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UNIVERSITÉ DE GENÈVE
Section de Physique
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FACULTÉ DES SCIENCES
Professeure Ruth DURRER

Cosmological perturbations in massive gravity and bigravity

THÈSE

présentée à la Faculté des sciences de l'Université de Genève
pour obtenir le grade de
Docteur ès sciences, mention physique

par

Pietro GUARATO

de
Valdagno (Italie)

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intitulée :

"Cosmological Perturbations in Massive Gravity and Bigravity"

La Faculté des sciences, sur le préavis de Madame R. DURRER, professeure ordinaire et directrice de thèse (Département de physique théorique), Monsieur M. MAGGIORE, professeur ordinaire (Département de physique théorique), Madame C. DE RHAM, professeure (Department of physics, Case Western Reserve University, Cleveland, U.S.A.) et Monsieur M. VON STRAUSS, docteur (Physique théorique: gravitation et cosmologie, Institut d'astrophysique de Paris, France), autorise l'impression de la présente thèse, sans exprimer d'opinion sur les propositions qui y sont énoncées.

Genève, le 7 septembre 2016

Thèse - 4984 -

Le Doyen

N.B. - La thèse doit porter la déclaration précédente et remplir les conditions énumérées dans les "Informations relatives aux thèses de doctorat à l'Université de Genève".

Abstract

This Thesis deals with two modified gravity theories which have received a considerable attention in the past years. The first one is the dRGT massive gravity theory proposed by C. de Rham, G. Gabadadze and A. Tolley, in which the graviton is considered as massive and an interaction potential is added to the usual Einstein-Hilbert action. This interaction potential has a very specific structure and requires the introduction of a second, non-dynamical metric. The Hassan-Rosen bigravity model basically extends the dRGT model by giving this "reference metric" its own dynamics. We derive a formalism to study perturbations in both these theories in complete generality in the first part of the Thesis. This formalism can in principle be applied to any cosmological settings. The second part is devoted to an analysis of perturbations for a specific sub-model of the Hassan-Rosen bimetric theory. In particular, we focus on gravitational waves, and we find that, despite the presence of an instability at late times, from a pure analysis of the tensor sector the model cannot be ruled out, since this instability is pushed towards the future due to the strong suppression of the tensor perturbations amplitude at the end of inflation.

Contents

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List of Publications

The following works have been considered as part of the Thesis:

[77] P. Guarato and R. Durrer, "Perturbations for massive gravity theories", Phys. Rev. D **89** (2014) 084016 [arXiv:1309.2245 [gr-qc]].

[34] G. Cusin, R. Durrer, P. Guarato and M. Motta, "Gravitational waves in bigravity cosmology", JCAP **1505** (2015) 030 [arXiv:1412.5979 [astro-ph]].

[35] G. Cusin, R. Durrer, P. Guarato and M. Motta, "Inflationary perturbations in bimetric gravity", JCAP **1509** (2015) 043 [arXiv:1505.01091 [astro-ph]].

[36] G. Cusin, R. Durrer, P. Guarato and M. Motta, "A general mass term for bigravity", JCAP **1604** (2016) 051 [arXiv:1512.02131 [astro-ph]].

Notation

Here we set some notation frequently used throughout this Thesis.

Constants and units. We will always (unless otherwise specified) assume natural units in which the reduced Planck constant $\hbar \equiv h/(2\pi)$, the speed of light c and the Boltzmann constant $k_{\text{Boltzmann}}$ are set to one: $\hbar = c = k_{\text{Boltzmann}} = 1$. G denotes the Newton's gravitational constant, and we never set $G = 1$. We make often use of the "reduced Planck mass" $M_g = 1/\sqrt{8\pi G} \equiv M_p \simeq 2.4 \times 10^{18}\text{GeV}$.

Tensor calculus. We work with the metric signature $(-, +, \ldots, +)$ and we restrict to $D = 4$ spacetime dimensions. Greek (Lorentz) indices, such as $\alpha, \beta, \ldots, \mu, \nu, \ldots$ take the values $0, \ldots, 3$, while spatial indices are denoted by Latin letters $i, j, \cdots = 1, 2, 3$. With $\cdot$ and with $'$ we indicate derivatives with respect to physical time t and to conformal time τ, respectively. We also define

$$x^\mu = (x^0, \mathbf{x}) , \qquad x^0 = c\,t , \qquad \partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\partial_t, \partial_i\right) \qquad \mathrm{d}^4x = \mathrm{d}x^0\mathrm{d}^3x = c\,\mathrm{d}t\,\mathrm{d}^3x .$$

Repeated upper and lower Lorentz (or spatial) indices are summed over, e. g. $A_\mu B^\mu \equiv \sum_{\mu=0}^{3} A_\mu B^\mu$. Our conventions on the metric signature and Riemann tensor are the same as Misner, Thorne and Wheeler [120]. We denote the curved space-time metric by $g_{\mu\nu}(x)$ and its determinant by $\det g$ (so $\det g < 0$). The Christoffel symbol is

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) .$$

The Riemann tensor is defined as

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\alpha\rho}\Gamma^\alpha_{\nu\sigma} - \Gamma^\mu_{\alpha\sigma}\Gamma^\alpha_{\nu\rho} .$$

The Ricci tensor is $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$, and the Ricci scalar is $R = g^{\mu\nu}R_{\mu\nu}$. The Einstein equations (without cosmological constant) read

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} ,$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor and $T_{\mu\nu}$ is the stress-energy tensor (or energy-momentum tensor). The (flat space) d'Alambertian operator is $\Box \equiv \partial_\mu \partial^\mu = -\partial_0^2 + \boldsymbol{\nabla}^2$. The notation $\mathbf{A}$ denotes a spatial vector whose components have upper indices, $\mathbf{A} = (A^1, A^2, A^3)$. We sometimes use $\boldsymbol{\nabla} \cdot \mathbf{v}$ as $\partial_i v^i$. The covariant derivative of a $(1,1)$ tensor is defined as $\nabla_\lambda A^\mu{}_\nu \equiv \partial_\lambda A^\mu{}_\nu + \Gamma^\mu_{\rho\lambda} A^\rho{}_\nu - \Gamma^\rho_{\lambda\nu} A^\mu{}_\rho$. We often denote the trace of a rank-2 tensor as $[\dots]$, as in e. g. $[\Pi] \equiv \Pi^\mu{}_\mu$, and the symmetrization of a tensor with respect to a couple of indices as $A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})$.

Abbreviations.

(C)DM	(Cold) dark matter
CMB	Cosmic microwave background
FLRW	Friedmann-Lemaître-Robertson-Walker homogeneous and isotropic metric
GR	General Relativity
IR	Infrared
Λ	Cosmological constant
ΛCDM	"Standard" cosmological model with CDM and cosmological constant
LSS	Large scale structure
UV	Ultraviolet

Résumé

Dans la description courante de la physique théorique, les forces fondamentales de la nature sont la force gravitationnelle, la force electromagnetique, la force nucléaire forte et la force nucléaire faible. Alors que les trois dernières forces ont une description cohérente dans le modèle standard de la physique des particules en tant que champs quantiques, la force gravitationnelle a une description en termes purement classiques donnée par la théorie de la Relativité Générale d'Einstein [66]. En outre, la gravité constitue la force la plus importante sur les larges échelles.

La Relativité Générale identifie le champ gravitationnel avec la «métrique» qui détermine la géométrie spatio-temporelle de l'Univers et fournit une connexion très élégante entre cette géométrie et la matière qui constitue notre Univers. Même si cette théorie a passé une série impressionnante de tests observationnels et experimentaux, elle est encore confrontée avec une série de problèmes, dans le régime *ultraviolet* (UV) tout comme dans le régime *infrarouge* (IR).

La phénoménologie de la Relativité Générale joue naturellement un rôle sur les échelles cosmologiques. Il y a quelques années, en 2011, le Prix Nobel pour la Physique a été attribué à S. Perlmutter, B. P. Schmidt et A. G. Riess pour les données de l'observation de Supernovae de type Ia (SN Ia) qui dans l'année 1998 ont mis en évidence que l'expansion actuelle de l'Univers est accélérée [129, 127], contrairement à ce qu'on s'attendrait si la gravité (en tant que force attractive) était la seule force dominante aux échelles cosmologiques. Cette découverte a amené la communauté scientifique à s'interroger sur la possible source de cette accélération, communément appelé «énergie sombre», une composante mystérieuse de notre Universe avec pression négative qui domine aujourd'hui.

La «constante cosmologique», Λ, à première vue semble être le candidat le plus simple pour l'énergie sombre. Les equations d'Einstein deviennent $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$, où $G_{\mu\nu}$ est le tenseur d'Einstein tensor et $T_{\mu\nu}$ est le tenseur énergie-impulsion. Dans un contexte cosmologique, ces équations montrent que Λ joue le rôle d'une composante avec pression négative, dont on a besoin pour reproduire l'accélération cosmique observée. Dans le «modèle standard de la cosmologie» (ou «modèle ΛCDM») la Relativité Générale est donc assumée être la théorie correcte de la gravitation et l'Univers contient de la matière noire froide et une constante cosmologique qui représente l'énergie

sombre. Ce modèle est en excellent accord avec les données cosmologiques actuellement à notre disposition.

Du point de vue de la physique des particules, Λ représente la densité d'énergie du vide. Le problème principal posé par cette hypothèse est qu'il y a un désaccord d'environ 120 ordres de grandeur entre la densité d'énergie du vide calculée en recourant à la théorie quantique des champs et la densité d'énergie sombre réellement observée. La valeur observée est très petite mais pas zéro, contrairement à ce que la plus part des gens pensait avant la découverte de l'expansion accélérée de l'Universe.

Si le problème de la constante cosmologique est résolu de façon telle que Λ est égale à zéro, il faut expliquer de manière différente l'accélération cosmique. Ces théories alternatives amènent à modifier les equations d'Einstein du côté gauche (géométrique) - on parle dans ce cas-là de «modèles de gravité modifiée» - ou du côté droite (matière) - et on parle alors de '«modèles de matière modifiée» [16].

Nous allons nous focaliser sur le premier type de théories, en particulier sur deux récentes théories qui modifient la Relativité Générale en introduisant un nouveau champ de spin 2 qui intéragit avec la métrique $g_{\mu\nu}$. Ces théories sont la *gravité massive* (dans la formulation donnée par de Rham, Gabadadze et Tolley [47, 49] et par Hassan et Rosen [82] respectivement, communément appelée *modèle dRGT*), dont l'idée originaire est due à M. Fierz et W. Pauli en 1939 [71, 126], et la théorie de la *bi-gravité* (ou *gravité bi-métrique*) créée par Hassan et Rosen [83]. En Relativité Générale, le graviton est considéré être une particule de spin 2 sans masse avec deux degrés de liberté. En gravité massive, on donne une masse au graviton[1] et par conséquence le nombre de degrés de libertés est augmenté à cinq. Cette théorie demande l'introduction d'un deuxième champ tensoriel, la «métrique de référence» $f_{\mu\nu}$, qui entre dans l'action dans un potentiel qui couple les deux champs tensoriels. La forme de ce potentiel est contraint par la nécessité d'éviter l'apparition de degrés de liberté instables appelés *ghosts*. En gravité massive la métrique de référence n'est pas déterminée dynamiquement mais elle est fixée *a priori*. La bi-gravité étend la théorie de la gravité massive en donnant à la deuxiéme métrique sa propre dynamique, introduisant un terme d'Einstein-Hilbert pour cette métrique aussi. La bi-gravité décrit sept degrés de liberté, dont deux associés à un champ de spin 2 sans masse et cinq associés à un champ de spin 2 massif. L'élaboration d'une théorie de gravité massive est intéressante en elle-même, mais aussi parce que, si le graviton possède une masse, alors les effets de la force gravitationnelle sont supprimés *à la Yukawa* sur des distances plus grandes que la longueur d'onde Compton de la particule. Donc, si $m \sim H_0$, la gravité est affaiblie sur des distances de l'ordre de la longueur de Hubble. Pour cette raison, la gravité massive est dans une bonne position pour expliquer l'expansion accélérée de l'Universe aujourd'hui.

Dans cette thése, nous visons à étudier les perturbations cosmologiques de la grav-

[1] Pour être compatible avec les contraintes phénoménologiques les plus directes, il faut que cette masse soit suffisamment petite, i.e. $m_g < 1.2 \times 10^{-22}$ eV [1, 2, 3, 4].

ité massive et de la bi-gravité, et à contraindre les modèles de bi-gravité de façon établir s'ils peuvent fournir une alternative valide au modèle ΛCDM ou pas. Nous sommes particulièrement intéressés aux instabilités qui peuvent surgir dans certaines cosmologies et qui peuvent potentiellement exclure le modèle de la liste des possibles alternatives au modèle standard de la cosmologie.

Dans le chapitre 2 nous exposons les principaux aspects théoriques de la gravité massive et de la bi-gravité, spécialement du point de vue de leur développement historique.

Le chapitre 3 présente en détail la détermination du terme de masse quadratique dans l'action pour le modéle dRGT. Ce résultat est valide pour des métriques complètement génériques et il peut donc être appliqué dans différentes cosmologies, e. g. pour étudier la présence d'instabilités.

Le chapitre 4 étend le formalisme développé dans le chapitre 3 à la théorie de bi-gravité de Hassan et Rosen. Ce formalisme est ensuite appliqué à une classe particulière de solutions cosmologiques de la bi-gravité, la «branch I» de solutions des équations de Friedmann. Nous trouvons que les perturbations tensorielles dans cette «branch» sont affectées par une instabilité, qui se développe lorsque la matrice de masse n'est plus définie positive.

Dans le chapitre 5 nous étudions la cosmologie de la théorie de bi-gravité de Hassan et Rosen dans l'autre classe de solutions («branch II») des équations de Friedmann et dans la situation où le rapport entre les facteurs d'échelle des deux métriques évolue de l'infini à une valeur finie et où les valeurs de tous les paramètres β_n de la théorie à l'exception de β_1 et β_4 sont nuls. En particulier, nous nous sommes focalisés sur les perturbations tensorielles, et nous avons trouvé qu'il y a une instabilité (de type loi de puissance) dans le secteur de $f_{\mu\nu}$ qui est transférée au secteur de la métrique physique par le moyen du couplage entre les deux métriques. Ceci pourrait en principe éliminer le modèle; de toute façon, si les conditions initiales pour l'évolution des perturbations tensorielles de $f_{\mu\nu}$ sont fortement supprimées par rapport à celles de $g_{\mu\nu}$, l'instabilité peut être repoussée vers l'avenir et on peut retrouver l'évolution habituelle des ondes gravitationnelles en ΛCDM.

Dans le chapitre 6 nous étudions la génération des perturbations primordiales durant l'inflation dans la même situation cosmologique de la bi-gravité du chapitre 5. Nous trouvons que l'amplitude des perturbations tensorielles générées durant l'inflation est suffisamment supprimée pour éviter (au moins jusqu'à aujourd'hui) l'apparition dans le secteur physique de l'instabilité analysée dans le chapitre 5 qui se développe durant l'évolution cosmologique. Nous commentons également brièvement les secteurs vectoriel et scalaire: les perturbations vectorielles amènent des contraintes faibles sur les échelles viables d'énergie inflationnaire pour ce modèle, lorsque les perturbations scalaires sont affectées par une instabilité (exponentielle) de type Higuchi.

Dans le chapitre 7 nous concluons et discutons des possibles développements futurs.

Chapter 1

Introduction

In the current picture of theoretical physics, the fundamental forces of nature are the gravitational, the electromagnetic, the strong nuclear, and the weak nuclear forces. While the last three forces are consistently described as quantum fields in a coherent framework given by the Standard Model of particle physics, the gravitational force is described as a purely classical field by Albert Einstein's theory of General Relativity [66]. Moreover, strong and weak interactions are short-range and hence they have effects on subatomic scales, whereas the electromagnetic and the gravitational interactions produce effects on macroscopic scales. Since large collections of objects tend to be electromagnetically neutral, gravity remains the main important force on large scales.

General Relativity identifies the gravitational field with the "metric" which determines the space-time geometry of our Universe and provides a very elegant connection between the matter content of the Universe and the space-time geometry. This theory has been able so far to remarkably pass an impressive series of experimental and observational tests to a very high precision. Despite these successes, General Relativity is still confronted with a series of issues, both in the Ultraviolet (UV) and in the Infrared (IR) regime, where the first one corresponds to high-energy scales and correspondingly (in natural units, in which $c = \hbar = 1$) short-distance scales, whereas the second regime corresponds to low-energy (large-distance) scales.

On UV scales, the main problem is that General Relativity and Quantum Mechanics are presently incompatible. More precisely, the theory which can be obtained by quantizing classical General Relativity is non-renormalizable, which implies that its validity is limited to scales up to a cut-off energy scale at which quantum effects become relevant, and that there is a loss of predictivity on energy scales higher than the cutoff (see e. g. [128]). However, since this Thesis does not concern the UV completion of General Relativity, we will not insist on this issue any further.

Naturally the IR behaviour of General Relativity plays a role on cosmological scales.

Some years ago, in 2011, the Nobel Prize in Physics was awarded to S. Perlmutter, B. P. Schmidt and A. G. Riess due to the observational data of Supernovae Type Ia (SN Ia) accumulated by the year 1998 which have been independently reported in [129] and [127] and which evidence that the expansion of the present Universe is accelerating, and not decelerating as it would to be the case if gravity (as an attractive force) were the only dominant force on cosmological scales. This fact has brought the scientific community to conjecture about the physical source of this acceleration, commonly denominated "dark energy", a mysterious component of our Universe with negative pressure which dominates today.

The "cosmological constant" (usually denoted by Λ) appears to be the simplest candidate for dark energy. It enters the Einstein field equations through an additive term which is simply proportional to the space-time metric $g_{\mu\nu}$: $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$, where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ the energy-momentum tensor which describes the matter content of the Universe. According to Lovelock's theorem [112, 113], these are the only second-order gravitational field equations involving the metric tensor and its derivatives only that we can construct in a four-dimensional Riemannian space from an action principle. In a cosmological setting, these equations of motion tell us that Λ plays the role of a negative pressure component, which is needed to get the observed cosmic acceleration.

From the particle physics point of view, it is natural to interpret Λ as the vacuum energy density. In fact, vacuum energy density and the cosmological constant have identical behaviour in General Relativity, as long as the vacuum energy density is identified with $\rho_{vac} = \Lambda/(8\pi G)$. However, there are two issues which plague the hypothesis of the cosmological constant as dark energy (see e. g. [16]).

The first problem, the so-called "fine-tuning problem" (or "old cosmological constant problem"), is the fact that the vacuum energy density evaluated on the theoretical side by summing the zero-point energy of a scalar field is about 10^{121} times larger than the observed dark energy density for a momentum cut-off around the Planck scale. This observed value is extremely tiny but not zero, as most people believed before the observational discovery of the cosmic acceleration. At that time people tried to explain why the cosmological constant was exactly zero on the theoretical side by relying on the existence of some symmetry, e. g. in the context of supersymmetric theories. The proof that Λ actually vanishes is however still missing.

The second problem, the "coincidence problem" (or "new cosmological constant problem"), is that the value of the cosmological constant is almost identical to an apparently unrelated number, the present matter energy density, for no obvious reason. In other terms, the observed value $\Omega_\Lambda^{(0)}$ is doubly unlikely: because it is too small compared with the theoretical prediction and because its value coincides (to a factor of about 2 or 3) with $\Omega_\mathrm{m}^{(0)}$.

If the old cosmological constant problem is solved in a way that Λ completely vanishes, we need to explain in a different way the late-time cosmic acceleration. Basically two approaches have been explored. The first one consists in searching for dynamical models of dark energy or "modified matter models" in which this component of the Universe is described by an exotic matter source with a negative pressure, which is contained in the energy-momentum tensor on the right-hand side of the Einstein field equations.

The second approach is based on "modified gravity models" in which the Einstein tensor on the left-hand side of the Einstein equations is modified. These models can hence be regarded as true modifications of Einstein's gravitational law which display their effects on large scales, so that the late-time accelerated expansion of the Universe is realized without recourse to an extra, exotic matter component.

A part of these models had been introduced even before the discovery of the cosmic acceleration, as attempts to extend Einstein's theory of gravity. A review of these theories can be found e. g. in [16] or in [28].

These models need of course to satisfy an important numbers of constraints in order to fit well the cosmological data. During the last decades, the main sources of these data, which have nowadays reached an impressive level of precision, have been the observations of the Cosmic Microwave Background (CMB) and of the Large Scale Structure (LSS) of the Universe, performed by satellites such as COBE, WMAP or Planck and galaxy redshift surveys such as SDSS, 2dF or 6dF, respectively. These observations have allowed to put increasingly tight constraints on a variety of cosmological models and parameters. According to the latest observational data released by the Planck mission [6], our Universe is spatially flat ($\Omega_k < 10^{-3}$) and composed by a small part of radiation ($\Omega_{rad} \sim 10^{-5}$), whereas dark energy ($\Omega_\Lambda \sim 0.7$) and matter ($\Omega_m \sim 0.3$, mostly composed of dark matter) are dominant today. These data are remarkably well fitted by the "standard model" of cosmology, the so-called ΛCDM model, in which General Relativity is assumed to be the correct theory of gravitation and in which the Universe contains cold dark matter and a cosmological constant as dark energy. One of the main challenges for alternative models of dark energy and modified gravity is therefore to be competitive with the ΛCDM model at the level of the agreement with observations.

On smaller scales, local gravity tests (that is, observational tests of gravity within the solar system) are true measurements of general relativistic effects. General Relativity has been tested accurately only at the present time, in laboratory experiments and within the solar system up to $10^{14}\,$m, but it is very weakly constrained on large scales: this leaves some room for consistent modifications of the gravitation theory in the IR. However, modified gravity models need to evade the tight constraints coming from local gravity tests in order to be reduced to the usual General Relativity at short distances.

Throughout this Thesis we will focus on two recent Lorentz-invariant theories which modify General Relativity by the introduction of an additional spin-2 field interacting with the metric $g_{\mu\nu}$, the *massive gravity* theory (in the formulations given by de Rham, Gabadadze and Tolley [47, 49] and by Hassan and Rosen [82] respectively) and the *bi-gravity* (or *bimetric gravity*) theory (as given by Hassan and Rosen [83]). In General Relativity the graviton is considered to be a massless spin-2 particle with two degrees of freedom. In massive gravity, the main idea consists in giving the graviton a mass[1], which increases the gravitational degrees of freedom to 5 and in general requires the introduction of a second tensor field, the "reference metric", through a special potential term in the action which couples the two tensor fields. The form of this potential is constrained to avoid the appearance of spurious, unstable degrees of freedom called 'ghosts'. This new potential term of course breaks the diffeomorphism invariance of General Relativity, since massive gravity is meant to describe the 5 degrees of freedom of a massive spin-2 particle, while gauge invariance reduces them to the two needed to describe a massless spin-2 particle. In massive gravity the reference metric is not dynamically determined (there is no kinetic term for this metric in the theory) but it is fixed by hand. Bigravity extends the massive gravity theory by giving the reference metric its own dynamics through the introduction in the action of an additional Einstein-Hilbert term for this metric, so that the main spirit of General Relativity is restored. The interaction potential, which is the same as in massive gravity, breaks the two diffeomorphism invariances (each for one metric) down to their diagonal subgroup, in such a way that this new theory describes 7 degrees of freedom, 2 associated to a massless and 5 to a massive spin-2 field.

The beginning of both these theories, massive gravity and bimetric gravity, can be traced further back in time. In fact, the first publications on the unique classically consistent linearized theory of a free massive spin-2 field propagating on a Minkowski space-time are due to Fierz and Pauli back in 1939 [71, 126], whereas bimetric theories of gravity where first introduced in [97] back in 1971. Although the question about the graviton having a mass being interesting on itself, massive gravity is also a 'nice' candidate for dark energy because, if the graviton has a mass, then the effects of gravity are Yukawa suppressed on distances larger than the Compton wavelength of the particle. Therefore, if $m \sim H_0$, gravity becomes weaker on distances of the order of the Hubble scale.

Another motivation to study massive gravity as an alternative to the cosmological constant is that, since its interaction potential breaks diffeomorphism invariance, the spin-2 mass scale is expected to be protected from receiving large quantum corrections. Therefore, one hopes that the interactions of the graviton could give rise to a small amount of cosmic acceleration, which, unlike the cosmological constant, can be consid-

[1] In order to be compatible with the present most direct phenomenological bounds, this mass should be sufficiently small, i.e. $m_g < 1.2 \times 10^{-22}$ eV [1, 2, 3, 4].

ered as "technically natural", according to the definition given by 't Hooft[2] [135, 27].

The results of the application of these theories to a cosmological setup have had the merit of bringing a new light on the issues and the possibilities that we have when dealing with the search for consistent and cosmologically viable modifications of gravity in the IR. This is what we want to reach with this Thesis: by studying cosmological perturbations of bigravity (in particular, gravitational waves) we want to put constraints on these models in such a way to find out whether they can provide a valid alternative to the ΛCDM model or they have to be ruled out. We are particularly interested in the instabilities which can arise in these cosmological settings and potentially rule out the model. We focus on the situation in which only one of the two metrics is coupled to the stress-energy tensor. Even if in the literature other possibilities have also been explored (see e. g. [11, 52, 89]), this is the only ghost-free coupling.

The main scopes of this Thesis are: first, to derive the full expression of the quadratic mass term for general perturbations at the level of the action both in the ghost-free massive gravity theory (with general reference metric) and in the ghost-free bimetric theory; and secondly, to investigate some cosmological applications of the linear perturbation theory to the case of bimetric gravity in order to see whether this theory can give rise to a viable description of our Universe, according to the current observational data. Part I of the Thesis is devoted to the first scope and part II to the second one.

In chapter 2 we expose the main theoretical aspects of both massive gravity and bigravity theories, especially from the point of view of their historical development.

Chapter 3 presents in detail the calculation of the quadratic mass term in the action for the dRGT theory of massive gravity. This result is valid for completely general physical and reference metrics and can therefore be applied to any cosmological setups, e. g. to investigate the presence of instabilities.

Chapter 4 extends the formalism of chapter 3 to the Hassan-Rosen bigravity theory, which shares with the dRGT massive gravity model the same structure of the interaction potential but has an additional Einstein-Hilbert term for the reference metric $f_{\mu\nu}$ in the action. This enriches considerably the dynamics of the theory. This formalism is then applied to a particular cosmological setting of bigravity, the so-called "branch I" (or "algebraic branch") of solutions to the Friedmann equations. We find that the tensor perturbations in this branch are affected by a late-time instability, which sets in when the mass matrix is no longer positive definite.

In chapter 5 we study the cosmology of Hassan-Rosen bigravity in the alternative

[2]The criterion of the *'t Hooft naturalness* states that when a theory has more symmetry when a small parameter vanishes than it has when it does not vanish, the enhanced symmetry prevents quantum corrections from generating a nonzero parameter if it is initially taken to be zero. This typically ensures that corrections to an initially small parameter must then themselves also be small. This is equivalent to say that a parameter of the theory is allowed to be small only if setting it to zero increases the symmetry of the theory: if this does not happen, the theory is technically unnatural.

“branch II”, in the specific setting in which the ratio between the scale factors of the two metrics evolves from infinity to a finite value and the values of all the β_n parameters of the theory but β_1 and β_4 are zero. We have studied this model both at the background and at the perturbations level. In particular, we have focused on tensor perturbations, and we have found that there is a Higuchi (power-law) instability in the f-sector which is transferred to the physical sector of the theory through the coupling between the two metrics. This could in principle rule out the model; however, if the initial conditions for the evolution of the tensor perturbations of $f_{\mu\nu}$ are highly suppressed with respect to the $g_{\mu\nu}$ ones, one can still push the instability towards the future and recover the usual behaviour of gravitational waves in ΛCDM.

In chapter 6 we study the generation of primordial perturbations during inflation in the same cosmological setting of bigravity as in chapter 5. We find that the amplitude of tensor perturbations generated during inflation is sufficiently suppressed to avoid (at least until today) the appearance in the physical sector of the instability analyzed in chapter 5 which develops during the cosmological evolution. We also comment briefly on the vector and scalar sectors: vector perturbations add some mild constraints on the viable inflationary energy scales for this model, whereas scalar perturbations are plagued by a Higuchi (exponential) instability.

In chapter 7 we conclude and discuss possible future developments.

In appendix A we review the main aspects of cosmological perturbations as it has been developed in the framework of the theory of General Relativity. We present some of the most important tools on which we relied to derive the main results of this Thesis.

Chapter 2

From massive gravity to bigravity

This chapter is devoted to a review of the main aspects of the theories of massive gravity and bigravity, which are among the most widely studied theories which attempt to modify Albert Einstein's theory of General Relativity (GR). The chapter is structured as follows: first, we review the analysis of both massless and massive representations in field theory according to group theory, which allows us to determine the exact number of propagating degrees of freedom for massless and massive particles of a given spin; secondly, we focus on the case of the spin-2 particle, the graviton, described by a tensor field $h_{\mu\nu}$ propagating in a flat space-time, for which we derive the linearized action for the kinetic term and then construct the only viable mass term at the linear level (found by Fierz and Pauli back in 1939); in doing so, we introduce the concepts of gauge symmetry (which helps to determine the correct number of propagating modes in a theory) and of Ostrogradskij instability or "ghost" (whose absence is required in order to have a consistent theory); then we discuss the issues that have plagued for a long time all the attempts of finding a consistent completion of the Fierz-Pauli theory, mainly concerning the appearance of a ghost at the full non-linear level; and afterwards, we deal with the recently proposed consistent non-linear model of massive gravity known as "dRGT model", both in the original formulation by de Rham, Gabadadze and Tolley [47, 49] and in the one by Hassan and Rosen [82]. In the last part of the chapter, we present the extension of the dRGT model in the context of a bimetric theory of gravity, the "Hassan-Rosen bigravity" [83], in which the reference metric, which was kept fixed in the dRGT theory, acquires its own dynamics.

2.1 Massless and massive particles in field theory

In order to understand the features of the graviton, the elementary spin-2 boson which, according to quantum field theory, mediates the gravitational interaction, we need to introduce the notion of “spin" of a particle. We start by writing the full space-time metric of General Relativity as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$: in this way, we can *linearize* General Relativity and hence we can treat it as a field theory of the field $h_{\mu\nu}$ living in flat space-time described by the Minkowski metric $\eta_{\mu\nu}$. In this approach, we partially forget about the beautiful geometric interpretation of General Relativity according to which an incoming gravitational wave described in terms of $h_{\mu\nu}$ induces perturbations in the space-time geometry whose metric is given by $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ is some curved, dynamical background metric and $|h_{\mu\nu}| \ll |\bar{g}_{\mu\nu}|$ is the perturbation. However, since a flat space-time background is an excellent approximation in many situations, it makes sense to look for a relativistic quantum field theory living in flat space-time, that in the non-relativistic limit reduces to Newtonian gravity and that is mediated by a spin-2 boson which propagates in this flat space-time.

As we know, fields can be classified accordingly to their transformation properties under the Lorentz group (or the Poincaré group, the group of space-time translations plus Lorentz transformations)[1] [114]; for example, fields which transform as scalars under the transformations of the Lorentz group are called *scalar fields*, fields which transform as vectors under the Lorentz group are the *vector fields*, and so on. The irreducible tensor representations of the Lorentz group are given by tensors that, with respect to any pair of indices, are either symmetric and traceless, or antisymmetric [114]. An irreducible representation of the Lorentz group provides of course also a representation of the spatial rotation subgroup $SO(3)$, and we can therefore ask what is the spin s of the various tensor representations. Moreover, since a representation

[1] A Lie group is a group whose elements g depend in a continuous and differentiable way on a set of real parameters θ_a. A (linear) representation R of a group is an operation which assigns to an abstract element g of a group a linear operator $D_R(g)$ defined on a linear space, $g \to D_R(g)$, with the properties that the identity element of the group goes into the identity operator and that the mapping preserves the group structure. A generic representation $D_R(g)$ of a group element can be written as

$$D_R(g(\theta)) = e^{i\theta_a T^a_R}\,, \tag{2.1}$$

where T^a_R is a representation of an element T^a of the *Lie algebra* associated to the Lie group. The algebra is defined through the commutation rules which are independent of the representation:

$$[T^a, T^b] = i f^{ab}_c T^c\,, \tag{2.2}$$

for some constants f^{ab}_c. The elements of a Lie algebra are called *generators* of the group.

The space on which the operators D_R act is called the *basis* for the representation R, and the *dimension* of the representation is defined as the dimension of the base space.

A representation R is called *reducible* if it has a non-trivial invariant subspace, i.e. if the action of any $D_R(g)$ on the vectors in the subspace gives another vector of the subspace. Contrariwise, a representation with no non-trivial invariant subspace is called *irreducible*.

The Lorentz and the Poincaré group are two of the most important Lie groups used in physics. In particular, fields provide an infinite-dimensional representation of the Lorentz group (or the Poincaré group), and the notion of mass emerges from the representations of the Poincaré group.

that is irreducible with respect to the full Lorentz group can be reducible if we limit ourselves to the spatial rotation subgroup, it can be decomposed into the direct sum of irreducible representations of the group $SO(3)$ (the only exception is given by the trivial case of a Lorentz scalar, which is of course also a scalar under spatial rotations, so it has $s = 0$). For example, a four-vector V_μ is an irreducible representation of the Lorentz group, but from the point of view of spatial rotations it decomposes into a scalar ($s = 0$) and a vector ($s = 1$) representation, that is, we can write

$$V_\mu \in \mathbf{0} \oplus \mathbf{1}\,, \tag{2.3}$$

where $\oplus$ denotes the direct sum of representations and $\mathbf{s}$ the representation of the rotation group $SO(3)$ corresponding to spin s (so $\mathbf{0}$ is the scalar and $\mathbf{1}$ is the vector representation). The representation $\mathbf{s}$ of $SO(3)$ has dimension $2s + 1$, so in particular the scalar is 1-dimensional and the vector is a 3-dimensional representation. When we consider tensors with two indices, an antisymmetric tensor $A_{\mu\nu}$ (which has 6 independent components) decomposes as two spatial vectors, and therefore

$$A_{\mu\nu} \in \mathbf{1} \oplus \mathbf{1}\,, \tag{2.4}$$

while a traceless symmetric tensor $S_{\mu\nu}$ (which has 9 independent components) decomposes with respect to $SO(3)$ as

$$S_{\mu\nu} \in \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}\,. \tag{2.5}$$

Hence, the simplest tensor that contains a spin-2 field is the traceless symmetric tensor $S_{\mu\nu}$. From eq. (2.5) we can see that the 9 components of a traceless symmetric tensor with two indices can be decomposed into a spin-0, the 3 components of a spin-1, and the 5 ($2s+1$ with $s = 2$) components of a spin-2 field: a spin-2 can therefore be described using $S_{\mu\nu}$ and imposing conditions to eliminate the extra degrees of freedom.

If we then want to introduce the notion of “mass" of a particle (from the point of view of quantum field theory), we need to refer ourselves to the representations of the Poincaré group and to study how this group can be represented using the Hilbert space of a free particle as a basis for the representation. The generators of the Poincaré group are given by the components of the four-momentum operator P^μ (generators of space-time translations) and the generators of the Lorentz group $J^{\mu\nu}$. The representations are labeled by the eigenvalues of the Casimir operators[2] which, for the Poincaré group, are $P^2 \equiv P_\mu P^\mu$ and $W^2 \equiv W^\mu W_\mu$, where $W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma$ is the *Pauli-Lubanski operator*.

We then distinguish two physically interesting representations of the Poincaré group:

[2] The Casimir operators of a Lie algebra are the operators which commute with all the generators of the algebra.

- *Massive representations*, which describe massive particles. In this case on the one-particle states we have[3] $-P^2 = m^2$ while $W^2 = m^2 s(s+1)$. Hence, the representations are labeled by the mass m and by the spin s. If $m \neq 0$, with a Lorentz transformation we can bring the four-momentum into the form $P^\mu = (m, 0, 0, 0)$ (in the rest frame of the particle). This particular choice of P^μ still leaves us with the freedom to perform spatial rotations. The group of transformations which leaves invariant a given form of P^μ is called the *little group*. However, since we want to include spinor representations of the Lorentz group, the little group in this case is[4] $SU(2)$ and not $SO(3)$. Therefore, the spin labelling the massive representations can take values $s = 0, 1/2, 1, 3/2, \dots$, and states within each representation are labeled by $s_z = -s, -s+1, \dots, s$. Hence, *massive particles of spin s have $2s+1$ degrees of freedom*;

- *Massless representations*, which describe massless particles. When $m = 0$, the rest frame of the particle does not exist, but we can still bring P^μ into the form $P^\mu = (\omega, 0, 0, \omega)$. The little group is then the set of Poincaré transformations which leaves this vector invariant. It is immediate to see that the rotations in the (x, y) plane (which is an $SO(2)$ group) leave this particular form of P^μ unchanged. The generator of the group $SO(2)$ of rotations in the (x, y) plane is the angular momentum J^3 and hence the irreducible representations of $SO(2)$, which are 1-dimensional, are labeled by the eigenvalue h of J^3, which represents the projection of the angular momentum in the direction of propagation of the particle (in this case, the z axis). It is called the *helicity*, and it can be shown that it can take the value $h = 0, \pm 1/2, \pm 1, \dots$. Massless representations are actually 2-dimensional instead of 1-dimensional if we also require that they are representations of parity (that is, if the interaction conserves parity), in which case it is natural to assemble together the two helicity states $h = \pm j$. To this day, the only massless particles which can be observed as free particles in nature are two bosons: the photon and the graviton[5]. Since both the electromagnetic interaction and the gravitational interaction conserve parity, we can conclude by saying that in this context *massless particles have two degrees of freedom, characterised by the two helicity (or polarisation) states $h = \pm j$.*

[3] We will restrict to m real and positive, even if, in principle, there is also the possibility of representations with $m^2 < 0$, which are called *tachyons*.

[4] The main difference between $SU(2)$ and $SO(3)$ is that the representations of $SO(3)$ are labeled by an index s which takes only integer values, whereas the representations of $SU(2)$ are labeled by an index s which takes also half-integer values, $s = 0, 1/2, 1, 3/2, \dots$ and gives the spin of the state, in units of $\hbar$. The spin-s representation has dimension $2s+1$, and the various states within it are labeled by s_z, which takes the values $-s, \dots, s$ in integer steps.

[5] Actually, even though gravitational waves have been directly detected in 2015 by the LIGO experiment [3], this has not lead us to detect individual quanta of graviton.

This signifies that a massive spin-1 and a massive spin-2 particle have three and five degrees of freedom respectively, whereas a massless spin-1 particle and a massless spin-2 particle have only two degrees of freedom, the former with helicities $h = \pm 1$ and the latter with $h = \pm 2$.

2.2 Massless spin-2 field: linear theory

In order to find the theory which describes a massless spin-2 particle (the graviton), we start with a traceless symmetric tensor as in (2.5), then we impose a local invariance to eliminate the extra degrees of freedom and keep only the components of the spin-2 field, and we write down a Lagrangian that respects this local symmetry. Then, using the equations of motion which can be derived from this Lagrangian, we can count the number of degrees of freedom in our theory. Actually it is technically simpler to start from a tensor $h_{\mu\nu}$ which is symmetric, but not traceless, so that from the point of view of spatial rotations it decomposes as $h_{\mu\nu} \in \mathbf{0} \oplus (\mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2})$, for a total of 10 degrees of freedom. Hence, we want to impose an invariance that eliminates eight spurious degrees of freedom. The gauge transformation for a field $h_{\mu\nu}$ with two indices is the linearized version of the diffeomorphism invariance (gauge symmetry) of General Relativity given by

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x) \tag{2.6}$$

for a transformation of coordinates

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)\,, \tag{2.7}$$

that is,

$$h_{\mu\nu}(x) \to h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu\,. \tag{2.8}$$

The most general form of the kinetic term (free action) that we can write down (composed by contributions which are quadratic in $h_{\mu\nu}$ and with two derivatives) is given by [115]

$$S_{\text{kin}} = \int \mathrm{d}^4x \left[a_1 \partial_\rho h_{\mu\nu} \partial^\rho h^{\mu\nu} + a_2 \partial_\rho h_{\mu\nu} \partial^\nu h^{\mu\rho} + a_3 \partial_\nu h^{\mu\nu} \partial_\mu h + a_4 \partial^\mu h \partial_\mu h \right]. \tag{2.9}$$

Here $h \equiv h^\mu{}_\mu$ denotes the trace of $h_{\mu\nu}$. The requirement of invariance of S_{kin} under the gauge transformation (2.8) fixes the values for all the coefficients a_1, a_2, a_3 and a_4, except of course for an overall normalization, that we choose to be such that $a_1 = -1/2$ (the sign is fixed by the requirement that the energy be positive definite). This yields:

$$S_{\text{kin}} = \frac{1}{2} \int \mathrm{d}^4x \left[-\partial_\rho h_{\mu\nu} \partial^\rho h^{\mu\nu} + 2\partial_\rho h_{\mu\nu} \partial^\nu h^{\mu\rho} - 2\partial_\nu h^{\mu\nu} \partial_\mu h + \partial^\mu h \partial_\mu h \right]. \tag{2.10}$$

After a rescaling $h_{\mu\nu} \to (M_g/2)\, h_{\mu\nu}$, we recover the *linearized Einstein-Hilbert action* that we know from General Relativity,

$$S_{\text{kin}} = \frac{M_g^2}{2} \int \mathrm{d}^4x \left\{ \frac{1}{4} \left[-\partial_\rho h_{\mu\nu} \partial^\rho h^{\mu\nu} + 2\partial_\rho h_{\mu\nu} \partial^\nu h^{\mu\rho} - 2\partial_\nu h^{\mu\nu} \partial_\mu h + \partial^\mu h \partial_\mu h \right] \right\}. \tag{2.11}$$

One can arrive to the same action if we had started from the untuned action (2.9) without requiring any gauge invariance, but instead requiring to avoid a ghost within the kinetic term [45]. A *ghost*, in this context, is an additional degree of freedom of the theory whose kinetic term in the Lagrangian has the wrong sign, which implies that the Hamiltonian (that is, the energy) associated to the Lagrangian taken into account is unbounded from below: this means that there is a linear instability in the Hamiltonian of this theory. According to Ostrogradskij's theorem [125, 144, 145], this wrong sign in the kinetic term is hidden in Lagrangians which depend upon more than one time derivative in such a way that higher order derivatives cannot be removed by partial integration[6].

In order to see how the ghost-freedom requirement brings the action into the form (2.10), we start by splitting the 10 components of $h_{\mu\nu}$ into a transverse tensor[7] $h^\perp_{\mu\nu}$ (which carries 6 components) and a four-vector field χ_μ (which carries 4 components):

$$h_{\mu\nu} = h^\perp_{\mu\nu} + 2\partial_{(\mu}\chi_{\nu)}\,. \tag{2.12}$$

From this, we can see that an arbitrary kinetic term of the form (2.9) with untuned coefficients would contain higher derivatives for χ_μ which in turn, according to Ostrogradskij's theorem, would imply a ghost. Preventing these higher derivative terms from arising sets the coefficients to be such that the action takes the form (choosing the overall normalization in such a way to follow standard conventions) of eq. (2.10). As a result, the kinetic term for $h_{\mu\nu}$ is invariant under the gauge transformation (2.8).

Now we want to find the number of independent degrees of freedom which are actually propagating in the theory. First, we can use the gauge-invariance (2.8) to choose the *Lorentz gauge*

$$\partial^\nu h_{\mu\nu} - \frac{1}{2}\partial_\mu h = 0. \tag{2.13}$$

This gauge-fixing gives four conditions which eliminate four of the 10 degrees of freedom which are present in the tensor $h_{\mu\nu}$, and still does not fix the gauge completely. Therefore, there is a residual gauge-invariance, that is, the transformation (2.8) with functions ξ_μ that satisfy $\Box\xi_\mu = 0$ (where $\Box \equiv \partial_\mu\partial^\mu$ represents the d'Alembertian in flat Minkowski space-time) does not spoil the condition (2.13). In the vacuum, where

[6]This fact proves that Newton was right to assume the laws of physics take the form of second order differential equations when expressed in terms of fundamental dynamical variables.

[7]That is, such that $\partial^\mu h^\perp_{\mu\nu} = 0$.

$T^{\mu\nu} = 0$, using the Lorentz gauge, the equations of motion which can be derived from the linearized Einstein-Hilbert action are $\Box(h_{\mu\nu} - (1/2)\eta_{\mu\nu}h) = 0$, hence four functions ξ_μ which satisfy $\Box\xi_\mu = 0$ can be used to eliminate four additional components of $h_{\mu\nu}$ to fix the gauge completely: in total, we arrive to the *transverse-traceless gauge*,

$$h^{0\mu} = 0\,, \qquad h^i{}_i = 0\,, \qquad \partial^j h_{ij} = 0\,, \tag{2.14}$$

and we remain with only two degrees of freedom. Hence, the requirement of gauge-invariance determines uniquely the linearized action and we obtain a massless spin-2 field with two transverse propagating degrees of freedom, the graviton [115].

2.3 Massive spin-2 field: linear theory

We now move to the construction of a linearized field theory for massive gravitons, the so-called *Fierz-Pauli theory* [71, 126]. We start from the action (2.10) and we add a mass term. The most general Lorentz-invariant mass term that we can build is a linear combination of the two scalar quantities $h_{\mu\nu}h^{\mu\nu}$ and h^2. The full action (without the source term), *without* fixing the gauge, reads

$$\begin{aligned} S = S_{\rm kin} + S_{\rm mass} = \frac{1}{2}\int \mathrm{d}^4x\,[&-\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + 2\partial_\rho h_{\mu\nu}\partial^\nu h^{\mu\rho} - 2\partial_\nu h^{\mu\nu}\partial_\mu h + \partial^\mu h\partial_\mu h \\ &+ m^2(c_1 h^2 + c_2 h_{\mu\nu}h^{\mu\nu})\,]\,, \end{aligned} \tag{2.15}$$

where m represents the mass of the graviton. Of course, the mass term breaks the gauge invariance (2.8) of the massless theory.

In order to find the numerical coefficients in front of $h_{\mu\nu}h^{\mu\nu}$ and h^2, we require the resulting mass term to be ghost-free. To do so, we start by introducing four Stückelberg fields χ_μ associated to the symmetric tensor field $h_{\mu\nu}$ (they are introduced in such a way to be patterned after the gauge symmetry (2.8) present in the massless case) [93]:

$$h_{\mu\nu} \to h_{\mu\nu} + 2\partial_{(\mu}\chi_{\nu)}\,, \tag{2.16}$$

The Stückelberg fields transform under linear diffeomorphism in such a way as to make the mass term gauge-invariant. That is, the mass term that we get after including the four linearized Stückelberg fields,

$$S_{\rm mass} = \frac{m^2}{2}\int \mathrm{d}^4x\,[\,c_1(h + 2\partial_\alpha\chi^\alpha)^2 + c_2(h_{\mu\nu} + 2\partial_{(\mu}\chi_{\nu)})^2\,]\,, \tag{2.17}$$

is invariant under the simultaneous transformations:

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_{(\mu}\xi_{\nu)}\,, \qquad \chi_\mu \to \chi_\mu - \frac{1}{2}\xi_\mu\,. \tag{2.18}$$

The mass term provides a kinetic term for the fields χ_μ:

$$S_{\text{kin}}^{\chi_\mu} = 2\,m^2 \int \mathrm{d}^4x\,[\,c_1(\partial_\mu\chi^\mu)(\partial_\nu\chi^\nu) + c_2(\partial_\mu\chi_\nu)(\partial^\mu\chi^\nu)\,]\,. \tag{2.19}$$

We can split the four degrees of freedom carried by χ_μ into a transverse contribution $\chi_\mu^\perp$ satisfying $\partial^\mu\chi_\mu^\perp = 0$, bearing *a priori* three degrees of freedom, and a longitudinal mode χ such that $\chi_\mu = \chi_\mu^\perp + \partial_\mu\chi$. Focusing on the longitudinal (or helicity-0) mode χ and integrating by parts, the kinetic term then takes the form [45]

$$S_{\text{kin}}^{\chi} = 2\,m^2 \int \mathrm{d}^4x\,[\,(c_1 + c_2)(\Box\chi)^2\,]\,. \tag{2.20}$$

From this we directly see that, unless $c_1 = -c_2$, the kinetic term for the field χ bears time (and space) derivatives of order higher than one. Hence, as a consequence of Ostrogradskij's theorem, two degrees of freedom are actually hidden in χ (which should in principle carry one degree of freedom) with an opposite sign kinetic term, and one of them is a ghost, that we need to get rid of by conveniently tuning the relative value of the two parameters c_1 and c_2. We can explicitly see the appearance of this instability by introducing a Lagrange multiplier $\tilde{\chi}(x)$, which allows us to write an action [45]

$$S_{\text{kin}}^{\chi} = 2\,m^2 \int \mathrm{d}^4x\,[\,(c_1 + c_2)(\tilde{\chi}\,\Box\,\chi - \frac{1}{4}\tilde{\chi}^2)\,]\,, \tag{2.21}$$

which is equivalent to (2.20) after integrating out the Lagrange multiplier $\tilde{\chi} \equiv 2\,\Box\,\chi$. After performing the change of variables $\chi = \phi_1 + \phi_2$ and $\tilde{\chi} = \phi_1 - \phi_2$, we obtain an action for the two scalar fields ϕ_1 and ϕ_2:

$$S_{\text{kin}}^{\chi} = 2\,m^2 \int \mathrm{d}^4x\,[\,(c_1 + c_2)(\phi_1\,\Box\,\phi_1 - \phi_2\,\Box\,\phi_2 - \frac{1}{4}(\phi_1 - \phi_2)^2)\,]\,. \tag{2.22}$$

This shows us that the two scalar fields ϕ_1 and ϕ_2 always enter with kinetic terms with opposite sign, which means that one of them is always a ghost. In order to prevent this instability to arise, we need to make the specific choice $c_1 + c_2 = 0$, which is the so-called *Fierz-Pauli tuning* [71, 126]. The only possible mass term is hence given by the action (2.17) with the Fierz-Pauli tuning:

$$S_{\text{mass}} = \frac{m^2}{2} \int \mathrm{d}^4x\,[\,(h + 2\partial_\mu\chi^\mu)^2 - (h_{\mu\nu} + 2\partial_{(\mu}\chi_{\nu)})^2\,]\,, \tag{2.23}$$

In the *unitary gauge*, that is, the gauge in which the Stückelberg fields χ^μ are set to zero, the Fierz-Pauli mass term takes the simple form

$$S_{\text{mass}} = \frac{m^2}{2} \int \mathrm{d}^4x\,(\,h^2 - h_{\mu\nu}h^{\mu\nu}\,)\,. \tag{2.24}$$

The full action (2.15) therefore reads

$$S_{\rm FP} = \frac{1}{2}\int \mathrm{d}^4x\,[-\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + 2\partial_\rho h_{\mu\nu}\partial^\nu h^{\mu\rho} - 2\partial_\nu h^{\mu\nu}\partial_\mu h + \partial^\mu h\partial_\mu h + m^2(h^2 - h_{\mu\nu}h^{\mu\nu})\,]\,. \tag{2.25}$$

After rescaling $h_{\mu\nu} \to (M_g/2)\,h_{\mu\nu}$, we get

$$S_{\rm FP} = \frac{M_g^2}{8}\int \mathrm{d}^4x\,[-\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + 2\partial_\rho h_{\mu\nu}\partial^\nu h^{\mu\rho} - 2\partial_\nu h^{\mu\nu}\partial_\mu h + \partial^\mu h\partial_\mu h + m^2(h^2 - h_{\mu\nu}h^{\mu\nu})\,]\,. \tag{2.26}$$

If we introduce the Lichnerowicz operator in a flat space-time, $\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}$, defined as [45]

$$\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} = -\frac{1}{2}\left(\Box h_{\mu\nu} - \partial_\mu\partial_\alpha h^\alpha_\nu - \partial_\nu\partial_\alpha h^\alpha_\mu + \partial_\mu\partial_\nu h - \eta_{\mu\nu}\Box h + \eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta}\right)\,, \tag{2.27}$$

the action (2.26) can be written (after integrating by parts) in a more compact form:

$$S_{\rm FP} = M_g^2\int \mathrm{d}^4x\left[-\frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} + \frac{m^2}{8}(h^2 - h_{\mu\nu}h^{\mu\nu})\right]. \tag{2.28}$$

As we already said, the mass term breaks the gauge invariance (2.8) of the massless theory, as it should: in fact, we want to describe the five degrees of freedom of a massive spin-2 particle, while we have seen that gauge invariance reduces the 10 components of the symmetric tensor $h_{\mu\nu}$ to just two independent degrees of freedom (in the massless case). However, in the massive theory, we still need to get rid of 5 out of 10 degrees of freedom. The appropriate number of conditions can emerge dynamically, as it is shown for example in [115]. Another way to identify the propagating degrees of freedom of the massive theory is to decompose the Stückelberg field χ^μ further into a transverse and a longitudinal mode[8],

$$\chi^\mu = \frac{1}{m}A^\mu + \frac{1}{m^2}\partial^\mu\pi\,, \tag{2.29}$$

where we have introduced normalization factors. In terms of $h_{\mu\nu}$ and the Stückelberg fields A_μ and π, the action (2.28) reads

$$\begin{aligned} S = M_g^2\int \mathrm{d}^4x\Big[&-\frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} - \frac{1}{2}h^{\mu\nu}\left(\Pi_{\mu\nu} - [\Pi]\eta_{\mu\nu}\right) - \frac{1}{8}F^2_{\mu\nu} \\ &+ \frac{1}{8}m^2(h^2 - h_{\mu\nu}h^{\mu\nu}) + \frac{1}{2}m(h\eta^{\mu\nu} - h^{\mu\nu})\partial_{(\mu}A_{\nu)}\Big], \end{aligned} \tag{2.30}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$, $\Pi_{\mu\nu} \equiv \partial_\mu\partial_\nu\pi$ and where we have defined $[\Pi] \equiv \mathrm{Tr}\Pi^\mu{}_\nu = \Pi^\mu{}_\mu$. We see that the kinetic term for the field π is hidden in the mixing with $h_{\mu\nu}$. To make it

[8] More details on the helicity decomposition in the context of massive gravity are given in [48].

explicit, we diagonalize this mixing through the field redefinition[9] [93] $h_{\mu\nu} = \tilde{h}_{\mu\nu} + \pi\eta_{\mu\nu}$. The linearized action then takes the form [48, 45]

$$S = M_g^2 \int \mathrm{d}^4x \bigg[-\frac{1}{4}\tilde{h}^{\mu\nu}\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}\tilde{h}_{\alpha\beta} - \frac{3}{4}(\partial\pi)^2 - \frac{1}{8}F^2_{\mu\nu} + \frac{1}{8}m^2(\tilde{h}^2 - \tilde{h}_{\mu\nu}\tilde{h}^{\mu\nu}) + \frac{3}{2}m^2\pi^2 + \frac{3}{2}m^2\pi\tilde{h} + \frac{1}{2}m(\tilde{h}\eta^{\mu\nu} - \tilde{h}^{\mu\nu})\partial_{(\mu}A_{\nu)} + 3m\pi\partial_\alpha A^\alpha \bigg]. \tag{2.31}$$

This decomposition allows us to identify the different degrees of freedom which are present in the Fierz-Pauli theory of massive gravity: $\tilde{h}_{\mu\nu}$ represents the helicity-2 mode which is already present in General Relativity (that is, the canonical massless graviton) and hence propagates two degrees of freedom, the four-vector A_μ represents the helicity-1 mode (a canonical massless vector) and propagates another 2 degrees of freedom, and π represents the helicity-0 mode (a canonical massless scalar) and propagates one degree of freedom. This makes a total of 5 propagating degrees of freedom, as expected for a massive spin-2 field. Moreover, by splitting the degrees of freedom into their mass eigenstates, one can verify that all the degrees of freedom have the same positive mass square m^2 (this is shown, for example, in [98] by using a (3+1) decomposition of the metric).

So far we have neglected the coupling of the massive graviton to matter, but of course it is present and it has the form $h_{\mu\nu}T^{\mu\nu} = \tilde{h}_{\mu\nu}T^{\mu\nu} + \pi T$, where T is the trace of the stress-energy tensor, which is assumed to be conserved[10]. This shows that the helicity-0 mode couples directly to the conserved matter sources, whereas the helicity-1 mode, represented by A_μ, does not. This allows us to neglect the helicity-1 mode in most of the applications of the theory. This also shows that the coupling of the scalar, π, to the trace of the stress-energy tensor survives the $m \to 0$ limit. This extra scalar potential exactly accounts for the discrepancy which appears when we calculate and compare the massless limit of the massive graviton propagator and the massless graviton propagator (and the corresponding gravitational exchange amplitude) for some non-relativistic sources [115]. This discrepancy is usually known as the *van Dam-Veltman-Zakharov (vDVZ) discontinuity* [138, 146] and it translates into a similar discrepancy for the Newtonian potentials in the non-relativistic limit, the potential in the massive theory

[9] This is just the linearization of a conformal transformation.

[10] If the external source were not conserved ($\partial_\nu T^{\mu\nu} \neq 0$), we would have additional terms involving the stress-energy tensor in the action, in particular a term $\propto m^{-1}A_\mu\partial_\nu T^{\mu\nu}$ (after integrating by parts) which shows that A_μ would become strongly coupled to the divergence of the source in the $m \to 0$ limit [93]. Moreover, the conservation of $T^{\mu\nu}$ is required in order to restore the gauge invariance (2.8) of the full action (including the coupling to matter $\propto h_{\mu\nu}T^{\mu\nu}$) in the massless limit of massive gravity [115].

being larger by a factor 4/3. This factor can be eliminated through a redefinition of the Newton gravitational constant G of the massive theory with respect to the one of the massless theory, but the discontinuity will then reappear in other observables, for example the deflection of light by a massive object (like the Sun) which will then be 25% smaller in the massive case than in the massless one. This discrepancy is much too large to be compatible with current measurements of the light bending by the Sun. This means that the vDVZ discontinuity is enough to rule out from standard solar system test of gravity any theory where it appears, as the Fierz-Pauli model of massive gravity [20].

This problem was addressed by A. I. Vainshtein back in 1972 [137]. He pointed out that, at a certain distance from the non-relativistic source, called the *Vainshtein radius*, r_V, the linear regime breaks down, and for any radius r below r_V the theory enters a non-linear regime. Since the Fierz-Pauli theory of massive gravity is, by construction, a linearized theory, it is perfectly normal that its predictions do not pass the observational tests inside the solar system. Therefore, we need to include non-linear corrections in order to restore agreement with solar system observations: the *Vainshtein mechanism* allows then to recover General Relativity predictions around massive bodies. The basic idea on which this mechanism relies is to decouple the additional modes from the gravitational dynamics via nonlinear interactions of the helicity-0 mode of the graviton. As a result, at the vicinity of matter, the non-linear interactions for the helicity-0 mode become large and hence suppress its coupling to matter. Therefore, the “fifth force" mediated by this scalar mode acts significantly only on large scales. Since the Vainshtein radius on a Schwarzschild background can be expressed as a function of the graviton mass as $r_V = (r_S m^{-4})^{1/5}$ [115], where r_S is the Schwarzschild radius of the source, we get that r_V diverges as $m \to 0$, as we expect.

The Vainshtein mechanism has been first discussed in the context of massive gravity, but it also applies to other modified gravity theories, like for example the DGP model [65] or the Galileon model [123].

2.4 The non-linear completion

2.4.1 Fundamental issues

We have introduced the spin-2 field $h_{\mu\nu}$ as the perturbation of the metric around flat space-time: in fact, it is natural to work with Minkowski as our space-time metric, since the notion of spin follows from that of Poincaré invariance. However, since the Vainshtein mechanism assures that non-linearities could restore agreement with observational tests, we would like to determine a consistent, fully non-linear extension of the Fierz-Pauli theory, which is, as we have seen, the unique ghost-free theory of non-interacting massive gravitons in a flat background at the linear level. That is, we would like no longer to investigate linear fluctuations $h_{\mu\nu}$ in a fixed flat background $\eta_{\mu\nu}$, but to combine the background and the fluctuation into a single non-linear field $g_{\mu\nu}$ and to construct a non-linear self-interaction potential without derivatives of the metric. Unfortunately, Boulware and Deser, who, in a paper of 1972 [26], studied the consistency of a wide class of possible non-linear completions of the Fierz-Pauli theory, concluded that any non-linear extension would inevitably reintroduce the sixth propagating degree of freedom, the scalar ghost which had been removed, in the linear case, by the Fierz-Pauli tuning. This scalar mode is hence also known as the *Boulware-Deser ghost*. The re-appearance of this ghost at the non-linear level implies that no consistent non-linear theory of massive spin-2 fields could exist. Nowadays, we know that this conclusion was wrong, for the following reasons [130]:

- Boulware and Deser actually did not consider the most general non-linear extensions of the Fierz-Pauli theory in their paper, but only non-derivative self-interactions of the field $h_{\mu\nu}$ of the form $f(h^2 - h_{\mu\nu}h^{\mu\nu})$ (where f is a general analytic function), which naturally expands in Taylor series to the Fierz-Pauli mass term at the lowest order. Today we know that, in the consistent theory of massive gravity, the self-interaction potential is a linear combination of the elementary symmetric polynomials, which in turn are constructed out of the matrix elements $[\sqrt{\mathbb{1} + \eta^{-1}h}]^{\mu}_{\nu}$, which is not of the form assumed by Boulware and Deser in their proof.

- The authors performed an Hamiltonian analysis in order to find out the degrees of freedom of the theory, and they found that, in the non-linear case, the lapse function of the metric is no longer a Lagrange multiplier and hence its variation does not provide anymore an additional constraint but rather determines the lapse itself, leaving us with six propagating modes instead of five (as in the linear case). Actually, it is not the equation of motion for the lapse variable, but a different equation (an artful combination of lapse and shift equations) that yields the constraint and consequently reduces the number of degrees of freedom of the consistent massive gravity theory from six to five, as expected.

Nevertheless, the conclusions provided by Boulware and Deser were widely accepted by the scientific community, and the consequence was that only in recent years a consistent non-linear theory of massive gravity has been fully developed.

2.4.2 Constraints on the mass term

In the action, the mass term for a tensor with two indices $g_{\mu\nu}$ has to be a scalar function $U(g)$ (which, by definition of a mass term, should not contain any derivatives), multiplied by the usual scalar density $\sqrt{-\det g_{\mu\nu}}$. The only possibility to construct a Lorentz scalar (with no loose covariant indices) would therefore be to contract the metric tensor $g_{\mu\nu}$ with its inverse, but the quantity $g^{\mu\nu}g_{\mu\nu} = \delta^\mu_\mu = 4$ leads to a trivial cosmological constant contribution. Hence, it is not possible to construct a covariant non-linear mass term for a spin-2 field using only the metric tensor. We need to introduce at least a second tensor field $f_{\mu\nu}$ (that we shall call the *reference metric*). The mass potential will be then constructed as $\sqrt{-\det g_{\mu\nu}}$ multiplying a scalar function of the quantity $g^{-1}f \equiv g^{\mu\rho}f_{\rho\nu}$ (as it is discussed in [82], this structure is completely adequate for describing ghost-free massive gravity actions). This means that the non-linear massive gravity action must be of the form

$$S_{\rm MG} = \frac{M_g^2}{2}\int \mathrm{d}^4x\,\sqrt{-\det g_{\mu\nu}}\left[R(g) - m^2 U(g^{-1}f)\right], \tag{2.32}$$

where m is, as usual, a parameter with the dimension of mass that sets the mass of the graviton.

