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ON ANGLES FORMED BY N POINTS OF THE EUCLIDEAN AND HYPERBOLIC PLANES

GHISLAIN JAUDON AND HUGO PARLIER

1. INTRODUCTION

Consider n points in the Euclidean plane, and consider the angles that are formed by all possible triples of points. In 1939, L. Blumenthal [1] proposed the problem of finding a sharp lower bound on the greatest of all these angles (which we shall refer to as the *mini-max problem*). The full solution was given by Bl. Sendov [3] more than a half century later, after several partial resolutions by a variety of authors. Recently, a very similar problem was proposed in [2] (Conjecture 6, p. 446) where the following is conjectured:

Among all angles formed by triples of n points in the plane there is an angle of at most $\frac{\pi}{n}$.

Furthermore, it is conjectured that the value $\frac{\pi}{n}$ is only obtained when the n points form a regular n -gon. At first glance, there is no apparent reason why the solution to finding the maximum least angle formed by n points (which we shall refer to as the *maxi-min problem*) would be any less difficult than the mini-max problem. The first goal of this note is to show that the conjecture is correct, and surprisingly the proof turns out to be elementary.

Subsequently, we treat the same question for n points in the hyperbolic plane. A similar proof shows that any configuration of n points of the hyperbolic plane form at least one angle strictly less than $\frac{\pi}{n}$. This value is also shown to be sharp, but this is slightly more complicated, because there is no maximal configuration as in the Euclidean case (and thus the “max” becomes a “sup”).

Before beginning with a proof of the conjecture, let us fix a few notations which we will carry with us throughout this note. For example, what we mean by the angle formed by a triple of points is the following, for both the Euclidean and hyperbolic plane. The angle associated to the triple (p, q, r) , denoted $\angle(p, q, r)$, is the positive interior angle of the triangle $\Delta(p, q, r)$ at point q . By our definition $\angle(p, q, r) = \angle(r, q, p) \in [0, \pi]$ and $\angle(p, q, r) \in]0, \pi[$ if $\Delta(p, q, r)$ is not degenerated. Let M be some metric space where the notions of triple of points and angle formed by a triple make

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some kind of sense. We shall denote a configuration of n points in M by \mathcal{C} and by $\alpha_{\min}(\mathcal{C})$ the smallest angle formed by all triples of \mathcal{C} . We further denote by $\mathcal{C}_n(M)$ the set of all possible configurations of n points. Finally, $\alpha_n(M) := \sup_{\mathcal{C} \in \mathcal{C}_n(M)} \alpha_{\min}(\mathcal{C})$. For instance, in the following section we shall show that $\alpha_n(\mathbb{R}^2) = \frac{\pi}{n}$ and that the “sup” is in fact a “max” attained by a unique type of configuration.

2. MINIMUM ANGLES IN THE PLANE

Our first goal is to show that n distinct points in the plane always form at least one angle which is at most $\frac{\pi}{n}$. Let us suppose that this is not the case, i.e., there exists a configuration \mathcal{C} of n points such that all the $3\binom{n}{3}$ angles formed by the $\binom{n}{3}$ triples of points are all strictly superior to $\frac{\pi}{n}$. In particular, every angle formed by three points in \mathcal{C} is in $]0, \pi[$.

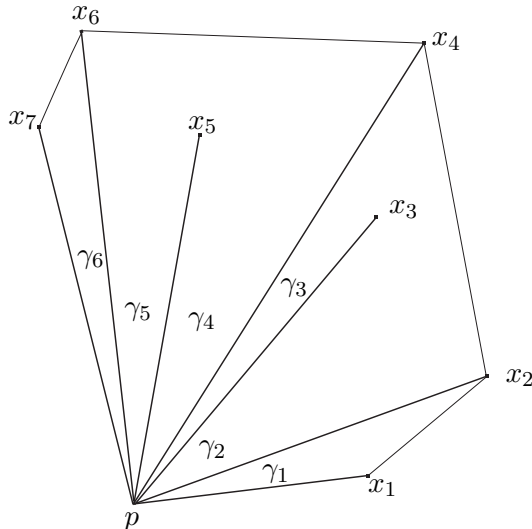


FIGURE 1. The convex hull of 8 points and the angles γ_i

Take an extremal point p in the convex hull X of \mathcal{C} . Let us denote by x_1 and x_{n-1} the two extremal points adjacent to p on the boundary of X (by hypothesis, X has at least three extremal points). Consider an enumeration x_2, \dots, x_{n-2} of the remaining points of \mathcal{C} such that $\angle(x_1, p, x_2) < \angle(x_1, p, x_3) < \dots < \angle(x_1, p, x_{n-1})$. We denote $\gamma_i := \angle(x_i, p, x_{i+1})$, for $i = 1, \dots, n-2$. Now, according to our assumption on \mathcal{C} , each angle γ_i is strictly greater than $\frac{\pi}{n}$. Therefore

$$(1) \quad \angle(x_1, p, x_{n-1}) = \sum_{i=1}^{n-2} \gamma_i > \frac{(n-2)\pi}{n}.$$

