On subgroups and Schreier graphs of finitely generated groups

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On Subgroups And Schreier Graphs Of Finitely Generated Groups

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Le Doyen

N.B. - La thèse doit porter la déclaration précédente et remplir les conditions énumérées dans les "Informations relatives aux thèses de doctorat à l'Université de Genève".

Die Grenzen meiner Sprache bedeuten
die Grenzen meiner Welt.

Ludwig Wittgenstein

Abstract

This thesis concerns combinatorial and geometric group theory — the interplay between the properties of groups and those of the geometric objects on which they act. More specifically, we will be interested in the subgroup structure of finitely generated groups, in particular of groups acting on trees, as well as in geometry of Schreier graphs associated with various group actions. Every subgroup of a group is naturally associated with a Schreier graph, and this correspondence is bijective as graphs are considered as labelled and rooted. This thesis evolves around three distinct but related topics.

The first one is transitivity of Schreier graphs. Our main result here is an algebraic characterization of transitive Schreier graphs in terms of associated subgroups. It is then applied to Tarski monster groups for which we show that they are simple in a strong "geometric" sense. This provides an answer to a question of Benjamini on coverings of transitive graphs.

We then turn our attention to limits of sequences of graphs. Whereas it is easy to understand convergence and limits of transitive graphs, the notion of limit of a sequence of arbitrary graphs is more complicated — as defined by Benjamini and Schramm it is an invariant measure on the space of rooted graphs. In the basis of this part of the thesis lies an unexpected observation: the limit of the sequence of de Brujin graphs $\mathcal{B}_{n,k}$ (a family of highly non-transitive graphs widely used in various branches of mathematics and in applications) is the Cayley graph of the lamplighter group also known as the Diestel-Leader graph $\mathrm{DL}(k,k)$. It can be described as the horospheric product of two $(k+1)$-regular trees. An interesting corollary of the connection of de Brujin graphs and the lamplighter group is that it allows us to compute the Fuglede-Kadison determinant of the lamplighter Cayley graph.

We then look at two different generalizations of de Brujin graphs: the spider-web graphs studied in statistical physics (this part of the thesis is based on a joint paper with Grigorchuk and Nagnibeda); and the Rauzy graphs studied in symbolic dynamics.

Our main theorem shows that Benjamini-Schramm limit of Rauzy graphs associated to a subshift of finite type over a finite alphabet is supported on horospheric products of trees.

Finally we study weakly maximal subgroups of branch groups. These are groups acting by automorphisms on rooted trees, with rich subgroup structure. Since geometric group theory is essentially interested in groups up to finite index subgroups, weakly maximal (i.e., maximal among subgroups of infinite index) subgroups is a natural object of study. In branch groups, they appear naturally as stabilizers of infinite rays in the rooted tree — such weakly maximal subgroups are called parabolic. In collaboration with Bou-Rabee and Nagnibeda we showed that in a regular branch group, every finite subgroup is contained in uncountably many weakly maximal subgroups. In particular, there exist uncountably many weakly maximal subgroups that are not parabolic. Branch actions give raise to weakly maximal subgroups. On the other hand, weakly maximal subgroups give raise to faithful and spherically transitive actions on rooted tree. These actions are in general not branch. We conjecture that stabilizer of rays for these actions are two-by-two distincts, thus giving raise to new invariant random subgroups. We prove that this conjecture is true if we pass to the profinite completion and give some partial results for the general case.

Résumé

Le sujet de cette thèse est la théorie géométrique et combinatoire des groupes — le lien entre les propriétés des groupes et celles des objets géométriques sur lesquels ils agissent. Plus précisément, nous nous intéressons à la structure des sous-groupes des groupes de type fini, en particulier des groupes agissant sur des arbres, ainsi qu'à la géométrie des graphes de Schreier associés à diverses actions de groupes. Tout sous-groupe d'un groupe est naturellement associé à un graphe de Schreier, et cette correspondance est bijective si l'on considère les graphes comme étiquetés et enracinés. Cette thèse s'intéresse tout particulièrement à trois sujets distincts mais reliés entre eux.

Le premier est la transitivité des graphes de Schreier. Notre principal résultat est une caractérisation algébrique de la transitivité d'un graphe de Schreier en terme du sous-groupe associé. Nous l'appliquons ensuite aux « monstres de Tarski » pour lesquels nous montrons qu'ils sont simples dans un sens « géométrique » très fort. Cela nous permet de répondre à une question de Benjamini sur les revêtements de graphes transitifs.

Nous tournons ensuite notre attention sur les limites de suites de graphes. Alors qu'il est aisé de comprendre la convergence et la limite d'une suite de graphes transitifs, la notion de limite pour une suite quelconque de graphes est plus compliquée — comme défini par Benjamini et Schramm, c'est une mesure de probabilité sur l'espace des graphes enracinés. Le point de départ de cette partie de la thèse provient d'une observation inattendue : la limite de la suite des graphes de de Bruijn $\mathcal{B}_{n,k}$ (une famille de graphes hautement non transitifs étudiés dans des domaines divers des mathématiques ainsi que dans d'autres sciences) est le graphe de Cayley du groupe de l'allumeur de réverbères, aussi connu comme le graphe de Diestel-Leader $\mathrm{DL}(k, k)$. Ce dernier graphe peut être décrit comme un produit horosphérique de deux arbres $k + 1$ réguliers. Un corollaire intéressant de ce lien entre les graphes de de Bruijn et le groupe de l'allumeur de réverbères est qu'il nous permet de calculer de déterminant de Fuglede-Kadison du graphe de Cayley de ce groupe.

Nous nous intéressons ensuite à deux différentes généralisations des graphes de de Bruijn : les graphes de toiles d'araignées étudiés en physique statistique (cette partie de la thèse est un travail en commun avec R. Grigorchuk et T. Nagnibeda), et les graphes de Rauzy étudiés en dynamique symbolique.

Notre théorème principal montre que la limite de Benjamini-Schramm des graphes de Rauzy associés à un sous-décalage de type fini sur un alphabet fini est supportée sur des produits horosphériques d'arbres.

Finalement, nous étudions les sous-groupes faiblement maximaux des groupes branchés. Ces groupes agissent par automorphismes sur des arbres enracinés et possèdent une riche structure de sous-groupes. Étant donné que la théorie géométrique des groupes s'intéresse surtout aux groupes à indice fini près, il est naturel d'étudier les sous-groupes faiblement maximaux (c'est-à-dire maximaux parmi les sous-groupes d'indice infini). Dans le cas des groupes branchés, ces sous-groupes apparaissent naturellement comme les stabilisateurs des rayons infinis dans l'arbre — de tels sous-groupes sont appelés paraboliques. En collaboration avec K. Bou-Rabee et T. Nagnibeda nous avons montré que dans un groupe régulièrement branché, tout sous-groupe fini est contenu dans un nombre non-dénombrable de sous-groupes faiblement maximaux. En particulier, il existe un nombre non-dénombrable de sous-groupes faiblement maximaux qui ne sont pas paraboliques. Les actions branchées donnent lieu à des sous-groupes faiblement maximaux. D'autre part, les sous-groupes faiblement maximaux donnent lieu a des actions fidèles et transitives sur des arbres enracinés. Ces actions ne sont pas en général branchées. Nous conjecturons que les stabilisateurs des rayons pour ces actions sont deux à deux distincts, donnant ainsi lieux à de nouveaux sous-groupes aléatoires. Nous prouvons que cette conjecture est vraie si l'on passe à la complétion profinie et donnons certains résultats partiels dans le cas général.

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Contents

List of Symbols

$\mathcal{A}$	—	Finite alphabet (finite set of symbols)
$\mathfrak{DiGraph}$, $\mathfrak{Graph}$	—	The category of (di)graphs
$\mathfrak{Gr}$	—	The category of groups
$\mathfrak{Set}$	—	The category of sets
$A \bigtriangleup B$	—	Symmetric difference of two sets: $A \bigtriangleup B = (A \setminus B) \cup (B \setminus A)$
$\mathscr{F}_N$	—	The N^{th} Fibonacci number
$\mathbf{N}, \mathbf{N}_0, \overline{\mathbf{N}}, \mathbf{Z}$	—	Sets of positive integers, non-negative integers, positive integers with $+\infty$ and relative integers
$\mathrm{Supp}(\mu)$	—	Support of the measure μ

Group theory

$\cong$	—	Isomorphism of groups
$G \curvearrowright X$	—	Group action of G on X
H^G	—	Normal closure of H in G
$\mathscr{C}$	—	Class of generalized multi-edge spinal groups
$\mathscr{G}$	—	Class of "trivial" generalized multi-edge spinal groups
$\mathcal{F}_{Z \sqcup Y}$	—	Group of the form $(*_{\lvert Z \rvert}\, \mathbf{Z}) * (*_{\lvert Y \rvert}\, \mathbf{Z}/2\mathbf{Z}) = \langle y_i \in Y,\ z_j \in Z \mid y_i^2 \rangle$
$\mathcal{G}$	—	First Grigorchuk group
$\mathcal{L}_k$	—	Lamplighter group
$\mathrm{rank}(G)$	—	Rank of G: the minimal number of generators of G
$\mathrm{Rist}_G(v)$	—	Rigid stabilizer of the vertex $v \in T$ for the action $G \curvearrowright T$
$\mathrm{Rist}_G(n)$	—	Rigid stabilizer of the level n of the rooted tree T for the action $G \curvearrowright T$
$\mathrm{Stab}_G(x)$	—	Stabilizer of a point for a group action of G
$\mathrm{Stab}_G(n)$	—	Stabilizer of the level n of the rooted tree T for the action $G \curvearrowright T$
$\mathrm{Sub}_{\mathrm{cl}}(G)$	—	Set of closed subgroups of G
wmc	—	Weakly maximal subgroup

Graph theory

$\mathfrak{h}(T)$	—	Busemann rank function of a tree T
$\simeq$	—	Isomorphism of (di)graphs
$\kappa(\Gamma)$	—	Complexity of the graph Γ
$\chi(\Gamma)$	—	Characteristic polynomial of the adjacency matrix of Γ
∂T	—	Boundary (set of infinite rays) of the rooted tree T
$T_\xi \circledh T_\eta$	—	Horospheric product of two trees with directing rays
$\mathrm{Aut}\,(\Gamma)$	—	Group of automorphisms of Γ
$\vec{\mathcal{B}}_N$	—	de Bruijn digraph
$\mathcal{B}_N$	—	de Bruijn graph
$\vec{\mathrm{Cay}}(G, X)$	—	Cayley digraph of a group G with generating system X
$\mathrm{Cay}(G, X^{\pm})$	—	Cayley graph
$\mathrm{der}(p), \mathrm{der}(\vec{\Gamma})$	—	Derangement of a path, of a digraph
$\vec{\mathrm{DL}}(p, q)$	—	Diestel-Leader digraph
$\vec{\mathcal{G}}_\bullet$, $\mathcal{G}_\bullet$	—	Space of rooted (di)graphs
$\mathscr{L}_n$	—	N^{th} level of a rooted tree
$\vec{R}_{k,F,N}$	—	Rauzy digraph
$R_{k,F,N}$	—	Rauzy graph
$\mathrm{rk}(v)$	—	Rank of the vertex v in $\vec{\Gamma}$
$\vec{\mathrm{Sch}}(G, H, X)$	—	Schreier digraph for a subgroup H of a group G with generating system X
$\mathrm{Sch}(G, H, X^{\pm})$	—	Schreier graph
$\mathrm{Star}(v)$	—	Star of a vertex
$\vec{\mathcal{S}}_{k,N,M}$	—	Spider-web digraph
$\mathcal{S}_{k,N,M}$	—	Spider-web graph

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Chapter 1

Introduction

This thesis evolves around two basic mathematical notions, that of a group and of a graph.

A group is a fundamental algebraic structure, and the modern group theory develops in many very different directions. In this thesis we will focus on geometric, combinatorial and asymptotic group theory that mainly studies infinite discrete groups as geometric objects, up to quasi-isometries (maps that preserve large-scale geometry). One subject of particular interest is groups that exhibit exotic geometrical properties, very different from the classical Euclidean, spherical and hyperbolic ones. For examples, groups with volume growth intermediate between polynomial and exponential. The existence of such geometries in the highly ordered algebraic world of groups, first established by Grigorchuk in 1980's, is a surprising and not yet fully understood phenomenon that generates a lot of interesting research.

A graph is a ubiquitous notion in mathematics and applications. Here, graphs will consist of a (possible infinite) set of vertices and a set of edges, such that only finitely many edges are incident with any given vertex. We will be interested in two classes of graphs that both stem from the following most basic example. Given a group G generated by finitely many elements (for example, a free group of finite rank F_d or a free abelian group of finite rank $\mathbf{Z}^d$, or the fundamental group of a compact Riemannian manifold, or any finite group), and a finite generating set S, one constructs the Cayley graph of the pair (G, S). It can be viewed as a model space for the group: the group acts freely (without fixed point) and transitively (with only one orbit) on the vertices of its Cayley graph. In the examples above, one gets the Bethe lattice as a Cayley graph of the free group, the cubic lattice as a Cayley graph for $\mathbf{Z}^d$, tesselations of the hyperbolic plane as Cayley graphs of the fundamental groups of hyperbolic surfaces. On one hand, Cayley graphs are typical (but not only!) examples of transitive graphs — a class of graphs widely studied in statistical physics as a reasonable class for defining various probabilistic models. Informally speaking, a transitive graph locally looks the same everywhere! On the other hand, whereas Cayley graphs correspond to free actions of groups, one can also construct similar graphs out of any group action — in this way one gets a much wider class of so-called Schreier graphs.

It is of great interest that the interplay between group theory and graph theory finds applications in problems from other areas of research: statistical physics, theory of dynamical systems, ergodic theory.

Some of the results of this thesis are joint work with other mathematicians and some of the results are already published. Sections 3.3 to 3.5 essentially come from [83]. Sections 4.2 to 4.5 are joint work with R. Grigorchuk and T. Nagnibeda and were published in [86] with slightly less details, while Section 4.6 is a joint work with T. Nagnibeda. Finally, Sections 5.5 and 5.6 are a common work with Bou-Rabee and Nagnibeda and were published in [24]. Precise information of collaborations and publication can be found at the end of the introductory part of each chapter.

1.1 Products and limits of graphs

Graphs are simple objects with both geometric and combinatorial meaning. They appear in many different areas as for example, topology, dynamics, discrete mathematics or group theory. As graphs may be used to model relations between objects, they have a lot of applications in different fields: physics, informatics, biology and chemistry (to study molecules) but also in sociology. Nevertheless, graph theory turned out to be of a great intrinsic interest.

The main flaw of graph theory is probably the numerous definitions of graphs that do not always agree. This is due to its versatility and long history going back to an Euler paper in 1736. In Chapter 2 we set the main definitions of graph theory that we will use and state some fundamental results that we will use later.

One important concept in graph theory is that the space $\mathcal{G}_\bullet$ of rooted (or marked) graphs is a metric space. Moreover, for any d, the subspace $\mathcal{G}_{\bullet,d}$ of graphs with maximum degree bounded by d is compact. This subspace contains in particular all Schreier graphs of group G with generating system X such that $|X| \leq d$. Therefore, the comprehension of limits in this space is of great interest. This limit is called local limit.

Sometimes, the graphs that we are interested in are not rooted and there is no canonical choice for the roots. If a graph Γ is finite, we can look at μ_Γ, the uniform probability distribution on the set of vertices. The set of probability measure on $\mathcal{G}_{\bullet,d}$ is compact for the weak convergence. In particular, any sequence of graphs of bounded degree admits a subsequence converging to some μ_Γ. This limit is called the Benjamini-Schramm limit.

There is a natural notion of morphisms of graphs: namely, maps that preserve the structure. This gives us a category $\mathfrak{Graph}$, or $\mathfrak{DiGraph}$ if we are interested in directed graphs. We may therefore asks for the existence of categorial notions as for example product, coproduct, pushout or pullback. In turns out that in $\mathfrak{DiGraph}$ all these notions exists (and all finite limit and colimits, see Scholion 2.1.6), but that this is not necessarily the case in $\mathfrak{Graph}$. Nevertheless, when these constructions exist, they are usually *obvious*, [115]. Where a categorial construction in $\mathfrak{Graph}$ (or in $\mathfrak{DiGraph}$) is obvious if it is obtain by taking the same construction on $\mathfrak{Set}$ for vertices and on $\mathbf{Z}/2\mathbf{Z} - \mathfrak{Set}$ for edges.

In Section 2.3 we turn our attention to the categorial product of graphs: the so-called tensor product. We prove that this product behaves well with coverings, the operation of taking the line graph and with the distance in $\mathcal{G}_\bullet$.

1.2 Coverings of transitive graphs and strong simplicity of groups

A continuous map $p\colon X \to Y$ between two topological spaces is a covering if for every point $y \in Y$, there exists an open neighbourhood U such that $p^{-1}(U)$ is a disjoint union of open sets of X. Coverings have always been a subject of great interest from a topological viewpoint and proved to be a useful tool in various areas of mathematics. In the special case of graphs, coverings admit a simple combinatorial description. In this context, it provides a powerful tool to explore property of subgroups of free groups, see for example [115]. There is an easy algorithm that takes a finite graph and an integer d and returns every covering of the graph of degree d. But the inverse problem: given a graph, to find all graphs that are covered by it — is harder to solve, especially for infinite graphs.

Motivated by the study of connectivity constants of transitive graphs, of utmost importance in the theory of self-avoiding walks in statistical physics and in probability theory, Benjamini asked the following

Conjecture 1.2.1 (Benjamini)**.** *There exists a constant M such that every infinite transitive graph (not quasi-isometric to* $\mathbf{Z}$*) covers an infinite vertex transitive graph of girth (that is the length of the shortest cycle) bounded by M.*

Where a *transitive graph* means a vertex-transitive graph: for every pair of vertices x and y, there exists an automorphisms f that sends x onto y. Informally, this signify that vertices are not distinguishable one from another. On the other hand, two spaces X and Y are *quasi-isometric* means, informally, that they have the same large scale geometry. It is easily seen that the only graphs covered by $\mathbf{Z}$ are itself and finite graphs (cycles). It is thus natural to exclude it from the conjecture.

The initial question about monotonicity of the connectivity constant was later answered by Grimmett and Li [61] by a different method. However, the conjecture is interesting in itself and even the following weaker version is still an open problem:

Conjecture 1.2.2. *Every infinite Cayley graph (not quasi-isometric to* $\mathbf{Z}$*) properly covers an infinite vertex transitive graph.*

This version of the conjecture is in fact more natural from the algebraic prospective. Indeed, if a Cayley graph of a group doesn't cover any distinct Cayley graph, then this exactly corresponds to the basic group property of being simple. The interest here lies in the fact that transitive graphs are not necessarily Cayley graphs, even if the latter are the most typical examples of the former. For example, the famous Diestel-Leader graphs — horospheric products of a two regular trees of different degrees — are transitive, but not even quasi-isometric to any Cayley graph [39].

In Chapter 3, we gave a partial negative answer to Conjecture 1.2.2: the conjecture is false if we restrict ourselves to coverings compatible with the algebraic nature of the Cayley graph, i.e., preserving the labelings of the edges by the generators of the group. The main idea was to reformulate the problem in group-theoretic terms and then to find a group with a suitable subgroup structure, in order to provide a negative answer to the conjecture. Firstly, we extend some classical arguments about finite graphs to show that every transitive graph is a Schreier graph over a (generalized) free group. Then we provide an algebraic characterization of transitivity of a Schreier graph which involves studying length functions defined by the choice of a generating set. To find a counterexample that would satisfy the criterion, we use Ol'shanskii's theory of Tarski monsters — groups in which every subgroup is cyclic. Precisely, we have the following

Theorem 1.2.3 ([83])**.** *Let G be a Tarski monster, S a generating set and Γ the corresponding Cayley graph. Suppose that $p\colon \Gamma \to \Delta$ is a covering such that for every edges e and f if $p(e) = p(f)$ then e and f have the same label by a generator. Then either $\Gamma = \Delta$, or Δ has only* 1 *vertex, or Δ is infinite and the orbit of each vertex under* $\mathrm{Aut}(\Delta)$ *is finite (in which case Δ is not transitive).*

Our criterion of transitivity of Schreier graphs and the Theorem above motivate the following new

Definition 1.2.4. A group with a Cayley graph that does not cover any transitive graph, is called *strongly simple.*

Such groups are obviously simple and Tarski monsters are so far the only known examples of such groups.

Since the 19th century and the beginnings of group theory simple groups are viewed as building blocks of the theory, and their classification is considered as one of the main problems in group theory, formulated in Jordan-Hölder program in 1880's. An impressive collective endeavor that took a big part of the 20th century resulted in the classification of finite simple groups including cyclic groups of prime order, alternating groups of even permutations $A_n, n \geq 5$, families of groups of Lie type and the sporadic ones. At present there is no hope of a similar classification of infinite simple groups, but they are in the center of an active area of research. Indeed, not so many sources of examples of infinite finitely generated simple groups are known: the first ones, constructed by Higman in 1950's, lattices acting of products of trees constructed by Burger and Mozes [26], recent examples of simple amenable groups by Juschenko and Monod [70]. Only few of them are finitely presented. "Quantifying" simplicity of these groups by refining the notion is potentially interesting in better understanding its nature.

In Section 3.5, we show that the alternating groups A_n are simple but not strongly simple for odd $n \geq 5$ and that Tarski monsters are strongly simple.

1.3 De Bruijn, spider-web and Rauzy graphs

The study of limits of finite graphs is in the center of attention of both graph theorists and probabilists, with interest in statistical physics. One of the basic tools is the notion of Benjamini-Schramm limit, introduced in [20]. It defines the limit of a sequence of finite graphs as an invariant probability distribution on the set of all possible rooted graphs that can be obtained as limits of our sequence with a random choice of root (once you choose roots in your graphs, you can define the limit by considering stabilizing balls of bigger and bigger radius around the root). If the graphs in the family were transitive, then in the limit we would get a Dirac measure concentrated on a single transitive graph. However in general it does not have to be the case.

One famous sequence of finite graphs is that of de Bruijn graphs — directed graphs representing overlaps between sequences of symbols in a given alphabet. The vertices of $\mathcal{B}_{k,n}$ are sequences of n letters on the alphabet $\{0, \dots, k-1\}$ and there is an oriented edge labeled by $x_1 \dots x_{n+1}$ from $x_1 \dots x_n$ to $x_2 \dots x_{n+1}$. They enjoy some nice graph-theoretic properties and are used in many different scientific areas: mathematics, where they serve as a discrete model of the Bernoulli map $x \mapsto kx \pmod 1$, but also in computer science and in bio-informatics [105, 31].

As, for k fixed, they form an infinite family on a growing number of vertices, it is of interest to find asymptotic properties of this family. In order to study asymptotic properties of a family of graphs, the best we can hope is to describe the limit.

In Chapter 4 we describe this limit as a well-known graph:

Theorem 1.3.1 ([86]). *The Benjamini-Schramm limit of de Bruijn graphs $(\mathcal{B}_{k,N})_{N\geq 0}$ is some (well-known) Cayley graph of the lamplighter group $\mathcal{L}_k$, which is the Diestel-Leader graph corresponding to the horospheric product of two regular trees of degree $k+1$.*

Interestingly, this theorem is obtained as a corollary of the fact, that we also prove, that de Bruijn are isomorphic to the Schreier graphs of the lamplighter group, realized as an automorphism group of the infinite binary tree.

In fact, in Chapter 4, we study the more general class of the so-called "spider-web networks" introduced in 1959 [67] in the context of systems of telephone exchange. Later these networks were shown to enjoy extremely nice properties in percolation [102, 101, 103] and also interesting spectral distributions [6]. One of the main problems acknowledged in these papers was the absence of the infinite limit lattice associated to the family of these networks. We show that the spider-web networks $\mathcal{S}_{k,n,m}$ are isomorphic to the tensor product (also called categorial product) of the (oriented versions of) de Bruijn graph $\mathcal{B}_{k,n}$ and the

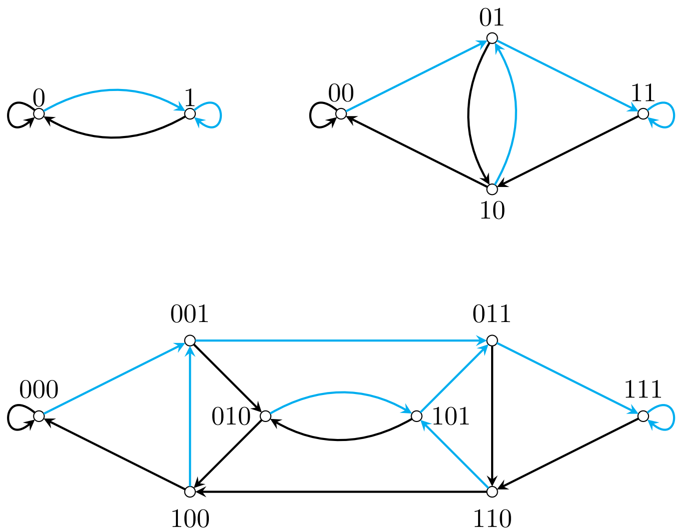


Figure 1.1: De Bruijn digraphs $\mathcal{B}_{2,1}$, $\mathcal{B}_{2,2}$ and $\mathcal{B}_{2,3}$.

cycle C_m on m vertices, thus providing the limit (given in the Theorem above) and also proving that spider-web networks are a natural two-parameter extension of the one-parameter family of de Bruijn graphs, which opens new possibilities for applications. This also allowe us to generalize many properties of de Bruijn graphs to spider-web graphs. For example, the fact that these graphs are Eulerian (there exists a closed path that visits each edge exactly once) and Hamiltonian (there exists a closed path that visits each vertex exactly once), the fact that they form a covering tower as well as the computation of the limit.

We also study spectral properties, like complexity or spectral zeta function of these graphs. In particular, this allows us to compute the Fuglede-Kadison determinant of the Laplacian of the Diestel-Leader graph $\mathrm{DL}(k,k)$. Indeed, this determinant is equal to $e^{-\zeta'_{\mathrm{DL}(k,k)}(0)}$ and $-\zeta'_{\mathrm{DL}(k,k)}(0) = \log(k) - (k-1)^2 \frac{d}{ds} \operatorname{Li}_s\left(\frac{1}{k}\right)\Big|_{s=0}$, where $\operatorname{Li}_s(z)$ is the polylogarithm.

