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Quot Schemes and Moduli Spaces

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Introduction

Moduli spaces are geometric constructions whose points represent geometric objects. For example, consider the set of lines going through the origin of a vector space. By considering the equivalence classes of non-zero vectors that lie on the same line, the set of lines can be given a manifold structure as the projectivisation of the initial vector space. Thus the projectivisation of a vector space is the moduli space of lines going through the origin, endowed with a natural manifold structure. Often moduli spaces will have a more abstract structure than that of a manifold, such as a scheme.

The central object in this thesis is the moduli space $\mathcal{M}$ of certain vector bundles over a smooth algebraic curve. Since we work over the complex numbers, these curves are, topologically, orientable surfaces.

One would be interested in calculating the number of global sections of line bundles over this moduli space. When the determinant of the vector bundles is fixed, every line bundle over $\mathcal{M}$ can be written as a tensor power of a single line bundle $\mathcal{L}$. Verlinde's formulæ describe the dimensions of the cohomologies of $\mathcal{L}^{\otimes k}$.

We would like to present a novel approach to this question. Instead of taking the moduli space of vector bundles $\mathcal{E}$, we can map these vector bundles via a sheaf embedding into a trivial bundle $\mathcal{O}$ over the curve. Then we can parametrise the pairs consisting of a vector bundle $\mathcal{E}$ and a sheaf embedding $f\colon \mathcal{E} \to \mathcal{O}$. This is called the **Quot scheme**, and we will denote it by $\mathcal{Q}$.

In general, any vector bundle has multiple ways to be embedded into the trivial bundle. To identify the different embeddings of isomorphic vector bundles, we can take a group acting on the Quot scheme. With the appropriate choice of parameters, the orbits on the Quot scheme under this action are exactly the pairs corresponding to all the embeddings of a certain vector bundle $\mathcal{E}$. Therefore the moduli space of vector bundles is the quotient of the Quot scheme under this group action.

In our approach, we will calculate the Euler characteristic, the alternating sum of the dimensions of the cohomology groups, of a line bundle $\mathcal{L}$ over the moduli space of vector bundles $\mathcal{M}$. The Atiyah-Bott localisation tells us that the Euler characteristic of $\mathcal{L}$ over a quotient of the Quot scheme may be expressed as the invariant part of an equivariant integral on $\mathcal{Q}$.

In the present work we develop the above method in cases when $\mathcal{M}$ is not the quotient of $\mathcal{Q}$, and evaluate the equivariant integral. In order to verify that the new formula gives the Euler characteristic, we look at another space that is birationally equivalent to the Quot scheme, where such calculations are already known. We compare the results over these two spaces for certain parameters. This leads us to our main conjecture that states that localising over these two spaces gives the same results.

Introduction en français

Les espaces de modules sont des espaces géométriques dont les points représentent des objets géométriques. Par exemple, considérons l'ensemble des droites passant par l'origine dans un espace vectoriel. Cet ensemble admet une structure de variété, donnée par la projectivisation de l'espace vectoriel initial. Ainsi la projectivisation d'un espace vectoriel est un espace de modules des droites passant par l'origine. Souvent les espaces de modules n'auront qu'une structure plus abstraite qu'une variété, comme celle d'un schéma.

L'objet central de cette thèse est l'espace de modules de certains fibrés vectoriels sur une courbe algébrique. Nous choisissons une courbe complexe lisse C. La topologie d'une telle courbe est celle d'une surface orientable de genre g. Deux paramètres particulièrement importants pour cette étude sont associés à un fibré vectoriel V, son rang $r = \operatorname{rk} V$ et son degré $d = \deg V$. On dit de V qu'il est stable si pour tout sous-fibré $U < V$, on a $\frac{\deg U}{\operatorname{rk} U} < \frac{\deg V}{\operatorname{rk} V}$. Comme les deux paramètres r et d sont à valeurs discrètes, on doit les fixer pour que l'espace de modules soit connexe. Par un résultat de Grothendieck, l'espace de modules $\mathcal{M}_{r,d}$ des fibrés stables de rang r et degré d sur la courbe C existe. Cet espace de modules est muni d'un fibré universel non-canonique $\mathcal{U}$ sur $\mathcal{M}_{r,d} \times C$ tel que pour chaque $x \in \mathcal{M}_{r,d}$ correspondant au fibré V, la restriction de $\mathcal{U}$ au $\{x\} \times C$ est isomorphe à V.

Fixons un fibré Δ de degré d, et considérons le sous-espace $\mathcal{M}_{r,d}^{\Delta}$ qui consiste en tous les fibrés dont le fibré déterminant est Δ. Les classes d'isomorphisme de fibrés sur un schéma forment un groupe pour le produit tensoriel, que l'on appelle le groupe de Picard. Pour $\mathcal{M}_{r,d}^{\Delta}$, ce groupe est isomorphe à $\mathbb{Z}$, et il est engendré par un fibré $\mathcal{L}$. Pour calculer les dimensions des groupes de cohomologie de $\mathcal{L}^{\otimes k}$, on peut remarquer que $\mathcal{M}_{r,d}^{\Delta}$ est une variété de Fano, et que $H^i(\mathcal{L}^{\otimes k}, \mathcal{M}) = 0$ pour tout $i > 0$. Les formules de Verlinde expriment les cohomologies de degré $i = 0$ pour les valeurs possibles de d dans le cas de $r = 2$.

Dans cette thèse on développe une nouvelle méthode qui nous permettra entre autres de recalculer ces formules. L'espace de modules $\mathcal{M} = \mathcal{M}_{r,d}^{\Delta}$ est un schéma lisse qui paramétrise certains fibrés E sur C. Au lieu de ces objets, considérons les injections de faisceaux $E \to \mathcal{O}^{\oplus n}$, où $\mathcal{O}^{\oplus n}$ est le fibré trivial de rang n sur C. Par le théorème de Grothendieck, il existe un schéma $\operatorname{Quot}_C(r, d, n)$ dont les points fermés correspondent à ces injections. On appelle ce schéma le **schéma de Quot**. Il possède une suite exacte universelle $\mathcal{E} \to \mathcal{O}^{\oplus n} \to \mathcal{F}$ sur $\operatorname{Quot}_C(r, d, n) \times C$, telle que pour chaque point $q \in \operatorname{Quot}_C(r, d, n)$ qui correspond à la suite $E_q \to \mathcal{O}^{\oplus n} \to F_q$, la restriction de la suite exacte universelle sur $\{q\} \times C$ donne $E_q \to \mathcal{O}^{\oplus n} \to F_q$.

Le schéma de Quot est birationnellement équivalent à l'espace $\mathbb{P} \operatorname{Hom}(\mathcal{O}^{\oplus n}, \mathcal{E}^{\vee})$, parce que chaque application $E \to \mathcal{O}^{\oplus n}$ admet un dual $\mathcal{O}^{\oplus n} \to E^{\vee}$. On abrégera cet espace par $\mathcal{P}$. Cet objet n'est pas en général isomorphe à $\mathcal{M}$, parce que chaque fibré peut s'injecter dans le fibré trivial de plusieurs façons. Néanmoins, on peut identifier certaines de ces injections par l'action de $\mathrm{PGL}(n)$ sur $\mathcal{O}^{\oplus n}$. Ce groupe agit aussi sur $\mathcal{P}$, et chaque orbite consiste en un même fibré avec ses différentes injections. La théorie géométrique des invariants nous explique que pour créer un espace topologique de Hausdorff dont les points correspondent aux orbites, il faut d'abord enlever certains points instables de l'espace. La stabilité de ces points dépend du choix d'un fibré en droite L sur $\mathcal{P}$ et du relèvement de l'action de $\mathcal{P}$ à L. On appelle ce fibré en droite une polarisation de $\mathcal{P}$. En choisissant bien cette polarisation, on obtient un quotient au sens de la théorie géométrique des invariants, qui correspond aux orbites de l'action de $\mathrm{PGL}(n)$.

À moins que $n = \dim \Gamma E^\vee$, le quotient n'est pas $\mathcal{M}$. Ses points s'identifient plutôt aux couples $(V = E^\vee, H \leq \Gamma V)$, qui forment une fibration sur $\mathcal{M}$ en fibres grassmanniennes, $\mathcal{G}r := \mathrm{Gr}(n, \Gamma_C \mathcal{U})$. Au lieu de calculer les cohomologies des fibrés sur $\mathcal{M}$, on peut aussi les calculer sur $\mathcal{G}r$, car les deux sont liées par le théorème Grothendieck-Riemann-Roch. En plus, le groupe de Picard de $\mathcal{G}r$ est engendré par le pullback du générateur de celui de $\mathcal{M}$, et par le fibré dual du déterminant du fibré tautologique des fibres de $\mathcal{G}r$.

La théorie géométrique des invariants nous fournit aussi un fibré $L_{\mathcal{G}r}$, qui dépend de la polarisation L sur $\mathcal{P}$. En définissant la caractéristique d'Euler par la formule $\chi(L) = \sum(-1)^i \dim H^i(L)$, et une forme équivariante analogue, le théorème de Teleman établit que $\chi(L_{\mathcal{G}r}) = \chi(L)^{\mathrm{PGL}(n)} := \sum(-1)^i \dim H^i(L)^{\mathrm{PGL}}$, où $H^i(L)^{\mathrm{PGL}}$ signifie la partie invariante de $H^i(L)$ sous l'action de $\mathrm{PGL}(n)$. Ainsi on peut calculer la caractéristique d'Euler de certains fibrés sur $\mathcal{G}r$ en cherchant un fibré correspondant sur $\mathcal{P}$.

Dans notre approche, on doit remarquer que $\mathrm{PGL}(n)$ agit aussi sur le schéma de Quot $\mathcal{Q} := \mathrm{Quot}_C(r, d, n)$. Pour calculer la partie invariante de $\chi(L_{\mathcal{Q}})$ pour un fibré en droite $L_{\mathcal{Q}}$ sur $\mathcal{Q}$, on peut utiliser la formule de localisation d'Atiyah-Bott, qui est une intégrale sur l'ensemble des points fixes de $\mathcal{Q}$. Le résultat principal de ce travail est ce calcul pour le cas $r = 2$. Afin d'utiliser cette approche pour calculer la caractéristique de $L_{\mathcal{G}r}$, il faut que $L_{\mathcal{Q}}$ sur $\mathcal{Q}$ et L sur $\mathcal{P}$ soient isomorphes sur $\mathcal{P} \cap \mathcal{Q}$, ce qui a du sens parce que $\mathcal{Q}$ et $\mathcal{P}$ sont birationnellement équivalents. Cela nous mène à notre principale conjecture:

Conjecture: Prenons un couple de polarisations $L_{\mathcal{P}}$ sur $\mathcal{P}$ et $L_{\mathcal{Q}}$ sur $\mathcal{Q}$, de façon que leurs restrictions sur $\mathcal{P} \cap \mathcal{Q}$ soient isomorphes. Supposons que les points stables dans $\mathcal{P}$ sous l'action de $\mathrm{PGL}(n)$ et la polarisation $L_{\mathcal{P}}$ sont contenus dans le sous-espace $\mathcal{P} \cap \mathcal{Q}$. Alors $\chi(L_{\mathcal{P}})^{\mathrm{PGL}(n)}$ et $\chi(L_{\mathcal{Q}})^{\mathrm{PGL}(n)}$ sont équivalentes.

Dans ce travail, on se concentre sur le cas où les paramètres r et d sont tels que $r = 2$ et $(r, d) = 1$. Pour vérifier cette conjecture dans certains cas, nous avons fixé $g = 2$, $n = 3$ et $d = 7$, et nous avons évalué les parties invariantes des deux caractéristiques d'Euler pour certaines polarisations L à l'aide de Maple. Chemin faisant, nous avons retrouvé les formules de Verlinde.

Overview

One of the main goals of this work is to develop a new method to calculate the Euler characteristic of line bundles over moduli spaces, in particular the moduli space of stable vector bundles over a smooth complex curve.

The first three chapters are preliminary informations that are needed in this thesis. In chapter 1, the preliminary objects and concepts are introduced, and conventions are established. Chapter 2 describes what is needed about moduli spaces and geometric invariant theory. Chapter 3 presents certain important prior results, including the Grothendieck-Riemann-Roch theorem, the cohomologies of symmetric spaces, and the Atiyah-Bott localisation.

In chapter 4, we introduce the main setup of this thesis, and our main conjecture is announced. In particular, following **[16]**, we study two objects, the Quot scheme $\mathcal{Q}$ over smooth complex curves and a projective space $\mathcal{P}$. Both of these describe certain pairs of a vector bundle V and n of its sections, and they are, under the proper conditions, birationally equivalent. There is a correspondence of polarisations $\mathcal{L}_{\mathcal{Q}}$ and $\mathcal{L}_{\mathcal{P}}$ over $\mathcal{Q}$ and $\mathcal{P}$, respectively, that are isomorphic on their intersection. There is also a compatible pair of natural $\mathrm{PGL}(n)$ actions on $\mathcal{Q}$ and $\mathcal{P}$, and it extends to certain polarisations $\mathcal{L}_{\mathcal{Q}}$ and $\mathcal{L}_{\mathcal{P}}$. Using the shorthand $\chi(L)^{\mathrm{PGL}(n)} = \sum(-1)^i \dim H^i(L)^{\mathrm{PGL}(n)}$, our conjecture is that $\chi(\mathcal{L}_{\mathcal{Q}})^{\mathrm{PGL}(n)} = \chi(\mathcal{L}_{\mathcal{P}})^{\mathrm{PGL}(n)}$ whenever $\mathcal{L}_{\mathcal{P}}$ is ample. We verify this for certain cases.

In chapter 5, we calculate $\chi_T(\mathcal{L}_{\mathcal{Q}}, t)$ for a maximal subtorus T of $\mathrm{PGL}(n)$ using the Atiyah-Bott localisation. This is the novel part of the work, and the main result is in **5.9**. Then in chapter 6, we calculate $\chi(\mathcal{L}_{\mathcal{P}})^{\mathrm{PGL}(n)}$ using already established methods, and compare numerically the values of the two characters for some cases.

This is a joint work with András Szenes and Zsolt Szilágyi.

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1. Preliminaries

1.1. Algebraic curve

Let us review a few definitions and classical results. For details, refer to [**9**].

In this dissertation, C will be a **smooth algebraic curve** over $\mathbb{C}$, the complex numbers, unless stated otherwise. Every function shall be complex analytic, that is **holomorphic**, unless stated otherwise.

Since topologically a smooth algebraic curve is a real orientable surface, it is homeomorphic to the connected sum of several tori (or to a sphere, in which case it is understood as the connected sum of zero tori). The number of such tori needed is called the **genus** of C, denoted by g.

1.2. Sheaves and sheaf cohomologies

Let us denote by $\mathcal{O}_C$ the *sheaf* of holomorphic functions over C. Every *vector bundle* V over C has a corresponding $\mathcal{O}_C$*-module* of (holomorphic) *sections*, itself a sheaf.

All *bundle maps* between vector bundles are *sheaf maps* between their sheaves of sections. However, a sheaf-map may be injective even when the corresponding bundle map is not: an injective sheaf map can be degenerate on a Zariski closed subset of C. In this work, unless specified, I will consider **all maps to be sheaf maps**.

Each vector bundle and its corresponding sheaf has its corresponding *sheaf cohomologies*, $H^n(\mathcal{F})$. We will abbreviate $\dim H^n(\mathcal{F})$ as $h^n(\mathcal{F})$. Since C is a curve embedded inside a projective space, it is covered by two affine spaces, and by Čech cohomology $H^i(\mathcal{F}) = 0$ if $i > 1$. In particular, $H^0(V) \cong \Gamma(V)$ is isomorphic to the group of global sections of V.

Since the sheaf of sections of V is an $\mathcal{O}_C$-module, its holomorphic sections that are not identically zero are in a bijection with injective sheaf morphisms from $\mathcal{O}$ to the sheaf of V.

The set of sheaf morphisms between two vector bundles U and V is denoted by $\mathrm{Hom}(U, V)$ and it is isomorphic to $\Gamma(U^* \otimes V)$. The sheaf of maps is defined as $\mathcal{H}om(U, V) = U^* \otimes V$.

Theorem 1: Over a smooth curve C, a sheaf $\mathcal{F}$ is a vector bundle if and only if for any $p \in C$, there is a Zariski open $p \in U$ such that $\mathcal{F}|_U$ is freely generated over $\mathcal{O}_C|_U$. This condition is called a **locally free** sheaf.

Theorem 2: Over a smooth curve C, a subsheaf of a locally free sheaf is locally free. Alternatively, subsheaves of a vector bundle may be represented as sheaf-embeddings of vector bundles.

1.3. Line bundles

Let $\mathcal{O}$ denote the trivial *line bundle.*

The set of isomorphism classes of *line bundles* over C is given by the first cohomology group $H^1(C, \mathcal{O}^\times)$, where $\mathcal{O}^\times$ is the multiplicative sheaf of nowhere zero functions (note that it is not an $\mathcal{O}_C$-module). There is a short exact sequence of sheaves

$$0 \to \mathbb{Z} \overset{\cdot 2\pi i}{\to} \mathcal{O} \overset{\exp}{\to} \mathcal{O}^\times \to 0.$$

In the corresponding long exact sequence of sheaf cohomologies, $H^0(\mathcal{O}) \to H^0(\mathcal{O}^\times)$ is surjective, and thus can be ignored. Furthermore, $H^1(\mathbb{Z}) \cong \mathbb{Z}^{2g}$, $H^2(\mathbb{Z}) \cong \mathbb{Z}$, and $H^1(\mathcal{O}) \cong \mathbb{C}^g$. This gives the sequence

$$0 \to \mathbb{Z}^{2g} \to \mathbb{C}^g \to J(C) \to \mathbb{Z}$$

where $J(C) = H^1(C, \mathcal{O}^\times)$ is the **Jacobian variety** of C, and it is denoted by $J(C)$. The map $J(C) \to \mathbb{Z}$ is the **degree map**. The kernel of the degree map is $J_0(C)$, the space of degree 0 line bundles.

An important example of a line bundle is the **canonical line bundle**, defined as T^*C, the cotangent line bundle of C. It is denoted by $\mathcal{K}$, and has degree $2(g-1)$.

1.4. Divisors

The group of **divisors** over C, denoted by $\mathrm{Div}(C)$, is the free Abelian group generated by the points of C. Divisors are thus of the form

$$\sum_{i=0}^{k} m_i p_i \quad \text{where } m_i \in \mathbb{Z}, p_i \in C.$$

A divisor is **effective** if all $m_i \geq 0$.

For a non-zero meromorphic section s of a line bundle L, we will denote by $[s]$ its corresponding divisor that is defined as the sum of the zeroes of s multiplied by the degrees of the zeroes, minus the poles multiplied by their degrees.

The **degree** of a divisor is defined via the map

$$\deg\colon \sum_{i=0}^{k} m_i p_i \longrightarrow \sum_{i=0}^{k} m_i.$$

Theorem 3: The degree of a line bundle is equal to the degree of the divisor of any of its meromorphic sections.

The holomorphic (and meromorphic) sections of $\mathcal{O}$ are in a one-to-one correspondence with the holomorphic (respectively meromorphic) functions over C.

Given two non-zero meromorphic sections s and t of two line bundles, K and L over C, $s \cdot t$ gives a well-defined section of $K \otimes L$. Multiplication of sections is additive on the divisors: $[st] = [s] + [t]$, and hence on their degrees: $\deg[st] = \deg[s] + \deg[t]$. Also, given a line bundle L over C with a meromorphic section s, $\frac{1}{s}$ gives a section of L^*, and so $\left[\frac{1}{s}\right] = -[s]$.

For a fixed point $p \in C$ of the curve, there is a degree 1 line bundle, denoted by $\mathcal{O}(p)$, such that it has a holomorphic section whose divisor is $[p]$.

For any divisor $D = \sum_i m_i p_i$, there is a line bundle who has a meromorphic section of divisor D:

$$\mathcal{O}(D) = \bigotimes_{m_i > 0} \mathcal{O}(p_i)^{\otimes m_i} \otimes \bigotimes_{m_i < 0} (\mathcal{O}(p_i)^*)^{\otimes m_i}$$

of degree $\sum_i m_i$. For any vector bundle V and divisor D, $V \otimes \mathcal{O}(D)$ will be shortened to $V(D)$.

Theorem 4: Denote by $\Gamma_{\text{mer}}(L)$ the vector space of meromorphic sections of a line bundle L, and let S be the set of pairs $\{(L, \gamma) \mid \gamma \in \mathbb{P}\Gamma_{\text{mer}}(L), L \in J(C)\}$. Then the map $[.]\colon S \to \mathrm{Div}(C)$ that sends a meromorphic section to its corresponding divisor is bijective.

The map arising from $(L, \gamma) \to L$ between $\mathrm{Div}(C)$ and $J(C)$ is the **Abel-Jacobi map**.

1.5. Vector bundles

Let us reserve the notation V^r for a vector bundle V of *rank* r. We will denote $V^{\oplus n} = \bigoplus_{i=1}^n V$ and $V^{\otimes n} = \bigotimes_{i=1}^n V$.

The top exterior product $\bigwedge^r V$ is a line bundle. In analogy with matrices, it is called the **determinant line bundle** of V, and it is denoted by $\det V$.

The **degree of a vector bundle** shall be defined as the degree of its determinant bundle: $\deg V = \deg \det V$. Then the degree of a tensor product of two vector bundles V and W is

$$\deg(V \otimes W) = \deg V \operatorname{rk} W + \operatorname{rk} V \deg W. \tag{1}$$

The **Euler class** $e(L)$ of a line bundle L is defined by taking a generic meromorphic section, and taking the difference of the Poincaré duals of the zeroset and poleset of the section, taken with multiplicity. The **first Chern class** of a vector bundle is defined to be $c_1(V) := e(\det V)$. In particular, for line bundles L, their first Chern class is equal to their Euler class, since $\det L = L$. Then $c_1(A \oplus B) = c_1(A) + c_1(B)$ and $c_1(A \otimes B) = \operatorname{rk} B c_1(A) + \operatorname{rk} A c_1(B)$ for any pair A, B of vector bundles.

1.6. Coherent sheaves

A **coherent sheaf** can be defined as the quotient of a locally free sheaf by a subsheaf. Every coherent sheaf over C decomposes as the direct sum of a vector bundle and its **torsion sheaf**, that is its maximal subsheaf *supported* on a finite set of points of C.

Subsheaves and quotients of coherent sheaves are also coherent. Furthermore, all standard operations on coherent sheaves, such as direct sum, tensor product and the Hom-functor give coherent sheaves. For this reason, we can restrict our study to coherent sheaves.

Given a short exact sequence of coherent sheaves

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

with $\mathcal{E}$ and $\mathcal{F}$ locally free, we define the **rank** and **degree** of $\mathcal{G}$ as $\operatorname{rk} \mathcal{F} - \operatorname{rk} \mathcal{E}$ and $\deg \mathcal{F} - \deg \mathcal{E}$, respectively.

The **Euler characteristic** of $\mathcal{F}$ is defined as

$$\chi(\mathcal{F}) = \sum_i (-1)^i \dim H^i(\mathcal{F})$$

([**19**], p24). Note that since C is a curve, only $\dim H^0(\mathcal{F}) - \dim H^1(\mathcal{F})$ appears in this formula.

1.7. Some essential theorems

Theorem 5: (Riemann-Roch theorem) For a vector bundle V,

$$\chi(V) = \deg V + (1-g) \operatorname{rk} V.$$

Theorem 6: (Serre duality) $H^i(V) \cong H^{\dim X - i}(\mathcal{K} \otimes V^*)^*$ where $\mathcal{K}$ is the canonical line bundle over the curve.

Recall the notation $\mathcal{F}(D) = \mathcal{F} \otimes \mathcal{O}(D)$ for some divisor $D \in \operatorname{Div}(C)$. Let us denote by Λ the divisor that is the intersection of the curve with a hyperplane in the projective space where C is embedded.

Theorem 7: (Cartan's theorem A and B) For a coherent sheaf $\mathcal{F}$, there is a large enough n such that $\mathcal{F}(n\Lambda)$ is generated by its sections, and $H^p(\mathcal{F}(n\Lambda)) = 0$ for $p > 0$. ([**19**], p25)

2. Moduli spaces

2.1. Introduction

Let us fix an algebraic curve C and two parameters d and r, and consider the set of isomorphism classes of vector bundles of degree d and rank r. In general, this set can not be given a variety structure. However, when restricting the set to certain vector bundles, called stable bundles, a variety structure exists. This variety is called the moduli space of stable vector bundles.

One possible way to construct this moduli space is via geometric invariant theory and the Quot scheme. In this chapter, I will give a brief review of these topics.

2.2. The Grassmannian

Let us fix a vector space V^n and an integer $0 < k < n$. The **Grassmannian** is a variety whose points are the k-dimensional subspaces. It is denoted by $\mathrm{Gr}(k, V)$ or $\mathrm{Gr}(V)$ when k is clear from the context. Equivalently, the points of a Grassmannian are the $n-k$ dimensional quotients of V.

The Grassmannian can be given a variety structure through the **Plücker embedding**. For each subspace $U \leq V$, we take a basis $u_1, \ldots, u_k$, and take their wedge product $u_1 \wedge \cdots \wedge u_k$. This identifies the subspace up to a scalar, defining the Grassmannian as a subspace of $\mathbb{P} \bigwedge^k V$.

Given $\mathcal{V} := V \times \mathrm{Gr}(V)$ the trivial bundle with fiber V over $\mathrm{Gr}(V)$, there is a subbundle $\mathcal{S} := \{(v, s) \mid v \in s\} \subset V \times \mathrm{Gr}(V)$. This gives us a universal sequence

$$0 \to \mathcal{S} \xrightarrow{f} \mathcal{V} \to \mathcal{Q} \to 0 \tag{2}$$

The bundle $\mathcal{S}$ is the **tautological vector bundle**. Taking $\mathcal{O}(1) := \det \mathcal{S}^*$, it can be seen through a routine calculation that the global sections of $\mathcal{O}(1)$ correspond to k-forms on V, hence $\Gamma(\mathcal{O}(1))$ is isomorphic to $\left(\bigwedge^k V\right)^*$. Thus $\mathrm{Gr}(k, V)$ can be embedded into $\bigwedge^k V \cong \mathbb{P}\Gamma(\mathcal{O}(1))^*$ via the Plücker embedding.

Theorem 8: The tangent bundle of $\mathrm{Gr}(V)$ is isomorphic to $\mathrm{Hom}(\mathcal{S}, \mathcal{Q})$ where $\mathcal{S} \to \mathcal{V} \to \mathcal{Q}$ is the universal sequence over the Grassmannian.

