption, that Alice and Bob share a pure state, can be easily dispensed with: if a strategy based on a mixed state is winning, then it is winning on all the pure states onto which the mixture can be decomposed; thus we are brought back to the previous case, and the proof is now complete. □

We want to stress that this Theorem does not exclude either (i) the existence of a pseudo-telepathy game with input cardinality $2 \times (n_{B_1} \times n_{B_2})$; nor (ii) the possibility that, in a game that cannot be won with probability 1 (and is therefore not a pseudo-telepathy game), the success probability may be larger using quantum than using classical strategies. For (i): if “Bob” is actually Bob₁ and Bob₂, the outcomes must also of the form $b^{(y)} = b_1^{(y)} + b_2^{(y)}$: it is this additional requirement is what makes the pseudo-telepathy possible, as in the GHZ-Mermin game [2, 3]. If this requirement would be removed (that is, if Bob₁ and Bob₂ would be allowed to communicate), the quantum strategy still exists of course, but our theorem proves that a classical strategy would become possible as well (again, a well-known statement for the example of the GHZ-Mermin game). For (ii), the CHSH Bell inequality provides such an example (see [15] and references therein): classical strategies succeed with $p = \frac{3}{4}$, quantum strategies with probability $p = \frac{2+\sqrt{2}}{4}$, but the game can be won with probability one only allowing the “PR-box” [16] as a resource.

Since games with input cardinality of 3×3 [14] and $2 \times 2 \times 2$ [3] are known to exist, Theorem 1 entails immediately the following

Corollary 2. *The minimal cardinality of the set of possible questions in pseudo-telepathy is 3×3 in the bipartite scenario and $2 \times 2 \times 2$ in the multipartite.*

To extend our result to multipartite pseudo-telepathy, one could reformulate the statement of Theorem 1 as:

For a pseudo-telepathy game to exist, every player must perceive the number of possible combinations of questions asked to the other player(s) to be greater than two.

$x \setminus y$	0	1	2
0	c_{00}	c_{01}	$c_{02} = c_{00} \oplus c_{01}$
1	c_{10}	c_{11}	$c_{12} = c_{10} \oplus c_{11}$
2	$c_{20} = c_{00} \oplus c_{10} \oplus 1$	$c_{21} = c_{01} \oplus c_{11} \oplus 1$	$c_{22} = ?$

Table 1: Example of a classical table for the Magic Square game. In the table, one reads the outputs $c_{xy} \in \{0, 1\}$. Upon receiving input x , Alice outputs the three bits in the corresponding row; upon receiving input y , Bob outputs the three bits in the corresponding column. The value of c_{22} cannot be chosen such as to fulfill all three requirements (2)-(4).

4 A Bell-type inequality from the Magic Square game

4.1 Definition of the game

We present here the pseudo-telepathy game generally known as the Magic Square game [14]. The participants, namely Alice and Bob, are each presented with a question: a random trit $x \in \{0, 1, 2\}$ and $y \in \{0, 1, 2\}$ respectively. They must produce three bits each, $(a_0^{(x)}, a_1^{(x)}, a_2^{(x)})$ and $(b_0^{(y)}, b_1^{(y)}, b_2^{(y)})$ respectively. In order for them to win, three requirements must be fulfilled:

$$\mathcal{R}_{L,A} : a_0^{(x)} \oplus a_1^{(x)} \oplus a_2^{(x)} = 0; \quad (2)$$

$$\mathcal{R}_{L,B} : b_0^{(y)} \oplus b_1^{(y)} \oplus b_2^{(y)} = 1; \quad (3)$$

$$\mathcal{R}_{AB} : a_y^{(x)} = b_x^{(y)}. \quad (4)$$

Note that the first two requirements are “local”, only the third one introduces a correlation between Alice and Bob.

