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Construction of spherical cubature formulas using lattices

Pierre DE LA HARPE, Claude PACHE, Boris VENKOV*

Contact author: Claude Pache (Claude.Pache@math.unige.ch)

3 November 2005

Abstract

We construct cubature formulas on spheres supported by homothetic images of shells in some Euclidian lattices. Our analysis of these cubature formulas uses results from the theory of modular forms. Examples are worked out on \mathbb{S}^{n-1} for $n = 4, 8, 12, 14, 16, 20, 23,$ and 24 , and the sizes of the cubature formulas we obtain are compared with the lower bounds given by Linear Programming.

1 Introduction

For a dimension $n \geq 2$ and a positive number r , let \mathbb{S}_r^{n-1} denote the sphere of equation $\langle x | x \rangle = r$ in the Euclidean space \mathbb{R}^n given with the canonical scalar product $\langle \cdot | \cdot \rangle$, and let σ denote the rotation-invariant measure on such a sphere, normalised by $\sigma(\mathbb{S}_r^{n-1}) = 1$; we write \mathbb{S}^{n-1} for \mathbb{S}_1^{n-1} . For an integer $t \geq 0$, a *cubature formula of strength t on \mathbb{S}_r^{n-1}* is a pair (X, W) , where X is a finite subset of \mathbb{S}_r^{n-1} and where $W : \mathbb{S}_r^{n-1} \rightarrow \mathbb{R}_{>0}$ is a positive-valued function such that

$$(CF) \quad \sum_{x \in X} W(x) f(x) = \int_{\mathbb{S}_r^{n-1}} f(x) d\sigma(x)$$

for every polynomial function $f : \mathbb{R}^n \rightarrow \mathbb{C}$ of degree at most t . The set X is the *support* of the cubature formula (X, W) , the points in X are its *nodes*, the values $W(x)$ are its *weights*, and the cardinality $|X|$ of X is its *size*. A *spherical t -design* on \mathbb{S}_r^{n-1} is a finite subset X of \mathbb{S}_r^{n-1} which, together with the constant weight (by necessity of value $|X|^{-1}$), is a cubature formula of strength t .

There are existence results according to which, for any dimension n and strength t , there exist cubature formulas, and indeed spherical designs, of strength t on \mathbb{S}^{n-1} . For cubature formulas, there are elementary arguments using general methods of convexity, which give some upper bounds on sizes (see Section 2); the existence of spherical designs is a particular case of results of [SeyZas84], which do not provide meaningful size bounds. There are no known straightforward constructions, except for $n = 2$, in which case, for any $t \geq 0$, the $(t + 1)$ th roots of unity in \mathbb{C} provide a spherical t -design on the circle; and in larger dimensions for low values

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Pierre de la Harpe and Claude Pache, Section de Mathématiques, Université de Genève, C.P. 64, 1211 Genève 4, Switzerland.

Boris Venkov, Petersburg Department of Steklov Institute of Mathematics, Fontanka 27, 191011 St. Petersburg, Russia.

Email: Pierre.delaHarpe@math.unige.ch, Claude.Pache@math.unige.ch, bbvenkov@yahoo.com

of t (antipodal pairs, regular simplices, and regular hyperoctahedra are respectively spherical 1-, 2-, and 3-designs, of the lowest possible sizes).

It is therefore a natural question to ask for more explicit constructions. Some have been given in terms of orbits of finite groups on the sphere (see [HarPac04]), and by other various methods (see [Bajn91] and later papers by Bajnok, [Kupe-a], [Kupe-b], and [HadSlo96] for \mathbb{S}^2).

The object of the present paper is to describe constructions involving *shells (or layers) of integral lattices* in \mathbb{R}^n . They are elaborations of constructions which can also be found in [MartV01], [Pach-a], and [HarPac-a].

Let E be a discrete subset of \mathbb{R}^n . For any $m > 0$, denote by E_m the *shell* $E \cap \mathbb{S}_m^{n-1}$. If m_1, \dots, m_r are pairwise distinct positive numbers such that each shell E_{m_j} is non-empty, the union of the sets $\frac{1}{\sqrt{m_j}} E_{m_j}$ is a finite subset (possibly with multiplicities) of the unit sphere in \mathbb{R}^n . Spherical designs and cubature formulas which appear below have supports of this kind, where E is either an integral lattice in \mathbb{R}^n or a union of lattices; observe that they are all *antipodal*, namely that $X = -X$ and $W(x) = W(-x)$ for all $x \in X$.

Results on \mathbb{S}^3 Let us describe some examples provided by this method when $n = 4$, and compare them to known lower bounds for sizes and to other known examples. For $n \geq 4$, the *root lattice of type \mathbf{D}_n* is the integral lattice

$$D_n = \{x \in \mathbb{Z}^n \mid x_1 + \dots + x_n \equiv 0 \pmod{2}\}.$$

(We denote *root systems* as \mathbf{D}_n by bold letters and *lattices* as D_n by thin letters.) It is important for our computations that D_4 is a 2-modular lattice (the definition is recalled in Section 4 below).

For $n \geq 5$, $n \neq 7$, it is known that each shell $(D_n)_{2m}$ is a spherical 3-design which is in general not of strength 4; for $n = 4$, each shell $(D_4)_{2m}$ is a spherical 5-design which is in general not of strength 6 (it has been checked in [Pach-a] that it is never of strength 6 for $2m \leq 1200$). The sizes of the shells of the root lattice of type \mathbf{D}_4 are given by the coefficients of the theta series

$$\begin{aligned} \Theta_{D_4} &= \sum_{m \geq 0} |(D_4)_{2m}| q^{2m} = \sum_{m \geq 0} r_4(2m) q^{2m} \\ &= 1 + 24q^2 + 24q^4 + 96q^6 + 24q^8 + 144q^{10} + 96q^{12} + 192q^{14} + \dots \end{aligned}$$

where $r_4(2m)$ denotes the number of ways of writing $2m$ as a sum of four squares.

The dual $D_4^* = \{y \in \mathbb{R}^n \mid \langle y \mid D_4 \rangle \subset \mathbb{Z}\}$ is similar to D_4 and the renormalized dual $D_4' = \sqrt{2}D_4^*$ is isometric to D_4 ; moreover the intersection $D_4 \cap D_4'$ is reduced to $\{0\}$, since the non-zero coordinates of the vertices in D_4' are never in \mathbb{Z} . For our analysis, we use some invariant theory of the finite group $\text{Aut}(D_4 \cup D_4')$.

THEOREM D_4 . Set $E = D_4 \cup D_4'$

- (i) For any even integer $2m \geq 2$, the shell E_{2m} is a spherical 7-design; in particular, E_2 is a spherical 7-design of size 48 in \mathbb{S}_2^3 .
- (ii) The two shells E_2, E_6 provide the support $X = \frac{1}{\sqrt{2}}E_2 \cup \frac{1}{\sqrt{6}}E_6$ of a cubature formula of strength 11 and size 240.
- (iii) Similarly, $\frac{1}{\sqrt{2}}E_2 \cup \frac{1}{6}E_6 \cup \frac{1}{\sqrt{10}}E_{10}$ is the support of a cubature formula of strength 15 and size 528.

Claim (i) for E_2 appears already in Section 6 of [GoeSei81], where the authors observe that E_2 is an orbit of the Weyl group of type \mathbf{F}_4 . Claim (i) of Theorem E_8

below and the analogous result involving the two shortest shells of a Leech lattice appear also in [GoeSei81].

For comparison, here are some known results concerning the 3-sphere.

Dickson reports on a formula due to Liouville (1859) which essentially shows that the system of roots $(D_4)_2$ of type \mathbf{D}_4 is a spherical 5-design, a formula due to Kempner (1912) which essentially show that $(D_4 \cup D'_4)_2$ is a spherical design of strength 7, and a formula due to Schur (1909) related to a cubature formula of strength 11 and size 144. See [Dick19, pp. 717–724].

There is a formula of strength 7 and size 46 described in [HadSlo94].

There is an exceptional regular polytope in dimension four known as the 600-cell. It has 120 vertices which constitute a spherical 11-design; it is an orbit of a Coxeter group of type \mathbf{H}_4 . Any orbit of this group is a spherical design of strength 11, but there is a special orbit of size 1440 which is of strength 19; see [GoeSei81, Section 5]. Another cubature formula of the same strength, 19, and of size 720 appears in [Sali75]: its support is the union of the 120 vertices and of the 600 centers of faces of the 600-cell.

The best lower bounds known to us for the size of cubature formulas in \mathbb{S}^3 are listed in Tables 6 and 7 of Section 7.

Results on \mathbb{S}^7 Let Γ_8 be a Korkine-Zolotareff lattice, namely an even unimodular lattice of dimension 8 (uniquely defined up to isometry by these properties). Such a lattice is generated by a root system of type \mathbf{E}_8 . The sizes of its shells are given by the coefficients of the theta series

$$\begin{aligned}\Theta_{\Gamma_8} &= \sum_{m \geq 0} |(\Gamma_8)_{2m}| q^{2m} = 240 \sum_{m \geq 0} \sigma_3(m) q^{2m} \\ &= 1 + 240 q^2 + 2160 q^4 + 6720 q^6 + 17520 q^8 + \dots\end{aligned}$$

where $\sigma_3(m) = \sum_{d|m} d^3$.