2.3. The Quot scheme

Let us fix a vector bundle W of rank n, in our case the trivial bundle $\mathcal{O}^n = \mathcal{O}^{\oplus n}$, and consider its sheaf quotients F of a certain fixed rank r and degree d:

$$0 \to E^{n-r} \to W^n \to F^r \to 0.$$

There are multiple conventions in the literature for denoting the rank and degree of E and F. In [**8**] and [**19**], F is of rank r and degree d, while E is of rank $n-r$ and degree $\deg W - d$ (when $W = \mathcal{O}^{\oplus n}$, $\deg E = -d$), and this is the convention chosen in this chapter. In [**15**], the subsheaves E are the center of attention, and there the rank and degree of E are denoted by r and d. In later chapters, we will choose E of rank r and degree $-d$.

Theorem 9: (Grothendieck's Theorem) [**8**] Given a projective space X and a vector bundle W over X, there is a projective variety called the **Quot scheme**, denoted by $\mathrm{Quot}_X^{r,d}(W)$, whose closed points correspond to rank r, degree d quotients of the vector bundle W. Furthermore there is a universal sequence

$$0 \to \mathcal{E} \to \mathcal{W} \to \mathcal{F} \to 0 \tag{3}$$

over $\mathrm{Quot}_X^{r,d}(W) \times X$, such that for a family of quotients $\tilde{\mathcal{W}} \to \tilde{\mathcal{F}}$ over $\mathcal{T} \times X$ with $\tilde{\mathcal{W}}$ being the pullback of W from X, there is a unique map from $\mathcal{T}$ to the Quot scheme where $\tilde{\mathcal{W}}$ and $\tilde{\mathcal{F}}$ are the pullbacks of $\mathcal{W}$ and $\mathcal{F}$, and the maps commute.

Grothendieck's theorem holds over any projective variety, however we will only consider sheaves over the curve C. Then, as a consequence of **Theorem 2**, E is always a vector bundle.

Theorem 10: In a smooth point $E \to W \to F$ of the Quot scheme, the tangent space is isomorphic to $\mathrm{Hom}(E, F)$. The whole tangent bundle over the subspace of smooth points is $\pi_*(\mathcal{E}^* \otimes \mathcal{F})$ where $\pi\colon \mathrm{Quot}_X^{r,d}(W) \times X \to \mathrm{Quot}_X^{r,d}(W)$.

There is a more general notion, the *Zariski tangent space*, which is described in [**8**] and [**16**] as $\mathrm{Hom}(E, F)$ in all points $E \to W \to F$. The smooth points are those where this is of minimal dimension.

Since $\chi(E^* \otimes F)$ clearly does not depend on the choice of E apart from its rank and degree, the smooth points are those E where $H^1(\mathcal{E}_E^* \otimes \mathcal{F}_E)$ is maximal. In particular, if $r = 0$, used in [**5**], $\mathcal{E}_E$ is always concentrated on a zero dimensional subset of C, and so $H^1(\mathcal{E}_E^* \otimes \mathcal{F}_E)$ is always zero. This gives us the following result:

Corollary 11: When $r = \operatorname{rk} F = 0$, the Quot scheme $\mathrm{Quot}_X^{0,d}(W)$ is smooth.

When $W = \mathcal{O}^{\oplus n}$, there is a different characterisation of the Quot scheme. A quotient $(F, \mathcal{O}^{\oplus n} \to F)$ can be completely described as F and n of its sections that generate it, up to a common scalar. Put differently, such a quotient is described by a pair $(F, f \in \mathbb{P}(H^0(F)^n))$, or F and some of its sections, with the condition that the sections f generate F.

2.4. Grothendieck's theorem

In this section, we will review the embedding of the Quot scheme into a Grassmannian for the case of $W = \mathcal{O}^n$ over the curve C.

First consider the projective line $\mathbb{P}^1$. Every line bundle over the projective line is completely determined by its degree. It is a simple calculation that $\dim H^0(\mathcal{O}(n)) = n+1$ for $n \geq 0$ and zero otherwise. Using the Riemann-Roch theorem, $\chi(\mathcal{O}(n)) = n+1$, and this implies that $\dim H^1(\mathcal{O}(n)) = -n-1$ for $n \leq -1$, and zero otherwise. For vector bundles of higher rank, we have the following theorem:

Theorem 12: (Grothendieck's theorem for vector bundles) Every vector bundle over $\mathbb{P}^1$ can be written uniquely as the direct sum of line bundles.

This makes it very easy to calculate properties of vector bundles over the projective line, such as the following:

Lemma 13: Given a vector bundle B over $\mathbb{P}^1$ with $h^0(B(-1)) = 0$ and a coherent subsheaf A, then $h^1(A(m)) = 0$ for $m \geq -\deg A + 1$.

Proof: By Grothendieck's theorem, every vector bundle decomposes as the sum of line bundles. Since B is a vector bundle, its subsheaves are vector bundles as well, including A. If $h^0(B(-1)) = 0$, then A has no positive degree component, and so the smallest degree component of A is at least $\deg A$. Therefore $h^1(A(m)) = 0$ for $m \geq -\deg A + 1$. □

Now consider an algebraic curve C of genus g, embedded inside a projective space as degree k curve. By taking hyperplanes passing through a hyperspace of codimension 2 that do not intersect C, we obtain a degree k map $f\colon C \to \mathbb{P}^1$. The dimensions of the cohomology groups of coherent sheaves are preserved via pushforward, that is $h^i(f_*F) = h^i(F)$ for a coherent sheaf F. In turn, the Euler characteristic is also preserved, and since $\operatorname{rk} f_*V = k \operatorname{rk} V$, this gives us $\deg f_*V = \deg V + (1-g-k)\operatorname{rk} V$. Let us denote by Λ the divisor that is the intersection of the curve C with an affine hyperplane in $\mathbb{P}^n$ with C. It is a degree k effective divisor. In fact, the line bundle $\mathcal{O}(\Lambda)$ is the same as the restriction of the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ over $\mathbb{P}^n$ (defined as the dual of the determinant of the tautological vector bundle) to C, and it does not depend on the choice of a hyperplane. This can also be expressed as $f^*\mathcal{O}_{\mathbb{P}^1}(1)$ and Λ is the preimage of a point on $\mathbb{P}^1$ via f. Consider the following lemma:

Lemma 14: $f_*(F(m\Lambda)) = (f_*F)(m)$ for any coherent sheaf F.

Using this lemma and **Lemma 13**, we arrive at the following result:

Lemma 15: Over C, given a coherent subsheaf $A \to \mathcal{O}$ with $\operatorname{rk} A = r$ and $\deg A = -d$, then $h^1(A(m\Lambda)) = 0$ for $m \geq 1 - d - (1-g-k)r$.

Proof: Consider that $h^0((f_*\mathcal{O})(-1)) = h^0(\mathcal{O}(-\Lambda))$ is zero since $\deg \mathcal{O}(-\Lambda) = -k < 0$. Therefore the conditions for the previous lemma are set for the bundle $f_*\mathcal{O}$ and its subbundle f_*A and $h^1((f_*A)(-\deg f_*A+1)) = 0$. Recall that $\deg f_*A = \deg A + (1-g-k)\operatorname{rk} A$, hence $h^1(A(m\Lambda)) = 0$ for $m \geq 1-d-(1-g-k)r$. $\square$

Lemma 16: Given a vector bundle A over C, if $h^1(A(m\Lambda)) = 0$ for every integer $m \geq 0$, then $A(\Lambda)$ is generated by its global sections.

Proof: First, we shall prove that for some m, $A(m\Lambda)$ is generated by its global sections, then that $m = 1$ suffices.

Consider the embedding of C into $\mathbb{P}^n$. By taking two affine hyperplanes H_1 and H_2, the complements of their intersections with C are two affine sets U_1 and U_2 that gives an affine covering of C. This intersection $H_i \cap C$ may also be described as the zeroes of a section f_i of $\mathcal{O}(\Lambda)$ for $i \in \{1,2\}$. For each open set U_i, the restriction $A|_{U_i}$ is generated by some holomorphic algebraic sections s that nonetheless extend as a meromorphic section of A. Then for some m, $f_i^m s$ is a holomorphic section of $A(m\Lambda)$.

We need to prove that this can be done with the choice $m = 1$. Take a global section s of $A(m\Lambda)$, and we will prove that it is the linear combinations of global sections of $A((m-1)\Lambda)$, provided that $m > 1$. Consider $\frac{s}{f_1 f_2}$, a holomorphic section of $A((m-2)\Lambda)$ over $U_1 \cap U_2$. Since $h^1(A((m-2)\Lambda)) = 0$, by the Čech cohomology this means that any section over $U_1 \cap U_2$ decomposes as the sum of sections over the two sets: $\frac{s}{f_1 f_2} = s_1 + s_2$ with s_1 and s_2 holomorphic over U_1 and U_2, respectively.

Multiplying the three by f_2, this gives $s/f_1 - f_2 s_1 = f_2 s_2$. Since the left-hand side is holomorphic over U_1, $f_2 s_2$ is as well, and so it is a global section of $A((m-1)\Lambda)$. Similarly for $f_1 s_1$. Now $s = f_2(f_1 s_1) + f_1(f_2 s_2)$, therefore s is generated by $f_1 s_1$ and $f_2 s_2$. Replacing all ocurring s with s_1 and s_2 for all sections of $A(m\Lambda)$, this gives a set of global sections of $A((m-1)\Lambda)$ that generate the sheaf. $\square$

Let us choose an integer m and consider the map from the Quot scheme $\operatorname{Quot}_C^{r,d}(\mathcal{O}^n)$ into a Grassmannian defined by taking the element $A \to \mathcal{O}^n \to B$ in the Quot scheme to the subspace $H^0(A(m\Lambda))$ of $H^0(\mathcal{O}^n(m\Lambda))$. We will show that this is in fact an embedding.

Consider the short exact sequence $A \to \mathcal{O}^n \to B$. Taking the cohomologies of their twists, this gives a long exact sequence

$$\begin{array}{ccccccccc} 0 & \to & H^0(A(m\Lambda)) & \to & H^0(\mathcal{O}^n(m\Lambda)) & \to & H^0(B(m\Lambda)) & \to \\ & \to & H^1(A(m\Lambda)) & \to & \dots & & & \end{array}$$

Provided that $H^1(A(m\Lambda)) = 0$, this gives once again a short exact sequence. Furthermore, the dimension of the subspace, $h^0(A(m\Lambda))$, is determined by r and $-d$, since $h^1(A(m\Lambda)) = 0$.

In order for this identification to give an embedding, any two subsheaves must have different images. Consider the following maps:

$$\begin{tikzcd} H^0(A(m\Lambda)) \otimes \mathcal{O} \arrow[r] \arrow[d] & H^0(\mathcal{O}^n(m\Lambda)) \otimes \mathcal{O} \arrow[d] \\ A(m\Lambda) \arrow[r] & \mathcal{O}^n(m\Lambda) \end{tikzcd}$$

where the diagram commutes. To make sure that the downward arrows are surjective, and thus generate the subsheaf $A(m\Lambda)$, both $A(m\Lambda)$ and $\mathcal{O}^n(m\Lambda)$ must be generated by their global sections. For $\mathcal{O}^n(m\Lambda)$, this holds whenever $m > 2g-2$ by Serre duality. For $A(m\Lambda)$, it is enough by **Lemma 16** if $H^1(A(m'\Lambda)) = 0$ for all $m' \geq m-1$. By **Lemma 15**, this holds whenever $m > 1-d-(1-g-k)r$. Therefore the image of a point of the Quot scheme in the Grassmannian determines the point on the Quot scheme.

This can be summed up in the following theorem:

Theorem 17: The Quot scheme of rank r degree $-d$ subsheaves of $\mathcal{O}^n$ can be embedded into $\mathrm{Gr}(-d+(1-g+mk)r, H^0(\mathcal{O}^n(m\Lambda)))$ provided that $m > \max(1-d-(1-g-k)r, 2g-2)$.

2.5. Proj construction

Let us consider a vector space V and its projectivisation $\mathbb{P}V$. Linear homogeneous functions over V correspond to sections of the bundle $\mathcal{T} := \mathcal{O}_{\mathbb{P}V}(1)$, and degree d homogeneous functions to sections of $\mathcal{T}^{\otimes d}$. The ring of functions $\mathcal{O}(V)$ is a graded ring that decomposes as $\bigoplus_{n\geq 0} \Gamma(\mathcal{T}^{\otimes n})$. Note that evaluating a polynomial in a point is not well defined, homogeneous polynomials have a well defined zero set in $\mathbb{P}V$.

For any pojective variety $Q \subseteq \mathbb{P}V$, there is a homogeneous ideal I_Q within $\mathcal{O}(V)$, consisting of polynomials that vanish in each point of Q. Then the ring of functions over Q may be defined as the graded ring $A = \mathcal{O}(V)/I_Q$.

The variety Q itself can be reconstructed from A using the **Proj construction**. The elements of $\operatorname{Proj} A$ are the *homogeneous prime ideals* that do not contain the *irrelevant ideal* A_+ consisting of functions with no degree-zero term. The topology is the same **Zariski topology** as for the Spec. ([**9**], p76) Alternatively, $\operatorname{Proj} A$ can be constructed as a *GIT quotient* of the complement of the irrelevant ideal within $\operatorname{Spec} A$ by the action of $\mathbb{C}^*$. ([**19**], p96)

For any embedding $\iota\colon Q \to \mathbb{P}V$, we can take the pullback of the bundle $\mathcal{O}_{\mathbb{P}V}(1)$ from over $\mathbb{P}V$, $\mathcal{L} := \iota^*\mathcal{T}$. Since sections of $\mathcal{L}$ are linear functions on $\mathbb{P}V$ restricted to Q, $\mathcal{L}$ is a separable bundle, and Q embeds into $\mathbb{P}\Gamma(\mathcal{L})^*$, itself a subspace of $\mathbb{P}V$.

In general, a line bundle $\mathcal{L}$ over Q is called the **polarisation** of the space Q. The bundle $\mathcal{L}$ is **very ample** if the natural map $Q \to \mathbb{P}\Gamma(\mathcal{L})^*$ is an embedding. ([**9**], p153) The ring of functions depends on this embedding, and since for very ample bundles $A_1 \cong \Gamma(\mathcal{L}^*)$, the Proj construction gives a natural linearisation as well. Projective subvarieties of the projective space are thus in a one-to-one correspondence with projective varieties with a very ample linearisation.

A line bundle is **ample** if it has a positive tensor power that is very ample. An important example of a very ample line bundle is the line bundle $\mathcal{O}_{\mathcal{G}}(1)$ over a Grassmannian $\mathcal{G}$.

A smooth curve C can always be embedded into a projective space $\mathbb{P}$, given by $\operatorname{Proj} \mathcal{K}^{\otimes n}$ where $\mathcal{K}$ is the canonical line bundle, provided that $n \geq 3$ for $g = 2$, $n \geq 2$ for hyper-elliptic curves and $n \geq 1$ otherwise. Such a curve is of degree $k = 2n(g-1)$, and the pullback of $\mathcal{O}_{\mathbb{P}}(1)$ onto C is $\mathcal{O}(\Lambda) = \mathcal{K}^{\otimes n}$

2.6. Geometric invariant theory

Definition 18: ([**19**], p85-p86) Let G be a reductive group. Given a algebraic variety Q and a G-action on it, a couple $(Z, \varphi\colon Q \to Z)$ is a **good quotient** of Q if Z is an algebraic variety with trivial G-action, φ is an equivariant map, and they satisfy the following three conditions:
(1) φ is affine and surjective;
(2) The image of a closed invariant subset of Q is closed in Z, and two such disjoint subsets have disjoint images in Z;
(3) The structure sheaf of Z is the sheaf of invariant sections of the structure sheaf of Q.

A good quotient for a projective variety $Q \subseteq \mathbb{P}V$ can not always be obtained. However, we can take the graded ring of functions $A = \mathcal{O}(Q)$, and consider the Proj of the invariant graded ring $\operatorname{Proj} A^G$, where A^G is the ring of G-invariant polynomials. Geometric invariant theory explains that this is a good quotient of a certain open subspace of Q.

Definition 19: An algebraic group G is **linearly reductive** if all finite dimensional G-modules are *semi-simple*. ([**19**], p88). From now on we will only consider such groups.

A point $q \in Q$ is **semi-stable** if there exists a homogeneous G-invariant polynomial with strictly positive degree which does not vanish at q. The semi-stable points of Q form an open subset Q^{ss}.

Theorem 20: There is a morphism $Q^{ss} \to \operatorname{Proj} A^G$ induced by the inclusion $A^G \subseteq A$ that is a good quotient of Q^{ss}. ([**19**], p98)

We will refer to this projective variety $\operatorname{Proj} A^G$ as the **GIT quotient** of the space Q. This quotient comes with a natural polarisation $\mathcal{P}$ via its embedding into a projective space. When $\mathcal{L}$ is the pullback of the line bundle $\mathcal{O}_{\mathbb{P}V}(1)$ via the embedding $Q \to \mathbb{P}V$, we will say that $\mathcal{L}$ **induces** to the polarisation $\mathcal{P}$ on $\operatorname{Proj} A^G$.

A point $v \in V$ is **semi-stable** if 0 does not appear in the closure of its orbit. It can be seen that v is semi-stable if and only if the point $[v] \in Q \subseteq \mathbb{P}V$ is semi-stable according to the above definition ([**19**], p98). The point v is defined to be **stable** if its orbit is closed and its stabiliser is finite. ([**19**], p100)

2.7. Mumford's criterion

To determine which points are semi-stable, we can use Mumford's criterion, as seen in [**19**], p101-102:

A **one parameter subgroup** of the group G is a multiplicative map $\lambda\colon \mathbb{C}^* \to G$. It can also refer to the image of the map, a subgroup of G.

Definition 21: Given a group G, a G-module V, and a one parameter subgroup $\lambda\colon \mathbb{C}^* \to G$ and an element $v \in V$, $\mu(\lambda, v)$ is defined in the following way: V decomposes in a unique way into subspaces $\bigoplus_{m\in\mathbb{Z}} V_m$, such that λ acts on V_m as $(t, u) \to t^m u$. Then any v decomposes as $\sum_{m\in\mathbb{Z}} v_m$, and $\mu(\lambda, v) = -\min\{m \mid v_m \neq 0\}$.

Theorem 22: (Hilbert-Mumford criterion) If the *linearly reductive* group G acts on a vector space V, then a point $v \in V$ is semi-stable (respectively stable) if and only if for every one parameter subgroup it is semi-stable (respectively stable).

If $Q \subseteq \mathbb{P}V$ is a closed G-invariant subvariety, then a point $q \in Q$ is semi-stable (respectively stable) if and only if for each one parameter subgroup λ, then $\mu(\lambda, q) \geq 0$ (respectively $\mu(\lambda, q) > 0$).

2.8. Moduli space of stable bundles

The set of isomorphism classes of vector bundles over C of a fixed rank r and degree d do not have a variety structure. However, when restricting the set to certain vector bundles, it may be given a variety structure via geometric invariant theory.

The **slope** of a coherent sheaf F is defined as $\mu(F) := \frac{\deg F}{\operatorname{rk} F}$. A vector bundle V is **semi-stable** if for any subbundle $U \leq V$, $\mu(U) \leq \mu(V)$, and it is **stable** if $\mu(U) < \mu(V)$, unless $U = V$. ([**21**], p14; [**19**], p74-75) Equivalently, a vector bundle V is stable if for all coherent subsheaves U, $\mu(U) < \mu(V)$, or for all quotient bundles or coherent sheaves U, $\mu(U) > \mu(V)$. ([**21**], p15)

Lemma 23: Given two semi-stable bundles U and V, either $\operatorname{Hom}(U, V) = 0$, or $\mu(U) \leq \mu(V)$. Furthermore, if they are stable and $\mu(U) = \mu(V)$, then either $\operatorname{Hom}(U, V) = 0$, or $\operatorname{Hom}(U, V)$ contains only isomorphisms between U and V. For a stable bundle U, $\operatorname{End}(U) = \operatorname{Aut}(U) = \mathbb{C}^*$. ([**19**], p74)

We will first suppose that $(r,d)=1$, so that semi-stability and stability are equivalent.

The moduli space of stable vector bundles is constructed in **[19]** as the good quotient of a Quot scheme as follows. Recall that points of the Quot scheme $\operatorname{Quot}^{r,d}(\mathcal{O}^n)$ can be described as pairs (F,f) of some sheaf F and n of its sections $f\colon \mathcal{O}\to F$, such that they generate F as a sheaf. The expected dimension of $H^0(F)$ is $\chi(F)=d+(1-g)r$, and it is equal if $H^1(F)=0$. Therefore if we choose $n=\chi(F)$, and act on the Quot scheme by $\operatorname{PGL}(H^0(F)^{\chi(F)})$, the orbits of these pairs should identify F.

Theorem 24: Fix an algebraic curve C and two relative prime parameters r and d and $\chi := d+(1-g)r$. If d is chosen large enough, the points of the GIT quotient of the Quot scheme $\operatorname{Quot}^{r,d}(\mathcal{O}^{\oplus\chi})$ under the natural $\operatorname{PGL}(\chi)$ action are in a one-to-one correspondence with rank r degree d stable vector bundles.

Quotienting the Quot scheme $\operatorname{Quot}^{d,r}(\mathcal{O}^{\chi})$ via the action by $\operatorname{PGL}(\chi)$ gives the **moduli space of semi-stable vector bundles** $\mathcal{M} := \operatorname{Quot}(W)^{ss}/\operatorname{PGL}(n)$.

The tangent space of $\mathcal{M}$ in $V\in\mathcal{M}$ is of dimension $h^1(\operatorname{End}V)$. Since $\chi(\operatorname{End}V)=(1-g)r^2$ and $h^0(\operatorname{End}V)=1$ for a semi-stable V, the dimension of the tangent space is $1-(1-g)r^2$ independently of V, and hence the dimension of the moduli space.

2.9. The universal bundle

Let us fix the parameters r and d such that $(r,d)=1$, and consider the moduli space of stable vector bundles $\mathcal{M}$.

Definition 25: A vector bundle $\mathcal{U}$ over the moduli space of stable vector bundles $\mathcal{M}$ is such that for any $V\in\mathcal{M}$, the pullback of $\mathcal{U}$ via the injection map $f_V\colon\{V\}\times C\to\mathcal{M}\times C$ gives $f_E^*(\mathcal{U})\cong V$.

Theorem 26: There is a universal bundle $\mathcal{U}$ over $\mathcal{M}\times C$.

This universal bundle can not be chosen in a unique way: for any line bundle L over $\mathcal{M}$, $L\otimes\mathcal{U}$ is also a universal bundle, and any two universal bundles differ in such a line bundle. However, this ambiguity may be removed in a well defined way. In **[2]**, a normalisation is described to identify the **normalised universal bundle**, that we will review here and give a generalisation.

First let us choose a bundle $\mathcal{U}$. Let $\mathcal{U}_p$ denote the pullback of $\mathcal{U}$ via the map $\mathcal{M}\times\{p\}\to\mathcal{M}\times C$ and $\pi_*\mathcal{U}$ the pushforward via the map $\mathcal{M}\times C\to\mathcal{M}\times\{*\}$. Let us introduce $a_1 := c_1(\mathcal{U}_p)$ and $d_1 := c_1(\pi_*\mathcal{U})$.

Whenever one takes a vector bundle $F=F(\mathcal{U})$ defined on $\mathcal{M}$ using the universal bundle $\mathcal{U}$, and we replace $\mathcal{U}$ by $\mathcal{U}'=\mathcal{U}\otimes\mathcal{L}$, this will change F to $F'=F(\mathcal{U}')=F\otimes\mathcal{L}^{\otimes A}$ for some $A\in\mathbb{Z}$. I will call this the **ambiguity** of F. Now if $X=X(\mathcal{U})$ is a vector bundle over $\mathcal{M}$ defined using the universal bundle $\mathcal{U}$, and $\det X$ has an ambiguity of A, replacing $\mathcal{U}$ with $\mathcal{U}'$ will change $\det X$ to $\det X' = \det X(\mathcal{U}') = \det X \otimes \mathcal{L}^{\otimes A}$. This results in $c_1(X)$ changing to $c_1(X') = c_1(X) + Ac_1(\mathcal{L})$.

The ambiguity of $\det \mathcal{U}_p$ is r, and that of $\det \pi_* \mathcal{U}$ is $\chi = d + (1-g)r$. Since r and d are relatively prime, so are r and χ, and there are integer coefficients κ and λ such that $\kappa r + \lambda \chi = 1$. Defining $\Lambda := (\det \mathcal{U}_p)^{\otimes \kappa} \otimes (\det \pi_* \mathcal{U})^{\otimes \lambda}$, the new universal bundle $\mathcal{V} := \mathcal{U} \otimes \Lambda^*$, the normalised universal bundle, has zero ambiguity. Therefore $\mathcal{V}$ only depends on the pair κ, λ, but not $\mathcal{U}$.

To calculate a_1 and d_1, consider the following. If we started out with the universal bundle $\mathcal{U}' := \mathcal{V}$, since all universal bundles are equivalent up to a tensoring by a line bundle, we would have $\mathcal{V} = \mathcal{U} \otimes L$, and $\mathcal{V}' = \mathcal{V} \otimes L^{\otimes 0} = \mathcal{V}$. Since $\mathcal{V}' = \mathcal{V} \otimes \Lambda'^*$, we would obtain that $\Lambda' := (\det \mathcal{V}_p)^{\otimes \kappa} \otimes (\det \pi_* \mathcal{V})^{\otimes \lambda}$ is the trivial bundle. Therefore $c_1(\Lambda') = \kappa a_1 + \lambda d_1 = 0$. Then the element $G = \chi a_1 - r d_1$ generates both a_1 and d_1, and by the previous theorem, the group $\mathrm{Pic}(\mathcal{M})$ is generated by $\mathcal{L}_{\mathrm{Ver}} = (\det \mathcal{V}_p)^{\otimes \chi} \otimes (\det \pi_* \mathcal{V})^{\otimes -r}$ and $a_1 = \lambda G$, $d_1 = \kappa G$.

