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On the von Neumann Algebras Generated by Field Operators.

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Summary. — It is shown that the von Neumann algebra \mathfrak{N}_Δ which is generated by field operators of an open space-time domain Δ is never a factor of type I_n (with $n < \infty$) or II_1 . An argument is sketched to show that the factor \mathfrak{N}_Δ associated with a bounded and open space-time domain Δ is not of type I_∞ either. It is noticed that a variant of this argument may be utilized for showing that the factors associated with special space-time domains are not of type II_∞ . The arguments for ruling out the possibility of « types » I_∞ and II_∞ are incomplete to the extent of two unproved mathematical propositions which however are likely to be true. Some of the known results pertaining to the structure of the algebra \mathfrak{N}_∞ which is generated by all field operators are also rederived from a slightly different point of view than the usual.

1. — Introduction.

The von Neumann algebras generated by the field operators are convenient tools for formulating and discussing the axioms of quantum field theory.⁽¹⁾ It is therefore of some interest to study the structures of these algebras. The questions concerning the structure of these algebras can be divided into two groups. The first group inquires about the structure of the algebra \mathfrak{N}_∞ generated by all field operators. This group of questions have been studied by several authors in recent years⁽²⁻⁴⁾. In particular BORCHERS has proved in

(*) Supported by the Swiss National Research Foundation.

(1) R. HAAG and B. SCHROER: *Journ. Mat. Phys.*, **3**, 248 (1962).

(2) H. J. BORCHERS: *Nuovo Cimento*, **24**, 214 (1962).

(3) T. F. JORDAN and E. C. G. SUDARSHAN: *Journ. Mat. Phys.*, **3**, 587 (1962).

(4) D. RUELE: *Helv. Phys. Acta*, **35**, 147 (1962).

the framework of Wightman's formalism, that the commutant \mathfrak{R}'_∞ of \mathfrak{R}_∞ is abelian. Furthermore it has been shown that \mathfrak{R}_∞ is irreducible if and only if the vacuum state (which is assumed to be cyclic for \mathfrak{R}_∞) is unique. In this paper we do not add any essentially new result concerning the first group of questions. However, we shall rederive (in Appendix I) some of the known results from a slightly different point of view.

The other group of questions, that can be asked, pertains to the structure of the von Neumann algebra \mathfrak{R}_Δ , which are generated by the field operators of an arbitrary open space-time domain Δ . REEH and SCHLIEDER have obtained the important result that the vacuum state (which is assumed to be cyclic for \mathfrak{R}_∞) is also cyclic for \mathfrak{R}_Δ , no matter how small Δ is ⁽⁵⁾. In a recent paper HAAG and SCHROER ⁽¹⁾ have outlined a method to prove that the algebras \mathfrak{R}_Δ are indeed factors. In this paper we shall investigate the structure of these factors.

It may be interesting to recall in this connection that the factors of « type II_1 » were a special favourite of von Neumann. He conjectured that all measurable field-theoretic quantities should be expressible in terms of operators in « type II_1 » factors. This conjecture would indeed be true in the factors \mathfrak{R}_Δ 's associated with bounded, open space-time domains were of type II_1 . For, in this case all, so-called, quasilocal observables would be expressible in terms of « type II_1 » factors.

Von Neumann's preference for « type II_1 » factors can be understood if one remembers that such factors have some very regular properties ⁽⁶⁾. For example, every closed, linear and hermitian operator affiliated to a factor of type II_1 is automatically self-adjoint. Furthermore all closed linear operators with domains of definition dense in the Hilbert space and affiliated to a « type II_1 » factor are defined in a common dense domain. Any polynomial of such operators is defined on a dense domain and admits of a closure. Thus the troubles connected with domain-questions of unbounded operators do not arise in the case of operators affiliated to a « type II_1 » factor.

However we shall see (in Section 3) that the factors \mathfrak{R}_Δ associated with open space-time domains are never of type II_1 or of type I_n (with $n < \infty$). In the same section we shall outline an argument for showing that the \mathfrak{R}_Δ 's associated with bounded, open space-time domains are not of type I_∞ . A variant of this argument may be used to rule out the possibility of « type II_∞ » factors in case of special space-time domains.

Our arguments for these cases are incomplete to the extent of an unproved mathematical lemma. Nevertheless we present this incomplete argument with the hope that the gap in it can be filled eventually.

⁽⁵⁾ H. REEH and S. SCHLIEDER: *Nuovo Cimento*, **22**, 1051 (1961).

⁽⁶⁾ J. VON NEUMANN: *Collected Works*, vol. **3** (New York, 1961), pp. 116-119.

Before entering into the discussion of the structure of \mathfrak{N}_d 's we shall begin (in the next section) with a rapid survey of some of the mathematical results that will be needed in the sequel.

2. - Mathematical preliminaries.

In this section we shall collect together some important definitions and theorems, all centering round the mathematical objects called « von Neumann algebras ». The literature about von Neumann algebras is voluminous ⁽⁶⁻⁸⁾ and the reader should refer to some of them for a more detailed discussion of the subject matter of the present section.

Let \mathfrak{S} be a set of (not necessarily bounded) closed linear operators in a Hilbert space \mathfrak{H} . Then \mathfrak{S}' will denote the set of all bounded linear operators T of \mathfrak{H} such that T as well as T^* commute (*) with every operator in \mathfrak{S} . If \mathfrak{S} is a *-set of operators, i.e., if with the operator S the set \mathfrak{S} contains the operator S^* also, then the set \mathfrak{S}' is identical with the set of all bounded linear operators which commute with every operator of \mathfrak{S} . The set \mathfrak{S}' is called the *commutant* of the set \mathfrak{S} of operators. We now state the

Definition 1. A set \mathfrak{N} of bounded linear operators in the Hilbert space \mathfrak{H} is called a *von Neumann algebra* if \mathfrak{N} is a *-set of operators and $\mathfrak{N}'' = \mathfrak{N}$.

It can be easily verified that whenever the operators T_1 and T_2 belong to a von Neumann algebra \mathfrak{N} , the operators $T_1 T_2$, $T_1 + T_2$ and αT (α , a complex number) are also in \mathfrak{N} . Evidently the unbounded operators cannot be members of a von Neumann algebra.

However one can introduce the weaker concept of the affiliation of an unbounded operator with a von Neumann algebra.

Definition 2. An operator T (which is not necessarily bounded) is said to be *affiliated* to the von Neumann algebra \mathfrak{N} if every operator of \mathfrak{N}' commutes with T .

The following theorems will illuminate the notion of affiliation:

Theorem 1. A self-adjoint operator T is affiliated to the von Neumann algebra \mathfrak{N} if and only if all the spectral projections of T are in \mathfrak{N} ⁽⁹⁾.