In this theory the reference metric $f_{\mu\nu}$ is a fixed field that needs to be put into the theory by hand. However, we will see that in the bimetric theory of gravity we will give $f_{\mu\nu}$ its own dynamical term in the action. Of course, we have to require that the mass potential contained in eq. (2.32) reduces to the well known Fierz-Pauli mass term once linearized around a Minkowski background (that is, when $f_{\mu\nu}$ is flat and when flat background solutions for $g_{\mu\nu}$ exist), in order to avoid the appearance of the Boulware-Deser ghost. But this, as we have previously seen, does not guarantee that the full non-linear theory is ghost-free. In order to construct a consistent (ghost-free) theory of massive gravity, we analyze the possibility of avoiding the Boulware-Deser ghost using the Hamiltonian formulation. This formulation has first been introduced for General Relativity by Arnowitt, Deser, and Misner back in 1962 [19]. According to this approach, the metric $g_{\mu\nu}$ is decomposed essentially into its time (0μ) and spatial (ij) components, precisely into a scalar N (the *lapse*), the three-dimensional spatial metric γ_{ij}, and a three-vector N_i (the *shift*), the so-called *ADM variables*:

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N_i\gamma^{ij}N_j & N_j \\ N_i & \gamma_{ij}\end{pmatrix}, \tag{2.33}$$

where γ^{ij} denotes the inverse of γ_{ij}. The inverse of the metric is therefore given by

$$g^{\mu\nu} = \frac{1}{N^2}\begin{pmatrix} -1 & N^j \\ N^i & N^2\gamma^{ij} - N^iN^j\end{pmatrix}, \tag{2.34}$$

where the index on N_i is raised using the inverse spatial metric γ^{ij}. In terms of the ADM variables, the Einstein-Hilbert action takes the following form (up to a boundary term which is not important for our discussion) [19]:

$$S_{\rm GR} = \frac{M_g^2}{2}\int \mathrm{d}^4x\,[\,\pi^{ij}\dot\gamma_{ij} - NR^0 - N_iR^i\,], \tag{2.35}$$

where the $R^\mu = (R^0, R^i)$ are functions of the γ_{ij} and their conjugate momenta π^{ij}, but not of the (N, N_i) [26]. This action does not contain dynamical terms for the lapse N nor for the shift N^i which, besides, enter only linearly and therefore are Lagrange multipliers whose equations of motion are independent of N and N^i themselves. This implies that these equations are just four constraints $R^\mu = 0$ on the remaining variables (six of γ_{ij} and six of π^{ij}). Then we can still use gauge transformations to remove another four degrees of freedom, and we end up with four dynamical variables (in the phase space of the coordinates and their conjugate momenta): the two helicity states of the massless graviton and their canonical momenta. This approach hence gives the correct counting of the degrees of freedom in General Relativity. In order to construct a consistent theory of massive gravity, we need to determine a potential $U(g^{-1}f)$ such that the resulting theory is ghost-free. Since the kinetic term for the physical metric $g_{\mu\nu}$ is the same as in General Relativity, the lapse and the shift of the physical metric will still enter without derivatives. However, the mass term will in general no longer be linear in these functions. Hence, their equations of motion, instead of imposing constraints on the remaining variables, will now determine N and N^i themselves. Since the theory is no longer gauge-invariant, the four gauge constraints are lost: this means that now there will generically be six propagating degrees of freedom (plus the corresponding canonical momenta), whereas we know that a healthy massive spin-2 field theory should bear only five degrees of freedom. Hence, we need an additional constraint in order to remove the sixth propagating degree of freedom, which is the Boulware-Deser ghost. Actually, two more conditions need to be met by a consistent theory of massive gravity:

- The conservation in time of the additional constraint must itself provide a constraint, in such a way that also the conjugate momentum can be removed and therefore the full phase space pair of fields associated with the Boulware-Deser ghost.

- The Hamiltonian must be positive definite within the regime of validity of this effective field theory, in order to assure that none of the surviving five propagating degrees of freedom gives rise to a ghost.

However, the second condition has never been proven in general for the non-linear theories we are going to discuss: as it is pointed out, for example, in [142], this condition would not seem to be valid without additional physical requirements. As we will see in the next chapters of this Thesis, ghosts different form the Boulware-Deser ghost can still raise and propagate in some cosmological backgrounds[11]. Therefore, the consistent theory of massive gravity we are going to develop is "ghost-free" only for what

[11] This is also the case for the Higuchi instability of the helicity-0 mode in a de Sitter spacetime [91, 69].

regards the Boulware-Deser ghost, but nothing can be said *a priori* (without applying the theory to some specific cosmological setups) about the possible presence of other "cosmological" ghosts. For our analysis, we need to introduce an ADM decomposition also for the reference metric,

$$f_{\mu\nu} = \begin{pmatrix} -L^2 + L_i \omega^{ij} L_j & L_j \\ L_i & \omega_{ij} \end{pmatrix}, \tag{2.36}$$

where ω^{ij} denotes the inverse of the three-dimensional spatial metric ω_{ij}, and L and L_i are the lapse and the shift functions of $f_{\mu\nu}$, respectively. We are hence able to write the generic mass potential in terms of the ADM variables for the physical metric $g_{\mu\nu}$ and the reference metric $f_{\mu\nu}$:

$$\sqrt{-\det g_{\mu\nu}}\, U(g^{-1}f) = N\sqrt{\det \gamma_{ij}}\, U(\gamma_{ij}, N, N^i; \omega_{ij}, L, L^i)\,. \tag{2.37}$$

As anticipated before, Boulware and Deser in [26] concluded that the right-hand side of eq. (2.37) needs to be linear in N, in such a way that the lapse becomes a Lagrange multiplier and therefore its variation provides an additional constraint to remove the sixth degree of freedom, and that in non-linear theories of massive gravity this is no longer the case. Actually this is not completely true, since the constraint can also be obtained by combining the equations of motion for N and N^i in such a way that the resulting equation does not depend on N.

This means that we need to perform a field-dependent redefinition of the shift function, $N^i \to n^i$, such that the resulting action is linear in the lapse N. Concretely, if we redefine N^i as $N^i(N, n^j, \dots) \equiv C^i{}_k(N, n^j, \dots) n^k$, the variation of the action with respect to N becomes

$$\begin{aligned} \frac{\delta S(N, n^j, \dots)}{\delta N} &= \frac{\delta S(N, N^j, \dots)}{\delta N}\bigg|_{N^i} + \frac{\delta S(N, N^j, \dots)}{\delta N^i}\bigg|_{N} \frac{\delta N^i(N, n^k, \dots)}{\delta N} \\ &= \frac{\delta S(N, N^j, \dots)}{\delta N}\bigg|_{N^i} + \frac{\delta S(N, N^j, \dots)}{\delta N^i}\bigg|_{N} \frac{\delta C^i{}_k(N, \dots)}{\delta N} n^k\,, \end{aligned} \tag{2.38}$$

where $|_{N^i}$ and $|_N$ mean that the shift and the lapse, respectively, are kept fixed while the functional derivative is taken. Since the redefinition $N^i \to n^i$ has made the action linear in N, then naturally the quantity $\delta S(N, N^j, \dots)/\delta N|_{N^i}$ does not depend on N. Moreover, the matrix $C^i{}_k(N, n^j, \dots)$ which enters the redefinition must be linear in N: in fact, the action contains the term $N^i R_i = N^i(N, n^j, \dots) R_i = C^i{}_k(N, n^j, \dots) n^k R_i$ and, since it is now linear in N, this implies that $C^i{}_k(N, n^j, \dots)$ is linear in N as well.

Since the function n^i is expected to appear in the constraint, it must be fully determined by its own equation of motion, which hence must not depend on N, in such a way that the equation of motion for n^i can be used to determine the n^i in terms of (γ_{ij}, π^{ij}) [86]. From (2.38), we can see that the constraint equation has then the form $\delta S/\delta N = 0$ or

$$\left.\frac{\delta S(N, N^j, \dots)}{\delta N}\right|_{N^i} + C^i \left.\frac{\delta S(N, N^j, \dots)}{\delta N^i}\right|_N = 0\,, \tag{2.39}$$

where we have chosen the redefined functions n^i in such a way that $C^i \equiv (\delta C^i{}_k(N, \dots)/\delta N)n^k$ is independent of N.

In conclusion, we need to make two requirements on the potential we want to construct, in order to make it ghost-free:

- the action must be linear in the lapse N;
- the equation of motion for n^i must not depend on N.

As we will see later, the potential constructed following requirement that the first condition is satisfied automatically satisfies the second condition as well. Hence, we will focus on the first requirement.

2.4.3 The construction of the interaction potential

In order to get a potential term $\sqrt{-\det g_{\mu\nu}}\, U(g^{-1}f)$ which is linear in N, since $\sqrt{-\det g_{\mu\nu}} = N\sqrt{\det \gamma_{ij}}$, we need to consider a function $U(g^{-1}f)$ which, written in redefined ADM variables, has the form

$$U(g^{-1}f) = \frac{1}{N} U_1 + U_2\,, \tag{2.40}$$

where U_1 and U_2 are functions only of γ_{ij}, n^i and of the components of the reference metric $f_{\mu\nu}$ and are independent of the lapse N. In order to obtain inverse powers of N as in eq. (2.40), since $g_{\mu\nu}$ is quadratic in N (as we can see from the ADM decomposition (2.33)), we need to consider the inverse metric $g^{\mu\nu}$, which is quadratic in $1/N$. Therefore, the only quantity that can give something linear in $1/N$ after a linear redefinition of the shift N^i is the square root of an object which is proportional to the inverse metric. This means that the potential U must be a function of the matrix $\mathbb{X} = \sqrt{g^{-1}f}$, defined as a matrix whose square is $\mathbb{X}^2 = g^{-1}f$. Before the redefinition of the function N^i, this square-root matrix is certainly highly non-linear in $1/N$, but through a special redefinition of the shift we can arrive to have the square-root matrix $\mathbb{X}$ of the form

$$\mathbb{X} = \frac{1}{N}\mathbb{A} + \mathbb{B}, \tag{2.41}$$

where $\mathbb{A}$ and $\mathbb{B}$ are functions only of γ_{ij}, n^i and of the components of the reference metric $f_{\mu\nu}$ and are independent of the lapse N. The derivation of the redefinition that leads to eq. (2.41) is given in [86], here we just give the result. The redefinition that makes the matrix $\mathbb{X}$ linear in $1/N$ is

$$N^i = Ln^i + L^i + ND^i{}_k n^k \,, \tag{2.42}$$

where we have defined $D \equiv \sqrt{\gamma^{-1}\omega Q}\, Q^{-1}$ and in turn the matrix Q is given by $Q \equiv \kappa\delta^i{}_j + n^i n^k \omega_{kj}$, where $\kappa \equiv 1 - n^i\omega_{ij}n^j$. Moreover, the existence of real solutions for the $3{\times}3$ square-root matrix D can also be demonstrated [130]. After some calculations [130] one can show that the matrices $\mathbb{A}$ and $\mathbb{B}$ can be written in terms of the redefined ADM variables as

$$\mathbb{A} = \frac{1}{\sqrt{\kappa}}\begin{pmatrix} L + n^k L_k & n^k\omega_{kj} \\ -(L + n^k L_k)(Ln^i + L^i) & -(Ln^i + L^i)n^k\omega_{kj} \end{pmatrix}, \quad \mathbb{B} = \sqrt{\kappa}\begin{pmatrix} 0 & 0 \\ D^i{}_j L^j & D^i{}_j \end{pmatrix}, \tag{2.43}$$

where the index on the shift vector of the reference metric, L_i, is raised using the inverse spatial metric ω^{ij}. In particular, the matrix $\mathbb{A}$ can be written as the outer product of two vectors,

$$\mathbb{A} = uw^T, \qquad \text{where} \quad u = \begin{pmatrix} 1 \\ -c^i \end{pmatrix} \qquad \text{and} \quad w = \frac{1}{\sqrt{\kappa}}\begin{pmatrix} a_0 \\ a_j \end{pmatrix}. \tag{2.44}$$

Here we have defined $c^i \equiv L\,n^i + L^i$, $a_0 \equiv L + n^k L_k$ and $a_j \equiv n^k\omega_{kj}$. As already explained, the potential that does not give rise to the Boulware-Deser ghost has to be a function of the square-root matrix $\mathbb{X} = \frac{1}{N}\mathbb{A} + \mathbb{B}$, multiplied by $\sqrt{-\det g_{\mu\nu}} = N\sqrt{\det \gamma_{ij}}$. We would like to determine the form of this function. We start by assuming that this function can be (at least formally) Taylor-expanded in $\mathbb{X}$: in general this will produce an expression which is non-linear in N. The only way to avoid non-linearities in N is to avoid the appearance of higher powers of $\frac{1}{N}\mathbb{A}$ in the Taylor expansion. At first sight, it would seem that all higher powers of $\mathbb{X}$ will involve higher powers of $1/N$. However, this is not correct, because of the peculiar structure of the matrix $\mathbb{A}$. We are now going to show how to construct a potential which is linear in $1/N$ despite the fact of having specific terms of higher order in $\mathbb{X}$. We consider a potential term of the form

$$U(\mathbb{X}) = \sum_{n=0}^{4} b_n \epsilon^{\mu_1\mu_2\ldots\mu_n\lambda_{n+1}\ldots\lambda_4}\epsilon_{\nu_1\nu_2\ldots\nu_n\lambda_{n+1}\ldots\lambda_4}\mathbb{X}^{\nu_1}_{\mu_1}\ldots\mathbb{X}^{\nu_n}_{\mu_n}\,, \tag{2.45}$$

with arbitrary coefficients b_n and totally antisymmetric Levi-Civita tensors $\epsilon_{\mu\nu\rho\ldots}$. At nth order, the product of $\mathbb{X}$ matrices under the sum can be written as

$$\sum_{l=0}^{n}\binom{n}{l}\left(\frac{1}{N}\right)^l \mathbb{A}^{\nu_1}{}_{\mu_1}\ldots\mathbb{A}^{\nu_l}{}_{\mu_l}\mathbb{B}^{\nu_{l+1}}{}_{\mu_{l+1}}\ldots\mathbb{B}^{\nu_n}{}_{\mu_n}\,. \tag{2.46}$$

Putting this expression into eq. (2.45) we get

$$U(\mathbb{X}) = \sum_{n=0}^{4} b_n \sum_{l=0}^{n}\binom{n}{l}\left(\frac{1}{N}\right)^l U_n(\mathbb{A},\mathbb{B})\,, \tag{2.47}$$

where

$$\begin{aligned} U_n(\mathbb{A},\mathbb{B}) &= \epsilon^{\mu_1\mu_2\ldots\mu_n\lambda_{n+1}\ldots\lambda_4}\epsilon_{\nu_1\nu_2\ldots\nu_n\lambda_{n+1}\ldots\lambda_4}\mathbb{A}^{\nu_1}{}_{\mu_1}\ldots\mathbb{A}^{\nu_l}{}_{\mu_l}\mathbb{B}^{\nu_{l+1}}{}_{\mu_{l+1}}\ldots\mathbb{B}^{\nu_n}{}_{\mu_n} \\ &= \epsilon^{\mu_1\mu_2\ldots\mu_n\lambda_{n+1}\ldots\lambda_4}\epsilon_{\nu_1\nu_2\ldots\nu_n\lambda_{n+1}\ldots\lambda_4}u^{\nu_1}w_{\mu_1}\ldots u^{\nu_l}w_{\mu_l}\mathbb{B}^{\nu_{l+1}}{}_{\mu_{l+1}}\ldots\mathbb{B}^{\nu_n}_{\mu_n} \end{aligned} \tag{2.48}$$

and where in the second equality of eq. (2.48) we have used eq. (2.44). Since all indices of the symmetric products $u^{\nu_1}\ldots u^{\nu_l}$ or $w_{\mu_1}\ldots w_{\mu_l}$ are contracted with the totally antisymmetric indices of the corresponding Levi-Civita tensors, we find that the only terms which contribute to $U_n(\mathbb{A},\mathbb{B})$ are terms with at most one matrix $\mathbb{A}$. This means that the sum over l which appears in eq. ((2.47) actually ends at $l=1$ and therefore $U(\mathbb{X})$ is linear in $1/N$, as we wanted. Hence, the most general form of the potential term which is linear in N after the redefinition of the shift vector (2.42) and therefore satisfies the first condition for its viability, is given by

$$\sqrt{-\det g_{\mu\nu}}U(\mathbb{X}) = \sqrt{-\det g_{\mu\nu}}\sum_{n=0}^{4} b_n \epsilon^{\mu_1\mu_2\ldots\mu_n\lambda_{n+1}\ldots\lambda_4}\epsilon_{\nu_1\nu_2\ldots\nu_n\lambda_{n+1}\ldots\lambda_4}\mathbb{X}^{\nu_1}_{\mu_1}\ldots\mathbb{X}^{\nu_n}_{\mu_n}\,. \tag{2.49}$$

It can finally be shown (see [85, 86] for more details) that the equations of motion for n^i that can be found by varying the action with the above potential term are automatically independent of the lapse N, without needing to impose further restrictions on the form of the potential. Moreover, one can show [84] that there is in fact a secondary constraint which arise from requiring the primary constraint to be constant in time. A secondary constraint is essential to prove the absence of the Boulware-Deser ghost: in fact, two constraints are needed in order to remove both the ghost mode and its canonical momentum from the phase space of dynamical variables.

2.4.4 Ghost-free non-linear massive gravity

If we introduce the elementary symmetric polynomials $e_n(\mathbb{X})$ of the matrix $\mathbb{X}$,

$$e_n(\mathbb{X}) = \frac{1}{n!(4-n)!}\epsilon^{\mu_1\mu_2\ldots\mu_n\lambda_{n+1}\ldots\lambda_4}\epsilon_{\nu_1\nu_2\ldots\nu_n\lambda_{n+1}\ldots\lambda_4}\mathbb{X}^{\nu_1}{}_{\mu_1}\mathbb{X}^{\nu_n}{}_{\mu_n} \tag{2.50}$$

(with $e_0(\mathbb{X})\equiv 1$), we can write the potential (2.49) in a more compact form, and therefore the general action for ghost-free massive gravity takes the form

$$S_{\mathrm{MG}} = \frac{M_g^2}{2}\int \mathrm{d}^4x\sqrt{-\det g_{\mu\nu}}\left[R(g) - 2\,m^2\sum_{n=0}^{4}\beta_n e_n(\mathbb{X})\right], \tag{2.51}$$

where $\mathbb{X} = \sqrt{g^{-1}f}$, m is the spin-2 mass scale and we have introduced the rescaled coefficients $\beta_n = b_n n!(4-n)!/2$ for $n = 0,\ldots,4$. The explicit expressions of the five elementary symmetric polynomials which enter this action are

$$e_0(\mathbb{X}) = 1\,, \qquad e_1(\mathbb{X}) = [\mathbb{X}]\,, \qquad e_2(\mathbb{X}) = \frac{1}{2}\left([\mathbb{X}]^2 - [\mathbb{X}^2]\right)\,, \tag{2.52}$$

$$e_3(\mathbb{X}) = \frac{1}{6}\left([\mathbb{X}]^3 - 3[\mathbb{X}][\mathbb{X}^2] + 2[\mathbb{X}^3]\right)\,, \tag{2.53}$$

$$e_4(\mathbb{X}) = \frac{1}{24}\left([\mathbb{X}]^4 - 6[\mathbb{X}]^2[\mathbb{X}^2] + 3[\mathbb{X}^2]^2 + 8[\mathbb{X}][\mathbb{X}^3] - 6[\mathbb{X}^4]\right)\,. \tag{2.54}$$

Here $[\mathbb{X}] \equiv X^\mu_\mu$ denotes the trace of $\mathbb{X}^\mu_\nu$. As we can see, each one of the free parameters β_n is associated with a higher level of non-linear complexity. Out of these five parameters, only β_1, β_2 and β_3 are truly measuring interactions strengths. In fact, since $e_0(\mathbb{X}) = 1$, the term proportional to β_0 in the sum is simply a cosmological constant term for $g_{\mu\nu}$, $\sqrt{-\det g_{\mu\nu}}\Lambda_g$. On the other hand, the term proportional to β_4 is a cosmological constant for $f_{\mu\nu}$, because $e_4(\mathbb{X}) = \det\mathbb{X}$ and therefore $\sqrt{-\det g_{\mu\nu}}\, e_4(\mathbb{X}) = \sqrt{-\det f_{\mu\nu}}$. This shows that the term proportional to β_4 is just a cosmological constant term for $f_{\mu\nu}$, $\sqrt{-\det f_{\mu\nu}}\Lambda_f$, because $e_4(\mathbb{X}) = \det\mathbb{X}$ and therefore $\sqrt{-\det g_{\mu\nu}}\, e_4(\mathbb{X}) = \sqrt{-\det f_{\mu\nu}}$. Hence, this term does not depend on the physical metric $g_{\mu\nu}$, which means that it does not contribute to the equations of motion. The only reason why we have introduced this term is because it will become relevant when we will make the reference metric $f_{\mu\nu}$ dynamical, in the context of the bimetric theory of gravity.

The equations of motion for $g_{\mu\nu}$ obtained from (2.51) are,

$$R_{\mu\nu}(g) - \frac{1}{2}g_{\mu\nu}R(g) + m^2 U^g_{\mu\nu}(g, f; \beta_n) = 0\,, \tag{2.55}$$

where the contribution from the interaction potential reads

$$U^g_{\mu\nu} = g_{\mu\rho}\sum_{n=0}^{3}(-1)^n\beta_n (Y_{(n)})^\rho{}_\nu(\mathbb{X})\,, \tag{2.56}$$

where

$$(Y_{(n)})^\rho{}_\nu(\mathbb{X}) \equiv \sum_{k=0}^{n}(-1)^k e_k(\mathbb{X})\,(\mathbb{X}^{n-k})^\rho{}_\nu\,. \tag{2.57}$$

There is no equation of motion for $f_{\mu\nu}$ whose form therefore needs to be put in by hand. Taking the divergence of the above equations and using the Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$ (where $G_{\mu\nu}$ is the Einstein tensor)[12], we get a set of Bianchi constraints,

$$\nabla^\mu U^g_{\mu\nu} = 0\,, \tag{2.58}$$

which eliminate four degrees of freedom while the remaining extra mode (the Boulware-Deser ghost) is removed by the additional constraint present in the special structure

[12] If we consider also the energy-momentum tensor in eq. (2.55), we also need to use the fact that the energy-momentum tensor is covariantly conserved, $\nabla^\mu T_{\mu\nu} = 0$, in order to get eq. (2.58).

of the potential in eq. (2.51). A covariant expression for this constraint is difficult, if not impossible, to obtain in general.

The action (2.51) is a non-trivial generalisation of the model of massive gravity called the de Rham-Gabadadze-Tolley (dRGT) model with a Minkowski reference metric [47, 49], and it was presented in this form for the first time in [82]. The advantage of presenting the massive gravity action in this form is that the construction exposed before is completely general, that is, valid for a generic reference metric $f_{\mu\nu}$, whereas, as already said, the construction of the dRGT model in [47, 49] relies on a flat reference metric, $f_{\mu\nu} = \eta_{\mu\nu}$.

The dRGT action can be written in the form [45, 130]

$$\begin{aligned} S_{\mathrm{dRGT}} &= \frac{M_g^2}{2}\int \mathrm{d}^4x\,\sqrt{-\det g_{\mu\nu}}\left[R(g) + 2m^2\sum_{n=0}^{4}\alpha_n e_n(\mathcal{K})\right] \\ &= \frac{M_g^2}{2}\int \mathrm{d}^4x\,\sqrt{-\det g_{\mu\nu}}\left[R(g) + 2m^2(e_2(\mathcal{K}) + \alpha_3 e_3(\mathcal{K}) + \alpha_4 e_4(\mathcal{K}))\right], \end{aligned} \tag{2.59}$$

where

$$\mathcal{K}^\mu{}_\nu = \delta^\mu{}_\nu - \left(\sqrt{g^{-1}\eta}\right)^\mu{}_\nu\,, \qquad \alpha_0 = \alpha_1 = 0\,, \qquad \alpha_2 = 1\,. \tag{2.60}$$

We can obtain this action from the one in the form (2.51) by setting $f_{\mu\nu} = \eta_{\mu\nu}$ and

$$\beta_n = (-1)^n \sum_{k=n}^{4}\binom{k}{n}\alpha_k\,. \tag{2.61}$$

Therefore, the consistency of massive gravity with general reference metric (2.51) implies the consistency (the absence of the Boulware-Deser ghost) of the dRGT action (2.59), whereas, on the other hand, there is no obvious way of getting the action (2.51) from the dRGT action (2.59) and hence we cannot automatically generalize to the former action the results that we obtain in the latter [130]. The fact that lowest-order parameters α_n are fixed is due to the requirements that $g_{\mu\nu} = \eta_{\mu\nu}$ is a solution for the equations of motion obtained by varying eq. (2.59), that terms linear in the fluctuation $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ do not appear in the quadratic action and that the parameter m^2 corresponds exactly to the squared mass which appears in the Fierz-Pauli action (2.28). In this way, the dRGT action (2.59) admits Minkowski solutions for $g_{\mu\nu}$ and the linearization of the theory around these backgrounds gives back the Fierz-Pauli action (2.28). In this sense the dRGT theory can be considered as the consistent non-linear extension of the action for a massive graviton propagating in a Minkowski background.

The knowledge of the full, consistent non-linear action (2.51) allows us to deduce the linear theory on an arbitrary background $\bar{g}_{\mu\nu}$. This calculation constitutes one of the main results of this Thesis, and it is exposed in detail in chapter 3.

2.4.5 The Stückelberg trick

Since massive gravity describes a massive spin-2 field which is expected carrying more degrees of freedom than the 2 of the massless field, the symmetry under diffeomorphisms (general coordinate transformations) of General Relativity must be broken by the mass term, as we have already said. This happens because of the presence of the fixed reference metric $f_{\mu\nu}$ which, contrary to the physical metric $g_{\mu\nu}$, does not transform as a tensor with two indices under diffeomorphisms if it is taken to be fixed in a given coordinate system. However, covariance is not a true symmetry, on the contrary it is completely artificial, and hence in this view it is possible to simply consider $f_{\mu\nu}$ as a fixed reference metric but still allow it to transform as a tensor under diffeomorphisms. A famous way to see this is the so-called *Stückelberg trick*, that we have already seen in 2.3: by representing the new degrees of freedom as Stückelberg fields, which have certain transformation properties under diffeomorphisms, we can restore gauge invariance and still keeping track of the redundant modes, whereas fixing the *unitary gauge* (or *physical gauge*) gives back the original action, in which the presence of the interaction potential breaks manifestly the gauge invariance. The Stückelberg trick is therefore an illustration of the fact that gauge invariance represents just a redundancy of description. In fact, we could take any theory and make it a gauge theory (that is, a gauge-invariant theory) by introducing redundant variables (the Stückelberg fields). On the other hand, given any gauge theory, we can always eliminate the gauge symmetry by eliminating the redundant degrees of freedom (of course, removing the redundancy is not always the smart thing to do, since it could bring the theory in a much less transparent form) [93].

In the case of massive gravity, if $f_{\mu\nu}$ is taken to not transform under diffeomorphisms (hence breaking gauge invariance), we can replace $f_{\mu\nu}$ by a quantity which is patterned after the transformation under diffeomorphisms of a gauge tensor with the two indices down[13],

$$f_{\mu\nu} \to \tilde{f}_{\mu\nu} = \partial_\mu \varphi^a \partial_\nu \varphi^b f_{ab} \,, \tag{2.62}$$

where the four dynamical fields[14] φ^a (the Stückelberg fields) transform as scalars under diffeomorphisms, in such a way that $\tilde{f}_{\mu\nu}$ transforms as a tensor under diffeomorphisms. f_{ab} is here a "metric" in the space of the Stückelberg fields. We can now construct the theory of massive gravity as the gauge-invariant action of the tensors $g_{\mu\nu}$ and $\tilde{f}_{\mu\nu}$. In unitary gauge, where $x^a = \varphi^a$, we simply recover $\tilde{f}_{\mu\nu} = f_{\mu\nu}$ [45]. The full system of equations of motion is given by the dynamical equations for the metric $g_{\mu\nu}$ and for the scalar fields φ^a (when the gauge invariance has been restored) or, equivalently,

[13] Here the latin indices $a, b, \dots = 0, 1, 2, 3$ label the dimensions in the space of the Stückelberg fields, whereas the Lorentz (greek) indices label, as usual, the four space-time dimensions. In the following calculations, however, one can identify the indices a, b with Lorentz indices as well for practical purposes.

[14] The kinetic term for the fields φ^a in the action comes from the time derivatives in eq. (2.62).

by the original dynamical equations for $g_{\mu\nu}$ without the Stückelberg trick: that is, the redundant gauge degrees of freedom φ^α do not introduce new dynamics into the theory. In fact, one can show that the equations of motion for the Stückelberg fields are actually equivalent to the Bianchi constraint of the theory [130].

It is important to recall that it is possible to perform the proof of the absence of the Boulware-Deser ghost in non-linear massive gravity also in the Stückelberg formulation. This result, even if expected (since the two formulations, with and without the Stückelberg fields, are completely equivalent), is absolutely non trivial and it is shown in [87].

2.4.6 The decoupling limit of massive gravity

The Stückelberg trick exposed above is useful when we consider the so-called *decoupling limit* of massive gravity, a special kind of limit of the theory in which some of the original degrees of freedom decouple from the remaining ones while keeping the total number of degrees of freedom identical [45]. In our case, concerning the reference metric, we start by taking $f_{\mu\nu} = \eta_{\mu\nu}$ in (2.62)[15] and we expand the Stückelberg fields around the identity:

$$\varphi^a = (x^\alpha - \pi^\alpha)\,\delta^a_\alpha\,. \tag{2.63}$$

We may further split the perturbations π^α into a transverse and a longitudinal component,

$$\pi^\alpha = \frac{\hat{\pi}^\alpha}{m\,M_g} + \frac{\partial^\alpha \pi}{m^2 M_g} \tag{2.64}$$

(with $\partial_\alpha \hat{\pi}^\alpha = 0$), where both π and $\hat{\pi}^\alpha$ have dimension of mass. The original construction of the ghost-free dRGT potential (2.59) was actually performed using the Stückelberg formalism and a decoupling limit to separate the interactions for the longitudinal (helicity-0) mode π and hence to make manifest its dynamics. A conjecture made in [18] states that the origin of the ghost instability of non-linear massive gravity lies in the presence of higher-derivative terms for the π field in flat space, where $f_{ab} = \eta_{ab}$, and therefore the longitudinal mode carries all information needed to construct a ghost-free massive gravity theory. The decoupling limit in this case is defined by taking

$$M_g \to \infty\,, \qquad m \to 0\,, \qquad \text{while keeping} \qquad \Lambda_3 \equiv (m^2 M_g)^{1/3} = \text{const.} \tag{2.65}$$

Concerning the physical metric, we decompose it into a Minkowski background and fluctuations:

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_g}\,. \tag{2.66}$$

[15]This method can in principle be used for any reference metric, but dealing with a general $f_{\mu\nu}$ does not allow to identify the proper physical degrees of freedom [50].

In a first approach the vector modes are set to zero. The matrix $\mathbb{X} = g^{-1}f$ which enters the interaction potential then reads, in matrix notation,

$$\mathbb{X} = g^{-1}f = \left(\mathbb{1} + \frac{\eta^{-1}h}{M_g}\right)^{-1}\left(\mathbb{1} + \frac{\Pi}{\Lambda_3^3}\right)^2 , \tag{2.67}$$

where we have defined $\Pi^\mu{}_\nu \equiv \eta^{\mu\rho}\partial_\rho\partial_\nu\pi$. We can see from (2.67) that the longitudinal mode π always appears with two derivatives and therefore its self-interactions generically give rise to terms with higher-order derivatives, which in turn lead to Ostrogradskij instabilities. The work of de Rham, Gabadadze and Tolley [47, 49] consists in tuning the coefficients of the Π interaction terms in such a way that the pathological terms with higher-order derivatives can be removed from the action: the resulting action for the π mode has exactly the same structure as the Galileon interactions in a flat space-time [123].

2.5 Is massive gravity viable?

Since the appearance of the seminal works by de Rham, Gabadadze and Tolley in 2010 and 2011 [47, 49], a huge amount of work has been done in order both to develop further the theoretical side and to investigate the possibility of massive gravity as a physical explanation for the late-time accelerated expansion of the Universe. However, some issues of the new theory have emerged quite soon. We have already mentioned in the Introduction that the fact that the reference metric is not dynamically determined but is fixed by hand (that is, its form cannot be determined by any obvious fundamental principle) somehow “spoils” the spirit of General Relativity and is difficult to motivate from a field theoretical point of view.

Focusing on the cosmological applications of this theory, at the origin the idea of *degravitation* [62, 63] brought to believe that massive gravity could be able to address the puzzle of cosmic acceleration: in fact whereas adding a small mass to the graviton keeps physics at small scales largely equivalent to General Relativity because of the Vainshtein mechanism, it modifies gravity in the IR and, since the cosmological constant is the most infrared source possible (since it is built entirely out of zero momentum modes) we may hope that gravitational effects are modified in a theory of massive gravity. In practice, gravity is weakened in the IR so that the effect of IR sources can be degravitated. This motivated the investigation of cosmological solutions in massive gravity.

However, it was actually found that there are no non-trivial spatially flat (and spatially closed) FLRW solutions in the dRGT theory of massive gravity with the reference metric taken to be Minkowski [39]. As a consequence, different alternatives have been explored in the literature to study the cosmology of this model. For example, spatially open FLRW solutions are allowed, but they are plagued by instabilities which rule them out [79]. On the other hand, taking the reference metric to be FLRW or de Sitter generally leads to unacceptable Higuchi bounds[16] [69]. However, despite the dynamics of massive gravity lacks the property to give rise to homogeneous and isotropic solutions, the Vainshtein mechanism guarantees that there exist inhomogeneous cosmological solutions which can approximate the normal FLRW solution of General Relativity in the limit $m \to 0$ [39].

These and other issues about the theory and its cosmological solutions (see also some related discussions in [45, 130]) have pushed the scientific community to look for possible extensions of the ghost-free massive gravity action. The most natural one consists in giving the reference metric its own dynamics, in the context of a fully dynamical bimetric theory.

2.6 The ghost-free bimetric theory

Our aim is to investigate whether adding a kinetic term for the reference metric could not only restore the main spirit of General Relativity but also lead to cosmological solutions able to mimic the ΛCDM model without needing to introduce a cosmological constant and the related problems which we reviewed in the Introduction. The fact that we can extend the previously discussed dRGT model with general $f_{\mu\nu}$ to a bigravity theory without re-introducing the Boulware-Deser ghost is a non-trivial issue. Hassan and Rosen constructed such a bimetric theory by augmenting the dRGT action by an Einstein-Hilbert term for the metric $f_{\mu\nu}$ which is no longer fixed by hand, but is dynamically determined [83]. It has also been proved that, assuming an Einstein-Hilbert term for $g_{\mu\nu}$ and an interaction potential of the form (2.51), the only possible kinetic term for $f_{\mu\nu}$ is also of the Einstein-Hilbert form [54, 55, 118].

According to this, the ghost-free fully non-linear action for Hassan-Rosen theory of bimetric gravity, published in [83] for the first time, has the form

$$S_{\rm BI} = \frac{M_g^2}{2}\int \mathrm{d}^4x\,\sqrt{-\det g_{\mu\nu}}\left[R(g) - 2\,m^2\sum_{n=0}^{4}\beta_n e_n(\mathbb{X})\right] + \frac{M_f^2}{2}\int \mathrm{d}^4x\,\sqrt{-\det f_{\mu\nu}}R(f)\,, \tag{2.68}$$

where we have introduced the reduced Planck mass for the metric $f_{\mu\nu}$, M_f. The

[16]The Higuchi bound (originally investigated in a de Sitter space-time) translates the requirement that the kinetic term in the action has the correct sign into a requirement on the parameters of the theory [91].

equations of motion for the two metrics which can be derived from this action are

$$\begin{aligned} \frac{1}{M_g^2} T_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{m^2}{2} \sum_{n=0}^{3} (-1)^n \beta_n \left[g_{\mu\lambda} Y^{(g)\lambda}_{(n)\nu} + g_{\nu\lambda} Y^{(g)\lambda}_{(n)\mu} \right], && (2.69) \\ 0 &= R^f_{\mu\nu} - \frac{1}{2} f_{\mu\nu} R^f + \frac{m^2}{2m_*^2} \sum_{n=0}^{3} (-1)^n \beta_{4-n} \left[f_{\mu\lambda} Y^{(f)\lambda}_{(n)\nu} + f_{\nu\lambda} Y^{(f)\lambda}_{(n)\mu} \right], && (2.70) \end{aligned}$$

where the superscript f indicates the curvature of the metric $f_{\mu\nu}$. The definition of the matrices $Y^{(g)\nu}_{(n)\mu} = Y^{\nu}_{(n)\mu}\left(\sqrt{g^{-1}f}\right)$ and $Y^{(f)\nu}_{(n)\mu} = Y^{\nu}_{(n)\mu}\left(\sqrt{f^{-1}g}\right)$ is as follows

$$\begin{aligned} Y_{(0)}(X) &= \mathbb{I}, \quad Y_{(1)}(X) = X - \mathbb{I}[X], \\ Y_{(2)}(X) &= X^2 - X[X] + \frac{1}{2}\mathbb{I}\left([X]^2 - [X^2]\right), \\ Y_{(3)}(X) &= X^3 - X^2[X] + \frac{1}{2}X\left([X]^2 - [X^2]\right) - \frac{1}{6}\mathbb{I}\left([X]^3 - 3[X][X^2] + 2[X^3]\right). \end{aligned}$$

As a consequence of the Bianchi identities and of the covariant conservation of $T_{\mu\nu}$, we obtain the following Bianchi constraints for each of the two metrics:

$$\nabla^g_\mu \sum_{n=0}^{3} (-1)^n \beta_n \left[Y^{(g)\nu\mu}_{(n)} + Y^{(g)\mu\nu}_{(n)} \right] = 0, \tag{2.71}$$

$$\nabla^f_\mu \sum_{n=0}^{3} (-1)^n \beta_{4-n} \left[Y^{(f)\nu\mu}_{(n)} + Y^{(f)\mu\nu}_{(n)} \right] = 0, \tag{2.72}$$

where we raise and lower indices of $Y^{(g)\cdots}_{(n)\cdots}$ and $Y^{(f)\cdots}_{(n)\cdots}$ with the metrics g and f respectively and the relevant metric is indicated in the covariant derivatives ∇^g and ∇^f. Both these constraints follow from the invariance of the interaction term under the diagonal subgroup of the general coordinate transformations of both metrics. Both constraints are equivalent and hence there is only one set of independent Bianchi constraints just as in massive gravity.

Whereas in the case of massive gravity (2.51) the β_4-term $\sqrt{-\det g_{\mu\nu}}\, e_4(\mathbb{X}) = \sqrt{-\det f_{\mu\nu}}$ did not contribute to the dynamics (and hence it could have been removed from the action (2.51)), in the case of the bigravity action (2.68) this term now contributes to the equations of motion for $f_{\mu\nu}$, since we have given $f_{\mu\nu}$ its own dynamics. We also notice that, due to the properties of the elementary symmetric polynomials, the structure of the action (2.68) is symmetric in the two metrics:

$$\sqrt{-\det g_{\mu\nu}}\, e_n\left(\sqrt{g^{-1}f}\right) = \sqrt{-\det f_{\mu\nu}}\, e_{4-n}\left(\sqrt{f^{-1}g}\right). \tag{2.73}$$

This means that the two metrics $g_{\mu\nu}$ and $f_{\mu\nu}$ are treated on the same footing and the “physical metric" is selected only via the coupling to matter.

The interaction potential of the theory breaks the two diffeomorphism invariances of $g_{\mu\nu}$ and $f_{\mu\nu}$ (that we would have if the action were composed only of the two Einstein-Hilbert terms) down to their diagonal subgroup. This means that the bigravity action (2.68) is not invariant under independent coordinate transformations of the two metrics but only under those ξ^{μ} that transform both metrics simultaneously in the same way: that is, the function ξ^{μ} must be such that $g'_{\mu\nu} = g_{\mu\nu} + \nabla^{g}_{\mu}\xi_{\nu} + \nabla^{g}_{\nu}\xi_{\mu}$ and $f'_{\mu\nu} = f_{\mu\nu} + \nabla^{f}_{\mu}\xi_{\nu} + \nabla^{f}_{\nu}\xi_{\mu}$ at the same time, where ∇^{g}_{μ} is the covariant derivative with respect to the metric $g_{\mu\nu}$ and ∇^{f}_{μ} is the covariant derivative with respect to the metric $f_{\mu\nu}$. The counting of the degrees of freedom in the theory is then straightforward. We start with 20 *a priori* independent components (10 for each of the two metrics), but 8 of these are removed by gauge constraints and gauge fixing coming from the diagonal subgroup of the two original diffeomorphisms. As we have already seen there is only one set of Bianchi constraints, which can be taken to be either (2.71) or (2.72) and which remove another four degrees of freedom. This leaves us with a total of eight propagating modes. These modes correspond to the two degrees of freedom of a massless spin-2 field, the five degrees of freedom of a massive spin-2 field and one additional scalar mode which is the Boulware-Deser ghost. In a consistent bimetric theory we therefore also need an additional constraint in order to eliminate the ghost mode.

Analogously to the massive gravity case, this means that, in terms of the ADM variables, we need the bigravity action to be linear in both lapses N and L of the two metrics $g_{\mu\nu}$ and $f_{\mu\nu}$. The action also needs to be linear in one three-dimensional vector (the shift L^{i} of the metric $f_{\mu\nu}$). In this way, we will have five non-dynamical variables whose equations of motion become constraints: four (the three associated to L^{i} and the one associated to one of the two lapses) corresponding to the gauge constraints associated to the surviving diffeomorphism invariance, plus one extra constraint which is needed to remove the Boulware-Deser ghost. Actually, it has been explicitly shown [83] that this is the case for the action (2.68), which therefore gives rise to the correct number of propagating modes: there are in total 7 degrees of freedom, corresponding to a massive (5 modes) and a massless (2 modes) spin-2 field.

2.6.1 The massive and massless eigenstates of the theory

In complete generality, at the fully non-linear level, there is no unique way to split the different degrees of freedom of the theory into “massive” and “massless” modes. In fact, the “mass” of a particle is exactly defined only in a space-time whose symmetry group is the Poincaré group (with 10 independent generators), since its notion arises as a Casimir invariant of this isometry group. It is possible to generalize that notion to spaces which enjoy the same number of symmetries (like for example the de Sitter or the Anti-de Sitter space-time) but, for spaces which have a lower number of symmetries (like for example the FLRW space-time), a clear identification of the mass becomes non-trivial. However, we can still classify a field as massless or massive according to its number of propagating degrees of freedom. In particular, we can identify with the "mass" that parameter of the theory such that, when it is taken to be zero, the amount of gauge redundancy of the theory is increased and hence the number of propagating degrees of freedom is reduced to that of a massless theory. This is the only way in which we can use the concepts of "mass" in massive gravity and "massless" and "massive" fields in bimetric gravity.

We can still determine well-defined mass eigenstates for the bimetric theory when we consider its linearization around the maximally symmetric background solutions of the theory. The simplest class of solutions is obtained by making the ansatz that the two background metrics are conformally related, $\bar{f}_{\mu\nu} = c(x)^2 \bar{g}_{\mu\nu}$, which because of the Bianchi constraint of the theory simply becomes $\bar{f}_{\mu\nu} = c^2 \bar{g}_{\mu\nu}$, with c = constant. This means that we are focusing on proportional backgrounds. When we consider small perturbations around these background metrics [80],

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{1}{M_g} h_{\mu\nu} \qquad f_{\mu\nu} = \bar{f}_{\mu\nu} + \frac{c}{M_f} l_{\mu\nu} = c^2 \bar{g}_{\mu\nu} + \frac{c}{M_f} l_{\mu\nu} \, , \tag{2.74}$$

we find that the linear perturbations equations are greatly simplified with respect to other possible choices of the background metrics. In particular, they can easily be diagonalized into an equation for a massless and a massive perturbation. In fact, considering the following combinations of metric fluctuations,

$$u_{\mu\nu} \equiv \frac{1}{\sqrt{c^2\alpha^2 + 1}} \left(h_{\mu\nu} + c\,\alpha\, l_{\mu\nu} \right) \, , \qquad v_{\mu\nu} \equiv \frac{1}{\sqrt{c^2\alpha^2 + 1}} \left(l_{\mu\nu} - c\,\alpha\, h_{\mu\nu} \right) \, , \tag{2.75}$$

where $\alpha \equiv M_f/M_g$, we can decouple the massless from the massive mode, and the corresponding massless and massive equation are[17] [80, 130]:

$$\mathcal{E}^{\rho\sigma}_{\mu\nu} u_{\rho\sigma} + \Lambda_g u_{\mu\nu} = 0 \, , \tag{2.76}$$

$$\mathcal{E}^{\rho\sigma}_{\mu\nu} v_{\rho\sigma} + \Lambda_g v_{\mu\nu} + \frac{m_{FP}^2}{2} \left(v_{\mu\nu} - \bar{g}_{\mu\nu} v \right) = 0 \, , \tag{2.77}$$

where $v \equiv \bar{g}^{\rho\sigma} v_{\rho\sigma}$,

$$\Lambda_g \equiv \frac{2\,m^4}{M_g^2} \left(\beta_0 + 3\,c\,\beta_1 + 3\,c^2\,\beta_2 + c^3\,\beta_3 \right) \, , \tag{2.78}$$

and where $\mathcal{E}^{\mu\nu\rho\sigma}$ is the covariant version of the Lichnerowicz operator in a Minkowski background (2.27). The Fierz-Pauli mass above is parametrized as

$$m_{FP}^2 \equiv 2\,m^4 (c\beta_1 + 2\,c^2 \beta_2 + c^3 \beta_3) \left(\frac{1}{c^2 M_f^2} + \frac{1}{M_g^2} \right) . \tag{2.79}$$

[17] In general, for backgrounds such that the matrix $\bar{g}^{-1}\bar{f}$ does not commute with the perturbations, the massive equation of motion has not the Fierz-Pauli structure, therefore it is difficult, or even impossible, to uniquely identify the massive field, whereas for proportional backgrounds we have that $\bar{g}^{-1}\bar{f} = c^2 \mathbb{1}$ which does commute with any other matrix.

From eqs. (2.76) and (2.77) we can see that $u_{\mu\nu}$ represents the massless fluctuation and hence carries two degrees of freedom, whereas $v_{\mu\nu}$ corresponds to the massive fluctuation and therefore carries five degrees of freedom.

2.6.2 Coupling to matter

We finally discuss the possible couplings of the spin-2 degrees of freedom to the matter fields. It has been found [52] that the only couplings that can be added to the bimetric gravity action (2.68) without reintroducing the Boulware-Deser ghost are given by

$$S_{\text{matter}} = \int \mathrm{d}^4x \sqrt{-\det g_{\mu\nu}}\, \mathcal{L}_{matter}(g, \Phi_g) + \int \mathrm{d}^4x \sqrt{-\det f_{\mu\nu}}\, \tilde{\mathcal{L}}_{matter}(f, \Phi_f)\,, \tag{2.80}$$

where both $\mathcal{L}_{matter}$ and $\tilde{\mathcal{L}}_{matter}$ are standard minimally coupled matter Lagrangians which have the same form as in General Relativity. There can be in principle two sectors of matter fields, the matter fields Φ_g which are only coupled to the metric $g_{\mu\nu}$ and the matter fields Φ_f which are only coupled to the metric $f_{\mu\nu}$. Φ_g and Φ_f are completely independent, since it is not possible to couple the same matter field to both metrics using minimal couplings: in fact, such coupling reintroduces the Boulware-Deser ghost. We can define the stress-energy tensors for both sectors as

$$\begin{aligned} T^g_{\mu\nu} &\equiv -\frac{1}{\sqrt{-\det g_{\mu\nu}}} \frac{\partial(\sqrt{-\det g_{\mu\nu}}\, \mathcal{L}_{matter}(g, \Phi_g))}{\partial g^{\mu\nu}}\,, \\ T^f_{\mu\nu} &\equiv -\frac{1}{\sqrt{-\det f_{\mu\nu}}} \frac{\partial(\sqrt{-\det f_{\mu\nu}}\, \mathcal{L}_{matter}(f, \Phi_f))}{\partial f^{\mu\nu}}\,. \end{aligned} \tag{2.81}$$

Due to the conceptual problems that plague a consistent physical explanation of the current observed accelerated expansion of the Universe within the framework of General Relativity (as we have seen in the Introduction), it is very interesting to try to see whether cosmological solutions of the Hassan-Rosen bigravity theory could provide this explanation without needing to introduce a cosmological constant. Differently from the case of the dRGT massive gravity model, in which (as we have seen) there are no stable homogeneous and isotropic solutions, in bimetric gravity with a dynamical reference metric $f_{\mu\nu}$ this kind of solutions does exist (see e. g. [139, 143]), therefore the hope to solve the puzzle of cosmic acceleration is justified. Moreover, as we will see, self-accelerating background solutions (that is, solutions which mimic the behaviour of dark energy) which fit the observational data as well as the ΛCDM model exist for different values of the β_n parameters. The best fit value for the mass m is of the order of the Hubble scale, $m \sim H_0$. Since in the limit $m \to 0$ the full diffeomorphism symmetry that transforms both the metrics separately is restored, this renders this tiny mass scale technically natural (according to the 't Hooft criterion), the existence of this symmetry protecting the spin-2 mass scale from receiving large quantum corrections [90].

In most of the analysis in the existing literature the source $T^f_{\mu\nu}$ of the $f_{\mu\nu}$ sector is set to zero, for sake of simplicity and in order to interpret $g_{\mu\nu}$ as the usual physical metric (provided that it has a standard matter coupling): this is what we shall assume throughout the following chapters of this Thesis, devoted to the analysis of cosmological solutions in bigravity, both at the background and at the perturbation level.

Part I

A general formalism for perturbations

Chapter 3

Perturbations for massive gravity theories

Based on:

[77] P. Guarato and R. Durrer, "Perturbations for massive gravity theories", Phys. Rev. D **89** (2014) 084016 [arXiv:1309.2245 [gr-qc]].

Abstract. A theory of massive gravity depends on a nondynamical "reference metric" $f_{\mu\nu}$ which is often taken to be the flat Minkowski metric. In this paper we examine the theory of perturbations on a background with metric $\bar{g}_{\mu\nu}$ which does not coincide with the reference metric $f_{\mu\nu}$. We derive the mass term for general perturbations on this background and show that it generically is not of the form of the Fierz-Pauli mass term. We explicitly compute it for some cosmological situations and show that it generically leads to instabilities.

3.1 Introduction

In recent years, interest in massive gravity theory has been rekindled. There are two main reasons for this: first, a graviton mass weakens gravity on large scales and provides a natural mechanism of "degravitation" which can solve the cosmological constant problem [64, 53, 44]. If the graviton is massive, the range of gravity is finite and a cosmological constant does not gravitate. Second, if one fine tunes the graviton mass to $m_g \sim H_0$, where $H_0 \simeq 1.5 \times 10^{-42}$GeV is the value of the Hubble constant, gravity weakens around this scale and such a modified gravity theory can explain the observed present accelerated expansion of the Universe [100, 134, 92, 8]; hence, it can play the role of dark energy [108, 79, 111, 74].

In order to give the graviton, i.e., the degrees of freedom of the metric of spacetime a mass, one has to introduce a reference metric in order to define a potential which gives energy to deviations away from the reference metric. For a scalar field or a vector field, this reference point is usually set to zero. For the metric this is not an option since the metric $f_{\mu\nu} = 0$ is singular.

There is also the possibility to avoid the reference metric but at the cost of nonlocal terms like for example $m^2 \Box^{-1} G_{\mu\nu}$ in the equations of motion [99]. Such theories are usually not ghost free, but recently a solution where massive gravity can mimic dark energy for such a theory has been found [116, 72, 117].

The most natural reference metric seems to be the Minkowski metric, $f_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, but in principle the reference metric is general [86]; also, other possibilities like a de Sitter reference metric [56, 111] have been considered. Moreover, since time translation invariance is broken at very low energy, i.e. on cosmological scales, this might be an indication for a more general, less symmetric reference metric.

A generic quadratic term in the "metric perturbations" gives rise not only to three additional propagating gravitational modes which are necessary to complete the two massless modes to a massive spin-2 particle, but to an additional helicity-zero mode which is a ghost. To avoid this ghost, one has to introduce a mass term of a very specific form, the so-called Fierz-Pauli mass term [126], but even in this case, as has been shown by Boulware and Deser [26], the ghost reappears at the nonlinear level.

Recently, de Rham, Gabadadze and Tolley (dRGT) [47, 49] have proposed a nonlinear, polynomial generalization of the Fierz-Pauli mass term which is ghost free for an arbitrary reference metric f and physical metric g. They have shown that the interactions between the different helicity modes can be at most fourth order in the Langrangian. The action is written in the form

$$S = \frac{M_P^2}{2} \int \sqrt{-\det g}\,[R(g) - U(f,g)]\ , \tag{3.1}$$

where the second term, added to the usual Einstein-Hilbert action, takes into account the mass potential of the graviton. This work has spurred a flurry of activity in massive gravity theories[1]. Especially, people want to investigate whether massive gravity can be at the origin of the observed accelerated cosmological expansion. For this, solutions which lead to an expansion history close to the one of the observable Universe have been studied [56, 111, 74].

To investigate cosmology in massive gravity, we of course cannot simply search for a background solution of massive gravity which reproduces the observed cosmological expansion history, but we also need to study perturbations on this cosmology which are relevant for the anisotropies of the cosmic microwave background and large-scale structure formation. This has been started for some specific cases e.g. in Refs. [79, 40, 38, 109].

This is also where the present work sets in. We derive the generic form of the graviton mass term in perturbation theory. For this we allow for an arbitrary reference

[1]Since 2010, 302 papers with "massive gravity" in the title have been submitted to the arXiv at the time of this writing.

metric $f_{\mu\nu}$ and a background solution $\bar{g}_{\mu\nu}$. We consider the true metric given by $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is a small perturbation which we want to study up to quadratic order in the Lagangian. The first-order terms vanish due to the fact that $\bar{g}$ solves the equations of motion, and we are only interested in the second order. For the perturbed potential we can write up to second order in $h_{\mu\nu}$

$$\sqrt{-\det g}U(f,g) = \sqrt{-\det \bar{g}}\left[U(f,\bar{g}) + \mathcal{M}^{\mu\nu\alpha\beta}h_{\mu\nu}h_{\alpha\beta}\right]. \tag{3.2}$$

The goal of this work is to determine the tensor $\mathcal{M}^{\mu\nu\alpha\beta}(f,\bar{g})$ for arbitrary reference metric $f_{\mu\nu}$ and background $\bar{g}_{\mu\nu}$. We will find that for $f = \bar{g}$, the mass term is, as expected, the Fierz-Pauli combination. In this case, we know that also the higher order terms in $h^{\mu}{}_{\nu}$ are ghost free by construction. We show that when $f \neq \bar{g}$ the quadratic mass term does not satisfy the Fierz-Pauli tuning. However, this does not imply the presence of a ghost. In this nonperturbative case, it has to be checked that the constraint equations still project out the ghost. This has been done previously in Ref. [86]. However, it has also been shown recently that even the second scalar mode, which is "healthy" in vacuum, can become ghostlike in certain cases, e.g. in cosmology [69].

We finally discuss our mass term in a cosmological setting, where we also solve the perturbation equations for a special case.

The rest of the paper is organized as follows. In Sec. 3.2 we derive the general form of $\mathcal{M}^{\alpha\beta\mu\nu}$. In Sec. 3.3 we apply our result in cosmology and discuss it. In Sec. 3.4 we conclude. Some lengthy calculations are deferred to appendixes.

Notation We use the metric signature $(-,+,+,+)$. The reduced Planck mass M_P is given by $M_P^2 = (8\pi G)^{-1}$, where G denotes Netwon's gravitational constant.
Matrices are often denoted without indices, $g \equiv (g_{\mu\nu})$. In order to avoid confusion, determinants and traces are always clearly indicated as such, $\det g$ and $\mathrm{tr}\mathcal{K} \equiv [\mathcal{K}]$.

3.2 Metric perturbations

Let us consider $\bar{g}_{\mu\nu}$ to be a solution to a given massive gravity theory with reference metric $f_{\mu\nu}$ and graviton potential

$$U(f,g) = -2m^2\left(e_2(\mathcal{K}) + e_3(\mathcal{K}) + e_4(\mathcal{K})\right) \tag{3.3}$$

where

$$\mathcal{K}^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} - (\sqrt{g^{-1}f})^{\mu}_{\nu} \qquad \text{and} \tag{3.4}$$

$$
\begin{aligned}
e_1(\mathcal{K}) &= [\mathcal{K}]\,, & (3.5)\\
e_2(\mathcal{K}) &= \frac{1}{2}\left([\mathcal{K}]^2-[\mathcal{K}^2]\right)\,, & (3.6)\\
e_3(\mathcal{K}) &= \frac{1}{6}\left([\mathcal{K}]^3-3[\mathcal{K}][\mathcal{K}^2]+2[\mathcal{K}^3]\right)\,, & (3.7)\\
e_4(\mathcal{K}) &= \frac{1}{24}\Big([\mathcal{K}]^4-6[\mathcal{K}]^2[\mathcal{K}^2]+3[\mathcal{K}^2]^2+\quad 8[\mathcal{K}][\mathcal{K}^3]-6[\mathcal{K}^4]\Big)=\det(\mathcal{K})\,. & (3.8)
\end{aligned}
$$

Here we use the notation $[\mathcal{K}]=\mathrm{tr}\mathcal{K}=\mathcal{K}^\mu{}_\mu$, $[\mathcal{K}^2]=\mathrm{tr}\mathcal{K}^2=\mathcal{K}^\mu{}_\nu\mathcal{K}^\nu{}_\mu$ and so forth. e_1 which does not appear in Eq. (3.3) has been defined for later convenience. Notice that $\mathcal{K}$ and therefore $U(f,g)$ vanish when $g=f$.

The square root of the matrix $g^{-1}f$ is just some matrix whose square is $g^{-1}f$. In general, this is not unique. However, if $g^{-1}f$ is close to the identity, $g^{-1}f=\mathbb{1}+\epsilon$ with $|\epsilon^\mu{}_\nu|<1/d$, where d denotes the dimension of the matrix, we want to choose the root given by the convergent Taylor series,

$$\sqrt{\mathbb{1}+\epsilon}=\mathbb{1}+\sum_{k=0}^{\infty}\frac{(\frac{1}{2}-k)(\frac{1}{2}-k+1)\cdots\frac{1}{2}}{(k+1)!}\epsilon^{k+1}\,. \tag{3.9}$$

The potential $U(f,g)$ can be deformed by introducing arbitrary coefficients in front of e_3 and e_4,

$$U(f,g)=-2m^2\left(e_2(\mathcal{K})+\alpha_3e_3(\mathcal{K})+\alpha_4e_4(\mathcal{K})\right)\,. \tag{3.10}$$

In Ref. [82] it is shown that this is the most general potential for a ghost-free theory of massive gravity in four dimensions.

We now want to consider linear perturbations around a background solution with $\bar{g}_{\mu\nu}\neq f_{\mu\nu}$ for the massive gravity theory with potential (3.10). To derive the linear perturbation equations we develop the Lagrangian

$$L(g)=\frac{M_P^2}{2}\sqrt{-\det g}\Big(R(g)-U(f,g)\Big) \tag{3.11}$$

to second order in $h_{\mu\nu}$, the deviation of the true metric g from the background, $g_{\mu\nu}=\bar{g}_{\mu\nu}+h_{\mu\nu}$. The kinetic term for $h_{\mu\nu}$ is determined by the Einstein operator, $\mathcal{E}^{\mu\nu\alpha\beta}$, in curved spacetime,

$$
\begin{aligned}
\sqrt{-\det g}R(g) &= \sqrt{-\det\bar{g}}\Big[R(\bar{g})-h_{\mu\nu}G^{\mu\nu}(\bar{g})\\
&\qquad +h_{\mu\nu}\mathcal{E}^{\mu\nu\alpha\beta}(\bar{g})h_{\alpha\beta}+\nabla_\mu V^\mu\Big]+\mathcal{O}(h^3)\,. & (3.12)
\end{aligned}
$$

with [93]

$$
\begin{aligned}
\mathcal{E}^{\mu\nu\alpha\beta}(\bar{g}) &= -\frac{1}{2}\Big[\left(\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta} - \bar{g}^{\mu\nu}\bar{g}^{\alpha\beta}\right)\Box \\
&\quad + \left(\bar{g}^{\mu\nu}\bar{g}^{\alpha\rho}\bar{g}^{\beta\sigma} + \bar{g}^{\alpha\beta}\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma} - \bar{g}^{\mu\beta}\bar{g}^{\nu\rho}\bar{g}^{\alpha\sigma}\right. \\
&\quad \left. -\bar{g}^{\alpha\nu}\bar{g}^{\beta\rho}\bar{g}^{\mu\sigma}\right)\nabla_\rho\nabla_\sigma\Big] + \frac{\bar{R}}{4}\left(\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta} - \frac{1}{2}g^{\mu\nu}\bar{g}^{\alpha\beta}\right). \qquad (3.13)
\end{aligned}
$$

Here the covariant derivatives are taken with respect to the background metric $\bar{g}$ and $\Box = \bar{g}^{\rho\sigma}\nabla_\rho\nabla_\sigma$ is the d'Alembertian operator. The kinetic term in square brackets in (3.13) is just the curved spacetime version of the well-known Einstein operator, see e.g. [47] and the term proportional to $\bar{R}$ gives a contribution to the potential for $h_{\mu\nu}$ which vanishes in a flat background. This term looks like a mass term which does not satisfy the Fierz-Pauli tuning; however, this term is usually not harmful. $G^{\mu\nu}$ in Eq. (3.12) is the Einstein tensor which solves the background equations of motion, and the total derivative $\nabla_\mu V^\mu$ is irrelevant for the equations of motion.

For $T^{\mu\nu} \neq 0$ there also comes a contribution to the mass term from the variation of the matter Langrangian which is of the form

$$
\begin{aligned}
\mathcal{M}^{\mu\nu\alpha\beta}_{\text{mat}} &= \frac{1}{2}\frac{1}{\sqrt{-\det g}}\left.\frac{\partial^2(\sqrt{-\det g}L_m)}{\partial g_{\mu\nu}\partial g_{\alpha\beta}}\right|_{g=\bar{g}} \\
&= \frac{1}{2}\frac{1}{\sqrt{-\det g}}\left.\frac{\partial(\sqrt{-\det g}T^{\mu\nu}/2)}{\partial g_{\alpha\beta}}\right|_{g=\bar{g}}, \qquad (3.14)
\end{aligned}
$$

where L_m denotes the matter Lagrangian. In the following we do not consider this model-dependent term. The result which we obtain is however strictly only valid in vacuum. This does not render it uninteresting as we expect that like the massless Einstein equations, also the massive equations have vacuum solutions where $\bar{g}$ differs widely from f at least in certain regions of spacetime, like, e.g., the Schwarzschild solution. However, in a cosmological context, this matter-induced mass term does in principle also contribute.

We note in passing that the only difference of massive gravity theory to a bimetric theory of gravity is that our Lagrangian does not contain a kinetic term for the reference metric f. Massive gravity is therefore a theory with a "frozen-in" second metric f which is not a dynamical element of the theory, but an "absolute spacetime". This is somewhat artificial. Actually, the beauty of general relativity where spacetime is dynamically determined by the matter content of the Universe is lost. Cosmological solutions for bimetric theories of gravity which add the term $(M_P^2/2)\sqrt{-\det f}R(f)$ to the above Langrangian have also been studied [140, 141].

The Einstein operator is symmetric under the exchange $(\mu\nu) \leftrightarrow (\alpha\beta)$. We could also symmetrize it in $\mu\nu$ and in $\alpha\beta$ but since we apply it only on the symmetric tensor

$h_{\mu\nu}$ this does note make a difference. Furthermore, we omit the total derivative in Eq. (3.12) for simplicity.

We want to determine the second-order perturbation of the potential. Up to second order in $h_{\mu\nu}$ the potential is of the form

$$\sqrt{-\det g}\,U(f,g) = \sqrt{-\det \bar{g}}\left[U(f,\bar{g}) + \mathcal{M}^{\mu\nu}(f,\bar{g})h_{\mu\nu} + \mathcal{M}^{\mu\nu\alpha\beta}(f,\bar{g})h_{\mu\nu}h_{\alpha\beta}\right], \tag{3.15}$$

where

$$\mathcal{M}^{\mu\nu}(f,\bar{g}) \equiv \frac{1}{\sqrt{-\det g}}\frac{\partial(\sqrt{-\det g}U(f,g))}{\partial g_{\mu\nu}}\bigg|_{g=\bar{g}}, \tag{3.16}$$

$$\mathcal{M}^{\mu\nu\alpha\beta}(f,\bar{g}) \equiv \frac{1}{2}\frac{1}{\sqrt{-\det g}}\frac{\partial^2(\sqrt{-\det g}U(f,g))}{\partial g_{\mu\nu}\partial g_{\alpha\beta}}\bigg|_{g=\bar{g}}. \tag{3.17}$$

We consider perturbations around a solution $\bar{g}$ of the equations of motion. The terms linear in $h_{\mu\nu}$ in the Lagrangian therefore cancel due to the background equations of motion and we omit them in our discussion.