Without resorting to quantum mechanics or communicating, Alice and Bob cannot win this game with probability one. In fact, for requirement (4) to be always fulfilled, Alice and Bob must draw their outcomes from a common 3×3 table of 0s and 1s with entries c_{xy} . But the other two requirements (2) and (3) say that the sum of the elements in each row must be even and the sum of the elements in each column must be odd: a simple parity argument shows that this is impossible. For instance, suppose Alice and Bob share a table of the form of Table 1. Then they succeed for eight possible inputs, but fail for the case $a = b = 2$: in fact, requirement $\mathcal{R}_{L,A}$ forces $a_2^{(2)} = c_{00} \oplus c_{10} \oplus c_{01} \oplus c_{11}$, requirement $\mathcal{R}_{L,B}$ forces $b_2^{(2)} = a_2^{(2)} + 1$, and requirement $\mathcal{R}_{AB}$ forces $a_2^{(2)} = b_2^{(2)}$.

On the other hand, if Alice and Bob are allowed to share entanglement, they exist an explicit successful strategy [14]. Success takes advantage of “contextuality”, i.e. the fact that “unmeasured quantities are not determined” in quantum physics: Alice and Bob perform only the measurements they are asked to perform, and each pair of measurements produces outcomes with the required relation, although the results cannot derive from an underlying classical table of pre-determined outcomes.

4.2 Study of the derived Bell-type inequality

4.2.1 The inequality

The classical bound for the Magic Square game can be re-written in the form of a Bell-type inequality. Any classical strategy is the convex mixture of extremal points which are deterministic strategies, that is, shared tables c_{xy} . From the example above in Table 1, it is clear that the game cannot be won for all inputs, but can be won for eight inputs out of nine; therefore the inequality

$$\sum_{x,y=0,1,2} \Pr [\mathcal{R}_{L,A}, \mathcal{R}_{L,B}, \mathcal{R}_{AB} | x, y] \leq 8, \quad (5)$$

holds for classical strategies. In this language, the existence of a winning quantum strategy means that quantum physics violates this inequality up to the maximal algebraic value of 9. This violation can be obtained the tensor product of two maximally entangled states of two qubits [14], which is equivalent to the maximally entangled state of two 4-dimensional systems; therefore, this inequality does not exhibit the “anomaly” reviewed in [17]. We set out to study whether this inequality is tight, that is, a facet of a polytope. The answer to this question requires a careful definition of the probability space.

Let’s first count how many deterministic strategies saturate the inequality. First, Alice and Bob agree on the pair of inputs $(x_?, y_?)$ for which they fail the game (the position of the “?” sign); there are 9 possibilities. Once this position fixed, 4 bits can be freely chosen and the others are determined by the local requirements. Note that Alice’s and Bob’s outputs at the position of “?” are fixed: in fact, for $x = x_?$ Alice must output the bit that fulfills $\mathcal{R}_{L,A}$; otherwise, she will fail as soon as $x = x_?$, independently of Bob’s input, and such a strategy does not saturate the inequality. A similar argument holds for Bob. In conclusion, the number of deterministic strategies that saturate the inequality is $9 \times 2^4 = 144$.

In full generality, we are studying a class of problems in which both Alice and Bob have 3 inputs and 8 outcomes (all the possible three-bit lists). In this general sense, the local polytope is therefore Ω_{38} . This polytope lives in a probability space of dimension 483; therefore, a hypersurface containing only 144 extremal points has too small dimensionality to be a facet. However, this general view puts all the requirements of the Magic Square game on the same footing: in particular, it supposes that Alice and Bob may also fail to fulfill their own local requirements $\mathcal{R}_{L,A}$ and $\mathcal{R}_{L,B}$, which is a rather uninteresting and inefficient way of losing the game, since it would guarantee a failure probability of at least 1/3. It is more natural to assume that *both Alice and Bob promise to fulfill their own local requirement*, and that the only requirement that can sometimes fail is $\mathcal{R}_{AB}$, which involves correlations. In this restricted sense, for each input, Alice and Bob have only 4 possible outcomes each, since as soon as they choose two bits, the value of the third one is fixed. The local polytope becomes then Ω_{34} , that lives in a probability space of dimension 99. The inequality reads

now

$$\sum_{x,y=0,1,2} \Pr [a_y^{(x)} = b_x^{(y)} | x, y, \mathcal{R}_{L,A}, \mathcal{R}_{L,B}] \leq 8. \quad (6)$$