This can be summarised in the following lemma:

Lemma 27: Denote $\chi = d + (1-g)r$ and consider some pair λ and κ where $\kappa r + \lambda \chi = 1$. The bundle $\mathcal{V} := \mathcal{U} \otimes ((\det \mathcal{U}_p)^{\otimes \kappa} \otimes (\det \pi_* \mathcal{U})^{\otimes \lambda})^*$ is a well-defined universal bundle that depends only on κ and λ. By introducing the line bundle

$$\mathcal{L}_{\mathrm{Ver}} := (\det \mathcal{V}_p)^{\otimes \chi} \otimes (\det \pi_* \mathcal{V})^{\otimes -r}$$

we have

$$\det \mathcal{V}_p = \mathcal{L}_{\mathrm{Ver}}^{\otimes \lambda} \qquad \det \pi_* \mathcal{V} = \mathcal{L}_{\mathrm{Ver}}^{\otimes -\kappa}.$$

Lemma 28: Suppose that $d \equiv 1 \pmod{r}$. Then there is a pair λ, κ such that

$$\det \mathcal{V}_p = \mathcal{L}_{\mathrm{Ver}} \quad \text{and} \quad \det \pi_* \mathcal{V} = \mathcal{L}_{\mathrm{Ver}}^{\otimes \frac{\chi - 1}{r}}.$$

Proof: Let us write d in the form $r\delta + 1$. Then we may choose $\lambda = 1$ and $\kappa = g - 1 - \delta$, giving the above equations. □

Atiyah (**[2]**) defines the **normalised universal bundle** this way for the case $r = 2$ and $d = 4g - 3$. I will refer to this construction as the normalised universal bundle in general, however we will only use it when $r = 2$ and $(r, d) = 1$, when the conditions of **Lemma 28** are satisfied.

3. Atiyah-Bott localisation

3.1. Characteristic classes

A **characteristic class** is a natural map from the set of coherent sheaves over a smooth space X to the set of its cohomologies. A characteristic class C is **additive** (or **multiplicative**) whenever for exact sequences $0 \to G \to E \to F \to 0$, $C(E) = C(G) + C(F)$ (or respectively, $C(E) = C(G) \cdot C(F)$).

The *total Chern class* is a multiplicative characteristic class, denoted by $c(V)$. Its coefficients in the cohomology ring $H^*(X)$ are the **Chern classes**: $c(V) = 1 + c_1(V) + \cdots + c_{\operatorname{rk} V}(V)$, where $1 \in H^0(X)$ and $c_i(V) \in H^{2i}(X)$. The first Chern class $c_1(V)$ is identical to the Euler class $e(\det V)$, as defined previously.

The *splitting principle* states that to prove an identity for the Chern classes of vector bundles, it is enough to prove it under the assumption that the vector bundle is a direct sum of line bundles. Hence we may introduce formal variables $\lambda_1, \ldots, \lambda_{\operatorname{rk} V}$ for a locally free sheaf V, to represent the Euler classes of the line bundles in a decomposition. These are the **Chern roots** of the locally free sheaf V. Then the Chern classes of V become the *symmetric polynomials* of these variables:

$$c_k(V) = \sum_{1 \leq i_1 < \ldots < i_k \leq \operatorname{rk} V} \lambda_{i_1} \cdot \ldots \cdot \lambda_{i_k}.$$

Since the algebra of symmetric polynomials is generated by these polynomials, any symmetric polynomial in the Chern roots gives a well defined class. In particular, if such a polynomial decomposes as the sum or product of a single variable polynomial evaluated in the Chern roots, the corresponding Chern class is additive or multiplicative, respectively.

Theorem 29: Any formal series $f \in \mathbb{C}[[x]]$ defines an additive (also a multiplicative) characteristic class that is defined for a line bundle L as $f(c_1(L))$.

We will review a few important characteristic classes that we will need. I will give their definitions for line bundles and whether they are additive or multiplicative.

The Euler class for line bundles is extended to other vector bundles multiplicatively.

The **Todd class** of a line bundle is $\operatorname{td}(L) = \frac{c_1(L)}{1 - e^{-c_1(L)}}$, and is multiplicative. ([**6**], p279)

The **Chern character** is $\operatorname{ch}(L) = e^{c_1(L)}$ for line bundles, and it is additive. ([**1**]). A neat feature of the Chern character is that $\operatorname{ch}(E \otimes F) = \operatorname{ch}(E) \cdot \operatorname{ch}(F)$.

For convenience, I introduce another class $\operatorname{ab}(L) = 1 - e^{-c_1(L)}$ (formally given by $\frac{c_1(L)}{\operatorname{td}(L)}$), which I will call the **AB-class**, used in the Atiyah-Bott localisation formula.

3.2. K-theory and direct image

Let us review the parts of K-theory relevant for our study. The **Grothendieck group** $K(X)$ is defined as a group generated by coherent sheaves, and for a short exact sequence $A \to B \to C$, we identify $[A] - [B] + [C]$ with zero. ([**9**], p148) Additive and multiplicative characteristic classes can be extended as functions over $K(X)$.

We will also need the **direct image** of sheaves. For a *proper* function $f\colon X \to Y$, when taking the topological pushforward $f_*(E)$, the stalk of $f_*(E)$ for $E \to X$ over a point y is going to correspond to the sections of E over $f^{-1}(y)$. Taking the derived functor $R^i f_*$ of this pushforward, we arrive at the direct image $f_!(E)$, from which

$$f_!(E) = \sum_{i=0}^{\dim X} (-1)^i \left[R^i f_*(E)\right].$$

([**9**], p436)

3.3. Grothendieck-Riemann-Roch theorem

The Grothendieck-Riemann-Roch theorem is stated in its full form in ([**9**], p436). Here, we will only need its following special case:

Theorem 30: Let $f\colon X \to Y$ be a bundle with smooth projective fibers over the smooth projective variety Y. Then for any locally free sheaf E,

$$\operatorname{ch}(f_!(E)) = f_*(\operatorname{ch}(E)\operatorname{td}(T_V X))$$

where $T_V X = \operatorname{Ker} Tf$ is the vertical tangent bundle and f_* is the *cohomological pushforward* $f_*\colon H^*(X) \to H^{*-\dim F}(Y)$.

In particular when $X = C$ is the curve and Y is a point, we obtain the classical Riemann-Roch theorem.

3.4. Cohomologies of symmetric products

For our main calculation, we will need certain properties of the cohomologies of symmetric products of the curve.

The **symmetric product** of the curve C, $\operatorname{Sym}^n C$ is the quotient of the space $C^{\times n}$ by the symmetric group S_n. The points of the symmetric product correspond to degree n effective divisors over the curve, each giving a line bundle with a section over C: $0 \to \mathcal{O}_C \to L$, or in its dual form, $0 \to L^* \to \mathcal{O}_C$. Therefore the symmetric product $\operatorname{Sym}^n C$ is isomorphic to the Quot scheme of rank 1, degree $-n$ subschemes of the rank 1 trivial scheme $\mathcal{O}_C$. It is a smooth space.

Let us denote the generators of $H^1(C)$ by δ_i and δ^i $(0 \leq i \leq g)$ and that of $H^2(C)$ by ω in a way that $\delta_i\delta^j = \begin{cases} \omega & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$ for each i and j. To calculate the cohomologies of $\operatorname{Sym}^n C$, note that there is a natural map $H^i(C^{\times n}) \cong H^i(C)^{\oplus n} \to H^i(\operatorname{Sym}^n C)$. Then for each factor C_k in the product $C^{\times n}$, we have the corresponding classes $\delta_{i,k}$, δ^i_k and ω_k that together generate $H^*(C)^{\oplus n}$. Since $\operatorname{Sym}^n C$ is a quotient by the action of S_n, we need to consider the S_n invariant cohomology classes, $H^i(C^{\times n})^{S_n}$, in particular

$$\xi_i = \sum_{k=1}^{n} \delta_{i,k} \qquad \xi^i = \sum_{k=1}^{n} \delta^i_k \qquad \eta = \sum_{k=1}^{n} \omega_k.$$

Theorem 31: The classes ξ_i, ξ^i and η generate $H^*(C^{\times n})^{S_n}$, and the map $H^*(C^{\times n})^{S_n} \to H^*(\operatorname{Sym}^n C)$ is an isomorphism. **[13]**

Let us also introduce

$$\theta = \sum_{i=1}^{g} \xi_i \xi^i.$$

Since $\operatorname{Sym}^n C$ is a Quot scheme, there is a universal sequence

$$0 \to \mathcal{L}^* \to \mathcal{O} \to \mathcal{F} \to 0$$

over $\operatorname{Sym}^n C \times C$, such that for any element $d \in \operatorname{Sym}^n C$, the dual of the above map restricted to this point, $\mathcal{O} \to \mathcal{L}_p$, has divisor d. $\mathcal{L}$ is the **universal line bundle**. Its first Chern class of degree 2 is found in $H^2(\operatorname{Sym}^n C \times C) \cong \big(H^2(\operatorname{Sym}^n C) \otimes H^0(C)\big) \oplus \big(H^1(\operatorname{Sym}^n C) \otimes H^1(C)\big) \oplus \big(H^0(\operatorname{Sym}^n C) \otimes H^2(C)\big)$, and so it can be written via the basis $\{1, \delta_i, \dots, \delta^i, \dots, \omega\}$ of $H^*(C)$. By a simple calculation, it takes the following form:

Proposition 32:

$$c_1(\mathcal{L}) = \eta \otimes 1 + \sum_i (\xi_i \otimes \delta^i + \xi^i \otimes \delta_i) + d \otimes \omega$$

Knowing that $\big(\sum(\xi_i \otimes \delta^i + \xi^i \otimes \delta_i)\big)^2 = -2\theta \otimes \omega$, since degree one elements of the tensor product anti-commute, we may calculate the Chern character via Grothendieck-Riemann-Roch:

$$\operatorname{ch}(\pi_*\mathcal{L}) = e^{\eta}(d - (g-1) - \theta). \tag{4}$$

We will need this formula in our main calculation. Note that $\operatorname{ch}(\pi_*\mathcal{L}^*) = e^{-\eta}(-d - (g-1) - \theta)$.

Consider the product $\mathrm{Sym}^{n_i} C \times \mathrm{Sym}^{n_j} C$ with $\mathcal{L}_i$ and $\mathcal{L}_j$ the two universal line bundles over $\mathrm{Sym}^{n_i} C$ and $\mathrm{Sym}^{n_j} C$. The line bundle $\mathcal{L}_i \otimes \mathcal{L}_j^*$ is defined over $\mathrm{Sym}^{n_i} C \times \mathrm{Sym}^{n_j} C \times C$. Following the steps of (**[16]**), we will be needing the characteristic classes of $\pi_*(\mathcal{L}_i \otimes \mathcal{L}_j^*)$ for $\pi\colon \mathrm{Sym}^{n_i} C \times \mathrm{Sym}^{n_j} C \times C \to C$. Let us introduce indices $\xi_{k,i}$, ξ_i^k, $\eta_i \in H^*(\mathrm{Sym}^{n_i} C)$ and $\xi_{k,j}$, ξ_j^k, $\eta_j \in H^*(\mathrm{Sym}^{n_j} C)$ and the class

$$\theta_{ij} = \sum_{k=1}^{g} (\xi_{k,i}\xi_j^k + \xi_{k,j}\xi_i^k)$$

note that this formula gives $\theta_{ii} = 2\theta_i$ for $i = j$. Then we can calculate

$$\mathrm{ch}(\pi_*(\mathcal{L}_i \otimes \mathcal{L}_j^*)) = e^{(\eta_i - \eta_j)}(d_i - d_j - (g-1) - (\theta_i + \theta_j - \theta_{ij})). \tag{5}$$

We will also need this formula in our main calculation.

3.5. The antidiagonal

The elements $(p, q) \in \mathrm{Sym}^{n_1} C \times \mathrm{Sym}^{n_2} C$ correspond to certain rank 2 vector bundles, which may be written as $\mathcal{L}_{1,p} \oplus \mathcal{L}_{2,q}$ using the universal line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$. For our main calculation, we will need the Poincaré dual of the set of those elements whose determinant $\det(\mathcal{L}_{1,p} \oplus \mathcal{L}_{2,q}) = \mathcal{L}_{1,p} \otimes \mathcal{L}_{2,q}$ is a fixed line bundle. The actual class will not depend on this bundle, so we may choose $\mathcal{O}$.

The Abel-Jacobi map sends $\mathrm{Sym}^{n_1} C \times \mathrm{Sym}^{n_2} C$ to $J \times J$, where $J = J(C)$ is the Jacobian of C, a torus of $2g$ real dimensions. The image in $J \times J$ of those points $(p, q) \in \mathrm{Sym}^{n_1} C \times \mathrm{Sym}^{n_2} C$ where $\mathcal{L}_{1,p} \otimes \mathcal{L}_{2,q}$ is the trivial line bundle correspond to the inverse image of $\{0\} \in J$ for the addition in $J \times J$. This is the set $\{(a, -a) \mid a \in J\} \subset J \times J$, which is the antidiagonal $\overline{\Delta}$ of $J \times J$.

Let us denote the generators of $H^1(J)$ by $x_1, \ldots, x_{2g}$. The Poincaré dual of the diagonal is in the group $H^*(J \times J) \cong H^*(J) \otimes H^*(J)$, given by

$$\Delta^* = \sum_{S \subset \{1 \ldots 2g\}} (-1)^{|\overline{S}| + \sigma_S} \cdot x_{S_1} \ldots x_{S_{|S|}} \otimes x_{\overline{S}_1} \ldots x_{\overline{S}_{|\overline{S}|}}$$

where $S = \{S_1, \ldots, S_{|S|}\}$ and the terms are ordered, $\overline{S} = \{\overline{S}_1, \ldots, \overline{S}_{|\overline{S}|}\}$ is the complement of S in $\{1 \ldots 2g\}$, while

$$\sigma_S = \#\{\text{permutations in } (1, 2, 3, \ldots) \to (S_1, S_2, \ldots, \overline{S}_1, \overline{S}_2, \ldots)\}$$

The anti-diagonal is obtained by flipping the signs of all the 1-classes inside the second J:

$$\overline{\Delta}^* = \sum_{S \subset \{1 \ldots 2g\}} (-1)^{\sigma_S} \cdot x_{S_1} \ldots x_{S_{|S|}} \otimes x_{\overline{S}_1} \ldots x_{\overline{S}_{|\overline{S}|}}$$

Note that using the notations of the previous section, $\delta_i \in H^1(C)$ is the preimage of $x_i \in J(C)$ under the Abel-Jacobi map, while $\delta^i \in H^1(C)$ is that of $x_{g+i} \in J(C)$. Hence the θ_1, θ_2 and θ_{12} defined in the previous section are the pullbacks via the Abel-Jacobi map of the following classes in the cohomology group $H^*(J) \otimes H^*(J)$:

$$\theta_1 = \sum_{i=1}^{g} x_i x_{g+i} \otimes 1 \qquad \theta_2 = \sum_{i=1}^{g} 1 \otimes x_i x_{g+i}$$

$$\theta_{12} = \sum_{i=1}^{g} (x_i \otimes x_{g+i} + x_{g+i} \otimes x_i)$$

With this identification, the following statement, announced only for $\operatorname{Sym}^{n_1} C \times \operatorname{Sym}^{n_2} C$, is true in $J \times J$ as well.

Proposition 33: Denoting by $\overline{\Delta}^*$ the Poincaré dual of the anti-diagonal in $\operatorname{Sym}^{n_1} C \times \operatorname{Sym}^{n_2} C$, it can be expressed as

$$\overline{\Delta}^* = \frac{1}{g!}(\theta_1 + \theta_2 + \theta_{12})^g$$

3.6. Representation theory

Let us consider a torus T with Lie algebra $\mathfrak{t}$ and exponential map $\exp\colon \mathfrak{t} \to T$. Let us fix an isomorphism $T \cong (\mathbb{C}^*)^k = \{(t_1, \ldots, t_k) \mid t_j \in \mathbb{C}^*\}$ that also identifies the Lie algebra $\mathfrak{t} \cong \mathbb{C}^k = \{(\tau_1, \ldots, \tau_k) \mid \tau_j \in \mathbb{C}\}$ and $\exp(\tau_1, \ldots, \tau_k) = (e^{2\pi i \tau_1}, \ldots, e^{2\pi i \tau_k})$. Consider T acting on a vector space V. Since T is commutative, its irreducible representations are one dimensional, and so V decomposes into one dimensional subspaces that are preserved under the action.

Let us consider the lattice $\mathfrak{t}^*_{\mathbb{Z}} \leq \mathfrak{t}^*$, identified with $\mathbb{Z}^k = \{(w_1, \ldots, w_k) \mid w_j \in \mathbb{Z}\}$. Then any element $w \in \mathfrak{t}^*_{\mathbb{Z}}$ gives a one dimensional representation on some one dimensional vector space L, given by $t \star v = t^w \cdot v = \left(\prod_{i=1}^k t_k^{w_k}\right) \cdot v$ for any $v \in L$ and $t \in T$. In fact, the one dimensional representations are in a bijection with elements of $\mathfrak{t}^*_{\mathbb{Z}}$, independently of the identification $T \cong (\mathbb{C}^*)^k$.

For a representation $\rho\colon G \to \operatorname{End}(V)$, let us denote the *invariant subspace* of V by V^G. Let us define the **trace of the action** ρ on V as $\operatorname{tr} \rho_V := \operatorname{tr} \circ \rho\colon G \to \mathbb{C}$.

Theorem 34: (Weyl character formula) Let G be a group acting on V and T a maximal torus of G, with $\mathfrak{t}$ the Lie algebra of T. Let $\mathfrak{R}^+$, $\mathfrak{R}^-$ be the positive and negative roots in $\mathfrak{t}^*$, respectively. Then $\dim V^G$ is the constant coefficient of the Laurent polynomial given by

$$\prod_{\alpha \in \mathfrak{R}^-} (1 - t^\alpha) \cdot \operatorname{tr}_V t.$$

where $\operatorname{tr}_V t$ is the trace of the T action. The coefficients of this Laurent polynomial with monomials t^λ having λ in the *dominant chamber* of the G action correspond to the multiplicities of the *irreducible representations* of G, parametrised by their *highest weight*, λ.

For $\mathrm{PGL}(n)$, choosing the maximal torus corresponds to diagonal matrices in a certain basis. We may assume that the diagonal elements are t_1 to t_n, defined up to a common scalar multiple. Then the Laurent polynomial $\prod_{\alpha \in \mathfrak{R}^-}(1 - t^\alpha)$ gives for the $n = 2$ case

$$1 - t_2/t_1 \tag{6}$$

and for the $n = 3$ case

$$(1 - t_2/t_1)(1 - t_3/t_1)(1 - t_3/t_2). \tag{7}$$

3.7. A few lemmata about the Grassmannian

A group acts on the trivial line bundle by a character $\chi\colon G \to \mathbb{C}^*$ when the image of $x \in \mathcal{O}_p$ by g is $\chi(g)x \in \mathcal{O}_{g\star p}$. When $G = \mathbb{C}^*$, $\chi(z)$ is of the form z^k, and it is called an action of weight $k \in \mathbb{Z}$.

Lemma 35: Take a vector space H with the scalar action by $\mathbb{C}^*$, whose quotient is $\mathbb{P}H$. Then acting on the trivial bundle $\mathcal{O}_H$ over H with weight 1 induces the polarisation $\mathcal{O}(1)_{\mathbb{P}H}$ over $\mathbb{P}H$.

Let us consider $H = \mathrm{Hom}(\mathbb{C}^n, V)$ for some vector space V. Under the actions $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$ that send $\varphi \in H$ to $\varphi \circ g$, and with respect to the polarisation $\mathcal{O}_H^{\det}$ with action $v \to \det g \cdot v$, the stable points contained in $H^s \subset H$, correspond to injective maps in $\mathrm{Hom}(\mathbb{C}^n, V)$.

When acted on by $\mathrm{SL}(n)$, the quotient of H^s is a subset $B \subset \bigwedge^n V$, while acting by $\mathrm{GL}(n)$ gives $Gr := \mathrm{Gr}(n, V)$. The projection $B \to \mathrm{Gr}(n, V)$ is a fibration whose fibers are $\mathbb{C} \setminus 0$, so it can be extended into a vector bundle $\overline{B}$ of rank 1.

Lemma 36: The vector bundle $\overline{B}$ over $Gr = \mathrm{Gr}(n, V)$ is isomorphic to $\mathcal{O}_{Gr}(-1)$, the determinant of the tautological vector bundle.

Lemma 37: The polarisation $\mathcal{O}_H^{\det}$, with $\mathrm{GL}(n)$ acting by the character det, induces the polarisation $\overline{B}^* \cong \mathcal{O}_{Gr}(1)$ over Gr.

We can collect these two lemmata in this illustration:

$$\begin{tikzcd}
\mathcal{O}(1)_{\mathbb{P}H} & \mathcal{O}_H \arrow[l, "/\mathbb{C}^*"'] & = & \mathcal{O}_H^{\det} \arrow[r, "/\,\mathrm{GL}(n)"] & \mathcal{O}(1)_{\mathcal{G}r}
\end{tikzcd}$$

The action by $\mathbb{C}^*$ can be identified by the action by diagonal matrices in GL. However, the determinant of this diagonal action is of weight n, so we need to change the diagram accordingly.

Corollary 38: The GIT quotient of $\mathbb{P}H := \mathbb{P}\operatorname{Hom}(\mathbb{C}^n, V)$ by the natural $\mathrm{PGL}(n)$ action and with polarisation $\mathcal{O}(n)_{\mathbb{P}H}$ is $Gr = \mathrm{Gr}(n, V)$ with the induced polarisation $\mathcal{O}(1)_{Gr}$.

This can be summed up in the following diagram, where arrows denote induced polarisations over quotients via the actions written on the arrows:

$$\begin{tikzcd}
\mathcal{O}_H^{\det} \arrow[r, "/\mathbb{C}^*"] \arrow[d, equal] & \mathcal{O}(n)_{\mathbb{P}H} \arrow[d, "/\,\mathrm{PGL}(n)"] \\
\mathcal{O}_H \arrow[r, "/\,\mathrm{GL}(n)"] & \mathcal{O}(1)_{\mathcal{G}r}
\end{tikzcd}$$

3.8. Atiyah-Bott localisation formula

We will denote by $\mathbb{C}[T]$ the ring of Laurent polynomial functions on the torus $T = \{(t_1, \dots, t_k) \mid t_i \in \mathbb{C}^*\}$, i.e. $\mathbb{C}[T] = \mathbb{C}[t_1, t_1^{-1}, \dots, t_k, t_k^{-1}]$.

The **equivariant Euler characteristic** of a coherent sheaf $\mathcal{F}$ that is acted on by T is defined as the Laurent polynomial

$$\chi_T(\mathcal{F}, t) = \sum_i (-1)^i \operatorname{tr} \rho_{H^i(\mathcal{F})}(t)$$

or $\chi_{\mathcal{F}}(t)$ for short. This generalises the non-equivariant case since $\operatorname{tr} \rho_V = \dim V$ for the trivial action.

Given a connected, smooth variety A and a torus T acting on a vector bundle V over A, if T acts on A trivially, then the fibers are preserved, and the weights on each fiber are the same. The *T-equivariant Chern class* is an additive class that is $\mathrm{ch}_T(L, t) = t^w \operatorname{ch}(L) \in \mathbb{C}[T] \otimes H^*(M)$ for line bundles L over A with a fiber preserving T-action, where w is the weight of the representation of T on the fibers. The *T-equivariant AB-class* is a multiplicative class that is $\mathrm{ab}_T(L, t) = 1 - \mathrm{ch}_T(L^*)$ for line bundles.

For a smooth variety A and a smooth subvariety X, the *normal bundle* of X in A, denoted by $N_A X$, is given by the short exact sequence $TX \to TA \to N_A X$.

Theorem 39: (Atiyah-Bott localisation, [3]) Given a torus action on A by the torus T and a line bundle L over A, the equivariant Euler characteristic can be calculated as follows: let us take $\mathrm{Fix}(A)$ the set of components of the fixpoint set A^T. For one such component $X \in \operatorname{Fix}(A)$, let $N_A X$ be the normal bundle of X within A. Then, by the Atiyah-Bott localisation

$$\chi_T(L,t) = \sum_{X \in \operatorname{Fix}(A)} \int_X \frac{\operatorname{ch}_T(L)\operatorname{td}(TX)}{\operatorname{ab}_T(N_A X)}.$$

Now let us suppose that A is a **complete intersection**, which means the following. A is embedded into some space Z in an equivariant manner, and there is a vector bundle B on Z with linearisation of the T action and an non-degenerate equivariant section $\sigma\colon Z \to B$ such that $A = \{p \in Z \mid \sigma(p) = 0\}$. Then there is a short exact sequence of tangent bundles $TA \to TZ \to B$. Then we can replace $\operatorname{ab}_T(N_A X)$ by $\frac{\operatorname{ab}_T(TZ)}{\operatorname{ab}_T(TX)\operatorname{ab}_T(B)}$, and so

$$\chi_T(L,t) = \sum_{X \in \operatorname{Fix}(A)} \int_X \operatorname{ch}_T(L)\operatorname{td}(TX)\frac{\operatorname{ab}_T(TX)\operatorname{ab}_T(B)}{\operatorname{ab}_T(TZ)}$$

This is true even when the space A is not smooth, and when the sections exist only locally, meaning that for a point p there is an open set $p \in U \subseteq Z$ and an equivariant section $\sigma\colon U \to B|_U$ such that $A \cap U$ is its zero set. **[7]**

Since the equivariant classes are defined for arbitrary elements of $K(A)$, we can replace TA with the formal difference $[TZ] - [B]$.

We will apply this formula in an even more general framework. Let us consider a space A endowed with a *perfect obstruction complex* $\mathcal{A}_0 \to \mathcal{A}_1$ (see **[12]** for details), generalising the previous sequence $TB \to Z$. This motivates the introduction of a virtual equivariant Euler characteristic that is equal to the equivariant Euler characteristic in the previous cases.

Definition 40: Let us consider a space A endowed with a perfect obstruction complex $\mathcal{A}_0 \to \mathcal{A}_1$, and a torus action on A by the torus T and a line bundle L over A. Let us denote by $\operatorname{Fix}(A)$ the set of components of the fixed point set of A under the action by T. Then the **virtual equivariant Euler characteristic** is

$$\chi_T^{\mathrm{virt}}(L,t) = \sum_{X \in \operatorname{Fix}(A)} \int_X \operatorname{ch}_T(L)\operatorname{td}(TX)\frac{\operatorname{ab}_T(TX)\operatorname{ab}_T(\mathcal{A}_1)}{\operatorname{ab}_T(\mathcal{A}_0)}.$$

For such a space, the **expected dimension** of the tangent space is $\dim \mathcal{A}_0 - \dim \mathcal{A}_1$. When the expected dimension is equal to the actual dimension of the space, the virtual equivariant Euler characteristic is equal to the equivariant Euler characteristic. For example, as seen in **[16]**, the Quot scheme of sequences $E \to \mathcal{O} \to F$ admits a perfect obstruction complex, and its expected dimension is $\chi(E^* \otimes F)$.