For noncommuting matrices $\sqrt{AB} \neq \sqrt{A}\sqrt{B}$, and we cannot simply expand $\sqrt{g^{-1}f} = \sqrt{(\mathbb{1}+h)^{-1}\bar{g}^{-1}f}$ in $h = (h^{\mu}{}_{\alpha}) = (\bar{g}^{\mu\nu}h_{\nu\alpha})$. Following [38], we therefore use the fact that the potential (3.10) can also be written in the form

$$U(f,g) = -2m^2\left[\beta_0 + \beta_1 e_1(\sqrt{g^{-1}f}) + \beta_2 e_2(\sqrt{g^{-1}f}) + \beta_3 e_3(\sqrt{g^{-1}f})\right], \tag{3.18}$$

with

$$\begin{aligned} \beta_0 &= 6 + 4\alpha_3 + \alpha_4\,, \qquad & \beta_1 &= -(3 + 3\alpha_3 + \alpha_4)\,, \\ \beta_2 &= 1 + 2\alpha_3 + \alpha_4\,, \qquad & \beta_3 &= -\alpha_3 - \alpha_4\,. \end{aligned} \tag{3.19}$$

Furthermore, as one can easily verify by bringing $\sqrt{g^{-1}f}$ into triangular form,

$$t_1 \equiv e_1(\sqrt{g^{-1}f}) = \sum_i \lambda_i^{1/2}, \tag{3.20a}$$

$$t_2 \equiv e_2(\sqrt{g^{-1}f}) = \sum_{i<k} \lambda_i^{1/2}\lambda_k^{1/2}, \tag{3.20b}$$

$$t_3 \equiv e_3(\sqrt{g^{-1}f}) = \sum_{i<k<l} \lambda_i^{1/2}\lambda_k^{1/2}\lambda_l^{1/2}, \tag{3.20c}$$

$$t_4 \equiv e_4(\sqrt{g^{-1}f}) = \sqrt{\lambda_1\lambda_2\lambda_3\lambda_4}, \tag{3.20d}$$

where λ_i are the eigenvalues of $g^{-1}f$, and $1 \leq i,k,l \leq 4$. Hence, we can write Eq. (3.18) as

$$U(f,g) \;=\; -2m^2\left[\beta_0 + \beta_1 t_1 + \beta_2 t_2 + \beta_3 t_3\right]. \tag{3.21}$$

We define

$$s_1 \equiv e_1(g^{-1}f) = \sum_i \lambda_i\,, \tag{3.22a}$$
$$s_2 \equiv e_2(g^{-1}f) = \sum_{i<j} \lambda_i\lambda_j\,, \tag{3.22b}$$
$$s_3 \equiv e_3(g^{-1}f) = \sum_{i<j<k} \lambda_i\lambda_j\lambda_k\,, \tag{3.22c}$$
$$s_4 \equiv e_4(g^{-1}f) = \lambda_1\lambda_2\lambda_3\lambda_4\,. \tag{3.22d}$$

We now use the following relations between the t_j and s_i $(1 \le j \le 3\,,\ \ 1 \le i \le 4)$:

$$t_1^2 = s_1 + 2t_2\,, \tag{3.23a}$$
$$t_2^2 = s_2 - 2\sqrt{s_4} + 2t_1t_3\,, \tag{3.23b}$$
$$t_3^2 = s_3 + 2t_2\sqrt{s_4}\,. \tag{3.23c}$$

With this we can write the perturbations of t_j in terms of perturbations of s_i which in turn can be obtained from $g^{-1}f = (\mathbb{1}+h)^{-1}\bar{g}^{-1}f$. We have to go to second order in the perturbations. The details of this lengthy calculation are given in Appendix 4.8, here we just present the result.

$$\sqrt{-\det g}\,U(f,g) = \sqrt{-\det \bar{g}}\left[U(f,\bar{g}) + \mathcal{M}^{\mu\nu\alpha\beta}(f,\bar{g})h_{\mu\nu}h_{\alpha\beta}\right] + \mathcal{O}(h^3)\,, \tag{3.24}$$
$$\text{with}\quad \mathcal{M}^{\mu\nu\alpha\beta} = -m^2\left[\beta_0\mathcal{M}_0^{\mu\nu\alpha\beta} + \beta_1\mathcal{M}_1^{\mu\nu\alpha\beta} + \beta_2\mathcal{M}_2^{\mu\nu\alpha\beta} + \beta_3\mathcal{M}_3^{\mu\nu\alpha\beta}\right]\,, \tag{3.25}$$
$$\mathcal{M}_0^{\mu\nu\alpha\beta} = \frac{1}{4}\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta} - \frac{1}{4}\left(\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta} + \bar{g}^{\mu\beta}\bar{g}^{\nu\alpha}\right), \tag{3.26}$$
$$\mathcal{M}_j^{\mu\nu\alpha\beta} = \bar{t}_j\mathcal{M}_0^{\mu\nu\alpha\beta} + \frac{1}{2}\left(\bar{g}^{\mu\nu}t_j^{\alpha\beta} + \bar{g}^{\alpha\beta}t_j^{\mu\nu}\right) + 2t_j^{\mu\nu\alpha\beta}\,, \quad 1 \le j \le 3\,, \tag{3.27}$$
$$t_j^{\mu\nu} = \left.\frac{\partial t_j}{\partial g_{\mu\nu}}\right|_{g=\bar{g}}\,, \qquad t_j^{\mu\nu\alpha\beta} = \left.\frac{1}{2}\frac{\partial^2 t_j}{\partial g_{\mu\nu}\partial g_{\alpha\beta}}\right|_{g=\bar{g}}\,. \tag{3.28}$$

Here $\mathcal{M}_0^{\mu\nu\alpha\beta}$ is the second-order perturbation of the determinant $\sqrt{-g}$ and the quantities $t_j^{\mu\nu}$ and $t_j^{\mu\nu\alpha\beta}$ are the first- and second-order derivatives of t_j with respect to the metric components $g_{\mu\nu}$. Their full expressions are very cumbersome, they are given in Appendix A.

Using the expressions given in the Appendix, as a first check one can verify that this new quadratic potential for $h_{\mu\nu}$ reduces to the Fierz-Pauli mass term if $\bar{g} = f$,

$$\mathcal{M}^{\mu\nu\alpha\beta}(\bar{g},\bar{g}) = -\frac{m^2}{4}\left[\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta} - \frac{1}{2}\left(\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta} + \bar{g}^{\mu\beta}\bar{g}^{\nu\alpha}\right)\right], \tag{3.29}$$

where we have explicitly symmetrized with respect to the exchanges $(\mu \leftrightarrow \nu)$, $(\alpha \leftrightarrow \beta)$.

Since the mass term given in Eq. (3.25) is so complicated, it is very unlikely that it is of the Fierz-Pauli form in general. Nevertheless, as explained in the introduction, this does not mean that the theory has a ghost, when $\bar{g} \neq f$.

3.3 Application to cosmology

In this section we apply our finding in a cosmological setting. To obtain a homogeneous and isotropic solution we first assume that both, $\bar{g}$ and f are of the Friedmann-Lemaître form with the same conformal time coordinate τ. To simplify the analysis we neglect curvature and set

$$\bar{g}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = a^2(\tau)(-\mathrm{d}\tau^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j), \tag{3.30}$$
$$f_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = b^2(\tau)(-\mathrm{d}\tau^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j)\,. \tag{3.31}$$

Since the two metrics are proportional to each other, the mass term can only be of the form

$$\mathcal{M}^{\mu\nu\alpha\beta}(f,\bar{g}) = -m^2\left[\alpha\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta} + \frac{\beta}{2}\left(\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta} + \bar{g}^{\mu\beta}\bar{g}^{\nu\alpha}\right)\right]. \tag{3.32}$$

In the cosmological situation α and β depend only on time, but the expressions below in terms of $r(t) = b(t)/a(t)$ are always correct when the two metrics $\bar{g}$ and f are conformally related by $f = r^2\bar{g}$.

Using the expressions in the Appendix and Eq. (3.25), one obtains

$$\alpha(\tau) = \frac{1}{4}\left[1 + (1-r)\left\{(5-r) + \alpha_3(4-2r) + \alpha_4(1-r)\right\}\right], \tag{3.33}$$
$$\beta(\tau) = -\frac{1}{4}\left[1 + (1-r)\left\{(11-4r) + \alpha_3\left(8-7r+r^2\right) + \alpha_4(1-r)(2-r)\right\}\right]. \tag{3.34}$$

Evidently, for $r(\tau) = 1$ or $a(\tau) = b(\tau)$ we recover the Fierz-Pauli mass term with $\alpha(\tau) = -\beta(\tau) = 1/4$, for arbitrary values of α_3 and α_4, but since r is time dependent, this value is not achieved in general. In Fig. 3.1 we show the behavior of α and β as functions of r for some special values for α_3 and α_4.

In [98], it has been shown that on a fixed background the mass term (3.32) for $\alpha \neq -\beta$ indicates the presence of a ghost with mass

$$m^2_{\rm ghost} = \frac{(\alpha+4\beta)}{2(\alpha+\beta)}m^2\,. \tag{3.35}$$

In our situation with $f \neq \bar{g}$ this is no longer true and the presence or absence of a ghost has to be investigated by other means. See e.g. Ref. [86].

Let us contrast this result with the alternative possibility that f and g have the same *physical* time, which of course is not equivalent,

$$\bar{g}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{3.36}$$
$$f_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + b^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\,. \tag{3.37}$$

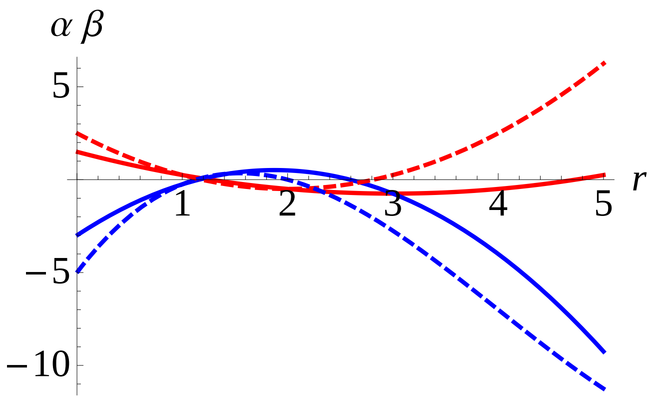


Figure 3.1: The functions $\alpha(r)$ (red) and $\beta(r)$ (blue) are shown for two cases: $\alpha_3 = \alpha_4 = 0$ (solid lines) and $\alpha_3 = 1\,,\ \alpha_4 = 0$ (dashed lines).

In this case the two metrics f and $\bar{g}$ are no longer proportional and the mass term takes the more complicated form

$$\mathcal{M}^{0000} = -m^2\gamma(t), \tag{3.38}$$
$$\mathcal{M}^{ij00} = -m^2\delta(t)\bar{g}^{ij}, \tag{3.39}$$
$$\mathcal{M}^{i0j0} = -m^2\epsilon(t)\bar{g}^{ij}, \tag{3.40}$$
$$\mathcal{M}^{ijkl} = -m^2\left\{\rho(t)\bar{g}^{ij}\bar{g}^{kl} + \frac{\sigma(t)}{2}\left[\bar{g}^{ik}\bar{g}^{jl} + \bar{g}^{il}\bar{g}^{jk}\right]\right\}. \tag{3.41}$$

Setting $r(t) = b(t)/a(t)$ we obtain

$$\gamma(t) = \frac{1}{4}\left[(1-r)\left\{(-6+3r) + \alpha_3\left(-4+5r-r^2\right) + \alpha_4\left(-1+2r-r^2\right)\right\}\right], \tag{3.42}$$
$$\delta(t) = -\frac{1}{4}\left[1 + (1-r)\left\{(5-r) + \alpha_3\,(4-2r) + \alpha_4\,(1-r)\right\}\right], \tag{3.43}$$
$$\epsilon(t) = \frac{1}{4}\frac{1}{(1+r)}\left[1 + (1-r)\left\{(5+2r-r^2) + \alpha_3\left(4-r-r^2\right) + \alpha_4\,(1-r)\right\}\right], \tag{3.44}$$
$$\rho(t) = \frac{1}{4}\left[1 + (1-r)\left\{2+\alpha_3\right\}\right], \tag{3.45}$$
$$\sigma(t) = -\frac{1}{4}\left[1 + (1-r)\left\{(5-r) + \alpha_3\,(2-r)\right\}\right]. \tag{3.46}$$

All other components of $\mathcal{M}^{\mu\nu\alpha\beta}$ are determined by its symmetry under exchange $\mu\nu \leftrightarrow \alpha\beta$, $\mu \leftrightarrow \nu$ and $\alpha \leftrightarrow \beta$. Again, when $r(t) = 1$ or $a(t) = b(t)$, we reach the Fierz-Pauli tuning which corresponds to $\gamma = 0$, $\rho = -\delta = -\sigma = 1/4$, $\epsilon = 1/8$. Note that in terms of the ratio $r\ \delta(r) = -\alpha(r)$ so that when writing $\mathcal{M}^{ij00} = -m^2\phi(r)\bar{g}^{ij}\bar{g}^{00}$, we obtain the same expression for ϕ in both cases, equivalent physical time and equivalent conformal time. Interestingly, α_4 does not enter the expressions for ρ and σ. In Fig. 3.2 we show the behavior of δ, γ, ϵ, ρ, and σ as functions of r for the special case $\alpha_3 = 1$, $\alpha_4 = 0$.

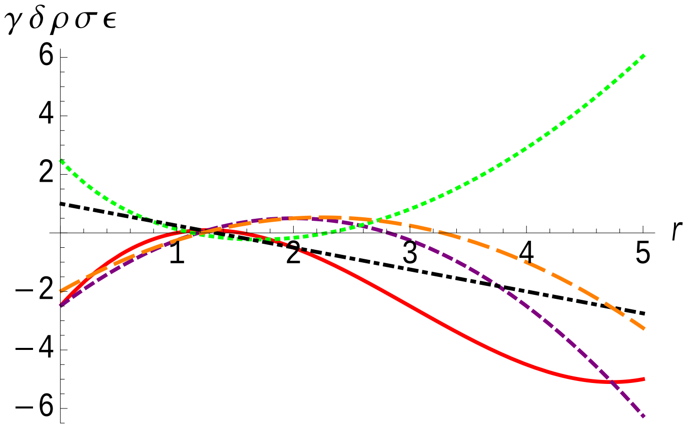


Figure 3.2: The functions $\gamma(r)$ (red, solid line), $\delta(r)$ (purple, dashed line), $\epsilon(r)$ (green, dotted line), $\rho(r)$ (black, dash-dotted line), and $\sigma(r)$ (orange, long-dashed line) are shown for the case $\alpha_3 = 1$, $\alpha_4 = 0$.

When $a(t) \neq b(t)$, the perturbations of these solutions again violate the Fierz-Pauli tuning.

For a cosmological situation where the time directions of f and $\bar{g}$ are boosted with respect to each other, the mass term is more complicated. However, this case would not allow for a homogeneous and isotropic solution and is therefore not relevant. The most general cosmological situation is $d\tau_f = r(\tau_g)d\tau_g$, where τ_f and τ_g denote the conformal times for the cosmological metrics f and $\bar{g}$.

3.3.1 Evolution of cosmological perturbations

From Eq. (3.11) we can derive the background equation of motion,

$$\bar{G}_{\mu\nu} + \bar{\mathcal{M}}_{\mu\nu} = M_P^{-2}\bar{T}_{\mu\nu}, \tag{3.47}$$

where $\bar{G}_{\mu\nu}$ is the Einstein tensor for $\bar{g}_{\mu\nu}$, $\bar{T}_{\mu\nu} \equiv T_{\mu\nu}(\bar{g})$ and $\bar{\mathcal{M}}_{\mu\nu}$ is the contribution from the mass term, which is calculated in Appendix 4.9. For the cosmological form of the metrics (3.36) and (3.37) and the energy momentum tensor

$$\bar{T}_{\mu\nu} = \begin{pmatrix} \bar{\rho} & 0 \\ 0 & a^2\bar{p}\delta_{ij} \end{pmatrix}, \tag{3.48}$$

where $\bar{\rho}$ and $\bar{p}$ are the background energy density and pressure, respectively, we obtain the Friedmann equations

$$\begin{aligned} &3H^2 + m^2\big[6 - 9r + 3r^2 + c_3(4 - 9r + 6r^2 - r^3) \\ &+ c_4(1 - 3r + 3r^2 - r^3)\big] = M_P^{-2}\bar{\rho} \end{aligned} \tag{3.49}$$

and

$$2\dot{H} + 3H^2 + m^2\big[3 - 4r + r^2 + c_3(1 - 2r + r^2)\big] = -M_P^{-2}\bar{p}, \tag{3.50}$$

where $H \equiv \dot{a}/a$ (the dot denotes the derivative with respect to physical time t).

We are interested in the question of whether perturbations of a cosmological solution have an instability due to the mass term, a ghost, in addition to the usual instability to gravitational clustering (Jeans instability). As is well known, the ghost always shows up in the scalar sector. Therefore, here we only analyze the scalar perturbation equations.

The most general scalar perturbations of the metric (in Fourier space) are of the form

$$h_{\mu\nu} \equiv \delta g_{\mu\nu} = \begin{pmatrix} -2\phi & iak_jB \\ iak_iB & 2a^2(\psi\delta_{ij} - k_ik_jE) \end{pmatrix}. \tag{3.51}$$

The perturbation equations resulting from this ansatz are Eqs. (3.79), (3.80), (3.81), and (3.82), given in Appendix 3.7. These equations are rather cumbersome. Here we simply analyze the presence of a ghost due to the mass term. For this, we simplify to the static solution $H \equiv 0$ and matter domination $\bar{p} = 0$. Inserting this in Eq. (3.50), we find two possible solutions for r,

$$r = \begin{cases} 1 \\ \frac{3+\alpha_3}{1+\alpha_3} = r_c \end{cases}. \tag{3.52}$$

The first is simply Minkowski space with the Fierz-Pauli tuning. For this case, a brief analysis of the perturbation equations shows that there is no ghost but just one massive degree of freedom, namely ψ, as expected, the helicity 0 mode of the massive graviton. For $r = r_c$, however, we obtain a static solution due to the presence of the mass term, which exists for $\alpha_3 \neq -1$. The positivity of the energy density $\bar{\rho}$ together with Eq. (3.49) then requires

$$P_1(\alpha_3, \alpha_4) = 3 + 2\alpha_3 + 3\alpha_3^2 - 4\alpha_4 > 0\,.$$

We can eliminate ϕ and B using the constraint Eqs. (3.79) and (3.80). We now consider the static case $r = r_c$ with vanishing matter perturbations $\delta\rho = \delta p = v - B = 0$ since we want to study the evolution of the free gravitational field. Inserting $H = 0$ and $r = r_c$ we obtain a system of the form

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\begin{pmatrix} \psi \\ \mathcal{E} \end{pmatrix} = \left(m^2A_0 + k^2A_2\right)\begin{pmatrix} \psi \\ \mathcal{E} \end{pmatrix}, \tag{3.53}$$

where $\mathcal{E} = m^2 E$. The matrices A_0 and A_2 are given by

$$A_0 = \begin{pmatrix} \frac{21+10\alpha_3+9\alpha_3^2-12\alpha_4}{4(1+\alpha_3)} & 0 \\ Q(\alpha_3) - \frac{4(1+\alpha_3)(2+\alpha_3)}{r_c P_1(\alpha_3,\alpha_4)} & r_c \end{pmatrix}, \tag{3.54}$$

$$A_2 = \begin{pmatrix} -\frac{1+\alpha_3}{2} & -\frac{P_1(\alpha_3,\alpha_4)}{4(1+c3)} \\ \frac{(1+\alpha_3)(-5+\alpha_3^2)}{r_c^2 P_1(\alpha_3,\alpha_4)} & \frac{-5+\alpha_3^2}{2r_c(3+\alpha_3)} \end{pmatrix},$$

where

$$Q(\alpha_3) = \frac{33+27\alpha_3-\alpha_3^2-3\alpha_3^3}{2(3+\alpha_3)^2}.$$

The eigenvalues of A_0 are

$$\lambda_{01} = \frac{3+\alpha_3}{1+\alpha_3} = r_c \tag{3.55}$$

$$\lambda_{02} = \frac{21+10\alpha_3+9\alpha_3^2-12\alpha_4}{4(1+\alpha_3)}, \tag{3.56}$$

with eigenvectors

$$v_{01} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.57}$$

$$v_{02} = \begin{pmatrix} \frac{3P_1(\alpha_3,\alpha_4)}{4(1+\alpha_3)} \\ (A_0)_{21} \end{pmatrix}. \tag{3.58}$$

The fact that $\lambda_{01} > 0$, indicates an exponential instability for small k.

The eigenvalues of A_2 are

$$\lambda_{21} = 0 \tag{3.59}$$

$$\lambda_{22} = -\frac{7+3\alpha_3}{r_c^2(1+\alpha_3)}, \tag{3.60}$$

with eigenvectors

$$v_{21} = \begin{pmatrix} -\frac{P_1(\alpha_3,\alpha_4)}{2(1+\alpha_3)^2} \\ 1 \end{pmatrix}, \tag{3.61}$$

$$v_{22} = \begin{pmatrix} -\frac{r_c^2 P_1(\alpha_3,\alpha_4)}{2(-5+\alpha_3^2)} \\ 1 \end{pmatrix}. \tag{3.62}$$

The nonvanishing eigenvalues are shown as functions of α_3 for $\alpha_4 = 0$ in Fig. 3.3. The situation for different values of α_4 is similar. Typically, one or both eigenvalues of A_0 are positive, which indicates an instability.

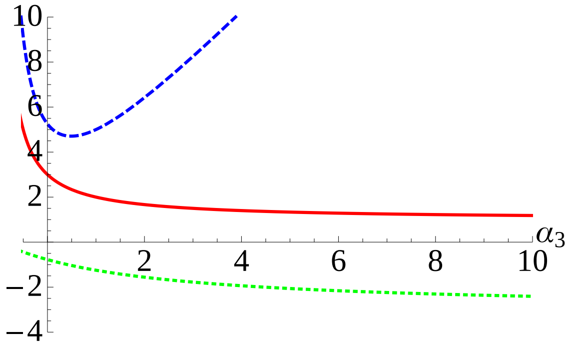


Figure 3.3: The eigenvectors λ_{01} (red, solid), λ_{02} (blue, dashed) and λ_{22} (green, dotted) are shown as functions of $-0.5 < \alpha_3 < 10$ for the case $\alpha_4 = 0$.

The eigenvalue λ_{22} is negative for $\alpha_3 > -1$ so that high momentum modes are stable. The value $\lambda_{21} = 0$ reflects the fact that in dRGT massive gravity, the second scalar mode does not really propagate [47, 49], but it also does not decouple as it does in the Fierz-Pauli tuning. This comes from the choice of the potential $U(f,g)$. Nevertheless, as we have seen in this analysis, the mass term still leads to exponential instabilities as the eigenmodes of Eq. (3.53) behave as $\exp(\pm\sqrt{\lambda_{0i}}mt)$ for small momenta.

At this point, it is not clear how the expansion of the Universe can mitigate this instability. When the eigenvalue for the momentum, λ_{22}, is negative, there is still the chance that damping terms reduce the instability to a power law as long as $m^2 \lesssim H^2$. Hence it may be that the instability found here is not a disaster for the phenomenology of the observable, expanding Universe.

3.4 Conclusions

In this paper we have determined the form of the mass matrix $\mathcal{M}^{\mu\nu\alpha\beta}(f,\bar{g})$ for fluctuations about some background solution $\bar{g}$. We have shown that for $\bar{g} = f$ we obtain the Fierz-Pauli mass term, whereas for $\bar{g} \neq f$ a more general mass term is found. In the simple case $f = r^2\bar{g}$ the mass term is of the form

$$\mathcal{M}^{\mu\nu\alpha\beta}(f,\bar{g}) = -m^2\left[\alpha\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta} + \frac{\beta}{2}(\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta} + \bar{g}^{\mu\beta}\bar{g}^{\nu\alpha})\right]. \tag{3.63}$$

We have calculated the functions α and β in terms of r and found that one recovers the Fierz-Pauli mass term only for $r = 1$. Even if r is a constant, $r = c \neq 1$, the mass term is different.

We have also calculated the mass term in the cosmological setting when f and $\bar{g}$ have the same physical time but different conformal time. Also, in this case, when

$\bar{g} \neq f$, the mass term differs from the Fierz-Pauli one.

We have briefly analyzed the consequence of this mass term in the case of "static cosmology" and have shown that even in this case, the mass term generically leads to instabilities.

The main point of the present paper is the full calculation of the mass term for perturbations around an arbitrary background which can be used to study linear perturbation theory around arbitrary backgrounds and for an arbitrary reference metric.

Acknowledgements. We thank Claudia de Rham for important clarifications, and Lavinia Heisenberg, Michele Maggiore and Mariele Motta for interesting discussions. This work is supported by the Swiss National Science Foundation.

3.5 Appendix: The computation of the perturbed potential

Here we present more details about the computation of $\mathcal{M}^{\mu\nu\alpha\beta}(f,\bar{g})$, and we give the detailed results. With the help of Eq. (3.23) we can express the first- and second-order perturbations of t_j in terms of those of s_i. Like for t_j we set

$$s_i^{\mu\nu} = \left.\frac{\partial s_i}{\partial g_{\mu\nu}}\right|_{g=\bar{g}} , \tag{3.64}$$

$$s_i^{\mu\nu\alpha\beta} = \frac{1}{2}\left.\frac{\partial^2 s_i}{\partial g_{\mu\nu}\partial g_{\alpha\beta}}\right|_{g=\bar{g}} . \tag{3.65}$$

The first derivatives of the t_j can be written as

$$\begin{aligned}
t_1^{\mu\nu} &= A\left\{\bar{t}_4\left(\bar{t}_1\bar{t}_4 - \bar{t}_2\bar{t}_3\right)s_1^{\mu\nu} - \bar{t}_3\bar{t}_4 s_2^{\mu\nu} - \bar{t}_1\bar{t}_4 s_3^{\mu\nu} + \left(\bar{t}_3 - \bar{t}_1\bar{t}_2\right)s_4^{\mu\nu}\right\} ,\\
t_2^{\mu\nu} &= A\left\{-\bar{t}_3^2\bar{t}_4 s_1^{\mu\nu} - \bar{t}_3\bar{t}_4\bar{t}_1 s_2^{\mu\nu} - \bar{t}_4\bar{t}_1^2 s_3^{\mu\nu} + \left(\bar{t}_3 - \bar{t}_1\bar{t}_2\right)\bar{t}_1 s_4^{\mu\nu}\right\} ,\\
t_3^{\mu\nu} &= A\left\{-\bar{t}_3\bar{t}_4^2 s_1^{\mu\nu} - \bar{t}_1\bar{t}_4^2 s_2^{\mu\nu} + \left(\bar{t}_3 - \bar{t}_1\bar{t}_2\right)\bar{t}_4 s_3^{\mu\nu} + \left(\bar{t}_2\bar{t}_3 + \bar{t}_1\left(\bar{t}_4 - \bar{t}_2^2\right)\right)s_4^{\mu\nu}\right\} ,\\
t_4^{\mu\nu} &= \frac{s_{4,\bullet}^{\mu\nu}}{2\bar{t}_4} ,
\end{aligned} \tag{3.66}$$

where

$$A = \left[2\,\bar{t}_4\left(\bar{t}_3^2 + \bar{t}_1^2\bar{t}_4 - \bar{t}_1\bar{t}_2\bar{t}_3\right)\right]^{-1} . \tag{3.67}$$

To obtain the second derivatives we have to derive Eq. (3.23) a second time. A rather cumbersome but straightforward calculation leads finally to

$$t_1^{\mu\nu\alpha\beta} = A\left\{\bar{t}_4\left(\bar{t}_1\bar{t}_4 - \bar{t}_2\bar{t}_3\right)S_1^{\mu\nu\alpha\beta} - \bar{t}_3\bar{t}_4 S_2^{\mu\nu\alpha\beta} - \bar{t}_1\bar{t}_4 S_3^{\mu\nu\alpha\beta} + \left(\bar{t}_3 - \bar{t}_1\bar{t}_2\right)S_4^{\mu\nu\alpha\beta}\right\}, \tag{3.68a}$$

$$t_2^{\mu\nu\alpha\beta} = A\left\{-\bar{t}_3^2\bar{t}_4 S_1^{\mu\nu\alpha\beta} - \bar{t}_1\bar{t}_3\bar{t}_4 S_2^{\mu\nu\alpha\beta} - \bar{t}_1^2\bar{t}_4 S_3^{\mu\nu\alpha\beta} + \bar{t}_1\left(\bar{t}_3 - \bar{t}_1\bar{t}_2\right)S_4^{\mu\nu\alpha\beta}\right\}, \tag{3.68b}$$

$$t_3^{\mu\nu\alpha\beta} = A\left\{-\bar{t}_3\bar{t}_4^2 S_1^{\mu\nu\alpha\beta} - \bar{t}_1\bar{t}_4^2 S_2^{\mu\nu\alpha\beta} + \left(\bar{t}_3 - \bar{t}_1\bar{t}_2\right)\bar{t}_4 S_3^{\mu\nu\alpha\beta} + \left[\bar{t}_2\bar{t}_3 + \bar{t}_1\left(\bar{t}_4 - \bar{t}_2^2\right)\right]S_4^{\mu\nu\alpha\beta}\right\}, \tag{3.68c}$$

where we have introduced

$$S_q^{\mu\nu\alpha\beta} \equiv s_q^{\mu\nu\alpha\beta} - \sum_{p\in\mathcal{I}}(-1)^{q+p}t_p^{\mu\nu}t_{2q-p}^{\alpha\beta}\,. \tag{3.69}$$

The above sum goes over $\mathcal{I} \subset \{1,2,3\}$ such that both, p and $2q-p$ are in $\{1,2,3\}$. With this we have expressed the derivatives of the quantities t_j in terms of those of the s_i, but the latter can be obtained directly by expanding the matrix

$$g^{-1}f = \left(\bar{g}(1+h)\right)^{-1}f \approx (1 - h + h^2)\bar{g}^{-1}f\,.$$

Here h denotes $(h^\mu{}_\nu) = (\bar{g}^{\mu\alpha}h_{\alpha\nu})$. We apply the formula (3.5) to (3.8) for $U_j(g^{-1}f)$. These are given in terms of $g^{-1} = (g^{\mu\nu})$.

To shorten the notation in what follows, we define $\bar{\Sigma}^\mu{}_\nu \equiv \bar{g}^{\mu\rho}\bar{f}_{\rho\nu}$, and also $\bar{\Sigma}^{\mu\nu} \equiv \bar{\Sigma}^\mu{}_\rho\bar{g}^{\rho\nu}$, $\left(\bar{\Sigma}^2\right)^{\mu\nu} \equiv \bar{\Sigma}^\mu{}_{\rho_1}\bar{\Sigma}^{\rho_1}{}_{\rho_2}\bar{g}^{\rho_2\nu}$, and generalizing $\left(\bar{\Sigma}^k\right)^{\mu\nu} \equiv \bar{\Sigma}^\mu{}_{\rho_1}\bar{\Sigma}^{\rho_1}{}_{\rho_2}\ldots\bar{\Sigma}^{\rho_{k-1}}{}_{\rho_k}\bar{g}^{\rho_k\nu}$. As usual, square brackets denote the trace, e.g. $[\bar{\Sigma}] \equiv \mathrm{Tr}\bar{\Sigma} = \bar{\Sigma}^\mu{}_\mu$.

A direct evaluation of s_i and their first and second derivatives leads to

$$
\begin{aligned}
\bar{s}_1 &= [\bar{\Sigma}]\,, && (3.70a)\\
s_1^{\mu\nu} &= -\bar{\Sigma}^{\mu\nu}\,, && (3.70b)\\
s_1^{\mu\nu\alpha\beta} &= \frac{1}{8}\left\{\left[\bar{g}^{\alpha\nu}\bar{\Sigma}^{\mu\beta} + (\mu\leftrightarrow\nu) + (\alpha\leftrightarrow\beta) + (\mu\leftrightarrow\nu)(\alpha\leftrightarrow\beta)\right] + [\cdots]\left((\mu,\nu)\leftrightarrow(\alpha,\beta)\right)\right\}\\
&\equiv \mathrm{Sym}\left\{\bar{g}^{\alpha\nu}\bar{\Sigma}^{\mu\beta}\right\}\,, && (3.70c)\\
\bar{s}_2 &= \frac{1}{2}\left([\bar{\Sigma}]^2 - [\bar{\Sigma}^2]\right)\,, && (3.70d)\\
s_2^{\mu\nu} &= \left(\bar{\Sigma}^2\right)^{\mu\nu} - [\bar{\Sigma}]\,\bar{\Sigma}^{\mu\nu}\,, && (3.70e)\\
s_2^{\mu\nu\alpha\beta} &= \mathrm{Sym}\left\{\bar{g}^{\mu\alpha}\left([\bar{\Sigma}]\,\bar{\Sigma}^{\beta\nu} - \left(\bar{\Sigma}^2\right)^{\beta\nu}\right) + \frac{1}{2}\bar{\Sigma}^{\mu\nu}\bar{\Sigma}^{\alpha\beta} - \frac{1}{2}\bar{\Sigma}^{\alpha\mu}\bar{\Sigma}^{\beta\nu}\right\}\,, && (3.70f)\\
\bar{s}_3 &= \frac{1}{6}\left([\bar{\Sigma}]^3 - 3[\bar{\Sigma}][\bar{\Sigma}^2] + 2[\bar{\Sigma}^3]\right)\,, && (3.70g)\\
s_3^{\mu\nu} &= [\bar{\Sigma}]\left(\bar{\Sigma}^2\right)^{\mu\nu} - \left(\bar{\Sigma}^3\right)^{\mu\nu} + \frac{1}{2}\bar{\Sigma}^{\mu\nu}\left([\bar{\Sigma}^2] - [\bar{\Sigma}]^2\right)\,, && (3.70h)\\
s_3^{\mu\nu\alpha\beta} &= \mathrm{Sym}\Bigg\{\bar{\Sigma}^{\mu\alpha}\left(\bar{\Sigma}^2\right)^{\beta\nu} - \left(\bar{\Sigma}^2\right)^{\mu\nu}\bar{\Sigma}^{\alpha\beta} + \frac{1}{2}[\bar{\Sigma}]\left(\bar{\Sigma}^{\mu\nu}\bar{\Sigma}^{\alpha\beta} - \bar{\Sigma}^{\mu\alpha}\bar{\Sigma}^{\nu\beta}\right)\\
&\qquad + \bar{g}^{\nu\alpha}\left(\left(\bar{\Sigma}^3\right)^{\mu\beta} - \left(\bar{\Sigma}^2\right)^{\mu\beta}[\bar{\Sigma}]\right) + \frac{1}{2}\bar{g}^{\mu\alpha}\bar{\Sigma}^{\nu\beta}\left([\bar{\Sigma}]^2 - [\bar{\Sigma}^2]\right)\Bigg\}\,, && (3.70i)\\
\bar{s}_4 &= \det\left(\bar{g}^{-1}\bar{f}\right)\,, && (3.70j)\\
s_4^{\mu\nu} &= -\bar{s}_4\bar{g}^{\mu\nu}\,, && (3.70k)\\
s_4^{\mu\nu\alpha\beta} &= \frac{\bar{s}_4}{2}\left(\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta} + \frac{1}{2}\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta} + \frac{1}{2}\bar{g}^{\nu\alpha}\bar{g}^{\mu\beta}\right)\,. && (3.70l)
\end{aligned}
$$

Here the operator Sym$\{\cdots\}$ indicates symmetrization in $(\mu\leftrightarrow\nu)$, $(\alpha\leftrightarrow\beta)$, $(\mu\leftrightarrow\nu)(\alpha\leftrightarrow\beta)$ and $(\mu,\nu)\leftrightarrow(\alpha,\beta)$ and we have used the fact that for an arbitrary function $F(g^{-1})$ we have

$$
\frac{\partial F}{\partial g_{\mu\nu}} = -g^{\mu\alpha}g^{\nu\beta}\frac{\partial F}{\partial g^{\alpha\beta}}\,. \tag{3.71}
$$

These are the expressions for the derivatives of the s_i which have to be inserted in the formulas for the variations of the t_j which in turn enter in the expression for $\mathcal{M}^{\mu\nu\alpha\beta}$. Not surprisingly, the expressions for the variations of s_2 and s_3 are quite cumbersome.

3.6 Appendix: The computation of $\bar{\mathcal{M}}^{\mu\nu}$

In this Appendix we present more details about the computation of the mass term $\bar{\mathcal{M}}^{\mu\nu} \equiv \mathcal{M}^{\mu\nu}(f,\bar{g})$ defined in Eq. (3.16). and used in Eq. (3.47). We have

$$\bar{\mathcal{M}}^{\mu\nu} \equiv \left\{ \frac{1}{\sqrt{-\det g}} \frac{\delta(\sqrt{-\det g}U(g,f))}{\delta g_{\mu\nu}} \right\}\Bigg|_{g=\bar{g}} = \frac{1}{2}U(f,\bar{g})\bar{g}^{\mu\nu} - 2m^2(a_1 t_1^{\mu\nu} + a_2 t_2^{\mu\nu} + a_3 t_3^{\mu\nu}), \tag{3.72}$$

where we have used $\frac{\delta\sqrt{-\det g}}{\delta g_{\mu\nu}} = \frac{1}{2}\sqrt{-\det g}g^{\mu\nu}$ and, from Eq. (3.21),

$$\frac{\delta U(f,g)}{\delta g_{\mu\nu}}\Bigg|_{g=\bar{g}} = -2m^2 \left(\frac{\delta(a_1 t_1 + a_2 t_2 + a_3 t_3)}{\delta g_{\mu\nu}} \right)\Bigg|_{g=\bar{g}} = -2m^2(a_1 t_1^{\mu\nu} + a_2 t_2^{\mu\nu} + a_3 t_3^{\mu\nu}). \tag{3.73}$$

The quantities $t_j^{\mu\nu}$ can be written in terms of $s_j^{\mu\nu}$, see Eqs. (4.94) which are given in Eqs. (3.70).

3.7 Appendix: The equations of motion for the cosmological perturbations

Here we present the derivation of the equations of motion for the perturbations at first order based on the second-order perturbed part of the action (3.11). In complete generality, these equations of motion have the form

$$\delta G^{\mu}{}_{\nu} + \delta\mathcal{M}^{\mu}{}_{\nu} = 8\pi G \delta T^{\mu}{}_{\nu}, \tag{3.74}$$

where $\delta G^{\mu}{}_{\nu}$, $\delta\mathcal{M}^{\mu}{}_{\nu}$ and $\delta T^{\mu}{}_{\nu}$ stand for the first-order perturbation of the usual Einstein tensor $G^{\mu}{}_{\nu}$, the first-order perturbation of the mass term and the first-order perturbation of the energy-momentum tensor, respectively. The perturbations $\delta G^{\mu}{}_{\nu}$ and $\delta T^{\mu}{}_{\nu}$ can be found in the literature (see e.g. [122, 103, 59]). For the mass term we have

$$\delta\mathcal{M}^{\mu}{}_{\nu} = \delta(\mathcal{M}^{\mu\rho}g_{\rho\nu}) = \delta\mathcal{M}^{\mu\rho}\bar{g}_{\rho\nu} + \bar{\mathcal{M}}^{\mu\rho}\delta g_{\rho\nu}, \tag{3.75}$$

where

$$\delta\mathcal{M}^{\mu\rho} = \frac{\delta}{\delta h_{\mu\rho}}(\mathcal{M}^{\tau\sigma\alpha\beta}h_{\tau\sigma}h_{\alpha\beta}) = 2\mathcal{M}^{\mu\rho\alpha\beta}h_{\alpha\beta}. \tag{3.76}$$

$\bar{\mathcal{M}}^{\mu\rho}$ has already been calculated in Appendix 4.9. We choose the background metric $\bar{g}_{\mu\nu}$ given by Eq. (3.36), while the metric $f_{\mu\nu}$ is given by Eq. (3.37) so that we can use Eq. (3.38) for the components of the mass tensor $\mathcal{M}^{\mu\nu\alpha\beta}$. We are interested in scalar perturbations of the metric $g_{\mu\nu}$ which we decompose into Fourier components that evolve independently. Note that we cannot fix a particular gauge since the mass term in the action is not gauge invariant a priori (see, however, the discussion about the "hidden symmetry" for perturbations on Minkowski or de Sitter spacetime in Ref. [98]). Gauge invariance can be restored by means of the Stückelberg trick [131, 45], but we are not doing this here. The metric perturbation of a Fourier component is

$$h_{\mu\nu}(t,k) \equiv \delta g_{\mu\nu} = \begin{pmatrix} -2\phi & iak_jB \\ iak_iB & 2a^2(\psi\delta_{ij} - k_ik_jE) \end{pmatrix}. \tag{3.77}$$

The energy-momentum tensor up to first order in scalar perturbations is given by

$$T^\mu{}_\nu = \begin{pmatrix} -\bar{\rho} - \delta\rho & -a(\bar{\rho}+\bar{p})(ik_jv - ik_jB) \\ a^{-1}(\bar{\rho}+\bar{p})ik^iv & (\bar{p}+\delta p)\delta^i_j \end{pmatrix}. \tag{3.78}$$

The first-order perturbation equation, $\delta G^0{}_0 + \delta\mathcal{M}^0{}_0 = M_P^{-2}\delta T^0{}_0$, then becomes

$$\begin{aligned}
&\left\{\frac{2k^2}{a^2} + 3m^2\Big(2\alpha_3(r-2)(r-1) + \alpha_4(r-1)^2 + r^2 - 6r + 6\Big)\right\}\psi + \frac{2Hk^2}{a}B \\
&- m^2\Big\{2\alpha_3(r-2)(r-1) + \alpha_4(r-1)^2 + r^2 - 6r + 6\Big\}k^2\mathrm{E} \\
&- \left\{6H^2 + m^2(r-1)\Big(\alpha_4(r-1)^2 + \alpha_3(r-4)(r-1) - 3r + 6\Big)\right\}\phi - 2Hk^2\dot{\mathrm{E}} + 6H\dot{\psi} = M_P^{-2}\delta\rho.
\end{aligned} \tag{3.79}$$

Equation $\delta G^0{}_i + \delta\mathcal{M}^0{}_i = M_P^{-2}\delta T^0{}_i$ is

$$m^2\frac{(r-1)r^2\big(\alpha_3(r-3) + \alpha_4(r-1)\big) + (3-2r)r^2}{r+1}B + \frac{2H}{a}\phi - \frac{2}{a}\dot{\psi} = M_P^{-2}\left(\bar{p}+\bar{\rho}\right)(v-B). \tag{3.80}$$

Equation $\delta G^i{}_i + \delta\mathcal{M}^i{}_i = M_P^{-2}\delta T^i{}_i$ reads

$$\begin{aligned}
&m^2\Big\{\alpha_3(r-3)(r-1) + r^2 - 8r + 9\Big\}k^2\mathrm{E} \\
&+ \left\{-3m^2\Big(2\alpha_3(r-2)(r-1) + \alpha_4(r-1)^2 + r^2 - 6r + 6\Big) + 12\dot{H} + 18H^2 - 2\frac{k^2}{a^2}\right\}\phi \\
&- \left\{3m^2\Big(\alpha_3(r-3)(r-1) + r^2 - 8r + 9\Big) + 2\frac{k^2}{a^2}\right\}\psi \\
&+ 2k^2\ddot{\mathrm{E}} - 18H\dot{\psi} + 6H\dot{\phi} - 6\ddot{\psi} + 6Hk^2\dot{\mathrm{E}} - \frac{2k^2}{a}\dot{B} - \frac{4Hk^2}{a}B = 3M_P^{-2}\delta\mathrm{p}.
\end{aligned} \tag{3.81}$$

Finally, the longitudinal, traceless part of the (ij) component of the equation of motion,

$$\left(\hat{k}_i\hat{k}^j - \frac{1}{3}\delta^j_i\right)\left(\delta G^i{}_j + \delta\mathcal{M}^i{}_j\right) = M_P^{-2}\left(\hat{k}_i\hat{k}^j - \frac{1}{3}\delta^j_i\right)\delta T^i{}_j \tag{3.82}$$

(where $\hat{k}_i$ is the unit wave vector), reads

$$m^2\left\{\alpha_3(r-1)r+r^2-2r\right\}\mathrm{E}+\frac{\dot{B}}{a}+\frac{\psi}{a^2}-\ddot{\mathrm{E}}-3H\dot{\mathrm{E}}+\frac{\phi}{a^2}+\frac{2H}{a}B=0. \tag{3.83}$$

For the static situation, $H=\dot{H}=0$ and vanishing matter perturbations, this system reduces to (3.53).

Chapter 4

A general mass term for bigravity

Based on:

[36] G. Cusin, R. Durrer, P. Guarato and M. Motta, "A general mass term for bigravity", JCAP **1604** (2016) 051 [arXiv:1512.02131 [astro-ph]].

Abstract. We introduce a new formalism to study perturbations of Hassan-Rosen bigravity theory, around general backgrounds for the two dynamical metrics. In particular, we derive the general expression for the mass term of the perturbations and we explicitly compute it for cosmological settings. We study tensor perturbations in a specific branch of bigravity using this formalism. We show that the tensor sector is affected by a late-time instability, which sets in when the mass matrix is no longer positive definite.

4.1 Introduction

The discovery of the accelerated expansion of the Universe [129, 127, 9] has reactivated attempts to modify gravity, i.e., General Relativity. Maybe the observed acceleration is not due to a highly fine tuned cosmological constant or some other form of dark energy, but to the fact that gravity becomes weaker on very large scales, see [61] for a review of the dark energy/modified gravity problem. This 'degravitation' [64, 53] can be achieved by different means, one of them is to give the graviton a mass of the order of the present Hubble parameter, H_0, so that gravity has finite range.

The idea of massive gravity goes back to Fierz and Pauli in 1939 [71, 126] who found the unique quadratic expression in the deviation of the metric from a flat background which is free of ghosts. Later, in the 70s, Boulware and Deser have shown that higher order terms in the 'metric potential' generally re-introduce a ghost [26]. Only a few years ago, de Rham, Gabadadze and Tolley have successfully derived the first metric potential which is ghost-free at least up to fourth order in perturbation theory around flat space and in the decoupling limit [49]. This potential has been proved to be the most general metric potential in [82] and the proof of the absence of the Boulware-Deser ghost in the full non-perturbative theory away from the decoupling limit has been worked out in [85], see also [86, 84] for the most general case.

A general feature of massive gravity is that the potential has to be formulated with respect to some reference metric, $f_{\mu\nu}$ as a function of $g^{\mu\alpha}f_{\alpha\nu}$. This reference metric is an absolute element of the theory, spoiling diffeomorphism invariance, one of the most attractive features of General Relativity. As a consequence, one can show that a flat reference metric, $f_{\mu\nu} = \eta_{\mu\nu}$, does not allow for cosmological solutions (apart for the Milne Universe which also represents flat space) [45]. But there is no obstruction to render also the reference metric dynamical by introducing an Einstein-Hilbert term in the action also for this metric. This leads in a natural way to a ghost-free bimetric theory of gravity [83]. Investigations of theoretical aspects of bimetric massive gravity can be found in [13, 88, 51, 37, 124, 11, 33, 46].

To avoid the re-appearance of the ghost from the coupling to matter, one has to request that matter fields couple to one of the two metrics but not to both. The equivalence principle then implies that all matter fields couple to the same metric. A violation of this could be ruled out experimentally. For simplicity, in this work we therefore assume that matter couples only to one metric which we call the physical metric, $g_{\mu\nu}$. This assumption, however, does not affect our main result, which does not involve the matter couplings. Relevant contribution to this bi-gravity sector are also given in [52, 81, 51, 90, 89, 95, 119]. For a review of recent results in bimetric theories see [130].

Cosmological solutions of bimetric theories can actually fit the expansion history of the accelerating Universe [139, 30, 106, 136, 70]. Perturbation theory and observational tests of several models of bigravity are presented in [132, 143, 21, 12, 107]. The cosmology of bigravity in various cosmological settings is studied in [10, 104] while in Refs. [78, 29, 67] the cosmology of models of bigravity where matter is coupled to a combination of the two metrics is investigated.

Previously, some of us have derived a general mass term for massive gravity perturbations on an arbitrary background and for an arbitrary reference metric [77], see also [24]. Here we want to generalize this work to bimetric gravity. In the bimetric context both the metrics admit fluctuations around their background configurations, $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ and $f_{\mu\nu} = \bar{f}_{\mu\nu} + \ell_{\mu\nu}$. The derivation of this mass term is the main goal of this paper. We then construct a convenient parametrization of the mass term for cosmological backgrounds and present the generic massive action for perturbations on a Friedmann-Lemaître spacetime. We express the mass term using the energy density and pressure as well as two new functions of the parameters of the theory. Finally, to illustrate our result, we apply our finding to a specific model and discuss tensor perturbations in this example.

The remainder of this paper is structured as follows: in the next section we briefly present the general setting of massive bigravity. In section 4.3 we derive the mass term. This section contains our main result. In section 4.4 we specialize to the cosmological setting and in section 4.5 we present an application. In section 4.6 we conclude. Some lengthy calculations are deferred to several appendices.

Notation: We set $c = \hbar = k_{\rm Boltzmann} = 1$. $M_g = 1/\sqrt{8\pi G} \equiv M_p \simeq 2.4\times 10^{18}$GeV is the reduced Planck mass. We work with the metric signature $(-,+,\dots,+)$ and we restrict to $D=4$ spacetime dimensions. With $\cdot$ and with $'$ we indicate derivatives with respect to physical time and to conformal time, respectively. We consider only one of the two metrics coupled to matter, and we restrict to minimal couplings. We normalize the scale factor a to be one at present time, i.e., $a_0 = 1$.

4.2 The bigravity action

The action for Hassan and Rosen bigravity theory is given by [83]

$$S = \frac{M_g^2}{2}\int \mathrm{d}^4x\sqrt{-\det g}\,\left[R(g) - U(f,g)\right] + \frac{M_f^2}{2}\int \mathrm{d}^4x\sqrt{-\det f}\,R(f) + S_{\rm matter}(g,\Phi)\,, \tag{4.1}$$

where $R(g)$ and $R(f)$ are the Ricci scalar for the physical metric $g_{\mu\nu}$ and the reference metric $f_{\mu\nu}$, respectively, M_g and M_f are the respective Planck masses and $S_{\rm matter}(g,\Phi)$ denotes the matter action. We assume the matter fields Φ to be coupled to g only. The potential $U(f,g)$ is given by

$$U(f,g) = 2m^2\left[\beta_0 + \beta_1 e_1(\mathbb{X}) + \beta_2 e_2(\mathbb{X}) + \beta_3 e_3(\mathbb{X}) + \beta_4 e_4(\mathbb{X})\right]\,, \tag{4.2}$$

where $\mathbb{X} \equiv \sqrt{g^{-1}f}$, β_i $(0 \le i \le 4)$ are arbitrary coefficients and

$$e_1(\mathbb{X}) = [\mathbb{X}]\,, \tag{4.3}$$

$$e_2(\mathbb{X}) = \frac{1}{2}\left([\mathbb{X}]^2 - [\mathbb{X}^2]\right)\,, \tag{4.4}$$

$$e_3(\mathbb{X}) = \frac{1}{6}\left([\mathbb{X}]^3 - 3[\mathbb{X}][\mathbb{X}^2] + 2[\mathbb{X}^3]\right)\,, \tag{4.5}$$

$$e_4(\mathbb{X}) = \frac{1}{24}\Big([\mathbb{X}]^4 - 6[\mathbb{X}]^2[\mathbb{X}^2] + 3[\mathbb{X}^2]^2 + 8[\mathbb{X}][\mathbb{X}^3] - 6[\mathbb{X}^4]\Big) = \det(\mathbb{X})\,. \tag{4.6}$$

Here we use the notation $[\mathbb{X}] = \mathrm{tr}\mathbb{X} = \mathbb{X}^\mu{}_\mu$, $[\mathbb{X}^2] = \mathrm{tr}\mathbb{X}^2 = \mathbb{X}^\mu{}_\nu\mathbb{X}^\nu{}_\mu$ and so forth.

Varying this action with respect to the metrics g and f one obtains the equations of motion

$$G_{\mu\nu}(g) = 8\pi G T_{\mu\nu} + \sqrt{-g^{-1}}\frac{\partial(\sqrt{-\det g}U(f,g))}{\partial g^{\mu\nu}}, \tag{4.7}$$

$$G_{\mu\nu}(f) = \sqrt{-f^{-1}}\frac{\partial(\sqrt{-\det g}U(f,g))}{\partial f^{\mu\nu}}. \tag{4.8}$$

Here $T_{\mu\nu}$ is the energy momentum tensor obtained by varying the matter action with respect to the metric $g^{\mu\nu}$, whereas $G_{\mu\nu}(g)$ and $G_{\mu\nu}(f)$ are the Einstein tensors for the metrics $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively.

The Bianchi identities for the two metrics together with energy momentum conservation imply

$$\nabla_f^{\nu}\left(\frac{\partial(\sqrt{-\det g}U(f,g))}{\partial f^{\mu\nu}}\right) = \nabla_g^{\nu}\left(\frac{\partial(\sqrt{-\det g}U(f,g))}{\partial g^{\mu\nu}}\right) = 0, \tag{4.9}$$

where ∇_f and ∇_g denotes the covariant derivatives w.r.t the metrics f and g respectively. Equations (4.9) are called the Bianchi constraints for f and g, respectively. One can show that both Bianchi constraints are equivalent.

Furthermore, using the fact that $\sqrt{g^{-1}f}$ can be written in triangular form, it is easy to verify that

$$t_1 \equiv e_1(\sqrt{g^{-1}f}) = \sum_i \lambda_i^{1/2}, \tag{4.10a}$$

$$t_2 \equiv e_2(\sqrt{g^{-1}f}) = \sum_{i<k} \lambda_i^{1/2}\lambda_k^{1/2}, \tag{4.10b}$$

$$t_3 \equiv e_3(\sqrt{g^{-1}f}) = \sum_{i<k<l} \lambda_i^{1/2}\lambda_k^{1/2}\lambda_l^{1/2}, \tag{4.10c}$$

$$t_4 \equiv e_4(\sqrt{g^{-1}f}) = \sqrt{\lambda_1\lambda_2\lambda_3\lambda_4}, \tag{4.10d}$$

where λ_i are the eigenvalues of $g^{-1}f$, and $1 \le i,k,l \le 4$. In terms of the t_i, the potential defined in eq. (4.2) can be written as

$$U(f,g) = 2m^2\left[\beta_0 + \beta_1 t_1 + \beta_2 t_2 + \beta_3 t_3 + \beta_4 t_4\right]. \tag{4.11}$$

We also introduce the quantities corresponding to the t_i for the metric $g^{-1}f$ without square root. They are easier to handle and will be used extensively in the rest of this work:

$$s_1 \equiv e_1(g^{-1}f) = \sum_i \lambda_i, \tag{4.12a}$$

$$s_2 \equiv e_2(g^{-1}f) = \sum_{i<j} \lambda_i\lambda_j, \tag{4.12b}$$

$$s_3 \equiv e_3(g^{-1}f) = \sum_{i<j<k} \lambda_i\lambda_j\lambda_k, \tag{4.12c}$$

$$s_4 \equiv e_4(g^{-1}f) = \lambda_1\lambda_2\lambda_3\lambda_4. \tag{4.12d}$$

The relations between the t_i and the s_i defined in eq. (4.10) and eq. (4.12) respectively are, see also [38]:

$$t_1^2 = s_1 + 2t_2\,, \tag{4.13a}$$
$$t_2^2 = s_2 - 2\sqrt{s_4} + 2t_1 t_3\,, \tag{4.13b}$$
$$t_3^2 = s_3 + 2t_2\sqrt{s_4}\,, \tag{4.13c}$$
$$t_4^2 = s_4\,. \tag{4.13d}$$

These relations (4.13) are easily derived from eqs. (4.10) and (4.12), by using the fact that every matrix can be written in triangular form. In any way, since both sides of (4.13) involve only the coordinate independent quantities t_i and s_i they are of course generally true. This has also been shown explicitly in Ref. [24].

4.3 Metric perturbations

We now expand the action for bigravity to second order in the metric perturbations, around generic background solutions $\bar{g}_{\mu\nu}$ and $\bar{f}_{\mu\nu}$. The perturbed metrics are defined as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}\,, \tag{4.14}$$
$$f_{\mu\nu} = \bar{f}_{\mu\nu} + \ell_{\mu\nu}\,. \tag{4.15}$$

From now on, the indices of the tensor $h_{\mu\nu}$ will be raised and lowered with the physical background metric $\bar{g}_{\mu\nu}$, whereas the indices of the tensor $\ell_{\mu\nu}$ will be raised and lowered with the background metric $\bar{f}_{\mu\nu}$.

4.3.1 Kinetic terms for $h_{\mu\nu}$ and for $\ell_{\mu\nu}$

The kinetic term for $h_{\mu\nu}$ is given by the Lichnerowicz operator, $\mathcal{E}^{\mu\nu\alpha\beta}(\bar{g})$, in curved spacetime

$$\begin{aligned}&\sqrt{-\det g}\,R(g)\\ &=\sqrt{-\det \bar{g}}\Big[R(\bar{g})-h_{\mu\nu}G^{\mu\nu}(\bar{g}) + h_{\mu\nu}\mathcal{E}^{\mu\nu\alpha\beta}(\bar{g})h_{\alpha\beta} + \nabla_\mu V_g^\mu\Big] + \mathcal{O}(h^3)\,,\end{aligned} \tag{4.16}$$

with

$$\begin{aligned}\mathcal{E}^{\mu\nu\alpha\beta}(\bar{g}) =& \frac{1}{4}\Big[\left(\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta} - \bar{g}^{\mu\nu}\bar{g}^{\alpha\beta}\right)\Box + \left(\bar{g}^{\mu\nu}\bar{g}^{\alpha\rho}\bar{g}^{\beta\sigma} + \bar{g}^{\alpha\beta}\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma} - \bar{g}^{\mu\beta}\bar{g}^{\nu\rho}\bar{g}^{\alpha\sigma} - \bar{g}^{\alpha\nu}\bar{g}^{\beta\rho}\bar{g}^{\mu\sigma}\right)\nabla_\rho\nabla_\sigma\Big] -\\ &- \frac{R(\bar{g})}{8}\left(\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta} + \bar{g}^{\nu\alpha}\bar{g}^{\mu\beta} - \bar{g}^{\mu\nu}\bar{g}^{\alpha\beta}\right) - \frac{1}{4}\left(\bar{g}^{\mu\nu}R^{\alpha\beta}(\bar{g}) + \bar{g}^{\alpha\beta}R^{\mu\nu}(\bar{g})\right) +\\ &+ \frac{1}{4}\left(\bar{g}^{\mu\alpha}R^{\beta\nu}(\bar{g}) + \bar{g}^{\mu\beta}R^{\alpha\nu}(\bar{g}) + \bar{g}^{\nu\alpha}R^{\beta\mu}(\bar{g}) + \bar{g}^{\nu\beta}R^{\alpha\mu}(\bar{g})\right)\,.\end{aligned} \tag{4.17}$$

Here the covariant derivatives are taken with respect to the background metric $\bar{g}$ and $\Box = \bar{g}^{\rho\sigma}\nabla_\rho\nabla_\sigma$ is the d'Alembertian operator. The kinetic term in square brackets in (4.17) is the curved spacetime version of the Lichnerowicz operator on flat space, see e.g. [47], and the terms proportional to $R(\bar{g})$ and $R^{\sigma\rho}$ contribute to the potential for $h_{\mu\nu}$ which vanishes on a flat background. This contribution has the form of a mass term (since it is quadratic in $h_{\mu\nu}$), which depends on the background solution. While probably not new, we have not found this general form of the Lichnerowicz operator in the literature[1]. An expression for Einstein spaces, for which $R^{\mu\nu}(\bar{g}) = \bar{g}^{\mu\nu}R(\bar{g})/4$ is given in Ref. [93], but also this does not quite agree with eq. (4.17), since it is written with a different index position like Ref. [130]. Note also that these references consider a mass as well defined only for Einstein spaces, since in more general backgrounds one no longer has a local Poincaré symmetry and hence the mass as a Casimir of the Poincaré group is not well defined. We take here a more naive point of view and just call 'mass term' the term quadratic in the perturbations $h_{\mu\nu}$ which does not contain any derivatives, by analogy to the scalar field case.

$G^{\mu\nu}(\bar{g})$ in eq. (4.16) is the Einstein tensor which solves the background equations of motion for $g_{\mu\nu}$ while the total derivative $\nabla_\mu V_g^\mu$ is irrelevant for the equations of motion. (Such a term can actually be removed on the level of the action by adding a suitable Gibbons-Hawking boundary term.)

The kinetic term for $\ell_{\mu\nu}$ is analogous to the one for $h_{\mu\nu}$,

$$\begin{aligned}&\sqrt{-\det f}\,R(f)\\&=\sqrt{-\det\bar{f}}\Big[R(\bar{f})-\ell_{\mu\nu}G^{\mu\nu}(\bar{f})+\ell_{\mu\nu}\mathcal{E}^{\mu\nu\alpha\beta}(\bar{f})\ell_{\alpha\beta}+\nabla_\mu V_f^\mu\Big]+\mathcal{O}(\ell^3)\,.\end{aligned} \tag{4.18}$$

In eq. (4.18), covariant derivatives are taken with respect to the background metric $\bar{f}$ and $\Box = \bar{f}^{\rho\sigma}\nabla_\rho\nabla_\sigma$ is the d'Alembertian operator. $\mathcal{E}^{\mu\nu\alpha\beta}(\bar{f})$ is the curved spacetime version of the Lichnerowicz operator, given by the analogous of eq. (4.17) for the $\bar{f}_{\mu\nu}$ background, $G^{\mu\nu}(\bar{f})$ is the Einstein tensor which solves the background equations of motion for $f_{\mu\nu}$ while the total derivative $\nabla_\mu V_f^\mu$ is irrelevant for the equations of motion.

4.3.2 The perturbed mass term

Making use of the definitions (4.10) and (4.12) and of the relations (4.13), it is possible to write the perturbations of t_i in terms of perturbations of s_i, which in turn can be obtained from the variation of $g^{-1}f \equiv g^{\mu\rho}f_{\rho\nu}$. We keep only terms up to second order in the metric perturbations. Therefore, for example, the fundamental quantity $g^{-1}f$ is expanded as

$$g^{\mu\rho}f_{\rho\nu} = (\delta^\mu_\alpha - h^\mu_\alpha + h^\mu_\gamma h^\gamma_\alpha)\bar{g}^{\alpha\rho}\bar{f}_{\rho\beta}(\delta^\beta_\nu + \ell^\beta_\nu) + \mathcal{O}(h^3)\,. \tag{4.19}$$

[1] The form given in eq. (2.12) of Ref. [130] is equivalent but it does not look the same since it uses $h^{\mu\nu}$ instead of $h_{\mu\nu}$ as independent variable.

Up to second order in the perturbations $h_{\mu\nu}$ and $\ell_{\mu\nu}$, the potential can be written as[2]

$$\sqrt{-\det g}\; U(f,g) = \sqrt{-\det \bar{g}}\Big[U(\bar{f},\bar{g}) + \mathcal{M}_g^{\mu\nu}(\bar{f},\bar{g})h_{\mu\nu} + \mathcal{M}_f^{\mu\nu}(\bar{f},\bar{g})\ell_{\mu\nu} + \\ + \mathcal{M}_{gg}^{\mu\nu\alpha\beta}(\bar{f},\bar{g})h_{\mu\nu}h_{\alpha\beta} + \mathcal{M}_{gf}^{\mu\nu\alpha\beta}(\bar{f},\bar{g})h_{\mu\nu}\ell_{\alpha\beta} + \mathcal{M}_{ff}^{\mu\nu\alpha\beta}(\bar{f},\bar{g})\ell_{\mu\nu}\ell_{\alpha\beta}\Big], \tag{4.20}$$

where

$$\mathcal{M}_g^{\mu\nu}(\bar{f},\bar{g}) \equiv \frac{1}{\sqrt{-\det g}} \frac{\partial(\sqrt{-\det g}\; U(f,g))}{\partial g_{\mu\nu}}\bigg|_{g=\bar{g},f=\bar{f}}, \tag{4.21}$$

$$\mathcal{M}_f^{\mu\nu}(\bar{f},\bar{g}) \equiv \frac{1}{\sqrt{-\det g}} \frac{\partial(\sqrt{-\det g}\; U(f,g))}{\partial f_{\mu\nu}}\bigg|_{g=\bar{g},f=\bar{f}}, \tag{4.22}$$

$$\mathcal{M}_{gg}^{\mu\nu\alpha\beta}(\bar{f},\bar{g}) \equiv \frac{1}{2}\frac{1}{\sqrt{-\det g}} \frac{\partial^2(\sqrt{-\det g}\; U(f,g))}{\partial g_{\mu\nu}\partial g_{\alpha\beta}}\bigg|_{g=\bar{g},f=\bar{f}}, \tag{4.23}$$

$$\mathcal{M}_{gf}^{\mu\nu\alpha\beta}(\bar{f},\bar{g}) \equiv \frac{1}{\sqrt{-\det g}} \frac{\partial^2(\sqrt{-\det g}\; U(f,g))}{\partial g_{\mu\nu}\partial f_{\alpha\beta}}\bigg|_{g=\bar{g},f=\bar{f}}, \tag{4.24}$$

$$\mathcal{M}_{ff}^{\mu\nu\alpha\beta}(\bar{f},\bar{g}) \equiv \frac{1}{2}\frac{1}{\sqrt{-\det g}} \frac{\partial^2(\sqrt{-\det g}\; U(f,g))}{\partial f_{\mu\nu}\partial f_{\alpha\beta}}\bigg|_{g=\bar{g},f=\bar{f}}. \tag{4.25}$$

Since we are considering perturbations around solutions of the background equations of motion, the terms linear in $h_{\mu\nu}$ and in $\ell_{\mu\nu}$ in the Lagrangian cancel on shell and in our discussion they can be omitted.