A natural generalization of de Bruijn graphs is the so called Rauzy graphs. In general these graphs are not regular, it is hence not possible to use a group-theoretic approach. However there is a more direct approach that leads to computation of the limit for the general case. It also implies an explicit isomorphism between de Bruijn graphs and the graphs of the action of the lamplighter group on a rooted tree.

In terms of symbolic dynamics, de Bruijn graphs are Rauzy graphs corresponding to the full shift and are therefore the simplest example of the theory. For the general case, let F be a set of words on the alphabet $\{0,\ldots,k-1\}$. We will consider F as a set of forbidden patterns defining a subshift. The corresponding Rauzy graph is obtained from $\mathcal{B}_{k,n}$ by erasing all edges and vertices containing subsequences in F. Hence $\mathcal{B}_{k,n}$ is simply the Rauzy graph for $F = \emptyset$. When F is finite (the corresponding subshift is of finite type), and under some additional technical assumptions, we compute the limit of the corresponding Rauzy graphs. This is not anymore a graph, but an invariant probability measure on the space of rooted graphs. This measure is concentrated on horospheric products of (not necessarily regular) trees. In this case, vertices of the trees correspond to infinite sequences without subwords in F Each tree appearing in the horospheric product can be seen as the tree of trajectories (of the subshift associated to F) that share a common "direction".

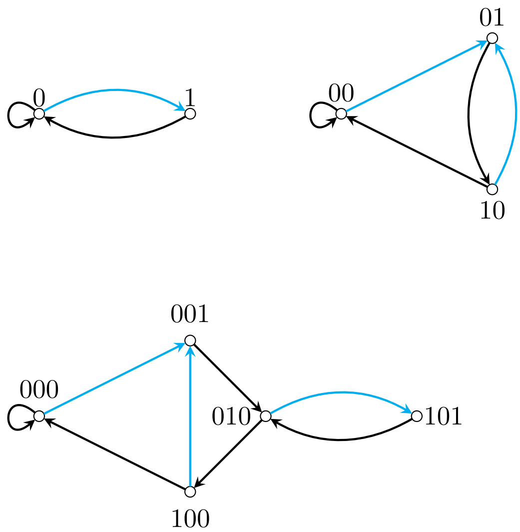


Figure 1.2: Rauzy graphs for $k = 2$, $N = 1, 2, 3$ and $F = \{11\}$.

1.4 Subgroups structure of branch groups

As mentioned above, infinite finitely generated simple groups are very far from being well understood. From the large-scale geometric prospective, it is natural to relax the condition of simplicity and to allow the group to have non-trivial quotients — but only finite ones. Such groups are called just-infinite. In the world of infinite groups, they retain the role of basic building blocks played by simple groups, in the sense that any finitely generated infinite group has a just-infinite quotient. A major discovery by Wilson in 1970's shows that the class of just-infinite groups consists of groups of three distinct types: roughly speaking, the simple groups, the hereditarily just-infinite groups (just-infinite groups whose finite index subgroups are also just-infinite), and the so-called branch groups. These are characterized by their rich subgroup structure, in spite of the fact that normal subgroups are all of finite index.

One aim of mathematics is to study general concepts and to prove theorems not about a singular object, but about many objects sharing the same properties. However, sometimes a single object is important enough in itself to attract attention of many mathematicians. People firstly study this particular object and then try to define a suitable generalization of it. This is the case for the first Grigorchuk group $\mathcal{G}$. Grigorchuk constructed it in [60] and proved in [47] that $\mathcal{G}$ has intermediate growth between polynomial and exponential, thus providing an answer to an open problem posed by John Milnor in 1968. Since then, this group continues to be extensively studied and gave rise to the definition of two important classes of groups: the branch groups (which naturally appear in the classification of just-infinite groups as described above) and the self-similar groups (which appear naturally as iterated monodromy groups of Julia sets in complex dynamics). All these groups appear as automorphism groups of infinite rooted trees.

Branch groups admit an intrinsic definition in terms of stabilizers of points and of stabilizers of rays (also called *parabolic subgroups*) for this action and a major open problem about them is to describe their lattice of subgroups. In the case of the Grigorchuk group or branch generalized multi-edge spinal groups, a deep result in this direction, see [97, 79], is the rigidity of maximal subgroups — a behavior which is completely different from the linear case of, say $SL(n, \mathbf{Z})$.

On the other hand, Grigorchuk [49] and Bartholdi and Grigorchuk [11, 10] showed that in a branch group, all the parabolic subgroups are infinite, pairwise distinct and weakly maximal; where a subgroup is said to be *weakly maximal* if it is of infinite index and maximal for this property. In view of this result, Grigorchuk asked [49]

Question 1.4.1 (Grigorchuk, [49])**.** *Does there exists a weakly maximal subgroup in $\mathcal{G}$ which is not parabolic?*

The main result of Chapter 5 is that there exist uncountably many such subgroups, and not only in $\mathcal{G}$ but in an arbitrary regularly branch group.

This discovery can be used to study the subgroup structure of branch groups from the statistical viewpoint. Namely, a very active area of research has lately developed around the notion of IRS — invariant random subgroup in a locally compact group. An IRS is a probability distribution on the space of all subgroups of the group which is required to be invariant under the natural action of the group on its subgroups by conjugation. This is a natural example of an invariant measure, but moreover it also has a nice algebraic meaning: it generalizes the notion of a normal subgroup that corresponds to the Dirac delta measures. It is very interesting and important to understand how the algebraic subgroup structure of the group is related to the variety of IRSs on it. For example, recent new examples of Juschenko and Monod [70] provide first examples of simple groups (i.e. groups that have no non-trivial normal subgroups) with non-trivial IRS. Benli, Grigorchuk and Nagnibeda [21] constructed first examples of groups of intermediate growth with uncountably many distinct IRS. It is still an open question whether a just-infinite group, and in particular the Grigorchuk group $\mathcal{G}$, can have uncountably many IRS (observe that being just-infinite it can only have countably many normal subgroups).

How is this problem connected to our result about variety of weakly maximal subgroups? There is indeed a strategy that potentially allows to construct an IRS out of a weakly maximal subgroup, as follows. Given a weakly maximal subgroups H in $\mathcal{G}$, it is possible to construct a binary rooted tree T_H on which $\mathcal{G}$ acts and such that H is the stabilizer of the leftmost ray. If H is not parabolic for the original action, then the action of $\mathcal{G}$ on T_H is not branched, but we can ask the following

Question 1.4.2. *Given the action of $\mathcal{G}$ on T_H, is it true that all parabolic subgroups for this action are*

1. *weakly maximal?*
2. *pairwise distinct?*

Positive answer to these questions will imply the construction of new non-trivial invariant random subgroups of $\mathcal{G}$.

In Section 5.7, we prove that both questions admit a positive answers if we pass to $\hat{\mathcal{G}}$, the profinite completion of $\mathcal{G}$. In this case, weakly maximal subgroups are replaced by closed subgroups of infinite that are maximal for this property and called wmc. The rigidity of maximal subgroups in $\mathcal{G}$ implies that weakly maximal subgroups are closed for the profinite topology. In Section 5.7, we also prove that there is a lattice isomorphism between finite index subgroups of $\mathcal{G}$ and finite index subgroups of $\hat{\mathcal{G}}$, given by $H \mapsto \bar{H}$ for $H \leq \mathcal{G}$, and $M \mapsto M \cap \mathcal{G}$ for $M \leq \hat{\mathcal{G}}$. There are good reasons to believe that these maps sends wmc subgroups to wmc subgroups, see Lemma 5.7.11 and Proposition 5.7.7. If this is the case, then in Question 1.4.2, both answers are positive.

1.5 Weakly maximal subgroups of the Grigorchuk group

As advocated in the previous section, the first Grigorchuk group $\mathcal{G}$ is an important and heavily studied example in geometric group theory. For example, it has intermediate growth between polynomial and exponential and it is amenable but not elementary amenable.

Its subgroup structure is widely studied and presents some interesting properties. For example, all non-trivial normal subgroups and all maximal subgroups are of finite index. Moreover, all normal subgroups are characteristic and the number b_n of normal subgroup of index 2^n is odd and there exists asymptotic bounds on it, [8]. It also has the congruence subgroup property: every finite index subgroup contains the stabilizer of a level for the (unique) branch action of $\mathcal{G}$. This implies that its completion in $\mathrm{Aut}(T)$ is the same as its profinite completion. A detailed study of the top of the lattice of normal subgroups can be found in [28] and [8].

Among other results, there is a description of finitely generated subgroups of $\mathcal{G}$ in [54] and a study of maximal locally finite subgroups in [108].

The geometrical properties of subgroups of $\mathcal{G}$ were also studied, via their Schreier graphs [118, 52]. In particular, Schreier graphs of parabolic subgroups of the original action are (after erasing loops and multiple edges) all isomorphic to $\mathbf{Z}$ or to $\mathbf{N}$. More precisely, after erasing loops and multiple edges, $\mathrm{Sch}(\mathcal{G}, \mathrm{Stab}_{\mathcal{G}}(\xi), \{a,b,c,d\}^{\pm})$ is isomorphic to $\mathbf{N}$ if ξ is in the $\mathcal{G}$-orbit of $\bar{1}$, and isomorphic to $\mathbf{Z}$ otherwise. All the $\mathrm{Sch}(\mathcal{G}, \mathrm{Stab}_{\mathcal{G}}(\xi), \{a,b,c,d\}^{\pm})$ are two-by-two non strongly isomorphic, but if we forget the labeling, there is only two isomorphism classes (corresponding to the cases $\mathbf{Z}$ and $\mathbf{N}$).

In Chapter 6, we show that there exists weakly maximal subgroups of $\mathcal{G}$ such that the corresponding Schreier graph is not linear, i.e. not isomorphic to $\mathbf{Z}$ or $\mathbf{N}$ after we had erased loops and multiple edges. We also prove results on the number of distinct classes of weakly maximal subgroups and give a detailed discussion of an example (due to Pervova) of a finitely generated weakly maximal subgroup and some generalization of it.

CHAPTER 2

Preliminaries on graphs

Graphs are simple objects with both geometric and combinatorial meaning. They were first studied by Leonhard Euler in 1736 in order to solve the problem of the Seven Bridges of Königsberg. The notion of graphs turned out to be of a great intrinsic interest, but also related to other branch of mathematics, as for example topology and group theory. In particular, many classical theorem about free groups admits graph theoretic proof that are usually simpler and more illuminating that their historical and combinatorial counterparts.

In this chapter we give all the basic definitions and some preliminaries results. In Section 2.1 we recall the main definitions. The next Section is about the space of marked graph and its metric. Section 2.3 is about tensor product of graphs and how it interacts with the limit; this was published as part of [86]. Finally, Section 2.4 presents the basics of Schreier graphs .

2.1 Definitions

The original sin of graph theory is the multiple and sometimes contradictory definitions of what is a graph; this is why we will write the definition of all notions used, even for the classical one. We will work with a generalization of the categorial definition of a graph coined by Serre [112]. Serre's definition turned out to be fruitful and was used with great success by Stalling [115] and others. This definition comes with two limitations: it is intended only for graphs and says nothing about digraphs (also called oriented graphs or quiver) and it does not allows "degenerate loops" which naturally arise in the context of Schreier graphs. Nevertheless, small modifications of the original definition of Serre allow to take in account both these cases.

Graphs and Digraphs For us, a graph is simply a digraph where every edge admits an inverse. This is the content of the following two definitions.

Definition 2.1.1. A *digraph* (for *directed graph*, also called a *quiver*) $\vec{\Gamma} = (V, \vec{E})$ consists of two sets $\vec{E}$ (*edges*) and V (*vertices*), and two functions $\iota, \tau \colon E \to V$ with no restrictions on it.

The vertices $\iota(e)$ and $\tau(e)$ are respectively the *initial* and *final* vertex of the edge e.

Observe that for us, digraphs may have *loops* (edge with the same initial and final vertex) and *multiple edges* (two (or more) edges that shares their initial vertex and their final vertex). In the literature they are sometimes called multidigraphs or quivers.

Definition 2.1.2. A *graph* $\Gamma = (V, E)$ is a digraph with a function $\bar{\ } \colon E \to E$ such that $\bar{\bar{e}} = e$ and $\tau(e) = \iota(\bar{e})$.

The edge $\bar{e}$ is called the *inverse edge* of e. An *unoriented edge* is a pair $\{e, \bar{e}\}$.

Observe that our definitions of graphs allow loops, multiple edges and also *degenerate edges* — edge e with $e = \bar{e}$. If an edge e is degenerate, then we have $\tau(e) = \iota(\bar{e}) = \iota(e)$ which means that e is a loop. Graphs without degenerate loops correspond to graphs as defined by Serre. Many results about such graphs from [115] can be extended easily to the general case.

Digraphs will be graphically represented by small circles for vertices and arrows between circles for edges. On the other hand, non-degenerate unoriented edges in graphs will be graphically represented by plain line while degenerate edges will be represented by dashed lines.

Definition 2.1.3. The *underlying graph* of a digraph $\vec{\Gamma} = (V, \vec{E})$ is the graph $\underline{\vec{\Gamma}} = (V, E)$, with $E := \vec{E} \sqcup \{\bar{e} \mid e \in \vec{E}\}$, where $\bar{e}$ is a formal inverse of e. For every $e \in \vec{E}$, we define $\iota(\bar{e}) := \tau(e)$, $\tau(\bar{e}) := \iota(e)$ and $\bar{\bar{e}} := e$.

An *orientation* $\mathcal{O}$ on a graph Γ is the choice of an edge in each of the pairs $\{e, \bar{e}\}$. For each choice of an orientation $\mathcal{O}$ on $\Gamma = (V, E)$, we define the digraph $\vec{\Gamma} = (V, \vec{E})$ where $\vec{E} = \mathcal{O}$ and ι and τ are restrictions on $\vec{E}$ of the original functions.

Observe that if Γ is an underlying graph of a digraph, then it has no degenerate loop. The operations of choosing an orientation on a graph and of taking the underlying graph of a digraph are mutually inverse in the following sense. Given a digraph $\vec{\Gamma}$, there exists an orientation on the underlying graph such that the resulting digraph is $\vec{\Gamma}$ itself. On the other hand, given a graph Γ without degenerate loops, the underlying graph of the digraph obtained by choosing an orientation on Γ, is Γ itself.

Let $\Gamma = (V, E)$ be a graph or a digraph. The *in-degree*, respectively the *out-degree*, of a vertex v is the number of edges e with initial vertex v, respectively end vertex v. If Γ is a graph, then both notions coincide and are simply called *degree*. Γ is said to be *locally finite* if every vertex has both finite in-degree and finite out-degree. Note that if e is a loop in a graph, it contributes 1 to the in-degree, but its inverse edge $\bar{e}$ also contributes 1. Therefore, the non-oriented loop $\{e, \bar{e}\}$ contributes 2 to the degree if and only if it is non-degenerate, otherwise it contributes to 1.

If Γ is a graph or a digraph, its *adjacency matrix* is the matrix $A_\Gamma = (a_{ij})_{i,j \in V}$ with a_{ij} the number of edges from i to j. Adjacency matrix of graphs are symmetric but this not necessarily the case for digraphs. If $\vec{\Gamma}$ is a digraph, we have $A_{\underline{\vec{\Gamma}}} = A_{\vec{\Gamma}} + A^T_{\vec{\Gamma}}$.

Let Γ be a graph or a digraph. A *path* p in Γ from v to w of *length* n is an ordered sequence of edges $(e_1, e_2, \ldots, e_n)$ such that $\iota(e_1) = v$, $\tau(e_n) = w$ and for all $1 \leq i < n$ we have $\tau(e_i) = \iota(e_{i+1})$. In particular, for any v there is a unique path of length 0 from v to v : the empty path. The inverse of the path $p = (e_1, e_2, \ldots, e_n)$ is the path $\bar{p} = (\bar{e}_n, \ldots, \bar{e}_1)$. A path p in a graph is said to be *reduced* if it does not contain subsequences of the form $e\bar{e}$. A *cycle* is a reduced path of length at least 1 with the same initial and final vertex.

A graph is *connected* if for all choices of two vertices v and w there exists a path from v to w. A digraph is *strongly connected* if for all choices of two vertices v and w there exists a path from v to w It is *weakly connected* (or simply *connected*) if its underlying graph is connected. The smallest example of a connected but not strongly connected digraph is $\circ\!\longrightarrow\!\circ$ since there is no path from the right vertex to the left one.

Important examples of connected graphs are tree. A *tree* is a connected graph such there is no cycle. In other words, between any two vertex there is exactly one reduced path. A *directed tree* is a digraph such that the underlying graph is a tree.

Another important connected graph is the *rose* with n petals. This is a graph with one vertex and d loops. Formally, we should speak of a (k, d)-rose: the only graph with one vertex, d non-degenerate loops and k degenerate loops.

Morphisms Following Serre, we want to look at graphs from a categorical viewpoint and therefore we need a notion of morphisms.

Definition 2.1.4. A *morphism of digraphs* $\varphi\colon \vec{\Gamma}_1 \to \vec{\Gamma}_2$ consist of a pair of functions $\varphi_V : V_1 \to V_2$ and $\varphi_E\colon \vec{E}_1 \to \vec{E}_2$ that preserves the structure. This mean that $\iota_{\Gamma_2} \circ \varphi_E = \varphi_v \circ \iota_{\Gamma_1}$ and $\tau_{\Gamma_2} \circ \varphi_E = \varphi_v \circ \tau_{\Gamma_1}$. A *morphism of graphs* is defined in the same way, with the additional requirement that $\varphi_E(\bar{e}) = \overline{\varphi_E(e)}$.

Remark 2.1.5. Since in a graph we have $\tau(e) = \iota(\bar{e})$ and $\iota(e) = \overline{\tau(\bar{e}),}$ in order to show that φ is a morphism of graphs, it is sufficient to check that $\varphi_E(\bar{e}) = \overline{\varphi_E(e)}$ and any one of the two other conditions.

We hence have two categories: $\mathfrak{DiGraph}$ and $\mathfrak{Graph}$. We also have two forgetful functors $F_V : \mathfrak{DiGraph} \to \mathfrak{Set}$ and $F_E : \mathfrak{DiGraph} \to \mathfrak{Set}$ defined by $F_V(V, \vec{E}) = V$ while $F_E(V, \vec{E}) = \vec{E}$. We can similarly define two functors F_V and F_E for $\mathfrak{Graph}$, but this time F_E has value in $\mathbf{Z}/2\mathbf{Z}$-$\mathfrak{Set}$ (the category of sets with an $\mathbf{Z}/2\mathbf{Z}$-action and equivariant maps as morphisms). If we restrict ourself to the full subcategory of $\mathfrak{Graph}$ consisting of graphs without degenerate loops, then F_E has values in free-$\mathbf{Z}/2\mathbf{Z}$-sets. See [115] for more on categorial aspects of graphs with no degenerate loops (most of the results and constructions admit easy analogues in $\mathfrak{Graph}$ and in $\mathfrak{DiGraph}$).

Scholion 2.1.6. The category $\mathfrak{DiGraph}$ is the category of functors from the category $E \rightrightarrows V$ (two object and two non-trivial morphsims) to $\mathfrak{Set}$. In particular it is the category of presheaves on the opposite category: $E \leftleftarrows V$. This turn out $\mathfrak{DiGraph}$ into an elementary topos and guaranty many good properties, as for example the existence of all finite limits and all finite colimits.

Let Γ be a graph or a digraph and let v any vertex of Γ. The *star* of v is the set $\{e \in E \mid \iota(e) = v\}$. Any morphism $\varphi\colon \Gamma_1 \to \Gamma_2$ induces, for any vertex v of Γ_1, a map: $\varphi_v\colon \mathrm{Star}_v \to \mathrm{Star}_{\varphi(v)}$.

Definition 2.1.7. A morphism $\varphi\colon \Gamma_1 \to \Gamma_2$ is a *covering* if all the induced maps φ_v are bijections. In this case, we say that Γ_1 *covers* Γ_2.

It follows immediately from the definition that a covering is onto as soon as Γ_2 is connected.

Every connected graph is covered by a tree; its universal cover. On the other hand, every $2d$-regular graph without degenerate loops covers the $(d, 0)$-rose, while a $2d + 1$-regular graph without degenerate loops covers the $(d, 1)$-rose if and only if it admits a 1-factor (see Subsection 2.4 and Proposition 3.2.8 on page 23).

Graphs with decorations Given a graph or digraph Γ, there is multiple way to "decorate" it. We will be interested in two of them.

A *labeled digraph* $(\vec{\Gamma}, f)$ is a digraph $\vec{\Gamma} = (V, \vec{E})$ with a map $f\colon \vec{E} \to X$ called the *labeling*. A graphical representation of $(\vec{\Gamma}, f)$ is a graphical representations of $\vec{\Gamma}$ where the label $f(e)$ is written over the arrow representing e. A *labeled graph* (Γ, f) is a graph $\Gamma = (V, E)$ with a map $f\colon \vec{E} \to X$, where X is a set with an involution $x \mapsto x^{-1}$ (i.e. a $\mathbf{Z}/2\mathbf{Z}$-set) such that $f(\bar{e}) = f(e)^{-1}$. A graphical representation of $(\Gamma = (V, E), f)$ is given by a choice of an orientation $\mathcal{O}$ on Γ. Vertices are represented by small circles, degenerate edge e by dashed loop labeled by $f(e)$ and for each $e \neq \bar{e} \in \mathcal{O}$ an arrow labeled by $f(e)$ from $i(e)$ to $\tau(e)$ if $f(e) \neq f(e)^{-1}$ and a line labeled by $f(e)$ between $i(e)$ and $\tau(e)$ otherwise.

If (Γ_1, f_1) and (Γ_2, f_2) are two graphs (respectively two) digraphs labeled by the same set X, a *X-morphism* (or *strong morphism*) is an equivariant morphism of graph (respectively of digraph), that is a morphism φ such that $f_2 = \varphi \circ f_1$. We hence have two categories $\mathfrak{Graph}_X$

and $\mathfrak{DiGraph}_X$ with obvious forgetful functors to $\mathfrak{Graph}$ and $\mathfrak{DiGraph}$. These functors are faithful. Therefore, any concept that can be expressed in $\mathfrak{Graph}$ (respectively in $\mathfrak{DiGraph}$) by using only morphisms admits an "X-analogue" in $\mathfrak{Graph}_X$ (respectively in $\mathfrak{DiGraph}_X$). For example, we can look at X-coverings and we can ask if a graph is X-transitive.

A *rooted digraph* $(\vec{\Gamma}, v)$, is a digraph with a distinguished vertex v, called the *root*. A *rooted graph* is defined similarly. The definitions of *rooted labeled digraph* and *rooted labeled graph* are obvious. For a rooted (di)graph (Γ, v) we will denote by $(\Gamma, v)^0$ the connected component containing v. Note that even for digraphs, we look at the connected component, and not at the srongly connected component. A *rooted morphism* of graphs or digraphs (labeled or not) is simply a morphism (in the appropriated category) that preserves the root.

2.2 Limits of graphs

We denote $\mathcal{G}_\bullet$ (respectively $\vec{\mathcal{G}}_\bullet$) the set of connected marked (di)graphs, up to rooted isomorphisms of (di)graphs. Observe that graphs in $\vec{\mathcal{G}}_\bullet$ are connected, but not necessarily strongly connected.

The set $\mathcal{G}_\bullet$ (respectively $\vec{\mathcal{G}}_\bullet$) can be topologized by considering for example the following distance: $d\big((\Gamma, v), (\Delta, w)\big) = \frac{1}{1+r}$, where r is the biggest integer such that the ball of radius r centered at v in Γ and the ball of the same radius centered at w in Δ are isomorphic as marked graphs (respectively digraphs). If the two graphs are isomorphic as rooted graphs, then the distance is defined to be 0. For a rooted digraph $(\vec{\Gamma}, v)$, the ball of radius r centered at v is the subdigraph of $\vec{\Gamma}$ such that its underlying graph is the ball of radius r centered at v in $\underline{\vec{\Gamma}}$. For any integer d, the subspaces $\mathcal{G}_{\bullet,d}$ of $\mathcal{G}_\bullet$ and $\vec{\mathcal{G}}_{\bullet,d}$ of $\vec{\mathcal{G}}_\bullet$ consisting of graphs with both maximal in-degree and out-degree bounded by d are compact.

If $(\vec{\Gamma}, v)$ is an element of $\vec{\mathcal{G}}_\bullet$, then its underlying rooted graph $\underline{\vec{\Gamma}}$ is an element of $\mathcal{G}_\bullet$. It is easy to check that, if $(\vec{\Gamma}, v)$ and $(\vec{\Delta}, w)$ are two rooted digraphs, then

$$d_{\vec{\mathcal{G}}_\bullet}\big((\vec{\Gamma}, v), (\vec{\Delta}, w)\big) \geq d_{\mathcal{G}_\bullet}\big((\underline{\vec{\Gamma}}, v), (\underline{\vec{\Delta}}, w)\big).$$

It immediately implies the following proposition.

Proposition 2.2.1. *If a sequence of rooted digraphs $(\vec{\Gamma}_n, v_n)$ converges to $(\vec{\Gamma}, v)$, then the sequence $(\underline{\vec{\Gamma}_n}, v_n)$ converges to $(\underline{\vec{\Gamma}}, v)$.*

Since $\mathcal{G}_\bullet$ and $\vec{\mathcal{G}}_\bullet$ are metric spaces, we have a notion of limit for rooted graphs. It is of interest to have a notion of limits for non-rooted (di)graphs. For a sequence of finite graphs, this is done via the Benjamini-Schramm limit.

Since $\mathcal{G}_{\bullet,d}$ is a compact metric space, the space of Borel probability measures on it is compact in the weak topology[1] — this follows from Riesz Representation Theorem. There is a natural way to attach a Borel probability measure to a finite graph Γ: by choosing the root uniformly at random. More formally, the measure associated to Γ is $\frac{1}{|V|}\sum_{v \in V} \delta_{(\Gamma, v)^0}$, where δ is a Dirac measure.

Definition 2.2.2 ([20]). Let Γ_n be a sequence of finite graphs and let λ_{Γ_n} be the Borel probability measures associated. We say that Γ_n is *Benjamini-Schramm convergent* with limit λ if λ_{Γ_n} converges to λ in the weak topology in the space of Borel probability measures on $\mathcal{G}_\bullet$.

In the particular case where λ is a Dirac measure concentrated on one transitive graph Γ, we say that Γ_n converges to Γ in the sense of Benjamini-Schramm.