(7) J. DIXMIER: *Les algèbres d'opérateurs dans l'espace Hilbertien* (Paris, 1957).

(8) M. A. NAIMARK: *Normed Rings* (Amsterdam, 1959).

(*) A bounded operator T is said to commute with an unbounded operator S if the operator ST is an extension of TS : $TS \subset ST$.

(9) J. VON NEUMANN: *Collected Works*, vol. 2 (New York, 1961).

As a simple corollary to Theorem 1, it follows that T is affiliated with \mathfrak{A} if and only if all bounded functions (in the sense of functional calculus ⁽¹⁰⁾) of T are in \mathfrak{A} . We have the following, slightly more general

Theorem 2. A closed linear operator T with nonempty resolvent set (*) is affiliated to the von Neumann algebra \mathfrak{A} if and only if all the resolvent operators of T are in \mathfrak{A} .

We shall formulate now another necessary and sufficient criterion for a closed operator to be affiliated with a von Neumann algebra.

For this purpose we recall that every closed linear operator T can be represented in the form $T = \Omega B$, where Ω and B have the following properties: the operator B is a positive definite self-adjoint operator such that $B^2 = T^*T$ the operator Ω is a partially isometric operator whose initial domain is the closure of the range of T^* and whose range is the closure of the range of T . It can be shown further that the decomposition $T = \Omega B$ having the properties just mentioned is unique. It is called the *polar decomposition* of the operator T ⁽¹¹⁾. The following theorem provides now the necessary and sufficient condition which was alluded to earlier.

Theorem 3. Let the closed linear operator T , with domain of definition dense in \mathfrak{H} , have the polar decomposition

$$T = \Omega B.$$

Then T is affiliated with a von Neumann algebra \mathfrak{A} if and only if Ω as well as all the spectral projections of B are in \mathfrak{A} ⁽¹²⁾.

As before, let \mathfrak{S} denote a given set of closed linear operators. It can be proven that the set \mathfrak{S}' is a von Neumann algebra of bounded operators ⁽¹³⁾. The following important theorem is an easy corollary of the preceding statement.

Theorem 4. Let \mathfrak{S} denote a set of closed linear operators. Then the set \mathfrak{S}'' is a von Neumann algebra with which every operator in \mathfrak{S} is affiliated. If \mathfrak{A} is a von Neumann algebra such that every operator of \mathfrak{S} is affiliated with it,

⁽¹⁰⁾ F. RIESZ and B. SZ.-NAGY: *Leçons d'analyse fonctionnelle* (Paris, 1955).

(*) A complex number z is said to belong to the resolvent set of T if the inverse $(T - zI)^{-1}$ of $(T - zI)$ exists and is a (everywhere defined) bounded linear operator. Every operator which is of the form $(T - zI)^{-1}$, where z belongs to the resolvent set of T , is called a resolvent operator of T .

⁽¹¹⁾ Ref. ⁽⁸⁾, p. 284.

⁽¹²⁾ F. MURRAY and J. VON NEUMANN: ref. ⁽⁶⁾, p. 33, lemma 4.4.1.

⁽¹³⁾ Ref. ⁽⁹⁾, p. 121.

then

$$\mathfrak{E}'' \subseteq \mathfrak{N}.$$

In other words \mathfrak{E}'' is the smallest von Neumann algebra with which every operator of \mathfrak{E} is affiliated. The set \mathfrak{E}'' will be referred to as the von Neumann algebra generated by the set \mathfrak{E} of operators.

Definition 3. Let \mathfrak{N} be a given set of linear operators of the Hilbert space \mathfrak{H} . A vector Ψ in \mathfrak{H} is called a *cyclic vector* of the set \mathfrak{N} if the smallest closed linear manifold spanned by all vectors of the form $T\Psi$ with $T \in \mathfrak{N}$ coincides with the space \mathfrak{H} . Another important concept is that of a *separating vector* for a set of operators.

Definition 4. A vector Ψ is called a separating vector of the set \mathfrak{N} of operators if $T \in \mathfrak{N}$ and $T\Psi = 0$ imply that $T = 0$.

In this connection we have the important

Theorem 5. The vector Ψ is a cyclic vector of the von Neumann algebra \mathfrak{N} if and only if it is a separating vector of \mathfrak{N}' ⁽¹⁴⁾.

The structure of von Neumann algebras (or more precisely factors) was first investigated by MURRAY and VON NEUMANN. What follows is a brief sketch of the Murray-von Neumann classification of factors into different types ⁽⁶⁻⁸⁾.

Definition 5. A von Neumann algebra \mathfrak{N} is called a factor if the center $\mathfrak{N} \cap \mathfrak{N}'$ of \mathfrak{N} is the multiplicity of the identity.

A trivial example of a factor is the algebra $\mathfrak{B}(\mathfrak{H})$ of all bounded (linear) operators in the Hilbert space \mathfrak{H} . The factors in a finite-dimensional vector space have a rather simple structure. It can be proved that every factor in a finite dimensional vector space \mathfrak{H}_n is (algebraically) isomorphic (*) to the algebra of all linear operators in some suitable vector space (which is in general different from \mathfrak{H}_n). The proof of this simple result hinges essentially on the fact that every factor in a finite-dimensional space possesses minimal projections. If the preceding assertions were also true of factors in the infinite-dimensional case the structure theory of factors in a Hilbert space of infinite dimensions would be the same as that in a finite-dimensional space. It was however discovered by MURRAY and VON NEUMANN that this is not the case and in

⁽¹⁴⁾ Ref. (7), p. 5-6.

(*) Two von Neumann algebras \mathfrak{N}_1 and \mathfrak{N}_2 are said to be (algebraically) isomorphic if there exists a one-to-one and linear mapping $\Phi: T \rightarrow \Phi(T)$, from \mathfrak{N}_1 onto \mathfrak{N}_2 such that $\Phi(TS) = \Phi(T)\Phi(S)$ and $\Phi(T^*) = (\Phi(T))^*$ for all $T, S \in \mathfrak{N}_1$.

consequence the factors in an infinite Hilbert space have in general a richer structure. In order to reveal the new possibilities, MURRAY and VON NEUMANN developed a theory of comparability of projections in the factors. In this connection the basic notion is that of the equivalence of two projections relative to a von Neumann algebra.

Definition 6. The projections P_1 and P_2 are said to be equivalent relative to a von Neumann algebra \mathfrak{A} if there exists a partially isometric operator Q in \mathfrak{A} such that

$$\widetilde{Q^*Q} = P_1 \quad \text{and} \quad Q\Omega^* = P_2 .$$

In this case we write $P_1 \sim P_2 (\dots \mathfrak{A})$.

