Spin Chains with Dynamical Lattice Supersymmetry

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Abstract Spin chains with exact supersymmetry on finite one-dimensional lattices are considered. The supercharges are nilpotent operators on the lattice of dynamical nature: they change the number of sites. A local criterion for the nilpotency on periodic lattices is formulated. Any of its solutions leads to a supersymmetric spin chain. It is shown that a class of special solutions at arbitrary spin gives the lattice equivalents of the $\mathcal{N} = (2, 2)$ superconformal minimal models. The case of spin one is investigated in detail: in particular, it is shown that the Fateev-Zamolodchikov chain and its off-critical extension possess a lattice supersymmetry for all its coupling constants. Its supersymmetry singlets are thoroughly analysed, and a relation between their components and the weighted enumeration of alternating sign matrices is conjectured.

Keywords Supersymmetry · Lattice models

1 Introduction

The aim of the present paper is to study models of two-dimensional statistical mechanics and related spin chains with exact space-time supersymmetry on the lattice. It is a systematic extension of the works on the XXZ and XYZ models investigated in [49, 96].

One of our main motivations is that many lattice models[1] are known to have continuum limits which possess supersymmetry. A rather prominent example is the tricritical Ising model: its scaling limit is described by a $\mathcal{N} = 1$ superconformal field theory [46, 81]. $\mathcal{N} = 1$ and $\mathcal{N} = 2$ were also observed the Ashkin-Teller model at special values of the coupling constants [4, 5, 94, 95]. Trigonometric vertex models with $U_q(\mathrm{sl}_2)$-symmetry were studied

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[1]We focus on two-dimensional lattice models which originate from statistical mechanics, and their one-dimensional quantum mechanical pendants (spin chains). Our discussion therefore excludes the wide field of lattice gauge theories with supersymmetry, pioneered in [33] (see e.g. [25] for a comprehensive overview).

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in [18, 44, 90]. The general outcome of these studies is that the supersymmetry occurs at some spin-anisotropy commensurable points—an observation which we will encounter in the present work, too. Models related to coset theories and their off-critical extensions are studied in [73, 76], and in the review paper [91], and height models in [27].

Even though this variety of examples has been known for a long time, it appears that an explicit lattice construction of the supercharges on the lattice has been accomplished in only rather few cases. After an early attempt in [77], the perhaps first successful construction were the so-called $\mathcal{M}_\ell$ models introduced in [40, 41] (an excellent and comprehensive introduction to the subject can be found in [54]). They describe spinless fermions on a one-dimensional lattice with the exclusion rule that at most ℓ consecutive sites may be occupied. The models possess an explicit supersymmetry whose supercharges insert or take out fermions. In particular, the $\mathcal{M}_1$ model (fermions with strong repulsion which forbids any two adjacent sites to be simultaneously occupied) has been studied for various boundary conditions [12, 42, 55]. Off-critical deformations are achieved through staggering of the coupling constants [13, 37, 38, 59], and their ground states appear to be related to classically integrable hierarchies. Moreover, higher-dimensional generalisations of these models were considered in [39, 56–58, 60], and display various exciting features such as extensive ground-state entropy, superfrustration and connections to rhombus tilings [62, 63].

The topic of the present work is explicit lattice supersymmetry for spin chains. Yet, the fermion models will serve as a great source of inspiration. The prime example for a connection between the two worlds is the spin-1/2 XXZ chain with anisotropy $\Delta = -1/2$ whose continuum limit corresponds to a superconformal field theory with central charge $c = 1$ (a free boson, compactified at a special radius). Using a mapping to the $\mathcal{M}_1$ model, the lattice supersymmetry of the spin chain was discovered by Yang and Fendley [96]. The supercharges have an unusual feature: they act non-locally and change the number of sites. Therefore, they are *dynamical*. It is worthwhile stressing that the concept of dynamical symmetries emerged also in the spin-chain description of scattering amplitudes and $\mathcal{N} = 4$ super Yang-Mills theory [14, 16, 17]. The XXZ results were subsequently extended to the spin-1/2 XYZ chain along an off-critical extension of the $\Delta = -1/2$ point: the lattice supersymmetry was constructed in [49], and shown to be manifest within the Bethe-ansatz for the eight-vertex model. It is a quite remarkable feature of the spin chains that they possess *two* sets of supercharges. Their algebraic relations yield a lattice counterpart of the $\mathcal{N} = (2, 2)$ supersymmetry algebra. Thus, the findings for the XYZ chain constitute a lattice equivalent of the non-local supersymmetry in the sine-Gordon field theory at the supersymmetric point [20]. The present article attempts to generalise the concept of dynamical lattice supersymmetry to spin chains with higher spin. This requires some general tools for the construction of nilpotent operators, and thus supercharges, on the lattice, and allows us to find spin chains with two copies of the supersymmetry algebra at arbitrary spin.

A very characteristic feature of supersymmetric theories is the special properties of their ground states. Indeed, only a short glimpse at the existing literature on both the XXZ chain at $\Delta = -1/2$ and the $\mathcal{M}_1$ model show that their ground-state components display various surprising relations with combinatorial problems such as the enumeration of alternating sign matrices, plane partitions, fully-packed loops [12, 28, 82, 83]. In the XXZ-case and for the related $O(1)$ loop model, these observations have pushed the development of rather powerful techniques, such as the application of the quantum Knizhnik-Zamolodchikov equation to combinatorial problems [29–31, 64, 85], and more combinatorial methods using dihedral symmetries [23, 24]. This allowed to prove the early conjectures. Off-critical extensions to the XYZ model are known, too [11, 75, 98]. Yet, in many cases a combinatorial interpretation is still missing (see however [87, 88]). Despite the omnipresence of lattice supersymmetry in these models its relation to combinatorics remains to be understood. The machinery

which is commonly used to analyse the ground states belongs rather to the world of quantum integrability. Yet, once established in a lattice model supersymmetry may be used as a heuristic tool in order to identify new interesting (and perhaps even combinatorial) ground states. In this work we present the (twisted) Fateev-Zamolodchikov spin chain as an example where this strategy is most fruitful. In this case, we demonstrate the existence of a lattice supersymmetry which does not present any obvious relation to the fermion models. Guided by this supersymmetry we point out various relations of its ground states and the weighted enumeration of alternating sign matrices [70].

The layout of this paper is the following. We start with a general considerations in Sect. 2, where we review the supersymmetry algebra, and outline the strategy to construct representations which are relevant for spin chains with periodic, twisted, and open boundary conditions. Section 3 discusses a special class of models for arbitrary spin, the so-called trigonometric models which are related to vertex models built from the six-vertex model through the fusion procedure. We proceed with a detailed discussion of the spin-one case in Sect. 4, and illustrate that the basic requirement for a supercharge to be nilpotent may lead to different families of inequivalent models. In particular, we show that the Fateev-Zamolodchikov spin chain and its off-critical deformation admit a lattice supersymmetry *for arbitrary coupling constants*. We discuss the zero-energy states of this model, and point out relations to the weighted enumeration of alternating sign matrices. Next, we present two other examples of supersymmetric spin-1 chains: on the one hand a spin chain which is closely related to the supersymmetric t–J model, for which we obtain an off-critical supersymmetry-preserving deformation, and on the other hand the so-called "mod-3 chain" which interpolates between a spin-1/2 and a spin-1 model. In Sect. 5 we present our conclusions as well as a variety of open problems and directions for further investigations. Some technical details are relegated to Appendices A–C.

Along the main text, we present several results based upon numerical analysis of small systems, for example exact diagonalisation. Many of them hint at structures which should be present for arbitrary system sizes, and therefore we formulate them as conjectures.

2 Lattice Supersymmetry

In this section we describe the concept of dynamical supersymmetry for spin chains. We start with a brief reminder of the well-known $\mathcal{N} = 2$ supersymmetry algebra in Sect. 2.1. In Sect. 2.2 we adopt this algebra to the setting of spin chains with periodic and twisted boundary conditions, and discuss the general structure of their Hamiltonians as well as basic symmetries. Moreover, we discuss the case when two mutually anticommuting copies of the supersymmetry algebra are present. Open boundaries are briefly treated in Sect. 2.3. Eventually, we review the example of the XYZ chain along its combinatorial line in Sect. 2.4.

2.1 Supersymmetry Algebra

The $\mathcal{N} = 2$ supersymmetry algebra is generated by two supercharges $Q, Q^\dagger$, the fermion number F and a Hamiltonian H. We denote by $\mathcal{H}$ the Hilbert space the algebra acts on. The fermion number induces a grading so that

$$\mathcal{H} = \bigoplus_{f=0}^{\infty} \mathcal{H}_f, \tag{1}$$

where $\mathcal{H}_f$ is the subspace of all states with f fermions. The supercharges are nilpotent mappings $Q : \mathcal{H}_f \to \mathcal{H}_{f+1}$, $Q^\dagger : \mathcal{H}_f \to \mathcal{H}_{f-1}$ which add or remove a fermion. We thus have the relations

$$Q^2 = \left(Q^\dagger\right)^2 = 0, \qquad [F, Q] = Q, \qquad \left[F, Q^\dagger\right] = -Q^\dagger.$$

Finally, the Hamiltonian $H : \mathcal{H}_f \to \mathcal{H}_f$ is given as an anticommutator of the supercharges

$$H = Q^\dagger Q + Q Q^\dagger.$$

It commutes with the supercharges Q, $Q^\dagger$ and the fermion number F.

The definition of the Hamiltonian implies that its spectrum is positive definite: any solution to $H|\psi\rangle = E|\psi\rangle$ has $E \geq 0$. Each eigenvalue $E > 0$ is necessarily doubly degenerate. It is easy to see that the corresponding eigenstates organise in pairs $(|\psi\rangle, Q|\psi\rangle)$ such that $Q^\dagger|\psi\rangle = 0$. They are called superpartners, and their fermion numbers differ by one. Conversely, zero-energy states must solve $Q|\psi\rangle = 0$ and $Q^\dagger|\psi\rangle = 0$, and therefore do not have superpartners. They are called *supersymmetry singlets*, and are known to be in a one-to-one correspondence with the elements of the quotient space $\ker Q / \operatorname{im} Q$ [92].

2.2 Supercharges and Spin Chains

Hilbert Space Our aim is to construct representations of the $\mathcal{N} = 2$ supersymmetry algebra for spin chains. The crucial idea is to identify the number of sites with the fermion number. Hence, the supercharges insert or remove a site. At each such site lives a spin-$\ell/2$ with $\ell = 1, 2, 3, \ldots$. We thus have a local quantum space $V \simeq \mathbb{C}^{\ell+1}$, and denote its canonical basis by $|m\rangle$, $m = 0, \ldots, \ell$ (we use the bra-ket-notation). The basis vectors are orthonormal with respect to the standard scalar product: $\langle m|n\rangle = \delta_{mn}$. The Hilbert space of a chain of length N is given by the N-fold tensor product $V^{\otimes N}$. The canonical choice for its basis consists of taking simple tensor products of local basis vectors. Hence, a basis vector is given by a sequence of integers $m_1, m_2, \ldots, m_N$. We use the common notation

$$|m_1, m_2, \ldots, m_N\rangle = \bigotimes_{j=1}^{N} |m_j\rangle.$$

The spin is measured by an operator s^3: on the basis $\{|m\rangle\}$ it acts according to $s^3|m\rangle = (m - \ell/2)|m\rangle$. Hence, s^3 is just the third member of the three su(2) generators at spin-$\ell/2$. If acting on the j-th factor of $V^{\otimes N}$ we write s_j^3 (and do so for any local operator). The total spin of any state in $V^{\otimes N}$ is measured by $S_N^3 = \sum_{j=1}^{N} s_j^3$. Here, the index N indicates the number of sites, what is sometimes necessary as we deal with operators which change the number of sites. Sometimes, we will also think of $M = S_N^3 + \ell N/2$ as the total number of particles as $|m\rangle$ can be interpreted as a state with m particles at a given sites. This will prove to be useful in Sect. 3.

Translation Invariance, Lattice Supercharges In order to implement the supersymmetry, we have to specify the spaces $\mathcal{H}_N$ in (1). They are subspaces of $V^{\otimes N}$ whose precise form depends on the boundary conditions. Most of this work is concerned with periodic or twisted boundary conditions (we restrict our considerations to diagonal twists). We denote by ϕ the twist angle, and introduce a shift operator $T_N(\phi) = T_N e^{i\phi s_N^3}$ where T_N is the usual translation operator acting on basis states for N sites according to

$$T_N|m_1, m_2, \ldots, m_{N-1}, m_N\rangle = |m_N, m_1, m_2, \ldots, m_{N-1}\rangle.$$

The $\mathcal{H}_N$ are related to the eigenspaces of the twisted shift operators. We will sometimes call them (pseudo)momentum sectors. In order to single out the relevant sectors, we need to explain the construction of the supercharges. These are built from local operations q_j : $V^{\otimes N} \to V^{\otimes(N+1)}$ which are related by translations. They are defined as

$$\mathsf{q}_j = (-1)^{j-1}(1 \otimes \cdots \otimes 1 \otimes \underbrace{\mathsf{q}}_{j} \otimes 1 \otimes \cdots \otimes 1), \quad j = 1, \ldots, N,$$

and therefore transform a spin at site j to a pair at sites $j, j + 1$. The matrix elements for this transformation are encoded in the *local supercharge* $\mathsf{q} : V \to V \otimes V$. Moreover, the operation is weighted by a site-dependent string $(-1)^{j-1}$ which will be crucial to make the supercharge nilpotent. We find

$$\mathsf{q}_{j+1} = -T_{N+1}(\phi)\mathsf{q}_j T_N(\phi)^{-1}, \quad j = 1, \ldots, N-1.$$

On a system with $N + 1$ sites, a pair has $N + 1$ possible positions. This motivates the introduction of another local operator q_0 through extension of the preceding equation to $j = 0$

$$\mathsf{q}_0 = -T_{N+1}(\phi)^{-1}\mathsf{q}_1 T_N(\phi) = (-1)^N T_{N+1}(\phi)\mathsf{q}_N.$$

This operator acts always on the last site (irrespectively of the number of sites). We see that there are two definitions for q_0: (i) shifting the action of q_1 to the left or (ii) shifting the action of q_N to the right. These definitions have to be compatible what is true only if the sign $(-1)^N$ is present in the second definition. Because of this requirement we find that the $\mathcal{H}_N$ are the eigenspaces of $T_N(\phi)$ in $V^{\otimes N}$ with eigenvalues $t_N = (-1)^{N+1}$. For a non-vanishing twist angle, this imposes a constraint on the magnetisation. Indeed, it follows from $T_N(\phi)^N = e^{i\phi S_N^3}$ that we need to restrict our considerations to subsectors where

$$\phi S_N^3 = 0 \quad \text{mod } 2\pi. \tag{2}$$

Often, a non-zero twist angle cannot be chosen arbitrarily. There are constraints coming from the local operator q: its components, defined through

$$\mathsf{q}|m\rangle = \sum_{j,k=0}^{\ell} a_{m,jk}|jk\rangle,$$

have to solve the equation

$$a_{m,jk}\left(e^{i\phi(\ell/2+m-j-k)} - 1\right) = 0. \tag{3}$$

Once again, this is a consequence of the two definitions of q_0 given above. This equation holds always for $\phi = 0$. Below, we will see that non-zero ϕ's are often quantised: only very particular values are compatible with this condition.

With all these preliminary definitions we are ready to define the *global supercharge* acting on $\mathcal{H}_N$ as the following sum of the local operators:

$$Q_N = \sqrt{\frac{N}{N+1}} \sum_{j=0}^{N} \mathsf{q}_j.$$

In all other (pseudo)momentum sectors we set $Q_N \equiv 0$. It is then straightforward to see that $T_{N+1}(\phi) Q_N T_N(\phi)^{-1} = -Q_N$ on $\mathcal{H}_N$. We thus see that Q_N does not only insert a site but also increases the (pseudo)momentum by π. It is worthwhile stressing that the construction of the supercharges, the restriction to special quantum sectors as well as the quantisation of twist angles fits well into the general framework of non-local currents in lattice models as developed by Bernard and Felder [19].

Nilpotent Operators   Our aim is to choose the local operator $\mathsf{q} : V \to V \otimes V$ so that Q_N is nilpotent in the sense that

$$Q_{N+1} Q_N = 0. \tag{4}$$

We give two very simple criteria for this property to hold: (i) the Proposition 1 rules out redundant parameters in q and leads to the concept of gauge-equivalent local supercharges; (ii) a criterion for the global supercharge to be nilpotent is given in Proposition 2. In both cases, the arguments are based on simple telescopic cancellations.

Proposition 1 *Suppose that there exists a $|\chi\rangle \in V$ such that $\mathsf{q}|m\rangle = |m\rangle \otimes |\chi\rangle + |\chi\rangle \otimes |m\rangle$ for every $|m\rangle \in V$. Then the global supercharge Q_N vanishes.*

Proof It is sufficient to verify the statement on basis vectors. Let us first consider a simple spin configuration $|\mu\rangle = |m_1, \ldots, m_N\rangle$ for any admissible choice of the labels m_j. Then

$$\mathsf{q}_i |\mu\rangle = (-1)^{i-1} \big(|m_1, \ldots, m_i\rangle \otimes |\chi\rangle \otimes |m_{i+1}, \ldots, m_N\rangle$$
$$+ |m_1, \ldots, m_{i-1}\rangle \otimes |\chi\rangle \otimes |m_i, \ldots, m_N\rangle \big)$$

and

$$\mathsf{q}_0 |\mu\rangle = - \big(\big(T_N(\phi) |\mu\rangle \big) \otimes |\chi\rangle + |\chi\rangle \otimes |\mu\rangle \big).$$

The sum over all q_j leads to telescopic cancellations, and we are left with

$$\left(\sum_{j=0}^{N} \mathsf{q}_j \right) |\mu\rangle = \big((-1)^{N+1} |\mu\rangle - T_N(\phi) |\mu\rangle \big) \otimes |\chi\rangle.$$

From any state $|\mu\rangle$ we can construct an eigenvector of the shift operator, given by

$$|\psi\rangle = \sum_{j=0}^{N-1} t_N^{-j} T_N(\phi)^j |\mu\rangle.$$

Using the preceding formula, we find that

$$Q_N |\psi\rangle = \sqrt{\frac{N}{N+1}} \big((-1)^{N+1} |\psi\rangle - T_N(\phi) |\psi\rangle \big) \otimes |\chi\rangle$$
$$= \sqrt{\frac{N}{N+1}} \big((-1)^{N+1} - t_N \big) |\psi\rangle \otimes |\chi\rangle$$

which vanishes for $t_N = (-1)^{N+1}$.   □

We conclude that any q contains $\ell + 1$ redundant parameters as long as no other restrictions are imposed. Moreover, we see that two local supercharges q, q' are *gauge-equivalent* in the sense that if

$$\mathsf{q}' = \mathsf{q} + \mathsf{q}_\chi, \quad \mathsf{q}_\chi|m\rangle = |m\rangle \otimes |\chi\rangle + |\chi\rangle \otimes |m\rangle,$$

then the global supercharges coincide $Q_N = Q'_N$. This provides a way to remove redundancies. Once this is done we need a more refined criterion in order to decide if (4) holds. Crucial for the property of Q_N to be nilpotent is the implicit string $(-1)^{j-1}$ in the definition of q_j. It leads to a set of anticommutation relations:

$$\mathsf{q}_i\mathsf{q}_j + \mathsf{q}_{j+1}\mathsf{q}_i = 0, \quad 1 \leq i < j \leq N, \tag{5a}$$

$$\mathsf{q}_0\mathsf{q}_j + \mathsf{q}_{j+1}\mathsf{q}_0 = 0, \quad j = 1, \ldots, N-1. \tag{5b}$$

They simplify considerably the expression $Q_{N+1}Q_N$, and leave only a few terms to be analysed. Guided by the nature of the preceding argument, which is based on telescopic cancellations, we are led to the following

Proposition 2 *The supercharge is nilpotent if there exists a vector* $|\chi\rangle \in V \otimes V$ *such that*

$$\big((\mathsf{q} \otimes 1)\mathsf{q} - (1 \otimes \mathsf{q})\mathsf{q}\big)|m\rangle = |\chi\rangle \otimes |m\rangle - |m\rangle \otimes |\chi\rangle, \quad \text{for all } |m\rangle \in V. \tag{6}$$

Proof Notice that because of the anticommutation relations (5) we can reduce the expression (4) to

$$Q_{N+1}Q_N = \sqrt{\frac{N}{N+2}}\left(\sum_{j=0}^{N}(\mathsf{q}_{j+1} + \mathsf{q}_j)\mathsf{q}_j + \mathsf{q}_{N+1}\mathsf{q}_0 + \mathsf{q}_0\mathsf{q}_N\right).$$

We investigate the action of the different terms on a simple basis vector $|\mu\rangle = |m_1, \ldots, m_N\rangle$. For $i = 1, \ldots, N$ we have

$$(\mathsf{q}_{i+1} + \mathsf{q}_i)\mathsf{q}_i|\mu\rangle = |m_1, \ldots, m_i\rangle \otimes |\chi\rangle \otimes |m_{i+1}, \ldots, m_N\rangle$$
$$- |m_1, \ldots, m_{i-1}\rangle \otimes |\chi\rangle \otimes |m_i, \ldots, m_N\rangle.$$

The remaining two terms are slightly different:

$$(\mathsf{q}_1 + \mathsf{q}_0)\mathsf{q}_0|\mu\rangle = T_{N+2}(\phi)\big((T_N(\phi)|\mu\rangle) \otimes |\chi\rangle\big) - |\mu\rangle \otimes |\chi\rangle$$
$$(\mathsf{q}_{N+1}\mathsf{q}_0 + \mathsf{q}_0\mathsf{q}_N)|\mu\rangle = (-1)^{N+1}\big((T_N(\phi)|\mu\rangle) \otimes |\chi\rangle - T_{N+2}(\phi)\big(|\mu\rangle \otimes |\chi\rangle\big)\big).$$

Hence, we find

$$\left(\sum_{j=0}^{N}(\mathsf{q}_{j+1} + \mathsf{q}_j)\mathsf{q}_j + \mathsf{q}_{N+1}\mathsf{q}_0 + \mathsf{q}_0\mathsf{q}_N\right)|\mu\rangle$$
$$= T_{N+2}(\phi)\big((T_N(\phi)|\mu\rangle - (-1)^{N+1}|\mu\rangle) \otimes |\chi\rangle\big)$$
$$+ \big((-1)^{N+1}T_N(\phi)|\mu\rangle - |\mu\rangle\big) \otimes |\chi\rangle.$$

From $|\mu\rangle$ we can construct a momentum state $|\psi\rangle$ like in the preceding proposition. As Q_N only acts on states with $T_N(\phi)|\psi\rangle = (-1)^{N+1}|\psi\rangle$ it follows that $Q_{N+1}Q_N|\psi\rangle = 0$. $\qquad\square$

Equation (6) leads to an efficient algorithm for finding supercharges at arbitrary spin. The general strategy is the following. First, choose a rectangular $(\ell + 1)^2 \times (\ell + 1)$ matrix q with possible restrictions (some entries might be zero due to desired symmetries such as spin-reversal symmetry, particle number conservation etc.). Second, determine $|\chi\rangle$ by solving the linear equations for its components, defined through $|\chi\rangle = \sum_{m,n=0}^{\ell} \chi_{mn} |m, n\rangle$. In terms of the components of the local supercharge $a_{m,ij}$ we find two expressions

$$\chi_{mn} = \begin{cases} \sum_{j=0}^{\ell} (a_{k,mj} a_{j,nk} - a_{k,jk} a_{j,mn}), & m \neq k, \\ \sum_{j=0}^{\ell} (a_{k,jn} a_{j,km} - a_{k,kj} a_{j,mn}), & n \neq k. \end{cases} \tag{7}$$

These expressions must hold for any allowed value of k. Third, one is generically left with a number of quadratic equations for the non-zero components of q whose solutions lead to a nilpotent supercharge. There may be many solutions, some of them inequivalent in the sense that q and q' cannot be related through $\mathsf{q}' = (u \otimes u) \mathsf{q} u^{-1}$ (where u is a unitary transformation on V) and/or a gauge transformation. It would certainly be interesting to classify them. In the following chapters, we give various interesting examples.

