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Multiplication in the center of the small quantum group and geometry

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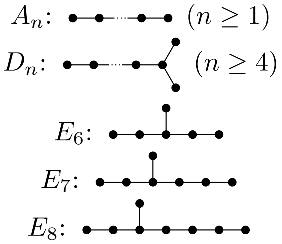
Contents

Chapter 1

Introduction

According to Igor Frenkel, there are many branches of mathematics but few fundamental objects, which tend to reappear everywhere under different names. A fundamental family of examples in mathematics is given by the *Dynkin diagrams* (simply laced, of finite type) which parametrise simple simply laced complex Lie algebras, finite subgroup of $SU(2)$, rational double points, quivers of finite type, conical symplectic complete intersections. Here is a complete list of such diagrams:



In the theory of Hopf algebras, all finite type Dynkin diagrams parametrise simple complex Lie algebras and a remarkable family of finite dimensional complex Hopf algebras, the *small quantum groups* (at a root of unity) $\mathsf{u}(\mathfrak{g})$. Introduced by Lusztig in [52] as a "quantum" analogue of the first Frobenius kernel for an algebraic group in positive characteristic, it turned out to be connected with many other areas of mathematics: of course representation theory, but also conformal field theory, knot theory, topological field theory, affine Lie algebras.

Let us gives an intuitive motivation for the study of the quantum universal enveloping algebras, especially the case when the deformation parameter q is a root of unity. Quantum groups were introduced by Drinfeld and Jimbo (see [22], [37]), where the original motivation was to find a deformation (called "deformation-quantization" since it is motivated by quantum mechanics) of the ring of function $\mathcal{O}(G)$, where G is a (semisimple) Lie group. In the dual setting, one needs to find a certain Hopf algebra $\mathsf{U}_q(\mathfrak{g})'$, defined over $\mathbb{Q}(q)$ (where q is a formal parameter), such that when $q \to 1$, $\mathsf{U}_q(\mathfrak{g})$ just becomes the ordinary universal enveloping algebra $\mathsf{U}(\mathfrak{g})$. This Hopf algebra was introduced by Jimbo and Drinfeld, and it is what usually is called a quantum group. Even though the motivation comes from quantum integrable systems, quantum groups found many other applications. A very beautiful application of quantum groups is the theory of crystal basis, initiated by Kashiwara ([39]) (Lusztig also introduced a "dual version" called canonical basis in [51]). Crystal basis are roughly obtained by making $q \to 0$. Taken litteraly, it doesn't make sense, but it is possible to turn this intuitive idea into a well defined procedure, which builts a bridge between combinatorics and representation theory. One might ask if other values of q are interesting. It is an important theorem by Drinfeld that if q is not a root of unity, then the the category of modules over the Drinfeld-Jimbo quantum group $\mathsf{U}_q(\mathfrak{g})'$ is equivalent to the ordinary category of representation of $\mathsf{U}(\mathfrak{g})$ (however, this equivalence is not monoidal. This observation is related to the beautiful theory of Drinfeld associators). In this thesis, we are concerned with the situation when q is a root of unity. In this case, the category of finite-dimensional representations of U becomes much more complicated. Lusztig made a deep study of quantum group at root of unity, that we will denote U, noticing the close relationship of its representaion theory with algebraic groups in positive characteristic. Such a connection is made even more precise in the work of [6], where is it proved that the representation theory of quantum groups a root of unity U is mostly "independent" of the order of the root of unity, and is related to the representation theory of algebraic groups in positive characteristic by a change of base (more precisely, there is an algebra defined over $\mathbb{Z}$ from which both the small quantum group and the first Frobenius kernel for an algebraic group can be obtained by base change). Quantum groups at root of unity are also related to the affine Lie algebra by the work of Kazdhan-Lusztig and Finkelberg ([42], [40], [43], [44], [24]). Finally, quantum groups at root of unity give rise to a family of topological quantum field theories ([64], [67], [68]).

The focus of this thesis is the small quantum group, built from the big quantum group. By definition it is an algebraic object, but work of Ginzburg–Kumar, Arkhipov–Bezrukavnikov–Ginzburg, Bezrukavnikov–Lachowka revealed a geometric meaning to the small quantum group through the nilpotent cone and the Springer resolution. New data obtained by geometric means in [48] lead the authors to formulate a serie of beautiful conjectures that would describe quite precisely the *center of the small quantum group*, the main object in this thesis. The affine Grassmannian and affine Springer fibers (we will define them later in the text) play an important role in this picture, as explained in the work of Arkhipov–Bezrukavnikov–Ginzburg ([8]), Boixeda-Alvarez–Losev. ([3]) and Bezrukavnikov–Boixeda-Alvarez–Shan–Vasserot ([14]). We also will look at the small quantum group through this perspective. Other geometric realizations of the small quantum group have been constructed before, for example via the semiinfinite flag variety in [7] or via factorizable sheaves on configuration spaces via [65].

The main result of [12] shows an isomorphism between graded vector spaces

$$\bigoplus_s HH^s(\mathsf{u}) \cong \bigoplus_s HH^s_{\mathbb{C}^*}(\widetilde{\mathcal{N}}) \cong \bigoplus_{i+j+k=s} H^i(\widetilde{\mathcal{N}}, \wedge^j T\widetilde{\mathcal{N}})^k \tag{1.1}$$

where $HH^*(\mathsf{u})$ is the Hochschild cohomology of $\mathsf{u} = \mathsf{u}(\mathfrak{g})$, and $HH^*_{\mathbb{C}}(\widetilde{\mathcal{N}})$ is the equivariant Hochschild cohomology of the Springer resolution for G. The second isomorphism is a version of the Hochschild-Kostant-Rosenberg. The isomorphism 1.1 also respects the multiplicative structure, which is stated as follows in [12]: the multiplication in $\mathrm{HH}^*(\mathsf{u})$ is determined by the exterior algebra structure in $\wedge^\bullet T\widetilde{\mathcal{N}}$. However, the precise meaning of how it is determined is not specified. Our first focus in this thesis is to clarify this statement and obtain an explicit formula for the multiplication in the Hochschild cohomology of the small quantum group.

It is apparent from the formula 1.1 that the center $\mathrm{HH}^0(\mathfrak{u})$ is bigraded. Recently, [14] proposed another geometric interpretation of the center in terms of the geometry of a certain affine Springer fiber. An affine Springer fiber is a infinite-dimensional space (we will define it later in the text). It can be thought to be the affine analogue (in the Lie algebra sense) to the ordinary Springer fibers (defined as zero set of invariant vector fields on G/B). An interesting feature of the affine Springers is that their cohomology groups are acted upon by the affine Weyl group $\widetilde{W} = W \ltimes P$.

Theorem 1.0.1 ([14]). *There is an algebra embedding* $\mathrm{H}^*(X)^{\widetilde{W}} \to \mathbf{z}^{G^\vee}$.

Conjecture 1.0.2 ([14]). *The previous map is an isomorphism.*

The second aspect of this thesis we are interested in is to determine the two gradings, that appear in 1.1, in the setting of affine Springer fibers. This bigrading comes from the geometry of the Springer resolution, appearing in the formula 1.1. Assuming conjecture 1.0.2, we try to construct the bigrading from the geometry of the Hitchin fiber. Our main tool here is a degeneration of spectral curves inside Hirzebruch surfaces. Using the product formula relating affine Springer fibers to Hitchin fibers, we can use tools available for Hitchin fibers. We can match one of the two grading on both side.

In the last chapter of the thesis, we study more algebraic and computational aspects of the center: in the last chapter we improve the sheaf-cohomology BGG algorithm (introduced in [32]). This algorithm computes cohomology of equivariant vector bundles on G/B in an efficient way, and can be used as tool to compute the cohomology appearing in formula 1.1. We then compute many Hochschild cohomology groups of small quantum groups, for example for $\mathfrak{g}$ of type G_2, A_4, B_3. We verify the conjectures of Lachowska-Qi in all the cases we computed. We also present a computation of the full Hochschild cohomology of the parabolic Springer resolution $\widetilde{\mathcal{N}}_P = T^*\mathbb{P}^2$. Finally we also present a cocycle computation related to the multiplicative structure of $\mathfrak{sl}_3$.

We now give a more precise outline of the thesis:

In chapter 2, we give the necessary background for the thesis. We tried to add many details and examples to make it self-contained and easy to read. The first section sets up the conventions and the notation. The second section gives basic definitions and theorems about quantum groups at root of unity (big and small), and also provides motivation to study this (complicated!) object. In the third section, we give a brief treatement of the relationship between the small quantum group and the Springer resolution, following [27], [8] and [12]. Section 4 gives a very detailled treatement of the BGG resolution and the sheaf-cohomology BGG algorithm of [48], that we develop further in chapter 5. Then we state the conjectures introduced by Lachowska-Qi, relating the center of the small quantum group with a combinatorial object constructed from the Weyl group, the so-called diagonal coinvariants algebra

$$DR_n = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]/(\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W_+)$$

Here, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, W acts diagonally on $\mathfrak{h} \oplus \mathfrak{h}^*$, and $+$ means that we take the augmentation ideal. The work of Lachowska-Qi is the background of chapter 3 and chapter 5. In the fifth section, we discuss the relationship between the small quantum group and affine Springer fibers, used in chapter 4, and the sixth section is about the Hitchin fibration and the perverse filtration, also needed for the chapter 4.

In chapter 3, we prove our first main result, a theorem which gives a multiplicative isomorphism between equivariant Hochschild cohomology of the Springer resolution and the Hochschild cohomology of the small quantum group. Section 2 is concerned with the proof of a torus-equivariant version of the Hochschild-Kostant-Rosenberg theorem , and recall some facts about equivariant sheaves and the theorem of Kontsevitch about the twisted Hochschikd-Kostant-Rosenberg theorem. Section 3 gives applications to the small quantum group.

In chapter 4, we deal with the question of finding a bigrading on the affine Springer side corresponding to the Springer resolution side. The method we use is a degeneration of spectral curves. Spectral curves can be seen as parametrizing Hitchin fibers, which are closely related to certain affine Springer fibers. These fibers are themselves related to the center of the small quantum group ([14]). This degeneration of spectral curves allows us to construct a map from the *diagonal coinvariants algebra* to the center of the small quantum group.

In chapter 5, we present several computations related to the Hochschild cohomology of small quantum groups. After the introduction, the second section explains how to improve the BGG sheaf-cohomology algorithm. Section 3 computes $HH^*_{\mathbb{C}^*}(\widetilde{\mathcal{N}}_P)$ where $\widetilde{\mathcal{N}}_P = T^*\mathbb{P}^2$ is the parabolic Springer resolution corresponding to a non-trivial singular block for $\mathfrak{g} = \mathfrak{sl}_3$. We describe this group both as G-module and as a module over $\mathbb{C}[\mathcal{N}]$. We also use our multiplicative HKR theorem proved in chapter 3 to compute some products inside this ring. In section 4, we compute the center (as bigraded vector space) of the principal blocks for type G_2, B_3, C_3, A_4 (and some singular blocks as well). This confirms the conjectures of Lachowska-Qi. In section 5, we compute some higher Hochschild cohomology group. Finally, in section 6 we look at the case where G/P is a projective space. We compute an interesting subalgebra of HH^1 and find its "positive part". We compute explicit cocycles that provide a basis for $\mathbf{z}_0(\mathfrak{sl}_3)$.

We hope that these three chapters will give more insight on the center of the small quantum group and its mysterious relationship with combinatorics.

1.1 Acknowledgements

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Chapter 2

Background

2.1 Conventions and notations

All the objects we will consider (different version of quantum groups, flag varieties, nilpotent cones, ...) usually depend on a complex semisimple Lie algebra that will be denoted by $\mathfrak{g}$. We will write G for the corresponding simple algebraic group (in certain references, e.g [12] it is assumed that G is of adjoint type).

We will often fix a Cartan torus inside a Borel subgroup $H \subset B$, and denote by $\mathfrak{h}, \mathfrak{b}$ the corresponding Lie algebra. The root lattice is written Q, the weight lattice is written P. The coroot/coweight lattices are denoted $Q^\vee, P^\vee$. The dominant weights is the monoid $P^+ \subset P$ spanned by the *fundamental weights* ϖ_i. The simple root $\alpha_i \in \mathfrak{h}^*$ is the eigenvalue of the Chevalley generators E_i. The Weyl group $W \subset GL(\mathfrak{h}^*)$ is the group generated by the reflection along the simple roots α_i. The affine Weyl group is defined as $\widehat{W}_\ell := W \ltimes \ell Q$. The extended affine Weyl group is $W \ltimes P$.

2.2 The big and small quantum groups

2.2.1 Motivation to study quantum groups at root of unity

2.2.2 The big quantum group

We will give definitions of several versions of quantum groups, following Lusztig [54]. Fix a semisimple Lie algebra $\mathfrak{g}$ with symmetric Cartan matrix $A = (a_{ij})$.

Definition 2.2.1 ([52], 0.4)**.** *We define U' to be the $\mathbb{Q}(v)$-algebra by generators E_i, F_i, K_i and K_i^{-1} with relations*

1. $K_iK_j = K_jK_i$, $K_iK_i^{-1} = K_i^{-1}K_i = 1$.
2. $K_iE_jK_i^{-1} = v^{a_{ij}}E_j$,
$K_iF_jK_i^{-1} = v^{-a_{ij}}F_j$.
3. $E_iF_j - F_jE_i = \delta_{ij}\frac{K_i-K_i^{-1}}{v-v^{-1}}$.

4. $E_iE_j = E_jE_i$,
   $F_iF_j = F_jF_i$, *if* $a_{ij} = 0$.
5. $E_i^2E_j - (v + v^{-1})E_iE_jE_i + E_jE_i^2 = 0$,
   $F_i^2F_j - (v + v^{-1})F_iF_jF_i + F_jF_i^2 = 0$, *if* $a_{ij} = -1$.

Remark 2.2.2. *One can define* U' *for non-symmetric Cartan matrix as well but the definition is slightly more complicated, we refer to e.g [54].*

Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$.

Definition 2.2.3. *The subalgebra* $\mathsf{U}_{\mathcal{A}}$ *is the* $\mathcal{A}$*-subalgebra of* U' *generated by the elements* $E_i^{(N)} := E_i^N/[N]!$, $F_i^{(N)} := F_i^N/[N]!$ *and* K_i, K_i^{-1}, *where* $[N]! = \prod_{s=1}^N \frac{v^s - v^{-s}}{v - v^{-1}} \in \mathbb{A}$.

Remark 2.2.4. *The subalgebra* $\mathsf{U}_{\mathcal{A}}$ *contains certain elements* $\begin{bmatrix} K, c \\ t \end{bmatrix}$ *introduced by Lusztig, that generalise* $\frac{K - K^{-1}}{v - v^{-1}}$. *When* v *is specialized to a* ℓ*-th root of unity* q, *we have* $K_i^{2\ell} = 1$.

Definition 2.2.5. *Fix a primitive* ℓ*-th root of unity* q, *we define* $\mathsf{U} = \mathsf{U}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{C}$, *where the* $\mathcal{A}$*-algebra structure on* $\mathbb{C}$ *is given by evaluating* v *at* q. *The algebra* U *is called* the big quantum group.

For $\lambda \in P^+$, we will define the *Weyl module* $W(\lambda)$. Let $Y(\lambda)$ be the U'-module isomorphic to U'^- (as a $\mathbb{Q}(v)$-vector space), where F_i acts by left multiplication, $E_i \cdot 1 = 0$ and $K_i \cdot 1 = v^{\lambda_i} \cdot 1$. Let $\overline{Y}(\lambda)$ be the unique simple quotient module, then we write $\overline{Y}_{\mathcal{A}}(\lambda)$ for its $\mathsf{U}_{\mathcal{A}}$-submodule generated by 1 and $W(\lambda)$ for its specialisation to $v = q$.

Definition 2.2.6. *The* Weyl module *of highest weight* λ *is* $W(\lambda)$.

We also want to define the dual Weyl modules $H(\lambda)$. To do this, we define the algebraic dual of a U'-module M to be the U submodule of $\mathrm{Hom}_{\mathbb{A}}(M, \mathsf{U})$ such that E_i, F_i acts locally nilpotently and K_i acts semisimply (see e.g [8]).

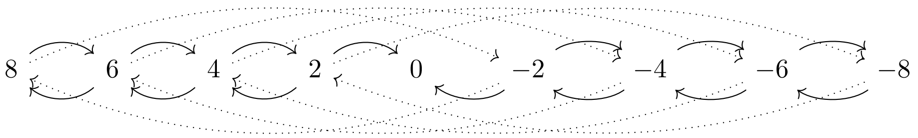
Definition 2.2.7. *The* dual Weyl module $H(\lambda)$ *is the algebraic dual of* $W(-w_0\lambda)$.

Remark 2.2.8. *This notation is because if* W *is a module of highest weight* μ, *then its algebraic dual has highest weight* $-w_0\lambda$, *where* w_0 *is the longest word in* W.

Remark 2.2.9. *As* $\mathfrak{h}$*-module (in particular, as vector space),* $W(\lambda)$ *is isomorphic to irreducible representation* $L_{\mathbb{C}}(\lambda)$ *of the corresponding complex Lie algebra* $\mathfrak{g}$.

Example 2.2.10. *Let* $\ell = 5$ *and* $\mathfrak{g} = \mathfrak{sl}_2$. *We will look at the Weyl module* $W(8)$. *As vector space, it is generated by the usual weight basis* $e_8, e_6, \dots, e_{-8}$. *Because* $E^5 = [5]_q E^{(5)} = 0$, *we have* $Ee_0 = E^5e_8 = 0$. *Similarly, we have* $Fe_0 = 0$. *If we draw the action of* E, F *by plain arrows and the action of* $E^{(5)}, F^{(5)}$ *by dashed arrows, it follows that* $W(8)$ *can be pictured as:*

$W(8)$:

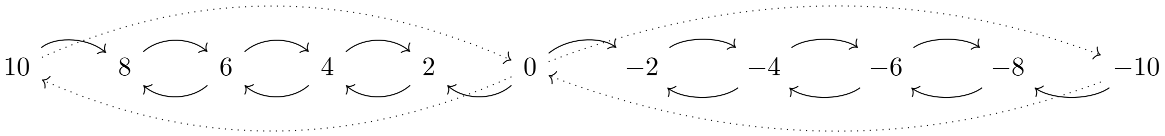


We see that the trivial module $L(0) = \mathbb{C}e_0$ *is a submodule of* $W(8)$*, and that the quotient module is ireducible. Hence we get a short exact sequence*

$$0 \to L(0) \to W(8) \to L(8) \to 0$$

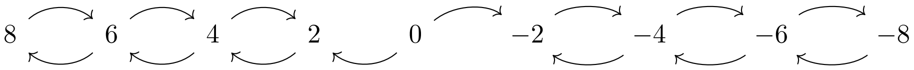
which is a general fact about Weyl module for $\mathsf{U}(\mathfrak{sl}_2)$*.*

Similarly, one can write $W(10)$ *as follows:*



(For readability, the actions of $E^{(5)}$ *and* $F^{(5)}$ *were restricted to* e_{10}, e_0 *and* e_{-10}*). This time, the simple submodule is* $L(8)$ *and the simple quotient is* $L(10)$*.*

The dual Weyl module $H(8)$ *is given by (*$E^{(5)}$*,* $F^{(5)}$ *not shown):*



Let us recall the strong linkage principle from [5] (an integrable module M is a module such that E, F acts locally nilpotently and each weight space is finite-dimensional).

Theorem 2.2.11 (Linkage principle)**.** *Let* V *be an integrable indecomposable module. If* $\lambda, \mu \in P^+$ *such that* $L(\lambda)$ *and* $L(\mu)$ *are composition factors of* V*, then* $\mu \in \widehat{W}_\ell \cdot \lambda$*.*

In particular, it follows that there is a decomposition $\mathsf{U} \cong \oplus_\mu \mathsf{U}_\mu$, where the U_μ are two-sided ideas called *blocks*, and that the blocks are parametrized by the orbits of P under the action of $\widehat{W}_\ell$.

Definition 2.2.12. *We call* U_0 *the* principal block*.*

Example 2.2.13. *For* $\mathfrak{g} = \mathfrak{sl}_2$*, the simple modules belonging to the principal block* U_0 *are exactly the* $L(2k\ell)$ *and* $L((2k+2)\ell - 2)$ *where* $k \in \mathbb{Z}_{\geq 0}$*.*

We now define a special class of modules called *tilting modules*, and explain the relation between projective modules and tilting modules following [4].

Definition 2.2.14. *Let M be a U-module. A* Weyl filtration *is a filtration $M_0 = 0 \subset M_1 \subset \cdots \subset M_n = M$ such that each subquotient M_i/M_{i-1} is isomorphic to a Weyl module $W(\lambda_i)$ for some $\lambda_i \in P^+$. A* dual Weyl module filtration *is a filtration $M^0 = 0 \subset M^1 \subset \cdots \subset M^n = M$ such that M^i/M^{i-1} is isomorphic to a dual Weyl module $H(\mu_i)$ for some $\mu_i \in P^+$.*

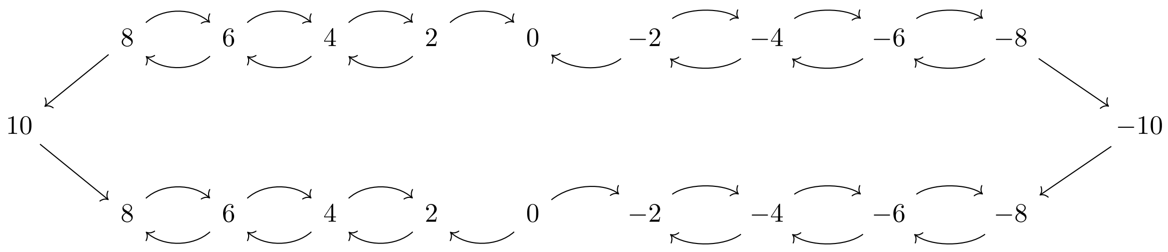
We now state important theorems about extensions and morphisms between Weyl and dual Weyl modules.

Theorem 2.2.15. *If λ, μ are dominant weights and $n > 0$, then $\mathrm{Ext}^n(W(\lambda), H(\mu)) = 0$. We have $\dim \mathrm{Hom}_{\mathsf{U}-\mathrm{mod}}(W(\lambda), H(\mu)) = \delta_{\lambda,\mu}$.*

Definition 2.2.16. *A U-module M is* tilting *if it admits both a filtration by Weyl modules and dual Weyl modules.*

Example 2.2.17. *Let us give an example of a module that is not tilting. For $\mathfrak{g} = \mathfrak{sl}_2$, both Weyl and dual Weyl modules has a 2-steps filtration by simple modules. Hence, if a dual Weyl module is tilting, we can deduce that $H(\lambda) = W(\mu)$ for some μ, λ. This is impossible in most cases, because $\mathrm{Hom}(W(\mu), H(\lambda)) = 0$ if $\mu \neq \lambda$ and $H(\lambda)$ is rarely isomorphic to $W(\lambda)$. Let us list such λ: for $\lambda = 0, \ldots, \ell - 1$, the Weyl and dual Weyl modules are actually simple (e.g we have $W(0) = H(0) = L(0)$). Moreover, for $\lambda = r\ell - 1$, all the modules coincide: $L(\lambda) = W(\lambda) = H(\lambda) = T(\lambda)$. For the other cases, the morphism $W(\lambda) \to H(\lambda)$ is not an isomorphism. This follows explicitely from the presentation of Weyl and dual Weyl modules. Hence, this gives for $\mathfrak{g} = \mathfrak{sl}_2$ the full list of dominant λ such that $W(\lambda)$ is tilting.*

Example 2.2.18. *Let us now give an example of a tilting module, $T(10)$ with $\ell = 5$:*



It is known that $\mathrm{Rep}(\mathsf{U})$ has enough projectives. Moreover, we have the following result:

Proposition 2.2.19. *All projective covers of finite-dimensional simple modules are given by indecomposable tilting modules of highest weight λ, written $T(\lambda)$, where $\lambda \in P^+ \backslash C_\ell$.*

Since the dual of a tilting modules is tilting, it follows that injective and projective module coincide. Let us state a more precise result, and for a dominant weight λ, we write $\lambda = \lambda_1 + \ell\lambda_0$ where $\lambda_1 \in P^+, \lambda_0 \in C_l$ (C_ℓ is the first ℓ-alcove). Let $\overline{\lambda} = 2 \cdot (\ell - 1) \cdot \rho + w_0(\lambda_0) + \ell \cdot \lambda_1$. We have the following theorem by Andersen:

Theorem 2.2.20 ([4])**.** *Let $I(\lambda)$ be the injective hull of $L(\lambda)$. Then $I(\lambda) \cong T(\overline{\lambda})$.*

Finally, to compute the multiplicities of Weyl modules inside tilting modules, one can use the following theorem:

Theorem 2.2.21. *For any dominant weights* μ, λ, *we have* $(I(\lambda) : W(\mu)) = [W(\mu) : L(\lambda)]$ *and* $(I(\lambda) : H(\mu)) = [W(\mu) : L(\lambda)]$.

2.2.3 The small quantum group

The small quantum group was introduced by Lusztig in [52], as a quantum analogue of the first Frobenius kernel of an algebraic group over a field of positive characteristic.

Consider the subalgebra $\mathsf{u}' \subset \mathsf{U}$ generated by E_i, F_i, K_i.

Definition 2.2.22. *[52] The small quantum group* u *is the quotient* $\mathsf{u} = \mathsf{u}'/\langle K_i^\ell - 1\rangle$.

Alternatively, one can consider the Kac-De Concini version of a quantum group at root of unity, U_{KdC} which is a specialisation of the usual Drinfeld-Jimbo quantum group at $v = q$ (factored by the ideal generated by the $K_i^\ell - 1$, where $q^\ell = 1$ is a primitive root of 1). There, the small quantum group is just the kernel of the map $\mathsf{U}_{KdC} \to \mathsf{U}$, sending $E_i \to E_i, F_i \to F_i$ and $K_i \to K_i$.

Proposition 2.2.23. *There is a short exact sequence of Hopf algebras*

$$0 \to (\mathsf{u}) \to \mathsf{U} \to \hat{U}(\mathfrak{g})$$

where $\hat{U}(\mathfrak{g})$ *is a certain completion of* $U(\mathfrak{g})$, *and* (u) *is the two-sided ideal generated by the small quantum group.*

Morally, we have $\mathsf{U}-\mathrm{mod} \cong \mathsf{u}-\mathrm{mod} \otimes \mathrm{Rep}(G)$. The precise relationship is explained by the work of Arkhipov-Gaitsgory [9] (see also [25] and [2]). A key ingredient is the *quantum Frobenius map* (see [53]), introduced by Lusztig.

Proposition 2.2.24. *The assignement* $E_i^{(\ell)} \mapsto e_i, F_i^{(\ell)} \to f_i$ *extends to an algebra morphism:* $\mathsf{U} \to \overline{U(\mathfrak{g})}$ *(a completion of the universal enveloping algebra).*

The Frobenius map can be interpreted as a functor $\mathrm{Fr}^* : \mathrm{Rep}(G) \to \mathsf{U}-\mathrm{mod}, V \mapsto \mathrm{Fr}^* V$, where $\mathrm{Fr}^* V = V$ as a vector space, and u acts on V trivially (i.e the K's act by 1 and the E, F's act by zero), and the divided powers acts via the Frobenius map. It gives a functor $\mathrm{Rep}(G) \times \mathsf{U}-\mathrm{mod} \to \mathsf{U}-\mathrm{mod}$, hence gives an action of the monoidal category $\mathrm{Rep}(G)$ on the category $\mathsf{U}-\mathrm{mod}$.

We state an important theorem relating modules over the big quantum group and modules over the small quantum group:

Definition 2.2.25. *A weight* λ_0 *is* ℓ*-reduced if* $0 \leq \langle \lambda_0, \alpha_i\rangle < \ell$ *for all simple root* α_i.

Theorem 2.2.26 (Lusztig's tensor product theorem)**.** *Let* λ *be a dominant weight. Write* $\lambda = \ell\lambda_1 + \lambda_0$, *where* λ_1 *is dominant and* λ_0 *is* ℓ*-reduced. Then,* $L(\lambda)_{|\mathsf{u}} \cong \mathrm{Fr}^* L_{\mathbb{C}}(\lambda_1) \otimes L(\lambda_0)$.

Here, $L(\mu)$ is the simple module of highest weight μ for U, and $L_{\mathbb{C}}(\mu)$ is the simple module of highest weight μ for $U(\mathfrak{g})$.

Finally, we state the linkage principle for the small quantum group.

Theorem 2.2.27. *The blocks of* u *are parametrized by* $P/\widetilde{W}_\ell$, *where* $\widetilde{W}_\ell = W \ltimes \ell P$.

Understanding the center of the small quantum group has been an important and complicated problem.

The "well-known" part of the center is the so-called *Harish-Chandra center* $\tilde{\mathbf{z}}$, and it is isomorphic as an algebra to the coinvariant algebra $\mathbb{C}[\mathfrak{h}^*]/\mathbb{C}[\mathfrak{h}^*]^W_+ \cong \mathrm{H}^*(G/B, \mathbb{C})$ ([17]).

The center for $\mathfrak{g} = \mathfrak{sl}_2$ was computed by Kerler in [45] (1994).

Theorem 2.2.28 (Kerler). *For $\mathfrak{g} = \mathfrak{sl}_2$, we have $\mathbf{z}_0 \cong \mathbb{C}^3$.*

Hence, for $\mathfrak{g} = \mathfrak{sl}_2$ we obtain $\mathbf{z} \cong \mathbb{C}^{3(\ell-1)/2+1}$ for the whole center (the last term comes from the singular block corresponding to the Steinberg module). However, even the case of $\mathfrak{g} = \mathfrak{sl}_3$ was not solved until 2016, by Lachowska-Qi [48], 20 years later.

In the meantime, the work of Lachowska [47] found a "large" subalgebra inside $\mathbf{z}$. Recall the homomorphism $\mathfrak{J} : \mathsf{u}^* \to \mathsf{u}, p \mapsto m(p \otimes \mathrm{id}(R_{21}R_{12})$ introduced by Drinfeld ([23]), where R is the classical R-matrix associated to u. The image of the complexified Grothendieck ring through $\mathfrak{J}$ is central, and the image is the Harisch-Chandra center $\tilde{\mathbf{z}}$. Then, for any finite-dimensional Hopf algebra H there is a map $\phi : H \to H^*$, constructed from the "right integral" element ([62]). Composing these maps one obtain a kind of "Fourier transform" $\mathcal{F} = \mathfrak{J} \circ \phi$. The main result of [47] reads as follows:

Theorem 2.2.29. *The intersection of $\tilde{\mathbf{z}} \cap \mathcal{F}(\tilde{\mathbf{z}})$ with each block is 1-dimensional.*

Hence the subalgebra $\tilde{\mathbf{z}} + \mathcal{F}(\tilde{\mathbf{z}})$ is almost "twice as big" as $\tilde{\mathbf{z}}$.

We will understand later a geometric description of these algebras, but it is still unkown how to relate the Fourier transform with the geometric description of the center obtained in [48]. For example, the geometric description of the center is only "block by block", but it is known that the Fourier transform is not block-preserving.

We now have a much better understanding of the center of the small quantum group: there are various conjectures (to be described in the next section) that describe it, and link it with other beautiful objects in mathematics (for example, the diagonal coinvariant algebra).

2.2.4 Relation between small and big quantum group (after Arkhipov-Gaitsgory)

We now describe a result due to Arkihpov and Gaitsgory ([9]), which explains how to describe the category $\mathsf{u}-$mod in term of the category big quantum group $\mathsf{U}-$mod. This result is crucial in order to deduce the Bezrukavnikov-Lachowska's equivalence from the Arkhipov-Bezrukavnikov-Gaitsgory [8] equivalence. In [9], ℓ is assumed to be even, but we will consider the case where ℓ is odd (the results still hold if one replaces the Langlands dual group $G^\vee$ by the original semisimple group G).

Let $\mathcal{A}_G$ be the Hopf algebra such that the category of comodules over $\mathcal{A}_G$ is equivalent to the category of modules over U.

Definition 2.2.30. *The category $\mathfrak{C}$ is the category where the objects are vector spaces M with an action of the algebra $\mathcal{O}_G$ and a co-action of the co-algebra $\mathcal{A}_G$, such that the co-action map $M \to \mathcal{A}_G \otimes M$ is a map of $\mathcal{O}_G$-modules.*

We first state their main theorem:

Theorem 2.2.31. *[9] The category of all modules over the small quantum group $\mathfrak{u}$ is equivalent to* *the category* $\mathfrak{C}$*. Finitely generated* $\mathcal{O}_G$*-modules correspond to finite-dimensional modules over* $\mathfrak{u}$*.*

Actually, this theorem follows from a even more general formalism, for example developped in the paper [25], or can also be seen as an instance of the Barr-Beck theorem (this was explained to us by Sergey Arkhipov).

In an equivalent way, one can describe the category $\mathfrak{C}$ as follows: objects are pairs (M, α_V), where $M \in \mathsf{U}-$mod and for each $V \in \mathrm{Irr(G)}$, we have a map $\alpha_V : \mathrm{Fr}^* V \otimes M \to \underline{V} \otimes M$ (where for a G-representation V, $\underline{V}$ is the underlying vector space), satisfying certain conditions. Let us sketch how the equivalence goes: the action map $\mathcal{O}_G \otimes M \to M$ can be thought by the Peter-Weyl theorem as a U-module map $\bigoplus_{V \in \mathrm{Irr}(G)} (V \otimes \underline{V}^*) \otimes M \to M$. We see that this is actually equivalent to maps $\alpha_V : \mathrm{Fr}^*(V) \otimes M \to \underline{V} \otimes M$ (here, we used that the U-module structure on V is compatible with the G-action via the Frobenius morphism).

2.3 Geometry of the Springer resolution and quantum groups at root of unity

2.3.1 The big quantum group

The origin of the geometric description of the (derived, mixed) category of modules over the big quantum group originates in the following result:

Theorem 2.3.1. *[27] There is a graded algebra isomorphism* $\mathrm{H}^{2\bullet}(\mathfrak{u}, \mathbb{C}) \cong \mathbb{C}^\bullet[\mathcal{N}]$*. Moreover* $\mathrm{H}^{2\bullet+1}(\mathfrak{u}, \mathbb{C}) = 0$*.*

Here, $\mathcal{N} \subset \mathfrak{g}$ is the set of nilpotent elements, and $\mathbb{C}^\bullet[\mathcal{N}]$ is the graded ring of poynomial functions on $\mathcal{N}$. The cohomology $\mathrm{H}^\bullet(H, M)$ of a module M over a Hopf algebra H is defined as follows ([27]):

Definition 2.3.2. *The cohomology groups* $H^i(H, M)$ *are defined as* $Ext^i_H(\mathbb{C}, M)$*, where* $\mathbb{C}$ *is seen as a trivial* H*-module via the augmentation.*

The Ginzburg-Kumar isomorphism was enhanced to an equivalence of derived categories by Arkhipov, Bezrukavnikov and Ginzburg in the paper [8].

Let G be the simple algebraic group associated to $\mathfrak{g}$, and B a Borel subgroup.

Definition 2.3.3. *The algebraic variety* $\widetilde{\mathcal{N}} = T^*(G/B)$ *is called the* Springer resolution.

Let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$. For varieties X, Y acted upon by a group H, let $X \times^H Y$ denote the quotient space (by the diagonal action) $(X \times Y)/H$.