Evidently, if two projections are equivalent relative to the algebra \mathfrak{A} then they belong to \mathfrak{A} . It is also evident that if two projections are equivalent relative to any von Neumann algebra \mathfrak{A} then the linear dimensions of the closed linear subspaces, onto which the projections project the entire Hilbert space are equal. The converse of this statement is however not true. One can now define an order-relation in the set of the projections of a factor.

Definition 7. Let P_1 and P_2 be two projections of a von Neumann algebra \mathfrak{A} . We say that P_1 is not greater than P_2 relative to \mathfrak{A} (and write $P_1 \not\gtrsim P_2 (\dots \mathfrak{A})$), if there exists a projection $P_0 \leq P_2$ (*) such that $P_0 \sim P_1 (\dots \mathfrak{A})$.

If the relation $P_1 \not\gtrsim P_2 (\dots \mathfrak{A})$ holds, but the relation $P_1 \sim P_2 (\dots \mathfrak{A})$ does not hold then we say P_1 is less than P_2 (relative to \mathfrak{A}) and write $P_1 \prec P_2 (\dots \mathfrak{A})$.

The set \mathfrak{A}' of all projections of a factor \mathfrak{A} is totally well-ordered with respect to the order relation ($\not\gtrsim$) introduced in Definition 7. This means that if P_1 and P_2 are any two projections in \mathfrak{A} then at least one of the relations $P \not\gtrsim P_2 (\dots \mathfrak{A}) : P_2 \not\gtrsim P_1 (\dots \mathfrak{A})$ holds. Furthermore if the relations $P_1 \not\gtrsim P_2 (\dots \mathfrak{A})$ and $P_2 \not\gtrsim P_1 (\dots \mathfrak{A})$ hold simultaneously then we have $P_1 \sim P_2 (\dots \mathfrak{A})$.

Definition 8. A projection P in a von Neumann algebra \mathfrak{A} will be called *infinite* relative to \mathfrak{A} if there exists a projection P_0 such that $P_0 \prec P$ and $P_0 \sim P (\dots \mathfrak{A})$. Otherwise it is said to be *finite* relative to \mathfrak{A} .

It can be proved that an infinite projection of a factor in a separable, Hilbert space is equivalent (relative to \mathfrak{A}) to the identity operator.

Definition 9. A von Neumann algebra \mathfrak{A} is said to be of *finite type* if every projection of \mathfrak{A} is finite relative to \mathfrak{A} . Otherwise it is said to be of *infinite type*.

(*) We write $P_0 \leq P_2$ if $P_0(\mathfrak{H}) \subseteq P_2(\mathfrak{H})$; where $P_0(\mathfrak{H})$ and $P_2(\mathfrak{H})$ denote the subspaces onto which P_0 and P_1 respectively project the Hilbert space \mathfrak{H} .

A trivial example of an algebra of infinite type is the algebra $\mathfrak{B}(\mathfrak{H})$ in an infinite-dimensional Hilbert space \mathfrak{H} . It can be verified that if \mathfrak{A}_1 is a finite (or infinite) von Neumann algebra then any other von Neumann algebra \mathfrak{A}_2 which is (algebraically) isomorphic to \mathfrak{A}_1 is also finite (or, respectively, infinite). Von Neumann algebras can also be grouped into two classes according as they contain minimal projections or not.

Definition 10. A projection $P \neq 0$ of the von Neumann algebra \mathfrak{A} is called a minimal projection of \mathfrak{A} if the conditions

- a) P_1 is a projection of \mathfrak{A} , and
- b) $P_1 \leq P$

imply that either $P_1 = 0$ or $P_1 = P$.

We have now all tools on hand to introduce the Murray-von Neumann classification of factors.

Definition 11.

a) A finite factor containing minimal projections is called a factor of type I_{finite} .

b) A finite factor which has no minimal projections is said to be of type II_1 .

c) An infinite factor having minimal projections is referred to as a factor of type I_∞ .

d) An infinite factor \mathfrak{A} which has no minimal projections and which contains at least one finite projection (other than zero) relative to \mathfrak{A} is called a factor of type II_∞ .

e) An infinite factor \mathfrak{A} is said to be of type III_∞ if every one of its nonzero projections is infinite (relative to \mathfrak{A}).

In their original papers MURRAY and VON NEUMANN arrived at the classification of factors through a different method. They introduced the important notion of a (relative) dimension function on the projections of factors.

Definition 12. Let \mathfrak{A} be a factor. A real-valued function $D_{\mathfrak{A}}(P)$ defined on the projections of \mathfrak{A} is said to be a *relative dimension function* on \mathfrak{A} if the following conditions are satisfied:

- 1) $D_{\mathfrak{A}}(P) = 0$ for $P = 0$ and $D_{\mathfrak{A}}(P) > 0$ for $P \neq 0$.
- 2) $D_{\mathfrak{A}}(P) = \infty$ if and only if P is infinite relative to \mathfrak{A} .
- 3) If $P_1 \sim P_2 (\dots \mathfrak{A})$ then $D_{\mathfrak{A}}(P) = D_{\mathfrak{A}}(P_2)$.
- 4) If $P_1 P_2 = 0$ then $D_{\mathfrak{A}}(P_1 + P_2) = D_{\mathfrak{A}}(P_1) + D_{\mathfrak{A}}(P_2)$.
- 5) If $P_1 \leq P_2$ and P_1 is finite relative to \mathfrak{A} , then $D_{\mathfrak{A}}(P_1) < D_{\mathfrak{A}}(P_2)$.

One has the important

Theorem 5. If \mathfrak{A} is a factor then there exists a relative dimension function on the projection of \mathfrak{A} . Any two relative dimension functions on the same factor can differ only by a constant multiplying factor.

In other words if D_1 and D_2 are two relative dimension functions on \mathfrak{A} , then we have

$$D_1(P) = \alpha D_2(P)$$

for all projections P in \mathfrak{A} with some constant real positive α . Murray-von Neumann's classification of factors follows from the following

Theorem 6. Let \mathfrak{A} be a given factor, and $D_{\mathfrak{A}}$ a relative dimension function of \mathfrak{A} . Then a constant real multiplying factor α can be chosen so that the set of all values which the function $\alpha D_{\mathfrak{A}}$ assumes will coincide with one and only one of the following sets:

- a) The set of numbers: $0, 1, 0, \dots, n$ (with $n < \infty$).
- b) The set of all numbers: $0, 1, 2, \dots, n, \dots, \infty$.
- c) The closed interval $[0, 1]$.
- d) The interval $[0, \infty]$.
- e) The numbers 0 and ∞ .

The factor \mathfrak{A} is of type $I_n, I_\infty, II_1, II_\infty$, or III_∞ according as case a), b), c), d) or e) holds (respectively). It can be verified that the classification of factors as given by Definition 11 is the same as the one just described.