Hamiltonian Given a solution to (4), we construct the Hamiltonian as the anticommutator

$$H_N = Q_N^\dagger Q_N + Q_{N-1} Q_{N-1}^\dagger. \tag{8}$$

Notice that its action is non-zero only when acting on the subsectors where the translation operator $T_N(\phi)$ has eigenvalue $(-1)^{N+1}$. On these sectors, we have all the features of a supersymmetric model explained in Sect. 2.1. The spectrum of the Hamiltonian is positive. Possible zero-energy eigenstates (supersymmetry singlets) are annihilated by both Q_N and $Q_{N-1}^\dagger$, and in one-to-one correspondence with the quotient space $\mathfrak{H}_N = \ker Q_N / \mathrm{im} \, Q_{N-1}$. All other eigenstates are organised in supersymmetry doublets $(|\psi\rangle, Q_N |\psi\rangle)$: due to the property that $H_{N+1} Q_N = Q_N H_N$ the superpartners have the same energy.

For later developments it will be useful to understand how the Hamiltonian H_N acts locally. In fact, Eq. (8) can be simplified as there exists a second set of anticommutation relations, this time between the local q_j and their adjoints:

$$\mathsf{q}_i \mathsf{q}_j^\dagger + \mathsf{q}_{j+1}^\dagger \mathsf{q}_i = 0, \quad 1 \leq i < j - 1 \leq N - 1,$$

$$\mathsf{q}_0 \mathsf{q}_j^\dagger + \mathsf{q}_{j+1}^\dagger \mathsf{q}_0 = 0, \quad 2 \leq i \leq N - 1.$$

As before, these relations are a simple consequence of the shift and the implicit string in the definition of the q_j, and do not depend on the actual structure of q. Using these, we find that the Hamiltonian can be written as sum over nearest-neighbour interactions:

$$H_N = \Pi_N \left(\sum_{j=1}^{N} h_{j,j+1} \right) \Pi_N.$$

Here, $\Pi_N = N^{-1} \sum_{j=0}^{N-1} (-1)^{(N+1)j} T_N(\phi)^j$ is the projector on the momentum spaces we are interested in, and $h_{j,j+1}$ is the Hamiltonian density acting on sites $j, j + 1$. We have $h_{j+1,j+2} = T_N(\phi) h_{j,j+1} T_N(\phi)^{-1}$ with the site identification $N + 1 \equiv 1$. As a matrix, the density is given by

$$h = -(\mathsf{q}^\dagger \otimes 1)(1 \otimes \mathsf{q}) - (1 \otimes \mathsf{q}^\dagger)(\mathsf{q} \otimes 1) + \mathsf{q}\mathsf{q}^\dagger + \frac{1}{2}(\mathsf{q}^\dagger \mathsf{q} \otimes 1 + 1 \otimes \mathsf{q}^\dagger \mathsf{q}). \tag{9}$$

This choice is of course not unique because of translation invariance, yet it appears natural because h becomes self-adjoint.

Given that the local supercharges q are not unique we ask how the Hamiltonian density h defined in (9) changes if we deform q. Proposition 1 provides a criterion for redundant parameters within the local supercharges: if we have $\mathsf{q}' = \mathsf{q} + \mathsf{q}_\chi$ with $\mathsf{q}_\chi |m\rangle = |\chi\rangle \otimes |m\rangle + |m\rangle \otimes |\chi\rangle$, $|\chi\rangle \in V$, then $Q_N = Q'_N$. Yet the local densities h and h' differ in general. Indeed we find

$$h' = h + (X \otimes 1 - 1 \otimes X), \tag{10a}$$

with

$$X = \frac{1}{2} \sum_{ijk} \chi_j (a_{i,jk} - a_{i,kj} + a_{k,ji} - a_{k,ij}) |i\rangle\langle j|. \tag{10b}$$

The term $X \otimes 1 - 1 \otimes X$ deserves the name 'local boundary' for obvious reasons. We see that it vanishes obviously if $a_{i,jk} = a_{i,kj}$. When summing over all sites the contributions from the local boundary terms cancel out telescopically and the total Hamiltonians for q and q' are the same. Yet, local boundary terms may be helpful in order to understand local symmetries. We will see this when studying quantum group symmetries in Sects. 3.3 and 4.1.3.

Basic Symmetries   If the local supercharge q has some symmetries, then they are naturally inherited by the Hamiltonian density h, and may lead to symmetries of the Hamiltonian H_N. We will be particularly interested in cases related to particle number conservation/conservation of the magnetisation. We thus analyse if there are suitable $\alpha \in (0, 2\pi)$ such that the equation

$$\left[e^{i\alpha(s^3 \otimes 1 + 1 \otimes s^3)}, h \right] = 0$$

holds. We distinguish two special cases. A supercharge is called *trigonometric* if this equation holds for arbitrary value of α in case of which the total magnetisation is conserved. This is the case if $(s^3 \otimes 1 + 1 \otimes s^3)\mathsf{q} = \mathsf{q}(s^3 + m)$ for a fixed number $m \in \mathbb{Z}/2$. It is called *elliptic* if the condition on h holds only for $\alpha = \pi$. In that case the total magnetisation is conserved mod 2. The denomination is clearly inspired from the context of integrable vertex models whose R-matrices have generically trigonometric weights if some particle number conservation is imposed, whereas the weights are expressed in terms of elliptic functions if this conservation holds only mod 2. However, these are not the only interesting cases. Indeed, in Sect. 4.3 we discuss a case where $\alpha = 2\pi/3$ so that the magnetisation is conserved mod 3.

We discuss briefly two other symmetries for closed chains with periodic boundary conditions and open chains. We start with the spin-reversal operator defined through

$$R|m\rangle = |\ell - m\rangle.$$

If q is invariant under spin reversal in the sense that $(R \otimes R)\mathsf{q}R = \pm\mathsf{q}$ then h will be invariant $[R \otimes R, h] = 0$, and thus $[R_N, H_N] = 0$ where $R_N = R^{\otimes N}$. While the local invariance under spin reversal remains valid in the case of non-zero twist angle, the global symmetry operator needs to be replaced by $R'_N = R_N K$ where K denotes complex conjugation. The simplicity of all these statements is due to the fact that we deal with a local operation. This is not the case for the reflection/parity operator acting on a state for N spins according to $P_N |m_1, m_2, \ldots, m_{N-1}, m_N\rangle = |m_N, m_{N-1}, \ldots, m_2, m_1\rangle$. We find

$$P_{N+1}\mathsf{q}_j = (-1)^{N+1}\mathsf{q}^{\mathrm{op}}_{N-j+1} P_N, \quad j = 1, \ldots, N, \qquad P_{N+1}\mathsf{q}_0 = \mathsf{q}^{\mathrm{op}}_0 P_N T_N.$$

The local supercharge q^{op} differs from the action of q by a transposition: $\mathsf{q}^{\mathrm{op}}|m\rangle = \sum_{jk} a_{m,jk}|kj\rangle$. The global supercharge Q_N will have a definite transformation behaviour with respect to reflection symmetry if $\mathsf{q}^{\mathrm{op}} = \epsilon \mathsf{q}$ with $\epsilon = \pm 1$ what is equivalent to $a_{m,jk} = \epsilon a_{m,kj}$. Then it follows that

$$P_{N+1} Q_N P_N = (-1)^{N+1} \epsilon Q_N \tag{11}$$

on $\mathcal{H}_N$ what implies the invariance of the Hamiltonian under reflection $[P_N, H_N] = 0$.

Extended Supersymmetry Given a spin chain whose Hamiltonian admits a dynamical supersymmetry on the lattice one may ask if it is part of a larger algebra, at least for certain choices of the parameters in the model. There are obviously many ways to imagine possible symmetry extensions. Examples are the $SU(2)$-extended algebra which was studied by in the context of lattice fermions with exclusion rules [89], or the $su(1, 1|2)$-sector of $\mathcal{N} = 4$ at one one-loop where the supercharges are part of a dynamical representation of a Lie superalgebra [15].

In both examples cases a second set of supercharges is present. Their algebraic relations amongst themselves and with other operators of the theory are model-specific. Here we consider only the minimal situation: we suppose that there are two copies of supercharges $Q_N, Q_N^\dagger$ and $\bar{Q}_N, \bar{Q}_N^\dagger$ which anticommute:

$$Q_N Q_{N-1} = \bar{Q}_N \bar{Q}_{N-1} = 0, \qquad Q_{N-1}^\dagger Q_N^\dagger = \bar{Q}_{N-1}^\dagger \bar{Q}_N^\dagger = 0, \tag{12a}$$

$$\bar{Q}_N^\dagger Q_N + Q_{N-1} \bar{Q}_{N-1}^\dagger = 0, \qquad Q_N^\dagger \bar{Q}_N + \bar{Q}_{N-1} Q_{N-1}^\dagger = 0, \tag{12b}$$

$$\bar{Q}_N Q_{N-1} + Q_N \bar{Q}_{N-1} = 0, \qquad Q_{N-1}^\dagger \bar{Q}_N^\dagger + \bar{Q}_{N-1}^\dagger Q_N^\dagger = 0. \tag{12c}$$

To each pair of supercharges is associated a Hamiltonian

$$H_N = Q_N^\dagger Q_N + Q_{N-1} Q_{N-1}^\dagger, \qquad \bar{H}_N = \bar{Q}_N^\dagger \bar{Q}_N + \bar{Q}_{N-1} \bar{Q}_{N-1}^\dagger. \tag{12d}$$

It follows that H_N (resp. $\bar{H}_N$) commutes with $\bar{Q}_N, \bar{Q}_N^\dagger$ (resp. $Q_N, Q_N^\dagger$), and furthermore $[H_N, \bar{H}_N] = 0$. Thus, they can be diagonalised simultaneously. The eigenvectors with strictly positive eigenvalues organise in *quadruplets* with one state at $N - 1$ sites, two at N sites, and one at $N + 1$ sites:

$$\left(|\psi\rangle, Q_{N-1}|\psi\rangle, \bar{Q}_{N-1}|\psi\rangle, \bar{Q}_N Q_{N-1}|\psi\rangle\right), \qquad Q_{N-2}^\dagger|\psi\rangle = \bar{Q}_{N-2}^\dagger|\psi\rangle = 0.$$

The two states $Q_{N-1}|\psi\rangle, \bar{Q}_{N-1}|\psi\rangle$ in the middle of the quadruplet can be mapped onto each other by the nilpotent operator

$$C_N = \bar{Q}_N^\dagger Q_N, \qquad C_N^2 = 0.$$

It commutes which both $H_N, \bar{H}_N$ as a consequence of (12a), (12b), (12c), (12d) and is generically a rather non-trivial (non-local) conserved charge. We will be interested in the case where the two supercharges are related by a similarity transformation U_N acting on $\mathcal{H}_N$: $\bar{Q}_N = U_{N+1} Q_N U_N^{-1}$. This implies that the two Hamiltonians are conjugate $\bar{H}_N = U_N H_N U_N^{-1}$. In Sect. 3 we will encounter a case, where U_N is a symmetry of the Hamiltonian, so that $H_N = \bar{H}_N$, but does not leave invariant the supercharges.

This set of relations (12a), (12b), (12c), (12d) is a lattice version of the $\mathcal{N} = (2, 2)$ supersymmetry algebra with zero centre which is well known from field theory (see e.g. [53]). In

the field-theory context, non-zero central extensions occur typically on non-compact spaces [93]. We do not expect them in the present setting of finite lattices. Moreover, in the field theory the operators Q and $\bar{Q}$ are usually associated with left- and right-movers, a concept which is absent on the lattice. The best we can hope is that in the scaling limit some suitable rescaled linear combinations of the lattice supercharges converges to the left and right field-theory supercharges. This might however make necessary to take linear combinations of the algebra at N and $N+1$ sites and then let $N \to \infty$ in order to obtain objects which are mapped onto each other through spatial reflection as one expects for left- and right-movers. The reason is that in all examples with two copies of the supersymmetry found below *all* supercharges have the same definite parity $(-1)^{N+1}$ at fixed N in the sense of (11).

2.3 Open Boundary Conditions

Consider a q such that (6) holds with $|\chi\rangle = 0$: $(q \otimes 1 - 1 \otimes q)q = 0$. This means that the mechanism which makes Q_N nilpotent is completely local. In this case there is an extension to open boundary conditions which has a fully supersymmetric Hamiltonian (no restriction to special subsectors is necessary). In the case of an open interval with sites $j = 1, 2, \ldots, N$ we define the global supercharge as

$$Q_N = \sum_{j=1}^{N} q_j,$$

which is nilpotent if $|\chi\rangle = 0$. As before the Hamiltonian is given by an anticommutator:

$$H_N = Q_N^\dagger Q_N + Q_{N-1} Q_{N-1}^\dagger = \sum_{j=1}^{N-1} h_{j,j+1} + \frac{1}{2}\big(q_1^\dagger q_1 + q_N^\dagger q_N\big)$$

where $h_{j,j+1}$ is the Hamiltonian density defined in (9). The boundary terms can be interpreted as surface magnetic fields. This is for example the case for the spin-1/2 XXZ chain [96]. At the end of the next section, we show how to extend this to the XYZ chain.

2.4 An Example: the XYZ Chain

We review the example of the XYZ spin chain at the combinatorial line [10, 11, 75, 84], discussed in [49] from the point of view of supersymmetry, within the present formalism. The supersymmetry is a generalisation of the XXZ-case which acts locally like $0 \to \emptyset$ and $1 \to 00$ [96]. The XYZ chain is an elliptic chain in the sense that it breaks particle number conservation mod 2. The local supercharge found in [49] acts according to

$$q|0\rangle = 0, \qquad q|1\rangle = |00\rangle - \zeta|11\rangle.$$

Indeed, one checks that (6) holds with $|\chi\rangle = -\zeta|00\rangle$. Thus, we have a bona fide supercharge Q_N on the lattice. It generates the XYZ-Hamiltonian

$$H_N = -\frac{1}{2} \sum_{j=1}^{N} \big(J_1 \sigma_j^1 \sigma_{j+1}^1 + J_3 \sigma_j^2 \sigma_{j+1}^2 + J_3 \sigma_j^3 \sigma_{j+1}^3\big) + E_0, \tag{13a}$$

with the coupling constants

$$J_1 = 1 + \zeta, \qquad J_2 = 1 - \zeta, \qquad J_3 = \frac{\zeta^2 - 1}{2}, \qquad E_0 = \frac{N(J_1 + J_2 + J_3)}{2}. \tag{13b}$$

Here σ^a, $a = 1, 2, 3$ denote the usual Pauli matrices. The Hamiltonian is invariant under spin reversal, and thus a second local supercharge $\bar{\mathsf{q}} = (R \otimes R)\mathsf{q}R$ can be defined. Thus, we have

$$\bar{\mathsf{q}}|0\rangle = |11\rangle - \zeta|00\rangle, \qquad \bar{\mathsf{q}}|1\rangle = 0.$$

It was shown in [49] that the corresponding global supercharges Q_N and $\bar{Q}_N$ anticommute like in (12a), (12b), (12c), (12d), and generate the same Hamiltonian. All this holds only for periodic boundary conditions as for non-zero ζ the condition (3) allows only the twist angle $\phi = 0$.

Rotational Invariance and Duality Transformation   We use the occasion to inspect a known duality transformation of the Hamiltonian (13a), (13b) in parameter space from the point of view of the supersymmetry algebra. We denote by $\rho^a(\theta) = \exp(i\theta\sigma^a/2)$ a local rotation about the a-axis. A global rotation on $\mathcal{H}_N$ then corresponds to $\Omega_N^a(\theta) = \prod_{j=1}^N \rho_j^a(\theta)$. Moreover, we write $\mathsf{q} = \mathsf{q}(\zeta)$ in order to stress the explicit dependence on ζ. Let us start with $a = 3$. We find that

$$\left(\rho^3(\theta) \otimes \rho^3(\theta)\right)\mathsf{q}(\zeta)\rho^3(-\theta) = e^{i\theta/2}\mathsf{q}\left(e^{-2i\theta}\zeta\right).$$

We see that the transformation $\zeta \to -\zeta$ is equivalent to setting $\theta = \pm\pi/2$. Next, let us consider rotations about the 2-axis. Choosing the same angle we find

$$\left(\rho^2\left(\pm\frac{\pi}{2}\right) \otimes \rho^2\left(\pm\frac{\pi}{2}\right)\right)\mathsf{q}(\zeta)\rho^2\left(\mp\frac{\pi}{2}\right) = \left(\frac{1-\zeta}{2}\right)\frac{\mathsf{q}(\zeta') \pm \bar{\mathsf{q}}(\zeta')}{\sqrt{2}} + \mathsf{q}_{\text{red}}$$

with

$$\zeta' = \frac{\zeta+3}{\zeta-1}.$$

Here q_{red} is redundant in the sense of Proposition 1 with vector $|\chi\rangle = -(1+\zeta)(|0\rangle + |1\rangle)/2\sqrt{2}$. Therefore, it does not contribute to the global supercharge and can be discarded. Thus we have

$$\Omega_{N+1}^3(\pm\pi/2)Q_N(\zeta)\Omega_N^3(\pm\pi/2)^{-1} = e^{\pm i\pi/4}Q_N(-\zeta),$$

$$\Omega_{N+1}^2(\pm\pi/2)Q_N(\zeta)\Omega_N^2(\pm\pi/2)^{-1} = \left(\frac{1-\zeta}{2}\right)\frac{Q_N(\zeta') \pm \bar{Q}_N(\zeta')}{\sqrt{2}}.$$

Hence these rotations imply simple transformations for the supercharges, what is of course compatible with the known invariance properties of the Hamiltonian [84]. The strategy to find invariances is generic and can be applied to other cases. We present an example for a spin-one model in Sect. 4.

Zero-Energy States   The supersymmetry singlets of the XYZ Hamiltonian (13a), (13b) were studied in [11, 37, 38, 75, 98]. These investigations rely on the following existence conjecture: if the number of sites $N = 2n + 1$ is odd then there are exactly two linearly independent zero-energy states. The statement is believed to hold for periodic boundary conditions, and the states are found to be invariant under translations. Here we give a short proof of this conjecture in the momentum sectors where the supersymmetry exists. It is based on Witten's conjugation argument, which states the following. Consider an invertible

transformation M on $\mathcal{H}$ on which acts a supersymmetry algebra with supercharges $Q, Q^\dagger$. If one introduces the conjugated supercharge

$$\tilde{Q} = MQM^{-1}$$

then the Hamiltonians $H = \{Q, Q^\dagger\}$ and $\tilde{H} = \{\tilde{Q}, \tilde{Q}^\dagger\}$ have the same number of zero-energy eigenstates. If M leaves invariant certain quantum numbers, such as the fermion number, then the result holds sector-wise. We refer to [92] for the details of the proof. The basic idea is to exploit that the zero-energy states of H are in one-to-one correspondence with the elements of the quotient space $\mathfrak{H}_Q = \ker Q / \mathrm{im}\, Q$, in other words the number of linearly independent ground states is equal to $\dim \mathfrak{H}_Q$. Using the conjugation by M one establishes a bijection between the elements of $\mathfrak{H}_Q$ and $\mathfrak{H}_{\tilde{Q}}$. This leads to the equality of their dimensions, and thus to the desired result. Notice that this does not imply that the singlets of $\tilde{H}$ can be obtained from the singlets of H through the transformation M however.