[1] The *weak topology* on the space of measures is in fact the weak* topology from the point of view of functional analysis.

It is also possible to describe Benjamini-Schramm convergence by convergence in neighborhood sampling statistics, see [20]. That is, a sequence $(\lambda_n)_n$ converges to λ, if and only if for all finite graph α and all $r \in \mathbf{N}$ we have

$$\mathbf{P}(\mathrm{Ball}_{\lambda_n}(r) \simeq \alpha) \xrightarrow{n\to\infty} \mathbf{P}(\mathrm{Ball}_{\lambda}(r) \simeq \alpha)$$

The same definitions hold in $\vec{\mathcal{G}}_\bullet$.

2.3 Tensor product of graphs

Given two (di)graphs Γ and Δ, there are multiple ways to define a "product": cartesian product, tensor product and many others. The one which will interests us is the tensor product. This is exactly the categorical product in the category of (di)graphs (or labeled (di)graphs, rooted (di)graphs, ...) and interacts nicely with respect to adjacency matrices.

Definition 2.3.1. Let $\Gamma = (V, E)$ and $\Delta = (W, F)$ be two (di)graphs. Their *tensor product* is the (di)graph $\Gamma \otimes \Delta$, with vertex set $V \times W$, where there is an edge (e, f) from (v_1, w_1) to (v_2, w_2) if e is an edge from v_1 to v_2 in $\vec{\Gamma}$ and f is an edge from w_1 to w_2 in $\vec{\Delta}$. If $\Gamma = (V, E)$ and $\Delta = (W, F)$ are graphs, then the inverse of the edge (e, f) is the edge $(\bar{e}, \bar{f})$.

If $\Gamma = (V, E)$ has labeling $l\colon E \to X$ and $\Delta = (W, F)$ has labeling $l'\colon F \to Y$, the tensor product has labeling $l \times l'\colon E \times F \to X \times Y$.

If Γ is rooted at v and Δ rooted at w, then $\Gamma \otimes \Delta$ is rooted at (v, w).

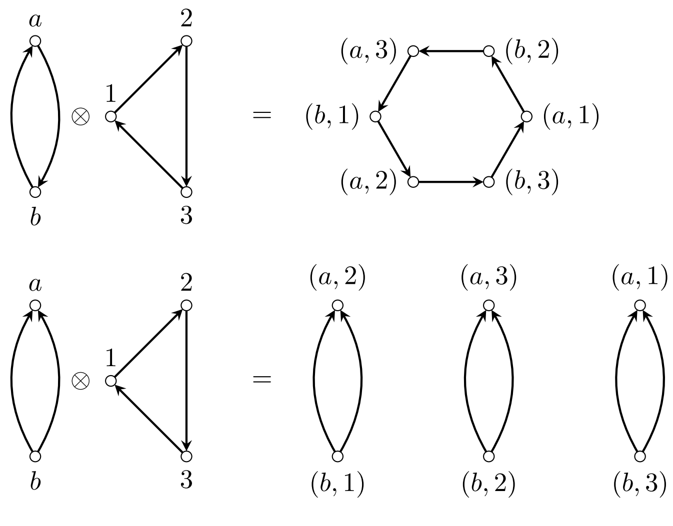


Figure 2.1: Two examples of tensor products of digraphs.

On the level of adjacency matrices, the tensor product correspond to the Kronecker product, where the Kronecker product of two matrices $A \otimes B$ is given by the block matrix with blocks $a_{ij}B$.

It directly follows from the categorial description of the tensor product that it is commutative and associative (up to isomorphisms). On the other hand, since the empty graph $\emptyset$ is the zero object in $\mathfrak{Graph}$ and in $\mathfrak{DiGraph}$, we have $\Delta \otimes \emptyset \simeq \emptyset$ for any (di)graph Δ.

Since the tensor product is the categorical product, for any pair of morphisms $\varphi\colon \vec{\Gamma} \to \vec{\Delta}$ and $\varphi'\colon \vec{\Gamma}' \to \vec{\Delta}'$, $\varphi \otimes \varphi'$ is a morphism from $\vec{\Gamma} \otimes \vec{\Gamma}'$ to $\vec{\Delta} \otimes \vec{\Delta}'$. Moreover, $\varphi \otimes \varphi'$ is an isomorphism if and only if φ and φ' are isomorphisms.

We will now prove that the tensor product behaves well with respect to coverings and line digraphs.

Lemma 2.3.2. *For $i = 1, 2$, let $\varphi_i \colon \Gamma_i \to \Delta_i$ be a covering. Then, $\varphi_1 \otimes \varphi_2 \colon \Gamma_1 \otimes \Gamma_2 \to \Delta_1 \otimes \Delta_2$ is a covering. The same result is true for digraphs.*

Proof. Let (v_1, v_2) be any vertex in $\Gamma_1 \otimes \Gamma_2$. Since the φ_i's are coverings, the induced morphisms $(\varphi_i)_{v_i} \colon \mathrm{Star}_{v_i} \to \mathrm{Star}_{\varphi_i(v_i)}$ are bijections. On the other hand, by definition of the tensor product, there is a natural bijection between $\mathrm{Star}_{v_1} \times \mathrm{Star}_{v_2}$ and $\mathrm{Star}_{(v_1,v_2)}$. Under this bijection, the map $(\varphi_1 \otimes \varphi_2)_{(v_1,v_2)}$ corresponds to $(\varphi_1)_{v_1} \times (\varphi_2)_{v_2}$ and is therefore a bijection. ☐

Definition 2.3.3. Let $\vec{\Gamma} = (V, \vec{E})$ be a digraph. The *line digraph* of $\vec{\Gamma}$ is the digraph $L(\vec{\Gamma})$ with vertex set $\vec{E}$ (the edge set of $\vec{\Gamma}$) and with an edge from e to f if we have $\tau(e) = \iota(f)$ (that is f "directly follows" e) in $\vec{\Gamma}$.

Lemma 2.3.4. *For $i = 1, 2$, let $\vec{\Gamma}_i = (V_i, \vec{E}_i)$ be a digraph. Then the graphs $L(\vec{\Gamma}_1) \otimes L(\vec{\Gamma}_2)$ and $L(\vec{\Gamma}_1 \otimes \vec{\Gamma}_2)$ are isomorphic.*

Proof. Vertices of $L(\vec{\Gamma}_1 \otimes \vec{\Gamma}_2)$ are in 1-to-1 correspondence with edges of $\vec{\Gamma}_1 \otimes \vec{\Gamma}_2$ and therefore in 1-to-1 correspondence with pairs of edges in $\vec{E}_1 \times \vec{E}_2$. On the other hand, vertices of $L(\vec{\Gamma}_1) \otimes L(\vec{\Gamma}_2)$ are in 1-to-1 correspondence with $\{(v_1, v_2) \mid v_i \text{ a vertex in } L(\vec{\Gamma}_i)\}$. Therefore, vertices of $L(\vec{\Gamma}_1) \otimes L(\vec{\Gamma}_2)$ are also in 1-to-1 correspondence with pairs of edges in $\vec{E}_1 \times \vec{E}_2$.

Now, in $L(\vec{\Gamma}_1 \otimes \vec{\Gamma}_2)$ there is an edge from (e_1, e_2) to (f_1, f_2) if and only if, for $i = 1, 2$, f_i directly follows e_i in Γ_i. The same relation holds in $L(\vec{\Gamma}_1) \otimes L(\vec{\Gamma}_2)$, which proves the isomorphism. ☐

Tensor product and convergence

Recall that for a rooted labeled digraph $(\vec{\Gamma}, v)$, we denote by $(\vec{\Gamma}, v)^0$ the connected component of $\vec{\Gamma}$ containing the root.

Since we are interested in connected components of digraphs, we have to look at paths in the corresponding underlying graph $\underline{\vec{\Gamma}}$. For such paths, it may be useful to describe how far they are to be actually in $\vec{\Gamma}$.

Definition 2.3.5. Let Γ be a graph, $p = (e_1, \dots, e_n)$ a path of length n in Γ and $\mathcal{O}$ an orientation on Γ. The *signature $\sigma(p)$ of p with respect to $\mathcal{O}$* is an ordered sequence of ± 1 of length n, where there is a 1 in the position i if and only if e_i belongs to $\mathcal{O}$ and a -1 otherwise.

The *derangement of p with respect to $\mathcal{O}$*, $\operatorname{der}(p)$, is the sum of the ± 1 in the signature of p. The derangement of a path of length 0 is 0. It follows from the definition that $\operatorname{der}(\bar{p}) = -\operatorname{der}(p)$ and that $\sigma(\bar{p})$ is the sequence $-\sigma(p)$ read backward.

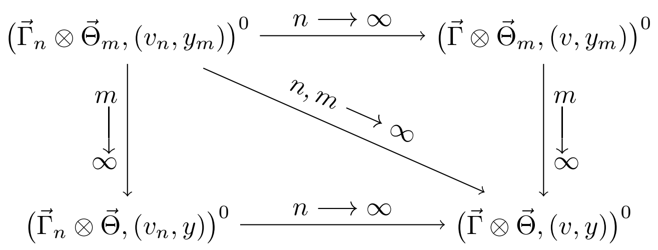
The *derangement of Γ with respect to $\mathcal{O}$* is

$$\operatorname{der}(\Gamma) \coloneqq \min\{|\operatorname{der}(p)| \mid p \text{ is a closed path in } \Gamma \text{ and } \operatorname{der}(p) \neq 0\},$$

where this minimum is defined to be 0 if there is no closed path in Γ with non-zero derangement.

If $\vec{\Gamma}$ is a digraph a p a path in $\underline{\vec{\Gamma}}$, we take its signature and derangement with respect to the orientation coming from Γ.

Theorem 2.3.6. *If* $(\vec{\Gamma}_n, v_n)_n$ *converges (in* $\vec{\mathcal{G}}_*$*) to* $(\vec{\Gamma}, v)$ *and* $(\vec{\Theta}_m, y_m)_m$ *converges to* $(\vec{\Theta}, y)$ *then the following diagram is commutative*



Proof. Take any $\epsilon > 0$. By convergence, there exists n_0 and m_0 such that for every $n \geq n_0$ the graphs $(\vec{\Gamma}_n, v_n)$ and $(\vec{\Gamma}, v)$ are at distance lesser than ϵ and such that for every $m \geq m_0$ the graphs $(\vec{\Theta}_m, v_m)$ and $(\vec{\Theta}, v)$ are too at distance lesser than ϵ.

Let $(\vec{\Delta}_1, v)$, $(\vec{\Delta}_2, w)$, $(\vec{\Pi}_1, x)$ and $(\vec{\Pi}_2, y)$ be four elements of $\vec{\mathcal{G}}_*$. We affirm that the distance between $\big(\vec{\Delta}_1 \otimes \vec{\Pi}_1, (v, x)\big)^0$ and $\big(\vec{\Delta}_2 \otimes \vec{\Pi}_2, (w, y)\big)^0$ is lesser or equal to the maximum of $d\big((\vec{\Delta}_1, v), (\vec{\Delta}_2, w)\big)$ and $d\big((\vec{\Pi}_1, x), (\vec{\Pi}_2, y)\big)$. Lemma 2.3.7 below implies in turn that $\big(\vec{\Gamma}_n \otimes \vec{\Theta}_m, (v_n, y_m)\big)^0$ and $\big(\vec{\Gamma} \otimes \vec{\Theta}, (v, y)\big)^0$ are at distance less than ϵ, which proves the convergence when both n and m grow together.

Now, if we take first the limit on n we can use this result with $\vec{\Theta}_m$ constant to find

$$\lim_{n\to\infty} \Big(\big(\vec{\Gamma}_n \otimes \vec{\Theta}_m, (v_n, y_m)\big)^0\Big) = \big(\vec{\Gamma} \otimes \vec{\Theta}_m, (v, y_m)\big)^0.$$

Taking then the limit on m (with $\vec{\Gamma}$ constant) we have that the upper right triangle is commutative. A similar argument proves the commutativity of the downer left triangle. □

Note that Theorem 2.3.6 holds also for rooted graphs as well as for labeled rooted (di)graphs with strong morphisms.

We will now prove the technical result used in the proof of Theorem 2.3.6.

Lemma 2.3.7. *Let* $\vec{\Gamma}$ *and* $\vec{\Delta}$ *be two digraphs and* p *be a path in* $\underline{\vec{\Gamma} \otimes \vec{\Delta}}$ *from* (x, v) *to* (y, w). *Then there exists paths* q *in* $\underline{\vec{\Gamma}}$ *from* x *to* y *and* r *in* $\underline{\vec{\Delta}}$ *from* v *to* w *with same signature as* p.

More precisely, given a non-negative integer n *and a sequence* σ *of* ± 1 *of length* n*, there is a bijection between the set of paths* p *from* (x, v) *to* (y, w) *in* $\underline{\vec{\Gamma} \otimes \vec{\Delta}}$ *of signature* σ *and the set of couples* (q, r) *where* q *is a path in* $\underline{\vec{\Gamma}}$ *from* x *to* y *and* r *a path in* $\underline{\vec{\Delta}}$ *from* v *to* w*, both of signature* σ.

Proof. It is obvious that the second statement implies the first one. By definition of the tensor product, we have a function φ from the set of paths from (x, v) to (y, w) to the set of couples (q, r) where q is a path in $\underline{\vec{\Gamma}}$ from x to y and r a path in $\underline{\vec{\Delta}}$ from v to w. Indeed, φ is the product of the left projection and the right projection. This function naturally preserves the signature and is injective. Now, if $q = (e_1, \ldots, e_n)$ and $r = (e'_1, \ldots, e'_n)$ have the same signature σ then either e_1 belongs to $\vec{\Gamma}$ and e'_1 belongs to $\vec{\Delta}$, in which case we have an edge (e_1, e'_1) in $\vec{\Gamma} \otimes \vec{\Delta}$, or $\bar{e}_1$ belongs to $\vec{\Gamma}$ and $\bar{e}'_1$ belongs to $\vec{\Delta}$, in which case we have an edge $(\bar{e}_1, \bar{e}'_1)$ in $\vec{\Gamma} \otimes \vec{\Delta}$. By induction, it is possible to construct a path p in $\underline{\vec{\Gamma} \otimes \vec{\Delta}}$ from (x, v) to (y, w) with signature σ. □

2.4 Schreier graphs

Some of the most important examples of graphs come from algebra as Cayley graphs. Cayley graphs are rigid in the sense that a locally finite graph Γ is isomorphic to a Cayley graph of a finitely generated group G if and only there exists a free and transitive action of G on Γ. Moreover, given a locally finite graph Γ with a free and transitive action of a finitely generated group G, Sabidussi showed in [110] an explicit way to put labels on edges of Γ so as to make it a Cayley graph of G.

There are two natural generalizations of Cayley graphs of finitely generated groups. The first one is geometric and consists to look at all locally finite vertex-transitive graphs. The second one is algebraic and consists of Schreier graphs of locally finite groups.

In 1977, Gross showed [62] that every finite regular graph of degree $2d$ without degenerate loop is a Schreier graph over F_d, the free group of rank d. On the other hand, Godsil and Royl showed in [45] that every finite transitive graph without degenerate loop admits a 1-factor (a perfect matching). This fact can be used to show that every finite transitive graph without degenerate loop of degree $2d+1$ is a Schreier graph over $F_d * (\mathbf{Z}/2\mathbf{Z})$. We are going to extend these results to all $2d$ regular graphs and all transitive graphs of degree $2d+1$. In particular, we will allow our graph to be infinite and to have degenerate loops. After that we will give an algebraic characterization (in term of the generating set and of the subgroup) of transitivity of a Schreier graph.

Definitions

Usually, Cayley and Schreier graphs of a group G are defined with respect to a generating set. For a lot of applications this is sufficient, but it is sometimes useful to allow to count the same generator more than once. One can imagine to replace the generating set by a generating multiset X of G which corresponds to see G as a quotient of $F_{|X|}$, the free group on X. This solution is not totally satisfactory since edges labeled by element of order 2 will play a special role. This motivates the following definition.

Definition 2.4.1. A *generating system* X of a group G is a set $X = Z \sqcup Y$ such that there exists a surjective homomorphism $\pi\colon F_Z * (\mathbf{Z}/2\mathbf{Z})^{*Y} \to G$, where F_Z is the free group on Z and $(\mathbf{Z}/2\mathbf{Z})^{*Y}$ is a free product of $|Y|$ copy of $\mathbf{Z}/2\mathbf{Z}$.

Definition 2.4.2. The (right) *Cayley digraph* of G with respect to a generating system X is the labeled digraph $\overrightarrow{\mathrm{Cay}}(G,X)$ with vertex set G, and for every $x \in X$, an edge from g to h labeled by x if and only if $h = g\pi(x)$.

The (right) *Cayley graph* is the labeled graph $\mathrm{Cay}(G, X^{\pm})$ with vertex set G, and for every $x \in X^{\pm} \coloneqq \{x \mid x \in X \text{ or } x^{-1} \in X\}$, an edge from g to h labeled by x if and only if $h = g\pi(x)$.

Definition 2.4.3. The (right) *Schreier digraph* of G with respect to a generating system X and a subgroup H is the labeled digraph $\overrightarrow{\mathrm{Sch}}(G,H,X)$ with vertex set the set of right cosets Hg, and for every x with $x \in X$, an edge from Hg to Hh labeled by x if and only if $Hh = Hgx$.

The (right) *Schreier graph* $\mathrm{Sch}(G,H,X^{\pm})$ has vertex set the set of right cosets Hg, and for every x with $x \in X$ or $x \in X^{-1}$, an edge from Hg to Hh labeled by x if and only if $Hh = Hgx$.

If G acts on the right on a set V, we can define the *digraph of the action with respect to the generating set* X as the labeled digraph with vertex set V and an edge from v to w labeled by x for every generator $x \in X$ such that $v.x = w$.

With these definitions, if $X \cap X^{-1}$ is empty, the underlying graph of $\overrightarrow{\mathrm{Cay}}(G, X)$ is exactly $\mathrm{Cay}(G, X^{\pm})$ and similarly for Schreier graphs.

For every vertex v in V, the connected component of the digraph of the action with root v is strongly isomorphic (as rooted labeled digraph) to the Schreier graph $\overrightarrow{\mathrm{Sch}}(G, \mathrm{Stab}_G(v), X)$.

Observe that Cayley and Schreier graphs (respectively digraphs) are labeled rooted graphs (respectively digraphs) with root 1 for Cayley graphs and with root H for Schreier graphs.

The graphical representations of Cayley and Schreier graphs will be as follows. Unoriented edges $\{e, \bar{e}\}$ labeled by elements in Y (elements of order 2) are represented by plain lines, while each unoriented edges $\{e, \bar{e}\}$ such that the label of e is in Z is represented by an arrow representing e. When we want to specify the label, we will either write it on the edges, or use colored edges. See Figure 2.2 on page 17 for concrete examples.

Basic facts about Schreier graphs

There is a well-know algorithm that given a subgroup H of G allows to read generators of H on the Schreier graph. The algorithm is the following. Note v_0 the root of the Schreier graph of H and choose a spanning tree T. For each pair of edges $\{e, \bar{e}\}$ not in T choose one of them, let say e. There is a unique path in T from v_0 to the initial vertex of e and a unique path in T from the final vertex of e to v_0. The set of labels of path $[v_0, \iota(e)]_T e[\tau(e), v_0]_T$ form a generating system for H. Note that this generating system need not to be minimal.

Another well-known fact is that changing the root in the Schreier graph corresponds to conjugating the subgroup. In particular, there is a $1-1$ correspondence between conjugacy class of subgroups of G and Schreier graphs of G, where two graphs are considered to be the same if they are strongly isomorphic.

Lemma 2.4.4. *If H is a normal subgroup of G, then* $\mathrm{Sch}(G, H, X^{\pm}) \simeq \mathrm{Cay}(H\backslash G, X^{\pm})$ *and* $\overrightarrow{\mathrm{Sch}}(G, H, X) \simeq \overrightarrow{\mathrm{Cay}}(H\backslash G, X)$ *as labeled rooted (di)graphs, where $H\backslash G$ is the right quotient of G by H and $H\backslash X$ is the image of the multiset X in this quotient.*

This result justifies the particular definition of a Cayley graph that we use (allowing loops and multiples edges).

Remark 2.4.5. While the lemma above is useful in practice, we want to draw attention to the fact that this point of view leads to some unusual description of Cayley graphs. See Figure 2.2 for an example of two Cayley graphs of **Z**2**Z**.

Figure 2.2: Two Cayley graphs of $\mathbf{Z}/2\mathbf{Z}$. On the left with the usual generating system $\emptyset \sqcup \{a\}$ (1 generator of order 2), on the right with generating system $\{x\} \sqcup \emptyset$ (when we see $\mathbf{Z}/2\mathbf{Z}$ as a quotient of $\mathbf{Z}$).

All Cayley and Schreier graphs are connected by definition.

It is well known that a graph Γ is a Cayley graph of a group G if and only if there exists a simply transitive action (that is: a free and transitive action) of G on Γ. Moreover, given a graph Γ with a simply transitive action of a group G, Sabidussi shows in [110] an explicit way to put labels on edges of Γ so as to make it a Cayley graph of G. Namely, choose any vertex v_0 as the root and for any neighbor w_i of v_0, label the edge from v_0 to w_i by the unique element x_i of G that sends v_0 on w_i. Then, use x_i to label the remaining edges. For example, the edge from w_i to some vertex u is labeled by x_j if and only if $x_j x_i$ sends v_0 to u. It is then easy to check that the action of G is (in fact) also X-transitive.

The above characterization of Schreier graphs implies that Γ is a Schreier graph of some group if and only if it is a Schreier graph of a free product of the form

$$(\star) \qquad \mathcal{F}_X \coloneqq \mathcal{F}_{Z\sqcup Y} = (\mathop{*}_{|Z|} \mathbf{Z}) * (\mathop{*}_{|Y|} \mathbf{Z}/2\mathbf{Z}) = \langle y_i \in Y,\ z_j \in Z \mid y_i^2 \rangle,$$

with generating system $X = Z \sqcup Y$. Observe that the generating system X of $\mathcal{F}_X$ is an actual subset of $\mathcal{F}_X$. Groups of the form $\mathcal{F}_X$ are exactly the groups that admit a Cayley graph isomorphic to a tree.

Scholion 2.4.6. Groups $\mathcal{F}_X$ look similar to free groups. The reason is that they come from the same categorial construction. Indeed, free groups are free $\mathfrak{Set}$-objects in the category of groups, $\mathfrak{Gr}$, for the usual forgetful functor $\mathfrak{Gr} \to \mathfrak{Set}$. But $\mathfrak{Gr}$ is also a category over $\mathfrak{Set}_2$, where objects in $\mathfrak{Set}_2$ are pairs (Z, Y) of sets and morphisms are functions $f\colon Z \sqcup Y \to Z' \sqcup Y'$ such that $f(Y) \subseteq Y'$. Indeed, we have a faithful functor $S\colon \mathfrak{Gr} \to \mathfrak{Set}_2$ that sends G to $(\{g \mid g^2 \neq 2\}, \{g \mid g^2 = 2\})$. Then, groups $\mathcal{F}_X$ are free $\mathfrak{Set}_2$-objects in $\mathfrak{Gr}$. More precisely, the functor $F\colon \mathfrak{Set}_2 \to \mathfrak{Gr}, (Y, Z) \mapsto \mathcal{F}_{Z\sqcup Y}$ is a left adjoint to the functor S.

Hence, groups of the form $\mathcal{F}_X$ arise naturally when we want to distinguish elements of order 2.

In fact, a graph is a Schreier graph if and only if it admits a decomposition into disjoint 1 and 2-factors, where an *n-factor* of a graph Γ is a subgraph Δ of Γ such that every vertex of Γ has degree n in Δ. Here, the 1-factors correspond to subgraphs consisting of edges labeled by a generator of order 2 and the 2-factors to subgraphs with edges labeled by a generator of infinite order in the group $\mathcal{F}_X$. Observe that 1-factor are often called *perfect matching* in the literature.

This fact had been used to show that every finite graph of even degree without degenerate loops is a Schreier graph over a free group [62]. On the other side, Godsil and Royl showed that every finite transitive graph admits a matching (a regular subgraph of degree 1) that is maximal (and thus misses at most one vertex), see [45]. In particular, every finite transitive graph of odd degree admits a 1-factor and is therefore a Schreier graph. This implies that all finite transitive graphs are Schreier graphs, while all finite Cayley graphs are transitive. In Chapter 3 we will, among other things, extends these results to infinite graphs and provide a characterization of Schreier graphs that are transitive.

Chapter **3**

Schreier graphs and transitivity

Itai Benjamini asked the following question. Is there a Cayley graph of an infinite finitely generated group other than $\mathbf{Z}$ that does not cover any infinite transitive graph other than itself? If we ask the same question with replacing "transitive graph" by "Cayley graph" then the answer is yes — it is enough to take any Cayley graph of a simple group (or more generally of a just infinite group). For the general case, any transitive graph is a Schreier graph of a free group, but the converse is not true. In this chapter we will prove that every transitive graph is a Schreier graph and give a characterization of transitivity for Schreier graphs of free group and then apply it to the question of Benjamini.

We first extend some results about finite graphs to all locally finite graphs. Namely, the facts that $2d$ regular graphs and transitive graphs are isomorphic to Schreier graphs. Then we give a characterization of isomorphisms between Schreier graphs in terms of the groups, subgroups and generating systems. In the case of regular graphs of even degree, this characterization may be formulated in purely graph theoretic terms, without prior realization of graphs as Schreier graphs, and thought of as a rigidity result "a la Mostow". As a corollary, we have a characterization of vertex-transitive Schreier graphs (by automorphisms that may not preserve the labeling) in terms of the subgroup H. Such subgroups will be called length-transitive. They generalize the notion of normal subgroups. We investigate the comportment of length-transitive subgroup and in particular prove that the intersection of two such subgroups is still length-transitive. We also give a new characterization of normal subgroups in term of Schreier graphs. This leads to a strengthening of the notion of simple group. We prove that this notion is not equivalent to simplicity, by showing that for odd $n \geq 5$ alternating groups A_n are not strongly simple in this sense. We also exhibit non-trivial examples of strongly simple groups, namely Tarski groups. These infinite strongly simple groups allow us to give a partial answer to the question of Benjamini: Tarski monsters can not X-cover an infinite transitive graph (distinct from itself). The result is only about X-coverings and not about all coverings, but de la Salle and Tessera showed that in the context of finitely presented Tarski monsters T_p, if a graph Δ has balls of radius big enough X-isomorphic to the balls in the Cayley graph Γ of T_p, then Γ X-covers Δ.