Remark 2.3.4. *We have* $\widetilde{\mathcal{N}} \cong G \times^B \mathfrak{n} := (G \times \mathfrak{n})/B$*, and the map* $\mu : (g, x) \mapsto Ad(g)(x)$ *is a resolution of singularities of the nilpotent cone* $\mathcal{N}$*, hence the name. We can also write* $\widetilde{\mathcal{N}} = \{(x, \mathfrak{b}') \in \mathcal{N} \times G/B : x \in \mathfrak{b}'\}$*. (Recall that* G/B *can be seen as parametrizing all Borel subalgebra* $\mathfrak{b}' \subset \mathfrak{g}$*).*

The Springer resolution is very important in geometric representation theory, see e.g [59]. Let us consider the natural G-action on $\widetilde{\mathcal{N}}$, and the $\mathbb{C}^*$ action on the cotangent fibers by $t \cdot (x, v) = (x, t^{-2} v)$. Let $\mathrm{Coh}^{G \times \mathbb{C}^*}(\widetilde{\mathcal{N}})$ be the category of coherent sheaves on $\widetilde{\mathcal{N}}$, that are $G \times \mathbb{C}^*$-equivariant.
The main relationship between the principal block of the big quantum group and the Springer resolution is as follows:

Theorem 2.3.5. *[8] There is a triangulated functor* $F : \mathrm{D}^b(\mathrm{Coh}^{G\times\mathbb{C}^*}(\widetilde{\mathcal{N}})) \to \mathrm{D}^b(\mathsf{U}_0\mathrm{-mod})$*, such that the image of* F *generates* $\mathrm{D}^b(\mathsf{U}_0\mathrm{-mod})$ *as triangulated categories, and for any* $\mathscr{F}, \mathscr{F}' \in \mathrm{D}^b(\mathrm{Coh}^{G\times\mathbb{C}^*}(\widetilde{\mathcal{N}}))$ *induces canonical isomorphism:*

$$\bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}^\bullet(\mathscr{F}, z^i \otimes \mathscr{F}^{'}) \cong \mathrm{Hom}^\bullet(F(\mathscr{F}), F(\mathscr{F}'))$$

2.3.2 The small quantum group

In [12], Bezrukavnikov and Lachowska used theorem 2.3.5 to obtain a similar relationship between $\mathrm{Coh}^{\mathbb{C}^*}(\widetilde{\mathcal{N}})$ and the principal block of the small quantum group $\mathfrak{u}_0$:

Theorem 2.3.6. *[12] There is a triangulated functor* $F : \mathrm{D}^b(\mathrm{Coh}^{\mathbb{C}^*}(\widetilde{\mathcal{N}})) \to \mathrm{D}^b(\mathsf{u}_0\mathrm{-mod})$*, such that the image of* F *generates* $\mathrm{D}^b(\mathsf{u}_0\mathrm{-mod})$ *as triangulated categories, and for any* $\mathscr{F}, \mathscr{F}' \in \mathrm{D}^b(\mathrm{Coh}^{\mathbb{C}^*}(\widetilde{\mathcal{N}}))$ *induces canonical isomorphism:*

$$\bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}^\bullet(\mathscr{F}, z^i \otimes \mathscr{F}^{'}) \cong \mathrm{Hom}^\bullet(F(\mathscr{F}), F(\mathscr{F}'))$$

Let us sketch the proof of this theorem (for details see [12]). The idea is that the category of modules over $\mathfrak{u}_0$ can be reconstructed from the category modules over $\mathcal{O}(G)$ in the category $\mathsf{U}\mathrm{-mod}$, i.e $\mathsf{u}_0\mathrm{-mod} \cong \mathrm{Mod}_{\mathsf{U}_0\mathrm{-mod}}(\mathcal{O}(\mathrm{G}))$. A similar relationship can be obtained between the category $\mathrm{Coh}^{G\times\mathbb{C}^*}(\widetilde{\mathcal{N}})$ and $\mathrm{Coh}^{\mathbb{C}^*}(\widetilde{\mathcal{N}})$, where sheaves in $\mathrm{Coh}^{\mathbb{C}^*}(\widetilde{\mathcal{N}})$ can be reconstructed from sheaves in $\mathrm{Coh}^{G\times\mathbb{C}^*}(\widetilde{\mathcal{N}})$ that are modules over the algebra $\mathcal{O}(G)$. Hence, one just need to check the compatibility between the G-action in theorem 2.3.5.

Theorem 2.3.7. *There is an isomorphism of algebras*

$$\mathbf{z}_0 \cong \bigoplus_{i+j+k=0} H^i(\widetilde{\mathcal{N}}, \wedge^j \mathrm{T}\widetilde{\mathcal{N}})^k$$

This theorem is basically an equivariant version of the usual Hochschild-Kostant-Rosenberg theorem. The technicality is that one has to extend the equivalence of categories to a "doubled" version of it, since the Hochschild cohomology is usually defined as derived endomorphisn of the structure sheaf of the diagonal $\Delta \subset X \times X$.

In chapter 3, we will prove that this isomorphism can be extended to an isomorphism of algebras. This is based on a theorem by Konsevitch, stating that twisting by the negative of the square root of the Todd class induces a *ring* isomorphism between the Hochschild cohomology of a smooth complex scheme, and the total cohomology groups of the poly-vector fields.

2.3.3 The center of the small quantum group: the Lachowska-Qi conjectures

We now explain how to compute the center of the small quantum group using the "sheaf-cohomology BGG algorithm" (to be described in the next subsection, and in more detail in chapter 6). As we know from 2.3.2, the center $\mathbf{z}_0$ is given by the equivariant Hochschild cohomology group $\bigoplus_{i+j+k=0} H^i(\widetilde{\mathcal{N}}, \wedge^j \mathrm{T}\widetilde{\mathcal{N}})^{-k}$.

By definition, these pieces correspond to the sheaf cohomology $\mathrm{H}^i(G/B, \mathcal{V}_{j,k})$ for a certain vector bundle $\mathcal{V}_{j,k}$. Hence, we can use the sheaf-cohomology BGG algorithm to compute these cohomology groups. In [48], the $\mathfrak{b}$-module $V_{j,k}$ corresponding to the vector bundle $\mathcal{V}_{j,k}$ has been determined.

Consider the coadjoint action $\mathrm{coadj} : \mathfrak{b} \to \mathrm{End}(\mathfrak{u}) \cong \mathfrak{u} \otimes \mathfrak{n}$. It induces a map $\Delta : \mathfrak{b} \to \mathfrak{g} \oplus \mathfrak{u} \otimes \mathfrak{n}$, where $\Delta = (\mathrm{incl}, \mathrm{coadj})$.

Definition 2.3.8. *The* k-grading *on the algebra* $\mathrm{Sym}(\mathfrak{u}) \otimes \wedge^* \mathfrak{g} \otimes \wedge^* \mathfrak{n}$*, is defined such that* $\deg(\mathfrak{g}) = 0$, $\deg(\mathfrak{n}) = -\deg(\mathfrak{u}) = -2$.

Let

$$V_1 := \frac{\mathfrak{g} \oplus \mathfrak{u} \otimes \mathfrak{n}}{\Delta(\mathfrak{b})}$$

Definition 2.3.9. *The modules $V_{j,k}$ are defined as the k-th graded part of* $V_j := \wedge^j_{\mathrm{Sym}(\mathfrak{u})} V_1$.

Theorem 2.3.10. *[48] We have* $\mathcal{V}_{j,k} \cong G \times^B V_{j,k}$.

From the formula 1.1, one can already see that $\mathbf{z}_0$ has a natural bigrading. There is also an $\mathfrak{sl}_2$-action on the center:

Theorem 2.3.11. *[48] Let* $\tau \in H^0(\widetilde{\mathcal{N}}, \wedge^2 \mathrm{T}\widetilde{\mathcal{N}})^{-2}$ *be the Poisson bivector field (dual to the natural symplectic form on* $\widetilde{\mathcal{N}} = T^*(G/B)$*). The wedge product of* $\bigoplus_{i+j+k=0} H^i(\widetilde{\mathcal{N}}, \wedge^j \mathrm{T}\widetilde{\mathcal{N}})^{-k}$ *with* τ *induces a* $\mathfrak{sl}_2$*-action on* $\mathbf{z}_0$*, where* τ *acts as* $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Using this theorem, Lachowska-Qi explicitely computed the center $\mathbf{z}_0(\mathfrak{sl}_3)$.

Theorem 2.3.12. *[48]*

As bigraded vector space, the center of the principal block $\mathbf{z}_0(\mathfrak{sl}_3)$ *is given by the following table:*

$h^{i,j}$	$j-i=0$	$j-i=2$	$j-i=4$	$j-i=6$
$i+j=0$	1			
$i+j=2$	2	1		
$i+j=4$	2	3	1	
$i+j=6$	1	2	2	1

Definition 2.3.13. *Let W be the Weyl group associated to a complex semisimple Lie algebra $\mathfrak{g}$. The diagonal coinvariant algebra is the algebra* $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]/(\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W_+)$*, where* $(\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W_+)$ *is the ideal generated by the non-constant invariant polynomials.*

We write DC_m or DR_m for $W = S_m$. It was noticed in [48] that the table for $\mathbf{z}_0(\mathfrak{sl}_3)$ coincide with the diagonal coinvariant algebra for A_2. Further computations (for $\mathfrak{g} = \mathfrak{sl}_4, \mathfrak{sl}_5, \mathfrak{b}_2$, and various parabolic versions) confirmed this coincidence. This motivated a set of conjecture from [48], [49] that we will present now.

Conjecture 2.3.14. *[48] In type A,* $\mathbf{z}_0(\mathfrak{sl}_m)$ *is isomorphic as bigraded vector space to the diagonal coninvariant algebra. More precisely:*

- *There is a natural symmetric group S_m action on $\mathbf{z}_0$, extending the action of S_m on the coinvariant algebra $C_m \subset \mathbf{z}_0$.*
- *The symmetric group action commutes with the $\mathfrak{sl}_2$-action.*
- *As bigraded representation, we have $\mathbf{z}_0 \cong DC_m \otimes \mathrm{sgn}$. The bigrading match as follows: the (i,j)-component*
$$\mathbf{z}_0^{i,j} \cong (DC_m \otimes \mathrm{sgn})^{\binom{\mathrm{m}}{2} - \frac{\mathrm{i+j}}{2}, \frac{\mathrm{j-i}}{2}}$$
In particular, the dimension of the center is $\dim \mathbf{z}_0 = (m+1)^{m-1}$.

Finally, in [49] the various blocks of the center has been computed for $\mathfrak{sl}_3, \mathfrak{sl}_4$. Summing up all the blocks with multiplicities, one obtains a rational Catalan number. This motivates the following conjecture:

Conjecture 2.3.15. *[49] We have* $\dim \mathbf{z}(\mathfrak{sl}_m) = c_{(m+1)\ell - m, m} := \frac{1}{(m+1)}\binom{(m+1)\ell}{m}$

Conjecture 2.3.16. *[48] The natural $\mathfrak{g}$-action on $\mathbf{z}_0$ is trivial if $\mathfrak{g} = \mathfrak{sl}_n$.*

Remark 2.3.17. *This has been checked for $\mathfrak{g} = \mathfrak{sl}_2$ ([45], $\mathfrak{g} = \mathfrak{sl}_3, \mathfrak{sl}_4$ ([48]), $\mathfrak{g} = \mathfrak{sl}_5$ ([49]). We will see more example in chapter 4.*

In this formulation, all the conjectures are currently open. However, significant progress has been made toward these conjectures, using another (conjectural) description of the center in term of affine Springer fibers. We will describe it in the next section. Let us mention than using a computer program implementation of the BGG sheaf-cohomology algorithm introduced in [32], we verified various cases of the conjectures in ([32] (this work is presented in chapter 5).

Let us briefly describe the importance of the diagonal coinvariant algebra (a beautiful survey of Haiman's work is [28]). There is a conjecture in combinatorics that certain (q,t)- symmetric polynomials have positive coefficients, known as *Macdonald positivity conjecture*. This conjecture would follows from the fact that certain bigraded S_n-modules M has character given by these polynomials. This conjecture is called the *n! conjecture*, because these modules would actually be the regular representation. Finally, the last conjecture gives a formula for the character of DC_n. It is complicated to state, but it implies that its dimension in type A is given by $(n+1)^{n-1}$, and it implies also the two last conjectures. All these conjectures have been proved by Haiman using combinatorics of DC_n and the geometry of the Hilbert scheme (a [30], [31]).

2.4 The Lachowska-Qi algorithm and the BGG resolution

Lachowska-Qi gave an algorithm to compute the center using the isomorphism

$$\mathbf{z}_0 \cong \bigoplus_{i+j+k=0} H^i(\widetilde{\mathcal{N}}, \wedge^j \mathrm{T}\widetilde{\mathcal{N}})^k$$

To do this, two steps are needed: first one needs to be able to compute efficiently sheaf cohomology on G/B, we will briefly explain how to do it here (since more details will be given in chapter 5. The answer is in term of Lie algebra cohomology, hence we need to have an algebraic precise definition of the modules associated to the vector bundles $(pr_* \wedge^j \mathrm{T}\widetilde{\mathcal{N}})^k$ (here, $pr : \widetilde{\mathcal{N}} \to G/B$ is the natural projection).

We recall the existence of the BGG resolution, a key ingredient:

Theorem 2.4.1 ([35], chapter 6)**.** *There is an exact sequence*

$$0 \to M(w_0 \cdot \lambda) \to \bigoplus_{\ell(w)=n-1} M(w \cdot \lambda) \to \cdots \bigoplus_{\ell(w)=k} M(w \cdot \lambda) \cdots \to M(\lambda) \to L(\lambda) \to 0$$

Let us explain the big picture of the work of Lachowska-Qi in [48]. Let E a B-module. We consider the vector bundle $\mathcal{E} := G \times_B E$ on the flag variety $X = G/B$. Since $\mathcal{E}$ is G-equivariant, the cohomology groups $\mathrm{H}^\bullet(G/B, G \times_B E)$ are naturally G-modules. In [48] these G-modules are computed using the BGG complex. To compute $\mathrm{H}^\bullet(X, \mathcal{E})$, we only need to compute $\mathrm{Hom}_G(L(\lambda), \mathrm{H}^\bullet(X, \mathcal{E}))$ for all $\lambda \in P^+$. To obtain a list of dominant weights λ that might contribute to $\mathrm{H}^\bullet(X, \mathcal{E})$, we use a filtration of E given by the $\mathfrak{n}$-action. The composition factors $\mathfrak{n}^k E/\mathfrak{n}^{k+1} E$ are direct sums of 1-dimensional weight spaces. It follows that $\mathcal{E}$ has a filtration with composition factors isomorphic to a direct sum of line bundles. At this point it is useful to recall the Borel-Weil-Bott theorem:

Definition 2.4.2. *We define $\mathscr{L}_\lambda$ be the homogeneous line bundle corresponding to the 1-dimensional B-module $\mathbb{C}_{-\lambda}$, (so H acts by the character $\chi_{-\lambda} : H \to \mathbb{C}$ and $U = B/T$ acts by the identity).*

Remark 2.4.3. *The sign is just for convention, and makes the Borel-Weil-Bott theorem simpler to state.*

Theorem 2.4.4 (Borel-Weil-Bott)**.** *Assume $\lambda \in P$ is dot-singular, then $\mathrm{H}^i(X, \mathscr{L}_\lambda) = 0$ for all $i \in \mathbb{N}$. If not, let $w \in W$ the unique element so that $w \cdot \lambda$ is dominant. Then, $\mathrm{H}^i(X, \mathscr{L}_\lambda) = L(\lambda)$ if $i = \ell(w)$ and 0 else.*

Let $\mathcal{E}$ be an homogeneous vector bundle. Then, $\mathcal{E}$ has a filtration, such that the composition factors are isomorphic to a direct sum of homogeneous line bundles. It follows that there is a surjection from the direct sum of cohomology groups of these line bundles onto $\mathrm{H}^\bullet(X, \mathcal{E})$. Hence, by the Borel-Weil-Bott theorem, the only representations that can appear in $\mathrm{H}^\bullet(X, \mathcal{E})$ are given by the dot-orbits of the set $\mathrm{wt}(E)$. Hence it is natural to try to find the kernel of this surjection to obtain the cohomology $\mathrm{H}^\bullet(X, \mathcal{E})$, but it boils down to compute the map in the associated spectral sequence which is not trivial. As an alternative road, it is possible to use the BGG resolution to explicitly compute the cohomology as was done in [48]. The main role is played by Bott's theorem relating the sheaf cohomology on X with the Lie algebra cohomology of E.

Theorem 2.4.5. *[16] Let $X = G/B$ and $\mathcal{E} = G \times_B E$ for a B-module E. For each $\lambda \in P^+$ there is a vector space isomorphism $\mathrm{Hom}_G(L(\lambda), \mathrm{H}^\bullet(X, \mathcal{E})) \cong \mathrm{H}^\bullet(\mathfrak{b}, \mathfrak{h}, \mathrm{Hom}_{\mathbb{C}}(L(\lambda), E))$.*

Here, $\mathrm{H}^\bullet(\mathfrak{b}, \mathfrak{h}, \mathrm{Hom}_{\mathbb{C}}(L(\lambda), E))$ is the $\mathfrak{h}$-equivariant Chevalley-Eilenberg cohomology of $\mathfrak{b}$ with coefficients in $E \otimes L(\lambda)^*$. By definition, this is the cohomology of the complex $\mathrm{Hom}(\wedge^\bullet(\mathfrak{b}/\mathfrak{h})^* \otimes \mathrm{Hom}_B(L(\lambda), E))^{\mathfrak{h}}$. We also need the following lemma, originally stated in [16]:

Lemma 2.4.6. *[16] For a $\mathfrak{b}$-module F there is an isomorphism $\mathrm{H}^\bullet(\mathfrak{b}, \mathfrak{h}, F) \cong \mathrm{H}^\bullet(\mathfrak{n}, F)^{\mathfrak{h}}$.*

Following [48], we will explain how to compute the cohomology groups $\mathrm{H}^\bullet(G/B, \mathcal{E})$ using the BGG complex. We need to compute $\mathrm{H}^\bullet(\mathfrak{b}, \mathfrak{h}, \mathrm{Hom}_{\mathbb{C}}(L(\lambda), E)) \cong \mathrm{H}^\bullet(\mathfrak{n}, \mathrm{Hom}_{\mathbb{C}}(L_\lambda, E))^{\mathfrak{h}}$. By definition, the latter is computed by picking an $\mathfrak{h}$-invariant resolution of $\mathrm{Hom}_{\mathbb{C}}(E, L(\lambda))$ as a $U(\mathfrak{n})$-module. We have

$\mathrm{H}^\bullet(\mathfrak{n}, \mathrm{Hom}_\mathbb{C}(L(\lambda), E))^\mathfrak{h} \cong \mathrm{Ext}^\bullet_\mathfrak{n}(\mathbb{C}, \mathrm{Hom}_\mathbb{C}(L(\lambda), E))^\mathfrak{h} \cong \mathrm{Ext}^\bullet_\mathfrak{n}(\mathbb{C}, E \otimes L(\lambda)^*)^\mathfrak{h} \cong \mathrm{Ext}^\bullet_\mathfrak{n}(E^* \otimes L(\lambda), \mathbb{C})^\mathfrak{h})$. Tensoring the BGG resolution for $L(\lambda)$ with E^* gives a projective resolution for $\mathrm{Hom}_B(E, L(\lambda))$. Hence a complex that computes $\mathrm{H}^\bullet(\mathfrak{n}, \mathrm{Hom}_B(L(\lambda), E))^\mathfrak{h}$ is given by $\mathrm{Hom}(M_\bullet(\lambda) \otimes E^*, \mathbb{C})^\mathfrak{h}$. To describe the terms in the previous complex, we use that for a Verma module $M(\mu)$ we have $\mathrm{Hom}_{U(\mathfrak{n})}(M(\mu), E)^\mathfrak{h} \cong E[\mu]$. Hence the k-th term of the complex computing $\mathrm{Hom}(L(\lambda), \mathrm{H}^\bullet(X, \mathcal{E}))$ is given by $\bigoplus_{\ell(w)=k} E[w \cdot \lambda]$.

Definition 2.4.7. *The BGG complex associated to E and λ, written $\mathrm{BGG}^\bullet(E, \lambda)$, is the complex $\mathrm{Hom}(M_\bullet(\lambda) \otimes E^*, \mathbb{C})^\mathfrak{h}$.*

By the discussion before, there is an identification $\mathrm{BGG}^k(E, \lambda) \cong \bigoplus_{\ell(w)=k} E[w \cdot \lambda]$. In this setting, the maps $E[x \cdot \lambda] \to E[w \cdot \lambda]$ are just given by multiplication by $\mathcal{F}(x, w)$. To summarize, the multiplicities $Hom_G(L(\lambda), \mathrm{H}^\bullet(X, \mathcal{E}))$ can be computed using the BGG complex as explained.

Let us explain how to get similar results for partial flag varieties $X = G/P$. We recall that to any subset $J \subset S$, where S is a set of simple roots, we obtain a corresponding standard parabolic subgroup P_J, which Lie algebra is given by $\mathfrak{p}_J := Lie(P_J) = \mathfrak{b} \oplus \langle f_i : i \in J \rangle$, where $\langle f_i : i \in J \rangle$ is the Lie subalgebra generated by the f_i for $i \in J$. A parabolic version of the BGG resolution exists but seems more complicated to implement. However, by the degeneration of the Leray spectral sequence for the projection $p : G/B \to G/P$, if $\mathcal{E}$ is a vector bundle on G/P one has an isomorphism of G-modules $H^i(G/B, p^*\mathcal{E}) \cong H^i(G/P, \mathcal{E})$. In practice, it means that for a $\mathfrak{p}$-module E, to compute the cohomology of the corresponding vector bundle we can compute the cohomology of E considered as a $\mathfrak{b}$-module by restriction. Hence, to compute sheaf cohomology we only need to work with G/B. We will give explicit examples of such computations in the last chapter.

2.5 The small quantum group and the affine Springer fibers

The main result of [8] actually gives two geometric pictures describing the category U_0−mod. The second picture uses the affine Grassmannian and constructible sheaves, and we will make it more precise here.

If H is a reductive group, the *affine* is an ind-scheme (we will define this later in the text) Gr_H such that $\mathrm{Gr}_H(\mathbb{C}) = H(\mathbb{C}((t)))/H(\mathbb{C}[[t]])$. Let $\mathrm{ev}_0 : H(\mathbb{C}[[t]]) \to H$ the evaluation map, and $I \subset H(\mathbb{C}((t)))$ be an *Iwahori subgroup*, that is the preimage $\mathrm{ev}_0^{-1}(B)$ of a Borel subgroup $B \subset H$. For a H-space Y, let $\mathrm{Perv}_{(H)}(Y)$ denote the abelian category of perverse sheaves on Y, constructible with respect to the stratification induced by the H-action. (We will recall some facts about perverse sheaves in the next section). We have the following result:

Theorem 2.5.1. *[8] Let G be a simple algebraic group of adjoint type. There is an equivalence of categories U_0−mod $\cong \mathrm{Perv}_{(I)}(\mathrm{Gr}_{G^\vee})$.*

Here, $G^\vee$ is the complex connected, simply connected semisimple Langlands dual group of G.

Hence one could try to understand the center using the geometry of the affine Grassmannian. Using the formalism from [9], it should be possible to obtain a categorical description of u_0−mod as the "de-equivariantisation" of the category $\mathrm{Perv}_{(I)}(\mathrm{Gr}_G)$, but to our knowledge this has not been done. Recent work where the relationship with affine flag varieties and the center of the small quantum group has been done in [15], [3], [14]. In [14] a certain subalgebra $H \subset \mathbf{z}^G$ is constructed with the property that $\dim H = c_{(m+1)\ell-m,m}$ (In [33], the dimension of H has been also computed independently). Conjecturally, in type A, we have $H = \mathbf{z}^G = \mathbf{z}$. These conjectures will mostly follow from the conjecture made in [14] that $H = \mathbf{z}^G$ in type A (see [3]).

2.6 The perverse filtration and the Hitchin fibration

We now turn to perverse sheaves and the Hitchin fibration, needed in chapter 4.

2.6.1 The perverse filtration

Let K be a complex of constructible sheaves on a complex algebraic variety X, and $\mathrm{H}^*(X, K)$ the corresponding hypercohomology group. Recall that there are perverse truncation functors ${}^p\tau^{\leq i} \in \mathrm{End}(D(X))$, where $D(X)$ is the derived category of constructible sheaves of finite-dimensional $\mathbb{Q}$-vector spaces (with respect to a given stratification).

Let us recall how to construct these: if a triangulated category D get a *t-structure*, that is a pair of full subcategories $(D^{\leq 0}, D^{\geq 0})$ satisfying certain axioms, then the inclusion functors $D^{\geq 0} \to D, D^{\leq 0} \to D$ admits adjoints functors ${}^p\tau^{\geq 0} : D \to D^{\geq 0}$, ${}^p\tau^{\leq 0} : D \to D^{\leq 0}$, called the truncation functors.

Definition 2.6.1. *The* perverse filtration *on $H^*(X, K)$ is the filtration defined as* $P^r K = \mathrm{im}(\mathrm{H}(X,{}^p \tau^{\leq r} K) \to \mathrm{H}^*(X, K))$

This definition is quite abstract, but a beautiful theorem of de Cataldo-Migliorini tells us that there is a geometric interpretation of this filtration. For simplicity, we assume that $X \subset A$ is embedded in an affine space. If $L_1 \subset L_2 \subset \ldots L_n = A$ is a complete flag of linear subspace, we also get a filtration on $\mathrm{H}^*(X, K)$ as $F^i \,\mathrm{H}^*(X, K) = \ker(H^*(X, K) \to H^*(L_{i-1}, K_{|L_{i-1}}))$.

Theorem 2.6.2. *[21] If the flag L_* is a generic flag, then the two filtrations coincide, i.e*

$$P^r \,\mathrm{H}^j(X, K) = F^{r+j} \,\mathrm{H}^j(X, K)$$

2.6.2 The Hitchin fibration

We introduce the Hitchin fibration, and the relationship with the affine Springer fibers defined earlier. Let C be a smooth, connected projective curve (in our case we will take $C = \mathbb{P}^1$), and let D be a divisor on C such that $\deg(D)$ is even and satisfies $\deg(D) > 2g(C)$. In our case, $D = 2[0]$.

Definition 2.6.3. *An twisted* Higgs bundle *on C is a pair (E, θ) where E is a vector bundle and θ is a map of vector bundles $\theta : E \to E \otimes \mathcal{O}(D)$.*

What is called a Higgs bundle in the litterature is a twisted Higgs bundle when $D = K_C$ is the canonical divisor of C. In this text, "Higgs bundle" means "twisted Higgs bundle" (for a fixed divisor D). We now define the characteristic polynomial of θ: we can look θ as a map of $\mathcal{O}_C$-modules $End(E) \to \mathcal{O}(D)$. Hence by looking at $\theta(id_E)$, we obtain a section of $H^0(C, \mathcal{O}(D))$.

Definition 2.6.4. *The trace of θ is $\theta(id_E)$. Similarly, we have higher trace for any $0 \leq i \leq \mathrm{rank(E)}$ by $\mathrm{trace}(\wedge^i \theta) \in H^0(C, \mathcal{O}(iD))$.*

Definition 2.6.5. *The characteristic polynomial* χ_θ *is the polynomial*

$$\chi_\theta(X) = X^n - \mathrm{trace}(\theta)X^{r-1} + \cdots + (-1)^n \mathrm{trace}(\wedge^r \theta) \in \bigoplus_{i=0}^{r} H^0(C, iD)$$

Definition 2.6.6. *Let us fix an integers* r, d*. The stack of Higgs bundle* $\mathscr{M} = \mathscr{M}_{d,r}$ *is the stack classifying the pairs* (E, θ) *up to isomorphism, where* E *has rank* r *and degree* d*.*

Formally, $\mathscr{M}$ is the functor sending a scheme S to the groupoid of pairs $(\mathcal{E}, \phi)$ where $\mathcal{E}$ is a G-torsor over $S \times X$ and $\phi \in H^0(S \times X, (\mathcal{E} \times^G \mathfrak{g}) \otimes \mathcal{O}(D)$.

Definition 2.6.7. *Let* C, D, r, d *as before. Hitchin fibration is the map* $f : \mathscr{M} \to A, f(E, \theta) = \chi_\theta$*, where* A *is the affine space* $\oplus_{i=0}^{r} H^i(C, rD)$*. The* Hitchin fibers $\mathscr{M}_a$ *are the fibers* $f^{-1}(a)$*.*

We now define a parabolic version of the Hitchin fibration:

Definition 2.6.8. *The stack of parabolic Higgs bundles is the stack classifying the tuples* (E, θ, x, ϕ_x) *where* (E, θ) *is a Higgs bundle,* $x \in C$ *and* ϕ_x *is a* B*-reduction of* E *along* x*.*

Recall that a vector bundle of rank r over a space Y can be though as maps $g_{ij} : U_i \cap U_j \to \mathrm{GL_r}$ called cocycles satisfying certain conditions, where U_i is some covering of Y. If $H \subset \mathrm{GL_r}$, a H-reduction at a point x is a choice of cocycle representative where $g_{ij}(x) \in H$. In our case, one can think of a B-reduction of a vector bundle E at a point $x \in C$ as the data of a complete flag on the fiber E_x.

We will use the geometry of Hitchin fibers and their relationship with affine Springer fibers in chapter 4.

Chapter 3

An equivariant twisted HKR isomorphism and the small quantum group

Recall that in [12] where the center was described geometrically via the Springer resolution. Before stating the main result, let us introduce some notations: z is the trivial line bundle on $\widetilde{\mathcal{N}}$ with the $\mathbb{C}^*$-action given by $(z, v) \mapsto z^2 v$, R is the principal block, viewed as bimodule over $\mathfrak{u}$. By definition $H^i(\widetilde{\mathcal{N}}, (\wedge^j T\widetilde{\mathcal{N}})^k) := H^i(G/B, (pr_* \wedge^j T\widetilde{\mathcal{N}})^k)$. By definition $HH^\bullet(\mathfrak{u}_0)$ is the graded ring

$$\bigoplus_m \operatorname{Hom}_{D^b(\mathfrak{u}_0 \otimes \mathfrak{u}_0^{op})}(R, R[m])$$

where the multiplication is given by the composition. Moreover, $HH^*_{\mathbb{C}^*}(\widetilde{\mathcal{N}})$ is by definition the ring

$$\bigoplus_{j+k=m} \operatorname{Hom}_{D^b(\operatorname{Coh}^{\mathbb{C}^*}(\widetilde{\mathcal{N}} \times \widetilde{\mathcal{N}}))}(\mathcal{O}_\Delta, \mathcal{O}_\Delta[j] \otimes z^k)$$

where the ring structure comes from the Yoneda product. The geometric multiplication comes from the exterior algebra structure. The main theorem in [12] can be formulated:

Theorem 3.0.1. *There is a ring isomorphism* $HH^\bullet(\mathfrak{u}_0) \cong HH^*_{\mathbb{C}^*}(\widetilde{\mathcal{N}})$.

The purpose of this chapter is to explain precisely how to obtain a multiplicative isomorphism, using a theorem by Kontsevitch. We will check the compatibility of the multiplicative version of the Hochschild-Kostant-Rosenberg theorem with a action of a torus, and prove the following version:

Theorem 3.0.2. *The composition*

$$HH^\bullet(\mathfrak{u}_0(\mathfrak{g})) \xrightarrow{HKR} \bigoplus_{i+j+k=\bullet} H^i(\widetilde{\mathcal{N}}, \wedge^j T\widetilde{\mathcal{N}})^k \xrightarrow{\langle -, Todd(\widetilde{\mathcal{N}})^{-1/2} \rangle} \bigoplus_{i+j+k=\bullet} H^i(\widetilde{\mathcal{N}}, \wedge^j T\widetilde{\mathcal{N}})^k$$

is a ring isomorphism.

Let T be a complex torus, and X be an irreducible smooth complex variety acted on by T. More generally, one can consider an arbitrary complex reductive group acting on a smooth variety X, if there exists an affine cover by invariant open subset. Let $HH^\bullet(X)$ be the Hochschild cohomology of X, defined as $Ext^\bullet_{Coh(X\times X)}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$, where $\Delta \subset X \times X$ is the diagonal (for a variety Z we denote $\mathcal{O}_Z$ the structure sheaf), and $HT^\bullet(X)$ denotes the bigraded vector space $\bigoplus_{i,j} H^i(X, \wedge^j TX)$. Our first result is:

Theorem 3.0.3. *The twisted HKR morphism $HH^\bullet(X) \to HT^\bullet(X)$ is T-equivariant.*

This result relies on the following two propositions:

Proposition 3.0.4. *In the derived category of coherent sheaves on X $D^b(Coh(X))$, the quasi-isomorphism*

$$\iota^* \mathcal{O}_\Delta \cong \bigoplus_{i \in X} \Omega^i_X[i]$$

is T-equivariant.

Here $\iota : \Delta \to X \times X$ is the inclusion map, Ω^i_X is the sheaf of differential forms of degree i, and $[i]$ is the shift functor in the derived category.

Proposition 3.0.5. *Let $t \in H\Omega^\bullet(X)$ be the Todd class of $\widetilde{\mathcal{N}}$. Then, twisting with $t^{-1/2}$ gives a T-equivariant multiplicative isomorphism $HH^\bullet(X) \cong HT^\bullet(X)$.*

When no group action is involved, the two statements are well-known ([26], section 9). The first proposition implies the famous Hochschild-Kostant-Rosenberg (HKR) theorem relating the Hochschild cohomology of X with sheaf cohomology of poly-vector fields. The second statement relates the multiplicative structures on both sides. Hence the essence of our results is the compatibility with the action of T. Using our first theorem, we can deduce an explicit way to make the Bezrukavnikov-Lachowska isomorphism multiplicative. Setting $X = \widetilde{\mathcal{N}}$ and $T = \mathbb{C}^*$, we will deduce in section 3 the following theorem, describing geometrically the multiplicative structure of $z_0(\mathfrak{g})$:

Theorem 3.0.6. *The composition*

$$HH^\bullet(\mathfrak{u}_0(\mathfrak{g})) \xrightarrow{HKR} \bigoplus_{i+j+k=\bullet} H^i(\widetilde{\mathcal{N}}, \wedge^j T\widetilde{\mathcal{N}})^k \xrightarrow{\langle -, Todd(\widetilde{\mathcal{N}})^{-1/2} \rangle} \bigoplus_{i+j+k=\bullet} H^i(\widetilde{\mathcal{N}}, \wedge^j T\widetilde{\mathcal{N}})^k$$

is a ring isomorphism.

Taking the zero degree part of the previous isomorphism gives the following corollary:

Corollary 3.0.7. *The natural bigraded vector space isomorphism*

$$HH^0(\mathfrak{u}_0(\mathfrak{g})) \cong \bigoplus_{i+j+k=0} H^i(\widetilde{\mathcal{N}}, \wedge^j T\widetilde{\mathcal{N}})^k$$

can be upgraded to a multiplicative isomorphism using a twist.

The theorem 3.0.6 and the corollary 3.0.7 also hold for singular blocks, where one needs to twist by the corresponding “parabolic Todd class”.

Let us briefly outline the contents of this chapter: in section 2 we study equivariant sheaves on smooth varieties, give background material for the Kontsevitch theorem and prove theorem 3.0.3. In section 3 we explain the connection with the small quantum group, and derive some consequences for the structure of $z_0(\mathfrak{g})$.

3.1 Geometry

We first recall a theorem by Sumihiro:

Theorem 3.1.1. *[66] Let T be a torus and X a complex T-algebraic variety. Then, there is an open cover $\mathfrak{U}$ of X, which is T-invariant (i.e for all $U \in \mathfrak{U}$, we have $TU \subset U$).*

This theorem reduces many statements about algebraic varieties with a T-action to an affine statement about graded R-modules that are easier to prove.

3.1.1 Equivariant sheaves

Let X be a variety over the complex numbers, with an algebraic action of a complex reductive group K (which is not related to the Lie algebra $\mathfrak{g}$ mentioned at the beginning of the introduction). We will use $\mathrm{QCoh}^{\mathrm{K}}(\mathrm{X})$ for the category of quasi-coherent K-equivariant sheaves. First we recall the definitions and standard facts of an equivariant sheaf:

Definition 3.1.2. *A sheaf $\mathscr{F} \in \mathrm{QCoh}(\mathrm{X})$ is K-equivariant if there is an isomorphism $\theta : m^*\mathscr{F} \cong pr_2^*\mathscr{F}$, where $m, pr_2 : K \times X \to X$ are respectively the action and the projection (moreover, θ should satisfies a cocycle condition).*

Taking the stalks of θ at $(g, x) \in K \times X$ gives an isomorphism $\theta_x : \mathscr{F}_{g \cdot x} \cong \mathscr{F}_x$. A K-equivariant sheaf $\mathscr{F}$ is equivalent to the data of a sheaf $\mathscr{F}$ with such isomorphisms. We also have

Lemma 3.1.3. *A K-equivariant sheaf $\mathscr{F}$ is equivalent to a sheaf $\mathscr{F}$ with maps $g : \Gamma(U, \mathscr{F}) \to \Gamma(g(U), \mathscr{F})$ for each $g \in T$ and U open, commuting with restrictions and such that $g_1 \cdot (g_2 \cdot s) = (g_1 \cdot g_2) \cdot s$ for all $g_1, g_2 \in K$ and all $s \in \mathscr{F}(U)$.*

Corollary 3.1.4. *There is a natural structure of K-modules on $\Gamma(X, \mathscr{F})$ for any equivariant sheaf $\mathscr{F}$. More generally, there is a structure of K-modules on $Hom_{\mathrm{QCoh(X)}}(\mathscr{F}_1, \mathscr{F}_2)$ if $\mathscr{F}_i \in \mathrm{QCoh}^{\mathrm{K}}(\mathrm{X})$.*

Proof. The first point is clear. The action in general is defined as follow: if $\varphi : \mathscr{F}_1(U) \to \mathscr{F}_2(U)$, we define $g \cdot \varphi = g \circ \varphi \circ g^{-1}$, where g denotes the map in lemma 3.1.3. □

Definition 3.1.5. *The category $\mathrm{QCoh}^{\mathrm{K}}(\mathrm{X})$ is the category where objects are K-equivariant quasi-coherent sheaves of $\mathcal{O}_X$-modules, and morphisms are equivariant morphisms of $\mathcal{O}_X$-modules, i.e. the maps $f : \mathscr{F}_1 \to \mathscr{F}_2$ such that that $f(g \cdot s) = g \cdot f(s)$ for all $s \in \mathscr{F}_1(U)$ and all $g \in T$.*

To give more intuition, we recall that there is an equivalence of categories between the category of locally free coherent K-equivariant sheaves and the category of finite-dimensional vector bundles with an action of K, linear on each fibers, where morphisms are K-equivariant morphisms of vector bundles (see [59], chapter 5).