We apply this argument to the XYZ spin chain by constructing a simple transformation M for its supercharge which allows to vary the parameter ζ. To this end, notice that for any real number $\mu \neq 0$ the two local supercharges $\mathsf{q}(\zeta)$ and $\mathsf{q}(\mu^2\zeta)$ can be related through conjugation as follows

$$\big(m(\mu) \otimes m(\mu)\big)\mathsf{q}(\zeta)m(\mu)^{-1} = \mu^{-1}\mathsf{q}\big(\mu^2\zeta\big), \quad \text{with } m(\mu) = |0\rangle\langle 0| + \mu|1\rangle\langle 1|.$$

If we set $M_N(\mu) = m(\mu)^{\otimes N}$ then the global supercharge is transformed under conjugation to

$$\tilde{Q}_N(\zeta, \mu) = M_{N+1}(\mu)Q_N(\zeta)M_N(\mu)^{-1} = \mu^{-1}Q_N\big(\mu^2\zeta\big).$$

These new supercharge generates the Hamiltonian $\tilde{H}_N(\zeta, \mu) = \mu^{-2}H_N(\mu^2\zeta)$. According to Witten's argument, it has the same number of zero-energy states as $H_N(\zeta)$. This statement holds for any $\mu \neq 0$. We suppose now that $\zeta > 0$ without loss of generality. Set $\mu = 1/\sqrt{\zeta}$, and observe that

$$\tilde{H}_N(\zeta, 1/\sqrt{\zeta}) = \zeta\, H_N(1) = \zeta \sum_{j=1}^{N}\big(1 - \sigma_j^1\sigma_{j+1}^1\big)$$

is, up to a multiplicative factor, the Hamiltonian for a one-dimensional *classical* Ising model. We introduce the states $|\rightarrow\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, $|\leftarrow\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$. In terms of these, the only zero-energy states are the fully-polarised configurations $|\leftarrow\leftarrow\cdots\leftarrow\rangle$ and $|\rightarrow\rightarrow\cdots\rightarrow\rangle$. Restricting to our momentum sectors with $t_N = (-1)^{N+1}$, we find that $H_N(1)$ has two zero-energy states for N odd, and none for N even. The same holds thus for $H_N(\zeta)$ for any $\zeta > 0$. Using the duality transformations, it is easy to see that this result extends to any ζ.

Open Boundaries   The model based on the preceding local supercharge cannot be defined on open intervals without breaking the supersymmetry unless $\zeta = 0$. The reason is that q solves (6) with $|\chi\rangle = -\zeta|00\rangle$ which is non-vanishing for non-zero ζ. In order to circumvent this problem we take advantage of the gauge transformations, and introduce the local supercharge defined through

$$\mathsf{q}|0\rangle = \lambda\big(|01\rangle + |10\rangle\big), \qquad \mathsf{q}|1\rangle = |00\rangle + (2\lambda - \zeta)|11\rangle.$$

The part proportional to λ is in fact of the form discussed in Proposition 1. It is easy to check that this q solves (6) with vector $|\chi\rangle = (\lambda - \zeta)(|00\rangle + \lambda|11\rangle)$. The Hamiltonian density h

is independent of λ, and therefore coincides with the one of the XYZ chain (13a), (13b). Setting $\lambda = \zeta$ we find $|\chi\rangle = 0$, and thus may define the model with manifest supersymmetry on an open interval. The Hamiltonian is given by

$$H_N(\zeta) = \sum_{j=1}^{N-1} h_{j,j+1} + \frac{1 + 3\zeta^2}{2} + \frac{\zeta^2 - 1}{4}(\sigma_1^3 + \sigma_N^3).$$

Hence, we find a diagonal boundary magnetic field. Its sign may be flipped through a spin reversal transformation which leaves the bulk part invariant. If the we write $\bar{H}_N(\zeta) = R_N H_N(\zeta) R_N$ then for any $0 \leq \alpha \leq 1$ then combination $\alpha H_N(\zeta) + (1 - \alpha)\bar{H}_N(\zeta)$ has positive spectrum. Its bulk Hamiltonian is independent of α, but the surface magnetic field is given by $(2\alpha - 1)(\zeta^2 - 1)(\sigma_1^3 + \sigma_N^3)/4$. The supersymmetry however is present only for $\alpha = 0, 1$.

It is possible to apply the same conjugation argument as for the periodic chain: for $\mu \neq 0$ the Hamiltonians $H_N(\zeta)$ and $\mu^{-2} H_N(\mu^2 \zeta)$ have the same number of zero-energy states. Again, we restrict to $\zeta > 0$ and choose $\mu = 1/\sqrt{\zeta}$ (the case $\zeta < 0$ is treated similarly). We consider thus $\zeta H_N(1) = \zeta \sum_{j=1}^{N-1}(1 - \sigma_j^1 \sigma_{j+1}^1) + 2\zeta$. The spectrum of this Hamiltonian is bounded from below by 2ζ for any number of sites. It follows that for all strictly positive ζ, the Hamiltonian has no zero-energy state for any finite number of sites. The point $\zeta = 0$ is special however as it cannot be reached through conjugation. Indeed, the exact diagonalisation for small N suggests that there is a single zero-energy state for every N at $\zeta = 0$ [96].

3 Trigonometric Models

In this section, we present a class of spin-$\ell/2$ chains with dynamical lattice supersymmetry. The construction is inspired from the $\mathcal{M}_\ell$-models for lattice fermions [40, 41]. They possess a (non-dynamic) $\mathcal{N} = 2$ supersymmetry on the lattice. The crucial observation here is that once reformulated in the language of spin chains one may impose spin-reversal symmetry, which has no obvious counterpart in the fermion models, and thus enforce the existence of a second copy of the supersymmetry algebra such that (12a), (12b), (12c), (12d) holds. In this way we arrive at lattice models for the $\mathcal{N} = (2, 2)$ superconformal minimal series with explicit supercharges on the lattice [91].

We proceed as follows. First, we review briefly the definition of the $\mathcal{M}_\ell$ models in Sect. 3.1, and discuss the supercharges for the related spin chains. In Sect. 3.2 we analyse the effect of spin-reversal symmetry, and show that all coupling constants of the theory are essentially fixed by this requirement. Next, we prove that suitably modified versions of the corresponding Hamiltonians possess a local quantum group symmetry. This leads us to an identification of the models with spin-$\ell/2$ chains constructed from the six-vertex model through the fusion procedure, with a particular anisotropy depending on ℓ. This is the spin-anisotropy commensurability mentioned in the introduction. In the last section, we address the computation of the Witten index, which indicates the existence of zero-energy states for the spin chains if periodic boundary conditions are imposed.

3.1 Supercharges

The $\mathcal{M}_\ell$ describe spinless fermions with the constraint that connected fermion clusters cannot contain more than ℓ particles. Their supercharges take out single fermions from a state.
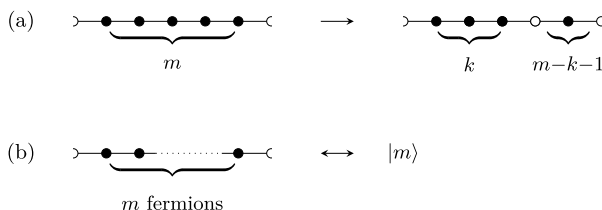
Fig. 1 (**a**) The local action of the supercharge in the fermion model: an occupied site is represented as •, whereas an empty site corresponds to ◦. The supercharge splits up a string of m particles into two strings with k and $m - k - 1$ particles. (**b**) Correspondence between string of m fermions and spin state $|m\rangle$

This operation is weighted by an amplitude $a_{m,k}$, $k = 0, \ldots, m - 1$ if the $(k + 1)$-th member in a string of m consecutive particles is removed. See Fig. 1(a) for an illustration. There is a natural identification of a string of m fermions between two empty sites with a basis vector $|m\rangle$ in the vector space $V \simeq \mathbb{C}^{\ell+1}$ as illustrated in Fig. 1(b). We translate the splitting process shown on Fig. 1(a) into the spin language: the local supercharge takes thus the basis vector $|m\rangle \in V$ to a pair $|k, m - k - 1\rangle \in V \otimes V$ with amplitude $a_{m,k}$. We set $a_{m,k} = 0$ for $k < 0$ and $k > m - 1$. Thus we have a local q with amplitudes $a_{m,jk} = \delta_{j+k+1,m} a_{m,j}$:

$$\mathsf{q}|m\rangle = \sum_{k=0}^{m-1} a_{m,k} |k, m - k - 1\rangle. \tag{14}$$

Notice that q changes the magnetisation by $-(\ell + 2)/2$. Combining this with (3) we conclude that the set of admissible twist angles is given by

$$\phi = \frac{4\pi m}{\ell + 2}, \quad m = 0, 1, \ldots, p, \quad p = \begin{cases} \ell + 1, & \ell \text{ odd}, \\ \ell/2, & \ell \text{ even}. \end{cases} \tag{15}$$

Next, we want the global supercharge Q_N built from q to be nilpotent. Hence we need to find a vector $|\chi\rangle$ such that (6) holds. Writing out the components in (7) with $a_{m,jk} = \delta_{j+k+1,m} a_{m,j}$ we find $|\chi\rangle = 0$. It is thus sufficient to solve the equation $(1 \otimes \mathsf{q})\mathsf{q} = (\mathsf{q} \otimes 1)\mathsf{q}$. In order to simplify the discussion, we impose another condition $a_{m,k} = a_{m,m-1-k}$. Hence the splitting depends only on the length of the subsequences but not on their order. This requirement implies $\mathsf{q} = \mathsf{q}^{\mathrm{op}}$ and therefore the Hamiltonian will automatically be parity symmetric. Using this we obtain the following recurrence relation:

$$a_{m,k} a_{m-k-1,n} = a_{m,n} a_{m-n-1,k}.$$

We set $k = 0$ and find by iteration the following expression:

$$a_{m,n} = \left(\frac{a_{m,0}}{a_{m-n-1,0}} \right) a_{m-1,n} = \prod_{j=1}^{m-n-1} \left(\frac{a_{j+n+1,0}}{a_{j,0}} \right) a_{n+1,0}. \tag{16}$$

Hence, it is sufficient to know the numbers $a_{k,0}$, $k = 1, \ldots, \ell$. This is of course the same result as for the $\mathcal{M}_\ell$ models [40]: up to a global factor there are $\ell - 1$ free parameters.

3.2 Hamiltonian and Spin-Reversal Symmetry

We fix the free parameters by imposing spin-reversal symmetry $m \leftrightarrow \ell - m$ on the Hamiltonian density h. This has no obvious equivalent in the fermion models as the exclusion rule does not admit a particle-hole symmetry (at least not for fixed length). For the matrix elements $h_{(r,s),(m,n)} = \langle r, s | h | m, n \rangle$ this implies

$$h_{(r,s),(m,n)} = h_{(\ell-r,\ell-s),(\ell-m,\ell-n)}. \tag{17}$$

It is straightforward to express them in terms of the amplitudes $a_{m,k}$:

$$h_{(r,s),(m,n)} = \frac{1}{2} \delta_{r,m} \delta_{s,n} \left(\sum_{k=0}^{m-1} a_{m,k}^2 + \sum_{k=0}^{n-1} a_{n,k}^2 \right)$$

$$+ \delta_{r+s,m+n} (a_{m+n+1,r} a_{m+n+1,m} - a_{n,m+n-r} a_{r,m} - a_{m,r} a_{m+n-r,n}).$$

Our aim is to find the numbers $a_{m,n}$ such that (17) holds. The structure of the Hamiltonian density implies that we need to solve a set of quadratic recursion relations what is done in detail in Appendix A. We write the result in terms of q-integers defined through

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

We find that supercharge defined leads to a spin-reversal invariant Hamiltonian density if we choose the coefficients

$$a_{m,n} = \begin{cases} \sqrt{\frac{[m+1]}{[m-n][n+1]}}, & \text{for} \quad m = 1, \ldots, \ell, \; n = 1, \ldots, m-1, \\ 0, & \text{otherwise}, \end{cases} \tag{18}$$

where the q-integers are evaluated at

$$q = \exp\big(i\pi/(\ell+2)\big).$$

Let us sketch how this statement is verified for the matrix elements of h. We start with the diagonal $b_{m,n} = h_{(m,n),(m,n)}$. Consider the difference

$$b_{\ell-m,\ell-n} - b_{m,n} = a_{2\ell-(m+n)+1,\ell-m}^2 - a_{m+n+1,m}^2 + a_{m+1,0}^2 + a_{n+1,0}^2 - a_{1,0}^2.$$

The construction given in Appendix A implies that expression is zero for $m + n = \ell$. We focus on $m + n < \ell$ (the case $m + n > \ell$ can be obtained through symmetry): the first term on the right-hand side vanishes because $2\ell - (m+n) + 1 > \ell + 1$. Hence we need to check only the equation $a_{m+1,0}^2 + a_{n+1,0}^2 - a_{1,0}^2 = a_{m+n+1,m}^2$. Indeed, it holds what can be shown from simple properties of the q-integers. Thus, $b_{\ell-m,\ell-n} = b_{m,n}$. Next, let us turn to the off-diagonal matrix elements for which $(r,s) \neq (m,n)$. Without loss of generality, we suppose that $r \leq m$. Using the explicit form of the matrix elements $h_{(r,s),(m,n)}$ given above, the spin-reversal symmetry is equivalent to

$$a_{m+n+1,r} a_{m+n+1,m} - a_{m,r} a_{m+n-r,n}$$

$$= a_{2\ell-(m+n)+1,\ell-r} a_{2\ell-(m+n)+1,\ell-m} - a_{\ell-r,\ell-m} a_{\ell-n,\ell+r-(m+n)}.$$

It is obvious that this relation holds if $m + n = \ell$. For symmetry reasons, it is sufficient to consider thus $m + n < \ell$. In this case the first term of the second line vanishes. Moreover, using $[m + 1]a_{\ell-r,\ell-m} = [r + 1]a_{m,r}$, and some basic identities for q-integers the equality follows. This concludes the proof of spin-reversal invariance for the constants (18).

In order to make it more explicit, we expand the Hamiltonian density h obtained from the trigonometric supercharge in components as follows:

$$h = \sum_{m_1,m_2=0}^{\ell} \sum_{n=-\min(m_1,\ell-m_2)}^{\min(m_2,\ell-m_1)} \beta^n_{m_1,m_2} |m_1 + n, m_2 - n\rangle\langle m_1, m_2|. \tag{19a}$$

For $n > 0$ the coefficients are given by

$$\beta^n_{m_1,m_2} = -\frac{1}{[n]}\sqrt{\frac{[M_1 + 1][M_2 - n + 1]}{[M_2 + 1][M_1 + n + 1]}}, \tag{19b}$$

where $M_1 = \min(m_1, \ell - m_2)$ and $M_2 = \min(m_2, \ell - m_1)$. The case $n < 0$ can easily be obtained from the symmetry of h:

$$\beta^n_{m_1,m_2} = \beta^{-n}_{m_2,m_1}. \tag{19c}$$

If $n = 0$ we have

$$\beta^0_{m_1,m_2} = c_{M_1+1} + c_{M_2+1}, \quad c_m = \frac{1}{2}\sum_{n=1}^{m}\frac{[n+1] - [n-1]}{[n]}. \tag{19d}$$

The constants c_m appear to be some q-analogues of the harmonic numbers, enjoying the property $c_m = c_{\ell+1-m}$. Notice the similarity of these matrix elements with those of the spin-$\ell/2$ XXX Hamiltonian studied in [26].

Symmetry Enhancement Imposing the symmetry under spin reversal leads in a natural way to the question if a second copy of the supersymmetry generators, constructed from $\bar{q} = (R \otimes R)qR$, leads to the relations (12a), (12b), (12c), (12d). This is indeed the case. In fact, it is sufficient to verify the relations locally and then reason similarly to Proposition 2. Consider for example the anticommutator between $\bar{Q}_N$ and Q_N. First we check that there is a vector $|\chi\rangle \in V \otimes V$ such that for all $|\psi\rangle \in V$ we have the form of a local boundary

$$\big((q \otimes 1 - 1 \otimes q)\bar{q} + (\bar{q} \otimes 1 - 1 \otimes \bar{q})q\big)|\psi\rangle = |\chi\rangle \otimes |\psi\rangle - |\psi\rangle \otimes |\chi\rangle$$

for the coefficients (18). A simple calculation leads to the following decomposition of $|\chi\rangle$ in terms of basis vectors of $V \otimes V$:

$$|\chi\rangle = -\sum_{i=0}^{\ell}\frac{1}{[i+1]}|i, \ell - i\rangle.$$

Second, we apply the argument of telescopic cancellations in order to prove the relations on the momentum spaces of interest. This proves $\bar{Q}_{N+1}Q_N + Q_{N+1}\bar{Q}_N = 0$. The other relations are proved similarly.

Examples We illustrate the cases $\ell = 1$ and $\ell = 2$. The Hamiltonian density is obtained from (9).

For $\ell = 1$ we find

$$
h = \begin{pmatrix} 1 & & & \\ & \frac{1}{2} & -1 & \\ & -1 & \frac{1}{2} & \\ & & & 1 \end{pmatrix} = -\frac{1}{2}\left(\sigma^1 \otimes \sigma^1 + \sigma^2 \otimes \sigma^2 - \frac{1}{2}\sigma^3 \otimes \sigma^3\right) + \frac{3}{4},
$$

where we indicated only non-zero matrix elements. This is just the local Hamiltonian for the spin-1/2 XXZ chain at $\Delta = -1/2$.

For $\ell = 2$, we obtain the 9×9 matrix

$$
h = \frac{1}{\sqrt{2}} \left(\begin{array}{ccc|ccc|ccc}
2 & & & & & & & & \\
& 2 & & -1 & & & & & \\
& & 1 & & -\sqrt{2} & & -1 & & \\ \hline
& -1 & & 2 & & & & & \\
& & -\sqrt{2} & & 2 & & -\sqrt{2} & & \\
& & & & & 2 & & -1 & \\ \hline
& & -1 & & -\sqrt{2} & & 1 & & \\
& & & & & -1 & & 2 & \\
& & & & & & & & 2
\end{array} \right).
$$

This expression reminds strongly of the local Hamiltonian for the Fateev-Zamolodchikov chain [35]. The latter is given by

$$
h_{\text{FZ}} = \sum_{a=1}^{3} J_a\left(s^a \otimes s^a + \left(s^a\right)^2 \otimes 1 + 1 \otimes \left(s^a\right)^2\right) - \sum_{a,b=1}^{3} A_{ab} s^a s^b \otimes s^a s^b \tag{20}
$$

with coupling constants J_a and a symmetric matrix $A_{ab} = A_{ba}$ such that $A_{aa} = J_a$, and

$$
J_1 = J_2 = 1, \qquad J_3 = \cos 2\eta \quad \text{and} \quad A_{12} = 1, \qquad A_{13} = A_{23} = 2\cos\eta - 1. \tag{21}
$$

The spin operators s^a, $a = 1, 2, 3$ are the usual spin-1 representation of su(2). We identify $\eta = \pi/4$ as a good candidate, but the matrices h and h_{FZ} do not quite coincide. In fact, the difference is just a simple local gauge transformation:

$$
h = \frac{1}{\sqrt{2}} u h_{\text{FZ}} u^{-1}, \quad u = 1 \otimes e^{i\pi s^3}. \tag{22}
$$

Therefore, we conclude that for spin one the supersymmetric Hamiltonian is the *twisted* Fateev-Zamolodchikov Hamiltonian (in the sense that boundary conditions are $s^{\pm}_{N+1} = (-1)^N s^{\pm}_1$, $s^3_{N+1} = s^3_1$, and thus antiperiodic for N odd) at $\eta = \pi/4$. We shall see later in Sect. 4 that this is not the only supersymmetry in the Fateev-Zamolodchikov chain.

3.3 Local Quantum Group Symmetry

We proceed with a more detailed analysis of the Hamiltonian density: we show that a suitably modified version of h commutes with the quantum group $U_q(\text{sl}_2)$ with

$q = \exp(i\pi/(\ell + 2))$. For generic $q = \exp(i\eta)$ the algebraic relations of its generators are given by

$$\left[\mathfrak{s}^+, \mathfrak{s}^-\right] = \frac{\sin(2\eta\mathfrak{s}^3)}{\sin\eta}, \qquad \left[\mathfrak{s}^3, \mathfrak{s}^\pm\right] = \pm\mathfrak{s}^\pm.$$

A highest-weight spin-$\ell/2$ representation acting on a vector space $V \simeq \mathbb{C}^{\ell+1}$ is given by the following action on the basis vectors $|m\rangle$, $m = 0, \ldots, \ell$:

$$\mathfrak{s}^3|m\rangle = (m - \ell/2)|m\rangle,$$

$$\mathfrak{s}^+|m\rangle = \sqrt{[\ell - m][m + 1]}|m + 1\rangle, \qquad \mathfrak{s}^-|m\rangle = \sqrt{[\ell - m + 1][m]}|m - 1\rangle.$$

In order to obtain the action of $U_q(\mathrm{sl}_2)$ on two sites we use its Hopf-algebra structure (see e.g. [65]). The comultiplication is defined on the generators according to

$$\Delta\left(\mathfrak{s}^\pm\right) = \mathfrak{s}^\pm \otimes q^{-\mathfrak{s}^3} + q^{\mathfrak{s}^3} \otimes \mathfrak{s}^\pm, \qquad \Delta\left(\mathfrak{s}^3\right) = \mathfrak{s}^3 \otimes 1 + 1 \otimes \mathfrak{s}^3. \qquad (23)$$

These few definitions are sufficient to prove the hidden quantum group invariance of h. The relations become more transparent if we consider a slightly modified (twisted) version h'. To define it, we need the unitary transformation $u = 1 \otimes \exp(i\pi\mathfrak{s}^3)$. We set

$$h' = uhu^{-1} + \lambda\left(\mathfrak{s}^3 \otimes 1 - 1 \otimes \mathfrak{s}^3\right),$$

where λ is a constant to be adjusted. We see that h' decomposes into a local bulk term uhu^{-1}, and a local boundary term $\lambda(\mathfrak{s}^3 \otimes 1 - 1 \otimes \mathfrak{s}^3)$. The density h' commutes with $U_q(\mathrm{sl}_2)$ at $q = \exp(i\pi/(\ell + 2))$, i.e. $[h', \Delta(\mathfrak{s}^\pm)] = [h', \Delta(\mathfrak{s}^3)] = 0$, provided that we choose

$$\lambda = i\sin\left(\frac{\pi}{\ell + 2}\right). \qquad (24)$$

The proof is elementary but tedious, and we only sketch it. The commutation relation with $\Delta(\mathfrak{s}^3)$ is trivial. We focus on $[h', \Delta(\mathfrak{s}^+)] = 0$. This amounts to prove that

$$\gamma_{m+1,n}^j q^{\ell/2-n}\sqrt{[m + 1][m + 2]} + \gamma_{m,n+1}^{j+1} q^{m-\ell/2}\sqrt{[n + 1][n + 2]}$$

$$= \gamma_{mn}^j q^{\ell/2+j-n}\sqrt{[m + j + 1][m + j + 2]}$$

$$+ \gamma_{mn}^{j+1} q^{m+j+1-\ell/2}\sqrt{[n - j][n - j + 1]},$$

where the γ_{mn}^j are defined through the decomposition of the density

$$h' = \sum_{m_1, m_2=0}^{\ell} \sum_{n=-\min(m_1, \ell-m_2)}^{\min(m_2, \ell-m_1)} \gamma_{m_1, m_2}^n |m_1 + n, m_2 - n\rangle\langle m_1, m_2|,$$

and given in terms of the components of h by

$$\gamma_{mn}^j = (-1)^j \beta_{mn}^j + \lambda\delta_{j,0}(m - n).$$

We start with $j = 0$ and fix the value of λ. For $m + n \leq \ell$ we find

$$\gamma_{m+1,n}^0 - \gamma_{m,n}^0 = q^{m+n-\ell}\sqrt{\frac{[n + 1]}{[m + 1][m + 2]}}\left(\sqrt{[n]}\gamma_{mn}^1 - \sqrt{[n + 2]}\gamma_{m,n+1}^1\right).$$

Using the simple identity

$$[m + n] = q^m[n] + q^{-n}[m], \tag{25}$$

the right-hand side simplifies to $q^{m+2}/[m + 2]$. The left-hand side is given by $\lambda + ([m + 3] - [m + 1])/(2[m + 2])$. Equating both sites leads to (24). The case $m + n \geq \ell$ is treated similarly, and gives of course the same result. The constant λ appears also if we set $j = -1$. Yet this case is equal to the one just considered because of symmetry of the coefficients γ_{mn}^j under $j \to -j$ (which they inherit from the numbers β_{mn}^j). Next, we have to need to check the equation for $j \geq 1$. In this case, we have $\gamma_{mn}^j = (-1)^j \beta_{mn}^j$. Grouping together the terms containing on the one hand $\beta_{m+1,n}^j$, β_{mn}^j and $\beta_{m,n+1}^{j+1}$, β_{mn}^{j+1} on the other, the identity is readily verified by using the explicit expressions (19a), (19b), (19c), (19d) as well as the identity (25). As before, the work for $j < -1$ can be reduced to this case by using the symmetry of the coefficients under $j \to -j$. Finally, the proof of $[h', \Delta(\mathfrak{s}^-)] = 0$ follows the same lines.