It is known that each shell $(\Gamma_8)_{2m}$ is a spherical 7-design which is in general not of strength 8. Indeed, the shell $(\Gamma_8)_{2m}$ would be a design of strength 8 if and only if the Ramanujan function τ would vanish at m ; it is a conjecture of Lehmer [Lehm47] that $\tau(m) \neq 0$ for all $m \geq 1$, and this has been checked for $m \leq 10^{15}$ [Serr85]; more on this in [Pach-a] and in Item 1.16 of [HarPac-a].

THEOREM E_8 . Set $E = \Gamma_8$.

- (i) The two shells E_2, E_4 provide the support $X = \frac{1}{\sqrt{2}}E_2 \cup \frac{1}{2}E_4$ of a cubature formula of strength 11 and size 2400.
- (ii) Similarly, $\frac{1}{\sqrt{2}}E_2 \cup \frac{1}{\sqrt{6}}E_6 \cup \frac{1}{2\sqrt{2}}E_8$ is the support of a cubature formula of strength 13 and size 24240.
- (iii) Similarly, $\frac{1}{\sqrt{2}}E_2 \cup \frac{1}{2}E_4 \cup \frac{1}{\sqrt{6}}E_6 \cup \frac{1}{2\sqrt{2}}E_8$ is the support of a cubature formula of strength 15 and size 26400.

The largest known lower bounds for the sizes of cubature formulas on \mathbb{S}^7 are given in Table 8 of Section 7.

Dimensions 12, 14, 16, 20, 23, and 24 There are similar constructions on \mathbb{S}^{23} in terms of a Leech lattice, an even unimodular lattice Λ of dimension 24 with empty shell Λ_2 (uniquely defined up to isometry by these properties). We also describe constructions with shells of

- the 3-modular Coxeter-Todd lattice K_{12} ,

- the 3-modular Quebbemann lattice Q_{14} ,
- the 2-modular Barnes-Wall lattice BW_{16} ,
- the 2-modular Nebe lattices N_{20} (three of them),
- the 3-modular Nebe lattice N_{24} , and
- the shorter Leech lattice O_{23} (which is up to isometry the unique unimodular integral lattice Λ in \mathbb{R}^{23} with $\Lambda_1 = \Lambda_2 = \emptyset$).

Constructions with one shell of these lattices are already described in [BacVen01]. Our results appear in Tables 9 to 14 of Section 7.

2 Bounds on the size of cubature formulas

For fixed values of n and t , we are interested in constructing cubature formulas on \mathbb{S}_r^{n-1} of strength t of small size; it is therefore useful to estimate the minimal size that such a cubature formula can have. We recall in this paragraph some general results on this subject.

Let $b(n, t)$ be the minimal size of a spherical cubature formula on \mathbb{S}_r^{n-1} of strength t .

Linear Programming bound [DeGoSe77, Theorem 5.10], see also [Yudi97]

Let $Q^{(k)}$, $k \in \mathbb{Z}_{\geq 0}$, be the orthogonal polynomials for the scalar product

$$\langle f | g \rangle := \int_{-1}^1 f(t) g(t) (1-t^2)^{(n-3)/2} dt$$

normalized to $\langle Q^{(k)} | Q^{(k)} \rangle = Q^{(k)}(1)$. For a continuous function $F : [-1, 1] \rightarrow \mathbb{C}$, we define the numbers $F_k \in \mathbb{C}$ by

$$F_k = \frac{\langle F | Q^{(k)} \rangle}{Q^{(k)}(1)},$$

so that $F = \sum_{k \geq 0} F_k Q^{(k)}$. Set

$$M(n, t) := \left\{ F : [-1, 1] \rightarrow \mathbb{R} \text{ continuous} \left| \begin{array}{l} F(u) \geq 0 \text{ for } u \in [-1, 1], \\ F \neq 0, \text{ and } F_k \leq 0 \text{ for } k > t \end{array} \right. \right\};$$

then $b(n, t) \geq F(1)/F_0$ for every $F \in M(n, t)$. In other words

$$b_{\text{LP}}(n, t) := \sup_{F \in M(n, t)} \frac{F(1)}{F_0}$$

is a lower bound for $b(n, t)$. Moreover, it is a consequence of [NikNik02, Theorems I and 2.7] that there exists a *polynomial* function $F \in M(n, t)$ for which $b_{\text{LP}}(n, t) = F(1)/F_0$.

When t is odd, we show:

1. PROPOSITION. *Let $t = 2s + 1$, and let*

$$\widehat{M}(n, t) := \{F \in M(n, t) \mid F(-u) = F(u)\}.$$

Then

$$b_{\text{LP}}(n, t) = \sup_{F \in \widehat{M}(n, t)} \frac{2F(1)}{F_0}.$$

Proof. Let $F \in M(n, t)$; set $\widehat{F}(u) = (F(u) + F(-u))/2$; we have $\widehat{F} \in \widehat{M}(n, t)$. Set $F^*(u) = (1 + u)\widehat{F}(u)$. One checks that

$$F^* \in M(n, t).$$

Yet $2\widehat{F}(1) = F(1) + F(-1) \geq F(1)$ and $F_0 = \widehat{F}_0 = F_0^*$, therefore

$$\frac{F(1)}{F_0} \leq \frac{2\widehat{F}(1)}{\widehat{F}_0} = \frac{F^*(1)}{F_0^*}.$$

The result follows. \square

In general, the exact value of $b_{\text{LP}}(n, t)$ is not known. The following bounds are obtained by choosing a particular F in $M(n, t)$:

Delsarte (or Fisher-type) bound [DeGoSe77, Theorems 5.11 and 5.12]

Choose $F(u) := (\sum_{k=0}^s Q^{(k)}(u))^2$ if $t = 2s$, and $F(u) := (1+u)(\sum_{j=0}^{\lfloor s/2 \rfloor} Q^{(s-2j)}(u))^2$ if $t = 2s + 1$. We obtain the bounds

$$\begin{aligned} b_{\text{D}}(n, 2s) &= \binom{n+s-1}{n-1} + \binom{n+s-2}{n-1}, \\ b_{\text{D}}(n, 2s+1) &= 2 \binom{n+s-1}{n-1}. \end{aligned}$$

Moreover, it is known that, if $n \geq 3$, the equality $b(n, t) = b_{\text{D}}(n, t)$ is possible only for some values of (n, t) : see [BanDam79], [BanDam80], and [BaMuVe02]. There also exists a criterion to decide when $b_{\text{D}}(n, t) = b_{\text{LP}}(n, t)$: see [NikNik02, Theorem I].

Yudin bound [Yudi97] Another choice of F gives the bound

$$b_{\text{Y}}(n, t) = \frac{\int_{-1}^1 (1-u^2)^{(n-3)/2} du}{\int_{\gamma}^1 (1-u^2)^{(n-3)/2} du},$$

where γ is the largest root of the polynomial $(Q^{(t+1)})'$.

Special cases For some values of (n, t) , it is possible to compute the exact value of $b(n, t)$. For example, it is shown in [Andr00] that $b(4, 11) = b_{\text{LP}}(4, 11) = 120$.

Numerical estimate of the Linear Programming bound For fixed values of (n, t) , $t = 2s + 1$, the following procedure gives an estimate of the Linear Programming bound $b_{\text{LP}}(n, t)$. (There is a similar procedure when t is even.)

- Choose a degree $d \geq s$, and a finite subset $A \subseteq [0, 1]$ of well-distributed points, for example $A = \{i/N \mid i = 0, 1, \dots, N\}$ with N large.
- By linear programming, find, among all polynomial F satisfying

$$F = 1 + \sum_{i=1}^d F_i Q^{(2i)}, \quad F(u) \geq 0 \text{ for } u \in A, \quad F_k \leq 0 \text{ for } k > t,$$

the polynomial minimizing $F(1)$; denote it by G .

- Set $\epsilon := -\inf_{x \in [0,1]} G(x) \geq 0$. The polynomial $\tilde{G} := G + \epsilon$ is in the set $\widehat{M}(n, t)$, and we have the estimate

$$\frac{2\tilde{G}(1)}{\tilde{G}_0} = \frac{2(G(1) + \epsilon)}{1 + \epsilon} \leq b_{\text{LP}}(n, t).$$

Note that we have also $b_{\text{LP}}^{(d)}(n, t) \leq 2G(1)$, where

$$b_{\text{LP}}^{(d)}(n, t) := \sup_{\substack{F \in M(n, t) \\ \deg f \leq d}} \frac{F(1)}{F_0}.$$

Yet, if d is sufficiently large and A sufficiently dense, $2\tilde{G}(1)/\tilde{G}_0$ is a good approximation of $b_{\text{LP}}(n, t)$. In practice, we first apply the procedure with a relatively large d and a relatively small set A ; we observe that $G_k = 0$ for $k > d_0$. Then we apply the procedure with $d = d_0$ and with a larger set A .

The procedure just described imitates the one used in [ConSlo99, Chap. 13] to compute bounds for kissing numbers.