The algebra $\mathfrak{B}(\mathfrak{H})$ in the n -dimensional (n may be ∞) Hilbert space \mathfrak{H} is an example of factor of type I_n . In fact every factor of type I_n is algebraically isomorphic to the algebra $\mathfrak{B}(\mathfrak{H})$ in an n -dimensional space \mathfrak{H} . Special examples of factors belonging to type II_1, II_∞ or III_∞ can also be constructed (6-8). Thus factors of all possible types actually occur.

We shall close this preparatory section with some additional definitions which will be needed later.

Definition 13. The von Neumann algebra \mathfrak{A}_1 in a Hilbert space \mathfrak{H} , is said to be *spatially isomorphic* to a von Neumann algebra \mathfrak{A}_2 in the Hilbert space \mathfrak{H}_2 if there exists a unitary mapping (i.e., a one-to-one, linear and isometric mapping) U from \mathfrak{H}_1 onto \mathfrak{H}_2 such that

$$U\mathfrak{A}_1U^{-1} = \mathfrak{A}_2.$$

Here $U\mathfrak{S}_1U^{-1}$ denotes the set of all operators (in \mathfrak{S}_2) of the form UTU^{-1} with $T \in \mathfrak{R}_1$.

It may be noted that if two von Neumann algebras are spatially isomorphic then they are also algebraically isomorphic. Two von Neumann algebras may however be (algebraically) isomorphic without being spatially isomorphic.

Definition 14. Let \mathfrak{N} be a von Neumann algebra and $\mathfrak{M} \neq 0$ a closed linear subspace of \mathfrak{S} . Consider those operators A of \mathfrak{N} which are reduced by \mathfrak{M} and form the restrictions $A_{(\mathfrak{M})}$ of A into \mathfrak{M} . The set of all such operators $A_{(\mathfrak{M})}$ will be denoted by $\mathfrak{N}_{(\mathfrak{M})}$ and will be called the reduction of the algebra \mathfrak{N} to the subspace \mathfrak{M} . If P is the projection operator onto \mathfrak{M} one writes sometimes \mathfrak{N}_P to denote $\mathfrak{N}_{(\mathfrak{M})}$.

3. – The von Neumann algebras of field operators associated with open space-time regions.

A field is usually defined as a linear and (weakly) continuous mapping of a certain space of functions (called the test-functions) defined on the space-time manifold, into the set of closed linear operators in a Hilbert space \mathfrak{S} ⁽¹⁵⁾. The field operator corresponding to the test-function $f(x_0, \mathbf{x})$ will be denoted by $A(f)$. Let Δ be an open domain of the space-time manifold. With the domain Δ we shall associate the von Neumann algebra \mathfrak{N}_Δ generated by all field operators $A(f)$ corresponding to test-functions which vanish outside the region Δ .

As mentioned in the introduction, HAAG and SCHROER have outlined an argument to show that the von Neumann algebras \mathfrak{N}_Δ are in fact factors. In this section we shall attempt to determine the « type » of these factors \mathfrak{N}_Δ .

Before proceeding with this task we shall first rapidly enumerate the postulates of quantum field theory that are used (either explicitly or implicitly) in our discussions:

- 1) The Hilbert space \mathfrak{S} is with positive definite metric.
- 2) For Φ and Ψ in the common domain of all field operators $A(f)$, the functional $(\Phi, A(f)\Psi)$ is a tempered distribution.
- 3) Relativistic invariance: There exists a unitary representation $(a, \Delta) \rightarrow U(a, \Delta)$ of the proper inhomogeneous Lorentz group such that

$$U(a, \Delta)\mathfrak{N}_\Delta U^{-1}(a, \Delta) = \mathfrak{N}_{L\Delta L^{-1}(a, \Delta)}.$$

⁽¹⁵⁾ A. S. WIGHTMAN: *Problèmes Mathématiques de la Théorie Quantique des Champs*, C.N.R.S. (1957).

Here $[\Delta](a, \Delta)$ denotes the set of all space-time points of the form $\Delta x + a$ with $x \in \Delta$.

4) The spectrum condition.

5) Local commutativity: If the space-time domain \mathfrak{S}_1 is totally spacelike with respect to the domain Δ_2 then

$$\mathfrak{R}_{\Delta_1} \subset \mathfrak{R}'_{\Delta_2}.$$

6) Existence of a cyclic vacuum state.

7) The completeness postulate: The algebra \mathfrak{R}_∞ generated by all field operators $A(f)$ is irreducible,

$$\mathfrak{R}'_\infty = \{\lambda I\}.$$

In addition to these postulates several restrictions of technical nature are imposed on the domains of definition of the field operators $A(f)$ ⁽¹⁵⁾. A detailed discussion of the postulates enumerated above can be found in the paper of HAAG and SCHROER (1) and in that of WHIGHTMAN ⁽¹⁵⁾.

We also formulate the additional postulate of

8) The *strong « primitive causality »*: the von Neumann algebra \mathfrak{R}_Δ associated with a spatially complete (*) open space-time region Δ is identical with the algebra \mathfrak{R}_∞ generated by all field operators $A(f)$.

We may note that the postulate of « primitive causality » (as formulated in the paper of HAAG and SCHROER (1)) is a special case of postulate (8).

It may be that postulate (8) can be derived from the usual postulates of quantum field theory and the postulate of « primitive causality ». In fact BORCHERS has proved that the algebra \mathfrak{R}_Δ associated with a special type of spatially complete region is identical with the algebra \mathfrak{R}_∞ ⁽¹⁶⁾. However we shall not discuss here further about the validity of postulate (8) and only remark that the property of being a factor, of the algebra \mathfrak{R}_Δ (associated with open domain Δ) is an immediate consequence of the postulates (5), (7) and (8) ⁽¹⁾. In the following discussion the postulate (8) is used only in so far as it is necessary to prove that the algebras \mathfrak{R}_Δ 's are factors.

After these preliminary remarks, we now prove the

(*) A space-time region Δ is said to be *spatially complete* if there exists no open space-time domain which is totally spacelike with respect to Δ . Otherwise a space-time region Δ is said to be *spatially incomplete*.

⁽¹⁶⁾ H. J. BORCHERS: *Nuovo Cimento*, **19**, 787 (1961).

Theorem A (*). The factor \mathfrak{N}_Δ associated with an opens pace-time region Δ is not of finite type. In other words \mathfrak{N}_Δ is not of type I_n (with $n < \infty$) nor of type II_1 .