This chapter is organized as follows. In Section 3.2, we prove that locally finite transitive graphs and locally finite $2d$-regular graphs are isomorphic to Schreier graphs. In Section 3.3, we prove Theorem 3.3.4 on isomorphisms between Schreier graphs. One direction is straightforward, but for the other direction we have to prove that the isomorphism between the subgroups can be extended to a bijection of the whole groups with good properties. This is showed in Lemmas 3.3.8 and 3.3.9 and in Proposition 3.3.10. Observe that in Theorem 3.3.4 we do not ask that the Schreier graphs are over the same group. We then give a reformulation (Theorem 3.3.5) of Theorem 3.3.4 to make the relation with Mostow's rigidity theorem more apparent. In the next section, we investigate coverings and label-preserving coverings (also called X-coverings) of Schreier graphs. We also prove small extension of the fact that if $N \leq A$ is a normal subgroup of finite index, then the Cayley graphs of A and A/N are quasi-isometric. In Section 3.5, we define a stronger notion of simplicity for groups and prove that this definition is not equivalent to simplicity. We use this to show in Theorem 3.5.10 that for any proper subgroup of a Tarski monster, the orbits under the automorphism group in the corresponding Schreier graph are finite. Therefore, Tarski monsters are potential counterexamples to the question of Benjamini. Finally, Section 3.6 contains all the open questions appearing in this chapter as well as ideas for further research.

Definitions and basic facts about Schreier graphs can be found in Section 2.4.

Apart for Sections 3.1, 3.2 and 3.6 and Proposition 3.5.1, material of this chapter was published in [83].

3.1 Preliminaries

All Cayley and Schreier graphs are connected by definition. Thus, from now on and unless otherwise specified, we will assume all graphs in this chapter to be connected.

Recall from Section 2.4, page 18, that a graph Γ is a Schreier graph of some group if and only if it is a Schreier graph of a free product of the form

$$(\star) \qquad \mathcal{F}_X \coloneqq \mathcal{F}_{Z \sqcup Y} = (\mathop{*}_{|Z|} \mathbf{Z}) * (\mathop{*}_{|Y|} \mathbf{Z}/2\mathbf{Z}) = \langle y_i \in Y,\ z_j \in Z \mid y_i^2 \rangle,$$

with generating system $X = Z \sqcup Y$. This is in turn equivalent to the fact that Γ admits a decomposition into disjoint 1 and 2-factors.

From now on, the letter $\mathcal{F}$ will always denote a group of the form $(\star)$. In such a group, the only cancellations that can occur are of the form ww^{-1} where w is one of the generators.

For such a group $\mathcal{F}$, we have the easy but useful following lemma (see proposition 1.3 in [50] for the special case of free groups).

Lemma 3.1.1. *Let H be any subgroup of $\mathcal{F} = \mathcal{F}_X$ and $\Gamma \coloneqq \mathrm{Sch}(\mathcal{F}, H, X^{\pm})$ be the corresponding Schreier graph. Then, for every vertex v in Γ, there is a bijection between reduced paths starting at v and elements of $\mathcal{F}$. This bijection restricts to a bijection between closed reduced paths starting at $v = Hg$ and elements of $g^{-1}Hg$.*

Proof. Observe that the presentation of $\mathcal{F}$ is chosen such that a word w is reduced if and only if it doesn't contain a subword of the form xx^{-1} or of the form $x^{-1}x$, where x is any generator, and a path in Γ is reduced if and only if it does not contains a subpath of the form $e\bar{e}$ or $\bar{e}e$. Moreover, at each vertex of Γ, for each generator $x \in X \cup X^{-1}$, there is exactly one outgoing edge labeled by x. The bijection therefore consists of reading the label of the path. □

In order to pass from the finite world to the infinite one, we will use on two occasions the compactness theorem from logic. See [32] for a detailed exposition of its history and link with the axiom of choice and [36] for its use in infinite graph theory.

Theorem 3.1.2 (Gödel, 1930)**.** *A set of first-order sentences has a model if and only if every finite subset of it has a model.*

3.2 Transitive graphs as Schreier graphs

In this section, we will prove that every connected locally finite transitive graph is isomorphic to a Schreier graph. For finite graphs, this result goes back to Gross [62] (for even degree) and Godsil and Royl [45] (for the general case). In this subsection and unless staten otherwise, all our graphs will be locally finite. In particular, every loop adds 2 to the degree of a vertex. The local finiteness condition is here because some of the proof use induction on the degree of a vertex.

Regular graphs

Firstly, we recall some classical results on finite regular graphs of even degree. In particular, every such graphs always admit a 2-*factorization*; a decomposition in 2-factors. This implies that every connected 2-regular graph without degenerate loop is isomorphic to a Schreier graph. Then, in Theorem 3.2.5 we extend this result to locally finite infinite graphs. Finally, we discuss in Proposition 3.2.8 the case of regular graphs of odd degree. Observe that the only non-empty connected graph that is 0-regular is the graph with one vertex and no edge. This graph is isomorphic to $\mathrm{Sch}(\{\mathrm{Id}\}, \{\mathrm{Id}\}, \emptyset^{\pm})$. We can therefore restrict our attention to $2d$-regular graphs with $d \geq 1$. In order to sho that every 2-regular graph admits a 2-factorization we will firstly show it for finite graph and then extend it to infinite, locally finite, graphs.

If Γ is a finite graph, we will use the following well-known relation $|E| = \sum_{v\in V} babel(v)$. In particular, if Γ is $2d$ regular, then $|E| = 2d$.

Remark 3.2.1. Recall that for us, in a graph $\Gamma = (V, E)$, the set E consist of oriented edges. Therefore, if there is no degenerate loops, then the number of unoriented edges is $\frac{|E|}{2}$.

We will use the following classical results.

Lemma 3.2.2 (Euler)**.** *Let Γ be a connected graph, without degenerate loop. Then Γ has an Eulerian cycle. That is a cycle that goes trough every unoriented edge $\{e\bar{e}\}$ exactly once.*

Proposition 3.2.3 (Petersen [99])**.** *Every $2d$-regular finite graph without degenerate loop has a 2-factorization.*

We now turn our attention to infinite graphs.

Theorem 3.2.4. *Every $2d$-regular graph without degenerate loop admits a 2-factorization.*

Proof. It is enough to show that every $2d$-regular graph Γ has a 2-factor Δ. Indeed, in this case, $\Gamma - E(\Delta)$ is a $2(d-1)$ regular graph and we can conclude by induction.

If $d = 1$, then Γ is a 2-factorization and there is nothing to do.

If $d \geq 2$, we will use the compactness theorem in the following way. For each unoriented edge $\{e, \bar{e}\}$ of Γ we introduced a logical variable w_e. An assignment of truth value to the w_e corresponds bijectively to the choice of a subgraph of Γ that contains all vertices of Γ. Such a subgraph is given by all unoriented edges $\{e, \bar{e}\}$ such that w_e is true.

For each vertex v of Γ we also introduce a sentence P_v saying that v has degree 2 in the chosen subgraph. Since P_v depends only on a finite number ($2d$) of variables w_e, it is a first order sentence. The existence of a 2-factor in $\Gamma = (V, E)$ is then equivalent to the existence of a model for the set $\{P_v\}_{v\in V}$. By compactness (Theorem 3.1.2), this is in turn equivalent to the existence of a model for every finite subset of $\{P_v\}_{v\in V}$. This last statement is itself equivalent to the fact that for every finite subset W of V, there exists a subgraph Θ of Γ such that every $w \in W$ has degree 2 in Θ.

Let W be a finite subset of V and Θ the subgraph of Γ consisting of all edges such that at least one vertex is in W. Let i be the number of unoriented edges of Θ with both extremities in W and j the number of unoriented edges of Θ with exactly one vertex in W. Then $2i + j = \sum_{w \in W} \deg(w) = 2d|W|$, which implies that j is even. Since j is even, we can replace these j unoriented edges by $j/2$ new unoriented edges between vertices in W to obtain $\tilde{\Theta}$ a $2d$-regular graph. By Proposition 3.2.3, $\tilde{\Theta}$ admits a 2-factor $\tilde{\Delta}$. This induces a subgraph Δ of Θ, by saying that an unoriented edge is in Δ if and only if the corresponding unoriented edge is in $\tilde{\Delta}$. It is clear that the vertex set of Δ contains W and that for each $w \in W$, the degree of w in Δ is 2. □

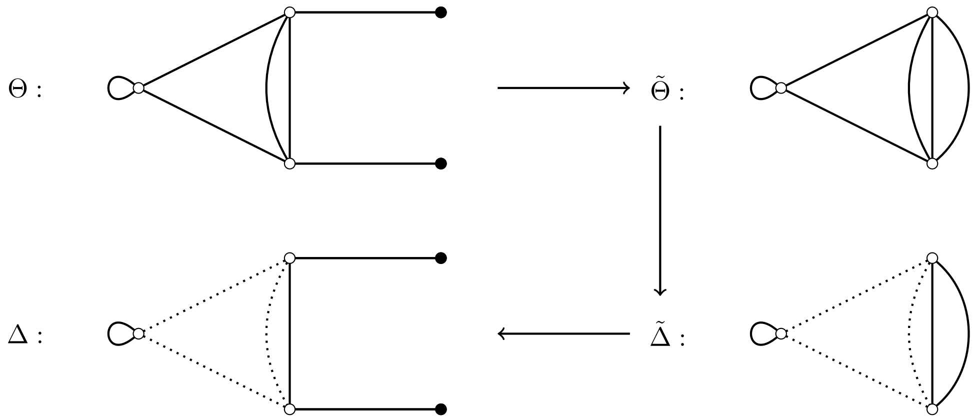


Figure 3.1: From Θ to $\tilde{\Theta}$ and from $\tilde{\Delta}$ to Δ. Vertices outside W are in black.

We can now prove the following generalization of Petersen's theorem.

Theorem 3.2.5. *Every $2d$-regular connected graph without degenerate loop is isomorphic to a Schreier graph of the free group of rank d.*

Proof. The graph Γ has a 2-factorization in d 2-factors. Let Δ_i be one of those factor; this is a 2-regular graph. Then Δ_i is a disjoint union of circles and of biinfinite lines. On each component of Δ_i, we choose an orientation $\mathcal{O}$ and label edges in $\mathcal{O}$ with x_i and edges outside $\mathcal{O}$ with x_i^{-1}.

Therefore, every vertex of Γ has exactly one outgoing and one ingoing edges labeled by x_i. Since Γ is connected, we have $\Gamma = \mathrm{Sch}(F_d, H, \{x_1, \ldots, x_d\}^{\pm})$ where H is the image of the fundamental group of Γ by the covering $\pi\colon \Gamma \to R_d$ onto the rose with d petals □

Remark 3.2.6. An independent proof of this theorem can be found in [27]. It uses König's lemma, which implies the compactness theorem.

Remark 3.2.7. In Theorem 3.2.5, the assumption that Γ has no degenerate loops is necessary. A counter-example is given in Figure 3.2. Indeed, if the graph in Figure 3.2 were a Schreier graph, then all degenerate loops would be labeled by generators of order 2, and would therefore be in one (or more) 1-factor. On the other hand, non-degenerate loops are necessarily in a 2-factor. Therefore, our graph would decompose into two 1-factors and one 2-factor which would contain all non-degenerate loops. This is clearly not possible.

We proved that $2d$ regular, connected regular graphs without degenerate loops are isomorphic to Schreier graphs. What happens if we replace $2d$-regular by $(2d+1)$-regular? Then, it is not true in general that these graphs will be isomorphic to Schreier graphs, but we have the following criterion, which for the finite case belongs to the folklore of graph theory.

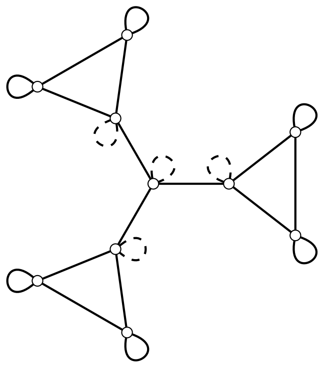


Figure 3.2: A 4-regular graph which is not isomorphic to a Schreier graph; dashed loops represent degenerate loops.

Proposition 3.2.8. *Let* Γ *be a connected graph of degree* $2d+1$ *without degenerate loops. Then* Γ *is isomorphic to a Schreier graph if and only if it admits a* 1*-factor.*

Proof. Since the degree of Γ is odd, if it is isomorphic to a Schreier graph, there is at least one generator a of order 2. Then edges labeled by $a = a^{-1}$ are a 1-factor of Γ.

On the other hand, if Γ has a 1-factor M, then the graph Γ minus the edges of M is $2d$-regular. Therefore, it admits a 2-factorization and Γ is isomorphic to a Schreier graph of $F_d * \mathbf{Z}/2\mathbf{Z}$. □

Using this characterization, we can exhibit an example of a 3-regular graph which is not isomorphic to a Schreier graph, see Figure 3.3.

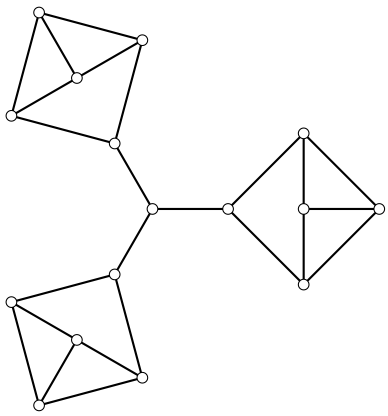


Figure 3.3: A 3-regular graph that does not admit a 1-factor.

Transitive graphs are isomorphic to Schreier graphs

We have seen in the last subsection that some regular graphs are isomorphic to Schreier graphs. The aim of this subsection is to prove that all locally finite connected transitive graphs are isomorphic to Schreier graphs, this is Theorem 3.2.18. In order to prove this theorem, we will first show in Proposition 3.2.17 that every transitive graph of odd degree admits a 1-factor. Our prove of this proposition is based on the one from Godsil and Royle from [45] which deals with the finite case.

It will be convenient to, in a first time, concentrate our attention on *simple graphs*, that is graphs without loops and without multiple edges (for any pair of vertices v and w, there is at most one edge from v to w).

Definition 3.2.9. A graph $\Gamma = (V, E)$ is *transitive* (or *vertex-transitive*) if for every pair of vertices v and w, there exists an automorphism of Γ sending v onto w — that is, if $\mathrm{Aut}(\Gamma)$ acts transitively on V. The graph $\Gamma = (V, E)$ is *almost transitive* if the number of orbits for the action of $\mathrm{Aut}(\Gamma)$ on V is finite.

Similarly, we can define X-transitivity for labeled graphs.

Recall that 1-factor are also called perfect matching. In order to show the existence of a perfect matching, we will need some notions on general matchings.

Definition 3.2.10. A *matching* M in a graph $\Gamma = (V, E)$ is a subgraph of the form (V, E') such that every $v \in V$ has degree at most 1 in M. A vertex $v \in V$ is *covered* by M if it has degree 1 in M. Vertices that are not covered by M are *missed* by it. A *perfect matching* is a matching that covers all vertices. This is what we called a 1-factor.

Since a matching $M = (V, E')$ is a subgraph of $\Gamma = (V, E)$ containing all vertices, we will sometimes identify M with its edges set E'. In particular, we will write $|M| = (V, E')$ instead of $|E|'$ and $M - N$ for the matching $(V, E(M) \setminus E(N)$.

Definition 3.2.11. A matching M of Γ is *strongly maximal* if there exists no matching N such that $|N - M| > |M - N|$. A vertex v of Γ is *critical* if it is covered by all strongly maximal matchings.

An *alternating path* for a matching M is a path in Γ that alternates between edges in M and edges not in M. An *augmenting path* is an alternating path that begins and ends on vertices that are not covered by M.

We now give a characterization of strongly maximal matchings in term of augmenting paths.

Lemma 3.2.12. *Let Γ be a simple connected locally finite graph. A matching M of Γ is strongly maximal if and only if there is no augmenting path in Γ. Moreover, if Γ is finite, then M is strongly maximal if and only if the number of vertices covered by M is maximal.*

Proof. Let P be an augmenting path for M et $N \coloneqq M \triangle P$ be the matching given by the symmetric difference of M and P. Then $N - M = P - M$ and $M - N = M \cap P$. Since P is augmenting we have $|P - M| = |P \cap M| + 1$ which implies that M is not strongly maximal.

On the other hand, suppose that there exists a matching N such that $|N - M| > |M - N|$. Since N and M are 1-regular, in $N \triangle M$ every vertex has degree at most 2. Therefore, connected components of $N \triangle M$ are isolated vertices, cycles or (possibly infinite) paths. Every such component is an alternating path for N and for M. In particular, each cycle is of even length. Since Γ is connected and locally finite, it has at most countably many edges and the relation $|N - M| > |M - N|$ implies that $|M - N|$ is finite. Therefore, paths in $N \triangle M$ are finite. Since all cycles in $N \triangle M$ are of even length and $|N - M| > |M - N|$, there exists a finite path P in $N \triangle M$ with more edges in $N - M$ than in $M - N$. This path is an augmenting path for M.

Finally, look at the case where Γ is finite. If M is not strongly maximal there exists an augmenting path P, then the matching $M \triangle P$ is such that $|M \triangle P| = |M| + 1 > |M|$, which implies that the number of vertices covered by M is not maximal. On the other hand, if the number of vertices covered by M is not maximal, there exists a matching N with $|N| > |M|$. But then we have

$$|N - M| = |N| - |N \cap M| > |M| - |N \cap M| = |M - N|$$

which implies that M is not strongly maximal. □

An immediate corollary of this lemma is that every finite graph admits a maximal matching. On the other hand, Aharoni showed in [3] that every simple connected locally finite graph admits a strongly maximal matching.

Remark 3.2.13. It is possible to partially order matching by inclusion. It is important to observe that for finite graphs, strongly maximal matchings are maximal for this order, but that the converse is not true as showed in Figure 3.4.

Figure 3.4: On left, in orange: a maximal matching of a graph Γ, which is not strongly maximal. On right, in orange: the unique strongly maximal of Γ; this is also a perfect matching.

Corollary 3.2.14. *Let M and N be two strongly maximal matching of Γ. Then every connected component of $M \triangle N$ is finite and is either a single vertex, a cycle of even length, or a path of even length.*

Proof. We already know that connected components are isolated vertices, cycle of even length or finite paths. If there was one path of odd length, then it would be augmenting for M or for N, which is absurd. □

We will show that for every transitive graph, there exists a matching that misses at most one vertex. In order to do that we first prove two preliminary lemmas.

Lemma 3.2.15. *Let u and v be two distinct vertices of Γ such that there is no strongly maximal matching missing both u and v. Let M_u, respectively M_v, be a strongly maximal matching that misses u, respectively v. Then there is a connected component of $M_u \triangle M_v$ which is a path with u and v has initial and final vertex.*

Proof. By hypothesis, u and v both have degree 1 in $M_u \triangle M_v$, therefore u is the initial vertex of a path P_u and v the final vertex of a path P_v. If $P_u = P_v$ there is nothing to prove.

Suppose now that $P_u \neq P_v$. The path P_u is alternating for M_v and $M_v \triangle P_u$ is a matching that misses both u and v. We will show that $M_v \triangle P_u$ is strongly maximal which will contradicts the hypothesis that $P_u \neq P_v$.

Suppose by contradiction that $M_v \triangle P_u$ is not strongly maximal. That is, there exists a matching N such that $|N - (M_v \triangle P_u)| > |(M_v \triangle P_u) - N|$. Figure 3.5 shows the Venn diagram corresponding to P_u, M_v and N. In order to simplify notation, we will write a for the cardinal of A and so on for the other sets appearing in Figure 3.5. The choice of N implies that $g + e > a + c$. We want to show that N contradicts the strong maximality of M_v, that is that $g + f > a + b$.

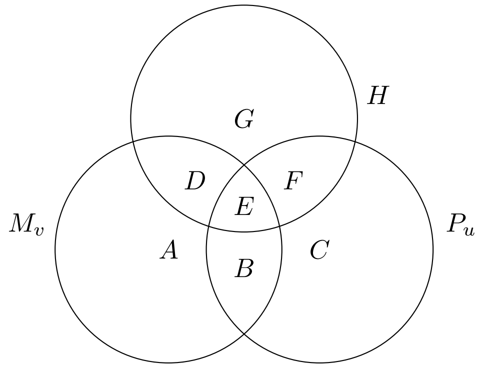


Figure 3.5: Venn diagram for N, M_v et P_u in the proof of Lemma 3.2.15.

Since P_u is finite, we have that b, c, e and f are finite. On the other hand, we know that P_u is of even length, which is equivalent to say that $b + e = c + f$. Therefore, $g + f = g + e + (f - e) > a + c + f - e = a + b$ and we obtain the desired contradiction. □

Lemma 3.2.16. *Let $u \neq v$ be two vertices of Γ and P a path from u to v. If there is no critical vertex in $V(P) \setminus \{u, v\}$, then every strongly maximal matching covers u or v.*

Proof. The proof is done by induction on the length l of P. If $l = 1$, then $P = e$ an edge between u and v. If M is a matching that misses u and v, then $N := M \cup \{e\}$ is a matching such that $|N - M| = 1 > 0 = |M - N|$. In particular, M is not strongly maximal.

If $l \geq 2$, then there exists a vertex x on P which is distinct from u and from v. But then, P induced a path from u to x, of length strictly smaller than l and (except maybe for u) with no critical vertex on it. By induction hypothesis, there is no strongly maximal matching missing both u and x. The same holds for x and v. But the vertex x is not critical. Therefore, there exists a strongly maximal matching M_x missing x. Such an M_x has to cover both u and v. Suppose now by contradiction that there exists a strongly maximal matching N that misses both u and v. In this case, lemma 3.2.15 implies that u and x are the two extremities of a connected component in $N \triangle M_x$, and the same is true for x and v. But this is possible if and only if $u = v$ which is absurd. □

It is obvious that if Γ is transitive and has a critical vertex, then all its vertices are critical and therefore any strongly maximal matching of Γ is a perfect matching. On the other hand, Lemma 3.2.16 implies that if Γ has no critical vertex, then every strongly maximal matching misses at most one vertex. We are going to show that if Γ is a transitive graph of odd degree, then every strongly maximal matching is a perfect matching, that is a 1-factor.

Proposition 3.2.17. *Every simple connected locally finite transitive graph of odd degree admits a 1-factor.*

Proof. Firstly look at finite Γ. We have that $E = (2d + 1)|V|$. But since Γ is simple, it has no degenerate loops and thus $|E|$ is even. In particular, the number of vertices is even. In a simple graph, every matching covers an even number of vertices; therefore it cannot miss only one vertex.

Now, for infinite Γ we will use compactness (Theorem 3.1.2). For such a Γ, there always exists a strongly maximal matching that misses at most one vertex of Γ. Let T be a finite subset of vertices of Γ. By transitivity, there exists a strongly regular matching M that covers all vertices of T. By compactness (we use here the fact that Γ is locally finite), there exists a matching that covers all vertices of Γ. □

We are now ready to prove the principal theorem of this subsection.

Theorem 3.2.18. *Every connected locally finite transitive graph is isomorphic to a Schreier graph.*

Proof. As for Theorem 3.2.5 and Proposition 3.2.8, it is enough to show that Γ admits a decomposition into disjoint 2-factors and 1-factors. Indeed, if such a decomposition exists, with d 2-factors and n 1-factors, then by labeling 2-factors with generator of infinite order and 1-factors by generator of order 2 we have that Γ is isomorphic to a Schreier graph of $F_d * (*_n \mathbf{Z}/2\mathbf{Z})$.

Since Γ is transitive, the number d_1 of non-degenerate loops and the number n_1 of degenerate loops at some vertex v do not depends on v. Therefore they give rises to d_1 2-factors and n_1 1-factors. We can thus suppose that Γ has no loops.

Suppose for a moment that Γ is simple. If Γ is of odd degree, then by Proposition 3.2.17 it has a 1-factor. Hence we can suppose that Γ is of even degree. In this case, by Theorem 3.1.2, it admits a 2-factorization.

Now, if Γ has multiple edges we do an induction on $n \geq 1$, the maximal number of edges between two vertices. If $n = 1$, then Γ is simple and we are done. If $n > 1$, then we can look at $S\Gamma$ the simple graph obtained from Γ be gluing together multiple edges. The graph $S\Gamma$ is simple, connected and transitive. Therefore, it admits a decomposition into 2-factors and 1-factors. Its complement $M\Gamma \coloneqq \Gamma - S\Gamma$ is transitive but not necessarily connected. All connected components have the same degree, and the maximal number of edges between two vertices in it is now $n - 1$. By induction, all connected component of $M\Gamma$ (and therefore $M\Gamma$ itself) admit a decomposition into 2-factors and 1-factors. The union of this decomposition together with the decomposition for $S\Gamma$ gives a decomposition into 2-factors and 1-factors for Γ. □

3.3 A criterion for transitivity

In Theorem 3.2.18 we have proven that locally finite transitive graphs are all (isomorphic to) Schreier graphs. On the other hand, it is easy to exhibit Schreier graphs that are not transitive, see for example Figure 3.7 on page 33. In this section we will provide a characterization of transitivity for Schreier graphs.

As said in Section 2.4 and repeated in Section 3.1, any Schreier graphs is isomorphic to s Schreier graph over

$$(\star) \qquad \mathcal{F}_X \coloneqq \mathcal{F}_{Y \sqcup Z} = (\mathop{*}_{|Y|} \mathbf{Z}) * (\mathop{*}_{|Z|} \mathbf{Z}/2\mathbf{Z}) = \langle y_i \in Y,\ z_j \in Z \mid z_j^2 \rangle,$$

with generating system $X = Y \sqcup Z$. Therefore, we can restrict our attention to groups of this form. In the following, we will use the notation $\mathcal{F} = \mathcal{F}_X$ for such groups.

Let A be any group with generating system X. The *degree* of A (with respect to X) is the degree of any vertex in the graph $\mathrm{Cay}(A, X^{\pm})$.

Observe that the degree of A depends on the choice of the generating system and could be infinite. The degree of A with respect to $X = Y \sqcup Z$, where Z is the subset of generators of order 2, is equal to $2 \cdot |Y| + |Z|$.