The explicit calculation of the matrix elements (4.21)-(4.25) is cumbersome but rather straightforward. We present here only the final result in a compact form, collecting the details of the calculation and the open expressions of the final result in Appendix 4.8. We find that the mass term expanded to quadratic order in the perturbation variables can be written as

$$\sqrt{-\det g}\; U(f,g) \\ = \sqrt{-\det \bar{g}}\left[U(\bar{f},\bar{g}) + \mathcal{M}_{gg}^{\mu\nu\alpha\beta}(\bar{f},\bar{g})h_{\mu\nu}h_{\alpha\beta} + \mathcal{M}_{gf}^{\mu\nu\alpha\beta}(\bar{f},\bar{g})h_{\mu\nu}\ell_{\alpha\beta} + \mathcal{M}_{ff}^{\mu\nu\alpha\beta}(\bar{f},\bar{g})\ell_{\mu\nu}\ell_{\alpha\beta}\right], \tag{4.26}$$

[2]We will always denote the indices of h with the letters $\mu\nu$ and the indices of ℓ with the letters $\alpha\beta$ in the mixed term, $\mathcal{M}_{gf}^{\mu\nu\alpha\beta}(\bar{f},\bar{g})h_{\mu\nu}\ell_{\alpha\beta}$.

with

$$\mathcal{M}^{\mu\nu\alpha\beta}_{gg} = m^2\left[\beta_0\mathcal{M}^{\mu\nu\alpha\beta}_{0,gg} + \beta_1\mathcal{M}^{\mu\nu\alpha\beta}_{1,gg} + \beta_2\mathcal{M}^{\mu\nu\alpha\beta}_{2,gg} + \beta_3\mathcal{M}^{\mu\nu\alpha\beta}_{3,gg}\right], \tag{4.27}$$

$$\mathcal{M}^{\mu\nu\alpha\beta}_{gf} = m^2\left[\beta_1\mathcal{M}^{\mu\nu\alpha\beta}_{1,gf} + \beta_2\mathcal{M}^{\mu\nu\alpha\beta}_{2,gf} + \beta_3\mathcal{M}^{\mu\nu\alpha\beta}_{3,gf} + \beta_4\mathcal{M}^{\mu\nu\alpha\beta}_{4,gf}\right], \tag{4.28}$$

$$\mathcal{M}^{\mu\nu\alpha\beta}_{ff} = m^2\left[\beta_1\mathcal{M}^{\mu\nu\alpha\beta}_{1,ff} + \beta_2\mathcal{M}^{\mu\nu\alpha\beta}_{2,ff} + \beta_3\mathcal{M}^{\mu\nu\alpha\beta}_{3,ff} + \beta_4\mathcal{M}^{\mu\nu\alpha\beta}_{4,ff}\right], \tag{4.29}$$

$$\mathcal{M}^{\mu\nu\alpha\beta}_{0,gg} = \frac{1}{\sqrt{-\det g}}\frac{\partial^2\sqrt{-\det g}}{\partial g_{\mu\nu}\partial g_{\alpha\beta}} = \frac{1}{4}\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta} - \frac{1}{4}\left(\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta} + \bar{g}^{\mu\beta}\bar{g}^{\nu\alpha}\right), \tag{4.30}$$

$$\mathcal{M}^{\mu\nu\alpha\beta}_{i,gg} = \bar{t}_i\mathcal{M}^{\mu\nu\alpha\beta}_{0,gg} + \frac{1}{2}\left(\bar{g}^{\mu\nu}t^{\alpha\beta}_{i,g} + \bar{g}^{\alpha\beta}t^{\mu\nu}_{i,g}\right) + 2t^{\mu\nu\alpha\beta}_{i,gg}, \tag{4.31}$$

$$\mathcal{M}^{\mu\nu\alpha\beta}_{i,gf} = \bar{g}^{\mu\nu}t^{\alpha\beta}_{i,f} + 2t^{\mu\nu\alpha\beta}_{i,gf}, \tag{4.32}$$

$$\mathcal{M}^{\mu\nu\alpha\beta}_{i,ff} = 2t^{\mu\nu\alpha\beta}_{i,ff}, \tag{4.33}$$

where $i = 1,2,3,4$ and to shorten the notation we have defined the following derivatives which are calculated explicitly in Appendix 4.8:

$$t^{\mu\nu}_{i,g} = \left.\frac{\partial t_i}{\partial g_{\mu\nu}}\right|_{g=\bar{g},f=\bar{f}}, \qquad t^{\mu\nu}_{i,f} = \left.\frac{\partial t_i}{\partial f_{\mu\nu}}\right|_{g=\bar{g},f=\bar{f}}, \qquad t^{\mu\nu\alpha\beta}_{i,gg} = \left.\frac{1}{2}\frac{\partial^2 t_i}{\partial g_{\mu\nu}\partial g_{\alpha\beta}}\right|_{g=\bar{g},f=\bar{f}} \tag{4,34}$$

$$t^{\mu\nu\alpha\beta}_{i,gf} = \left.\frac{\partial^2 t_i}{\partial g_{\mu\nu}\partial f_{\alpha\beta}}\right|_{g=\bar{g},f=\bar{f}}, \qquad t^{\mu\nu\alpha\beta}_{i,ff} = \left.\frac{1}{2}\frac{\partial^2 t_i}{\partial f_{\mu\nu}\partial f_{\alpha\beta}}\right|_{g=\bar{g},f=\bar{f}}. \tag{4.35}$$

All these objects are computed explicitly in terms of the background metrics in Appendix 4.8. It is easy to check that this potential correctly reduces to the massive gravity one calculated in [77], once the limits $\ell_{\mu\nu} \to 0$ and $f_{\mu\nu} \to \bar{f}_{\mu\nu}$ are taken. Furthermore, one can check that

$$\begin{aligned} \sqrt{-\det\bar{g}}\,\mathcal{M}^{\mu\nu\alpha\beta}_{i,ff}(\bar{f},\bar{g}) &= \sqrt{-\det\bar{f}}\,\mathcal{M}^{\mu\nu\alpha\beta}_{4-i,gg}(\bar{g},\bar{f}) \quad \text{and} \\ \sqrt{-\det\bar{g}}\,\mathcal{M}^{\mu\nu\alpha\beta}_{i,gg}(\bar{f},\bar{g}) &= \sqrt{-\det\bar{f}}\,\mathcal{M}^{\mu\nu\alpha\beta}_{4-i,ff}(\bar{g},\bar{f}). \end{aligned} \tag{4.36}$$

This is a consequence of the fact that for $M_g = M_f$, the gravitational action (4.1) is invariant under the simultaneous exchange $f \leftrightarrow g$ and $\beta_i \to \beta_{4-i}$.

Using the results above, we can write the general expression for the perturbed action, quadratic in the metric perturbations $h_{\mu\nu}$ and $\ell_{\mu\nu}$

$$\begin{aligned} S^{(2)} =& \frac{M_g^2}{2}\int d^4x\sqrt{-\det\bar{g}}\,h_{\mu\nu}\mathcal{E}^{\mu\nu\alpha\beta}(\bar{g})h_{\alpha\beta} + \frac{M_f^2}{2}\int d^4x\,\sqrt{-\det\bar{f}}\,\ell_{\mu\nu}\mathcal{E}^{\mu\nu\alpha\beta}(\bar{f})\ell_{\alpha\beta} \\ &-\frac{M_g^2}{2}\int d^4x\sqrt{-\det\bar{g}}\left[\mathcal{M}^{\mu\nu\alpha\beta}_{gg}(\bar{f},\bar{g})h_{\mu\nu}h_{\alpha\beta} + \mathcal{M}^{\mu\nu\alpha\beta}_{gf}(\bar{f},\bar{g})h_{\mu\nu}\ell_{\alpha\beta} + \mathcal{M}^{\mu\nu\alpha\beta}_{ff}(\bar{f},\bar{g})\ell_{\mu\nu}\ell_{\alpha\beta}\right]. \end{aligned} \tag{4.37}$$

Eq. (4.37) together with the expressions for the mass terms $\mathcal{M}_{\bullet\bullet}^{\mu\nu\alpha\beta}$ in Appendix A and the Einstein operator given in eq. (4.17) are the main result of our paper. It allows to write down the perturbation equations of bimetric massive gravity on an arbitrary background. In the following we apply it to cosmology.

4.4 Mass term on cosmological backgrounds

In this and the following sections, we specify to background solutions where both the metrics exhibit spatial isotropy and homogeneity. For simplicity, we assume that both the metrics have a flat spatial section, $K = 0$. Then, modulo time re-parameterizations, the most general form for the metrics (in conformal time τ) is

$$\bar{g}_{\mu\nu}dx^\mu dx^\nu = a^2(\tau)\left(-d\tau^2 + \delta_{ij}dx^i dx^j\right) , \tag{4.38}$$

$$\bar{f}_{\mu\nu}dx^\mu dx^\nu = b^2(\tau)\left(-c^2(\tau)d\tau^2 + \delta_{ij}dx^i dx^j\right) . \tag{4.39}$$

It is convenient to define the conformal Hubble parameter ($\mathcal{H}$) and the physical one (H) for both metrics:

$$H = \frac{\mathcal{H}}{a} = \frac{a'}{a^2} , \qquad H_f = \frac{\mathcal{H}_f}{b} = \frac{b'}{b^2\, c} , \tag{4.40}$$

where with $'$ we denote the derivative with respect to the conformal time τ. We introduce also the ratio between the two conformal scale factors

$$r = \frac{b}{a} . \tag{4.41}$$

The Friedmann equations can then be written as

$$3H^2 = 8\pi G\left(\rho + \rho_g\right) , \quad \text{with } \rho_g = \tfrac{m^2}{8\pi G}\left(\beta_0 + 3\beta_1 r + 3\beta_2 r^2 + \beta_3 r^3\right) , \tag{4.42}$$

$$3H_f^2 = 8\pi G\,\rho_f , \quad \text{with } \rho_f = \tfrac{m^2}{8\pi G M_*^2}\left(\beta_4 + 3\beta_3 r^{-1} + 3\beta_2 r^{-2} + \beta_1 r^{-3}\right) , \tag{4.43}$$

$$\begin{aligned} &3H^2 + 2\dot{H} = -8\pi G\left(p + p_g\right) , \text{ with} \\ &p_g = -\frac{m^2}{8\pi G}\left(\beta_0 + \beta_1(c+2)r + \beta_2\left(2c+1\right)r^2 + \beta_3 c r^3\right) , \end{aligned} \tag{4.44}$$

$$\begin{aligned} &3cH_f^2 + 2\dot{H}_f = -8\pi G\, p_f , \text{ with} \\ &p_f = -\frac{m^2}{8\pi G M_*^2}\left(c\beta_4 + \beta_3(2c+1)r^{-1} + \beta_2(c+2)r^{-2} + \beta_1 r^{-3}\right) . \end{aligned} \tag{4.45}$$

Here $(8\pi G)^{-1/2} = M_P = M_g$ is the physical Planck mass and M_* is the ratio between the two Planck masses, $M_* \equiv M_f/M_g$, while ρ and p are the usual matter energy density and pressure. The quantities ρ_g, p_g and ρ_f, p_f play the role of gravitational 'energy densities' and 'pressure'. They come from the mass term.

With the ansatz (4.38, 4.39) for the background metrics, homogeneity and isotropy request that the mass tensor in eq. (4.26) admit the following general parametrization. For the gg and ff terms

$$\mathcal{M}^{0000}_{\bullet\bullet}(\bar{f},\bar{g}) = m^2 a^{-4}\alpha_\bullet(\tau)\,, \tag{4.46}$$
$$\mathcal{M}^{ij00}_{\bullet\bullet}(\bar{f},\bar{g}) = \mathcal{M}^{00ij}_{\bullet\bullet}(\bar{f},\bar{g}) = m^2 a^{-4}\gamma_\bullet(\tau)\delta^{ij}\,, \tag{4.47}$$
$$\mathcal{M}^{i0j0}_{\bullet\bullet}(\bar{f},\bar{g}) = \mathcal{M}^{0i0j}_{\bullet\bullet}(\bar{f},\bar{g}) = m^2 a^{-4}\epsilon_\bullet(\tau)\delta^{ij}\,, \tag{4.48}$$
$$\mathcal{M}^{ijkl}_{\bullet\bullet}(\bar{f},\bar{g}) = m^2 a^{-4}\left\{\eta_\bullet(\tau)\delta^{ij}\delta^{kl} + \frac{\sigma_\bullet(\tau)}{2}\left(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right)\right\}\,. \tag{4.49}$$

where $\bullet$ stands for either g or f. And for the mixed terms gf the parametrization takes the form

$$\mathcal{M}^{0000}_{gf}(\bar{f},\bar{g}) = m^2 a^{-4}\alpha_{gf}(\tau)\,, \tag{4.50}$$
$$\mathcal{M}^{ij00}_{gf}(\bar{f},\bar{g}) = m^2 a^{-4}\gamma_{gf}(\tau)\delta^{ij}\,, \tag{4.51}$$
$$\mathcal{M}^{00ij}_{gf}(\bar{f},\bar{g}) = m^2 a^{-4}\gamma_{fg}(\tau)\delta^{ij}\,, \tag{4.52}$$
$$\mathcal{M}^{i0j0}_{gf}(\bar{f},\bar{g}) = \mathcal{M}^{0i0j}_{gf}(\bar{f},\bar{g}) = m^2 a^{-4}\epsilon_{gf}(\tau)\delta^{ij}\,, \tag{4.53}$$
$$\mathcal{M}^{ijkl}_{gf}(\bar{f},\bar{g}) = m^2 a^{-4}\left\{\eta_{gf}(\tau)\delta^{ij}\delta^{kl} + \frac{\sigma_{gf}(\tau)}{2}\left(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right)\right\}\,. \tag{4.54}$$

The functions $\alpha_\bullet$, $\gamma_\bullet$, $\epsilon_\bullet$, $\sigma_\bullet$ and $\eta_\bullet$ (with $\bullet = g, f, gf$ or fg) depend on conformal time through the ratio between the two scale factors, r, and the lapse function c. Their explicit expressions are given in Appendix 4.11. Note that contrary to gg and ff, $\mathcal{M}^{ij00}_{gf} \neq \mathcal{M}^{00ij}_{gf}$ and we have introduced $\gamma_{gf} \neq \gamma_{fg}$.

Given this parametrization it is straightforward to write the mass term for any type of perturbations on a cosmological background,

$$\mathcal{S}^{(2)}_m = -\frac{M_g^2}{2}\int d^4x\left[\mathcal{L}^{(2)}_{gg} + \mathcal{L}^{(2)}_{ff} + \mathcal{L}^{(2)}_{gf}\right], \tag{4.55}$$

$$\mathcal{L}^{(2)}_{gg} = m^2\left[\alpha_g h_{00}^2 + \gamma_g h_{00}h_{ij}\delta^{ij} + 2\epsilon_g h_{0i}h_{0j}\delta^{ij} + \eta_g h_{ij}h_{kl}\delta^{ij}\delta^{kl} + \frac{\sigma_g}{2}h_{ij}h_{kl}\left(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right)\right] \tag{4.56}$$

$$\mathcal{L}^{(2)}_{ff} = m^2\left[\alpha_f \ell_{00}^2 + \gamma_f \ell_{00}\ell_{ij}\delta^{ij} + 2\epsilon_f \ell_{0i}\ell_{0j}\delta^{ij} + \eta_f \ell_{ij}\ell_{kl}\delta^{ij}\delta^{kl} + \frac{\sigma_f}{2}\ell_{ij}\ell_{kl}\left(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right)\right] \tag{4.57}$$

$$\begin{aligned}\mathcal{L}^{(2)}_{gf} = m^2\Big[&\alpha_{gf}h_{00}\ell_{00} + \gamma_{fg}h_{00}\ell_{ij}\delta^{ij} + \gamma_{gf}\ell_{00}h_{ij}\delta^{ij} + 2\epsilon_{gf}h_{0i}\delta^{ij}\ell_{0j} + \eta_{gf}h_{ij}\ell_{kl}\delta^{ij}\delta^{kl} \\ &+ \frac{\sigma_{gf}}{2}h_{ij}\ell_{kl}\left(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right)\Big]\end{aligned} \tag{4.58}$$

Even though this parametrization contains 16 functions of the background, these functions can be expressed solely in terms of two combinations of the β_i parameters, which we denote as σ_1 and σ_2 together with the 'densities' and 'pressures' ρ_g, ρ_f and p_g, p_f. This is because one can use the Friedmann equations to simplify the mass term by combining it with contributions from the Einstein-Hilbert and the matter action. We present this in detail in Appendix 4.11.

4.5 Cosmological application: Algebraic branch

As an illustration of our formalism, we now apply our general result for the mass term to a specific cosmological setting, the algebraic branch of solutions of bigravity with matter minimally coupled to the g metric.

The Bianchi constraint (4.9) for the cosmological ansatz takes the form

$$m^2 \left(\beta_1 + 2\beta_2\, r + \beta_3\, r^2\right) \left(\mathcal{H} - \mathcal{H}_f\right) = 0\,. \tag{4.59}$$

We distinguish two branches of solutions, depending on how the Bianchi constraint (4.59) is solved. Either there is an algebraic constraint for r

$$\text{Branch I} \qquad \left(\beta_1 + 2\beta_2\, r + \beta_3\, r^2\right) = 0\,, \tag{4.60}$$

or

$$\text{Branch II} \qquad \mathcal{H}_f = \mathcal{H}\,. \tag{4.61}$$

At the background level the first branch, also called *algebraic* branch, is equivalent to GR with an effective cosmological constant, while the solution II gives rise to a richer cosmology.

Since branch II has been studied in detail in Refs. [110, 34, 101, 35], here we consider branch I. We shall show, that the claim which can be found in the literature [143, 110], that in this branch not only the background but also the perturbations are identical to ΛCDM is not strictly true, but that tensor perturbations, i.e. gravitational waves in this branch actually develop a tachyonic instability at late time. The Bianchi constraint in branch I is realized by setting $r = \bar{r}$ =const. such that

$$\beta_3\, \bar{r}^2 + 2\beta_2\, \bar{r} + \beta_1 = 0\,, \tag{4.62}$$

hence

$$H_f = \frac{1}{\bar{r}c} H\,. \tag{4.63}$$

Solving the Bianchi constraint, eq. (4.62), $\bar{r}$ can be expressed as a functions of the β_i

$$\bar{r} = \frac{-\beta_2 \pm \sqrt{\beta_2^2 - \beta_1\beta_3}}{\beta_3}\,, \qquad \text{for } \beta_3 \neq 0 \text{ or, if } \beta_3 = 0, \quad \bar{r} = -\frac{\beta_1}{2\beta_2}. \tag{4.64}$$

In this case, the gravitational 'energy densities' in eqs. (4.42) and (4.43) become constant,

$$\Lambda_{\rm eff} = 8\pi G\rho_g(\bar r)\,, \qquad \Lambda_c = 8\pi G\rho_f(\bar r)\,. \tag{4.65}$$

It is easy to check that (independently of the value of c), eq. (4.62) implies $p_g(\bar r) = -\rho_g(\bar r)$ and the physical sector at the background level is equivalent to GR with an effective cosmological constant.

From the Friedmann equation (4.43), we can extract the lapse function c, with the result[3]

$$c^2 = \frac{8\pi G\rho + \Lambda_{\rm eff}}{\bar r^2\Lambda_c} = \frac{H^2}{\bar r^2 H_f^2} \propto H^2\,. \tag{4.66}$$

In order to have a viable background evolution, we have to impose conditions on the parameters such that $\Lambda_{\rm eff} \geq 0$ and $c^2 \geq 0$, i.e., respectively

$$\beta_0 + 3\beta_1 r + 3\beta_2 r^2 + \beta_3 r^3 \geq 0\,, \qquad \text{and} \qquad \beta_4 + 3\beta_3 r^{-1} + 3\beta_2 r^{-2} + \beta_1 r^{-3} \geq 0\,, \tag{4.67}$$

and we have to request that $\bar r$ given by eq. (4.64) is non-vanishing

$$\frac{-\beta_2 \pm \sqrt{\beta_2^2 - \beta_1\beta_3}}{\beta_3} \neq 0\,, \qquad \text{or if} \qquad \beta_3 = 0 \qquad \frac{\beta_1}{2\beta_2} \neq 0\,. \tag{4.68}$$

We want now to find a minimal model (i.e. with the minimal number of non-vanishing parameters) in this branch satisfying the conditions (4.67) and (4.68) above. We restrict to the case $\beta_0 = 0$, since this parameter simply represents a *traditional* cosmological constant, and we know that the model $\beta_0 = \Lambda/m^2$, $\beta_i = 0$ for $i \neq 0$ works well, it is simply the standard ΛCDM cosmology which we do not want to investigate here. Single-parameter models are not viable since they give $\bar r = 0$ or $\bar r = \infty$. (The case where only β_4 or β_0 is non-vanishing leads to no coupling between the two metrics.) It is easy to check that none of the $\beta_3\beta_i$ models satisfy the viability requirements. Also the $\beta_1\beta_2$ model alone is not feasible since in this case eq. (4.67) requires at the same time $\beta_2 < 0$ and $\beta_2 > 0$. All three-parameter models, $\beta_i\beta_j\beta_k$ are viable if the parameters are chosen to satisfy the following conditions:

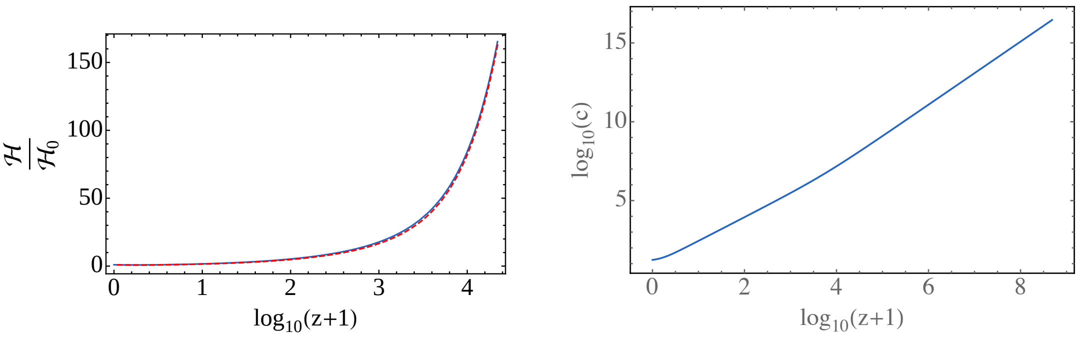
$$\beta_1\beta_2\beta_4 \qquad \bar r = \frac{-\beta_1}{2\beta_2}\,, \quad 0 < -\beta_2 < \left(\frac{\beta_1^2\beta_4}{4}\right)^{1/3}\,, \tag{4.69}$$

$$\beta_1\beta_3\beta_4 \quad \bar r = -\sqrt{\frac{-\beta_1}{\beta_3}}\,, \quad 0 \leq \sqrt{-\beta_1\beta_3} \leq \frac{1}{2}\frac{\beta_1\beta_4}{\beta_3}\,, \quad \text{or} \quad \bar r = \sqrt{\frac{-\beta_1}{\beta_3}}\,, \quad \frac{1}{2}\frac{\beta_1\beta_4}{\beta_3} \leq \sqrt{-\beta_1\beta_3} \leq 0\,, \tag{4.70}$$

$$\beta_2\beta_3\beta_4 \qquad \bar r = -2\frac{\beta_2}{\beta_3}\,, \quad 0 \leq \frac{3}{4}\frac{\beta_3^2}{\beta_4} \leq \beta_2\,. \tag{4.71}$$

[3]In the decoupling limit $M_*^2 \to \infty$, eq. (4.43) implies that $H_f^2 \to 0$ and, since r is a constant, from eq. (4.63) it follows $H^2/c^2 \to 0$ which is a consequence of $c \to \infty$. The Friedmann eq. (4.42) is not affected by the decoupling limit.

In what follows we set $M_* = 1$. This is not a restriction since a finite M_* can always be absorbed in the normalisation of the metric $f_{\mu\nu}$ and the parameters β_i. The three 'minimal' models above have equations for the background and for the perturbations which are of the same structure, hence we expect them to be qualitatively similar both at the level of the background and of the perturbations. For definitiveness, but no other more physical motivation, in the rest of this work, we focus on the $\beta_2\beta_3\beta_4$ model. We choose β_i's satisfying the inequalities (4.71) and such that $\Omega_\Lambda = \frac{\Lambda_{\rm eff}}{3H_0^2}$ has the observed value of about 0.73. We set $\beta_3^2 = 2\,\beta_2^3$ and $\beta_4 \geq 3/2\,\beta_2$. With the choice $\beta_4 \simeq 3/2\,\beta_2$, we obtain $\Lambda_c \simeq 3/2\beta_2(1-\beta_2)\,m^2$. Setting $m^2\beta_2 = 0.997\,\mathcal{H}_0^2$, this results in $\bar{r} \simeq -1.4$.[4] With this choice of parameters, the background evolution of the physical quantities is the one of ΛCDM with $\Omega_\Lambda = 0.73$. In Fig. 4.1 we show the evolution of the conformal Hubble parameter and of the lapse c as functions of the redshift.



(a) (b)

Figure 4.1: Evolution of the lapse function c and of the conformal Hubble parameter in the $\beta_2\beta_3\beta_4$ bigravity model with $m^2\beta_2 = 0.997\,\mathcal{H}_0^2$, $m^2\beta_3 = 1.408\,\mathcal{H}_0^2$ and $m^2\beta_4 = 1.496\,\mathcal{H}_0^2$. In Fig 4.1(a) the evolution of the conformal Hubble parameter in bigravity is plotted together with the one of ΛCDM with $\Omega_\Lambda = 0.73$ (in red). The two quantities perfectly agree.

The equations for the perturbations in this background branch of bigravity are derived in [31]. We recall that this branch is affected by a strong coupling problem: neglecting matter, from the canonical analysis 7 degrees of freedom are expected (one massive and one massless graviton) but only 4 are accounted for. The helicity zero and helicity 1 modes of the massive graviton are not present and they are probably infinitely strongly coupled, i.e. have vanishing kinetic terms. The same problem of infinite strong coupling appears in the analogous branch in massive gravity.

[4]With this choice of parameters, at late times $c^2 \simeq \Lambda_{\rm eff}/(\bar{r}^2\Lambda_c) \gg 1$, hence, as we will see in the equations for tensor perturbations, the coupling between the two propagating tensor modes will be not negligible at late times.

Recently, in [43], a 'minimal' model of massive gravity has been proposed which propagates at the fully non linear level the same number of degrees of freedom as linearized dRGT massive gravity in the algebraic branch. In this model, at every order in the perturbative expansion, only the helicity-2 modes are propagating. In principle it may be possible to extend the formalism developed in [43] to the bigravity case, building a *minimal model of bigravity*: this model would then propagate at every order in the perturbative expansion only 4 degrees of freedom, two belonging to the massless graviton and two the massive one. In particular, this model would have the same perturbation equations as the present algebraic bigravity branch, without being affected by the strong coupling problem.

In the following we focus on the study of the tensor sector in the algebraic branch, showing how the machinery developed in the first part of this paper can be applied to this case. We recall that the results of this study have to be taken *cum grano salis*, as we do not take into account problems which may be present as a consequence of the infinitely strong coupling of the helicity 0 and 1 modes. Our study would be realistic if these modes do not affect the evolution of the helicity 2 modes, e.g. because they remain small or because they even vanish as in a *would be minimal model of bigravity.*

4.5.1 New formalism applied to the study of tensor perturbations

In the following, we study tensor perturbations in the algebraic branch, both numerically and analytically. Making use of the formalism developed in this work, we show that the late time instability that we find in the tensor sector is due to the fact that an eigenvalue of the mass matrix becomes negative during the cosmological evolution.

Tensor perturbations of a given $\mathbf{k}$-mode are composed of two independent helicity modes,

$$h_{ij}^{TT} = h^+ e_{ij}^{(+2)} + h^- e_{ij}^{(-2)} \tag{4.72}$$

where + and − denote the two helicity-2 modes of the gravitational wave. For a plane wave with wave vector $\mathbf{k}$ in the orthonormal system $(\widehat{\mathbf{k}}, \mathbf{e}^{(1)}, \mathbf{e}^{(2)})$ we have

$$\mathbf{e}^{\pm} = \frac{1}{\sqrt{2}}\left(\mathbf{e}^{(1)} \pm i\mathbf{e}^{(2)}\right) \quad \text{and} \quad e_{ij}^{(+2)} = \mathbf{e}_i^+\mathbf{e}_j^+ , \quad e_{ij}^{(-2)} = \mathbf{e}_i^-\mathbf{e}_j^- . \tag{4.73}$$

For parity invariant perturbations

$$\langle h^+(\mathbf{k})(h^+(\mathbf{k}'))^*\rangle = \langle h^-(\mathbf{k})(h^-(\mathbf{k}'))^*\rangle = \delta(\mathbf{k}-\mathbf{k}')2\pi^2 P_h(k)\,,$$

and $\langle h^+h^-\rangle = 0$. Here $\langle\cdots\rangle$ denotes a stochastic expectation value. We consider a stochastic ensemble of parity invariant Gaussian gravitational waves. We discuss just one mode, say $h^+ = hG_h$ for the g tensor perturbations and $\ell^+ = \ell G_\ell$ for the f tensor perturbations. Here G_a, with $a = h, \ell$ are independent Gaussian random variables with vanishing mean and with variance $\langle G_a(\mathbf{k})G_b(\mathbf{k}')\rangle = \delta_{ab}\delta(\mathbf{k}-\mathbf{k}')2\pi^2$, so that h, ℓ is the square root of the power spectrum. All what follows is also valid for the modes h^-, ℓ^- which are not correlated with h^+, ℓ^+ in the parity symmetric situation which we consider.

In the tensor sector, for the first order modified Einstein equations with a perfect fluid source, i.e. no anisotropic stress, in the algebraic background branch, from eq. (4.37) we obtain

$$S^{(\pm 2)} = \frac{1}{2}M_g^2 \int d^4x\, a^2 \left\{ (h')^2 + \frac{\bar{r}^2}{c}(\ell')^2 - k^2h^2 - k^2 c\,\bar{r}^2\ell^2 + a^2m^2\sigma_2\bar{r}\,(h-\ell)^2 \right\}, \quad (4.74)$$

where

$$m^2\bar{r}\sigma_2 = m^2\left[\beta_2\bar{r}^2 + \beta_1\bar{r} + \left(\beta_3\bar{r}^3 + \beta_2\bar{r}^2\right)c(\tau)\right] \equiv m^2\left[C + D\,c(\tau)\right], \quad (4.75)$$

is the mass term for the helicity 2 modes. Notice that the mass term in the above action is much simpler than the *original* one obtained from the action (4.55) for tensor perturbations. This is a result of replacing the Friedmann equations, which simplify contributions of the *original* mass term by combining them with contributions from the kinetic and the matter parts of the action, as explained in Appendix 4.11. This simplification is general for the helicity-2 mode in cosmological solutions of bigravity and does not depend on our choice of branch. While the kinetic term is as expected diagonal in h and ℓ, the mass term acts on $h - \ell$.

Varying this action with respect to h and ℓ, we find the following equations of motion for the two tensor modes

$$h'' + 2\mathcal{H}\,h' + k^2h + m^2a^2\sigma_2\bar{r}\,(h-\ell) = 0\,, \quad (4.76)$$

$$\ell'' + \left(2\mathcal{H} - \frac{c'}{c}\right)\ell' + c^2k^2\,\ell - m^2\,\sigma_2\,\frac{c\,a^2}{\bar{r}}\,(h-\ell) = 0\,. \quad (4.77)$$

The kinetic term in eq. (4.74) is diagonal and, for the background under study, positive definite. The tensor sector is therefore free of (Higuchi) ghost instabilities [70, 34].

We introduce canonically normalized variables associated with each of the two tensor modes

$$Q_h \equiv M_g\,a\,h\,, \qquad Q_\ell \equiv M_g\,a\,\frac{\bar{r}}{\sqrt{c}}\ell\,. \quad (4.78)$$

The action (4.74) for the canonically normalized variables can be written in matrix form as

$$S^{(\pm 2)} = \frac{1}{2}\int d^4x \left\{ \begin{pmatrix} Q_h', & Q_\ell' \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Q_h' \\ Q_\ell' \end{pmatrix} - \begin{pmatrix} Q_h, & Q_\ell \end{pmatrix} \mathcal{M} \begin{pmatrix} Q_h \\ Q_\ell \end{pmatrix} \right\}, \quad (4.79)$$

where the 2×2 matrix $\mathcal{M}$ is given by

$$\mathcal{M} = \begin{pmatrix} \mathcal{E}(a) + k^2 + a^2 m^2 \sigma_2 \bar{r} & -a^2 m^2 \sigma_2 \sqrt{c} \\ -a^2 m^2 \sigma_2 \sqrt{c} & \mathcal{F}(a,c) + k^2 c^2 + a^2 m^2 \sigma_2 \frac{c}{\bar{r}} \end{pmatrix}, \tag{4.80}$$

and

$$\mathcal{E}(a) = -\frac{a''}{a}, \qquad \mathcal{F}(a,c) = -\frac{a''}{a} + \frac{1}{2}\frac{c''}{c} + \frac{c'}{c}\frac{a'}{a} - \frac{3}{4}\left(\frac{c'}{c}\right)^2. \tag{4.81}$$

The action for the canonically normalized modes, eq. (4.79), is the action for two coupled harmonic oscillators with potential given by the matrix $\mathcal{M}$. From now on, we will refer to this matrix as the *mass matrix*, since it has dimension of mass square. This matrix collects the contributions from the original mass term in the bigravity action (i.e. the terms proportional to m^2), the gradient terms (proportional to k^2) and contributions generated by the change of variable (4.78) from the original action (4.74) which canonically normalizes the variables and removes all damping terms (first order derivatives).

In matrix notation, in terms of $Q = (Q_h, Q_\ell)^T$, the equations of motion now reduce to

$$Q'' = -\mathcal{M} Q. \tag{4.82}$$

By construction, the kinetic matrix is simply the identity. The fact that we can choose it positive means that there is no ghost present. The eigenvalues of the mass matrix are

$$\lambda_{1,2} = p \pm q, \tag{4.83}$$

with

$$p = \frac{1}{2\bar{r}^2}\left[\left(k^2(c^2+1) + \mathcal{E} + \mathcal{F}\right)\bar{r}^2 + a^2 m^2 \sigma_2 \bar{r}\left(c + \bar{r}^2\right)\right], \tag{4.84}$$

$$q = \frac{1}{2\bar{r}^2}\Bigg(\left(k^2(c^2-1) - \mathcal{E} + \mathcal{F}\right)^2 \bar{r}^4 \\ + 2a^2 m^2 \sigma_2 \bar{r}^3 \left(k^2 c^2 - k^2 - \mathcal{E} + \mathcal{F}\right)\left(c - \bar{r}^2\right) + a^4 m^4 (\sigma_2 \bar{r})^2 \left(c + \bar{r}^2\right)^2\Bigg)^{1/2}. \tag{4.85}$$

For our choice of parameters with $m^2\beta_i \simeq \mathcal{H}_0^2$ and $c \gg \bar{r}^2 \simeq 2$, $c^2 = H^2/(\bar{r}H_f)^2 = 3\mathcal{H}^2/(a^2\bar{r}^2\Lambda_c)$, in the various epochs the eigenvalues can be approximated in compact form as

$$\lambda_1 \equiv p - q \simeq k^2 + \mathcal{E}(a) = k^2 - \frac{a''}{a} = k^2 - \frac{\alpha(\alpha-1)}{\tau^2}, \tag{4.86}$$

$$\begin{aligned} \lambda_2 \equiv p + q &\simeq k^2 c^2 + \mathcal{F}(a,c) + \frac{a^2}{\bar{r}^2} m^2 D c^2 \\ &= k^2 c^2 + \frac{1}{\tau^2}\left(\left(-\frac{9}{4} + \frac{\beta_3 \bar{r} + \beta_2}{3(1-\beta_2)}\right)\alpha^2 + 2\alpha - \frac{3}{4}\right), \end{aligned} \tag{4.87}$$

where D is defined in eq. (4.75) and we have considered a power law background expansion, $a \propto \tau^\alpha$ in the last equalities for λ_1 and λ_2. We call Q_1 and Q_2 the mass eigenstates associated to λ_1 and λ_2, respectively. The eigenvalue λ_1 is the expression for the squared frequency of the standard canonically normalized tensor mode in GR. Hence, one of the two mass eigenstates is exactly the massless graviton of GR and, as is well known, it is (marginally) stable. On super Hubble scales $Q_1 \propto a$, such that Q_1/a =constant while for sub-Hubble scales Q_1 oscillates at constant amplitude. The more interesting eigenvalue is λ_2. For our choice of the parameters β_i and m , we obtain

$$\lambda_2 \simeq k^2c^2 - \frac{359\alpha^2 - 2\alpha + 3/4}{\tau^2}\,. \tag{4.88}$$

For the stability of the second mass eigenstate we request $\lambda_2 \geq \lambda_1$. In the various cosmological epochs, from the expressions for $\mathcal{H}$ one can easily derive

$$k^2 - \frac{\alpha(\alpha-1)}{\tau^2} \lesssim k^2c^2 - \frac{\xi}{\tau^2}\,, \tag{4.89}$$

where $\xi = 357 + 3/4\,,\ 1432 + 3/4\,,\ 361 + 3/4$ in radiation ($\alpha = 1$), matter ($\alpha = 2$) and de Sitter ($\alpha = -1$) eras respectively.

At early times, condition (4.89) is satisfied for all modes of cosmological interest, $k^2 \gtrsim H_0^2 \simeq 1/\tau_0^2$, since $c \gg 1$. At low redshift, instead, a tachyonic instability appears which sets-in earlier for lower frequency modes.

Both tensor modes of the g- and f- metrics develop an instability since they are given by a mixture of the two mass eigenstates. To quantify the effects of the instability on the physical g sector, we need to go beyond this qualitative analysis and to explicitly solve the equations for perturbations in the various cosmological epochs.

In Appendix 4.12 we determine analytical solutions during a de Sitter like inflationary stage, during radiation domination, matter domination and during a late de Sitter phase in the limit where the couplings between h and ℓ can be neglected. We see that in this case, no instabilities occur. We also find that for an inflationary Hubble parameter $H_I \gg H_0$ the power spectrum of the ℓ-mode is severely suppressed with respect to the one of h for all modes of cosmological interest. Therefore, we expect that at high redshift, the evolution of the physical tensor mode h will be not affected by the coupling with ℓ. We explicitly verify this, by solving analytically the decoupled differential equations describing the evolution of gravitational waves in the radiation- and matter-dominated epochs and by estimating the amplitude of the coupling. During the radiation and matter era at sufficiently high redshift h evolves like in ΛCDM while ℓ has a constant and a decaying mode. Only at late times the coupling term in the equation for the physical tensor mode becomes relevant and deviation from the GR evolution can be expected.

The scale of inflation may lie above the strong coupling scale[5]. For massive gravity on a flat background this scale is $\Lambda_3 \equiv (m^2 M_{Pl})^{1/3}$ but for cosmological backgrounds it is not known neither for massive gravity nor for bigravity. For this reason the results for the inflationary power spectra may not be representative. In the following numerical treatment we shall not consider this point and we do not make any assumptions on the initial conditions but just vary them and study their effect.

4.5.2 Numerical results

In this section we report the results of a numerical integration of the tensor mode evolution equations. In particular we want to test the appearance of a tachyonic instability at late times for low-k modes. For this, we numerically evolve the system of differential equations (4.76), (4.77), starting from the redshift of matter-radiation equality. We focus on modes that become sub-horizon during the late matter-dominated era.

Since during inflation and radiation, the physical tensor mode h evolves like in GR, see Appendix 4.12, we choose GR-like initial conditions for h at the redshift of equality

$$h(z_{eq}) = 1\,, \qquad h'(z_{eq}) = 0\,. \tag{4.90}$$

The mode ℓ at the end of inflation is suppressed by a factor $(k/H_0)(H_0/H_I)^{1/2}$ with respect to h and during radiation ℓ has a constant and a decaying mode, which oscillates with a very high frequency. In our numerical study we choose generic initial condition for ℓ, parametrized by two constants (A, B) as

$$\ell(z_{eq}) = A\,, \qquad \ell'(z_{eq}) = \mathcal{H}_0\, B\,, \tag{4.91}$$

and we explore how the result of the numerical integration is affected changing the parameters A and B.

Fig. 4.2 shows the result of the numerical integration for different modes for the case $A = B = 1$. The evolution of the physical tensor mode h is plotted together with the corresponding result in ΛCDM (red, dotted line). A deviation from the ΛCDM evolution appears at late times and becomes more important for small values of $k/\mathcal{H}_0$. This is in line with what we find in Appendix 4.12: at late times the coupling terms in the equations for the tensor modes (4.167) and (4.168) grow like $\mathcal{H}^2$ and become relevant when they reach the order of magnitude of the k-term. Therefore, the effect of the coupling becomes relevant earlier for larger modes. In other words, at late times, modes with $k \simeq \mathcal{H}_0$ will experience a larger deviation from the ΛCDM evolution. This is visible in Fig. 4.3, where for different values of $k/\mathcal{H}_0$ the evolution of tensor perturbations is plotted into the future, until negative redshift. We see that the tensor

[5]The scale of self-interactions of the helicity-0 mode of the massive graviton, at which perturbativity breaks down.

sector exhibits an instability and the instant at which this instability shows up in the physical sector is pushed towards future time for smaller wave lengths, $k \gg \mathcal{H}_0$.

In Fig. 4.4 we show the evolution of tensor perturbations keeping the mode k fixed and we vary the initial conditions for the ℓ mode, A and B. For h we again choose for GR-like initial conditions $h(z_{eq}) = 1$ and $h'(z_{eq}) = 0$. The presence of the coupling of the ℓ mode to h affects the evolution of the physical mode only at late time and only if the ratio ℓ/h at the beginning of matter domination is not too small(i.e. A and $B \gtrsim 1$), see Figs. 4.4(a) and 4.4(c). The evolution of h at the beginning of matter domination is not affected by the coupling, even if initially $\ell/h \gg 1$. Figs. 4.4(f) and 4.4(h) show that when h enters the horizon and starts oscillating, this oscillation is transferred to the ℓ sector trough the coupling between the two modes [6].

4.6 Discussion and conclusion

In this work we have developed a new formalism that allowed us to write the general expression for the bi-gravity action, perturbed to second order in the metric perturbations around generic backgrounds. We explicitly give this mass term of the perturbations in terms of the arbitrary background metrics $\bar{g}$ and $\bar{f}$. Even though we use a special form (upper triangular) to derive the expressions, we finally give them in terms of the metric components valid in arbitrary coordinates.

We then apply our formalism to cosmology. We present the generic action of cosmological perturbation which can be expressed in terms of the energy densities and pressures, ρ_g, ρ_h and p_g, p_f and two additional functions σ_1 and σ_2 of the background variables r and c.

We finally study the evolution of tensor perturbations in the algebraic branch of bigravity cosmology. Analyzing the positivity condition of the mass matrix for the canonically normalized tensor modes, we find that at late times, the tensor sector is affected by a tachyonic instability, which sets in earlier for smaller wave numbers. This result has been checked both analytically, solving the equations for tensor perturbations in the various cosmological epochs, and numerically. We find that the evolution of tensor perturbations in the algebraic branch manifests a late time instability in both the $g-$ and the $f-$ sectors. The order of magnitude of the instability strongly depends on the ratio between the two tensor modes at the beginning of matter domination. In particular if ℓ is of the same order as h, this instability manifests itself at present time in the physical sector for k of the order of the Hubble parameter today. If instead initially ℓ is significantly suppressed with respect to h the initial time of the instability is pushed towards the future. This last situation $\ell(z_e) \ll h(z_e)$ is the one which

[6] In the cases examined in Figs. 4.4(f) and 4.4(h), the oscillation in the ℓ sector after horizon-crossing is visible since ℓ is initially suppressed.

appears in our analytic study of the dynamics of the tensor sector during de Sitter inflation. However, since inflation lies most probably in the strong coupling regime where the effective bimetric massive gravity theory studied in this paper is no longer valid, we have discussed the instability for arbitrary initial conditions. Furthermore, independent of the initial conditions, for every mode k, there is a redshift (in the past or in the future) at which this mode will become unstable.

Finally, some comments on our choice of the β_i parameters are in order. We have chosen the values of the β_i such that the constraints for the viability of the background evolution are satisfied (see section 4.5) and such that they reproduce the observed late time evolution of the Universe. The remaining choice of the values for the β_i does not affect the qualitative result found for the evolution of tensor perturbations and the appearance of a late-time instability. Indeed, in our analysis the physical quantity which is sensitive to a change in the parameters is the redshift at which a given mode becomes unstable, given by eq. (4.89) for $m^2\beta_i \simeq \mathcal{H}_0^2$. The redshift at which an instability shows up depends on the values of the β_i and only if we choose a special tuning, e.g., such that the right hand side of eq. (4.89) is vanishing, the tensor sector is stable on all scales and redshifts.

Of course, the analysis presented here is a linear perturbation analysis and it cannot determine what happens at higher order. Whether a positive h^4 term or couplings to scalar and vector perturbations re-install stability cannot be decided with our analysis.

Another point in our analysis where the values of the β_i enter is the expression for c at the end of inflation. Our estimate $c \simeq \sqrt{H_I/H_0} \gg 1$ is valid for $m^2\beta_i \simeq \mathcal{H}_0^2$. The fact that c is so large at the end of inflation determines the suppression of the power spectrum of the f tensor mode with respect to the one of the physical mode and it is crucial to push the beginning of the instability towards future times. This suppression can be reduced by tuning the value of $m^2\beta_i$ to lower the value of c at the end of inflation, e.g. $c \simeq 1$ for $m^2\beta_i \simeq H_I^2$. However, this choice of the β_i looks rather contrived. Indeed, if we take $m^2\beta_i \simeq H_I^2 \gg H_0^2$, in eq. (4.65) a very significant fine tuning is needed to obtain the correct dark energy density today.

The mass term computed in this paper can be used for all future application of massive bigravity for cosmology or to discuss perturbations on an arbitrary background solution. It can also be used to investigate whether tachyonic modes are present for a given solution, rendering the theory unstable.

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4.8 Appendix: Details of the mass term

In this appendix, we present the details of the computation of the mass term $\mathcal{M}^{\mu\nu\alpha\beta}(\bar{g}, \bar{f})$ and the final result given in section 4.3, written explicitly in terms of the background metrics.

In the main text the mass matrix elements are given in terms of first and second derivatives of the functions $t_i(g, f)$ in eqs. (4.27 – 4.33). Here we compute these derivatives.

Making use of eq. (4.13), we can express the first- and second-order perturbations of t_i in terms of those of s_i. The variables t_i and s_i are defined in eqs. (4.10) and (4.12), respectively. We first introduce the derivatives of s_i in the same way as those for the functions t_i given in eqs. (4.34) and (4.35),

$$s_{i,g}^{\mu\nu} = \left.\frac{\partial s_i}{\partial g_{\mu\nu}}\right|_{g=\bar{g}, f=\bar{f}}, \qquad s_{i,f}^{\mu\nu} = \left.\frac{\partial s_i}{\partial f_{\mu\nu}}\right|_{g=\bar{g}, f=\bar{f}}, \tag{4.92}$$

$$s_{i,gg}^{\mu\nu\alpha\beta} = \frac{1}{2}\left.\frac{\partial^2 s_i}{\partial g_{\mu\nu}\partial g_{\alpha\beta}}\right|_{g=\bar{g}, f=\bar{f}}, \quad s_{i,gf}^{\mu\nu\alpha\beta} = \left.\frac{\partial^2 s_i}{\partial g_{\mu\nu}\partial f_{\alpha\beta}}\right|_{g=\bar{g}, f=\bar{f}}, \quad s_{i,ff}^{\mu\nu\alpha\beta} = \frac{1}{2}\left.\frac{\partial^2 s_i}{\partial f_{\mu\nu}\partial f_{\alpha\beta}}\right|_{g=\bar{g}, f=\bar{f}}, \tag{4.93}$$

for $i \in \{1, 2, 3, 4\}$.

The first derivatives of t_i, which have been defined in eq. (4.34), can then be written as

$$\begin{aligned}
t_{1,\bullet}^{\mu\nu} &= A\left\{\bar{t}_4\left(\bar{t}_1\bar{t}_4 - \bar{t}_2\bar{t}_3\right)s_{1,\bullet}^{\mu\nu} - \bar{t}_3\bar{t}_4 s_{2,\bullet}^{\mu\nu} - \bar{t}_1\bar{t}_4 s_{3,\bullet}^{\mu\nu} + \left(\bar{t}_3 - \bar{t}_1\bar{t}_2\right)s_{4,\bullet}^{\mu\nu}\right\}, \\
t_{2,\bullet}^{\mu\nu} &= A\left\{-\bar{t}_3^2\bar{t}_4 s_{1,\bullet}^{\mu\nu} - \bar{t}_3\bar{t}_4\bar{t}_1 s_{2,\bullet}^{\mu\nu} - \bar{t}_4\bar{t}_1^2 s_{3,\bullet}^{\mu\nu} + \left(\bar{t}_3 - \bar{t}_1\bar{t}_2\right)\bar{t}_1 s_{4,\bullet}^{\mu\nu}\right\}, \\
t_{3,\bullet}^{\mu\nu} &= A\left\{-\bar{t}_3\bar{t}_4^2 s_{1,\bullet}^{\mu\nu} - \bar{t}_1\bar{t}_4^2 s_{2,\bullet}^{\mu\nu} + \left(\bar{t}_3 - \bar{t}_1\bar{t}_2\right)\bar{t}_4 s_{3,\bullet}^{\mu\nu} + \left(\bar{t}_2\bar{t}_3 + \bar{t}_1\left(\bar{t}_4 - \bar{t}_2^2\right)\right)s_{4,\bullet}^{\mu\nu}\right\}, \\
t_{4,\bullet}^{\mu\nu} &= \frac{s_{4,\bullet}^{\mu\nu}}{2\bar{t}_4},
\end{aligned} \tag{4.94}$$

where

$$A = \left[2\,\bar{t}_4\left(\bar{t}_3^2 + \bar{t}_1^2\bar{t}_4 - \bar{t}_1\bar{t}_2\bar{t}_3\right)\right]^{-1}. \tag{4.95}$$

Here $\bullet$ denotes g or f.

The mass term is well defined only if A is finite, i.e., $\bar{t}_3^2 + \bar{t}_1^2\bar{t}_4 - \bar{t}_1\bar{t}_2\bar{t}_3 \neq 0$ ($\bar{t}_4$, being the determinant of $\sqrt{\bar{g}^{-1}\bar{f}}$ is of course non-zero). If the eigenvalues λ_i of $\bar{g}^{-1}\bar{f}$ are all positive, it is easy to check that A is always negative[7]. This means that we have to require that the time directions of f and of g are sufficiently closely aligned such that

[7] To do so, we can use the expressions of the $\bar{t}_i$ functions in terms of the $\lambda_i^{1/2}$ given in Eqs. (4.10): it turns out that A is just the sum of negative terms.

$\bar{g}^{-1}\bar{f}$ is positive definite. This is also requested for our formalism to make sense, as otherwise the λ_i might not have real roots.

To obtain an explicit expression for the second derivatives of t_i in terms of these first derivatives and the second derivatives of s_i eq. (4.93), we have to derive the relation between s_i and t_i, eq. (4.13), a second time. A rather cumbersome but straightforward calculation leads finally to the following expressions:

$$t^{\mu\nu\alpha\beta}_{1,\bullet\bullet} = A\left\{\bar{t}_4\left(\bar{t}_1\bar{t}_4 - \bar{t}_2\bar{t}_3\right)S^{\mu\nu\alpha\beta}_{1,\bullet\bullet} - \bar{t}_3\bar{t}_4 S^{\mu\nu\alpha\beta}_{2,\bullet\bullet} - \bar{t}_1\bar{t}_4 S^{\mu\nu\alpha\beta}_{3,\bullet\bullet} + \left(\bar{t}_3 - \bar{t}_1\bar{t}_2\right)S^{\mu\nu\alpha\beta}_{4,\bullet\bullet}\right\}, \tag{4.96a}$$

$$t^{\mu\nu\alpha\beta}_{2,\bullet\bullet} = A\left\{-\bar{t}_3^2\bar{t}_4 S^{\mu\nu\alpha\beta}_{1,\bullet\bullet} - \bar{t}_1\bar{t}_3\bar{t}_4 S^{\mu\nu\alpha\beta}_{2,\bullet\bullet} - \bar{t}_1^2\bar{t}_4 S^{\mu\nu\alpha\beta}_{3,\bullet\bullet} + \bar{t}_1\left(\bar{t}_3 - \bar{t}_1\bar{t}_2\right)S^{\mu\nu\alpha\beta}_{4,\bullet\bullet}\right\}, \tag{4.96b}$$

$$t^{\mu\nu\alpha\beta}_{3,\bullet\bullet} = A\left\{-\bar{t}_3\bar{t}_4^2 S^{\mu\nu\alpha\beta}_{1,\bullet\bullet} - \bar{t}_1\bar{t}_4^2 S^{\mu\nu\alpha\beta}_{2,\bullet\bullet} + \left(\bar{t}_3 - \bar{t}_1\bar{t}_2\right)\bar{t}_4 S^{\mu\nu\alpha\beta}_{3,\bullet\bullet} + \left[\bar{t}_2\bar{t}_3 + \bar{t}_1\left(\bar{t}_4 - \bar{t}_2^2\right)\right]S^{\mu\nu\alpha\beta}_{4,\bullet\bullet}\right\}, \tag{4.96c}$$

$$t^{\mu\nu\alpha\beta}_{4,\bullet\bullet} = \frac{S^{\mu\nu\alpha\beta}_{4,\bullet\bullet}}{2\bar{t}_4}, \tag{4.96d}$$

where $\bullet\bullet$ denotes gg, ff or gf, and where we have introduced

$$S^{\mu\nu\alpha\beta}_{q,\bullet\bullet} \equiv s^{\mu\nu\alpha\beta}_{q,\bullet\bullet} - \xi\cdot\sum_{p\in\mathcal{I}}(-1)^{q+p}t^{\mu\nu}_{p,\bullet}t^{\alpha\beta}_{2q-p,\bullet}. \tag{4.97}$$

The above sum goes over $\mathcal{I}\subset\{1,2,3,4\}$ such that both, p and $2q-p$ are in $\{1,2,3,4\}$ and the parameter ξ is $\xi=1$ when $\bullet\bullet = gg$ or $\bullet\bullet = ff$ (since the symmetrization with respect to the exchange of the indices $(\mu,\nu)\leftrightarrow(\alpha,\beta)$ applies in this case), whereas $\xi = 2$ when $\bullet\bullet = gf$ (since the same symmetrization does not apply in this case).

With this we have expressed the derivatives of the variables t_i in terms of those of the s_i. Recalling the definition of the s_i, eq. (4.12) and the definition of the potential, eqs. (4.3)-(4.6), the derivatives of s_i can be obtained directly by expanding the matrix

$$g^{-1}f = \left(\bar{g}(1+h)\right)^{-1}\bar{f}(1+\ell) = (1-h+h^2)\bar{g}^{-1}\bar{f}(1+\ell) + \mathcal{O}(h^3), \tag{4.98}$$

where h denotes $(h^\mu{}_\nu) = (\bar{g}^{\mu\alpha}h_{\alpha\nu})$ and ℓ denotes $(\ell^\mu{}_\nu) = (\bar{f}^{\mu\alpha}\ell_{\alpha\nu})$.

We have also used the variations of the metric determinant, which appear in expressions (4.30-4.31):

$$\begin{aligned}&\frac{\partial^2\sqrt{-\det g}}{\partial g_{\mu\nu}\partial g_{\alpha\beta}}\Big|_{g=\bar{g},f=\bar{f}}\\ &\equiv\left(\sqrt{-\det g}\right)^{\mu\nu\alpha\beta}_{,gg} = \frac{1}{4}\sqrt{-\det g}\left[\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta} - \left(\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta} + \bar{g}^{\mu\beta}\bar{g}^{\nu\alpha}\right)\right],\end{aligned} \tag{4.99}$$

$$\frac{\partial\sqrt{-\det g}}{\partial g_{\mu\nu}}\Big|_{g=\bar g,f=\bar f} \equiv \left(\sqrt{-\det g}\right)_{,g}^{\mu\nu} = \frac{1}{2}\sqrt{-\det g}\, g^{\mu\nu}\,. \tag{4.100}$$

To shorten the notation in what follows, we define $\bar\Sigma^\mu{}_\nu \equiv \bar g^{\mu\rho}\bar f_{\rho\nu}$, and also $\bar\Sigma^{\mu\nu} \equiv \bar\Sigma^\mu{}_\rho \bar g^{\rho\nu}$, $\left(\bar\Sigma^2\right)^{\mu\nu} \equiv \bar\Sigma^\mu{}_{\rho_1}\bar\Sigma^{\rho_1}{}_{\rho_2}\bar g^{\rho_2\nu}$, and generalizing $\left(\bar\Sigma^k\right)^{\mu\nu} \equiv \bar\Sigma^\mu{}_{\rho_1}\bar\Sigma^{\rho_1}{}_{\rho_2}\dots\bar\Sigma^{\rho_{k-1}}{}_{\rho_k}\bar g^{\rho_k\nu}$. As usual, square brackets denote the trace, e.g. $[\bar\Sigma] \equiv \mathrm{Tr}\bar\Sigma = \bar\Sigma^\mu{}_\mu$.

A direct evaluation of s_i and their first and second derivatives leads to[8]

$$\bar s_1 = [\bar\Sigma]\,, \tag{4.101a}$$
$$s_{1,g}^{\mu\nu} = -\bar\Sigma^{\mu\nu}\,, \tag{4.101b}$$
$$s_{1,f}^{\mu\nu} = \bar g^{\mu\nu}\,, \tag{4.101c}$$
$$\begin{aligned} s_{1,gg}^{\mu\nu\alpha\beta} &= \frac{1}{8}\left\{\left[\bar g^{\alpha\nu}\bar\Sigma^{\mu\beta} + (\mu\leftrightarrow\nu) + (\alpha\leftrightarrow\beta) + (\mu\leftrightarrow\nu)(\alpha\leftrightarrow\beta)\right] + [\cdots]\left((\mu,\nu)\leftrightarrow(\alpha,\beta)\right)\right\} \\ &\equiv \mathrm{Sym}\left\{\bar g^{\alpha\nu}\bar\Sigma^{\mu\beta}\right\} \end{aligned} \tag{4.101d}$$
$$s_{1,gf}^{\mu\nu\alpha\beta} = -\frac{1}{2}\left(\bar g^{\mu\alpha}\bar g^{\nu\beta} + \bar g^{\mu\beta}\bar g^{\nu\alpha}\right)\,, \tag{4.101e}$$
$$s_{1,ff}^{\mu\nu\alpha\beta} = 0\,, \tag{4.101f}$$

$$\bar s_2 = \frac{1}{2}\left(\left[\bar\Sigma\right]^2 - \left[\bar\Sigma^2\right]\right)\,, \tag{4.101g}$$
$$s_{2,g}^{\mu\nu} = \left(\bar\Sigma^2\right)^{\mu\nu} - \left[\bar\Sigma\right]\bar\Sigma^{\mu\nu}\,, \tag{4.101h}$$
$$s_{2,f}^{\mu\nu} = \bar g^{\mu\nu}\left[\bar\Sigma\right] - \bar\Sigma^{\mu\nu}\,, \tag{4.101i}$$
$$s_{2,gg}^{\mu\nu\alpha\beta} = \mathrm{Sym}\left\{\bar g^{\mu\alpha}\left(\left[\bar\Sigma\right]\bar\Sigma^{\beta\nu} - \left(\bar\Sigma^2\right)^{\beta\nu}\right) + \frac{1}{2}\bar\Sigma^{\mu\nu}\bar\Sigma^{\alpha\beta} - \frac{1}{2}\bar\Sigma^{\alpha\mu}\bar\Sigma^{\beta\nu}\right\}\,, \tag{4.101j}$$
$$\begin{aligned} s_{2,gf}^{\mu\nu\alpha\beta} &= \frac{1}{4}\left[\bar g^{\beta\mu}\bar\Sigma^{\alpha\nu} + \bar g^{\alpha\nu}\bar\Sigma^{\beta\mu} - \bar g^{\alpha\beta}\bar\Sigma^{\mu\nu} - \bar g^{\alpha\mu}\bar g^{\beta\nu}\left[\bar\Sigma\right] + (\mu\leftrightarrow\nu) + (\alpha\leftrightarrow\beta) + (\mu\leftrightarrow\nu)(\alpha\leftrightarrow\beta)\right] \\ &\equiv \mathrm{sym}\left\{\bar g^{\beta\mu}\bar\Sigma^{\alpha\nu} + \bar g^{\alpha\nu}\bar\Sigma^{\beta\mu} - \bar g^{\alpha\beta}\bar\Sigma^{\mu\nu} - \bar g^{\alpha\mu}\bar g^{\beta\nu}\left[\bar\Sigma\right]\right\}\,, \end{aligned} \tag{4.101k}$$
$$s_{2,ff}^{\mu\nu\alpha\beta} = \frac{1}{2}\bar g^{\alpha\beta}\bar g^{\mu\nu} - \frac{1}{4}\left(\bar g^{\alpha\nu}\bar g^{\beta\mu} + \bar g^{\alpha\mu}\bar g^{\beta\nu}\right)\,, \tag{4.101l}$$

[8] We use the well known identity $\frac{\partial F}{\partial g_{\mu\nu}} = -g^{\mu\alpha}g^{\nu\beta}\frac{\partial F}{\partial g^{\alpha\beta}}$, valid for an arbitrary function $F(g^{-1})$.

$$\bar{s}_3 = \frac{1}{6}\left(\left[\bar{\Sigma}\right]^3 - 3\left[\bar{\Sigma}\right]\left[\bar{\Sigma}^2\right] + 2\left[\bar{\Sigma}^3\right]\right), \tag{4.101m}$$

$$s^{\mu\nu}_{3,g} = \left[\bar{\Sigma}\right]\left(\bar{\Sigma}^2\right)^{\mu\nu} - \left(\bar{\Sigma}^3\right)^{\mu\nu} + \frac{1}{2}\bar{\Sigma}^{\mu\nu}\left(\left[\bar{\Sigma}^2\right] - \left[\bar{\Sigma}\right]^2\right), \tag{4.101n}$$

$$s^{\mu\nu}_{3,f} = \left(\bar{\Sigma}^2\right)^{\mu\nu} - \bar{\Sigma}^{\mu\nu}\left[\bar{\Sigma}\right] + \frac{1}{2}\bar{g}^{\mu\nu}\left(\left[\bar{\Sigma}\right]^2 - \left[\bar{\Sigma}^2\right]\right), \tag{4.101o}$$

$$\begin{aligned} s^{\mu\nu\alpha\beta}_{3,gg} = \mathrm{Sym}\Big\{&\bar{\Sigma}^{\mu\alpha}\left(\bar{\Sigma}^2\right)^{\beta\nu} - \left(\bar{\Sigma}^2\right)^{\mu\nu}\bar{\Sigma}^{\alpha\beta} + \frac{1}{2}\left[\bar{\Sigma}\right]\left(\bar{\Sigma}^{\mu\nu}\bar{\Sigma}^{\alpha\beta} - \bar{\Sigma}^{\mu\alpha}\bar{\Sigma}^{\nu\beta}\right) \\ &+ \bar{g}^{\nu\alpha}\left(\left(\bar{\Sigma}^3\right)^{\mu\beta} - \left(\bar{\Sigma}^2\right)^{\mu\beta}\left[\bar{\Sigma}\right]\right) + \frac{1}{2}\bar{g}^{\mu\alpha}\bar{\Sigma}^{\nu\beta}\left(\left[\bar{\Sigma}\right]^2 - \left[\bar{\Sigma}^2\right]\right)\Big\}, \end{aligned} \tag{4.101p}$$

$$\begin{aligned} s^{\mu\nu\alpha\beta}_{3,gf} = \mathrm{sym}\Big\{&2\bar{g}^{\nu\beta}\left(\bar{\Sigma}^{\mu\alpha}\left[\bar{\Sigma}\right] - \left(\bar{\Sigma}^2\right)^{\mu\alpha}\right) + \bar{g}^{\alpha\beta}\left(\left(\bar{\Sigma}^2\right)^{\mu\nu} - \bar{\Sigma}^{\mu\nu}[\bar{\Sigma}]\right) + \bar{\Sigma}^{\mu\nu}\bar{\Sigma}^{\alpha\beta} - \bar{\Sigma}^{\mu\beta}\bar{\Sigma}^{\alpha\nu} \\ &+ \frac{1}{2}\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\left(\left[\bar{\Sigma}^2\right] - \left[\bar{\Sigma}\right]^2\right)\Big\}, \end{aligned} \tag{4.101q}$$

$$s^{\mu\nu\alpha\beta}_{3,ff} = \mathrm{Sym}\left\{\frac{1}{2}\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta}\left[\bar{\Sigma}\right] - \frac{1}{2}\bar{g}^{\mu\beta}\bar{g}^{\nu\alpha}\left[\bar{\Sigma}\right] + \bar{g}^{\mu\alpha}\bar{\Sigma}^{\nu\beta} - \bar{g}^{\alpha\beta}\bar{\Sigma}^{\mu\nu}\right\},$$

$$\bar{s}_4 = \det\left(\bar{g}^{-1}\bar{f}\right), \tag{4.101r}$$

$$s^{\mu\nu}_{4,g} = -\bar{s}_4\bar{g}^{\mu\nu}, \tag{4.101s}$$

$$s^{\mu\nu}_{4,f} = \bar{s}_4\bar{f}^{\mu\nu}, \tag{4.101t}$$

$$s^{\mu\nu\alpha\beta}_{4,gg} = \frac{\bar{s}_4}{2}\left(\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta} + \frac{1}{2}\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta} + \frac{1}{2}\bar{g}^{\nu\alpha}\bar{g}^{\mu\beta}\right), \tag{4.101u}$$

$$s^{\mu\nu\alpha\beta}_{4,gf} = -\bar{s}_4\bar{g}^{\mu\nu}\bar{f}^{\alpha\beta}, \tag{4.101v}$$

$$s^{\mu\nu\alpha\beta}_{4,ff} = \frac{\bar{s}_4}{2}\left(\bar{f}^{\mu\nu}\bar{f}^{\alpha\beta} - \frac{1}{2}\bar{f}^{\mu\alpha}\bar{f}^{\nu\beta} - \frac{1}{2}\bar{f}^{\mu\beta}\bar{f}^{\nu\alpha}\right). \tag{4.101w}$$

The operator Sym$\{\cdots\}$ indicates symmetrization in $(\mu \leftrightarrow \nu)$, $(\alpha \leftrightarrow \beta)$, $(\mu \leftrightarrow \nu)(\alpha \leftrightarrow \beta)$ and $(\mu,\nu) \leftrightarrow (\alpha,\beta)$, whereas the operator sym$\{\cdots\}$ indicates symmetrization in $(\mu \leftrightarrow \nu)$, $(\alpha \leftrightarrow \beta)$ and $(\mu \leftrightarrow \nu)(\alpha \leftrightarrow \beta)$ but not $(\mu,\nu) \leftrightarrow (\alpha,\beta)$.