Now we can state the following

Proposition 3. *Inequality (6) is tight, i.e. it defines a facet of Ω_{34} .*

Proof. We must prove that the 144 extremal points that saturate inequality (6) define a hypersurface of dimension 98. Each of the points is written as a 99-component vector, following e.g. the parametrization introduced in [12] (where the vectors are written as square tables, but this ordering is just for convenience). The 144 vectors are then arranged in a matrix; the Proposition follows if this matrix has full rank, because 99 linearly independent vectors are needed to define a hyperplane of dimension 98. This verification was made with a computer by two of us independently and without any approximation. $\square$

A remark: it is known that any facet of a polytope can always be “lifted” in a natural way and give a facet of a polytope with more parties and/or more inputs and/or more outcomes [18]. This is not a contradiction with the fact that the Magic Square game defines a facet of Ω_{34} and not of Ω_{38} : the lifted version of inequality (6) to eight outcomes on each side is *not* inequality (5).

4.2.2 Abstract version of the inequality

Inequality (6) defines a facet of Ω_{34} which had not been listed before to our knowledge; remember that quantum physics violates it up to the algebraic maximum. It is instructive to rewrite the inequality by removing any reference to the Magic Square game: after all, this inequality is a facet of Ω_{34} , it must then be possible to write it as a formula involving only the ternary inputs and quaternary outputs.

Alice receives input $x \in \{0, 1, 2\}$ and produces the output $a^{(x)} \in \{0, 1, 2, 3\}$. We can decide conventionally that the bits $a_0^{(x)}$ and $a_1^{(x)}$ of the Magic Square game are defined through $a^{(x)} = 2a_0^{(x)} + a_1^{(x)}$, and then $a_2^{(x)} = a_0^{(x)} \oplus a_1^{(x)}$ (but any other convention would be valid, and would simply define an equivalent facet). Therefore $a_0^{(x)} = \left(\frac{a^{(x)} - a^{(x)} \bmod 2}{2} \right) \bmod 2$, $a_1^{(x)} = a^{(x)} \bmod 2$ and $a_2^{(x)} = \left(\frac{a^{(x)} + a^{(x)} \bmod 2}{2} \right) \bmod 2$. We make a similar definition for Bob; note that $b_2^{(y)} = \left(\frac{b^{(y)} + b^{(y)} \bmod 2}{2} + 1 \right) \bmod 2$. Thus, denoting equality modulo 2 by the symbol $\equiv_2$,

the inequality reads as follows:

$$\begin{aligned}
& \Pr \left[\frac{a^{(0)} - a^{(0)} \bmod 2}{2} \stackrel{2}{\equiv} \frac{b^{(0)} - b^{(0)} \bmod 2}{2} \right] + \Pr \left[\frac{a^{(1)} - a^{(1)} \bmod 2}{2} \stackrel{2}{\equiv} b^{(0)} \bmod 2 \right] \\
& + \Pr \left[\frac{a^{(2)} - a^{(2)} \bmod 2}{2} \stackrel{2}{\equiv} \frac{b^{(0)} + b^{(0)} \bmod 2}{2} + 1 \right] + \Pr \left[a^{(0)} \bmod 2 \stackrel{2}{\equiv} \frac{b^{(1)} - b^{(1)} \bmod 2}{2} \right] \\
& + \Pr \left[a^{(1)} \bmod 2 \stackrel{2}{\equiv} b^{(1)} \bmod 2 \right] + \Pr \left[a^{(2)} \bmod 2 \stackrel{2}{\equiv} \frac{b^{(1)} + b^{(1)} \bmod 2}{2} + 1 \right] \\
& \Pr \left[\frac{a^{(0)} + a^{(0)} \bmod 2}{2} \stackrel{2}{\equiv} \frac{b^{(2)} - b^{(2)} \bmod 2}{2} \right] + \Pr \left[\frac{a^{(1)} + a^{(1)} \bmod 2}{2} \stackrel{2}{\equiv} b^{(2)} \bmod 2 \right] \\
& + \Pr \left[\frac{a^{(2)} + a^{(2)} \bmod 2}{2} \stackrel{2}{\equiv} \frac{b^{(2)} + b^{(2)} \bmod 2}{2} + 1 \right] \leq 8. \quad (7)
\end{aligned}$$

This is how the inequality looks like (up to symmetries) if the outputs of Alice and Bob appear explicitly as quaternary values: the expression is less elegant than Eq. (6), and the origin is completely hidden.