Proof of theorem A. We first remark that it is sufficient to prove the theorem for a (arbitrarily small) spatially incomplete open domain Δ . This is because any open domain Δ_1 will always contain a spatially incomplete open domain, say Δ_0 , and if \mathfrak{N}_{Δ_0} is not finite then the algebra \mathfrak{N}_{Δ_0} which contains \mathfrak{N}_{Δ_0} can not be finite either. Furthermore, for the sake of easy visualization, we shall consider bounded cylindrical regions of space-time; *i.e.*, collection of all space-time points (\mathbf{x}, t) such that

$$|\mathbf{x}| < r \quad \text{and} \quad |t| < \tau, \quad (r, \tau > 0 \text{ and finite}).$$

One may convince oneself as before that this is not an essential restriction.

In order to complete the proof of the theorem *A* we need the following lemmata.

Lemma 1. The factor \mathfrak{N}_Δ associated with a spatially incomplete open domain Δ is of finite type if and only if its commutant \mathfrak{N}'_Δ is of finite type.

Proof of Lemma 1. We recall that the vacuum state Ψ_0 is cyclic for \mathfrak{N}_Δ (*); since the domain Δ is spatially incomplete there exists an open domain Δ_1 which is totally space-like with respect to Δ . The vacuum state is also cyclic for \mathfrak{N}_{Δ_1} and hence it is, *a fortiori*, cyclic for \mathfrak{N}'_{Δ_1} which contains \mathfrak{N}_{Δ_1} . Thus Ψ_0 is both a cyclic and a separating vector of \mathfrak{N}_{Δ_1} as well as \mathfrak{N}'_{Δ_1} . It is well known that if there exists a vector which is a cyclic as well as a separating vector of a von Neumann algebra (say \mathfrak{N}) then either, both \mathfrak{N} and \mathfrak{N}' are finite algebras or both are infinite ⁽¹⁷⁾. Hence either, both \mathfrak{N}_{Δ_1} and \mathfrak{N}'_{Δ_1} are finite or both are infinite.

Lemma 2. Let \mathfrak{N} and \mathfrak{N}_1 be two factors such that a) $\mathfrak{N}_1 \subset \mathfrak{N}$; b) $\mathfrak{N}'_1 \cap \mathfrak{N} \neq \{\lambda I\}$. Further let there be a vector, say Ψ , which is a cyclic vector for \mathfrak{N}_1 and a separating vector for \mathfrak{N} . Then there exists a vector f ($\neq 0$) such that

$$\overline{\{\mathfrak{N}_1 f\}} \subsetneq \overline{\{\mathfrak{N} f\}} (**).$$

(*) The proof given here was stimulated by a communication from Professor WIGHTMAN where it was mentioned that this theorem had been proved by Professor KADISON. The proof given here is however independent of Kadison's.

(**) If \mathfrak{N} is a von Neumann algebra then we denote by $\overline{\{\mathfrak{N}\psi\}}$ the smallest closed linear manifold spanned by the vectors of the form $T\psi$ with $T \in \mathfrak{N}$.

⁽¹⁷⁾ Ref. (7), p. 250, ex. 2.

Proof of Lemma 2. Let $P (\neq 0, \neq I)$ be a projection of the algebra $\mathfrak{N}'_1 \cap \mathfrak{N}$. We shall show that the vector $P\Psi \equiv f$ has the desired properties.

Since $P (\neq 0)$ belongs to \mathfrak{N} and Ψ is a separating vector for \mathfrak{N} , we cannot have $f (\equiv P\Psi) = 0$. Furthermore it is evident that $\overline{\{\mathfrak{N}_1 f\}} \subseteq \overline{\{\mathfrak{N} f\}}$. Therefore we have only to prove the existence of a vector in $\overline{\{\mathfrak{N} f\}}$ which does not belong to $\overline{\{\mathfrak{N}_1 f\}}$. Let $U (\neq I)$ be a unitary operator of \mathfrak{N} which does not commute with P . (Such unitary U exists, for otherwise the projection $P (\neq 0, \neq I)$ of \mathfrak{N} would commute with every unitary operator of \mathfrak{N} and hence it would belong to the center of the factor \mathfrak{N} , which is not possible.) Evidently both Uf and U^*f are in $\overline{\{\mathfrak{N} f\}}$. We now show that at least one of them is not in $\overline{\{\mathfrak{N}_1 f\}}$.

Suppose, to the contrary, that both Uf and U^*f are in $\overline{\{\mathfrak{N}_1 f\}}$. We have

$$(1) (**) \quad \overline{\{\mathfrak{N}_1 f\}} = \overline{\{\mathfrak{N}_1 P\Psi\}} = \overline{\{P\mathfrak{N}_1 \Psi\}} = P(\mathfrak{N}).$$

Therefore we should find

$$(2) \quad P U f = P U P \Psi = U f = U P \Psi$$

and

$$(3) \quad P U^* f = P U^* P \Psi = U^* f = U^* P \Psi.$$

It follows that

$$(4) \quad (P U P - U P) \Psi = 0$$

and

$$(5) \quad (P U^* P - U^* P) \Psi = 0.$$

Since the operators $(P U P - U P)$ and $(P U^* P - U^* P)$ belong to \mathfrak{N} and Ψ is a separating vector for \mathfrak{N} , we should have:

$$(6) \quad P U P = U P$$

and

$$(7) \quad P U^* P = U^* P.$$

Taking the adjoint of the last equation we obtain $P U P = P U$. Thus we are led to the impossible conclusion that U and P commute. Hence the hypothesis that both Uf and U^*f belong to $\overline{\{\mathfrak{N}_1 f\}}$ must be false.

Lemma 3. Let \mathfrak{N} be a finite factor and $D_{\mathfrak{N}}$ a (relative) dimension function of \mathfrak{N} . If \mathfrak{N}_1 is a subfactor of \mathfrak{N} then the restriction of $D_{\mathfrak{N}}$ to the set of projec-

(*) As before $P(\mathfrak{N})$ denotes the range of the projection operator P .

tions of \mathfrak{N}_1 is again a (relative) dimension function on \mathfrak{N}_1 . In other words if we define a function $D_{\mathfrak{N}_1}$ by setting $D_{\mathfrak{N}_1}(P) = D_{\mathfrak{N}}(P)$, for all $P \in \mathfrak{N}_1$ then $D_{\mathfrak{N}_1}$ is a (relative) dimension function of \mathfrak{N}_1 .

The proof of Lemma 3 consists in verifying that the conditions 1 through 5 of the definition 12 (Section 2) are fulfilled by the function $D_{\mathfrak{N}_1}$.

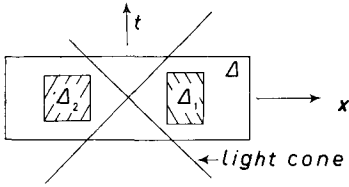


Fig. 1.

We can now complete the proof of Theorem A as follows. Consider a bounded and open space-time cylinder Δ . Every open space-time cylinder contains two open domains which are totally spacelike to each other. Let Δ_1 and Δ_2 be two such open sub-domains of Δ (Fig. 1).