Lemma 3.1.6. *The K-module $V := Hom_{\mathrm{QCoh(X)}}(\mathscr{F}_1, \mathscr{F}_2)$ is semisimple. Moreover there is an isomorphism $Hom_{\mathrm{QCoh^K(X)}}(\mathscr{F}_1, \mathscr{F}_2) = (Hom_{\mathrm{QCoh(X)}}(\mathscr{F}_1, \mathscr{F}_2))^K$.*

Proof. The induced K-action is algebraic, i.e given by a ring map $V \to V \otimes \mathcal{O}(K)$. It follows easily that V is the union of its finite-dimensional K-modules, and hence semi-simple. The second part follows from the definition of the K-action. □

We now investigate basic properties of K-equivariant sheaves in order to be able to describe derived functors, now computed in the equivariant category.

Remark 3.1.7. *Let us emphasize that in geometric representation theory, the usual definition of the "equivariant derived category" (as defined e.g in Bernstein-Lunts:[38]) does not coincide with the derived category of equivariant sheaves (the one we consider here).*

Lemma 3.1.8. *For each $\mathscr{F} \in \mathrm{QCoh^K(X)}$, there is an injective object $I \in \mathrm{QCoh^K(X)}$ and a K-equivariant monomorphism $\mathscr{F} \to I$. Moreover $\mathrm{Fgt}(I) \in \mathrm{QCoh(X)}$ is still injective, where $\mathrm{Fgt} : \mathrm{QCoh^K(X)} \to \mathrm{QCoh(X)}$ is the forgetful functor.*

Proof. The first part is proposition 5.1.2 in [29]. The second part follows from the explicit construction of I in the same paper. □

This lemma ensure that we can consider right derived functors of $Hom(\mathscr{F}_1, -)$.

Corollary 3.1.9. *Let $\mathscr{F}_1 \in \mathrm{Coh(X)^K(X)}$ and $\Gamma_1 = Hom_{\mathrm{QCoh^K(X)}}(\mathscr{F}_1, -) : \mathrm{Coh^K(X)} \to \mathrm{K-Mod}$. Then there is a canonical isomorphism $R(\Gamma_1 \circ \mathrm{Fgt}) \cong R\Gamma_1 \circ \mathrm{Fgt}$.*

Corollary 3.1.10. *For any equivariant sheaf $\mathscr{F}_1$, there is an isomorphism between the functors $\mathrm{QCoh^K(X)} \to \mathrm{Vect}$:*

$$R(\mathrm{Inv} \circ \Gamma_{\mathscr{F}_1}) \cong \mathrm{Inv} \circ \mathrm{R}\Gamma_{\mathscr{F}_1}$$

where $\Gamma_{\mathscr{F}_1} := Hom_{\mathrm{QCoh(X)}}(\mathrm{Fgt}(\mathscr{F}_1), \mathrm{Fgt}(-))$, and $Inv : K - Mod \to \mathrm{Vect}$, $Inv(M) = M^K$. In particular if $\mathscr{F}_1, \mathscr{F}_2$ are K-equivariant sheaves on X, there is an isomorphism

$$Ext^q_{\mathrm{QCoh^K(X)}}(\mathscr{F}_1, \mathscr{F}_2) \cong (Ext^q_{\mathrm{QCoh(X)}}(\mathscr{F}_1, \mathscr{F}_2))^K$$

Proof. Recall the Grothendieck formula to derive compatible left-exact functors: $R(F \circ F') \cong RF \circ RF'$. Taking invariant is exact on the subcategory of semisimple K-representations, and by lemma 3.1.6 $\Gamma(Hom_{\mathrm{QCoh(X)}}(\mathscr{F}_1, \mathscr{F}_2))$ is semisimple. Hence $R\mathrm{Inv} = \mathrm{Inv}$ and the formula follows. □

Corollary 3.1.11. *If $\mathscr{F}$ is a K-equivariant sheaf, then there is an isomorphism*

$$RHom_{\mathrm{QCoh^K(X)}}(\mathcal{O}_X, \mathscr{F}) \cong (RHom_{\mathrm{QCoh(X)}}(\mathcal{O}_X, \mathscr{F}))^K$$

Lemma 3.1.12. *If $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ is a K-equivariant sequence of K-equivariant sheaves, and $\mathscr{F}_3$ is locally free, then the corresponding cohomology class in $H^1(X, \mathscr{F}_1 \otimes \mathscr{F}_3^*)$ is K-invariant.*

Proof. By definition the short exact sequence gives a map $\mathscr{F}_3 \to \mathscr{F}_1[1]$ in $D^b(\mathrm{QCoh^K(X)})$. This is the same as an element in $Ext^1_{\mathrm{QCoh^K(X)}}(\mathscr{F}_3, \mathscr{F}_1)$. Since $Ext^1_{\mathrm{QCoh^K(X)}}(\mathscr{F}_3, \mathscr{F}_1) \cong Ext^1_{\mathrm{QCoh^K(X)}}(\mathcal{O}_X, \mathscr{F}_1 \otimes$

$\mathscr{F}_3^*$) we can apply the corollary 3.1.10 and compute the latter as $H^1(R\Gamma(X, \mathscr{F}_1 \otimes \mathscr{F}_3^*)^K)$ which is K-invariant. □

Now we specialize to the case where $K = T$ is a complex algebraic torus.

Lemma 3.1.13. *Assume that the T-action on X is trivial and that $\mathcal{E}$ is a T-equivariant vector bundle on X. Then, there is a decomposition into eigenspace $\mathcal{E} = \oplus_\lambda \mathcal{E}[\lambda]$ where $\lambda : T \to \mathbb{C}^*$ is a character and we have $H^i_T(X, \mathcal{E}) = H^i(X, \mathcal{E}[0])$.*

Proof. If $U \subset X$ is affine, then $\mathcal{E}_{|U} \cong E \times U$ for some vector space E. The action of T is given by an action on the first component, and we get a decomposition $E \cong \oplus_\lambda E[\lambda]$ into T-eigenspaces. Hence $\mathcal{E}_{|U} \cong \oplus_\lambda \mathcal{E}_{|U}[\lambda]$, and we get a global decomposition $\mathcal{E} \cong \oplus_\lambda \mathcal{E}[\lambda]$ since the T -action does not depend on U. Moreover, there is an isomorphism of functors $\Gamma_T(X, \mathcal{E}) \cong \Gamma(X, \mathcal{E}[0])$. It follows that $H^i_T(X, \mathcal{E}) = H^i(X, \mathcal{E}[0])$ by taking derived functors. □

Most of these results were used implicitly in [12].

3.1.2 Kontsevitch's theorem

Let X be a smooth algebraic variety over the complex numbers. The Hochschild cohomology of X is defined as $HH^\bullet(X) = \bigoplus_i \operatorname{Ext}^i_{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$ where $\Delta \subset X \times X$ is the diagonal. It contains information about deformations of X, for example $H^1(X, TX)$ is the space parametrizing complex deformations of X. Let us recall two important theorems about the Hochschild cohomology of a smooth complex algebraic variety. The first theorem is the *Hochschild-Kostant-Rosenberg isomorphism*

Theorem 3.1.14. *([26],) Let $\iota : \Delta \to X \times X$ be the diagonal inclusion. There is a quasi-isomorphism*

$$\iota^* \mathcal{O}_\Delta \cong \bigoplus_{i \in \mathbb{N}} \Omega^i_X[i]$$

in the derived category $D^b(Coh(X))$.

Here Ω^i_X is the sheaf of differential forms on X of degree i, and $[i]$ is the cohomological shift functor.

This is a stronger version of the HKR theorem (that appeared first in [34]) stating that there is an isomorphism of graded vector spaces

$$I_{HKR} : HH^\bullet(X) \to HT^\bullet(X) := \bigoplus_{i+j=\bullet} H^i(X, \wedge^j TX))$$

Let us recall the proof of theorem 3.1.14: a non-trivial argument using sheaves (see [69], [26]) reduces it to the statement when $X = \operatorname{Spec}(R)$ is affine. This is the case treated in [34]. We use the *bar complex* $\mathscr{C}_\bullet(R)$, which is a flat resolution of $\mathcal{O}_\Delta$ by $\mathcal{O}_{X\times X}$-modules. Recall that the bar resolution is given by $\mathscr{B}_i(R) := R^{\otimes(i+2)}$. We define $\mathscr{C}_i(R) = R \otimes_{R\times R} \mathscr{B}_i(R)$. The differential on $\mathscr{B}_\bullet(R)$ is given by

$$d(1 \otimes a_1 \otimes \cdots \otimes a_i \otimes 1) = a_1 \otimes a_2 \cdots \otimes a_i \otimes 1 - 1 \otimes a_1 a_2 \otimes a_3 \otimes \cdots \otimes a_i \otimes 1 + \cdots + (-1)^{i+1} 1 \otimes a_0 \otimes \cdots \otimes a_i$$

A quasi-isomorphism $I : \mathscr{C}_\bullet(R) \to (\bigwedge^\bullet(\Omega^1_R[1]), 0)$ is given by

$$1 \otimes (1 \otimes a_1 \otimes \cdots \otimes a_i \otimes 1) \mapsto da_1 \wedge da_2 \cdots \wedge da_i$$

Since the bar complex resolves R as R-bimodule, this bar complex is isomorphic to $\mathcal{O}_\Delta$ in $D^b(X \times X)$. This finishes the proof.

In general, I_{HKR} is not multiplicative. To understand how the multiplications are related, we need to introduce the Todd class. We use the formalism of Atiyah class explained in section 1 of [58].

There is an exact sequence

$$0 \to \Omega_X \otimes TX \to J^1(TX) \to TX \to 0$$

where $J^1(TX)$ is the bundle of first-order jet of TX. Such sequence gives a class in $Ext^1(TX, \Omega_X \otimes TX) = H^1(X, \Omega_X \otimes End(TX))$.

Definition 3.1.15. *The* Atiyah class *of X is defined as the class of this extension: $At(X) \in H^1(X, \Omega_X \otimes End(TX))$.*

Definition 3.1.16. *The* Todd class *of X is defined as*

$$Td(X) = \det\left(\frac{At(X)}{1 - e^{-At(X)}}\right)$$

A more familiar definition of the Todd class involves Chern classes of TX. Our definition is equivalent to that, according to the following statement (see [58], formula 1.4.1):

Proposition 3.1.17. *We have $tr(\wedge^i At(X)) = c_i(X)$ where $c_i(X)$ is the i-th Chern class of the tangent bundle TX.*

Now we can state Kontsevitch's theorem:

Theorem 3.1.18. *[46][19] Let X be as before and t be the Todd class of X. Then, the composition*

$$\langle t^{-1/2}, - \rangle \circ I_{HKR} : HH^\bullet(X) \to HT^\bullet(X)$$

is a ring isomorphism.

3.1.3 Equivariance of the twisted HKR isomorphism

We now assume that T is a complex torus acting on a smooth variety X.

Lemma 3.1.19. *If X is affine, the isomorphism I_{HKR} is T-equivariant.*

Proof. Say $X = \mathrm{Spec}(R)$. Let us keep notation from the discussion after 3.1.14. The differential of the bar-complex is T-equivariant, hence the bar-complex gives a flat T-equivariant resolution of $\mathcal{O}_\Delta$. Moreover the map $I : A \to \Omega_R, a \mapsto da$ is T-equivariant, where Ω_R is the module of Kähler differential. This is because there is a natural identification $\Omega_R \cong J/J^2, da \mapsto a \otimes 1 - 1 \otimes a$, where J is the kernel of the multiplication map $R \otimes R \to R$. Under this identification, the map $a \mapsto da$ is T-equivariant. It follows that the quasi-isomorphism $I : \mathscr{C}_\bullet(R) \to (\bigwedge^\bullet(\Omega^1_R[1]), 0)$ is also T-equivariant. $\square$

Lemma 3.1.20. *If X is a quasi-projective smooth variety acted upon by T then the previous lemma holds.*

Proof. By Sumihiro's theorem, there is an T-invariant affine open cover. So we can check the statement on each affine open set $U \subset X$ that is T-invariant. But recall that in the discussion sketching the proof of theorem 3.1.14, we saw that we could glue together the bar complex to construct a global quasi-isomorphism I_{HKR}. So it means that the invariance can be only checked for the bar complex $\mathscr{C}_\bullet(R)$, that was precisely given by our previous lemma. Hence I_{HKR} is T-equivariant. $\square$

Lemma 3.1.21. *If X is as before, then the Atiyah class of X is T-invariant. Therefore the Todd class of X is also T-invariant.*

Proof. This follows from lemma 3.1.12 because the exact sequence defining the Atiyah class is T-equivariant. $\square$

It follows that the twisted HKR morphism $HH^\bullet(X) \to HT^\bullet(X)$ is T-equivariant, which completes the proof of our theorem 3.0.3. It would be interesting to generalize theorem 3.0.3 and relate it with the geometry of X/G, where G is a complex reductive group. However it is known that even when G is a finite group, the analogue of Kontsevitch's theorem does not hold for the stack $[X/G]$, see [60]. Without Sumihiro's theorem,(i.e without the existence of an invariant cover) it seems non-trivial to prove that I_{HKR} is G-equivariant.

3.2 Applications to the small quantum group

We prove theorem 3.0.6. As an application, we show that the multiplication on the subalgebra generated by the Harish-Chandra center and the Poisson bi-vector field τ is untwisted.

We can now give a proof of the theorem 3.0.6:

Theorem 3.2.1. *The composition*

$$HH^\bullet(\mathfrak{u}_0(\mathfrak{g})) \overset{BL}{\to} \bigoplus_{i+j+k=\bullet} H^i(\widetilde{\mathcal{N}}, \wedge^j T\widetilde{\mathcal{N}})^k \overset{\langle -, Todd(\widetilde{\mathcal{N}})^{-1/2}\rangle}{\to} \bigoplus_{i+j+k=\bullet} H^i(\widetilde{\mathcal{N}}, \wedge^j T\widetilde{\mathcal{N}})^k$$

is a ring isomorphism.

Proof. Let us recall how to deduce the isomorphism 1.1 from the theorem 3.0.1. There are isomorphisms

$$HH^\bullet(\mathfrak{u}_0) \overset{3.0.1}{\cong} \bigoplus_{q+k=\bullet} Ext^q_{\mathrm{Coh}^{\mathbb{C}^*}(\widetilde{\mathcal{N}}\times\widetilde{\mathcal{N}})}(\mathcal{O}_\Delta, \mathcal{O}_\Delta \otimes z^k) \cong \bigoplus_{i+j+k=\bullet} H^i_{\mathbb{C}^*}(\widetilde{\mathcal{N}}, \wedge^j T\widetilde{\mathcal{N}} \otimes z^k)$$

Here $H^i_{\mathbb{C}^*}(-) = R^i(Hom_{\mathbb{C}^*}(\mathcal{O}_X, -))$. The second isomorphism follows by taking the $\mathbb{C}^*$ invariant on both side of the usual HKR isomorphism (and using corollary 3.1.10). Hence we know that

$$H^i_{\mathbb{C}^*}(\widetilde{\mathcal{N}}, \wedge^j T\widetilde{\mathcal{N}} \otimes z^k) = (H^i(\widetilde{\mathcal{N}}, \wedge^j T\widetilde{\mathcal{N}} \otimes z^k))^{\mathbb{C}^*} = (H^i(G/B, pr_* \wedge^j T\widetilde{\mathcal{N}} \otimes z^k))^{\mathbb{C}^*}$$

since $pr : \widetilde{\mathcal{N}} \to G/B$ is an affine $\mathbb{C}^*$-morphism. Since the $\mathbb{C}^*$-action is trivial on G/B by proposition 3.1.13 we obtain an isomorphism

$$(H^i(G/B, pr_* \wedge^j T\widetilde{\mathcal{N}} \otimes z^k))^{\mathbb{C}^*} \cong H^i(G/B, (pr_* \wedge^j T\widetilde{\mathcal{N}} \otimes z^k)[0]) \cong H^i(G/B, (pr_* \wedge^j T\widetilde{\mathcal{N}})^{-k})$$

Putting everything together we obtain indeed the isomorphism stated in Theorem 3.0.6:

$$HH^\bullet(u_0) \cong \bigoplus_{q+k=\bullet} Ext^q(\mathcal{O}_\Delta, \mathcal{O}_\Delta \otimes z^k) \cong \bigoplus_{i+j+k=\bullet} H^i(\widetilde{\mathcal{N}}, (\wedge^j T\widetilde{\mathcal{N}})^k)$$

Now the isomorphism $HH^\bullet(\mathfrak{u}_0) \cong \bigoplus_{q+k=\bullet} Ext^q(\mathcal{O}_\Delta, \mathcal{O}_\Delta \otimes z^k)$ is multiplicative. Since the Todd class is T-invariant by lemma 3.1.21, and that I_{HKR} is T-equivariant by lemma 3.1.20 it follows that the usual multiplicative HKR theorem restricts to a multiplicative isomorphism

$$\bigoplus_{q+k=\bullet} Ext^q(\mathcal{O}_\Delta, \mathcal{O}_\Delta \otimes z^k) \cong \bigoplus_{i+j+k=\bullet} H^i_{\mathbb{C}^*}(\widetilde{\mathcal{N}}, \wedge^j T\widetilde{\mathcal{N}} \otimes z^k)$$

Finally the isomorphism

$$\bigoplus_{i+j+k=\bullet} H^i_{\mathbb{C}^*}(\widetilde{\mathcal{N}}, \wedge^j T\widetilde{\mathcal{N}} \otimes z^k) \cong \bigoplus_{i+j+k=\bullet} H^i(G/B, (pr_* \wedge^j T\widetilde{\mathcal{N}})^k)$$

is multiplicative, because for an affine map $q : X \to Y$, the equivalence $\mathrm{Coh}(X) \cong \mathrm{Coh}(Y) - q_*\mathcal{O}_X$ is monoidal, hence respect the cup-product. □

3.2.1 Multiplicative structure of $HC[\tau]$

We will compute the multiplicative structure of a subalgebra of $z_0(\mathfrak{g})$, already noticed in [48], where the arguments was given at level of sheaf cohomology.

Lemma 3.2.2. *If $Y = T^*X$ is a cotangent bundle of a smooth algebraic variety X, with projection morphism $pr : Y \to X$, then the Todd class of Y is given by the pullback to Y of*

$$\prod_{\alpha,\beta \in C} \frac{-\alpha\beta}{(1-e^{-\alpha})(1-e^{-\beta})} \in H\Omega^\bullet(X)$$

where C is the set of Chern roots of TY. In particular, this Todd class can be written $1 + pr^*a$ where $a \in H^{\geq 2}(X)$.

Proof. The map $pr : Y \to X$ induces a short exact sequence

$$0 \to T_{Y/X} \to TY \to pr^*TX \to 0$$

Moreover, $T_{Y/X} \cong pr^*(T^*X)$. Using the naturality of the Todd class and the additivity on short exact sequences the result easily follows. □

For example if $X = G/B$ then C is the set of positive roots of $\mathfrak{g}$ and the Todd class of G/B is given by

$$Todd(G/B) = \prod_{\alpha\in\Phi^+} \frac{\alpha}{1-e^{-\alpha}}$$

Let τ be the the generator of $H^0(\widetilde{\mathcal{N}}, End(T\widetilde{\mathcal{N}}))$ given by the identity map.

Definition 3.2.3. *We define $HC[\tau]$ to be the subalgebra of $z_0(\mathfrak{g})$ generated by the Harish-Chandra center HC_λ and τ.*

Of course, for singular blocks $\mathfrak{u}_\lambda(\mathfrak{g})$ there is a similar subalgebra inside $z_\lambda(\mathfrak{g})$, also written $HC[\tau]$ if there is no confusion.

Proposition 3.2.4. *For any block, the multiplicative structure of $HC[\tau]$ is untwisted.*

Proof. Let us take $U \subset G/P$ an affine open set isomorphic to an affine space. If $pr : \widetilde{\mathcal{N}}_P \to G/P$ is the projection, then $pr^{-1}(U)$ is an affine space with coordinates (x, y). In these coordinates the Poisson bivector field is $\tau = \sum_i \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$ and the Todd class is $t = 1 + \sum a_{ij} dx_i \wedge dx_j$ + higher order terms. It follows that $(1 - t, \tau) = 0$ where $(\cdot, \cdot)$ is the pairing $T^*\widetilde{\mathcal{N}}_P \otimes T\widetilde{\mathcal{N}}_P \to \mathcal{O}_{\widetilde{\mathcal{N}}}$. In the exact sequence

$$0 \to pr^*T^*(G/P) \to \widetilde{\mathcal{N}}_P \to pr^*T(G/P) \to 0$$

the left hand-side corresponds to the fiber of pr since $\widetilde{\mathcal{N}}_P = T^*(G/P)$. In particular an element $\alpha \in HT^\bullet(\widetilde{\mathcal{N}})$ coming from $H^\bullet(X, \Omega^\bullet)$ can we written in local coordinates as a polynomial in $\frac{\partial}{\partial y_i}$. For such a class it is again clear that $(1 - t, \alpha) = 0$. In particular it holds for the Harish-Chandra center $HC \cong H^\bullet(G/P)$. □

Corollary 3.2.5. *If G/P is a projective space $\mathbb{P}^n$, then $HH^0(\mathfrak{u}_\lambda(\mathfrak{g})) = z_\lambda(\mathfrak{g})$ is the polynomial algebra $\mathbb{C}[x, \tau]/(x^a\tau^b : a + b > n)$.*

Proof. It follows immediately from the previous proposition and the fact (proved in [49]) that $HC[\tau] = z_\lambda(\mathfrak{g})$ if G/P_λ is a projective space. □

Chapter 4

Spectral curves, affine Springer fibers and a bigrading

The starting point of the relationship between small quantum group and affine Springer fibers is a conjecture from Ginzburg–Kumar ([27]), relating the affine Grassmannian with the cohomology of the small quantum group. A refined result was obtained by Arkhipov–Bezrukavnikov–Ginzburg ([8]), namely an equivalence of abelian categories between Iwahori-constructible perverse sheaves on the affine flag variety and the category of modules of the principal block of the small quantum group. In the work by Bezrukavnikov–Boixeda-Alvarez–Shan–Vasserot ([14]), the center of the small quantum group is investigated from this viewpoint. They use deformation technics, to "degenerate" a certain algebra to the small quantum group. For the corresponding algebra, the center has a description in term of the cohomology of a certain affine Springer fiber, defined in the next paragraph. One of the main theorem from [14] states as follow (for simplicity we restrict ourselves to type A):

Theorem 4.0.1. *There is an injective map of algebra* $H^*(\mathcal{F}\ell_{\mathbf{I}}^{\gamma})^{\Lambda} \to \mathbf{z}^{G^\vee}$.

Here, $\mathcal{F}\ell_{\mathbf{I}}^{\gamma}$ is a certain affine Springer fiber, γ is a certain element in $\mathfrak{sl}_n((t))$, and Λ is the centralizer of γ. We will carefully define these objects in the next section.

Conjecture 4.0.2. *The map* $H^*(\mathcal{F}\ell_{\mathbf{I}}^{\gamma})^{\Lambda} \to \mathbf{z}^{G^\vee}$ *is an isomorphism.*

Motivated by this conjecture, it is natural to try to understand the center of the small quantum group from the perspective of affine Springer fibers. In this chapter, we are using the relationship between Hitchin fibers and affine Springer fibers to obtain a natural geometric bigrading on the center. Our main construction is a degeneration of spectral curves, inducing a map from the diagonal coinvariants algebra to the center of the small quantum group. We conjecture that this map is an isomorphism. Our geometric map preserves the multiplication, so it would conjecturally give another approach to understand the multiplicative structure of the small quantum group. Our main theorem can be stated as follows:

Theorem 4.0.3. *Let* $G = \mathsf{SL}_n$*. There is a degeneration of spectral curves* $\mathfrak{X} \to \mathbb{A}^1$*, which induces a map* $R_W \to \mathbf{z}_0^G$.

We can interpret this degeneration as an explicit geometric construction of a map $R_W \to \mathbf{z}_0^\Lambda$. We expect this map to be an isomorphism of bigraded vector spaces. It is not a ring isomorphism but our construction also explains where precisely this map fails to be multiplicative.

4.1 Affine Springer fibers

We give formally definitions of affine Springer fibers, affine Grassmannian and explain a precise relationship with the small quantum group, following [14].

Definition 4.1.1. *Let G be a reductive group over the complex numbers. The affine Grassmannian is the ind-scheme Gr, such that the set of $\mathbb{C}$-points are given by $Gr(\mathbb{C}) = G(\mathbb{C}((t))/G(\mathbb{C}[[t]])$.*

An ind-scheme can be understood as an inductive limit of schemes, see [71] for a precise definition. For example, the Laurent polynomial ring $\mathbb{C}[t, t^{-1}]$ is not a subscheme of the infinite-dimensional affine space $\mathbb{A}^\infty = \prod_{i \in \mathbb{Z}} \mathbb{C}t^i$, because the condition of being a Laurent polynomial is not cut by polynomial equations. However, $U_i = \mathbb{C}[t^j : |j| \leq i]$ is a scheme. Taking the inductive limit of these schemes gives the ind-scheme $\mathbb{C}[t, t^-1]$. In a similar way, we can write the affine Grassmannian (or more general affine flag varieties that will be introduced a bit after) as increasing union of various affine spaces, this is the so-called *Bruhat decomposition* (first introduced in [36]).

If $G = GL_n$, there is a geometric interpretation of the affine Grassmannian: one can see $GL_n(\mathbb{C}((t))$ as the space of trivial GL_n-torsors on the formal punctured disk, i.e a vector bundle of rank n. Moreover $GL_n(\mathbb{C}[[t]])$ can be seen similary as the space of vector bundles over the formal disks. Hence, the affine Grassmannian is the space of trivial vector bundle on the punctured disk, modulo the bundles that extends to the full disk.

Let $B \subset G$ be a Borel subgroup of a reductive subgroup G. The preimage of B by the evaluation map $ev_0 : G(\mathbb{C}[[t]]) \to G$ is called an *Iwahori* group, usually denoted I.

Definition 4.1.2. *The* affine flag variety *is the space $\mathcal{F}\ell_G = G(\mathbb{C}((t)))/I$.*

The affine flag variety is closely related to the affine Grassmannian: by definition there is a fiber bundle $\mathcal{F}\ell \to Gr$, with fiber G/B. There are also partial affine flag varieties $\mathcal{F}\ell_J$, where J is a subset of the (affine) Dynkin diagram associated to G. There is also a notion of affine Springer fibers:

Definition 4.1.3. *Let $\gamma \in \mathfrak{g}((t))$ be regular semisimple.The* affine Springer fiber *Gr^γ associated to γ is the subset*

$$\{gG(\mathbb{C}[[t]]) : Ad(g^{-1}(\gamma) \in \mathfrak{g}[[t]]\}$$

In this context, regular semisimple means that over an algebraic closure of $\mathbb{C}((t))$, γ is regular semisimple in the usual sense.

Finite-dimensional Springer fibers are classified by the nilpotent orbits. However, for affine Springer fibers the situation is quite different (for example they can be empty!).

Example 4.1.4. *Let us explicitely compute an affine Springer fiber for* $G = \mathsf{SL}_2$. *We pick* $\gamma = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \in \mathfrak{sl}_2((t))$. *We use the non-archimedian Iwasawa decomposition ([18]) to write* $\mathsf{SL}_2(\mathbb{C}((t)) = \sqcup_{n\in\mathbb{Z}}\ N \begin{pmatrix} t^n & 0 \\ 0 & t^{-n} \end{pmatrix} \mathsf{SL}_2(\mathbb{C}[[t]])$, *where* $N \subset \mathsf{SL}_2(\mathbb{C}((t)))$ *is the subgroup* $N = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{C}((t))\}$. *In this case, an easy calculation gives that for an arbitrary element* $g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^n & 0 \\ 0 & t^{-n} \end{pmatrix}$ *we have* $Ad(g^{-1})\gamma = \begin{pmatrix} t & 2t^{1-2n}x \\ 0 & -t \end{pmatrix}$. *Hence, we want* $x \in t^{2n-1}\mathbb{C}[[t]]$. *It is easy to see that* x *actually belongs to* $\mathbb{C}((t))/t^{2n}\mathbb{C}[[t]]$, *appropriately left-multiplying by an element of* $\mathsf{SL}_2(\mathbb{C}[[t]])$. *Hence, we have* $x \in \mathbb{C}$ *which gives exactly an affine line for each* $n \in \mathbb{Z}$.

By definition, the affine Springer fiber associated to γ is included in the affine Grassmannian, but for each partial affine flag variety $\mathcal{F}\ell_J$ there is a corresponding affine Springer fiber. These varieties are actually "finite-dimensional up to a lattice action", by an important theorem by Kazhdan and Lusztig. Before stating it, let us define this lattice. We assume now that γ is regular semisimple. It is known that the centralizer G_γ is a torus. For $\lambda \in X_*(G_\gamma)$ a cocharacter, we consider $\lambda(t) \in G_\gamma$. We get an injective morphism $\lambda \to \lambda(t)$, from $X_*(G_\gamma)$ to G_γ. Its image is denoted by Λ_γ.

Theorem 4.1.5. *[41] Let* $\gamma \in \mathfrak{g}(\mathbb{C}((t)))$ *be semi-simple regular, then* Gr^γ/Λ_γ *is a finite-dimensional projective variety.*

The dimension was conjectured by Kazhdan-Lusztig and the formula was proved by Bezrukavnikov [13].

A very important fact about affine Springer fibers is that their cohomology groups are acted upon by the affine Weyl group:

Theorem 4.1.6. *[55] There is an affine Weyl group action on* $H^*(Gr^\gamma)$.

This action is constructed by using the ordinary Weyl group action of finite Springer fibers.

We explain the set-up of [14]. Recall that ℓ is the order of the root of unity of our quantum group. There is a $\mathbb{G}_m \subset G_\gamma(\mathbb{C}((t)))$ action on the affine Springer fiber, hence we can look at the fixed point of Gr_γ under the action of the group generated by $\zeta \in \mathbb{G}_m$, denoted by $Gr_\gamma^\zeta = Gr_\gamma \cap Gr^\zeta$

Theorem 4.1.7 ([14])**.** *There is an injection* $H^*(Gr_\gamma^\zeta)^\Lambda \to Z(\mathsf{U})^{G^\vee}$, *where* $\gamma = s \otimes t^\ell$. *Here,* $s \in \mathfrak{h}$ *is a semisimple regular element.*

Under the action by ζ, is it actually possible to write down a decomposition of Gr_γ^ζ into various "parahoric" (affine version of parabolic) ordinary affine Springer fibers. The induced decomposition on the cohomology corresponds to the block decomposition for the small quantum group by results from [14].

4.2 The main construction

In this section we explain how to get a bigrading on the space $\mathrm{H}^*(\mathcal{F}\ell_{\mathbf{I}}^{\gamma})^{\widetilde{W}} \cong \mathrm{H}^*(\mathcal{F}\ell_{\mathbf{I}}^{\gamma})^{\Lambda}$ for $G = \mathsf{GL}_n$, which corresponds to the principal block of the center under Conjecture 4.0.2, using the perverse filtration on the parabolic Hitchin fibration. The advantage of using the Hitchin fibration is that there is a natural "Lefschetz element" coming from the relatively ample determinant bundle on the parabolic Hitchin fibration. We conjecture that the $\mathfrak{sl}_2$-action on the center obtained this way coincides with the one given by the wedge product with the Poisson bivector field on the Springer resolution. Further, we conjecture the bigradings obtained in these two ways are the same (up to an explicit change of variables).

4.2.1 The parabolic Hitchin fibration

First, we want to construct a particular compactification of the singular curve given by $x^n + y^n = 0 \subset \mathbb{C}^2$, inside a Hirzebruch surface Σ. Most importantly, this compactification will be irreducible, i.e. a spectral curve for the anisotropic locus of the Hitchin fibration, and have only an isolated singular point which is an ordinary n-uple point.

Let $\Sigma_r = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(r) \oplus \mathcal{O}_{\mathbb{P}^1})$ be the r-th Hirzebruch surface. The Picard group of Σ_r is generated by the zero section E_r and the class of a fiber F, with intersection form determined by $F^2 = 0, E_r^2 = -r$ and $E_r F = 1$.

Recall that there is a birational map from Σ_r to Σ_{r+1}, called an "elementary transform" (see [11, Chapter 3]), constructed as follows. We choose some fiber F, and consider the surface Σ_r', the blow-up of Σ_r at $p := F \cap E_r$. Let F', E_r' the strict transforms of F, E_r and $\tilde{E}$ be the exceptional divisor of this blow-up. Then we have

$$0 = F^2 = (F' + \tilde{E})^2 = (F')^2 + 2 - 1$$

hence F' is a (-1)-curve and can be contracted, the resulting surface being Σ_{r+1}. See Figure 4.2.1 for the toric picture, where the red line is the contracted curve.

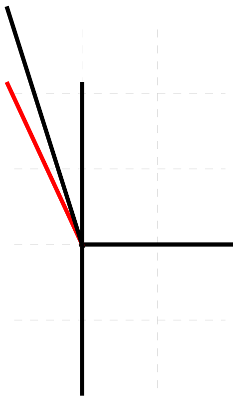


Figure 4.1: The toric blow-up and contraction giving a birational map $\Sigma_r \to \Sigma_{r+1}$.

Now we can prove:

Lemma 4.2.1. *For all $n \geq 0$, there is a curve $C \subset \Sigma_2$ such that C is irreducible, has a unique singular point, with singularity type $x^n + y^n = 0$.*

Proof. Let $C_2 \subset \mathbb{P}^2$ be a smooth curve of degree n, and Σ_1 be the blow-up of $\mathbb{P}^2$ at a point $a \notin C_2$. We denote by C_1 the strict transform of C_2. Consider a generic fiber F_0 and the corresponding elementary transform.

The strict transform C' of C_1 inside Σ_1' is isomorphic to C_1. Denote by C the image of C' under the contraction of F'. Since $F \cap C_1$ is given by n points, we see that C is analytically isomorphic to C_2 where n points have been glued transversally together, resulting in an ordinary n-uple point q. It's clear that $C \backslash \{q\}$ is smooth. Since $C \backslash \{q\}$ is connected, C is irreducible. ☐

Remark 4.2.2. *Since C_2 is the normalization of C, the geometric genus of C is $g_g = \binom{n-1}{2}$. Since the blowdown introduces $\binom{n}{2}$ nodes to C', the arithmetic genus is $g_a = g_g + \binom{n}{2} = (n-1)^2$.*

Definition 4.2.3. *Let $X/\mathbb{C}$ be a smooth projective curve, G a split reductive group, and L a line bundle on X with $\deg L \geq g_X$. The* Hitchin moduli stack *is the functor*

$$\mathcal{M} : Sch_{\mathbb{C}} \to Grpd$$

sending

$$S \mapsto \{(E, \varphi) | E \textit{ is a } G-\textit{torsor over } S \times X, \varphi \in H^0(\mathrm{Ad}(E) \otimes L)\}$$

Definition 4.2.4. *Let X, G, L be as above. The* parabolic Hitchin moduli stack *is the functor*

$$\widetilde{\mathcal{M}} : Sch_{\mathbb{C}} \to Grpd$$

sending

$$S \mapsto \{(E, \varphi, x, E_x) | (E, \varphi) \in \mathcal{M}, x \in X, E_x \textit{ is a } B-\textit{reduction along } \Gamma(x) \textit{ of } E\}$$

Let D be a divisor so that $\mathcal{O}(D) = L$. The Hitchin moduli stack can be interpreted as classifying sections

$$a : X \to \mathcal{O}(D) \times^{\mathbb{G}_m} [\mathfrak{g}/G]$$

, see [61, Lemme 2.4.].

Definition 4.2.5. *The morphism*

$$\mathcal{M} \to \mathcal{A} := \bigoplus_{i=1}^{n} H^0(X, \mathcal{O}(d_i D))$$

sending a section a to its image in $\mathcal{O}(D) \times^{\mathbb{G}_m} \mathrm{Sym}(\mathfrak{t}^*)^W$ is called the Hitchin fibration. *The base $\mathcal{A}$ is called the* Hitchin base. *The composition*

$$\widetilde{\mathcal{M}} \to \mathcal{M} \times X \to \mathcal{A} \times X$$

is called the parabolic Hitchin fibration.

Let now $G = \mathsf{SL}_n$ and L be a line bundle of degree ≥ 0 on $\mathbb{P}^1$. By the BNR correspondence [63], we may realize the curve C from Lemma 4.2.1, or rather its intersection with $\mathrm{Tot}(\mathcal{O}(2))$ as a spectral curve $\{\det(xI - \varphi) = 0\}$ for the Hitchin fibration

$$\mathcal{M} \to \mathcal{A}$$

associated to the data of $\mathbb{P}^1, G, L$. Let $a \in \mathcal{A}$ be such that C is the associated spectral curve. Note that we in fact have $a \in \mathcal{A}^{ani} \subset \mathcal{A}^{\heartsuit}$, the locus where the spectral curves are irreducible, resp. reduced (we will not need a more general definition of $\mathcal{A}^{ani}$ or $\mathcal{A}^{\heartsuit}$ here, for that see [61, § 6.1]).