4.3 Error tolerance

For $n_A = n_B = 4$, unstructured noise is the probability distribution $P(a, b|x, y) = \frac{1}{16}$ (random uncorrelated outputs for any input). The amount p_n of unstructured noise that one needs to add to the quantum correlations in order to make them local is a measure of non-locality called “resistance to noise” [19, 17]. Denoting I_{QM} , I_{LV} and I_{noise} the values that an inequality achieves with QM, local variables and unstructured noise respectively, p_n is defined by the linear relation $(1 - p_n)I_{QM} + p_n I_{\text{noise}} = I_{LV}$ that is

$$p_n = \frac{I_{QM} - I_{LV}}{I_{QM} - I_{\text{noise}}}. \quad (8)$$

For our new inequality (6)-(7), we know $I_{QM} = 9$ and $I_{LV} = 8$, and it is easy to compute that $I_{\text{noise}} = \frac{9}{2}$: in fact, since bits computed from random a and b are random bits, all the nine terms of the inequality become just the probability that two random bits are equal. All in all, we obtain $p_n = \frac{2}{9} = 0.\bar{2}$.

This value can be compared to the value obtained for other inequalities with the same number of outcomes. The best known one is I_{2244} [12]. For this inequality, one has $I_{LV} = 0$, $I_{\text{noise}} = -\frac{3}{4}$ and $I_{QM} \approx 0.3648$ — this last value comes from the value $I'_{QM} = 2.9727$ given in Table I of [20] for the equivalent inequality I' defined by $I = \frac{4-1}{2 \times 4}(I' - 2)$. Therefore $p_n \approx 0.3272$. We see that an inequality, which is not related to any pseudo-telepathy game (because of Theorem 1 in this paper, has a stronger resistance to noise than the one based

on the Magic Square game. Another inequality that can be checked is I_{3344} ; we have done it numerically and found a smaller p_n than for our new inequality.

In summary, no clear-cut picture can be derived from the comparison of inequalities based on the resistance to noise (see also the discussion in [20]); but to our present knowledge, pseudo-telepathy does not provide the best possible resistance to noise.

5 Conclusion

With the first part of this study (Section 3), we now have a complete picture of the minimal requirements for pseudo-telepathy:

- Bipartite scenario.
 - Minimal entanglement: 3×3 [7].
 - Minimal output cardinality: 3×2 [10].
 - Minimal input cardinality: 3×3 .
- Multipartite scenario.
 - Minimal entanglement: $2 \times 2 \times 2$.
 - Minimal output cardinality: $2 \times 2 \times 2$.
 - Minimal input cardinality: $2 \times 2 \times 2$.

It is somewhat peculiar that there is so much symmetry in the minimal requirements, up to the 2 in the minimal output cardinality. It would thus be interesting to know why we can “save” only in output cardinality. To further understand pseudo-telepathy, we would need to characterize the potential role of POVMs in this framework. While they have been taken into account for the studies of the minimal requirements discussed here, it is not clear whether the use of POVMs could lower the maximal probability of a classical strategy to win an instance of a pseudo-telepathy game for certain fixed dimensions. Another open question of interest is whether we can find a bipartite pseudo-telepathy game which only uses the minimal requirements or if there some kind of trade-off between these properties.

The second part of this work is a contribution to the study of pseudo-telepathy games in the broader context of non-local probability distributions. We have shown that a tight inequality is associated to the bipartite Magic Square game, but the question remains open, whether all pseudo-telepathy games define tight Bell-type inequalities. The resistance to noise of this new inequality is not the highest known to date for four-dimensional outputs.

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