3.4 Relation to Higher-Spin XXZ Chains

Looking at the two examples $\ell = 1$ and $\ell = 2$ given at the end of Sect. 3.2, it seems natural that the generic case leads to spin-$\ell/2$ versions of the XXZ spin chain. The Hamiltonian densities of these spin chains with arbitrary anisotropy were computed explicitly in [22]. Let $C = \mathfrak{s}^-\mathfrak{s}^+ + [\mathfrak{s}^3][\mathfrak{s}^3 + 1]$ be the quadratic Casimir of $U_q(\mathrm{sl}_2)$. Define $X_\ell = \Delta(C)$, acting on $V \otimes V \simeq \mathbb{C}^{\ell+1} \otimes \mathbb{C}^{\ell+1}$. In terms of the generators it is given by

$$X_\ell = \frac{q^{\mathfrak{s}^3 \otimes 1}}{2} \left[\mathfrak{s}^+ \otimes \mathfrak{s}^- + \mathfrak{s}^- \otimes \mathfrak{s}^+ + \frac{1}{4\sin^2 \eta} \left(\cos \eta \left(q^{-\mathfrak{s}^3} \otimes q^{\mathfrak{s}^3} + q^{\mathfrak{s}^3} \otimes q^{-\mathfrak{s}^3} \right) \right.\right.$$
$$\left.\left. - \cos\left((\ell + 1)\eta\right) \left(q^{\mathfrak{s}^3} \otimes q^{\mathfrak{s}^3} + q^{-\mathfrak{s}^3} \otimes q^{-\mathfrak{s}^3} \right) \right) \right] q^{-1 \otimes \mathfrak{s}^3}.$$

The Hamiltonian density of the higher-spin XXZ chain is a polynomial of order ℓ in X_ℓ. Explicitly it is given by

$$h_{\mathrm{XXZ}_{\ell/2}} = 2 \sum_{j=1}^{\ell} c_j \prod_{m=0, m\neq j}^{\ell} \frac{2X_\ell - [m][m + 1]}{[j - m][j + m + 1]},$$

where c_j is the coefficient defined in (19a), (19b), (19c), (19d). We see that the direct (naive) evaluation of this expression at $q = \exp \mathrm{i}\pi/(\ell + 2)$ is problematic some of the q-integers in the denominator become zero. Yet, writing out the products it is easy to show that for $\ell = 1$ and $\ell = 2$ this expression coincides with the modified local Hamiltonian h'. For $\ell = 3$, one checks that both Hamiltonians coincide at $q = \exp \mathrm{i}\pi/5$ by using the explicit expressions given in [22]. Beyond this value, we verified through explicit numerical comparison up to $\ell = 6$ the following statement:

Conjecture 1 *We have*

$$h' = h_{\mathrm{XXZ}_{\ell/2}} \quad at \ q = \exp\left(\mathrm{i}\pi/(\ell + 2)\right).$$

We have little doubt that this relation holds for general ℓ. Hence, we propose the ℓ-th trigonometric model as lattice equivalent of the ℓ-th model in the $\mathcal{N} = (2, 2)$ superconformal series with central charge $c_\ell = 3\ell/(\ell + 2)$. This is compatible with the central charges

for spin-$\ell/2$ XXZ chain [3, 61]. The genuinely new feature of the present investigation is the construction of the two copies of the $\mathcal{N} = 2$ supersymmetry algebra on the lattice. As mentioned in the introduction, this phenomenon appears only at very specific spin-anisotropy commensurable points. The two supersymmetries at these points are not only a feature of the Hamiltonian: the supercharges have definite commutation relations with the transfer matrix of the corresponding spin-$\ell/2$ fused vertex models as can quite directly be seen from the fusion equations and the algebraic Bethe ansatz [66]. Eventually, notice that all the preceding expressions make formally sense in the limit $\ell \to \infty$. This is equivalent to the limit $q \to 1$ in the above expressions where $[n] \to n$. The quantum group reduces to sl_2 and we obtain a non-compact spin chain [15]. On the field-theory side this limit gives raise to a non-rational superconformal field theory studied in [45]. It was shown in [47] that this type of non-rational limit can be related to free theories with continuous orbifolds. The fact that in the limit $\ell \to \infty$ we have a continuum of admissible twist angles, as can be seen from (15), supports the conjecture.

3.5 Witten Index

We would like to understand if the models defined here above have zero-energy states. In general, if the Hilbert space $\mathcal{H}$ is finite, then a sufficient criterion for the existence of supersymmetry singlets is a non-vanishing Witten index [92]

$$W = \mathrm{tr}_{\mathcal{H}}(-1)^F e^{-\beta H}.$$

Indeed, as all excited states come in doublets with fermion number differing by one they cancel out of the trace. Only zero-energy states contribute. Therefore we find $W = \mathrm{tr}_{\mathcal{H}}(-1)^F$ and thus $|W|$ provides a lower bound to their number. In the present case however, the dynamic nature of the supercharges makes a direct evaluation of the Witten index as defined above difficult because $\mathcal{H}$ is infinite-dimensional. Yet the trigonometric models allow to take advantage of the conservation of S_N^3, and introduce and compute a Witten index-type object [96].

Recall that the trigonometric supercharges Q_N at spin-$\ell/2$ change the total magnetisation by $-(\ell + 2)/2$. Hence we conclude that the value of $J = (\ell + 2)N/2 + S_N^3$ is conserved by their action. Moreover, we introduce the total particle number $M = J - N$ (this is the actual particle number in the fermion model). Then for a doublet $(|\psi\rangle, Q_N|\psi\rangle)$ we find the values $(M, M - 1)$. Hence we may introduce a version of the Witten index, given as the trace of $(-1)^M$ over subspaces of $\mathcal{H}$ with constant $J > 0$:

$$W_J = \sum_{N=1}^{\infty} \mathrm{tr}_{\mathcal{H}_N}(-1)^M \delta_{M+N,J} = (-1)^J \sum_{N=1}^{\infty} (-1)^N \mathrm{tr}_{\mathcal{H}_N} \delta_{M+N,J}. \tag{26}$$

Thus, we have to evaluate the number ν_{MN} of states in $\mathcal{H}_N$ with total particle number M. We introduce its generating function $f_N(z) = \sum_{M=0}^{\infty} \nu_{MN} z^M$ with respect to M (for given N the sum is actually finite). Assuming convergence, we can then write the generating function $\mathcal{W}(z) = \sum_{J=0}^{\infty} W_J z^J$ for the Witten index (26) in terms of the infinite series

$$\mathcal{W}(z) = \sum_{N=1}^{\infty} f_N(-z) z^N. \tag{27}$$

Open Chains The trigonometric models may be defined on an open interval because their supercharges are a solution of (6) with $|\chi\rangle = 0$. For an open chain it is straightforward to evaluate the dimensions of the relevant subspaces as no translation invariance needs to be taken into account. The generating function is given by the simple expression

$$f_N(z) = \left(\frac{1 - z^{\ell+1}}{1 - z}\right)^N.$$

Thus, we find the generating function for the Witten index to be

$$\mathcal{W}(z) = \frac{1 + z}{1 - (-z)^{\ell+2}} - 1,$$

wherefrom it follows that for $J > 0$ we have

$$W_J = (-1)^J \times \begin{cases} 1, & J = 0 \bmod \ell + 2, \\ -1, & J = 1 \bmod \ell + 2, \\ 0, & \text{otherwise.} \end{cases}$$

Closed Chains: Periodic Boundary Conditions This case is more difficult as one has to take into account translation invariance. More precisely, we would like to count the number of states with momentum zero for N odd, and momentum π for N even, with total particle number M fixed. Each such state can be generated from a representative $|\mu\rangle = |m_1, \ldots, m_N\rangle$ such that $\sum_{j=1}^N m_j = M$. For N odd we just have to count the number of inequivalent representatives where two configurations are considered to be equivalent when they are connected by a rotation. For N even we have an additional constraint. In fact, each simple configuration $|\mu\rangle = |m_1, \ldots, m_N\rangle$ possesses a symmetry factor which is the smallest non-zero integer r such that $T_N^r|\mu\rangle = |\mu\rangle$. The requirement to have states with momentum π restricts r to be even. Therefore, we count the number of inequivalent representatives in this subset of configurations. This is a classical enumeration problem which can be solved by applying Burnside's lemma to the cyclic group. We defer the details to Appendix C. There, we derive the generating function for the dimensions v_{MN}:

$$f_N(z) = \frac{1}{N} \sum_{m=0}^{N-1} (-1)^{(N+1)m} \left(\frac{1 - z^{N(\ell+1)/\gcd(N,m)}}{1 - z^{N/\gcd(N,m)}}\right)^{\gcd(N,m)}. \tag{28}$$

Here $\gcd(a, b)$ denotes the greatest common divisor of a and b. For even N the factor $(-1)^{(N+1)m} = (-1)^m$ subtracts the number of configurations with odd symmetry factors. Notice in particular that for $z = 1$ we obtain the total number of configurations without restrictions on the particle number: $f_N(1) = N^{-1} \sum_{m=0}^{N-1} ((-1)^{N+1}(\ell + 1))^{\gcd(N,m)}$. Given $f_N(z)$ we compute the generating function of the Witten index using (27), and obtain the surprisingly simple result

$$\mathcal{W}(z) = \frac{1}{1 - (-z)^{\ell+2}} - \frac{1}{1 + z}.$$

This implies that

$$W_J = \begin{cases} 0, & J = 0 \bmod \ell + 2, \\ (-1)^{J+1}, & \text{otherwise.} \end{cases}$$

We present a proof of the closed expression for $\mathcal{W}(z)$ due to Don Zagier.[2] It is based on the simple observation

$$(-1)^{(N+1)m} = (-1)^{N+\gcd(N,m)}.$$

This implies that in (28) the different terms in the sum depend on m only through $d = \gcd(N, m)$ what suggests a change of variables and sum over all divisors of N. Indeed, one finds

$$f_N(z) = \frac{(-1)^N}{N} \sum_{d|N} (-1)^d \phi\left(\frac{N}{d}\right) \left(\frac{1 - z^{N(\ell+1)/d}}{1 - z^{N/d}}\right)^d.$$

Here $\phi(n)$ is Euler's totient function which counts the number of positive integers $k < n$ such that k and n are coprime. Hence $\phi(N/d)$ equals the number of integers $0 \leq m < N$ such that $d = \gcd(N, m)$. For the generating function $\mathcal{W}(z)$ we obtain:

$$\begin{aligned}
\mathcal{W}(z) &= \sum_{N=1}^{\infty} \frac{(-z)^N}{N} \sum_{d|N} (-1)^d \phi\left(\frac{N}{d}\right) \left(\frac{1 - (-z)^{N(\ell+1)/d}}{1 - (-z)^{N/d}}\right)^d \\
&= \sum_{k,d=1}^{\infty} (-1)^d \frac{(-z)^{kd}\phi(k)}{kd} \left(\frac{1 - (-z)^{(\ell+1)k}}{1 - (-z)^k}\right)^d \\
&= -\sum_{k=1}^{\infty} \frac{\phi(k)}{k} \ln\left(\frac{1 - (-z)^{(\ell+2)k}}{1 - (-z)^k}\right).
\end{aligned}$$

From the second to the third line, we introduced a new variable $k = N/d$. It allows to perform the summation over d which yields the third line. The latter can be simplified with the help of the following the identity [52]:

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n} \ln(1 - x^n) = -\frac{x}{1 - x}.$$

From this the expression for $\mathcal{W}(z)$ given above follows immediately.

For both open and closed chains the Witten index calculations show that the trigonometric models possess exact zero-energy ground states because W_J is non-vanishing for certain choices of J. Unfortunately, it is not clear how many ground states are present as a function of the number of sites as J mixed N and the magnetisation. In the case of periodic boundary conditions the most likely scenario compatible with the result for the Witten index is the existence of $\ell + 1$ ground states with total magnetisation $S_N^3 = -\ell/2, -\ell/2 + 1, \ldots, \ell/2 - 1, \ell/2$ if the number of sites N is odd, and none if N is even. Moreover, our consideration do not cover the case of twisted boundary conditions for which the values for W_J will be different.

4 Spin-1 Models

In this section we present different spin chains for spin one with lattice supersymmetry. These were obtained by solving (6) with various additional requirements. The aim here is

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not to exhaust all possible solutions but rather to present some examples related to known spin chains. We start with the Fateev-Zamolodchikov spin chain and its elliptic generalisation in Sect. 4.1. We point out that it possesses a supersymmetry for any choice of its coupling constants. Moreover, we discuss some special points in the space of couplings were additional features emerge. In Sect. 4.2 we proceed with the analysis of a supercharge which generates a spin chain related to the supersymmetric $t-J$ model. In particular, we will present a supersymmetry-preserving elliptic extension of this model. In Sect. 4.3, we discuss a spin chain with particle number conservation mod 3 which is related to combinatorial problems, and interpolates between the trigonometric models for $\ell = 1$ and $\ell = 2$.

4.1 The (Elliptic) Fateev-Zamolodchikov Chain

In this section we investigate in detail the spin-1 chain introduced by Fateev and Zamolodchikov [97], and its elliptic extension [35]. Heuristically this is motivated by the following observation. The trigonometric model with $\ell = 2$ coincides with the twisted Fateev-Zamolodchikov chain at the special point $\eta = \pi/4$ as we saw in the last section. Numerical simulations [8, 44] show that the low-energy states in the antiferromagnetic regime (the one studied here) are in one-to-one correspondence with the spectrum and field content of a $c = 3/2$ superconformal field theory. It is the second member of the $\mathcal{N} = 2$ superconformal minimal series which can be described in terms of a free boson and a free fermion. The two sets of supercharges carry a $U(1)$-charge ± 1 (with respect to the standard $U(1)$-symmetry which is part of every $\mathcal{N} = (2, 2)$ supersymmetry algebra). This is similar to the fact that the trigonometric supercharges for $\ell = 2$ change the magnetisation by ± 2. Yet, this particular $c = 3/2$ theory is known to have a further supercharge which is $U(1)$-neutral and generates an *additional* $\mathcal{N} = 1$ superconformal symmetry [32]. Here we show that this supersymmetry has a lattice predecessor. It is quite different from the trigonometric supercharges for spin one because it does—quite consistently—not change the magnetisation.

We proceed as follows. First, we will prove the existence of the lattice supersymmetry in Sect. 4.1.1, show that it is present even off the critical point and furthermore not restricted to any particular anisotropy of the chains. Sections 4.1.2 and 4.1.3 deal with some local and global symmetries of the Hamiltonian, and their interplay with the supercharges. The existence of supersymmetry singlets is discussed in 4.1.4. In Sect. 4.1.5 we make a series of conjectures on the nature of these zero-energy states: in particular, we conjecture a relation to the weighted enumeration of alternating sign matrices. Eventually we come back to the point $\eta = \pi/4$ in Sect. 4.1.6, where we show that the lattice version of the $\mathcal{N} = (2, 2)$ supersymmetry is present even away from the trigonometric point.

4.1.1 Supercharges and Hamiltonian

The type of Hamiltonian which we would like to study is locally of the form (20), and therefore

$$H_N = \sum_{j=1}^N \left(\sum_{a=1}^3 J_a \left(s_j^a s_{j+1}^a + 2 \left(s_j^a \right)^2 \right) - \sum_{a,b=1}^3 A_{ab} s_j^a s_j^b s_{j+1}^a s_{j+1}^b \right). \tag{29}$$

Here J_a and A_{ab}, $a, b = 1, 2, 3$ are constants to be fixed. The operators s^a, $a = 1, 2, 3$ are the spin-1 representation of su(2). In the usual basis $\{|0\rangle, |1\rangle, |2\rangle\}$ where s^3 is diagonal they

are given by 3×3 matrices

$$s^3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad s^+ = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad s^- = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

with $s^\pm = s^1 \pm is^2$. Thus we have $s^3|m\rangle = (m-1)|m\rangle$, and furthermore $s^+|0\rangle = \sqrt{2}|1\rangle$ and $s^+|1\rangle = \sqrt{2}|2\rangle$.

We want to show that there is a local supercharge q which generates the local Hamiltonian density of (29). In order to construct it, we need the three states

$$|\psi_0\rangle = |10\rangle - |01\rangle, \qquad |\psi_1\rangle = |20\rangle - |02\rangle, \qquad |\psi_2\rangle = |21\rangle - |12\rangle,$$

which span a spin-1 module of su(2) on two sites. The local operation q is defined through

$$q|0\rangle = x|\psi_0\rangle - y|\psi_2\rangle,$$

$$q|1\rangle = |\psi_1\rangle,$$

$$q|2\rangle = x|\psi_2\rangle - y|\psi_0\rangle,$$

where x and y are two real parameters. An explicit check shows that it leads to a nilpotent Q_N as (6) holds with vector $|\chi\rangle = (x^2 - y^2)|11\rangle - x(|20\rangle + |02\rangle) - y(|22\rangle + |00\rangle)$. Writing out the Hamiltonian density h, we thus identify the coupling constants. We find that $A_{ab} = A_{ba}$. The diagonal elements are given by $A_{aa} = J_a$. The remaining couplings are quadratic polynomials in x, y:

$$J_1 = 1 - 4xy, \qquad J_2 = 1 + 4xy, \qquad J_3 = 2(x^2 + y^2) - 1,$$

$$A_{12} = 1 - 4y^2, \qquad A_{13} = (2x-1)(1-2y), \qquad A_{23} = (2x-1)(1+2y).$$

The Hamiltonian density h is invariant under spin reversal as it should be because $(R \otimes R)qR = -q$. This implies that no second copy of the supersymmetry algebra can be constructed from q through spin reversal. From (3) we conclude that the only possible twist angles are $\phi = 0, \pi$ (the latter will play an important role in what follows). The case $y = 0$, which corresponds to the trigonometric limit, however admits *arbitrary* twist angles, and the supersymmetry is realised provided that (2) holds. In terms of the spin chain, twisted boundary conditions correspond to

$$s^3_{N+1} = s^3_1, \qquad s^\pm_{N+1} = e^{\pm i\phi}s^\pm_1.$$

The fact that ϕ is arbitrary comes from the property that the local supercharge q is neutral for $y = 0$: it does not change the magnetisation unlike the trigonometric supercharge at spin one defined in Sect. 3.

Elliptic Parametrisation, Integrability Fateev showed that the Hamiltonian (29) is integrable along a two-parameter submanifold in the space of couplings [35]. The coupling constants are parametrised by Jacobian elliptic functions. The J_a take the values

$$J_1 = \frac{2}{1 + \mathrm{dn}\,2\gamma}, \qquad J_2 = \frac{2\,\mathrm{dn}\,2\gamma}{1 + \mathrm{dn}\,2\gamma}, \qquad J_3 = \frac{2\,\mathrm{cn}\,2\gamma}{1 + \mathrm{dn}\,2\gamma}.$$

Furthermore, the symmetric 3×3 matrix A has diagonal elements $A_{aa} = J_a$, and off-diagonal elements

$$A_{12} = \frac{2 \operatorname{sn} 2\gamma}{1 + \operatorname{dn} 2\gamma} \left(\frac{\operatorname{cn} \gamma}{\operatorname{sn} \gamma} - \frac{\operatorname{cn} 2\gamma}{\operatorname{sn} 2\gamma} \right), \qquad A_{13} = \frac{2 \operatorname{sn} 2\gamma}{1 + \operatorname{dn} 2\gamma} \left(\frac{1}{\operatorname{sn} \gamma} - \frac{1}{\operatorname{sn} 2\gamma} \right),$$

$$A_{23} = \frac{2 \operatorname{sn} 2\gamma}{1 + \operatorname{dn} 2\gamma} \left(\frac{\operatorname{dn} \gamma}{\operatorname{sn} \gamma} - \frac{\operatorname{dn} 2\gamma}{\operatorname{sn} 2\gamma} \right).$$

Here the two relevant parameters are thus $\gamma = 2K\eta/\pi$, where $0 \leq \eta \leq \pi$ is the so-called *crossing parameter*, and the *elliptic nome* $p = \exp(-\pi K'/K)$, where K, iK' are the quarter-periods of the elliptic functions [1]. We call the generic case with non-vanishing nome p elliptic. The trigonometric limit corresponds to $p \to 0$ (where $K \to \pi/2$): the coupling constants reduce to the values (21).