4. Moduli spaces and polarisations

4.1. Introduction

Verlinde's formulæ express the number of sections of the powers of an ample line bundle over the moduli space of vector bundles over a curve. In this thesis, we present a novel approach to calculate this number using localisation.

This chapter describes the setup used here, which is similar to that found in **[15]** and **[4]**. We introduce two birationally equivalent spaces, a Quot scheme $\mathcal{Q}$ and a projective bundle $\mathcal{P}$ over the moduli space of vector bundles. For a certain choice of parameters, certain line bundles over the two spaces can be identified in a canonical manner. There is a compatible group action on both of these spaces and line bundles. The GIT quotient of $\mathcal{P}$ is a bundle $\mathcal{G}r$ over the moduli space with Grassmannian fibers, and the ample line bundles over $\mathcal{P}$ induce line bundles over $\mathcal{G}r$ as well. The number of sections, or rather the Euler characteristic, of an ample line bundle $\mathcal{L}$ over the moduli space or $\mathcal{G}r$ can be obtained by taking the invariant part of the equivariant Euler characteristic of a polarisation $\mathcal{L}_{\mathcal{P}}$ over $\mathcal{P}$ that induces $\mathcal{L}$. Our conjecture is that by taking another line bundle $\mathcal{L}_{\mathcal{Q}}$ over $\mathcal{Q}$, we may instead count the invariant part of the equivariant Euler characteristic of $\mathcal{L}_{\mathcal{Q}}$, which then may be done through localisation.

4.2. Moduli space and fixed determinant

Let us fix the curve C. Let us denote the moduli space of rank r and degree d stable vector bundles E by $\mathcal{M}_{r,d}$ or $\mathcal{M}$ when the parameters are clear from the context. Note that $\mathcal{M}_{r,d} \cong \mathcal{M}_{r,d+kr}$, as any vector bundle may be multiplied by a fixed line bundle of degree k. Hence we may assume $0 \leq d < r$.

When $r = 1$, the moduli space is isomorphic to the Jacobian $\mathcal{J}_d$ of degree d line bundles. These Jacobians are isomorphic for all values of d.

When the determinant bundle $\det E = \bigwedge^r E = \Delta$ is fixed, there is a moduli space of stable vector bundles of rank r, degree d and determinant Δ, denoted by $\mathcal{M}_\Delta$. For any two Δ and Δ', the moduli spaces $\mathcal{M}_\Delta$ and $\mathcal{M}_{\Delta'}$, since there is a Γ such that $\Gamma^{\otimes r} = \Delta' \otimes \Delta^*$, and $\det(E \otimes \Gamma) = \Delta \otimes \Gamma^{\otimes r} = \Delta'$.

The determinant map $\det\colon \mathcal{M} \to \mathcal{J}$ sends a bundle to its determinant line bundle. The fiber of this morphism is isomorphic to $\mathcal{M}_{\Delta,r,d}$ or $\mathcal{M}_\Delta$ when the parameters are clear from the context.

We will consider the $(r, d) = 1$ case, and fix $r = 2$ and d for most of this work.

We will denote the map $\pi\colon \bullet \times C \to \bullet$ for some variety $\bullet$. For the same variety, $\star_p\colon \bullet \times \{p\} \to \bullet \times C$ is the embedding of a point $p \in C$.

4.3. Verlinde's formulæ

Since the Picard group of $\mathcal{M}_\Delta$ is of rank 1, we may choose an ample line bundle $\mathcal{L}_{\mathrm{Ver}}$ that generates it. For $r = 2$, $d = 0$, Verlinde's formulæ ([**25**]) are as follows:

$$\dim H^0(\mathcal{L}_{\mathrm{Ver}}^{\otimes k}, \mathcal{M}_{\Delta,r=2,d=0}) = \left(\frac{k+2}{2}\right)^{g-1} \sum_{j=1}^{k+1} \sin^{2-2g} \frac{\pi j}{k+2},$$

$$\dim H^0(\mathcal{L}_{\mathrm{Ver}}^{\otimes k}, \mathcal{M}_{\Delta,r=2,d=1}) = \left(\frac{2k+2}{2}\right)^{g-1} \sum_{j=1}^{2k+1} (-1)^{j-1} \sin^{2-2g} \frac{\pi j}{2k+2}.$$

These formulæ have a residue form ([**22**]) as well, which is closer to what we expect to have after localisation:

$$\dim H^0(\mathcal{L}_{\mathrm{Ver}}^{\otimes k}, \mathcal{M}_{\Delta,r=2,d=0}) = -(2(k+2))^{g-1} \operatorname*{Res}_{x=0} \frac{(k+2)\cot((k+2)x)dx}{(2\sin x)^{2(g-1)}},$$

$$\dim H^0(\mathcal{L}_{\mathrm{Ver}}^{\otimes k}, \mathcal{M}_{\Delta,r=2,d=1}) = -(2(2k+2))^{g-1} \operatorname*{Res}_{x=0} \frac{(2k+2)\csc((2k+2)x)dx}{(2\sin x)^{2(g-1)}}.$$

Since the Moduli space is a *Fano variety*, there are no higher cohomologies, and $H^0(\mathcal{M}_\Delta) = \chi(\mathcal{M}_\Delta)$.

4.4. The universal bundle

Recall from **Theorem 26** that if $(r,d) = 1$, there is a universal bundle $\mathcal{U}$ over $\mathcal{M} \times C$, such that for any point $E \in \mathcal{M}$, the pullback via the injection map $f_E\colon \{E\} \times C \to \mathcal{M} \times C$, $f_E^*(\mathcal{U}) \to C$ is isomorphic to E. This bundle can be restricted to the fixed determinant moduli space, $\mathcal{M}_\Delta$. Consider the following theorem, a special case of a more general theorem found in [**2**].

Theorem 41: $a_1 = c_1(\det \mathcal{U}_p)$ and $d_1 = c_1(\det \pi_* \mathcal{U})$ generate $H^2(\mathcal{M}_\Delta, \mathbb{Z})$.

The bundle $\mathcal{U}$ is not uniquely defined, but only up to tensoring with a line bundle $\mathcal{L}$ over $\mathcal{M}_\Delta$. That is, $\mathcal{L} \otimes \mathcal{U}$ is just as well a universal bundle as $\mathcal{U}$ is, and any other potential universal bundle $\mathcal{U}'$ can be expressed as $\mathcal{U} \otimes \mathcal{L}$ for some $\mathcal{L}$. However, as seen in **Lemma 27**, by fixing a pair of numerical parameters, there is a canonical way to choose a normalisation of the universal bundle $\mathcal{V}$, such that $\det \mathcal{V}_p$ and $\det \pi_* \mathcal{V}$ are tensor powers of a certain line bundle, denoted there by $\mathcal{L}_{\mathrm{Ver}}$.

Since a_1 and d_1 are integer multiples of $c_1(\mathcal{L}_{\mathrm{Ver}})$, the bundle $\mathcal{L}_{\mathrm{Ver}}$ generates $\mathrm{Pic}(\mathcal{M}_\Delta)$. A corollary of **Lemma 27** and **Lemma 28** for the fixed determinant case is the following:

Lemma 42: Denote $\chi = d + (1-g)r$ and consider some pair λ and κ where $\kappa r + \lambda\chi = 1$. The bundle $\mathcal{V} := \mathcal{U} \otimes ((\det \mathcal{U}_p)^{\otimes\kappa} \otimes (\det \pi_*\mathcal{U})^{\otimes\lambda})^*$ is a well-defined universal bundle that depends only on κ and λ. By introducing the line bundle

$$\mathcal{L}_{\mathrm{Ver}} = (\det \mathcal{V}_p)^{\otimes\chi} \otimes (\det \pi_*\mathcal{V})^{\otimes -r}$$

we have

$$\det \mathcal{V}_p = \mathcal{L}_{\mathrm{Ver}}^{\otimes\lambda} \qquad \det \pi_*\mathcal{V} = \mathcal{L}_{\mathrm{Ver}}^{\otimes-\kappa}.$$

Furthermore, if $d \equiv 1 \pmod r$, then λ, κ can be chosen in a way that

$$\det \mathcal{V}_p = \mathcal{L}_{\mathrm{Ver}} \quad \text{and} \quad \det \pi_*\mathcal{V} = \mathcal{L}_{\mathrm{Ver}}^{\otimes\frac{\chi-1}{r}}.$$

Since we will only consider the case $r = 2$, $(r,d) = 1$, we can apply the last statement to all our calculations.

Lemma 43: When $d \geq 4g-3$, a stable bundle V of degree d and rank 2 has $h^1(V) = 0$.

Proof: By Serre duality we have $h^1(V) = h^0(V^* \otimes \mathcal{K})$ where $\mathcal{K}$ is the canonical bundle of degree $2g-2$. The bundle $V^* \otimes \mathcal{K}$ is stable, so if its degree is negative, it does not have non-zero sections. Hence it is enough to choose $d > 4g-4$. □

From now on, we will only consider $d \geq 4g-3$, and in this case $\pi_*\mathcal{U}$ is a vector bundle for the projection map $\pi\colon \mathcal{M} \times C \to \mathcal{M}$.

4.5. Spaces of pairs

Let us consider the set Sect^n of isomorphism classes of pairs (V, f) that consist of a vector bundle V over the curve and n non-zero sections of V, represented by $f\colon \mathcal{O}^n \to V$. We will consider two particular subsets that can be endowed with a projective scheme structure.

Recall from **Theorem 9** that the Quot scheme consists of exact sequences of sheaf homomorphisms

$$E^r \to \mathcal{O}^n \to F^{n-r}$$

where $r \geq n$. Over curves, subsheaves of locally free sheaves are locally free, hence E is also a vector bundle and $E \to \mathcal{O}^n$ is a generically injective vector bundle map. So by denoting $V := E^*$, we may consider the Quot scheme as a subset of Sect^n:

Lemma 44: The Quot scheme $\widetilde{\mathcal{Q}} := \mathrm{Quot}^{n-r,d}(\mathcal{O}^n)$ parametrises maps $\mathcal{O}^n \to E^* = V$ that are generically surjective.

The expected dimension of $\widetilde{\mathcal{Q}}$ is $\chi(E^* \otimes F) = nd + (1-g)(n-r)r$, attained when d is large relative to n, r, g ([**4**], **Theorem 4.28**) and there is a universal exact sequence over $\widetilde{\mathcal{Q}}$

$$0 \to \mathcal{E}^r \to \mathcal{O}^n \to \mathcal{F}^{n-r} \to 0. \tag{3}$$

On the other hand, we may restrict Sect^n to stable bundles V, but permit all non-zero maps $f\colon \mathcal{O}^n \to V$, or equivalently n sections $\mathbb{C}^n \to \Gamma E^* = \Gamma V$ of ΓV that are not all zero.

Lemma 45: Suppose that $d \geq 4g-3$. Then the projective space $\widetilde{\mathcal{P}} := \mathbb{P}\,\mathrm{Hom}(\mathbb{C}^n, \pi_* \mathcal{U})$ where π denotes the map $\widetilde{\mathcal{P}} \times C \to \widetilde{\mathcal{P}}$, parametrises all stable bundles with n sections that are not all zero.

It is a fibration over $\mathcal{M}$, and is of dimension $\dim \mathcal{M} + nh^0(V) - 1 = 1 - (1-g)r^2 + nd + (1-g)nr - 1$. Each fiber is a projective space, and so there is a universal sequence over the fibration:

$$S^1 \to \mathrm{Hom}(\mathbb{C}^n, \pi_* \mathcal{U}) \to Q. \tag{2}$$

We will denote the line bundle $\Pi_{\mathcal{P}}(\mathcal{U}) := S^*$, and $\mathcal{O}_{\mathcal{P}}(1) := \Pi_{\mathcal{P}}(\mathcal{V})$ for the normalised universal bundle $\mathcal{V}$.

The same spaces can be defined for vector bundles of fixed determinant Δ as the fibers of the fibrations $\widetilde{\mathcal{Q}} \to \mathcal{J}_d$ and $\widetilde{\mathcal{P}} \to \mathcal{J}_d$, where $\mathcal{J}_d$ is the Jacobian of degree d line bundles. Since the complex dimension of the Jacobian is g, their dimensions can be calculated easily.

Lemma 46: Let us define $\mathcal{Q} := \widetilde{\mathcal{Q}} \cap \mathrm{Sect}^n_\Delta$ and $\mathcal{P} := \widetilde{\mathcal{P}} \cap \mathrm{Sect}^n_\Delta$ (where Sect^n_Δ is the fiber of the map $\mathrm{Sect}^n \to \mathcal{J}_d$), the fibers of the map $\mathrm{Sect}^n \to \mathcal{J}_d$. The dimension of $\mathcal{P}$ and the expected dimension of $\mathcal{Q}$ is

$$nd + (1-g)(n-r)r - g.$$

The intersection of $\mathcal{Q}$ and $\mathcal{P}$ within Sect^n is open in both of them, with the same topology. This intersection will be denoted by $\mathcal{Q} \cap \mathcal{P}$.

4.6. The Grassmannian over the moduli space

The points of the space $\mathcal{P}$ correspond to pairs of some vector bundle $V \in \mathcal{M}_\Delta$, and an indexed set of n vectors in ΓV, given up to a common scalar. This set gives a subspace of ΓV with a fixed basis (up to scalar).

There is a natural $\mathrm{PGL}(n)$ action that removes the choice of the generators of the subspace of ΓV generated by the n sections. When these sections are linearly independent, each orbit gives an n dimensional subspace of ΓV, or an element of $\mathrm{Gr}(n, \Gamma V)$. This

Grassmannian is defined universally for all $V \in \mathcal{M}_\Delta$ by creating a Grassmannian bundle over $\mathcal{M}_\Delta$: $\mathcal{G}r := \mathrm{Gr}(n, \pi_*\mathcal{U})$. This space does not depend on the choice of the universal bundle, since multiplying a vector space by a scalar gives the same Grassmannian, and so tensoring $\mathcal{U}$ with a line bundle over $\mathcal{M}_\Delta$ will not change $\mathcal{G}r$.

We will construct this Grassmannian as a GIT quotient of $\mathcal{P}$ under the action $\mathrm{PGL}(n)$. Recall that by **Corollary 38**, we have a natural $\mathrm{PGL}(n)$ action on $\mathcal{O}_\mathcal{P}(n) := \mathcal{O}_\mathcal{P}(1)^{\otimes n}$. Since $\mathcal{L}_{\mathrm{Ver}}$ is ample over $\mathcal{M}_\Delta$ and $\mathcal{O}_\mathcal{P}(1)$ is ample along the fibers of $\mathcal{P} \to \mathcal{M}_\Delta$, an appropriate tensor combination of these two is ample over $\mathcal{P}$.

Lemma 47: Consider the fibration $\mathcal{P}$ over $\mathcal{M}_\Delta$. Let us denote the pullback of $\mathcal{L}_{\mathrm{Ver}}$ from $\mathcal{M}_\Delta$ to $\mathcal{P}$ by $\mathcal{L}_{\mathrm{Ver}}$ as well. Then for a large enough K, the set of stable points of $\mathcal{P}$ under the action $\mathrm{PGL}(n)$ and polarisation $\mathcal{L}_{\mathrm{Ver}}^{\otimes K} \otimes \mathcal{O}_\mathcal{P}(n)$, denoted by $\mathcal{P}^s$, is such that the closed points correspond to those maps $\mathcal{O}^n \to V$ where the n sections are linearly independent, and the GIT quotient is $\mathcal{G}r$.

Since each fiber is a Grassmannian, there is a universal sequence over the fibration $\mathcal{G}r$, where π denotes the map $\mathcal{G}r \times C \to \mathcal{G}r$.

$$S^n \to \pi_*\mathcal{U} \to Q \tag{2}$$

We will denote the line bundle $\Pi_{\mathcal{G}r}(\mathcal{U}) := \det S^*$, and $\mathcal{O}_{\mathcal{G}r}(1) := \Pi_{\mathcal{G}r}(\mathcal{V})$ for the normalised universal bundle $\mathcal{V}$.

Note: We could also define the Grassmannian over $\mathcal{M}$, denoted by $\widetilde{\mathcal{G}r}$. Most of the results of this chapter hold also when replacing $\mathcal{M}_\Delta$, $\mathcal{Q}$, $\mathcal{P}$, $\mathcal{G}r$ by $\mathcal{M}$, $\widetilde{\mathcal{Q}}$, $\widetilde{\mathcal{P}}$ and $\widetilde{\mathcal{G}r}$, with a few exceptions, among others those about their dimensions and cohomology numbers.

4.7. Birational equivalence

In this section, we will construct explicit parameters where $\mathcal{P}$ and $\mathcal{Q}$ are birationally equivalent. Furthermore, we will choose the parameters in a way such that for any pair of line bundles $\mathcal{L}_\mathcal{P}$ and $\mathcal{L}_\mathcal{Q}$ over $\mathcal{P}$ and $\mathcal{Q}$, respectively, that are isomorphic over $\mathcal{Q} \cap \mathcal{P}$, any section over $\mathcal{Q} \cap \mathcal{P}$ of one of them extends to a global section of $\mathcal{L}_\mathcal{P}$ and $\mathcal{L}_\mathcal{Q}$ as well.

We have denoted by $\mathcal{P}^s$ the stable points of $\mathcal{P}$ under the action $\mathrm{PGL}(n)$, according to the polarisation $\mathcal{L}_{\mathrm{Ver}}^{\otimes K} \otimes \mathcal{O}_\mathcal{P}(n)$. With the proper conditions, $\mathcal{P}^s$ is a subset of $\mathcal{Q} \cap \mathcal{P}$.

Lemma 48: Suppose that $r = 2$, $g \geq 2$ and $d \geq 4g-3$. Then if $\max(\chi/2, 2g-2) < n \leq \chi$, $\mathcal{P}^s$ is a subset of $\mathcal{Q} \cap \mathcal{P}$.

Proof: Clearly n must be at most $\dim \Gamma V = \chi$, otherwise the Grassmannian does not exist.

Let us recall from **Lemma 47** that the stable points of $\mathcal{P}$ are those maps $\mathcal{O}^n \to V$ where V is stable and the n sections are linearly independent. On the other hand, points in $\mathcal{Q}$ are maps $\mathcal{O}^n \to V$ that are generically surjective.

Suppose that there is a map $\mathcal{O}^n \to V$ that is a stable element of $\mathcal{P}$ but does not appear in $\mathcal{Q}$. This means that this map is not generically surjective. Therefore the n sections generate a subbundle of rank lower than 2, i.e. 1. Let us denote this line bundle by L. Since the map $\mathcal{O}^n \to V$ is an element of $\mathcal{P}$, V is stable, and so $\deg L \leq \frac{1}{2}d$. Furthermore, the n sections are linearly independent, since they correspond to a stable element of $\mathcal{P}$. These are sections of L, and so $h^0(L) \geq n$. There are three cases:

1. $\deg L \geq 2g-2$ and L is not the canonical line bundle. Then by Serre's duality, $h^1(L) = 0$, and so by Riemann-Roch, $h^0(L) = \chi(L) = \deg L + 1 - g \geq n$. So if $\frac{\chi}{2} < n$, this is a contradiction.
2. L is the canonical line bundle of degree $2g-2$. Then $h^1(L) = 1$, and $h^0(L) = g \geq n$. Since $d > 4g-4$, we have $\frac{\chi}{2} > 2g-2 \geq g$, and so if $\frac{\chi}{2} < n$, this gives a contradiction as well.
3. $\deg L < 2g-2$. In general, $h^0(L) \leq \max(0, \deg L + 1)$, so we may suppose $0 \leq \deg L$. Then $h^0(L) \leq \deg L + 1 < 2g-1$. Hence if $n > 2g-2$, we obtain a contradiction in this case as well. $\square$

Theorem 49: Suppose that $g \geq 2$, $(r,d) = 1$, $r = 2$, $d > 6g-6$, $\max(\chi/2, 2g-2) < n \leq \chi$, $\mathcal{P}^s \subseteq \mathcal{Q}$, and the following conditions are satisfied:

$$d + 2g - ng \geq 3 \tag{8}$$

$$n \leq d - 2g + 1 \tag{9}$$

$$d \geq (n+3)(g-1). \tag{10}$$

Then the dimension of the Quot scheme $\mathcal{Q}$ is equal to that of $\mathcal{P}$, and they are birationally equivalent.

Furthermore, given a pair of line bundles $\mathcal{L}_{\mathcal{P}}$ and $\mathcal{L}_{\mathcal{Q}}$ over $\mathcal{P}$ and $\mathcal{Q}$, respectively, that are isomorphic over $\mathcal{Q} \cap \mathcal{P}$, then any section over $\mathcal{Q} \cap \mathcal{P}$ extends to a section of both $\mathcal{L}_{\mathcal{P}}$ and $\mathcal{L}_{\mathcal{Q}}$, and so $H^0(\mathcal{L}_{\mathcal{P}}) \cong H^0(\mathcal{L}_{\mathcal{Q}})$

Proof: Since $\mathcal{P}^s$ is included in $\mathcal{Q}$, and it is open in Sect^n with respect to the topology of $\mathcal{Q}$, the dimension of the corresponding component is that of $\mathcal{P}^s$. We will show that the dimensions of the other components are smaller then this, thereby proving that $\mathcal{Q}$ is of the same dimension as $\mathcal{P}$. In fact, we will show that the codimensions of the other components are at least 2, thereby proving that $\mathcal{Q}$ and $\mathcal{P}$ are birationally equivalent. Also by Hartog's theorem, any section of a $\mathcal{L}_{\mathcal{P}}$ will extend to one over $\mathcal{L}_{\mathcal{Q}}$, and vice versa.

Let us define $V = E^*$ with $\operatorname{rk} V = r = 2$ and $\deg V = d$, $\chi(V) = d + 2 - 2g$ for an element $E \to \mathcal{O} \to F$. Since $\mathcal{O} \to E^*$ is surjective, V must be generated by n sections. The expected dimension of the Quot scheme is $3g - 4 + n(d + 2 - 2g)$.

We will choose a component of the Quot scheme that does not contain a stable bundle. Choose an unstable bundle V.

Lemma 50: Given an unstable bundle V of rank 2, there is a **unique** destabilising line bundle L, i.e. $L \to V \to M$ with $\mu(L) > \mu(V)$.

Since we fixed the determinant of V, $\det V = \Delta$, given a line bundle L, it determines $M = \det L^* \otimes \Delta$, and vice versa. Then the extensions of L and M are in a bijection with the elements of $H^1(M^* \otimes L)$ up to isomorphism, with 0 corresponding to the splitting extension $V = L \oplus M$.

Here are a few lemmata about the relation between cohomologies and the degree for line bundles:

Lemma 51: If $\deg L > 2g - 2$, then $h^1(L) = 0$.

Lemma 52: If $\deg L > 0$, then $h^0(L) \leq \deg L$. If $\deg L = 0$, then $h^0(L) = 1$ if and only if $L = \mathcal{O}$, otherwise $h^0(L) = 0$.

Corollary 53: By Serre duality, if $\deg L = 2g - 2$, then $h^1(L) = 1$ if and only if $L = \mathcal{K}$, otherwise $h^1(L) = 0$.

Lemma 54: If $\deg L > 2g - 2$, then $H^0(V) \cong H^0(L) \oplus H^0(M)$.

Proof: If the condition holds, then by **Lemma 51**, $H^1(L) = 0$. The long exact sequence of cohomologies then becomes a short exact sequence $H^0(L) \to H^0(V) \to H^0(M)$. $\square$

We will take each component of the Quot scheme, and calculate the codimension of this component. For this, we will first need the dimension. Let us denote by $\mathcal{S}_{\deg M}$ the moduli space of degree $\deg M$ line bundles that admit a non-zero section.

Lemma 55: Consider a subvariety $\mathcal{S} \subseteq \mathcal{S}_{\deg M}$ of certain line bundles, and fix an extension $L \to V \to M$ with $L \in \mathcal{S}$. The dimension of the corresponding component in the Quot scheme is then $\dim \mathcal{S} + h^1(M^* \otimes L) - \dim \operatorname{Aut} V + nh^0(V)$.

Lemma 56: Let $\varepsilon = 2$ if the short exact sequence $L \to V \to M$ splits, $\varepsilon = 1$ otherwise. Then $\dim \operatorname{Aut} V = h^0(M^* \otimes L) + \varepsilon$.

Proof: Since L is the unique destabiliser by **Lemma 50**, an automorphism of V preserves L. Supppose the elements of L are fixed. Then there is a natural map $\alpha\colon M \to L$ corresponding to such an automorphism, and if $\pi\colon M \to L$, $\varphi(v) := v + \alpha(\pi v)$ defines an automorphism. The action on L gives a multiplication by a scalar on the whole bundle V.

Since the representant in $H^1(M^* \otimes L)$ must be preserved, the action on L determines the generated action on M, unless the representant is zero, hence $\varepsilon = 1$. Otherwise the two are independent, giving $\varepsilon = 2$. $\square$

Corollary 57: Let us use the notations in the previous two lemmata. Supposing that $H^1(L) = 0$, the dimension of the component of the Quot scheme, if not empty, is $\dim \mathcal{S} - \chi(M^* \otimes L) - \varepsilon + n(\chi(V) + h^1(M))$.

We will use this formula to calculate an upper bound for the dimension of the components of the Quot scheme.

Since L is a destabiliser, clearly $\deg L > \frac{1}{2}d \geq 2g - 2$, and so $H^0(L) = 0$. In order for the component not be empty, we need $h^0(M) \neq 0$, and so $\deg M \geq 0$. We will parametrise the components by $\delta := \deg M$, and since $\deg L = d - \delta$, we have $0 \leq \delta < \frac{1}{2}d$.

The expression $\dim \mathcal{S}_\delta$ is bounded. On one hand, it is part of the Jacobian, hence at most g. On the other hand, since $h^0(M) \neq 0$, this M is defined by an effective divisor, that appears as the image of $\operatorname{Sym}^\delta C$ inside the Jacobian, of dimension at most $\deg M$. Therefore $\dim \mathcal{S}_\delta \leq \min(\deg M, g)$.

Remark that $\deg(M^* \otimes L) = d - 2\deg M$ and $\chi(M^* \otimes L) = d - 2\deg M + 1 - g$. Whenever $\delta < \frac{d}{2} - g + 1$ (which is larger than $2g - 2$ since $d > 6g - 6$), the extension splits, since $h^1(M^* \otimes L) = 0$. However, we can say slightly more when $\deg(M^* \otimes L) = g - 1$ because of the following lemma:

Lemma 58: If L is a line bundle and $\deg L < g$, then $H^0(L) = 0$ generically.