We end this section with another kind of bound. Set

$$B(n, t) := \binom{n+t-1}{n-1} + \binom{n+t-2}{n-1}.$$

Note that $B(n, t) = b_{\text{D}}(n, 2t)$. This number is the dimension of the space of the restrictions to the sphere of the polynomial functions of degree at most t .

2. PROPOSITION. *For every n and t , there exist cubature formulas with at most $\max\{1, B(n, t) - 1\}$ nodes.*

3. PROPOSITION. *Let (X, W) be a cubature formula on \mathbb{S}_r^{n-1} of strength t . Then there exist a subset $X' \subseteq X$ and a weight function $W' : X' \rightarrow \mathbb{R}_{>0}$ such that $|X'| \leq B(n, t)$ and (X', W') is a cubature formula of degree t .*

See [HarPac-a, Proposition 2.6]. The second proposition is a consequence of the proof of the first one and provides an algorithm for computing (X', W') from (X, W) .

Note however that these propositions do not hold for spherical designs (case of constant weights).

3 Harmonic polynomials

It is sufficient to check condition (CF) in the definition of cubature formulas for harmonic polynomials only. Recall that a smooth function f on \mathbb{R}^n is *harmonic* if $\Delta f = 0$, where $\Delta = \sum_{i=1}^n (\partial/\partial x_i)^2$ is the usual Laplacian. We denote by $\mathcal{H}^{(k)}(\mathbb{R}^n)$ the space of harmonic polynomial functions $\mathbb{R}^n \rightarrow \mathbb{C}$ that are homogenous of degree k . It is a classical fact that the space of restrictions to \mathbb{S}^{n-1} of homogenous polynomials of degree at most t coincides with the space of restrictions of the direct sum $\bigoplus_{k=0}^t \mathcal{H}^{(k)}(\mathbb{R}^n)$. This implies the following well-known criterion; see for example [VenMar01, Theorem 3.2] (details are written there for spherical designs, but the proof carries over to cubature formulas).

4. PROPOSITION. *Let X be a finite subset of some sphere \mathbb{S}_m^{n-1} in \mathbb{R}^n , let $W : X \rightarrow \mathbb{R}_{>0}$ be a weight function, and let $t \geq 0$ be an integer. Then (X, W) is a cubature formula of strength t on \mathbb{S}_m^{n-1} if and only if $\sum_{x \in X} W(x) = 1$ and*

$$\sum_{x \in X} W(x) P(x) = 0 \quad \text{for all } P \in \mathcal{H}^{(k)}(\mathbb{R}^n), 1 \leq k \leq t.$$

Let $O(n)$ be the orthogonal group of the Euclidean space \mathbb{R}^n ; this group acts by isometries on the sphere \mathbb{S}_r^{n-1} . Moreover, $O(n)$ acts naturally on $\mathcal{H}^{(k)}(\mathbb{R}^n)$ by $g \cdot f = f \circ g^{-1}$.

Whenever a group G acts on a space V , we denote by V^G the subspace of all elements of V fixed by G .

Proposition 4 above can be refined in the case the cubature formula is invariant by a finite subgroup of G . Namely, we have the following result, known as *Sobolev's theorem* ([Sobo62], see also [Sobo96, Chap. 2, § 2, Theorem 2.3]):

5. PROPOSITION. *Let X be a finite subset of \mathbb{S}_r^{n-1} and $W : X \rightarrow \mathbb{R}_{>0}$ be a weight function. Let G be a finite subgroup of $O(n)$ leaving (X, W) invariant, that is*

$$gx \in X \text{ and } W(gx) = W(x) \quad \forall x \in X, \forall g \in G.$$

Then (X, W) is a cubature formula of strength t if and only if

$$\sum_{x \in X} W(x) P(x) = 0 \quad \forall P \in \mathcal{H}^{(k)}(\mathbb{R}^n)^G, 1 \leq k \leq t.$$

Proof. For $P \in \mathcal{H}^{(k)}(\mathbb{R}^n)$, let $P^G := |G|^{-1} \sum_{g \in G} g \cdot f$. On the one hand, the G -invariance of (X, W) implies

$$\sum_{x \in X} W(x) P^G(x) = \sum_{x \in X} W(x) P(x),$$

and, on the other hand, the G -invariance of the Lebesgue measure implies

$$\int_{\mathbb{S}_r^{n-1}} P^G(x) d\sigma(x) = \int_{\mathbb{S}_r^{n-1}} P(x) d\sigma(x).$$

The result follows from Proposition 4. □

4 Modular forms

This section is a reminder from [BacVen01]. Consider a lattice Λ in \mathbb{R}^n and a homogeneous harmonic polynomial $P \in \mathcal{H}^{(k)}(\mathbb{R}^n)$ of degree k . The *theta series of Λ with harmonic coefficient P* is the formal power series $\Theta_{\Lambda, P}$ defined by

$$\Theta_{\Lambda, P} := \sum_{x \in \Lambda} P(x) q^{\langle x | x \rangle} = P(0) + \sum_{m > 0} \left(\sum_{x \in \Lambda_m} P(x) \right) q^m.$$

In case $P = 1$, we write simply Θ_{Λ} . Set

$$q := e^{i\pi z}, \quad z \in \mathbb{H},$$

with $\mathbb{H} = \{z \in \mathbb{C} \mid \Im z > 0\}$ the Poincaré half-plane; the theta series $\Theta_{\Lambda, P}$ converges uniformly on every compact subset of \mathbb{H} and defines consequently a holomorphic function on \mathbb{H} . When the lattice Λ satisfies appropriate conditions, this function is a *modular form of weight ω* , and we can use results on modular forms for the computation of the theta series.

Let us first describe the class of lattices which plays the most important role below. A lattice Λ in \mathbb{R}^n is *even* if $\langle x | x \rangle \in 2\mathbb{Z}$ for all $x \in \Lambda$; such a lattice is contained in its dual $\Lambda^* = \{y \in \mathbb{R}^n \mid \langle \Lambda | y \rangle \subseteq \mathbb{Z}\}$, and there are integers $\ell \geq 1$ such that $\sqrt{\ell} \Lambda^*$ is again even (ℓ is not unique since, if $\sqrt{\ell} \Lambda^*$ is even, so is $\sqrt{k^2 \ell} \Lambda^*$ for $k \geq 1$). For given integers n and ℓ , denote by

$$\mathcal{L}at_n(\ell)$$

the class of even lattices Λ such that $\Lambda' := \sqrt{\ell} \Lambda^*$ is even and $\det(\Lambda') = \det(\Lambda)$; the latter condition implies that $\det(\Lambda) = \ell^{n/2}$. Clearly, if $\Lambda \in \mathcal{L}at_n(\ell)$, then $\Lambda' \in \mathcal{L}at_n(\ell)$. A lattice $\Lambda \in \mathcal{L}at_n(\ell)$ is ℓ -modular if Λ' is equivalent to Λ .

Now, we describe the modular forms associated to these lattices. For ω a nonnegative integer and $\epsilon \in \{+, -\}$, we define

$$\widehat{\mathcal{M}}_\omega^\epsilon(\ell)$$

as the space of holomorphic function $f : \mathbb{H} \rightarrow \mathbb{C}$ that verify

$$(*) \quad \begin{aligned} f(z+1) &= f(z) \\ f\left(-\frac{1}{\ell z}\right) &= \epsilon \left(\frac{i}{\sqrt{\ell} z}\right)^\omega f(z) \end{aligned}$$

for all $z \in \mathbb{H}$, and that are holomorphic at infinity, i.e., bounded on $\{z \in \mathbb{H} \mid \Im z > y_0\}$ for $y_0 > 0$. This means that f can be written as

$$f(z) = \sum_{m \geq 0} a_m q^{2m}, \quad q = e^{i\pi z}.$$

Let $f(\infty) := \lim_{\Im z \rightarrow \infty} f(z) = a_0$. We define

$${}^0\widehat{\mathcal{M}}_\omega^\epsilon(\ell) = \{f \in \widehat{\mathcal{M}}_\omega^\epsilon(\ell) \mid f(\infty) = 0\}.$$

We give here the classical terminology: A *modular form* of weight $\omega \geq 0$ for a discrete group $\Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$ and a character $\chi : \Gamma \rightarrow \mathbb{R}$ is a holomorphic function $f : \mathbb{H} \rightarrow \mathbb{C}$ that is holomorphic at infinity and that satisfy

$$\frac{\chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)}{(cz+d)^\omega} f\left(\frac{az+b}{cz+d}\right) = f(z), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

A *parabolic form* is a modular form f that is zero at infinity, namely such that $f(\infty) = 0$.

For ℓ a positive integer, let $\Gamma_*(\ell)$ be the subgroup of $\mathrm{SL}_2(\mathbb{R})$ generated by

$$\Gamma_0(\ell) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{\ell} \right\} \quad \text{and} \quad t_\ell = \begin{pmatrix} 0 & 1/\sqrt{\ell} \\ -\sqrt{\ell} & 0 \end{pmatrix}.$$

For any integer s , let χ_s be the multiplicative character of $\Gamma_*(\ell)$ defined by

$$\begin{aligned} \chi_s\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \left(\frac{(-\ell)^s}{d}\right) \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\ell), \\ \chi_s(t_\ell) &= i^s, \end{aligned}$$

where $(-)$ denotes the Kronecker symbol. Note that χ_s depends only of the class of s modulo 4. Denote by $\mathcal{M}_\omega^+(\ell)$ the space of modular forms of weight ω for the group $\Gamma_*(\ell)$ and the character χ_ω , and by $\mathcal{M}_\omega^-(\ell)$ the space of modular forms of weight ω for the same group and the character $\chi_{\omega+2}$; denote by ${}^0\mathcal{M}_\omega^\pm(\ell)$ the space of corresponding parabolic forms.