In fact, it is just a matter of comparison of coefficients that the choice

$$x = \frac{(1 + \operatorname{dn} \gamma) \operatorname{sn} 2\gamma}{2(1 + \operatorname{dn} 2\gamma) \operatorname{sn} \gamma}, \qquad y = \frac{(1 - \operatorname{dn} \gamma) \operatorname{sn} 2\gamma}{2(1 + \operatorname{dn} 2\gamma) \operatorname{sn} \gamma}$$

leads to supercharges that generate the Hamiltonian density of the spin chain (29) along the integrable line. Hence we conclude that the Hamiltonian of the (elliptic) Fateev-Zamolodchikov spin chain is part of an $\mathcal{N} = 2$ lattice supersymmetry algebra *for any choice of the parameters x and y*, in sharp contradistinction to the XYZ chain at spin-1/2 where the lattice supersymmetry is only present at a the special point $\eta = \pi/3$.

4.1.2 Rotational Invariance and Duality Transformations

Motivated by the discussion on duality transformations for the spin-1/2 XYZ chain at the combinatorial point given in Sect. 2.4, we discuss the symmetries of the present supercharge with respect to global rotations. As before, we denote by $\rho^a(\theta) = \exp(i\theta s^a)$ a local rotation about the a-axis b, where s^a is one of the su(2)-generators in the spin-1 representation, and by $\Omega_N^a(\theta) = \prod_{j=1}^N \rho_j^a(\theta)$ a global rotation. We start with generic values for x, y for which no simple rotational invariance is expected. Writing $\mathsf{q} = \mathsf{q}(x, y)$ we find that

$$\left(\rho^1 \left(\frac{\pi}{2} \right) \otimes \rho^1 \left(\frac{\pi}{2} \right) \right) \mathsf{q}(x, y) \rho^1 \left(-\frac{\pi}{2} \right) = (x - y) \mathsf{q}(x_+, y_+), \tag{30a}$$

$$\left(\rho^2 \left(\frac{\pi}{2} \right) \otimes \rho^2 \left(\frac{\pi}{2} \right) \right) \mathsf{q}(x, y) \rho^2 \left(-\frac{\pi}{2} \right) = (x + y) \mathsf{q}(x_-, y_-), \tag{30b}$$

$$\left(\rho^3 \left(\frac{\pi}{2} \right) \otimes \rho^3 \left(\frac{\pi}{2} \right) \right) \mathsf{q}(x, y) \rho^3 \left(-\frac{\pi}{2} \right) = \mathsf{q}(x, -y) \tag{30c}$$

where we abbreviated

$$x_\pm = \frac{x \pm y + 1}{2(x \mp y)}, \qquad y_\pm = -\frac{x \pm y - 1}{2(y \mp x)}.$$

In the case of untwisted boundary conditions $\phi = 0$ we conclude thus that the Hamiltonian transforms according to

$$\Omega_N^1(\pi/2) H_N(x, y) \Omega_N^1(-\pi/2) = (x - y)^2 H_N(x_+, y_+),$$

$$\Omega_N^2(\pi/2)H_N(x,y)\Omega_N^2(-\pi/2) = (x+y)^2 H_N(x_-, y_-),$$

$$\Omega_N^3(\pi/2)H_N(x,y)\Omega_N^3(-\pi/2) = H_N(x,-y).$$

We see that up to rescaling the spectrum is invariant under the transformations in parameter space induced by these rotations. Unfortunately, if we choose the twist angle $\phi = \pi$, which will prove to be the most interesting choice, then only the third transformation survives. We discuss briefly some special points: let us suppose that x is left invariant by (30a). Then we have either (i) $x = y + 1$, which can be transformed by (30b) according to $(y+1, y) \to (1/(2y+1), 0)$ or (ii) $x = -1/2$. If we set $y = \zeta/2$ in the second case then we see that (30a) leads to $\zeta \to (3-\zeta)/(1+\zeta)$, and (30b) to $\zeta \to (\zeta+3)/(\zeta-1)$. We recognise here the homographic transformation which appears in the spin-1/2 XYZ chain at the combinatorial point as discussed in Sect. 2.4.

Eventually, there is a symmetry relating x and $-x$. Consider the projector $p^\pm = s^3(s^3 \mp 1)/2$ on the spin states 0 and 2 respectively. It follows that

$$\left(e^{i\pi p^\pm} \otimes e^{i\pi p^\pm}\right)\mathsf{q}(x,y)e^{-i\pi p^\pm} = -\mathsf{q}(-x,y),$$

what implies that the Hamiltonian enjoys the transformation property

$$\Omega_N^\pm H_N(x,y)\left(\Omega_N^\pm\right)^{-1} = H_N(-x,y), \qquad \Omega_N^\pm = \prod_{j=1}^N e^{i\pi p_j^\pm}.$$

All these symmetries are simple in the sense that they can be built as tensor products of local transformations.

Less obvious seems the following statement:

Conjecture 2 *For the twist angle $\phi = \pi$ the spectrum is invariant under the exchange of x and y.*

This statement implies the existence of a unitary transformation $V(x,y)$ with the intertwining property $V(x,y)H_N(x,y) = H_N(y,x)V(x,y)$ with $V(x,y)^{-1} = V(x,y)^\dagger = V(y,x)$. The explicit construction of the simplest examples with $N = 2$ and $N = 3$ sites shows that $V(x,y)$ is not a tensor product of local transformations, even though the eigenvalues of the Hamiltonian density are invariant under the exchange of x and y.

Finally, we turn to the trigonometric limit $y = 0$. We have a full invariance with respect to rotations about the 3-axis: $(\rho^3(\theta) \otimes \rho^3(\theta))\mathsf{q}\rho^3(-\theta) = \mathsf{q}$. This is nothing but the U(1)-symmetry which leads to the conservation law for the total magnetisation in the Fateev-Zamolodchikov chain. It becomes a full SU(2) symmetry at the point $x = 1$, $y = 0$ which corresponds to the XXX chain at spin one [6, 7]. Indeed, the supercharges itself are invariant because $(\rho^a(\theta) \otimes \rho^a(\theta))\mathsf{q}(1,0)\rho^a(-\theta) = \mathsf{q}(1,0)$ for all $a = 1, 2, 3$ at this point. The invariance is due to the fact that the supercharge maps a spin-1 module onto a spin-1 module, the latter being realised on two sites.

4.1.3 Local Quantum Group Symmetry

The fact that the local supercharge q makes explicit the SU(2) symmetry at $x = 1$, $y = 0$ leads quite naturally to the question if for $x = (q + q^{-1})/2$, $y = 0$ the local quantum group invariance of the Fateev-Zamolodchikov chain may be seen from the supersymmetry. This

is indeed the case as we show now. To this end, we consider the gauge-equivalent local supercharge

$$\mathsf{q}' = \mathsf{q} + \mathsf{q}_\beta, \qquad \mathsf{q}_\beta |m\rangle = |\beta\rangle \otimes |m\rangle + |m\rangle \otimes |\beta\rangle \quad \text{with } |\beta\rangle = \frac{1}{2}\big(q^{-1} - q\big)|1\rangle.$$

As noted before, we add only a redundant part which has no global effect. The new local supercharge acts according to

$$\mathsf{q}'|0\rangle = q^{-1}|10\rangle - q|01\rangle,$$
$$\mathsf{q}'|1\rangle = |20\rangle - |02\rangle + \big(q^{-1} - q\big)|11\rangle,$$
$$\mathsf{q}'|2\rangle = q^{-1}|21\rangle - q|12\rangle.$$

The new supercharge q' solves (6) with $|\chi\rangle = -(q|02\rangle + q^{-1}|20\rangle - |11\rangle)$ which is the quantum-group singlet. Moreover, on the right-hand side of the preceding equations, we recognise the basis states of the spin-1 module of $\mathrm{U}_q(\mathrm{sl}_2)$ realised on two sites. We thus see that here the supersymmetry is a consequence of the structure of the q-Clebsch-Gordon coefficients. Moreover, the given structure implies immediately that q' commutes with its generators in the sense that

$$\mathsf{q}'\mathsf{s}^{\pm} = \Delta\big(\mathsf{s}^{\pm}\big)\mathsf{q}', \quad \mathsf{q}'\mathsf{s}^3 = \Delta\big(\mathsf{s}^3\big)\mathsf{q}',$$

where Δ is the comultiplication defined in (23). According to (10a), (10b) the corresponding Hamiltonian densities h and h' differ by a local boundary term. Indeed, we find

$$h' = h + \frac{q^{-2} - q^2}{2}\big(\mathsf{s}^3 \otimes 1 - 1 \otimes \mathsf{s}^3\big).$$

This is precisely the Hamiltonian density of the quantum group invariant Fateev-Zamolodchikov spin chain studied in [80].

4.1.4 Supersymmetry Singlets

Given that the elliptic Fateev-Zamolodchikov spin chain has a supersymmetry for any choice of the parameters x, y we ask if any solutions to the equations $Q_N|\Phi\rangle = 0$ and $Q_{N-1}^\dagger|\Phi\rangle = 0$ exist, and how the space of solutions depends on the parameters. We use the conjugation technique described in Sect. 2.4 in order to relate different values of x, y. Consider the operator $m = m(\lambda, \theta)$ defined through

$$m(\lambda, \theta)|0\rangle = \frac{1}{\sqrt{\lambda}}\left(\cosh\left(\frac{\theta}{2}\right)|0\rangle + \sinh\left(\frac{\theta}{2}\right)|2\rangle\right),$$
$$m(\lambda, \theta)|1\rangle = |1\rangle,$$
$$m(\lambda, \theta)|2\rangle = \frac{1}{\sqrt{\lambda}}\left(\sinh\left(\frac{\theta}{2}\right)|0\rangle + \cosh\left(\frac{\theta}{2}\right)|2\rangle\right).$$

It follows that

$$\big(m(\lambda, \theta) \otimes m(\lambda, \theta)\big)\mathsf{q}(x, y)m(\lambda, \theta)^{-1} = \lambda^{-1}\mathsf{q}\big(x', y'\big),$$

where

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \lambda \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{31}$$

We thus see that λ plays the role of a dilatation parameter, whereas θ is the angle of a "Lorentz transformation" on the parameter space. We allow negative values for λ, and thus include reflections with respect to the origin. As in the case of the XYZ chain studied in Sect. 2.4, the transformation can be promoted to the full chain: we have $M_N(\lambda, \theta) = m(\lambda, \theta)^{\otimes N}$. The conjugation of the supercharges gives

$$M_{N+1}(\lambda, \theta) Q_N(x, y) M_N(\lambda, \theta)^{-1} = \lambda^{-1} Q_N(x', y').$$

Notice that $M_N(\lambda, \theta)$ preserves the momentum spaces where $T_N(\phi) \equiv (-1)^{N+1}$ for $\phi = 0, \pi$. Hence, from the conjugation argument given in Sect. 2.4 we conclude that $H_N(x, y)$ and $H_N(x', y')$ have the same number of zero-energy states. Thus, we need to understand which points can be mapped onto each other. It is easy to see that $(x')^2 - (y')^2 = \lambda^2(x^2 - y^2)$. This divides the (x, y)-plane in (i) "time-like" points $x^2 - y^2 < 0$, (ii) "light-like" points $x^2 = y^2$ and (iii) "space-like" points $x^2 - y^2 > 0$ with respect to the origin $x = y = 0$. The latter plays a distinguished rule, as it is the only fixed point. The transformation (31) preserves the nature of the points, i.e. if (x, y) is time-like then (x', y') is, and so on. It follows that for all time-like choices of the coupling constants x, y, the Hamiltonian has the same number of zero-energy states. The same applies to all space-like choices for x, y. Conjecture 2 implies then that for all choices with $|x| \neq |y|$ the Hamiltonian has the same number of supersymmetry singlets. Therefore, it is sufficient to analyse a single, not light-like point in the (x, y)-plane. As opposed to the XYZ chain studied in Sect. 2.4 there does not seem to be a simple choice however. We content ourselves with the following statement:

Conjecture 3 *For any number of sites N and twist angle $\phi = \pi$ there exists a choice of parameters x, y with $x \neq y$ where the Hamiltonian has a single zero-energy state in the sector $S_N^3 = 0 \bmod 2$. Furthermore, for the twist angle $\phi = 0$ there is a choice of parameters x, y with $x \neq y$ where the Hamiltonian has no zero-energy states.*

If this conjecture holds then it implies the existence of exactly one zero-energy state for all choices with $|x| \neq |y|$ if the twist angle is set to $\phi = \pi$, and none if $\phi = 0$. The existence of a singlet for $x \neq 0, y = 0$ for the twisted chain was already observed in [43] where a staggered spin chain based on the quantum group $U_q(\mathfrak{sl}(2|1))$ was studied. Its spectrum contains the spectrum of the twisted Fateev-Zamolodchikov spin chain.

Next, let us consider the light-like points with $|x| = |y|$, $x \neq 0$. We show in this case that there is at least one ground state for the twist angle $\phi = \pi$. We start with even N. Invoking Conjecture 2 we see that it is sufficient to consider the half-line $x = y$, $x > 0$. All choices for $x > 0$ lead to the same number of singlets. The point $x = y = 1/2$ is particularly convenient because most of the terms in the Hamiltonian cancel as can be see from Sect. 4.1.1. Moreover, if restricted to the subspace of states containing only 0's and 2's the sum of the entries of the Hamiltonian density along each row and column is zero. Hence, the Hamiltonian has the zero-energy state $|\Psi\rangle = \sum_\mu |\mu\rangle$ where μ runs over all spin configurations which do not contain any 1's in the desired momentum sector because $T_N(\phi = \pi)|\Psi\rangle = -|\Psi\rangle$. Next, let us turn to odd N. In this case it turns out to be convenient to choose the half-line $x = -y$, $x > 0$, and pick the point $x = -y = 1/4$. Instead of H_N, we consider the unitarily

equivalent Hamiltonian $H'_N = U H_N U^{-1}$ with $U = \exp(i\pi \sum_{j=1}^{N} j s_j^3)$. H'_N is invariant under translations (without twists), and we are interested in its zero-energy states in the translationally invariant sector. It turns out that if we set $x = 1/4$, $y = -1/4$ then the corresponding Hamiltonian density has again the property that the sum of its components along each row or column is zero. This time, no restriction to any other subspace than the translation sector is necessary. Hence, a zero-energy state of H'_N is given by $|\Psi'\rangle = \sum_\mu |\mu\rangle$ what implies that H_N has at least one zero-energy state. Using the conjugation argument we conclude that for all choices of x, y with $|x| = |y|$ and $x, y \neq 0$ there is at least one zero-energy states in the momentum sectors of interest.

We are now left with the point $x = y = 0$. It is known that in this case the Fateev-Zamolodchikov Hamiltonian can be mapped onto the supersymmetric $t-J$ model. This case will be analysed in Sect. 4.2, where we will show that H_N possesses zero-energy states even for $\phi = 0$. This demonstrates that their number may change when parameters of the Hamiltonian are varied.

4.1.5 Combinatorial Nature of the Singlet States

It is natural to ask if for the twist angle $\phi = \pi$ the singlet state and its components, defined through $|\Psi\rangle = \sum_\mu \psi_\mu |\mu\rangle$ where μ runs over all spin configurations, possess interesting properties. Indeed this seems to be the case. For example if we choose the components to be coprime polynomials in x, y then we observe a quite non-trivial sum rule:

Conjecture 4 *The square norm of the supersymmetry singlet eigenvector factorises into two polynomials in x, y. At odd N, one of these polynomials is the component $\psi_{11\cdots1}$.*

Trigonometric Case   We observed a refinement of this conjecture in the trigonometric case when $y = 0$ by exact diagonalisation of the Hamiltonian. To state it, we need to specify the normalisation of the ground state vector $|\Psi\rangle$ in a more precise way. As the Hamiltonian is a quadratic polynomial in x we may choose its components to be coprime polynomials in x. This fixes the state up to an overall multiplicative constant. We adjust its value through the asymptotic the limit $x \to \infty$. The Hamiltonian reduces to

$$\lim_{x \to \infty} x^{-2} H_N = 2 \sum_{j=1}^{N} \left(s_j^3 s_{j+1}^3 + \left(s_j^3 \right)^2 + \left(s_{j+1}^3 \right)^2 \right), \tag{32}$$

and therefore does not depend on the twist. It is easy to see that the Hamiltonian is positive and annihilates the states $|11\cdots1\rangle$ for odd N, and $|0202\cdots02\rangle + |2020\cdots20\rangle$ for even N. These are the only zero-energy states in the sectors where $T_N(\phi = \pi) \equiv (-1)^{N+1}$. It follows that in our normalisation for odd N the component $\psi_{11\cdots1}$ must have the highest degree in x amongst all the components of the singlet vector. The same applies to the components $\psi_{0202\cdots02}$ and $\psi_{2020\cdots20}$ for even N. Let us denote in both cases this maximal degree by d_N. We fix the normalisation by adjusting the leading coefficient as x goes to infinity:

$$\psi_{11\cdots1} \underset{x \to \infty}{\sim} (2x)^{d_N} \left(1 + o(1) \right) \quad \text{for } N \text{ odd}, \tag{33a}$$

$$\psi_{0202\cdots02} \underset{x \to \infty}{\sim} (2x)^{d_N} \left(1 + o(1) \right) \quad \text{for } N \text{ even}. \tag{33b}$$

$$\begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix} \quad \begin{pmatrix} 1\,0\,0 \\ 0\,0\,1 \\ 0\,1\,0 \end{pmatrix} \quad \begin{pmatrix} 0\,1\,0 \\ 1\,0\,0 \\ 0\,0\,1 \end{pmatrix} \quad \begin{pmatrix} 0\,1\,0 \\ 0\,0\,1 \\ 1\,0\,0 \end{pmatrix} \quad \begin{pmatrix} 0\,0\,1 \\ 1\,0\,0 \\ 0\,1\,0 \end{pmatrix} \quad \begin{pmatrix} 0\,0\,1 \\ 0\,1\,0 \\ 1\,0\,0 \end{pmatrix} \quad \begin{pmatrix} 0\,1\,0 \\ 1\,-1\,1 \\ 0\,1\,0 \end{pmatrix}$$

$$1 \quad + \quad 1 \quad + \quad 1 \quad + \quad 1 \quad + \quad 1 \quad + \quad 1 \quad + \quad t \quad = 6 + t$$

Fig. 2 All 3×3 alternating sign matrices: six permutation matrices with weight 1 and a single matrix with weight t. The generating function for these is $A_3(t) = 6 + t$. The last matrix is the only 3×3 vertically symmetric alternating sign matrix. Therefore we have $T_1(t) = 1$

Conjecture 5 (*Degree*) *With this choice of normalisation all components are polynomials in x with integer coefficients if we adjust the degree to*

$$d_N = \begin{cases} n^2, & N = 2n + 1, \\ n(n-1), & N = 2n. \end{cases}$$

We analyse the states in this normalisation. It turns out that both their components, and square norms are related to problems of enumerative combinatorics. This is rather similar to the case of the XXZ chain with anisotropy $\Delta = -1/2$ whose combinatorial features were first observed in [28, 82, 83]. In order to understand the combinatorial nature of the supersymmetry singlet in the present context, we need some basic notions about the weighted enumeration of alternating sign matrices. Consider the set of $n \times n$ alternating sign matrices. Their entries take the values 0 or ± 1. The first non-zero element in each row and column is 1, and the non-zero elements alternate in sign. We assign the weight t^k to a matrix if it contains k -1's. The generating function for given size is $A_n(t)$. The example of 3×3 matrices is shown in Fig. 2, and gives $A_3(t) = 6 + t$. Next, restrict to $(2n + 1) \times (2n + 1)$ alternating sign matrices which are symmetric about the vertical axis. Assign weight t^k to any such matrix if it contains k -1's within the first n columns. Then, the generating function is given by $T_n(t)$. From Fig. 2 we infer that for $n = 1$ there is only one vertically-symmetric alternating sign matrix which gives $T_1(t) = 1$. In general, both $T_n(t)$ and $A_n(t)$ are given in terms of determinantal formulae. The relevant polynomials can be found in Robbins' [86] and Kuperberg's work [70]. We need the following expressions:

$$T_n(t) = \det_{0 \leq i, j < n} \sum_{k=0}^{2n-2} \binom{i+1}{k-i} \binom{j}{2j-k} t^{2j-k},$$

$$R_n(t) = \det_{0 \leq i, j < n} \sum_{k=0}^{2n-1} Y(i, k, 1) Y(j, k, 0) t^{2j+1-k}.$$

The coefficients $Y(i, j, k)$ are

$$Y(i, j, k) = \binom{i+k}{2i+1+k-j} + \binom{i+k+1}{2i+1+k-j}.$$

The polynomial $A_n(t)$ is then given by

$$A_n(t) = \begin{cases} T_n(t) R_n(t), & N = 2n + 1, \\ 2 T_n(t) R_{n-1}(t) & N = 2n. \end{cases}$$

These polynomials appear in the supersymmetry singlet provided that we identify $t = 4x^2$. Indeed, up to $N = 11$ sites we checked the following statement:

Conjecture 6 *Choose the normalisation as in* (33a), (33b). *For odd* $N = 2n + 1$ *we have*

$$\psi_{11\cdots1} = (2x)^n T_n(4x^2), \quad \|\Psi\|^2 = A_{2n+1}(4x^2),$$

whereas for even $N = 2n$

$$\psi_{0202\cdots02} = T_n(4x^2), \quad \|\Psi\|^2 = A_{2n}(4x^2).$$

We see that in both cases there are distinguished components related to vertically-symmetric alternating sign matrices whereas the square norm gives the polynomials for unrestricted alternating sign matrix enumeration. For odd $N = 2n + 1$ we found another sum rule related which is related to the polynomials $R_n(t)$. To state it, we transform the state according to

$$|\Phi\rangle = U|\Psi\rangle, \quad U = \exp\left(i\pi \sum_{j=1}^{N} j s_j^3\right).$$