This, when applied to $M^* \otimes L$, gives $h^1(M^* \otimes L) = -\chi(M^* \otimes L) = 0$ generically. Therefore, for this $\deg M$, we can consider the two cases when the extension splits and when it does not. When it splits, we have $\varepsilon = 2$, and we look at the whole $\mathcal{S}_\delta$. The non-splitting case, with $\varepsilon = 1$, corresponds to a subvariety $\mathcal{S} \subset \mathcal{S}_\delta$ with $\dim \mathcal{S} \leq \dim \mathcal{S}_\delta - 1$. Since $\dim \mathcal{S}_\delta - 2 = (\dim \mathcal{S}_\delta - 1) - 1$, we may assume that the extension splits for this δ as well when looking for the largest dimensional components.

The formula for the dimension becomes

$$\dim \mathcal{S} + 2\delta - d + g - 1 - \varepsilon + n(d + 2 - 2g + h^1(M))$$

Supposing that the dimension of the Quot scheme is equal to the expected dimension, we shall calculate the expected codimension of this part instead:

$$2g - 3 + \varepsilon + d - \dim \mathcal{S} - 2\delta - nh^1(M) \tag{11}$$

and this must be at least 0 to be the correct formula for the codimension. Also, it must be at least 2 for Hartog's theorem.

Now let us consider the following components, separated according to $\deg M = \delta$. We will look at only the smallest values of the codimensions, as that is what determines whether we may use Hartog's theorem.

- $\delta = 0$

Since $h^0(M) \neq 0$ if and only if $M = \mathcal{O}$, $\dim \mathcal{S} = 0$ and $h^1(M) = g$. The codimension formula (11) becomes

$$2g - 1 + d - ng$$

If (8) is satisfied, this is at least 2.

- $g \leq \delta < 2g - 2$

This case only happens when $g > 2$. We have $\varepsilon = 2$. In this case, $h^0(M) \leq \delta$, thus $h^1(M) = h^0(M) - \chi(M) \leq \deg M - (\deg M + 1 - g) = g - 1$. Also, $\dim \mathcal{S} \leq g$. The codimension (11) is

$$g - 1 + d - 2\delta - n(g - 1)$$

which is the smallest when $\delta = 2g - 3$:

$$-3g + 5 + d - n(g - 1)$$

If (10) is satisfied, this is at least 2. Note that (10) is equivalent to (9) when $g = 2$.

- $\delta = 2g - 2$

Since $2g - 2 \geq g$, $\dim \mathcal{S} = g$. We have $\varepsilon = 2$.

In the generic case, $M \neq \mathcal{K}$, and so $h^1(M) = 0$. The codimension formula (11) becomes

$$d - 3g + 3$$

Since $d \geq 4g - 3$, this is at least $g \geq 2$. Hence we don't need to consider it.

Otherwise, $h^1(M) = 1$, but $\dim \mathcal{S} = 0$, since only $\mathcal{K}$ is considered.

$$2\delta - d + g - 3 + n(d + 3 - 2g)$$

The codimension (11) is

$$d - 2g + 3 - n$$

If (9) is satisfied, this is at least 2.

- $0 < \delta \leq g$ and $\delta < 2g - 2$

We have $\varepsilon = 2$. $h^0(M) \leq \delta$, thus $h^1(M) = h^0(M) - \chi(M) \leq \deg M - (\deg M + 1 - g) = g - 1$. Also, $\dim \mathcal{S} \leq \delta$. The codimension formula (11) becomes at least

$$2g - 1 + d - 3\delta - n(g - 1)$$

If $g > 2$, this is the smallest when $\delta = g$:

$$-g - 1 + d - n(g-1)$$

However, this is strictly larger than the codimension in the case $g \leq \delta < 2g - 2$.

Otherwise, if $g = 2$, it becomes the smallest when $\delta = 2g - 3 = 1$:

$$2g - 4 + d - n(g-1) = d - n$$

However, for the $M = \mathcal{K}$ case, the codimension is $d - 2g + 3 - n = d - n - 1$, which is strictly smaller.

Hence this case is subsumed by the previous two cases.

- $2g - 2 < \delta < \frac{1}{2}d$

This case only happens when $4g - 1 \geq d$. In this case, $h^1(M) = 0$, $\dim \mathcal{S} = g$.

There are two cases: when $\delta \leq \frac{d-1}{2} - g + 1$ or $\delta = \frac{d-g+1}{2}$, the extension V splits (in the later case, only generically), and $\varepsilon = 2$. Otherwise $\varepsilon = 1$. The codimension (11) is

$$g - 3 + \varepsilon + d - 2\delta$$

For the non-splitting cases, this is the smallest for $\delta = \frac{1}{2}(d-1)$ when $g > 2$, becoming $g - 1$, and this is always at least 2. It always splits for $g = 2$.

For the splitting case, if $2 \mid g$, then the codimension is the smallest for $\delta = \frac{d-g+1}{2}$, becoming $2g - 2$, and this is at least 2.

When $2 \nmid g$, the codimension is the smallest for $\delta = \frac{d+1-2g}{2}$, becoming $3g - 2$, which is also at least 2. □

Note: It is easy to check that the conditions are satisfied for the case $g = 2$, $d = 7$, $n = 3$, for instance.

4.8. The universal sequence

In the following sections, we will construct a correspondence of certain line bundles over $\mathcal{P}$ and $\mathcal{Q}$ that are isomorphic over $\mathcal{Q} \cap \mathcal{P}$. In this section in particular, we will give a pair of vector bundles over these two spaces that are isomorphic over the intersection.

Recall from **Theorem 9** that $\mathcal{Q}$ has a universal sequence $\mathcal{E} \to \mathcal{O}^{\oplus n} \to \mathcal{F}$ over it. Recall also from **Theorem 26** that there is a universal bundle $\mathcal{U}$ over $\mathcal{M}_\Delta$. We will denote by $\mathcal{U}$ the pullbacks of the universal bundle onto both $\mathcal{P}$ and $\mathcal{G}r$.

Theorem 59: The universal bundle $\mathcal{E}_\mathcal{Q}$ over $\mathcal{Q} \times C$ and the bundle $\Pi_\mathcal{P}(\mathcal{U})^* \otimes \mathcal{U}^*$ over $\mathcal{P} \times C$ have isomorphic restrictions to $\mathcal{Q} \cap \mathcal{P}$, and $\Pi_\mathcal{P}(\mathcal{U})^* \otimes \mathcal{U}^* = \mathcal{O}_\mathcal{P}(-1) \otimes \mathcal{V}^*$ does not depend on $\mathcal{U}$.

Proof: Since $\Pi_{\mathcal{P}}(\mathcal{U} \otimes \mathcal{L}) = \Pi_{\mathcal{P}}(\mathcal{U}) \otimes \mathcal{L}^*$, $\Pi_{\mathcal{P}}(\mathcal{U} \otimes \mathcal{L})$ does not depend on $\mathcal{U}$, and we may assume $\mathcal{U} = \mathcal{V}$ and $\Pi_{\mathcal{P}}(\mathcal{U}) = \mathcal{O}_{\mathcal{P}}(1)$.

Let us consider $\mathcal{P} = \mathbb{P}\operatorname{Hom}(\mathbb{C}^n, \pi_*\mathcal{V})$. There is a universal sequence over each fiber, each isomorphic to a projective space, that gives a sequence over $\mathcal{P}$:

$$0 \to \mathcal{O}_{\mathcal{P}}(-1) \to \mathcal{O}_{\mathcal{P}} \otimes \operatorname{Hom}(\mathbb{C}^n, \pi_*\mathcal{V}) \to Q \to 0 \tag{12}$$

where the first morphism gives a canonical global section of the Hom-bundle $\mathcal{O}_{\mathcal{P}}(1) \otimes \operatorname{Hom}(\mathbb{C}^n, \pi_*\mathcal{V})$. Since $\operatorname{Hom}(\mathbb{C}^n, \pi_*\mathcal{V}) = \operatorname{Hom}_{\mathcal{P}\times C}(\mathcal{O}^n, \mathcal{V})$, this gives a global section of $\operatorname{Hom}_{\mathcal{P}\times C}(\mathcal{O}^n, \mathcal{O}_{\mathcal{P}}(1) \otimes \mathcal{V})$, or a map

$$\mathcal{O}^n \xrightarrow{\mathcal{F}} \mathcal{O}_{\mathcal{P}}(1) \otimes \mathcal{V}.$$

Now consider a point $(V, f\colon \mathcal{O}^n \to V) \in \mathcal{P}$. Restricting the universal sequence to this point, the first morphism in (12) becomes $\mathcal{O} \to \operatorname{Hom}(\mathbb{C}^n, \pi_*\mathcal{V})$, which is equal to f, and hence $\mathcal{F}|_{(V,f)}$ as well. If such a point appears also in $\mathcal{Q}$, the maps $f = \mathcal{F}|_{(V,f)}$ are generically surjective. Thus $\mathcal{O}_{\mathcal{P}}(1) \otimes \mathcal{V}$ is a quotient of the trivial bundle $\mathcal{O}^n$ over the subset of stable points.

Let us consider the dual map $\mathcal{F}^*$ over $\mathcal{Q} \cap \mathcal{P}$:

$$0 \to \mathcal{O}_{\mathcal{P}}(-1) \otimes \mathcal{V}^* \xrightarrow{\mathcal{F}^*} \mathcal{O}^n \to R \to 0.$$

The universal sequence over $\mathcal{Q}$ restricts to the intersection as well:

$$0 \to \mathcal{E}|_{\mathcal{Q}\cap\mathcal{P}} \to \mathcal{O}^n \to \mathcal{F}|_{\mathcal{Q}\cap\mathcal{P}} \to 0. \tag{3}$$

Since the first arrow is identical to $\mathcal{F}^*$, the two sequences are isomorphic, and the subsheaf $\mathcal{O}_{\mathcal{P}}(-1) \otimes \mathcal{V}^*$ of $\mathcal{O}^n$ is isomorphic to $\mathcal{E}$ on $(\mathcal{Q} \cap \mathcal{P}) \times C$. $\square$

4.9. Correspondence of line bundles

Let us fix $\mathcal{U} := \mathcal{V}$ to be the normalised universal bundle. By identifying $\mathcal{E}_{\mathcal{Q}}$ and $\mathcal{O}_{\mathcal{P}}(-1) \otimes \mathcal{V}^*_{\mathcal{P}}$ through **Theorem 59**, we may create corresponding line bundles over $\mathcal{Q}$ and $\mathcal{P}$ that are isomorphic over $\mathcal{Q} \cap \mathcal{P}$. To better understand this correspondence, let us consider two line bundles over $\mathcal{P}$, and another one parametrised by an integer (and the corresponding line bundles over $\mathcal{Q}$):

over $\mathcal{Q}$	over $\mathcal{P}$	
$\det \mathcal{E}_p^*$	$\mathcal{O}_{\mathcal{P}}(r) \otimes \det \mathcal{V}_p$	$=: \mathcal{L}_1$
$\det(\pi_*\mathcal{E}^*)$	$\mathcal{O}_{\mathcal{P}}(\chi) \otimes \det(\pi_*\mathcal{V})$	$=: \mathcal{L}_3$
$\det(\pi_*\mathcal{E}(m\Lambda))^*$	$\mathcal{O}_{\mathcal{P}}(\chi_m) \otimes \det(\pi_*\mathcal{V}^*(m\Lambda))^*$	$=: \mathcal{L}_{2,m}$

where $\chi := \chi(V) = d + (1-g)r$ and $\chi_m := \chi(V^*(m)) = mkr - d + (1-g)r$.

Furthermore, there is a compatible pair of natural $\mathrm{SL}(n)$ actions on $\mathcal{E}_{\mathcal{Q}}$ and $\mathcal{O}_{\mathcal{P}}(k)$ (and the action is trivial on $\mathcal{V}_{\mathcal{P}}$). Whenever we have a polarisation on $\mathcal{P}$, it induces a polarisation on $\mathcal{G}r$. By **Corollary 38**, this polarisation takes the following form:

Lemma 60: Suppose that $\mathcal{O}_{\mathcal{P}}(k) \otimes \Phi_{\mathcal{P}}$ is ample over $\mathcal{P}$ for some $n \mid k > 0$, where $\Phi_{\mathcal{P}}$ denotes the pullback of a line bundle $\Phi_{\mathcal{M}}$ on $\mathcal{M}_\Delta$, with trivial action. Then there is an induced polarisation $\mathcal{O}(k/n) \otimes \Phi_{\mathcal{G}r}$ on $\mathcal{G}r$, where $\Phi_{\mathcal{G}r}$ is the pullback of the bundle $\Phi_{\mathcal{M}}$.

A polarisation over $\mathcal{G}r$ induced from $\mathcal{O}_{\mathcal{P}}(k)$ is only well defined when $n \mid k$. None of $\mathcal{L}_1$, $\mathcal{L}_3$ and $\mathcal{L}_{2,m}$ give actual line bundles over $\mathcal{G}r$, however their product, with appropriate powers, may give one, if the formal degree of the bundle $\mathcal{O}_{\mathcal{G}r}(k/n)$ is an integer. To show this correspondence, I will use $\mathcal{O}_{\mathcal{G}r}(k/n)$ as a shorthand even when $n \nmid k$. Then the corresponding line bundles over $\mathcal{Q}$ and $\mathcal{G}r$ can be written as in the following table:

	over $\mathcal{Q}$	over $\mathcal{P}$	over $\mathcal{G}r$
$\mathcal{L}_1$	$\det \mathcal{E}_p^*$	$\mathcal{O}_{\mathcal{P}}(r) \otimes \det \mathcal{V}_p$	$\mathcal{O}_{\mathcal{G}r}(r/n) \otimes \det \mathcal{V}_p$
$\mathcal{L}_3$	$\det(\pi_* \mathcal{E}^*)$	$\mathcal{O}_{\mathcal{P}}(\chi) \otimes \det(\pi_* \mathcal{V})$	$\mathcal{O}_{\mathcal{G}r}(\chi/n) \otimes \det(\pi_* \mathcal{V})$
$\mathcal{L}_{2,m}$	$\det(\pi_* \mathcal{E}(m\Lambda))^*$	$\mathcal{O}_{\mathcal{P}}(\chi_m) \otimes \det(\pi_* \mathcal{V}^*(m\Lambda))^*$	$\mathcal{O}_{\mathcal{G}r}(\chi_m/n) \otimes \det(\pi_* \mathcal{U}^*(m\Lambda))^*$

The Picard group of $\mathcal{G}r$ is $\operatorname{Pic} \mathcal{G}r = \mathbb{Z}^{\oplus 2}$, and it is generated by the pullback of the generator of $\operatorname{Pic} \mathcal{M}_\Delta$, $\mathcal{L}_{\mathrm{Ver}}$, and the bundle $\mathcal{O}_{\mathcal{G}r}(1)$.

Lemma 61: Let r and d be such that $d \equiv 1 \pmod{r}$ and define $\chi = d + (1-g)r$. Let $\mathcal{V}$ denote the normalised universal bundle and its pullback to $\mathcal{G}r = \mathrm{Gr}(2, \pi_* \mathcal{V})$ as well. Then $\operatorname{Pic} \mathcal{G}r$ is freely generated by the line bundles

$$\mathcal{L}_{\mathrm{Ver}} = \mathcal{L}_1^{\otimes \chi} \otimes \mathcal{L}_3^{\otimes -r} \quad \text{and} \quad \mathcal{L}_{\mathrm{Taut}} := \mathcal{O}_{\mathcal{G}r}(1) = \mathcal{L}_1^{\otimes n \frac{1-\chi}{r}} \otimes \mathcal{L}_3^{\otimes n}.$$

Proof: Let us now consider the normalised universal bundle $\mathcal{V}$, defined for some pair of κ and λ such that $\kappa r + \lambda \chi = 1$, and recall from **Lemma 42** that the group $\operatorname{Pic}(\mathcal{M}_\Delta)$ is generated by

$$\mathcal{L}_{\mathrm{Ver}} := (\det \mathcal{V}_p)^{\otimes \chi} \otimes (\det \pi_* \mathcal{V})^{\otimes -r}$$

where $\chi = r(g-1) + d$. Furthermore, we have $c_1(\mathcal{V}_p) = \mathcal{L}_{\mathrm{Ver}}^{\otimes \lambda}$ and $c_1(\det \pi_* \mathcal{V}) = \mathcal{L}_{\mathrm{Ver}}^{\otimes(-\kappa)}$. We will compare the equivariant line bundles $\mathcal{L}_1$ and $\mathcal{L}_3$ with the two generators of $\operatorname{Pic} \mathcal{G}r$: $\mathcal{L}_{\mathrm{Ver}}$, the pullback of the generator of $\operatorname{Pic} \mathcal{M}_\Delta$, and $\mathcal{L}_{\mathrm{Taut}} = \mathcal{O}_{\mathcal{G}r}(1)$.

Recall from **Lemma 28** and **Lemma 42** that when d is of the form $r\delta + 1$ (i.e. $d \equiv 1 \pmod{r}$), $\chi = r(1 - g + \delta) + 1$, we can choose $\lambda = 1$ with $\kappa = g - 1 - \delta$ to obtain the normalised universal bundle. Then $\mathcal{L}_{\mathrm{Ver}} = \det \mathcal{V}_p$ generates the Picard group, and

$\det \pi_* \mathcal{V} = (\det \mathcal{V}_p)^{\otimes 1-g+\delta}$ (with $\delta = (d-1)/r$). Consider the line bundles $\mathcal{L}_1$ and $\mathcal{L}_3$ on $\mathcal{G}r$. By taking their appropriate combination so that $\mathcal{O}_{\mathcal{G}r}(1)$ falls out, we obtain the pullback to $\mathcal{G}r$ of the generator of $\operatorname{Pic}\mathcal{M}_\Delta$ we just described: $\mathcal{L}_1^{\otimes\chi} \otimes \mathcal{L}_3^{\otimes -r} = \mathcal{L}_{\mathrm{Ver}}$. We also know that $\mathcal{L}_1^{\otimes n\kappa} \otimes \mathcal{L}_3^{\otimes n\lambda} = \mathcal{O}_{\mathcal{G}r}(n(\kappa r + \lambda\chi)/n) = \mathcal{O}_{\mathcal{G}r}(1)$, as $(\det \mathcal{V}_p)^{\otimes\kappa} \otimes (\det(\pi_*\mathcal{V}))^{\otimes\lambda} = \mathcal{O}_{\mathcal{M}}$. Then $\kappa = \frac{1-\chi}{r}$, giving the formula in the lemma. □

The conditions of this lemma are satisfied in the case $r = 2$, $(r, d) = 1$.

Since $\mathcal{L}_1$ and $\mathcal{L}_3$ have corresponding equivariant line bundles over $\mathcal{Q}$ as well, for any element $\alpha \in \operatorname{Pic}\mathcal{G}r \cong \mathbb{Z}^{\oplus 2}$, we have a triplet of corresponding line bundles $\mathcal{L}_{\mathcal{Q}}^{\alpha}$, $\mathcal{L}_{\mathcal{P}}^{\alpha}$ and $\mathcal{L}_{\mathcal{G}r}^{\alpha}$, where the first two are isomorphic over $\mathcal{Q}\cap\mathcal{P}$, while whenever $\mathcal{L}_{\mathcal{P}}^{\alpha}$ is ample, it induces $\mathcal{L}_{\mathcal{G}r}^{\alpha}$ when quotiented by $\mathrm{PGL}(n)$. To any such triplet, I will use the same notation for all three line bundles, hence $\mathcal{L}_1$, $\mathcal{L}_3$, $\mathcal{L}_{2,m}$, $\mathcal{L}_{\mathrm{Ver}}$ and $\mathcal{L}_{\mathrm{Taut}}$ can be line bundles over all three spaces. To obtain Verlinde's formulæ, we must choose the tensor powers of $\mathcal{L}_{\mathrm{Ver}}$ as the polarisation.

4.10. The main conjecture

Our goal is to calculate the cohomologies of line bundles over $\mathcal{M}_\Delta$, or more generally over $\mathcal{G}r = \mathrm{Gr}(n, \pi_*\mathcal{V})$. The Picard group of $\mathcal{G}r$ is $\operatorname{Pic}\mathcal{G}r = \mathbb{Z}^{\oplus 2}$, and by **Lemma 61**, $\operatorname{Pic}\mathcal{G}r$ is generated by the pullback of the generator of $\operatorname{Pic}\mathcal{M}_\Delta$, $\mathcal{L}_{\mathrm{Ver}}$, and the line bundle $\mathcal{L}_{\mathrm{Taut}} := \mathcal{O}_{\mathcal{G}r}(1)$ over $\mathcal{G}r$.

Acting on $\mathcal{P}$ via the action of $\mathrm{PGL}(n)$, the GIT quotient of $\mathcal{P}$ is the Grassmannian fibration $\mathcal{G}r$ over $\mathcal{M}_\Delta$. Consider the following theorem (see Teleman's theorem in [**24**]):

Theorem 62: Choose an ample line bundle $\mathcal{L}$ over $\mathcal{P}$ with a $\mathrm{PGL}(n)$ action. Since $\mathcal{G}r$ is the GIT quotient of $\mathcal{P}$ under such a polarisation, there is an induced line bundle $\mathcal{L}_{\mathcal{G}r}$ over $\mathcal{G}r$. Denoting $\chi(\mathcal{L})^{\mathrm{PGL}(n)} := \sum_i (-1)^i \dim H^i(\mathcal{L})^{\mathrm{PGL}(n)}$, we have

$$\chi(\mathcal{L}_{\mathcal{G}r}) = \chi(\mathcal{L})^{\mathrm{PGL}(n)}.$$

We will refer to $\chi(\mathcal{L})^{\mathrm{PGL}(n)}$ as the invariant part of $\chi(\mathcal{L})$. Using this theorem, we can calculate the Euler characteristic of a line bundle $\mathcal{L}_{\mathcal{G}r}$ over $\mathcal{G}r$ by taking a polarisation $\mathcal{L}_{\mathcal{P}}$ over $\mathcal{P}$, such as the one described in **Lemma 60**, and calculating $\chi(\mathcal{L}_{\mathcal{P}})^{\mathrm{PGL}(n)}$ instead.

Conjecture 63: Suppose that $\mathcal{P}^s$ is contained within $\mathcal{Q}$. Consider a pair of line bundles $\mathcal{L}_{\mathcal{Q}}$ and $\mathcal{L}_{\mathcal{P}}$ over $\mathcal{Q}$ and $\mathcal{P}$, respectively, that are isomorphic over the common part. If $\mathcal{L}_{\mathcal{P}}$ is ample over $\mathcal{P}$, then the invariant parts of the equivariant characteristics are equal: $\chi(\mathcal{L}_{\mathcal{P}})^{\mathrm{PGL}(n)} = \chi(\mathcal{L}_{\mathcal{Q}})^{\mathrm{PGL}(n)}$.

If this conjecture is true, then we can calculate $\chi(\mathcal{L}_{\mathcal{Q}})^{\mathrm{PGL}(n)}$ instead of $\chi(\mathcal{L}_{\mathcal{P}})^{\mathrm{PGL}(n)}$ and obtain the same result. In this thesis, we will check that for certain pairs of $\mathcal{L}_{\mathcal{Q}}$ and $\mathcal{L}_{\mathcal{G}r}$, $\chi(\mathcal{L}_{\mathcal{Q}})^{\mathrm{PGL}(n)} = \chi(\mathcal{L}_{\mathcal{G}r})$.

5. Computation via localisation

5.1. Introduction

In this section, we will fix an equivariant line bundle $\mathcal{L}$ over the Quot scheme $\mathcal{Q}$ of fixed determinant subsheaves, and calculate the invariant part of the equivariant Euler characteristic $\chi_T(\mathcal{L},t)^T$ where $T = \mathbb{T}^{n-1} \leq \mathrm{SL}(n)$ is the subgroup of diagonal matrices, which is a maximal torus in $\mathrm{SL}(n)$. Using the Atiyah-Bott localisation (**Theorem 39** and its virtual form **Definition 40**)

$$\chi_T(\mathcal{L},t) = \sum_{X \in \mathrm{Fix}(\mathcal{Q})} \int_X \frac{\mathrm{ch}_T(L)\,\mathrm{td}(TX)}{\mathrm{ab}_T(N_{\mathcal{Q}}X)}$$

where the sum goes over the connected components of the fixed point set, we can calculate this value through an integral over the fixed point set $\mathrm{Fix}(\mathcal{Q})$ of $\mathcal{Q}^T$.

In the first few sections, we will consider the case of an arbitrary parameter r in the Quot scheme of rank r embeddings, provided that $(r,d) = 1$, and later we will return to our concrete case when $r = 2$.

5.2. Fixed point set

Let us choose d and r such that $(r,d) = 1$, fix an n and consider the Quot scheme $\widetilde{\mathcal{Q}}$ of degree $-d$, rank r subsheaves of $\mathcal{O}^n$ of arbitrary determinant (or equivalently, degree d, rank $n-r$ quotients). First we will consider this space instead of $\mathcal{Q}$, the space of subsheaves whose determinant is fixed.

Let us write $\mathcal{O}^n$ as the direct sum $\bigoplus_{i=1}^n \mathcal{O}_i$. As seen in **[16]**, the points of the fixed point set correspond to those

$$0 \to E \to \mathcal{O}^n \to F \to 0 \tag{3}$$

in $\widetilde{\mathcal{Q}}$ where the whole sequence decomposes as n exact sequences

$$0 \to E_i \to \mathcal{O}_i \to F_i \to 0$$

with $E = \bigoplus E_i$ and $F = \bigoplus F_i$. Since E_i is a vector bundle, it is either a line bundle or $E_i = 0$. When E_i is a line bundle, the subsheaf $E_i \to \mathcal{O}_i$ is determined by an effective divisor given by the section $\mathcal{O}_i^* \to E_i^*$. When $E_i = 0$, we obtain the degenerate map

$$0 \to 0 \to \mathcal{O}_i \to \mathcal{O}_i \to 0$$

with $F_i = \mathcal{O}_i$, so to characterise one fixed point, it is enough to consider the set of indices where $E_i \neq 0$, and the corresponding divisors. As a set, $\mathrm{Fix}(\widetilde{\mathcal{Q}})$ consists of pairs

$(I, (D_i)_{i=1}^r)$ where $I \subseteq \{1, \dots, n\}$ is a subset of r elements, and for each $i \in I$, $D_i \in \operatorname{Div} C$ is an effective divisor, such that $\sum_{i=1}^r \deg D_i = d$.