The relationship to the affine Springer fibers we are considering is as follows. The curve C may be chosen so that the unique singularity is over $0 \in X$. Its local form corresponds to $\gamma = st \in \mathfrak{g}(\mathcal{K})$ as before, for $s = \mathrm{diag}(1, \rho, \dots, \rho^{n-1})$ where ρ is a primitive n-th root of unity. Let $(a, 0) \in \mathcal{A}^{\heartsuit} \times X$. Then [70, Proposition 2.4.1] says that

$$\mathcal{P}_a \times^{P_0^{red}(J_a)} \mathcal{F}\ell_{\mathbf{I}}^{\gamma} \to \widetilde{\mathcal{M}}_a \tag{4.1}$$

is a homeomorphism of stacks.

Here $\mathcal{P}_a$ is the generalized Picard stack, $P_0^{red}(J_a)$ the reduced quotient of the local Picard stack at 0. Modding out by $\mathcal{P}_a$, the left-hand side of Eq. (4.1) simplifies to $\mathcal{F}\ell_{\mathbf{I}}^{\gamma}/P_0^{red}(J_a)$. By taking $\gamma = st$ for $s \in \mathfrak{t}^{reg}$ as above, it is easy to compute by hand in this case that $P_0(J_a) = T(\mathbb{C}) \times \Lambda$ where T is the diagonal torus in GL_n and $\Lambda = X^*(T) \cong \mathbb{Z}^n$ is the lattice part of the centralizer.

Modifying the proof of [61, Proposition 4.13.1] slightly, we can write the following variant of Eq. (4.1):

$$\widetilde{\mathcal{M}}_a/\mathcal{P}_a^{\flat} \cong \mathcal{F}\ell_{\mathbf{I}}^{\gamma}/\Lambda \tag{4.2}$$

where $\mathcal{P}_a^{\flat}$ is the Picard group of the normalization of C as in [61, 4.7.3].

The upshot of this analysis is that we may define the *perverse filtration* on $H^*(\mathcal{F}\ell_{\mathbf{I}}^{\gamma}/\Lambda)$. Namely, if $\pi : \widetilde{\mathcal{M}} \to \mathcal{M} \times \{0\} \to \mathcal{A}^{ani}$ denotes the restriction of the parabolic Hitchin fibration to the locus of irreducible spectral curves and with the parabolic reduction at $0 \in X$, $\pi_*\mathbb{C}$ acquires a filtration from the t-structure on the base as

$$P_{\leq i} := \mathrm{im}({}^p\tau_{\leq i}\pi_*\mathbb{C} \to {}^p\tau_{\leq i+1}\pi_*\mathbb{C}).$$

Restricting to the stalk at a, we get a filtration $P_{\leq i}$ on $H^*(\widetilde{\mathcal{M}}_a/\mathcal{P}_a^{\flat}) \cong \mathrm{H}^*(\mathcal{F}\ell_{\mathbf{I}}^{\gamma}/\Lambda)$. By results of Maulik-Yun [56] this filtration is independent of the choice of deformation of C used here (we only require the total space to be smooth and a codimension estimate on the base, handled in this case by [61]). See also [56, Section 3.1.3.].

We make the following conjecture, which holds for $G = \mathsf{SL}_2$.

Conjecture 4.2.6. *As bigraded vector spaces*

$$\mathrm{DR}_n \cong \mathrm{gr}^P\, \mathrm{H}^*(\mathcal{F}\ell_{\mathbf{I}}^{\gamma})^{\Lambda} \tag{4.3}$$

Proposition 4.2.7. *The conjecture 4.2.6 is true for $G = SL_2$.*

Proof. In this case, the two vector spaces are equal to $\mathbb{C}^3$, hence we just need to check that the gradings agree. The affine Springer fiber $\mathcal{F}\ell_{\mathbf{I}}^{\gamma}$ can be identified with an infinite chains of $\mathbb{P}^1$, and the lattice action is obtained by translation by 2 ([70]). Hence the quotient $X_0 = \mathcal{F}\ell_{\mathbf{I}}^{\gamma}/\Lambda$ is isomorphic to an elliptic curve with a singularity of type I_2 (i.e two $\mathbb{P}^1$ glued transversally twice). By the discussion before, this curve also appears as a spectral curve inside a cotangent bundle of $\mathbb{P}^1$, hence its compactified Jacobian is a Hitchin fiber inside the corresponding Hitchin fibration. Since X_0 has arithmetic genus 1, it is isomorphic to its own compactified Jacobian. It follows by versality of the Hitchin map in this case that the restriction of this fibration to a generic line is simply a smoothing of X_0, say $f : X \to L = \mathbb{C}$. Let $L^* = L\backslash\{0\}$. By the decomposition theorem, we have

$$f_*\underline{\mathbb{C}}_X = \underline{\mathbb{C}}_L \oplus \underline{\mathbb{C}}_L[-2] \oplus \underline{\mathbb{C}}_0[-2] \oplus \mathscr{L}[-1]$$

where $\mathscr{L}$ is the rank 2 local system on L^* given by the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. The pure part is given by $\underline{\mathbb{C}}_L \oplus \underline{\mathbb{C}}_L[-2] \oplus \underline{\mathbb{C}}_0[-2]$. The perverse degree are $-2, 0, 2$. Up to renormalisation, we obtained the same bigrading as the diagonal coinvariants in this case. □

We now turn to the $\mathfrak{sl}_2$-action. Let L_{det} be the determinant line bundle on $\mathcal{M}$. The iterated cup product by $c_1(L_{det})$ induces a map

$$\cup c_1(L_{det})^{g_a - i} : {}^pH^{\dim \mathcal{A}+i}\pi_*\mathbb{C} \to {}^pH^{\dim \mathcal{A}+2g_a-i}\pi_*\mathbb{C}$$

and therefore maps

$$\cup c_1(L_{det}) : \mathrm{gr}^P H^*(\mathcal{F}\ell_{\mathbf{I}}^{\gamma}/\Lambda) \to \mathrm{gr}^p\, \mathrm{H}^*(\mathcal{F}\ell_{\mathbf{I}}^{\gamma}/\Lambda)$$

of bidegrees $(a, b) = (2, 2)$, where a is the cohomological degree and b is the perverse degree.

The first Chern class $c_1(L_{det})$ coincides with a certain polynomial in the ring of diagonal coinvariants, under Conjecture 4.2.6 (see next theorem). We can hope that $c_1(L_{det})$ also coincides with the Poisson bivector field on the Springer resolution as explained in more detail in Conjecture 4.2.9.

Theorem 4.2.8. *Under the identification Eq.* (4.3)*, the element* $c_1(L_{det}) \in \mathrm{gr}^P\, \mathrm{H}^*(\mathcal{F}\ell_{\mathbf{I}}^{\gamma})^{\Lambda}$ *corresponds up to a nonzero scalar to the "Haiman determinant"* $\Delta_{(n-1,1)} \in DR_n$ *given by*

$$\Delta_{(n-1,1)} = \det(y_i^{p_j} x_i^{q_j})_{1 \le i,j \le n}$$

where $(p_1, q_1), \dots, (p_n, q_n)$ *is any ordering of* $(0, 0), (0, 1), \dots, (0, n-1), (1, 0) \in \mathbb{Z}^2_{\ge 0}$

The theorem is clear because both bigraded pieces are 1-dimensional.

Finally, note that by the Jacobson-Morozov theorem, the nilpotent action of $e = \cup c_1(L_{det})$ extends to an $\mathfrak{sl}_2$-triple (e, f, h) acting on $\mathrm{gr}^p\, \mathrm{H}^*(\mathcal{F}\ell_{\mathbf{I}}^{\gamma}/\Lambda)$. By [56, Conjecture 2.17.] the Jacobson-Morozov filtration induced by $c_1(L_{det})$ on $\mathrm{H}^*(\mathcal{F}\ell_{\mathbf{I}}^{\gamma}/\Lambda)$ is opposite to the perverse filtration. It is clear that the Jacobson-Morozov filtration induced by $\Delta_{(n-1,1)}$ on the diagonal coinvariants induces the filtration by antidiagonals.

Recall that the Combining Theorem 2.3.6 and Conjecture 4.0.2, we also have the following conjecture.

Conjecture 4.2.9. *There is a bigraded algebra isomorphism*

$$\oplus_{i+j+k=0} H^i(\widetilde{\mathcal{N}}, \wedge^j T\widetilde{\mathcal{N}})^k \cong \mathrm{H}^*(\mathcal{F}\ell_{\mathbf{I}}^{\gamma})^{\Lambda}$$

where the bigrading on the right will explained later. The bigraded pieces (i,j) on the left should go to the piece $gr^i\,\mathrm{H}^j(\mathcal{F}\ell_{\mathbf{I}}^{\gamma})^{\Lambda}$. Moreover, the element τ on the left should correspond up to a scalar to the polynomial $\Delta_{(n-1,1)}$ introduced in Theorem 4.2.8 on the right, or in the second version equivalently to $c_1(L_{det})$.

Combined with the Carlsson-Mellit conjecture, this Conjecture would imply [48, Conjecture 4.9(3)].

4.2.2 A degeneration of spectral curves

We define certain elements $\gamma_{n+1/n}$. Let Φ_r the set of roots of height r. Let $m = ah+b$ where $0 \leq b < h$ (here, h is the Coxeter number associated to $\mathfrak{g}$). Define

$$\gamma_{m/h} = t^a \left(t \sum_{\phi \in \Phi_{h-b}} e_\phi + \sum_{\phi \in \Phi_{-b}} e_\phi \right)$$

To propose yet another model for the center in type A, we will study the elliptic homogeneous affine Springer fibers of slope $(n+1)/n$ associated to the elements $\gamma_{n+1/n}$ and their relation to $\mathcal{F}\ell_{\mathbf{I}}^{\gamma}$, where γ is as in the introduction. It is known by e.g. [20] that

$$\mathrm{gr}^P\,\mathrm{H}^*(\mathcal{F}\ell^{\gamma_{n+1/n}}) \cong \mathrm{D}R_n$$

We will construct a a family of irreducible spectral curves $C_t \subset \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(2))$, such that the associated family of parabolic Hitchin fibers models the degeneration of affine Springer fibers of slope n/n to the one of slope $(n+1)/n$. One can then ask whether the specialization map from the cohomology of the total family (which turns out to be just that of the central fiber) gives an injection to the cohomology of the special fiber, respecting the perverse filtration.

Theorem 4.2.10. *There exists a family of irreducible curves $\mathcal{C} \to \mathbb{A}^1$, arising as a restriction of the Hitchin system to a line on the Hitchin base, so that the spectral curve $C_t, t \in \mathbb{A}^1$ will have two singular points: one "constant" (i.e independent of t) singular point with equation $y^n = z^{n-1}$, and another singular point of the form $y^n = tx^n + x^{n+1}$.*

Remark 4.2.11. *Let us notice that if $t \neq 0$, this singular point is isomorphic to the singularity $y^n + x^n = 0$. Indeed, in $\mathbb{C}[[x,y]]$ the equation can be written $(t+x)^{-1}y^n = x^n$. Taking a n-th root of the unit $(t+x)$, say a, we can use the coordinate change $Y = a^{-1}y$ and $X = x$ to get $Y^n = X^n$. Hence, around this second singular point we are degenerating the singularity $y^n = x^n$ into the singularity $y^n = x^{n+1}$.*

Proof. We construct the family of spectral curves realizing this degeneration as follows: let $E \subset \Sigma_1$ be the exceptional section inside the first Hirzebruch surface, and $F \subset \Sigma_1$ some fiber, which we will call "the fiber at infinity". Let $U = \Sigma_1 \backslash (E \cup F)$. Take coordinates x, y on U such that the straight lines $x =$ constant are the fibers of the projection $U \subset \Sigma_1 \to \mathbb{P}^1$. Let us consider the curve $\hat{C}_t \subset U$ given by the equation $y^n = t + x$. The effect of a positive elementary transform $\varphi : \Sigma_r \dashrightarrow \Sigma_{r+1}$ is given by the change of variables $u = y/x, v = x$.

Hence the strict transforms of $\hat{C}_t$ (inside $\varphi(U)$) have local equation given by $u^n = tv^n + v^{n+1}$, giving the desired degeneration. Now let us describe the singular point at infinity (i.e compute the closure of these curves inside Σ_2), and prove that C_t is irreducible for all $t \in \mathbb{A}^1$.

First, we claim that the closure of $\hat{C}_t$ doesn't intersect E. Indeed, recall that Σ_1 is the blow-up of $\mathbb{P}^2$ at a point. Hence, it's enough to take the closure of the preimage of $\hat{C}_t$ inside $\mathbb{P}^2$ (call this curve $\tilde{C}_t$) and check that $\tilde{C}_t$ doesn't intersect the center of the blow-up. On U, we have coordinates x, y, that form a dense open of $\mathbb{P}^2$ (recall that U and E are disjoint by definition). Because $U \cong \mathbb{A}^2$, we can take homogeneous coordinates $[x : y : z]$ on $\mathbb{P}^2$. The fiber at infinity is given by $z = 0$ and U is given by $z = 1$. The fiber $x = 0$ and $z = 0$ both contains the center of the blow-up which is therefore $[0 : 1 : 0]$.

The closure of $\tilde{C}_t$ has equation $y^n = tz^n + xz^{n-1}$, which clearly doesn't contain $[0 : 1 : 0]$.
Since the elementary transforms are isomorphism outside the exceptional locus, it follows that the closure of C_t coincide with the closure of $\tilde{C}_t$ inside $\mathbb{P}^2$, i.e the curve with equation $y^n = tz^n + xz^{n-1}$. The only point at infinity is $[1 : 0 : 0]$, and has local equation $y^n = tz^n + z^{n-1}$ as claimed. To check that C_t is irreducible, it's enough to check that C_t is irreducible on the chart $x \neq 0$. On this chart, C_t is isomorphic to the curve given by $y^n = z^{n-1}$, which is irreducible. □

Consider the associated family of parabolic Hitchin fibers, which is a restriction of the family in 4.2.5 to a line. Using Eq. (4.1), we note that the only affine Springer fibers contributing to the cohomology are the ones coming from the singularities described above. We will ignore the one which is constant, for there is an injective map in cohomology sending the cohomology classes $\alpha \in \mathrm{H}^*(\mathcal{F}\ell_{\mathbf{I}}^{\eta}/\Lambda_\eta)$ of interest to

$$\alpha \otimes 1 \in \mathrm{H}^*(\mathcal{F}\ell_{\mathbf{I}}^{\eta}/\Lambda_\eta) \otimes \mathrm{H}^*(\mathcal{F}\ell_{\mathbf{I}}^{\gamma_{n-1/n}}) \cong \mathrm{H}^*(\widetilde{\mathcal{M}}_a)$$

where η is either γ or $\gamma_{n+1/n}$.

In particular, by Theorem 4.2.10, we get a pullback map $i^* : \mathrm{H}^*(\mathcal{F}\ell_{\mathbf{I}}^{\gamma_{n+1/n}}) \to \mathrm{H}^*(\mathcal{F}\ell_{\mathbf{I}}^{\gamma}/\Lambda)$. Now, both of these spaces are endowed with the perverse filtration, as the family comes by restriction of the Hitchin fibration and on the locus of interest the map is proper, so that the decomposition theorem applies. Note however that it is unclear how this filtration compares to that induced by the t-structure on $\mathbb{A}^1$, as the pullback along the inclusion to the base is in general only right t-exact.

Remark 4.2.12. *It seems likely that the map i^* is injective and its image is exactly $H^*(\mathcal{F}\ell_{\mathbf{I}}^{\gamma})^{\Lambda}$. Moreover, the map respects the perverse filtration. Note that as the map is a pullback in cohomology, it automatically respects the multiplicative structure.*

The only supporting evidence for this remark is that these properties are true for $G = SL_2$, where they are easy to check. In general, we observe that $\mathcal{F}\ell_{\mathbf{I}}^{\gamma_{n+1/n}}$ has only even-dimensional cohomology as it is paved by affines. It is also known that it has $n!$ components, as does $\mathcal{F}\ell_{\mathbf{I}}^{\gamma}/\Lambda$. On the level of top cohomology, it is clear that the map is injective, but in general it seems hard to control the associated vanishing-nearby cycles-central fiber exact sequence in cohomology.

Let us recall the main result of [20]:

Theorem 4.2.13. *There is an isomorphism* $DC_n \cong \mathrm{H}^*(\mathcal{F}\ell_{\mathbf{I}}^{\gamma_{n+1/n}})$*, and a filtration such that the natural* $\mathbb{Q}[x]$*-action is equivariant on the subquotients of the filtration.*

Using this result, we obtain a map $DC_n \to \mathbf{z}^G$. Our conjecture would imply that this map is an isomorphism, and understanding the y-multiplication on $\mathrm{H}^*(\mathcal{F}\ell_{\mathbf{I}}^{\gamma_{n+1/n}})$ could relates the multiplication on both sides.

Chapter 5

The BGG sheaf-cohomology algorithm and examples

5.1 Overview

In [48], the *sheaf-cohomology BGG algorithm* was introduced, and was used to compute the center for $\mathfrak{g} = \mathfrak{sl}_3$. Unfortunately, the computations for higher ranks are too complicated to be done by hand, even for type $\mathfrak{g}_2$. Later in [32], we developed the algorithm further and implemented it as a software package. We provided several geometric applications of the method, but the original motivation was to study the center of the small quantum group. This is what we will do in this chapter.

In Section 5.2 we recall some facts about the small quantum group. In Section 5.3 we compute the full ring $\mathrm{HH}^\bullet(\mathfrak{u}_1)$, where $\mathfrak{u}_1$ is the non-trivial singular block for $\mathfrak{g} = \mathfrak{sl}_3$. Using the BGG sheaf cohomology method, we express this ring as $\mathfrak{g}$-module and as a module over the functions on the nilpotent cone $\mathbb{C}[\mathcal{N}]$. The $\mathbb{C}[\mathcal{N}]$-module structure of $\mathrm{HH}^\bullet(\mathfrak{u}_0)$ was computed in [50] for $\mathfrak{g} = \mathfrak{sl}_2$. In Section 5.4 we compute the center for all blocks of G_2, B_3, C_3 and A_4, confirming the conjectures by Lachowska-Qi for each case. In Section 5.5 we discuss higher Hochschild cohomology groups and give several examples. Finally, in Section 5.6 we consider the case when $G/P \cong \mathbb{P}^n$, for which HH^0 was previously computed in [49]. We present partial results for higher Hochschild cohomology.

5.2 The small quantum group and the BGG complex

In this section we recall the algorithm of Lachowska-Qi to compute its center. We will then give explicit formulas for the BGG maps of the higher cohomology of $\mathfrak{sl}_3$, and we give a simplified description of the modules used to compute the Hochschild cohomology of blocks of the small quantum group.

Let us recall the main theorem of [12]:

Theorem 5.2.1 ([12])**.** *There is an isomorphism* $\mathrm{HH}^s_{\mathbb{C}^\times}(\widetilde{\mathcal{N}}) \cong \mathrm{HH}^s(\mathfrak{u}_0)$*, where the left-hand side is the* $\mathbb{C}^*$*-equivariant Hochschild cohomology of* $\widetilde{\mathcal{N}}$*.*

This isomorphism is compatible with the $\mathfrak{g}$-module structure [50]. In [48] and [49], the left-hand side was explicitely computed in terms of the BGG resolution associated to a finite-dimensional irreducible representation of $\mathfrak{g}$. Let us recall briefly how to do this. There are certain G-equivariant vector bundles $\mathcal{V}_{j,k} = G \times^B V_{j,k}$ (where j, k are integers) on G/B such that the Hochschild cohomology can be obtained from sheaf cohomology of the $\mathcal{V}_{j,k}$. We will describe the B-modules $V_{j,k}$ in section 5.2.2. The precise relation is

$$\mathrm{HH}^s_{\mathbb{C}^*}(\widetilde{\mathcal{N}}) \cong \bigoplus_{i+j+k=s} \mathrm{H}^i(G/B, \mathcal{V}_{j,k}). \tag{5.1}$$

It follows that the center has a natural bigrading. One can reduce the computation of the right-hand side of equation (5.1) to a Lie algebra cohomology computation [48]. The *BGG resolution* is a convenient tool for this computation, and we will recall its basic properties in the next subsection. These observations lead to the following structure result:

Theorem 5.2.2 ([48], Theorem 4.3)**.** *For any $s \geq 0$, there is an $\mathfrak{sl}_2$-action on the Hochschild cohomology $\mathrm{HH}^s(\mathfrak{u}_0)$, where the generator $e \in \mathfrak{sl}_2$ acts as a homogeneous element of bidegree $(i, j) = (0, 2)$.*

More precisely, this homogenous element is the Poisson bivector field $\tau \in H^0(\widetilde{\mathcal{N}}, \wedge^2 T\widetilde{\mathcal{N}})^{-2}$ (which is the dual of the canonical symplectic form $\omega \in H^2(\widetilde{\mathcal{N}}, \Omega^2)^2)$). The action of e is hence given by the map

$$\tau \wedge - \colon\ \mathrm{H}^i(\widetilde{\mathcal{N}}, \wedge^j T\widetilde{\mathcal{N}})^k \to \mathrm{H}^i(\widetilde{\mathcal{N}}, \wedge^{j+2} T\widetilde{\mathcal{N}})^{k-2}.$$

In particular, there are canonical isomorphisms ([48], Corollary 4.4)

$$\tau^j \wedge - \colon\ \mathrm{H}^i(\widetilde{\mathcal{N}}, \wedge^{n-j} T\widetilde{\mathcal{N}})^k \to \mathrm{H}^i(\widetilde{\mathcal{N}}, \wedge^{n+j} T\widetilde{\mathcal{N}})^{k-2j}.$$

We can represent the action of τ on the bigraded table as follows (we took $\mathfrak{g} = \mathfrak{sl}_3$ and $s = 0$):

$$\begin{tikzcd}
i+j=0 & \mathrm{H}^0(\widetilde{\mathcal{N}}, \wedge^0 T\widetilde{\mathcal{N}})^0 \arrow[dr, dashed, "\tau\wedge-"] & & & \\
i+j=2 & \mathrm{H}^1(\widetilde{\mathcal{N}}, \wedge^1 T\widetilde{\mathcal{N}})^{-2} \arrow[dr, dashed, "\tau\wedge-"] & \mathrm{H}^0(\widetilde{\mathcal{N}}, \wedge^2 T\widetilde{\mathcal{N}})^{-2} \arrow[dr, dashed, "\tau\wedge-"] & & \\
i+j=4 & \mathrm{H}^2(\widetilde{\mathcal{N}}, \wedge^2 T\widetilde{\mathcal{N}})^{-4} \arrow[dr, dashed, "\tau\wedge-"] & \mathrm{H}^1(\widetilde{\mathcal{N}}, \wedge^2 T\widetilde{\mathcal{N}})^{-4} \arrow[dr, dashed, "\tau\wedge-"] & \mathrm{H}^0(\widetilde{\mathcal{N}}, \wedge^4 T\widetilde{\mathcal{N}})^{-4} \arrow[dr, dashed, "\tau\wedge-"] & \\
i+j=6 & \mathrm{H}^3(\widetilde{\mathcal{N}}, \wedge^3 T\widetilde{\mathcal{N}})^{-6} & \mathrm{H}^2(\widetilde{\mathcal{N}}, \wedge^4 T\widetilde{\mathcal{N}})^{-6} & \mathrm{H}^1(\widetilde{\mathcal{N}}, \wedge^5 T\widetilde{\mathcal{N}})^{-6} & \mathrm{H}^0(\widetilde{\mathcal{N}}, \wedge^6 T\widetilde{\mathcal{N}})^{-6} \\
h^{i,j} & j-i=0 & j-i=2 & j-i=4 & j-i=6
\end{tikzcd}$$

To compute the bigraded table corresponding to $\mathrm{HH}^s_{\mathbb{C}^*}(\widetilde{\mathcal{N}})$, we hence just need to compute the top-left half of the table (including the diagonal starting at the bottom left), and use the $\mathfrak{sl}_2$-action to obtain the full table.

In [48], the center of $\mathfrak{u}_0(\mathfrak{sl}_3)$ was obtained as bigraded vector space, and it was noticed that, as a bigraded vector space, $\mathrm{HH}^0(\mathfrak{sl}_3)$ is isomorphic to the double coinvariant algebra DC_3, where

$$\mathrm{DC}_m = \mathbb{C}[x_1, \dots, x_m, y_1, \dots, y_m]/I,$$

and I is the ideal generated by invariant polynomials (for the diagonal $\mathfrak{S}_m$-action). This motivated the following conjecture:

Conjecture 5.2.3 ([48])**.** *As a bigraded vector space, there is an isomorphism* $\mathbf{z}(\mathfrak{u}_0(\mathfrak{sl}_m)) \cong \mathrm{DC}_m$*. In particular,* $\dim \mathbf{z}(\mathfrak{u}_0(\mathfrak{sl}_m)) = (m+1)^{m-1}$*.*

It is also conjectured that, as $\mathfrak{sl}_m$-module, $\mathbf{z}(\mathfrak{u}_0(\mathfrak{sl}_m))$ only contains the trivial representation. In other types, the presence of non-trivial representations made the formulation of a general conjecture more difficult. Computations for $\mathfrak{b}_2$ showed that the $\mathfrak{g}$-invariant part was also isomorphic as a bigraded vector space to the double-coinvariant algebra (which is defined for any Weyl group W). One could thus still hope for the following:

Conjecture 5.2.4. *[49] Let* $\mathfrak{g}$ *be a semisimple Lie algebra with Weyl group* W*. Then,* $\mathbf{z}(\mathfrak{u}_0(\mathfrak{g}))^{\mathfrak{g}} \cong \mathrm{DC}(W)$*.*

For $\mathfrak{g} = \mathfrak{sl}_n$, it is expected that the whole center is $\mathfrak{g}$-invariant:

Conjecture 5.2.5. *[48] Let* $\mathfrak{g} = \mathfrak{sl}_n$*. Then,* $\mathbf{z}(\mathfrak{u}_q(\mathfrak{sl}_n)) = \mathbf{z}(\mathfrak{u}_q(\mathfrak{sl}_n))^{\mathfrak{sl}_n}$*.*

Finally, outside the simply-laced case, non-trivial representations appear. Based on our computations, we make the following conjecture:

Conjecture 5.2.6. *Let* L *be an irreducible non-trivial representation appearing in* $z(\mathfrak{u}_q(\mathfrak{g}))$*, and let* h *be the Coxeter number of* $\mathfrak{g}$*. Then,* $h+1$ *divides* $\dim L$*.*

5.2.1 The BGG complex

We recall the BGG complex and some of its basic properties. Fix a triangular decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{h} \oplus \mathfrak{n}$, and let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. Recall that, for any $\mu \in \mathfrak{h}^*$, the Verma module is defined as $M(\mu) = \mathrm{Ind}_{U(\mathfrak{h}\oplus\mathfrak{n})}^{U(\mathfrak{g})} \mathbb{C}_\mu$, where $\mathbb{C}_\mu$ is a one-dimensional $\mathfrak{b}$-module, and where $\mathfrak{h}$ acts by μ and $\mathfrak{n}$ by zero. Let P^+ be the set of dominant weights, and let $\lambda \in P^+$.

Theorem 5.2.7 ([35], chapter 6)**.** *There is an exact sequence*

$$0 \to M(w_0 \cdot \lambda) \to \bigoplus_{\ell(w)=n-1} M(w \cdot \lambda) \to \cdots \bigoplus_{\ell(w)=k} M(w \cdot \lambda) \cdots \to M(\lambda) \to L(\lambda) \to 0$$

We now state the following facts needed to understand the maps in this complex:

Proposition 5.2.8 ([35], p.75)**.** *For all* $\mu, \lambda \in P$ *we have* $\dim \mathrm{Hom}_{\mathfrak{g}}(M(\mu), M(\lambda)) \leq 1$*. Moreover, if* λ *is dominant, then for any* $x, w \in W$ *a morphism* $M(w \cdot \mu) \to M(x \cdot \mu)$ *exists if and only if* $x < w$*. Such a morphism is always an embedding.*

It follows that the maps in the BGG complex are essentially given by scalars. As we will see later, it is proved in [57] that this complex is independent of the choices of the scalars.

Proposition 5.2.9 ([1], Lemma 10.3)**.** *If* $w, w' \in W$ *are such that there is* $x \in W$ *with* $w' \to x \to w$*,* *then there are exactly two such elements, say x and y. We call such a quadruple of elements a* square *in W and denote it by (w', x, y, w).*

Proposition 5.2.10 ([35] p.118)**.** *Let α be a simple root, β be a positive root and $x, y \in W$. The first diagram exists if and only if the second diagram does:*

$$\begin{tikzcd}
 & y & y \arrow[dr, "\alpha"] & \\
s_\alpha x \arrow[ur, "\beta"] \arrow[dr, "\alpha"] & & & s_\alpha y \\
 & x & x \arrow[ur, "s_\alpha\beta"] &
\end{tikzcd}$$

We also require the following lemma:

Lemma 5.2.11. *For each edge $x \xrightarrow{t} w$, there is a square (w', x, y, w) and a simple reflection s so that either $w = sx$ or $w = sy$.*

Proof. The lemma is obvious if $w = w_0$ so we assume $w \neq w_0$. Since we assumed $w \neq w_0$ there is a simple root α so that $y := s_\alpha w \to w$. Taking $\beta = s_\alpha \gamma$ (where γ is the positive root corresponding to t) proves the existence of the right part of the diagram in proposition 5.2.10, so the proposition 5.2.10 finishes the proof. □

In light of proposition 5.2.8 and theorem 5.2.7, the maps in the BGG complex exactly correspond to pairs (x, w) with $x, w \in W$, such that $x \to w$, i.e. there is a reflection $t \in W$ so that $w = tx$ and $\ell(w) = \ell(x) + 1$. Hence, the maps in the BGG complex exactly corresponds to the edges of the Bruhat graph $\mathscr{B}$ of the corresponding Weyl group W.

We recall the definition of the Bruhat graph: the vertices of $\mathscr{B}$ are given by W, and there is an edge from x to y if and only if $x \to w$, i.e. there is a reflection $t \in W$ with $w = tx$ and $\ell(w) = \ell(x) + 1$. For example, for $\mathfrak{sl}_3$ the BGG complex associated to λ is represented by the following diagram:

$$\begin{tikzcd}
 & M(s_1 s_2 \cdot \lambda) \arrow[r, "s_1 s_2 s_1"] \arrow[dr, "s_2"] & M(s_1 \cdot \lambda) \arrow[dr, "s_1"] & & \\
M(s_1 s_2 s_1 \cdot \lambda) \arrow[ur, "s_2"] \arrow[dr, "s_1"] & & & M(\lambda) \arrow[r] & L(\lambda) \\
 & M(s_2 s_1 \cdot \lambda) \arrow[r, "s_2 s_1 s_2"'] \arrow[ur, "s_1"] & M(s_2 \cdot \lambda) \arrow[ur, "s_2"] & &
\end{tikzcd}$$

Here each column represents a term of the BGG complex, and each edge $\sigma(x, w)$ corresponds to an element $t \in W$ so that $tx = w$. For example, for the lower horizontal arrow we have $x = s_2, w = s_2 s_1$ and $t = s_2 s_1 s_2$.

Now let us describe in more detail the maps associated to edges in the Bruhat graph. A non-zero map

$M(w\cdot\lambda)\to M(x\cdot\lambda)$ is injective (by 5.2.8). Hence we can write $M(w\cdot\lambda)=U(\mathfrak{n})u$ where $u\in M(x\cdot\lambda)$ is a highest weight vector of weight $w\cdot\lambda$. The map $M(x\cdot\lambda)\to M(w\cdot\lambda)$ is determined by the image of u, which is a highest weight vector of weight $w\cdot\lambda-x\cdot\lambda$, so it can be written as $\mathcal{F}u$ for a unique $\mathcal{F}:=\mathcal{F}(x,w)\in U(\mathfrak{n})[x\cdot\lambda-w\cdot\lambda]$. We emphasize that $\mathcal{F}(x,w)$ depends on λ. Explicitly computing all these elements $\mathcal{F}(x,w)$ is an important part of the algorithm. An easy case is when $w\cdot\lambda-x\cdot\lambda$ is a multiple of a simple root α_i (i.e. when the element $t\in W$ associated to the edge $\sigma(x,w)$ correspond to a simple reflection). In this case, the element $\mathcal{F}$ will simply be a scalar multiple of $f_i^{1+\langle w\cdot\lambda,\alpha_i^\vee\rangle}$.

Now assume that we found all the elements $\mathcal{F}(x,w)$. These elements are well-defined up to scalar multiplication. To obtain a complex, we should pick scalars for each edge to ensure the equation $d^2=0$. In view of propositions 5.2.9 and 5.2.10, to check that $d^2=0$ it is enough to check it 'square-wise'. That is, for each square $w\to x\to w', w\to y\to w'$ we require the equality $\mathcal{F}(w,x)\mathcal{F}(x,w')+\mathcal{F}(w,y)\mathcal{F}(y,w')=0$. In practice it is easier to solve the equation $\mathcal{F}(w,x)\mathcal{F}(x,w')=\mathcal{F}(w,y)\mathcal{F}(y,w')$ and then assign signs $\sigma(w,w')$ to each edge so that $\sigma(w,x)\sigma(x,w')+\sigma(w,y)\sigma(y,w')=0$. Note that then the maps $\sigma(x,w)\mathcal{F}(x,w)$ will form a complex. The two following results ensure that the BGG resolution is in fact exact, and moreover unique:

Theorem 5.2.12. *[1] For any choice of scalars such that $d^2=0$, the corresponding BGG complex is exact.*

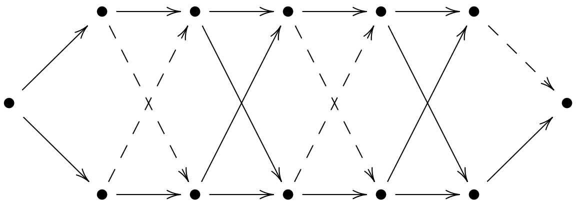
Theorem 5.2.13. *[57] Different choices of scalars give isomorphic complexes.*

So we can just pick an arbitrary choice of scalars such that $d^2=0$. Finally we explain how we compute the maps. For each square, we can recursively solve the equation $fg=hk$ where we know all but one map. We surely know the first column since each map corresponds to a monomial. For any edge t we can then find a square where the opposite edge is a simple reflection by lemma 5.2.11 (hence we know all but one map and can compute the map corresponding to t). Once we have solved each system, we just distribute signs for each edge so that each signed square has 1 or 3 negative signs.