These are the expressions for the derivatives of the functions s_i which enter the expressions for the derivatives of the t_i, eqs. (4.96), which in turn enter in the expression for $\mathcal{M}^{\mu\nu\alpha\beta}$, eqs. (4.27)-(4.33). Not surprisingly, the expressions for the second derivatives of s_2 and s_3 are lengthy. The $s^{\mu\nu}_{i,g}$ and the $s^{\mu\nu\alpha\beta}_{i,gg}$ have already been computed in Ref. [77] and the $s^{\mu\nu}_{i,f}$ and the $s^{\mu\nu\alpha\beta}_{i,ff}$ are less complicated. The terms $s^{\mu\nu\alpha\beta}_{i,gf}$ are quite cumbersome and they are new.

4.9 Appendix: The computation of $\bar{\mathcal{M}}^{\mu\nu}$

In this Appendix we present more details about the computation of the mass term $\bar{\mathcal{M}}^{\mu\nu} \equiv \mathcal{M}^{\mu\nu}(f,\bar{g})$ defined in Eq. (3.16). We have

$$\bar{\mathcal{M}}_g^{\mu\nu} \equiv \left\{\frac{1}{\sqrt{-\det g}}\frac{\delta(\sqrt{-\det g}U(g,f))}{\delta g_{\mu\nu}}\right\}\Bigg|_{g=\bar{g}} = \frac{1}{2}U(f,\bar{g})\bar{g}^{\mu\nu} - 2m^2(\beta_1 t_{1,h}^{\mu\nu} + \beta_2 t_{2,h}^{\mu\nu} + \beta_3 t_{3,h}^{\mu\nu} + \beta_4 t_{4,h}^{\mu\nu}), \tag{4.102}$$

where we have used $\frac{\delta\sqrt{-\det g}}{\delta g_{\mu\nu}} = \frac{1}{2}\sqrt{-\det g}g^{\mu\nu}$ and, from Eq. (4.11),

$$\frac{\delta U(f,g)}{\delta g_{\mu\nu}}\Bigg|_{g=\bar{g}} = -2m^2\left(\frac{\delta(\beta_1 t_1 + \beta_2 t_2 + \beta_3 t_3 + \beta_4 t_4)}{\delta g_{\mu\nu}}\right)\Bigg|_{g=\bar{g}} = -2m^2(\beta_1 t_{1,h}^{\mu\nu} + \beta_2 t_{2,h}^{\mu\nu} + \beta_3 t_{3,h}^{\mu\nu} + \beta_4 t_{4,h}^{\mu\nu}). \tag{4.103}$$

We also have

$$\bar{\mathcal{M}}_f^{\mu\nu} \equiv \left\{\frac{1}{\sqrt{-\det f}}\frac{\delta(\sqrt{-\det g}U(g,f))}{\delta f_{\mu\nu}}\right\}\Bigg|_{f=\bar{f}} = -\frac{\sqrt{-\det g}}{\sqrt{-\det f}}2m^2(\beta_1 t_{1,\ell}^{\mu\nu} + \beta_2 t_{2,\ell}^{\mu\nu} + \beta_3 t_{3,\ell}^{\mu\nu} + \beta_4 t_{4,\ell}^{\mu\nu}), \tag{4.104}$$

where we have used

$$\frac{\delta U(f,g)}{\delta f_{\mu\nu}}\Bigg|_{f=\bar{f}} = -2m^2\left(\frac{\delta(\beta_1 t_1 + \beta_2 t_2 + \beta_3 t_3 + \beta_4 t_4)}{\delta f_{\mu\nu}}\right)\Bigg|_{f=\bar{f}} = -2m^2(\beta_1 t_{1,\ell}^{\mu\nu} + \beta_2 t_{2,\ell}^{\mu\nu} + \beta_3 t_{3,\ell}^{\mu\nu} + \beta_4 t_{4,\ell}^{\mu\nu}). \tag{4.105}$$

The quantities $t_j^{\mu\nu}$ can be written in terms of $s_j^{\mu\nu}$, see Eqs. (4.94) which are given in Eqs. (4.101).

4.10 Appendix: The equations of motion for the cosmological perturbations

Here we present the derivation of the equations of motion for the perturbations at first order based on the second-order perturbed part of the action. In complete generality, these equations of motion have the form

$$\delta G^{\mu}_{g\,\nu} + \delta\mathcal{M}^{\mu}_{g\,\nu} = 8\pi G\delta T^{\mu}{}_{\nu} \tag{4.106}$$

for $h_{\mu\nu}$, and

$$\delta G^{\mu}_{f\,\nu} + \delta\mathcal{M}^{\mu}_{f\,\nu} = 0 \tag{4.107}$$

for $\ell_{\mu\nu}$. Here $\delta G^{\mu}_{g\,\nu}$, $\delta G^{\mu}_{f\,\nu}$, $\delta\mathcal{M}^{\mu}_{g\,\nu}$, $\delta\mathcal{M}^{\mu}_{f\,\nu}$ and $\delta T^{\mu}{}_{\nu}$ stand for the first-order perturbation of the Einstein tensor $G^{\mu}_{g\,\nu}$ for the metric $g_{\mu\nu}$, he first-order perturbation of the Einstein tensor $G_{f\ \nu}^{\mu}$ for the metric $f_{\mu\nu}$, the first-order perturbation of the mass term in $h_{\mu\nu}$ and in $\ell_{\mu\nu}$ and the first-order perturbation of the energy-momentum tensor, respectively. The perturbations $\delta G_{g\ \nu}^{\mu}$ (and equivalently $\delta G_{f\ \nu}^{\mu}$) and $\delta T^{\mu}{}_{\nu}$ can be found in the literature (see e.g. [122, 103, 59]). For the mass term we have

$$\delta \mathcal{M}_{g\ \nu}^{\mu} = \delta(\mathcal{M}_g^{\mu\rho} g_{\rho\nu}) = \delta\mathcal{M}_g^{\mu\rho}\bar{g}_{\rho\nu} + \bar{\mathcal{M}}_g^{\mu\rho}\delta g_{\rho\nu}, \tag{4.108}$$

where

$$\delta\mathcal{M}_g^{\mu\rho} = \frac{\delta}{\delta h_{\mu\rho}}(\mathcal{M}_{gg}^{\tau\sigma\alpha\beta} h_{\tau\sigma}h_{\alpha\beta} + \mathcal{M}_{gf}^{\tau\sigma\alpha\beta} h_{\tau\sigma}\ell_{\alpha\beta}) = 2\,\mathcal{M}_{gg}^{\mu\rho\alpha\beta} h_{\alpha\beta} + \mathcal{M}_{gf}^{\mu\rho\alpha\beta}\ell_{\alpha\beta}\,, \tag{4.109}$$

and

$$\delta \mathcal{M}_{f\ \nu}^{\mu} = \delta(\mathcal{M}_f^{\mu\rho} f_{\rho\nu}) = \delta\mathcal{M}_f^{\mu\rho}\bar{f}_{\rho\nu} + \bar{\mathcal{M}}_f^{\mu\rho}\delta f_{\rho\nu}, \tag{4.110}$$

where

$$\delta\mathcal{M}_f^{\mu\rho} = \frac{\delta}{\delta \ell_{\mu\rho}}(\mathcal{M}_{ff}^{\tau\sigma\alpha\beta} \ell_{\tau\sigma}\ell_{\alpha\beta} + \mathcal{M}_{gf}^{\tau\sigma\alpha\beta} h_{\tau\sigma}\ell_{\alpha\beta}) = 2\,\mathcal{M}_{ff}^{\mu\rho\alpha\beta} \ell_{\alpha\beta} + \mathcal{M}_{gf}^{\mu\rho\alpha\beta} h_{\alpha\beta}\,. \tag{4.111}$$

4.11 Appendix: Parametrization of the cosmological mass term

We give here the explicit expressions for the functions which parametrize the mass tensor on cosmological backgrounds, as presented in Section 4.4:

$$
\begin{aligned}
\alpha_g &= -\frac{1}{2}\left(\beta_0+\beta_3 r^3+3\beta_2 r^2+3\beta_1 r\right), && (4.112)\\
\gamma_g &= -\frac{1}{2}\left(\beta_0+\beta_2 r^2+2\beta_1 r\right), && (4.113)\\
\epsilon_g &= \frac{1}{2}\left(\beta_0+\frac{(3c+2)r}{c+1}\beta_1+\beta_2\frac{(3c+1)r^2}{c+1}+\frac{\beta_3 c r^3}{c+1}\right), && (4.114)\\
\eta_g &= \frac{1}{2}\left(\beta_0+\beta_2 c r^2+\beta_1(c+1)r\right), && (4.115)\\
\sigma_g &= -\frac{1}{2}\left(2\beta_0+\beta_3 c r^3+\beta_2(3c+1)r^2+\beta_1(2c+3)r\right), && (4.116)
\end{aligned}
$$

$$
\begin{aligned}
\alpha_f &= -\frac{1}{2c^3}\left(\beta_4+3\beta_3 r+3\beta_2 r^{-2}+\beta_1 r^{-3}\right), && (4.117)\\
\gamma_f &= -\frac{1}{2c}\left(\beta_4+2\beta_3 r^{-1}+\beta_2 r^{-2}\right), && (4.118)\\
\epsilon_f &= \frac{1}{2c}\left(\beta_4+\frac{\beta_3(2c+3)r^{-1}}{(c+1)}+\frac{\beta_2(c+3)r^{-2}}{(c+1)}+\frac{\beta_1 r^{-3}}{(c+1)}\right), && (4.119)\\
\eta_f &= \frac{1}{2}\left(\beta_4 c+\beta_3(c+1)r^{-1}+\beta_2 r^{-2}\right), && (4.120)\\
\sigma_f &= -\frac{1}{2}\left(2\beta_4 c+\beta_3(3c+2)r^{-1}+\beta_2(c+3)r^{-2}+\beta_1 r^{-3}\right), && (4.121)
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{gf} &= 0, && (4.122)\\
\gamma_{gf} &= -\frac{1}{r}\left(\beta_1+2\beta_2 r+\beta_3 r^2\right), && (4.123)\\
\gamma_{fg} &= -\frac{1}{rc}\left(\beta_1+2\beta_2 r+\beta_3 r^2\right), && (4.124)\\
\epsilon_{gf} &= \frac{1}{(1+c)r}\left(\beta_1+2\beta_2 r+\beta_3 r^2\right), && (4.125)\\
\eta_{gf} &= \frac{1}{r}\left(\beta_1+\beta_3 c r^2+\beta_2(c+1)r\right), && (4.126)\\
\sigma_{gf} &= -\frac{1}{r}\left(\beta_1+\beta_3 c r^2+\beta_2(c+1)r\right). && (4.127)
\end{aligned}
$$

Note that for $c = 1$ an arbitrary coefficient μ_f can be obtained from the corresponding μ_h by replacing $\beta_i \to \beta_{4-i}$ and $r \to r^{-1}$ as a consequence of the symmetry (4.36). As c has been set to 1 for the metric $\bar{g}$ the behaviour with c is less evident. Setting $c = 1$ gives the general mass term for conformally related metrics g and f. In this case, in the algebraic branch all coupling terms μ_{gf} vanish and the mass matrix completely decouples. For conformally related metrics the algebraic branch reduces to two independent copies of Einstein gravity.

We can conveniently express all parametrization functions in terms of ρ_g, ρ_f, p_g, p_f introduced in eqs. (4.42-4.45) and two functions $\sigma_1 \equiv \beta_1 + 2\beta_2 r + \beta_3 r^2$ and $\sigma_2 \equiv \beta_1 + \beta_2(c+1)r + \beta_3 c r^2$ as follows:

$$\alpha_g = -\frac{8\pi G\rho_g}{2m^2}\,, \quad \epsilon_g = -\frac{r\sigma_1}{2(c+1)} + \frac{8\pi G\rho_g}{2m^2}, \quad \eta_g = -\frac{r\sigma_2}{2} - \frac{8\pi G p_g}{2m^2}\,, \tag{4.128}$$

$$\sigma_g = \frac{r\sigma_2}{2} + \frac{8\pi G p_g}{m^2}\,, \quad \gamma_g = \frac{r\sigma_1}{2} - \frac{\rho_g}{2m^2}\,, \tag{4.129}$$

$$\alpha_f = -\frac{8\pi G\rho_f}{2m^2c^3}\,, \quad \epsilon_f = -\frac{\sigma_1}{2(c+1)r^3} + \frac{8\pi G\rho_f}{2m^2c}, \quad \eta_f = -\frac{\sigma_2}{2r^3} - \frac{8\pi G p_f}{2m^2}\,, \tag{4.130}$$

$$\sigma_f = \frac{\sigma_2}{2r^3} + \frac{8\pi G p_f}{m^2}\,, \quad \gamma_f = \frac{\sigma_1}{2cr^3} - \frac{8\pi G\rho_f}{2m^2c}\,, \tag{4.131}$$

$$\alpha_{gf} = 0\,, \quad \epsilon_{gf} = \frac{\sigma_1}{r(1+c)}, \quad \eta_{gf} = \frac{\sigma_2}{r}\,, \tag{4.132}$$

$$\sigma_{gf} = -\frac{\sigma_2}{r}\,, \quad \gamma_{gf} = c\gamma_{fg} = -\frac{\sigma_1}{r}\,. \tag{4.133}$$

Once the Friedmann equations (4.42-4.45) are used to replace $\rho_\bullet$ and $p_\bullet$ by $\mathcal{H}_\bullet$ and $\mathcal{H}'_\bullet$, all these terms cancel with contributions from the in the kinetic part of the action and σ_1 and σ_2 are the only functions of the β_i which remain[9]. The final action for scalars, vectors and tensors after this simplification is given in [41] for a specific gauge choice.

4.12 Appendix: Analytic results for the evolution of tensor perturbations

In this appendix we collect the results obtained by solving analytically the equations of tensor perturbations (4.76) and (4.77) in the early de Sitter, radiation- , matter and late de Sitter-dominated epochs.

4.12.1 Analytic results during de Sitter inflation

We start by embedding the model in inflation. We consider a model of inflation with a single scalar filed coupled to the g-metric, with potential $V(\phi) = M_\phi^2\phi^2/2$. Deep in the inflationary era the inflaton is slowly rolling. Since $p_\phi = -\rho_\phi$, it is legitimate to model

[9] The functions σ_1 and σ_2 correspond respectively to f_2 and f_1 in [31].

this period as a de Sitter phase with constant Hubble parameter $H \simeq H_I = const.$ It follows that

$$c \simeq \sqrt{\frac{\rho_\phi}{\Lambda_c}} \approx \frac{H_I}{H_0}\,. \tag{4.134}$$

With this and $(a'/a)^2 = \frac{1}{2}a''/a = 1/\tau^2$, the equations of motion for the canonically normalized variables become

$$Q_h'' - \frac{2}{\tau^2}Q_h + k^2 Q_h + \frac{1}{\tau^2}\left(\frac{H_0}{H_I}\right)\left(Q_h - \sqrt{\frac{H_I}{H_0}}Q_\ell\right) = 0\,, \tag{4.135}$$

$$Q_\ell'' - \frac{2}{\tau^2}Q_\ell + \left(\frac{H_I}{H_0}\right)^2 k^2 Q_\ell - \frac{1}{\tau^2}\sqrt{\frac{H_0}{H_I}}\left(Q_h - \sqrt{\frac{H_I}{H_0}}Q_\ell\right) = 0\,. \tag{4.136}$$

Taking into account that $H_I \gg H_0$, these equations decouple and they can be approximated as

$$Q_h'' - \frac{2}{\tau^2}Q_h + k^2 Q_h = 0\,, \tag{4.137}$$

$$Q_\ell'' - \frac{1}{\tau^2}Q_\ell + \left(\frac{H_I}{H_0}\right)^2 k^2 Q_\ell = 0\,. \tag{4.138}$$

Eq. (4.137) is equal to the one for the canonically normalized tensor mode in GR. Requiring that for $|k\tau| \gg 1$ we recover the quantum vacuum solution

$$Q_h = \frac{1}{\sqrt{2k}}e^{-ik\tau}\,, \tag{4.139}$$

we find the standard result

$$Q_h = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right)e^{-ik\tau}\,. \tag{4.140}$$

On super-Hubble scales, $|\tau k| < 1$, the canonical variable grows, $Q_h \propto -1/\tau \propto a$ so that $h \propto a^{-1}Q_h$ remains constant.

In eq. (4.138) the mass term $-1/\tau^2$ becomes relevant only for

$$\frac{1}{\tau^2} = H_I^2 a^2 = \frac{H_I^2}{(1+z)^2} > k^2\frac{H_I^2}{H_0^2}\,, \qquad \text{hence for} \quad k \lesssim \frac{H_0}{(1+z)} < \frac{H_0}{(1+z_e)}\,, \tag{4.141}$$

where z_e is the redshift at the end of inflation. This corresponds to a huge scale irrelevant for cosmological observations. On presently measurable scales, $k \gtrsim H_0$, eq. (4.138) can be approximated as an harmonic oscillator equation, with solution

$$Q_\ell = \frac{1}{\sqrt{2ck}}e^{-ikc\tau}\,, \quad c \approx \frac{H_I}{H_0}\,. \tag{4.142}$$

As c is very large, this solution oscillates rapidly with small amplitude.

To conclude this analysis, we determine the power spectra for super horizon modes $|k\tau| \ll 1$ at the end of inflation. We find

$$P_h(k) = 2 \cdot \frac{k^3|Q_h|^2}{a^2M_P^2} \simeq \left(\frac{H_I}{M_P}\right)^2 , \tag{4.143}$$

$$P_\ell(k) = 2 \cdot c \cdot \frac{k^3|Q_\ell|^2}{a^2M_P^2} \simeq \left(\frac{H_I}{M_P}\right)^2 \left(\frac{k}{H_0}\right)^2 \left(\frac{H_0}{H_I}\right)^2 (1+z_e)^2 \simeq P_h(k) \left(\frac{k}{H_0}\right)^2 \left(\frac{H_0}{H_I}\right) , \tag{4.144}$$

where we have approximated at the end of inflation $(1+z_e)^2 \simeq H_I/H_0$.

The power spectrum of the ℓ-mode is highly suppressed for all modes of cosmological interest. Therefore, we expect that at very high redshift, the evolution of the physical tensor mode h will not be affected by the coupling with ℓ.

However, typical inflation scales are expected to lie above the strong coupling scale and so the results for the inflationary power spectra may not be relevant.

4.12.2 Analytic results in radiation-dominated epoch

In the radiation dominated era $a \propto \tau$, $\mathcal{H} = 1/\tau$ and

$$c \simeq \sqrt{\frac{\rho_r}{\Lambda_c}} \approx \sqrt{\frac{3\mathcal{H}_0^2\,\Omega_{r0}}{m^2\left(\beta_4\bar{r}^2 + 2\bar{r}\beta_3 + \beta_2\right)}}\,\frac{1}{a^2} \equiv \frac{\bar{c}_r}{a^2} , \qquad \frac{c'}{c} = -2\mathcal{H} , \tag{4.145}$$

where $\bar{c}_r \sim \sqrt{3H_0^2\Omega_{r0}/\Lambda_c} \simeq 0.12$ for our parameter choice. The two equations for the tensor modes (4.76) and (4.77) can now be written as

$$h'' + \frac{2}{\tau}h' + k^2 h + \mathcal{R}_r\,(h - \ell) = 0 , \tag{4.146}$$

$$\ell'' + \frac{4}{\tau}\ell' + \frac{\mathcal{K}_r^2}{a^4}\ell - \mathcal{S}_r\,(h - \ell) = 0 , \tag{4.147}$$

where

$$\mathcal{K}_r^2 = \bar{c}_r^2\,k^2 , \tag{4.148}$$

$$\mathcal{R}_r = m^2a^2C + m^2D\,\bar{c}_r \approx m^2D\,\bar{c}_r \simeq -0.2H_0^2 , \tag{4.149}$$

$$\mathcal{S}_r = \frac{m^2C}{\bar{r}^2}\,\bar{c}_r + \frac{m^2D}{\bar{r}^2}\,\frac{\bar{c}_r^2}{a^2} \approx \frac{m^2D}{\bar{r}^2}\,\frac{\bar{c}_r^2}{a^2} \equiv \frac{\bar{\mathcal{S}}_r}{a^2} \simeq -0.014H_0^2/a^2 . \tag{4.150}$$

Here we have used that $a = 1/(1+z) \ll 1$ with our normaization of $a_0 = 1$.

If we consider super-Hubble modes and we neglect the k-term in eq. (4.146), also the term proportional to $\mathcal{R}_r\,h$ has to be neglected since $k^2\,h \geq |\mathcal{R}_r h| \approx 0.2\mathcal{H}_0^2\,h$. Therefore, recalling that at the end of inflation $h > \ell$, eq. (4.146) is decoupled and given by

$$h'' + \frac{2}{\tau}h' = 0 , \tag{4.151}$$

with solution

$$h = d_1 + d_2 \frac{\tau_0}{\tau} \,. \tag{4.152}$$

In eq. (4.147), the coupling is negligible if

$$\left(\frac{\bar{\mathcal{S}}_r}{a^2} h\right) / \left(\frac{\mathcal{K}_r^2}{a^4} \ell\right) \simeq \frac{10}{(1+z)^2} \left(\frac{H_0}{k}\right)^2 \left(\frac{h}{\ell}\right) \approx \frac{10}{(1+z)^2} \frac{H_0^2}{kH_I} \ll 1 \,. \tag{4.153}$$

In the last inequality we have used the inflationary result for h/ℓ. This is certainly satisfied early on but may be violated for large scales at late time, when $z \to 3000$. If we set $h/\ell \sim 1$ it is even satisfied until equality. As long as eq. (4.153) is satisfied also eq. (4.147) is decoupled and for $\ell \ll h$, it can be approximated as

$$\ell'' + \frac{4}{\tau} \ell' + \frac{T^2}{\tau^4} \ell = 0 \,, \tag{4.154}$$

where $T \equiv \bar{c}_r \,(k/\mathcal{H}_0)\,(\Omega_{r0}\mathcal{H}_0)^{-1} \simeq 3 \times 10^3 k/H_0^2$. The analytic solution is given by

$$\ell = c_1 \left(\frac{T}{\tau} + i\right) e^{i\frac{T}{\tau}} + c_2 \left(\frac{T}{\tau} - i\right) e^{-i\frac{T}{\tau}} \simeq c_1 \frac{T}{\tau} e^{i\frac{T}{\tau}} + c_2 \frac{T}{\tau} e^{-i\frac{T}{\tau}} \,. \tag{4.155}$$

Summarizing, we found that in radiation for super-horizon scales, h evolves as in GR while ℓ has a constant and a decaying mode and it oscillates with the phase $3 \times 10^3 k/(\tau H_0^2)$.

4.12.3 Analytic results in matter-dominated epoch

In the matter dominated era $a \propto \tau^2$, $\mathcal{H} = 2/\tau$ and

$$c \simeq \sqrt{\frac{\rho_m}{\Lambda_c}} \approx \sqrt{\frac{3\mathcal{H}_0^2\,\Omega_{m0}}{m^2\,(\beta_4 \bar{r}^2 + 2\bar{r}\beta_3 + \beta_2)}} \frac{1}{a^{3/2}} \equiv \frac{\bar{c}_m}{a^{3/2}} \,, \qquad \frac{c'}{c} = -\frac{3}{2}\mathcal{H} \,. \tag{4.156}$$

With our choice of parameters $\bar{c}_m \simeq 10$. The two equations for the tensor modes (4.76) and (4.77) can be written as

$$h'' + \frac{4}{\tau} h' + k^2 h + \mathcal{R}_m \,(h - \ell) = 0 \,, \tag{4.157}$$

$$\ell'' + \frac{7}{\tau} \ell' + \frac{\mathcal{K}_m^2}{a^3} \ell - \mathcal{S}_m \,(h - \ell) = 0 \,, \tag{4.158}$$

where

$$\mathcal{K}_m^2 = \bar{c}_m^2 \, k^2 \,, \tag{4.159}$$

$$\mathcal{R}_m = m^2 a^2 C + m^2 D \,\bar{c}_m \, a^{1/2} \approx m^2 D \,\bar{c}_m \, a^{1/2} \simeq -19 H_0^2 a^{1/2} \,, \tag{4.160}$$

$$\mathcal{S}_m = \frac{m^2 C}{\bar{r}^2}\,\bar{c}_m\,a^{1/2} + \frac{m^2 D}{\bar{r}^2}\,\frac{\bar{c}_m^2}{a} \approx \frac{m^2 D}{\bar{r}^2}\,\frac{\bar{c}_m^2}{a} \equiv \frac{\bar{\mathcal{S}}_m}{a} \simeq -\frac{95 H_0^2}{a}\,. \tag{4.161}$$

In eq. (4.157), $\mathcal{R}_m\,h \simeq 19 H_0^2\,a^{1/2}\,h \simeq (\mathcal{H}_0/\mathcal{H})\;\mathcal{H}_0^2\,h$. As long as this is smaller than $k^2\,h$, and the term $\mathcal{R}_m\,h$ is subdominant, since we expect $\ell \le h$, eq. (4.157) reduces to

$$h'' + \frac{4}{\tau}\,h' + k^2 h = 0\,. \tag{4.162}$$

This is the same equation that we find in GR during matter domination. On super-horizon scales, it has the following solution

$$h = d_1 + \frac{d_2}{\tau^3}\,. \tag{4.163}$$

At late time, when $a^{1/2} \sim 1$, the coupling becomes significant and this approximation no longer holds on large scales. In eq. (4.158), the coupling is negligible as long as

$$\left(\frac{\bar{\mathcal{S}}_m}{a}\,h\right) / \left(\frac{\mathcal{K}_m^2}{a^3}\,\ell\right) \simeq \frac{10 H_0^2}{k^2}\,\frac{1}{(1+z)^2}\left(\frac{h}{\ell}\right) \ll 1\,. \tag{4.164}$$

This condition is satisfied for $h/\ell \ll (1+z)^2$. If we neglect the coupling, eq. (4.158) becomes

$$\ell'' + \frac{7}{\tau}\,\ell' + \frac{T^4}{\tau^6}\,\ell = 0\,, \tag{4.165}$$

where $T^2 = 8\,\bar{c}_m\,\frac{k}{\mathcal{H}_0}\,\mathcal{H}_0^{-2}\,\Omega_{m0}^{-3/2} \simeq 490 k/H_0^3$. This equation is decoupled and can be solved analytically. The solution is given by

$$\begin{aligned}\ell &= \sqrt{4 + \frac{T^4}{\tau^4}}\left[c_1\,e^{-i\arctan\left(\frac{T^2}{2\tau^2}\right)} e^{\frac{i}{2}\left(\frac{T}{\tau}\right)^2} + c_2\,e^{i\arctan\left(\frac{T^2}{2\tau^2}\right)}\,e^{-\frac{i}{2}\left(\frac{T}{\tau}\right)^2}\right] \\ &\simeq \sqrt{4 + \frac{T^4}{\tau^4}}\left[c_1\,e^{\frac{i}{2}\left(\frac{T}{\tau}\right)^2} + c_2\,e^{-\frac{i}{2}\left(\frac{T}{\tau}\right)^2}\right].\end{aligned}$$

Summarizing, we find that also during matter domination, as long as couplings can be neglected, h evolves as in GR while ℓ has a constant and a decaying mode and it oscillates rapidly. At late times, especially for the large modes with $k \sim H_0$ coupling can in general not be neglected and the system has to be solved numerically.

4.12.4 Analytic results in the late de Sitter phase

Let us finally find approximate solutions for eqs. (4.76) and (4.77) valid in the late de Sitter phase, where $\Lambda_{\rm eff} \gg 8\pi G\rho \to 0$. In this phase $a \simeq -(\mathcal{H}_0\,\tau)^{-1}$, $\mathcal{H} \simeq a\,\mathcal{H}_0 \simeq -1/\tau$ and

$$c \simeq \sqrt{\frac{\Lambda_{\rm eff}}{\Lambda_c}} \approx 15\,, \qquad \bar{r}\sigma_2(\beta_i, c) \simeq \text{const.} \simeq -30\,. \tag{4.166}$$

Therefore, the two equations for the tensor modes can be written as

$$h'' - \frac{2}{\tau} h' + k^2 h + \frac{\mathcal{R}_\Lambda}{\tau^2} (h - \ell) = 0 \,, \tag{4.167}$$

$$\ell'' - \frac{2}{\tau} \ell' + c^2 k^2 \ell - \frac{\mathcal{R}_\Lambda}{\tau^2} \frac{c}{\bar{r}^2} (h - \ell) = 0 \,, \tag{4.168}$$

where

$$\mathcal{R}_\Lambda = \frac{m^2}{\mathcal{H}_0^2} \bar{r} \sigma_2 \simeq -30 \,. \tag{4.169}$$

For super-Hubble scales, $k^2 \ll 1/\tau^2$, eqs. (4.167) and (4.168) can be diagonalized in terms of $H_+ \equiv h + \ell$ and $H_- \equiv h - \ell$.

$$H_+'' - \frac{2}{\tau} H_+' = 0 \,, \tag{4.170}$$

$$H_-'' - \frac{2}{\tau} H_-' + \frac{\mathcal{R}_-}{\tau^2} H_- = 0 \,, \tag{4.171}$$

where $\mathcal{R}_- \equiv \left(1 + \frac{c}{r^2}\right) \mathcal{R}_\Lambda \simeq -255$. These equations are solved by

$$H_+ = c_1 + c_2 \tau^3 \,, \tag{4.172}$$

$$H_- = d_1 \tau^{\frac{1}{2}(3 - \sqrt{9 - 8\mathcal{R}_-})} + d_2 \tau^{\frac{1}{2}(3 + \sqrt{9 - 8\mathcal{R}_-})} \,. \tag{4.173}$$

The physical tensor modes h and ℓ can be expressed as linear combinations of H_+ and H_-

$$h = \frac{1}{2} (H_+ + H_-) = a_1 + a_2 \tau^3 + a_3 \tau^{\frac{1}{2}(3 - \sqrt{9 - 8\mathcal{R}_-})} + a_4 \tau^{\frac{1}{2}(3 + \sqrt{9 - 8\mathcal{R}_-})} \,, \tag{4.174}$$

$$\ell = \frac{1}{2} (H_+ - H_-) = a_1 + a_2 \tau^3 - a_3 \tau^{\frac{1}{2}(3 - \sqrt{9 - 8\mathcal{R}_-})} - a_4 \tau^{\frac{1}{2}(3 + \sqrt{9 - 8\mathcal{R}_-})} \,. \tag{4.175}$$

Since $\mathcal{R}_- \ll 0$ the a_3-mode is growing while all other modes are decaying in time (we recall that in the parametrization chosen, conformal time $\tau \in [-\infty, 0]$).

For sub-Hubble scales, we expect the couplings in eqs. (4.167) and (4.168) to be suppressed with respect to the k-terms. If this is the case, eqs. (4.167) and (4.168) decouple and are the same as in GR

$$h'' - \frac{2}{\tau} h' + k^2 h = 0 \,, \tag{4.176}$$

$$\ell'' - \frac{2}{\tau} \ell' + c^2 k^2 \ell = 0 \,, \tag{4.177}$$

with solutions, $|k\tau| \gg 1$,

$$h = c_1 \tau \left(\sin(k\tau) + \frac{\cos(k\tau)}{k\tau} \right) + c_2 \tau \left(\frac{\sin(k\tau)}{k\tau} - \cos(k\tau) \right) \,, \tag{4.178}$$

$$\ell = d_1\,\tau\left(\sin\left(kc\tau\right) + \frac{\cos\left(kc\tau\right)}{kc\,\tau}\right) + d_2\,\tau\left(\frac{\sin\left(kc\tau\right)}{kc\,\tau} - \cos\left(kc\tau\right)\right). \tag{4.179}$$

As $|\tau|$ decreases the coupling terms in eqs. (4.167) and (4.168) can become relevant, when $\mathcal{R}_\Lambda/\tau^2 \simeq m^2\,\sigma_2 \bar{r}\,\left(\frac{\mathcal{H}}{\mathcal{H}_0}\right)^2 \gg \mathcal{H}_0^2 \simeq k^2$. More precisely

$$\left(\frac{|\mathcal{R}_\Lambda|}{\tau^2}\,\ell\right)/\left(k^2\,h\right) \simeq 30\left(\frac{\mathcal{H}}{k}\right)^2\left(\frac{\ell}{h}\right). \tag{4.180}$$

Hence, once the initial conditions for the two tensor perturbations are fixed, modes with smaller k will experience the effects of the coupling earlier. On the other hand, for a given mode k, the effects of the coupling at late times will be proportional to the ratio (ℓ/h). This result is in line with what we have found in section 4.12.1 by examining the positivity of the mass matrix for the canonically normalized tensor modes. At the end of inflation, the amplitude of the tensor mode ℓ is suppressed by a factor $\left(H_0/H_I\right)^{1/2}$ with respect to the one of h. In the following epochs, h evolves like in ΛCDM while ℓ has a constant and a decaying mode during the radiation and matter era. We therefore expect that the ratio ℓ/h during radiation and matter will stay almost constant and, if the inflationary result is to be trusted, of the order of $\left(H_0/H_I\right)^{1/2}$.

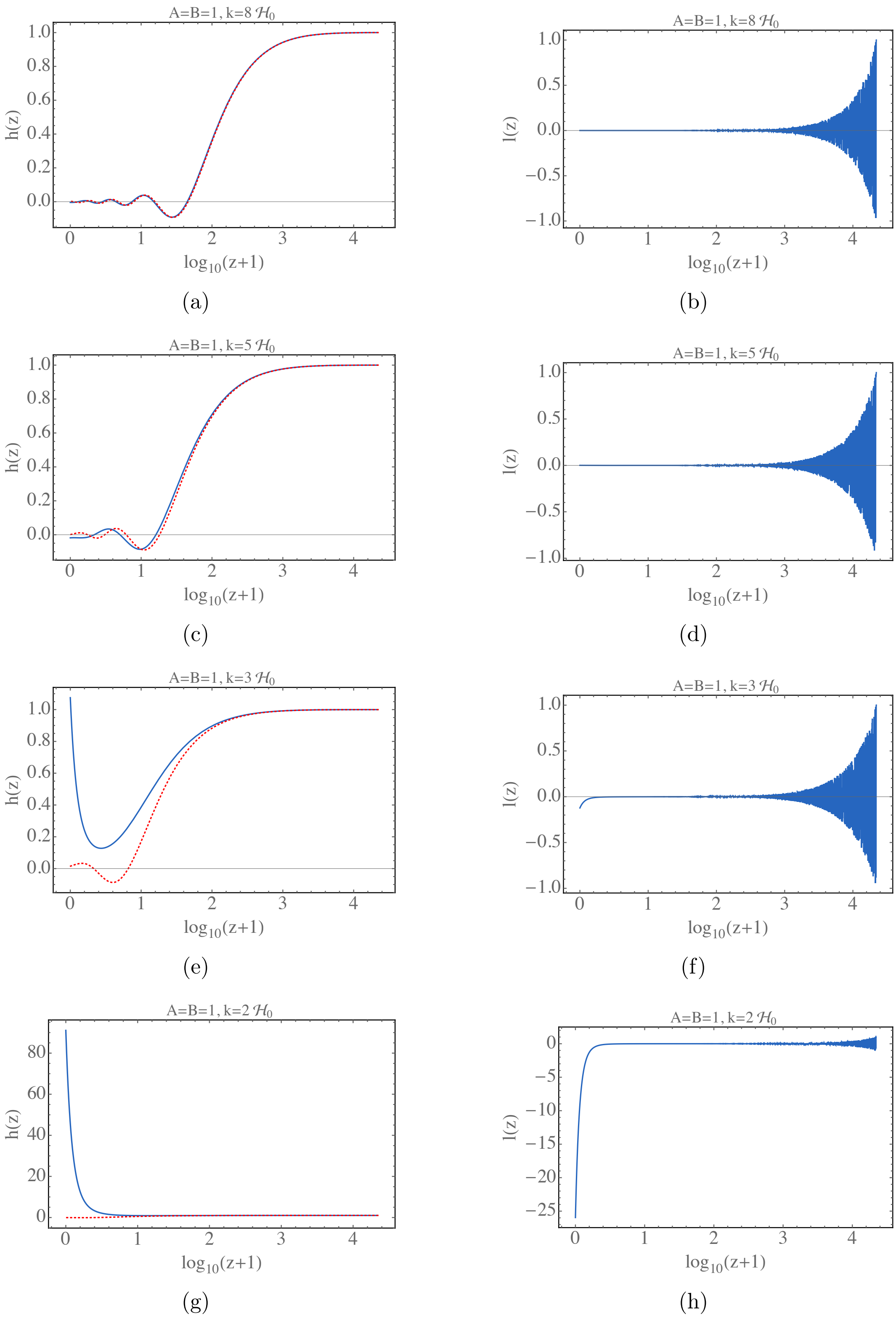


Figure 4.2: Tensor perturbations for GR-like type initial conditions in the physical sector, $h(z_{eq}) = 1\,,\ h'(z_{eq}) = 0$ and $\ell(z_{eq}) = A\,,\ \ell'(z_{eq}) = H_0\,B$. The evolution of tensor perturbations in the physical sector is plotted together with the one of ΛCDM with the same initial conditions (red, dotted line).

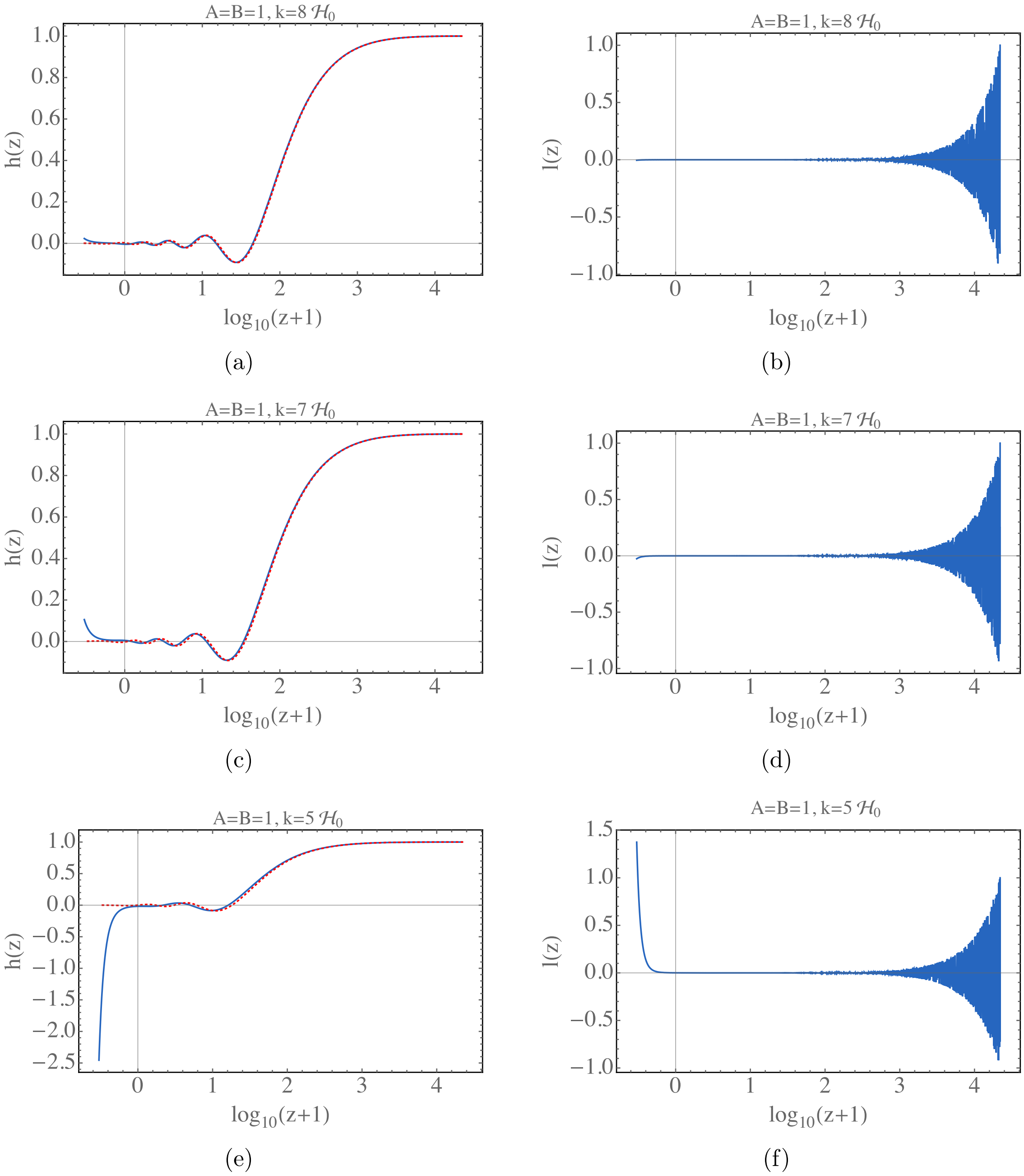


Figure 4.3: Tensor perturbations for GR-like type initial conditions in the physical sector, $h(z_{eq}) = 1\,,\ h'(z_{eq}) = 0$ and $\ell(z_{eq}) = A\,,\ \ell'(z_{eq}) = H_0\,B$. The evolution of tensor perturbations in the physical sector is plotted together with the one of ΛCDM with the same initial conditions (red, dotted line). The system is evolved into the future to show the appearance of an instability for $k \simeq \mathcal{H}$.

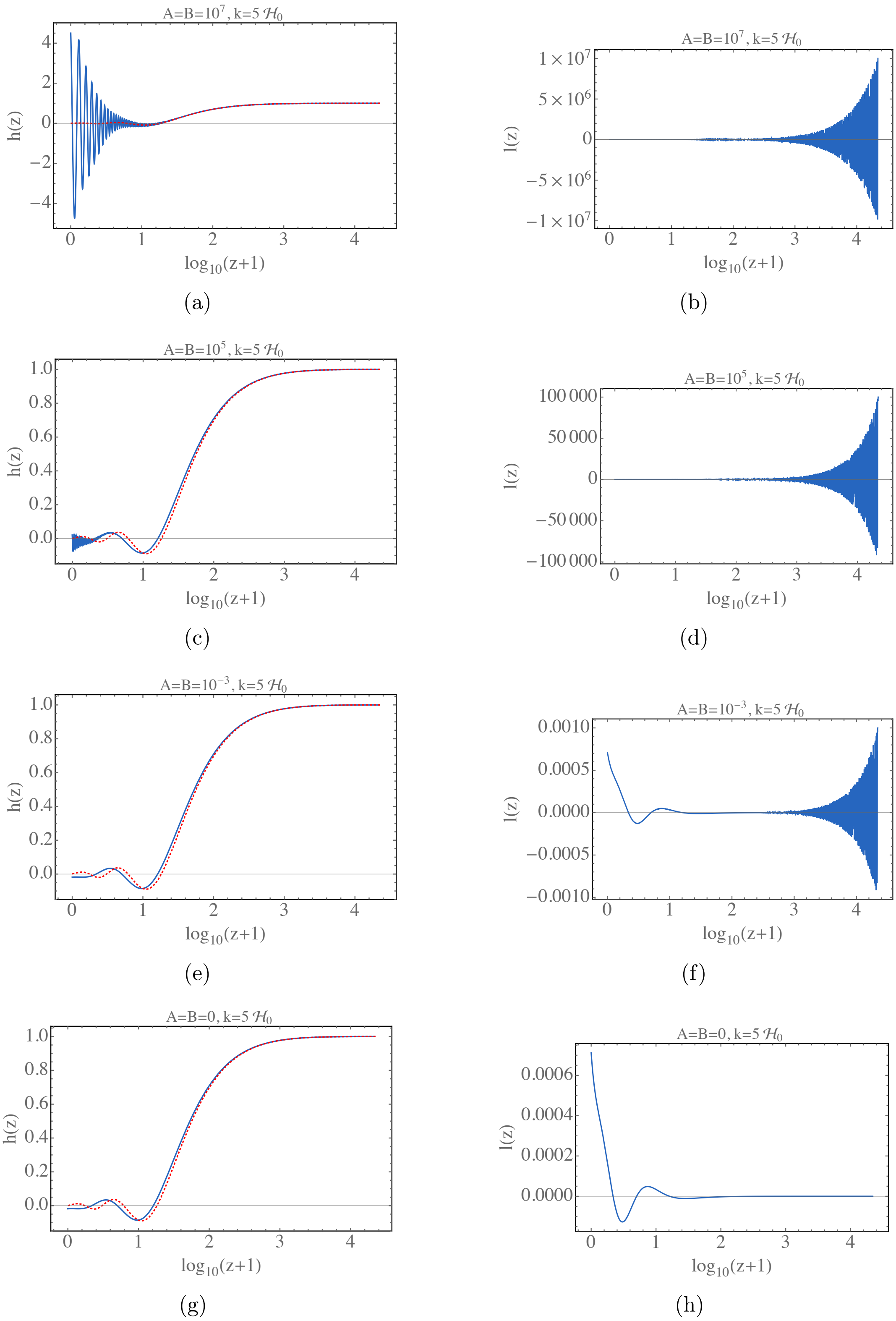


Figure 4.4: Tensor perturbations for initial conditions $h(z_{eq}) = 1$, $h'(z_{eq}) = 0$ and $\ell(z_{eq}) = A$, $\ell'(z_{eq}) = H_0 B$. The evolution of tensor perturbations in the physical sector is plotted together with the ΛCDM one with the same initial conditions (red, dotted line). The result of the numerical integration is shown for various choices of $A \simeq B$. Note that even for $A = B = 0$, the coupling generates an ℓ-mode at late time, which has a (small) effect on the physical gravitational wave amplitude $h(z)$.

Part II

Bigravity cosmology

Chapter 5

Gravitational waves in bigravity cosmology

Based on:

[34] G. Cusin, R. Durrer, P. Guarato and M. Motta, "Gravitational waves in bigravity cosmology", JCAP **1505** (2015) 030 [arXiv:1412.5979 [astro-ph]].

Abstract. In this paper we study gravitational wave perturbations in a cosmological setting of bigravity which can reproduce the ΛCDM background and large scale structure. We show that in general gravitational wave perturbations are unstable and only for very fine tuned initial conditions such a cosmology is viable. We quantify this fine tuning. We argue that similar fine tuning is also required in the scalar sector in order to prevent the tensor instability to be induced by second order scalar perturbations. Finally, we show that due to this power law instability, models of bigravity can lead to a large tensor to scalar ratio even for low scale inflation.

5.1 Introduction

The problem of dark energy is one of the fundamental problems not only of cosmology but of theoretical physics. How can vacuum energy, or equivalently a cosmological constant, be as small as the measured density of dark energy? (Fine tuning problem.) And why should it be of the same order of magnitude as the present mean density of matter in the Universe? (Coincidence problem.) These questions have led cosmologists to search for alternative explanations of the accelerated expansion of the Universe.

The possibility of giving the graviton a mass has attracted considerable attention. Such a mass leads to 'degravitation' which can solve the cosmological constant problem [64, 53, 44]. If the graviton is massive, the range of gravity is finite and a cosmological constant does not gravitate. Furthermore, if one fine tunes the graviton mass to $m_g \sim H_0$, where $H_0 \simeq h \times 2.1 \times 10^{-42}$ GeV is the value of the Hubble constant and $h = H_0/(100\text{km/sec/Mpc})$, gravity weakens around this scale, which might lead to accelerated expansion.

Adding a mass term to gravity is a non-trivial problem. It removes diffeomorphism invariance and hence the metric has six degrees of freedom (four being absorbed by the Bianchi identities). Five of these are the massive graviton while the sixth is usually a ghost, the so-called Boulware-Deser ghost [26]. To remove this ghost one has to make sure to obtain an additional constraint. This was shown to be possible with a very specific form of the potential for the gravitational field, the dRGT (de Rham, Gabadadze, Tolley) potential [47, 49, 85], which has been the basis for a large amount of work on this topic (see, e.g., [82, 86, 108, 77] and refs. therein). Applications to cosmology, however, have shown that it is not possible to find homogeneous and isotropic solutions of massive gravity which resemble our Universe. In massive gravity, a fixed reference metric has to be chosen with respect to which the mass term is defined. The possible solutions of course strongly depend on this reference metric, but even when choosing the reference metric to be Friedmann, the resulting solutions either do not show the well known cosmological behavior going from a radiation dominated to a matter dominated Universe followed by a late dark energy dominated phase, or they are unstable [79, 111, 69], see [45] for a review and [33, 46] for the study of the cosmology in the context of the so-called *generalized* massive gravity models.

Apart from these problems, it is somewhat disappointing to introduce the reference metric as an 'absolute element', i.e., a non-dynamical field in the theory. From this point of view, bimetric (or more general multi-metric) theories, where also the reference metric is dynamical are better motivated [83, 84, 80]. Interesting discussions about theoretical aspects of bimetric massive gravity can be found in [13, 88, 51, 37, 124, 11].

It has been shown that bigravity theories can by stable and well behaved [70], and that cosmological solutions of bimetric theories can actually fit the expansion history of the accelerating Universe [139, 30, 106, 136]. Observational tests of various models of bigravity are discussed in [132, 143, 21, 12, 107]. The study of the cosmology of models of bigravity where matter is coupled to a combination of the two metrics is addressed in a series of recent papers [78, 29, 67]. The analysis of cosmological perturbations have been studied in different settings and different models of bigravity in [31, 32, 42]. Recently, scalar perturbations of these models have been investigated and it has been shown that there exists a sub-class of models of bigravity that admit solutions with well behaved scalar perturbations of the physical metric, while other sub-classes of models are unstable [105].

In this paper we want to study the behavior of tensor perturbations, i.e., gravitational waves in bimetric gravity theories which fit the expansion history of the Universe and which lead to physically acceptable scalar perturbations, i.e., scalar perturbations which just exhibit the usual Jeans instability of Newtonian gravity which leads to cosmological structure formation but no significant additional instability. A more generic study of instabilities in bimetric theories can be found in Ref. [109].

Examining cosmological tensor perturbations, we show that gravitational waves are unstable in this theory. We study this instability in detail. We find that it changes the spectrum and strongly boosts the amplitude of gravitational waves. In order to prevent conflict with observations we have to fine tune the initial conditions for tensor perturbations.

While we were working on this, a preliminary study of gravitational waves in this model has appeared [110]. In Ref. [110] the authors investigate all perturbation modes, scalar, vector and tensor for two cosmological solutions, namely the expanding branch and the bouncing branch, called *infinite-branch bigravity* (IBB) in Ref. [105]. This latter has cosmologically acceptable scalar perturbations and is therefore of particular interest.

Our analysis goes beyond the results presented in [110]. We numerically solve the gravitational wave equations and study the resulting gravitational wave spectrum for different initial conditions for the perturbations. We finally argue that even if we would initially set gravitational wave perturbations to zero, non-linearities which induce small tensor perturbations are sufficient to trigger the instability and the model can only be saved if we also fine tune the initial conditions of scalar perturbations.

The rest of the paper is organised as follows. In the next section we write the Lagrangian and the equations of motion of bimetric gravity in general and for cosmological (i.e. homogeneous and isotropic) spacetimes. We then specialise to the physically viable model which gives an acceptable expansion history. In Section 5.3 we comment on the scalar sector of perturbations and in Section 5.4 we study tensor perturbations and compute the gravitational wave spectrum for different initial conditions. In Section 5.5 we discuss our results and conclude.

Notation: We set $c = \hbar = k_{\mathrm{Boltzmann}} = 1$. $M_g = 1/\sqrt{8\pi G} \equiv M_p \simeq 2.4 \times 10^{18}\mathrm{GeV}$ is the reduced Planck mass.

5.2 Cosmological solutions of massive bigravity

5.2.1 The Lagrangian

We start from a massive bigravity theory defined by the action

$$S = -\int d^4x\sqrt{-\det g}\left[\frac{M_g^2}{2}(R(g) - 2m^2V(g,f)) + \mathcal{L}_m(g,\Phi)\right] - \int d^4x\sqrt{-\det f}\,\frac{M_f^2}{2}R(f)\,. \tag{5.1}$$

Here f and g are the two metrics while M_f and $M_g = 1/\sqrt{8\pi G} \equiv M_p$ are the respective Planck masses with dimensionless ratio $m_* = M_f/M_g$. We assume the matter fields Φ to be coupled to g only. We use the notation of [143]. The potential is a function of the tensor field $X = \sqrt{g^{-1}f}$ given by

$$V(g,f) = \sum_{n=0}^{4} \beta_n e_n(X)\,, \tag{5.2}$$

where the polynomials $e_i(X)$ are

$$\begin{aligned}
e_0 &= \mathbb{1}, \quad e_1 = [X], && (5.3)\\
e_2 &= \frac{1}{2}([X]^2 - [X^2]), && (5.4)\\
e_3 &= \frac{1}{6}([X]^3 - 3[X][X^2] + 2[X^3], && (5.5)\\
e_4 &= \frac{1}{24}([X]^4 - 6[X]^2[X^2] + 8[X][X^3] + 3[X^2]^2 - 6[X^4]) = \det X\,. && (5.6)
\end{aligned}$$

The square bracket $[\cdots]$ denotes the trace. The equations of motion of this theory are

$$\begin{aligned}
\frac{1}{M_g^2} T_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{m^2}{2} \sum_{n=0}^{3} (-1)^n \beta_n \left[g_{\mu\lambda} Y^{(g)\lambda}_{(n)\nu} + g_{\nu\lambda} Y^{(g)\lambda}_{(n)\mu} \right], && (5.7)\\
0 &= R^f_{\mu\nu} - \frac{1}{2} f_{\mu\nu} R^f + \frac{m^2}{2m_*^2} \sum_{n=0}^{3} (-1)^n \beta_{4-n} \left[f_{\mu\lambda} Y^{(f)\lambda}_{(n)\nu} + f_{\nu\lambda} Y^{(f)\lambda}_{(n)\mu} \right], && (5.8)
\end{aligned}$$

where the superscript f indicates the curvature of the metric $f_{\mu\nu}$. The definition of the matrices $Y^{(g)\nu}_{(n)\mu} = Y^{\nu}_{(n)\mu}\left(\sqrt{g^{-1}f}\right)$ and $Y^{(f)\nu}_{(n)\mu} = Y^{\nu}_{(n)\mu}\left(\sqrt{f^{-1}g}\right)$ is as follows

$$\begin{aligned}
Y_{(0)}(X) &= \mathbb{1}\,, \quad Y_{(1)}(X) = X - \mathbb{1}[X]\,,\\
Y_{(2)}(X) &= X^2 - X[X] + \frac{1}{2}\mathbb{1}\left([X]^2 - [X^2]\right),\\
Y_{(3)}(X) &= X^3 - X^2[X] + \frac{1}{2}X\left([X]^2 - [X^2]\right) - \frac{1}{6}\mathbb{1}\left([X]^3 - 3[X][X^2] + 2[X^3]\right).
\end{aligned}$$

As a consequence of the Bianchi identities and of the covariant conservation of $T_{\mu\nu}$, we obtain the following Bianchi constraints for each of the two metrics

$$\nabla^g_\mu \sum_{n=0}^{3} (-)^n \beta_n \left[Y^{(g)\nu\mu}_{(n)} + Y^{(g)\mu\nu}_{(n)} \right] = 0\,, \tag{5.9}$$

$$\nabla^f_\mu \sum_{n=0}^{3} (-)^n \beta_{4-n} \left[Y^{(f)\nu\mu}_{(n)} + Y^{(f)\mu\nu}_{(n)} \right] = 0\,, \tag{5.10}$$

where we raise and lower indices of $Y^{(g)\cdots}_{(n)\cdots}$ and $Y^{(f)\cdots}_{(n)\cdots}$ with the metrics g and f respectively and the relevant metric is indicated in the covariant derivatives ∇^g and ∇^f. Both these constraints follow from the invariance of the interaction term under the diagonal subgroup of the general coordinate transformations of both metrics. Both constraints are equivalent and we will use only the first one.

5.2.2 Cosmological equations of motion

We now consider solutions which are spatially homogeneous and isotropic. For simplicity we neglect spatial curvature. If curvature is included, it is easy to see that it has to be the same for both metrics. The metrics can be written in the form

$$g_{\mu\nu}dx^\mu dx^\nu = a^2(\tau)\left(-d\tau^2 + \delta_{ij}dx^i dx^j\right), \tag{5.11}$$

$$f_{\mu\nu}dx^\mu dx^\nu = b^2(\tau)\left(-c^2(\tau)d\tau^2 + \delta_{ij}dx^i dx^j\right), \tag{5.12}$$

where τ is conformal time for the g-metric and $c(\tau)$ is a lapse function which parametrizes the difference between the conformal time τ_f for the f-metric and τ, $d\tau_f = c(\tau)d\tau$.

It is convenient to define the conformal Hubble parameter ($\mathcal{H}$) and the standard one (H) for the two metrics

$$H = \frac{\mathcal{H}}{a} = \frac{a'}{a^2}, \qquad H_f = \frac{\mathcal{H}_f}{b} = \frac{b'}{b^2\, c}, \tag{5.13}$$

where with $'$ we denote the derivative with respect to the conformal time τ. We also introduce the ratio between the two scale factors

$$r = \frac{b}{a}. \tag{5.14}$$

For symmetry reasons, the energy-momentum tensor has the form of a perfect fluid with equation of state $p = w\rho$. Explicitly

$$\begin{aligned} T_{\mu\nu} &= (p+\rho)\, u_\mu u_\nu + p\, g_{\mu\nu}\,, && (5.15)\\ \rho' &= -3(\rho+p)\,\mathcal{H}\,, && (5.16)\\ p &= w\rho\,. && (5.17) \end{aligned}$$

Introducing a 'gravity fluid' which represents the mass term in the Einstein equations, we define

$$\begin{aligned} \rho_g &= \frac{m^2}{8\pi G}\left(\beta_3\, r^3 + 3\beta_2\, r^2 + 3\beta_1\, r + \beta_0\right), && (5.18)\\ p_g &= -\frac{m^2}{8\pi G}\Big(\beta_3 c\, r^3 + \beta_2(2c+1)r^2 + \beta_1(c+2)r + \beta_0\Big). && (5.19) \end{aligned}$$

Note that the gravity fluid becomes like a cosmological constant when $c = 1$.

The Bianchi constraint in the cosmological ansatz can then be written as

$$\rho_g' = -3\mathcal{H}\left(\rho_g + p_g\right), \tag{5.20}$$

This constraint is equivalent to

$$m^2\left(\beta_3 r^2 + 2\beta_2 r + \beta_1\right)\left(c\, b\, a' - a\, b'\right) = 0\,, \qquad \text{or} \qquad m^2\left(\beta_3 r^2 + 2\beta_2 r + \beta_1\right)\left(\mathcal{H} - \mathcal{H}_f\right) = 0\,. \tag{5.21}$$

The full set of equations of motion is given by the time-time and the space-space components of the modified Einstein equations. For the metric g they are

$$3H^2 = 8\pi G\,(\rho + \rho_g)\,, \tag{5.22}$$

$$3H^2 + 2\dot{H} = -8\pi G\,(p + p_g)\,. \tag{5.23}$$

The modified Friedmann equations for the f metric become

$$3H_f^2 = 8\pi G\,\rho_f\,, \quad \text{with} \;\; \rho_f = \frac{m^2}{8\pi G M_*^2}\left(\beta_4 + 3\beta_3 r^{-1} + 3\beta_2 r^{-2} + \beta_1 r^{-3}\right)\,, \tag{5.24}$$

$$\begin{aligned} &3cH_f^2 + 2\dot{H}_f = -8\pi G\,p_f\,, \;\; \text{with} \\ &p_f = -\frac{m^2}{8\pi G M_*^2}\left(c\beta_4 + \beta_3(2c+1)r^{-1} + \beta_2(c+2)r^{-2} + \beta_1 r^{-3}\right)\,. \end{aligned} \tag{5.25}$$

We consider the first Friedmann equation for both metrics, the Bianchi constraint, and in the matter sector, the 'energy conservation' equation and the equation of state as independent equations which determine the 5 functions $a(\tau)$, $b(\tau)$, $c(\tau)$, $\rho(\tau)$ and $p(\tau)$,

$$H^2 = \frac{8\pi G}{3}\,(\rho + \rho_g)\,, \tag{5.26}$$

$$H_f^2 = \frac{m^2}{3\,m_*^2}\left(\frac{\beta_1}{r^3} + \frac{3\beta_2}{r^2} + \frac{3\beta_3}{r} + \beta_4\right)\,, \tag{5.27}$$

$$\left(\beta_3\,r^2 + 2\beta_2\,r + \beta_1\right)(\mathcal{H} - \mathcal{H}_f) = 0\,, \tag{5.28}$$

$$\rho' = -3(\rho + p)\,\mathcal{H}\,, \qquad p = w\rho\,. \tag{5.29}$$

Eq. (5.28) allows for two branches of solutions. Either $r = \bar{r}$ is constant given by the solution of the quadratic equation

$$\beta_3\bar{r}^2 + 2\beta_2\bar{r} + \beta_1 = 0\,, \tag{5.30}$$

or the second factor of eq. (5.28) vanishes implying

$$\mathcal{H}_f = \mathcal{H}\,, \qquad \text{or} \qquad r H_f = H\,. \tag{5.31}$$

According to [143, 42] the first branch is equivalent to general relativity with an effective cosmological constant. Therefore we expect the usual ΛCDM phenomenology for this branch. We now concentrate on the second branch which is more interesting, hence we request $rH_f = H$. Inserting this in (5.27) yields

$$H^2 = \frac{m^2}{3\,m_*^2}\left(\frac{\beta_1}{r} + 3\beta_2 + 3\beta_3\,r + \beta_4\,r^2\right)\,. \tag{5.32}$$

With this, the Friedmann equation for g can be written as

$$-\frac{\beta_3}{3}r^3 + r^2\left(\frac{\beta_4}{3}\frac{1}{m_*^2} - \beta_2\right) + r\left(\frac{\beta_3}{m_*^2} - \beta_1\right) + \frac{\beta_1}{3m_*^2 r} + \frac{\beta_2}{m_*^2} - \frac{\beta_0}{3} = \frac{8\pi G}{3m^2}\rho\,. \tag{5.33}$$

This is a polynomial equation which can be solved for r in terms of the energy density ρ and the constants β_i, m^2, M_g^2 and m_*^2. The detailed evolution depends on the parameters but we can already observe that at late time, when $\rho \to 0$, $r \to$ const so that also $H^2 \to$ const. Therefore we have a late-time de Sitter phase, independent of the choice of parameters, as long as these admit a real and positive solution $\bar{r}$ of the fourth order equation given by (5.33) with $\rho = 0$.