Note that Equations (*) say that $\widehat{\mathcal{M}}_\omega^+(\ell)$, respectively $\widehat{\mathcal{M}}_\omega^-(\ell)$, is the space of modular forms of weight ω for the subgroup generated by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S_\ell = \begin{pmatrix} 0 & 1/\sqrt{\ell} \\ -\sqrt{\ell} & 0 \end{pmatrix},$$

and for the character χ_ω , respectively $\chi_{\omega+2}$.

6. LEMMA. For $\ell \in \{1, 2, 3\}$, the group $\Gamma_*(\ell)$ is generated by T and S_ℓ . In particular, for these values of ℓ , we have

$$\widehat{\mathcal{M}}_\omega^\epsilon(\ell) = \mathcal{M}_\omega^\epsilon(\ell) \quad \text{and} \quad {}^0\widehat{\mathcal{M}}_\omega^\epsilon(\ell) = {}^0\mathcal{M}_\omega^\epsilon(\ell).$$

Proof. Let $\ell \in \{1, 2, 3\}$ and $\gamma \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_*(\ell)$. We show, by induction on $c^2 \in \mathbb{N}$, that $\gamma \in \langle T, S_\ell \rangle$, where $\langle T, S_\ell \rangle$ denotes the group generated by T and S_ℓ .

If $c = 0$, then $\gamma = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = T^m$ for an integer m ; thus $\gamma \in \langle T, S_\ell \rangle$.

If $c \neq 0$, let $k \in \mathbb{Z}$ such that $|d + kc| \leq |c/2|$. We have

$$\gamma T^k S_\ell = \begin{pmatrix} -(b + ka)\sqrt{\ell} & a/\sqrt{\ell} \\ -(d + kc)\sqrt{\ell} & c/\sqrt{\ell} \end{pmatrix},$$

and $(-(d + kc)\sqrt{\ell})^2 \leq c^2(\ell/4) < c^2$. By induction, $\gamma T^k S_\ell \in \langle T, S_\ell \rangle$, therefore $\gamma \in \langle T, S_\ell \rangle$. \square

7. PROPOSITION. Let $\Lambda \in \mathcal{L}at_n(\ell)$, where ℓ is a positive integer. Then

$$\begin{aligned} \Theta_\Lambda + \Theta_{\Lambda'} &\in \mathcal{M}_{n/2}^+(\ell), \\ \Theta_\Lambda - \Theta_{\Lambda'} &\in {}^0\mathcal{M}_{n/2}^-(\ell). \end{aligned}$$

Let moreover $P \in \mathcal{H}^{(2h)}(\mathbb{R}^n)$, $h \geq 1$. Then, if $2h \equiv 0 \pmod{4}$,

$$\begin{aligned} \Theta_{\Lambda, P} + \Theta_{\Lambda', P} &\in {}^0\mathcal{M}_{n/2+2h}^+(\ell), \\ \Theta_{\Lambda, P} - \Theta_{\Lambda', P} &\in {}^0\mathcal{M}_{n/2+2h}^-(\ell), \end{aligned}$$

and, if $2h \equiv 2 \pmod{4}$,

$$\begin{aligned} \Theta_{\Lambda, P} + \Theta_{\Lambda', P} &\in {}^0\mathcal{M}_{n/2+2h}^-(\ell), \\ \Theta_{\Lambda, P} - \Theta_{\Lambda', P} &\in {}^0\mathcal{M}_{n/2+2h}^+(\ell). \end{aligned}$$

Partial proof. We give the proof only for $\ell \in \{1, 2, 3\}$. The general case is more complicated: see [Ebel94, §3.1] together with [VenMar01, Chapter 2].¹

It is straightforward that the theta series involved are holomorphic in \mathbb{H} and holomorphic at infinity. Now, by Lemma 6, it is sufficient to check that the theta series satisfy Equations (*).

As Λ and Λ' are even, we have clearly $\Theta_{L, P}(z + 1) = \Theta_{L, P}(z)$ for $L = \Lambda$ or Λ' . Now, as a direct consequence of the Poisson Summation Formula—see for example [Ebel94, Prop. 3.1, p. 87]—we have, for any lattice L of rank n and any $P \in \mathcal{H}^{(2h)}(\mathbb{R}^n)$,

$$\Theta_{L^*, P}(z) = (\det L)^{1/2} (-1)^h (i/z)^{n/2+2h} \Theta_{L, P}(-1/z).$$

From this formula, we deduce, for $L = \Lambda$ or Λ' ,

$$\Theta_{L', P}(z) = (-1)^h (i/\sqrt{\ell}z)^{n/2+2h} \Theta_{L, P}(-1/\ell z).$$

The result follows. \square

Let

$$\begin{aligned} \mathcal{M}^+(\ell) &:= \bigoplus_{\omega \geq 0} \mathcal{M}_\omega^+(\ell), \\ {}^0\mathcal{M}^+(\ell) &:= \bigoplus_{\omega > 0} {}^0\mathcal{M}_\omega^+(\ell), \quad {}^0\mathcal{M}^-(\ell) := \bigoplus_{\omega > 0} {}^0\mathcal{M}_\omega^-(\ell). \end{aligned}$$

These are $\mathcal{M}^+(\ell)$ -modules graded by the weight. For some values of ℓ , the structure of these algebras is known:

¹Note however that our examples of constructions of designs in Section 6 involve lattices in $\mathcal{L}at_n(\ell)$ for $\ell \in \{1, 2, 3\}$ only.

Remark. In general, the numerical subscript of modular forms given below indicates the *double* of its weight; for example $\Delta_{24} \in {}^0\mathcal{M}_{12}(1)$ is of weight 12, and $\Theta_{\mathbf{E}_8} \in \mathcal{M}_4(1)$ is of weight 4. We have made an exception for the Eisenstein series $E_{(k)}$, for which we have retained the traditional notation.

8. THEOREM. *Let ℓ be 1 or a prime number such that $\ell + 1$ divides 24. Let k_0, k_1 and k_2 given by*

$$k_0 = \begin{cases} 4 & \text{if } \ell = 1, \\ 2 & \text{if } \ell \equiv 1 \text{ or } 2 \pmod{4}, \text{ and } \ell \neq 1, \\ 1 & \text{if } \ell \equiv 3 \pmod{4}, \end{cases}$$

$$k_1 = \frac{24}{\ell + 1}, \quad k_2 = k_0 + k_1 + 2.$$

Then, we have

$$\begin{aligned} \mathcal{M}^+(\ell) &= \mathbb{C}[\theta_{2k_0}, \Delta_{2k_1}], \\ {}^0\mathcal{M}^+(\ell) &= \Delta_{2k_1} \mathbb{C}[\theta_{2k_0}, \Delta_{2k_1}], \\ {}^0\mathcal{M}^-(\ell) &= \Phi_{2k_2} \mathbb{C}[\theta_{2k_0}, \Delta_{2k_1}], \end{aligned}$$

where

$$\begin{aligned} \theta_{2k_0} &= \Theta_{L_0} \in \mathcal{M}_{k_0}^+(\ell), \text{ with } L_0 \in \mathcal{L}at_{2k_0}(\ell), \\ \Delta_{2k_1} &= (\eta(z)\eta(\ell z))^{k_1} \in {}^0\mathcal{M}_{k_1}^+(\ell), \\ \Phi_{2k_2} &\in {}^0\mathcal{M}_{k_2}^-(\ell), \end{aligned}$$

and where L_0 is given in Table 1.

For a proof, see [BacVen01, Section 2] and [Queb95, Section 3]. Recall that

$$\eta(z) = q^{1/12} \prod_{m=1}^{\infty} (1 - q^{2m}) = q^{1/12} (1 - q^2 - q^4 + O(q^{10})), \quad q = e^{i\pi z}.$$

For $\ell = 1$, the modular forms of the theorem are $\Delta_{24} = \eta(z)^{24}$, whose Fourier coefficients are the Ramanujan numbers,

$$\Theta_{\mathbf{E}_8} = E_{(4)}, \quad \text{and} \quad \Phi_{36} = \Delta_{24} E_{(6)},$$

where $E_{(k)}$ is the Eisenstein series of weight k .

Tables 1 to 3 give the theta series θ_{2k_0} , Δ_{2k_1} and Φ_{2k_2} for all possible values of ℓ . In the third column of Table 1, the lattice L_0 is designated either by a root system that generates it (e.g., \mathbf{E}_8), or by a quadratic form (e.g., $\begin{pmatrix} 2 & \\ & 1 \end{pmatrix}$). In Table 3, P_k is a suitable element of $\mathcal{H}^{(k)}(\mathbb{R}^n)$, the lattice L_4 is the unique (up to isometry) 4-dimensional lattice of minimum 4 and determinant 11^2 , and the lattices $L_0^{(1)}$ and $L_0^{(2)}$ are the two lattices given in the column L_0 of Table 1. We normalize Φ_{2k_2} by $\Phi_{2k_2} = q^2 + O(q^4)$.