In one connected component of the fixed point set, the set of indices I such that E_i is a line bundle may not change. Also, the degrees $d_i = \deg D_i$ of the divisors are fixed within each connected component. The effective divisors of degree d_i are parametrised by $\operatorname{Sym}^{d_i} C$. Hence we obtain the following characterisation:

Lemma 64: The space $\operatorname{Fix}(\widetilde{\mathcal{Q}})$ is the union of connected components $X_{I,(d_i)_{i=1}^r}$ such that $I \subseteq \{1, \dots, n\}$ is a subset of r elements, d_i are non-negative integers such that $d = \sum_{i \in I} d_i$, and $X_{I,(d_i)_{i=1}^r} \cong \prod_{i \in I} \operatorname{Sym}^{d_i} C$.

(3)

Over each connected component X of the fixed point set, we have universal bundles $\mathcal{E}_i := \mathcal{L}_i^*$, and the universal sequence (3) takes the form

$$0 \to \bigoplus_{i=1}^r \mathcal{E}_i \to \mathcal{O}^n \to \bigoplus_{i=1}^r \mathcal{F}_i \oplus \mathcal{O}^{n-r} \to 0.$$

When we consider one connected component, we may suppose that $I = \{1, \dots, r\}$ and that $\mathcal{E}_i = 0$ and $\mathcal{F}_i = \mathcal{O}_i$ whenever $i > r$.

5.3. Virtual tangent space and normal space

The tangent space of $\widetilde{\mathcal{Q}}$ at a smooth point $E \to \mathcal{O}^n$ is isomorphic to $\operatorname{Hom}(E, F)$ by **Theorem 10**. When considering a point in the fixed point set, this space decomposes into $\bigoplus_{i,j} \operatorname{Hom}(E_i, F_j)$.

Lemma 65: Consider a connected component X of the fixed point set. Then the tangent space of X and the normal bundle in $\widetilde{Q}$ at a smooth point are

$$\bigoplus_{i=1}^r \operatorname{Hom}(E_i, F_i) \qquad \text{and} \quad \bigoplus_{\substack{1 \le i \le r \\ 1 \le j \le n \\ i \ne j}} \operatorname{Hom}(E_i, F_j)$$

respectively.

A space is smooth when the tangent space has the same dimension at all of its points. However, the Quot scheme is not smooth, and the above spaces are not of the same dimension at all points, hence the Atiyah-Bott localisation formula may not be used in its classical form. As seen in **[16]**, we may replace the tangent space of the Quot scheme by a virtual tangent space and calculate the virtual Euler classes via the virtual Atiyah-Bott localisation.

Recall from **[16]** that the virtual tangent bundle of the Quot scheme may be constructed as a two-step complex of vector bundles $\mathcal{A}_0 \to \mathcal{A}_1$ with the property that for all sheaves $\mathcal{I}$ on the Quot scheme,

$$0 \to \mathrm{Hom}(\mathcal{E}, \mathcal{F} \otimes \pi^*\mathcal{I}) \to \mathcal{A}_0 \otimes \mathcal{I} \to \mathcal{A}_1 \otimes \mathcal{I} \to \mathrm{Ext}^1(\mathcal{E}, \mathcal{F}) \otimes \mathcal{I} \to 0.$$

Then K-theoretically $\mathcal{A}_0 - \mathcal{A}_1 = \mathrm{Hom}(\mathcal{E}, \mathcal{F}) - \mathrm{Ext}^1(\mathcal{E}, \mathcal{F})$ at any point, which is the direct image $\pi_!(\mathcal{E}^* \otimes \mathcal{F})$ at that point. Therefore, instead of using the characteristic classes of the complex $\mathcal{A}_0 - \mathcal{A}_1$, we may replace them with the characteristic classes of $\pi_!(\mathcal{E}^* \otimes \mathcal{F})$. When applying the Atiyah-Bott localisation, we replace the tangent bundle with the virtual bundle $\pi_!(\mathcal{E}^* \otimes \mathcal{F})$.

Following the setup in **[16]**, we obtain via calculation the virtual tangent space of a connected component X of the fixed point set

$$TX = \bigoplus_{i=1}^{r} T_{ii}, \tag{13}$$

and the virtual normal bundle in $\widetilde{Q}$

$$NX = N_{\widetilde{Q}}X = \bigoplus_{\substack{1 \le i \le r \\ 1 \le j \le n \\ i \ne j}} T_{ij}. \tag{14}$$

Then we can use the following form of the virtual Atiyah-Bott localisation (**Definition 40**)

$$\chi_T^{\mathrm{virt}}(L, t) = \sum_{X \in \mathrm{Fix}(\widetilde{Q})} \int_X \frac{\mathrm{ch}_T(L)\,\mathrm{td}(TX)}{\mathrm{ab}_T(NX)}$$

by replacing $[\mathcal{A}_0] - [\mathcal{A}_1] - [TX]$ with $[NX]$, according to (13) and (14).

5.4. The characteristic classes over the fixed point set

Let us fix a connected component X of the fixed point set. To calculate the Atiyah-Bott formula, we need to calculate $\mathrm{td}(TX)$ and $\mathrm{ab}_T(NX)$ where TX and NX are the virtual tangent and virtual normal bundles of X, respectively. Recall that the characteristic classes are defined on elements in $K(\mathrm{Quot})$ via $\mathrm{td}(A - B) = \mathrm{td}(A)/\,\mathrm{td}(B)$ and $\mathrm{ab}_T(A - B) = \mathrm{ab}_T(A)/\,\mathrm{ab}_T(B)$, and as seen in the comments after **Theorem 39**, we may apply the Atiyah-Bott localisation to the characteristic classes of the virtual bundles, $\mathrm{td}(TX)$ and $\mathrm{ab}_T(NX)$ to obtain the virtual Atiyah-Bott formula.

Recall that the virtual normal bundle is $n = \bigoplus_{i \neq j} T_{ij}$, while the virtual tangent bundle is $T = \bigoplus_i T_{ii}$ where $T_{ij} = \pi_!(\mathcal{E}_i^* \otimes \mathcal{F}_j) = \pi_!(\mathcal{E}_i^* \otimes \mathcal{O}_j) - \pi_!(\mathcal{E}_i^* \otimes \mathcal{E}_j)$. Since both td and ab_T are multiplicative, it is enough to calculate $\mathrm{td}(T_{ii})$ and $\mathrm{ab}_T(T_{ij})$.

Recall that $\mathcal{E}_i^*$ is none other than the universal line bundle $\mathcal{L}_i$ over $\mathrm{Sym}^{d_i} C$, and so we can write the needed equivariant Chern characters over the fixed point set

$$\mathrm{ch}_T(\pi_!(\mathcal{E}_i^* \otimes \mathcal{O}_j)) = e^{\eta_i}(d_i - (g-1) - \theta_i) \cdot (t_j/t_i) \tag{4}$$

$$\mathrm{ch}_T(\pi_!(\mathcal{E}_i^* \otimes \mathcal{E}_j)) = e^{\eta_i - \eta_j}(d_i - d_j - (g-1) - (\theta_i + \theta_j - \theta_{ij})) \cdot (t_j/t_i). \tag{5}$$

([**16**])

To obtain $\mathrm{td}(TX)$ and $\mathrm{ab}_T(NX)$, we will apply the following proposition to these Chern classes.

Lemma 66: Given k variables x_i in a commutative ring, if $\sum x_i^n = 0$ for all $k \geq n > 1$ and $\sum x_i = \theta$, then a symmetric sum $\sum x_{i_1}^{k_1} \cdots x_{i_n}^{k_n}$ is 0 if and only if any of the k_j are greater than 1, and $\sum x_{i_1} \cdots x_{i_n} = \frac{1}{n!}\theta^n$.

Proposition 67: Let us suppose that a (possibly virtual) vector bundle V has a T-equivariant Chern character of the form $\mathrm{ch}^T(V,t) = t^w e^\eta (n + \theta)$ with η and θ degree 2 cohomology classes and $n \in \mathbb{Z}$. Then the Todd and equivariant AB-classes are as follows:

$$\mathrm{td}(V) = \left(\frac{\eta}{1 - e^{-\eta}}\right)^n \cdot \exp\left(\frac{\theta}{\eta} - \frac{\theta}{e^\eta - 1}\right)$$

$$\mathrm{ab}_T(V) = (1 - t^{-w}e^{-\eta})^n \exp\left(\frac{\theta}{t^w e^\eta - 1}\right).$$

Proof: With this condition, the symmetric polynomials constructed from the Chern roots of this vector bundle behave in the way described in **Lemma 66**.

Let us consider the Chern character of the form $q^{-1}(n + \theta)$ where $q = t^{-1}e^{-\eta}$ is a degree 2 equivariant cohomology class. Suppose that the roots are λ_i. Then the AB-class is $\prod(1 - qe^{-\lambda_i})$. It can be expanded as $\prod\left(1 - q + q\lambda_i - \frac{1}{2}q\lambda_i^2 + \ldots\right)$. Expanding the product gives us a symmetric polynomial. Because of the **Lemma 66**, any higher power in the λs will disappear, so we can suppose that $\lambda_i^2 = 0$, and we look at $\prod\left((1-q) + q\lambda_i\right)$.

Then the AB-class is equal to

$$\left(\sum_k (1-q)^{n-k} \cdot q^k \frac{1}{k!}\theta^k\right) = (1-q)^n \exp\left(\frac{q\theta}{1-q}\right).$$

or

$$(1 - t^{-w}e^{-\eta})^n \exp\left(\frac{t^{-w}e^{-\eta}\theta}{1 - t^{-w}e^{-\eta}}\right).$$

In a similar way, assuming that the Chern character is $e^{\eta}(n+\theta)$, the Todd class would be $\prod \frac{\eta+\lambda_i}{1-e^{-(\eta+\lambda_i)}}$. Looking at only the denominator, one factor can be then rewritten as $\sum_{k=0}^{\infty} \frac{1}{(k+1)!}(-1)^k(\eta+\lambda_i)^k$. As before, we can ignore the higher powers of λ_i, which gives us $\sum \frac{1}{(k+1)!}(\eta^k + k\eta^{k-1}\lambda_i)$.

Let us group the terms of this sum using the notation $f(\eta) = \sum \frac{1}{(k+1)!}\eta^k$, $g(\eta) = \sum \frac{1}{(k+1)!}k\eta^{k-1}$. The reciprocal of the Todd class can thus be rewritten as $\prod(f(\eta)+g(\eta)\lambda_i)$. Once again, we can rewrite in terms of the symmetric polynomials, the denominator becoming $\sum_{k=0}^{n} f(\eta)^{n-k}g(\eta)^k\frac{1}{k!}\theta^k$, which is equal to $f(\eta)^n \exp\left(\frac{g(\eta)\theta}{f(\eta)}\right)$.

Since $f(\eta) = \frac{1-e^{-\eta}}{\eta}$ and $g(\eta) = f'(\eta) = \frac{-1+(1+\eta)e^{-\eta}}{\eta^2}$, the complete formula for the Todd class becomes as in the proposition. □

Using these results, we can give the expression for $\mathrm{ab}_T(NX)$:

$$\mathrm{ab}_T(T_{ij}) = (1 - (t_i/t_j)e^{-\eta_i})^{d_i-(g-1)} \exp\left(-\frac{(t_i/t_j)e^{-\eta_i}}{1-(t_i/t_j)e^{-\eta_i}} \cdot \theta_i\right)$$

if $j > r$, otherwise:

$$\mathrm{ab}_T(T_{ij}) = \frac{(1 - (t_i/t_j)e^{-\eta_i})^{d_i-(g-1)} \exp\left(-\frac{(t_i/t_j)e^{-\eta_i}}{1-(t_i/t_j)e^{-\eta_i}} \cdot \theta_i\right)}{(1 - (t_i/t_j)e^{-\eta_i+\eta_j})^{d_i-d_j-(g-1)} \exp\left(-\frac{(t_i/t_j)e^{-\eta_i+\eta_j}}{1-(t_i/t_j)e^{-\eta_i+\eta_j}} \cdot (\theta_i+\theta_j-\theta_{ij})\right)}$$

and $\mathrm{td}(TX)$:

$$\mathrm{td}(T_{ii}) = \left(\frac{1-e^{-\eta_i}}{\eta_i}\right)^{-d_i+(g-1)} \cdot \exp\left(\left(\frac{e^{-\eta_i}}{1-e^{-\eta_i}} - \frac{1}{\eta_i}\right)\cdot\theta_i\right).$$

The complete characteristic classes are then the products of these formulæ:

$$\mathrm{ab}_T(NX) = \prod_{i\neq j} \mathrm{ab}_T(T_{ij}) \qquad \mathrm{td}(TX) = \prod_i \mathrm{td}(T_{ii}).$$

Note that $\mathrm{ch}(\pi_!(\mathcal{E}_i^* \otimes \mathcal{E}_i)) = -(g-1)$, so $\mathrm{td}(\pi_!(\mathcal{E}_i^* \otimes \mathcal{E}_i)) = 1$.

5.5. Line bundles over the symmetric products

To calculate the localisation formula, we need to fix an equivariant line bundle $\mathcal{L}$ over the Quot scheme. Among others, we are going to look at these three particular virtual equivariant line bundles: $\det \mathcal{E}_p$, $\det \pi_! \mathcal{E}(m\Lambda)$ and $\det \pi_! \mathcal{E}^*$ (where Λ is a degree k divisor in C).

Lemma 68: If we restrict $\mathcal{E}$ to the fixed point set $\prod \operatorname{Sym}^{d_i} C$, we obtain

$$\operatorname{ch}_T(\det \mathcal{E}_p) = \prod_i t_i e^{-\eta_i}, \tag{15}$$

$$\operatorname{ch}_T(\det \pi_! \mathcal{E}(m\Lambda)) = \prod_i (t_i e^{-\eta_i})^{mk-d_i-(g-1)} \cdot e^{-\sum_i \theta_i}, \tag{16}$$

$$\operatorname{ch}_T((\det \pi_! \mathcal{E}^*)^*) = \prod_i (t_i e^{-\eta_i})^{d_i-(g-1)} \cdot e^{\sum_i \theta_i}. \tag{17}$$

Proof: Let us consider $\det \mathcal{E}$, which becomes $\bigotimes_i \mathcal{E}_i$ over the fixed point set. Given the line bundle $\det \mathcal{E}_p$, which is the pullback of $\det \mathcal{E}$ via the injection $\{p\} \to C$, and knowing that the first Chern class is $-\eta$, the equivariant Chern character becomes $\prod t_i e^{-\eta_i}$.

Taking the line bundle $\det \pi_! \mathcal{E}(m\Lambda)$, then according to the Grothendieck-Riemann-Roch theorem (and knowing that $c_1(\mathcal{O}_C(\Lambda)) = k \cdot 1 \otimes \omega$ over $H^*(\operatorname{Sym} C) \otimes H^*(C)$), $\operatorname{ch}(\pi_! \mathcal{E}_i(m\Lambda)) = t_i e^{-\eta_i}(mk - d_i - (g-1) - \theta_i)$. Knowing that for $\operatorname{ch}(\mathcal{X}) = te^{\pm\eta}(n \pm \theta)$, $\operatorname{ch}(\det \mathcal{X}) = te^{\pm n\eta} e^{\pm\theta}$, we obtain (16).

To prove (17), we can use the same argument, using that $\operatorname{ch}(\pi_! \mathcal{E}_i^*) = t_i^{-1} e^{\eta_i}(d_i - (g-1) - \theta_i)$. $\square$

Comparing the Chern characters of these three line bundles, we see that we may choose two parameters μ and ν as follows:

		ch
$\mathcal{L}_1^* = \widehat{\mathcal{L}}_{-1,0}$	$\det \mathcal{E}_p$	$\prod t_i e^{-\eta_i}$
$\mathcal{L}_{2,m}^* = \widehat{\mathcal{L}}_{-mk+g-1,-1}$	$\det(\pi_! \mathcal{E}(m\Lambda))$	$(\prod t_i e^{-\eta_i})^{mk-(g-1)} \cdot \left(\prod t_i^{d_i} e^{-d_i\eta_i+\theta_i}\right)^{-1}$
$\mathcal{L}_3^* = \widehat{\mathcal{L}}_{g-1,1}$	$\det(\pi_! \mathcal{E}^*)^*$	$(\prod t_i e^{-\eta_i})^{-(g-1)} \cdot \left(\prod t_i^{d_i} e^{-d_i\eta_i+\theta_i}\right)$
$\widehat{\mathcal{L}}_{-\mu,\nu}$	general case	$(\prod t_i e^{-\eta_i})^{\mu} \cdot \left(\prod t_i^{d_i} e^{-d_i\eta_i+\theta_i}\right)^{\nu}$

The Chern character calculation suggests that by introducing $\widehat{\mathcal{L}}_{-\mu,\nu} = \mathcal{L}_1^{\otimes(\mu-\nu)} \otimes \mathcal{L}_3^{\otimes\nu}$, the line bundle $\mathcal{L}_{2,m}^* = \widehat{\mathcal{L}}_{-mk+g-1,-1}$. We will see this proven in a later chapter.

Note that we can conclude from **Lemma 61** that the Verlinde polarisation is $\mathcal{L}_{\mathrm{Ver}} = \widehat{\mathcal{L}}_{d,r}$.

5.6. Calculating the localisation formula

We will calculate the Atiyah-Bott localisation formula (**Theorem 39**) for one connected component of the fixed point set $X \in \mathrm{Fix}(\widetilde{\mathcal{Q}})$ over the Quot scheme $\widetilde{\mathcal{Q}}$, in the non-fixed determinant case:

$$\int_X \frac{\mathrm{ch}_T(L)\,\mathrm{td}(TX)}{\mathrm{ab}_T(NX)}$$

choosing the general case $L = \widehat{\mathcal{L}}_{-\mu,\nu}$.

Let us plug in the values of these classes for Quot $\times$ C for one connected component of the fixed point set X:

$$\int_X \underbrace{\left(\prod_{i=1}^{r} t_i e^{-\eta_i}\right)^{\mu} \cdot \left(\prod_{i=1}^{r} t_i^{d_i} e^{-d_i\eta_i+\theta_i}\right)^{\nu}}_{\mathrm{ch}_T(L)} \cdot$$

$$\underbrace{\prod_{i=1}^{r}\left(\frac{1-e^{-\eta_i}}{\eta_i}\right)^{-d_i+(g-1)} \exp\left(\sum_{i=1}^{r}\left(\frac{e^{-\eta_i}}{1-e^{-\eta_i}} - \frac{1}{\eta_i}\right)\cdot\theta_i\right)}_{\mathrm{td}(TX)} \cdot$$

$$\underbrace{\prod_{i=1}^{r}\prod_{\substack{j=1\\ i\neq j}}^{n} (1-(t_i/t_j)e^{-\eta_i})^{-d_i+(g-1)} \cdot \prod_{i=1}^{r}\prod_{\substack{j=1\\ i\neq j}}^{r} (1-(t_i/t_j)e^{-\eta_i+\eta_j})^{d_i-d_j-(g-1)}}_{\text{part of } \mathrm{ab}_T(NX)} \cdot$$

$$\underbrace{\exp\left(\sum_{i=1}^{r}\sum_{\substack{j=1\\ i\neq j}}^{n} \frac{(t_i/t_j)e^{-\eta_i}}{1-(t_i/t_j)e^{-\eta_i}}\cdot\theta_i - \sum_{i=1}^{r}\sum_{\substack{j=1\\ i\neq j}}^{r} \frac{(t_i/t_j)e^{-\eta_i+\eta_j}}{1-(t_i/t_j)e^{-\eta_i+\eta_j}}\cdot(\theta_i+\theta_j-\theta_{ij})\right)}_{\text{part of } \mathrm{ab}_T(NX)}.$$

Henceforth I will only write the limits of the summation when it is for the index j, as i is always summed from 1 to r.

The sum of $(\theta_i + \theta_j - \theta_{ij})$ can be regroupped by $i < j$, and since

$$\frac{(t_i/t_j)e^{-\eta_i+\eta_j}}{1-(t_i/t_j)e^{-\eta_i+\eta_j}} + \frac{(t_j/t_i)e^{-\eta_j+\eta_i}}{1-(t_j/t_i)e^{-\eta_j+\eta_i}} = -1,$$

the exponential part can be rewritten as

$$\sum_i \left(\sum_{j\neq i}^{n} \frac{(t_i/t_j)e^{-\eta_i}}{1-(t_i/t_j)e^{-\eta_i}} + \frac{e^{-\eta_i}}{1-e^{-\eta_i}} - \frac{1}{\eta_i}\right)\cdot\theta_i + \sum_{i<j}^{r}(\theta_i+\theta_j-\theta_{ij}) =$$

$$\sum_i \left(\sum_{j \neq i}^{n} \frac{(t_i/t_j)e^{-\eta_i}}{1-(t_i/t_j)e^{-\eta_i}} + \frac{e^{-\eta_i}}{1-e^{-\eta_i}} - \frac{1}{\eta_i} + (r-1) \right) \cdot \theta_i - \sum_{i<j}^{r} \theta_{ij}.$$

This way the formula can be rewritten as

$$\int_X \prod_i t_i^{\mu+d_i\nu} \cdot e^{-\sum_i (\mu+d_i\nu)\eta_i + \nu\sum_i \theta_i}$$

$$\prod_i \left(\frac{1-e^{-\eta_i}}{\eta_i}\right)^{-d_i+(g-1)} \prod_{i\neq j}^{n} (1-(t_i/t_j)e^{-\eta_i})^{-d_i+(g-1)} \prod_{i\neq j}^{r} (1-(t_i/t_j)e^{-\eta_i+\eta_j})^{d_i-d_j-(g-1)}$$

$$\exp\left(\sum_i \left(\sum_{j\neq i}^{n} \frac{(t_i/t_j)e^{-\eta_i}}{1-(t_i/t_j)e^{-\eta_i}} + \frac{e^{-\eta_i}}{1-e^{-\eta_i}} - \frac{1}{\eta_i} + (r-1) \right) \cdot \theta_i - \sum_{i<j}^{r} \theta_{ij} \right).$$

By properly combining terms, we can get rid of the conditions $i \neq j$, giving the following result.

Proposition 69: In the non-fixed determinant case, taking a connected component of the fixed point set in the Quot scheme, $X_{I,(d_i)_i} \in \mathrm{Fix}(\widetilde{\mathcal{Q}})$, corresponding to the index set $I = \{1, \dots, r\}$ and degrees $\det \mathcal{E}_i = -d_i$, the Atiyah-Bott localisation integral over this connected component is

$$\int_X \prod_i t_i^{\mu+d_i\nu} \cdot e^{-\sum_i (\mu+d_i\nu)\eta_i + \nu\sum_i \theta_i}$$

$$\prod_i \eta_i^{d_i-(g-1)} \prod_{i,j}^{n} (1-(t_i/t_j)e^{-\eta_i})^{-d_i+(g-1)} \prod_{i\neq j}^{r} (1-(t_i/t_j)e^{-\eta_i+\eta_j})^{d_i-d_j-(g-1)}$$

$$\exp\left(\sum_i \left(\sum_{j=1}^{n} \frac{(t_i/t_j)e^{-\eta_i}}{1-(t_i/t_j)e^{-\eta_i}} - \frac{1}{\eta_i} + (r-1) \right) \cdot \theta_i - \sum_{i<j}^{r} \theta_{ij} \right).$$

5.7. Fixed determinant case

Up till now, we have considered the case when the determinant of E is not fixed. The Quot scheme of all degree $-d$, rank r subsheaves, which we denoted by $\widetilde{\mathcal{Q}}$, contains the Quot scheme of those subsheaves of a fixed determinant D, denoted by $\mathcal{Q}$, as a subvariety. Since the T action on the two spaces are compatible, $\mathrm{Fix}(\mathcal{Q}) = \mathrm{Fix}(\widetilde{\mathcal{Q}}) \cap \mathcal{Q}$. The localisation formula over $\mathcal{Q}$ is then equal to that of $\widetilde{\mathcal{Q}}$, but with an additional factor corresponding to the Poincaré dual of $\mathcal{Q}$.

Lemma 70:

$$\chi_T(L,t) = \sum_{X \subset \mathrm{Fix}(\widetilde{\mathcal{Q}})} \int_X \frac{\mathrm{ch}_T(L)\,\mathrm{td}(TX)}{\mathrm{ab}_T(N_{\widetilde{\mathcal{Q}}}X)} [\mathcal{Q} \cap X]^*.$$

Now let us restrict ourselves to rank $r = 2$ bundles. Such a bundle in the fixed point set is one that decomposes as a sum $E = E_1 \oplus E_2$. Its determinant, which is now fixed, becomes a product $D = \det E = \det E_1 \otimes \det E_2$. Under the determinant map, this becomes $[D] = [E_1] + [E_2] \in J$. This means that the fixed determinant part of the fixed point set is the preimage via the determinant map of the set of pairs $(\det E_1, \det E_2) = (a, b) \in J \times J$ such that $a - b = [D] \in J$. This is a translation of the anti-diagonal in $J \times J$.

This anti-diagonal is calculated in **Proposition 33**, and since $\mathcal{Q} \cap X = \overline{\Delta}$, the integration becomes

$$\int_X \frac{\mathrm{ch}_T(L)\,\mathrm{td}(TX)}{\mathrm{ab}_T(N_{\widetilde{\mathcal{Q}}}X)} \overline{\Delta}^* \quad \text{where} \quad \overline{\Delta}^* = \frac{1}{g!}(\theta_1 + \theta_2 + \theta_{12})^g.$$

5.8. The Hausel-Szenes formula over symmetric products

From now on, we will only consider the $r = 2$ case.

In the article of Hausel and Szenes ([**10**]), a consequence of theorem 8.4 is the following:

Proposition 71: Given polynomials A and B_{ij} in several variables,

$$\int A(\vec{\eta}) \exp(\mathrm{tr}(\mathbf{B}(\vec{\eta}) \cdot \mathbf{\Theta}^T)) = \operatorname*{Res}_{\vec{x}=0} A(\vec{x}) \cdot \frac{\det(\mathbf{D}(\vec{x}))^g}{\prod_i x_i^{d_i - (g-1)}}$$

for $\mathbf{D} = \mathbf{B} + \mathrm{diag}\left(\frac{1}{\vec{x}}\right)$, where $\mathbf{B}_{ij} = B_{ij}$ and $\mathbf{\Theta}_{ij} = \begin{cases} \frac{1}{2}\theta_{ij} & \text{if } i \neq j \\ \theta_i & \text{if } i = j \end{cases}$ are symmetric matrices, while $\vec{\eta} = \eta_i$ is a vector. Also, $\mathrm{diag}\left(\frac{1}{\vec{x}}\right)$ is the diagonal matrix with entries $\frac{1}{x_i}$. The notation $\operatorname*{Res}_{\vec{x}=0}$ is shorthand for the sequence $\operatorname*{Res}_{x_1=0} \ldots \operatorname*{Res}_{x_n=0}$.