Definition 3.3.1. Let $H_1 = \langle X_1 \mid \mathcal{R}_1 \rangle$ and $H_2 = \langle X_2 \mid \mathcal{R}_2 \rangle$ be two arbitrary groups. A morphism $\alpha\colon H_1 \to H_2$ *preserves lengths* (with respect to X_1 and X_2) if for every h in H_1 we have $|\alpha(h)|_{X_2} = |h|_{X_1}$. If α is an isomorphism, we say that H_1 and H_2 are *length-isomorphic*.

Definition 3.3.2. Let $\mathcal{F}$ be as in ($\star$). A subgroup H of $\mathcal{F}$ is *length-transitive* if it is length-isomorphic to all its conjugates. That is, if for every g in $\mathcal{F}$, there exists a group isomorphism $\alpha_g\colon \mathcal{H} \to g^{-1}Hg$ that preserves lengths.

Remark 3.3.3. In this definition, α_g is only defined on H, not on $\mathcal{F}$ itself.

In general, we have $H \cong g^{-1}Hg$, but the conjugation homomorphism does not preserve lengths unless $\mathcal{F} = \mathbf{Z}$ or $\mathcal{F} = \mathbf{Z}/2\mathbf{Z}$.

We are now able to state the main result of this chapter.

Theorem 3.3.4. *Let $\mathcal{F}_1 = \mathcal{F}_{X_1}$ and $\mathcal{F}_2 = \mathcal{F}_{X_2}$ be two groups of the type ($\star$). Suppose that $\Gamma_i \coloneqq \mathrm{Sch}(\mathcal{F}_i, H_i, X_i^{\pm})$ for $i = 1, 2$. Then, there exists a graph isomorphism from Γ_1 to Γ_2 that respects roots (the image of the vertex H_1 is the vertex H_2) if and only if $\mathcal{F}_1$ and $\mathcal{F}_2$ have same degree and H_1 and H_2 are length-isomorphic.*

Moreover, there exists an X-isomorphism from Γ_1 to Γ_2 that respects roots if and only if $\mathcal{F}_1 = \mathcal{F}_2$ and $H_1 = H_2$.

Observe that it is possible that $\mathcal{F}_1$ and $\mathcal{F}_2$ are not isomorphic, for example if $\mathcal{F}_1 = \mathbf{Z} * \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$ and $\mathcal{F}_2 = \mathbf{Z} * \mathbf{Z} = F_2$

It is possible to reformulate this theorem in order to have a rigidity theorem "a la Mostow". Let Γ_1 and Γ_2 be two $2d$-regular graphs without degenerate loop. If they are isomorphic, then their fondamental groups $\pi_1(\Gamma_1)$ and $\pi_1(\Gamma_2)$ are isomorphic as abstract groups, but the converse is not necessarily true. On the other hand, since the graphs are $2d$-regular without degenerate loop, we have two coverings $p_i\colon \Gamma_i \to R_d$, where R_d is the unique graph with one vertex and d loops — see section 3.4 for more on coverings. These two coverings induce two injections $p_{i*}\colon \pi_1(\Gamma_i) \to \pi_1(R_d) = \langle X\rangle$, where loops of R_d are in bijection with elements of X. The situation for odd regular graphs is more complex since the projections p_i may not exist. More precisely, if Γ is a $2d+1$-regular graph without degenerate loop and $R_{d,1}$ denotes the graph with one vertex, d loops and 1 degenerate loop, then there exists a covering $p\colon \Gamma \to R_{d,1}$ if and only if Γ admits a perfect matching, if and only if Γ is isomorphic to a Schreier graph. This gives us the alternative formulation (for regular graphs) of Theorem 3.3.4.

Theorem 3.3.5 (Rigidity theorem for regular graphs)**.** *Let Γ_1 and Γ_2 be two locally finite regular graphs without degenerate loop.*

If Γ_1 and Γ_2 are $2d$-regular, then they are isomorphic as graphs if and only if $p_{1*}\big(\pi_1(\Gamma_1)\big)$ is length-isomorphic to $p_{2*}\big(\pi_1(\Gamma_2)\big)$.

If Γ_1 and Γ_2 are $(2d+1)$-regular and both admit a perfect matching, then they are isomorphic as graphs if and only if $p_{1*}\big(\pi_1(\Gamma_1)\big)$ is length-isomorphic to $p_{2*}\big(\pi_1(\Gamma_2)\big)$.

Moreover, these statements are independent of the choice of the coverings p_1 and p_2.

To prove Theorem 3.3.4, we need to prove two implications. The first one is easy and proven in the next proposition. The second one is a little harder and is the subject of Proposition 3.3.10.

Proposition 3.3.6. *Let $\mathcal{F}_i$, H_i, X_i and Γ_i be as in Theorem 3.3.4, for $i = 1, 2$. Recall that the graph Γ_i is naturally a rooted graph, with root (the vertex) H_i. Suppose that there exists an isomorphism $\beta\colon \Gamma_1 \to \Gamma_2$ such that $\beta(H_1) = H_2$, then H_1 and H_2 are length-isomorphic and $\mathcal{F}_1$ and $\mathcal{F}_2$ have same degree.*

Moreover, if β preserves labeling, then $\mathcal{F}_1 = \mathcal{F}_2$ and $H_1 = H_2$.

Proof. The graphs Γ_1 and Γ_2 being isomorphic, they have same degree. Hence, the groups $\mathcal{F}_1$ and $\mathcal{F}_2$ have same degree too.

Now, pick an element h of H_1. By Lemma 3.1.1, h is represented by a closed path p with base-point H_1. If we apply β we have a closed path $\beta(p)$ with base-point H_2. Define a map from H_1 to $\mathcal{F}_2$ by

$$\alpha(h) := \text{label in } \Gamma_2 \text{ of } \beta(p).$$

Since β induces a bijection between closed paths with base-point H_1 and closed paths with base-point H_2, α is a bijection between H_1 and H_2. Moreover, $\alpha(1) = 1$ and $\alpha(h^{-1}) = \alpha(h)^{-1}$ (the path is read backward). We also have $\alpha(h_1h_2) = \alpha(h_1)\alpha(h_2)$ (the paths are read one after the other). This proves that α is a group isomorphism between H_1 and H_2. The fact that α preserves lengths is trivial.

Now, suppose that β preserves labelings. In this case, we immediately have $\mathcal{F}_1 = \mathcal{F}_2$ and $\alpha = \mathrm{Id}$. □

Example 3.3.7. The Petersen graph is 3-regular and hence can be seen has a Schreier graph: $\Gamma = \mathrm{Sch}(\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}, H, X^{\pm})$, see Figure 3.6. It is a well-known fact that it is transitive, but not a Cayley graph. Now, let us denote by H (resp. M) the group of labels of closed reduced paths based at v_1 (resp. w_1) in Figure 3.6. We have $M = aHa$. The element $xax^{-2}a$ belongs to H but not to M, therefore H and M are not equal and both are not normal. This means that there exists no X-automorphism of Γ sending v_1 to w_1. But there exists an automorphism β that does the job. And therefore there exists an isomorphism $\alpha\colon H \to M$ that preserves lengths. We want to compute $\alpha(xax^{-2}a)$. The automorphism β is given by:

$$\begin{aligned} v_1 &\mapsto w_1 & w_1 &\mapsto v_1 \\ v_2 &\mapsto w_3 & w_2 &\mapsto v_3 \\ v_3 &\mapsto w_5 & w_3 &\mapsto v_5 \\ v_4 &\mapsto w_2 & w_4 &\mapsto v_2 \\ v_5 &\mapsto w_4 & w_5 &\mapsto v_4. \end{aligned}$$

For two adjacent edges a and b in Γ, let us denote the unique edge from a to b by $[ab]$. Then $xax^{-2}a \in H$ corresponds to the path $[v_1v_3][v_3w_3][w_3w_2][w_2w_1][w_1v_1]$. This path is sent by β to the path $[w_1w_5][w_5v_5][v_5v_3][v_3v_1][v_11_1]$, which has label $x^{-1}ax^{-2}a \in M$. Therefore, $\alpha(xax^{-2}a) = x^{-1}ax^{-2}a$.

We are going to prove the converse of Proposition 3.3.6. Namely, that if H_1 and H_2 are length-isomorphic by an isomorphism α, then there exists an isomorphism between their Schreier graphs that preserves roots. For that, we first extend α to a bijection (not a group homomorphism) from $\mathcal{F}_1$ to $\mathcal{F}_2$ and see that it is possible to find such an extension with good properties. Then we will use such an extension to find an isomorphism β from Γ_1 to Γ_2 such that $\beta(H_1) = H_2$ (as vertices).

Lemma 3.3.8. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two groups with the same degree, and H_i be any subgroup of $\mathcal{F}_i$ for $i = 1, 2$. Then, every isomorphism $\alpha\colon H_1 \to H_2$ which preserves lengths can be extended to a bijection $\gamma\colon \mathcal{F}_1 \to \mathcal{F}_2$ such that γ preserves lengths and initial segments. That is: for every f, $g \in \mathcal{F}_1$ such that fg is reduced (i.e. $|fg|_{X_1} = |f|_{X_1} + |g|_{X_1}$), there exists w such that $\gamma(fg) = \gamma(f)w$ with $|w|_{X_2} = |\gamma(g)|_{X_2}$.*

Proof. Clearly, α preserves lengths and initial segments if we restrict it to $f, g \in H_1$. So let $\gamma|_{H_1} := \alpha$. We are now going to look at the set of initial segments of H_1:

$$C := \{f \in \mathcal{F} \mid \exists w \in \mathcal{F} : fw \in H_1 \text{ and } fw \text{ is reduced}\}.$$

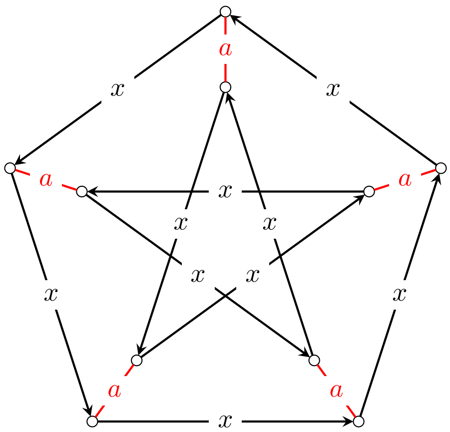


Figure 3.6: The Petersen graph viewed as a Schreier graph on $\langle x, a \mid a^2 \rangle \cong \mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$.

Let $c \in C$ be an initial segment of length n of $h \in H_1$. Define $\gamma(c)$ as the initial segment of length n of $\gamma(h) = \alpha(h)$. We need to check that $\gamma(c)$ is well defined. Firstly, cw is reduced and h and $\gamma(h)$ are of length at least n, so it is possible to choose an initial segment of length n of $\gamma(h)$. Secondly, we need to check that $\gamma(c)$ does not depend on the particular choice of h. Let h_1 and h_2 be elements of H_1 and let c be their maximal common initial segment. Then, if c is of length n:

$$\begin{aligned} |h_1|_1 + |h_2|_1 - 2n &= |h_1^{-1}h_2|_1 && c \text{ is maximal} \\ &= |\alpha(h_1^{-1}h_2)|_2 && \alpha \text{ preserves lengths} \\ &= |\alpha(h_1)^{-1}\alpha(h_2)|_2 && \alpha \text{ is a homomorphism} \\ &= |\alpha(h_1)^{-1}|_2 + |\alpha(h_2)|_2 - 2n' \\ &= |h_1|_1 + |h_2|_1 - 2n', \end{aligned}$$

where n' is the length of the maximal initial segment common to $\alpha(h_1)$ and $\alpha(h_2)$, and $|\cdot|_i$ is short for $|\cdot|_{X_i}$. So $n = n'$, hence $\gamma(c)$ does not depend on the choice of $h = cw$. Moreover, it is trivial that for $c_1, c_2 \in C$, if c_1 is an initial segment of c_2, then $\gamma(c_1)$ is an initial segment of $\gamma(c_2)$. We have thus a bijection between C and $\gamma(C)$ which preserves lengths and initial segments. The groups $\mathcal{F}_1$ and $\mathcal{F}_2$ having same degree, they are length-isomorphic. This induces a bijection which preserves lengths:

$$\gamma' \colon D \coloneqq \mathcal{F}_1 \setminus C \longrightarrow \mathcal{F}_2 \setminus \gamma(C).$$

We now need to define γ on D. The set C being closed under the operation "initial segment", no elements of D are initial segments of elements of C. We can thus define γ on D from the "bottom". Let D_n be the set of elements of D of length n and let n_0 be the smallest integer such that D_{n_0} is non-empty — it is also the smallest integer such that $\gamma'(D_{n_0})$ is non-empty. Let $d \in D_{n_0}$. By minimality of n_0, there exists $c \in C$ and $x \in X_1^{\pm}$ such that $d = cx$ is reduced. Similarly, for $d' \in \gamma'(D_{n_0})$ there exists $c' \in C$ and $y \in X_2^{\pm}$ such that $d' = \gamma'(c')y$ is reduced.

We are now going to prove that the following two sets are in bijection:

$$\begin{aligned} E &\coloneqq \{x \in X_1^{\pm} \mid cx \in D_{n_0}\} \\ F &\coloneqq \{y \in X_2^{\pm} \mid \gamma(c)y \in \gamma'(D_{n_0})\}. \end{aligned}$$

To show that, we are going to prove that their complements $\bar{E} \subset {X_1}^{\pm}$ and $\bar{F} \subset {X_2}^{\pm}$ are in bijection. These complements are exactly

$$\bar{E} = \{x \in X_1^{\pm} \mid cx \in C\}$$
$$\bar{F} = \{y \in X_2^{\pm} \mid \gamma(c)y \in \gamma(C)\}.$$

For $x \in \bar{E}$, we have $\gamma(cx) = \gamma(c)y$ for a unique $y \in X_2^{\pm}$. This defines a map $\theta\colon \bar{E} \to \bar{F}$ by $\theta(x) = y$. This map is injective because $\theta(x) = \theta(x')$ if and only if $\gamma(cx) = \gamma(cx')$ and so if and only if $x = x'$. On the other hand, θ is also surjective. Indeed, if y is in $\bar{F}$, then $\gamma(c)y$ is an element of $\gamma(C)$. Hence, there exists $c' \in C$ such that $\gamma(c') = \gamma(c)y$. But γ preserves initial segments on C, thus c is an initial segment of c', hence $c' = cx$ for some x. This finishes the proof of the existence of a bijection between $\bar{E}$ and $\bar{F}$ and therefore between E and F.

This bijection allows us to define $\gamma(d) = \gamma(cx) \coloneqq \gamma(c)y$ for $d \in D_{n_0}$. We have thus extended γ to $C \cup D_{n_0}$ such that γ is a bijection which preserves lengths and initial segments. Finally, we put $C_0 \coloneqq C$ and conclude by induction on $C_i \coloneqq C_{i-1} \cup D_{n_i}$ and D_{n_i} where n_i is the smallest integer greater than n_{i-1} such that D_{n_i} is non-empty. □

Lemma 3.3.9. *The bijection $\gamma\colon \mathcal{F}_1 \to \mathcal{F}_2$ of the preceding lemma has the following properties:*

1. *For all $fg^{-1} \in H_1$ reduced, $\gamma(fg^{-1}) = \gamma(f)\gamma(g)^{-1}$;*
2. *γ^{-1} preserves lengths and initial segments;*
3. *For all $fg^{-1} \in \gamma(H_1) = H_2$ reduced, $\gamma^{-1}(fg^{-1}) = \gamma^{-1}(f)\gamma^{-1}(g)^{-1}$.*

Proof. If fg^{-1} is reduced and is an element of H_1, the same is true for its inverse gf^{-1}. But then, there exists w and w' in H_2 such that $\gamma(fg^{-1}) = \gamma(f)w$ and $\gamma(gf^{-1}) = \gamma(g)w'$ are reduced. The bijection γ being a group homomorphism on H_1, we have:

$$e = \gamma(e) = \gamma(fg^{-1}gf^{-1}) = \gamma(fg^{-1})\gamma(gf^{-1}) = \gamma(f)w \cdot \gamma(g)w'.$$

The only reductions possible are between w and $\gamma(g)$, which are of the same length. Hence $w = \gamma(g)^{-1}$, which is what we wanted to prove.

For the second part, it is trivial that γ^{-1} preserves lengths. For the initial segments part, let $fg \in H_2$ be reduced. Then $\gamma^{-1}(fg) = f'g'$ is reduced, where $|f'|_1 = |\gamma^{-1}(f)|_1 = |f|_2$ (and analogously for $|g'|_1$). If we apply γ to both sides of the equality, we have $fg = \gamma(f'g') = \gamma(f')w$ reduced, for some $w \in H_2$. We conclude that $f = \gamma(f')$. Therefore, $\gamma^{-1}(fg) = \gamma^{-1}(f)g'$ is reduced.

The last point can be proved in the same way as for γ, using the fact that the restriction of γ^{-1} to $\gamma(H_1)$ is a group homomorphism. □

Proposition 3.3.10. *Let $\mathcal{F}_i$, H_i, X_i and Γ_i be as in Theorem 3.3.4. If $\mathcal{F}_1$ and $\mathcal{F}_2$ have same degree and H_1 and H_2 are length-isomorphic, then there exists a graph isomorphism $\beta\colon \Gamma_1 \to \Gamma_2$ that respects roots (i.e. β maps the vertex H_1 to the vertex H_2).*

Proof. Let $\alpha\colon H_1 \to H_2$ be an isomorphism which preserves lengths. We extend α to $\gamma\colon \mathcal{F}_1 \to \mathcal{F}_2$ as in Lemma 3.3.8. We define β on vertices by $\beta(H_1 f) \coloneqq H_2\gamma(f)$. It is trivial that $\beta(H_1) = H_2$. Moreover, β is well defined and injective. Indeed, $H_1 f = H_1 g$ if and only if fg^{-1} is an element of H_1. In the same way, $H_2\gamma(f) = H_2\gamma(g)$ if and only if $\gamma(f)\gamma(g)^{-1} = \gamma(fg^{-1})$ is an element of $H_2 = \gamma(H_1)$. Hence, $H_1 f = H_1 g$ if and only if $H_2\gamma(f) = H_2\gamma(g)$. Finally, for every vertex $H_2 h$ of Γ_2 (h an element of $\mathcal{F}_2$),

$H_2h = H_2\gamma(\gamma^{-1}(h))$ with $\gamma^{-1}(h) \in \mathcal{F}_1$. Hence, $H_2h = \beta(H_1\gamma^{-1}(h))$ and β is surjective on vertices.

Instead of describing β explicitly on edges, we are going to show that for every pair of vertices H_1f and H_1g, the edges between H_1f and H_1g are in bijection with the edges between $\beta(H_1f)$ and $\beta(H_1g)$. Taking this bijection as a definition of β on edges makes β an isomorphism from Γ_1 to Γ_2. Firstly, suppose that H_1f and H_1g are joined by at least one edge, labeled by $x_0 \in X_1^{\pm}$, such that $H_1fx_0 = H_1g$. Then the set of all edges from H_1f to H_1g is

$$A \coloneqq \{x \in X_1^{\pm} \mid H_1fx = H_1fx_0\} = \{x \in X_1^{\pm} \mid fx_0x^{-1}f^{-1} \in H_1\}.$$

On the other hand, we have $\beta(H_1fx_0) = H_2\gamma(fx_0) = H_2\gamma(f)y_0 = \beta(H_1f)y_0$ for a unique $y_0 \in X_2^{\pm}$. Thus, there is at least one edge from $\beta(H_1f)$ to $\beta(H_1g)$, labeled by y_0. The set of all edges from $\beta(H_1f)$ to $\beta(H_1g)$ is

$$\begin{aligned} B &\coloneqq \{y \in X_2^{\pm} \mid \gamma(f)y_0y^{-1}\gamma(f)^{-1} \in H_2\} \\ &= \{y \in X_2^{\pm} \mid \gamma(fx_0)y^{-1}\gamma(f)^{-1} \in H_2\}. \end{aligned}$$

Take any x in A. By Lemma 3.3.9 we have $\gamma(fx_0x^{-1}f^{-1}) = \gamma(fx_0x^{-1})\gamma(f)^{-1}$ and that there exists a unique $y \in X_2^{\pm}$ such that $\gamma(fx_0x^{-1}) = \gamma(fx_0)y^{-1}$. Moreover, this particular y belongs to B, thus we have a map from A to B and we need to show that this map is bijective. The map is injective. Indeed, if $y = y'$, then $\gamma(fx_0x^{-1}) = \gamma(fx_0x'^{-1})$ and (γ is a bijection) thus $x = x'$. For the surjectivity, we know that for every $y \in X_2^{\pm}$ there is a unique $x \in X_1^{\pm}$ such that $\gamma(fx_0x^{-1}) = \gamma(fx_0)y^{-1}$, so we only need to show that if y belongs to B then x belongs to A. If y is in B, we have that $\gamma(fx_0)y^{-1}\gamma(f)^{-1}$ belongs to H_2. Then by Lemma 3.3.9 we have $\gamma(fx_0)y^{-1}\gamma(f)^{-1} = \gamma(fx_0x^{-1}f^{-1})$. This implies that $fx_0x^{-1}f^{-1}$ belongs to H_1 and finally that x is in A.

We now need to show that if H_1f and H_1g are not connected by any edge, then neither are $\beta(H_1f)$ and $\beta(H_1g)$. But the same argument as before shows that if $\beta(H_1f)$ and $\beta(H_1g)$ are connected by at least one edge, then H_1f and H_1g are connected by an edge.

This concludes the existence of a bijection between edges from $\beta(H_1f)$ to $\beta(H_1g)$ and edges from H_1f to H_1g. Since this bijection preserves initial and final vertices, we can take it as the definition of β on edges. Defining β in such a way makes it an isomorphism from Γ_1 to Γ_2 that sends H_1 on H_2.

Now, if $\mathcal{F}_1 = \mathcal{F}_2$ and $H_1 = H_2$, the existence of an X-automorphism between the two Schreier graphs is trivial. ☐

This finishes the prove of Theorem 3.3.4.

Here are two easy applications of this theorem.

Corollary 3.3.11. *Suppose that $\Gamma \coloneqq \mathrm{Sch}(\mathcal{F}, H, X^{\pm})$. Then the graph Γ is transitive if and only if the subgroup H is length-transitive.*

Proof. The graph Γ is transitive if and only if for all $g \in \mathcal{F}$ there exists an automorphism of the graph that sends the vertex H to the vertex Hg. But the graph Γ rooted in Hg is exactly the graph $\mathrm{Sch}(\mathcal{F}, g^{-1}Hg, X^{\pm})$. ☐

Corollary 3.3.12. *For A a group with generating system X, K a subgroup and Γ the corresponding Schreier graph, the number of X-orbits is $[A : N_A(K)]$, each X-orbit has $[N_A(K) : K]$ elements and each orbit is a union of X-orbits.*

Proof. We have $A = \mathcal{F}/N$ for some normal subgroup of $\mathcal{F}$ and K corresponds to a $N \leq H \leq \mathcal{F}$. Two vertices Hf and Hg are in the same X-orbit if and only if $f^{-1}Hf = g^{-1}Hg$. Therefore, the number of vertices in one orbit is $[N_{\mathcal{F}}(H) : H] = [N_A(K) : K]$ and the number of orbits is $[A : N_A(K)]$. Since an X-automorphism is an automorphism of the graph, the orbits are unions of X-orbits. □

The following lemma comes to simplify the application of the transitivity criterion of Corollary 3.3.11.

Lemma 3.3.13. *Let $\mathcal{F}$, H and X be as before. Then H is length-transitive if and only if for every $x \in X^{\pm} = X \cup X^{-1}$ the group H is length-isomorphic to $x^{-1}Hx$.*

Proof. The proofs of Theorem 3.3.4 and Corollary 3.3.11 show in fact that for every g in $\mathcal{F}$, the existence of a length-preserving isomorphism $\alpha_g\colon H \to g^{-1}Hg$ is equivalent to the existence of an automorphism of $\Gamma = \mathrm{Sch}(\mathcal{F}, H, X^{\pm})$ that sends the vertex H to the vertex Hg. On the other hand, α_x exists for every $x \in X^{\pm}$ if and only if it is possible to send the vertex H to each of its neighbours by an automorphism of Γ. Since Γ is connected, this last condition is equivalent to the transitivity of Γ, and hence to the length-transitivity of H. □

Remark 3.3.14. It is important to notice that, in order to ensure the length-transitivity of H, we need to check the existence of α_x for all generators $x \in X$ and their inverses. For example, the graph in Figure 3.7 is non-transitive, even if α_a and α_x exist. Indeed, $\alpha_{x^{-1}}$ does not exist.

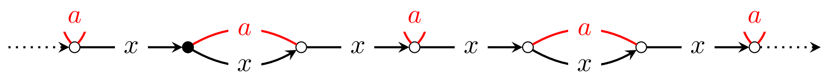


Figure 3.7: A non-transitive Schreier graph over $\langle a, x \mid a^2 \rangle$. The root is marked in black.

Observe that if Γ is X-transitive, then H is normal and therefore Γ is a Cayley graph. However, the converse does not hold. More precisely, let $\Gamma \coloneqq \mathrm{Sch}(A, H, X^{\pm})$ be a Schreier graph that is isomorphic to a Cayley graph. Then it is in general not true that Γ is X-transitive (and that H is normal). All we can say is the following, which characterizes Cayley graphs among Schreier graphs.

Theorem 3.3.15. *Let $\Gamma = \mathrm{Sch}(\mathcal{F}, H, X^{\pm})$ be any Schreier graph over a group $\mathcal{F}$. Then Γ is (isomorphic to) a Cayley graph if and only if there exists a group $\mathcal{F}_1 = \langle X_1 \rangle$ that has the same degree as $\mathcal{F}$ and a normal subgroup $N \trianglelefteq \mathcal{F}_1$ which is length-isomorphic to H.*

Proof. If there exists such $\mathcal{F}_1$ and N, then the graph Γ is isomorphic to $\mathrm{Sch}(\mathcal{F}_1, N, X_1^{\pm})$, which is a Cayley graph.

On the other hand, if Γ is isomorphic to a Cayley graph Γ_1, then Γ_1 is a Schreier graph $\mathrm{Sch}(\mathcal{F}_1, N, X_1^{\pm})$ over some group $\mathcal{F}_1$ which has the same degree as $\mathcal{F}$, and for some normal subgroup N. Moreover, the isomorphism between Γ and Γ_1 implies that H is length-isomorphic to a conjugate of N. □

3.4 Coverings

In this section, we give a criterion for the existence of coverings and of X-coverings of Schreier graphs. We also give some relations between X-coverings and quasi-isometries.