As an example we compute the first non-trivial map in the BGG resolution for $\mathfrak{g}=\mathfrak{g}_2$ and $\lambda=0$ (note that the maps depend on the weights in general). We write α_1 for the short root and α_2 for the long root, and obtain $s_1\cdot 0=-\alpha_1, s_2\cdot 0=-\alpha_2, s_2s_1\cdot 0=-\alpha_1-2\alpha_2$ and $s_1s_2\cdot 0=-4\alpha_1-\alpha_2$. Therefore the beginning of the complex is:

$$\begin{tikzcd}
{} \arrow[r] & M(-4\alpha_1-\alpha_2) \arrow[r] \arrow[rdd] & M(-\alpha_1) \arrow[rd] & & \\
\cdots & & & M(0) \arrow[r] & L(0) \\
{} \arrow[r] & M(-\alpha_1-2\alpha_2) \arrow[r] \arrow[ruu] & M(-\alpha_2) \arrow[ru] & &
\end{tikzcd}$$

Write $M(0)=U(\mathfrak{n})v_0$ where $v_0=1$ and $M(-\alpha_1)=U(\mathfrak{n})v_{-\alpha_1}$. Looking at the weight, it is clear that the map $M(-\alpha_1)\to M(0)$ is given by $v_{-\alpha_1}\mapsto f_1v_0$, up to scaling. Similarly, the map $M(-4\alpha_1-\alpha_2)\to M(-\alpha_2)$ is given by $v_{-4\alpha_1-\alpha_2}\mapsto f_1^4v_{-\alpha_2}$ up to scaling. Since we require the upper square to commute, we need to solve the equation $f_1^4f_2=\mathcal{F}f_1$ where $\mathcal{F}$ is the unknown. For this particular case, the solution follows easily from the Serre relation $\mathrm{ad}(f_1)^4(f_2) = 0$, giving $\mathcal{F} = 4f_1^3f_2 - 10f_1^2f_2f_1 + 4f_1f_2f_1^2 - f_2f_1^3$. Similarly, the map $M(-\alpha_1 - \alpha_2) \to M(-\alpha_1)$ is obtained from the other Serre relation $\mathrm{ad}(f_2)^2(f_1) = 0$. A possible choice of signs for $\mathfrak{g}_2$ is shown below (solid arrows correspond to $+$ and dashed arrows to $-$):



Here, $w \cdot \mu = w(\mu + \rho) - \rho$ is the *dot-action*, where ρ is half the sum of positive roots. We denote the resulting complex made of Verma modules by $\mathrm{BGG}^\bullet(\lambda)$. The maps of this complex are given by monomials in $U(\mathfrak{n})$. For a finite-dimensional $\mathfrak{b}$-module E, there is an associated complex $\mathrm{BGG}^\bullet(E, \lambda) := (\mathrm{Hom}_{\mathfrak{n}}(\mathrm{BGG}^\bullet(\lambda), E))^{\mathfrak{h}}$. Using that Verma modules are free $U(\mathfrak{n})$-modules, we can identify

$$\mathrm{BGG}^k(E, \lambda) = \bigoplus_{\ell(w)=k} E[w \cdot \lambda],$$

where $E[\mu]$ denotes the weight space corresponding to the weight μ. This complex can be used to compute certain sheaf cohomology groups:

Proposition 5.2.14 ([48]). *Let $\mathcal{E} = G \times^B E$ be the associated vector bundle on G/B. Then,*

$$\mathrm{Hom}_G(L(\lambda), \mathrm{H}^\bullet(G/B, \mathcal{E})) \cong \mathrm{H}^\bullet(\mathrm{BGG}(E, \lambda)).$$

For example, if $\mathfrak{g} = \mathfrak{sl}_3$, and $\lambda = n_1\alpha_1 + n_2\alpha_2$ is a dominant root weight, i.e. $n_1, n_2 \geq 0$, $2n_2 \geq n_1$ and $2n_1 \geq n_2$, then the corresponding BGG complex is:

$$\begin{tikzcd}
 & E\begin{bmatrix} -n_1+n_2-1 \\ n_2 \end{bmatrix} \arrow[r, "s_1s_2s_1"] \arrow[drr, "s_2"] & E\begin{bmatrix} -n_2-2 \\ n_1-n_2-1 \end{bmatrix} \arrow[dr, "s_1"] & \\
E\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \arrow[ur, "s_2"] \arrow[dr, "s_1"'] & & & E\begin{bmatrix} -n_2-2 \\ -n_1-2 \end{bmatrix} \\
 & E\begin{bmatrix} n_1 \\ n_1-n_2-1 \end{bmatrix} \arrow[r, "s_2s_1s_2"] \arrow[urr, "s_1"] & E\begin{bmatrix} -n_1+n_2-1 \\ -n_1-2 \end{bmatrix} \arrow[ur, "s_2"'] &
\end{tikzcd}$$

Here $E\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ denotes the weight space of weight $a_1\alpha_1 + a_2\alpha_2$.

This for example implies that the map $E[\lambda] \to E[s_1 \cdot \lambda]$ is given by multiplication by $f_1^{2n_2-n_1+1}$, since it has to be $\mathfrak{h}$-equivariant. Up to symmetry, the only non-trivial map is the map $E[s_1 \cdot \lambda] \to E[s_1 s_2 \cdot \lambda]$. Let us compute this map as an example. We know that this map is given by multiplication by an element $f_{1\to 12} \in U(\mathfrak{n})$ of weight $(n_2+1)\alpha_1 + (2n_2 - n_1 + 1)\alpha_2$, and which satisfies the equation

$$f_{1\to 12} f_1^{2n_1-n_2+1} = f_1^{n_1+n_2+2} f_2^{-n_1+2n_2+1}.$$

If we assume $m \geq n$, we then have

$$f_1^m f_2^n = \sum_{0\leq r\leq n} \binom{n}{r} \left(\prod_{j=0}^{r-1} (m-j) \right) f_{12}^r f_2^{n-r} f_1^{m-r}.$$

It follows that $f_1^m f_2^n$ is right divisible by f_1^a if and only if $m \geq a + n$. This is the case for the operator $f_{1\to 12}$ in our BGG complex, hence we obtain:

Proposition 5.2.15. *The BGG operator $f_{1\to 12}$ is given by*

$$f_{1\to 12} = \sum_{0\leq r\leq -n_1+2n_2+1} \binom{-n_1+2n_2+1}{r} \left(\prod_{j=0}^{r-1} (n_1+n_2+2-j) \right) f_{12}^r f_2^{-n_1+2n_2+1-r} f_1^{-n_1+2n_2+1-r}.$$

These formula will be used later in calculations.

We will use the BGG complex in the next section to compute Hochschild cohomology of the block $\mathfrak{u}_1(\mathfrak{sl}_3)$. Let us mention that a parabolic version of the BGG resolution exists (cf. [35, §9.16]), but implementing a parabolic version of the sheaf-cohomology BGG algorithm is not easy. However, since we are only interested in computing sheaf cohomology, we can simply consider a $\mathfrak{p}$-module E as a $\mathfrak{b}$-module by restriction and compute its cohomology as $\mathfrak{b}$-module. This gives the correct result because the spectral sequence associated to the projection $G/B \to G/P$ degenerates. Finally let us present some concrete computations with the BGG resolution:

Below we present two examples that can be computed by hand, using the algorithm previously described. The computations were done by hand and confirmed by the computer implementation of the BGG algorithm presented in [32]. Our first example is the cohomology of the flag variety G/B for G of type G_2. Unlike the traditional approach via the Chevalley-Eilenberg complex, the computation using the BGG resolution is straightforward. In our second example (which is more involved), we compute the Hochschild cohomology of the complete flag variety $X = G/B$ for $G = \mathsf{SL}_4$.

Cohomology of G/B, $G = G_2$

Let X be the complete flag variety of type $G = G_2$. We recall that any partial flag variety $X = G/P$ is a smooth projective algebraic variety, with Picard rank given by the number of simple roots corresponding to P. Hence here, our X has dimension 6 and Picard rank 2.

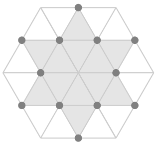


Figure 5.1: Root system of G_2.

We would like to compute $\mathrm{H}^q(X,\mathbb{C})$. By the Hodge decomposition and the fact that Schubert classes are algebraic, this group is isomorphic to $\mathrm{H}^q(X,\Omega^q_X)$. Moreover, thanks to Poincaré duality we only need to compute it for $q = 1, 2, 3$. Finally, it is a well-known fact that the $\mathfrak{g}$-structure on $\mathrm{H}^q(X,\Omega^q_X)$ is trivial, hence we can focus solely on the multiplicity of the trivial representation.

Let α_1 denote the short root and α_2 the long root. We pick a Chevalley basis of $\mathfrak{n}$ with elements $f_1, f_2, f_{12}, f_{112}, f_{1112}$ and f_{11122} where the subscript indicates the weight. The following diagram represents the dot-orbit of 0, indexed by the Weyl group of G_2. It shows which weights will appear in the BGG complex associated to $\lambda = 0$.

$$\begin{tikzcd}
 & -9\alpha_1-6\alpha_2 \arrow[r] \arrow[ddr] & -9\alpha_1-4\alpha_2 \arrow[r] \arrow[ddr] & -4\alpha_1-4\alpha_2 \arrow[r] \arrow[ddr] & -4\alpha_1-\alpha_2 \arrow[r] \arrow[ddr] & -\alpha_2 \arrow[dr] & \\
-10\alpha_1-6\alpha_2 \arrow[ur] \arrow[dr] & & & & & & 0 \\
 & -10\alpha_1-5\alpha_2 \arrow[r] \arrow[uur] & -6\alpha_1-5\alpha_2 \arrow[r] \arrow[uur] & -6\alpha_1-2\alpha_2 \arrow[r] \arrow[uur] & -\alpha_1-2\alpha_2 \arrow[r] \arrow[uur] & -\alpha_1 \arrow[ur] &
\end{tikzcd}$$

In order to compute $\mathrm{H}^1(X,\Omega^1_X)$ we use that $\Omega^1_X = G\times_B\mathfrak{n}$. Hence, by our previous discussion we need to compute $\mathrm{H}^1\,\mathrm{BGG}^\bullet(\mathfrak{n},0)$. We have $\mathrm{BGG}^2(\mathfrak{n},0) = \mathfrak{n}[-\alpha_1-2\alpha_2]\oplus\mathfrak{n}[-4\alpha_1-\alpha_2]$. However it's clear that these weight spaces are zero, and it's obvious that $\mathrm{BGG}^0(\mathfrak{n},0) = 0$. Hence we get $\mathrm{H}^1(X,\Omega_X) = L(0)^{\oplus 2}$, generated by f_1 and f_2. That was the expected answer since the Picard group of G_2/B is generated by the divisors associated to the line bundles $\mathscr{L}_{\alpha_1}$ and $\mathscr{L}_{\alpha_2}$. Let us compute $\mathrm{H}^2(X,\mathbb{C}) = \mathrm{H}^2(X,\Omega^2_X)$. This time our vector bundle is $G\times_B\wedge^2\mathfrak{n}$, so we would like to compute $\mathrm{H}^2\,\mathrm{BGG}^\bullet(\wedge^2\mathfrak{n},0)$. Clearly, we have $\wedge^2\mathfrak{n}[-\alpha_1] = \wedge^2\mathfrak{n}[-\alpha_2] = 0$ and $\wedge^2\mathfrak{n}[-6\alpha_1-2\alpha_2] = \wedge^2\mathfrak{n}[-4\alpha_1-4\alpha_2] = 0$ by direct inspection. So our cohomology is generated by $\wedge^2\mathfrak{n}[-4\alpha_1-\alpha_2]$ (generated by $f_1\wedge f_{1112}$) and $\wedge^2\mathfrak{n}[-\alpha_1-2\alpha_2]$ (generated by $f_1\wedge f_{12}$). As a last example, let us compute $\mathrm{H}^3(X,\Omega^3_X)$. Once more it is clear that $\wedge^3\mathfrak{n}[-9\alpha_1-4\alpha_2] = 0 = \wedge^3\mathfrak{n}[-6\alpha_1-5\alpha_2]$, and similarly $\wedge^3\mathfrak{n}[-\alpha_1-2\alpha_2] = 0 = \wedge^3\mathfrak{n}[-4\alpha_1-\alpha_2]$. It follows that $\mathrm{H}^3(X,\Omega^3_X)\cong\wedge^3\mathfrak{n}[-6\alpha_1-2\alpha_2]\oplus\wedge^3\mathfrak{n}[-4\alpha_1-4\alpha_2]$, generated by $f_1\wedge f_{12}\wedge f_{112}$ and $f_2\wedge f_{12}\wedge f_{11122}$ respectively. Other computations are similar. We present the final result in a table 5.1.

One can notice the Poincaré duality through the BGG complex. For example f_1 and $f_2\wedge f_{12}\wedge f_{112}\wedge f_{1112}\wedge f_{11122}$ are Poincaré dual to each other. This duality is specific to the case where $\lambda = 0$. In the next example, such duality does not appear.

Hochschild cohomology of $G/B, G = \mathsf{SL}_4$

Recall that the Hochschild cohomology of a smooth scheme X over a field is defined as $\mathrm{HH}^\bullet(X) := \mathrm{Ext}^\bullet_{X\times X}(\mathcal{O}_\Delta,\mathcal{O}_\Delta)$, where $\Delta\subset X\times X$ is the diagonal. By the Hochschild-Kostant-Rosenberg theorem

q	$\dim \mathrm{H}^q(X, \Omega_X^q)$	Explicit BGG generators
0	1	1
1	2	$f_1,\ f_2$
2	2	$f_1 \wedge f_{1112},\ f_2 \wedge f_{12}$
3	2	$f_1 \wedge f_{112} \wedge f_{1112},\ f_2 \wedge f_{12} \wedge f_{11122}$
4	2	$f_2 \wedge f_{12} \wedge f_{112} \wedge f_{11122}$ $f_1 \wedge f_{112} \wedge f_{1112} \wedge f_{11122}$
5	2	$f_2 \wedge f_{12} \wedge f_{112} \wedge f_{1112} \wedge f_{11122}$ $f_1 \wedge f_{12} \wedge f_{112} \wedge f_{1112} \wedge f_{11122}$
6	1	$f_1 \wedge f_2 \wedge f_{12} \wedge f_{112} \wedge f_{1112} \wedge f_{11122}$

Table 5.1: Cohomology of G/B for $G = G_2$

there is an isomorphism $\mathrm{HH}^\bullet(X) \cong \bigoplus_{i+j=\bullet} \mathrm{H}^i(X, \wedge^j \mathrm{T}_X)$.

In this paragraph we compute the Hochschild cohomology of G/B for G of type A_3. It is a classical result (see [10]) that $\mathrm{H}^0(G/P, T_{G/P}) = \mathfrak{g}$ if G is of type A. Hence we just need to compute $\mathrm{H}^0(X, \wedge^j \mathrm{T}_X)$ for $j = 2, 3, 4, 5, 6$. We have $\mathrm{T}_X = \mathsf{SL}_4 \times_B \mathfrak{u}$ where $\mathfrak{u} \cong \mathfrak{g}/\mathfrak{b}$ is the Lie subalgebra generated by e_1, e_2, e_3. In order to not duplicate computations we will use the symmetry exchanging α_1 and α_2. Below we list the cohomology $\mathrm{H}^0(X, \wedge^j \mathrm{T}_X)$ for each j.

- $\mathrm{H}^0(X, \mathcal{O}_X) = \mathbb{C}$
- $\mathrm{H}^0(X, \mathrm{T}_X) = L(\alpha_1 + \alpha_2 + \alpha_3)$
- $\mathrm{H}^0(X, \wedge^2 \mathrm{T}_X) = L(\alpha_1 + \alpha_2 + \alpha_3) \oplus L(2\alpha_1 + 2\alpha_2 + \alpha_3) \oplus L(\alpha_1 + 2\alpha_2 + 2\alpha_3)$
- $\mathrm{H}^0(X, \wedge^3 \mathrm{T}_X) = L(\alpha_1 + \alpha_2 + \alpha_3) \oplus L(2\alpha_1 + 2\alpha_2 + \alpha_3) \oplus L(\alpha_1 + 2\alpha_2 + 2\alpha_3) \oplus L(2\alpha_1 + 2\alpha_2 + 2\alpha_3) \oplus L(\alpha_1 + 2\alpha_2 + \alpha_3)^{\oplus 2} \oplus L(3\alpha_1 + 2\alpha_2 + \alpha_3) \oplus L(\alpha_1 + 2\alpha_2 + 3\alpha_3) \oplus L(2\alpha_1 + 3\alpha_2 + 2\alpha_3)$
- $\mathrm{H}^0(X, \wedge^4 \mathrm{T}_X) = L(2\alpha_1 + 2\alpha_2 + \alpha_3) \oplus L(\alpha_1 + 2\alpha_2 + 2\alpha_3) \oplus L(2\alpha_1 + 2\alpha_2 + 2\alpha_3)^{\oplus 2} \oplus L(2\alpha_1 + 3\alpha_2 + 2\alpha_3)^{\oplus 2} \oplus L(3\alpha_1 + 3\alpha_2 + 2\alpha_3) \oplus L(2\alpha_1 + 3\alpha_2 + 3\alpha_3) \oplus L(2\alpha_1 + 4\alpha_2 + 2\alpha_3)$
- $\mathrm{H}^0(X, \wedge^5 \mathrm{T}_X) = L(2\alpha_1 + 3\alpha_2 + 2\alpha_3) \oplus L(3\alpha_1 + 3\alpha_2 + 2\alpha_3) \oplus L(2\alpha_1 + 3\alpha_2 + 3\alpha_3) \oplus L(3\alpha_1 + 3\alpha_2 + 3\alpha_3) \oplus L(3\alpha_1 + 4\alpha_2 + 2\alpha_3) \oplus L(2\alpha_1 + 4\alpha_2 + 3\alpha_3)$
- $\mathrm{H}^0(X, \wedge^6 \mathrm{T}_X) = L(3\alpha_1 + 4\alpha_2 + 3\alpha_3)$

We consider the computation for $\mathrm{H}^0(X, \wedge^2 \mathrm{T}_X)$ in some more detail. We fix the notation and take the following basis of $\mathfrak{h}$: $h_1 = \mathrm{diag}(1, -1, 0, 0)$, $h_2 = \mathrm{diag}(1, 0, -1, 0)$ and $h_3 = \mathrm{diag}(1, 0, 0, -1)$. For $i = 1, 2, 3$ we take $f_i, e_i \in \mathfrak{sl}_4$ so that f_i is lower-triangular and each (e_i, f_i, h_i) is a $\mathfrak{sl}_2$-triple. We define $f_{12} := [f_1, f_2], f_{23} := [f_2, f_3]$ and $f_{123} := [f_1, f_{23}] = [f_{12}, f_3]$ and $e_{12} = [e_2, e_1], e_{23} = [e_3, e_2]$ and $e_{123} = [e_3, e_{12}]$. Let us compute the coadjoint action of $\mathfrak{b}$ on $\mathfrak{u}$. Note that the $\mathfrak{h}$-action is the same as the adjoint action. Here we list the non-zero coadjoint actions of $\mathfrak{n}$ on $\mathfrak{u}$:

	e_{12}	e_{23}	e_{123}
f_1	e_2	0	e_{23}
f_2	$-e_1$	e_3	0
f_3	0	$-e_2$	$-e_{12}$
f_{12}	0	0	e_3

Since $\mathfrak{u}$ is nilpotent there is a natural filtration on it, and each composition factor splits as a direct sum of 1-dimensional weight spaces. Geometrically it means that the vector bundle T_X has a filtration with direct sums of line bundles as composition factors. For example, in our case the composition factors for T_X are $\mathscr{L}_{\alpha_1} \oplus \mathscr{L}_{\alpha_2} \oplus \mathscr{L}_{\alpha_3}, \mathscr{L}_{\alpha_1+\alpha_2} \oplus \mathscr{L}_{\alpha_2+\alpha_3}$ and $\mathscr{L}_{\alpha_1+\alpha_2+\alpha_3}$, where for $\mu \in P$, $\mathscr{L}_\mu$ is the line bundle $G \times_B \mathbb{C}_\mu$. By considering the associated long exact sequences it is clear that there is a surjection $\bigoplus_{\mu \in \mathtt{wt}(\wedge^2 \mathfrak{u})} \mathrm{H}^0(X, \mathscr{L}_\mu) \to \mathrm{H}^0(X, \wedge^2 \mathrm{T}_X)$. However, since line bundles might have higher cohomology, there could be some cancellation in the long exact sequence. Using the Borel-Weil theorem it is at least clear that dot-singular weights will not contribute, hence we can restrict ourselves to line bundles so that the corresponding weight is dot-regular. For each such weight λ, the cancellation will come from a dot-regular, non-dominant weight μ in the same dot-orbit as λ. Hence, if there is no such μ we know that the $L(\lambda)$-isotypical component of $\mathrm{H}^0(X, \wedge^2 \mathrm{T}_X)$ is given by the λ-weight space of $\wedge^2 \mathfrak{n}$. In most cases we don't know how to compute the maps in the long exact sequence explicitly. However, we can still compute the multiplicity of $L(\lambda)$ in $\mathrm{H}^0(X, \wedge^2 \mathrm{T}_X)$ using the BGG complex.

The dominant, dot-regular weights appearing in $\wedge^2 \mathfrak{u}$ are $\alpha_1+\alpha_2+\alpha_3$ with multiplicity 2, $\alpha_1+2\alpha_2+\alpha_3$ with multiplicity 2, $2\alpha_1+2\alpha_2+\alpha_3$ and $\alpha_1+2\alpha_2+2\alpha_3$ with multiplicity 1. The dot-regular weights that are non-dominant are $\alpha_1+\alpha_3$, $\alpha_1+2\alpha_2$ and $2\alpha_2+\alpha_3$. We have $s_2 \cdot (\alpha_1+\alpha_3) = \alpha_1+\alpha_2+\alpha_3$ and $s_3 \cdot (\alpha_1+2\alpha_2) = \alpha_1+2\alpha_2+\alpha_3$. This means that these two dominant weights might appear in the cohomology.

To compute the multiplicity of $L(\alpha_1+\alpha_2+\alpha_3)$ in $\mathrm{H}^1(X, \wedge^2 \mathrm{T}_X)$, we use the BGG complex $\mathrm{BGG}^\bullet(\wedge^2 \mathfrak{u}, \alpha_1+\alpha_2+\alpha_3)$. Since $s_1 \cdot (\alpha_1+\alpha_2+\alpha_3) = -\alpha_1+\alpha_2+\alpha_3$ and $s_2 \cdot (\alpha_1+\alpha_2+\alpha_3) = \alpha_1+\alpha_3$. We see that $\mathrm{BGG}^1(\wedge^2 \mathfrak{u}, \alpha_1+\alpha_2+\alpha_3) = \mathbb{C}\{e_1 \wedge e_3\}$. Clearly $\mathrm{BGG}^0(\wedge^2 \mathfrak{u}, \alpha_1+\alpha_2+\alpha_3)$ is spanned by $e_1 \wedge e_{23}$ and $e_{12} \wedge e_3$. The differential given by the coadjoint action is $x \mapsto f_2 x$ and it is clearly surjective, hence $\mathrm{Hom}(L(\alpha_1+\alpha_2+\alpha_3), \mathrm{H}^1(X, \wedge^2 \mathrm{T}_X)) = 0$ and $\mathrm{Hom}(L(\alpha_1+\alpha_2+\alpha_3), \mathrm{H}^0(X, \wedge^2 \mathrm{T}_X)) = \mathbb{C}$.

To compute $\mathrm{Hom}(L(\alpha_1+2\alpha_2+\alpha_3), \mathrm{H}^1(X, \wedge^2 \mathrm{T}_X))$ we also use the BGG complex:

$$\wedge^2 \mathfrak{u}[\alpha_1+2\alpha_2+\alpha_3] \to \wedge^2 \mathfrak{u}[2\alpha_2+\alpha_3] \oplus \wedge^2 \mathfrak{u}[\alpha_1+2\alpha_2] \to 0 \tag{5.2}$$

A basis of $\wedge^2 \mathfrak{u}[\alpha_1+2\alpha_2+\alpha_3]$ is given by $e_{12} \wedge e_{23}$ and $e_2 \wedge e_{123}$ and $\wedge^2 \mathfrak{u}[2\alpha_2+\alpha_3] \oplus \wedge^2 \mathfrak{u}[\alpha_1+2\alpha_2]$ has basis spanned by $e_2 \wedge e_{12}$ and $e_2 \wedge e_{23}$. The differential is

$$d(e_2 \wedge e_{123}) = e_2 \wedge e_{23} - e_2 \wedge e_{12}$$
$$d(e_2 \wedge e_{123}) = e_2 \wedge e_{23} + e_2 \wedge e_{12}$$

meaning that d is surjective so again there is no cohomology. We get

$$\mathrm{Hom}_G(L_{\alpha_1+2\alpha_2+\alpha_3}, \mathrm{H}^1(X, \wedge^2 \mathrm{T}_X)) = 0, \text{ and } \mathrm{Hom}_G(L_{\alpha_1+2\alpha_2+\alpha_3}, \mathrm{H}^0(X, \wedge^2 \mathrm{T}_X)) = \mathbb{C}.$$

To summarize, $\mathrm{H}^\bullet(X, \wedge^2 \mathrm{T}_X)$ is concentrated in degree 0, and

$$\mathrm{H}^0(X, \wedge^2 \mathrm{T}_X) \cong L(\alpha_1+\alpha_2+\alpha_3) \oplus L(2(\alpha_1+\alpha_2)+\alpha_3) \oplus L(\alpha_1+2(\alpha_2+\alpha_3)). \tag{5.3}$$

5.2.2 An improvement of the BGG sheaf-cohomology algorithm using the modules $V_{j,k}$

To compute $\mathrm{HH}^s_{\mathbb{C}^*}(\widetilde{\mathcal{N}})$ using (5.1) we need a description of the modules $V_{j,k}$. Let us recall the construction of $V_{j,k}$ from [48] and [49]. Let P be a standard parabolic subgroup, $\mathfrak{p} = \mathrm{Lie}(P)$, $\mathrm{pr} : \widetilde{\mathcal{N}}_P = T^*G/P \to G/P$ be the projection, $\mathfrak{n}_\mathfrak{p}$ the nilradical of P, and $\mathfrak{u}_\mathfrak{p}$ its dual seen as a subset of $\mathfrak{g}$ using the Killing form. Recall that $\mathfrak{p}$ acts on its nilradical by the adjoint action, and on $\mathfrak{u}_\mathfrak{p}$ by the coadjoint action. We hence have a map $\mathrm{ad}\colon \mathfrak{p} \to \mathrm{End}(\mathfrak{n}_\mathfrak{p}) \cong \mathfrak{n}_\mathfrak{p} \otimes \mathfrak{u}_\mathfrak{p}$. Let $\Delta = (\iota, \mathrm{ad})\colon \mathfrak{p} \to \mathfrak{g} \oplus \mathfrak{n}_\mathfrak{p} \otimes \mathfrak{u}_\mathfrak{p}$, where ι is the inclusion. We also write Δ for the induced map of $\mathrm{Sym}^\bullet(\mathfrak{u}_\mathfrak{p})$-modules

$$\Delta\colon \mathrm{Sym}^\bullet(\mathfrak{u}_\mathfrak{p}) \otimes \mathfrak{p} \to \mathrm{Sym}^\bullet(\mathfrak{u}_\mathfrak{p}) \otimes (\mathfrak{g} \oplus \mathfrak{n}_\mathfrak{p} \otimes \mathfrak{u}_\mathfrak{p}). \tag{5.4}$$

Here the $\mathfrak{p}$-action on $\mathrm{Sym}^\bullet(\mathfrak{u}_\mathfrak{p})$ is induced from the coadjoint action, and the action on $\mathfrak{g}$ and $\mathfrak{n}_\mathfrak{p}$ is the adjoint action.

Proposition 5.2.16 ([49]). *Let* $\mathrm{pr}_* T_{\widetilde{\mathcal{N}}_P} = G \times^P V_1$.*, then we have an isomorphism of* $\mathfrak{p}$*-modules*

$$V_1 \cong \frac{\mathrm{Sym}^\bullet(\mathfrak{u}_\mathfrak{p}) \otimes \mathfrak{g} \oplus \mathrm{Sym}^\bullet(\mathfrak{u}_\mathfrak{p}) \otimes \mathfrak{n}_\mathfrak{p}}{\mathrm{Sym}^\bullet(\mathfrak{u}_\mathfrak{p}) \otimes \Delta(\mathfrak{p})}. \tag{5.5}$$

All exterior powers of $\mathrm{pr}_* T_{\widetilde{\mathcal{N}}_P}$ are described in [49] as well. For example, if $V_2 \colon = \wedge^2_{\mathrm{Sym}^\bullet(\mathfrak{u}_\mathfrak{p})} V_1$ then

$$V_2 \cong \frac{\mathrm{Sym}^\bullet(\mathfrak{u}_\mathfrak{p}) \otimes (\wedge^2 \mathfrak{g} \oplus \mathfrak{g} \otimes \mathfrak{n}_\mathfrak{p} \oplus \wedge^2 \mathfrak{n}_\mathfrak{p})}{\Delta(\mathrm{Sym}^\bullet(\mathfrak{u}_\mathfrak{p}) \otimes \mathfrak{p}) \wedge (\mathrm{Sym}^\bullet(\mathfrak{u}_\mathfrak{p}) \otimes (\mathfrak{g} \oplus \mathfrak{n}_\mathfrak{p}))}.$$

We now recall that in order to apply this to studying the small quantum group, we need a $\mathbb{C}^*$-grading (which we will call the *k-grading*).

Definition 5.2.17. *The* k-grading *on* $\mathrm{Sym}^\bullet(\mathfrak{u}_\mathfrak{p}) \otimes \mathfrak{g} \oplus \mathrm{Sym}^\bullet(\mathfrak{u}_\mathfrak{p}) \otimes \mathfrak{n}_\mathfrak{p}$ *is defined by* $\deg(\mathfrak{n}_p) = -2$, $\deg(\mathrm{Sym}^m(\mathfrak{u}_\mathfrak{p})) = 2m$ *and* $\deg(\mathfrak{g}) = 0$. *We write* $V_{j,k}$ *for the k-th graded part of* $\wedge^j_{\mathrm{Sym}^\bullet(\mathfrak{u}_\mathfrak{p})}(V_1)$ *(for the natural induced grading). Note that k has to be even.*

We can give an alternative description of the $\mathfrak{p}$-modules $\wedge^j_{\mathrm{Sym}^\bullet(\mathfrak{u}_\mathfrak{p})} V_1$ and the graded pieces $V_{j,k}$ that is more convenient algorithmically. This description essentially follow from the short exact sequence of vector bundles (recall that Ω_X is the cotangent sheaf of X):

$$0 \to \mathrm{pr}_* \mathcal{O}_{\widetilde{\mathcal{N}}_P} \otimes \Omega_{G/P} \to \mathrm{pr}_* T(\widetilde{\mathcal{N}}_P) \to \mathrm{pr}_* \mathcal{O}_{\widetilde{\mathcal{N}}_P} \otimes T_{G/P} \to 0$$

Lemma 5.2.18 ([48]). *Let* $S_\mathfrak{p} := S^\bullet(\mathfrak{u}_\mathfrak{p})$. *There is a vector space isomorphism* $\psi\colon S_\mathfrak{p} \otimes (\mathfrak{u}_\mathfrak{p} \oplus \mathfrak{n}_\mathfrak{p}) \to V_1$.

Proof. Note that the map

$$\widetilde{\psi} = (\iota, \mathrm{id})\colon S_\mathfrak{p} \otimes \mathfrak{u}_\mathfrak{p} \oplus S_\mathfrak{p} \otimes \mathfrak{n}_\mathfrak{p} \to S_\mathfrak{p} \otimes \mathfrak{g} \oplus S_\mathfrak{p} \otimes \mathfrak{n}_\mathfrak{p}$$

induces a map $\psi\colon S_\mathfrak{p} \otimes (\mathfrak{u}_\mathfrak{p} \oplus \mathfrak{n}_\mathfrak{p}) \to V_1$. To show that ψ is surjective, let $p \otimes x + q \otimes y \in V_1$ where $p, q \in S_\mathfrak{p}$ and $x \in \mathfrak{g}, y \in \mathfrak{n}_\mathfrak{p}$. Let us write $x = u + w$ where $u \in \mathfrak{u}_\mathfrak{p}$ and $w \in \mathfrak{p}$. Using the relations in V_1 we get $w = \sum_i p_i \otimes w_i$ where $w_i \in \mathfrak{n}_\mathfrak{p}$, giving $p \otimes w = \sum p p_i \otimes w_i$. Hence $p \otimes x + q \otimes y = p \otimes u +$

$\sum_i pp_i \otimes w_i + q \otimes y$ is in the image of ψ. To prove injectivity of ψ, suppose that $\psi(p \otimes u + q \otimes y) = 0$. This means that $\tilde{\psi}(p \otimes u + q \otimes y) \in S_{\mathfrak{p}} \otimes \Delta(\mathfrak{p})$, hence there are polynomials $r_i \in S_{\mathfrak{p}}$ and elements $x_i \in \mathfrak{p}$ (we can assume that x_i are linearly independent) such that $p \otimes u + q \otimes y + \sum r_i \otimes (x_i + \mathrm{ad}(x_i)) = 0$. Now it is easy to see that we should have $r_i = 0$, and it follows that $p = q = 0$ since $u \in \mathfrak{u}_{\mathfrak{p}}$ and $y \in \mathfrak{n}_{\mathfrak{p}}$. □

Corollary 5.2.19. *For any $j \geq 0$, the map $\wedge^j \psi \colon \wedge^j_{S_{\mathfrak{p}}} (S_{\mathfrak{p}} \otimes (\mathfrak{u}_{\mathfrak{p}} \oplus \mathfrak{n}_{\mathfrak{p}})) \to \wedge^j_{S_{\mathfrak{p}}} V_1$ is an isomorphism of vector spaces.*

This isomorphism preserves the k-grading. Hence we obtain:

Corollary 5.2.20. *There is a $\mathfrak{h}$-equivariant, k-graded vector space isomorphism*

$$V_{j,k} \cong \bigoplus_{r_2+r_3=j, r_1-r_3=k/2} S^{r_1}(\mathfrak{u}_{\mathfrak{p}}) \otimes \bigwedge^{r_2} \mathfrak{u}_{\mathfrak{p}} \otimes \bigwedge^{r_3} \mathfrak{n}_{\mathfrak{p}}$$

The isomorphism is also compatible with the $\mathfrak{p}$-structure in the following sense:

Lemma 5.2.21. *There is a $\mathfrak{p}$-module structure on $\wedge^j_{S_{\mathfrak{p}}} (S_{\mathfrak{p}} \otimes (\mathfrak{u}_{\mathfrak{p}} \oplus \mathfrak{n}_{\mathfrak{p}}))$ such that $\wedge^j \psi$ is $\mathfrak{p}$-equivariant.*

Proof. It is sufficient to prove the case $j = 1$. For $u \in \mathfrak{u}_{\mathfrak{p}}$ and $x \in \mathfrak{p}$, write $[x, u] = u_1 + x_1$ where $u_1 \in \mathfrak{u}_{\mathfrak{p}}$ and $x_1 \in \mathfrak{p}$. We then define a $\mathfrak{p}$-module structure on $S_{\mathfrak{p}} \otimes (\mathfrak{u}_{\mathfrak{p}} \oplus \mathfrak{n}_{\mathfrak{p}})$ by

$$x \cdot (u, x') = (u_1, -\mathrm{ad}(x_1) + [x, x']).$$

This induces an obvious $\mathfrak{p}$-module structure on $S_{\mathfrak{p}} \otimes (\mathfrak{u}_{\mathfrak{p}} \oplus \mathfrak{n}_{\mathfrak{p}})$ and on $\wedge^j_{S_{\mathfrak{p}}} (S_{\mathfrak{p}} \otimes (\mathfrak{u}_{\mathfrak{p}} \oplus \mathfrak{n}_{\mathfrak{p}}))$. By construction ψ (resp. $\wedge^j \psi$) is $\mathfrak{p}$-equivariant. □

Example. We take $\mathfrak{g} = \mathfrak{sl}_3$ with the usual Chevalley generators e_i, h_i, f_i. Pick the vector $e_2^2 \otimes e_1 \otimes f_1 \wedge f_2 \in S^2(\mathfrak{u}) \otimes \mathfrak{u} \otimes \wedge^2 \mathfrak{n}$. We have $f_1 \cdot e_2 = 0$, $f_1 \cdot f_1 = 0$ and $f_1 \cdot f_2 = f_{12}$. We also have $[f_1, e_1] = -h_1$ and $\mathrm{ad}_{\mathfrak{n}}(h_1) = -2e_1 \otimes f_1 + e_2 \otimes f_2 - e_{12} \otimes f_{12}$. Hence we get

$$f_1 \cdot (e_2^2 \otimes e_1 \otimes f_1 \wedge f_2) = e_2^2 e_{12} \otimes f_1 \wedge f_2 \wedge f_{12} + e_2^2 \otimes e_1 \otimes f_1 \wedge f_{12}.$$

5.2.3 The improved sheaf-cohomology BGG algorithm

The presentation of $V_{j,k}$ of corollary 5.2.20 is very helpful for computing the cohomology of the BGG resolution by hand. However, the resulting $\mathfrak{p}$-module structure on $V_{j,k}$ is difficult to implement algorithmically. Instead, we will consider $V_{j,k}$ as cokernel of the map between spaces with simple $\mathfrak{p}$-module structure. We define

$$M_j = \bigoplus_{r=0}^{j} \mathrm{Sym}^{\bullet}(\mathfrak{u}_{\mathfrak{p}}) \otimes \bigwedge^{r} \mathfrak{g} \otimes \bigwedge^{j-r} \mathfrak{n}_{\mathfrak{p}}.$$

Note that the map Δ defined in (5.4) extends to a map

$$\Delta \colon M_{j-1} \otimes \mathfrak{p} \to M_{j-1} \otimes (\mathfrak{g} \oplus \mathfrak{n}_{\mathfrak{p}} \otimes \mathfrak{u}_{\mathfrak{p}}) \subset M_j.$$

Note that now per definition $V_j := M_j/\Delta(M_{j-1} \otimes \mathfrak{p}) = \operatorname{coker}\Delta$. If we define the k-grading of $\mathfrak{p}$ by 0, then Δ respects the k-grading and thus

$$V_{j,k} = \operatorname{coker}\left(\Delta\colon M_{j-1,k} \otimes \mathfrak{p} \to M_{j,k}\right).$$

The map Δ therefore induces a short exact sequence of $\mathfrak{p}$-modules

$$\begin{tikzcd} 0 \arrow[r] & M_{j-1,k} \otimes \mathfrak{p} \arrow[r, "\Delta"] & M_{j,k} \arrow[r, "\varpi"] & V_{j,k} \arrow[r] & 0. \end{tikzcd}$$

Note that the $U(\mathfrak{n})$ action on $V_{j,k}$ is defined in terms of that on $M_{j,k}$; suppose $\mathcal{F} \in U(\mathfrak{n})$, and $X \in M_{j,k}$, then we have that $\mathcal{F} \cdot \varpi(X) = \varpi(\mathcal{F} \cdot X)$. Now let us choose a linear section σ of ϖ (respecting the $\mathfrak{h}$-grading). Then if $Y \in V_{j,k}$ we have that $\mathcal{F} \cdot Y = \varpi(\mathcal{F} \cdot \sigma(Y))$. Since the differential in the BGG complex is given by multiplication by elements of $U(\mathfrak{n})$ on weight components of $V_{j,k}$ and $M_{j,k}$, we have the equality $d_V = \varpi d_M \sigma$, with d_V and d_M the respective differentials on $\mathrm{BGG}^\bullet(V_{j,k}, \lambda)$ and $\mathrm{BGG}^\bullet(M_{j,k}, \lambda)$.