In our case,

$$\mathbf{D} = \begin{bmatrix} \sum_{j=1}^n \frac{1}{(t_j/t_1)e^{\eta_1}-1} + 1 + \nu & -1 \\ -1 & \sum_{j=1}^n \frac{1}{(t_j/t_2)e^{\eta_2}-1} + 1 + \nu \end{bmatrix}.$$

To restrict this integral to the fixed determinant subset of the fixed point set, that is the anti-diagonal $\overline{\Delta}$, we need to multiply the integrand by $\overline{\Delta}^*$. In order to write

$$\int A(\vec{\eta})(\theta_i + \theta_j + \theta_{ij})^g \exp(\mathrm{tr}(\mathbf{B}(\vec{\eta}) \cdot \mathbf{\Theta}^T))$$

we will write $\mathbf{B}' = \mathbf{B} + \epsilon \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ in the place of $\mathbf{B}$, and derive it g times. Since $\det \mathbf{D}' = \det \mathbf{D} + \epsilon \cdot \tilde{D}$ where $\tilde{D} = D_{11} - D_{12} - D_{21} + D_{22}$, the right hand side becomes

$$g! \cdot \operatorname*{Res}_{\vec{x}=0} A(\vec{x}) \cdot \frac{\tilde{D}(x)^g}{\prod_i x_i^{d_i-(g-1)}}.$$

For our $\mathbf{D}$

$$\tilde{D} = \left(\sum_{j=1}^{n} \frac{1}{(t_j/t_1)e^{\eta_1} - 1} + 1 + \nu \right) + \left(\sum_{j=1}^{n} \frac{1}{(t_j/t_2)e^{\eta_2} - 1} + 1 + \nu \right) + 2 =$$

$$= \sum_{j=1}^{n} \frac{1}{(t_j/t_1)e^{\eta_1} - 1} + \sum_{j=1}^{n} \frac{1}{(t_j/t_2)e^{\eta_2} - 1} + 2\nu + 4.$$

Then, after multiplying the body of the integral by $\overline{\Delta}^*$, and regrouping the θ_i variables:

$$\int_X \frac{1}{g!}(\theta_1 + \theta_2 + \theta_{12})^g \cdot \prod_i (t_i e^{-\eta_i})^{\mu + d_i \nu}$$

$$\prod_i \eta_i^{d_i-(g-1)} \prod_{i,j}^{n} (1 - (t_i/t_j)e^{-\eta_i})^{-d_i+(g-1)} \prod_{i \neq j}^{2} (1 - (t_i/t_j)e^{-\eta_i+\eta_j})^{d_i-d_j-(g-1)}$$

$$\exp\left(\sum_i \left(\sum_{j=1}^{n} \frac{1}{(t_j/t_i)e^{\eta_i} - 1} - \frac{1}{\eta_i} + \nu + (2-1) \right) \cdot \theta_i - \sum_{i<j}^{r} \theta_{ij} \right) =$$

$$= \operatorname*{Res}_{x_1=0} \operatorname*{Res}_{x_2=0} \prod_i (t_i e^{-x_i})^{\mu + d_i \nu}$$

$$\prod_{i,j}^{n} (1 - (t_i/t_j)e^{-x_i})^{-d_i+(g-1)} \prod_{i \neq j}^{2} (1 - (t_i/t_j)e^{-x_i+x_j})^{d_i-d_j-(g-1)}$$

$$\left(\sum_{j=1}^{n} \frac{1}{(t_j/t_1)e^{x_1} - 1} + \sum_{j=1}^{n} \frac{1}{(t_j/t_2)e^{x_2} - 1} + 2\nu + 4 \right)^g \mathrm{d}x_2 \mathrm{d}x_1.$$

After changing variables $X_i = e^{-x_i}$, $\mathrm{d}x_i$ becomes $-\frac{\mathrm{d}X_i}{X_i}$, and the residue may be taken in $X_1 = Y_1 = 1$, giving the following result:

Proposition 72: In the fixed determinant case, taking a connected component of the fixed point set in the Quot scheme, $X \in \mathrm{Fix}(\mathcal{Q})$ a connected component, corresponding to the index set $I = \{1, \ldots, r\}$ and degrees $\det \mathcal{E}_i = -d_i$, the Atiyah-Bott localisation integral over this connected component is

$$
\begin{gathered}
\operatorname*{Res}_{X_1=1} \operatorname*{Res}_{X_2=1} \prod_i (t_i X_i)^{\mu + d_i \nu} \\
\prod_{i,j}^{n} (1 - (t_i/t_j) X_i)^{-d_i + (g-1)} \prod_{i \neq j}^{2} \left(1 - \frac{t_i X_i}{t_j X_j}\right)^{d_i - d_j - (g-1)} \\
\left(\sum_{j=1}^{n} \frac{1}{(t_j/t_1) X_1^{-1} - 1} + \sum_{j=1}^{n} \frac{1}{(t_j/t_2) X_2^{-1} - 1} + 2\nu + 4 \right)^{g} \frac{\mathrm{d}X_2}{X_2} \frac{\mathrm{d}X_1}{X_1}.
\end{gathered}
$$

5.9. Summation over the connected components

First, let us rewrite the terms $\prod_{i \neq j}^{2} \left(1 - \frac{t_i X_i}{t_j X_j}\right)^{d_i - d_j - (g-1)}$ as

$$
\prod_{i \neq j}^{2} (t_j X_j - t_i X_i)^{d_i - d_j - (g-1)} \cdot \prod_i (t_i X_i)^{-d + 2d_i + (g-1)}
$$

and combine the second factor with the first line; then rewrite the term $\prod_{i \neq j}^{2} (t_j X_j - t_i X_i)^{d_i - d_j - (g-1)}$ as $(-1)^{d + (g-1)} \cdot (t_1 X_1 - t_2 X_2)^{2(1-g)}$. Also note that in the last line, we can replace the terms $\sum_j ((t_j/t_1) X_1^{-1} - 1)^{-1}$ by $\sum_j (1 - (t_1/t_j) X_1)^{-1} - n$.

$$
\begin{gathered}
\mathcal{I}_{d_1, d_2}(t_1, t_2; t_3, \ldots, t_n) := \operatorname*{Res}_{X_1=1} \operatorname*{Res}_{X_2=1} \prod_{i=1}^{2} \left((t_i X_i)^{\mu - d + 2d_i + (g-1) + d_i \nu} \cdot X_i^{-1} \right) \\
\prod_{i,j}^{n} (1 - (t_i/t_j) X_i)^{-d_i + (g-1)} \cdot (-1)^{d + (g-1)} \cdot (t_1 X_1 - t_2 X_2)^{2(1-g)} \\
\left(\sum_{j=1}^{n} \frac{1}{1 - (t_1/t_j) X_1} + \sum_{j=1}^{n} \frac{1}{1 - (t_2/t_j) X_2} - 2n + 2\nu + 4 \right)^{g} \mathrm{d}X_2 \mathrm{d}X_1
\end{gathered}
$$

To calculate the contribution of all connected components, we have to sum these formulæ for all $d_1 + d_2 = d$ and for all choice of the subset $\{t_{i_1}, t_{i_2}\}$ within $\{t_1, \ldots, t_n\}$. One way to write this formally is

$$
\mathcal{I} = \sum_{d_1 + d_2 = d} \sum_{\substack{\sigma \in S_n \\ \sigma(1) < \sigma(2) \\ \sigma(3) < \ldots < \sigma(n)}} \mathcal{I}_{d_1, d_2}(t_{\sigma(1)}, t_{\sigma(2)}; t_{\sigma(3)}, \ldots, t_{\sigma(n)}).
$$

Let us fix σ and sum these for d_1 and d_2. Since $d_1 + d_2 = d$, we can introduce a single variable, and write $\delta := d_1$ and $d_2 = d - \delta$:

$$\sum_{\delta=0}^{d} \operatorname*{Res}_{X_1=1} \operatorname*{Res}_{X_2=1} (t_1 t_2 X_1 X_2)^{\mu-d+(g-1)} (t_1 X_1)^{2\delta+\delta\nu} (t_2 X_2)^{2d+d\nu-2\delta-\delta\nu} \frac{1}{X_1 X_2} \cdot$$
$$\left(\prod_{i,j}^{n} (1 - (t_i/t_j) X_i)\right)^{g-1} \cdot \prod_{i=1}^{n} (1 - (t_1/t_i) X_1)^{-\delta} \cdot \prod_{i=1}^{n} (1 - (t_2/t_i) X_2)^{\delta-d} \cdot$$
$$(-1)^{d+(g-1)} \cdot (t_1 X_1 - t_2 X_2)^{2(1-g)} \cdot$$
$$\left(\sum_{j=1}^{n} \frac{1}{1 - (t_1/t_j) X_1} + \sum_{j=1}^{n} \frac{1}{1 - (t_2/t_j) X_2} - 2n + 2\nu + 4\right)^{g} \mathrm{d}X_2 \mathrm{d}X_1.$$

Then regroup the factors according to whether they depend on δ or not:

$$\operatorname*{Res}_{X_1=1} \operatorname*{Res}_{X_2=1} (t_1 t_2 X_1 X_2)^{\mu-d+(g-1)} (t_2 X_2)^{2d+d\nu} \frac{1}{X_1 X_2} \cdot$$
$$\prod_{i,j}^{n} (1 - (t_i/t_j) X_i)^{g-1} \cdot \prod_{i=1}^{n} (1 - (t_2/t_i) X_2)^{-d} \cdot (-1)^{d+(g-1)} \cdot (t_1 X_1 - t_2 X_2)^{2(1-g)} \cdot$$
$$\left(\sum_{j=1}^{n} \frac{1}{1 - (t_1/t_j) X_1} + \sum_{j=1}^{n} \frac{1}{1 - (t_2/t_j) X_2} - 2n + 2\nu + 4\right)^{g} \cdot$$
$$\sum_{\delta=0}^{d} \left(\left(\frac{t_1 X_1}{t_2 X_2}\right)^{2+\nu} \cdot \prod_{i=1}^{n} \frac{1 - (t_2/t_i) X_2}{1 - (t_1/t_i) X_1} \right)^{\delta} \mathrm{d}X_2 \mathrm{d}X_1.$$

We can proceed to sum the geometric series in

$$\left(\frac{t_1 X_1}{t_2 X_2}\right)^{2+\nu} \cdot \prod_{i=1}^{n} \frac{1 - (t_2/t_i) X_2}{1 - (t_1/t_i) X_1}$$

giving:

$$\operatorname*{Res}_{X_1=1} \operatorname*{Res}_{X_2=1} (t_1 t_2 X_1 X_2)^{\mu-d+(g-1)} (t_2 X_2)^{2d+d\nu} \frac{1}{X_1 X_2} \cdot$$
$$\prod_{i=1}^{n} (1 - (t_1/t_i) X_1)^{g-1} \cdot \prod_{i=1}^{n} (1 - (t_2/t_i) X_2)^{g-1-d} \cdot (-1)^{d+(g-1)} \cdot (t_1 X_1 - t_2 X_2)^{2(1-g)} \cdot$$
$$\left(\sum_{j=1}^{n} \frac{1}{1 - (t_1/t_j) X_1} + \sum_{j=1}^{n} \frac{1}{1 - (t_2/t_j) X_2} - 2n + 2\nu + 4\right)^{g} \cdot$$
$$\frac{\left(\left(\frac{t_1 X_1}{t_2 X_2}\right)^{2+\nu} \cdot \prod_{i=1}^{n} \frac{1-(t_2/t_i) X_2}{1-(t_1/t_i) X_1} \right)^{d+1} - 1}{\left(\frac{t_1 X_1}{t_2 X_2}\right)^{2+\nu} \cdot \prod_{i=1}^{n} \frac{1-(t_2/t_i) X_2}{1-(t_1/t_i) X_1} - 1} \mathrm{d}X_2 \mathrm{d}X_1.$$

Let us consider the new term

$$\left(\left(\frac{t_1X_1}{t_2X_2}\right)^{2+\nu}\cdot\prod_{i=1}^{n}\frac{1-(t_2/t_i)X_2}{1-(t_1/t_i)X_1}\right)^{d+1}.$$

When we calculate the inner residue in $X_2 = 1$, we will be looking at the order of terms in some expansion of $1 - X_2$. Without actually calculating the expansion, we can see that in this term the smallest exponent is $(1 - X_2)^{d+1}$. However, combining all other factors in the formula, the smallest exponent there may appear is $-d - 1$, found in

$$\ldots\cdot(1-(t_2/t_2)X_2)^{g-1-d}\cdot\ldots\cdot\left(\ldots+\frac{1}{1-(t_2/t_2)X_2}+\ldots\right)^{g}\cdot\ldots$$

Therefore this factor can be eliminated.

This gives us the main result of the thesis:

Theorem 73: Consider the Quot scheme $\mathcal{Q}$ of fixed determinant subsheaves of rank $r = 2$, and fix the line bundle $L = \widehat{\mathcal{L}}_{-\mu,\nu}$. Then the Atiyah-Bott localisation integral over the complete fixed point set is

$$\chi(\widehat{\mathcal{L}}_{-\mu,\nu}) = \sum_{1\le i_1<i_2\le n}\operatorname*{Res}_{X_1=1}\operatorname*{Res}_{X_2=1}(-1)^{d+(g-1)}\cdot(t_{i_1}t_{i_2}X_1X_2)^{\mu-d+(g-1)}(t_{i_2}X_2)^{2d+d\nu}\frac{1}{X_1X_2}\cdot$$
$$\prod_{j=1}^{n}(1-(t_{i_1}/t_j)X_1)^{g-1}\cdot\prod_{j=1}^{n}(1-(t_{i_2}/t_j)X_2)^{g-1-d}\cdot(t_{i_1}X_1-t_{i_2}X_2)^{2(1-g)}\cdot$$
$$\left(\sum_{j=1}^{n}\frac{1}{1-(t_{i_1}/t_j)X_1}+\sum_{j=1}^{n}\frac{1}{1-(t_{i_2}/t_j)X_2}-2n+2\nu+4\right)^{g}\cdot$$
$$\left(1-\left(\frac{t_{i_1}X_1}{t_{i_2}X_2}\right)^{2+\nu}\cdot\prod_{j=1}^{n}\frac{1-(t_{i_2}/t_j)X_2}{1-(t_{i_1}/t_j)X_1}\right)^{-1}\mathrm{d}X_2\mathrm{d}X_1.$$

5.10. A special case: $n = 2$

In **[5]**, a Quot scheme is studied with parameters $r = n = 2$, which is a smooth space, since by **Corollary 11**, the dimension of the Zariski tangent space is constant throughout the space. We will give a more concrete expression of the previous integral for this case.

We may calculate the localisation for $n = 2$ for the SL-action. In that case, $t_1t_2 = 1$, so we can replace t_1 by t and t_2 by t^{-1}. To simplify the notation, we will also replace X_1

by X and X_2 by Y.

$$\begin{gathered}\operatorname*{Res}_{X=1}\operatorname*{Res}_{Y=1}(-1)^{d+(g-1)}\cdot(XY)^{\mu-d+(g-1)}(t^{-1}Y)^{2d+d\nu}\cdot\frac{1}{XY}\cdot\\ (1-X)^{g-1}\cdot(1-t^2X)^{g-1}\cdot(1-t^{-2}Y)^{g-1-d}\cdot(1-Y)^{g-1-d}\cdot(tX-t^{-1}Y)^{2(1-g)}\cdot\\ \left(\frac{1}{1-X}+\frac{1}{1-t^2X}+\frac{1}{1-t^{-2}Y}+\frac{1}{1-Y}+2\nu\right)^g\cdot\\ \left(1-\left(t^2\frac{X}{Y}\right)^{2+\nu}\cdot\frac{1-t^{-2}Y}{1-t^2X}\cdot\frac{1-Y}{1-X}\right)^{-1}\mathrm{d}Y\mathrm{d}X\end{gathered}$$

Then combine $(1-X)^g\cdot(1-t^2X)^g\cdot(1-t^{-2}Y)^g\cdot(1-Y)^g$ with the second to last line, to obtain the following form:

Proposition 74: Consider the Quot scheme of fixed determinant quotients $\mathcal{Q}$ with $r=n=2$. Then the Atiyah-Bott localisation integral over the complete fixed point set is

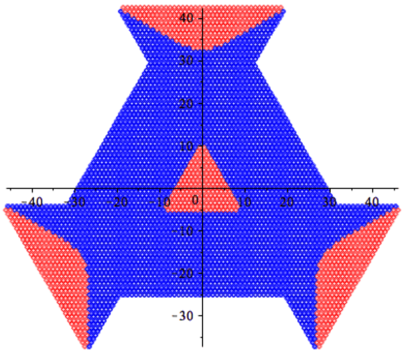
$$\begin{gathered}\operatorname*{Res}_{X=1}\operatorname*{Res}_{Y=1}(-1)^{d+(g-1)}\cdot t^{-2d-d\nu}X^{\mu-d+(g-1)-1}Y^{\mu+d+(g-1)+d\nu-1}\cdot\\ (1-X)^{-1}\cdot(1-t^2X)^{-1}\cdot(1-t^{-2}Y)^{-1-d}\cdot(1-Y)^{-1-d}\cdot(tX-t^{-1}Y)^{2(1-g)}\cdot\\ \big((1-X+1-t^2X)(1-t^{-2}Y)(1-Y)+(1-t^{-2}Y+1-Y)(1-t^2X)(1-X)+\\ +2\nu(1-t^{-2}Y)(1-Y)(1-t^2X)(1-X)\big)^g\cdot\\ \left(1-\left(t^2\frac{X}{Y}\right)^{2+\nu}\cdot\frac{1-t^{-2}Y}{1-t^2X}\cdot\frac{1-Y}{1-X}\right)^{-1}\mathrm{d}Y\mathrm{d}X.\end{gathered}$$

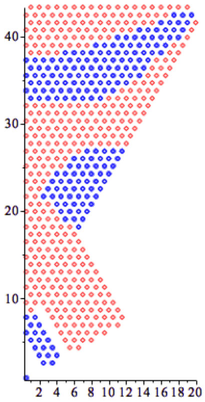
5.11. Computer calculation for $n=3$

Together with András Szenes and Zsolt Szilágyi, we have written a Maple program that calculates the formula in **Theorem 73**. We have chosen the parameters $r=2$, $n=3$, $g=2$, $d=7$, and considered the Verlinde direction $(5\kappa,-2\kappa)$ (**Lemma 61**) in coordinates $\mathcal{L}_1$ and $\mathcal{L}_3$ (note, this is not the basis we chose for calculating the Atiyah-Bott localisation). We obtain a homogeneous Laurent polynomial in three variables t_1, t_2, t_3, corresponding to the diagonal entries of SL(3).

In the first image ($\kappa=7$), we show the signs of the coefficients of the monomials (blue for positive, red for negative, empty if zero). The coordinates are t_1/t_2 and t_2/t_3, skewed by rotating the vertical axis 30° to the left.

The multiplicities of the irreducible representations of the SL(3) action are shown in the second image. These can be derived using the polynomial in **Theorem 73** via **Theorem 34**, by multiplying the polynomial with the Laurent polynomial (7) and taking the coefficients in the dominant chamber.





By taking the invariant part of the SL(3) representation — appearing as the origin in the diagram — we were able to numerically verify Verlinde's formula for these parameters.

6. Calculation on the Grassmannian

6.1. Introduction

To verify **Conjecture 63**, we need to calculate $\chi(\mathcal{L}_{\mathcal{P}})^{\mathrm{PGL}(n)}$ (where $\chi(L)^{\mathrm{PGL}(n)}$ is defined as $\sum_i (-1)^i H^i(\mathcal{L})^{\mathrm{PGL}(n)}$) and compare it to $\chi(\mathcal{L}_{\mathcal{Q}})^{\mathrm{PGL}(n)}$ for a certain pair of line bundles $\mathcal{L}_{\mathcal{P}}$ and $\mathcal{L}_{\mathcal{Q}}$, isomorphic over $\mathcal{Q} \cap \mathcal{P}$. As we have seen in **Theorem 62**, $\chi(\mathcal{L}_{\mathcal{P}})^{\mathrm{PGL}(n)} = \chi(\mathcal{L}_{\mathcal{G}r})$ when $\mathcal{L}_{\mathcal{P}}$ is ample, so we will calculate the Euler characteristic of line bundles over $\mathcal{G}r$.

To do this, in this chapter we will review some results on the Grassmannian, mostly from **[26]**, and compare them with the equivariant integrals of the previous chapter.

6.2. Basics

Consider the algebraic curve C of genus $g \geq 2$ and let $1, e_1, \ldots, e_{2g}, \sigma$ be a basis of $H(C)$ with $1 \in H^0(C)$, $\sigma \in H^2(C)$, $e_i \in H^1(C)$ and $e_i e_j = \frac{j-i}{g}\delta_{|i-j|-g}\sigma$. We will denote the moduli space of rank $r = 2$, degree d vector bundles of determinant Δ by $\mathcal{M}^g_\Delta = \mathcal{M}_\Delta$. Recall the notation $\mathcal{U}$ for the rank $r = 2$, degree d universal bundle over $\mathcal{M}_\Delta \times C$ and the projection $\pi\colon \bullet \times C \to \bullet$.

Let $\mathcal{W} \subset \mathrm{End}(\mathcal{U})$ be the subbundle of traceless endomorphisms, and assume that $d \geq 4g - 3$. Let us introduce the classes $\alpha \in H^2(\mathcal{M}_\Delta)$, $\beta \in H^4(\mathcal{M}_\Delta)$, $\psi_i \in H^3(\mathcal{M}_\Delta)$ in the Künneth decomposition of the second Chern class of $\mathcal{W}$:

$$c_2(\mathcal{W}) = 2\alpha \otimes \sigma + 4\sum_{i=1}^{2g} \psi_i \otimes e_i - \beta \otimes 1 \in H^4(\mathcal{M}_\Delta \times C)$$

and the abbreviated classes

$$\Psi = 4\sum_{i=1}^{2g} \psi_i \otimes e_i \qquad \gamma = -2\sum_{i=1}^{2g} \psi_i \psi_{i+g} \in H^6(\mathcal{M}_\Delta).$$

Note that $\Psi^2 = \gamma \otimes \sigma$.

6.3. Chern classes of the universal bundle

Once again, we will look at the universal bundle $\mathcal{U}$. Recall from **Lemma 28** and **Lemma 42** that we may choose the normalised universal bundle $\mathcal{V}$, for which $c_1(\mathcal{V}_p)$ generates $\operatorname{Pic}\mathcal{M}$

$$(g-1)c_1(\mathcal{V}_p) = c_1(\pi_! \mathcal{V}).$$

Consider the first Chern class of the universal bundle:

$$c_1(\mathcal{U}) = d \otimes \sigma + y \otimes 1 \in H^2(\mathcal{M}_\Delta \times C)$$

with $y = \alpha$ in the case of $\mathcal{U} = \mathcal{V}$ (note that $y = c_1(\mathcal{U}_p)$); also its total Chern class:

$$c(\mathcal{U}) = 1 + c_1(\mathcal{U}) + \frac{1}{4}c_1(\mathcal{U})^2 + \frac{1}{4}c_2(\mathcal{W}).$$

Introducing the formal parameter $\sqrt{\beta}$, we can formally decompose this formula as

$$c(\mathcal{U}) = \left(1 + \frac{c_1(\mathcal{U}) + \sqrt{\beta} \otimes 1}{2}\right) \cdot \left(1 + \frac{c_1(\mathcal{U}) - \sqrt{\beta} \otimes 1}{2}\right) + \delta \qquad \delta = \frac{1}{2}\alpha \otimes \sigma + \Psi.$$

Lemma 75: ([**26**]) Let ξ be a vector bundle over X with the (formal) decomposition $c(\xi) = (1 + x_1)(1 + x_2) + \delta$ with x_1, $x_2 \in H^2(X)$ and $\delta \in H^4(X)$ and $\delta^3 = 0$. Then

$$\operatorname{ch}(\xi) = e^{x_1} + e^{x_2} + \frac{e^{x_1} - e^{x_2}}{x_1 - x_2}\delta - \left(\frac{e^{x_1} - e^{x_2}}{(x_1 - x_2)^3} - \frac{e^{x_1} + e^{x_2}}{2(x_1 - x_2)^2}\right)\delta^2.$$

Then if ξ has the decomposition $c(\xi) = (1 + x_1 + x_2)(1 + x_1 - x_2) + \delta$, the formula becomes.

$$\operatorname{ch}(\xi) = e^{x_1}\left(2\cosh x_2 - \frac{\sinh x_2}{x_2}\delta - \left(\frac{\sinh x_2}{4x_2^3} - \frac{\cosh x_2}{4x_2^2}\right)\delta^2\right)$$

In particular, for $\xi = \mathcal{U}$,

$$\operatorname{ch}(\mathcal{U}) = e^{\frac{c_1(\mathcal{U})}{2}} \cdot$$
$$\left(2\cosh(\sqrt{\beta}/2) - \frac{\sinh(\sqrt{\beta}/2)}{\sqrt{\beta}/2}\delta - \left(\frac{\sinh(\sqrt{\beta}/2)}{\beta\sqrt{\beta}/2} - \frac{\cosh(\sqrt{\beta}/2)}{\sqrt{\beta}}\right)\gamma \otimes \sigma\right)$$

6.4. Functors of the universal bundle

One can calculate the total Chern character of $\pi_! \mathcal{U}$ via the Grothendieck-Riemann-Roch formula

$$\operatorname{ch}(\pi_! \mathcal{U}) = \int_C \operatorname{ch}(\mathcal{U}) \operatorname{td}(TC) =$$
$$e^{\frac{y}{2}}\left(\left(d + 2 - 2g + \frac{\gamma}{\beta}\right)\cosh(\sqrt{\beta}/2) - \left(\frac{\alpha}{2} + \frac{\gamma}{\beta}\right)\frac{\sinh(\sqrt{\beta}/2)}{\sqrt{\beta}/2}\right)$$

or, by introducing the notation $\mathrm{ch}(\xi,t)=\sum e^{t\theta_i}$ when θ_i are the Chern roots of ξ, and $\gamma^*=2\gamma+\alpha\beta$,

$$\mathrm{ch}(\pi_!\mathcal{U},t)=e^{\frac{y}{2}t}\left(\left(d+2-2g+\frac{\gamma^*}{2\beta}t-\frac{\alpha}{2}t\right)\cosh(\sqrt{\beta}t/2)-\frac{\gamma^*}{2\beta}\frac{\sinh(\sqrt{\beta}t/2)}{\sqrt{\beta}/2}\right)$$

Then its determinant has Chern character

$$\mathrm{ch}(\det(\pi_!\mathcal{U}),t)=\exp\left(\frac{y(d+2-2g)-\alpha}{2}t\right)$$

On the other hand,

$$\mathrm{ch}(\mathcal{U}_p,t)=2e^{\frac{y}{2}t}\cosh(\sqrt{\beta}t/2)$$

and so

$$\mathrm{ch}(\det(\mathcal{U}_p),t)=e^{yt}$$

6.5. Push-forward to the Grassmannian bundle

Proposition 76: Let $\xi\to X$ be a complex vector bundle of rank n. Consider the Grassmannian bundle $f\colon \mathrm{Gr}(n,\xi)\to X$. Let $S(\xi)\to\mathrm{Gr}(n,\xi)$ be the tautological bundle and denote $\mathcal{L}(\xi)=\det(S(\xi)^*)$. Then

$$\mathrm{ch}(f_!(\mathcal{L}(\xi)^{\otimes k}))=\mathrm{Schur}_{(k^n)}(e^{-\nu_1},\dots,e^{-\nu_n})$$

where $\mathrm{Schur}_{(k^n)}$ is the *Schur polynomial* associated with partition $(k,\dots,k)$ (n times) and $\nu_1,\dots,\nu_n$ are the Chern roots of $V\to X$.