See Definition 2.1.7 on page 11 for the definition of covering.

If Γ_1 is a labeled graph, a morphism $\varphi\colon \Gamma_1 \to \Gamma_2$ is *consistent with the labeling* if for any two edges e and f, the fact that $\varphi(e) = \varphi(f)$ implies that e and f have same label. Every X-covering is consistent with the labeling. Moreover, every covering $\varphi\colon \Gamma_1 \to \Gamma_2$ consistent with labeling induces a labeling on Γ_2 such that φ is an X-morphism for this labeling. On the other hand, if Γ_2 is labeled by X, then every covering $\varphi\colon \Gamma_1 \to \Gamma_2$ induces a labeling on Γ_1 such that φ is an X-covering.

Lemma 3.4.1. *Let $\Gamma_i \coloneqq \mathrm{Sch}(A, H_i, X^{\pm})$ for $i = 1, 2$ be two Schreier graphs over the same group. Then there exists an X-covering from Γ_1 to Γ_2 if and only if H_1 is a subgroup of a conjugate of H_2.*

Proof. We have $A = \mathcal{F}_X/N$. Since the correspondence between subgroups of A and subgroups of $\mathcal{F}_X$ containing N preserves inclusions and conjugations and induces an isomorphism $\mathrm{Sch}(\mathcal{F}_X, M, X^{\pm}) \simeq \mathrm{Sch}(A, M/N, X^{\pm})$, it is sufficient to prove the result for $\mathcal{F}_X$.

Let $\varphi\colon \Gamma_1 \to \Gamma_2$ be an X-covering and let $v_0 \coloneqq \varphi(H_1)$ be the image of the base-vertex of Γ_1. The group H_1 is isomorphic (see Lemma 3.1.1) to the group of closed paths based at the vertex H_1. This group is itself isomorphic to its image under φ, which is a subgroup of the group of closed paths based at the vertex v_0. This last group is isomorphic to gHg^{-1}, where g is the label of the path between v_0 and H_2.

For the converse, let $H_1 \leq H = gH_2g^{-1}$, and $\Gamma \coloneqq \mathrm{Sch}(\mathcal{F}_X, H, X^{\pm})$. It is obvious that Γ and Γ_2 are X-isomorphic; indeed, we only changed the root. Hence, to conclude the proof, we only need to show that there exists an X-covering from Γ_1 to Γ. Define $\varphi\colon \Gamma_1 \to \Gamma$ on the vertices by $\varphi(H_1g) \coloneqq Hg$. We need to check that φ is well defined. But $H_1g = H_1f$ if and only if $gf^{-1} \in H_1 \leq H$, which implies that $Hg = Hf$. Now, define φ on edges by sending the unique edge leaving H_1g and labeled by x to the unique edge leaving Hg and labeled by x. With this definition, all the φ_v are bijections and φ preserves the labeling. All that remains to check is that φ is a morphism of graphs. It is immediate from the definition that φ preserves initial vertices. Now, let e be an edge in Γ_1 with initial vertex H_1g and label x. The inverse edge $\bar{e}$ has initial vertex H_1gx and label x^{-1}. Therefore, $\varphi(e)$ has initial vertex Hg and label x, and its inverse has initial vertex Hgx and label x^{-1}. That is $\overline{\varphi(e)} = \varphi(\bar{e})$. ☐

At this point, an obvious but important remark is the fact that if one of the H_i is normal, the existence of an X-covering is equivalent to the fact that $H_1 \leq H_2$. This is also true if we ask that the covering preserves roots. As an immediate corollary we have:

Proposition 3.4.2. *Let A be a group with generating system X. Then for any $X^{\pm}$-labeled graph Γ, there is an X-covering from $\mathrm{Cay}(A, X^{\pm})$ to Γ if and only if Γ is a Schreier graph over A.*

Proof. We have $A = \mathcal{F}/N$ with N normal and $\mathrm{Cay}(A, X^{\pm}) \simeq \mathrm{Sch}(\mathcal{F}, N, X^{\pm})$. There is an X-covering if and only if, up to a choice of base point, Γ is a Schreier graph of $\mathcal{F}$ for some subgroups H containing N. Therefore, Γ is a Schreier graph of H/N in $A = \mathcal{F}/N$. ☐

Proposition 3.4.3. *Let $\Gamma_i \coloneqq \mathrm{Sch}(\mathcal{F}_i, H_i, X_i^{\pm})$ for $i = 1, 2$ be two Schreier graphs. Then there exists a covering from Γ_1 to Γ_2 if and only if $\mathcal{F}_1$ and $\mathcal{F}_2$ are of the same degree and H_1 is length-isomorphic to a subgroup of a conjugate of H_2.*

Proof. Suppose that there exists a covering. Then both graphs (and therefore groups) have the same degree. Moreover, we can pullback by φ the labeling of Γ_2 onto Γ_1. Let us denote by $\Gamma = \mathrm{Sch}(\mathcal{F}_2, H, X_2^{\pm})$ the graph obtained in this way. Apart from the labeling, it is the graph Γ_1. Therefore, H is length-transitive to H_1. Moreover, due to this new labeling, $\varphi\colon \Gamma \to \Gamma_2$ preserves the labels. Hence we can use the last lemma to prove that H is a subgroup of H_2.

The converse is quite obvious. Let H be the subgroup of $\mathcal{F}_2$ which is length-transitive to H_1. By Lemma 3.4.1, there exists an X-covering from Γ to Γ_2. The graph Γ being isomorphic to Γ_1 (only the labeling changes), we have the desired covering. □

The following lemma is an easy adaptation of a well-known fact about coverings of topological spaces.

Lemma 3.4.4. *For a covering $\varphi\colon \Gamma_1 \to \Gamma_2$ and two vertices w and u in the same connected component of Γ_2, fibers over w and over u have same cardinality.*

Definition 3.4.5. If $\varphi\colon \Gamma_1 \to \Gamma_2$ is a covering with Γ_2 connected, the cardinality of the fibers is called the *degree* of the covering.

Lemma 3.4.6. *Let $H_1 \leq H_2$ be two subgroups of a group A. Then the index of H_1 in H_2 is equal to the degree of the X-covering $\mathrm{Sch}(A, H_1, X^{\pm}) \to \mathrm{Sch}(A, H_2, X^{\pm})$.*

Proof. Let us look at the fiber F over the vertex H_2. It is exactly the set $\{H_1 g \mid H_2 g = H_2\} = \{H_1 g \mid g \in H_2\}$. This corresponds to the decomposition of H_2 into right H_1-coset. □

This lemma will allow us to make a link between X-covering and quasi-isometries.

Definition 3.4.7. Let (M_1, d_1) and (M_2, d_2) be two metric spaces. A *quasi-isometry* from (M_1, d_1) to (M_2, d_2) is a function $f\colon M_1 \to M_2$ such that there exists constants $A \leq 1$, $B \leq 0$ and $C \leq 0$ for which

1. For every two points x and y in M_1 we have

$$A^{-1} d_1(x, y) - B \leq d_2(f(x), f(y)) \leq A d_1(x, y) + B.$$

2. For every point z in M_2, $d_2(z, f(M_1)) \leq C$.

The spaces M_1 and M_2 are called *quasi-isometric* if there exists a quasi-isometry from M_1 to M_2.

The first point means that even if the function f does not necessarily preserve distances, it does not change them too much. The second point says that f is close to being surjective: every point in M_2 is at a bounded distance from the image. This notion naturally arises in the study of Cayley graphs, since two different finite generating systems for the same group give quasi-isometric Cayley graphs. Note that every two finite graphs are quasi-isometric.

For a covering $\varphi\colon \Gamma_1 \to \Gamma_2$ and a vertex $v \in \Gamma_2$, the *diameter of the fiber of* v, $\mathrm{diam}(\varphi^{-1}(v))$, is the maximal distance in Γ_1 between two preimages of v. For coverings of finite degree, the quasi-isometry of the two graphs follows from one simple condition.

Lemma 3.4.8. *Let $\varphi\colon \Gamma_1 \to \Gamma_2$ be a covering of finite degree, with Γ_2 connected. Suppose that there exists a constant B such that for every vertex v in Γ_2, the diameter $\mathrm{diam}(\varphi^{-1}(v))$ is at most B. Then φ is a quasi-isometry with $A = 1$ and $C = 0$.*

Proof. Since φ is surjective, we have $C = 0$ and we only need to check the first condition in the definition of quasi-isometry. Moreover, since φ maps paths from v to w to paths from $\varphi(v)$ to $\varphi(w)$ we always have

$$d_2(\varphi(v), \varphi(w)) \leq d_1(v, w).$$

For the other inequality, take a path p in Γ_2 that realizes the distance between $\varphi(v)$ and $\varphi(w)$. This path lifts to a path $\tilde{p}$ in Γ_1 from v to z with z in the same fiber as w. This give us the desired inequality:

$$d_1(v, w) \leq d_1(v, z) + d_1(z, w) \leq d_2(\varphi(v), \varphi(z)) + B = d_2(\varphi(v), \varphi(w)) + B.$$

□

Lemma 3.4.9. *Let $L \leq H \leq A$ be two subgroups of A such that L has finite index in H. This induces an X-covering of finite degree $\varphi\colon \mathrm{Sch}(A, L, X^{\pm}) \to \mathrm{Sch}(A, H, X^{\pm})$. Then, for all $l \in A$, the supremum*

$$\sup\{\mathrm{diam}(\varphi^{-1}(Hf)) \mid f \in A, fl^{-1} \in N_A(H) \cap N_A(L)\}$$

is finite.

Proof. Firstly, note that $fl^{-1} \in N_A(H) \cap N_A(L)$ if and only if $f^{-1}Hf = l^{-1}Hl$ and $f^{-1}Lf = l^{-1}Ll$.

Let k be the degree of the covering φ. Therefore, we have $H = Lg_1 \sqcup \cdots \sqcup Lg_k$ for some $g_i \in H$. The fiber over the vertex Hl is

$$\begin{aligned}\{Llg \mid Hlg = Hl\} &= \{Llg \mid g \in l^{-1}Hl\} \\ &= \{Lgl \mid g \in H\} \\ &= \{Lg_1 l, \ldots, Lg_k l\} \\ &= \{Llg'_1, \ldots, Llg'_k\},\end{aligned}$$

for $g'_i = l^{-1}g_i l$. Therefore, the distance between two vertices in the fiber is at most $2 \cdot \max\{|g'_i|\}$. Indeed, the distance between Llg'_i and Llg'_j is by the triangular inequality less than or equal to $d(Llg'_i, L) + d(Llg'_j, L)$. On the other side, the fiber over the vertex Hf is

$$\begin{aligned}\{Lfg \mid Hf = Hfg\} &= \{Lfg \mid g \in f^{-1}Hf = l^{-1}Hl\} \\ &= \{Lfl^{-1}gl \mid g \in H\} \\ &= \{Lfg'_1, \ldots, Lfg'_k\}.\end{aligned}$$

Indeed, we have $Lfg = Lfh$ if and only if $fgh^{-1}f^{-1}$ belongs to L, if and only if gh^{-1} belongs to $f^{-1}Lf = l^{-1}Ll$ if and only if $Llg = Llh$. Therefore, the Lfg'_i's consists of k distinct points, and they are all the fiber over Hf. Hence the distance between two vertices in the fiber over Hf is also at most $2 \cdot \max\{|g'_i|\}$. □

We are now going to use these two lemmas to prove a classical result about Cayley graphs and small extensions of it. Recall that a subgroup is *almost normal* if it has only finitely many conjugates and *nearly normal* if it has finite index in its normal closure. Let us call an automorphism φ of Γ_2 *compatible with the covering* $\pi\colon \Gamma_1 \twoheadrightarrow \Gamma_2$ if there exists an automorphism $\tilde{\varphi}$ of Γ_1 such that $\varphi\pi = \pi\tilde{\varphi}$.

Theorem 3.4.10. *Let A be a group with generating system X. Let $N \leq H \leq A$ be two subgroups such that N has finite index in H. Then the graphs* $\mathrm{Sch}(A, N, X^{\pm})$ *and* $\mathrm{Sch}(A, H, X^{\pm})$ *are quasi-isometric if one of the following assumptions holds:*

1. *N is almost normal and H is nearly normal;*
2. *N and H are almost normal;*
3. $\mathrm{Sch}(A, H, X^{\pm})$ *is almost transitive by automorphisms compatible with the covering;*
4. *N is normal, H/N is cyclic of prime order and* $\mathrm{Sch}(A, H, X^{\pm})$ *is almost transitive.*

For the proofs, we have $A = \mathcal{F}_X/L$ and, using the correspondence theorem, it is therefore sufficient to prove the assertion when $A = \mathcal{F}_X$ is a free product of copies of $\mathbf{Z}$ and of $\mathbf{Z}/2\mathbf{Z}$. See Lemmas 3.4.12 to 3.4.14 for the proofs in this case. As an immediate corollary of the theorem, we obtain

Proposition 3.4.11. *Let A be a group with generating system X and B a quotient by a finite normal subgroup. Then, Cayley graphs* $\mathrm{Cay}(A, X^{\pm})$ *and* $\mathrm{Cay}(B, X^{\pm})$ *are quasi-isometric.*

Proof. Simply apply part 2 of the theorem with $N = \{1\}$ and H normal such that $B = A/H$. □

Lemma 3.4.12. *Let $N \leq H \leq \mathcal{F}$ with N of finite index in H, N almost normal and H almost normal or nearly normal. Then, Schreier graphs* $\mathrm{Sch}(\mathcal{F}, N, X^{\pm})$ *and* $\mathrm{Sch}(\mathcal{F}, H, X^{\pm})$ *are quasi-isometric.*

Proof. By Lemma 3.4.9, for all $l \in \mathcal{F}$,

$$B_l = \sup\{\mathrm{diam}(\varphi^{-1}(Hf)) \mid f \in \mathcal{F}, f^{-1}Hf = l^{-1}Hl, f^{-1}Nf = l^{-1}Nl\}$$

is finite. If H is almost normal, and since N is almost normal too, there is only finitely many couples of the form $(l^{-1}Hl, l^{-1}Nl)$. Therefore, in $B \coloneqq \sup\{B_l \mid l \in \mathcal{F}\}$, we only have finitely many different terms and B is finite. We conclude using Lemma 3.4.8.

If H is nearly normal, let M denote its normalizer. We have a sequence of subgroups with finite index inclusion

$$N \hookrightarrow H \hookrightarrow M$$

with N almost normal and M normal. By Lemmas 3.4.1 and 3.4.6, this is equivalent to the following sequence of coverings of finite degree

$$\mathrm{Sch}(G, N, X^{\pm}) \xrightarrow{\psi} \mathrm{Sch}(G, H, X^{\pm}) \xrightarrow{\varphi} \mathrm{Sch}(G, M, X^{\pm}).$$

Therefore, the first part of this lemma gives us

$$d_N(v, w) - B \leq d_M(\varphi \circ \psi(v), \varphi \circ \psi(w)) \leq d_H(\psi(v), \psi(w)) \leq d_N(v, w),$$

where $d_N(v, w)$ is the distance in $\mathrm{Sch}(G, N, X^{\pm})$. □

Lemma 3.4.13. *Let $N \leq H \leq \mathcal{F}$ with N normal and H/N cyclic of prime order in $\mathcal{F}/N$. If the graph* $\mathrm{Sch}(\mathcal{F}, H, X^{\pm})$ *is almost transitive, then it is quasi-isometric to* $\mathrm{Sch}(\mathcal{F}, N, X^{\pm})$.

Proof. By Lemma 3.4.8, it is sufficient to find a universal bound B on the distance between vertices in the same fiber. The graph being almost transitive, there is a finite number of classes of vertices under the action of its automorphism group. It is thus enough to find bounds for fibers over vertices in the same class, the bound B being the maximum over all these bounds.

Choose a vertex Hh in $\mathrm{Sch}(\mathcal{F}, H, X^{\pm})$ and let g be an element of minimal length in $h^{-1}Hh - N$. Due to the structure of H, we have that $h^{-1}Hh = N\langle g\rangle$ and that the fiber over the vertex Hh is $\{Nh, Nhg, \dots, Nhg^{p-1}\}$. Hence the distance between two of its elements is at most $(p-1)|g|$. Indeed, the distance between Nhg^i and Nhg^j is at most $|i-j||g|$. If Hf is in the same transitivity class as Hh, then there is a bijection that preserves lengths between $h^{-1}Hh$ and $f^{-1}Hf$. Hence, we can choose g' in $f^{-1}Hf - N$ of same length as g. Since H/N is cyclic of prime order, we have $f^{-1}Hf = N\langle g'\rangle$ and, as before, the distance between two elements of the fiber is at most $(p-1)|g'| = (p-1)|g|$. □

Lemma 3.4.14. *Let $N \leq H \leq \mathcal{F}$ be as in Theorem 3.4.10. If the graph $\mathrm{Sch}(G, H, X^{\pm})$ is almost transitive by automorphisms compatible with the covering, then it is quasi-isometric to $\mathrm{Sch}(G, N, X^{\pm})$.*

Proof. Thinking in terms of subgroups, the automorphism φ of $\mathrm{Sch}(G, H, X^{\pm})$ corresponds to a length-preserving isomorphism $\alpha\colon H \to f^{-1}Hf$. The compatibility with the covering is then equivalent to $\alpha(N) = N$. Fibers over vertices H and Hf are respectively $\{Ng \mid g \in H\}$ and $\{Nf\alpha(g) \mid g \in H\}$.

We have

$$\begin{aligned} N = Ng &\Leftrightarrow g \in N \\ &\Leftrightarrow \alpha(g) \in \alpha(N) = N \\ &\Leftrightarrow Nf = Nf\alpha(g). \end{aligned}$$

Hence, the application $Ng \mapsto Nf\alpha(g)$ is a well defined bijection between the fibers. Therefore, if the fiber over H is given by $\{Ng_1, \dots, Ng_k\}$, the fiber over $f^{-1}Hf$ is the set $\{Nf\alpha(g_1), \dots, Nf\alpha(g_k)\}$ with $|g_i| = |\alpha(g_i)|$.

Once again, we conclude using Lemma 3.4.8. □

It is natural to ask if Theorem 3.4.10 can be extended. It may be possible, but not in full generality. Indeed, there are examples of subgroups $N \leq H$ with N of finite index in H but such that $\mathrm{Sch}(\mathcal{F}_X, N, X^{\pm})$ and $\mathrm{Sch}(\mathcal{F}_X, H, X^{\pm})$ are not quasi-isometric. There are even such examples with N or H normal.

In order to show that some graphs are not quasi-isometric, we will use the notion of ends. There are different equivalent definitions for the ends of a graph, but for our purpose it is sufficient to know that the number of ends of a locally finite graph Γ is the maximal number of infinite connected components of $\Gamma - \Delta$ where Δ is a finite subgraph (not necessarily connected). The number of ends is invariant under quasi-isometries. For a Cayley graph, the number of ends is either 0 (if and only if the graph is finite), 1 ($\mathbf{Z}^d$ with $d \geq 2$ for example), 2 (if and only if the group is virtually $\mathbf{Z}$) or uncountable (F_n for $n \geq 2$ for example).

We now exhibit two examples of $N \leq H$ such that N is of finite index in H but the graphs $\mathrm{Sch}(\mathcal{F}_X, N, X^{\pm})$ and $\mathrm{Sch}(\mathcal{F}_X, H, X^{\pm})$ are not quasi-isometric. Instead of describing the subgroups H and N explicitly, we will simply describe their Schreier graphs and show that there exists an X-covering of finite degree between them. Indeed, by Lemmas 3.4.1 and 3.4.6, this implies that N is a subgroup of finite index of H.

Example 3.4.15. The graphs of Figure 3.8 correspond to subgroups $N \leq H \leq \langle x, y \rangle = F_2$ with N of index two in H. Since $\mathrm{Sch}(F_2, H, \{x, y\}^{\pm})$ is X-transitive, the subgroup H is normal. But $\mathrm{Sch}(F_2, N, \{x, y\}^{\pm})$ has four ends while the graph $\mathrm{Sch}(F_2, H, \{x, y\}^{\pm})$ has only two ends. Therefore, the two graphs are not quasi-isometric.

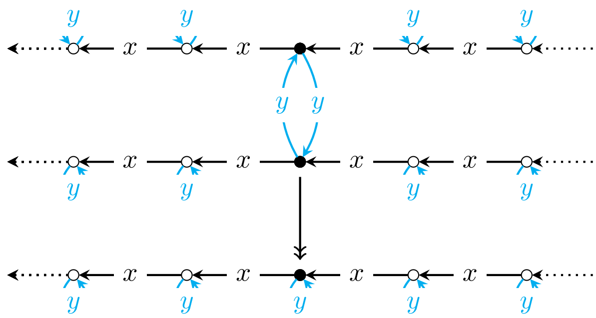


Figure 3.8: An X-covering between two Schreier graphs over the free group of rank two. The root of the base graph and its two preimages are marked in black.

Example 3.4.16. The graphs of Figure 3.9 correspond to subgroups $N \leq H \leq \langle x, a \mid a^2 \rangle = \mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$ with N of index two in H. The covering is given by the central inversion in respect to the middle point between the two black vertices. Since $\mathrm{Sch}(\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}, N, \{x, a\}^{\pm})$ is X-transitive, the subgroup N is normal and $\mathrm{Sch}(\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}, N, \{x, a\}^{\pm}) \simeq \mathrm{Cay}((\mathbf{Z} * \mathbf{Z}/2\mathbf{Z})/N, \{x, a\}^{\pm})$. But $\mathrm{Sch}(F_2, N, \{x, y\}^{\pm})$ has two ends while $\mathrm{Sch}(F_2, H, \{x, y\}^{\pm})$ has only one end. Therefore, the two graphs are not quasi-isometric.

Graphs of Figure 3.10 shows a similar example, with $N \leq H \leq \langle x, y \rangle = F_2$. Since $\mathrm{Sch}(F_2, N, \{x, y\}^{\pm})$ is almost X-transitive, the subgroup N is this time almost normal. In fact, N has only two conjugates: itself (corresponding to the black vertex in Figure 3.10) and $y^{-1}Ny$ (corresponding to the dark gray vertex in Figure 3.10).

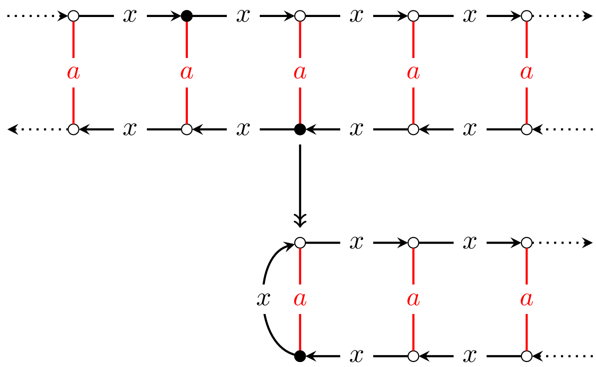


Figure 3.9: An X-covering between two Schreier graphs over $\langle x, a \mid a^2 \rangle$. The root of the base graph and its two preimages are marked in black.

Since people are usually interested in Schreier graphs over free groups and Cayley graphs without loops or multiple edges, it is natural to ask the following question.

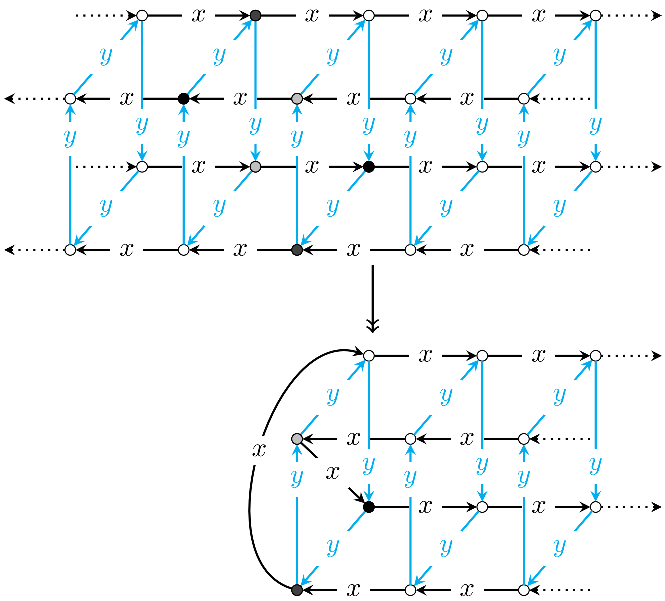


Figure 3.10: An X-covering between two Schreier graphs over the free group of rank two. The root of the base graph and its two preimages are marked in black.

Question 3.4.17. *Is it possible to find $N \leq H \leq F_n$ such that N has finite index in H, the Schreier graphs $\mathrm{Sch}(F_n, N, X^{\pm})$ and $\mathrm{Sch}(F_n, H, X^{\pm})$ are both simple (without loop or multiple edges) and non quasi-isometric and such that at least one of N or H is normal?*

The second part of Example 3.4.16 shows that this is possible if we replace normality by almost normality and Example 3.4.15 shows that this is possible if we do not ask the graphs to be simple.

3.5 Applications to groups

We have seen that for a subgroup H of $\mathcal{F}$, length-transitivity is a weak version of normality. More generally, for any group A and subgroup H, asking for the transitivity of $\mathrm{Sch}(A, H, X^{\pm})$ is a weak version of the normality. Since the normality does not depend on the generating system, it is natural to ask if the same is true for the transitivity of the Schreier graph. It turns out that this is not the case. We will prove in Proposition 3.5.3 that for all subgroups, there exists a (big) generating system such that the corresponding Schreier graph is transitive. Moreover, even if we restrict ourselves to "reasonable" generating systems, the only subgroups such that all Schreier graphs are transitive are the normal subgroups (Proposition 3.5.5).

Normal subgroups are now to be a modular sublattice of the lattice of subgroups. In particular, intersection and join of normal subgroups are still normal. This partially extends to length-transitive subgroups.