5 Construction of cubature formulas using shells of lattices

Let G be a finite subgroup of $O(n)$, and let X_1, X_2, \dots, X_r be nonempty finite subsets of \mathbb{S}^{n-1} such that $G X_j = X_j$. Consider the set

$$X := \bigcup_{j=1}^r X_j \subseteq \mathbb{S}^{n-1}.$$

ℓ	k_0	L_0	$\theta_{2k_0} = \Theta_{L_0}$
1	4	\mathbf{E}_8	$1 + 240q^2 + 2160q^4 + 6720q^6 + O(q^8)$
2	2	\mathbf{D}_4	$1 + 24q^2 + 24q^4 + 96q^6 + O(q^8)$
3	1	$\mathbf{A}_2 \approx \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$	$1 + 6q^2 + 6q^6 + O(q^8)$
5	2	\mathbf{A}_4	$1 + 20q^2 + 30q^4 + 60q^6 + O(q^8)$
7	1	$\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$	$1 + 2q^2 + 4q^4 + O(q^8)$
11	1	$\begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix}$	$1 + 2q^2 + 4q^6 + O(q^8)$
23	1	$\begin{pmatrix} 4 & 1 \\ 1 & 6 \end{pmatrix}$ or $\begin{pmatrix} 2 & 1 \\ 1 & 12 \end{pmatrix}$	$1 + 2q^2 + O(q^8)$ or $1 + 2q^4 + 2q^6 + O(q^8)$

Table 1: Modular forms $\theta_{2k_0} = \Theta_{L_0}$ of weight k_0

ℓ	k_1	$\Delta_{2k_1} = (\eta(z)\eta(\ell z))^{k_1}$
1	12	$q^2 - 24q^4 + 252q^6 - 1472q^8 + O(q^{10})$
2	8	$q^2 - 8q^4 + 12q^6 + 64q^8 + O(q^{10})$
3	6	$q^2 - 6q^4 + 9q^6 + 4q^8 + O(q^{10})$
5	4	$q^2 - 4q^4 + 2q^6 + 8q^8 + O(q^{10})$
7	3	$q^2 - 3q^4 + 5q^6 - 7q^8 + O(q^{10})$
11	2	$q^2 - 2q^4 - q^6 + 2q^8 + O(q^{10})$
23	1	$q^2 - q^4 - q^6 + O(q^{10})$

Table 2: Modular forms $\Delta_{2k_1} = (\eta(z)\eta(\ell z))^{k_1}$ of weight k_1

ℓ	k_2	Φ_{2k_2} (definition)	Φ_{2k_2} (expansion)
1	18	$\Theta_{\mathbf{E}_8, P_{14}}$	$q^2 - 528q^4 - 4284q^6 + 147712q^8 + O(q^{10})$
2	12	$\Theta_{\mathbf{D}_4^2, P_8} - \Theta_{(\mathbf{D}_4^2)', P_8}$	$q^2 - 88q^4 + 252q^6 + 64q^8 + O(q^{10})$
3	9	$\Theta_{\mathbf{A}_2^3, P_6} + \Theta_{(\mathbf{A}_2^3)', P_6}$	$q^2 - 14q^4 + 48q^6 + 68q^8 + O(q^{10})$
5	8	$\Theta_{L_0, P_6} + \Theta_{L_0', P_6}$	$q^2 - 14q^4 - 48q^6 + 68q^8 + O(q^{10})$
7	7	$\Theta_{L_0^2, P_4} - \Theta_{(L_0^2)', P_4}$	$q^2 - 10q^4 - 14q^6 + 68q^8 + O(q^{10})$
11	5	$\Theta_{L_0 \perp L_4, P_2} + \Theta_{(L_0 \perp L_4)', P_2}$	$q^2 - 6q^4 - 3q^6 - 14q^8 + O(q^{10})$
23	4	$\Theta_{L_0^{(1)} \perp L_0^{(2)}, P_2} + \Theta_{(L_0^{(1)} \perp L_0^{(2)})', P_2}$	$q^2 - 2q^4 - 5q^6 - 4q^8 + O(q^{10})$

Table 3: Modular forms Φ_{2k_2} of weight k_2

The aim is to find numbers $W_1, W_2, \dots, W_r \in \mathbb{R}_{>0}$ such that (X, W) is a cubature formula of high strength, where W is the weight function given by

$$W(x) = \sum_{j: x \in X_j} W_j.$$

By Proposition 4, (X, W) is of strength t if and only if the following conditions on W_1, \dots, W_r are satisfied:

$$\begin{aligned} \sum_{j=1}^r W_j |X_j| &= 1, \\ \sum_{j=1}^r W_j P(X_j) &= 0, \quad \forall P \in \mathcal{H}^{(k)}, \quad 1 \leq k \leq t, \end{aligned}$$

where

$$P(X_j) := \sum_{x \in X_j} P(x).$$

The following statement, together with Proposition 5, is fundamental in our analysis.

9. LEMMA. *Let G be a finite subgroup of $O(n)$, and let $E \subseteq \mathbb{R}^n$ be a discrete set that is invariant by G . Let m_1, m_2, \dots, m_r be pairwise distinct positive numbers, let*

$$X_j = \frac{1}{\sqrt{m_j}} E_{m_j} \subseteq \mathbb{S}^{n-1}, \quad j = 1, \dots, r,$$

where $E_m = \{x \in E \mid \langle x \mid x \rangle = m\}$. Let $X = \bigcup_{j=1}^r X_j$ and $W_1, \dots, W_r > 0$, and let k be a positive integer.

Assume that there exist formal series

$$\Theta_i = \sum_{m>0} a_i(m) q^m, \quad i = 1, \dots, N$$

such that, for every $P \in \mathcal{H}^{(k)}(\mathbb{R}^n)^G$, the theta series $\Theta_{E,P}$ is of the form

$$\Theta_{E,P} = c_1(P) \Theta_1 + c_2(P) \Theta_2 + \dots + c_N(P) \Theta_N$$

for some $c_i(P) \in \mathbb{C}$.

Then, for the condition

$$\sum_{j=1}^r W_j P(X_j) = 0 \quad \forall P \in \mathcal{H}^{(k)}(\mathbb{R}^n)^G$$

to hold, it suffices that

$$(\#) \quad \sum_{j=1}^r \frac{a_i(m_j)}{m_j^{k/2}} W_j = 0, \quad i = 1, \dots, N.$$

Proof. Assume that Equation (#) holds, and let $P \in \mathcal{H}^{(k)}(\mathbb{R}^n)^G$. By hypothesis, there exist $c_1, \dots, c_N \in \mathbb{C}$ such that

$$\sum_{m>0} P(E_m) q^m = \Theta_{E,P} = \sum_{i=1}^N c_i \Theta_i = \sum_{m>0} \left(\sum_{i=1}^N c_i a_i(m) \right) q^m.$$

Since P is homogeneous of degree k , we have $P((1/\sqrt{m})E_m) = m^{-k/2}P(E_m)$. Therefore,

$$P(X_j) = m_j^{-k/2}P(E_{m_j}) = \sum_{i=1}^N c_i \frac{a_i(m_j)}{m_j^{k/2}},$$

hence

$$\sum_{j=1}^r W_j P(X_j) = \sum_{i=1}^N c_i \sum_{j=1}^r \frac{a_i(m_j)}{m_j^{k/2}} W_j = 0. \quad \square$$

6 Applications

If not otherwise specified, Lemma 9 is applied with the trivial group $G = \{\text{id}\}$. In this section, we do not give the details for all lattices we have considered, but we have made a selection which reflects most situations that may occur. All cubature formulas we have obtained are listed in Section 7 below.

6.1 The Leech lattice

Let Λ be the Leech lattice, namely the unique (up to isometry) even unimodular lattice of dimension 24 and of minimum 4. Let $P \in \mathcal{H}^{(2h)}(\mathbb{R}^n)$. From Proposition 7 and Theorem 8, we deduce (using $\Lambda' = \Lambda$):

$$\Theta_{\Lambda, P} = \begin{cases} \Theta_{\mathbf{E}_8}^3 - 720\Delta_{24} & \text{if } P = 1, \\ 0 & \text{if } P \in \mathcal{H}^{(2h)}(\mathbb{R}^{24}), 2h = 2, 4, 6, 8, 10, 14, \\ c_1(P)\Delta_{24}^2 & \text{if } P \in \mathcal{H}^{(12)}(\mathbb{R}^{24}), \\ c_2(P)\Theta_{\mathbf{E}_8}\Delta_{24}^2 & \text{if } P \in \mathcal{H}^{(16)}(\mathbb{R}^{24}), \end{cases}$$

where c_1 and c_2 are linear forms on $\mathcal{H}^{(12)}(\mathbb{R}^{24})$ and $\mathcal{H}^{(16)}(\mathbb{R}^{24})$ respectively. The first coefficients of these theta series are

$$\begin{aligned} \Theta_{\mathbf{E}_8}^3 - 720\Delta_{24} &= 1 + 196\,560q^4 + 16\,752\,960q^6 + O(q^8), \\ \Delta_{24}^2 &= q^4 - 48q^6 + O(q^8), \\ \Theta_{\mathbf{E}_8}\Delta_{24}^2 &= q^4 + 192q^6 + O(q^8). \end{aligned}$$

Since $\Theta_{\Lambda, P} = 0$ for $P \in \mathcal{H}^{(2h)}(\mathbb{R}^{24})$, $2 \leq 2h \leq 10$, it follows from Proposition 4 that every shell of the Leech lattice is a spherical 11-design.