We will describe how one can find the map ϖ and the associated σ. We will always use a basis of $M_{j,k}$ and $T_{j,k}$ induced from a Chevalley basis on $\mathfrak{g}$. In practice we realize $V_{j,k}$ as $\operatorname{coker}\Delta = \ker \Delta^\top$, where the transposition is with respect to the Chevalley basis. For practical reasons we choose a basis of $\operatorname{coker}\Delta$ with integer coefficients to describe the map ϖ. Furthermore instead of using a true differential we take,

$$\widetilde{d}_V := \varpi d_M \varpi^\top. \tag{5.6}$$

A priori the rows of ϖ are just a basis, and we thus have that $\varpi\varpi^\top \neq \mathrm{Id}$. Therefore $\widetilde{d}_V^2 \neq 0$, and $\widetilde{d}_V$ is hence not a differential. We could choose ϖ such that $\varpi\varpi^\top = \mathrm{Id}$ but this is in general impossible with integer entries, and by using integer entries we get a big computational benefit for exact linear algebraic operations. Nevertheless we do have that

$$\dim\ker \widetilde{d}_V - \dim\operatorname{im} \widetilde{d}_V = \dim\ker d_V - \dim\operatorname{im} d_V,$$

and therefore $\widetilde{d}_V$ can be used to compute the cohomology of $\mathrm{BGG}^\bullet(V_{j,k}, \lambda)$. This is because $\varpi^\top$ can be transformed to a section of ϖ by the isomorphism $(\varpi\varpi^\top)^{-1}$. The images of $\widetilde{d}_V$ and d_V are hence identical, while the kernels have the same dimension.

The cokernel of Δ can be efficiently computed using the isomorphism ψ of lemma 5.2.18. The fact that ψ is an isomorphism gives us a convenient subspace $\operatorname{im}\psi \subset M_{j,k}$ such that $\varpi|_{\operatorname{im}\psi}$ is an isomorphism, but this is not yet a full description of $\ker\Delta^\top$. The subspace $\operatorname{im}\psi$ has a basis $\{x_i\}_{i\in I}$, and we can define $\varpi(x_i) = e_i$ with e_i an elementary basis vector. Each element $i \in I$ then corresponds to a row of $\Delta^\top$. To obtain a full description of ϖ we simply need to write the remaining rows of $\Delta^\top$ as a linear combination of the columns corresponding to the indices in I, i.e. if $j \notin I$ then we write

$$\Delta_j^\top = \sum_{i\in I} c_i \Delta_i, \qquad \varpi(x_j) := \sum_{i\in I} c_i \varpi(x_i) \tag{5.7}$$

Thus to compute ϖ we need to solve a linear system, which is much cheaper in integer arithmetic than directly computing the kernel of $\Delta^\top$. In summary:

Proposition 5.2.22. *To compute the BGG cohomology of $V_{j,k}$ we use*

$$\dim \mathrm{H}^i(\mathrm{BGG}^\bullet(V_{j,k})) = \dim\ker \varpi d_{M^i_{jk}} \varpi^\top - \dim \mathrm{im}\, \varpi d^{i-1}_{M_{j,k}} \varpi^\top$$

with ϖ as in (5.7).

5.3 Equivariant Hochschild cohomology of the non-trivial singular block of $\mathfrak{u}_q(\mathfrak{sl}_3)$

In this section we will compute the equivariant Hochschild cohomology of the non-trivial singular block of $\mathfrak{u}_1(\mathfrak{sl}_3)$. This corresponds geometrically to the equivariant Hochschild cohomology of $T^*\mathbb{P}^2$. A good understanding of these Hochschild cohomology groups is highly desirable since it could lead to an explicit description of the geometric cup-product and Gerstenhaber bracket. The Hochschild cohomology $\mathrm{HH}^\bullet_{\mathbb{C}^*}(\mathbb{P}^1)$ is already described in [50] (we will recall their results later), and $\mathbb{P}^2$ is the next non-trivial case.

Let us fix the notation for this section: we take $\mathfrak{g} = \mathfrak{sl}_3$, $\mathfrak{p} = \mathfrak{b} \oplus \mathbb{C}\{e_1\}$, and P a parabolic subgroup such that $\mathsf{SL}_3/P \cong \mathbb{P}^2$. As a result $\mathfrak{n}_\mathfrak{p} = \mathbb{C}f_2 \oplus \mathbb{C}f_{12}$ and $\mathfrak{u}_\mathfrak{p} \cong \mathfrak{g}/\mathfrak{p} \cong \mathbb{C}e_2 \oplus \mathbb{C}e_{12}$. We use the same convention as [48], that is, $e_1, e_2, f_1, f_2, h_1, h_2$ are the usual Chevalley generators, and we define $f_{12} := [f_1, f_2]$ and $e_{12} := [e_2, e_1]$. Furthermore for a root weight $\lambda = n_1\alpha_1 + n_2\alpha_2$, let L_{n_1,n_2} be the irreducible $\mathfrak{sl}_3$-representation with highest weight λ.

We now state an elementary but useful lemma about root weights of $\mathfrak{sl}_3$:

Lemma 5.3.1. *Let $\lambda = n_1\alpha_1 + n_2\alpha_2$ be a root weight, and assume $n_2 \geq n_1 \geq 0$. Then exactly one of the three cases occur:*

- *λ is dominant.*
- *$s_1 \cdot \lambda = (n_2 - n_1 - 1)\alpha_1 + n_2\alpha_2$ is dominant.*
- *λ is dot-singular. (In this case, $2n_1 = n_2 - 1$.)*

In particular, the cohomology of the corresponding line bundles is concentrated in degree 0 *and* 1.

Proof. If λ is not dominant then $n_2 \geq 2n_1 + 1$. We then obtain $s_1 \cdot \lambda = (n_2 - n_1 - 1)\alpha_1 + n_2\alpha_2$. This is dominant if $2(n_2 - n_1 - 1) \geq n_2$, i.e, if $n_2 \geq 2n_1 + 2$. Hence, if λ and $s_1\lambda$ are not dominant, we should have $n_2 = 2n_1 + 1$, which implies that $s_1 \cdot \lambda = \lambda$. □

5.3.1 Description of $\mathrm{HH}^\bullet_{\mathbb{C}^*}(T^*\mathbb{P}^2)$ as $\mathfrak{g}$-module

The purpose of this subsection is to prove the following theorem:

Theorem 5.3.2. *Let $\mathfrak{u}_{\lambda_\mathfrak{p}}(\mathfrak{sl}_3)$ be the block of the small quantum group corresponding to $\mathfrak{p}$. For $s \geq 2$, $\mathrm{HH}^s(\mathfrak{u}_{\lambda_\mathfrak{p}}(\mathfrak{sl}_3))$ is given by the following tables:*

For $s = 2m+1$ odd:

$i+j=1$	$L_{m,m} \oplus L_{m+1,m} \oplus L_{m,m+1} \oplus L_{m+1,m+1}$	
$i+j=3$	0	$L_{m,m} \oplus L_{m+1,m} \oplus L_{m,m+1} \oplus L_{m+1,m+1}$
$h^{i,j}$	$j-i=1$	$j-i=3$

For $s = 2m$ even:

$i+j=0$	$L_{m,m}$		
$i+j=2$	0	$L_{m,m-1} \oplus L_{m-1,m} \oplus L^2_{m,m} \oplus L_{m,m+1} \oplus L_{m+1,m}$	
$i+j=4$	0	0	$L_{m,m}$
$h^{i,j}$	$j-i=0$	$j-i=2$	$j-i=4$

For $s = 1$:

$i+j=1$	$L_{0,0} \oplus L_{1,1}$	
$i+j=3$	$L_{0,0}$	$L_{0,0} \oplus L_{1,1}$
$h^{i,j}$	$j-i=1$	$j-i=3$

For $s = 0$ this was computed more generally for all projective space in [49]. For $s = 1$ this is an easy computation, hence we assume now that $s \geq 2$ and divide the proof into a series of lemmas. Notice that using the $\mathfrak{sl}_2$-action, we only need to compute half of the table, as explained in section 2.

Lemma 5.3.3. *We have* $\mathrm{H}^0(\mathbb{P}^2, \mathrm{Sym}^m(T_{\mathbb{P}^2})) \cong L_{m,m}$.

Proof. We need to compute the BGG complex associated to $\mathrm{Sym}^m(\mathfrak{u}_{\mathfrak{p}})$. A basis of $\mathrm{Sym}^m(\mathfrak{u}_{\mathfrak{p}})$ is given by $\{e_2^{m-i} e_{12}^i\}_{0 \leq i \leq m}$. If $0 \leq a \leq m$ then, $e_1^{m-a} e_{12}^a$ and $e_1^{a+1} e_{12}^{m-a-1}$ have weights in the same dot-orbit. By lemma 5.3.1, one of the weights μ is dominant and the other weight μ' is such that $s_1 \cdot \mu'$ dominant (if they coincide, then by definition μ is dot-singular). It is easy to see that the corresponding map in the BGG complex is non-zero, hence the contributions cancel for $a = 0, \ldots, m-1$. For $a = m$ the BGG complex is $\mathbb{C} \to 0$, giving the result. □

Corollary 5.3.4 ($i = 0$, $j = 0$)**.** *There is an isomorphism of G-modules* $\mathrm{H}^0(\widetilde{\mathcal{N}}_P, \mathcal{O}_{\widetilde{\mathcal{N}}_P}) \cong \bigoplus_{m \geq 0} L_{m,m}$. *If $k = 2m$, the k-th graded part corresponds to $L_{m,m}$.*

Proof. For a vector bundle $\mathrm{pr}\colon E \to X$ we have $\mathrm{pr}_* \mathcal{O}_E \cong \mathrm{Sym}^\bullet(E^\vee)$. Hence,

$$\mathrm{H}^0(E, \mathcal{O}_E) \cong \mathrm{H}^0(X, \mathrm{Sym}^\bullet(E^\vee)),$$

and thus

$$\mathrm{H}^0(\widetilde{\mathcal{N}}_P, \mathcal{O}_{\widetilde{\mathcal{N}}_P}) = \mathrm{H}^0(\mathbb{P}^2, \mathrm{pr}_* \mathcal{O}_{\widetilde{\mathcal{N}}_P}) = \mathrm{H}^0(\mathbb{P}^2, \mathrm{Sym}^\bullet(T\mathbb{P}^2)).$$

The result now follows from the previous lemma. □

Proposition 5.3.5 ($i = 0$, $j = 1$)**.** *Let $2m = k$. If $m \geq 2$, we have*

$$\mathrm{H}^0(\widetilde{\mathcal{N}}_P, T_{\widetilde{\mathcal{N}}_P})^k \cong L_{m,m} \oplus L_{m,m+1} \oplus L_{m+1,m} \oplus L_{m+1,m+1}.$$

Proof. Let us first write the weight spaces of the corresponding module $S^m(\mathfrak{u}_\mathfrak{p}) \otimes \mathfrak{u}_\mathfrak{p} \oplus S^{m+1}(\mathfrak{u}_\mathfrak{p}) \otimes \mathfrak{n}_\mathfrak{p}$ in a table:

Weight	Modules	Vector	Value of a
$a\alpha_1 + m\alpha_2$	$S^{m+1}(\mathfrak{u}_\mathfrak{p}) \otimes \mathfrak{n}_\mathfrak{p}$	$e_2^{m+1-a} e_{12}^a \otimes f_2$	$0 \leq a \leq m+1$
$(a-1)\alpha_1 + m\alpha_2$	$S^{m+1}(\mathfrak{u}_\mathfrak{p}) \otimes \mathfrak{n}_\mathfrak{p}$	$e_2^{m+1-a} e_{12}^a \otimes f_{12}$	$0 \leq a \leq m+1$
$a\alpha_1 + (m+1)\alpha_2$	$S^m(\mathfrak{u}_\mathfrak{p}) \otimes \mathfrak{u}_\mathfrak{p}$	$e_2^{m-a} e_{12}^a \otimes e_2$	$0 \leq a \leq m$
$(a+1)\alpha_1 + (m+1)\alpha_2$	$S^m(\mathfrak{u}_\mathfrak{p}) \otimes \mathfrak{u}_\mathfrak{p}$	$e_2^{m-a} e_{12}^a \otimes e_{12}$	$0 \leq a \leq m$

The set of admissible dominant weights are hence of the form $n_1\alpha_1 + m\alpha_2$ and $n_1\alpha_1 + (m+1)\alpha_2$ (for appropriate values of n_1). Notice that we have $s_2 \cdot (a\alpha_1 + m\alpha_2) = a\alpha_1 + (a - m - 1)\alpha_2$. It is obvious that there are no vectors with this weight (and similarly for the weight $s_2 \cdot (a\alpha_1 + (m+1)\alpha_2)$). Hence the differential $E[\lambda] \to E[s_2 \cdot \lambda] = 0$ is zero, and therefore the BGG complex in our case is simply given by $f_1^t \colon E[\lambda] \to E[s_1 \cdot \lambda]$ for some integer t.

We begin with the case $\lambda = n_1\alpha_1 + m\alpha_2$, where $2n_1 \geq m$, which means that λ is dominant. For now wee assume that $n_1 \leq m - 1$. The BGG complex is then $d \colon \mathbb{C}^2 \to \mathbb{C}^2$, where the differential is given by multiplication by $f_1^{2n_1 - m - 1}$. Let us write the formula for d. In a basis, we have

$$d(e_2^{m+1-n_1} e_{12}^{n_1} \otimes f_2) = \frac{n_1!}{(m - n_1 - 1)!} \left(e_2^{n_1+2} e_{12}^{m-n_1-1} \otimes f_2 + (2n_1 - m + 1) e_2^{n_1+1} e_{12}^{m-n_1} \otimes f_{12} \right),$$

and

$$d(e_2^{m-n_1} e_{12}^{n_1+1} \otimes f_{12}) = \frac{(n_1+1)!}{(m - n_1 - 1)!} e_2^{n_1+1} e_{12}^{m-n_1} \otimes f_{12}.$$

Indeed, $f_1 \cdot f_2 = f_{12}$, $f_1 \cdot f_{12} = f_1 \cdot e_2 = 0$, and the action of f_1 is given by

$$f_1 \cdot e_2^{r_1} e_{12}^{r_2} \otimes f_2 = r_2 e_2^{r_1+1} e_{12}^{r_2-1} \otimes f_2 + e_2^{r_1} e_{12}^{r_2}, \otimes f_{12}$$

and by induction the formula for the differential follows. It also follows easily that d is an isomorphism, thus these weights do not contribute to the cohomology. We look at the remaining case: $n_1 \in \{m, m+1\}$. For $n_1 = m$ the complex is given by $\mathbb{C}^2 \to \mathbb{C}$, where the basis of $\mathbb{C}^2$ is given by $e_2 e_{12}^m \otimes f_2$ and $e_{12}^{m+1} f_{12}$, and the basis of $\mathbb{C}$ by $e_2^{m+1} \otimes f_{12}$. Since $f_1^{m+1} \cdot e_{12}^{m+1} \otimes f_{12} = e_2^{m+1} \otimes f_{12}$, it follows that the differential is non-zero. Hence $L_{m,m}$ appears in $\mathrm{H}^0(\widetilde{\mathcal{N}}_P, T_{\widetilde{\mathcal{N}}_P})^k$ with multiplicity one. For the weight $(m+1)\alpha_1 + m\alpha_2$, the complex is $\mathbb{C} \to 0$, therefore $L_{m+1,m}$ appears in $\mathrm{H}^0(\widetilde{\mathcal{N}}_P, T_{\widetilde{\mathcal{N}}_P})^k$ with multiplicity one.

Now we consider the case $\lambda = n_1\alpha_1 + (m+1)\alpha_2$. We take λ dominant and first consider $n_1 \leq m - 1$. Let us recall that, by definition, the action of f_1 on $e_{12}, e_2 \in \wedge^1 \mathfrak{u}_\mathfrak{p}$ is given by $f_1 \cdot e_{12} = e_2$ and $f_1 \cdot e_2 = 0$. The differentials are given by

$$d(e_2^{m+1-n_1} e_{12}^{n_1} \otimes e_{12}) = \frac{n_1!}{(m - n_1 - 1)!} \left(e_2^{n_1+2} e_{12}^{m-n_1-1} \otimes e_{12} + (2n_1 - m + 1) e_2^{n_1+1} e_{12}^{m-n_1} \otimes e_2 \right),$$

and

$$d(e_2^{m-n_1}e_{12}^{n_1+1}\otimes e_2)=\frac{(n_1+1)!}{(m-n_1-1)!}e_2^{n_1+1}e_{12}^{m-n_1}\otimes e_2.$$

It follows again that all these weights do not contribute to the cohomology. We are left with the weights $m\alpha_1+(m+1)\alpha_2$ and $(m+1)\alpha_1+(m+1)\alpha_2$. In the former case, the complex is $\mathbb{C}^2\to\mathbb{C}$, where a basis of $\mathbb{C}^2$ is given by $e_{12}^m\otimes e_2$ and $e_2e_{12}^{m-1}\otimes e_{12}$, and where a basis of $\mathbb{C}$ is given by $e_2^m\otimes e_2$. The differential is again surjective, thus $L_{m,m+1}$ appears with multiplicity one in $\mathrm{H}^0(\widetilde{\mathcal{N}}_P,T_{\widetilde{\mathcal{N}}_P})^k$. Finally, the BGG complex in the latter case is given by $\mathbb{C}\to 0$, and the proposition follows. □

Lemma 5.3.6 ($i=1$, $j=1$)**.** *We have* $\mathrm{H}^1(\widetilde{\mathcal{N}}_P,T_{\widetilde{\mathcal{N}}_P})=0$.

Proof. It follows from the previous proof that $E[\lambda]\to E[s_1\cdot\lambda]$ is surjective for each λ that can possibly contribute to the cohomology. □

Proposition 5.3.7 ($i=0$, $j=2$)**.** *Write* $s=2m$*. We have*

$$\mathrm{H}^0(\widetilde{\mathcal{N}}_P,\wedge^2T_{\widetilde{\mathcal{N}}_P})^k=L_{m,m-1}\oplus L_{m-1,m}\oplus L_{m,m}^2\oplus L_{m,m+1}\oplus L_{m+1,m}$$

Proof. We have $i+j+k=s$, hence $k=2(m-1)$. As before, let us first split the module into different weight spaces:

Weight	Modules	Vector	Value of a
$(a+1)\alpha_1+(m+1)\alpha_2$	$S^{m-1}(\mathfrak{u}_\mathfrak{p})\otimes\wedge^2\mathfrak{u}_\mathfrak{p}$	$e_2^{m-a-1}e_{12}^a\otimes e_2\wedge e_{12}$	$0\leq a\leq m-1$
$(a+1)\alpha_1+m\alpha_2$	$S^m(\mathfrak{u}_\mathfrak{p})\otimes\mathfrak{u}_\mathfrak{p}\otimes\mathfrak{n}_\mathfrak{p}$	$e_2^{m-a}e_{12}^a\otimes e_{12}\otimes f_2$	$0\leq a\leq m$
$a\alpha_1+m\alpha_2$	$S^m(\mathfrak{u}_\mathfrak{p})\otimes\mathfrak{u}_\mathfrak{p}\otimes\mathfrak{n}_\mathfrak{p}$	$e_2^{m-a}e_{12}^a\otimes e_2\otimes f_2$	$0\leq a\leq m$
$a\alpha_1+m\alpha_2$	$S^m(\mathfrak{u}_\mathfrak{p})\otimes\mathfrak{u}_\mathfrak{p}\otimes\mathfrak{n}_\mathfrak{p}$	$e_2^{m-a}e_{12}^a\otimes e_{12}\otimes f_{12}$	$0\leq a\leq m$
$(a-1)\alpha_1+m\alpha_2$	$S^m(\mathfrak{u}_\mathfrak{p})\otimes\mathfrak{u}_\mathfrak{p}\otimes\mathfrak{n}_\mathfrak{p}$	$e_2^{m-a}e_{12}^a\otimes e_2\otimes f_{12}$	$0\leq a\leq m$
$(a-1)\alpha_1+(m-1)\alpha_2$	$S^{m+1}(\mathfrak{u}_\mathfrak{p})\otimes\wedge^2\mathfrak{n}_\mathfrak{p}$	$e_2^{m+1-a}e_{12}^a\otimes f_2\wedge f_{12}$	$0\leq a\leq m+1$

As in the previous case, for all the weights λ that appear, there is no element of weight $s_2\cdot\lambda$. First we compute the cohomology for a dominant weight of the form $\lambda=n_1\alpha_1+(m+1)\alpha_2$. We then have $s_1\cdot(n_1\alpha_1+(m+1)\alpha_2)=(m-n_1)\alpha_1+(m+1)\alpha_2$. If we assume $n_1\neq m$, then $n_1\geq 2(m+1)$ and the BGG complex is $\mathbb{C}\to\mathbb{C}, e_2^{m-n_1}e_{12}^{n_1}\otimes e_2\wedge e_{12}\mapsto e_2^{n_1}e_{12}^{m-n_1}\otimes e_2\wedge e_{12}$, and hence these weights do not contribute to the cohomology. Finally, for $n_1=m$ we get the BGG complex $\mathbb{C}\to 0$, therefore $L_{m,m+1}$ contributes with multiplicity one.

We now consider dominant weights of the form $\lambda=n_1\alpha_1+(m-1)\alpha_1$. We know that $-1\leq n_1\leq m$. Moreover, $s_1\cdot\lambda=(m-n_1-2)\alpha_1+(m-1)\alpha_2$, and for these weights the BGG complex is $1\colon\mathbb{C}\to\mathbb{C}$. For $-1\leq n_1\leq m-1$ the weights hence do not contribute to the cohomology. The only remaining weight is $\lambda=m\alpha_1+(m-1)\alpha_2$, where the BGG complex is $\mathbb{C}\to 0$. Therefore $L_{m,m-1}$ appears with multiplicity one.

Next we consider the case where $\lambda=n_1\alpha_1+m\alpha_2$ is dominant. If $n_1\leq m-2$, then we claim that d is invertible, and these weights do not contribute to the cohomology. Indeed, the matrix of d is given by

$$\begin{pmatrix} S & 0 & 0 & 0 \\ -T & S & 0 & 0 \\ T & 0 & S & 0 \\ -\frac{(2n_1-m)T}{m-n_1+2} & -T & -T & S \end{pmatrix},$$

where $T = (2n_1 - m + 1)(n_1 - 1)\dots(m - n_1 + 2)$ and $S = (n_1 - 1)\dots(m - n_1 + 1)$. Since $S \neq 0$ it follows that d is an isomorphism. The entries of d are obtained by induction. For example the coefficients in the first column can be deduced from

$$\begin{aligned} &f_1 \cdot (e_2^{m-n_1} e_{12}^{n_1-1} \otimes e_{12} \otimes f_2) \\ &\quad = (n_1 - 1) e_2^{m-n_1+2} e_{12}^{n_1-2} \otimes e_{12} \otimes f_2 - e_2^{m-n_1} e_{12}^{n_1-1} \otimes e_2 \otimes f_2 + e_2^{m-n_1} e_{12}^{n_1-1} \otimes e_{12} \otimes f_{12}. \end{aligned}$$

The remaining cases are $n_1 \in \{m-1, m, m+1\}$. For the case $\lambda = (m+1)\alpha_1 + m\alpha_2$, the BGG complex is $\mathbb{C} \to 0$.

If $\lambda = (m-1)\alpha_1 + m\alpha_2$, then $s_1 \cdot \lambda = m\alpha_2$ and there is no vector of this weight on the form $e_2^{m-a} e_{12}^{a} \otimes e_{12} \otimes f_2$. Hence the BGG complex is $d \colon \mathbb{C}^4 \to \mathbb{C}^3$ and using the previous formula we easily see that d is surjective. This means that $L_{m-1,m}$ also appears with multiplicity one.

Finally, if $\lambda = m\alpha_1 + m\alpha_2$, the BGG complex is $\mathbb{C}^3 \to \mathbb{C}$, with basis of $\mathbb{C}^3$

$$e_2 e_{12}^{m-1} \otimes e_{12} \otimes f_2, \quad e_{12}^{m} \otimes e_2 \otimes f_2, \quad e_{12}^{m} \otimes e_{12} \otimes f_{12},$$

and for $\mathbb{C}^1$ basis $e_2^m \otimes f_{12}$. It follows again from the explicit formula that d is surjective, hence $L_{m,m}$ appears with multiplicity two. ☐

Lemma 5.3.8 ($i = 1$, $j = 2$)**.** *We have* $\mathrm{H}^1(\widetilde{\mathcal{N}}_P, \wedge^2 T\widetilde{\mathcal{N}}_P) = 0$.

Proof. It follows from the previous proof that $\mathrm{H}^1 = 0$, since for each dominant weight d was always surjective. ☐

Lemma 5.3.9 ($i = 2$, $j = 2$)**.** *We have* $\mathrm{H}^2(\widetilde{\mathcal{N}}_P, \wedge^2 T\widetilde{\mathcal{N}}_P) = 0$.

Proof. The module $V_{2,m}$ is given by

$$(\mathrm{Sym}^m(\mathfrak{u}_\mathfrak{p}) \otimes \wedge^2 \mathfrak{u}_\mathfrak{p}) \oplus (\mathrm{Sym}^{m+1}(\mathfrak{u}_\mathfrak{p}) \otimes \mathfrak{u}_\mathfrak{p} \otimes \mathfrak{n}_\mathfrak{p}) \oplus (\mathrm{Sym}^{m+1}(\mathfrak{u}_\mathfrak{p}) \otimes \wedge^2 \mathfrak{n}_\mathfrak{p}).$$

By lemma 5.3.1, we only need to check weight $\lambda = n_1\alpha_1 + n_2\alpha_2$ with $n_1 < 0$ or $n_2 < 0$. There is a single weight μ with such coefficients, given by $\mu = -\alpha_1 + (m+1)\alpha_2$. Since $s_1 \cdot \mu = (m+1)\rho$ is dominant, $\mathrm{H}^2 = 0$ as claimed. (If $m = 0$, there is also the exceptional case $e_2 \otimes f_2 \wedge f_{12}$ with weight $-\alpha_1 - \alpha_2 = -\rho$. But such a weight is dot-singular.) ☐

5.3.2 Relation to the nilpotent cone

In this subsection we describe $\mathrm{HH}_{\mathbb{C}^*}(T^*\mathbb{P}^2)$ as a module over $\mathbb{C}[\mathcal{N}]$, where $\mathcal{N} \subset \mathfrak{sl}_3$ is the nilpotent cone. The following theorem relates $\mathfrak{u}_q(\mathfrak{g})$ with $\mathcal{N}$:

Theorem 5.3.10 ([27]). *There is an isomorphism of algebras* $\mathrm{H}^{2\bullet}(\mathfrak{u}_q(\mathfrak{g}), \mathbb{C}) \cong \mathbb{C}[\mathcal{N}]$. *Moreover we have that* $\mathrm{H}^{2\bullet+1}(\mathfrak{u}_q(\mathfrak{g}), \mathbb{C}) = 0$.

Since $\mathrm{H}^\bullet(\mathfrak{u}_q(\mathfrak{g}), \mathbb{C})$ acts on $\mathrm{HH}^\bullet(\mathfrak{u}_q(\mathfrak{g}))$, the G-representation $\mathrm{HH}^\bullet(\mathfrak{u}_{\lambda_\mathfrak{p}}(\mathfrak{g}))$ is actually a $G \times \mathbb{C}^*$-equivariant coherent sheaf on the nilpotent cone for each block. There is a $\mathrm{H}^0(G/P, \mathrm{pr}_*\mathcal{O}_{\widetilde{\mathcal{N}}_P})$-module structure on $\mathrm{HH}^\bullet(\mathfrak{u}_{\lambda_\mathfrak{p}}(\mathfrak{g}))$, hence a $\mathbb{C}[\mathcal{N}]$-module structure. For $\mathfrak{g} = \mathfrak{sl}_2$ the module structure is known:

Theorem 5.3.11 ([50]). *As module over* $\mathbb{C}[\mathcal{N}]$, *the even Hochschild cohomology of* $\mathfrak{u}_0(\mathfrak{sl}_2)$ *is given by the following table:*

$i+j=0$	$\mathbb{C}[\mathcal{N}]$	
$i+j=2$	$\mathbb{C}$	$\mathbb{C}[\mathcal{N}]$
$h^{i,j}$	$j-i=0$	$j-i=2$

The odd Hochschild cohomology is concentrated in bidegree $(0,1)$ *and given by*

$$\mathrm{HH}^{2\bullet+1}(\mathfrak{u}_0(\mathfrak{sl}_2)) \cong \mathbb{C}[\mathcal{N}]_+[1] \oplus \mathbb{C}[\mathcal{N}][-1],$$

where $[1]$ *is a shift in the* s*-grading and* $\mathbb{C}[\mathcal{N}]_+$ *is the augmentation ideal.*

We would like to obtain a similar result for $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathfrak{p} = \mathfrak{b} \oplus \mathbb{C}e_1$, i.e., when $G/P \cong \mathbb{P}^2$. Let $\mathfrak{u}_1 := \mathfrak{u}_{\lambda_\mathfrak{p}}(\mathfrak{g})$ and $Y := \mathcal{O}_\mathfrak{p}$. Corollary 5.3.4 and the previous discussion imply:

Proposition 5.3.12. *There is a graded* G*-module isomorphism,* $\mathbb{C}[Y] \cong \bigoplus_{m\geq 0} L(m,m)$.

We want to understand the $\mathbb{C}[Y]$-module structure on $\mathrm{HH}^\bullet(\mathfrak{u})$. Using that $\mathbb{C}[\widetilde{\mathcal{N}}_P] \cong \mathbb{C}[Y]$, we can reduce the problem to understanding the $\mathcal{O}_{\widetilde{\mathcal{N}}_P}$-module structure on the cohomology $\mathrm{HH}^\bullet_{\mathbb{C}^*}(\widetilde{\mathcal{N}}_P)$. Using the equivalence $\mathcal{O}_{\widetilde{\mathcal{N}}_P}$–mod $\cong \mathrm{pr}_*\mathcal{O}_{\widetilde{\mathcal{N}}_P}$–mod, we can reduce the problem to understanding the maps

$$\mathrm{H}^0(\mathrm{pr}_*\mathcal{O}_{\widetilde{\mathcal{N}}_P}) \otimes \left(\bigoplus_{i+j+k=s} \mathrm{H}^i(\mathbb{P}^2, \mathcal{V}_{j,k}) \right) \to \bigoplus_{i+j+k=s+2} \mathrm{H}^i(\mathbb{P}^2, \mathcal{V}_{j,k}),$$

where $\mathcal{V}_{j,k}$ is the vector bundle associated to the module $V_{j,k}$ introduced earlier. Since $\mathbb{C}[Y]$ is generated by its degree one elements $\mathbb{C}[Y]_1 \cong L_{1,1} \cong \mathfrak{sl}_3$, it is enough to compute the maps

$$L_{1,1} \otimes \left(\bigoplus_{i+j+k=s} \mathrm{H}^i(\mathbb{P}^2, \mathcal{V}_{j,k}) \right) \to \bigoplus_{i+j+k=s+2} \mathrm{H}^i(\mathbb{P}^2, \mathcal{V}_{j,k}).$$

We notice that, for $m \geq 2$, the only non-zero contribution to $\mathrm{HH}^m(\mathfrak{u}_1)$ is when $i = 0$. By definition, if E is a $\mathfrak{p}$-module and $v \in E^{\mathfrak{n}_P}[\mu]$ for some $\mu \in P^+$, then v is the BGG representative corresponding to an irreducible representation $L_\mu \subset \mathrm{H}^0(G/P, \mathcal{E})$. This vector can be found from the representation as follows: L_μ has a highest weight vector w, corresponding to a section $s_w : G/P \to E$. Then, $s_w(eP) = v$ is the corresponding vector. This is by definition, because $\mathrm{H}^0(G/P, E) = \mathrm{Ind}_P^G E$. In particular, it follows that when multiplying global sections we can multiply the corresponding BGG representative together (since the multiplication is an equivariant map, we only need to know what happens at one point, for example at eP).

Let $W_m = L_{m,m} \oplus L_{m+1,m} \oplus L_{m,m+1} \oplus L_{m+1,m+1}$. This is the summand of bidegree $(i,j) = (0,1)$ in $\mathrm{HH}^{2m+1}(\mathfrak{u}_1)$. Since the $\mathfrak{sl}_2$-action obtained from the Poisson bivector field commutes with the $\mathbb{C}[Y]$-action, we just need to understand the $\mathbb{C}[Y]$-module structure on this summand, since the summand of bidegree $(i,j) = (0,3)$ will be isomorphic to W_m. Let $W_m^1 = L_{m,m}, W_m^2 = L_{m+1,m}, W_m^3 = L_{m,m+1}$ and $W_m^4 = L_{m+1,m+1}$.

Proposition 5.3.13. *The map* $L_{1,1} \otimes W_m \to W_{m+1}$ *is given by* $L_{1,1} \otimes W_m^l \to W_{m+1}^l \subset W_{m+1}$*, where the first map is the projection onto the corresponding isotopic component.*

Proof. This directly follows from the representatives we computed in the proof of Theorem 5.3.2 and the discussion before. Let us remark that the projection onto the isotopic component is only defined up to multiplication by a non-zero scalar, but it can be fixed by choosing a highest weight vector v_m in each W_m^l such that the projection sends v_m to v_{m+1}. □

A similar proposition holds for $s = 2m$. We now describe the corresponding $\mathbb{C}[Y]$-modules. We define the module M_1 as $(M_1)_m = L_{m+1,m}$ if $m \geq 1$ and $(M_1)_0 = 0$. The $\mathbb{C}[Y]$-module structure is obtained from the maps $L_{1,1} \otimes W_m^2 \to W_{m+1}^2$. Let M_2 be the module such that $(M_2)_m = W_m^3$ if $m \geq 1$ and 0 otherwise, with similar $\mathbb{C}[Y]$-module structure. We also define shifted modules N_1, N_2 by $(N_1)_m = (M_1)_{m-1}$ and $(N_2)_m = (M_2)_{m-1}$, with the same $\mathbb{C}[Y]$-module structure. Finally, let $\mathbb{C}[Y]_+$ be the ideal sheaf of the origin $0 \in Y$, and $\mathbb{C}_0$ be the skyscraper sheaf at 0. We obtain the following description of the Hochschild cohomology as $\mathbb{C}[Y]$-modules:

Proposition 5.3.14. *The following tables give the even (resp. odd) Hochschild cohomology groups of* $\mathfrak{u}_1(\mathfrak{sl}_3)$*, as* $\mathbb{C}[Y]$*-modules:*

$i+j=0$	$\mathbb{C}[Y]$		
$i+j=2$	$\mathbb{C}_0$	$\mathbb{C}[Y] \oplus N_1 \oplus N_2 \oplus \mathbb{C}[Y]_+ \oplus M_1 \oplus M_2$	
$i+j=4$	$\mathbb{C}_0$	$\mathbb{C}_0$	$\mathbb{C}[Y]$
$h^{i,j}$	$j-i=0$	$j-i=2$	$j-i=4$

$i+j=1$	$\mathbb{C}[Y] \oplus M_1 \oplus M_2 \oplus \mathbb{C}[Y]_+$	
$i+j=3$	$\mathbb{C}_0$	$\mathbb{C}[Y] \oplus M_1 \oplus M_2 \oplus \mathbb{C}[Y]_+$
$h^{i,j}$	$j-i=1$	$j-i=3$

Let us now investigate the multiplicative structure. A presentation by generators and relations would be cumbersome, but we can describe the multiplication by looking at each pair of bidegree. For degree reason, we see that the last lines in the two previous tables are included in the annhilator of the nilradical (i.e the multiplication with the positive degree part is zero), so we can only look at the remaining pair of relevant bidegree. Inspecting at the proof, we see that all the relevant Hochschild cohomology is concentrated when $i = 0$, or comes from the Harish-Chandra center. Hence our theorem implies that the original HKR isomorphism is already multiplicative. Let us give an example how to compute the multiplication:

$\mathrm{HH}_{\mathbb{C}^*}(T^*\mathbb{P}^2)^{(1,1)} \otimes \mathrm{HH}_{\mathbb{C}^*}(T^*\mathbb{P}^2)^{(1,1)} \to \mathrm{HH}_{\mathbb{C}^*}(T^*\mathbb{P}^2)^{(2,2)}$

The multiplication is by definition the cup-product in de Rham cohomology $H^1(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}) \otimes H^1(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}) \to H^2(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2})$, i.e this the map $\mathbb{C} \otimes \mathbb{C} \to \mathbb{C}, 1 \otimes 1 \mapsto 1$.