Proposition 3.5.1. *Let $A, B \leq G$ be length-transitive subgroups for a given generating system X. Then $A \cap B$ is length-transitive for X.*

Proof. By the correspondence theorem, it is enough to prove the assertion for $G = \mathcal{F}_X$. In [115], Stallings showed that for free groups, intersection of subgroups correspond to (any connected component of) pullback of immersions for the Schreier graphs. This result extends to all group of the form $\mathcal{F}_X$. Therefore, $A \cap B$ is represented by any connected component of the pullback of $\mathrm{Sch}(\mathcal{F}, A, X^{\pm})$ and $\mathrm{Sch}(\mathcal{F}, B, X^{\pm})$ over $\mathrm{Sch}(\mathcal{F}, \mathcal{F}, X^{\pm})$. But this last graph is a rose: which is the terminal object in the category of Schreier graphs of $\mathcal{F}_X$ with X-morphisms. Therefore, the pullback is simply the categorial product of $\mathrm{Sch}(\mathcal{F}, A, X^{\pm})$ and $\mathrm{Sch}(\mathcal{F}, B, X^{\pm})$, that is the tensor product.

Now, if A and B are length-transitive, their Schreier graphs are transitive. This implies that their tensor product is also transitive. Indeed, for any (v_1, w_1) and (v_2, w_2) in the tensor product, there exists ψ an automorphism of $\mathrm{Sch}(\mathcal{F}, A, X^{\pm})$ sending v_1 to v_2 and φ an automorphism of $\mathrm{Sch}(\mathcal{F}, B, X^{\pm})$ sending w_1 to w_2. In particular, $\psi \otimes \varphi$ sends (v_1, w_1) to (v_2, w_2). Finally, since $\mathrm{Sch}(\mathcal{F}, A \cap B, X^{\pm})$ is transitive, then by definition $A \cap B$ is length-transitive. ☐

Remark 3.5.2. Similar arguments show that the graph $\mathrm{Sch}(\mathcal{F}, \langle A, B\rangle, X^{\pm})$ is the pushout of $\mathrm{Sch}(\mathcal{F}, A, X^{\pm})$ and $\mathrm{Sch}(\mathcal{F}, B, X^{\pm})$ with respect to $\mathrm{Sch}(\mathcal{F}, \{1\}, X^{\pm})$. In this case, the graph $\mathrm{Sch}(\mathcal{F}, \{1\}, X^{\pm})$ is a tree and hence the initial object in the category of Schreier graphs of $\mathcal{F}_X$ with X-morphisms that preserves the root. Therefore, $\mathrm{Sch}(\mathcal{F}, \langle A, B\rangle, X^{\pm})$ is the coproduct $\mathrm{Sch}(\mathcal{F}, A, X^{\pm}) \coprod \mathrm{Sch}(\mathcal{F}, B, X^{\pm})$ in the category of X-labeled graphs with X-morphisms that preserves the root. Observe that since we ask our morphisms to preserves the root, the coproduct is not the disjoint union. In general it is unclear if such a graph is transitive.

For a group A, denote by $d(A)$ the number of elements of order 2 plus half of the number of elements of order at least 3, if A is infinite, then $d(A) = |A|$.

Proposition 3.5.3. *Let A be a group (not necessarily finitely generated) and H any subgroup. Then there exists a generating system X of size $d(A)$ such that $\mathrm{Sch}(A, H, X^{\pm})$ is transitive.*

Proof. Any group A can be decomposed as $A = \{1\} \sqcup S \sqcup T \sqcup T^{-}$, where S consists of elements of order 2, T of half of elements of order at least 3 and T^{-} of inverses of elements in T. Let $X \coloneqq S \cup T$. Then, $X^{\pm} = A \setminus \{1\}$ and $|X| = d(A)$.

Let $A = \bigsqcup Hg_i$ be the decomposition into right cosets. For each i and j, there is an edge labeled by g from Hg_i to Hg_j if and only if $g \in g_i^{-1}Hg_j$. Since $|g_i^{-1}Hg_j| = |H|$, for any two vertices in $\mathrm{Sch}(A, H, A)$, there is exactly $|H|$ edges going from v to w and this graph is transitive (it is a thick complete graph). The edges labeled by 1 being always loop, the graph $\mathrm{Sch}(A, H, X^{\pm})$ is also transitive. ☐

We will prove that even if we restrict ourselves to generating systems of size at most $\mathrm{rank}(A) + 1$, the fact that $\mathrm{Sch}(A, H, X^{\pm})$ is transitive does depend on X if H is not normal. Before that we prove the following technical lemma.

Lemma 3.5.4. *Let A be a group (not necessarily finitely generated) and let H be a proper subgroup. Then there exists a generating system X of A such that $X \cap H$ is empty and $|X| = \mathrm{rank}(A)$.*

Proof. Let X be a generating system of A such that $|X| = \mathrm{rank}(A)$. If $X \cap H$ is empty, the assertion is true. Therefore, we can suppose that $X \cap H$ is not empty. Since H is a proper subgroup, we have $X \cap H \neq X$. Thus, we can order the elements of X and find an $x_0 \in X$ such that $x \in X$ belongs to H if and only if $x < x_0$. Now, take $Y \coloneqq \{x \mid x \in X, x \geq x_0\} \sqcup \{xx_0 \mid x \in X, x < x_0\}$. This is trivially a generating system of the same cardinality as X, and $Y \cap H$ is empty. Indeed, if $x \geq x_0$ then $x \notin H$. But if $x < x_0$ and xx_0 belongs to H, we have $x_0 = x^{-1}xx_0 \in H$, which is absurd. ☐

Proposition 3.5.5. *Let A be a group and H a subgroup of A. If, for all generating systems X of size at most* $\operatorname{rank}(A)+1$*, the graph* $\operatorname{Sch}(A,H,X^{\pm})$ *is transitive, then H is a normal subgroup of A.*

Proof. If $H = A$, there is nothing to prove. Therefore, we can suppose that H is a proper subgroup and find, by the preceding lemma, a generating system X such that $|X| = \operatorname{rank}(A)$ and $X \cap H = \emptyset$. The Schreier graph $\operatorname{Sch}(A,H,X^{\pm})$ is transitive by assumption and does not have loops since $X \cap H = \emptyset$. For any $h \in H$, let $X_h := X \sqcup \{h\}$; a generating system of size $\operatorname{rank}(A)+1$. The graph $\operatorname{Sch}(A,H,X_h^{\pm})$ is transitive and has a unique loop (labeled by h) at the vertex H. Therefore, for all $g \in A$, the vertex Hg as a unique loop. The label of this loop is h since the graph $\operatorname{Sch}(A,H,X^{\pm})$ has no loops at the vertex Hg. But this implies that for all $g \in A$, $Hgh = Hg$. Therefore, for all $h \in H$ and $g \in A$, ghg^{-1} belongs to H and we have just proven that H is normal. □

In the following, we will only take in account locally finite graphs and finite generating systems of groups. This is justified by the fact that if A is not finitely generated, then $|A| = d(A) = \operatorname{rank}(A)$, but has other important consequences for the study of Schreier graphs. For example, every connected, locally finite transitive graph is a Schreier graph, see Section 3.1.

Due to Proposition 3.5.5, we know that the transitivity of $\operatorname{Sch}(A,H,X^{\pm})$ does not only depends on H, but on X too if H is non-normal. This and Proposition 3.5.3 motivate the following definition.

Definition 3.5.6. A finitely generated A is *strongly simple* if for any generating system X of size at most $\operatorname{rank}(A)+1$, and any proper subgroup $\{1\} < H < A$, the graph $\operatorname{Sch}(A,H,X^{\pm})$ is not transitive.

It is immediate that strong simplicity implies simplicity and that cyclic groups of prime order C_p are strongly simple. Indeed, such groups do not have proper subgroups.

Theorem 3.5.10 shows the existence of infinite strongly simple groups, proving that the class of strong simple groups is not reduced to cyclic groups. On the other hand, the following proposition shows that there exists (finite) simple groups which are not strongly simple.

Proposition 3.5.7. *For odd $n \geq 7$, let H_n be the subgroup of A_n consisting of elements fixing n and let $a_n := (1,3,4,5,\dots n,2)$ and $b_n := (2,4,6,\dots n-1,1,n,n-2,\dots,5,3)$. Then $\{a_n,b_n\}$ generates A_n and the graph* $\operatorname{Sch}(A_n,H_n,\{a_n,b_n\}^{\pm})$ *is transitive.*

In particular, A_n is simple but not strongly simple.

Proof. We have that H_n is isomorphic to A_{n-1} and has index n. Moreover, the right cosets for H_n depend only on the preimage of n. Therefore, the action of A_n on A_n/H_n is isomorphic to the action of A_n on $\{1,\dots,n\}$. It is then easy to see that $\operatorname{Sch}(A_n,H_n,\{a_n,b_n\}^{\pm})$ is isomorphic to the circulent graph $C_n^{1,2}$ (see Figure 3.11 for an example): each vertex i has 4 neighborhood: $i\pm 1, i\pm 2$. Such a graph is obviously transitive. In order to finish the proof, we need to show that a_n and b_n generate A_n. For $n \geq 7$, a direct computation gives $b_n a_n^{-2} b_n^{-1} a_n^2 = (4,3,n-1)$. We conclude using the fact that $(4,3,n-1)$ and a_n generates A_n for odd $n \geq 5$ (see [100]). □

The above proof does not work in the case where $n = 5$ or n is even. For $n = 5$, the graph is still transitive, but $b_5^2 = a_5$ and therefore $\{a_5,b_5\}$ does not generate A_5. A careful check shows that if $\operatorname{Sch}(A_5,H_5,X^{\pm})$ is transitive, then X has at least 3 elements and 3 is possible (take a_5, b_5 and $(1,2,3,4,5)$). For $n = 6$, we even have that if $\operatorname{Sch}(A_6,H_6,X^{\pm})$ is transitive, then X has at least 4 elements. More generally, for even n, let $c_i = (1,2,3,\dots,\hat{\imath},\dots,n)$ (the cycle $(1,2,\dots,n)$ without i). Then the c_i's generate A_n and $\operatorname{Sch}(A_n,H_n,\{c_i\}^{\pm})$ is transitive.

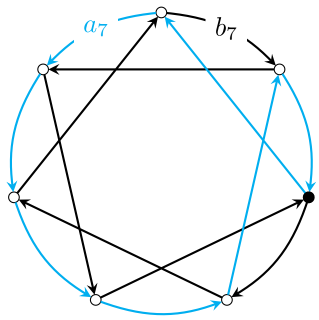


Figure 3.11: The graph $\mathrm{Sch}(A_7, H_7, \{a_7, b_7\}^{\pm})$. The root (the vertex H_n) is marked in black.

In this case, we have a generating set of size n, which is small if we compare it to $d(A_n) > \frac{n!}{4}$ but big if we compare it to $\mathrm{rank}(A_n) = 2$.

We now turn our attention on the cyclic subgroups of prime order.

Proposition 3.5.8. *Let A be finitely generated group, X a finite generating system, and K a cyclic subgroup of prime order. Then, in the graph $\mathrm{Sch}(A, K, X^{\pm})$, each orbit is a finite union of X-orbits.*

Proof. We have $A = \mathcal{F}/N$, with $\mathcal{F}$ finitely generated. The subgroup K corresponds to a subgroup H of $\mathcal{F}$ containing N. For any vertex Hg in the graph $\mathrm{Sch}(\mathcal{F}, H, X^{\pm}) \simeq \mathrm{Sch}(A, K, X^{\pm})$, its orbit $[Hg]$ is the set of all vertices Hf such that there exists an automorphism mapping Hg to Hf. Since K is cyclic of prime order, we have $g^{-1}Hg = N\langle h\rangle$ for every h in $g^{-1}Hg - N$. We can choose h to have minimal length among elements of $g^{-1}Hg - N$. For any vertex Hf in $[Hg]$, there exists a bijection from $g^{-1}Hg$ to $f^{-1}Hf$ that preserves lengths. Hence, there exists h' in $f^{-1}Hf - N$ which has same length as h; and we have $f^{-1}Hf = N\langle h'\rangle$. Since $\mathcal{F}$ is finitely generated, its set of elements of length $|h|$ is finite. Thus, we have that the set of subgroups $\{f^{-1}Hf \mid Hf \in [Hg]\}$ is finite. We conclude the proof using the fact that the X-orbit of Hg consists exactly of vertices Hf such that $f^{-1}Hf = g^{-1}Hg$. □

Corollary 3.5.9. *Let A be a infinite simple group, X a finite generating system, K a proper non-trivial subgroup and $\Gamma := \mathrm{Sch}(A, K, X^{\pm})$. Then $\mathrm{Aut}_X(\Gamma)$ has an infinite number of orbits. Moreover, if K is cyclic of prime order, then $\mathrm{Aut}(\Gamma)$ has an infinite number of orbits and therefore, Γ is not almost transitive.*

Proof. Since A is infinite simple, it does not have any finite index subgroups. Therefore, the number of X-orbits, which is $[A : N_A(K)]$, is infinite. The last proposition implies that if K is cyclic of prime order, then the number of orbits is also infinite. □

Recall that a *Tarski monster T_p* is an infinite group such that every proper subgroup is isomorphic to a cyclic group of order p, for p a fixed prime. It follows directly from the definition that every such group has rank 2 and is simple. Ol'shanskii proved in [96] that there exists uncountably many non-isomorphic Tarski monsters for every p greater that 10^{75}.

Theorem 3.5.10. *Let T_p be a Tarski monster, X a finite generating system, K a proper non-trivial subgroup and Γ the corresponding Schreier graph. Then $\mathrm{Aut}_X(\Gamma) = \{1\}$ and each orbit of $\mathrm{Aut}(\Gamma)$ is finite. In particular, Γ is not almost transitive and Tarski monsters are strongly simple.*

Proof. Due to the particular structure of subgroups in T_p, we have $N_{T_p}(K) = K$, which implies that all X-orbits are singletons. Thus, $\mathrm{Aut}_X(\Gamma) = \{1\}$. Since K is cyclic of prime order, each orbit is a finite union of X-orbits and therefore finite. □

The existence of strongly simple infinite groups partially answer a question of Benjamini:

Question 3.5.11. *Does there exists a constant M such that every infinite transitive (Cayley) graph Γ, not quasi-isometric to $\mathbf{Z}$, covers an infinite transitive graph Δ of girth at most M and such that Δ is non quasi-isometric to Γ?*

The original motivation for this question was a conjecture off Benjamini and Duminil-Copin about the connective constant of transitive graphs. This conjecture was solved by Grimmet and Li in [61], but the question of Benjamini remains and the following weak form is still an open problem:

Conjecture 3.5.12. *Every infinite Caley graph Γ, not quasi-isometric to $\mathbf{Z}$, covers an infinite transitive graph $\Delta \not\simeq \Gamma$.*

If we ask for X-coverings, the conjecture is false, with Cayley graph of infinite strongly y simple groups as counter-examples.

On the other hand, de la Salle and Tessera showed in [33] that for p large enough, every Tarski monster T_p admits a Cayley graph whith a discrete group of automorphisms. Moreover, if the Tarski monster in question is finitely presented, then there exists a finite generating set X and a constant $R = R(T_p, X)$ such that if Δ is a graph with R-balls X-isomorphic to R-balls in $\mathrm{Cay}(T_p, X^{\pm})$, then Δ is X-covered by $\mathrm{Cay}(T_p, X^{\pm})$. More generally, both statements remain true for any finitely presented group with an element of sufficiently big order (depending on the size of the generating set).

This is a reason to believe that the Conjecture 3.5.12 itself is false, with Cayley graph of finitely presented strongly simple groups as possible counter-examples. Nevertheless, it is important to note that the existence of finitely presented Tarski monsters is still an open problem. A possibility to circumvent this problem would be to look at finitely presented extensions of T_p and ensure that they do not have too many subgroups.

Finally, it is possible to translate Question 3.5.11 in a group theoretic question. Indeed, suppose $A = \mathcal{F}/N$ and $\varphi\colon \Gamma = \mathrm{Cay}(A, X^{\pm}) \to \Delta$ is a covering with $\Delta \not\simeq \Gamma$ an infinite transitive graph. Then Δ being transitive, is a Schreier graph of some infinite index and length-transitive $H < \mathcal{F}'$. This endows Δ with a labeling that can be pullback by φ to Γ. Therefore φ is a X-covering and the new labeling of Γ correspond to a subgroup $M < H$ which is length isomorphic to N (but not necessarily normal). We just showed

Lemma 3.5.13. *Let $A = \mathcal{F}/N$. Then $\Gamma = \mathrm{Cay}(A, X^{\pm})$ covers an infinite transitive graph $\Delta \not\simeq \Gamma$ if and only if there exists $M < H < \mathcal{F}'$ with $\mathcal{F}'$ of same degree as $\mathcal{F}$, M length-isomorphic to N and H length-transitive and of infinite index.*

3.6 Open questions and further research directions

In this section, we list the open questions that appeared during this chapter and propose further direction of research.

The original question that motivated this work is still open. Solving the easy version presented in Conjecture 3.5.12 would lead to a better understanding of covering between transitive graphs and thus would be of great interest. We conjecture that the answer to this question is no.

Question 3.6.1. *Does every infinite Caley graph Γ, not quasi-isometric to $\mathbf{Z}$, covers an infinite transitive graph $\Delta \not\simeq \Gamma$?*

Another question about quasi-isometry is Question 3.4.17 which is motivated by Theorem 3.4.10.

Question 3.6.2. *Is it possible to find $N \leq H \leq F_n$ such that N has finite index in H, the Schreier graphs $\mathrm{Sch}(F_n, N, X^{\pm})$ and $\mathrm{Sch}(F_n, H, X^{\pm})$ are both simple (without loop or multiple edges) and non quasi-isometric and such that at least one of N or H is normal?*

Another direction of research would be the better understanding of length-transitive subgroup and strongly simple groups. The following is motivated by Proposition 3.5.1

Question 3.6.3. *Given two length-transitive subgroups A and B of $\mathcal{F}_X$. Is it true that $\langle A, B\rangle$ is length-transitive?*

If yes, then length-transitive subgroups are a sublattice of the lattice of subgroups. Is this lattice modular?

Proposition 3.5.5 says that for G a group and H a non-normal subgroup there exists a generating system X of G of size at most $\mathrm{rank}(G) + 1$ such that $\mathrm{Sch}(G, H, X^{\pm})$ is not transitive. In particular, length-transitivity does depend on the generating set.

Question 3.6.4. *For H non-normal, is it always possible to find such a generating system X of size $\mathrm{rank}(G)$ such that the corresponding Schreier graph is not transitive?*

What happens in the case where H is length-transitive?

Propositions 3.5.7 and Theorem 3.5.10 give example of strongly simple groups (non cyclic of prime order) and of simple but not strongly simple groups. A complete classification of strongly simple groups (non cyclic of prime order) is probably impossible at the time, but we may ask the following

Question 3.6.5.

1. *Does there exist infinite finitely presented strongly simple groups?*
2. *Does there exist finite strongly simple groups (non cyclic of prime order)?*
3. *Does there exist infinite simple groups that are not strongly simple?*

Chapter **4**

De Bruijn graphs and generalizations

4.1 Introduction

Spider-web networks were introduced by Ikeno in 1959 [67] in order to study systems of telephone exchanges. They were later shown to enjoy interesting properties in percolation, see [102], [101] and [103]. This work stems from the paper [6] by Balram and Dhar where they are interested in the asymptotic properties of the sequence of spider-web graphs $\{\mathcal{S}_{k,N,M}\}$, for $k = 2$. In particular, they find, using an interesting approach based on symmetries, the spectra of graphs $\mathcal{S}_{2,N,M}$ and observe that they converge to a discrete limiting distribution as $M, N \to \infty$.

Here, we develop a method that leads to the full understanding of this infinite discrete model, including its spectral characteristics, via finite approximations, using the notion of Benjamini-Schramm limit of graphs that has lately become very important in probability theory. A remarkable feature of the model that we discover is that it is related to one of the most interesting and important test-cases in combinatorial group theory, both algebraically and from the spectral and probabilistic viewpoints, the lamplighter groups.

The aim of the present chapter is to provide a unified rigorous framework for studying spider-web graphs $\mathcal{S}_{2,N,M}$, for any $k \geq 2$, and their spectra, and to identify their limit, as $M, N \to \infty$, as a particular Cayley graph of the lamplighter group $\mathcal{L}_k = \mathbf{Z}/k\mathbf{Z} \wr \mathbf{Z}$ (see Subsection 4.2 for the definition) known (see [121]) as the Diestel-Leader graph .

Convergence of spider-web graphs to this graph comes from the following structural result that we prove. For any $k \geq 2$, the spider-web digraph $\vec{\mathcal{S}}_{k,N,M}$ decomposes into the tensor product of the digraph $\vec{\mathcal{S}}_{k,N,1}$ and the oriented cycle $\vec{C}_M$ of length M. It is then useful to note that the sequence $\vec{\mathcal{S}}_{k,N,1}$ is nothing else than the well-studied sequence of de Bruijn digraphs, see Subsection 4.3. De Bruijn graphs are famous for their useful connectivity properties and, being both Hamiltonian and Eulerian, are used both in mathematics, where they represent word overlaps in symbolic dynamical systems, and in applications, as for example for the discrete model for the Bernoulli map or for genome assembly in bioinformatics [31]. Our results imply that, for each $k \geq 2$, the two-parameter family of spider-web graphs $\mathcal{S}_{k,N,M}$ is in fact a natural extension of the family of de Bruijn graphs $\mathcal{B}_{k,N}$.

We then prove a result of independent interest, that de Bruijn graphs are isomorphic to another well known sequence of finite graphs provided by a self-similar action of the lamplighter group $\mathcal{L}_k$ by automorphisms on the k-regular rooted tree, see [59]. Our main result then follows: the sequence of spider-web digraphs $\vec{\mathcal{S}}_{k,N,M}$ (respectively $\mathcal{S}_{k,N,M}$) converges, as $M, N \to \infty$ to the Cayley graph of the lamplighter group $\mathcal{L}_k$, see Theorem 4.3.17 (respectively Corollary 4.3.18).

The spectra of de Bruijn graphs have been computed by Delorme and Tillich in [34]. We extend this computation to all spider-web graphs by using their tensor product structure. The spectral approximation in the context of Benjamini-Schramm limits (see Definition 2.2.2) then ensures that the spectra of finite spider-web graphs converge to the spectral distribution corresponding to the limit graph. As mentioned above, this spectral distribution coincides with one of those associated with the lamplighter group.

The spectral theory of discrete Laplacians on lattices and on Cayley graphs is a very popular topic related to the theory of random walks on groups initiated by Kesten, Atiyah's theory of L^2-invariants, Kadison-Kaplansky Conjecture and many more. The lamplighter group is a very interesting object from the viewpoint of spectral theory. It was open for a longtime whether the Laplacian spectrum on a Cayley graph can have a discrete component. This was answered in [59] where it was shown that the spectrum of a certain Cayley graph of the lamplighter group is pure point.

On the other hand, it follows from [38] by Elek that for the "standard" generating set (the one that corresponds to the algebraic structure of the lamplighter group), the spectrum contains no eigenvalue. This is illustrated on Figure 4.1, where the left column corresponds to the Diestel-Leader graph $\vec{\mathrm{DL}}(2,2)$ which, as we have already mentioned, is isomorphic to a specific Cayley graph of $\mathcal{L}_2$, see (‡) on page 54, whereas the right column corresponds to the Cayley graph of $\mathcal{L}_2$ with respect to the standard generating set (see (†) on page 54).

This is the first example of a dramatic change that the Laplacian spectrum can undergo under local perturbations, even in the presence of a large underlying group of symmetries. Recently other examples of this type were discovered by Grigorchuk, Lenz and Nagnibeda, [53, 52], in the context of group actions with aperiodic order.

The first two lines of Figure 4.1 are the histograms of the spectral measure (respectively for linear and logarithmic y-axes) and the last line shows the corresponding density functions. In both cases the graphics correspond to approximations of the infinite graph by graphs with 2^N vertices (provided by the action of $\mathcal{L}_2$ on the infinite full binary tree, see Subsection 4.3). On the left, the Diestel-Leader graph is approximated by de Bruijn (and equivalently spider-web $\mathcal{S}_{2,N,M}$ for any M) graphs (see Remark 4.4.4); in this case the exact spectral measure is known ([59], see also (#) on page 67). It is not known for the Cayley graph of $\mathcal{L}_k$ with respect to the standard generators.

We also briefly study the complexity (the number of covering tree) of spider-web (di)graphs and compute the spectral zeta function of $\mathcal{S}_{k,N,M}$. We then show that the spectral zeta functions of spider-web graphs converge to the spectral zeta function of $\mathrm{DL}(k,k)$. This allows us to compute the Fuglede-Kadison determinant of $\mathrm{DL}(k,k)$.

Rauzy digraphs appear in dynamics and are a natural generalization of de Bruijn digraphs. They are obtained as subdigraphs of de Bruijn graphs by looking only at vertices $v \in \{0,\ldots,k-1\}^N$ not containing "forbidden subwords". Therefore, they are finite models of the subshift given by the set of forbidden subwords, where de Bruijn graphs correspond to the full shift. Using totally different methods that for the case of de Bruijn graphs, we show that for Rauzy digraphs corresponding to subshift of finite type, their limit is a horospheric product of two trees; a generalization of Diestel-Leader digraphs. Moreover, the trees appearing in the description of the limits are also related to the subshift: they corresponds to "directions" of the trajectories.