However we have assumed that $\angle(x_1, x_{n-1}, p) > \frac{\pi}{n}$ and $\angle(p, x_1, x_{n-1}) > \frac{\pi}{n}$, and thus we get the following contradiction:

$$\pi = \angle(x_1, x_{n-1}, p) + \angle(x_1, p, x_{n-1}) + \angle(p, x_1, x_{n-1}) > \frac{\pi}{n} + \frac{(n-2)\pi}{n} + \frac{\pi}{n} = \pi.$$

This shows that any configuration of n points in the plane necessarily contains an angle smaller than or equal to $\frac{\pi}{n}$. Moreover, if (C) is a regular n -gon, all points on the boundary of X , and for any choice of point p , the angles γ_i are all equal to $\frac{\pi}{n}$. The smallest angle is exactly equal to $\frac{\pi}{n}$ which shows that the bound is sharp. The next step is to show that if all of the angles are greater than or equal to $\frac{\pi}{n}$, then this is the only possible configuration, i.e., the n points are the vertices of a regular n -gon.

To show this, let us backtrack a little supposing this time that $\alpha_{\min}(C) = \frac{\pi}{n}$. We can consider the convex hull X of C , an extremal point p of X , and the same enumeration of the $n - 1$ remaining points. In this case, equation (1) becomes $\angle(x_1, p, x_{n-1}) = \sum_{i=1}^{n-2} \gamma_i \geq \frac{(n-2)\pi}{n}$. By the same reasoning, if $\angle(x_1, p, x_{n-1}) > \frac{(n-2)\pi}{n}$, then either angle $\angle(x_1, x_{n-1}, p)$ or angle $\angle(p, x_1, x_{n-1})$ is strictly less than $\frac{\pi}{n}$, a contradiction. Thus $\angle(x_1, p, x_{n-1}) = \frac{(n-2)\pi}{n}$, $\angle(x_1, x_{n-1}, p) = \angle(p, x_1, x_{n-1}) = \frac{\pi}{n}$ and the triangle $\triangle(p, x_1, x_{n-1})$ is isosceles. This reasoning can be applied to every extremal point on the convex hull, i.e., every extremal point q of X forms an isosceles triangle with its neighbors of the convex hull with interior angle in q equal to $\frac{(n-2)\pi}{n}$. It follows that the extremal points of the convex hull form a regular N -gon, with interior angles equal to $\frac{(n-2)\pi}{n}$. As the interior angle of a regular N -gon is exactly equal to $\frac{(N-2)\pi}{N}$, it follows that $N = n$, and thus the points of C form a regular n -gon. This proves the conjecture.

3. ANGLES OF POINTS IN THE HYPERBOLIC PLANE

Interestingly, the solution to the problem in the hyperbolic plane \mathbb{H} is very similar to the problem in the Euclidean plane. Our goal is not to fully present this complicated and beautiful metric space, but for clarity, let us recall a few facts concerning the hyperbolic plane. One way of viewing the hyperbolic plane is as the interior of the unit disk $\mathbb{D} = \{x \in \mathbb{R}^2 \mid \|x\| < 1\}$ endowed with a metric of constant curvature -1 . The traces of its geodesics are the intersection with \mathbb{D} of the circles and lines of \mathbb{R}^2 which are perpendicular to the boundary of \mathbb{D} (i.e., $\partial\mathbb{D} = S^1$) each time they intersect it. In particular, these circles and lines cross $\partial\mathbb{D}$ exactly twice, and the lines pass through the origin. As in \mathbb{R}^2 , there is a unique geodesic between two distinct points, and thus a unique way of defining the triangle associated to a triple of points (a, b, c) . Hyperbolic triangles differ from Euclidean ones in that the sum of their interior angles, say \angle_a , \angle_b and \angle_c , verifies

$$(2) \quad \angle_a + \angle_b + \angle_c < \pi.$$

Conversely, for any triple of real positive numbers $(\theta_a, \theta_b, \theta_c)$ whose sum is strictly less than π , there exists a hyperbolic triangle whose interior angles are exactly equal

to $(\theta_a, \theta_b, \theta_c)$ and this triangle is defined uniquely up to isometry. Furthermore, the area of a hyperbolic triangle Δ of interior angles $(\theta_a, \theta_b, \theta_c)$ is given by

$$(3) \quad \text{area}(\Delta) = \pi - (\theta_a + \theta_b + \theta_c).$$

Essentially, the smaller the interior of hyperbolic triangle gets, the more Euclidean it looks. For simplicity, our proof will only make use of these facts.