A detailed study of the background cosmology in this cosmological setting can be found in [30, 107]. Several possible branches of the solutions are possible, depending on the initial values for r. In [107], a series of conditions defining a viable cosmological evolution are elaborated. Violations of these conditions do not necessarily imply contradiction with observations if they occur outside the observable range, hence in principle they can be relaxed or lifted. However, when these conditions are satisfied, the cosmological evolution requires no special tuning and it is much safer.

In order not to mimic a cosmological constant we set $\beta_0 = 0$. We now study in detail the case $\beta_0 = \beta_2 = \beta_3 = 0$ (we call it the 'β_1-β_4 model') which has been identified as the only one which gives both, an acceptable background solution and viable scalar perturbations [107, 105].

We consider a Universe containing matter and radiation with densities ρ_m and ρ_r and pressure $p_m = 0$ and $p_r = \rho_r/3 = w_r\rho_r$ which we assume to be separately conserved such that $\rho_m = \rho_{m0}a^{-3}$ and $\rho_r = \rho_{r0}a^{-4}$. Here we have normalized the scale factor to unity today, $a_0 = 1$. The Bianchi constraint can be rewritten as

$$\frac{r'}{r} = (c-1)\,\mathcal{H}\,. \tag{5.34}$$

Furthermore, under the rescaling $f_{\mu\nu} \to m_*^{-2}f_{\mu\nu}$ and $\beta_n \to m_*^n\beta_n$ the equations become independent of m_* so that we can simply set $m_* = 1$, see [21] for a more detailed discussion. With this, the background equations can be written as

$$3\mathcal{H}^2 = a^2\left(3m^2\beta_1 r + M_p^{-2}(\rho_m + \rho_r)\right)\,, \tag{5.35}$$

$$a^2m^2\left(\beta_1 + \beta_4 r^3\right) - 3\mathcal{H}^2 r = 0\,, \tag{5.36}$$

$$\mathcal{H}^2 + 2\mathcal{H}' = a^2\left(3m^2\beta_1 r - M_p^{-2}\rho_r/3 + m^2\beta_1\frac{r'}{\mathcal{H}}\right)\,. \tag{5.37}$$

We solve the first three equations for $\mathcal{H}$, ρ_m and r',

$$\mathcal{H}^2 = a^2m^2\frac{\beta_1 + \beta_4 r^3}{3r}\,, \tag{5.38}$$

$$\rho_m = M_p^2 m^2 \left(\frac{\beta_1}{r} - 3\beta_1 r + \beta_4 r^2 \right) - \rho_r \,. \tag{5.39}$$

$$\frac{r'}{r} = \frac{-9\beta_1 r^2 + 3\beta_1 + 3\beta_4 r^3 + r M_p^{-2} m^{-2} \rho_r}{3\beta_1 r^2 + \beta_1 - 2\beta_4 r^3} \mathcal{H} \,. \tag{5.40}$$

We want to solve eq. (5.40) numerically for a given present value of r. Let us divide eq. (6.32) by $\mathcal{H}_0 = H_0$ so that

$$\frac{\mathcal{H}}{\mathcal{H}_0} = a \frac{\sqrt{\beta_1 + \beta_4 r^3}}{\sqrt{3r}} \left(\frac{m}{\mathcal{H}_0} \right) . \tag{5.41}$$

We now evaluate eq. (5.41) at τ_0 and we solve the resulting equation expressing $r_0 = r(\tau_0)$ as a function of the constants β_i. This equation has three real solutions. We choose the only one that, when used as 'final' condition in eq. (5.40), gives an evolution for r starting at very large values and decreasing to a finite value at late times. In [107] and [105], it has been shown that this solution is the only one able to give rise to both a viable background cosmology and viable scalar perturbations in the β_1-β_4 model.

We choose the best-fit values $\beta_1 m^2 = 0.48 H_0^2$ and $\beta_4 m^2 = 0.94 H_0^2$ obtained in [107] and [105] fitting measured growth data and type Ia supernovae. We can then solve eq. (5.40) numerically. The evolution of r is shown in Fig. 5.1. We observe that r is very

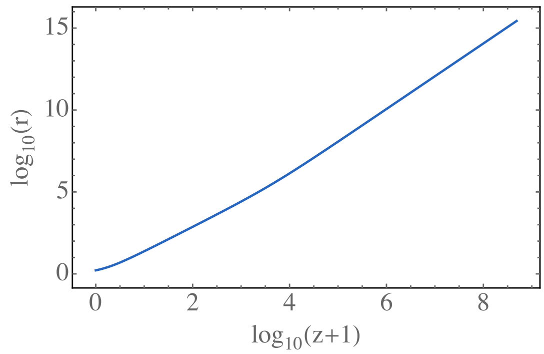


Figure 5.1: The evolution of the ratio of the two scale factors, $r = b/a$ is shown as function of the redshift $z + 1 = a^{-1}$. We have chosen the parameters $m^2\beta_1 = 0.48\,\mathcal{H}_0^2$, $m^2\beta_4 = 0.94\,\mathcal{H}_0^2$ and $\beta_0 = \beta_2 = \beta_3 = 0$.

large at early times (the vertical axis is not r but $\log_{10}(r)$), at the present time $r \simeq 1$ while the value $r = 1$ is a future attractor.

The lapse c of the f-metric is given by the Bianchi constraint (5.34). Its time evolution is presented in Fig. 5.2.

The lapse function is $c \simeq -1 \simeq$ constant in the radiation dominated phase, so that $r'/r = -2\mathcal{H}$, hence $r \propto a^{-2}$, see Eq. (5.34). It grows to a new plateau at the matter-radiation transition and stays $c \simeq -1/2$ during the matter dominated phase. Then again

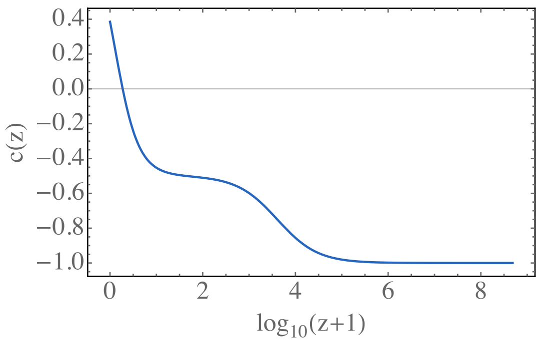


Figure 5.2: The evolution of the lapse of the f metric for $m^2\beta_1 = 0.48\,\mathcal{H}_0^2$, $m^2\beta_4 = 0.94\,\mathcal{H}_0^2$ and $\beta_0 = \beta_2 = \beta_3 = 0$.

at the transition to the gravity-dominated phase in the future, $z < 0$ the lapse grows to $c = 1$ which is reached in the future de Sitter phase. Interestingly, the lapse function changes sign roughly at redshift $z_c \simeq 0.9$. This in principle signals a singularity in the f-metric where, for example, its determinant vanishes. However, since the Bianchi constraint requires that also $b' = 0$ when $c = 0$ and $a' \neq 0$, $\mathcal{H}_f = b'/(bc) = \mathcal{H} = a'/a$ remains finite and no physical observable diverges[1].

At this point we want to direct the attention of the reader to the fact that even though in the Langrangian the lapse function c only appears as $\sqrt{c^2}$ which one naively might replace by $|c|$, this cannot lead to the background phenomenology needed to mimic dark energy. At early times, when $r \gg 1$ and the Universe is radiation dominated, Eq. (5.40) becomes

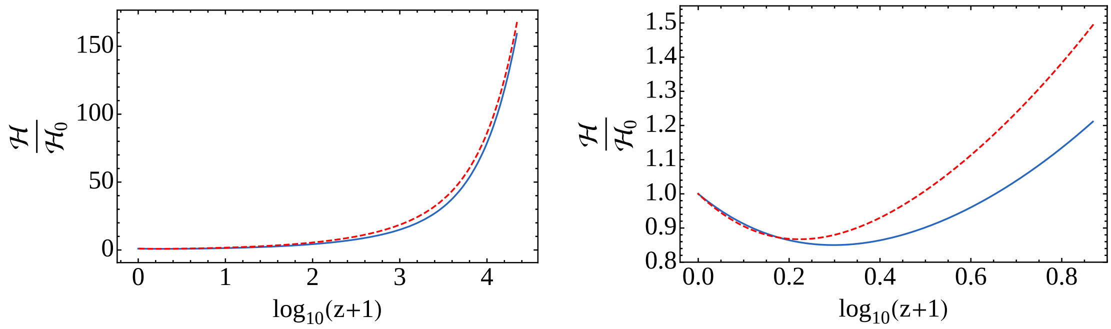
$$\frac{r'}{r} = -\frac{3m^2\beta_4 r^2 + M_p^{-2}\rho_r}{2m^2\beta_4 r^2}\mathcal{H} = -2\mathcal{H}\,. \tag{5.42}$$

For the second equal sign we have used Eq.(5.38) in the limit of large r. In order to satisfy both, Eq. (5.34) and Eq. (5.40), we therefore need $c = -1$ in the radiation dominated era and we cannot replace c by $|c|$ in Eq. (5.34). We can also not replace it by $-|c|$ since we need the factor $(c-1) \to 0$ when the Universe becomes dark energy dominated. With our choice of the parameters β_i, a radiation dominated Universe at early time and a dark energy like solution at late time, requires that c passes through zero, which $\pm|c|$ cannot.

Substituting the numerical solution for r in eq. (5.41), we obtain the evolution of $\mathcal{H}/\mathcal{H}_0$. In Fig. 5.3 we have plotted $\mathcal{H}/\mathcal{H}_0$ as a function of redshift in β_1-β_4 bigravity and in standard ΛCDM.[2]

[1]This would be different if we would couple matter to the f-metric since e.g. its Ricci scalar R_f which might then become observable diverges.

[2]We have considered a scenario with radiation, matter and a cosmological constant with $\Omega_\Lambda = 0.7$.



(a) Complete evolution

(b) Low redshift behaviour

Figure 5.3: We show the evolution of the Hubble parameter $\mathcal{H}(z)$ for a ΛCDM background with $\Omega_\Lambda = 0.7$ (red, dashed) and for a bimetric cosmology (blue, solid) with $m^2\beta_1 = 0.48\,\mathcal{H}_0^2$, $m^2\beta_4 = 0.94\,\mathcal{H}_0^2$ and $\beta_0 = \beta_2 = \beta_3 = 0$.

In Fig. 5.4 we compare the background evolution of the comoving distance $d(z) = \int_0^z H(z')^{-1}dz'$ for the β_1-β_4 model with the best fit parameters $m^2\beta_1 = 0.48\mathcal{H}_0^2$, $m^2\beta_4 = 0.94\mathcal{H}_0^2$ (which we shall also consider in the perturbation analysis) and for a ΛCDM model with $\Omega_\Lambda = 0.7$.

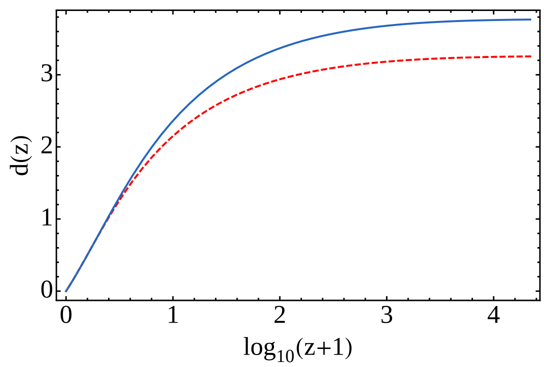


Figure 5.4: We show the evolution of the comoving distance $d(z)$ for a ΛCDM background with $\Omega_\Lambda = 0.7$ (red, dashed) and for a bimetric cosmology (blue, solid) with $m^2\beta_1 = 0.48\,\mathcal{H}_0^2$, $m^2\beta_4 = 0.94\,\mathcal{H}_0^2$ and $\beta_0 = \beta_2 = \beta_3 = 0$. (We do not optimise the cosmological parameters to obtain a good fit since this is not the point of the present work. The similarity of the behaviour obtained here is sufficient for our purpose.)

5.3 Comments on scalar perturbations

Let us now consider perturbations of this cosmology around the homogeneous and isotropic background

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + a^2\, h_{g\mu\nu}\,, \tag{5.43}$$

$$f_{\mu\nu} = \bar{f}_{\mu\nu} + b^2\, h_{f\mu\nu}\,. \tag{5.44}$$

From now on, the use of an overbar indicates background quantities. We parametrize the perturbations as follows

$$(h_g)_{\mu\nu} = \begin{pmatrix} -2A_g & C_j^{(g)} - \partial_j B_g \\ C_i^{(g)} - \partial_i B_g & h_{ij}^{(g)TT} + \partial_i \mathcal{V}_j^{(g)} + \partial_j \mathcal{V}_i^{(g)} + 2\partial_i\partial_j S_g + 2\delta_{ij} F_g \end{pmatrix}, \tag{5.45}$$

$$(h_f)_{\mu\nu} = \begin{pmatrix} -2c^2 A_f & C_j^{(f)} - \partial_j B_f \\ C_i^{(f)} - \partial_i B_f & h_{ij}^{(f)TT} + \partial_i \mathcal{V}_j^{(f)} + \partial_j \mathcal{V}_i^{(f)} + 2\partial_i\partial_j S_f + 2\delta_{ij} F_f \end{pmatrix}, \tag{5.46}$$

with

$$\partial_i C^i_\bullet = \partial_i \mathcal{V}^i_\bullet = \partial_i h^{TTij}_\bullet = 0\,, \qquad \delta^{ij} h^{TT}_{\bullet ij} = 0\,. \tag{5.47}$$

Spatial indices are raised and lowered using the flat spatial metric, δ_{ij}. There are eight scalar perturbations, $A_\bullet$, $B_\bullet$, $S_\bullet$ and $F_\bullet$, eight vector perturbations, $C_{\bullet j}$ and $\mathcal{V}_{\bullet j}$ and four tensor perturbations $h_{ij}^{\bullet TT}$. Here $_\bullet$ denotes $_g$ or $_f$. Two scalar and two vector modes can be removed by coordinate transformations, leaving six scalar six vector and four tensor degrees of freedom.

In Ref. [105] scalar perturbations of the viable β_1-β_4 model have been analysed for perfect fluid matter (i.e. matter without anisotropic stress and with adiabatic perturbations) and it has been found that they can fit the growth rate of the observed perturbations during the matter and dark energy dominated eras[3]. In Ref. [110] a preliminary analysis of all, scalar, vector and tensor perturbations is presented and analytic solutions in limiting regimes are found, which all do not show exponential instabilities.

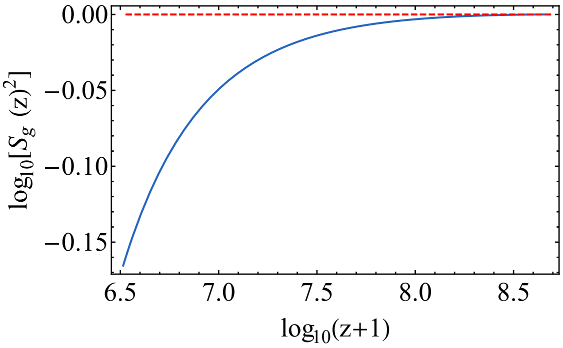
In this section we discuss briefly scalar perturbations while the rest of this work is devoted to a detailed study of tensor perturbations. For the scalar sector, we derive analytic solutions for the propagating degrees of freedom valid in the radiation era and we compare them with the results of the numerical integration of the perturbation equations in radiation. The result of this analysis differs from the one of Ref. [110] and we find that an instability in the scalar sector of the f metric shows up at early times and, if sufficiently large, it is transferred to the physical sector of the g metric through the coupling between the two sectors.

[3]One of the main conclusions of Ref. [105] is that during *matter domination* scalar perturbations do not exhibit *exponential* instabilities. In this context, however, the stability of scalar perturbations at early times and the absence of power-low instabilities during matter is not analysed in this work.

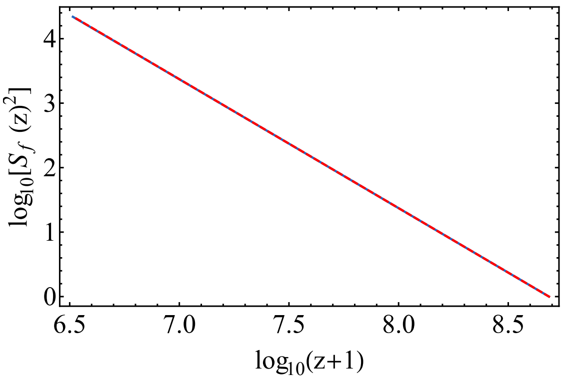
The equations for the two propagating scalar degrees of freedom in the radiation dominated era can be approximated by [4]

$$S_g'' + 2\mathcal{H}S_g' + \frac{9\beta_1\mathcal{H}^3 S_f'}{2\beta_4 k^2 r} - \frac{k^2 S_g}{3} - \frac{1}{2}a^2 m^2 \beta_1 r S_f = 0\,, \tag{5.48}$$

$$S_f'' + \frac{6\beta_1\mathcal{H}S_f'}{\beta_4 r} - \frac{\beta_1 k^2 S_g'}{\beta_4\mathcal{H}r} + \frac{k^2 S_f}{3} - \frac{2a^2 m^2 \beta_1 k^2 r S_g}{3\mathcal{H}^2} = 0\,. \tag{5.49}$$



(a)



(b)

Figure 5.5: Evolution of scalar perturbations for the metric g and f, in the case $A = B = 1$ for $k \simeq \mathcal{H}_0$. The red-dashed line represent the analytical solution for the equation of scalar perturbations valid in the radiation dominated era. For S_f there is perfect agreement between the analytical and numerical solutions.

For super-Hubble modes, $k\tau \ll 1$, we can neglect terms proportional to k^2. Furthermore, in the background branch under study, in the radiation era we have $r \gg 1$,

[4] We adopt here the gauge choice of Ref. [110] to eliminate the redundant degrees of freedom in the scalar sector.

$r' \simeq -2\mathcal{H}r$ and $m^2\beta_1 a^2 r \simeq 0.48\mathcal{H}_0^2\sqrt{\Omega_r/0.94} \simeq 0.24\times 10^{-2}\mathcal{H}_0^2 = \text{constant}$. Hence, in this regime, the last three terms in (5.48) and (5.49) can be dropped and the two equations decouple.

The solutions of the resulting approximated equations can be written as [5]

$$S_g = c_1 + c_2\frac{\tau_{\text{in}}}{\tau}\,, \qquad\qquad S_f = c_3 + c_4\frac{\tau}{\tau_{\text{in}}}\,. \tag{5.50}$$

In the physical sector we recover the usual behavior of super-Hubble scalar perturbation in the radiation dominated era, while the perturbations of f grow linearly. Neglecting the constant mode for S_f and the subdominant decaying mode for S_g, we have

$$S_g = A\,, \qquad\qquad S_f = B\frac{\tau}{\tau_{\text{in}}}\,. \tag{5.51}$$

We solve Eqs. (5.48) and (5.49) numerically with initial conditions (5.51) and we compare the result with the analytical solution valid in the radiation era, see Fig. 5.5. The analytical and numerical solutions for S_f are in very good agreement. The solution for S_g, however is soon affected by the coupling term $\frac{9\beta_1\mathcal{H}^3}{2\beta_4 k^2 r}S_f'$ in (5.48) which can be large for small values of k.

We then choose the initial condition for scalar perturbations in the physical sector compatible with the observational constraints from structure formation, $A = 10^{-5}$, and we explore how the evolution changes varying B, i.e., the initial condition for S_f, see Fig. 5.6. If the ratio between the initial condition of S_f and S_g is big, i.e. $B/A \gtrsim 1$, the solution for S_g develops a growing mode in the radiation dominated epoch. In Fig. 5.7 we plot the amplification of S_g at the end of the radiation era ($z \sim 10^4$) as a function of B. We see that the amplification is roughly proportional to the initial condition of S_f for $B/A \gg 1$. The amplification during the radiation era is absent for $B/A < 1$.

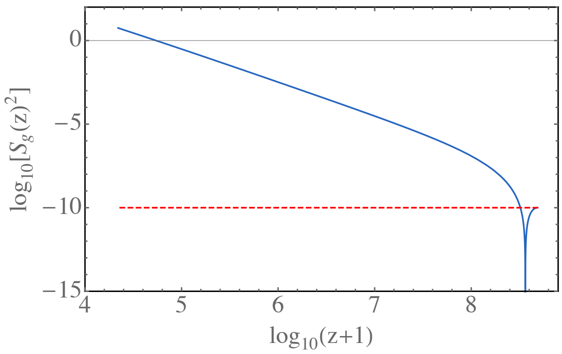
Comparing the order of magnitude of the terms in eq. (5.48) we find in order for the instability to develop during the radiation dominated era we need

$$\frac{B}{A} \gtrsim 100\frac{(1+z_{\text{eq}})^2}{1+z_{\text{in}}}\,. \tag{5.52}$$

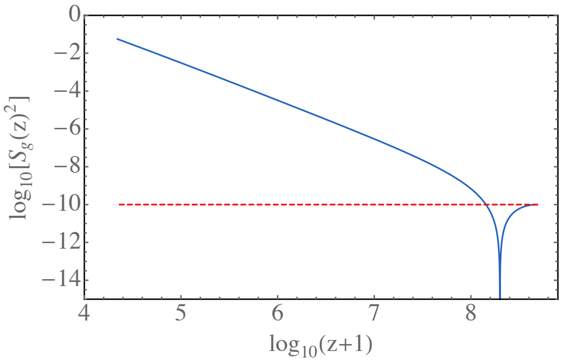
For a realistic value of $1+z_{\text{eq}} \simeq 3\times 10^3$ and our example plotted in Fig. 5.6, i.e., $1+z_{\text{in}} = 10^9$, this requires $B/A > 1$. For an early inflationary phase with reheat temperature $T_{\text{in}} \simeq 10^{10}$GeV we obtain $1+z_{\text{in}} \simeq 10^{23}$, hence in order to avoid this mild instability we need to require that

$$B < 10^{-14}A\,. \tag{5.53}$$

[5] More precisely, the exact solution of the decoupled eq. (5.49) has a constant mode and a growing one proportional to $\text{erf}\left(\sqrt{\frac{1.2\times10^3\beta_1^2}{\beta_4}}\frac{m}{\mathcal{H}_0(1+z)}\right)$. This function is growing roughly like $(1+z)^{-1} \propto \tau$ as long as the argument is smaller than 1, hence during the entire radiation dominated epoch. It can therefore be approximated with the growing mode in (5.50) to good precision.



(a)



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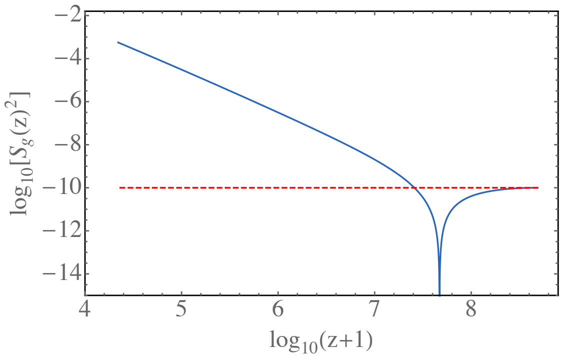
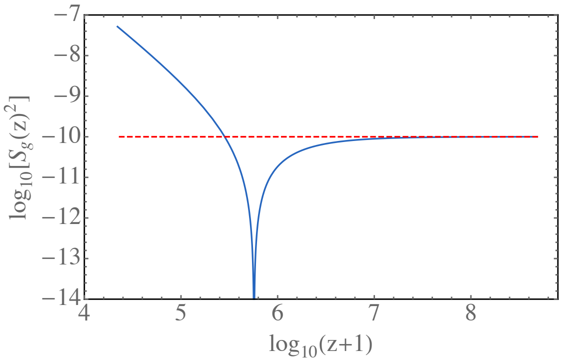


(c)



(d)

Figure 5.6: Evolution of scalar perturbations for the metric g for the case $A = 10^{-5}$ and $k \simeq 10\mathcal{H}_0$, varying the initial condition for S_f. We have chosen the cases $B = 10, 1, 10^{-1}, 10^{-3}$, in Fig. 5.6(a), 5.6(b), 5.6(c) and 5.6(d), respectively. The red-dashed line represent the analytical solution for the equation of scalar perturbations valid in the radiation dominated era.

Hence for early inflation, only very fine tuned initial condition can avoid to be affected by this instability in the scalar sector.

5.4 Gravitational waves in massive bigravity cosmology

Tensor perturbations of a given **k**-mode can be written as

$$h_{ij}^{TT} = h^{+} e_{ij}^{(+2)} + h^{-} e_{ij}^{(-2)} \tag{5.54}$$

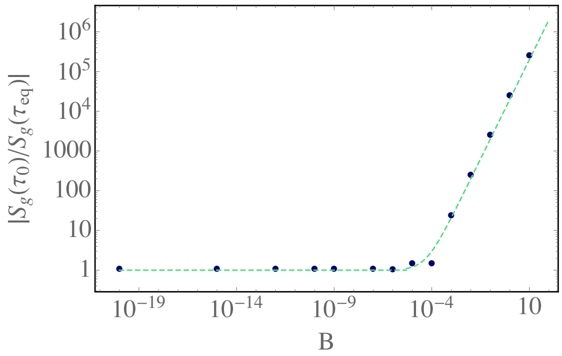


Figure 5.7: Amplification of scalar perturbation S_g at equality as a function of B. The dotted line is the interpolating function $i(B) = 0.3\,\frac{B}{A}\cdot\theta(B-A)+1$, where θ denotes the Heaviside function. We have chosen $A = 10^{-5}$ and a mode with $k \simeq 10\mathcal{H}_0$.

where + and − denote the two helicity-2 modes of the gravitational wave. For an orthonormal system $\widehat{\mathbf{k}}, \mathbf{e}^{(1)}, \mathbf{e}^{(2)}$ we have

$$\mathbf{e}^{\pm} = \frac{1}{\sqrt{2}}\left(\mathbf{e}^{(1)} \pm i\mathbf{e}^{(2)}\right) \quad \text{and} \quad e_{ij}^{(+2)} = \mathbf{e}_i^{+}\mathbf{e}_j^{+}\,, \quad e_{ij}^{(-2)} = \mathbf{e}_i^{-}\mathbf{e}_j^{-}\,. \tag{5.55}$$

For parity invariant perturbations

$$\langle h^{+}(\mathbf{k})(h^{+}(\mathbf{k}'))^{*}\rangle = \langle h^{-}(\mathbf{k})(h^{-}(\mathbf{k}'))^{*}\rangle = \delta(\mathbf{k}-\mathbf{k}')2\pi^2 P_h(k)/k^3\,,$$

and $\langle h^{+}h^{-}\rangle = 0$. This is what we shall assume in the following and we shall consider just one mode, say $h_f^{+} = h_f G$ and $h_g^{+} = h_g G$. Here G is a Gaussian random variable with vanishing mean and with variance $\langle G(\mathbf{k})G(\mathbf{k}')\rangle = \delta(\mathbf{k}-\mathbf{k}')2\pi^2/k^3$, so that $h_\bullet$ is the square root of the power spectrum. All what follows is also valid for the modes $h_\bullet^{-}$ which are not correlated with $h_\bullet^{+}$ in the parity symmetric situation which we consider.

For the first order modified Einstein equation with a perfect fluid source term, i.e. no anisotropic stress, we obtain the following tensor perturbation equations for our bimetric cosmology

$$h_g'' + 2\mathcal{H}\,h_g' + k^2 h_g + m^2 a^2 r\,\beta_1\,(h_g - h_f) = 0\,, \tag{5.56}$$

$$h_f'' + \left[2\left(\mathcal{H} + \frac{r'}{r}\right) - \frac{c'}{c}\right] h_f' + c^2k^2\,h_f - m^2\beta_1\frac{c\,a^2}{r}\,(h_g - h_f) = 0\,. \tag{5.57}$$

At very early times, in the radiation dominated Universe where we want to define our initial conditions, r is very large and $m^2\beta_1 a^2 r = 0.48\mathcal{H}_0^2\sqrt{3\rho_{r0}/(M_p^2 m^2\beta_4)} = 0.48\mathcal{H}_0^2\sqrt{\Omega_r/0.94} = 0.24\times10^{-2}\mathcal{H}_0^2 =$ constant. Furthermore, $c \simeq -1 \simeq$ constant. This implies that in this limit the square bracket of eq. (5.57) becomes $-2\mathcal{H}$ and the coupling term is suppressed by a factor $1/r^2$ with respect to the coupling term in eq. (5.56) and can be neglected. Choosing a super Hubble mode, $k\tau \ll 1$ and recalling that in the radiation era $\mathcal{H} = 1/\tau$, we can neglect the term proportional to k^2 in both the equations. To be consistent, in eq. (5.56), we then have to neglect also the coupling term, since $K^2 = m^2\beta_1 a^2 r \simeq (0.05\mathcal{H}_0)^2 < k^2$ for the best fit parameters with $m^2\beta_1 = 0.48\mathcal{H}_0^2$. On super Hubble scales in the radiation era we then obtain the solutions

$$h_g = c_1 + c_2\left(\frac{\tau_{\rm in}}{\tau}\right), \tag{5.58}$$

$$h_f = c_3(k\tau)^2 y_1(ck\tau) - 3c_4\frac{(k\tau)^2}{(k\tau_{\rm in})^3} j_1(ck\tau) \simeq c_3 + c_4\left(\frac{\tau}{\tau_{\rm in}}\right)^3. \tag{5.59}$$

The solution for h_g differs from the one found in Ref. [110]: in this work when deriving the approximated equation valid in the radiation era for super Hubble modes, the term proportional to K^2 in eq. (5.58) is not neglected. As explained above, this approximation is not completely consistent.[6]

Interestingly, when neglecting the coupling term which for h_f is never relevant, the first expression for the solution (5.59) is valid both in the radiation and matter era on all scales as long as c =constant. Actually, in the matter dominated era the anti-damping term in eq. (5.57) becomes $2\left(\mathcal{H} + \frac{r'}{r}\right) - \frac{c'}{c} \simeq -\mathcal{H}$ and with $\mathcal{H} = 2/\tau$, the h_f equation remains unchanged. The functions y_1 and j_1 denote the spherical Bessel functions [5] and $c = -1$ in the radiation era while $c = -1/2$ in the matter era. Considering the growing mode proportional to c_4 we find that h_f grows like τ^3 on super Hubble scales and like τ on sub Hubble scales.

However, in general we can no longer neglect the coupling term in the solution for h_g since, depending on the initial condition h_f may have grown too large to be neglected in its coupling to h_g. In contrary, since h_g cannot grow more than h_f and since the pre factor of the coupling term remains small, the coupling can be neglected in the h_f equation and (5.59) remains a good approximation on super Hubble scales.

The solution for h_g in the radiation dominated era agrees with the well know GR solution, but h_f has a growing mode which indicates the presence of an instability. Neglecting the decaying modes we choose the initial conditions

$$h_g(\tau) = A, \qquad h_f(\tau) = B\left(\frac{\tau}{\tau_{\rm in}}\right)^3. \tag{5.60}$$

The behaviour of the solution depends very sensitively on the initial condition, in the following we explore different possibilities. Naively, we might argue that initially,

[6] This can be checked substituting the solutions found in Ref. [110] with coefficients c_i expressed as functions of the initial conditions after inflation in the full equations for perturbations: the terms which do not cancel are negligible only in the specific case in which the initial condition for h_f' after inflation is fine-tuned to be very small, $h_f'(\tau_{\rm in}) \ll \tau_{\rm in}^3 K^4$.

e.g., after inflation, both h_f and h_g are of the same order of magnitude, $A \simeq B$. The gravitational waves $h_g(k,\tau)$ and $h_f(k,\tau)$ for these initial conditions found by solving numerically Eqs. (5.56) and (5.57) for the wave numbers $k \simeq 10\mathcal{H}_0$, $100\mathcal{H}_0$, $200\mathcal{H}_0$ are shown in Fig. 5.8. In a linear plot it looks as if h_g and h_f would be nearly constant during radiation, then h_f starts oscillating with frequency $\omega^2 = c^2k^2 + cK^2/r^2$ and with increasing amplitude at redshift corresponding to the horizon crossing for the mode chosen. The instability is transferred to the h_g mode trough the coupling. In Fig. 5.9 we present a log-plot for the same modes together with the analytic solutions (5.58,5.59). The analytic solution for h_f is a very good approximation on super-Hubble scales. There one sees that the τ^3 growth on super Hubble scales turns into the milder growth $\propto \tau$ after Hubble entry.

When $A \simeq B$, the gravitational wave amplitude today is amplified tremendously, for a mode with wavenumber k, roughly by a factor $f(k) = \mathcal{H}(\tau_0)\mathcal{H}(\tau_i)^3/k^4$, as shown in Fig. 5.10. Therefore, in any case, if the initial amplitudes are not very small, gravitational wave perturbations will grow very large at late time.

We want to check whether there exists a choice of the initial conditions such that we recover an evolution of tensor perturbations similar to the one of ΛCDM and an amplitude of tensor perturbations today which is of order of the one of ΛCDM. We find that if we tune the initial conditions for h_f to be very small, i.e. $B \ll A$, the instability can be avoided and we can recover an evolution of tensor perturbations at late times (i.e. during the matter era and later) that is similar to the standard gravitational wave evolution of General Relativity. In Fig. 5.11 we show how the evolution of tensor perturbations is affected by decreasing B. The evolution of tensor perturbation h_g in the bigravity model for different initial values B at fixed $A = 1$ is superimposed to the ΛCDM result with initial condition $h_{GR}(\tau_{\rm in}) = 1$, $h'_{GR}(\tau_{\rm in}) = 0$.

The amplitude of h_g at late times is proportional to B for values of $B/A \gtrsim 10^{-16}$. For smaller values of B it converges to the GR result and becomes independent of B. In other words, if h_f is not about 16 orders of magnitude smaller than h_g initially, the value of the latter at late times is entirely determined by h_f.

The small shift in redshift of the bimetric h_g spectrum for $B = 0$ with respect to the one of ΛCDM is due to the presence of a slight difference between the evolution of the scale factor in the β_1-β_4 bigravity model compared to ΛCDM (see Fig. 5.3), while the coupling of the tensor mode h_g with h_f in the perturbation equation (5.56) is effectively negligible. This can be checked easily comparing the spectrum of tensor perturbations of the $B = 0$ bigravity model with the one of ΛCDM, calculated on a bigravity background: the two spectra overlap with a very good precision[7], as shown

[7] In other words, if we choose fine-tuned initial conditions for tensor perturbations, $B < 10^{-16}A$, the coupling between the two tensor modes in (5.56) is effectively negligible and the fact that the evolution of tensor perturbations of the physical metric differs form the one in ΛCDM can be simply ascribed to a slightly different background evolution.

in Fig. 5.13.

In Fig. 5.14 we show the energy spectrum of the gravitational waves h_g in units of the critical density

$$\frac{d\Omega_{GW}}{d\log k}(\tau,k) \equiv \frac{k^5|h_g|^2(\tau,k)}{12\,H_0^2}\,, \tag{5.61}$$

for the cases $B = 0$ and $B = 10^{-15}A$ and $A = 0$ at the redshift of decoupling and today.

Let us also make the following remark: one might worry about the singularity of the term c'/c in eq. (5.28) when the lapse function of the f metric, c, passes through zero (see Fig. 5.2). It can actually be shown by a simple analytic argument that this singularity is just an apparent one. First, by using eq. (5.34), we find that eq. (5.57) can also be written as

$$h_f'' + \left[2\,c\,\mathcal{H} - \frac{c'}{c}\right] h_f' + c^2k^2\,h_f - m^2\beta_1\frac{c\,a^2}{r}\,(h_g - h_f) = 0\,. \tag{5.62}$$

When $c \sim 0$, eq. (5.62) can be approximated by

$$h_f'' - \frac{c'}{c}\,h_f' = 0\,, \tag{5.63}$$

which is solved by $h_f' \propto c$. Therefore the singularity in the term c'/c in eq. (5.63) when $c = 0$ is cancelled by the factor $h_f' \propto c$ and the differential equation for h_f is regular for all values of τ. The fact that h_f' passes through zero when $c = 0$ at $z = z_c \simeq 0.9$ is well visible in Fig. 5.15.

5.5 Discussion and conclusions

5.5.1 Higuchi bounds

In cosmology and in particular in theories of modified gravity, it is important to check whether the theory may contain 'ghosts'. In this context a ghost is a degree of freedom with a kinetic term of the wrong sign. The energy of such a degree of freedom is not bounded from below and via its coupling to other degrees of freedom it can pass to them unlimited amounts of energy, rendering the theory unstable and therefore unphysical. In a static spacetime this instability is exponential. In an expanding spacetime it is typically a milder power law instability.

As pointed out for the first time by Higuchi [91], even though the sixth degree of freedom of generic massive gravity, which is always a ghost, is absent, in the dRGT theory of massive gravity the helicity-0 mode of the massive graviton can behave like a ghost for particular values of the theory on a de Sitter background, leading to instabilities of the theory beyond the classical linear regime. The condition for having the kinetic term positive definite is known as Higuchi bound. The study of the stability of massive gravity linearized around a de Sitter background (with flat reference metric) was continued in [58] where it has been shown that the helicities (±1 and ±2) of the massive graviton are stable and unitary since they are immune to the helicity-0 constraint.

The requirement that the helicity-0 mode on a FRW background has a positive-definite kinetic term is referred to as the *generalized* Higuchi bound. This has been studied for the first time in the bigravity theory in [69] and in [70] (see also [42] for an alternative analysis of the scalar sector).

In the background branch with $\mathcal{H}_f = \mathcal{H}$, the generalised Higuchi bound for the helicity-0 mode can be written as

$$\tilde{m}^2\left(1+\frac{1}{r^2}\right) - 2H^2 \geq 0\,, \tag{5.64}$$

where

$$\tilde{m}^2 \equiv m^2 r\left(\beta_1 + 2\beta_2 r + \beta_3 r^2\right)\,. \tag{5.65}$$

For the vector modes we find instead the condition

$$\tilde{m}^2 \geq 0\,, \tag{5.66}$$

which is always satisfied in the β_1-β_4 model. This is not the case for the Higuchi bound for the helicity-0 mode as has been noted also in Ref. [110]. Indeed, using the background constraints, eqs. (5.35-5.37), the bound (5.64) can be written as

$$\beta_1 r - \frac{2}{3}\beta_4 r^2 + \frac{1}{3}\frac{\beta_1}{r} \geq 0\,. \tag{5.67}$$

For the best-fit values with $2\beta_1 \simeq \beta_4$, this constraint is satisfied only in the asymptotically de Sitter phase of the cosmological expansion, where $r = 1$ so that the bound is saturated. Hence, the scalar sector is affected by a ghost instability. In an expanding Universe with time dependent Hubble parameter, this instability is not exponential like in the de Sitter case but it manifests itself by the presence of a power law growing mode in the scalar sector of perturbations, as found in Sec. 5.3.

In the context of bigravity, the Higuchi bound in the tensor sector has not been properly addressed in the literature. If we write the quadratic kinetic part of the action for the tensor modes from eq. (5.1), we find

$$S^{(\pm2)}_{\rm kin} \propto M_g^2 \int d^4x\, a^2\left((h'_g)^2 + r^2\,\frac{\sqrt{c^2}}{c^2}(h'_f)^2\right)\,, \tag{5.68}$$

where $\sqrt{c^2}$ comes from the square root of the determinant of the f-metric. Here we can choose either c or $-c$ for $\sqrt{c^2}$, but we are not allowed to choose $|c|$ in order to have a differentiable action[8]. To reproduce the phenomenology discussed in this paper we have to choose the positive square root[9]. Only with this we obtain the correct equations of motion, e.g. eq. (5.34). Therefore the correct action is

$$S_{\rm kin}^{(\pm 2)} \propto M_g^2 \int d^4x\, a^2 \left((h_g')^2 + r^2 \frac{1}{c} (h_f')^2 \right) , \tag{5.69}$$

and the kinetic term for the tensor mode of the f metric is positive definite only if $c \geq 0$.

In the background branch that we consider, c is negative and crosses zero at recent time, $z_c \sim 0.9$. This means that along the entire cosmological evolution, the helicity-2 sector is affected by a ghost instability. This instability is connected with the one we have observed in the study of perturbations. Actually, writing the h_f-equation in the form (5.62) shows that in the epochs of $c \simeq$ constant, the sign of c indicates whether we have a damped ($c > 0$) or anti-damped ($c < 0$) evolution. At late time, $z < z_c \simeq 0.9$ the lapse function c changes sign and the tensor sector becomes healthy. This is clearly visible in the numerical solutions shown in Fig. 5.8 where one sees a decay of the amplitude of h_f at very late times.

The physical interpretation to the negative sign of the lapse c is that the time for the f-metric sector goes in the opposite direction with respect to the time for the physical sector. The scale factor b is decreasing when a is increasing since $\mathcal{H}_f = b'/bc = \mathcal{H} = a'/a$. As a consequence, instead of decreasing, the amplitude of tensor perturbations for the f-metric are growing in time.

We have chosen the lapse c negative at early times and crossing zero going to positive value only at very recent times. We observe that we could have done the opposite choice, taking $c > 0$ at early times. This choice however does not give rise to a viable cosmological evolution.

Finally, we stress that a violation of the generalized Higuchi bound in a Friedmann universe is not as devastating as it is in a de Sitter universe since the instability it gives rise to is power law and not exponential. Nevertheless, in order to agree with observations which are well reproduced with the GR behaviour, we need to fine tune these unstable modes so that their initial conditions are significantly suppressed compared to the usual GR modes.

5.5.2 Non-linearities

There is an additional subtlety which becomes relevant as soon as there are unstable modes in a theory which is intrinsically non-linear. It is a simple choice of initial

[8] For a detailed discussion of this point, see also Refs. [75, 76].

[9] We could also choose $-c$, but then we would have to change the sign of c to reproduce the ΛCDM phenomenology, so that in the end it does not change our finding that there is a ghost in the tensor sector.

conditions to set the unstable modes to zero initially and within linear perturbation theory we have found that their coupling to the other modes is sufficiently suppressed so that they are not generated significantly.

However, once we go beyond linear perturbation theory it is to be expected that the unstabler modes should acquire amplitudes of the order of Φ^2 where $\Phi \sim 10^{-5}$ is a typical linear mode which we expect to couple to all other modes at the next order. Therefore, even if a given inflationary model does not generate any tensor perturbations we expect tensor perturbations induced from scalar perturbations on the level of 10^{-10}. For the case of general relativity these induced perturbations have been calculated in 2nd order perturbation theory and numerically [17, 7].

However, the coupling of the g-metric to the f-metric is suppressed by a factor $m^2 \sim \mathcal{H}_0^2$ which makes it very small. As we have seen, at least at linear order the coupling of the g-metric to f-perturbations is nearly always negligible. Therefore, an inflationary model with nearly vanishing initial conditions for the f-metric may actually remain viable.

5.5.3 Conclusions

We have found that in bimetric cosmology the tensor perturbations of the second metric, the one that does not couple to matter, exhibits a power law instability, $h_f \propto \tau^3$ on super Hubble scales and $h_f \propto \tau$ on sub Hubble scales. For 'natural' initial conditions with $h_g \sim h_f$, the time evolution of h_g is very different from the behavior in ΛCDM cosmology. Due to its coupling to h_f it grows rapidly and the final gravitational wave spectrum is determined entirely by the initial amplitude of h_f. Only if the initial amplitude of h_f is suppressed by a factor of about $\tau_{\rm in}^3\tau_0/k^4$ w.r.t h_g we can recover the standard behavior of gravitational waves. This opens up new possibilities to test bimetric cosmology via the gravitational wave sector. Not only the final gravitational wave spectrum shown in Figs. 5.14(a) to 5.14(f) can be very different from the standard GR result, but also its time evolution differs leading to a different signature in the CMB.

To determine the initial conditions A and B we would have to specify an inflationary phase which generates them. Assuming an agnostic point of view as we have done in this work, no firm predictions can be made. Nevertheless, if inflation reheats to about $T_{\rm in} = 10^{10}$GeV, the gravitational wave amplitude on very large scales $k \sim \mathcal{H}_0$ at late times is of the order of $B(T_{\rm in}/T_{\rm eq})^3(T_{\rm eq}/T_0)^{3/2} \simeq 10^{32}B$ unless $B < 10^{-32}A$. In other words, unless there is a very significant suppression of gravitational waves of the f-metric, their amplitude and time evolution will completely dominate the gravitational wave signal and show up in the CMB.

This finding has yet another consequence: we may obtain a significant gravitational wave signal even from low energy inflation. For an inflationary Hubble parameter $H_{\rm in}$, the gravitational wave amplitude is typically $A \simeq H_{\rm in}/M_p$, leading to a tensor to scalar ratio $r = 16\epsilon$. Assuming a bimetric theory with $A \sim B$ we now obtain a scalar to tensor ratio from inflation given by

$$r = 16\epsilon \left(\frac{T_{\rm in}}{T_{\rm eq}}\right)^6 \left(\frac{T_{\rm eq}}{T_0}\right)^3 . \tag{5.70}$$

Since the scalar perturbation amplitude is

$$A_s^2 \simeq \frac{H_{\rm in}^2}{\epsilon M_p^2} \simeq 10^{-9}$$

this requires

$$\epsilon \simeq 10^9 \frac{H_{\rm in}^2}{M_p^2} .$$

For standard inflation $r = 16\epsilon$ requires $H_{\rm in} \simeq 10^{-3} M_p$ for a tensor to scalar ratio of $r \sim 0.1$.

Setting $T_{\rm in}^2 \simeq H_{\rm in} M_p$ we obtain for our bimetric cosmology

$$r \simeq 2 \times 10^{10} \frac{T_{\rm in}^4}{M_p^4} \left(\frac{T_{\rm in}}{T_{\rm eq}}\right)^6 \left(\frac{T_{\rm eq}}{T_0}\right)^3 \simeq 0.3 \left(\frac{T_{\rm in}}{1{\rm GeV}}\right)^{10} . \tag{5.71}$$

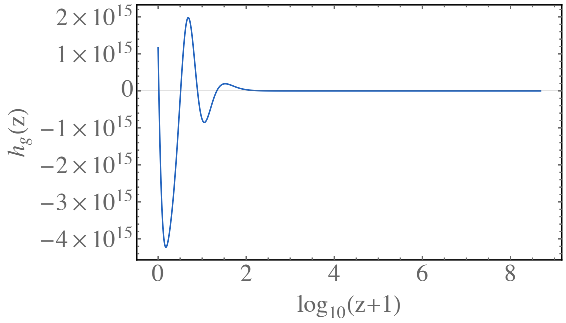
For arbitrary values of B we obtain correspondingly

$$r \simeq 0.3 \left(\frac{T_{\rm in}}{1{\rm GeV}}\right)^{10} \left[\frac{B}{H_{\rm in}/M_p}\right]^2 . \tag{5.72}$$

This rules out all simple well motivated inflationary models which cannot provide a mechanism to suppress the generation of f-perturbations during inflation.

To conclude, we have found that both, the scalar and the tensor sectors of β_1-β_4 bimetric theories, exhibit a power law instability which is related to the Higuchi ghost. Depending on the inflationary model, this instability can render the theory in serious conflict with observation. On the other hand, it may also open a new possibility to obtain significant tensor perturbations from low scale inflation.

Acknowledgments. We thank Julian Adamek, Jens Chluba, Yves Dirian, Stefano Foffa, Michele Maggiore and Ignacy Sawicki for interesting discussions and suggestions. This work is supported by the Swiss National Science Foundation.



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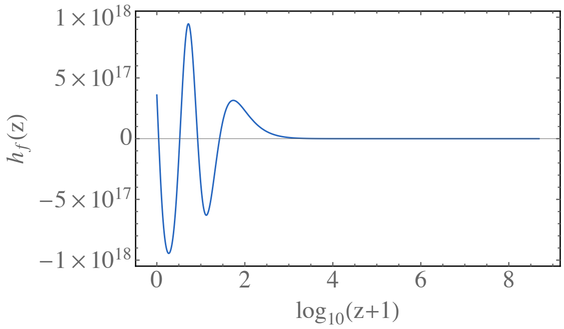

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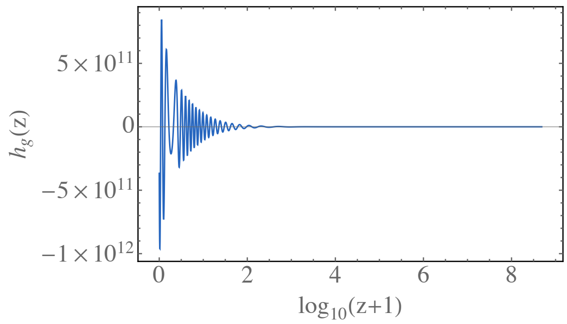

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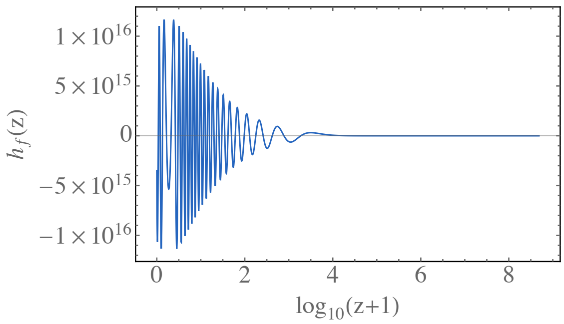

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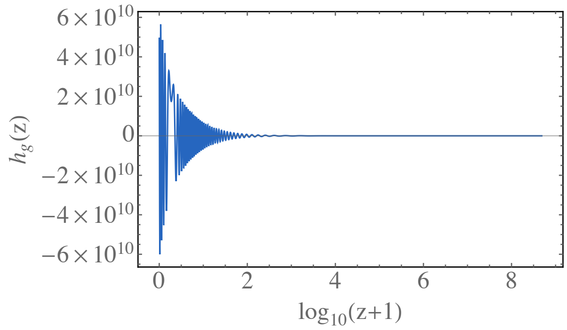

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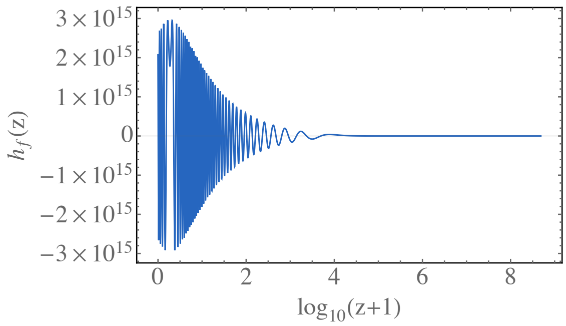

(f)

Figure 5.8: Evolution of tensor perturbations for the metrics g and f, in the case $A = B = 1$ for the modes $k \simeq 10\mathcal{H}_0$, Figs. 5.8(a), 5.8(b), $k \simeq 100\mathcal{H}_0$ Figs. 5.8(d), 5.8(d) and $k \simeq 200\mathcal{H}_0$ Figs. 5.8(f), 5.8(f).

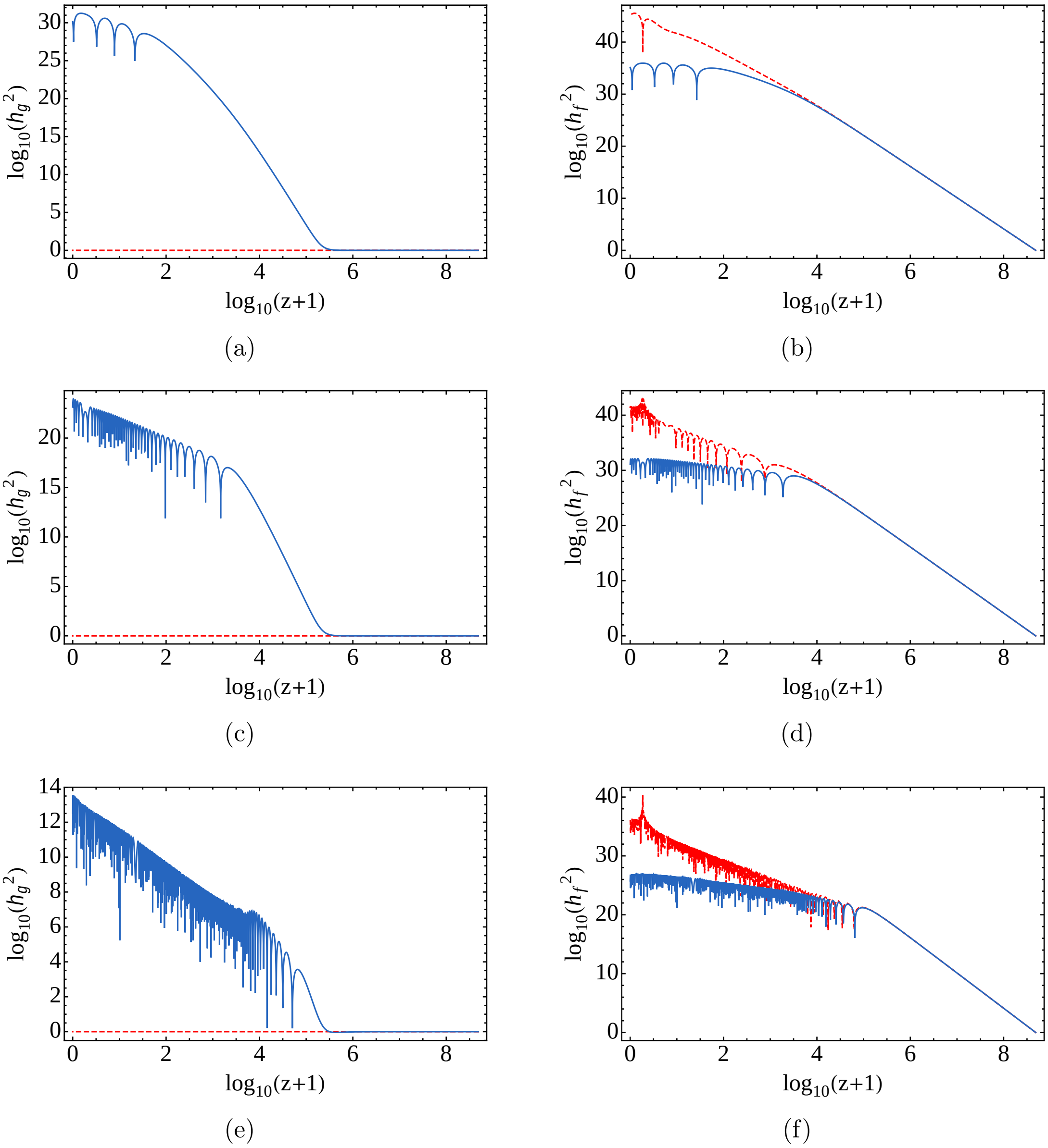

Figure 5.9: Evolution of tensor perturbations for the metrics g and f, in the case $A = B = 1$ for the modes $k \simeq 10\mathcal{H}_0$, Figs. 5.9(a), 5.9(b), $k \simeq 100\mathcal{H}_0$ Figs. 5.9(c), 5.9(d) and $k \simeq 2000\mathcal{H}_0$ Figs. 5.9(e), 5.9(f). The result of the numerical integration (blue, solid) is plotted together with the analytic approximation valid in the radiation era for h_g and on super-Hubble scales for h_f (red, dashed).

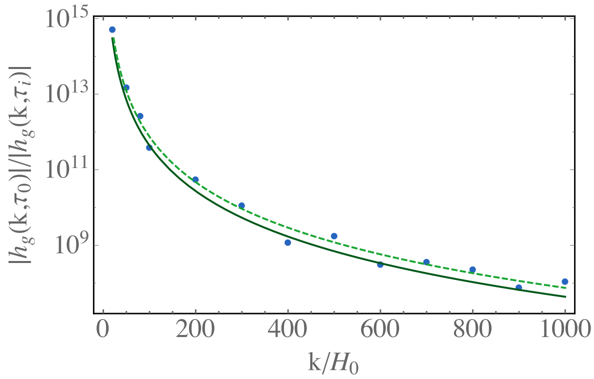

Figure 5.10: Amplification of tensor perturbations of the g metric in the case $A = B$. The dots represent the value of the amplifications for different modes, the blue solid line is the interpolating function $f(k) = \mathcal{H}(\tau_0)\mathcal{H}(\tau_i)^3/k^4$, while the green dashed line is the best polynomial fit of the data points given by the software Mathematica.

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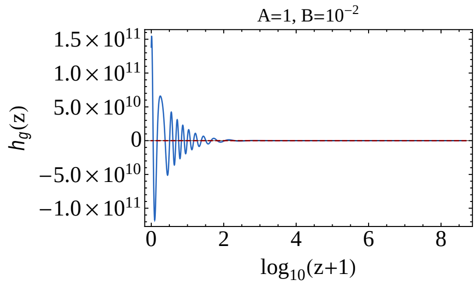

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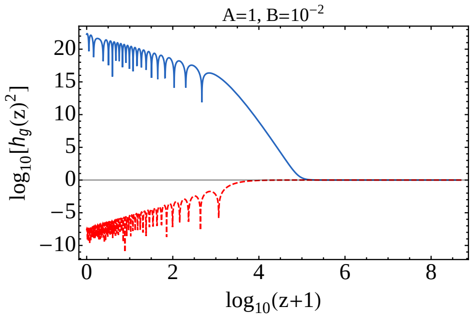

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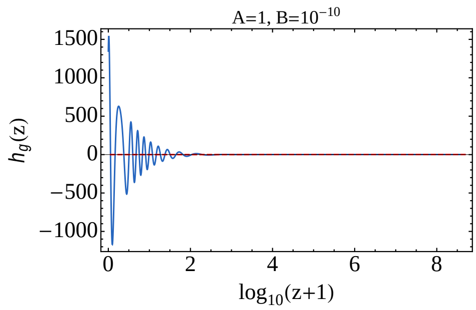

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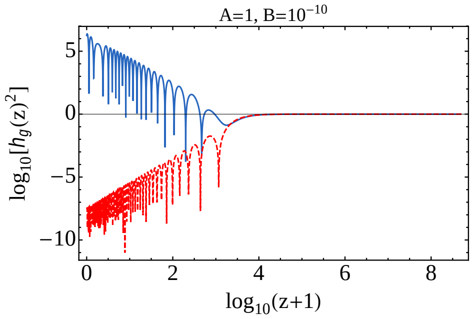

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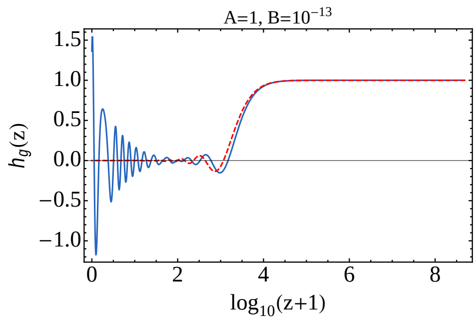

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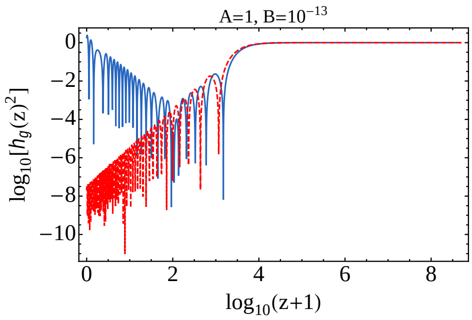

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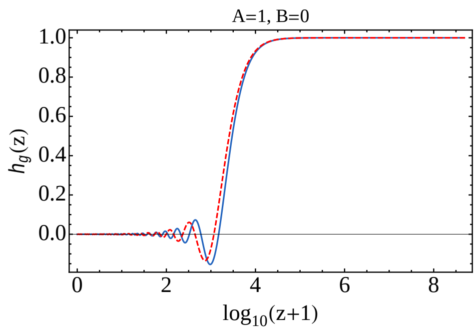

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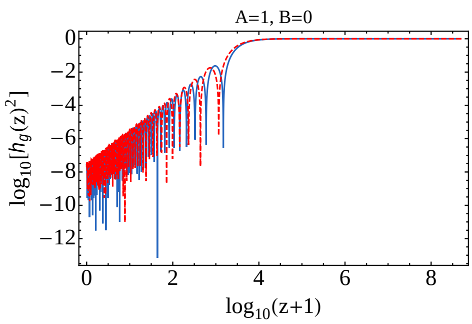

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Figure 5.11: Evolution of tensor perturbations with wave vector $k = 50H_0$ for the metric g for $A = 1$ and $B = 10^{-2}, B = 10^{-10}, B = 10^{-13}$ and $B = 0$. In Figs. 5.11(a) and Figs. 5.11(c), 5.11(e) and 5.11(g) respectively . The red dashed line represents the evolution of tensor perturbation in ΛCDM, with initial condition after inflation $h_{GR}(\tau_{\text{in}}) = 1$, $h'_{GR}(\tau_{\text{in}}) = 0$.

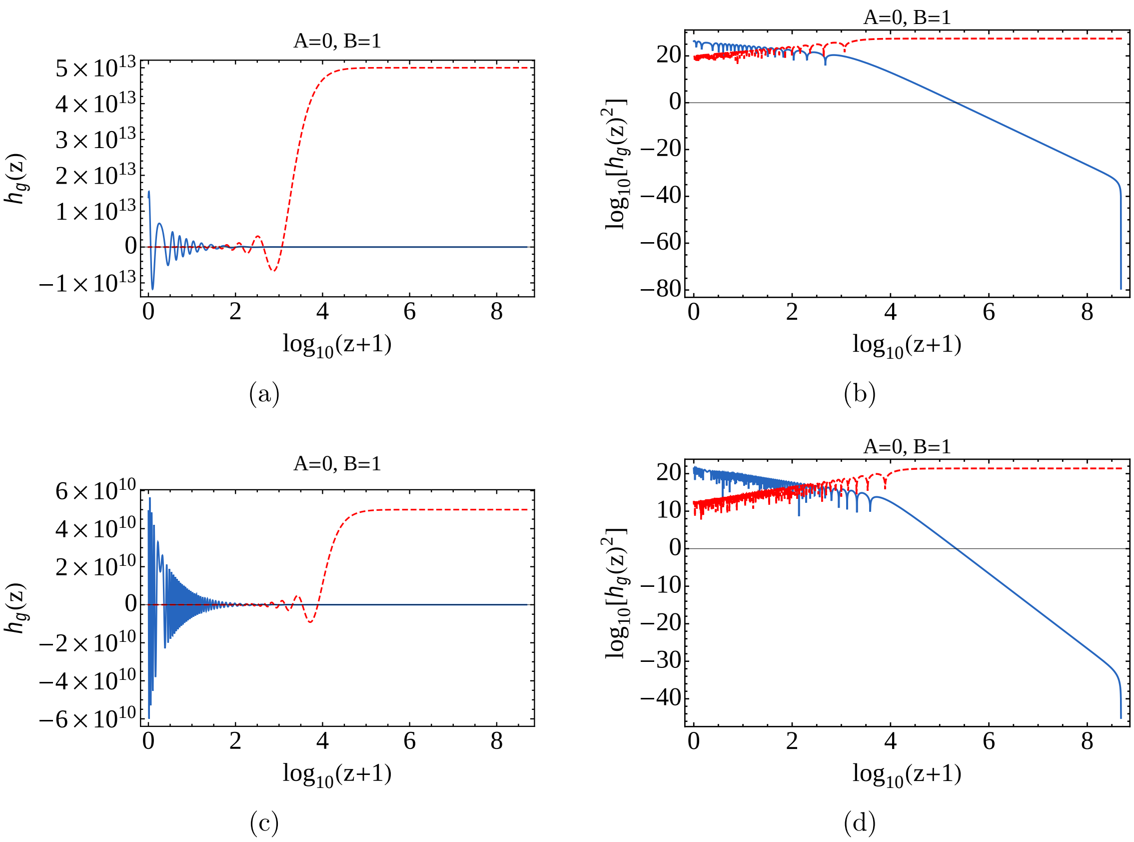
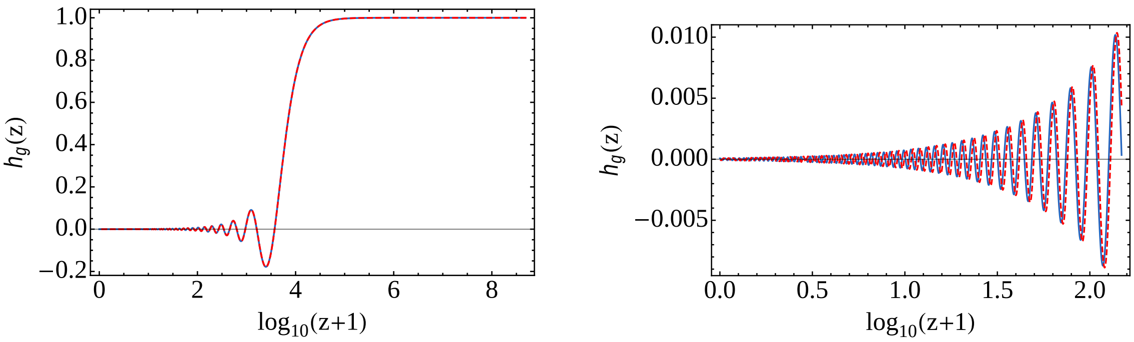


Figure 5.12: Evolution of tensor perturbations for the metric g for the case $A = 0$, and $B = 1$ for $k = 50H_0$ (Figs. 5.12(a) and 5.12(b)) and $k = 200H_0$ (Figs. 5.12(c) and 5.12(d)). The red dashed line represents the rescaled evolution of tensor perturbation in ΛCDM, with initial condition after inflation $h_{GR}(\tau_{\rm in}) = 1$, $h'_{GR}(\tau_{\rm in}) = 0$. The rescaling is $5 \cdot 10^{13}$ and $5 \cdot 10^{10}$ for $k = 50H_0$ and $k = 200H_0$, respectively.