Now, we want to find cubature formulas of higher strengths using the construction of Section 5. Consider the set

$$X = X_1 \cup X_2, \quad \text{where } X_1 = \frac{1}{\sqrt{4}}\Lambda_4, \quad X_2 = \frac{1}{\sqrt{6}}\Lambda_6.$$

The two sets X_1 and X_2 are disjoint. Indeed, if we had $x \in X_1 \cap X_2$, we would have $\sqrt{4}x \in \Lambda_4$ and $\sqrt{6}x \in \Lambda_6$; but $\langle \sqrt{4}x \mid \sqrt{6}x \rangle = 2\sqrt{6}\langle x \mid x \rangle = 2\sqrt{6}$, which is impossible since Λ is integral.

We want to find out the numbers W_1 and W_2 such that (X, W) is a cubature of strength 15 (same notation as in Section 5). For this, we have to fulfill the conditions

$$\begin{aligned} \sum_{x \in X} W(x) &= 1, \\ \sum_{x \in X} W(x) P(x) &= 0 \text{ for } P \in \mathcal{H}^{(12)}(\mathbb{R}^n). \end{aligned}$$

The first condition is equivalent to $|X_1|W_1 + |X_2|W_2 = 1$, that is

$$196\,560\,W_1 + 16\,752\,960\,W_2 = 1.$$

By Lemma 9 the second condition is equivalent to

$$\frac{1}{4^6}W_1 - \frac{48}{6^6}W_2 = 0,$$

or $W_2 = (3/4)^5W_1$. The solution of this set of two linear equations is

$$W_1 \approx 2.394 \times 10^{-7}, \quad W_2 \approx 0.568 \times 10^{-7}.$$

So we get a cubature formula of size $196\,560 + 16\,752\,960 = 16\,949\,520$.

We can obtain in the same way a cubature formula of strength 17 by using the shells of norm 4, 6, and 8, and a cubature formula of strength 19 by using the shells of norm 4, 6, 8, and 10 of the Leech lattice.

6.2 The Korkine-Zolotareff lattice

Let Λ be the Korkine-Zolotareff lattice, namely the unique (up to isometry) even unimodular lattice of dimension 8. It is the lattice generated by the root system of type \mathbf{E}_8 . From Proposition 7 and Theorem 8, we deduce (using $\Lambda' = \Lambda$):

$$\Theta_{\Lambda, P} = \begin{cases} \Theta_{\mathbf{E}_8} & \text{if } P = 1, \\ 0 & \text{if } P \in \mathcal{H}^{(2h)}(\mathbb{R}^8), 2h = 2, 4, 6, 10, \\ c_1(P)\Delta_{24} & \text{if } P \in \mathcal{H}^{(8)}(\mathbb{R}^8), \\ c_3(P)\Theta_{\mathbf{E}_8}\Delta_{24} & \text{if } P \in \mathcal{H}^{(12)}(\mathbb{R}^8), \\ c_2(P)\Phi_{36} & \text{if } P \in \mathcal{H}^{(14)}(\mathbb{R}^8). \end{cases}$$

So, by Proposition 4, any shell of the Korkine-Zolotareff lattice is a spherical 7-design, and we can obtain a cubature formula of strength 11 by combining the shells of norms 2 and 4.

But if we try to obtain a cubature formula of strength 13 of nodes

$$X = X_1 \cup X_2 \cup X_3, \quad \text{where } X_1 = \frac{1}{\sqrt{2}}\Lambda_2, X_2 = \frac{1}{\sqrt{4}}\Lambda_4, X_3 = \frac{1}{\sqrt{6}}\Lambda_6,$$

we obtain the weights

$$W_1 \approx -0.744 \times 10^{-4}, \quad W_2 \approx 1.587 \times 10^{-4}, \quad W_3 \approx 1.005 \times 10^{-4},$$

with W_1 negative. A similar problem occurs if we try to use the shells of norm 2, 4 and 8. Therefore, in order to get a true cubature formula with positive weights, we have to use

$$X = X_1 \cup X_2 \cup X_3, \quad X_1 = \frac{1}{\sqrt{2}}\Lambda_2, X_2 = \frac{1}{\sqrt{6}}\Lambda_6, X_3 = \frac{1}{\sqrt{8}}\Lambda_8.$$

We obtain the weights

$$W_1 \approx 0.792 \times 10^{-4}, \quad W_2 \approx 0.457 \times 10^{-4}, \quad W_3 \approx 0.385 \times 10^{-4}.$$

Here, we have $X_1 \subseteq X_3$, since $2x \in \Lambda_8$ if $x \in \Lambda_2$. Therefore, we obtain a cubature formula of strength 13 of size

$$|X| = |X_2| + |X_3| = 6\,720 + 17\,520 = 24\,240$$

and of weights

$$W(x) = \begin{cases} W_2 \approx 0.457 \times 10^{-4} & \text{if } x \in X_2, \\ W_3 \approx 0.385 \times 10^{-4} & \text{if } x \in X_3 \setminus X_1, \\ W_1 + W_3 \approx 1.177 \times 10^{-4} & \text{if } x \in X_1. \end{cases}$$

6.3 The Barnes-Wall lattice of dimension 16

Let Λ be the Barnes-Wall lattice of dimension 16. It is a 2-modular lattice, that is $\Lambda' := \sqrt{2}\Lambda$ is equivalent to Λ , and it is of minimum 4. From Proposition 7 and Theorem 8, we have:

$$\Theta_{\Lambda,P} + \Theta_{\Lambda',P} = \begin{cases} 2\Theta_{\mathbf{D}_4}^4 - 192\Delta_{16} & \text{if } P = 1, \\ 0 & \text{if } P \in \mathcal{H}^{(2h)}(\mathbb{R}^{16}), 2h = 2, 4, 6, 10, \\ c_1(P)\Delta_{16}^2 & \text{if } P \in \mathcal{H}^{(8)}(\mathbb{R}^{16}), \\ c_2(P)\Delta_{16}^2\Theta_{\mathbf{D}_4}^2 & \text{if } P \in \mathcal{H}^{(12)}(\mathbb{R}^{16}), \\ c_3(P)\Phi_{24}\Delta_{16}\Theta_{\mathbf{D}_4} & \text{if } P \in \mathcal{H}^{(14)}(\mathbb{R}^{16}), \end{cases}$$

$$\Theta_{\Lambda,P} - \Theta_{\Lambda',P} = \begin{cases} 0 & \text{if } P \in \mathcal{H}^{(2h)}(\mathbb{R}^{16}), 2h = 0, 2, 4, 6, 8, 12, \\ c_4(P)\Delta_{16}^2\Theta_{\mathbf{D}_4} & \text{if } P \in \mathcal{H}^{(10)}(\mathbb{R}^{16}), \\ c_5(P)\Phi_{24}\Delta_{16} & \text{if } P \in \mathcal{H}^{(12)}(\mathbb{R}^{16}), \end{cases}$$

and therefore

$$\Theta_{\Lambda,P} = \begin{cases} \Theta_{\mathbf{D}_4}^4 - 96\Delta_{16} & \text{if } P = 1, \\ 0 & \text{if } P \in \mathcal{H}^{(2h)}(\mathbb{R}^{16}), 2h = 2, 4, 6, \\ c_6(P)\Delta_{16}^2 & \text{if } P \in \mathcal{H}^{(8)}(\mathbb{R}^{16}), \\ c_7(P)\Delta_{16}^2\Theta_{\mathbf{D}_4} & \text{if } P \in \mathcal{H}^{(10)}(\mathbb{R}^{16}). \end{cases}$$

Every shell of Λ is a spherical 7-design, and we obtain a cubature formula of strength 9 by combining two different shells of Λ , for example Λ_4 and Λ_6 . We also obtain a cubature formulas of strength 11 of support $\frac{1}{\sqrt{4}}(\Lambda \cup \Lambda')_4 \cup \frac{1}{\sqrt{6}}(\Lambda \cup \Lambda')_6$, and a cubature formulas of strength 13 of support $\frac{1}{\sqrt{4}}(\Lambda \cup \Lambda')_4 \cup \frac{1}{\sqrt{6}}(\Lambda \cup \Lambda')_6 \cup \frac{1}{\sqrt{10}}(\Lambda \cup \Lambda')_{10}$.