$\mathrm{HH}_{\mathbb{C}^*}(T^*\mathbb{P}^2)^{(1,1)} \otimes \mathrm{HH}_{\mathbb{C}^*}(T^*\mathbb{P}^2)^{(0,2)} \to \mathrm{HH}_{\mathbb{C}^*}(T^*\mathbb{P}^2)^{(1,3)}$

Since $\mathrm{HH}_{\mathbb{C}^*}(T^*\mathbb{P}^2)^{(1,3)} = \mathbb{C}$, and that $\mathrm{HH}_{\mathbb{C}^*}(T^*\mathbb{P}^2)^{(1,1)} \otimes \mathrm{HH}_{\mathbb{C}^*}(T^*\mathbb{P}^2)^{(0,2)}$ contains only a single copy of the trivial representation, we conclude that the product is at most non-zero with a single element inside a weight basis of $\mathrm{HH}_{\mathbb{C}^*}(T^*\mathbb{P}^2)^{(1,1)}$. Such element is given by the Poisson bivector field, and the product is non-zero because we know that cup-product with the Poisson bivector field induces a $\mathfrak{sl}_2$-action.

$\mathrm{HH}_{\mathbb{C}^*}(T^*\mathbb{P}^2)^{(0,2)} \otimes \mathrm{HH}_{\mathbb{C}^*}(T^*\mathbb{P}^2)^{(0,2)} \to \mathrm{HH}_{\mathbb{C}^*}(T^*\mathbb{P}^2)^{(0,4)}$

Because each $\mathbb{C}[Y]$-submodule of $\mathrm{HH}_{\mathbb{C}^*}(T^*\mathbb{P}^2)^{(0,2)}$ is generated in degree 1, we just need to compute the degree 1 multiplication. Now, let us write explicitly the vectors in the BGG complex that span the modules $\mathbb{C}[Y], M_1, M_2, N_1, N_2, \mathbb{C}_+[Y]$:

$\mathbb{C}[Y]$	
M_1	$e_{12}^m \otimes e_{12} \otimes f_2$
M_2	$e_{12}^{m-1} \otimes e_2 \wedge e_{12}$
N_1	$e_{12}^{m+1} \otimes f_2 \wedge f_{12}$
N_2	
$\mathbb{C}_+[Y]$	

5.4 Computation of $\mathbf{z}_0(\mathfrak{u}_q(\mathfrak{g}))$ for type G_2, B_3, C_3 and A_4

We now present our results obtained using the algorithm described in [32], using the description of $V_{j,k}$ introduced in section 5.2.3.

5.4.1 Type G_2

For type G_2, we let α_1 be the short root and α_2 be the long root.

Theorem 5.4.1. *The center of the principal block of the small quantum group for $\mathfrak{g}$ of type G_2 has dimension* 91. *The bigraded dimensions are given by the following table:*

$i+j=0$	$\mathbb{C}$						
$i+j=2$	$\mathbb{C}^2$	$\mathbb{C}$					
$i+j=4$	$\mathbb{C}^2$	$\mathbb{C}^2$	$\mathbb{C}$				
$i+j=6$	$\mathbb{C}^2$	$\mathbb{C}^2$	$\mathbb{C}^2$	$\mathbb{C}$			
$i+j=8$	$\mathbb{C}^2$	$\mathbb{C}^2 \oplus L_{2,1}^2$	$\mathbb{C}^2 \oplus L_{2,1}$	$\mathbb{C}^2$	$\mathbb{C}$		
$i+j=10$	$\mathbb{C}^2$	$\mathbb{C}^3 \oplus L_{2,1}$	$\mathbb{C}^2 \oplus L_{2,1}^2$	$\mathbb{C}^2$	$\mathbb{C}^2$	$\mathbb{C}$	
$i+j=12$	$\mathbb{C}$	$\mathbb{C}^2$	$\mathbb{C}^2$	$\mathbb{C}^2$	$\mathbb{C}^2$	$\mathbb{C}^2$	$\mathbb{C}$
$h^{i,j}$	$j-i=0$	$j-i=2$	$j-i=4$	$j-i=6$	$j-i=8$	$j-i=10$	$j-i=12$

The trivial representation has multiplicity 49 in total. The table confirms the conjecture 5.2.4 for $\mathfrak{g}$

of type G_2, and hence for all complex simple Lie algebras of rank 2 using previous work from [48] and [49]. We also computed the center of the singular blocks of type $\mathfrak{u}_1(\mathfrak{g}_2)$:

$h^{i,j}$	$j-i=0$	$j-i=2$	$j-i=4$	$j-i=6$	$j-i=8$	$j-i=10$
$i+j=0$	$\mathbb{C}$					
$i+j=2$	$\mathbb{C}$	$\mathbb{C}$				
$i+j=4$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$			
$i+j=6$	$\mathbb{C}$	$\mathbb{C}\oplus L_{2,1}$	$\mathbb{C}$	$\mathbb{C}$		
$i+j=8$	$\mathbb{C}$	$\mathbb{C}\oplus L_{2,1}$	$\mathbb{C}\oplus L_{2,1}$	$\mathbb{C}$	$\mathbb{C}$	
$i+j=10$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$

and similarly for $\mathfrak{u}_2(\mathfrak{g}_2)$:

$h^{i,j}$	$j-i=0$	$j-i=2$	$j-i=4$	$j-i=6$	$j-i=8$	$j-i=10$
$i+j=0$	$\mathbb{C}$					
$i+j=2$	$\mathbb{C}$	$\mathbb{C}$				
$i+j=4$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$			
$i+j=6$	$\mathbb{C}$	$\mathbb{C}\oplus L_{2,1}$	$\mathbb{C}\oplus L_{2,1}$	$\mathbb{C}$		
$i+j=8$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}\oplus L_{2,1}$	$\mathbb{C}$	$\mathbb{C}$	
$i+j=10$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$

5.4.2 Type B_3 and C_3

Definition 5.4.2. *For a subset $J \subset I$, let $\mathfrak{u}_J$ be the block corresponding to a singular weight λ with stabiliser generated by $\langle s_j : j \in J\rangle$.*

Theorem 5.4.3. *In type B_3, the center of the principal block is given by the following table:*

$h^{i,j}$	$j-i=0$	$j-i=2$	$j-i=4$	$j-i=6$	$j-i=8$	$j-i=10$	$j-i=12$	$j-i=14$	$j-i=16$	$j-i=18$
$i+j=0$	$\mathbb{C}$									
$i+j=2$	$\mathbb{C}^3$	$\mathbb{C}$								
$i+j=4$	$\mathbb{C}^5$	$\mathbb{C}^3$	$\mathbb{C}$							
$i+j=6$	$\mathbb{C}^7$	$\mathbb{C}^6$	$\mathbb{C}^3$	$\mathbb{C}$						
$i+j=8$	$\mathbb{C}^8$	$\mathbb{C}^{10}$	$\mathbb{C}^6$	$\mathbb{C}^3$	$\mathbb{C}$					
$i+j=10$	$\mathbb{C}^8$	$\mathbb{C}^{14}\oplus L_{1,1,1}$	$\mathbb{C}^{10}$	$\mathbb{C}^6$	$\mathbb{C}^3$	$\mathbb{C}$				
$i+j=12$	$\mathbb{C}^7$	$\mathbb{C}^{15}\oplus L_{1,1,1}^3$	$\mathbb{C}^{14}\oplus L_{1,1,1}^4$	$\mathbb{C}^{10}\oplus L_{1,1,1}$	$\mathbb{C}^6$	$\mathbb{C}^3$	$\mathbb{C}$			
$i+j=14$	$\mathbb{C}^5$	$\mathbb{C}^{12}\oplus L_{1,1,1}^2$	$\mathbb{C}^{15}\oplus L_{1,1,1}^6\oplus L_{1,2,2}$	$\mathbb{C}^{14}\oplus L_{1,1,1}^4$	$\mathbb{C}^{10}$	$\mathbb{C}^6$	$\mathbb{C}^3$	$\mathbb{C}$		
$i+j=16$	$\mathbb{C}^3$	$\mathbb{C}^8$	$\mathbb{C}^{12}\oplus L_{1,1,1}^2$	$\mathbb{C}^{15}\oplus L_{1,1,1}^3$	$\mathbb{C}^{14}\oplus L_{1,1,1}$	$\mathbb{C}^{10}$	$\mathbb{C}^6$	$\mathbb{C}^3$	$\mathbb{C}$	
$i+j=18$	$\mathbb{C}$	$\mathbb{C}^3$	$\mathbb{C}^5$	$\mathbb{C}^7$	$\mathbb{C}^8$	$\mathbb{C}^8$	$\mathbb{C}^7$	$\mathbb{C}^5$	$\mathbb{C}^3$	$\mathbb{C}$

The invariant part of the center coincides with the invariant part of the center of the principal block for type C_3.

We also computed the corresponding diagonal coinvariants for type B_3, and confirmed Conjecture 5.2.4 for B_3 and C_3. We note that the vector space isomorphism $\mathbf{z}(\mathfrak{u}_0(\mathfrak{so}_7))^{\mathfrak{so}_7} \cong \mathbf{z}(u_0(\mathfrak{sp}_6))^{\mathfrak{sp}_6}$ was also predicted by the conjecture since they have isomorphic Weyl group. We now give the answer for other blocks, and for simplicity we only give the result for $\mathfrak{g}$ of type B_3.

Theorem 5.4.4. *The centers of $\mathfrak{u}_1(\mathfrak{so}_7)$ and $\mathfrak{u}_2(\mathfrak{so}_7)$ are isomorphic as bigraded vector spaces, and* *given by the following table:*

$h^{i,j}$	$j-i=0$	$j-i=2$	$j-i=4$	$j-i=6$	$j-i=8$	$j-i=10$	$j-i=12$	$j-i=14$	$j-i=16$
$i+j=0$	$\mathbb{C}$								
$i+j=2$	$\mathbb{C}^2$	$\mathbb{C}$							
$i+j=4$	$\mathbb{C}^3$	$\mathbb{C}^2$	$\mathbb{C}$						
$i+j=6$	$\mathbb{C}^4$	$\mathbb{C}^4$	$\mathbb{C}^2$	$\mathbb{C}$					
$i+j=8$	$\mathbb{C}^4$	$\mathbb{C}^6$	$\mathbb{C}^4$	$\mathbb{C}^2$	$\mathbb{C}$				
$i+j=10$	$\mathbb{C}^4$	$\mathbb{C}^7 \oplus L_{1,1,1}$	$\mathbb{C}^6 \oplus L_{1,1,1}$	$\mathbb{C}^4$	$\mathbb{C}^2$	$\mathbb{C}$			
$i+j=12$	$\mathbb{C}^3$	$\mathbb{C}^6 \oplus L_{1,1,1}$	$\mathbb{C}^7 \oplus L_{1,1,1}^3 \oplus L_{1,2,2}$	$\mathbb{C}^6 \oplus L_{1,1,1}$	$\mathbb{C}^4$	$\mathbb{C}^2$	$\mathbb{C}$		
$i+j=14$	$\mathbb{C}^2$	$\mathbb{C}^4$	$\mathbb{C}^6 \oplus L_{1,1,1}$	$\mathbb{C}^7 \oplus L_{1,1,1}$	$\mathbb{C}^6$	$\mathbb{C}^4$	$\mathbb{C}^2$	$\mathbb{C}$	
$i+j=16$	$\mathbb{C}$	$\mathbb{C}^2$	$\mathbb{C}^3$	$\mathbb{C}^4$	$\mathbb{C}^4$	$\mathbb{C}^4$	$\mathbb{C}^3$	$\mathbb{C}^2$	$\mathbb{C}$

The center of $\mathfrak{u}_3(\mathfrak{so}_7)$ is given by the following table:

$h^{i,j}$	$j-i=0$	$j-i=2$	$j-i=4$	$j-i=6$	$j-i=8$	$j-i=10$	$j-i=12$	$j-i=14$	$j-i=16$
$i+j=0$	$\mathbb{C}$								
$i+j=2$	$\mathbb{C}^2$	$\mathbb{C}$							
$i+j=4$	$\mathbb{C}^3$	$\mathbb{C}^2$	$\mathbb{C}$						
$i+j=6$	$\mathbb{C}^4$	$\mathbb{C}^4$	$\mathbb{C}^2$	$\mathbb{C}$					
$i+j=8$	$\mathbb{C}^4$	$\mathbb{C}^6 \oplus L_{1,1,1}$	$\mathbb{C}^4$	$\mathbb{C}^2$	$\mathbb{C}$				
$i+j=10$	$\mathbb{C}^4$	$\mathbb{C}^7 \oplus L_{1,1,1}^2$	$\mathbb{C}^6 \oplus L_{1,1,1}^2$	$\mathbb{C}^4$	$\mathbb{C}^2$	$\mathbb{C}$			
$i+j=12$	$\mathbb{C}^3$	$\mathbb{C}^6 \oplus L_{1,1,1}^2$	$\mathbb{C}^7 \oplus L_{1,1,1}^4$	$\mathbb{C}^6 \oplus L_{1,1,1}^2$	$\mathbb{C}^4$	$\mathbb{C}^2$	$\mathbb{C}$		
$i+j=14$	$\mathbb{C}^2$	$\mathbb{C}^4$	$\mathbb{C}^6 \oplus L_{1,1,1}^2$	$\mathbb{C}^7 \oplus L_{1,1,1}^2$	$\mathbb{C}^6 \oplus L_{1,1,1}$	$\mathbb{C}^4$	$\mathbb{C}^2$	$\mathbb{C}$	
$i+j=16$	$\mathbb{C}$	$\mathbb{C}^2$	$\mathbb{C}^3$	$\mathbb{C}^4$	$\mathbb{C}^4$	$\mathbb{C}^4$	$\mathbb{C}^3$	$\mathbb{C}^2$	$\mathbb{C}$

The center of $\mathfrak{u}_{12}(\mathfrak{so}_7)$ is as follows:

$h^{i,j}$	$j-i=0$	$j-i=2$	$j-i=4$	$j-i=6$	$j-i=8$	$j-i=10$	$j-i=12$
$i+j=0$	$\mathbb{C}$						
$i+j=2$	$\mathbb{C}$	$\mathbb{C}$					
$i+j=4$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$				
$i+j=6$	$\mathbb{C}^2$	$\mathbb{C}^2$	$\mathbb{C}$	$\mathbb{C}$			
$i+j=8$	$\mathbb{C}$	$\mathbb{C}^2$	$\mathbb{C}^2 \oplus L_{1,1,1} \oplus L_{1,2,2}$	$\mathbb{C}$	$\mathbb{C}$		
$i+j=10$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}^2$	$\mathbb{C}^2$	$\mathbb{C}$	$\mathbb{C}$	
$i+j=12$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}^2$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$

The center of $\mathfrak{u}_{13}(\mathfrak{so}_7)$ *is as follows:*

$h^{i,j}$	$j-i=0$	$j-i=2$	$j-i=4$	$j-i=6$	$j-i=8$	$j-i=10$	$j-i=12$	$j-i=14$
$i+j=0$	$\mathbb{C}$							
$i+j=2$	$\mathbb{C}$	$\mathbb{C}$						
$i+j=4$	$\mathbb{C}^2$	$\mathbb{C}$	$\mathbb{C}$					
$i+j=6$	$\mathbb{C}^2$	$\mathbb{C}^3$	$\mathbb{C}$	$\mathbb{C}$				
$i+j=8$	$\mathbb{C}^2$	$\mathbb{C}^3 \oplus L_{1,1,1}$	$\mathbb{C}^3$	$\mathbb{C}$	$\mathbb{C}$			
$i+j=10$	$\mathbb{C}^2$	$\mathbb{C}^3 \oplus L_{1,1,1}$	$\mathbb{C}^3 \oplus L_{1,1,1}^2$	$\mathbb{C}^3$	$\mathbb{C}$	$\mathbb{C}$		
$i+j=12$	$\mathbb{C}$	$\mathbb{C}^2$	$\mathbb{C}^3 \oplus L_{1,1,1}$	$\mathbb{C}^3 \oplus L_{1,1,1}$	$\mathbb{C}^3$	$\mathbb{C}$	$\mathbb{C}$	
$i+j=14$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}^2$	$\mathbb{C}^2$	$\mathbb{C}^2$	$\mathbb{C}^2$	$\mathbb{C}$	$\mathbb{C}$

The center of $\mathfrak{u}_{23}(\mathfrak{so}_7)$ *is as follows:*

$h^{i,j}$	$j-i=0$	$j-i=2$	$j-i=4$	$j-i=6$	$j-i=8$	$j-i=10$
$i+j=0$	$\mathbb{C}$					
$i+j=2$	$\mathbb{C}$	$\mathbb{C}$				
$i+j=4$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$			
$i+j=6$	$\mathbb{C}$	$\mathbb{C} \oplus L_{1,1,1}$	$\mathbb{C} \oplus L_{1,1,1}$	$\mathbb{C}$		
$i+j=8$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C} \oplus L_{1,1,1}$	$\mathbb{C}$	$\mathbb{C}$	
$i+j=10$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$

The isomorphism of bigraded vector spaces $\mathbf{z}(\mathfrak{u}_1((\mathfrak{so}_7))) \cong \mathbf{z}(\mathfrak{u}_2((\mathfrak{so}_7)))$ is reminiscent of a similar isomorphism found in [49], where two non-conjugated blocks were found to have isomorphic centers. This suggests that the corresponding categories might be Morita equivalent.

5.4.3 Type A_4

We check the conjectures 5.2.4 and 5.2.5 for $\mathfrak{g} = \mathfrak{sl}_5$. Since the center is $\mathfrak{g}$-invariant, we only report the dimensions.

Theorem 5.4.5. *There is an isomorphism of bigraded vector space* $\mathbf{z}(\mathfrak{u}_0(\mathfrak{sl}_5)) \cong \mathbf{z}(\mathfrak{u}_0(\mathfrak{sl}_5)^{\mathfrak{g}} \cong \mathrm{DC}_5$. *The table is as follows:*

$h^{i,j}$	$j-i=0$	$j-i=2$	$j-i=4$	$j-i=6$	$j-i=8$	$j-i=10$	$j-i=12$	$j-i=14$	$j-i=16$	$j-i=18$	$j-i=20$
$i+j=0$	1										
$i+j=2$	4	1									
$i+j=4$	9	5	1								
$i+j=6$	15	14	5	1							
$i+j=8$	20	29	15	5	1						
$i+j=10$	22	44	33	15	5	1					
$i+j=12$	20	51	54	34	15	5	1				
$i+j=14$	15	46	66	58	34	15	5	1			
$i+j=16$	9	31	56	66	54	33	15	5	1		
$i+j=18$	4	15	31	46	51	44	29	14	5	1	
$i+j=20$	1	4	9	15	20	22	20	15	9	4	1

Let us present the other bigraded tables. We label the singular blocks as before, i.e as a subset $J \subset \{1,2,3,4\}$. However we consider them up to the involution $i \mapsto 5-i$ (i.e up to conjugacy in the extended affine Weyl group). There are also non-conjugated blocks giving isomorphic centers: $\mathbf{z}(\mathfrak{u}_1(\mathfrak{sl}_5)) \cong \mathbf{z}(\mathfrak{u}_2(\mathfrak{sl}_5))$, $\mathbf{z}(\mathfrak{u}_{12}(\mathfrak{sl}_5)) \cong z(\mathfrak{u}_{23}(\mathfrak{sl}_5))$ and $\mathbf{z}(\mathfrak{u}_{13}(\mathfrak{sl}_5)) \cong \mathbf{z}(\mathfrak{u}_{14}(\mathfrak{sl}_5))$. We also know that the block $\mathfrak{u}_{123}(\mathfrak{sl}_5)$ correspond to a projective space and was computed in [49]. Hence we only need to present $\mathfrak{u}_1(\mathfrak{sl}_5), \mathfrak{u}_{12}(\mathfrak{sl}_5), \mathfrak{u}_{13}(\mathfrak{sl}_5)$ and $\mathfrak{u}_{124}(\mathfrak{sl}_5)$. We will only present their dimension table, since for all blocks, the center is $\mathfrak{g}$-invariant.

Theorem 5.4.6. *The center of* $\mathfrak{u}_1(\mathfrak{sl}_5)$ *is as follows:*

$h^{i,j}$	$j-i=0$	$j-i=2$	$j-i=4$	$j-i=6$	$j-i=8$	$j-i=10$	$j-i=12$	$j-i=14$	$j-i=16$	$j-i=18$
$i+j=0$	1									
$i+j=2$	3	1								
$i+j=4$	6	4	1							
$i+j=6$	9	10	4	1						
$i+j=8$	11	18	11	4	1					
$i+j=10$	11	23	21	11	4	1				
$i+j=12$	9	23	29	22	11	4	1			
$i+j=14$	6	17	28	29	21	11	4	1		
$i+j=16$	3	9	17	23	23	18	10	4	1	
$i+j=18$	1	3	6	9	11	11	9	6	3	1

The center of $\mathfrak{u}_{12}(\mathfrak{sl}_5)$ *is as follows:*

$h^{i,j}$	$j-i=0$	$j-i=2$	$j-i=4$	$j-i=6$	$j-i=8$	$j-i=10$	$j-i=12$	$j-i=14$
$i+j=0$	1							
$i+j=2$	2	1						
$i+j=4$	3	3	1					
$i+j=6$	4	5	3	1				
$i+j=8$	4	7	6	3	1			
$i+j=10$	3	6	8	6	3	1		
$i+j=12$	2	4	6	7	5	3	1	
$i+j=14$	1	2	3	4	4	3	2	1

The center of $\mathfrak{u}_{13}(\mathfrak{sl}_5)$ *is as follows:*

$h^{i,j}$	$j-i=0$	$j-i=2$	$j-i=4$	$j-i=6$	$j-i=8$	$j-i=10$	$j-i=12$	$j-i=14$	$j-i=16$
$i+j=0$	1								
$i+j=2$	2	1							
$i+j=4$	4	3	1						
$i+j=6$	5	7	3	1					
$i+j=8$	6	10	8	3	1				
$i+j=10$	5	11	12	8	3	1			
$i+j=12$	4	9	14	12	8	3	1		
$i+j=14$	2	5	9	11	10	7	3	1	
$i+j=16$	1	2	4	5	6	5	4	2	1

The center of $\mathfrak{u}_{124}(\mathfrak{sl}_5)$ *is as follows:*

$i+j=0$	1						
$i+j=2$	1	1					
$i+j=4$	2	2	1				
$i+j=6$	2	3	2	1			
$i+j=8$	2	3	4	2	1		
$i+j=10$	1	2	3	3	2	1	
$i+j=12$	1	1	2	2	2	1	1
$h^{i,j}$	$j-i=0$	$j-i=2$	$j-i=4$	$j-i=6$	$j-i=8$	$j-i=10$	$j-i=12$

5.5 Higher Hochschild cohomology groups

We present some tables of higher Hochschild cohomology groups, obtained by our computer algorithm. These results are not complete, and there are additional results available on the repository of the algorithm: `https://github.com/RikVoorhaar/bgg-cohomology`.

Recall that $\mathrm{HH}^\bullet(\mathfrak{u})$ is a Gerstenhaber algebra, and that the bracket is of degree -1. In particular, $\mathrm{HH}^1(\mathfrak{u}_q(\mathfrak{g}))$ acts on $\mathrm{HH}^0(\mathfrak{u}_q(\mathfrak{g}))$. This gives a large group of symmetries, and understanding it could lead to a better understanding of the center.

Proposition 5.5.1. *For type* A_2*, for* $4 \leq s \leq 14$ *and* $s = 2k$ *even we get the following bigraded table:*

$i+j=2$	$L\left(k,k\right)^2$		
$i+j=4$	0	$L(k,k-1)^2 \oplus L(k-1,k)^2 \oplus L(k,k)^4 \oplus L(k+1,k)^2 \oplus L(k,k+1)^2$	
$i+j=6$	0	0	$L(k,k)^2$
$h^{i,j}$	$j-i=0$	$j-i=2$	$j-i=4$

For $3 \leq s \leq 15$ *and* $s = 2k+1$ *odd (the second column of the table can be deduced from the* $\mathfrak{sl}_2$ *symmetry) we get:*

$i+j=3$	$L(k,k)^2 \oplus L(k+1,k)^2 \oplus L(k,k+1)^2 \oplus L(k+1,k+1)$
$i+j=5$	0
$h^{i,j}$	$j-i=1$

In type G_2, $\mathrm{HH}^1(\mathfrak{u}_0)$ is given by the following table:

$i+j=1$	$\mathbb{C} \oplus L_{3,2}$					
$i+j=3$	$\mathbb{C}^2 \oplus L_{2,1} \oplus L_{3,2}^2$	$\mathbb{C} \oplus L_{3,2}$				
$i+j=5$	$\mathbb{C}^2 \oplus L_{2,1}^2 \oplus L_{3,2}^2$	$\mathbb{C}^2 \oplus L_{2,1} \oplus L_{3,2}^2$	$\mathbb{C} \oplus L_{3,2}$			
$i+j=7$	$\mathbb{C}^2 \oplus L_{2,1}^2 \oplus L_{3,2}$	$\mathbb{C}^3 \oplus L_{2,1}^6 \oplus L_{3,2}^2 \oplus L_{4,2}$	$\mathbb{C}^2 \oplus L_{2,1} \oplus L_{3,2}^2$	$\mathbb{C} \oplus L_{3,2}$		
$i+j=9$	$\mathbb{C}^3 \oplus L_{2,1}$	$\mathbb{C}^4 \oplus L_{2,1}^6 \oplus L_{3,2} \oplus L_{4,2}^2$	$\mathbb{C}^3 \oplus L_{2,1}^6 \oplus L_{3,2}^2 \oplus L_{4,2}$	$\mathbb{C}^2 \oplus L_{2,1} \oplus L_{3,2}^2$	$\mathbb{C} \oplus L_{3,2}$	
$i+j=11$	$\mathbb{C}^2$	$\mathbb{C}^3 \oplus L_{2,1}$	$\mathbb{C}^2 \oplus L_{2,1}^2 \oplus L_{3,2}$	$\mathbb{C}^2 \oplus L_{2,1}^2 \oplus L_{3,2}^2$	$\mathbb{C}^2 \oplus L_{2,1} \oplus L_{3,2}^2$	$\mathbb{C} \oplus L_{3,2}$
$h^{i,j}$	$j-i=1$	$j-i=3$	$j-i=5$	$j-i=7$	$j-i=9$	$j-i=11$

As noticed in [50], we obtain that $\mathfrak{g}$ is a subalgebra of HH^1 sitting in bidegree $(1,1)$. Their results imply that this copy of $\mathfrak{g}$ is acting in a compatible way on the center with both the algebraic and geometric $\mathfrak{g}$-action. We hope to understand the full action of HH^1 on HH^0, by understanding the geometric action of $\mathrm{HH}^1_{\mathbb{C}^*}(\widetilde{\mathcal{N}})$ on $\mathrm{HH}^0_{\mathbb{C}^*}(\widetilde{\mathcal{N}})$.

We also computed $\mathrm{HH}^s(\mathfrak{u}_q(\mathfrak{g}))$ for $s \leq 3$ and $\mathfrak{g} = \mathfrak{sl}_4$. We present the tables for $s = 1$:

Proposition 5.5.2. *The group* $\mathrm{HH}^1(\mathfrak{u}_q(\mathfrak{sl}_4))$ *is given by the following table:*

$i+j=1$	$\mathbb{C} \oplus L_{1,1,1}$					
$i+j=3$	$\mathbb{C}^4 \oplus L^3_{1,1,1}$	$\mathbb{C} \oplus L_{1,1,1}$				
$i+j=5$	$\mathbb{C}^9 \oplus L^5_{1,1,1}$	$\mathbb{C}^5 \oplus L^4_{1,1,1}$	$\mathbb{C} \oplus L_{1,1,1}$			
$i+j=7$	$\mathbb{C}^{11} \oplus L^3_{1,1,1}$	$\mathbb{C}^{13} \oplus L^8_{1,1,1} \oplus L^3_{1,2,1}$	$\mathbb{C}^5 \oplus L^4_{1,1,1} \oplus L_{1,2,1}$	$\mathbb{C} \oplus L_{1,1,1}$		
$i+j=9$	$\mathbb{C}^8$	$\mathbb{C}^{17} \oplus L^5_{1,1,1} \oplus L^3_{1,2,1}$	$\mathbb{C}^{13} \oplus L^8_{1,1,1} \oplus L^3_{1,2,1}$	$\mathbb{C}^5 \oplus L^4_{1,1,1}$	$\mathbb{C} \oplus L_{1,1,1}$	
$i+j=11$	$\mathbb{C}^3$	$\mathbb{C}^8$	$\mathbb{C}^{11} \oplus L^3_{1,1,1}$	$\mathbb{C}^9 \oplus L^5_{1,1,1}$	$\mathbb{C}^4 \oplus L^3_{1,1,1}$	$\mathbb{C} \oplus L_{1,1,1}$
$h^{i,j}$	$j-i=1$	$j-i=3$	$j-i=5$	$j-i=7$	$j-i=9$	$j-i=11$

5.5.1 Limits of the algorithm

With the computational resources available to the authors, it is not feasible to do the computation of the center of the principal block for type D_4 and B_4 with the current implementation of the algorithm. Hence, it seems that most of the computations are not accessible in rank ≥ 4. For higher Hochschild cohomology, in type A_2 we could go up to $s \leq 13$, but for type B_2 only $s \leq 7$. However, considering singular blocks makes the computation easier, and even for A_4 we could compute the Hochschild cohomology of the projective spaces up to $s \leq 6$ (this specific computation will be detailed in the next section).

5.6 Remarks on projective spaces

We now look at $X = \mathbb{P}^n = G/P$. For $s = 0$, the Hochschild cohomology was computed in [49]. Let us call a bidegree (i, j) *positive* if $i > 0$, and let us call the positive part of $\mathrm{HH}^s_{\mathbb{C}^*}(T^*(G/P))$ the direct sum of the positive bigraded summands. For $s = 1$, we show that the positive Hochschild cohomology is generated by the first column under the $\mathfrak{sl}_2$-action. For $s \geq 2$, we prove a *vanishing property*, and we compute several Hochschild cohomology groups for $T^*\mathbb{P}^3$ and $T^*\mathbb{P}^4$.

5.6.1 A subalgebra of HH^1

In this subsection we will compute the positive part of $\mathrm{HH}^1_{\mathbb{C}^*}(\mathbb{P}^n)$, and we present the full computation for $3 \leq n \leq 6$. We will need the following lemma:

Lemma 5.6.1 ([49])**.** *Let* $p' \neq p \leq n$*. Then* $\mathrm{H}^{p'}(\mathbb{P}^n, \Omega^{\otimes p}_{\mathbb{P}^n}) = 0$*. Moreover,* $\mathrm{H}^p(\mathbb{P}^n, \Omega^{\otimes p}_{\mathbb{P}^n}) = \mathbb{C}$.

Let us emphasize that the non-trivial cohomology comes from the split summand $\wedge^p \Omega_{\mathbb{P}^n} \subset \Omega^{\otimes p}$. We now want to compute the first column of $\mathrm{HH}^1_{\mathbb{C}^*}(\mathbb{P}^n)$. For this, we introduce $1 \leq p \leq n$ such that $j = i + 1 = p$. It follows that $k = -2p$, and $s = i + j + k = 1$.

Proposition 5.6.2. *Let* $1 \leq p \leq n$*. Then* $\mathrm{H}^{p-1}(\mathbb{P}^n, \wedge^p \mathrm{pr}_*(T_{\widetilde{\mathcal{N}}_P})^{-2p}) = \mathbb{C}$ *if* $p \neq 1$ *and* $\mathbb{C} \oplus \mathfrak{g}$ *if* $p = 1$.

Proof. First, we assume $p > 1$. By definition, there is a short exact sequence

$$0 \longrightarrow T_{\mathbb{P}^n} \otimes \wedge^p \Omega_{\mathbb{P}^n} \longrightarrow \wedge^p \mathrm{pr}_*(T_{\widetilde{\mathcal{N}}_P})^{-2p} \longrightarrow T_{\mathbb{P}^n} \otimes \wedge^{p+1} \Omega_{\mathbb{P}^n} \longrightarrow 0.$$

As usual, let $\omega_{\mathbb{P}^n} = \det(\Omega_{\mathbb{P}^n})$ be the canonical bundle on $\mathbb{P}^n$. Using $\wedge^p \Omega_{\mathbb{P}^n} \cong \wedge^{n-p} T_{\mathbb{P}^n} \otimes \omega_{\mathbb{P}^n}$, we obtain by Serre duality that

$$\mathrm{H}^p(\mathbb{P}^n, T_{\mathbb{P}^n} \otimes \wedge^p \Omega_{\mathbb{P}^n}) \cong \mathrm{H}^{n-p}(\Omega_{\mathbb{P}^n} \otimes \wedge^{n-p} \Omega_{\mathbb{P}^n})^*.$$

By lemma 5.6.1, we know that the cohomology group $\mathrm{H}^q(\mathbb{P}^n, \Omega_{\mathbb{P}^n} \otimes \wedge^m \Omega_{\mathbb{P}^n})$ is nonzero if and only if $q = m + 1$. Hence we have that

$$\mathrm{H}^{p-1}(\mathbb{P}^n, T_{\mathbb{P}^n} \otimes \wedge^p \Omega_{\mathbb{P}^n}) = \mathbb{C}, \quad \mathrm{H}^{p-1}(\mathbb{P}^n, T_{\mathbb{P}^n} \otimes \wedge^{p+1} \Omega_{\mathbb{P}^n}) = 0, \quad \text{and} \quad \mathrm{H}^{p-2}(\mathbb{P}^n, T_{\mathbb{P}^n} \otimes \wedge^{p+1} \Omega_{\mathbb{P}^n}) = 0.$$

It follows that the long exact sequence in cohomology gives the short exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathrm{H}^{p-1}(\mathbb{P}^n, \wedge^p \mathrm{pr}_*(T_{\widetilde{\mathcal{N}}_P})^{-2p}) \longrightarrow 0,$$

proving the proposition if $p > 1$.

If $p = 1$, then the composition factors of $(\mathrm{pr}_* T_{\widetilde{\mathcal{N}}_P})^0$ are $T_{\mathbb{P}^n} \otimes \Omega_{\mathbb{P}^n}$ and $T_{\mathbb{P}^n}$. We have

$$\mathrm{H}^0(\mathbb{P}^n, T_{\mathbb{P}^n} \otimes \Omega_{\mathbb{P}^n}) = \mathbb{C}, \quad \text{respectively} \quad \mathrm{H}^0(\mathbb{P}^n, T_{\mathbb{P}^n}) = \mathfrak{g}.$$

By the Borel-Bott-Weil theorem, $\mathrm{H}^1(\mathbb{P}^n, T_{\mathbb{P}^n}) = 0$ so the result follows using the long exact sequence. □

Corollary 5.6.3. *There is a subalgebra of* $\mathrm{HH}^1_{\mathbb{C}^*}(T\mathbb{P}^n)$ *of dimension* $(n+1)n/2 + (n^2-1)n$.

Proof. This follows easily from the existence of the $\mathfrak{sl}_2$-action on $\mathrm{HH}^1_{\mathbb{C}^*}(\mathbb{P}^n)$, together with the fact that the Poisson bivector field has bidegree $(i,j) = (0,2)$. □

Proposition 5.6.4. *The positive part of* $\mathrm{HH}^1_{\mathbb{C}^*}(\mathbb{P}^n)$ *coincides with the positive part of the subalgebra from the previous corollary, i.e for* (i,j) *with* $i \geq 1$ *we have* $\mathrm{HH}^1_{\mathbb{C}^*}(\mathbb{P}^n)^{(i,j)} = \mathbb{C}$.