(a) (b)

Figure 5.13: Evolution of tensor perturbations for the metric g in the case $A = 1$ and $B = 0$. The red dashed line represents the evolution of pure GR tensor perturbation, with initial condition after inflation $h_{GR}(\tau_{\rm in}) = 1$, $h'_{GR}(\tau_{\rm in}) = 0$ calculated on a bigravity background (i.e. we choose the evolution of the scale factor to be the one of the β_1-β_4 model).

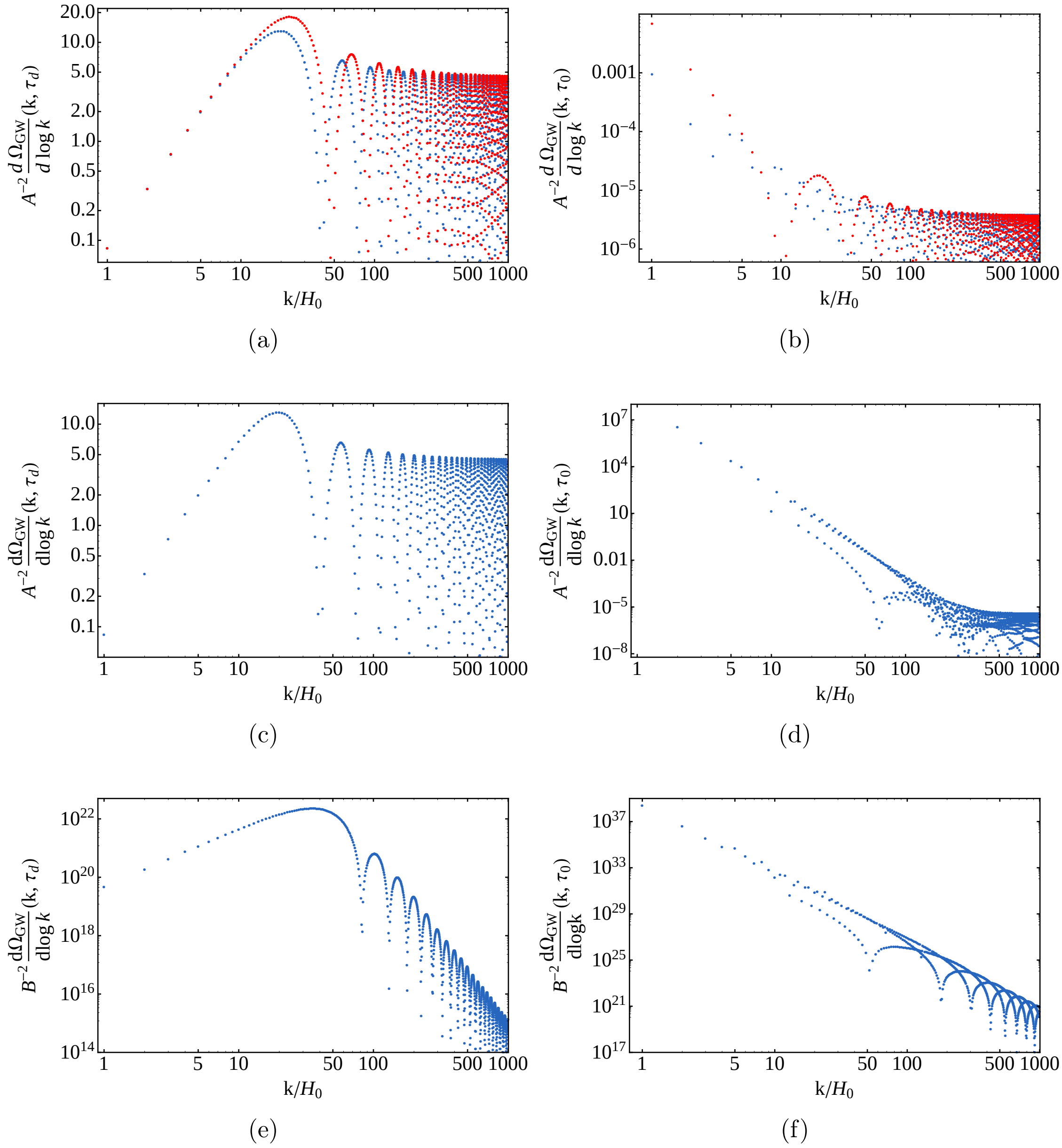


Figure 5.14: Spectra of the energy of gravitational waves h_g at the redshift of decoupling (left) and today (right) for the cases $B = 0$ and $B = 10^{-15}A$ and $A = 0$, in Figs. 5.14(a), 5.14(b), Figs. 5.14(c), 5.14(d) and Figs. 5.14(e), 5.14(f), respectively. In the first case, the spectrum for the bigravity model at a given redshift is superimposed to the spectrum of ΛCDM at the same redshift (red points).

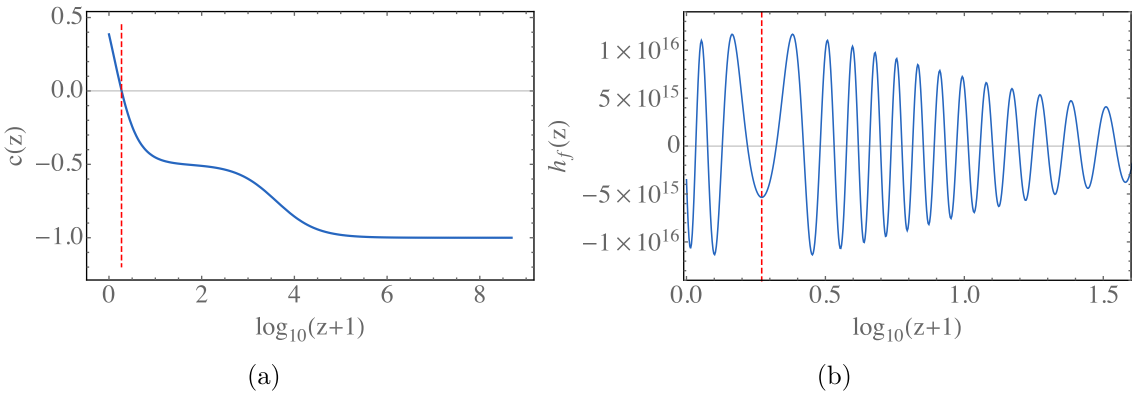


Figure 5.15: The plots of $c(z)$ and $h_f(z)$ in Fig. 5.15(a) and 5.15(b) respectively, for a mode $k \simeq 100\mathcal{H}_0$, are shown together with the value of z for which c becomes zero (indicated by the red, dashed line in both the plots), $z_c \sim 0.9$, which corresponds to the value for which the derivative of h_f changes sign.

Chapter 6

Inflationary perturbations in bimetric gravity

Based on:

[35] G. Cusin, R. Durrer, P. Guarato and M. Motta, "Inflationary perturbations in bimetric gravity", JCAP **1509** (2015) 043 [arXiv:1505.01091 [astro-ph]].

Abstract. In this paper we study the generation of primordial perturbations in a cosmological setting of bigravity during inflation. We consider a model of bigravity which can reproduce the ΛCDM background and large scale structure and a simple model of inflation with a single scalar field and a quadratic potential. Reheating is implemented with a toy-model in which the energy density of the inflaton is entirely dissipated into radiation. We present analytic and numerical results for the evolution of primordial perturbations in this cosmological setting. We find that the amplitude of tensor perturbations generated during inflation is sufficiently suppressed to avoid the effects of the tensor instability discovered in Refs. [110, 34] which develops during the cosmological evolution in the physical sector. We argue that from a pure analysis of the tensor perturbations this bigravity model is compatible with present observations. However, we derive rather stringent limits on inflation from the vector and scalar sectors.

6.1 Introduction

The question whether the graviton may have a mass has attracted considerable attention in the last decade. However, constructing a viable theory of massive gravity is a non-trivial problem since the presence of a mass term in the gravity action removes diffeomorphism invariance: the metric has six degrees of freedom (four being absorbed by the Bianchi identities), five of these represent the massive graviton while the sixth is usually a ghost, the so-called Boulware-Deser ghost [26, 57]. To remove this ghost one needs an additional constraint. This has been achieved with a very specific form of the potential for the gravitational field, the dRGT (de Rham, Gabadadze, Tolley) potential [47, 49, 85], which has been the basis for much of the recent work on this topic (see, e.g., [82, 86, 108, 77] and refs. therein).

In massive gravity theories, the mass term is defined with respect to a fixed reference metric and the possible solutions of course strongly depend on this reference metric. Moreover, even when choosing the reference metric to be Friedmann, the resulting solutions either do not show the well known cosmological behavior, or they are unstable [79, 39, 111, 69, 133], see [45] for a review.

Also from a theoretical point of view, it is rather unsatisfactory to introduce the reference metric as an 'absolute element', i.e., a non-dynamical field in the theory. For this reason, bimetric (or more general multi-metric) theories, with a dynamical reference metric, are more natural [83, 84, 80]. Investigations of theoretical aspects of bimetric massive gravity can be found in [13, 88, 51, 37, 124, 11, 33, 46].

Cosmological solutions of bimetric theories can actually fit the expansion history of the accelerating Universe [139, 30, 106, 136, 70]. Observational tests of several models of bigravity are presented in [132, 143, 21, 12, 107]. The cosmology of bigravity in various cosmological settings is studied in [10, 104] while in Refs. [78, 29, 67] the cosmology of models of bigravity where matter is coupled to a combination of the two metrics is investigated.

Cosmological perturbations in bigravity have been studied in different settings and for different models in [31, 102, 32, 41]. A more generic study of instabilities in bimetric theories can be found in Refs. [109, 110]. Recently, scalar perturbations of these models have been investigated and it has been shown that there exists a class of models of bigravity that admit solutions with scalar perturbations free of exponential instabilities at all times, while the other models do exhibit exponential instabilities in the scalar sector [105, 104].

The evolution of tensor perturbations in this particular class of models has been first studied in [110] and in more detail in [34]. The problem of how cosmological observations are affected by these tensor instabilities and possible ways out are discussed in [15]. In [68] a general analysis of the tensor sector in models which are free from known instabilities is presented and it is discussed how measurements of the amplitude of primordial gravitational waves can be used to constrain them.

In Ref. [34] it has been found that tensor perturbations are affected by a power-law instability connected with the violation of the Higuchi bound [91] in the sector not-coupled to matter. This instability is then transferred to the physical sector through the coupling between the two tensor modes. By fine-tuning the amplitude of the unstable tensor mode to be highly suppressed with respect to the one of the physical sector at the end of inflation, one can achieve that the instability does not show up in the physical metric until today. In [34] this fine-tuning is explicitly quantified. The problem of the viability of the model is therefore translated into the question: is the amplitude of the uncoupled tensor mode after inflation sufficiently suppressed with respect to the one of the graviton coupled to matter?

In this paper we address this question in detail, i.e., we embed the model of bi-gravity studied in [34] in an inflationary scenario and determine the amplitude and the spectrum of primordial tensor perturbations generated at the end of inflation. We find an expression for the ratio between the amplitude of the two tensor modes at the beginning of the radiation era as a function of the reheating temperature of the inflationary model. We argue that however, the amplitude of the non-physical metric is nearly entirely in the constant mode and the growing mode amplitude is much smaller, actually too small to affect the 'physical metric', i.e. the metric which is coupled to matter in the way discussed in [34]. From this linear analysis, the model is therefore not in conflict with observations.

We also investigate the vector and the scalar sector. We find that in the model considered, in addition to the nearly scale invariant inflaton mode, very large vector perturbations and a very red spectrum of scalar perturbations are generated during inflation. The condition for the vector mode not to spoil perturbation theory (i.e. back-react) during inflation constrains the scale of inflation substantially.

While we were working on this, a study of primordial gravitational waves in this model has appeared in [101] in a larger context. Our analysis goes beyond the results presented in [101]. We perform an analytical study of primordial perturbations in all the sectors. We study in detail, both numerically and analytically, tensor perturbations during inflation and reheating. The main results of [101] are confirmed. We also find that tensor perturbations of the second metric which are generated during inflation are too small to affect the observable gravitational waves in the physical metric.

The paper is organised as follows. In the next section we present the equations of motion of bimetric gravity for cosmological (i.e. homogeneous and isotropic) space-times. We then specialise to a model which gives an acceptable expansion history. In Sec. 6.3 we present our model of inflation and reheating and we study its background evolution. In Sec. 6.4 we briefly review the perturbation equations of bimetric gravity. The study of tensor perturbations is presented in Sec. 6.5 and discussed in Sec. 6.6. In Sec. 6.7 we study the generation of vector perturbations during inflation and in Sec. 6.8 we discuss scalar perturbations. Finally, in Sec. 6.9 we conclude.

Notation: We set $c = \hbar = k_{\text{Boltzmann}} = 1$. $M_g = 1/\sqrt{8\pi G} \equiv M_p \simeq 2.4 \times 10^{18}$GeV is the reduced Planck mass. We work with the metric signature $(-,+,+,+)$, and we restrict to 4 spacetime dimensions. With $\cdot$ and with $'$ we indicate derivatives with respect to physical time and to conformal time, respectively. The conventions for the bigravity action are those of [83, 84, 80]. We consider only one of the two metrics coupled to matter, and we restrict to minimal couplings. For self-consistency, the Hassan-Rosen bi-metric action and the general equations of motion are given explicitly in Appendix 6.10.

6.2 Cosmological ansatz and background equations

We consider solutions of bigravity where both metrics are spatially isotropic and homogeneous. For simplicity, we also assume that both metrics have flat spatial sections, $K = 0$. Modulo time reparameterizations, the most general form for the metrics (in conformal time τ) is

$$g_{\mu\nu}dx^\mu dx^\nu = a^2(\tau)\left(-d\tau^2 + \delta_{ij}dx^i dx^j\right) , \tag{6.1}$$

$$f_{\mu\nu}dx^\mu dx^\nu = b^2(\tau)\left(-c^2(\tau)d\tau^2 + \delta_{ij}dx^i dx^j\right) . \tag{6.2}$$

Here a and b are the scale factors of the two metrics and c is a lapse function for f. It is convenient to define both the conformal Hubble parameter ($\mathcal{H}$) and the standard one (H) for both metrics

$$H = \frac{\mathcal{H}}{a} = \frac{a'}{a^2} , \qquad H_f = \frac{\mathcal{H}_f}{b} = \frac{b'}{b^2\, c} , \tag{6.3}$$

where $'$ denotes the derivative with respect to the conformal time τ. We introduce also the ratio between the two scale factors

$$r = \frac{b}{a} . \tag{6.4}$$

In the matter sector, we consider the energy-momentum tensor of a covariantely conserved perfect fluid with equation of state $p = w\rho$ and 4-velocity u^μ. Explicitly,

$$T_{\mu\nu} = (p + \rho)\; u_\mu u_\nu + p\, g_{\mu\nu} , \tag{6.5}$$

$$\rho' = -3\,\mathcal{H}\,(\rho + p), \tag{6.6}$$

$$p = w\rho . \tag{6.7}$$

The general Lagrangian of bimetric gravity and the resulting modified Einstein equations for the metrics g and f are presented in the Appendix 6.10.

The Bianchi constraint in the cosmological ansatz can be written as

$$\rho_G' = -3\,\mathcal{H}\,\left(\rho_G + p_G\right) , \tag{6.8}$$

where we have introduced a 'gravity fluid' with density and pressure given by

$$\rho_G = \frac{m^2}{8\pi G}\left(\beta_3\, r^3 + 3\beta_2\, r^2 + 3\beta_1\, r + \beta_0\right) , \tag{6.9}$$

$$p_G = -\frac{m^2}{8\pi G}\left(\beta_3 c\, r^3 + \beta_2(2c+1)r^2 + \beta_1(c+2)r + \beta_0\right) . \tag{6.10}$$

Here the β_i are the parameters of the bigravity potential, see Appendix 6.10. It is easy to show that the Bianchi constraint (6.8) is equivalent to

$$m^2\left(\beta_3\, r^2 + 2\beta_2\, r + \beta_1\right)\left(c\, b\, a' - a\, b'\right) = 0 . \tag{6.11}$$

The equations of motion (the Friedmann equation and the acceleration equation) for the metric g are

$$3H^2 = 8\pi G\,\left(\rho + \rho_G\right) , \tag{6.12}$$

$$3H^2 + 2\dot{H} = -8\pi G\,\left(p + p_G\right) , \tag{6.13}$$

while for the f metric we find the equations of motion

$$3\, H_f^2 = \frac{m^2}{\alpha^2}\left(\frac{\beta_1}{r^3} + \frac{3\beta_2}{r^2} + \frac{3\beta_3}{r} + \beta_4\right) , \tag{6.14}$$

$$\dot{H}_f = \frac{1}{2}\frac{m^2}{\alpha^2}(1-c)\left(\frac{\beta_1}{r^3} + \frac{2\beta_2}{r^2} + \frac{\beta_3}{r}\right) , \tag{6.15}$$

where α is a dimensionless parameter, $\alpha \equiv M_f/M_g$ (see also Appendix 6.10). Under the rescaling $f_{\mu\nu} \to \alpha^{-2} f_{\mu\nu}$ and $\beta_n \to \alpha^n\, \beta_n$, the equations of motion become independent of α [21, 80], which has motivated many works on the cosmology of bigravity to simply set $\alpha = 1$, as we shall do here. Recently, however, it has been argued that this choice actually hides the possibility to recover General Relativity (GR) with a cosmological constant in the limit $\alpha \to 0$, see [10] for a detailed discussion.

We distinguish two branches of solutions, depending on how the Bianchi constraint (6.11) is implemented. Either there is an algebraic constraint for r,

$$\beta_3\, r^2 + 2\beta_2\, r + \beta_1 = 0 , \tag{6.16}$$

or

$$\mathcal{H}_f = \mathcal{H} , \qquad r\, H_f = H . \tag{6.17}$$

At the background level the first branch with constant r is equivalent to GR with an effective cosmological constant, while the second one gives rise to a richer cosmology. We will focus on the second branch in the rest of this work. The Bianchi constraint in the second branch can be re-written as

$$c = \frac{r' + r\mathcal{H}}{r\mathcal{H}} . \tag{6.18}$$

This fixes c as a function of $\mathcal{H}$, r and r'.

From now on, we will focus on the so-called '$\beta_1\beta_4$ model' of bigravity, where all the β_n parameters but β_1 and β_4 are set to zero. This model is also called the 'infinite branch $\beta_1\beta_4$ model' or 'infinite branch bigravity' in Ref. [105], referring to the fact that the initial condition for r has to be chosen in such a way that r evolves from infinity to a finite value during the cosmological evolution, in order for the exponential instabilities in the scalar sector not to show up. As already mentioned, this model is the only one free of these instabilities. The study of the cosmological evolution of this model has been addressed in a series of recent papers [34, 110, 105].

6.3 Scalar field inflation and reheating

6.3.1 General setting

In this work we focus on the evolution of the $\beta_1\beta_4$ model of bigravity during the inflationary period, where the dynamics of the universe is dominated by a scalar field ϕ, the inflaton, minimally coupled to the physical metric g. We consider a simple model of inflation with a single scalar field with mass M_ϕ and quadratic potential. We choose the best-fit values $\beta_1 m^2 = 0.48\, H_0^2$ and $\beta_4 m^2 = 0.94\, H_0^2$ obtained in [107] and [105] by fitting measured growth data and type Ia supernovae.

The Lagrangian density for the inflaton can be written as

$$\mathcal{L}_\phi = -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\,, \qquad V(\phi) = \frac{1}{2}M_\phi^2\,\phi^2\,. \tag{6.19}$$

The field ϕ can in principle interact with other fields such as fermions, gauge bosons, etc., but we assume that this interaction can be neglected during inflation and that energy and pressure are dominated by the contribution from the inflaton. The energy-momentum tensor of ϕ is given by

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi + g_{\mu\nu}\,\mathcal{L}_\phi = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\,g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi + V(\phi)\right)\,. \tag{6.20}$$

For the energy density and pressure this yields

$$\rho_\phi = -T^0_0 = \frac{\phi'^2}{2a^2} + \frac{1}{2a^2}\,(\nabla\phi)^2 + V(\phi) \simeq \frac{\phi'^2}{2a^2} + V(\phi) \simeq V(\phi)\,, \tag{6.21}$$

and

$$p_\phi \equiv \omega_\phi\,\rho_\phi = \frac{T^i_i}{3} = \frac{\phi'^2}{2a^2} - \frac{1}{6a^2}\,(\nabla\phi)^2 - V(\phi) \simeq \frac{\phi'^2}{2a^2} - V(\phi) \simeq -V(\phi)\,. \tag{6.22}$$

The first approximation in eqs. (6.21,6.22) is due to the fact that here we suppose that there exists some (sufficiently large) region of space within which we may neglect the spatial derivatives of ϕ at some initial time τ_i, explicitly $\nabla\phi(\mathbf{x},\tau_\mathrm{i}) \ll \phi'(\mathbf{x},\tau_\mathrm{i})$. The second approximation is due to the fact that we also suppose that in this region of space also the time derivative is much smaller than the potential, $\dot{\phi}(\mathbf{x},\tau_\mathbf{i}) \ll V^{1/2}(\phi)$. These slow-roll conditions are such that we have $p_\phi \equiv \omega_\phi \, \rho_\phi \simeq -\rho_\phi$ and $\rho_\phi + 3p_\phi \simeq -2\,V(\phi) < 0$.

At early times in some sufficiently large patch, the Universe is dominated by the potential of a slowly varying (slow rolling) scalar field, and hence it is in an inflationary phase. As time goes on, the scalar field starts evolving faster and inflation eventually comes to an end when the time derivative of ϕ grows to the order of $V^{1/2}$. When inflation ends, ϕ decays rapidly and starts oscillating about its minimum. At the end of inflation, the inflaton oscillates as

$$\phi \simeq \phi_0 \cos(M_\phi \, \tau)\,. \tag{6.23}$$

The field ϕ is a damped harmonic oscillator with frequency M_ϕ. For a harmonic oscillator, when averaging over one period we have

$$\langle V \rangle = \frac{\langle \phi'^2 \rangle}{2a^2}\,, \tag{6.24}$$

so that

$$\langle p_\phi \rangle = \left\langle \frac{\phi'^2}{2a^2} - V \right\rangle = 0\,, \quad \text{and hence} \quad \langle \rho_\phi \rangle \propto a^{-3}\,. \tag{6.25}$$

We assume that during these oscillations the coupling of ϕ to other degrees of freedom than $g_{\mu\nu}$ becomes relevant and the inflaton finally decays into a mix of elementary particles which rapidly thermalize. As a simple approximation to this complicated and model dependent reheating process, we describe the coupling with the other degrees of freedom by means of a dissipative term $\propto \Gamma\phi'$ in the equation of motion. In physical time, the equation of motion for the inflaton becomes

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma\dot{\phi} = -V_{,\phi}(\phi)\,. \tag{6.26}$$

During inflation $H \gg \Gamma$ and particle production is negligible. When $H \simeq \Gamma$, reheating takes place and the inflaton energy is rapidly dissipated into other particles which couple to the inflaton.

We consider a toy-model of reheating in which the energy density of the inflaton is entirely dissipated into radiation. In this setting, the total energy momentum tensor has a contribution given by the inflaton and one given by radiation,

$$T_{\mu\nu} = T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(r)}\,. \tag{6.27}$$

Initially $T_{\mu\nu}^{(r)} = 0$. The total energy momentum is covariantly conserved:

$$\nabla_\mu T^{\mu\nu} = 0 \qquad \Longrightarrow \qquad \nabla_\mu T^{\mu\nu\,(\phi)} = -\nabla_\mu T^{\mu\nu\,(r)}\,. \tag{6.28}$$

In our cosmological setting, eq. (6.28) is equivalent to the following set of equations:

$$\dot{\rho_\phi} + 3\,\frac{\dot{a}}{a}\,(\rho_\phi + p_\phi) = -\Gamma\,(\rho_\phi + p_\phi)\,, \tag{6.29}$$

$$\dot{\rho}_r + 3\,\frac{\dot{a}}{a}\,(\rho_r + p_r) = \Gamma\,(\rho_\phi + p_\phi)\,. \tag{6.30}$$

It is easy to check that eq. (6.29) is equivalent to the equation of motion for the inflaton, eq. (6.26).

6.3.2 Background evolution during inflation

To study the background evolution during inflation, we consider as a complete set of independent equations the two Friedmann equations, (6.12), (6.14), and the acceleration equation for the g-metric, (6.13), the Bianchi constraint in the second branch, (6.17), the equation of state for the inflaton, (6.22) and the equation of motion for the inflaton, (6.26). Solving the two Friedmann equations together with the acceleration equation for g, we can express r', H and ρ_r as functions of r and ρ_ϕ

$$\frac{\dot{r}}{H} = \frac{r'}{\mathcal{H}} = \frac{-3\,r\,(1+\omega_r)\,(\beta_4\,r^3 - 3\beta_1 r^2 + \beta_1) + 3r^2\,(\omega_r - \omega_\phi)\,8\pi G\,m^{-2}\,\rho_\phi}{2\beta_4\,r^3 - 3\beta_1 r^2 - \beta_1}\,, \tag{6.31}$$

$$H^2 = \frac{\mathcal{H}^2}{a^2} = m^2\,\frac{\beta_1 + \beta_4 r^3}{3\,r}\,, \tag{6.32}$$

$$8\pi G\rho_r = m^2\frac{\beta_1}{r} - 3\,m^2\beta_1 r + m^2\beta_4\,r^2 - 8\pi G\rho_\phi\,, \tag{6.33}$$

where the suffixes 'r' and 'ϕ' refer to radiation and to the inflaton, respectively; with $\rho_\phi = \frac{1}{2a^2}\,\phi'^2 + \frac{1}{2}M_\phi^2\phi^2$ and $\omega_\phi \equiv p_\phi/\rho_\phi = -1 + \phi'^2/(a^2\rho_\phi)$. The equation of motion for the inflaton, eq. (6.26), in conformal time can be written as

$$\phi'' + 2\mathcal{H}\phi' + a\Gamma\phi' + a^2 V_{,\phi}(\phi) = 0\,. \tag{6.34}$$

The two differential equations (6.31) and (6.34) are coupled and we solve them together with initial conditions given deep in the inflationary epoch. We choose the expectation value of the inflaton at the beginning of inflation to be of order $10\,M_p$. Since during inflation the slow-roll condition holds and $\Gamma \ll H$, the initial conditions for eq. (6.34) can then be parametrized as

$$\phi(\tau_i) = 10 M_p\,, \tag{6.35}$$

$$\frac{\phi'}{a}(\tau_i) = -\frac{V_{,\phi}}{3H}(\tau_i) = -\frac{M_\phi^2\,\phi}{3H}(\tau_i)\,, \tag{6.36}$$

where we choose for the mass of the inflaton $M_\phi \simeq 0.2$ eV.[1] Therefore, the state parameter for the inflaton can also be written as

$$\omega_\phi|_{\tau\approx\tau_i} = -1 + \frac{2M_\phi^2}{3\beta_4 m^2 r^2}\,, \tag{6.37}$$

[1]Therefore, from $H(\tau_i)^2 \simeq \frac{8\pi G}{3}\,V(\tau_i) \simeq \left(\frac{M_\phi}{M_p}\right)^2\,\frac{\phi(\tau_i)^2}{6}$ it follows that $H(\tau_i) \simeq 1$ eV.

where in the last equality we have used the fact that during inflation $r \gg 1$ (as follows from eq. (6.32)) and $\rho_r \simeq 0$.

Eq. (6.32) and $r \gg 1$ also imply that during inflation

$$r(\tau_i) \simeq \sqrt{\frac{3\,H(\tau_i)^2}{m^2\,\beta_4}} \sim \mathcal{O}\left(\frac{H}{H_0}\right). \tag{6.38}$$

Once the coupled differential equations (6.31) and (6.34) are solved with initial conditions (6.35), (6.36) and (6.38), the evolution of the Hubble parameter, of the lapse c and of ρ_r (via eq. (6.33)) can be derived.

The results of the numerical integration are shown in Figs. 6.1, 6.2 and 6.3. For the numerical integration, the parameter Γ in eq. (6.34) has been chosen such that $\Gamma = H(z_{\rm reh})$, where $z_{\rm reh} = 5 \cdot 10^{17}$ is the reheating redshift. Fig. 6.1 shows that the inflaton starts oscillating at the end of inflation, and that this oscillation is transferred to ω_ϕ and c, which starts from the value $c = 1$ during inflation and becomes $c = -1$ in radiation domination. The variable r is almost constant during inflation ($r_I \sim 10^{33}$) and it starts to decay rapidly in the radiation dominated era. Fig. 6.2(a) shows that the physical Hubble parameter is almost constant during inflation and then starts to decrease. Fig. 6.3 shows that at the end of inflation the energy density of the inflaton is matter-like while the energy density of radiation produced by the decaying of the inflaton has the usual evolution with time[2] $\propto a^{-4}$.

6.4 Analysis of perturbations: gauge invariant variables

We consider perturbations around the Friedmann backgrounds,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + a^2\,h_{g\,\mu\nu}\,, \qquad\qquad f_{\mu\nu} = \bar{f}_{\mu\nu} + b^2\,h_{f\,\mu\nu}\,. \tag{6.39}$$

In this section, background quantities are indicated with an overbar. We parametrize the perturbations in as follows:

$$(h_{g\,\mu\nu}) = \begin{pmatrix} -2A_g & C_{gj} - \partial_j B_g \\ C_{gi} - \partial_i B_g & h^{TT}_{gij} + \partial_i \mathcal{V}_{gj} + \partial_j \mathcal{V}_{gi} + 2\partial_i\partial_j E_g + 2\delta_{ij} F_g \end{pmatrix}, \tag{6.40}$$

$$(h_{f\,\mu\nu}) = \begin{pmatrix} -2c^2 A_f & C_{fj} - \partial_j B_f \\ C_{fi} - \partial_i B_f & h^{TT}_{fij} + \partial_i \mathcal{V}_{fj} + \partial_j \mathcal{V}_{fi} + 2\partial_i\partial_j E_f + 2\delta_{ij} F_f \end{pmatrix}, \tag{6.41}$$

[2] We have also checked that the evolution of ρ_r from eq. (6.33) is equivalent to the one obtained solving the differential equation (6.30), with vanishing initial condition for ρ_r at early times.

with

$$\partial_i C^i_{g,f} = \partial_i \mathcal{V}^i_{g,f} = \partial_i h^{TTij}_{g,f} = 0\,, \qquad \delta^{ij} h^{TT}_{g,fij} = 0\,. \tag{6.42}$$

Spatial indices are raised and lowered using the flat spatial metric δ_{ij}.

In the scalar sector we have 8 fields and 2 gauge freedoms, hence we can form 6 gauge invariant combinations which can be chosen as [31, 132]

$$\begin{aligned}
&\Psi_g = A_g - \mathcal{H}\,\Gamma_g A_g - \Gamma'_g\,, \quad \Psi_f = A_f + c^{-2}\left(\frac{c'}{c} - c\,\mathcal{H}_f\right)\Gamma_f - c^{-2}\,\Gamma'_f\,,\\
&\Phi_g = F_g - \mathcal{H}\,\Gamma_g\,, \quad \Phi_f = F_f - c^{-1}\,\mathcal{H}_f\,\Gamma_f\,,\\
&\mathcal{E} = E_g - E_f\,, \quad \mathcal{B} = B_f - c^2\,B_g + (1 - c^2)E'_g\,,
\end{aligned} \tag{6.43}$$

where $\Gamma_{g,f} \equiv B_{g,f} + E'_{g,f}$. In the vector sector we have 4 fields and 1 gauge freedom and hence we can form 3 gauge-invariant combinations which we choose as follows [31, 41]

$$V_{g,fi} = C_{g,fi} - \mathcal{V}'_{g,fi}\,, \qquad \chi_i = C_{gi} - C_{fi}\,. \tag{6.44}$$

The energy-momentum tensor for the perturbed universe is

$$T^\mu_\nu = \bar{T}^\mu_\nu + \delta T^\mu_\nu\,. \tag{6.45}$$

The perturbations can be divided in perfect-fluid and non-perfect-fluid ones, with 5+5 dof (degrees of freedom). The perfect fluid dof in δT^μ_ν are those which keep T^μ_ν in the perfect fluid form:

$$T^\mu_\nu = (p + \rho)\, u^\mu\, u_\nu + p\, \delta^\mu_\nu\,. \tag{6.46}$$

We suppose here that the perturbations are only of this type. Thus, they are given by the density perturbation, the pressure perturbation and the velocity perturbation. Explicitly:

$$p = \bar{p} + \delta p\,, \qquad \rho = \bar{\rho} + \delta\rho\,, \qquad u^i = \bar{u}^i + \delta u^i = \delta u^i \equiv \frac{1}{a} v_i\,. \tag{6.47}$$

The δu^0 is not an independent dof, it is fixed by the normalisation $u_\mu u_\nu\, g^{\mu\nu} = -1$.

We can now write the perturbed Einstein equations for the two metrics. In the following we will use the Fourier transform of perturbations with respect to x^i, the corresponding 3-momentum will be k^i and $k^2 \equiv k_i k^i$. To keep the notation simple, the Fourier transform will be denoted by the same symbol as the original function.

6.5 Tensor perturbations

Tensor perturbations of a given **k**-mode are composed of two independent helicity modes,

$$h^{TT}_{ij} = h^+ e^{(+2)}_{ij} + h^- e^{(-2)}_{ij} \tag{6.48}$$

where + and − denote the two helicity-2 modes of the gravitational wave. For an orthonormal system $(\widehat{\mathbf{k}}, \mathbf{e}^{(1)}, \mathbf{e}^{(2)})$ we have

$$\mathbf{e}^{\pm} = \frac{1}{\sqrt{2}}\left(\mathbf{e}^{(1)} \pm i\mathbf{e}^{(2)}\right) \quad \text{and} \quad e_{ij}^{(+2)} = \mathbf{e}_i^{+}\mathbf{e}_j^{+}\,, \quad e_{ij}^{(-2)} = \mathbf{e}_i^{-}\mathbf{e}_j^{-}\,. \tag{6.49}$$

In what follows we assume parity invariant perturbations,

$$\langle h^{+}(\mathbf{k})(h^{+}(\mathbf{k}'))^{*}\rangle = k^3\langle h^{-}(\mathbf{k})(h^{-}(\mathbf{k}'))^{*}\rangle = \delta(\mathbf{k}-\mathbf{k}')2\pi^2 P_h(k)\,,$$

and $\langle h^{+}h^{-}\rangle = 0$. And we shall consider just one mode, say $h_f^{+} = h_f G_f$ and $h_g^{+} = h_g G_g$, where G_f and G_g are uncorrelated Gaussian random variables with vanishing mean and variance $\langle G_{g,f}(\mathbf{k})G_{g,f}(\mathbf{k}')\rangle = \delta(\mathbf{k}-\mathbf{k}')2\pi^2$, so that $h_{g,f}$ is the square root of the power spectrum. All the following is also valid for the modes $h_{g,f}^{-}$ which are not correlated with $h_{g,f}^{+}$ in the parity symmetric situation which we consider.

With a perfect fluid source term, i.e. no anisotropic stress, in the first order modified Einstein equation, we obtain the following tensor perturbation equations for our bimetric cosmology [34].

$$h_g'' + 2\mathcal{H}\,h_g' + k^2 h_g + m^2a^2 r\,\beta_1\,(h_g - h_f) = 0\,, \tag{6.50}$$

$$h_f'' + \left[2\left(\mathcal{H} + \frac{r'}{r}\right) - \frac{c'}{c}\right] h_f' + c^2k^2\,h_f - m^2\beta_1\frac{c\,a^2}{r}\,(h_g - h_f) = 0\,. \tag{6.51}$$

In Ref. [34] we have solved these coupled differential equations in the radiation era and have found that h_f has a growing mode, $h_f \propto \tau^3$ on large scales which via the coupling enhances also the mode h_g of the physical metric. Here we solve these equations numerically and analytically in the inflationary regime, where sensible approximations can be introduced to simplify the system.

6.5.1 Analytical results during inflation

Deep in the inflationary epoch, the potential $V(\phi)$ is very flat and the inflaton is slowly rolling. Since $p_\phi \simeq -\rho_\phi$, it is legitimate to model this period as a de Sitter phase with constant Hubble parameter $H = H_I \simeq$ const. (where the suffix 'I' hereafter stands for inflation). From eq. (6.32) it follows that during inflation $r = r_I =$ const., with $r_I^2 \simeq 3H_I^2/(m^2\beta_4) \simeq 3\,(H_I/H_0)^2$ and eq. (6.18) gives for the lapse function of the f-metric $c \simeq$ const $\simeq 1$. With the parametrization $\mathcal{H} = -1/\tau$ and $a = -1/(\tau\,H_I)$ (note that with this choice $\tau < 0$ during inflation), and $m^2\beta_1 a^2 \simeq (H_0/H_I)^2\tau^{-2}$. eqs. (6.50) and (6.51) can be approximated in a de Sitter universe as

$$h_g'' - \frac{2}{\tau}\,h_g' + k^2 h_g + \left(\frac{H_0}{H_I}\right)\frac{1}{\tau^2}\,(h_g - h_f) = 0\,, \tag{6.52}$$

$$h_f'' - \frac{2}{\tau}\, h_f' + k^2\, h_f - \left(\frac{H_0}{H_I}\right)^3 \frac{1}{\tau^2}\,(h_g - h_f) = 0\,. \tag{6.53}$$

These equations can be solved exactly in terms of oscillating and decaying modes.

We want to choose as initial conditions the quantum vacuum of the graviton degree of freedom. For tensor perturbations the canonically normalised variables (recall that $M_f = M_g = M_p$) are given by

$$(Q_g)_{ij} = e_{ij}\, Q_g = M_p\, a\, \left(h_{ij}^{TT}\right)_g\,, \qquad (Q_f)_{ij} = e_{ij}\, Q_f = M_p\, b\, \left(h_{ij}^{TT}\right)_f\,. \tag{6.54}$$

Equations (6.52) and (6.53) in terms of these new variables and recalling that $b = r\, a$ become

$$Q_g'' + \left(k^2 - \frac{2}{\tau^2}\right) Q_g + \left(\frac{H_0}{H_I}\right) \frac{1}{\tau^2} \left(Q_g - \frac{1}{r} Q_f\right) = 0\,, \tag{6.55}$$

$$Q_f'' + \left(k^2 - \frac{2}{\tau^2}\right) Q_f - \left(\frac{H_0}{H_I}\right)^3 \frac{1}{\tau^2}\,(r\, Q_g - Q_f) = 0\,. \tag{6.56}$$

Since during inflation $r_I \simeq H_I/H_0 \gg 1$, eqs. (6.55) and (6.56) can be approximated as

$$Q_g'' + \left(k^2 - \frac{2}{\tau^2}\right) Q_g + \left(\frac{H_0}{H_I}\right) \frac{1}{\tau^2}\, Q_g = 0\,, \tag{6.57}$$

$$Q_f'' + \left(k^2 - \frac{2}{\tau^2}\right) Q_f - \left(\frac{H_0}{H_I}\right)^2 \frac{1}{\tau^2}\, Q_g = 0\,. \tag{6.58}$$

For sub-horizon scales, $|k\tau| \gg 1$, eqs. (6.57) and (6.58) reduce to two copies of the same equation for a harmonic oscillator with frequency k. The quantum vacuum solutions are

$$Q_g = \frac{1}{\sqrt{2\,k}} \exp\left(-i\, k\tau\right), \qquad Q_f = \frac{1}{\sqrt{2\,k}} \exp\left(-i\, k\tau\right), \qquad \text{for} \qquad |k\tau| \gg 1\,. \tag{6.59}$$

We want to solve eqs. (6.57) and (6.58) with initial conditions (6.59). These equations can be decoupled introducing the new variable $Q_+ \equiv Q_f + \left(\frac{H_0}{H_I}\right) Q_g$

$$Q_g'' + \left(k^2 - \frac{2}{\tau^2}\right) Q_g + \left(\frac{H_0}{H_I}\right) \frac{1}{\tau^2}\, Q_g = 0\,, \tag{6.60}$$

$$Q_+'' + \left(k^2 - \frac{2}{\tau^2}\right) Q_+ = 0\,. \tag{6.61}$$

Eqs. (6.60) and (6.61) can be solved in terms of Bessel functions. Requiring that the asymptotic behavior (6.59) is recovered for $|k\,\tau| \gg 1$, we find the following solutions for the canonically normalized variables Q_g and Q_f

$$Q_g = -\sqrt{\frac{\pi}{2}}\,\sqrt{\frac{k\,\tau}{2\,k}}\, J_{\frac{1}{2}\sqrt{9-4\gamma}}\,(k\tau) + i\,\sqrt{\frac{\pi}{2}}\,\sqrt{\frac{k\,\tau}{2\,k}}\, Y_{\frac{1}{2}\sqrt{9-4\gamma}}\,(k\tau)\,, \tag{6.62}$$

$$Q_f = \frac{1}{\sqrt{2\,k}}\left(1+\frac{H_0}{H_I}\right)\left(1-\frac{i}{k\,\tau}\right)e^{-i\,k\tau}-\left(\frac{H_0}{H_I}\right)Q_g\,. \tag{6.63}$$

To simplify the notation, we have introduced the tiny constant $\gamma \equiv H_0/H_I$.[3] In the limit $\gamma \to 0$ we have $Q_g = Q_f = Q_+$.

The canonically normalized variables Q_g and Q_f are connected to the power spectrum by

$$P_{h_g}(k) = k^3\,|h^{TT}_{ij\,g}\,h^{ij\,TT}_g| = 2\cdot\frac{k^3\,|Q_g|^2}{a^2\,M_p^2}\,, \tag{6.64}$$

$$P_{h_f}(k) = k^3\,|h^{TT}_{ij\,f}\,h^{ij\,TT}_f| = \frac{2}{r_I^2}\cdot\frac{k^3\,|Q_f|^2}{a^2\,M_p^2}\,, \tag{6.65}$$

where the factor of 2 is due to the two helicity modes in each tensor sector. Hence from eqs. (6.62) and (6.63) we can find the solutions of eqs. (6.52) and (6.53) making use of the relations

$$h_g = \frac{1}{a\,M_p}\,k^{3/2}\,Q_g\,, \qquad\qquad h_f = \frac{1}{r_I}\,\frac{1}{a\,M_p}\,k^{3/2}\,Q_f\,. \tag{6.66}$$

6.5.2 Numerical results during inflation and reheating

The asymptotic behavior of the solutions (6.66) for $|k\tau| \gg 1$ during inflation is given by

$$h_g = -\frac{k}{\sqrt{2}M_p}\,H_I\,\tau\,e^{-i\,k\tau}\,, \qquad h_f = -\frac{k}{\sqrt{2}M_p}\,\frac{H_I\,\tau}{r_I}\,e^{-i\,k\tau}\,, \qquad \text{for} \qquad |k\tau| \gg 1\,. \tag{6.67}$$

These functions and their first derivatives can be evaluated at $\tau = \tau_i$ to find the initial conditions for the numerical evolution of the full tensor perturbation equations, (6.50) and (6.51). The results of the numerical integration are shown in Figs. 6.4 and 6.5, for four different k-modes.

Independently of the mode k, the agreement between the analytical solutions for h_g and h_f, obtained from (6.62) and (6.63), and the numerical one is reasonably good in the regime in which the slow-roll condition holds and the background is well approximated by de Sitter. As expected, both modes h_g and h_f oscillate in the redshift range in which $k > \mathcal{H}(z)$, see eq. (6.67). Furthermore, the mode h_f develops an instability at the end of the inflationary period, where it oscillates with increasing amplitude. This is due to the fact that the damping term in eq. (6.51) becomes an anti-damping term

[3] Given that for $|k\tau| \gg 1$, the behavior of the Bessel functions is $J_{\frac{1}{2}\sqrt{9-4\gamma}}(k\tau) \to -\sqrt{\frac{2}{\pi k\tau}}\cos(k\tau)$ and $Y_{\frac{1}{2}\sqrt{9-4\gamma}}(k\tau) \to -\sqrt{\frac{2}{\pi k\tau}}\sin(k\tau)$, the asymptotic behavior of (6.62) and (6.63) for $|k\tau| \gg 1$ is exactly of the type (6.59).

at the end of the inflationary stage. Indeed, using eq. (6.18), eq. (6.51) can be written as

$$h_f'' + \left[2\,c\,\mathcal{H} - \frac{c'}{c}\right] h_f' + c^2 k^2\, h_f - m^2 \beta_1 \frac{c\,a^2}{r}\,(h_g - h_f) = 0\,. \tag{6.68}$$

Since $c = 1$ at the beginning of the inflationary era whereas $c = -1$ in the radiation era, the term in square bracket changes sign when inflation ends, going from $2\mathcal{H}$ to $-2\mathcal{H}$, i.e. from a positive damping term to a negative anti-damping term.

From eqs. (6.62) and (6.63), it follows that on super horizon scales the power spectra at the end of inflation are scale invariant and given by

$$P_{h_g}(z,k) \simeq \left(\frac{H_I}{M_p}\right)^2 \simeq r_I^2\, P_{h_f}(z,k)\,, \qquad |k\,\tau| \ll 1\,. \tag{6.69}$$

This result has also been derived in [101]. P_{h_g} hence has the same behaviour as the standard (i.e., GR) tensor power spectrum, whereas P_{h_f} is suppressed with respect to P_{h_g} by a huge factor r_I^2. The numerical results for the power spectra at the end of inflation are shown in Fig. 6.6. In the analytical result (6.69) slow roll corrections are neglected since in this context we are mainly interested in orders of magnitude and not in very precise results. The power spectra shown in Fig. 6.6, however, are numerically calculated with the full model.

6.6 Discussion: is the model still viable?

In [34], the cosmological evolution of tensor perturbations in the $\beta_1\beta_4$ model of bigravity has been addressed and the condition needed for the instability not to show-up until present times has been quantified in terms of a fine-tuning of the amplitude of the two tensor modes after inflation. We have found that during the radiation dominated era tensor fluctuations of the f metric can grow like a^3 on super horizon scales. Furthermore, if they become larger than those of the g metric, $h_f(t_0) > h_g(t_0)$, they can influence the latter via the coupling term and show up e.g. in the CMB.

Considering naively, as from eq. (6.69), $h_f/h_g(z_e) = r_I^{-1} \simeq H_0/H_I$ at the end of inflation and that h_f grows by an amount $(T_{\rm eq}/T_e)^3 \simeq ((1+z_{\rm eq})/(1+z_e))^3$, the condition for the instability not to show-up $h_f(t_0) < h_g(t_0)$ [34] implies a very low bound on the value of H_I and therefore on the scale of inflation. Here the index $_e$ denotes the value of a quantity at the end of inflation.

However, this naive argument is misleading since at the end of inflation, when $|k\tau_e| \ll 1$ for the relevant modes,

$$h_f \sim r_I^{-1} \frac{H_I}{M_p} \left(1 + |k\tau_e|^2\right)\,. \tag{6.70}$$

Here, $\tau_e \simeq -(a_e H_I)^{-1} \simeq -(1+z_e)/H_I$ is the conformal time at the end of inflation. At the beginning of the radiation era the constant mode of h_f turns into the constant mode and only the very severely suppressed decaying mode, $\propto |k\tau_e|^2$ turns into the growing mode. Therefore, considering only this decaying mode for h_f at the end of inflation (when we put our initial conditions on the amplitudes of the tensor modes), our bound will be reduced by a factor $|k\tau_e|^2 \simeq ((1+z_e)k/H_I)^2$. This bound is valid for perturbations on very large scales with wave number $k \sim H_0$. Inserting this value for k, and replacing $((1+z_{\rm eq})/(1+z_e))^3 \simeq (H_{\rm eq}/H_I)^{3/2}$ by $(H_0/H_I)^{3/2}$ for simplicity[4], we obtain

$$\left(\frac{h_f}{h_g}\right)(t_0) \; \lesssim \; r_I^{-1}\left(\frac{H_0}{H_I}\right)^{-1/2} \simeq \left(\frac{H_0}{H_I}\right)^{1/2} \ll 1\,. \tag{6.71}$$

For the last $\simeq$ sign we used that $r_I^{-1} \simeq H_0/H_I$. Therefore, no meaningful bound is obtained. Also for the smallest scales, with wave number $k \simeq H_I/(1+z_e)$, there is no significant bound since inside the horizon the growth of h_f is only linear in a so that the factor $(T_{\rm eq}/T_e)^3$ has to be replaced by $(T_{\rm eq}/T_e)$ and the same result is obtained.

One may finally ask how model independent is the fact that the dominant constant mode at the end of inflation only transits into the constant mode of the radiation era. On super horizon scales and for negligible couplings the mode equations (6.50) and (6.51) are always of the form

$$h'' + \alpha(\tau)h' = 0, \tag{6.72}$$

so they always have a constant mode. The general solution of (6.72) in fact is simply

$$h(\tau) = A_1 \int^{\tau} d\tau' \left[\exp\left(-\int^{\tau'} \alpha(\tau'')d\tau''\right)\right] + A_2\,. \tag{6.73}$$

The first mode is decaying or growing, depending on the sign of α while the second mode is always constant. Hence even if there is a relatively long reheating phase, there will also be a constant mode during this phase.

Therefore, from a pure analysis of tensor perturbations, the model cannot be ruled out.

Note also that our inflationary model is typically above the strong coupling scale, Λ_3, of the theory [18]. For $T_{\rm reh} \simeq 10$ MeV we have $H_I \simeq T_{\rm reh}^2/M_P < 10^{-22}$ GeV $\simeq \Lambda_3 = (m^2 M_P)^{1/3} \sim 2\times 10^{-22}$ GeV. Therefore, if the reheating temperature is above 10MeV, bigravity becomes strongly coupled and it is not granted that cosmological perturbation theory still applies. However we underline that, since the strong coupling scale is derived in a Minkowski background, it is not entirely clear whether this represents an upper limit on the Hubble scale or just on the energy of the perturbations on a given background. For the sake of the argument, our approach here is to simply analyse the classical bigravity Lagrangian in a Friedmann Universe with small perturbations.

[4] The last replacement is an approximation which corresponds to consider radiation domination lasting until present time.

6.7 Vector perturbations

Vector perturbations of a given **k**-mode can be decomposed as

$$\mathcal{V}_i = \mathcal{V}^{(1)} e_i^{(1)} + \mathcal{V}^{(2)} e_i^{(2)} \,, \tag{6.74}$$

where the two orthonormal vectors $e_i^{(1)}$ and $e_i^{(2)}$ are defined in Sec. 6.5. In what follows we shall consider just one mode, say $\mathcal{V}^{(1)}$, since the situation is perfectly symmetric for the other mode and suppress the superscript, so that $\mathcal{V}_i \equiv e_i \mathcal{V}$. If the background is pure de Sitter, in the vector sector only the mode $\mathcal{V}_i \equiv \mathcal{V}_{gi} - \mathcal{V}_{fi}$ propagates [31]. The action for this vector mode in Fourier space can then be written as [5]

$$S_{\mathcal{V}} = \frac{M_p^2}{2} \int \mathrm{d}\tau \, \mathrm{d}^3k \, \frac{k^2 \, a^4 \, r \, m^2 \, \beta_1}{k^2 + a^2 \, r \, m^2 \, \beta_1} \left(|\mathcal{V}_i'|^2 - \left(k^2 + a^2 \, r \, m^2 \, \beta_1\right) |\mathcal{V}_i|^2 \right) , \tag{6.75}$$

where e.g. $|\mathcal{V}_i|^2 \equiv \mathcal{V}_i^* \mathcal{V}^i$. The canonically normalized variable in this case is defined as

$$\mathcal{Q}_i \equiv e_i \, \mathcal{Q} = M_p \, a^2 \, k \sqrt{\frac{r \, m^2 \, \beta_1}{k^2 + a^2 \, r \, m^2 \, \beta_1}} \, \mathcal{V}_i \,. \tag{6.76}$$

After integration by parts, the action (6.75) for the variable $\mathcal{Q}$ can be written as

$$S_{\mathcal{V}} = \int d\tau \, \mathrm{d}^3k \, \frac{1}{2} \left[(\mathcal{Q}')^2 - \mathcal{C}(k\,,\tau) \, \mathcal{Q}^2 \right] , \tag{6.77}$$

where, in order to simplify the notation, we have introduced

$$\mathcal{C}(k\,,\tau) = k^2 + \beta_1 \, m^2 \, r \, a^2 - 2\,\mathcal{H}^2 - \mathcal{H}' \left(\frac{k^2}{\beta_1 \, m^2 \, r \, a^2 + k^2} \right) - 3\,\mathcal{H}^2 \left(\frac{k^2}{\beta_1 \, m^2 \, r \, a^2 + k^2} \right)^2 . \tag{6.78}$$

Using that $\beta_1 m^2 \sim H_0^2$ and that in pure de Sitter Universe with Hubble constant H_I, $a = -1/(H_I \tau)$ and $r_I \simeq H_I/H_0$, the previous expression can be approximated as

$$\mathcal{C}(k\,,\tau) \simeq k^2 + \left(\frac{H_0}{H_I} \right) \frac{1}{\tau^2} - \left(\frac{(k\tau)^2}{H_0/H_I + (k\tau)^2} \right) \frac{1}{\tau^2} - \left(\frac{(k\tau)^2}{H_0/H_I + (k\tau)^2} \right)^2 \frac{3}{\tau^2} - \frac{2}{\tau^2} \simeq k^2 - \frac{6}{\tau^2} \,, \tag{6.79}$$

where in the last equality we have used that $H_0/H_I \ll 1$ and we have assumed that also $\beta_1 m^2 a^2 r \ll k^2$ holds, or equivalently $H_0/H_I \ll (k\tau)^2$. This second inequality is valid for the following reason: during the radiation era ra^2 =const.$\simeq \sqrt{3\Omega_{\mathrm{rad}}}$ (This can be derived from the two Friedmann equations, see [34].). Since this quantity is growing during inflation we have $r_I a^2 \leq \sqrt{3\Omega_{\mathrm{rad}}}$. Therefore $\beta_1 m^2 r a^2 \simeq H_0^2 r_I a^2 \simeq (H_0/H_I)\tau^{-2} < H_0^2 \sqrt{3\Omega_{\mathrm{rad}}} < k^2$ for all values $k \gtrsim H_0$ which are observable.

[5] This action is equivalent to the action (63) in ref. [41].

The equation of motion derived from the action (6.77) with $\mathcal{C}(k\,,\tau) \simeq k^2 - \frac{6}{\tau^2}$ is then [6]

$$\mathcal{Q}'' + \left(k^2 - \frac{6}{\tau^2}\right)\mathcal{Q} = 0\,. \tag{6.80}$$

For $|k\tau| \gg 1$, Eq. (6.80) reduces to a harmonic oscillator equation with frequency k, and has the vacuum solution

$$\mathcal{Q} = \frac{1}{\sqrt{2k}}\,e^{-ik\tau}\,, \qquad\qquad |k\tau| \gg 1\,. \tag{6.81}$$

Eq. (6.80) can be solved exactly. Asking that the asymptotic behavior (6.81) is recovered for $|k\tau| \gg 1$ the solution is given by

$$\mathcal{Q} = \frac{ik\tau}{\sqrt{2k}} h_2^{(2)}(k\tau)\,, \tag{6.82}$$

where $h_\ell^{(2)}$ is the spherical Hankel function of the second kind of order ℓ, see [5]. Substituting eq. (6.82) in eq. (6.76), the evolution of the physical variable $\mathcal{V}$ can be written in terms of Q. Using again that $r_I \simeq H_I/H_0$, $\beta_1 m^2 a^2 r \ll k^2$ and that $\beta_1 m^2 \sim H_0^2$, we obtain from eq. (6.76)

$$\mathcal{V}_i \simeq \frac{1}{M_p}\frac{1}{a^2}\frac{1}{H_I}\sqrt{\frac{H_I}{H_0}}\,\mathcal{Q}_i\,. \tag{6.83}$$

For superhorizon modes $|k\tau| \ll 1$, using the asymptotic behaviour of the spherical Hankel function, $h_2^{(2)}(x) \simeq -3/x^3$ for $|x| \ll 1$, we find

$$\mathcal{V}_i \simeq -\frac{3}{\sqrt{2}}\frac{H_I}{M_p}\sqrt{\frac{H_I}{H_0}}\,k^{-5/2}\,e_i\,. \tag{6.84}$$

Therefore, for super-Hubble scales, the vector power spectrum can be written as

$$P_{\mathcal{V}}(k\,,\tau) \equiv k^3|k\,\mathcal{V}_i|^2 \simeq 2\cdot\frac{9}{2}\left(\frac{H_I}{M_p}\right)^2\frac{H_I}{H_0}\,, \qquad\qquad |k\tau| \ll 1\,, \tag{6.85}$$

where the multiplication by a factor 2 in the last expression is due to the two vector modes and the powers of k are introduced to make the power spectrum dimensionless. Note the large enhancement by a factor H_I/H_0 with respect to the standard tensor spectrum which is of order $(H_I/M_p)^2$.

In order for linear perturbation theory to remain viable, one has to request at least that $P_{\mathcal{V}} < 1$, which means that our inflation model must be such that $H_I < 10^{-2}$ GeV

[6] One can also verify that the exact equation of motion for $\mathcal{Q}$ which can be derived from the action (6.77) with the exact expression for $\mathcal{C}(k\,,\tau)$ coincides with eq. (7.9) in [31], once written in terms of the original variable $\mathcal{V}$.

(where we have used the fact that $H_0 \sim 10^{-42}$ GeV and that $M_p \sim 2.4 \cdot 10^{18}$ GeV). This requires a rather low scale of inflation which is however acceptable.

Asking that vector perturbation should not be larger than scalar perturbations after inflation, $P_{\mathcal{V}} < 10^{-9}$ [9] would require an inflationary Hubble scale of $H_I < 10^{-5}$ GeV which corresponds to a reheating temperature of $T_{\rm reh} \lesssim (H_I M_p)^{1/2} \sim 10^6$ GeV. This inflationary scale is not excluded, see [73, 14] but rather low. However, we have not studied the evolution of vector perturbation during the radiation era. If they decay, as in GR, this second limit is not relevant. It only applies if vector perturbations in bigravity stay constant during the radiation dominated Universe.

6.8 Scalar perturbations

Let us now turn to scalar perturbations. For this we assume that, like for the other degrees of freedom, the difference to GR during inflation mainly comes from the existence of additional modes, but that the GR-modes are not strongly affected, since the coupling between the GR-modes and the additional modes is suppressed by H_0/H_I. We therefore assume that the inflaton mode leads to a nearly scale invariant spectrum like in GR, and we only study the additional helicity-0 mode of the massive graviton. For this we work in a pure de Sitter background and neglect the slow roll and the inflaton perturbation. In this situation the helicity-0 mode of the massive graviton is the only dynamical scalar degree of freedom. It is given by a linear combination of the two Bardeen potentials [31],

$$\Phi \equiv \Phi_g - 2r_I^2\Phi_f \,. \tag{6.86}$$

Its evolution is governed by the equation

$$\begin{aligned} \Phi'' + 2\mathcal{H}\Phi'\left[\frac{2k^4}{9a^2\mathcal{H}^2m_\Phi^2 + k^4 - 18\mathcal{H}^4} - 1\right] + \\ \frac{1}{3}\Phi\left[\frac{4\,(k^6 - 3k^4\mathcal{H}^2)}{9a^2\mathcal{H}^2m_\Phi^2 + k^4 - 18\mathcal{H}^4} + 3a^2m_\Phi^2 - k^2 - 6\mathcal{H}^2\right] &= 0, \end{aligned} \tag{6.87}$$

where

$$m_\Phi^2 = m^2\beta_1\left(\frac{1}{r_I^2} + 1\right) \simeq m^2\beta_1 \sim H_0^2 \,. \tag{6.88}$$

Using the same approximations as for the vector mode, eq. (6.87) can be approximated on sub-Hubble scales by

$$\Phi'' + 2\mathcal{H}\Phi' + k^2\Phi = 0\,, \qquad |k\tau| \gg 1\,. \tag{6.89}$$

Analogously to tensors, we quantize the scalar perturbations in order to find the initial conditions. The canonical variable is given by $\phi = M_p\, a\, \Phi$. In terms of this variable, eq. (6.89) reduces to a harmonic oscillator equation with vacuum solution $\phi(\tau) = e^{-ik\tau}/\sqrt{2k}$. It follows that

$$\Phi(\tau) = -\frac{H_I}{M_p}\frac{e^{-ik\tau}}{\sqrt{2k}}\tau\,, \qquad |k\tau| \gg 1\,. \tag{6.90}$$

On super-Hubble scales eq. (6.87) can be approximated by

$$\Phi'' - 2\mathcal{H}\Phi' - 2\mathcal{H}^2\Phi = 0\,, \qquad |k\tau| \ll 1\,, \tag{6.91}$$

with general solution

$$\Phi = c_1\tau + \frac{c_2}{\tau^2}\,, \qquad |k\tau| \ll 1\,, \tag{6.92}$$

where c_1 and c_2 are integration constants, which can be fixed by matching the sub-Hubble solution and its derivative with the one in the super-Hubble regime.

Note that the mode $\propto c_2$ manifests an instability on super Hubble scales since $|\tau|$ is decreasing during inflation. This is the manifestation of the fact that also during inflation the Higuchi bound is violated in the scalar sector. Indeed, for the scalar sector (helicity-0 mode) the Higuchi bound of the $\beta_1\beta_4$ model is given by [70, 110]

$$\beta_1\left(3r + \frac{1}{r}\right) - 2\beta_4 r^2 > 0\,,$$

which is violated for $r > 1.02$. In our treatment, neglecting couplings of the scalar mode to other modes, the instability coming from this violation only shows up as a growth of perturbations on super-Hubble scales which leads to a red spectrum as we now show. Note also that the growth is exponential in physical time since $\tau^{-2} \propto a^2 \propto \exp(2H_I t)$.

Working out the matching conditions explicitly we obtain

$$\Phi = -e^{i}\frac{H_I}{M_p}\frac{1}{\sqrt{2k}}\left(\left(\frac{i}{3}+1\right)\tau + \frac{i}{3}\frac{1}{\tau^2 k^3}\right)\,, \qquad |k\tau| \ll 1\,, \tag{6.93}$$

The power spectrum for scalar perturbations on super-horizon scales can then be expressed as

$$P_\Phi(\tau,k) = k^3|\Phi|^2 \simeq \frac{1}{18}\left(\frac{H_I}{M_p}\right)^2\left(\frac{\mathcal{H}}{k}\right)^4 \propto k^{-4}\,, \qquad |k\tau| \ll 1\,. \tag{6.94}$$

This very red power spectrum is strongly enhanced on large scales, $|k\tau| \ll 1$. Comparing it to the standard inflationary scalar power spectrum which is of the order of [9]

$$P_s(z,k) \simeq \left(\frac{H_I}{M_p}\right)^2\frac{1}{\epsilon} \simeq 2\cdot 10^{-9}\,,$$

where $\epsilon < 1$ denotes the slow roll parameter, one must conclude that this mode, if it transits to the radiation era completely spoils the observed large scale structure.

However, for scalar perturbations the matching from inflation to the radiation era has to be studied carefully, it can even lead to a change in the power spectrum as found, e.g., for the inflationary magnetic mode studied in Ref. [25]. For this reason, we shall not draw strong conclusions from this result.

Nevertheless, we request that $P_\Phi(z,k) < 1$ for perturbation theory to remain valid during inflation, so that we can neglect back-reaction of the perturbation to the cosmic evolution. At the end of inflation we have $r_I a_{\rm end}^2 \simeq (ra^2)_{\rm rad} \simeq \sqrt{3\Omega_r}$ so that $\mathcal{H}_{\rm end} = |\tau_{\rm end}|^{-1} = H_I a_{\rm end} \sim (H_I H_0)^{1/2}(3\Omega_r)^{1/4}$. Inserting $k \sim H_0$ in $(\mathcal{H}_{\rm end}/k)^4$ we obtain $(\mathcal{H}_{\rm end}/H_0)^4 \sim 3\Omega_r (H_I/H_0)^2$ which leads to

$$P_\Phi(\tau_{\rm end}, H_0) \simeq \frac{3\Omega_r}{18}\left(\frac{H_I^2}{M_p H_0}\right)^2 . \tag{6.95}$$

The condition $P_\Phi(\tau_{\rm end}, H_0) < 1$ then becomes

$$H_I < \left[\frac{18 M_P^2 H_0^2}{3\Omega_r}\right]^{1/4} \sim 10^{-11}\,{\rm GeV}\,, \qquad V_I^{1/4} \sim (H_I M_P)^{1/2} \lesssim 10^4\,{\rm GeV}\,. \tag{6.96}$$

Also this is indeed a rather low inflation scale. Requesting $P_\Phi < 10^{-9}$ would reduce it by another two orders of magnitude.