In the following we write, for the sake of convenience, \mathfrak{N} , \mathfrak{N}_1 and \mathfrak{N}_2 in place of \mathfrak{N}_Δ , \mathfrak{N}_{Δ_1} and \mathfrak{N}_{Δ_2} , respectively. Now we first note the following facts:

a) $\mathfrak{N}_1 \subset \mathfrak{N}$; and $\mathfrak{N}'_1 \cap \mathfrak{N} \supset \mathfrak{N}_2 \neq \{\lambda I\}$.

b) The vacuum state Ψ_0 is a cyclic vector as well as a separating vector for \mathfrak{N}_1 and also for \mathfrak{N} . We have already noticed this fact in the course of the proof of Lemma 1.

Our task now will be to show that the facts a) and b) are inconsistent with the assumption that \mathfrak{N} is a finite factor. If \mathfrak{N} is of finite type then its subfactor \mathfrak{N}_1 is also finite. From Lemma 1 it follows that \mathfrak{N}'_1 is also finite. Let D and D'_1 be two relative dimension functions on \mathfrak{N} and \mathfrak{N}'_1 respectively. The restriction of D onto \mathfrak{N}_1 and that of D'_1 onto \mathfrak{N}' , will be respectively denoted by D_1 and D' (note that $\mathfrak{N}' \subset \mathfrak{N}'_1$). According to Lemma 3 D_1 and D' are relative dimension functions on \mathfrak{N}_1 and \mathfrak{N}' respectively.

The desired contradiction will now result from an application of the following well-known.

Theorem 7 ⁽¹⁸⁾. Let D and D' be relative dimension functions of a factor \mathfrak{N} and its commutant \mathfrak{N}' , respectively. Then there exists a (real, positive) constant c such that

$$D(\overline{\{\mathfrak{N}'f\}}) = cD'(\overline{\{\mathfrak{N}f\}}),$$

for all vectors f of the underlying Hilbert space (*).

Before applying the Theorem 7 to the case in hand it may be well to add the following explanatory remarks. Except for the case when \mathfrak{N} and \mathfrak{N}' are

⁽¹⁸⁾ Ref. ⁽⁶⁾, p. 72, theorem X.

(*) It can be easily verified that the projection P , onto $\overline{\{\mathfrak{N}'f\}}$ belongs to \mathfrak{N} . Here we write $D(\overline{\{\mathfrak{N}'f\}})$ to denote $D(P)$. The notation $D'(\overline{\{\mathfrak{N}f\}})$ has similar significance.

of type III_∞ , the constant c of Theorem 7 is uniquely determined by the given pair D and D' of relative dimension functions. In particular when both \mathfrak{R} and \mathfrak{R}' are finite (so that D and D' assume only finite values), the constant c is obtained by forming the ratio

$$\frac{D(\overline{\{\mathfrak{R}'f\}})}{D'(\overline{\{\mathfrak{R}f\}})} = c,$$

with any vector $f \neq 0$.

We now choose of vector $f (\neq 0)$ such that $\overline{\{\mathfrak{R}_1 f\}} \subsetneq \overline{\{\mathfrak{R}f\}}$. Lemma 2 guarantees the possibility of such a choice. Since both \mathfrak{R}_1 and \mathfrak{R}'_1 are finite, the ratio

$$\frac{D_1(\overline{\{\mathfrak{R}'_1 f\}})}{D'_1(\overline{\{\mathfrak{R}_1 f\}})}$$

is a finite real number, say c_1 . According to Theorem 7 this ratio is independent of the vector f and hence we can write

$$(8) \quad c_1 = \frac{D_1(\overline{\{\mathfrak{R}'_1 \Psi_0\}})}{D'_1(\overline{\{\mathfrak{R}_1 \Psi_0\}})} = \frac{D(\overline{\{\mathfrak{R}'_1 \Psi_0\}})}{D_1(\overline{\{\mathfrak{R}_1 \Psi_0\}})}.$$

The last equality in relation (8) holds because D_1 is a restriction of D . Similarly we can also show that

$$(9) \quad c = \frac{D(\overline{\{\mathfrak{R}'f\}})}{D'(\overline{\{\mathfrak{R}f\}})} = \frac{D(\overline{\{\mathfrak{R}'_1 \Psi_0\}})}{D_1(\overline{\{\mathfrak{R}_1 \Psi_0\}})}.$$

Since Ψ_0 is a cyclic vector for every one of the factors \mathfrak{R} , \mathfrak{R}' , \mathfrak{R}_1 and \mathfrak{R}'_1 we obtain

$$(10) \quad c = c_1 = \frac{D(\mathfrak{S})}{D_1(\mathfrak{S})}.$$

On the other hand we have chosen f such that $\overline{\{\mathfrak{R}f\}} \supsetneq \overline{\{\mathfrak{R}_1 f\}}$. Therefore we should find

$$(11) (*) \quad D'(\overline{\{\mathfrak{R}f\}}) = D'_1(\overline{\{\mathfrak{R}f\}}) > D'_1(\overline{\{\mathfrak{R}_1 f\}}).$$

(*) The equality of the relation (11) is due to the fact that D' is the restriction of D'_1 . The inequality follows from property (5) in definition 12 (of Section 2).

Evidently $\overline{\{\mathfrak{N}'_1 f\}} \supseteq \overline{\{\mathfrak{N}' f\}}$ (recall that $\mathfrak{N}'_1 \supset \mathfrak{N}'$). Hence we also have

$$(12) \quad D_1(\overline{\{\mathfrak{N}'_1 f\}}) = D(\overline{\{\mathfrak{N}'_1 f\}}) \geq D(\overline{\{\mathfrak{N}' f\}}).$$

Relations (11) and (12) now yield $c_1 > c$, which contradicts relation (10). Thus the assumption that \mathfrak{N} is finite leads to contradictory conclusions and hence must be false.

We now mention a simple corollary of Theorem A. For this purpose we introduce the

Definition. A bounded linear operator T is said to be a « quasi-local » operator if there exists a bounded and open space-time region Δ such that $T \in \mathfrak{N}_\Delta$. More generally an (not necessarily bounded) operator S is said to be a quasi-local operator if it is affiliated to the algebra \mathfrak{N}_Δ of a bounded space-time region Δ .

We can now formulate the

Corollary to the Theorem A. A quasi-local operator is never a compact (completely continuous) operator.

Proof of the Corollary. Suppose, to the contrary, that a compact operator T_1 belongs to the factor \mathfrak{N}_Δ of a bounded space-time region Δ . It would imply that \mathfrak{N}_Δ contains at least one projection, say P which projects the Hilbert space onto a finite-dimensional subspace. It is however not possible since the reduction $(\mathfrak{N}'_\Delta)_P$ of \mathfrak{N}'_Δ onto the range of P would be algebraically isomorphic to \mathfrak{N}'_Δ which is a factor of infinite type.