Lemma 77: The degree zero and degree one parts of the following polynomial are

$$\mathrm{Schur}_{(k^n)}(e^{-\nu_1},\dots,e^{-\nu_n})=D-Dn\frac{k}{n}\nu+\dots$$

for some D and for $\nu:=\sum_{i=1}^n\nu_i$.

Proof: Since we are not interested in higher degree terms, we may assume that $\nu_i\nu_j=0$ for all ν_i and ν_j. Then the Schur polynomial takes the form

$$\begin{vmatrix} h_k & h_{k+1} & \dots & h_{k+n-1}\\ h_{k-1} & h_k & \dots & h_{k+n-2}\\ \vdots & \vdots & & \vdots\\ h_{k+n-1} & h_{k+n-2} & \dots & h_k \end{vmatrix}$$

where

$$h_\lambda = \binom{\lambda+n-1}{\lambda} - \binom{\lambda+n-1}{\lambda-1}\nu$$

or $h_\lambda = a_\lambda\left(1-\frac{\lambda}{n}\nu\right)$ with the shorthand notation $a_\lambda = \binom{\lambda+n-1}{\lambda}$.

To calculate the determinant of the above matrix A, we may consider a permutation $\sigma \in S_n$ and define $L^\sigma := \operatorname{sgn}(\sigma)\prod_{i=1}^n A_{\sigma(i),i}$, a summand in $\det A$. Since $\nu^2 = 0$, and each term h_λ decomposes as $a_\lambda - \frac{\lambda}{n}a_\lambda\nu$, instead of considering all the expansions in L^σ, we only have to treat those that contain a single ν.

Denote by L_0^σ the same summand with ν replaced by 0 (h_λ replaced by a_λ). Consider $A_{ji} = h_{k+i-j}$, and write

$$L^\sigma = L_0^\sigma\left(1-\nu\cdot\sum_{i=1}^n\frac{k+i-\sigma(i)}{n}\right) = L_0^\sigma\cdot\left(1-n\frac{k}{n}\nu\right)$$

where each index i corresponds to the decomposition with ν chosen from column i. Finally, when summed for all σ, we obtain $D\cdot\left(1-n\frac{k}{n}\nu\right)$, where D is the determinant of the A matrix with all h_λ replaced by a_λ. □

6.6. Integration on $\mathcal{M}_\Delta$

We want to calculate the formula $\int_{\mathcal{M}_\Delta} F(\beta)\alpha^m(\gamma^*)^p e^\alpha$. Let us first review some results of Zagier.

Zagier introduced the class $\gamma^* := 2\gamma + \alpha\beta \in H^6(\mathcal{M}_\Delta)$. Let $R = \mathbb{Q}[\alpha,\beta,\gamma] = \mathbb{Q}[\alpha,\beta,\gamma^*]$. For every g let us define the linear function $E_g\colon R \to \mathbb{Q}$ by specifying its values on the monomials

$$E_g[\alpha^m\beta^n\gamma^p] := \int_{\mathcal{M}_\Delta^g}\alpha^m\beta^n\gamma^p$$

and summing for g to obtain $\mathcal{E}_T\colon R \to \mathbb{Q}[T]$:

$$\mathcal{E}_T[x] := \sum_{g=1}^\infty E_g[x]\left(-\frac{1}{4}T\right)^{g-1}$$

which is a polynomial, since for large enough g the integral $E_g[x]$ is zero.

Proposition 78: ([**26**], Proposition 2) Let F, u, $w \in \mathbb{Q}[[\beta]]$ such that $u(0) \neq 0$. Then defining the power series Q through its inverse $Q^{-1}(\beta) = \frac{\beta}{u(\beta)}$, we obtain

$$E_T\left[F(\beta)e^{u(\beta)\alpha+w(\beta)\gamma^*}\right] = \left.\frac{\sqrt{\beta}F(\beta)Q'(T)}{\sinh[\sqrt{\beta}(u(\beta)+\beta w(\beta))]}\right|_{\beta=Q(T)}.$$

Or, in residual form:

$$\operatorname*{Res}_{S=0} \frac{(-4)^{g-1}F(S^2)u(S^2)^g}{S^{2g-2}\sinh[S(u(S^2)+S^2w(S^2))]}\mathrm{d}S$$

Proposition 79: ([**26**], Proposition 3) Let F, h, u, $w \in \mathbb{Q}[[\beta]]$ such that $h(0)u(0) \neq 0$. Then defining the power series q through its inverse $q^{-1} = \frac{\beta}{u(\beta)h(\beta)}$, we obtain

$$E_T\left[F(\beta)h(\beta)e^{u(\beta)\alpha+w(\beta)\gamma^*}\right] = \left.\frac{\sqrt{\beta}F(\beta)q'(T)}{\sinh[\sqrt{\beta}(u(\beta)+\beta w(\beta))]}\right|_{\beta=q(T)}.$$

Or, in residual form:

$$\operatorname*{Res}_{S=0} \frac{(-4)^{g-1}F(S^2)u(S^2)^g h(S^2)^g}{S^{2g-2}\sinh[S(u(S^2)+S^2w(S^2))]}\mathrm{d}S$$

To calculate $\int_{\mathcal{M}_\Delta} F(\beta)\alpha^m(\gamma^*)^p e^\alpha$, consider

$$\int_{\mathcal{M}_\Delta} F(\beta)\alpha^m(\gamma^*)^p e^\alpha = m!\ p!\ \mathrm{Coeff}_{x^m,z^p}\int_{\mathcal{M}_\Delta} F(\beta)e^{(1+x)\alpha+z\gamma^*},$$

so replacing $u(\beta) = 1+x$ and $w(\beta) = z$ in the first proposition gives

$$\int_{\mathcal{M}_\Delta} F(\beta)e^{(1+x)\alpha+z\gamma^*} = \operatorname*{Res}_{S=0} \frac{(-4)^{g-1}F(S^2)(1+x)^g}{S^{2g-2}\sinh[S+xS+zS^3]}\mathrm{d}S$$

thus

$$\int_{\mathcal{M}_\Delta} F(\beta)\alpha^m(\gamma^*)^p e^\alpha = m!\ p!\ \operatorname*{Res}_{x=0}\operatorname*{Res}_{z=0}\operatorname*{Res}_{S=0} \frac{1}{x^{m+1}z^{p+1}}\frac{(-4)^{g-1}F(S^2)(1+x)^g}{S^{2g-2}\sinh[S+xS+zS^3]}\mathrm{d}S\mathrm{d}z\mathrm{d}x$$

6.7. Line bundles

Proposition 80: Denote by $f\colon \mathcal{G}r \to \mathcal{M}$ the projection. Then $\det f_!(\mathcal{L}^{\otimes K} \otimes \mathcal{O}(M))$ is positive over $\mathcal{M}$ if $2K\chi > nM(\chi-1)$, where $\chi = d + 2(1-g)$ and $\mathcal{L} = \mathcal{L}_{\mathrm{Ver}}$.

Proof: Let us consider a general line bundle $\mathcal{B} := \mathcal{L}^{\otimes K} \otimes \mathcal{O}(M)$. By Grothendieck-Riemann-Roch, we know that $\operatorname{ch} f_!(\mathcal{B}) = f_*(\operatorname{ch}\mathcal{B}\operatorname{td}TF)$ where $f\colon \mathcal{G}r \to \mathcal{M}$, and $TG = T_V\mathcal{G}r$ is the vertical tangent bundle via the projection. Since $\mathcal{L}$ is a pullback,

$$f_*(\operatorname{ch}\mathcal{L}^{\otimes K}\operatorname{ch}\mathcal{O}(M)\operatorname{td}TG) = \operatorname{ch}\mathcal{L}^{\otimes K} f_*(\operatorname{ch}\mathcal{O}(M)\operatorname{td}TG).$$

The line bundle $\det f_!\mathcal{B}$ is determined by the degree 1 term in $\operatorname{ch} f_!(\mathcal{B})$.

Recall from **Proposition 76** and **Lemma 77** that

$$f_*(\operatorname{ch}\mathcal{O}(M)\operatorname{td}T) = D - Dn\frac{M}{\chi}\nu + \dots$$

for some D and for $\nu := \sum_{i=1}^n \nu_i$ the Chern roots of $\mathcal{V}$. We know that $\operatorname{ch}\pi_*\mathcal{V} = d+2-2g+\frac{d+1-2g}{2}\alpha+\dots$, hence $\nu = \frac{d+1-2g}{2}\alpha = \frac{\chi-1}{2}\alpha$. Therefore

$$\operatorname{ch} f_!(\mathcal{B}) = (\operatorname{ch}\mathcal{L})^K f_*(\operatorname{ch}\mathcal{O}(M)\operatorname{td}TG) =$$
$$= (1+\alpha K)D\left(1 - n\frac{M}{\chi}\frac{\chi-1}{2}\alpha+\dots\right) = D + D\alpha\left(K - nM\frac{\chi-1}{2\chi}\right)+\dots$$

Since the constant term must be positive, $\operatorname{ch} f_!(\mathcal{B})$ is positive if $2K\chi > nM(\chi-1)$. □

Corollary 81: If $\mathcal{L}^{\otimes K}\otimes\mathcal{O}(M)$ is ample over $\mathcal{G}r$, then its pushforward to $\mathcal{M}$ must be positive, hence $2K\chi > nM(\chi-1)$, where $\chi = d+2(1-g)$.

Lemma 82: The line bundle $\mathcal{L}_{2,m}$, defined as $\mathcal{O}_{\mathcal{G}r}(\chi_m/n)\otimes\det(\pi_*\mathcal{U}^*(m\Lambda))$ with $\chi_m = mkr-d+(1-g)r$, can be decomposed in $\operatorname{Pic}\mathcal{G}r$ as $[\mathcal{L}_{2,m}] = (mk+2(1-g))[\mathcal{L}_1]-[\mathcal{L}_3]$.

Proof: It is a simple calculation that the powers of $\mathcal{O}_{\mathcal{G}r}(1)$ add up, hence it suffices to verify that $c_1(\pi_*\mathcal{U}^*(m\Lambda)) = (mk+2(1-g))c_1(\pi_*\mathcal{U})_p - c_1(\pi_*\mathcal{U})$, as $\operatorname{Pic}\mathcal{M}$ is generated by a single element. This is a simple consequence of the Grothendieck-Riemann-Roch theorem. □

Recall the notation $\mathcal{L}_{\mathrm{Taut}}$ for the line bundle $\mathcal{O}_{\mathcal{P}}(n)$. Recall also from **Lemma 61** that $[\mathcal{L}_{\mathrm{Ver}}] = \chi[\mathcal{L}_1]-2[\mathcal{L}_3]$ and $[\mathcal{L}_{\mathrm{Taut}}] = n\frac{1-\chi}{2}[\mathcal{L}_1]+n[\mathcal{L}_3]$, giving $[\mathcal{L}_1] = [\mathcal{L}_{\mathrm{Ver}}]+\frac{2}{n}[\mathcal{L}_{\mathrm{Taut}}]$ and $[\mathcal{L}_3] = \frac{\chi}{n}[\mathcal{L}_{\mathrm{Taut}}]+\frac{\chi-1}{2}[\mathcal{L}_{\mathrm{Ver}}]$.

Recall from **Theorem 17** that over $\mathcal{Q}$, $\mathcal{L}_{2,m}$ is ample whenever $m > \max(1-d-2(1-g-k), 2g-2)$. Recall that C can be embedded into $\operatorname{Proj}\mathcal{K}^{\otimes n}$ provided that $n\geq 3$ for $g=2$, $n\geq 2$ for hyper-elliptic curves and $n\geq 1$ otherwise. Then $k = \deg\Lambda = 2n(g-1)$. Therefore:

Lemma 83: Let us fix a parameter $k = 2n(g-1)$ such that C can be embedded into $\operatorname{Proj}\mathcal{K}^{\otimes n}$. Then whenever $m > \max(1-d-2(1-g-k), 2g-2)$, the line bundles

$$(mk+2(1-g))[\mathcal{L}_1]-[\mathcal{L}_3]$$

are ample over $\mathcal{Q}$.

On the other hand, if a line bundle is ample on $\mathcal{G}r$, then its push forward must be positive on $\mathcal{M}$ by **Corollary 81**, giving the following condition:

Lemma 84: If the line bundle $A[\mathcal{L}_1] + B[\mathcal{L}_3]$ over $\mathcal{G}r$ is ample, then $A > 0$.

6.8. Computer calculation

We have checked together with András Szenes and Zsolt Szilágyi in Maple several values for the case $g = 2$, $r = 2$, $d = 7$, $\chi = 5$.

Over the Quot scheme, we evaluate the localisation formula for different equivariant line bundles, and take the SL(3)-invariant part. We also calculate the corresponding integral over the space $\mathcal{G}r$ for the corresponding line bundles. We obtain two tables with similar entries. The left-top corner starts at $\mathcal{O}$, while the horizontal coordinates are the negative powers of $\mathcal{L}_3$, and the vertical ones are the positive powers of $\mathcal{L}_1$.

According to **Conjecture 63**, we expect that in this diagram, in the ample chamber of the space $\mathcal{P}$, the values will match in the two diagrams. This chamber is delimited by the Verlinde direction, which is $(\chi, -r) = (5, -2)$ by **Lemma 61**, marked like **this**. The values that match are marked like this, and the first entries in each column where the two tables diverge is marked like *this*.

We can verify two important properties from this diagram: the ample cone must contain only positive line bundles. By **Lemma 84**, these are the elements where $\mathcal{L}_1$ has positive powers. The matching part is contained within.

Furthermore, the line bundles that are ample over $\mathcal{Q}$ may not appear inside the ample cone of $\mathcal{P}$. Recall from **Lemma 83** that for $m > \max(2(k-2), 4)$, $(mk-2, -1)$ is in the ample cone of $\mathcal{Q}$. For $g = 2$, we may choose $k = \nu \cdot 2(g-1) = 2\nu$ with $\nu \geq 3$. Hence for $m > \max(4(\nu-1), 4)$, $(2\nu m - 2, -1)$ is in the ample cone of $\mathcal{Q}$. These appear outside the matching part in fact for any $\nu \geq 1$.

	$\mathcal{O}$	$\mathcal{L}_3^*$	$(\mathcal{L}_3^*)^{\otimes 2}$	$(\mathcal{L}_3^*)^{\otimes 3}$	$(\mathcal{L}_3^*)^{\otimes 4}$	$(\mathcal{L}_3^*)^{\otimes 5}$	$(\mathcal{L}_3^*)^{\otimes 6}$	$(\mathcal{L}_3^*)^{\otimes 7}$	$(\mathcal{L}_3^*)^{\otimes 8}$
$\mathcal{O}$	**1**	69	213529	10856560	159439822	1241803165	6557484419	26606035859	89198380777
$\mathcal{L}_1$	20	0	63176	5907352	107368640	932136700	5252743496	22243653152	76845639108
$\mathcal{L}_1^{\otimes 2}$	*175*	0	4426	2612533	65628849	658693900	4031540106	18003106996	64523990726
$\mathcal{L}_1^{\otimes 3}$	980	134	−3410	778850	35465716	435975812	2964529304	14123682704	52884841526
$\mathcal{L}_1^{\otimes 4}$	4116	2945	0	3225	15764320	265866978	2078455904	10726448130	42310813310
$\mathcal{L}_1^{\otimes 5}$	14112	21504	**44**	−131516	4492976	144121160	1374427016	7854964876	32994078572
$\mathcal{L}_1^{\otimes 6}$	41580	*96546*	2905	24671	−603676	63609755	839858463	5505154259	24998102805
$\mathcal{L}_1^{\otimes 7}$	108900	326018	34944	0	−1667736	16156260	454824568	3645547994	18303712570
$\mathcal{L}_1^{\otimes 8}$	259545	911050	208332	1275	−536699	−6453911	196092380	2228515821	12841339875
$\mathcal{L}_1^{\otimes 9}$	572572	2223296	827640	27480	0	−11884248	39559728	1198077384	8509439520
$\mathcal{L}_1^{\otimes 10}$	1184183	4899183	*2553188*	218484	**231**	−6841516	−38244736	494967109	5188281166
$\mathcal{L}_1^{\otimes 11}$	2318680	9967382	6632206	1053360	14070	3242560	−59153188	59925098	2749387994
$\mathcal{L}_1^{\otimes 12}$	4331600	19016613	15209885	3729583	162736	0	−42581434	−164069710	1061970532
$\mathcal{L}_1^{\otimes 13}$	7768320	34411736	31733560	10670660	963072	4504	−4882024	−230138556	−2597460
$\mathcal{L}_1^{\otimes 14}$	13441968	59566962	61464766	*26215360*	3907695	92335	0	−186133127	−567489775
$\mathcal{L}_1^{\otimes 15}$	22535064	99285942	112115742	57506512	12399530	711816	**670**	−73454698	−747803490
$\mathcal{L}_1^{\otimes 16}$	36729945	160179458	194629127	115604151	33040280	3377880	38915	73887737	−648301397
$\mathcal{L}_1^{\otimes 17}$	58373700	251172448	324119496	216849184	77183808	11923340	435176	0	−361420296
$\mathcal{L}_1^{\otimes 18}$	90684055	384113147	520997225	384505475	*162921795*	34412287	2516148	10901	34384350
$\mathcal{L}_1^{\otimes 19}$	138003404	574498218	812296602	650710310	317510180	85753440	10058730	216410	0
$\mathcal{L}_1^{\otimes 20}$	206108980	842328881	1233231571	1058765600	580278499	190955016	31671783	1628508	**1469**

Values of $\chi(\mathcal{L}_{\mathcal{Q}})^{\mathrm{inv}}$

	$\mathcal{O}$	$\mathcal{L}_3^*$	$(\mathcal{L}_3^*)^{\otimes 2}$	$(\mathcal{L}_3^*)^{\otimes 3}$	$(\mathcal{L}_3^*)^{\otimes 4}$	$(\mathcal{L}_3^*)^{\otimes 5}$	$(\mathcal{L}_3^*)^{\otimes 6}$	$(\mathcal{L}_3^*)^{\otimes 7}$	$(\mathcal{L}_3^*)^{\otimes 8}$
$\mathcal{O}$	**1**	1	−6804	−518661	−8302392	−65033144	−335815389	−1318191424	−4259587371
$\mathcal{L}_1$	20	0	−320	−174240	−4585920	−45733380	−272259000	−1172727520	−4042746540
$\mathcal{L}_1^{\otimes 2}$	*−1120*	0	61	−28908	−1834833	−25683345	−183996780	−890803179	−3327370288
$\mathcal{L}_1^{\otimes 3}$	−20232	134	0	1120	−493922	−11735712	−107652006	−603274230	−2482916480
$\mathcal{L}_1^{\otimes 4}$	−161535	2945	0	1055	−51480	−4145960	−54404505	−368218500	−1706216595
$\mathcal{L}_1^{\otimes 5}$	−847704	21504	**44**	44	20832	−922064	−22930416	−201244296	−1082475576
$\mathcal{L}_1^{\otimes 6}$	−3396120	*83160*	2905	0	9095	30855	−7261040	−96067895	−629694000
$\mathcal{L}_1^{\otimes 7}$	−11236320	179322	34944	0	914	127080	−1048138	−37668906	−329820920
$\mathcal{L}_1^{\otimes 8}$	−32200839	46137	208332	1275	0	51296	564707	−9973968	−149516675
$\mathcal{L}_1^{\otimes 9}$	−82483940	−1430720	827640	27480	0	8260	526680	313040	−53002020
$\mathcal{L}_1^{\otimes 10}$	−193051144	−7511688	*2496637*	218484	**231**	231	217812	2422277	−9093096
$\mathcal{L}_1^{\otimes 11}$	−419497000	−26067810	6107920	1053360	14070	0	47490	1738770	5828960
$\mathcal{L}_1^{\otimes 12}$	−856589175	−73852471	12498672	3729583	162736	0	3639	754812	7489573
$\mathcal{L}_1^{\otimes 13}$	−1659003696	−183414880	21430404	10670660	963072	4504	0	203784	4907760
$\mathcal{L}_1^{\otimes 14}$	−3070049040	−413712000	29398985	*26053520*	3907695	92335	0	26985	2245320
$\mathcal{L}_1^{\otimes 15}$	−5460495744	−865723950	25603600	56139984	12399530	711816	**670**	670	711144
$\mathcal{L}_1^{\otimes 16}$	−9379973295	−1704652335	−14800500	109063515	33040280	3377880	38915	0	133765
$\mathcal{L}_1^{\otimes 17}$	−15623767260	−3190581504	−141270480	193563216	77183808	11923340	435176	0	9316
$\mathcal{L}_1^{\otimes 18}$	−25318246928	−5719809392	−443626659	315971700	*162550719*	34412287	2516148	10901	0
$\mathcal{L}_1^{\otimes 19}$	−40028577160	−9879407370	−1074628640	474552000	314559630	85753440	10058730	216410	0
$\mathcal{L}_1^{\otimes 20}$	−61892817679	−16517947407	−2281328192	650041119	566863704	190955016	31671783	1628508	**1469**

Values of $\chi(\mathcal{L}_{\mathcal{G}_r})$

References

[1] M. F. Atiyah: Vector Bundles over an Elliptic Curve, *Proc. London Math. Soc* (1957)

[2] M. F. Atiyah, Raoul Bott: The Yang-Mills Equations over Riemann Surfaces, *Phil. Trans. R. Soc. Lond. A*, **308** (1983), 523–615

[3] M. F. Atiyah, Raoul Bott: A Lefschetz Fixed Point Formula for Elliptic Differential Operators, *Bull. Am. Math. Soc.*, **72** (1966), 245–250

[4] Aaron Bertram, Georgios Daskalopoulos, Richard Wentworth: Gromov Invariants for Holomorphic Maps from Riemann Surfaces to Grassmannians, `arxiv:alg-geom/9306005v2`

[5] Emili Bifet, Franco Ghione, Maurizio Letizia: On the Abel-Jacobi map for divisors of higher rank on a curve, `arxiv:alg-geom/9203004v1`

[6] Raoul Bott, Loring W. Tu: *Differential Forms in Algebraic Topology*, Springer Verlag, New York, Heidelberg, Berlin, 1924

[7] Thomas Graber, Rahul Pandharipande: Localization of Virtual Classes, *Invent. Math.*, **135** (1999), 487–518, `arxiv:alg-geom/9708001v2`

[8] Alexander Grothendieck: Techniques de construction et théorèmes d'existence en géométrie algébrique IV : les schémas de Hilbert, *Séminaire N. Bourbaki*, **6** (1961), 249–276

[9] Robin Hartshorne: *Algebraic Geometry*, Springer Verlag, 1977

[10] Tamás Hausel, András Szenes: Equivariant Intersection Theory of Higgs Moduli, *manuscript*

[11] Máté Lehel Juhász: Kévekohomológiák (Sheaf Cohomologies), *master thesis*, 2009

[12] J. Li, G. Tian: Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties, *J. Amer. Math. Soc.*, **11** (1998), 119–174

[13] I. G. Macdonald: Symmetric Products of an Algebraic Curve, *Topology*, **1** (1962), 319–343

[14] Alina Marian: On the Intersection Theory of Quot Schemes and Moduli of Bundles with Sections, *PhD thesis*

[15] Alina Marian, Dragos Oprea: On the Intersection Theory of Moduli Space of Rank Two Bundles, *Topology*, **45** (2006), 531–541

[16] Alina Marian, Dragos Oprea: Virtual Intersections on the Quot Scheme and Vafa-Intriligator Formulas, *Duke Math. J.*, **136** (2007), 81–113

[17] Nitin Nitsure: Construction of Hilbert and Quot Schemes, `arxiv:math/0504590v1`

[18] Nitin Nitsure: Moduli Space of Semistable Pairs on a Curve, *J. Topology*, **S3-62** (1991), 275–300

[19] Joseph Le Potier: *Lectures on Vector Bundles*, 1997, Cambridge

[20] Jean-Pierre Serre: Coherent Algebraic Sheaves, *Annals of Mathematics*, **61** (1955), 197–278 (in French)

[21] C. S. Seshadri: *Fibrés Vectoriels sur les Courbes Algébriques*, Société mathématique de France, 1982

[22] András Szenes: The Combinatorics of the Verlinde formulas, `arxiv:alg-geom/9402003v3`

[23] András Szenes, Michèle Vergne: $[Q,R]$=0 and Kostant Partition Functions, `arxiv:alg-geom/1006.4149v3`

[24] Chris Woodward: Moment maps and geometric invariant theory, `arxiv:09121132v6`

[25] Don Zagier: Elementary Aspects of the Verlinde Formula and of the Harder-Narasimhan-Atiyah-Bott Formula, *Israel Mathematical Conference Proceedings*, **9** (1996), 445–462

[26] Don Zagier: On the cohomology of moduli spaces of rank two vector bundles over curves, *Progress in Mathematics*, **129** (1995), 533–563

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