6.9 Conclusions

In this paper we have studied the generation of perturbations during inflation in a bimetric theory of gravity. We have analysed the evolution of the two tensor modes and we have found that both acquire a scale invariant spectrum with $h_f = h_g/r_I$, where the ratio $r_I = (b/a)|_I \simeq H_I/H_0 \gg 1$ is nearly constant during inflation. In addition to this constant tensor mode, the f-metric sector has also a decaying mode, which is the one that turns into the growing mode during the subsequent radiation era. Nevertheless, this mode is so severely suppressed that it does not lead to a significant amplification of the physical tensor mode as discussed in Ref. [34]. Therefore, looking at the tensor sector alone, the $\beta_1\beta_4$ model of bimetric gravity cannot be ruled out despite the fact that the Higuchi bound of the tensor sector of the f-metric is violated. Note that this Higuchi bound on a Friedmann background does not lead to an exponential instability but only to power law growth of fluctuations. For this reason, the detailed analysis of the initial conditions from inflation carried-out in this work was needed to decide whether the model is ruled out or not.

We have also analysed the vector (helicity 1) and scalar sectors. Also vector perturbations are generated during inflation leading to a scale invariant vector spectrum with an amplitude which is boosted by a factor r_I with respect to the tensor spectrum. Requiring vector fluctuations to remain perturbative gives an upper limit to the scale of inflation, $H_I < 10^{-2}$ GeV which translates to an inflationary energy scale $V_I^{1/4} < 10^8$ GeV. In principle, even if the scale of inflation is lower, these vector mode will source tensor modes in the f-metric at second order which then can feed into the growing mode. Anyway, a detailed calculation of this is beyond the scope of the present work.

In the scalar sector we have not discussed the inflaton perturbations, assuming that they are not modified due to the very weak coupling of bigravity during inflation. However, in a bigravity theory we have the helicity-0 mode of the graviton as a second scalar mode. As the Higuchi bound in the scalar sector is violated this mode is growing on super Hubble scales during inflation. We have computed its spectrum at the end of inflation and have found that it is very red, $\propto k^{-4}$. Requesting that these fluctuations remain perturbative also on the largest scales $k \sim H_0$ gives stringent constraints on the scale of inflation, $H_I < 10^{-11}$ GeV, which translates to an inflationary energy scale $V_I^{1/4} \lesssim 10^4$ GeV.

We conclude that the $\beta_1\beta_4$ model studied in this paper cannot be ruled out from the analysis of the tensor sector alone. Nevertheless, it is strongly constraint due to the large vector perturbations which are generated during inflation and due to the very red spectrum of scalar perturbations. However, considering that all the problematic scales of inflation have $H_I > 10^{-11}$ GeV, which is far higher than the strong coupling scale, Λ_3, these results have to be taken with a grain of salt. Still, we recall that this strong coupling limit is derived in a Minkowski background and it is not clear that it should invalidate quantum perturbations on classical Friedmann solutions with a Hubble scale which is larger than Λ_3.

All other models of bigravity where matter only couples to one of the metrics (the g metric in this work) suffer from exponential instabilities of scalar perturbations on a FLRW background. This makes these models less attractive as candidate solutions to the dark energy problem. Due to the breakdown of linearity one has to work out the theory at higher orders and hope to cure the instabilities, possibly through the Vainshtein mechanism [137]. A possible way out is to push the gradient instability to very early times, rendering it unobservable. This can be achieved by lowering the value of the Planck mass of the metric which does not couple to matter [10]. In addition, there remain a multitude of massive (bigravity) models whose cosmology deserves further investigation, e.g., where matter, or even different matter sectors, can couple to both metrics [52, 51, 90, 124]. Alternatively one could also consider non-FLRW backgrounds or change the status of the parameters of the theory, e.g. by promoting the β_i coefficients to functions of the helicity-0 mode [46], or of some independent scalar field.

Acknowledgments

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6.10 Appendix: Hassan-Rosen bigravity model: general aspects

The conventions used for the bigravity action are those of [80]. Only one of the two metrics is coupled with matter, and we restrict to minimal couplings. The action is given by

$$S = -\int d^4x\sqrt{-\det g}\left[\frac{M_g^2}{2}(R(g) - 2m^2V(g,f)) + \mathcal{L}_m(g,\Phi)\right] - \int d^4x\sqrt{-\det f}\,\frac{M_f^2}{2}R(f)\,, \tag{6.97}$$

where g is the physical metric (the one coupled to matter), f is the second metric, and $M_g = 1/\sqrt{8\pi G} \equiv M_p$ and M_f are the respective Planck masses with dimensionless ratio $\alpha = M_f/M_g$. We assume the matter fields Φ to be coupled to g only. The potential is given in terms of the tensor field $\mathbb{X} = \sqrt{g^{-1}f}$:

$$V(g,f) = \sum_{n=0}^{4}\beta_n e_n(\mathbb{X})\,, \tag{6.98}$$

where the coefficients β_n are constants and the polynomials $e_n(X)$ are

$$e_0 = \mathbb{I}, \quad e_1 = [\mathbb{X}], \tag{6.99}$$

$$e_2 = \frac{1}{2}([\mathbb{X}]^2 - [\mathbb{X}^2]), \tag{6.100}$$

$$e_3 = \frac{1}{6}([\mathbb{X}]^3 - 3[\mathbb{X}][\mathbb{X}^2] + 2[\mathbb{X}^3], \tag{6.101}$$

$$e_4 = \frac{1}{24}([\mathbb{X}]^4 - 6[\mathbb{X}]^2[\mathbb{X}^2] + 8[\mathbb{X}][\mathbb{X}^3] + 3[\mathbb{X}^2]^2 - 6[\mathbb{X}^4]) = \det\mathbb{X}\,. \tag{6.102}$$

The square bracket $[\cdots]$ denotes the trace. The equations of motions for $g_{\mu\nu}$ and $f_{\mu\nu}$ are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{m^2}{2}\sum_{n=0}^{3}(-)^n\beta_n\left[g_{\mu\lambda}Y^{\lambda}_{(n)\nu}\left(\sqrt{g^{-1}f}\right) + g_{\nu\lambda}Y^{\lambda}_{(n)\mu}\left(\sqrt{g^{-1}f}\right)\right] = \frac{1}{M_g^2}T_{\mu\nu}\,, \tag{6.103}$$

$$\bar{R}_{\mu\nu} - \frac{1}{2}f_{\mu\nu}\bar{R} + \frac{m^2}{2\alpha^2}\sum_{n=0}^{3}(-)^n\beta_{4-n}\left[f_{\mu\lambda}Y^{\lambda}_{(n)\nu}\left(\sqrt{f^{-1}g}\right) + f_{\nu\lambda}Y^{\lambda}_{(n)\mu}\left(\sqrt{f^{-1}g}\right)\right] = 0\,, \tag{6.104}$$

where the overbar indicates $f_{\mu\nu}$ curvature. The definition of the $Y^{\nu}_{(n)\mu}(\mathbb{X})$ matrices is as follows:

$$Y_{(0)}(\mathbb{X}) = \mathbb{I}\,, \quad Y_{(1)}(\mathbb{X}) = \mathbb{X} - \mathbb{I}[\mathbb{X}]\,, \tag{6.105}$$

$$Y_{(2)}(\mathbb{X}) = \mathbb{X}^2 - \mathbb{X}[\mathbb{X}] + \frac{1}{2}\mathbb{I}\left([\mathbb{X}]^2 - [\mathbb{X}^2]\right)\,, \tag{6.106}$$

$$Y_{(3)}(\mathbb{X}) = \mathbb{X}^3 - \mathbb{X}^2[\mathbb{X}] + \frac{1}{2}\mathbb{X}\left([\mathbb{X}]^2 - [\mathbb{X}^2]\right) - \frac{1}{6}\mathbb{I}\left([\mathbb{X}]^3 - 3[\mathbb{X}][\mathbb{X}^2] + 2[\mathbb{X}^3]\right)\,. \tag{6.107}$$

As a consequence of the Bianchi identity and of the covariant conservation of $T_{\mu\nu}$, we find the following Bianchi constraints (for each one of the two metrics)

$$\nabla_\mu \sum_{n=0}^{3} (-)^n \, \beta_n \left[g_{\mu\lambda} \, Y^\lambda_{(n)\nu}\left(\sqrt{g^{-1}f}\right) + g_{\nu\lambda} \, Y^\lambda_{(n)\mu}\left(\sqrt{g^{-1}f}\right)\right] = 0 \,, \tag{6.108}$$

$$\overline{\nabla}^\mu \sum_{n=0}^{3} (-)^n \, \beta_{4-n} \left[f_{\mu\lambda} \, Y^\lambda_{(n)\nu}\left(\sqrt{f^{-1}g}\right) + f_{\nu\lambda} \, Y^\lambda_{(n)\mu}\left(\sqrt{f^{-1}g}\right)\right] = 0 \,, \tag{6.109}$$

where the overbar indicates covariant derivatives with respect to the f metric. Both these constraints follow from the invariance of the interaction term under the diagonal subgroup of the general coordinate transformations of the two metrics. They are equivalent and in this work we focus on the first one.

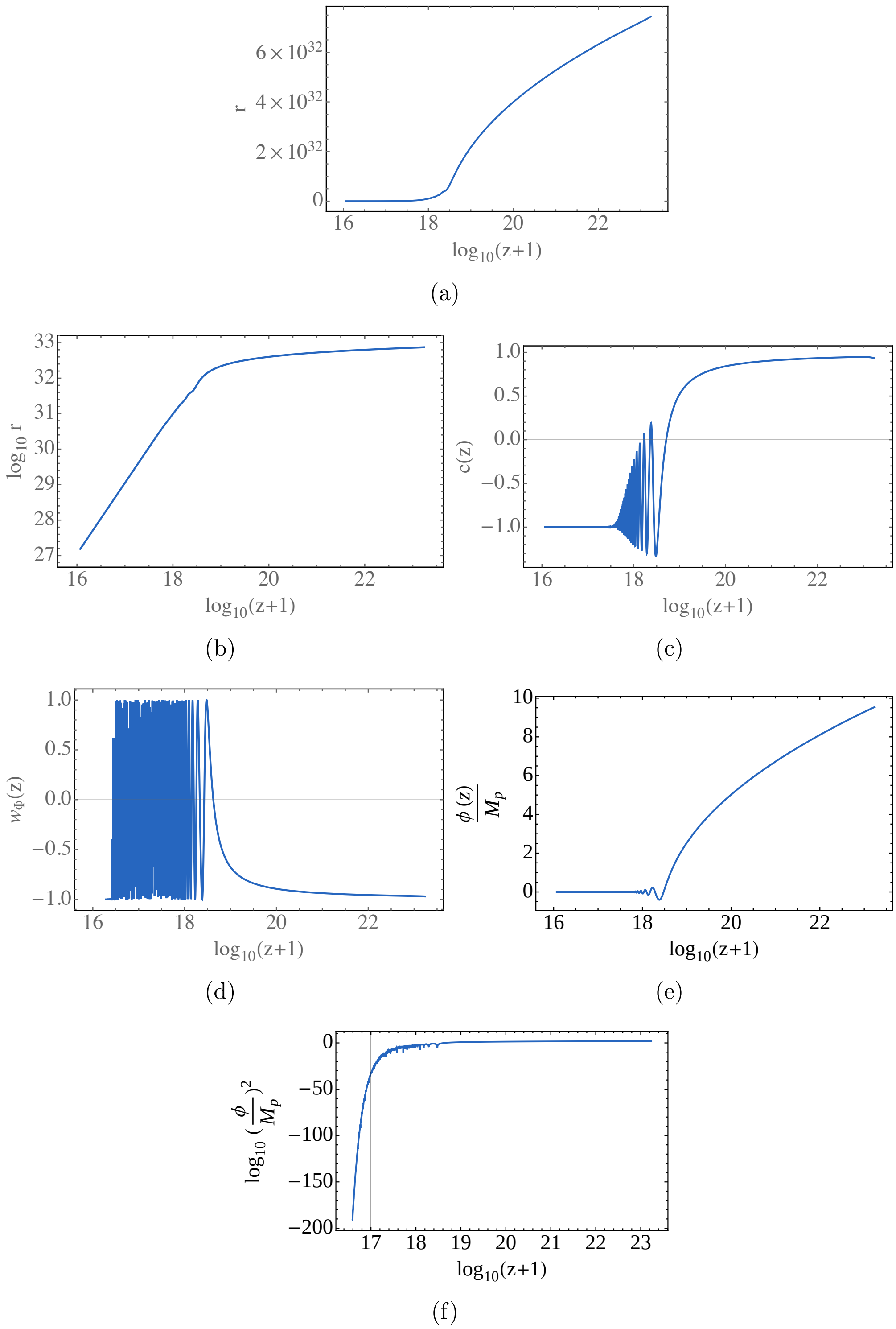


Figure 6.1: We show the ratio between the scale factors r, the lapse of the f-metric c, the equation of state parameter of the inflaton ω_ϕ and the expectation value of the inflaton as functions of redshift during inflation. We have chosen $\phi(z_i) = 10\, M_p$ and $H(z_i) = 1$ eV. The parameter Γ in eq. (6.34) has been chosen such that $\Gamma = H(z_{\rm reh})$ with $z_{\rm reh} = 5 \cdot 10^{17}$. Note how the oscillations of ϕ lead to strong oscillations of the lapse function c and the equation of state parameter ω_ϕ at the end of inflation.

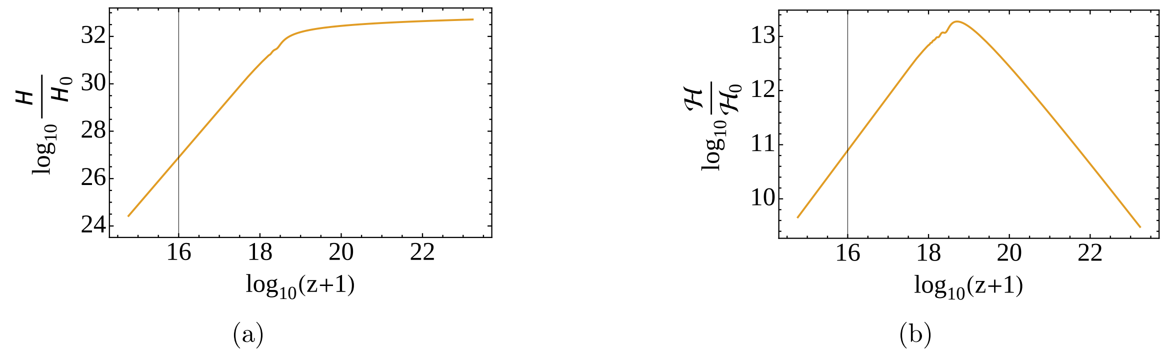


Figure 6.2: The physical and the conformal Hubble parameters as functions of redshift, panels 6.2(a) and 6.2(b) respectively.

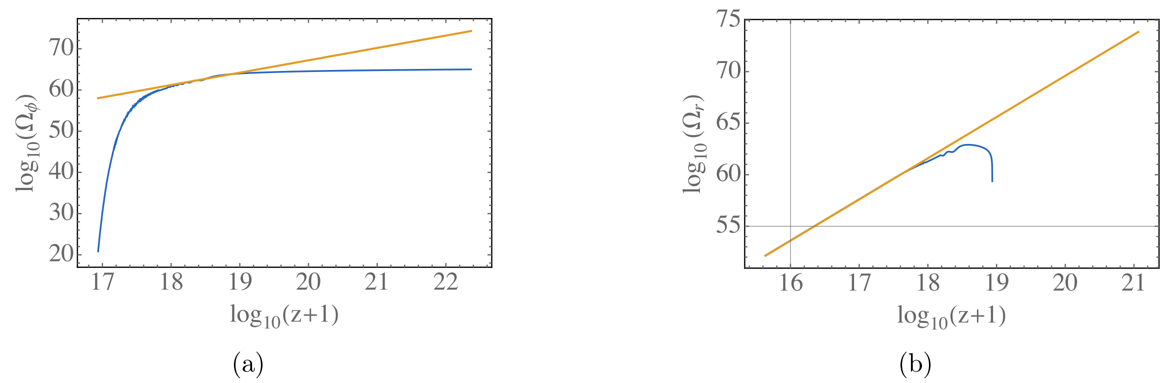


Figure 6.3: Evolution of the energy density of the inflaton and of radiation (in blue), normalized with respect to the critical energy density of the universe today. The yellow curves in panels 6.3(a) and 6.3(b) are $\propto a^{-3}$ and $\propto a^{-4}$, respectively.

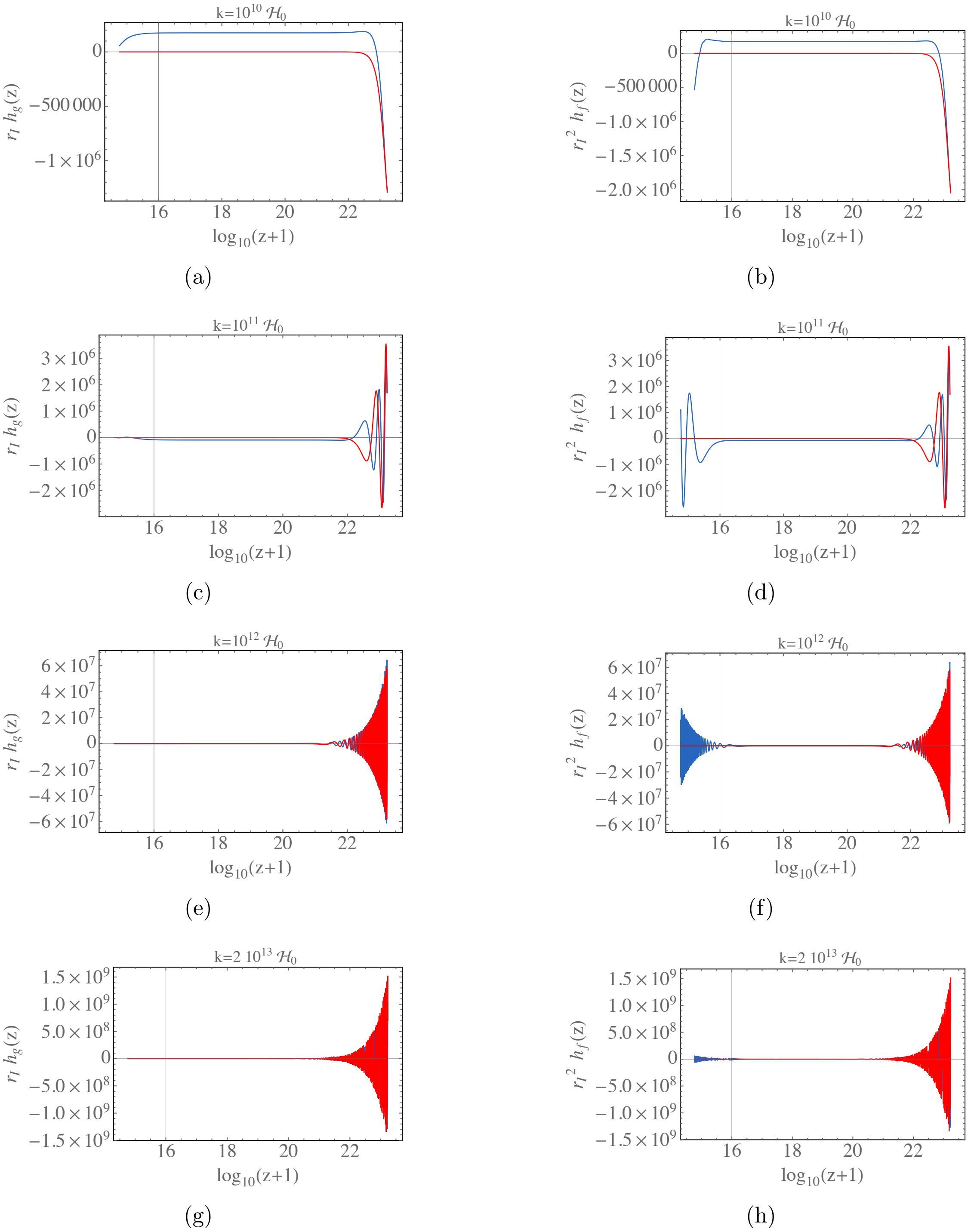


Figure 6.4: Evolution of tensor perturbations of the metrics g and f as functions of redshift. The numerical solution (blue) is plotted together with the analytical one (in red) valid in the inflationary era, i.e., in the regime in which the hypothesis of slow-rolling holds. We have chosen $k = 10^{10}\,\mathcal{H}_0$, $k = 10^{11}\,\mathcal{H}_0$, $k = 10^{12}\,\mathcal{H}_0$ and $k = 2 \cdot 10^{13}\,\mathcal{H}_0$ in the panels 6.4(a)-6.4(b), 6.4(c)-6.4(d), 6.4(e)-6.4(f) and 6.4(g)-6.4(h), respectively. The spectrum for the g-mode is rescaled with a factor $r_I \simeq 10^{33}$ while the spectrum for the f-mode is rescaled with a factor r_I^2. Note that in our model inflation ends roughly at $\log_{10}(1+z) \simeq 19.0$ while radiation domination is established at $\log_{10}(1+z) \simeq 17.5$.

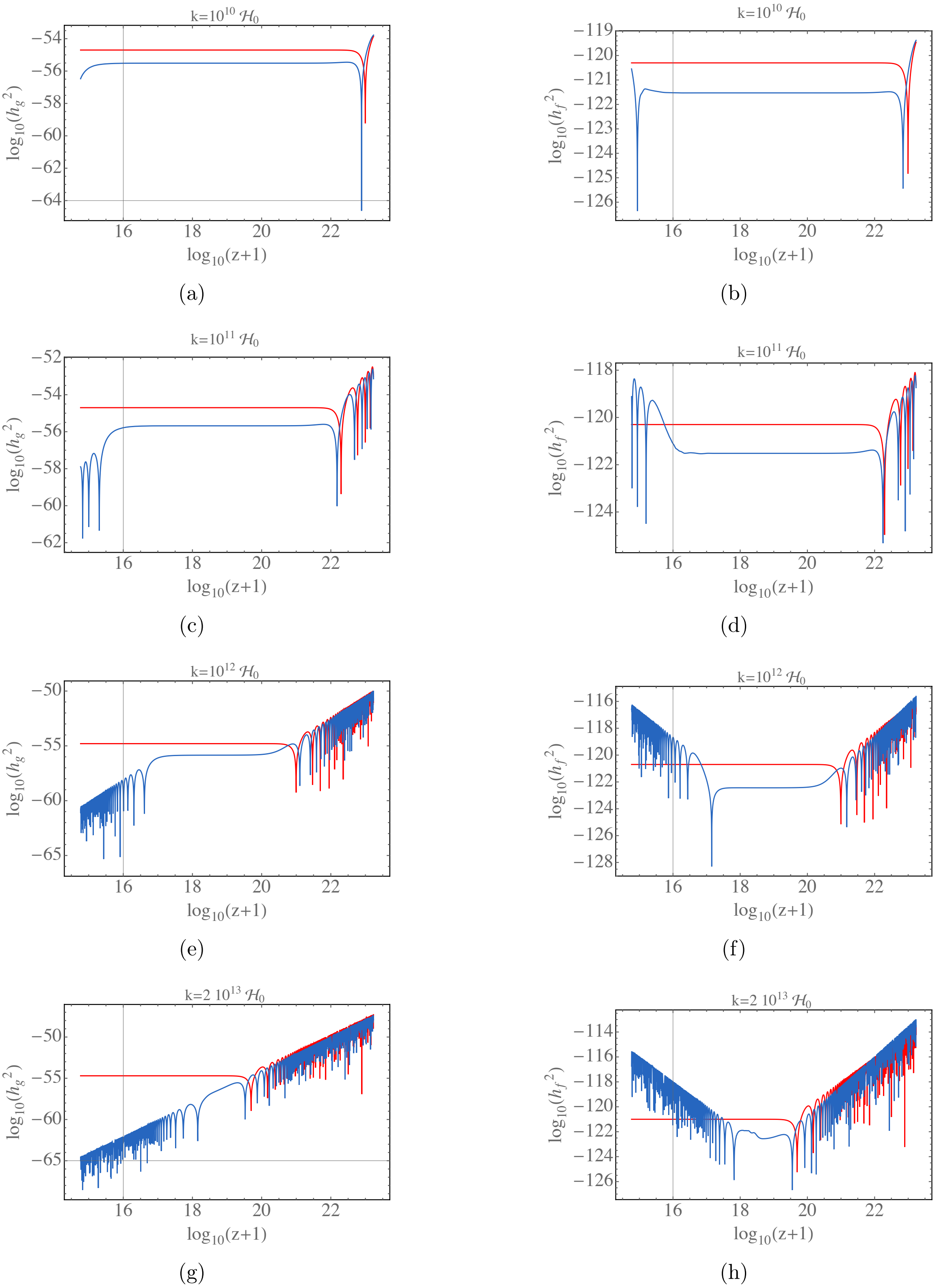
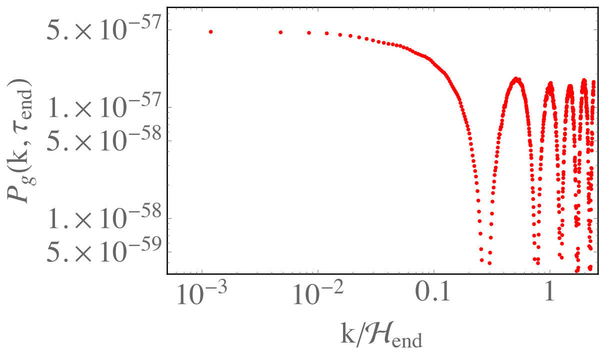
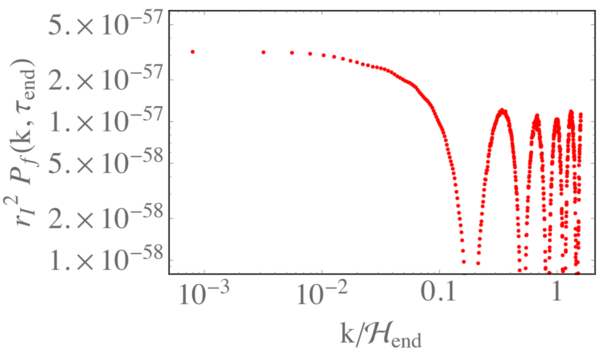


Figure 6.5: We show the same plots as in Fig. 6.4 but in log scale for more detailed apprehension of the decay and growth of perturbations. One sees that the analytical solution during inflation is out of phase with the numerical solution. The reason is that the space-time background is somewhat different in the two cases, in fact, for our analytical solution the background is pure de Sitter, whereas for the numerical solution we have taken into account the full evolution of the background.



(a)



(b)

Figure 6.6: Power spectra at the end of inflation, $z \simeq 10^{19}$. The power spectrum for the h_f mode has been rescaled by r_I^2, with $r_I \simeq 10^{33}$ for easier comparison with the spectrum of the physical mode h_g.

Chapter 7

Overview and conclusions

This Thesis deals with two recent theories of gravity, the de Rham-Gabadadze-Tolley (dRGT) theory of massive gravity [47, 49], and (mostly) the Hassan-Rosen bimetric theory [83], which, since their formulation, have been two of the most widely studied models attempting to modify Einstein's theory of General Relativity in the IR. In particular, the original contributions can be divided in two parts: a part mainly dealing with formalism and the other one focused on applications to cosmology.

More precisely, the first part of the Thesis consists in the derivation of the mass term at second order in perturbations for completely general metrics $g_{\mu\nu}$ and $f_{\mu\nu}$: this has been done for the actions of both massive gravity and bigravity theories. The main difference in the two cases is the fact that in bimetric theory we allow also the reference metric to be perturbed whereas in massive gravity it is taken to be fixed and unperturbed. Another difference is due to the fact that the term proportional to the symmetric polynomial $e_4(\sqrt{g^{-1}f})$ becomes dynamical in bigravity, whereas it can be neglected in massive gravity. This translates of course into a greater complexity of the expression for the mass term in bigravity, which however correctly reduces to the one in massive gravity once the limit of vanishing $f_{\mu\nu}$ perturbations is taken. This has been the first computation of the general expression for the interaction potential in the action: in fact, due to the difficulty in perturbing the square root quantity $\sqrt{g^{-1}f}$ in full generality, this had only been done before for explicit choices of the two metrics (see e. g. [41]). Our expressions also allow to derive quite straightforwardly the linearised equations of motion around general background metrics. Later on, Bernard *et al.* have performed an alternative derivation of the same second-order interaction potential in the context of massive gravity [23, 24] and bigravity [22]. Our approach is based on the algebraic relations which exist between the elementary symmetric polynomials of the matrix $\sqrt{g^{-1}f}$ which enter the interaction potential and the same polynomials of the matrix $g^{-1}f$. The approach of Bernard *et al.* makes use of the Cayley-Hamilton theorem in order to compute the variation of the square root matrix $\sqrt{g^{-1}f}$ in terms of the variation of the matrix $g^{-1}f$ in the massive gravity case, and of a subtle redefinition of variables in the bimetric case. Both approaches are fully covariant and lead to the same results. We stress that the perturbed mass term that we have computed has a vast range of possible future applications for the study of perturbation-related issues in massive gravity and bigravity theories (for any kind of couplings to matter). It can be used to study any kinds of perturbations on an arbitrary background solutions or to investigate the presence of unstable modes for a given solution in the theory.

In the second part of the Thesis we have focused on the study of the cosmology of the bigravity model, in order to get predictions to be tested against cosmological data. In particular, we wanted to investigate whether there is a sub-class of these bigravity models with an evolution of perturbations which is compatible with the current constraints coming from the CMB. We have therefore studied the cosmological perturbations in the context of Hassan-Rosen bigravity in a FLRW Universe. In this case there are two different branches of solutions to the Friedmann equations ("branch I" and "branch II"), depending on how the Bianchi constraint is implemented.

We have first restricted our analysis (both numerical and analytical) to the "branch II", more precisely to the so-called '$\beta_1\beta_4$' model, where all the β_n parameters but β_1 and β_4 are set to zero. In fact, this has been proven to be the only self-accelerating model with a cosmological evolution free of exponential (gradient) instabilities during matter domination. We have considered the best-fit values for β_1 and β_4, obtained from fitting measured growth data and type Ia supernovae [107, 105]. We have then focused on tensor perturbations and found that in general gravitational waves are unstable and therefore can lead to a signature in CMB which is very different from the one predicted by the ΛCDM model. This is due to an instability in the f-sector of tensor perturbations (related to the violation of the generalized Higuchi bound on FLRW backgrounds) which is transferred to the physical sector via the coupling between the two metrics. However, we have also found numerically that the instability does not show up until today in the g-sector and that we recover ΛCDM-like gravitational waves if we take the initial conditions for the time evolution of tensor perturbations in the f-sector to be greatly suppressed with respect to the ones in the g-sector. Since initial conditions are settled at the beginning of the radiation domination era, we have studied the generation of tensor perturbations in the $\beta_1\beta_4$ model during inflation. We have found that the instability is not effective and that the evolution of gravitational waves is ΛCDM-like for every typical inflationary scale. Therefore, the $\beta_1\beta_4$ model cannot be ruled out from a pure analysis of the tensor sector. This is in agreement with the conclusions of Johnson and Terrana [101]. We have also briefly considered vector and scalar perturbations during the inflationary era. We have found that there are mild constraints on the inflation scale coming from the vector sector. However, this model suffers from a serious problem coming from the presence of an Higuchi ghost in the scalar sector, even during the de Sitter inflation.

As a direct application of the formalism presented in the first part of the Thesis, we have also studied the "branch I". Our aim was mainly to check the statement, which can be found in the previous literature [143, 110], that this branch is completely degenerate with ΛCDM not only at the background but also at the perturbations level. We have found that for tensor perturbations this is not actually true: in fact, gravitational waves in this branch can develop a tachyonic instability at late times. However, these results are valid only if the strong coupling scale of the theory is above the typical inflationary energy scales, which is not granted. A fundamental problem which plagues this branch is also given by the fact that the helicity-0 and the helicity-1 degrees of freedom of the massive graviton are absent at linear level and are probably infinitely strongly coupled.

Many challenging issues related to the work presented in this Thesis are still unsolved and could open new interesting perspectives. For example, a calculation of the strong coupling energy scale Λ_3 for the interaction potential has been performed in a Minkowski background only (see e. g. [93]). Therefore, it is not legitimate to use this result to decide whether our analysis of inflationary perturbations in bigravity, is valid (in the sense that, if the standard inflationary energy scales are above the strong coupling scale, it is not granted that perturbation theory is still valid). In fact, our calculation uses FLRW as the background metric, and it is not clear whether we could directly apply the number found in the flat case, $\Lambda_3 = (M_g m^2)^{1/3}$, to this cosmological setup. It is then necessary to calculate the strong coupling scale for the interaction potential in the case of a FLRW background.

One could also address the issue of the Higuchi ghost in the scalar sector of the $\beta_1\beta_4$ model by trying to modify just this sector of the theory. One possibility could be to introduce a varying graviton mass, by promoting the graviton mass to a potential for an external scalar field ψ (which has its own dynamics), $m \to m(\psi)$, as it has already been performed in the massive gravity case [94]. This formulation can also be promoted to varying parameters $\beta_n \to \beta_n(\psi)$ but also to multiple scalar fields ψ_A, with $A = 1, \dots, N$ [96].

Another open issue is the fact that, contrarily to what is expected, scalar and vector linear perturbations are absent from a canonical counting of the degrees of freedom of the branch I solutions. One could address this problem e. g. by trying to build a *minimal model of bigravity* along the lines of the minimal model of massive gravity proposed in [43]. The aim would be to have a bigravity model whose background and tensor perturbations equations are exactly the same as in branch I but in which the scalar and vector gravitational degrees of freedom are absent at the fully nonlinear level. In this way, this absence of scalar and vector modes could be explained at a fundamental level and there would not be any strong coupling problem for these missing degrees of freedom.

Appendix A

Cosmological perturbation theory in General Relativity

This appendix is devoted to a review of the main aspects of the cosmological perturbation theory which is a very important framework for this work. We also fix part of the notation and of the conventions that we use throughout this Thesis. This appendix is mainly based on [121] and on [60].

A.1 The background cosmology

As we know, our Universe looks very close to being spatially isotropic (which means, independent of direction) and homogeneous (independent of position) when averaged over large scales. To a first approximation, we can hence treat it as perfectly homogeneous and isotropic: this means that it can be described by the *Friedmann-Lemaître-Robertson-Walker* (FLRW) metric, whose line element is

$$\mathrm{d}s^2 \equiv g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + \mathrm{d}l^2 = -\mathrm{d}t^2 + a^2(t)\gamma_{ij}\mathrm{d}x^i \mathrm{d}x^j\,, \tag{A.1}$$

where $\mathrm{d}l^2 = a^2\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j$ is the line element of the spatial part of the metric, the scale factor $a(t)$ a takes into account the expansion of the Universe and

$$\gamma_{ij} \equiv \delta_{ij} + k\,\frac{x_i x_j}{1-kx^2}, \qquad \text{for} \qquad k \equiv \begin{cases} 0 & \text{Euclidean} \\ +1 & \text{spherical} \\ -1 & \text{hyperbolic} \end{cases} \tag{A.2}$$

It is often convenient to use spherical polar coordinates, (r, θ, ϕ), in such a way to make the symmetry of the space manifest:

$$\mathrm{d}l^2 = a^2 \left[\frac{\mathrm{d}r^2}{1 - k\, r^2} + r^2 \mathrm{d}\Omega^2 \right] , \tag{A.3}$$

where $\mathrm{d}\Omega^2 \equiv \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2$. Moreover, if we introduce the *conformal time* τ, defined with respect to physical time t as

$$\mathrm{d}\tau \equiv \frac{\mathrm{d}t}{a(t)} , \tag{A.4}$$

eq. (A.1) becomes

$$\mathrm{d}s^2 = a^2(\tau) \left[-\mathrm{d}\tau^2 + \mathrm{d}l^2 \right] = a^2(\tau) \left[-\mathrm{d}\tau^2 + \gamma_{ij} \mathrm{d}x^i \mathrm{d}x^j \right] . \tag{A.5}$$

The dynamics of the Universe is determined by the Einstein equations,

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu} , \tag{A.6}$$

which relate the Einstein tensor $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ (a measure of the "space-time curvature" of the Universe) to the stress-energy tensor $T_{\mu\nu}$ (a measure of the "matter content" of the Universe).

We want to determine what form of the stress-energy tensor $T_{\mu\nu}$ is consistent with the requirements of homogeneity and isotropy. First, we decompose $T_{\mu\nu}$ into a 3-scalar, T_{00}, 3-vectors, T_{i0} and T_{0j}, and a 3-tensor, T_{ij}. Isotropy requires the mean values of 3-vectors to vanish, i.e. $T_{i0} = T_{0j} = 0$. Moreover, isotropy around a point $x = 0$ requires the mean value of any 3-tensor, such as T_{ij}, at that point to be proportional to δ_{ij} and hence to g_{ij} (which equals $a^2\delta_{ij}$ at $\mathbf{x} = 0$),

$$T_{ij}(\mathbf{x} = 0) \propto \delta_{ij} \propto g_{ij}(\mathbf{x} = 0) . \tag{A.7}$$

Homogeneity requires the proportionality coefficient to be only a function of time. Since this is a proportionality between two 3-tensors, T_{ij} and g_{ij}, it must remain unaffected by an arbitrary transformation of the spatial coordinates, including those transformations that preserve the form of g_{ij} while taking the origin into any other point. Hence, homogeneity and isotropy require the components of the stress-energy tensor everywhere to take the form

$$T_{00} = \rho(t) , \qquad T_{i0} = T_{0j} = 0 , \qquad T_{ij} = p(t)\, g_{ij}(t, \mathbf{x}) . \tag{A.8}$$

With mixed upper and lower indices, the stress-energy tensor looks like

$$T^{\mu}{}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} . \tag{A.9}$$

This is the stress-energy tensor of a *perfect fluid* as seen by a comoving observer. More generally, the stress-energy tensor can be written in the following, explicitly covariant, form:

$$T^{\mu}{}_{\nu} = (\rho + p)u^{\mu}u_{\nu} + p\,\delta^{\mu}{}_{\nu}\,, \tag{A.10}$$

where $u^{\mu} \equiv \mathrm{d}x^{\mu}/\mathrm{d}s$ is the relative four-velocity between the fluid and the observer, while ρ and p are the energy density and the pressur in the rest-frame of the fluid. We recover the result (A.9) for a comoving observer, $u^{\mu} = (1, 0, 0, 0)$.

How do the density and pressure evolve with time? In Minkowski space, energy and momentum are conserved. This translates into a four-component conservation equation for the stress-energy tensor,

$$\partial_{\mu} T^{\mu}{}_{\nu} = 0\,, \tag{A.11}$$

which in General Relativity is promoted to the covariant conservation equation

$$\nabla_{\mu} T^{\mu}{}_{\nu} = 0\,, \tag{A.12}$$

which reduces to eq. (A.11) in the local inertial frame. The evolution of the energy density ρ is determined by the $\nu = 0$ equation, which reduces to the *continuity equation*

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0\,. \tag{A.13}$$

Now we want to discuss the different matter components which fill the Universe. It is common to classify the different sources by their contribution to the pressure.

- *Matter.* We will use the term "matter" to refer to all forms of matter for which the pressure is much smaller than the energy density, $|p| \ll \rho$. This is the case for a gas of non-relativistic particles (where the energy density is dominated by the mass). Setting $p = 0$ in eq. (A.13) gives

  $$\rho \propto a^{-3}\,. \tag{A.14}$$

  This dilution of the energy density is simply a reflection of the expansion of the volume, $V \propto a^3$. Most of the matter in the Universe is in the form of invisible *dark matter*, whose real nature is still unknown (even if this is usually thought to be a new heavy particle species). Cosmologists refer to ordinary matter (nuclei and electrons) as *baryons*.

- *Radiation.* We will use the term "radiation" to denote anything for which the pressure is about a third of the energy density, $p = \frac{1}{3}\rho$. This is the case for a gas of relativistic particles (like photons and neutrinos), for which the energy density is dominated by the kinetic energy (i.e. the momentum is much bigger than the mass). In this case, eq. (A.13) implies

$$\rho \propto a^{-4} \tag{A.15}$$

The dilution now includes the redshifting of the energy, since $E \propto a^{-1}$. The early universe was dominated by photons: being massless, they are always relativistic. Today, we detect those photons in the form of the Cosmic Microwave Background (CMB). Neutrinos behaved like radiation for most of the history of the universe. Only recently their small masses became relevant and they started to behave like matter.

- *Dark energy.* As we know, the universe today seems to be dominated by a mysterious negative pressure component, $p = -\rho$. From eq. (A.13), we find that the energy density is constant,

$$\rho \propto a^0 \,. \tag{A.16}$$

Most cosmological fluids can be parameterised in terms of a constant equation of state: $w = p/\rho$. This includes cold dark matter ($w = 0$), radiation ($w = 1/3$) and dark energy ($w = -1$). In that case, the solutions to eq. (A.13) scale as

$$\rho \propto a^{-3(1+w)} \,, \tag{A.17}$$

and therefore

$$\rho \propto \begin{cases} a^{-3} & \text{matter} \\ a^{-4} & \text{radiation} \\ a^{0} & \text{dark energy} \end{cases} \,. \tag{A.18}$$

We want to relate these matter sources to the evolution of the scale factor in the FLRW metric (A.1). To do so, we compute the Einstein field equations (A.6) in this metric. The final result is expressed by the *Friedmann equations*

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \,, \tag{A.19}$$

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3p) \,, \tag{A.20}$$

which are sometimes denoted as the Friedmann equation and the acceleration equation, respectively. Here, ρ and p should be understood as the sum of all contributions to the energy density and pressure in the universe. We write ρ_r for the contribution from radiation (with ρ_γ for photons and ρ_ν for neutrinos), ρ_m for the contribution by matter (with ρ_{CDM} for cold dark matter and ρ_b for baryons) and ρ_Λ for the dark energy contribution. The Friedmann equation is often written in terms of the Hubble parameter, $H \equiv \dot{a}/a$,

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}\,. \tag{A.21}$$

The left-hand side of Einstein equations (A.6) is not uniquely defined. We can add the term $\Lambda\, g_{\mu\nu}$, where Λ is called the "cosmological constant", without changing the conservation of the stress-energy tensor (A.12), since $\nabla^\mu g_{\mu\nu} = 0$. In other words, we could have written the Einstein equations as

$$G_{\mu\nu} + \Lambda\, g_{\mu\nu} = 8\,\pi\, G\, T_{\mu\nu}\,. \tag{A.22}$$

In this case, the Friedmann equations read

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}\,, \tag{A.23}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \Lambda\,, \tag{A.24}$$

and describe the gravitational dynamics at the background level in the ΛCDM model. The subscript '0' is commonly used to denote quantities which are evaluated today, at $t = t_0$. A flat universe ($k = 0$) corresponds to the following *critical density* today, which can be read from the Friedmann equation (A.21):

$$\rho_{\mathrm{crit},0} = \frac{3\,H_0^2}{8\pi\, G}\,, \tag{A.25}$$

We use this quantity to define the dimensionless density parameters

$$\Omega_{i,0} \equiv \frac{\rho_{i,0}}{\rho_{\mathrm{crit},0}}\,, \qquad \text{where} \qquad i = r, m, \Lambda, \ldots \tag{A.26}$$

The Friedmann equation (A.21) can then be written as

$$H^2(a) = H_0^2\left[\Omega_{r,0}\left(\frac{a_0}{a}\right)^4 + \Omega_{m,0}\left(\frac{a_0}{a}\right)^3 + \Omega_{k,0}\left(\frac{a_0}{a}\right)^2 + \Omega_{\Lambda,0}\right], \tag{A.27}$$

where we have defined a "curvature" density parameter, $\Omega_{k,0} \equiv -k/(a_0 H_0)^2$. If we drop the subscript '0' on the density parameters and use the conventional normalization $a_0 \equiv 1$ for the scale factor, eq. (A.27) then becomes

$$\frac{H^2}{H_0^2} = \Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda\,. \tag{A.28}$$

Observations show that the Universe is filled with radiation ('r'), matter ('m') and dark energy ('Λ'):

$$|\Omega_k| \leq 0.01\,, \qquad \Omega_r = 9.4 \times 10^{-5}\,, \qquad \Omega_m = 0.32\,, \qquad \Omega_\Lambda = 0.68\,. \tag{A.29}$$

The equation of state of dark energy seems to be that of a cosmological constant, $w_\Lambda = -1$. The matter splits into 5% of ordinary matter (baryons, 'b') and 27% of (cold) dark matter ('CDM'):

$$\Omega_b = 0.05\,, \qquad \Omega_{\rm CDM} = 0.27\,. \tag{A.30}$$

We see that even today curvature makes up less than 1% of the total energy of the Universe. At earlier times, the effects of curvature are then completely negligible (if we recall that matter and radiation scale as a^{-3} and a^{-4}, respectively, while the curvature contribution only scales as a^{-2}). The different scalings of radiation, matter and dark energy imply that for most of its history the universe was dominated by a single component (first radiation, then matter, then dark energy). Parameterising this component by its equation of state w_I captures all cases of interest. For a flat, single-component universe, the Friedmann equation (A.28) reduces to

$$\frac{\dot a}{a} = H_0\sqrt{\Omega_I}\, a^{-\frac{3}{2}(1+w_I)}\,. \tag{A.31}$$

Integrating this equation, we obtain the time dependence of the scale factor:

$$a(t) \propto \begin{cases} t^{2/3(1+w_I)} & w_I \neq -1 \\ e^{Ht} & w_I = -1 \end{cases} \quad \begin{cases} t^{2/3} & MD \\ t^{1/2} & RD \\ & \Lambda D \end{cases}\,. \tag{A.32}$$

or, in conformal time,

$$a(\tau) \propto \begin{cases} \tau^{2/(1+3w_I)} & w_I \neq -1 \\ (-\tau)^{-1} & w_I = -1 \end{cases} \quad \begin{cases} \tau^{2} & MD \\ \tau & RD \\ & \Lambda D \end{cases}\,. \tag{A.33}$$

Matter and radiation were equally important at $a_{\rm eq} \equiv \Omega_r/\Omega_m \approx 3\times 10^{-4}$, which was shortly before the cosmic microwave background was released (at $a_{\rm rec} \approx 9\times 10^{-4}$).

A.2 Perturbations

The Universe in which we live is not perfectly homogeneous and isotropic and hence only in a first approximation it can be described by using the FLRW metric (A.1). In order to understand the formation and evolution of large-scale structure and the anisotropies of the CMB, however, we have to introduce deviations from the background picture, and we will do this perturbatively. Depending on the aim pursued, one could use Newtonian perturbation theory, which is based on Newtonian Gravity (which is an adequate approximation of General Relativity in cosmology on scales well inside the Hubble radius and when describing non-relativistic matter), or relativistic perturbation theory, which makes use of the full toolbox of General Relativity and furnishes the correct description on scales larger than the Hubble radius and for relativistic fluids. We will henceforth focus on the fully relativistic treatment.

A.2.1 The perturbations of the space-time

We start by considering small perturbations $h_{\mu\nu}$ around the FLRW background metric $\bar{g}_{\mu\nu}$,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \,. \tag{A.34}$$

Through the Einstein equations, $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, the metric perturbations will be coupled to perturbations in the matter sector. In the case of a spatially flat ($k = 0$) FLRW background spacetime,

$$\mathrm{d}s^2 = a^2(\tau)\left[-\mathrm{d}\tau^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right] . \tag{A.35}$$

The perturbed metric then reads

$$\mathrm{d}s^2 = a^2(\tau)\left[-(1+2A)\mathrm{d}\tau^2 - 2B_i\mathrm{d}x^i\mathrm{d}\tau + (\delta_{ij} + h_{ij})\mathrm{d}x^i\mathrm{d}x^j\right] , \tag{A.36}$$

where A, B_i and h_{ij} are functions of the space-time coordinates. We shall adopt the convention that Latin indices on spatial vectors and tensors are raised and lowered with δ_{ij}, e. g. $h^i{}_i = \delta^{ij}h_{ij}$. It is very useful to perform a scalar-vector-tensor (SVT) decomposition of the perturbations. For three-vectors (as we have already seen), it simply means that we can split them into the gradient of a scalar B and a divergenceless vector $\hat{B}_i$,

$$B_i = \partial_i B + \hat{B}_i \,, \tag{A.37}$$

with $\partial^i\hat{B}_i = 0$. Similarly, any rank-2 symmetric tensor can be written as

$$h_{ij} = \underbrace{2\,C\,\delta_{ij} + 2\left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)E}_{\text{scalar}} + \underbrace{2\,\partial_{(i}\hat{E}_{j)}}_{\text{vector}} + \underbrace{\hat{h}_{ij}}_{\text{tensor}} \,. \tag{A.38}$$

As before, the hatted quantities are divergenceless, i. e. $\partial^i\hat{E}_i = 0$ and $\partial^i\hat{h}_{ij} = 0$. The tensor perturbation is traceless, $\hat{h}^i{}_i = 0$. The 10 degrees of freedom of the metric $g_{\mu\nu}$ have hence been decomposed into 4 scalar degrees of freedom (A, B, C, E), 4 vector degrees of freedom ($\hat{B}_i$ and $\hat{E}_i$ with the two divergenceless conditions) and 2 tensor degrees of freedom (the symmetric tensor $\hat{h}_{ij}$ with the divergenceless and the traceless conditions). What makes the SVT decomposition so powerful is the fact that the Einstein equations for scalar, vector, and tensor perturbations don't mix at linear order and can therefore be treated separately.

Before going on, we want to address an important subtlety. The metric perturbations in eq. (A.36) are not uniquely defined, but depend on our choice of coordinates, also called the *gauge choice*. In particular, when we wrote down the perturbed metric, we implicitly chose a specific time slicing of the space-time and defined specific spatial coordinates on these time slices. Making a different choice of coordinates can change the values of the perturbation variables. It may even introduce fictitious perturbations. These are fake perturbations that can arise by an inconvenient choice of coordinates even if the background is perfectly homogeneous.

For example, consider the homogeneous FLRW space-time given in eq. (A.35) and make the following change of the spatial coordinates, $x^i \to \tilde{x}^i = x^i + \xi^i(\tau, \mathbf{x})$. We assume that the function ξ^i is small, so that it can also be treated as a perturbation. Using $\mathrm{d}x^i = \mathrm{d}\tilde{x}^i - \partial_\tau \xi^i \mathrm{d}\tau - \partial_k \xi^i \mathrm{d}\tilde{x}^k$, eq. (A.35) becomes

$$\mathrm{d}s^2 = a^2(\tau)\left[-\mathrm{d}\tau^2 - 2\xi_i' \mathrm{d}\tilde{x}^i \mathrm{d}\tau + (\delta_{ij} - 2\,\partial_{(i}\xi_{j)})\mathrm{d}\tilde{x}^i \mathrm{d}\tilde{x}^j\right], \tag{A.39}$$

where we have dropped terms that are quadratic in ξ^i and defined $\xi_i' \equiv \partial_\tau \xi_i$. We apparently have introduced the metric perturbations $B_i = \xi_i'$ and $\hat{E}_i = -\xi_i$. But these are just fictitious *gauge modes* that can be removed by going back to the old coordinates. Similarly, we can change our time slicing, $\tau \to \tau + \xi^0(\tau, \mathbf{x})$. The homogeneous density of the universe then gets perturbed, $\rho(\tau) \to \rho(\tau + \xi^0(\tau, \mathbf{x})) = \bar{\rho}(\tau) + \bar{\rho}' \xi^0$. So even in an unperturbed universe, a change of the time coordinate can introduce a fictitious density perturbation

$$\delta\rho = \bar{\rho}' \xi^0 \,. \tag{A.40}$$

Similarly, we can remove a real perturbation in the energy density by choosing the hypersurface of constant time to coincide with the hypersurface of constant energy density. Then $\delta\rho = 0$ although there are real inhomogeneities. These examples illustrate that we need a more physical way to identify true perturbations. One way to do this is to define perturbations in such a way that they do not change under a change of coordinates. Consider the coordinate transformation

$$x^\mu \to \tilde{x}^\mu \equiv x^\mu + \xi^\mu(\tau, \mathbf{x}), \qquad \text{where} \qquad \xi^0 \equiv T\,, \quad \xi^i \equiv L^i = \partial^i L + \hat{L}^i\,, \tag{A.41}$$

where we have split the spatial shift L^i into a scalar, L, and a divergenceless vector, $\hat{L}^i$. We would like to know how the metric transforms under this change of coordinates. In general, metric perturbations transform as

$$h_{\mu\nu} \to h_{\mu\nu} + \mathcal{L}_\xi \bar{g}_{\mu\nu}\,, \tag{A.42}$$

where $\mathcal{L}_\xi$ is a *Lie derivative*,

$$\mathcal{L}_\xi \bar{g}_{\mu\nu} \equiv -\xi^\lambda \partial_\lambda \bar{g}_{\mu\nu} - \partial_\mu \xi^\lambda \bar{g}_{\lambda\nu} - \partial_\nu \xi^\lambda \bar{g}_{\lambda\mu}\,. \tag{A.43}$$

Evaluating the Lie derivatives for the components of the FLRW metric (A.35), we obtain

$$\mathcal{L}_\xi \bar{g}_{00} = -2\, a^2 \left[\mathcal{H}\, T + T'\right] , \tag{A.44}$$
$$\mathcal{L}_\xi \bar{g}_{i0} = a^2 \left[-\partial_i T + L_i'\right] , \tag{A.45}$$
$$\mathcal{L}_\xi \bar{g}_{ij} = 2\, a^2 \left[\mathcal{H}\, T\, \delta_{ij} + \partial_{(i} L_{j)}\right] , \tag{A.46}$$

where primes are derivatives with respect to τ and $\mathcal{H} \equiv a'/a$. This implies that the metric perturbations in eq. (A.36) transform as

$$A \to A - \mathcal{H}\, T - T' , \tag{A.47}$$
$$B \to B_i + \partial_i T - L_i' , \tag{A.48}$$
$$h_{ij} \to h_{ij} - 2\, \partial_{(i} L_{j)} - 2\, \mathcal{H}\, T\, \delta_{ij} . \tag{A.49}$$

In terms of the SVT decomposition, we get

$$A \to A - \mathcal{H}\, T - T' , \tag{A.50}$$
$$B \to B + T - L' , \qquad \hat{B}_i \to \hat{B}_i - \hat{L}_i' , \tag{A.51}$$
$$C \to C - \mathcal{H}\, T - \frac{1}{3}\nabla^2 L , \tag{A.52}$$
$$E \to E - L , \qquad \hat{E}_i \to \hat{E}_i - \hat{L}_i , \qquad \hat{h}_{ij} \to \hat{h}_{ij} . \tag{A.53}$$

One way to avoid the gauge problems is to define special combinations of metric perturbations that do not transform under a change of coordinates. These are the so-called *Bardeen variables*:

$$\Psi \equiv A + \mathcal{H}\,(B - E') + (B - E')' , \qquad \hat{\Phi}_i \equiv \hat{E}_i' - \hat{B}_i , \qquad \hat{h}_{ij} , \tag{A.54}$$
$$\Phi \equiv -C - \mathcal{H}\,(B - E') + \frac{1}{3}\nabla^2 E . \tag{A.55}$$

These gauge-invariant variables can be considered as the "real" space-time perturbations since they cannot be removed by a gauge transformation. This means that we can immediately distinguish physical perturbations from fictitious ones: if the Bardeen variables are equal to zero, then the metric perturbations (if they are present) are fictitious and can be removed by a suitable change of coordinates.

Of course we can construct an infinite number of gauge-invariant variables, since any combination of Ψ and Φ will also be gauge-invariant. Our choice of these variables is justified only by reason of convenience.

An alternative (but related) solution to the gauge problem is to fix the gauge and keep track of all perturbations (metric and matter). For example, we can use the freedom in the gauge functions T and L in (A.41) to set two of the four scalar metric perturbations to zero:

- *Newtonian (or longitudinal) gauge.* The choice

$$B = E = 0\,, \tag{A.56}$$

gives the metric

$$\mathrm{d}s^2 = a^2(\tau)\left[-(1+2\,\Psi)\mathrm{d}\tau^2 + (1-2\Phi)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right]. \tag{A.57}$$

Here, we have renamed the remaining two metric perturbations, $A \equiv \Psi$ and $C \equiv -\Phi$, in order to make contact with the Bardeen potentials in (A.54) and (A.55). For perturbations that decay at spatial infinity, the Newtonian gauge is unique, i. e. the gauge is fixed completely[1]. In this gauge, the physics appears rather simple since the hypersurfaces of constant time are orthogonal to the worldlines of observers at rest in the coordinates (since $B = 0$) and the induced geometry of the constant-time hypersurfaces is isotropic (since $E = 0$). In the absence of anisotropic stress, we get $\Psi = \Phi$. Note that, due to the similarity of the metric to the usual weak-field limit of General Relativity about Minkowski space-time, Ψ plays the role of the gravitational potential.

- *Spatially flat gauge.* A convenient gauge for computing inflationary perturbations is

$$C = E = 0\,. \tag{A.58}$$

A.2.2 The perturbations in the matter sector

We now consider small perturbations of the stress-energy tensor,

$$T^{\mu}{}_{\nu} = \bar{T}^{\mu}{}_{\nu} + \delta\,T^{\mu}{}_{\nu}\,, \tag{A.59}$$

where $\bar{T}^{\mu}{}_{\nu}$ has the form of the stress-energy tensor of a perfect fluid (A.10), since it represents matter in a homogeneous and isotropic Universe. In a perturbed Universe, however, the energy density ρ, the pressure p and the four-velocity u^μ can be functions of the position. Moreover, the stress-energy tensor can now have a contribution from the *anisotropic stress*, $\Pi^{\mu}{}_{\nu}$. The perturbation of the stress-energy tensor is

$$\delta T^{\mu}{}_{\nu} = (\delta\rho + \delta p)\bar{u}^\mu\bar{u}_\nu + (\bar{\rho}+\bar{p})(\delta u^\mu\bar{u}_\nu + \bar{u}^\mu\delta u_\nu) + \delta p\,\delta^{\mu}{}_{\nu} + \Pi^{\mu}{}_{\nu}\,, \tag{A.60}$$

where we have splitted the four-velocty into a background and a perturbed part, $u^\mu = \bar{u}^\mu + \delta u^\mu$, and we have done the same for the energy density and the pressure of the

[1] In fact, from eq. (A.53) we see that the condition $E = 0$ is violated by any $L \neq 0$, and using this result and eq. (A.51) we see that any time transformation with $T \neq 0$ destroys the condition $B = 0$. Hence, there is no extra coordinate freedom which preserves $B = E = 0$.

fluid. To find the explicit expressions of the $\delta T^{\mu}{}_{\nu}$, we have to compute the perturbed four-velocity in the perturbed metric (A.36). What we finally get is

$$\delta T^{0}{}_{0} = -\delta\rho\,, \tag{A.61}$$
$$\delta T^{i}{}_{0} = -(\bar\rho + \bar p)v^i \tag{A.62}$$
$$\delta T^{0}{}_{j} = (\bar\rho + \bar p)(v_j + B_j) \tag{A.63}$$
$$\delta T^{i}{}_{j} = +\delta p\,\delta^{i}{}_{j} + \Pi^{i}{}_{j}\,. \tag{A.64}$$

where $v^i \equiv \mathrm{d}x^i/\mathrm{d}\tau \equiv a\,\delta u^i$ is the *coordinate velocity*, so that

$$u^{\mu} = a^{-1}\left[1 - A, v^i\right]\,, \qquad u_{\mu} = a\left[1 + A, -(v_i + B_i)\right]\,. \tag{A.65}$$

We will use q^i for the momentum density $(\bar\rho + \bar p)v^i$. If there are several contributions to the stress-energy tensor (e. g. photons, baryons, dark matter, etc.), they are added: $T_{\mu\nu} = \sum_I T^I_{\mu\nu}$. This implies

$$\delta\rho = \sum_I \delta\rho_I\,, \qquad \delta p = \sum_I \delta p_I\,, \qquad q^i = \sum_I q^i_I\,, \qquad \Pi^{ij} = \sum_I \Pi^{ij}_I\,. \tag{A.66}$$

We see that the perturbations in the density, pressure and anisotropic stress simply add. The velocities do not add, but the momentum densities do. Finally, we note that the SVT decomposition can also be applied to the perturbations of the stress-energy tensor: $\delta\rho$ and δp have scalar parts only, q^i has scalar and vector parts and Π_{ij} has scalar, vector and tensor parts.

Under the coordinate transformation (A.41), the perturbations of the stress-energy tensor transform as

$$\delta T^{\mu}{}_{\nu} \to \delta T^{\mu}{}_{\nu} + \mathcal{L}_{\xi}\bar T^{\mu}{}_{\nu} \tag{A.67}$$

where the Lie derivative is

$$\mathcal{L}_{\xi}\bar T^{\mu}{}_{\nu} \equiv -\xi^{\lambda}\partial_{\lambda}\bar T^{\mu}{}_{\nu} + \partial_{\lambda}\xi^{\mu}\bar T^{\lambda}{}_{\nu} - \partial_{\nu}\xi^{\lambda}\bar T^{\mu}{}_{\lambda}\,. \tag{A.68}$$

Evaluating this for the different components of the stress-energy tensor, we find

$$\delta\rho \to \delta\rho - T\,\bar\rho'\,, \tag{A.69}$$
$$\delta p \to \delta p - T\,\bar p' \tag{A.70}$$
$$q_i \to q_i + (\bar\rho + \bar p)\,L_i' \tag{A.71}$$
$$v_i \to v_i + L_i'\,, \tag{A.72}$$
$$\Pi_{ij} \to \Pi_{ij}\,. \tag{A.73}$$

To solve the gauge problem that we discussed above, we can consider gauge-invariant quantities that can be formed from metric and matter variables. One useful combination is

$$\bar\rho\Delta \equiv \delta\rho + \bar\rho'\,(v + B)\,, \tag{A.74}$$

where $v_i = \partial_i v$. The quantity $\triangle$ is called the *comoving gauge density perturbation.* The alternative solution to the gauge problem, the gauge fixing, can also apply in the matter sector:

- *Uniform density gauge.* We can use the freedom in the time-slicing to set the total density perturbation to zero, $\delta\rho = 0$.
- *Comoving gauge.* Similarly, we can ask for the scalar momentum density to vanish, $q = 0$.

A.2.3 The linearized Einstein equations

By using the tools developed in the previous sections, one can finally get the perturbed Einstein equations, $\delta G_{\mu\nu} = 8\pi G\,\delta T_{\mu\nu}$, by using the perturbed metric and the perturbed stress-energy tensor. We will show only the result in the simple case of scalar perturbations, as an example. We work in Newtonian gauge, where the metric has the following form:

$$\mathrm{d}s^2 = a^2(\tau)\left[-(1+2\Psi)\mathrm{d}t^2 + (1-2\Phi)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right] . \tag{A.75}$$

Moreover, if the anisotropic stress does not play a significant role (as it is expected in the standard model of cosmology), we can set $\Psi = \Phi$. The Einstein equations then read

$$\nabla^2\Phi - 3\,\mathcal{H}(\Phi' + \mathcal{H}\,\Phi) = 4\pi G a^2\,\delta\rho\,, \tag{A.76}$$

$$\Phi' + \mathcal{H}\,\Phi = -4\pi G a^2(\bar\rho + \bar p)v\,, \tag{A.77}$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2\,\delta p\,. \tag{A.78}$$

The source terms on the right-hand side should be interpreted as the sum over all relevant matter components (e. g. photons, dark matter, baryons, etc.). The Poisson equation takes a particularly simple form if we introduce the comoving gauge density contrast $\bar\rho\triangle \equiv \delta\rho - 3\mathcal{H}(\bar\rho + \bar p)v$:

$$\nabla^2\Phi = 4\pi G a^2\bar\rho\,\triangle\ . \tag{A.79}$$

From the conservation of the stress energy-tensor, we derive the relativistic generalisations of the continuity equation and the Euler equation in Newtonian gravity:

$$\delta' + 3\,\mathcal{H}\left(\frac{\delta p}{\delta\rho} - \frac{\bar p}{\bar\rho}\right)\delta = -\left(1 + \frac{\bar p}{\bar\rho}\right)(\nabla\cdot\mathbf{v} - 3\,\Phi')\ , \tag{A.80}$$

$$\mathbf{v}' + 3\,\mathcal{H}\left(\frac{1}{3} - \frac{\bar p'}{\bar\rho'}\right)\mathbf{v} = -\frac{\nabla\delta p}{\bar\rho + \bar p} - \nabla\Phi\,, \tag{A.81}$$

where we have defined the *density contrast* $\delta \equiv \delta\rho/\bar\rho$. These equations apply for the total matter and velocity, and also separately for any non-interacting components so that the individual stress-energy tensors are separately conserved.

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