Theorem A leaves open the possibility of the types I_∞ , II_∞ and III_∞ . We now briefly sketch an argument to show that the factor \mathfrak{N}_{Δ_1} which is associated with a bounded and open (space-time) domain Δ_1 is not of type I_∞ : It is clear that by translating (spatially) the region Δ_1 we can obtain another space-time region Δ_2 which is totally spacelike with respect to Δ_1 .

Evidently we have

$$(13) \quad \mathfrak{N}_{\Delta_2} \subset \mathfrak{N}'_{\Delta_1}$$

and

$$(14) \quad \mathfrak{N}_{\Delta_2} = U(\mathbf{a})\mathfrak{N}_{\Delta_1}U^{-1}(\mathbf{a}).$$

If \mathfrak{N}_{Δ_1} is of type I_∞ then \mathfrak{N}_{Δ_2} (which is spatially isomorphic to \mathfrak{N}_{Δ_1}) would also be of type I_∞ . We consider now the von Neumann algebra $\{\mathfrak{N}_{\Delta_1} \cup \mathfrak{N}_{\Delta_2}\} \equiv \mathfrak{N}$.

The relation (13) and the assumption that \mathfrak{N}_{Δ_1} and \mathfrak{N}_{Δ_2} are factors of type I_∞ imply that \mathfrak{N} is also a factor of type I_∞ ⁽¹⁹⁾. Let now P be a minimal projec-

⁽¹⁹⁾ Ref. (6), p. 81, lemma 11.5.2.

tion of \mathfrak{N}_{Δ_1} . Then the reduction \mathfrak{N}_P of \mathfrak{N} to the range of P is (algebraically) isomorphic to \mathfrak{N}_{Δ_2} ⁽²⁰⁾. Since \mathfrak{N}_{Δ_2} is an infinite factor, the minimal projection P of \mathfrak{N}_{Δ_1} , and therefore *a fortiori* all projections of \mathfrak{N}_{Δ_1} , will be infinite relative to \mathfrak{N} . This conclusion which is derived here under the assumption that \mathfrak{N}_{Δ_1} is of type I_∞ , remains true in general. In other words it can be shown that if Δ_1 is any spatially incomplete (or bounded) region then there exists another spatially incomplete (or bounded) region Δ such that $\Delta_1 \subset \Delta$ and every projection in \mathfrak{N}_{Δ_1} is infinite relative to \mathfrak{N}_Δ ⁽²¹⁾. However this conclusion and the assumption that \mathfrak{N}_{Δ_1} is of type I_∞ are in contradiction with the following plausible

Proposition B₁ ()*. Let \mathfrak{N} and \mathfrak{N}_1 be factors of type I_∞ such that $\mathfrak{N}_1 \subsetneq \mathfrak{N}$. Furthermore let there be a vector Ψ which is a cyclic vector of \mathfrak{N}_1 and a separating vector for \mathfrak{N} . Then a minimal projection of \mathfrak{N}_1 is a finite projection relative to \mathfrak{N} .

Therefore if proposition B_1 turns out to be true then we would have a proof that \mathfrak{N}_{Δ_1} is not of type I_∞ .

The following generalization of the proposition B_1 suggests itself immediately.

Proposition B₂. Let \mathfrak{N} and \mathfrak{N}_1 be two factors which fulfil the following conditions

- a) $\mathfrak{N}_1 \subsetneq \mathfrak{N}$.
- b) There exists a vector Ψ which is a cyclic vector for \mathfrak{N}_1 and a separating vector for \mathfrak{N} .
- c) \mathfrak{N}_1 and \mathfrak{N} are spatially isomorphic, and
- d) $\{(\mathfrak{N}'_1 \cap \mathfrak{N}) \cup \mathfrak{N}_1\}'' = \mathfrak{N}$.

Then a projection (of \mathfrak{N}_1) which is finite relative to \mathfrak{N}_1 , is finite relative to \mathfrak{N} also. Proposition B_1 is a special case of proposition B_2 because when both \mathfrak{N} and \mathfrak{N}_1 are « type I_∞ » factors the conditions (c) and (d) are implied by the conditions (a) and (b) ⁽²²⁾. We remark now that if proposition B_2 is true then one can prove that the factor \mathfrak{N}_{Δ_1} of a space-time cylinder Δ_1 , which

⁽²⁰⁾ Ref. (6), p. 80, lemma 11.5.1.

⁽²¹⁾ B. MISRA: *On the algebra of quasi-local operators of quantum field theory*, to be published.

(*) See note added in proof.

⁽²²⁾ Ref. (7), p. 121 (lemma 6-7), p. 233 (theorem 3) and p. 307 (ex. 13).

extends (spatially) to infinity on one side of the light cone (see Fig. 2) is not of type \mathbb{I}_∞ . Before ending this paper we may add the following additional remark: if the factor \mathfrak{N}_{Δ_0} associated with a bounded space-time cylinder Δ^0

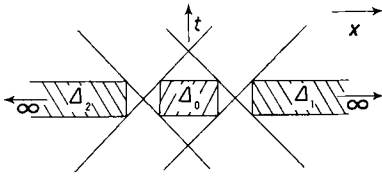


Fig. 2.

is not of type \mathbb{I}_∞ , then the factor \mathfrak{N}_{Δ_1} (or \mathfrak{N}_{Δ_2}) of the unbounded cylinder Δ_1 (or Δ_2) (see Fig. 2) can not be of type \mathbb{I}_∞ either. In order to prove this remark we note that the union of Δ_0 , Δ_1 and Δ_2 is a spatially complete open space-time domain. Hence we may assume

$$(15) \quad \{\mathfrak{N}_{\Delta_0} \cup (\mathfrak{N}_{\Delta_1} \cup \mathfrak{N}_{\Delta_2})\}'' = \mathfrak{N}_\infty$$

Evidently we also have

$$(16) \quad \{\mathfrak{N}_{\Delta_0} \cup (\mathfrak{N}_{\Delta_1} \cup \mathfrak{N}_{\Delta_2})\}'' = \{\mathfrak{N}_{\Delta_0} \cup \mathfrak{N}\}'' ,$$

where we have written \mathfrak{N} to denote the algebra $\{\mathfrak{N}_{\Delta_1} \cup \mathfrak{N}_{\Delta_2}\}''$. If \mathfrak{N}_{Δ_1} is of type \mathbb{I}_∞ then by reason of symmetry \mathfrak{N}_{Δ_2} is also of type \mathbb{I}_∞ . Hence \mathfrak{N} is of type \mathbb{I}_∞ . Therefore the algebra $\{\mathfrak{N}_{\Delta_0} \cup \mathfrak{N}\}'' = \mathfrak{N}_\infty$ will be of the same type as \mathfrak{N}_{Δ_0} (19). Since \mathfrak{N}_∞ is assumed to be irreducible it is of type \mathbb{I}_∞ and so should be \mathfrak{N}_{Δ_0} . Thus we have proved that \mathfrak{N}_{Δ_1} is of type \mathbb{I}_∞ only if \mathfrak{N}_{Δ_1} is also of type \mathbb{I}_∞ .