Our goal is to prove that $\alpha_n(\mathbb{H}) = \frac{\pi}{n}$. Let us begin by showing that for any given configuration \mathcal{C} , $\alpha_{\min}(\mathcal{C}) < \frac{\pi}{n}$. To show this, suppose that there is a configuration \mathcal{C} with $\alpha_{\min}(\mathcal{C}) \geq \frac{\pi}{n}$, and let us adopt the same notations as in the Euclidean case: the convex hull of our points will be denoted X , an extremal point of the convex hull p and the rest of the points of \mathcal{C} denoted x_i , $i = 1, \dots, n - 1$. (In fact, figure 1 remains valid up, albeit that the geodesics shouldn't be so straight.) We can now mimick the proof in the Euclidean case, and we obtain that $\angle(x_1, p, x_{n-1}) \geq \frac{(n-2)\pi}{n}$. Now considering the triangle $\Delta(p, x_1, x_{n-1})$ and by equation (2), $\min\{\angle(x_1, x_{n-1}, p), \angle(p, x_1, x_{n-1})\} < \frac{\pi}{n}$.

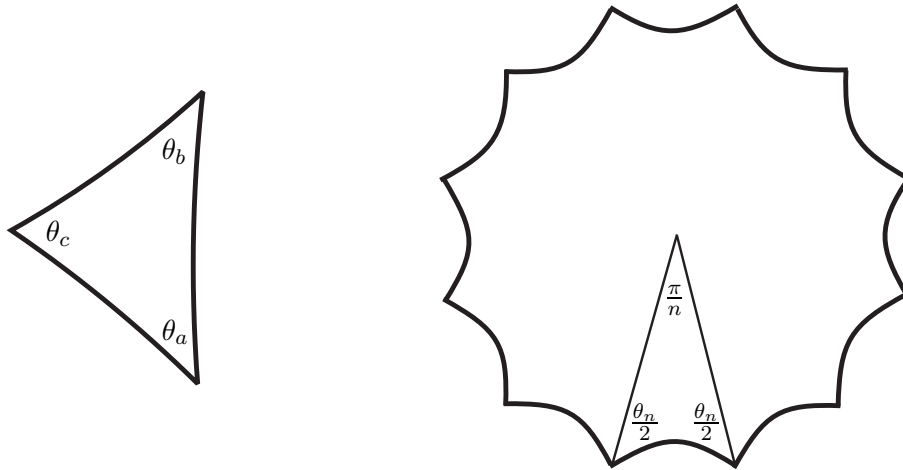


FIGURE 2. A hyperbolic triangle and a regular n -gon with $n = 12$

In order to prove $\alpha_n(\mathbb{H}) = \frac{\pi}{n}$, we need to show that for any $\varepsilon > 0$, there exists a configuration \mathcal{C}_ε of n points with $\angle_{\min}(\mathcal{C}_\varepsilon) \geq \frac{\pi}{n} - \varepsilon$. The configuration that will prove this is a regular n -gon inscribed in a small disk D_ε of area ε . (For those who are interested, the radius r_ε of such a disk is equal to $\text{arccosh} \frac{\varepsilon + 2\pi}{2\pi}$.) Any triangle Δ of points found inside or on the boundary of the disk will be entirely contained in the closed disk, and thus

$$\text{area}(\Delta) < \text{area}(D_\varepsilon) = \varepsilon.$$

By equation (3) it follows that the interior angles $(\theta_a, \theta_b, \theta_c)$ of \triangle verify

$$(4) \quad \pi - \varepsilon < \theta_a + \theta_b + \theta_c < \pi.$$

Now for $n \geq 3$, consider a regular n -gon inscribed in D_ε , which can be constructed as follows. First chose a point on the boundary of D_ε , take the unique geodesic segment between this point and the center of the disk, and take the image of this segment by rotations of angle $\frac{2\pi}{n}$ around the center of the disk. The intersection point of these segments and the boundary of the disks give you the n vertices of the regular n -gon. The polygon, denoted P_n , is then obtained by taking the n geodesic segments between successive vertices.

Consider the triangle formed by two successive vertices and the center of the disk (see figure 2). By what precedes, the triangle is formed by an edge of the polygon and two radii of the disk which meet at an angle of $\frac{2\pi}{n}$, and is thus isosceles. As we know the the length of the two radii, using hyperbolic trigonometry we could calculate the remaining lengths and angles of this triangle, but in fact we already know enough to prove our point. If we denote by θ_n the interior angle of our n -gon, the two equal interior angles of our triangle have value $\frac{\theta_n}{2}$. By equation (4) applied to this triangle, we have