6.4 The \mathbf{D}_4 root lattice

Let Λ be the lattice generated by the root system \mathbf{D}_4 . It is a 2-modular lattice. Here, we will apply Lemma 9 with

$$G = \text{Aut}(\Lambda \cup \Lambda').$$

We now describe Λ , Λ' and G . We have

$$\Lambda = \{x = (x_1, x_2, x_3, x_4) \mid x_i \in \mathbb{Z}, x_1 + x_2 + x_3 + x_4 \in 2\mathbb{Z}\},$$

$$\Lambda' = \{x = (x_1, x_2, x_3, x_4) \mid x_i \in \frac{1}{\sqrt{2}}\mathbb{Z}, x_i - x_j \in \sqrt{2}\mathbb{Z}\}.$$

The group $\text{Aut}(\Lambda)$ contains (actually, is) a reflection group, denoted by $W(\mathbf{F}_4)$, which is generated by the 24 reflections $x \mapsto x - 2\frac{\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$, where α is one of the following vectors:

$$(\pm 1, 0, 0, 0), \quad (\pm 1, \pm 1, 0, 0), \quad (\pm 1, \pm 1, \pm 1, \pm 1)$$

(all choices of signs and all permutations of coordinates). The invariant polynomials of this group are known [Smit95, §7.4, pp. 217–218]; they form a polynomial \mathbb{C} -algebra of basis $\{h_2, h_6, h_8, h_{12}\}$, where

$$h_2 = \text{sym}(x_1^2), \quad h_6 = \text{sym}(x_1^4 x_2^2) - 3 \text{sym}(x_1^2 x_2^2 x_3^2),$$

$$h_8 = \text{sym}(x_1^8) + 14 \text{sym}(x_1^4 x_2^4) + 168 x_1^2 x_2^2 x_3^2 x_4^2,$$

$$h_{12} = \text{sym}(x_1^{12}) + 22 \text{sym}(x_1^6 x_2^6) + 165 \text{sym}(x_1^4 x_2^4 x_3^4)$$

$$+ 330 \text{sym}(x_1^6 x_2^2 x_3^2 x_4^2) + 330 \text{sym}(x_1^4 x_2^4 x_3^2 x_4^2),$$

and where

$$\text{sym}(p) = \frac{1}{|(\mathfrak{S}_4)_p|} \sum_{\sigma \in \mathfrak{S}_4} \sigma \cdot p, \quad (\mathfrak{S}_4)_p := \{\sigma \in \mathfrak{S}_4 \mid \sigma \cdot p = p\},$$

where the symmetric group \mathfrak{S}_4 acts by permuting the coordinates. For our purpose, it is better to choose the basis

$$\begin{aligned} H_2 &:= h_2, & H_6 &:= 8h_6 - h_2^3, & H_8 &:= 10h_8 - 7h_2^4, \\ H_{12} &:= 64h_{12} - 55h_8h_2^2 - 176h_6^2 + 220h_6h_2^3 - 11h_2^6. \end{aligned}$$

The orthonormal transformation

$$T := \begin{pmatrix} -\sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 \\ -\sqrt{2}/2 & -\sqrt{2}/2 & 0 & 0 \\ 0 & 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$$

exchanges Λ and Λ' . So, $G = \text{Aut}(\Lambda \cup \Lambda')$ is generated by $\text{Aut}(\Lambda)$ and T . The action of T on the polynomials H_k is:

$$TH_2 = H_2, \quad TH_6 = -H_6, \quad TH_8 = H_8, \quad TH_{12} = -H_{12}.$$

Therefore, the polynomials that are invariant by G are linearly generated by the polynomials

$$H_2^\alpha H_6^\beta H_8^\gamma H_{12}^\delta, \quad \beta + \delta \equiv 0 \pmod{2}.$$

Let $\mathcal{P}^{(k)}(\mathbb{R}^n)$ be the space of homogeneous polynomial functions on \mathbb{R}^n of degree k . Since $\mathcal{H}^{(k)}(\mathbb{R}^n)$ is the kernel of the surjective G -equivariant Laplacian $\Delta : \mathcal{P}^{(k)}(\mathbb{R}^n) \rightarrow \mathcal{P}^{(k-2)}(\mathbb{R}^n)$, we have

$$d_k^G := \dim \mathcal{H}^{(k)}(\mathbb{R}^n)^G = \dim \mathcal{P}^{(k)}(\mathbb{R}^n)^G - \dim \mathcal{P}^{(k-2)}(\mathbb{R}^n)^G.$$

We can compute

$$\sum_{k \geq 0} d_k^G X^k = \frac{1 + X^{18}}{(1 - X^8)(1 - X^{12})(1 - X^{24})} = 1 + X^8 + X^{12} + X^{16} + O(X^{18}).$$

So, we have by Proposition 7,

$$\Theta_{\Lambda, P} + \Theta_{\Lambda', P} = \begin{cases} 2\Theta_{\mathbf{D}_4} & \text{if } P = 1, \\ 0 & \text{if } P \in \mathcal{H}^{(2h)}(\mathbb{R}^8)^G = \{0\}, 2h = 2, 4, 6, 10, 14, \\ c_1(P) \Delta_{16} \Theta_{\mathbf{D}_4} & \text{if } P \in \mathcal{H}^{(8)}(\mathbb{R}^8)^G, \\ c_2(P) \Delta_{16}^2 \Theta_{\mathbf{D}_4}^3 & \text{if } P \in \mathcal{H}^{(12)}(\mathbb{R}^8)^G. \end{cases}$$

Thus, using Lemma 9 together with Proposition 5, we obtain cubature formulas of strengths 11 and 15 by combining two or three shells of $\Lambda \cup \Lambda'$. Note however that certain combinations of shells, such as $\frac{1}{\sqrt{2}}(\Lambda \cup \Lambda')_2 \cup \frac{1}{\sqrt{4}}(\Lambda \cup \Lambda')_4$ are not possible, because Lemma 9 can provide a degenerate linear equations system. (In the present example, we have indeed $\frac{1}{\sqrt{2}}(\Lambda \cup \Lambda')_2 = \frac{1}{\sqrt{4}}(\Lambda \cup \Lambda')_4$.)

6.5 The shorter Leech lattice

Let Λ be the unique (up to isometry) odd unimodular integral lattice of dimension 23 and of minimum 3; it is the *shorter Leech lattice*. As for even ones, the theta series of odd unimodular lattices can be computed: see [Pach-a]. So, we can apply our method to construct cubature formulas from the shells of Λ .

7 Cubature formulas obtained by our construction

The cubature formulas we obtain by our method are summarized in Tables 6 to 14. These tables involved the lattices listed in Tables 4 and 5.

In Tables 6 to 14, the column “bound” indicates the lower bound of the size of cubature formulas of strength t on \mathbb{S}^{n-1} obtained by Linear Programming. The indication in parentheses has the following meaning:

- (LP d) refers to the estimate of the Linear Programming bound obtained by the procedure described in Section 2, paragraph “Numerical estimate of the Linear Programming bound”, where $2d$ is the degree of the polynomial used for the estimate. The calculations have been performed with “Maple” software, using the “Optimization” package. Because of limitations in time and memory of our computer, we have not always chosen an optimal d , in which case we write (LP $\geq d$); similarly, when the bound we have found can almost certainly be improved, we put the sign \geq before it.
- (T) refers to the Delsarte or (Fisher-type) bound; it is the same as (LP d) when $2d + 1 = t$ (the strength).
- When it is known that tight spherical designs with the corresponding parameters (n, t) do not exist, we note (D) to indicate the Delsarte bound increased by 1

The columns “set” and “shells” indicate respectively the set E and the shells m_j used in the construction of Section 5, according to the notation of Lemma 9. The column “size” gives the size of the cubature formula so constructed.

In the case $n = 4$, we have also listed in Table 7 the cubature formulas of smallest known sizes for fixed values of t ;

Most spherical designs with $n \leq 24$ described in [BacVen01, Table 3, p. 108] appear also in our tables.

n	ℓ	notation	name
4	2	D_4	\mathbf{D}_4 root lattice
8	1	Γ_8	Korkine-Zolotareff
12	3	K_{12}	Coxeter-Todd
14	3	Q_{14}	Quebbemann
16	2	BW_{16}	Barnes-Wall
20	2	$N_{20}^{(1)}, N_{20}^{(2)}, N_{20}^{(3)}$	Nebe
24	3	N_{24}	Nebe
24	1	Λ_{24}	Leech

Table 4: Even ℓ -modular lattices involved in Tables 6 to 14

n	notation	name
23	O_{23}	shorter Leech lattice

Table 5: Odd unimodular lattice involved in Table 13

strength	set	shells	size	bound
5	D_4	2	24	21 (D)
7	$D_4 \cup D'_4$	2	48	42 (LP5)
11	$D_4 \cup D'_4$	2, 6	240	120 (LP11)
15	$D_4 \cup D'_4$	2, 6, 10	528	267 (LP22)

Table 6: Cubature formulas for $n = 4$ —see also Table7

strength	best known cubature formula	size	bound
3	root system \mathbf{A}_1^4	8	8 (T)
5	root system \mathbf{D}_4	24	21 (D)
7	[HadSlo94]	46	42 (LP5)
9	announced in [HadSlo94]	86	74 (LP7)
11	vertices of the 600-cell	120	120 (LP11)
19	vertices of the 600-cell and of its dual, the 120-cell [Sali75]	720	500 (LP \geq 18)

Table 7: Best known cubature formulas for $n = 4$

strength	set	shells	size	bound
7	Γ_8	2	240	240 (T)
11	Γ_8	2, 4	2 400	1 856 (LP8)
13	Γ_8	2, 6, 8	24 240	4 361 (LP12)
15	Γ_8	2, 4, 6, 8	26 400	9 190 (LP16)