* * *

We thank Prof. J. M. JAUCH for his encouraging interest in this work and also for many constructive criticisms and suggestions. Prof. J. DIXMIER has kindly communicated to us some useful hints concerning the propositions B_1 and B_2 . We take this opportunity to thank him. We also thank Dr. EPSTEIN from CERN for several interesting discussions.

APPENDIX

It is amusing to note that some of the well-known results about the von Neumann algebra \mathfrak{N}_∞ are immediate consequences of the following postulates:

Postulate a). — Let Q denote the von Neumann algebra generated by all observables. Then

$$(A.1) \quad Q \subseteq \mathfrak{N}_\infty .$$

The meaning of postulate *a*) is that every observable can be expressed in terms of field operators.

Postulate b). – There exists a complete set of commuting observables⁽²³⁾. It has been shown^(24,25) that postulate *b*) is equivalent to the assumption that the commutant Q' of Q is abelian.

$$(A.2) \quad Q' \subset Q .$$

The later assumption is often referred to as the « assumption of commutative superselection rules ».

It follows now immediately from (A.1) and (A.2) that

$$(A.3) \quad \mathfrak{N}'_{\infty} \subset Q' \subset Q \subset \mathfrak{N}_{\infty} .$$

Thus the commutant \mathfrak{N}'_{∞} of \mathfrak{N}_{∞} is abelian; a result which had been established by BORCHERS⁽²⁾ in the framework of Wightman's formalism.

Another result which has been proved by BORCHERS⁽²⁾ is that the unitary operators $U(a, \Lambda)$ representing the connected component of the inhomogeneous Lorentz group, belong to \mathfrak{N}_{∞} . We shall rederive here this result. To this end we assume that the superselection observables have purely discrete spectra so that the Hilbert space \mathfrak{H} of state vectors can be written as a countable direct sum of coherent subspaces⁽²³⁻²⁶⁾.

If $T \in Q'$ and Ψ is any vector which lies wholly in any one of the coherent subspaces then we should have

$$T\Psi = \mu\Psi ,$$

for some complex number μ . Therefore we obtain

$$U(a, \Lambda) T\Psi = \mu U(a, \Lambda) \Psi .$$

On the other hand since we are considering only the connected component of the Lorentz group, the vector $U(a, \Lambda)\Psi$ will lie in the same coherent subspace as Ψ ⁽²⁶⁾. Hence we have

$$TU(a, \Lambda) \Psi = \mu U(a, \Lambda) \Psi .$$

Thus we obtain

$$U(a, \Lambda) T\Psi = TU(a, \Lambda) \Psi ,$$

for all Ψ which lie wholly in one of the coherent subspaces. Since the set of these vectors is dense in \mathfrak{H} we have

$$U(a, \Lambda) T = TU(a, \Lambda) .$$

⁽²³⁾ For a discussion of this postulate see J. M. JAUCH: *Helv. Phys. Acta*, **33**, 711 (1960).

⁽²⁴⁾ J. M. JAUCH and B. MISRA: *Helv. Phys. Acta*, 699 (1961).

⁽²⁵⁾ A. GALINDO, A. MORALES and R. NUÑEZ-LAGOS: *Journ. Math. Phys.*, **3**, 324 (1962).

⁽²⁶⁾ Cf. A. S. WIGHTMAN: *Suppl. Nuovo Cimento*, **14**, 81 (1959).

Thus

$$(A.4) \quad U(a, \Delta) \in Q'' = Q \subset \mathfrak{N}_\infty .$$

BORCHERS has also shown that the algebra \mathfrak{N}_∞ is irreducible if and only if the invariant vector (vacuum), (which is assumed to be a cyclic vector of \mathfrak{N}_∞) is unique. We shall prove here the following statement which is related to, though slightly different from, Borchers's. The algebra \mathfrak{N}_∞ is irreducible if and only if the invariant vector (which is assumed to be a physically realizable vector) is cyclic.

If \mathfrak{N}_∞ is irreducible then every vector, and hence a fortiori the invariant vector also, is cyclic for \mathfrak{N}_∞ ⁽²⁷⁾. Suppose conversely that the invariant vector Ψ_0 is cyclic for \mathfrak{N}_∞ . Since Ψ_0 is assumed to be a physically realizable vector it lies wholly in one and only one of the coherent subspaces. Therefore if $T \in \mathfrak{N}'_\infty \subset Q'$ we should obtain

$$(A.5) \quad T\Psi_0 = \mu\Psi_0 ,$$

for some complex number μ . The relation (A.5) implies that

$$ST\Psi_0 = TS\Psi_0 = \mu S\Psi_0 , \quad \text{for all } S \in \mathfrak{N}_\infty .$$

Since Ψ_0 is assumed to be a cyclic vector of \mathfrak{N}_∞ , the set of all vectors of the form $S\Psi_0$ with $S \in \mathfrak{N}_\infty$, is dense in \mathfrak{H} . Therefore it follows that $T = \mu I$ and hence \mathfrak{N}_∞ is irreducible.

⁽²⁷⁾ Ref. (8), p. 255.

Note added in proof.

We have received a communication from Prof. H. ARAKI containing a counter-example to our proposition B_1 . He also has given a proof that the algebras \mathfrak{N}_Δ are not of type I for a special class of domains Δ .

RIASSUNTO (*)

Si dimostra che l'algebra di von Neumann \mathfrak{N}_Δ , che è generata dagli operatori di campo di un dominio aperto Δ dello spazio-tempo, non è mai un fattore di tipo I_n (con $n < \infty$) o II_1 . Si abbozza un ragionamento per dimostrare che il fattore \mathfrak{N}_Δ associato con un dominio limitato ed aperto Δ dello spazio-tempo non è neanche di tipo I_∞ . Si nota che una variante di questo ragionamento può essere usata per dimostrare che i fattori associati con domini speciali dello spazio-tempo non sono del tipo II_∞ . I ragionamenti per escludere la possibilità dei tipi I_∞ e II_∞ sono incompleti per quanto riguarda le implicazioni di due proposizioni matematiche non dimostrate ma che probabilmente sono vere. Si deducono nuovamente da un punto di vista lievemente diverso da quello usuale alcuni risultati noti relativi alla struttura dell'algebra \mathfrak{N}_∞ generata da tutti gli operatori di campo.

(*) Traduzione a cura della Redazione.