In fact, U is the unitary transformation which spreads out the twist over the whole system so that for N odd $U H_N U^{-1}$ is invariant under the usual translations. Next, we denote by $\nu_0(\mu)$ the number of spins with value 0 in the configuration μ. For example $\nu_0(0100221) = 3$. If the number of sites is odd $N = 2n + 1$ and the states under consideration have $S_N^3 = 0$ then $0 \le \nu_0(\mu) \le n$. With these preliminary definitions we claim the following:

Conjecture 7 *For the components of the transformed states defined through* $|\Phi\rangle = \sum_\mu \phi_\mu |\mu\rangle$ *we have the linear sum rule*

$$\sum_\mu (2x)^{n-\nu_0(\mu)} \phi_\mu = R_n(4x^2).$$

Elliptic Case, Odd Length It is quite natural to ask how the combinatorial content gets deformed as we go off the critical line $y = 0$. We have not been able to formulate a conjecture on this deformation except for the point $\eta = \pi/3$, corresponding to $x = -1/2$ and $y = \zeta/2$ which we encountered already when studying the duality transformations. At this point we find the coupling constants:

$$J_1 = 1 + \zeta, \quad J_2 = 1 - \zeta, \quad J_3 = \frac{\zeta^2 - 1}{2},$$

$$A_{12} = -2J_3, \quad A_{13} = -2J_2, \quad A_{23} = -2J_1.$$

As in the spin-1/2 case (13a), (13b) the Hamiltonian is a quadratic polynomial in ζ. Thus the components of the singlet for $\phi = \pi$ can be chosen as mutually coprime polynomials in ζ. This fixes the normalisation up to an overall factor. We investigated this state up to $N = 7$ sites, and put forward the following conjecture:

Conjecture 8 *For* $\eta = \pi/3$ *and* $N = 2n + 1$ *sites we find in suitable normalisation the component*:

$$\psi_{11\cdots1}(\zeta) = \left(\frac{3+\zeta}{2}\right)^{n(n+1)} P_{-n-1}\left(\frac{1-\zeta}{3+\zeta}\right),$$

where the $p_n(z)$ are related to the polynomials found by Bazhanov and Mangazeev [11, 75]
for the spin-1/2 XYZ chain (*we recall them in Appendix B, see also* [98]). *The second factor
in the square norm can also be written in terms of the polynomials* $p_n(\zeta)$:

$$\left\|\Psi(\zeta)\right\|^2 = \psi_{11\cdots 1}(\zeta) \times \left(\frac{3+\zeta}{2}\right)^{n(n+1)} p_n\left(\frac{1-\zeta}{3+\zeta}\right).$$

*For $N = 2n$ sites, the normalisation can be adjusted in such a way that the square norm is
given by*

$$\left\|\Psi(\zeta)\right\|^2 = \left(\frac{3+\zeta}{2}\right)^{2n^2} p_{n-1}\left(\frac{1-\zeta}{3+\zeta}\right) p_{-n-1}\left(\frac{1-\zeta}{3+\zeta}\right).$$

4.1.6 Extended Supersymmetry

One of the main motivations to study in detail the Fateev-Zamolodchikov spin chain was the
coincidence of the second member of $\mathcal{N} = 2$ superconformal unitary series with a $c = 3/2$
superconformal theory which contains an additional charge-neutral superconformal sym-
metry as described above. At the level of spin chains the latter is present for any anisotropy
whereas the non-neutral supercharges are only part of the theory at the point $\eta = \pi/4$ in
the trigonometric case. Given that on the lattice the neutral supercharge exists even in the
elliptic case, i.e. for non-zero y, it is fairly natural to ask if the non-neutral lattice super-
charges are also present at $\eta = \pi/4$ off the critical point. This question is furthermore mo-
tivated by the following heuristic consideration. Off criticality, the long-range properties of
the spin chain are expected to be described by the super-sine-Gordon field theory. It was
found in [2, 21] that the soliton scattering matrix of this field theory factorises according to
$\mathcal{S}(\theta) = \mathcal{S}_{\text{RSG}}^{K=2}(\theta) \otimes \mathcal{S}_{\text{SG}}(\theta, \beta_{SG})$ where θ is the standard rapidity variable. The factor $\mathcal{S}_{\text{RSG}}^K(\theta)$
is the $\mathcal{S}$-matrix of the so-called K-th restricted sine-Gordon field theory, and here only rel-
evant insofar as for $K = 2$ it corresponds to the $U(1)$-neutral supersymmetry. The second
factor $S_{\text{SG}}(\theta, \beta_{SG})$ is the scattering matrix of the usual sine-Gordon field theory. The latter
is known to admit an $\mathcal{N} = (2, 2)$ supersymmetry if the sine-Gordon coupling takes the value
$\beta_{SG}^2 = 16\pi/3$, as was pointed out in [20]. This point is equivalent to $\eta = \pi/4$ in the lattice
model.

We will show now that this intuition is indeed correct: there is an elliptic extension of
the spin-1 trigonometric supercharges presented in Sect. 3. There, we found two local super-
charges q and q̄ which changed the magnetisation by $\Delta S^3 = -2$ and $\Delta S^3 = +2$ respectively.
Changing this to $\Delta S^3 = \pm 2 \bmod 2 = 0 \bmod 2$ destroys unfortunately the natural distinction
of q and q̄ (this is a general phenomenon for even ℓ). We circumvent this problem by con-
structing local supercharges with built-in spin-reversal symmetry $(R \otimes R)q^{\pm}R = \pm q^{\pm}$ and
require that they both generate the same local Hamiltonian $h^+ = h^-$.

We start with a supercharge with the property $(R \otimes R)q^-R = -q^-$: we determine con-
stants a_1, a_2, a_3, a_4 such that the action

$$q^-|0\rangle = a_3(|21\rangle + |12\rangle) - a_2(|01\rangle + |10\rangle),$$

$$q^-|1\rangle = a_1(|22\rangle - |00\rangle) + a_4(|20\rangle - |02\rangle),$$

$$q^-|2\rangle = a_2(|21\rangle + |12\rangle) - a_3(|01\rangle + |10\rangle)$$

leads to a nilpotent Q_N. We prefer to keep $a_2, a_3 \neq 0$ in order to assure that in the trigono-
metric limit the solution reduces to a suitable odd combination of the trigonometric super-

charges. The local supercharge solves (6) with vector $|\chi\rangle = a_1 a_2(|00\rangle + |22\rangle) - a_1 a_3(|02\rangle + |20\rangle) + (a_3^2 - a_2^2)|11\rangle$ if $a_4 = 0$. Thus, we have three free parameters a_1, a_2, a_3.

Next, let us construct a supercharge such that $(R \otimes R)\mathsf{q}^+ R = \mathsf{q}^+$. In components, we write

$$\mathsf{q}^+|0\rangle = b_3\big(|21\rangle + |12\rangle\big) + b_2\big(|01\rangle + |10\rangle\big),$$
$$\mathsf{q}^+|1\rangle = b_1\big(|22\rangle + |00\rangle\big) + b_4\big(|20\rangle + |02\rangle\big) + b_5|11\rangle,$$
$$\mathsf{q}^+|2\rangle = b_2\big(|21\rangle + |12\rangle\big) + b_3\big(|01\rangle + |10\rangle\big).$$

It solves (6) with $|\chi\rangle = (b_1 b_2 - b_3 b_4)(|00\rangle + |22\rangle) - (b_1 b_3 - b_2 b_4)(|02\rangle + |20\rangle) + (b_2^2 - b_3^2)|11\rangle$ provided that we set $b_5 = 2b_2$. This leads to a gauge contribution and hence the Hamiltonian density will not depend on b_2. Hence, we may impose $b_2 = b_5 = 0$ and are left with three free parameters b_1, b_3, b_4.

Eventually, we demand the Hamiltonian densities associated to the two supercharges coincide: $h^+ = h^-$. A solution for the resulting system of equations for the remaining free coefficients is given by

$$b_1 = -\sqrt{2}a_3, \qquad b_3 = \frac{1}{\sqrt{2}}, \qquad b_4 = -\sqrt{2}a_2, \qquad 2\big(a_2^2 + a_3^2\big) = 1, \qquad a_1 = 1.$$

It is convenient to introduce a parameter $-1 \le \zeta \le 1$ such that

$$a_2 = \frac{\sqrt{1+\zeta} - \sqrt{1-\zeta}}{2\sqrt{2}}, \qquad a_3 = -\frac{\sqrt{1-\zeta} + \sqrt{1+\zeta}}{2\sqrt{2}}.$$

This fixes all coefficients as functions of ζ. Moreover, the $\mathsf{q}^\pm$ lead to supercharges with the mutual anticommutation relations of the algebra (12a), (12b), (12c), (12d). As the particle number conservation holds for generic values of ζ only mod 2, the only possible twist angles are $\phi^\pm = 0, \pi$. The local Hamiltonian $h^+ = h^-$ generated from these two supercharges corresponds to a particular version of the Hamiltonian density h of (29) up to a similarity transformation

$$h^\pm = \big(1 \otimes e^{i\pi s^3}\big) h \big(1 \otimes e^{-i\pi s^3}\big),$$

provided that we set

$$\zeta = \frac{1 - \mathrm{dn}\, 2\gamma}{1 + \mathrm{dn}\, 2\gamma}, \qquad \gamma = \frac{2K\eta}{\pi} = \frac{K}{2}.$$

This result is the elliptic version of (22): the fact that the both spaces $h^\pm$ acts on are transformed differently implies that $H_N^\pm$ is a twisted version of H_N. With the unitary transformation

$$U = \exp\left(i\pi \sum_{j=1}^{N} j s_j^3\right)$$

we find $H_N^\pm = U H_N U^{-1}$. We denote by $\phi_\pm$ the twist angle for the present supercharge, and by ϕ the twist angle for the supercharge of (29). Then we obtain that $\phi_\pm = \phi + N\pi$ mod 2π.

We have thus established that at $\eta = \pi/4$ the twisted Fateev-Zamolodchikov spin chain possesses additional lattice supersymmetries even away from the critical point. It is quite

natural to ask if the enhanced symmetry leads to any new zero-energy states at this point. The exact diagonalisation for $N = 3, 5, 7$ sites suggests that the answer is affirmative:

Conjecture 9 *For an odd number of sites $N = 2n + 1$, twist angle $\phi = \pi$ and crossing parameter $\eta = \pi/4$ the Hamiltonian possesses two additional zero-energy states in the sector with $S_N^3 = 1$ mod 2. In the trigonometric limit these states can be found in the sector where $S_N^3 = \pm 1$.*

Notice that this is not in contradiction with the general discussion in Sect. 4.1.4 whose considerations applied only to the sector with $S_N^3 = 0 \mod 2$.

4.2 Spin Chains and the Supersymmetric t–J Model

For general spin-$\ell/2$ there is a simple case where (6) is satisfied with $|\chi\rangle = 0$: suppose that q is only non-zero when acting on a fixed component $|m\rangle$. Moreover assume that the resulting states $|jk\rangle$ are all such that $j, k \neq m$. One encounters such a case for the Fateev-Zamolodchikov chain at the point $\eta = \pi/2$, i.e. $x = y = 0$, were the supercharge acts like

$$\mathsf{q}|0\rangle = 0, \qquad \mathsf{q}|1\rangle = |20\rangle - |02\rangle, \qquad \mathsf{q}|2\rangle = 0.$$

The local Hamiltonian is then given by $h = 1 + \Pi$ where Π is the graded permutation operator defined through

$$\Pi|1, 1\rangle = |1, 1\rangle, \qquad \Pi|m, 1\rangle = |1, m\rangle, \qquad \Pi|m, m'\rangle = -|m', m\rangle, \quad m, m' = 0, 2.$$

If we consider the 0, 2 degrees as particles, then their individual numbers n_0 and n_2 is are conserved. The local Hamiltonian permutes them like fermions with strong repulsion (i.e. a site can either be empty or occupied by a single particle) on a neutral (bosonic) background of 1's. This is precisely the local operation of the supersymmetric t–J model [34, 67]. The only difference here is that, as opposed to fermionic variables, the spin variables commute for different sites. In fact, the spin chain presented here differs from the supersymmetric t–J model by a generalised Jordan-Wigner transformation [9]. The dynamical lattice supersymmetry was found in [96] in the fermion language. Here we prefer to work in the spin language.

We would like to study a simple deformation of this supercharge. Hence we choose $\mathsf{q}|0\rangle = 0$, $\mathsf{q}|2\rangle = 0$ and $\mathsf{q}|1\rangle = a_{00}|00\rangle + a_{20}|20\rangle + a_{02}|02\rangle + a_{22}|22\rangle$. It is then trivial to see that (4) holds. We restrict our considerations to a special case

$$\mathsf{q}|1\rangle = |02\rangle - |20\rangle + \zeta\big(|00\rangle - |22\rangle\big), \quad \zeta \in \mathbb{R}.$$

We see that $(R \otimes R)\mathsf{q}R = -\mathsf{q}$ and hence we have spin-reversal symmetry. We see that the Hamiltonian generated from this supercharge is a quadratic polynomial in ζ. Moreover, Eq. (3) shows that the only admissible twist angles for non-zero ζ are given by $\phi = 0, \pi$. We focus on periodic boundary conditions and thus $\phi = 0$. Switching on the parameter ζ breaks particle number conservation for the 0s and 2s but not the 1s. In general, we expect the singlet states to be rather complicated superpositions of polynomials in ζ (as is the case for the XYZ chain along its combinatorial line). Yet, we show here below that some of them are robust against this elliptic deformation.

Witten Index We start the study of the ground state by evaluating a Witten index which is related to a conservation law like in the trigonometric models of Sect. 3. Let us denote by n_j the total number of particles of type $j = 0, 1, 2$. The supercharge decreases $n_1 \to n_1 - 1$, and increases the number of sites $N \to N + 1$. Thus, $J = N + n_1$ is conserved, and we may introduce a Witten index W_J as in Sect. 3.5:

$$W_J = \sum_{N=0}^{J} \mathrm{tr}_{\mathcal{H}_N} (-1)^{n_1} \delta_{J,N+n_1} = (-1)^J \sum_{N=0}^{J} (-1)^N \mathrm{tr}_{\mathcal{H}_N} \delta_{J,N+n_1}.$$

We denote again by $\nu_{n_1,N}$ the number of distinct momentum states with zero momentum as N is odd, and momentum π for N even, containing n_1 particles of type 1. A slight variation of the argument given in Sect. 3.5 and Appendix C shows that the generating function $f_N(z) = \sum_{n_1=0}^{\infty} \nu_{n_1,N} z^{n_1}$ is given by

$$f_N(z) = \frac{1}{N} \sum_{m=0}^{N-1} (-1)^{(N+1)m} \left(2 + z^{N/\gcd(N,m)}\right)^{\gcd(N,m)}.$$

Computing the generating function $\mathcal{W}(z)$ of the Witten index, we find again a surprisingly simple and regular series expansion as in Sect. 3.5. Using a calculation which is similar to the one for the trigonometric models we obtain the closed expression

$$\mathcal{W}(z) = \sum_{J=0}^{\infty} W_J z^J = \sum_{J=0}^{\infty} f_J(-z) z^J = \frac{2z}{1 - z^2},$$

and thus conclude that

$$W_J = \begin{cases} 2, & J \text{ odd}, \\ 0, & J \text{ even}. \end{cases} \tag{34}$$

There are thus at least two zero-energy states for given odd $J = N + n_1$ for periodic boundary conditions.

Zero-Energy States As opposed to generic spin-1 chains, the present model possesses zero-energy states which can easily be determined. We consider only periodic boundary conditions. We start with zero-momentum states at odd N. Observe that any state which is only build from particles 0 and 2 is annihilated by Q_N. Therefore, we restrict our considerations to the subspace in which no local spin is in the state $|1\rangle$. On this subspace the Hamiltonian density h has two ζ-independent null vectors which are simple tensor-product states $|\chi_{\pm}\rangle = |v_{\pm}\rangle \otimes |v_{\pm}\rangle$ with $|v_{\pm}\rangle = (|0\rangle \pm |2\rangle)/\sqrt{2}$. There are thus two zero-energy states of H_N

$$\left|\Psi_N^{\pm}\right\rangle = |v_{\pm}\rangle \otimes \cdots \otimes |v_{\pm}\rangle. \tag{35}$$

These states are eigenstates of the spin-reversal operator: $R_N |\Psi_N^{\pm}\rangle = \pm|\Psi_N^{\pm}\rangle$, and belong to the sector with $S_N^3 = 1 \mod 2$. If $\zeta = 0$ then the Hamiltonian conserves S_N^3. On the subspace considered here, we have $S_N^3 = -N, -N+2, \ldots, N-2, N$. By projection of $|\Psi_N^{\pm}\rangle$ on these subsectors we obtain therefore $N + 1$ zero-energy states for $\zeta = 0$ (not $2(N + 1)$) because of the spin-reversal symmetry.

Next, let us consider states with momentum π at even N. We develop a suitable modification of the states for zero momentum by allowing a *single* site with state $|1\rangle$. Hence, no

pairs $|1\rangle \otimes |1\rangle$ are present. The Hamiltonian density at $\zeta = 0$ has the zero-energy states $|v_\pm\rangle$, and $|v_0\rangle = |01\rangle - |10\rangle$, $|v_2\rangle = |21\rangle - |12\rangle$. Using this fact, it is easy to see that the states

$$\left|\Psi_N^\pm\right\rangle = \sum_{j=1}^N (-1)^j |v_\pm\rangle \otimes \cdots \otimes |v_\pm\rangle \otimes \underbrace{|1\rangle}_{j} \otimes |v_\pm\rangle \otimes \cdots \otimes |v_\pm\rangle$$

are annihilated by the Hamiltonian with $\zeta = 0$. Upon projection we conclude that there is a single momentum state in each subsector with $S_N^3 = -N + 1, -N + 3, \ldots, N - 3, N - 1$, thus in total N zero-energy states for $\zeta = 0$. If however, $\zeta \neq 0$ this construction is not possible as the spectrum of the Hamiltonian density changes. Nonetheless, both cases $\zeta = 0$ and $\zeta \neq 0$ lead to the same Witten index. To see this consider first $\zeta = 0$ and fix $J = N + n_1$ to be odd. We thus have $J + 1$ zero-energy states for $n_1 = 0$, and $J - 1$ zero-energy states for $n_1 = 1$. This gives $W_J = (J + 1) - (J - 1) = 2$. Next, fix J even, then $W_J = 0$ trivially. For $\zeta \neq 0$ there are only two zero-energy states at N odd with $n_1 = 0$. Thus W_J is the same as in the case of $\zeta = 0$.

Despite the simplicity of this example, it has an interesting feature: the zero-energy states (35) are present for any ζ, and thus stable against a certain supersymmetry-preserving perturbation away from the critical point.

Conserved Quantities The supersymmetric $t-J$ model is known to be integrable. In particular, its conserved quantities were systematically determined by Essler and Korepin [34]. It is thus natural to investigate the question if the integrability of the present model survives if the parameter ζ is switched on.

We give an argument in favour of this conjecture using Reshetikhin's criterion [69] for the existence of a higher conserved quantity

$$H_N^{(3)} = \sum_{j=1}^N [h_{j,j+1}, h_{j+1,j+2}].$$

Obviously, it commutes with the translation operator $H_N^{(1)} = T_N$. The criterion states that commutation with the Hamiltonian $H_N^{(2)} = H_N$ is assured if there is an operator X acting on $V \otimes V$ such that

$$\left[h_{j,j+1} + h_{j+1,j+2}, [h_{j,j+1}, h_{j+1,j+2}]\right] = X_{j,j+1} - X_{j+1,j+2}.$$

The right-hand side is a local boundary term in the sense defined above, and thus cancels out in the summation over all sites. In the present case, a non-trivial X can indeed be found:

$$X = -2(1 - \zeta^2)^2 (h + X'),$$

where X' itself commutes with h. In terms of the su(2) generators at spin one we find

$$X' = \frac{\zeta^2}{(1 - \zeta^2)} \left(s^+ \otimes s^- + s^- \otimes s^+ + \left\{s^3 \otimes s^3, s^+ \otimes s^- + s^- \otimes s^+\right\}\right).$$

It is known that the existence of $H_N^{(3)}$ implies that there is another operator $H_N^{(4)} = [B, H_N^{(3)}]$, where $B = \sum_{j=1}^N j h_{j,j+1}$ is the boost operator. It commutes with $H_N^{(2)}$ [48] what follows from the Jacobi identity. The explicit construction of higher conserved charges is a non-trivial problem. Yet the simple result presented here makes plausible their existence. This

problem might find an elegant solution by relating the present Hamiltonian to a transfer matrix constructed from an R-matrix which will most likely be an elliptic extension of the solution to the Yang-Baxter equation presented in [34].

4.3 The mod-3 Chain

In this section, we introduce another set of supercharges for spin one. So far, our discussion focused on trigonometric and elliptic supercharges. The latter conserve the number of particles mod 2. If $\ell + 1 = 0 \mod p$ then it is possible to modify this to a mod p-conservation. Here we concentrate on $\ell = 2$, $p = 3$.

We present a solution of (6) with the requirement $m + 1 = j + k \mod 3$ (and the condition that the components which violate exact equality are non-zero). It is parametrised by two coupling constants λ, ζ, and its action is given by

$$
\begin{aligned}
\mathsf{q}|0\rangle &= \zeta |22\rangle, \\
\mathsf{q}|1\rangle &= \lambda \zeta \big(|02\rangle + |20\rangle\big) - |11\rangle, \\
\mathsf{q}|2\rangle &= \lambda |00\rangle.
\end{aligned}
\tag{36}
$$

Indeed, one may verify that the condition (6) holds if we set $|\chi\rangle = \lambda\zeta(|02\rangle + |20\rangle)$. Writing $\mathsf{q} = \mathsf{q}(\lambda, \zeta)$ we have the property $(R \otimes R)\mathsf{q}(\lambda, \zeta)R = \mathsf{q}(\zeta, \lambda)$ under spin reversal. Hence for periodic boundary conditions the line $\lambda = \zeta$ defines a spin-reversal symmetric Hamiltonian. Moreover, we have $\mathsf{q}^{\mathrm{op}} = \mathsf{q}$ and therefore reflection symmetry of the Hamiltonian. The mod-3 conservation implies that the supersymmetry admits the twist angles $\phi = 0, \pm 2\pi/3$. Here we focus only on periodic boundary conditions.