Table 8: Cubature formulas for $n = 8$

strength	set	shells	size	bound
5	K_{12}	4	756	157 (D)
7	$K_{12} \cup K'_{12}$	4	1 512	729 (D)
9	$K_{12} \cup K'_{12}$	6	8 064	2 940 (LP6)
11	$K_{12} \cup K'_{12}$	4, 6, 8	50 400	10 604 (LP7)

Table 9: Cubature formulas for $n = 12$

strength	set	shells	size	bound
5	Q_{14}	4	756	211 (D)
7	$Q_{14} \cup Q'_{14}$	4	1 512	1 121 (D)
9	$Q_{14} \cup Q'_{14}$	4, 8	89 964	4 902 (LP6)
11	$Q_{14} \cup Q'_{14}$	4, 6, 8	107 436	20 817 (LP7)

Table 10: Cubature formulas for $n = 14$

strength	set	shells	size	bound
7	BW_{16}	4	4 320	1 633 (D)
9	BW_{16}	4, 6	65 560	7 753 (D)
11	$BW_{16} \cup BW'_{16}$	4, 6	131 520	37 166 (LP7)
13	$BW_{16} \cup BW'_{16}$	4, 6, 10	4 555 200	$\geq 146 153$ (LP8)

Table 11: Cubature formulas for $n = 16$

strength	set	shells	size	bound
5	N_{20}	4	3 960	421 (D)
7	$N_{20} \cup N'_{20}$	4	7 920	3 081 (D)
9	$N_{20} \cup N'_{20}$	4, 6	345 840	17 711 (D)
11	$N_{20} \cup N'_{20}$	4, 6, 8	4 527 600	95 309 (LP7)

In this table, N_{20} denotes any of the three lattices $N_{20}^{(1)}$, $N_{20}^{(2)}$, and $N_{20}^{(3)}$.

Table 12: Cubature formulas for $n = 20$

strength	set	shells	size	bound
7	O_{23}	3	4 600	4 600 (T)
9	O_{23}	3, 5	958 458	29 901 (D)
11	O_{23}	3, 4, 6	6 574 550	166 808 (LP7)

Table 13: Cubature formulas for $n = 23$

strength	set	shells	size	bound
5	N_{24}	6	26 208	601 (D)
7	$N_{24} \cup N'_{24}$	6	52 416	5 201 (D)
11	Λ_{24}	4	196 560	196 560 (T)
15	Λ_{24}	4, 6	16 969 680	$\geq 6\,179\,991$ (LP9)
17	Λ_{24}	4, 6, 8	415 003 680	$\geq 27\,131\,261$ (LP13)
19	Λ_{24}	4, 6, 8, 10	5 044 384 800	$\geq 116\,303\,274$ (LP15)

Table 14: Cubature formulas for $n = 24$

References

- [Andr00] N.N. Andreev, *A minimal design of order 11 on the 3-sphere*, Mat. Zametki **67** (4) (2000) 489–497 (Russian) = Math. Notes **67** (4) (2000) 417–424.
- [BacVen01] Ch. Bachoc, B.B. Venkov, *Modular forms, lattices and spherical designs*, Chap. 2 of [MartV01].
- [BanDam79] E. Bannai, R. Damerell, *Tight spherical designs, I*, J. Math. Soc. Japan **31** (1979) 199–207.
- [BanDam80] E. Bannai, R. Damerell, *Tight spherical designs, II*, J. London Math. Soc. (2) **21** (1980) 13–30.
- [BaMuVe02] E. Bannai, A. Munemasa, B. Venkov, appendix by Y-F.S. Pétermann, *The nonexistence of certain tight spherical design*, Algebra i Analiz **16** (4) (2004) 1–23.
- [Bajn91] B. Bajnok, *Chebyshev-type quadrature formulas on the sphere*, Congr. Numer. **85** (1991) 214–218.
- [ConSlo99] J.H. Conway, N.J.A. Sloane, *Sphere packings, lattices and groups*, third edition, Springer (New York) 1999.
- [DeGoSe77] P. Delsarte, J.-M. Goethals, J.J. Seidel, *Spherical codes and designs*, Geometriae Dedicata **6** (1977) 363–388.
- [Dick19] L.E. Dickson, *History of the theory of numbers, Vol. II*, Carnegie Institution of Washington 1919, reprinted by Chelsea 1966.
- [Ebel94] W. Ebeling, *Lattices and codes, a course partially based on lectures by F. Hirzenbruch*, Vieweg (Braunschweig) 1994. Second revised edition: 2002.
- [GoeSei79] J.-M. Goethals, J.J. Seidel, *Spherical Designs*, Proc. Sympos. Pure Math., XXXIV, 255–272, Amer. Math. Soc. (Providence R.I.) 1979.
- [GoeSei81] J.-M. Goethals, J.J. Seidel, *Cubature formulas, polytopes, and spherical designs*, in: *The geometric vein, the Coxeter Festschrift*, Springer (New York, Berlin) (1981) 203–218.

- [HadSlo94] R.H. Hardin, N.J.A. Sloane, *Expressing $(a^2 + b^2 + c^2 + d^2)^3$ as a sum of 23 sixth powers*, J. Combin. Theory, Series A, **68** (1994) 481–485.
- [HadSlo96] R.H. Hardin, N.J.A. Sloane, *McLaren’s improved snub cube and other new spherical designs in three dimensions*, Discrete Comput. Geom. **15** (1996) 429–441.
- [HarPac04] P. de la Harpe, C. Pache, *Spherical designs and finite group representations (some results of E. Bannai)*, European J. Combin. **25** (2) (2004) 213–227.
- [HarPac-a] P. de la Harpe, C. Pache, *Cubature formulas, geometrical designs, reproducing kernels, and Markov operators*, to appear in the Proceedings of the International Conference on Group Theory, Gaeta (Italy) 1–6 June 2003. Preprint: arXiv:math.CO/0502312.
- [Kupe-a] G. Kuperberg, *Numerical cubature using error-correcting codes*, arXiv:math.NA/0402047.
- [Kupe-b] G. Kuperberg, *Numerical cubature from Archimedes’ hat-box theorem*, arXiv:math.NA/0405366.
- [Lehm47] D.H. Lehmer, *The vanishing of Ramanujan’s function $\tau(n)$* , Duke Math. J. **14** (1947) 429–433.
- [MartV01] J. Martinet ed., *Réseaux Euclidiens, designs sphériques et formes modulaires, Autour des travaux de B. Venkov*, L’Enseignement Mathématique, monographie n° 37 (Genève) 2001.
- [NikNik02] S. Nikova, V. Nikov, *Improvement of the Delsarte bound for τ -designs when it is not the best bound possible*, Des. Codes Cryptogr. **28** (2) (2003) 201–222.
- [Pach-a] C. Pache, *Selfdual lattices viewed as spherical designs*, Intern. J. Algebra Comput., to appear. Preprint: math.CO/0502313.
- [Queb95] H.-G. Quebbemann, *Modular lattices in Euclidean spaces*, J. Number Theory **54** (2) (1995) 190–202.
- [Rank77] R.A. Rankin, *Modular forms and functions*, Cambridge University Press, 1977.
- [Sali75] G.N. Salihov, *Cubature formulas for a hypersphere that are invariant with respect to the group of the regular 600-face*, Dokl. Akad. Nauk. SSSR **16** (1975) 1046–1049 (Russian) = Soviet Math. Dokl. **16** (4) (1975) 1046–1050.
- [Serr85] J.-P. Serre, *Sur la lacunarité des puissances de η* , Glasgow Math. J. **27** (1985) 203–221 = Œuvres, Volume IV, 66–84, see also 640.
- [SeyZas84] P.D. Seymour, T. Zaslavsky, *Averaging sets: a generalization of mean values and spherical designs*, Adv. in Math. **152** (1984) 213–240.
- [Smit95] L. Smith, *Polynomial invariants of finite groups*, A.K. Peters (Wellesley) 1995.
- [Sobo62] S.L. Sobolev, *Cubature formulas on the sphere which are invariant under transformations of finite rotation groups*, Dokl. Akad. Nauk SSSR **146** (1962), 310–313 (Russian).

- [Sobo96] S.L. Sobolev, V.L. Vaskevich, *The theory of cubature formulas*, translated from the 1996 Russian original and with a foreword by S.S. Kutateladze, Kluwer Academic Publishers Group (Dordrecht) 1997 (original in Russian).
- [Venk84] B.B. Venkov, *Even unimodular extremal lattices*, Tr. Mat. Inst. Steklova **165** (1984) 43–48 (Russian) = Proc. Steklov Inst. Math. **165** (3) (1985) 47–52.
- [VenMar01] B.B. Venkov (notes by J. Martinet), *Réseaux et designs sphériques*, Chap. 1 of [MartV01].
- [Vile68] N.Ya. Vilenkin, *Special functions and the theory of group representations*, Transl. Math. Monographs 22, Amer. Math. Soc. (Providence R.I.) 1968 (original in Russian; translated in French).
- [Yudi97] V.A. Yudin, *Lower bounds for spherical designs*, Investiya Math. **61** (3) (1997) 213–223.