Proof. We fix $1 \leq i \leq n \leq j \leq 2n$, where $i + j = 1 \mod 2$ and let $t = j - i$, and $r = (t+1)/2$. The corresponding cohomology group is $\mathrm{HH}^1_{\mathbb{C}^*}(\mathbb{P}^n)^{(i,j)} = H^i(\widetilde{\mathcal{N}}_P, \bigwedge^{i+t} T\widetilde{\mathcal{N}}_P)^{-2i-t+1}$. The sheaf $\mathrm{pr}_*(\bigwedge^{i+t} T\widetilde{\mathcal{N}}_P)^{-2i-t+1}$ has a filtration by factors $\mathcal{F}_l$ (where $0 \leq l \leq r$) given by

$$\mathcal{F}_l := S^{r-l} T_{\mathbb{P}^n} \otimes \bigwedge^l T_{\mathbb{P}^n} \otimes \bigwedge^{i+t-l} \Omega_{\mathbb{P}^n}.$$

We want to compute $\mathrm{H}^i(\mathcal{F}_l)$. Using Serre duality we obtain an isomorphism

$$\mathrm{H}^i(\mathcal{F}_l) \cong \mathrm{H}^{n-i}\left(S^{r-l}\Omega_{\mathbb{P}^n} \otimes \bigwedge^l \Omega_{\mathbb{P}^n} \otimes \bigwedge^{n+l-i-t} \Omega_{\mathbb{P}^n}\right)^*.$$

Denote the last factor in the tensor product above by $\mathcal{Q}_l$. Note that $\mathcal{Q}_l$ is then a direct summand of $\Omega^{\otimes(n+l+1-i-r)}$. Notice that $l+1-r-i \leq 0$, because $l \leq r$ and $i \geq 1$ by hypothesis, hence we can apply Lemma 5.6.1. When $l = r-1$, we have

$$\mathcal{Q}_l = \Omega_{\mathbb{P}^n} \otimes \bigwedge^{r-1} \Omega_{\mathbb{P}^n} \otimes \bigwedge^{n-i-r} \Omega_{\mathbb{P}^n},$$

hence $\mathcal{Q}_l$ contains the direct summand $\wedge^{n-i}\Omega_{\mathbb{P}^n}$, thus by Lemma 5.6.1 we obtain $\mathrm{H}^{n-i}(\mathcal{F}_{r-1}) = \mathbb{C}$. For other values of l, $\mathcal{Q}_l$ does not contribute. Therefore, again by Lemma 5.6.1, we have $\mathrm{H}^{n-i}(\mathcal{F}_l) = 0$. This shows that $\dim \mathrm{HH}^1_{\mathbb{C}^*}(\mathbb{P}^n)^{(i,j)} \leq 1$, and using the $\mathfrak{sl}_2$-action and Proposition 5.6.2 we conclude that $\dim \mathrm{HH}^1_{\mathbb{C}^*}(\mathbb{P}^n)^{(i,j)} = 1$. ☐

One can visualize this result as follows (we let $\mathfrak{g}' := \mathbb{C} \oplus \mathfrak{g}$, and **?** denotes an unknown factor):

$i+j=1$	$\mathfrak{g}'$						
$i+j=3$	$\mathbb{C}$	$\mathfrak{g}' \oplus$ **?**					
$i+j=5$	$\mathbb{C}$	$\mathbb{C}$	$\mathfrak{g}' \oplus$ **?**				
$i+j=7$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathfrak{g}' \oplus$ **?**			
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\mathfrak{g}' \oplus$ **?**	
$i+j=2n-1$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\cdots$	$\mathbb{C}$	$\mathfrak{g}'$
$h^{i,j}$	$j-i=1$	$j-i=3$	$j-i=5$	$j-i=7$	$\ldots$	$j-i=2n-3$	$j-i=2n-1$

We computed the bigraded summands of $\mathrm{HH}^1_{\mathbb{C}^*}(\mathbb{P}^n)$ for $i > 0$ in the previous proposition. We cannot compute these summands for $i = 0$ in general, but we present the missing bigraded summand for $3 \leq n \leq 6$ (giving the complete HH^1 using the $\mathfrak{sl}_2$-symmetry), obtained using our computer algorithm:

Proposition 5.6.5. *For $3 \leq n \leq 6$, the bigraded summands $\mathrm{HH}^1_{\mathbb{C}^*}(\mathbb{P}^n)^{(0,3)}$ and $\mathrm{HH}^1_{\mathbb{C}^*}(\mathbb{P}^n)^{(0,5)}$ are given by the following table:*

$n=3$	$\mathfrak{g}' \oplus L_{1,2,1}$	$\mathfrak{g}'$
$n=4$	$\mathfrak{g}' \oplus L_{1,2,2,1}$	$\mathfrak{g}'$
$n=5$	$\mathfrak{g}' \oplus L_{1,2,2,2,1}$	$\mathfrak{g}' \oplus L_{1,2,2,2,1} \oplus L_{1,2,3,2,1}$
$n=6$	$\mathfrak{g}' \oplus L_{1,2,2,2,1}$	$\mathfrak{g}' \oplus L_{1,2,2,2,1} \oplus L_{1,2,3,2,1}$
$i=0$	$j=3$	$j=5$

5.6.2 Higher Hochschild cohomology groups

We prove that $\mathrm{HH}^s_{\mathbb{C}^*}(\mathbb{P}^n)^{(i,j)} = 0$ if $s \geq 5$ and $i \geq s-2$. We also present some low-degree Hochschild cohomology groups for $n = 3, 4$. We begin by a lemma:

Lemma 5.6.6. *Let a, b, c be positive integers where $a \geq 3$ and $a+b+c = j$. Then, if V is a finite-dimensional vector space, the $\mathsf{SL}(V)$-module $\mathrm{Sym}^a(V) \otimes \wedge^b(V) \otimes \wedge^c(V)$ does not contain a copy of $\wedge^j(V)$.*

Proof. Let $V = \mathbb{C}^{n+1}$, and $\Gamma_{a_1,\ldots,a_n}$ be the irreducible $\mathsf{SL}(V)$-representation of highest weight $\sum_i a_i\varpi_i$, where ϖ_i are the fundamental weights. We have $\Gamma_{a,0,0,\ldots} \cong \mathrm{Sym}^a(V)$ and $\Gamma_{0,0,\ldots,0,1,0,\ldots,0} \cong \wedge^b(V)$ (where the 1 is at the b-th position). We recall that as a special case of the Littlewood-Richardson rule we have,

$$\Gamma_{a_1,\ldots,a_n} \otimes \wedge^k V \cong \bigoplus_{(b_1,\ldots,b_n)\in\mathcal{B}} \Gamma_{b_1,\ldots,b_n},$$

where $\mathcal{B}$ is a certain subset of $\mathbb{N}^n$, such that, if $(b_1,\ldots,b_n) \in \mathcal{B}$, then $|b_i - a_i| \leq 1$. We get

$$\mathrm{Sym}^a(V) \otimes \wedge^b(V) \otimes \wedge^c(V) \cong \bigoplus \Gamma_{c_1,\ldots,c_n},$$

where $(c_1,\ldots,c_n)$ are certain integers such that $c_1 \geq 1$, and thus the lemma follows. $\square$

Proposition 5.6.7. *If $s \geq 4$, and $s-2 \leq i$, we have* $\mathrm{HH}^s_{\mathbb{C}^*}(\mathbb{P}^n)^{(i,j)} = 0$.

Proof. As before, we fix $1 \leq i \leq n \leq j \leq 2n$, and $1 \leq r \neq n$ denotes the column in the bigraded table. We want to compute $\mathrm{H}^i(\widetilde{\mathcal{N}}_P, (\wedge^{s+i+2(r-1)}T\widetilde{\mathcal{N}}_P)^{-2i-2r+2}$. As usual $\mathrm{pr}_* \wedge^{s+i+2(r-1)} T\widetilde{\mathcal{N}}_P)^{-2i-2r+2}$ has a filtration with summands

$$\mathcal{F}_l := S^{r-l+s-2}T_{\mathbb{P}^n} \otimes \bigwedge^l T_{\mathbb{P}^n} \otimes \bigwedge^{s+i+2(r-1)-l} \Omega_{\mathbb{P}^n}.$$

Again, by Serre duality we have

$$\mathrm{H}^i(\mathbb{P}^n, \mathcal{F}_l) \cong \mathrm{H}^{n-i}(\mathbb{P}^n, S^{r-l+s-2}\Omega_{\mathbb{P}^n} \otimes \bigwedge^l \Omega_{\mathbb{P}^n} \otimes \bigwedge^{n-s-i-2(r-1)+l} \Omega_{\mathbb{P}^n})^*.$$

Denote the last sheaf by $\mathcal{Q}_l$, and note that it is a direct summand of $\Omega^{\otimes(n-i+l-r)}$. We know $l \leq r+s-2$, and, by hypothesis, $s-2 \leq i$, hence $l \leq i+r$. Therefore we can apply Lemma 5.6.1 to $p = n-i+l-r$, and get $\mathrm{H}^i(\mathbb{P}^n, \mathcal{F}_l) = 0$ if $l \neq r$. If $l = r$, the only summand that can contribute to the cohomology is $\wedge^p\Omega$. But $\mathcal{F}_r$ does not contains $\wedge^r\Omega_{\mathbb{P}^n}$ by Lemma 5.6.6, hence we have $\mathrm{H}^i(\mathbb{P}^n, \mathcal{F}_l) = 0$ as well. $\square$

We think that this vanishing holds more generally for all $i > 0$ and $s \geq 2$, based on our computer computations. This should hold only for projective spaces: already for Grassmannians we found several counterexamples.

We conclude this section by presenting tables of $\mathrm{HH}^s_{\mathbb{C}^*}(\mathbb{P}^3)$ for $0 \leq s \leq 9$ and $\mathrm{HH}^s_{\mathbb{C}^*}(\mathbb{P}^3)$ for $0 \leq s \leq 6$. We present truncated tables for readability, but the remaining part can be deduced from the $\mathfrak{sl}_2$-action, as explained in Section 5.2.

Proposition 5.6.8. *The group* $\mathrm{HH}^6_{\mathbb{C}^*}(\mathbb{P}^3)$ *is given by the following table:*

$i+j=0$	$L_{3,3,3}$	
$i+j=2$	0	$L_{3,3,2}L_{2,3,3}L_{4,3,2}L^2_{3,3,3}L_{2,3,4}L_{3,4,3}L_{4,4,3}L_{3,4,4}$
$i+j=4$	0	0
$i+j=6$	0	0
$h^{i,j}$	$j-i=0$	$j-i=2$

The group $\mathrm{HH}^7_{\mathbb{C}^*}(\mathbb{P}^3)$ *is given by the following table:*

$i{+}j{=}1$	$L_{3,3,3}L_{4,4,3}L_{3,4,4}L_{4,4,4}$	
$i{+}j{=}3$	0	$L_{4,3,2}L_{3,3,3}L_{2,3,4}L_{3,4,3}L^2_{4,4,3}L^2_{3,4,4}L_{5,4,3}L_{4,4,4}L_{3,4,5}L_{4,5,4}$
$i{+}j{=}5$	0	0
$h^{i,j}$	$j{-}i{=}1$	$j{-}i{=}3$

The group $\mathrm{HH}^2_{\mathbb{C}^*}(\mathbb{P}^4)$ *is given by the following table:*

$i{+}j{=}0$	$L_{1,1,1,1}$		
$i{+}j{=}2$	0	$L^2_{1,1,1,1}L_{1,2,2,1}L_{2,2,2,1}L_{1,2,2,2}$	
$i{+}j{=}4$	0	0	$L^2_{1,1,1,1}L^2_{1,2,2,1}L_{2,2,2,1}L_{1,2,2,2}L_{2,3,2,1}L_{1,2,3,2}$
$i{+}j{=}6$	0	0	0
$i{+}j{=}8$	0	0	0
$h^{i,j}$	$j{-}i{=}0$	$j{-}i{=}2$	$j{-}i{=}4$

The group $\mathrm{HH}^3_{\mathbb{C}^*}(\mathbb{P}^4)$ *is given by the following table:*

$i{+}j{=}1$	$L_{1,1,1,1}L_{2,2,2,1}L_{1,2,2,2}L_{2,2,2,2}$	
$i{+}j{=}3$	0	$L_{1,1,1,1}L_{1,2,2,1}L^2_{2,2,2,1}L^2_{1,2,2,2}L_{2,3,2,1}L_{2,2,2,2}L_{1,2,3,2}L_{3,3,2,1}L_{1,2,3,3}L_{2,3,3,2}$
$i{+}j{=}5$	0	0
$i{+}j{=}7$	0	0
$h^{i,j}$	$j{-}i{=}1$	$j{-}i{=}3$

5.7 Multiplicative structure of $z_0(\mathfrak{sl}_3)$

We now look at the principal block $\mathfrak{u}_0(\mathfrak{sl}_3)$. We let $X = \mathsf{SL}_3/B$. We recall the additive structure of $z_0(\mathfrak{sl}_3)$:

Theorem 5.7.1. *[48] As a vector space,* $z_0(\mathfrak{sl}_3) = HC[\tau] \oplus \mathbb{C}\{\xi\}$*, where* $\xi \in H^1(\widetilde{\mathcal{N}}, \wedge^3(T\widetilde{\mathcal{N}}))^{-4}$*.*

To study the multiplicative structure, we can look at G/B rather than $\widetilde{\mathcal{N}}$. This is clear that the multiplicative structure is preserved. Hence, if x, y is a basis of $H^1(X, \Omega_X)$ and τ is as before, we need to compute $\xi \cdot x, \xi \cdot y$ and $\xi \cdot \tau$. Actually, ξ is determined only up scaling and adding $x\tau$ and $y\tau$.

5.7.1 The local Cech complex and function on G (after Steven Jackson)

We explain a construction, due to Steven Jackson, which compute cohomology of homogeneous vector bundles on $X = G/B$, using explicit Cech cocycles. We will then compute explicit cocycles basis of $\mathbf{z}_0(\mathfrak{sl}_3)$. We expect that these computations can be used to compute the ring structure of $\mathbf{z}_0(\mathfrak{sl}_3)$. Let V be a finite-dimensional vector space over a field of characteristic zero k (in our case, V is indexed by W, thought as indexing an open cover $\{U_w\}_{w \in W}$ of a given open covering of G/B, and $k = \mathbb{C}(G)$). Pick a basis $e_1, \ldots, e_n$ of V and let $x_1, \ldots, x_n$ be the dual basis. Let $e = \sum_i e_i$ and $x = \sum_i x_i$.

Definition 5.7.2. *The local Cech complex of* V *is the complex* $C^i := \wedge^{i+1} V$*, with cohomological differential* $d(\omega) = e \wedge \omega$*.*

Theorem 5.7.3. *We have* $H^i(C^*) = k$ *if* $i = 0$*, and* 0 *otherwise.*

Proof. We notice that C^i can be made into an homological complex by $\partial(\omega) = \iota_x \omega$. Then, we have the identity $\partial \circ d + d \circ \partial = n$. It follows that if ω is a cocycle, then $(\partial\omega/n)$ is a primitive of ω (except in degree 0). □

Hence, if $d(c) = \omega$, and $\deg(\omega) \neq 1$ we deduce than $c = \partial(\omega/n) + d(\Xi)$ for $\Xi \in C^{i-2}$. If $\deg \omega = 1$ then $c = \partial(\omega/n) + ke$.
Recall that we are especially interested in cohomology of homogeneous vector bundles $E = G \times^B M$ on $X = G/B$, for a B-module M.

Proposition 5.7.4. *With same notation as before, there is a G-equivariant isomorphism $\Gamma(X, E) \cong \mathrm{Hom}_B(M^*, \mathbb{C}[G])$.*

Actually, such a description works also for $\Gamma(U, E)$ where U is open, but we need to replace $\mathbb{C}[G]$ by $\mathbb{C}[\pi^{-1}(U)]$ where $\pi : G \to G/B$ is the projection.

Theorem 5.7.5. *There is a cover $\mathfrak{U}_W = \{U_w\}_{w \in W}$ of G/B such that for $G = \mathsf{GL}_n$ we have*

$$\mathbb{C}[\pi^{-1}(U_1)] = \mathbb{C}[X_{1,1}, \dots, X_{n,n}, \det^{-1}, (\Delta_1^1)^{-1}, (\Delta_{12}^{12})^{-1}, \dots, (\Delta_{12\dots n-1}^{12\dots n-1})^{-1}]$$

where if $S, T \subset \{1, \dots, n\}$ have same cardinality, Δ_T^S is the minor with columns indexed by S and lines indexed by T. Functions on U_w are obtained by inverting minors where the lower entries are permuted by W.

Proposition 5.7.6. *For the regular left and right G-action on $\mathbb{C}[G]$, the functions $\Delta_T^{1,\dots,i}$ are right U-invariant, have right weight ω_i. According to the left action, the functions $\Delta_T^{1,\dots,i}$ span a copy of the representation $\wedge^i(\mathbb{C}^n)^*$.*

Let us also explicitely write the action of X_1 and X_2:

Proposition 5.7.7. *The action of X_1, X_2 on the set of minors Δ_S^T is as follows:*

$$\Delta_j^1 \cdot X_1 = \Delta_j^3 \cdot X_1 = 0, \Delta_j^2 \cdot X_1 = \Delta_j^1, \Delta_{kl}^{12} \cdot X_1 = \Delta_{kl}^{13} \cdot X_1 = 0, \Delta_{kl}^{23} \cdot X_1 = \Delta_{kl}^{13}$$

$$\Delta_j^1 \cdot X_2 = \Delta_j^2 \cdot X_2 = 0, \Delta_j^3 \cdot X_2 = \Delta_j^2, \Delta_{kl}^{12} \cdot X_2 = \Delta_{kl}^{23} \cdot X_2 = 0, \Delta_{kl}^{13} \cdot X_2 = \Delta_{kl}^{12}$$

Now, let us explain how to encode the cocycles using this data.

Proposition 5.7.8. *The data of a i-th cochain $c \in C^*(\mathfrak{U}_W, E)$ is the same as B-equivariant maps $c' : M^* \to \mathbb{C}[\pi^{-1}(U_{j_1} \cap \dots \cap U_{j_i})]$ for each $j_1 < \dots < j_i$.*

Let us relate the Cech complex $C^*(\mathfrak{U}_W, E)$ to the local Cech complex. The rings $\mathbb{C}[U_{j_1} \cap \dots \cap U_{j_i}]$ are all subrings of $k = \mathbb{C}(G)$, hence we obtain that $C^*(\mathfrak{U}_W, E)$ is a subcomplex of $\bigoplus_{\dim M^*} C$, where C is the local Cech complex associated to $k = \mathbb{C}(G)$ and $V = \oplus_{w \in W} k$. An open $U_{k_1} \cap \dots \cap U_{k_i}$ correspond to the element $e_{k_1} \wedge \dots \wedge e_{k_i}$. However, ∂ is not a map compatible with the global Cech complex, hence even if we know all local primitive of a given cocycle ω, maybe none of them are global primitive.

Hence, if $E = G\times^B M$, a cochain $c \in C^i(\mathfrak{U}_W, E)$ is the same as a set of B-equivariant map maps $M^* \to \mathbb{C}[U_{j_1}\cap\cdots\cap U_{j_i}]$ for each i-uple $j_1 < \cdots < j_i$. We can think of it as a list of sums $\sum_{j_1<\cdots<j_i} f_{j_1,\ldots,j_i} e_{j_1} \wedge \cdots \wedge e_{j_i}$, indexed by a basis of M^*.

5.7.2 Some useful tables

From now on, $G = SL_3$.We list here useful tables for us later. We enumerate first the Weyl group S_3:

Index	W
1	e
2	(12)
3	(13)
4	(23)
5	(123)
6	(132)

We write the set of elements A such that $\mathbb{C}[U_i \cap U_j] = \mathbb{C}[G][A^{-1}]$ and S for the function $\Delta_S^{1,\ldots,i}$.

Indices	A
12	$\{1\}, \{12\}, \{2\},$
13	$\{1\}, \{12\}, \{3\}, \{23\}$
14	$\{1\}, \{12\}, \{13\}$
15	$\{1\}, \{12\}, \{2\}, \{23\}$
16	$\{1\}, \{12\}, \{3\}, \{13\},$
23	$\{2\}, \{12\}, \{3\}, \{23\}$
24	$\{2\}, \{12\}, \{1\}, \{13\}$
25	$\{2\}, \{12\}, \{23\}$
26	$\{2\}, \{12\}, \{3\}, \{13\}$
34	$\{3\}, \{23\}, \{1\}, \{13\}$
35	$\{3\}, \{23\}, \{2\}$
36	$\{3\}, \{23\}, \{13\}$
45	$\{1\}, \{13\}, \{2\}, \{23\}$
46	$\{1\}, \{13\}, \{3\},$
56	$\{2\}, \{23\}, \{3\}, \{13\},$

5.7.3 The cocycles needed for the small quantum group

In this subsection, we will compute all the cocycles necessary to compute the products $\xi\tau, \xi x$ and ξy. It is especially useful to remember how we encode a cocycle using proposition 5.7.8 and the discussion after. It is also useful to remember the $\mathfrak{b}$-module V_j^k constructed in [48] such that $\bigoplus_{i+j+k=0} H^i(\mathfrak{n}, V_j^k) = z_0(\mathfrak{g})$.

The line bundle $\mathscr{L}_{-\alpha_1}$

We have $\mathscr{L}_{-\alpha_1} = G \times^B M$ where M is the 1-dimensional B-module, with trivial U-action and weight $\alpha_1 = (1, -1, 0)$. A 1-cocycle associate to each pair of elements $i, j \in W$ a section $s_{ij} \in \mathbb{C}[\pi^{-1}(U_i \cap U_j)]$ such that: 1) the left weight is $(0, 0, 0)$ (this is because we know $H^1(X, L)$ is a trivial representation) and 2) the right weight is $(-1, 1, 0)$ (the weight of M^*), and that the sections are right U-invariant (this is the B-equivariance condition). Hence candidates to construct the cocycles are the functions $f_1 = \frac{\{23\}}{\{2\}\{3\}}$ and f_2, f_3 defined similarly. Here, we abreviate the function $\Delta_S^{1,\dots,i}$ as S. Now, these functions are not regular everywhere, and we can make the list of pairs i, j such that f_k is regular on $\pi^{-1}(U_i \cap U_j)$. Hence, a candidate cocycle is

$$c(1) = (a_1e_{23}+a_2e_{26}+a_3e_{35}+a_4e_{56})f_1+(b_1e_{13}+b_2e_{16}+b_3e_{34}+b_4e_{46})f_2+(c_1e_{12}+c_2e_{15}+c_3e_{24}+c_4e_{45})f_3$$

Recall the Plucker relation $f_1 - f_2 + f_3 = 0$. Using this relation, we can solve the equation $dc = 0$ and obtain up to a nonzero scalar multiple

$$\tilde{x} := c(1) = (e_{23} + e_{26} - e_{35} + e_{56})f_1 + (e_{13} + e_{16} - e_{34} + e_{46})f_2 + (e_{12} + e_{15} - e_{24} + e_{45})f_3$$

Now, primitive of $c(1)$ are on the form $\partial c(1) + fe$, for $f \in \mathbb{C}(G)$. To be admissible, such a primitive should be regular on each open, and it's easy by looking at poles at the divisor $\{1\}$ for example than such f does not exist. Hence $c(1)$ is a basis of the space $H^1(G/B, \mathscr{L}_{-\alpha_1})$.

The line bundle $\mathscr{L}_{-\alpha_2}$

We omit details and just write the final answer for the cocycle:

$$\tilde{y} := c(2) = (e_{14} + e_{16} + e_{24} + e_{26})g_1 + (e_{13} + e_{15} + e_{23} + e_{25})g_2 + (-e_{34} - e_{36} + e_{45} - e_{56})g_3$$

where $g_1 = \frac{\{1\}\{123\}}{\{12\}\{13\}}$, $g_2 = \frac{\{2\}\{123\}}{\{12\}\{23\}}$ and $g_3 = \frac{\{3\}\{123\}}{\{13\}\{23\}}$.

The bundle Ω

We now need to lift our cocycle $c(1)$ and $c(2)$ found earlier to obtain representatives in $H^1(X, \Omega)$. For example, let us try to lift $c(1)$. We have $\Omega = G \times^B \mathfrak{n}$. The module $\mathfrak{n}$ is given by

Basis	Weights	X_1	X_2
X_1	$(1, -1, 0)$	0	$-X_{12}$
X_{12}	$(1, 0, -1)$	0	0
X_2	$(0, 1, -1)$	X_{12}	0

and the dual module is given by

Basis	Weights	X_1	X_2
Y_1	$(-1,1,0)$	0	0
Y_{12}	$(-1,0,1)$	$-Y_2$	Y_1
Y_2	$(0,-1,1)$	0	0

Recall that each coefficients in front of e_{ij} in the expression of $c(1)$ is a B-equivariant map from the 1-dimensional vector space $\mathbb{C}\{Y_1\}$ to $\mathbb{C}[\pi^{-1}(U_i \cap U_j)]$. To extend it in a B-equivariant way to $M^* = \mathbb{C}\{Y_1\} \oplus \mathbb{C}\{Y_{12}\} \oplus \mathbb{C}\{Y_2\}$, for each open $U_i \cap U_j$ we need to find an element F_{ij} such that $c(1)_{ij} = F_{ij} \cdot X_2$. Then, we can find the value of Y_{12}, and applying X_2 will gives us the value of Y_2. The non-trivial coefficients appearing in $c(1)$ are f_1, f_2 and f_3. We want to lift $f_1 = \frac{\{23\}}{\{2\}\{3\}}$, $f_2 = \frac{\{13\}}{\{1\}\{3\}}$ and $f_3 = \frac{\{12\}}{\{1\}\{2\}}$. Using proposition 5.7.7, we see that $F_1 = \frac{\Delta^{13}_{23}}{\{2\}\{3\}}$, $F_2 = \frac{\Delta^{13}_{13}}{\{1\}\{3\}}$ and $F_3 = \frac{\Delta^{13}_{12}}{\{1\}\{2\}}$ works. We obtain

$$\hat{x} := \tilde{c}(1)(y_{13}) = (e_{23} + e_{26} - e_{35} + e_{56})F_1 + (e_{13} + e_{16} - e_{34} + e_{46})F_2 + (e_{12} + e_{15} - e_{24} + e_{45})F_3$$

Hence the lift $\tilde{c}(1)$ is fully computed. Since f_i and F_i have the same poles for $i = 1, 2, 3$, we see that these lifts are still regular. It turns out that the analoguous of Plücker relations still holds: $\{1\}\Delta^{13}_{23} - \{2\}\Delta^{13}_{13} + \{3\}\Delta^{13}_{12} = 0$. It follows that $d\tilde{c}(1) = 0$, hence our lift of $c(1)$ is a cocycle as well. It's easy to check that $\tilde{c}(1)$ admits no primitive.

Completely similar computations apply for $c(2)$. The value of $\hat{y} := \tilde{c}(2)(Y_{12})$ is obtained by replacing the coefficients g_i by $-G_i$, where $G_i = \frac{\Delta^2_i\{123\}}{\{ij\}\{ik\}}$ (where $\{i,j,k\} = \{1,2,3\}$). Again, a Plücker-like relation $\{12\}\Delta^2_3 - \{13\}\Delta^2_2 + \{23\}\Delta^2_1 = 0$ ensures that $\tilde{c}(2)$ is indeed a cocycle, and it's easy to check that $\tilde{c}(2)$ is non-trivial.

The bundle $\Omega \otimes T$ and the products $x\tau, y\tau$

The module is $M = \mathfrak{n}^* \otimes \mathfrak{n}$. We pick the standard basis X_1, X_2, X_{12} of $\mathfrak{n}$, with dual basis Y_1, Y_2, Y_{12} and the cocycle is given by the constant function $\tau = \sum_{w \in W, I \in \{1,2,12\}} \delta_{II} e_w$, where $\delta_{IJ}(Y_K \otimes X_L) = \delta_{IK}\delta_{JL}$. Let us compute the product $x\tau$.

The bundle $\Omega^2 \otimes T$

A basis of $\Omega^2 \otimes T$ is given by $U_{I,J,K}$ where $I \neq J$ and $I, J, K \in \{1, 2, 12\}$, for example $U_{1,2,1}$ corresponds to $X_1 \wedge X_2 \otimes Y_1$. We notice that the $\mathfrak{b}$ module $\mathfrak{n}^2 \otimes \mathfrak{n}^*$ has a one-dimensional quotient with basis $U_{1,2,12}$. Let $M = \wedge^2\mathfrak{n} \otimes \mathfrak{n}^*/U_{1,2,12}$. The action of X_1 and X_2 on M^* are given by the following table (we pick the basis $V_{I,J,K}$ dual to $U_{I,J,K}$):

$$\begin{tikzcd}
 & V_{1,12,2} \arrow[dl, "X_1"'] \arrow[dr, "-X_1"] & \\
V_{1,12,12} & & V_{1,2,2} \\
V_{1,12,1} \arrow[u, "-X_2"'] \arrow[d, "-X_1"'] & & V_{2,12,2} \arrow[u, "X_2"] \arrow[d, "X_1"] \\
V_{1,2,1} & & V_{2,12,12} \\
 & V_{2,12,1} \arrow[ul, "X_2"] \arrow[ur, "-X_2"'] &
\end{tikzcd}$$

For example, the first line means that $X_2 \cdot V_{1,12,2} = 0, X_1 \cdot V_{1,12,2} = V_{1,12,12} - V_{1,2,2}$. Let M_1 be the submodule generated by $V_{1,12,12}, V_{1,2,2}, V_{1,2,1}$ and $V_{2,12,12}$, and $M_2 = M/M_1$. We see that $M_2 = L(1)^{\oplus 2} \oplus L(2)^{\oplus 2}$, where $L(i)$ is the 1-dimensional $\mathfrak{b}$-module with weight $-\alpha_i$. In particular, lifting the cocycle that corresponds to $(1,1,0,0) \in L(1)^{\oplus 2} \oplus L(2)^{\oplus 2}$ gives a cocycle (say ξ). Let us write the cocycles $x\tau, y\tau$ and ξ in $H^1(G/B, \Omega^2 \otimes T)$:

Cocycles	$V_{1,2,12}$	$V_{1,12,2}$	$V_{2,12,1}$	$V_{1,12,12}$	$V_{2,12,12}$	$V_{1,2,1}$	$V_{1,2,2}$	$V_{1,12,1}$	$V_{2,12,2}$
$x\tau$	0	0	0	$\tilde{x}$	0	0	$\tilde{x}$	$-\hat{x}$	$\hat{x}$
$y\tau$	0	0	0	0	$\tilde{y}$	$-\tilde{y}$	0	$-\hat{y}$	$-\hat{y}$
ξ	0	0	0	$\tilde{x}$	0	0	$\tilde{x}$	$-\hat{x}$	$\hat{x}$

The bundle $(pr_* \wedge^3 T\widetilde{\mathcal{N}})^{-4}$

Let V_3^{-4} the $\mathfrak{b}$-module corresponding to the vector bundle $pr_*(\wedge^3 T\widetilde{\mathcal{N}})^{-4}$ (for more details, see [48] section 2.2 and 2.3). We have a short exact sequence

$$0 \to \wedge^3 \mathfrak{n} \otimes \mathfrak{u} \to V_3^{-4} \to \wedge^2 \mathfrak{n} \otimes \mathfrak{u} \to 0$$

and we went to lift the cocycles x, y, ξ to cocycles in $H^1(G/B, (pr_* \wedge^3 T\widetilde{\mathcal{N}})^{-4})$. Let us write $u_I := X_1 \wedge X_2 \wedge X_{12} \otimes Y_I$ $(I \in \{1,2,12\})$ for a basis of $\wedge^3 \mathfrak{n} \otimes \mathfrak{u}$, and v_I for the corresponding dual basis. The vectors $U_{I,J,K}$ such that u_L appear in $X_i \cdot U_{I,J,K}$ are given by $X_1 \cdot U_{1,2,1} = U_{1,12,1} - u_{12}, X_1 \cdot U_{1,12,1} = -u_2, X_1 \cdot U_{2,12,1} = -2u_1, X_2 \cdot U_{1,2,2} = -U_{2,12,2} - u_{12}, X_2 \cdot U_{2,12,2} = u_1$ and $X_2 \cdot U_{1,12,2} = -2u_2$. We also have the natural $\mathfrak{b}$-module structure on $\wedge^3 \mathfrak{n} \otimes \mathfrak{u}$ given by the coadjoint action. It follows that the dual $\mathfrak{b}$-action on v_I is given by the following table:

Vector	X_1	X_2
v_1	$2V_{2,12,1}$	$v_{12} - V_{2,12,2}$
v_2	$V_{1,12,1} - v_{12}$	$2V_{1,12,2}$
v_{12}	$V_{1,2,1}$	$V_{1,2,2}$

We obtain the following cocycles:

Cocycles	v_1	v_2	v_{12}
$x\tau$	?	?	?
$y\tau$	0	0	$-\tilde{y}$
ξ	$-2\tilde{x}$	0	$-2\hat{x}$

5.7.4 Multiplicative structure of $z_0(\mathfrak{sl}_3)$

A basis of the module $(\wedge^3\mathfrak{n} \otimes \wedge^2\mathfrak{u})^*$ (corresponding to the vector bundle $\Omega^3 \otimes T^2$) is given by $W_1 = Y_1 \wedge Y_2 \wedge Y_{12} \otimes X_2 \wedge X_{12}, W_2 = Y_1 \wedge Y_2 \wedge Y_{12} \otimes X_1 \wedge X_{12}, W_{12} = Y_1 \wedge Y_2 \wedge Y_{12} \otimes X_1 \wedge X_2$. We have the following cocycles:

Cocycles	W_1	W_2	W_{12}
$x\tau^2$	$\tilde{x}$	0	$\hat{x}$
$y\tau^2$	0	$-\tilde{y}$	$\hat{y}$
$\xi\tau$	$-2\tilde{x}$	0	$-2\hat{x}$

Finally, for the bundle $\Omega^3 \otimes T$, we get the basis $Z_I = Y_1 \wedge Y_2 \wedge Y_{12} \otimes X_I$ $(I = 1, 2, 12)$:

Cocycles	Z_1	Z_2	Z_{12}
$x^2\tau$	0	0	0
$y^2\tau$	0	0	0
$x\xi$	0	$2\tilde{x}\hat{x}$	0
$y\xi$	$-\tilde{y}\hat{x}$	$\hat{y}\tilde{x}$	$-\tilde{x}\tilde{y}$

Hence we obtained explicit cocycles that represent the product, which is a significant progress toward understanding the multiplicative structure of the center. Indeed, it is easy to see that the Bott isomorphism relating sheaf cohomology to Lie algebra cohomology doesn't respect the product. However, these cocycles we computed with algebraic methods are exactly cocycles in the Cech complex associated to our bundles, hence the cup-product is the right one. We hope that this method can lead to a full description of the ring structure on $\mathbf{z}_0(\mathfrak{sl}_3)$.

Chapter 6

Table of symbols

- $\mathfrak{g}$: Lie algebra of the corresponding Lie group G
- $B, \mathfrak{b}$: a fixed Borel subgroup and its corresponding Lie algebra
- $\mathfrak{n}$: the nilradical of $\mathfrak{b}$
- $T/H, \approx, \mathfrak{h}$: a fixed torus inside B, and its corresponding Lie algebra
- P: the weight lattice
- Q: the root lattice
- ρ: half of the sum of all positive root
- $U(\mathfrak{g})$: the universal enveloping algebra of the Lie algebra $\mathfrak{g}$
- U': Drinfeld-Jimbo quantum group
- U: the big quantum group
- u: the small quantum group
- $L(\lambda)$: simple U-module of highest weight λ
- $\widetilde{W}_\ell$: the affine extended Weyl group $\widetilde{W}_\ell = W \ltimes \ell P$
- $\mathbf{\mathsf{z}}(\mathsf{u})$: the center of u
- u_0: the principal block of u
- u_λ: a block of u
- $\mathbf{\mathsf{z}}_0$: the principal block of the center
- z_λ: the corresponding block of the center
- $\mathcal{N}$: the nilpotent cone (associated to a Lie algebra $\mathfrak{g}$)

- $\widetilde{\mathcal{N}}$: the Springer resolution
- $\mathrm{Coh}(X)$: the category of coherent sheaves on an algebraic variety X
- $\mathrm{QCoh}(X)$: the category of quasi-coherent sheaves on an algebraic variety X
- $w \cdot \lambda$: the dot-action, given by $w \cdot \lambda = w(\lambda + \rho) - \rho$
- $\mathscr{L}_\lambda$: the line bundle on G/B associated to $\lambda \in P$
- $E[\lambda]$: the λ-weight space of the $\mathfrak{g}$-module E.
- $V_{j,k}$: certain $\mathfrak{g}$-modules used in chapter 5
- $\tau \in \wedge^2 T^*(\widetilde{\mathcal{N}})$: the Poisson bi-vector field
- DC_m: the diagonal coinvariant algebra associated to $\mathfrak{sl}_m$.
- $c_{a,b}$: the Catalan number $\frac{1}{a}\binom{a}{b}$
- Gr_G: the affine Grassmannian associated to G
- $\mathrm{Perv}_{(H)}(X)$: Category of perverse sheaves on X, constructible along the H-orbits
- $\mathscr{M}$: the moduli stack of twisted Higgs bundles
- $\widetilde{\mathscr{M}}$: the moduli stack of parabolic Higgs bundles
- $A = \bigoplus_{i=1}^r H^0(C, K^i)$: the base of the Hitchin fibration
- $f : \mathscr{M} \to A$: the Hitchin fibration
- $\Delta \subset X \times X$: the diagonal
- $t \in H\Omega^\bullet(X)$: the Todd class of X
- $\mathrm{Coh}^H(X)$: H-equivariant coherent sheaves on X
- $\mathcal{F}\ell_{\mathbf{I}}$: the affine flag variety
- $\mathcal{F}\ell_{\mathbf{I}}^\gamma$: an affine Springer fiber associated to $\gamma \in \mathfrak{g}((t))$.
- $\mathcal{P}_a$: the generalised Picard stack associated to $a \in A$
- $P_0^{red}(J_a)$: the local Picard stack at 0.
- $\mathrm{gr}^P H^*(X)$: the perverse filtration.
- Σ_r: the r-th Hirzebruch surface.

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