Asymptotic Limits Let us investigate some special limits in order to understand the structure of the ground states of the associated spin chain for an odd number of sites. As we may exchange λ and ζ by spin-reversal, we focus mainly on ζ. We consider three cases:

– At $\zeta = 0$ we find $\mathsf{q}|1\rangle = -|11\rangle$ and $\mathsf{q}|0\rangle = 0, \mathsf{q}|2\rangle = \lambda|00\rangle$. Hence the supercharge acts trivially on $|1\rangle$, and decouples it from $|0\rangle$, $|2\rangle$. Its action on the subspace spanned by the latter coincides with the one for the XXZ chain at $\Delta = -1/2$. For N odd, there is a trivial ground state $|\Psi_N^0\rangle = |11\cdots 1\rangle$, and the two ground states of the XXZ chain, $|\Psi_N^\pm\rangle$, in the sectors with $S_N^3 = \pm 1$, which can be mapped onto each other through spin reversal. These are the Razumov-Stroganov states [82] which can be normalised in such a way that all components are integers with the smallest one being 1, and the largest one A_n for $N = 2n + 1$ where A_n is the number of $n \times n$ alternating sign matrices.
– If $\zeta \to \infty$ we find, upon appropriate rescaling, that $\mathsf{q}|0\rangle = |22\rangle, \mathsf{q}|1\rangle = \lambda(|02\rangle + |20\rangle)$, and $\mathsf{q}|2\rangle = 0$. Up to a cyclic shift, given by the unitary transformation

$$
\Omega = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},
$$

it coincides with the trigonometric spin-1 supercharge of Sect. 3 *before* imposing the spin-reversal symmetry. If $\lambda = 1/\sqrt{2}$ we recover the twisted Fateev-Zamolodchikov spin chain at $\eta = \pi/4$. In our previous discussion we argued that for N odd there are *three* zero-energy states, one in each of the sectors $S_N^3 = 0, \pm 1$. Hence, we see that in some sense the present model provides a supersymmetry-preserving interpolation between the first two members of the trigonometric models studied in Sect. 3.

– If $\lambda = \zeta = 1$ then the model has a full cyclic symmetry: the Hamiltonian is invariant with respect to the cyclic shift $(\Omega \otimes \Omega)h(\Omega^{-1} \otimes \Omega^{-1}) = h$. In fact, the Hamiltonian density becomes $h = 2 - \mathsf{p}$ with the operator p acting according to

$$\mathsf{p}|j, k\rangle = |j + 1, k - 1\rangle + |j - 1, k + 1\rangle,$$

where we identify values of j, k mod 3. The Hamiltonian is trivial as it can be diagonalised by a simple Fourier transformation. We introduce the states

$$|\hat{p}\rangle = \sum_{j=0}^{2} U_{pj}|j\rangle, \qquad U_{pj} = \frac{e^{\mathrm{i}\pi/6}}{\sqrt{3}}e^{-2\pi\mathrm{i}pj/3}, \quad p = 0, 1, 2,$$

and build the Hilbert space for a chain with N sites from tensor products $|\hat{p}_1, \dots, \hat{p}_N\rangle$. Then, the Hamiltonian $H_N = \sum_{j=1}^{N} h_{j,j+1}$ is diagonal:

$$H_N|\hat{p}_1, \dots, \hat{p}_N\rangle = 4\sum_{j=1}^{N} \sin^2\left(\frac{\pi(p_j - p_{j+1})}{3}\right)|\hat{p}_1, \dots, \hat{p}_N\rangle.$$

The zero-energy states are thus given by the states with $p_j = p = 0, 1, 2$ for all $j = 1, \dots, N$. For N odd, they are precisely the supersymmetry singlets (as the latter exists only in the momentum space with $t_N = (-1)^{N+1}$). Thus we find again *three* supersymmetry singlets for N odd, and none for N even.

Zero-Energy States   The last case in the preceding list is interesting because the general case can be related to it by conjugation, following the strategy outlined in Sect. 2.4. We restrict our considerations to $\zeta, \mu > 0$ (other cases can be related to this one through simple unitary transformations). We introduce the diagonal matrix $m = \mathrm{diag}((\lambda^2\zeta)^{-1/3}, 1, (\lambda\zeta^2)^{-1/3})$, and find

$$(m \otimes m)\mathsf{q}(\lambda, \zeta)m^{-1} = \mathsf{q}(1, 1).$$

Thus, using $M_N = m^{\otimes N}$, one has $M_{N+1}Q_N(\lambda, \zeta)M_N^{-1} = Q_N(\lambda = 1, \zeta = 1)$. As we know the ground state structure in the case $\lambda = 1, \zeta = 1$ in the supersymmetry-relevant momentum sectors, we conclude that for odd N the Hamiltonian H_N generated from the supercharge (36) has three zero-energy states which are invariant under translation. They can be chosen according to the values of $S_N^3 = 0, 1, 2$ mod 3.

It would certainly be interesting to understand if the components of these vectors relate in any respect to enumeration problems. This is the case if $\zeta = 0$ (or $\lambda = 0$) because the XXZ ground state at $\Delta = -1/2$ relates to the alternating sign matrix numbers. For general values of λ, ζ we normalise the singlets so that their components are coprime polynomials in the parameter. The inspection of small systems suggests that a combinatorial interpretation might not be straightforward, since not all the coefficients of these polynomials are positive (what is already seen for $N = 5$ sites). Nevertheless, there seems to be an interesting underlying structure. The following conjecture, again based on exact diagonalisation up to $N = 7$ sites, remind us of the sum rules for the Fateev-Zamolodchikov singlet:

Conjecture 10 (*Sum rules*) *With the given normalisation the square norm of the singlet state for odd N factorises into two polynomials in ζ, λ with positive integer coefficients.*

Along the line $\lambda = \zeta$ we find more explicitly

$$\left\| \Psi_N^a(\zeta) \right\|^2 = 3^{-2n} \left(\sum_\mu \zeta^{v_1(\mu)} \psi_\mu^a \right) \times \left(\sum_\mu \zeta^{-v_1(\mu)} \psi_\mu^a \right), \quad a = 0, \pm,$$

where $v_1(\mu)$ is the number of 1's in the spin configuration μ. For the state Ψ_N^0 we find along the line $\lambda = \zeta$ the sum rule:

$$\sum_\mu \zeta^{v_1(\mu)} \psi_\mu^0 = 3^{2n} \zeta^{2n+1} \psi_{11\ldots1}^0.$$

5 Conclusion

In this article we presented a general framework for dynamical supersymmetry of spin chains, and worked out the supercharges for some particular models such as the trigonometric models, and the Fateev-Zamolodchikov spin chain, a spin chain related to the supersymmetry $t-J$ model, and a model with mod-3 conservation of the particle number. Moreover, we made a number of conjectures on the existence of zero-energy states and their properties. The most remarkable observations are certainly the sum rules for the singlets of the Fateev-Zamolodchikov chain, and a relation of its components to the weighted enumeration of alternating sign matrices.

The present work leads to many open questions. We have seen that the solution of (6) was instrumental in order to find spin chains with supersymmetry. Already at spin one, we observed the existence of several *inequivalent* solutions to the condition on the supercharges to be nilpotent. It is natural to ask if there is a way to classify all possible solutions for higher spin. Some of these solutions yield Hamiltonian densities for integrable/Bethe-ansatz-solvable spin chains. We observe this phenomenon in cases where additional constraints such as spin-reversal symmetry are imposed. It seems natural to ask if it is possible to set up a criterion for a supercharge to generate an integrable spin chain. Certainly, the analysis of Reshetikhin's criterion for the two-site Hamiltonians would be an interesting starting point, yet more refined tools to prove integrability should be developed.

For spin chains with lattice supersymmetry the possible existence of exact zero-energy states is certainly the most interesting phenomenon. Even though the conjugation argument provides a tool to prove their presence in some cases, it does not cover all of them. For example, the XXZ spin chain with twisted boundary conditions cannot be analysed in this way as we do not know an off-critical extension which preserves the supersymmetry and the twist. Therefore, alternative existence proofs could put conjectures for twisted spin chains on solid grounds. The existence of supersymmetry singlets is intimately related to the cohomology $\ker Q_N / \operatorname{im} Q_{N-1}$ [92]. The number of zero-energy states coincides with its dimension. Looking at the list of known cohomology theories, the present study shows striking similarities with the structures used in the theory of Hochschild (co)homology and cyclic (co)homology [72]. It would be interesting to establish a consistent connection with these theories, in particular in the presence of twists. This might provide an algebraic tool to prove the existence of the zero-energy states.

A symmetry is typically useful in order to determine quantities of physical interests such as ground-state correlation functions. Hence, it would be interesting to understand to what extend the supersymmetry presented here may simplify to the explicit evaluation of ground-state correlators. An example was provided in [37] where so-called scale-free expansions

around trivially solvable points were pointed out. These allow to guess closed formulas for lattice correlation functions from finite-size data (such as the spin two-point correlations of the XYZ chain along the supersymmetric line at various distances [36]). For a direct application of the supersymmetry, a natural place to start would be infinitely long spin chains where the distinction between N and $N + 1$ sites is not an issue. Besides, the analysis of an infinite system $N \to \infty$ might prove to be interesting by itself: for example is it possible and/or meaningful to find a central extension of the lattice version of the $\mathcal{N} = (2, 2)$ supersymmetry algebra?

Finally, there is an obvious generalisation of the type of supersymmetry discussed in this article. We presented a theory based on a local operation $\mathsf{q} : V \to V \otimes V$. It is natural to generalise it to $\mathsf{q} : V^{\otimes n} \to V^{\otimes m}$ with arbitrary $n \leq m$. In this case, the supercharges add $m - n$ sites to the system. The corresponding Hamiltonians will lead to interactions which involve in general more than just two neighbouring spins.

Let us now turn to the specific models presented in this paper. We studied the trigonometric models, which were inspired from supersymmetric fermion models with exclusion rules. The sectors where the supersymmetry holds qualify as lattice-pendants of the Ramond sectors in conformal field theory. Quite natural is the question: can we identify features of the Neveu-Schwarz sectors on the lattice? A glance at the spectrum of the spin-1/2 and spin-1 chains with periodic boundary conditions for small system sizes shows that they possess a *single negative eigenvalue* in the translationally-invariant sector if the number of sites is *even*. Hence this sector is a good candidate, and clearly deserves further investigation. Moreover, in this context the study of spectral flow (which will necessarily break the supersymmetry on the lattice), which may serve as a connection between the Ramond and Neveu-Schwarz sectors, appears natural [71] (see [58] for an application of spectral flow to the supersymmetric fermion models). A further challenge is to construct in a systematic way the elliptic extensions of the trigonometric supercharges presented in this work, so that the lattice version of the $\mathcal{N} = (2, 2)$ supersymmetry algebra still holds. This was done in Sect. 4 for the case of spin one. It seems natural to expect that these generate spin chains related to the spin-$\ell/2$ versions of the eight-vertex model at the special values $\eta = \pi/(\ell + 2)$ of the crossing parameter, obtained through the fusion procedure. Indeed, the existence of a lattice supersymmetry at these points can be proved from the fusion equations, but it remains to work out the explicit supercharges. There seem to be at least two possible routes to achieve this: either one uses the wave functions obtained from the algebraic Bethe ansatz, or one tries to solve the condition (6) for a local supercharge q which changes the magnetisation by $-(\ell + 2)/2 \mod 2$. The off-critical models are intimately related to the corresponding deformation in the $\mathcal{M}_\ell$ models. It was shown in [37] that for the spin-1/2 the off-criticality on the spin chain side corresponds to stagger the fermion models with period 3 on systems of length $L = 3n$. It is natural to extend this for higher spin to a staggering with period $\ell + 2$ on $L = (\ell + 2)n$ sites.

For the twisted Fateev-Zamolodchikov spin chain, the origin of the combinatorial features should be elucidated. A first step will be a proof of the sum rules, which seem to be a rather common features of both spin chains and fermion models with lattice supersymmetry [13]. Moreover, our findings hint at a simple eigenvalue for the transfer matrix of the 19-vertex model and its elliptic generalisation (a 41-vertex model [35, 68]) for any value of the crossing parameter and the elliptic nome. The exact diagonalisation of the transfer matrix for small system sizes confirms this conjecture. This allows to apply the machinery of the quantum Knizhnik-Zamolodchikov equation without any restriction to a special point in parameter space. This problem will be addressed in a future publication [50]. Eventually, as already mentioned in the introduction and further supported by the examples given in

this work, there appears to be a relation between combinatorics and supersymmetry. This is not the first time that this phenomenon is observed as shows the example of the Veneziano-Wosiek model [79]. It seems that this is not only a superficial coincidence and remains to be understood.

Appendix A: Trigonometric Supercharges

In this appendix we explain a constructive way to arrive at (18). Our strategy is the following. We specialise first the matrix elements $h_{(r,s),(m,n)}$ to the diagonal case $(r, s) = (m, n)$, and construct a particular solution which is invariant under spin reversal. Second, in the main text, we show that this solution satisfies indeed the spin-reversal symmetry even for off-diagonal matrix elements. We start thus by considering the matrix element $b_{m,n} = h_{(m,n),(m,n)}$, given by

$$b_{m,n} = a_{m+n+1,m}^2 + \frac{1}{2}\left(\sum_{k=0}^{m-1} a_{m,k}^2 + \sum_{k=0}^{n-1} a_{n,k}^2 \right).$$

First, set $m = n = 0$. We impose $b_{00} = b_{\ell\ell}$, and thus have

$$a_{1,0}^2 = \sum_{k=0}^{\ell-1} a_{\ell,k}^2.$$

Next, consider the equation $b_{0,\ell-n} = b_{\ell,n}$. Using the previous result we find that

$$a_{\ell-n+1,0}^2 + \frac{1}{2}\sum_{k=0}^{\ell-n-1} a_{\ell-n,k}^2 = \frac{1}{2}a_{1,0}^2 + \frac{1}{2}\sum_{k=0}^{n-1} a_{n,k}^2.$$

Replacing now $n \to \ell - n$ and subtracting both equations, we find a useful symmetry property:

$$a_{m,0}^2 + a_{\ell+2-m,0}^2 = a_{1,0}^2, \quad m = 2, \dots \ell. \tag{37}$$

Now impose $b_{m,1} = b_{\ell-m,\ell-1}$. For $m = 0, \dots, \ell - 2$ it follows that

$$a_{m+2,1}^2 = \frac{1}{2}\left(\sum_{k=0}^{\ell-m-1} a_{\ell-m,k}^2 + \sum_{k=0}^{\ell-2} a_{\ell-1,k}^2 - \sum_{k=0}^{m-1} a_{m,k}^2 \right) - \frac{1}{2}a_{1,0}^2.$$

In particular, the case $m = 0$ leads to $a_{2,1}^2 = a_{2,0}^2 = \sum_{k=0}^{\ell-2} a_{\ell-1,k}^2/2$, and thus to

$$a_{m+2,1}^2 = \frac{1}{2}\left(\sum_{k=0}^{\ell-m-1} a_{\ell-m,k}^2 - \sum_{k=0}^{m-1} a_{m,k}^2 \right) + a_{2,0}^2 - \frac{1}{2}a_{1,0}^2.$$

Again, we notice that the expression in brackets is antisymmetric under $m \to \ell - m$. This allows to derive the equation

$$a_{m+2,1}^2 + a_{\ell+2-m,1}^2 = 2a_{2,0}^2 - a_{1,0}^2.$$

The left-hand side can be reduced with the help of the recursion relation (16), and thus everything can be rewritten in terms of $a_{m,0}^2$. After a little algebra, we find the non-linear recursion relation

$$a_{m+2,0}^2 a_{m+1,0}^2 + \left(a_{1,0}^2 - a_{m+1,0}^2\right)\left(a_{1,0}^2 - a_{m,0}^2\right) = a_{1,0}^2\left(2a_{2,0}^2 - a_{1,0}^2\right).$$

We will now show that it admits a particular solution related to a linear recursion relation. To this end, set $a_{m,0}^2 = z_m/z_{m-1}$ and normalise $z_0 = 1$. We find that

$$z_{m-1}\left(z_{m+2} - z_1 z_{m+1} + z_m\right) + z_m\left(z_{m+1} - z_1 z_m + \left(2\left(z_1^2 - z_2\right) - 1\right)z_{m-1}\right) = 0.$$

This relation holds certainly if we choose for z_m as solution of the equation

$$z_{m+2} - z_1 z_{m+1} + z_m = 0, \quad \text{with } z_0 = 1, \ z_0 = 0.$$

This is the recursion relation for the Chebyshev polynomials of the second kind. We prefer to use q-integers and thus write

$$z_m = [m+1], \qquad [n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad \text{with } q = e^{i\eta}.$$

The phase η is not arbitrary as we have to respect the symmetry relation (37). After some algebra, one finds that this relation implies $\sin(\ell+2)\eta = 0$. The unique choice for η to keep all the $a_{m,0}^2$ positive is $\eta = \pi/(\ell+2)$. Hence, q takes the root-of-unity value

$$q = e^{i\pi/(\ell+2)}.$$

Thus, we have $[m+\ell+2] = -[m] = [-m]$. Hence, we find

$$a_{m,0}^2 = \frac{[m+1]}{[m]}.$$

The general coefficients $a_{m,k}$ are obtained from the recursion relation (16). This completes the derivation of the constants (18).

Appendix B: Special Polynomials

Bazhanov and Mangazeev introduced in [11, 75] a series of polynomials $s_n(z)$, $n \in \mathbb{Z}$, which appear as components in the ground-state vectors of the spin-1/2 XYZ Hamiltonian (13a), (13b), and sum rules (see also the related work of Razumov and Stroganov [84]). A sum rule for the square norm was recently proved by Zinn-Justin [98].

These polynomials appear to be not only off-critical generalisations of the combinatorial numbers found in the ground states of various variants of the spin-1/2 XXZ chain at $\Delta = -1/2$, but display also an interesting relation to classical integrability, namely to certain

tau-function hierarchies associated to the Painlevé VI equation. They are solutions to the non-linear difference-differential equation

$$2z(z-1)(9z-1)^2\big(\ln s_n(z)\big)'' + 2(3z-1)^2(9z-1)\big(\ln s_n(z)\big)'$$

$$+ 8(2n+1)^2\frac{s_{n+1}(z)s_{n-1}(z)}{s_n(z)^2} - \big(4(3n+1)(3n+2)+(9z-1)n(5n+3)\big) = 0$$

with $s_0(z) = s_1(z) \equiv 1$. This equation can be understood as a result of special Bäcklund transformations for Painlevé VI [74, 78]. It is far from being evident that its solutions are polynomials for the given initial conditions. Various properties of the solutions to this equation were conjectured in [75]. For our purposes, the factorisation properties are relevant. In fact, the authors claim that for any $k \in \mathbb{Z}$ one has

$$s_{2k+1}\big(w^2\big) = s_{2k+1}(0)\,p_k(w)\,p_k(-w), \quad p_k(0) = 1, \tag{38}$$

where the $p_k(w)$ are polynomials of degree $\deg p_k(w) = k(k+1)$ with positive integer coefficients. We have the transformation symmetry

$$p_k(w) = \left(\frac{1+3w}{2}\right)^{k(k+1)} p_k\left(\frac{1-w}{1+3w}\right).$$

The $p_k(w)$'s are the polynomials appearing in the zero-energy states of the twisted elliptic Fateev-Zamolodchikov chain at $x = \pm 1/2$, $y = \pm\zeta/2$.

Appendix C: Counting Momentum States

In this appendix, we use the classical theory of necklace enumeration in order to count certain momentum states for a spin-$\ell/2$ chain with N sites [51].

We consider thus a chain/necklace with N beads or equivalently a periodic lattice with N sites. Each site of this graph, labelled by some integer $j = 1, \ldots, N$, can be coloured with a colour $m_j = 0, 1, \ldots, \ell$. Thus, a colouring/configuration is given by an N-tuple $\mu = (m_1, \ldots, m_{N-1}, m_N)$. We denote by X the set of all possible configurations. There is a natural action of the cyclic group, generated by the cyclic shift T, on X through $T \cdot (m_1, \ldots, m_{N-1}, m_N) = (m_N, m_1, \ldots, m_{N-1})$. We will study the fixed points of this group action. This is motivated by the fact that the elements of X are labels of the Hilbert space of the spin chain.

Momentum States To each $\mu = (m_1, \ldots, m_{N-1}, m_N) \in X$ we associate a basis vector $|\mu\rangle = |m_1, \ldots, m_{N-1}, m_N\rangle$ in $V^{\otimes N}$ where $V \simeq \mathbb{C}^{\ell+1}$ as in Sect. 2. The translation operator on $V^{\otimes N}$ acts according to $T|\mu\rangle = |T \cdot \mu\rangle$ (we use the same notation for it acting on the vector space and the set X). Out of any basis state $|\mu\rangle$ we build a momentum state

$$|\psi_\mu\rangle = \sum_{j=0}^{N-1} e^{-ijk} T^j |\mu\rangle, \qquad T|\psi_\mu\rangle = e^{ik}|\psi_\mu\rangle,$$

where $k = 2\pi m/N$, $m = 0, 1, \ldots, N-1$ is the momentum. $|\psi_\mu\rangle$ may be zero what implies that the configuration μ is incompatible with the given momentum k. In fact, for each configuration $\mu \in X$ there is a *minimal* positive integer r such that $T^r \cdot \mu = \mu$. We call r the

symmetry factor of μ. For the momentum state to be non-zero we need

$$r \in p\mathbb{N} \quad \text{with } p = \frac{N}{\gcd(N, m)}.$$

Motivated by the main text, our aim is to count all momentum states with $k = 0$ for N odd, and $k = \pi$ for N even, with given total particle number

$$M = \sum_{j=1}^{N} m_j.$$

Chains of Odd Length We start with N odd. A momentum state may be generated by two different configurations μ, μ' if they can be mapped onto each other through a suitable translation. This induces an equivalence relation on the set of configurations, and thus our task consists of counting the number ν_{MN} of equivalence classes (or equivalently the number of orbits in the set of configurations generated by the action of T).