$$\frac{n-2}{n}\pi - \varepsilon < \theta_n < \frac{n-2}{n}\pi.$$

Now we've constructed our n -gon, and we have a lower bound on the interior angles, but what remains to be proved is that any angle γ formed by a triple of vertices of the n -gon verifies $\gamma > \frac{\pi}{n} - \varepsilon$. Let us place ourselves in a vertex p of P_n , and consider our usual enumeration of our remaining vertices, i.e., x_1 and x_{n-1} the two vertices closest to p , x_2, \dots, x_{n-2} of the remaining vertices of P_n such that $\angle(x_1, p, x_2) < \angle(x_1, p, x_3) < \dots < \angle(x_1, p, x_{n-1}) = \theta_n$, and denote by γ_i the angle $\angle(x_i, p, x_{i+1})$. First of all, in the isosceles triangle $\triangle(x_1, p, x_2)$, we have two angles of value γ_1 , and by equation (4), we obtain

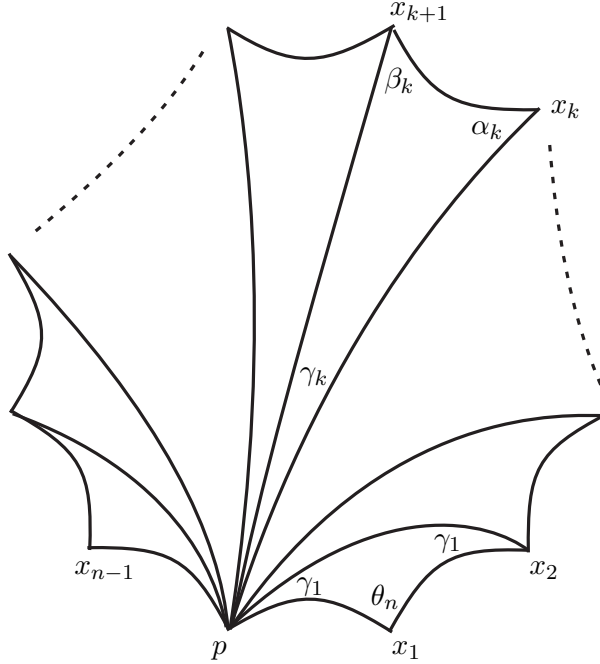
$$2\gamma_1 + \theta_n > \pi - \varepsilon,$$

and thus using what we know about θ_n

$$\gamma_1 > \frac{\pi}{n} - \varepsilon.$$

If, for a regular n -gon in the hyperbolic plane, we had $\gamma_i = \gamma_1$ for $i = 2, \dots, n-1$ as we do in the Euclidean plane, the proof would be finished, but in the hyperbolic plane this is not the case. We could prove this, but we definitely won't be using it, so we won't.

For $k > 1$, consider the triangle $\triangle(x_k, p, x_{k+1})$ which, in virtue of the regularity of P_n , has interior angles of values γ_k , $\alpha_k := \theta_n - \sum_{l=1}^{k-1} \gamma_l$ and $\beta_k := \sum_{l=1}^k \gamma_l$ (see figure 3). We can use equation (4) to obtain

FIGURE 3. A regular n -gon

$$\gamma_k + \sum_{l=1}^k \gamma_l + \theta_n - \sum_{l=1}^{k-1} \gamma_l = 2\gamma_k + \theta_n > \pi - \varepsilon$$

and as before

$$\gamma_k > \frac{\pi}{n} - \varepsilon.$$

All other angles formed by triples of the vertices are some sum of these angles and also verify the same inequality. For any n and any ε , we've shown the existence of a configuration \mathcal{C}_ε of n points of \mathbb{H} such that $\angle_{\min}(\mathcal{C}_\varepsilon) \geq \frac{\pi}{n} - \varepsilon$, which proves $\alpha_n(\mathbb{H}) = \frac{\pi}{n}$.

4. CONCLUDING REMARKS

One could ask the question of finding $\alpha_n(M)$ for other types of metric spaces or manifolds. When $M = \mathbb{R}^d$, there are a few immediate observations that one could make. Notice that for a configuration \mathcal{C} of n points in a Euclidean space, the inequality $\alpha_{\min}(\mathcal{C}) \leq \frac{\pi}{3}$ always holds and the equality $\alpha_{\min}(\mathcal{C}) = \frac{\pi}{3}$ is equivalent to having a configuration of n points all at an equal distance from one another. For \mathbb{R}^d , this is possible if and only if $n \leq d + 1$. (To see this consider a generalized tetrahedron, or formally a full-dimensional simplex.) It follows that $\alpha_n(\mathbb{R}^d) = \frac{\pi}{3}$ if and only if $n \leq d + 1$. Another problem which seems similar in nature to finding $\alpha_n(\mathbb{R}^3)$ is for $M = \mathbb{S}^2$. This problem is related to the problem of finding, for n points

on the sphere, a sharp upper bound on the minimum distance between them, which is a known difficult problem.

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