We denote by $X_M^m \subset X$ the set of all configurations with given M which are stable under the action of T^m. According to Burnside's lemma, we have

$$\nu_{MN} = \frac{1}{N} \sum_{m=0}^{N-1} |X_M^m|. \tag{39}$$

Therefore, we need to evaluate $|X_M^m|$. The case $m = 0$ is the simplest: we just need to count all possible configurations for given M. Let us thus consider $m = 1$: in this case all sites need to have equal colour because a single translation (cyclic shift) has only one cycle. Hence, X_M^1 contains a single configuration of colour j if and only if $M = jN$ for some integer $j = 0, 1, \ldots, \ell$, in other words if $N|M$ as long as $M \leq \ell N$. We abbreviate $a_N(M) = |X_M^1|$. For $m > 1$ there various distinct cycles. It is a classical result that their number is given by $\gcd(N, m)$, and therefore they have length $r_m = N/\gcd(N, m)$. For each cycle, we have however the same problem as for $m = 1$. Hence, we find the general result

$$|X_M^m| = \sum_{M_1, \ldots, M_{\gcd(N,m)}=0}^{\infty} \left(\prod_{j=1}^{\gcd(N,m)} a_{r_m}(M_j) \right) \delta_{M, \sum_{j=1}^{\gcd(N,m)} M_j}. \tag{40}$$

We notice a typical convolution structure. It is therefore useful to introduce the generating function $\mathcal{A}_N(z)$ for the numbers $a_N(M)$ where z is the conjugate variable of M. We obtain

$$\mathcal{A}_N(z) = \sum_{M=0}^{\infty} a_N(M) z^M = \frac{1 - z^{(\ell+1)N}}{1 - z^N}.$$

We use this in (40) and find the generating series

$$\sum_{M=0}^{\infty} |X_M^m| z^M = \mathcal{A}_{r_m}(z)^{\gcd(N,m)} = \left(\frac{1 - z^{(\ell+1)N/\gcd(N,m)}}{1 - z^{N/\gcd(N,m)}} \right)^{\gcd(N,m)}.$$

Performing the sum with respect to m leads to the generating function $f_N(z)$ given in the main text for odd N.

Chains of Even Length　　Next, we treat the case of even N and momentum $k = \pi$. Thus, we want to count the number of all representatives with even symmetry factor r. It seems natural to modify (39) in such a way that only fixed points of T^m with even m are retained. Thus, we attempt to discard all contributions with odd m and write $1/N \sum_{m=0}^{N/2-1} |X_M^{2m}|$. Yet, in this way we count some unwanted configurations. Indeed for $m > 0$, if we decompose $2m = 2^{k_m} b_m$ with b_m an odd integer, and $k_m \geq 1$ some integer, then the fixed points of T^{2m} contain the fixed points of T^{b_m}. These need to be subtracted. For $m = 0$ we need to subtract the fixed points of $T^{b_{N/2}}$. Thus we arrive at

$$\nu_{MN} = \frac{1}{N} \left(\sum_{m=0}^{N/2-1} |X_M^{2m}| - \sum_{m=1}^{N/2} |X_M^{b_m}| \right) \quad \text{for even } N.$$

The sum can be rearranged in a convenient way. To this end observe in (40) that $|X_M^m|$ depends on m only through $\gcd(N, m)$. In order to rearrange the sum we need the following lemma:

Lemma 1 *Let N be even and consider $c_m = \gcd(N, b_m)$ for $m = 1, 2, \ldots, N/2$. We have $c_{m+N/2} = c_m$. Furthermore, let $p = b_{N/2}$ then it follows that $c_{m+(p-1)/2} = \gcd(N, 2m - 1)$.*

Proof The proof is entirely based on two simple identities. Let m and n be positive integers. (i) if $m > n$ then we may write $\gcd(m, n) = \gcd(m - n, n)$; (ii) if n is odd then $\gcd(m, n) = \gcd(2m, n)$. From the second identity it follows in particular that $\gcd(b_m, n) = \gcd(m, n)$ for odd n. Furthermore we find $\gcd(b_m, n) = \gcd(m, b_n) = \gcd(b_m, b_n)$.

For the first part, use $b_N = b_{N/2}$ and write

$$c_{m+N/2} = \gcd(N, b_{m+N/2}) = \gcd(b_{N/2}, m + N/2) = \gcd(b_{N/2}, m) = \gcd(b_N, m)$$
$$= \gcd(N, b_m) = c_m.$$

For the second part, use $b_N = b_{N/2} = p$ and write

$$c_{m+(p-1)/2} = \gcd(b_{m+\frac{p-1}{2}}, N) = \gcd\left(m + \frac{p-1}{2}, b_N\right) = \gcd(2m + p - 1, p)$$
$$= \gcd(2m - 1, p) = \gcd(2m - 1, b_N) = \gcd(2m - 1, N).$$

This proves the claim.　　　　　　　　　　　　　　　　　　　　　　　　　　　$\square$

Combining the implicit dependence of $|X_M^{b_m}|$ on $c_m = \gcd(N, b_m)$ with the periodicity and shift properties given by this lemma, we obtain

$$\sum_{m=1}^{N/2} |X_M^{b_m}| = \sum_{m=1}^{N/2} |X_M^{2m-1}|.$$

Putting everything together we thus find

$$\nu_{MN} = \frac{1}{N} \left(\sum_{m=0}^{N/2-1} |X_M^{2m}| - \sum_{m=1}^{N/2} |X_M^{b_m}| \right) = \frac{1}{N} \sum_{m=0}^{N-1} (-1)^m |X_M^m| \quad \text{for even } N.$$

Hence, we proved the extra factor $(-1)^m$ for even N. Computing the generating function we obtain the result given in the main text.

References

1. Abramowitz, M., Stegun, I.: Handbook of Mathematical Functions. Dover, New York (1970)
2. Ahn, C., Bernard, D., Leclair, A.: Fractional supersymmetries in perturbed coset CFTs and integrable soliton theory. Nucl. Phys. B **346**, 409–439 (1990)
3. Alcaraz, F.C., Martins, M.J.: The spin-s XXZ quantum chain with general toroidal boundary conditions. J. Phys. A, Math. Gen. **23**, 1439 (1990)
4. Baake, M., von Gehlen, G., Rittenberg, V.: Operator content of the Ashkin-Teller quantum chain, superconformal and Zamolodchikov-Fateev invariance. I. Free boundary conditions. J. Phys. A, Math. Gen. **20**, L479 (1987)
5. Baake, M., von Gehlen, G., Rittenberg, V.: Operator content of the Ashkin-Teller quantum chain, superconformal and Zamolodchikov-Fateev invariance. II. Boundary conditions compatible with the torus. J. Phys. A, Math. Gen. **20**, L487 (1987)
6. Babujian, H.M.: Exact solution of the one-dimensional isotropic Heisenberg chain with arbitrary spins S. Phys. Lett. A **90**, 479–482 (1982)
7. Babujian, H.M.: Exact solution of the isotropic Heisenberg chain with arbitrary spins: thermodynamics of the model. Nucl. Phys. B **215**, 317–336 (1983)
8. Baranowski, D., Rittenberg, V.: The operator content of the ferromagnetic and antiferromagnetic spin-1 Zamolodchikov-Fateev quantum chain. J. Phys. A, Math. Gen. **23**, 1029 (1990)
9. Batista, C.D., Ortiz, G.: Generalized Jordan-Wigner transformations. Phys. Rev. Lett. **86**, 1082–1085 (2001)
10. Bazhanov, V.V., Mangazeev, V.V.: Eight-vertex model and non-stationary Lamé equation. J. Phys. A, Math. Gen. **38**, L145–L153 (2005)
11. Bazhanov, V.V., Mangazeev, V.V.: The eight-vertex model and Painlevé VI. J. Phys. A, Math. Gen. **39**, 12235–12243 (2006)
12. Beccaria, M., de Angelis G.F.: Exact ground state and finite-size scaling in a supersymmetric lattice model. Phys. Rev. Lett. **94**, 100401 (2005)
13. Beccaria, M., Hagendorf, C.: A staggered fermion chain with supersymmetry on open intervals. arXiv:1206.4194 (2012)
14. Beisert, N.: The su(2|3) dynamic spin chain. Nucl. Phys. B **682**, 487–520 (2004)
15. Beisert, N.: The dilatation operator of $N = 4$ super Yang Mills theory and integrability. Phys. Rep. **405**, 1–202 (2004)
16. Beisert, N.: The su(2|3) undynamic spin chain. Prog. Theor. Phys. Suppl. **177**, 1–11 (2009)
17. Beisert, N., Zwiebel, B.I.: On symmetry enhancement in the psu(1, 1|2) sector of $\mathcal{N} = 4$ SYM. J. High Energy Phys. **10**, 31 (2007)
18. Bérkovich, A., Gómez, C., Sierra, G.: Spin-anisotropy commensurable chains. Quantum group symmetries and $N = 2$ SUSY. Nucl. Phys. B **415**, 681–733 (1994)
19. Bernard, D., Felder, G.: Quantum group symmetries in two-dimensional lattice quantum field theory. Nucl. Phys. B **365**, 989–1120 (1991)
20. Bernard, D., LeClair, A.: Quantum group symmetries and non-local currents in 2D QFT. Commun. Math. Phys. **142**, 99–138 (1990)
21. Bernard, D., LeClair, A.: The fractional supersymmetric sine-Gordon models. Phys. Lett. B **247**, 309–316 (1990)
22. Bytsko, A.G.: On integrable Hamiltonians for higher spin XXZ chain. J. Math. Phys. **44**, 3698–3717 (2003)
23. Cantini, L., Sportiello, A.: Proof of the Razumov-Stroganov conjecture. J. Comb. Theory **118**, 1549–1574 (2011)
24. Cantini, L., Sportiello, A.: A one-parameter refinement of the Razumov-Stroganov correspondence. arXiv:1202.5253 (2012)
25. Catterall, S., Kaplan, D.B., Ünsal, M.: Exact lattice supersymmetry. Phys. Rep. **484**, 71–130 (2009)
26. Crampé, N., Ragoucy, E., Alonzi, L.: Coordinate Bethe ansatz for spin-s XXX model. SIGMA **7**, 6 (2011)
27. Date, E., Jimbo, M., Kuniba, A., Miwa, T., Okado, M.: Exactly solvable SOS models: local height probabilities and theta function identities. Nucl. Phys. B **290**, 231–273 (1987)
28. de Gier, J., Batchelor, M.T., Nienhuis, B., Mitra, S.: The XXZ spin chain at $\Delta = -1/2$: Bethe roots, symmetric functions, and determinants. J. Math. Phys. **43**, 4135–4146 (2002)
29. Di Francesco, P., Zinn-Justin, P.: Around the Razumov-Stroganov conjecture: proof of a multi-parameter sum rule. Electron. J. Comb. **12**, R6 (2005)
30. Di Francesco, P., Zinn-Justin, P.: Quantum Knizhnik-Zamolodchikov equation, totally symmetric self-complementary plane partitions and alternating sign matrices. Theor. Math. Phys. **154**, 331–348 (2008)

31. Di Francesco, P., Zinn-Justin, P., Zuber, J.: Sum rules for the ground states of the O(1) loop model on a cylinder and the XXZ spin chain. J. Stat. Mech. **8**, 11 (2006)
32. Dixon, L., Ginsparg, P., Harvey, J.: $\hat{c} = 1$ superconformal field theory. Nucl. Phys. B **306**, 470–496 (1989)
33. Dondi, P.H., Nicolai, H.: Lattice supersymmetry. Nuovo Cim., A **41**, 1–13 (1977)
34. Essler, F.H.L., Korepin, V.E.: Higher conservation laws and algebraic Bethe ansätze for the supersymmetric t–J model. Phys. Rev. B **46**, 9147–9162 (1992)
35. Fateev, V.A.: A factorized S-matrix for particles of opposite parities and an integrable 21-vertex statistical model. Sov. J. Nucl. Phys. **33**, 761–766 (1981)
36. Fendley, P.: Private communication (2012)
37. Fendley, P., Hagendorf, C.: Exact and simple results for the XYZ and strongly interacting fermion chains. J. Phys. A, Math. Theor. **43**, 402004 (2010)
38. Fendley, P., Hagendorf, C.: Ground-state properties of a supersymmetric fermion chain. J. Stat. Mech. **1102**, P02014 (2011)
39. Fendley, P., Schoutens, K.: Exact results for strongly correlated fermions in $2 + 1$ dimensions. Phys. Rev. Lett. **95**, 046403 (2005)
40. Fendley, P., Nienhuis, B., Schoutens, K.: Lattice fermion models with supersymmetry. J. Phys. A, Math. Gen. **36**, 12399–12424 (2003)
41. Fendley, P., Schoutens, K., de Boer, J.: Lattice models with $N = 2$ supersymmetry. Phys. Rev. Lett. **90**, 120402 (2003)
42. Fendley, P., Schoutens, K., van Eerten, H.: Hard squares with negative activity. J. Phys. A, Math. Gen. **38**, 315–322 (2005)
43. Frahm, H., Martins, M.J.: Finite size properties of staggered $U_q[sl(2|1)]$ superspin chains. Nucl. Phys. B **847**, 220–246 (2011)
44. Francesco, P.D., Saleur, H., Zuber, J.B.: Generalized Coulomb-gas formalism for two-dimensional critical models based on SU(2) coset construction. Nucl. Phys. B **300**, 393–432 (1988)
45. Fredenhagen, S., Restuccia, C., Sun, R.: The limit of $N = (2, 2)$ superconformal minimal models. arXiv:1204.0446 (2012)
46. Friedan, D., Qiu, Z., Shenker, S.: Superconformal invariance in two dimensions and the tricritical Ising model. Phys. Lett. B **151**, 37–43 (1985)
47. Gaberdiel, M.R., Suchanek, P.: Limits of minimal models and continuous orbifolds. J. High Energy Phys. **2012**, 1–30 (2012)
48. Grabowski, M.P., Mathieu, P.: Integrability test for spin chains. J. Phys. A, Math. Gen. **28**, 4777 (1995)
49. Hagendorf, C., Fendley, P.: The eight-vertex model and lattice supersymmetry. J. Stat. Phys. **146**, 1122–1155 (2012)
50. Hagendorf, C.: (2013, to be published)
51. Harary, F., Palmer, E.M.: Graphical Enumeration. Academic Press, San Diego (1973)
52. Hardy, G.H., Wright, E.M.: An Introduction to the Theory of Numbers. Oxford University Press, London (2008)
53. Hori, K., Katz, S., Klemm, A., Pandharipande, R., Thomas, R., Vafa, C., Vakil, R., Zaslow, E.: Mirror Symmetry. Am. Math. Soc., Providence (2003)
54. Huijse, L.: A supersymmetric model for lattice fermions. PhD thesis, Universiteit van Amsterdam (2010)
55. Huijse, L.: Detailed analysis of the continuum limit of a supersymmetric lattice model in 1D. J. Stat. Mech. **2011**(04), P04004 (2011)
56. Huijse, L., Schoutens, K.: Superfrustration of charge degrees of freedom. Eur. Phys. J. B **64**, 543–550 (2008)
57. Huijse, L., Schoutens, K.: Supersymmetry, lattice fermions, independence complexes and cohomology theory. Adv. Theor. Math. Phys. **14**, 643–694 (2010)
58. Huijse, L., Halverson, J., Fendley, P., Schoutens, K.: Charge frustration and quantum criticality for strongly correlated fermions. Phys. Rev. Lett. **101**, 146406 (2008)
59. Huijse, L., Moran, N., Vala, J., Schoutens, K.: Exact ground states of a staggered supersymmetric model for lattice fermions. Phys. Rev. B **84**, 115124 (2011)
60. Huijse, L., Mehta, D., Moran, N., Schoutens, K., Vala, J.: Supersymmetric lattice fermions on the triangular lattice: superfrustration and criticality. New J. Phys. **14**, 073002 (2012)
61. Johannesson, H.: Central charge for the integrable higher-spin XXZ model. J. Phys. A, Math. Gen. **21**, L611 (1988)
62. Jonsson, J.: Hard squares with negative activity and rhombus tilings of the plane. Electron. J. Comb. **13**, R67 (2006)
63. Jonsson, J.: Certain homology cycles of the independence complex of grids. Discrete Comput. Geom. **43**, 927–950 (2010)

64. Kasatani, M., Pasquier, V.: On polynomials interpolating between the stationary state of a $O(n)$ model and a Q.H.E. ground state. Commun. Math. Phys. **276**, 397 (2007)
65. Kassel, C.: Quantum Groups. Springer, Berlin (1994)
66. Kirillov, A.N., Reshetikhin, N.Y.: Exact solution of the integrable XXZ Heisenberg model with arbitrary spin. I. The ground state and the excitation spectrum. J. Phys. A, Math. Gen. **20**, 1565–1585 (1987)
67. Klümper, A., Schadschneider, A., Zittartz, J.: Equivalence and solution of anisotropic spin-1 models and generalized $t-J$ fermion models in one dimension. J. Phys. A, Math. Gen. **24**, L955 (1991)
68. Konno, H.: Fusion of Baxter's elliptic R-matrix and the vertex-face correspondence. Ann. Henri Poincaré **7**, 1555–1567 (2006)
69. Kulish, P., Reshetikhin, N., Sklyanin, E.: Yang-Baxter equation and representation theory: I. Lett. Math. Phys. **5**, 393–403 (1981)
70. Kuperberg, G.: Symmetry classes of alternating-sign matrices under one roof. Ann. Math. **156**, 835–866 (2002)
71. Lerche, W., Vafa, C., Warner, N.P.: Chiral rings in $N = 2$ superconformal field theories. Nucl. Phys. B **324**, 427–474 (1989)
72. Loday, J.L.: Cyclic Cohomology. Springer, Berlin/Heidelberg (1992)
73. Maassarani, Z., Nemeschansky, D., Warner, N.P.: Lattice analogues of $N = 2$ superconformal models via quantum-group truncation. Nucl. Phys. B **393**, 523–546 (1993)
74. Mangazeev, V.V.: Picard solution of Painlevé VI and related tau-functions. arXiv:1002.2327 (2010)
75. Mangazeev, V.V., Bazhanov, V.V.: The eight-vertex model and Painlevé VI equation II: Eigenvector results. J. Phys. A, Math. Theor. **43**, 085206 (2010)
76. Nemeschansky, D., Warner, N.P.: Off-critical lattice analogues of $N = 2$ supersymmetric quantum integrable models. Nucl. Phys. B **413**, 629–652 (1994)
77. Nicolai, H.: Supersymmetry and spin systems. J. Phys. A, Math. Gen. **9**, 1497–1506 (1976)
78. Okamoto, K.: Studies on the Painlevé equations. I: Sixth Painlevé equation PVI. Ann. Mat. Pura Appl. **146**, 337–381 (1987)
79. Onofri, E., Veneziano, G., Wosiek, J.: Supersymmetry and combinatorics. Commun. Math. Phys. **274**, 343–355 (2007)
80. Pasquier, V., Saleur, H.: Common structures between finite systems and conformal field theories through quantum groups. Nucl. Phys. B **330**, 523–556 (1990)
81. Qiu, Z.: Supersymmetry, two-dimensional critical phenomena and the tricritical Ising model. Nucl. Phys. B **270**, 205–234 (1986)
82. Razumov, A.V., Stroganov, Y.G.: Spin chains and combinatorics. J. Phys. A, Math. Gen. **34**, 3185–3190 (2001)
83. Razumov, A.V., Stroganov, Y.G.: Spin chains and combinatorics: twisted boundary conditions. J. Phys. A, Math. Gen. **34**, 5335–5340 (2001)
84. Razumov, A.V., Stroganov, Y.G.: A possible combinatorial point for the XYZ spin chain. Theor. Math. Phys. **164**, 977–991 (2010)
85. Razumov, A.V., Stroganov, Y.G., Zinn-Justin, P.: Polynomial solutions of qKZ equation and ground state of XXZ spin chain at $\Delta = -1/2$. J. Phys. A, Math. Gen. **40**, 11827–11847 (2007)
86. Robbins, D.P.: Symmetry classes of alternating sign matrices. arXiv:math.CO/0008045 (2000)
87. Rosengren, H.: An Izergin-Korepin-type identity for the 8VSOS model, with applications to alternating sign matrices. Adv. Appl. Math. **43**, 137–155 (2009)
88. Rosengren, H.: The three-colour model with domain wall boundary conditions. Adv. Appl. Math. **46**, 481–535 (2011)
89. Santachiara, R., Schoutens, K.: Supersymmetric model of spin-1/2 fermions on a chain. J. Phys. A, Math. Gen. **38**, 5425–5439 (2005)
90. Saleur, H.: Geometric lattice models for $N = 2$ supersymmetric theories in two dimensions. Nucl. Phys. B **382**, 532–560 (1992)
91. Saleur, H., Warner, N.: Lattice models and $N = 2$ supersymmetry. In: String Theory, Conformal Models and Topological Field Theories. Nato ASI Series B. Plenum, New York (1995)
92. Witten, E.: Constraints on supersymmetry breaking. Nucl. Phys. B **202**, 253–316 (1982)
93. Witten, E., Olive, D.: Supersymmetry algebras that include topological charges. Phys. Lett. B **78**, 97–101 (1978)
94. Yang, S.K.: Modular invariant partition function of the Ashkin-Teller model on the critical line and $N = 2$ superconformal invariance. Nucl. Phys. B **285**, 183–203 (1987)
95. Yang, S.K., Zheng, H.B.: Superconformal invariance in the two-dimensional Ashkin-Teller model. Nucl. Phys. B **285**, 410–422 (1987)
96. Yang, X., Fendley, P.: Non-local spacetime supersymmetry on the lattice. J. Phys. A, Math. Gen. **37**, 8937–8948 (2004)
97. Zamolodchikov, A.B., Fateev, V.A.: A model factorized S-matrix and an integrable spin-1 Heisenberg chain. Sov. J. Nucl. Phys. **32**, 298–303 (1981)
98. Zinn-Justin, P.: Sum rule for the eight-vertex model on its combinatorial line. arXiv:1202.4420 (2012)