The chapter is organized as follows. In Section 4.2 we turn our attention to tensor product when one of the factor is an oriented cycle or a line. We then specialize to the case of graphs defined by a group action and further to the case of the lamplighter group. The structure of spider-web graphs is analysed in Section 4.3. In particular we establishe a connection between spider-web graphs and lamplighter groups. Section 4.4 contains spectral computations on spider-web graphs. In Section 4.5 we provide some further results about

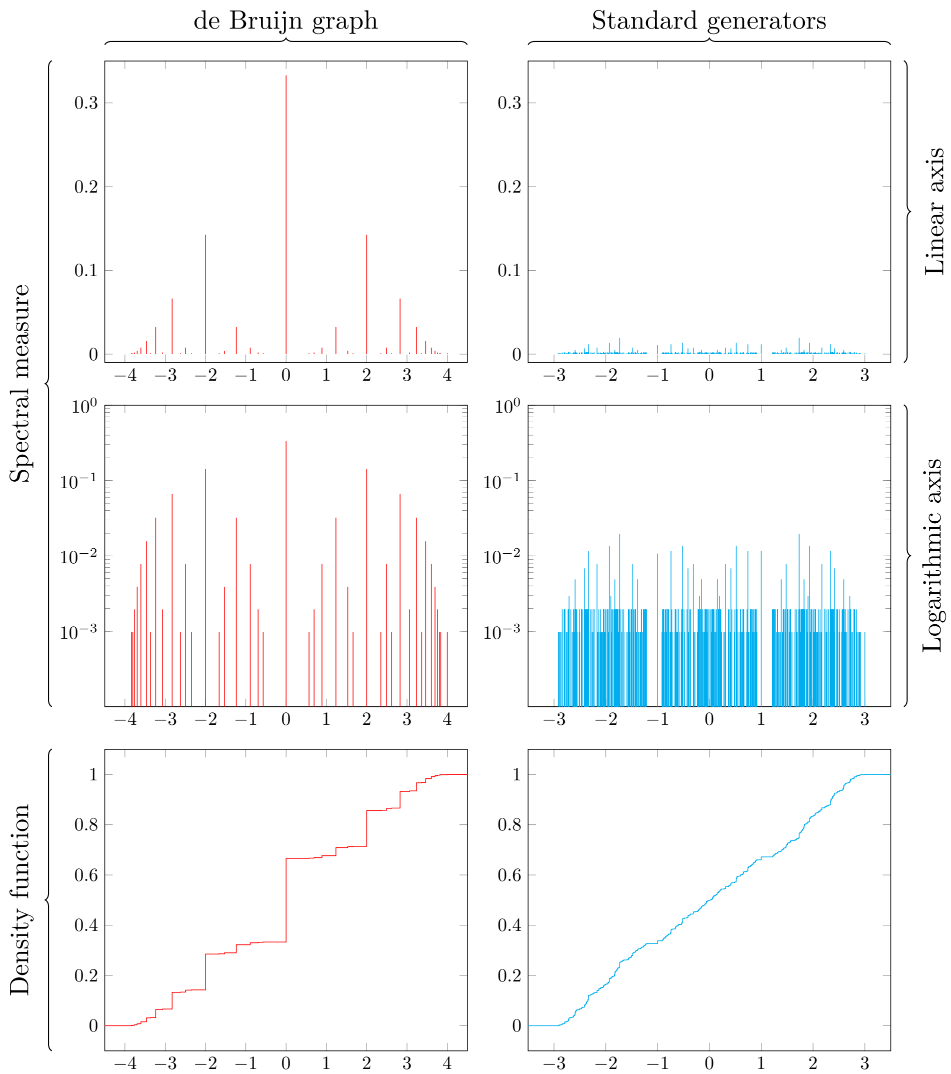


Figure 4.1: Approximations of the spectral measure and of the density function of two Cayley graphs of the lamplighter group $\mathcal{L}_2$. The standard one on the right (see Subsection 4.2), and the Diestel-Leader graph $\vec{\mathrm{DL}}(2,2)$ approximated by de Bruijn (and spider-web $\mathcal{S}_{2,N,M}$, any M) graphs on the left. The histograms correspond to graphs with 2^N, $N = 10$, vertices.

spider-web graphs and their relation to lamplighters. It turns out that all $\vec{\mathcal{S}}_{k,N,M}$ are Schreier graphs of the lamplighter group $\mathcal{L}_k$ (Theorem 4.3.14) and we identify the subgroups to which they correspond (Theorems 4.3.14 and 4.5.3). It is then shown in Theorem 4.5.10 that for all k, the graph $\vec{\mathcal{S}}_{k,N,M}$ is transitive if and only if $M \geq N$. In Theorem 4.5.8, we show that if moreover N divides M, it is a Cayley graph of a finite quotient of the lamplighter group $\mathcal{L}_k$. In Section 4.6 we study the complexity and spectral zeta function of spider-web graphs. This implies the computation of the Fuglede-Kadison determinant of $\mathrm{Cay}(\mathcal{L}_k, X_k^{\pm})$. In Section 4.7 we turn our attention on Rauzy digraphs on subshifts of finite type and prove that in general there limit is supported on horospheric product of trees. Finally, Section 4.8 contains all the open questions appearing in this chapter as well as ideas for further research.

Sections 4.2 to 4.5 are joint work with R. Grigorchuk and T. Nagnibeda and were published in [86] with slightly less details. Section 4.6 is a joint work with T. Nagnibeda.

4.2 More on tensor product of graphs

In Section 2.3 we defined the tensor product of (di)graphs as the categorial product and showed some preliminaries results on it. In this section, we will investigate more on the tensor product, when one of the factors is a cycle or line.

Tensor product with an oriented cycle and the oriented line

Let us now consider the special case when one of the factors in the tensor product is $\vec{C}_\infty$ or $\vec{C}_M$, where $\vec{C}_\infty$ is the "oriented line" with $V_{\vec{C}_\infty} = \mathbf{Z}$ (the set of integers) and for each vertex i there is a unique oriented edge from i to $i+1$, and $\vec{C}_M$ is the "oriented cycle of length M": $V_{\vec{C}_M} = \mathbf{Z}/M\mathbf{Z}$ and for each i there is a unique oriented edge from i to $i+1$ modulo M. Below, we will write $M \in \overline{\mathbf{N}} = \{1, 2, \ldots, \infty\}$ and $i \equiv j \pmod{\infty}$ will mean $i = j$.

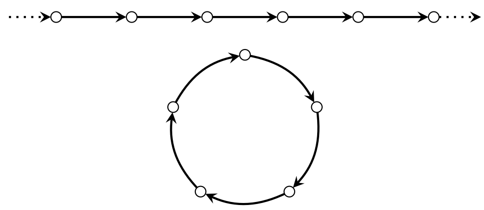


Figure 4.2: The digraphs $\vec{C}_\infty$ (on top) and $\vec{C}_5$ (on bottom).

In this subsection we will only consider oriented connected graphs $\vec{\Gamma}$. Recall the notion of derangement of a path from Definition 2.3.5 that we will need here.

Proposition 4.2.1. *For any connected digraph $\vec{\Gamma}$ and any $M \in \overline{\mathbf{N}}$, all connected components of $\vec{\Gamma} \otimes \vec{C}_M$ are isomorphic.*

Proof. Fix a vertex v of $\vec{\Gamma}$. Since $\vec{\Gamma}$ is connected, for any vertex w there is a path q from v to w in $\underline{\vec{\Gamma}}$, with signature $\sigma(q)$. For any integer i, there exists a path r from i to $i + \mathrm{der}(q) \pmod{M}$ in $\underline{\vec{C}_M}$ with signature $\sigma(q) = \sigma(r)$. Therefore, there is a path p in $\underline{\vec{\Gamma} \otimes \vec{C}_M}$ from (v, i) to $\big(w, i + \mathrm{der}(q)\big)$. Hence, for any vertex (w, j) in $\underline{\vec{\Gamma} \otimes \vec{C}_M}$, there exists an integer i such that (w, j) is in the connected component of (v, i).

On the other hand, since for any integers i and j, the rooted graphs $(\vec{C}_M, i)$ and $(\vec{C}_M, j)$ are isomorphic, say by an isomorphism $\varphi_{i,j}$, we have connected components $\big(\vec{\Gamma} \otimes \vec{C}_M, (v, i)\big)^0$

and $\big(\vec{\Gamma}\otimes\vec{C}_M,(v,j)\big)^0$ are isomorphic by $\mathrm{Id}\otimes\varphi_{i,j}$. This implies that all connected components are isomorphic. □

Theorem 4.2.2. *Let $\vec{\Gamma}$ be a connected locally finite digraph. For any $M\in\overline{\mathbf{N}}$ and any vertex v in $\vec{\Gamma}$, the rooted digraph $(\vec{\Gamma},v)$ is isomorphic (as rooted digraph) to $\big(\vec{\Gamma}\otimes\vec{C}_M,(v,0)\big)^0$ if and only if* $\operatorname{der}(\vec{\Gamma})\equiv 0 \pmod M$.

Proof. Suppose that $\operatorname{der}(\vec{\Gamma})\equiv 0 \pmod M$. For any vertex w of $\vec{\Gamma}$ define $\operatorname{rk}(w)$, the *rank* of w, to be the derangement of any path in $\underline{\Gamma}$ from v to w taken modulo M. This is well defined since for two such paths p and q, the concatenated path $p\bar{q}$ is a closed path based at v with derangement $0 \pmod M$. We define a morphism from $(\vec{\Gamma},v)$ to $\big(\vec{\Gamma}\otimes\vec{C}_M,(v,0)\big)^0$ by $w\mapsto\big(w,\operatorname{rk}(w)\big)$ for vertices. For the edges, it maps an edge e from w to x to an edge from $\big(w,\operatorname{rk}(w)\big)$ to $\big(x,\operatorname{rk}(x)\big)$. Note that the vertices $\big(w,\operatorname{rk}(w)\big)$ and $\big(x,\operatorname{rk}(x)\big)$ are indeed connected by an edge in the tensor product since $\operatorname{rk}(x)=\operatorname{rk}(w)+1$. It is easy to see that this morphism is surjective and injective, and hence is an isomorphism.

Suppose now that $\operatorname{der}(\vec{\Gamma})\not\equiv 0 \pmod M$. This implies the existence of a closed path q_0 in $\underline{\Gamma}$ from v to v with non-zero $\pmod M$ derangement and length n. By the second part of Lemma 2.3.7, the set of closed paths q based at v and of length n is in bijection with the set of (non necessarily closed) paths p from $(v,0)$ to $\big(v,\operatorname{der}(q)\big)$, where we used the fact that for every signature σ, there is a unique path in C_M with initial vertex 0 and signature σ. Hence, the number of closed paths in $\underline{\vec{\Gamma}\otimes\vec{C}_M}$ of length n based at $(v,0)$ is at most the number of closed paths of length n based at v, minus one (namely the path q_0). If $\vec{\Gamma}$ is locally finite (note that local finiteness of $\vec{\Gamma}$ is used only in this direction of the proof), there is only a finite number of such paths. In this case, $(\vec{\Gamma},v)$ and $\big(\vec{\Gamma}\otimes\vec{C}_M,(v,0)\big)^0$ cannot be isomorphic (as rooted digraphs). □

Remark 4.2.3. Proposition 4.2.1 and Theorem 4.2.2 (and their proofs) are still true in the category of labeled digraphs (with strong morphisms) if we identify the labeling $(l\times l')$ of the tensor product with its first coordinate l, which is the labeling of $\vec{\Gamma}$. In the following, we will always use this identification for tensor product of the form $\vec{\Gamma}\otimes\vec{C}_M$.

We know by Proposition 4.2.1 and Theorem 4.2.2 that all connected components of $\vec{\Gamma}\otimes\vec{C}_M$ are isomorphic and we are able, in the locally finite case, to decide when they are isomorphic (as rooted graphs) to $\vec{\Gamma}$. To complete the description of $\vec{\Gamma}\otimes\vec{C}_M$ it remains to count the number of connected components. This is the subject of the next proposition.

Proposition 4.2.4. *For any connected digraph $\vec{\Gamma}$, and any $M\in\mathbf{N}$ and any $i\in\mathbf{Z}$, let $[i]$ denotes the unique representative of i modulo M such that $-M/2<[i]\le M/2$. For $M=\infty$, we define $[i]\coloneqq i$. For any connected graph $\vec{\Gamma}$, the number of connected components of $\vec{\Gamma}\otimes\vec{C}_M$ is M if and only if* $\operatorname{der}(\vec{\Gamma})\equiv 0 \pmod M$. *Otherwise it is equal to the absolute value of* $[\operatorname{der}(\vec{\Gamma})]$.

In particular, the number of connected component of $\vec{\Gamma}\otimes\vec{\mathbf{Z}}$ is infinite if $\operatorname{der}(\vec{\Gamma})=0$ *and* $\operatorname{der}(\vec{\Gamma})$ *otherwise.*

Proof. Choose a vertex v_0 in $\vec{\Gamma}$. For every vertex w of $\vec{\Gamma}$ there is a path q in $\underline{\Gamma}$ from w to v_0, of length n and signature σ. For any i there is obviously a path r in $\underline{\vec{C}_M}$ of length n and signature σ with initial vertex $[i]$ and final vertex $\big[i+\operatorname{der}(r)\big]$. Hence, for every vertex $(w,[i])$ of the tensor product, there is a path p from $(w,[i])$ to $\big(v_0,[i+\operatorname{der}(r)\big)]$ in $\underline{\vec{\Gamma}\otimes\vec{C}_M}$. Therefore, to count the number of connected components of $\vec{\Gamma}\otimes\vec{C}_M$ it is sufficient to know when two vertices $(v_0, [i])$ and $(v_0, [k])$ are connected. But they are connected if and only if $(v_0, [0])$ and $(v_0, [k-i])$ are connected.

Let i_0 be the non-zero integer with the smallest absolute value such that $(v_0, 0)$ and $(v_0, [i_0])$ are connected by a path in $\underline{\vec{\Gamma} \otimes \vec{C}_M}$. If such an integer does not exist, put $i_0 = M$. The previous discussion implies that $i_0 = M$ if and only if the number of connected components of $\vec{\Gamma} \otimes \vec{C}_M$ is M. On the other hand, $i_0 = M$ if and only if every path p in $\underline{\vec{\Gamma} \otimes \vec{C}_M}$ with initial vertex $(v_0, 0)$ and final vertex (v_0, j) satisfies $j = 0$, in which case $\operatorname{der}(p) \equiv 0 \pmod M$. But this is equivalent (by Lemma 2.3.7 and by the existence in $\underline{\vec{C}_M}$ of a path with arbitrary signature) to every closed path q in $\underline{\vec{\Gamma}}$ with initial vertex v_0 having $\operatorname{der}(q) \equiv 0 \pmod M$, which is equivalent to $\operatorname{der}(\vec{\Gamma}) \equiv 0 \pmod M$.

If $i_0 \neq M$, we have either $M = \infty$ or $M/2 < i_0 \leq M/2$. In both cases $i_0 = [i_0]$. For every integer j, since $(v_0, 0)$ and (v_0, i_0) are connected, their images $(v_0, [j])$ and $(v_0, \left[[j] + i_0\right])$ by the automorphism $\operatorname{Id} \otimes \varphi_{0,[j]}$ are connected, where $\varphi_{0,[j]}$ is the automorphism of $\vec{C}_M$ sending i on $i + [j]$. Hence the vertices $(v_0, \left[[j] - i_0\right])$ and $(v_0, \left[[j] - i_0 + i_0\right])$ are also connected. As a special case we have that $(v_0, 0)$ and $(v_0, [-i_0])$ are connected. Therefore we can suppose that i_0 is strictly positive and $0 < i_0 \leq M/2$. We also have by induction that for all j, $(v_0, [j])$ is connected to (v_0, k) for some $0 \leq k \leq i_0$. On the other hand, $(v_0, 0)$, $(v_0, 1)$, $\ldots$ and $(v_0, i_0 - 1)$ are in different connected components by minimality of i_0. Hence, the number of connected components of $\vec{\Gamma} \otimes \vec{C}_M$ is i_0.

Let us now show that i_0 is equal to the absolute value of $[\operatorname{der}(\vec{\Gamma})]$. Take a path q in $\underline{\vec{\Gamma}}$ with initial vertex v_0 and such that $|\operatorname{der}(q)| = \operatorname{der}(\vec{\Gamma})$. By Lemma 2.3.7 this gives a path p in $\underline{\vec{\Gamma} \otimes \vec{C}_M}$ with $\operatorname{der}(p) = \operatorname{der}(\vec{\Gamma})$, initial vertex $(v_0, 0)$ and final vertex $(v_0, [\operatorname{der}(p)])$. This implies (by minimality of i_0) that the absolute value of $[\operatorname{der}(\vec{\Gamma})]$ is bigger or equal to i_0, which is the number of connected components of $\vec{\Gamma}$.

It remains to show that i_0 is bigger or equal to the absolute value of $[\operatorname{der}(\vec{\Gamma})]$. Now, if $(v_0, 0)$ is connected by p to (v_0, i_0), the derangement of p is equal to i_0 modulo M. This gives us a closed path q (from v_0 to v_0) in $\underline{\vec{\Gamma}}$ with derangement $i_0 + aM$ for some integer a. Since $[i_0 + aM] = [i_0] = i_0$, we have found a path q in $\vec{\Gamma}$ such that $[\operatorname{der}(q)] = i_0$. On the other hand, we have $\operatorname{der}(\vec{\Gamma}) \geq |\operatorname{der}(q)|$. We still have to show that $|\operatorname{der}(q)| \geq [\operatorname{der}(q)]$. But the stronger inequality $|i| \geq |[i]|$ is true for every integer i. Indeed, if $-M/2 < i \leq M/2$ we have $i = [i]$ and therefore $|i| = |[i]|$. Otherwise, $|i| > M/2 \geq |[i]|$. ☐

An analogous proposition holds for graphs, where the derangement is replaced by the length of a path and minimum is replaced by greatest common divisor. This gives a refinement of the following proposition: $\Gamma \otimes \Delta$ is connected if and only if Γ and Δ are connected and at least one factor is non-bipartite ([68], Theorem 5.29).

Tensor product of a Schreier graph and an oriented cycle

Here we keep $\vec{C}_M$, $M \in \overline{\mathbf{N}} = \{1, 2, \ldots, \infty\}$, as one factor of the tensor product and take the other one to be a Schreier graph. In the following, we will always assume that generating system of a groups do not contain element of order one or two. Under this assumption, Schreier and Cayley graphs do not contain degenerate loop and if e is an edge with label x, then $\bar{e}$ has label $x^{-1} \neq x$.

Definition 4.2.5. Let w be a word in the alphabet $X \sqcup X^{-1}$. For $x \in X$, the *exponent* of x in w, $\exp_x(w)$ is the number of times x appears in w minus the number of times x^{-1} appears in w. We also define the *exponent* of X as the sum of exponents:

$$\exp_X(w) := \sum_{x \in X} \exp_x(w).$$

The definition immediately implies

Lemma 4.2.6. *Let $G = \langle X \mid R \rangle$ be a group presentation. Then the derangement of a path in $\vec{\mathrm{Cay}}(G, X)$ is exactly the exponent of its label.*

Proposition 4.2.7. *Fix $M \in \overline{\mathbf{N}}$ and let $G = \langle X \mid R \rangle$ be a group presentation such that $\exp_X(r) \equiv 0 \pmod{M}$ for every relator $r \in R$. Then $\vec{\mathrm{Cay}}(G, X)$ is strongly isomorphic to any connected component of $\vec{\mathrm{Cay}}(G, X) \otimes \vec{C}_M$.*

Proof. Let p be a path with initial vertex 1 in $\mathrm{Cay}(G, X^{\pm})$ and let w be its label. Then $w = 1$ in G if and only if p is closed. But $w = 1$ in G if and only if $w = \prod h_i r_i h_i^{-1}$, where the r_i are relators and the h_i are words in $X \sqcup X^{-1}$.

On the other hand, by the previous lemma the derangement of p is equal to

$$\begin{aligned}\exp_X(w) &= \exp_X\big(\prod h_i r_i h_i^{-1}\big)\\ &= \sum \big(\exp_X(h_i) + \exp_X(r_i) - \exp_X(h_i)\big)\\ &\equiv 0 \pmod{M}.\end{aligned}$$

We conclude using Theorem 4.2.2 and Remark 4.2.3. □

Lemma 4.2.8. *Fix $M \in \overline{\mathbf{N}}$ and let $G = \langle X \mid R \rangle$ be a group presentation such that $\exp_X(r) \equiv 0 \pmod{M}$ for every relator $r \in R$. A labeled digraph $\vec{\Gamma}$ is the digraph of an action of G if and only if $\vec{\Gamma} \otimes \vec{C}_M$ is also the digraph of an action of G.*

Proof. Let $\vec{\Theta}$ be any X-labeled graph such that for each $x \in X$ and each vertex v, there is exactly one outgoing and one ingoing edge with label x. It is clear that $\vec{\Theta}$ is (strongly isomorphic to) a digraph of an action of $G = \langle X \mid R \rangle$ if and only if for every $r \in R$, and for every vertex v, the unique path with initial vertex v and label r is closed.

Now, fix v a vertex in $\vec{\Gamma}$, r a word on $X \sqcup X^{-1}$ and $0 \leq i < M$. There is a unique path p with initial vertex v and label r in $\vec{\Gamma}$ and a unique path q with initial vertex (v, i) and label r in $\vec{\Gamma} \otimes \vec{C}_M$. We have that $\tau(q) = \big(\tau(p), i + \mathrm{der}(p)\big)$. Therefore, if r is a relator we have $\tau(q) = (\tau(p), i)$ and p is closed if and only if q is closed. □

Using this lemma and Proposition 4.2.1 we have the following.

Proposition 4.2.9. *Fix $M \in \overline{\mathbf{N}}$ and let $G = \langle X \mid R \rangle$ be a group presentation such that $\exp_X(r) \equiv 0 \pmod{M}$ for every relator $r \in R$. Let H be a subgroup of G and let $\vec{\Gamma} := \vec{\mathrm{Sch}}(G, H, X)$ be the corresponding Schreier graph. Then, every connected component of $\vec{\Gamma} \otimes \vec{C}_M$ is the Schreier graph of G with respect to X and to the subgroup $H_M := \{h \in H \mid \exp_X(h) \equiv 0 \pmod{M}\}$.*

Proof. First, note that since $\exp_X(r) \equiv 0 \pmod{M}$ for every relator r, the exponent of $g \in G$ is well defined modulo M. By Proposition 4.2.1 and Remark 4.2.3, all connected components of $\vec{\Gamma} \otimes \vec{C}_M$ are strongly isomorphic. By the previous lemma, $\vec{\Gamma} \otimes \vec{C}_M$ is a digraph of an action of G and therefore all its connected components are Schreier graphs of G.

Now, let v be a vertex in $\vec{\Gamma}$ corresponding to the subgroup H. The subgroup H_M consists of labels of paths from $(v, 0)$ to $(v, 0)$ in $\underline{\vec{\Gamma} \otimes \vec{C}_M}$. By Lemma 2.3.7, for any signature σ, there is a bijection between the set of closed paths p with initial vertex $(v, 0)$ and signature σ and the set of couples (q, r) where q is a closed path with initial vertex v, r a closed path with initial vertex 0, both of signature σ. But there is a path r from 0 to 0 with signature σ in $\vec{C}_M$ if and only if $\mathrm{der}(r) \equiv 0 \pmod{M}$, and in this case there is a unique such path. Finally, we conclude using the fact that the labeling of $\vec{\Gamma} \otimes \vec{C}_M$ is inherited from the labeling of $\vec{\Gamma}$. □

Observe that if in Proposition 4.2.9, $\Gamma = \vec{\mathrm{Cay}}(G, X)$ then it corresponds to a Schreier graph with $H = \{1\}$ and thus $H_M = \{1\}$ and every connected component of $\vec{\Gamma} \otimes \vec{C}_M$ is isomorphic to $\vec{\Gamma}$ itself.

Remark 4.2.10. Lemma 4.2.8 and Proposition 4.2.9 have a geometrical meaning. If for every relator r, $\exp_X(r) \equiv 0 \pmod M$, then G naturally acts on $\vec{C}_M$ by $i.g := i + \exp_X(g)$. Therefore, the action of G on $\vec{\Gamma} \otimes \vec{C}_M$ is the product of the actions. In term of Schreier graphs, that exactly means that $H_M = H \cap L_M$, where L_M is the subgroup of G which stabilizes the vertex $i \in \vec{C}_M$.

The case of lamplighter groups

In this subsection, we will look at tensor product where on of the factor is $\vec{C}_M$, $M \in \overline{\mathbf{N}}$, and the other one a Schreier digraph of the Lamplighter groups. We turn our attention on this particular example because it will be of great interest later.

By the lamplighter group $\mathcal{L}_k$, for $k \geq 2$, we mean the restricted wreath product $\mathbf{Z}/k\mathbf{Z} \wr \mathbf{Z} \cong \left(\bigoplus_{\mathbf{Z}} \mathbf{Z}/k\mathbf{Z}\right) \rtimes \mathbf{Z}$ where $\mathbf{Z}$ acts on the normal subgroup $A_k := \bigoplus_{\mathbf{Z}} \mathbf{Z}/k\mathbf{Z}$ by shifting the coordinates. It is easy to see that it is given by the presentation

$$(\dagger) \qquad \mathcal{L}_k := \langle b, c \mid c^k, [c, b^n c b^{-n}]; n \in \mathbf{N} \rangle,$$

where $[x, y] = x^{-1}y^{-1}xy$ is the commutator of x and y. Observe that this in particular implies $[b^m c b^{-m}, b^n c b^{-n}] = 1$ in $\mathcal{L}_k$ for all m and n in $\mathbf{Z}$.

The subgroup $A_k = \oplus_{\mathbf{Z}} \mathbf{Z}/k\mathbf{Z}$, called the base of the wreath product, is generated by $\{b^i c b^{-i}\}_{i \in \mathbf{Z}}$, while $\mathbf{Z}$ is generated by b. The (right) action of b on A_k is by shift, $b^i c b^{-i}.b = b^{i-1} c b^{-i+1}$.

Following [59], instead of the "classical" presentation (†) of $\mathcal{L}_k$, we will use the automaton presentation. Consider the set $X_k = \{b, \bar{c}_1, \ldots, \bar{c}_{k-1}\}$, where $\bar{c}_i = bc^i$. Note that $c = b^{-1}\bar{c}_1$, so X_k does generate $\mathcal{L}_k$. It will be convenient to write $\bar{c}_0$ for $b = bc^0$, so that $X_k = \{\bar{c}_i\}_{i=0}^{k-1}$.

$$(\ddagger) \qquad \mathcal{L}_k = \langle X_k \mid (b^{-1}\bar{c}_1)^k, b(b^{-1}\bar{c}_1)^i \bar{c}_i^{-1}, [b^{-1}\bar{c}_1, b^{n-1}\bar{c}_1 b^{-n}]; n \in \mathbf{N}, 2 \leq i \leq k-1 \rangle.$$

It is possible to check that 1 does not belong to X_k and that $x \in X_k$ implies that $x^{-1} \notin X_k$ and that the generators are two by two distincts. In particular, the graph $\mathrm{Cay}(\mathcal{L}_k, X_k^{\pm})$ is strongly isomorphic to $\underline{\vec{\mathrm{Cay}}(\mathcal{L}_k, X_k)}$.

Remark 4.2.11. With this particular choice of generators, the graph $\mathrm{Cay}(\mathcal{L}_k, X_k^{\pm})$ is weakly isomorphic to the Diestel-Leader graph $\mathrm{DL}(k, k)$ (see [121]).

Remark 4.2.12. It is easy to see that, if G is any finite group of order k and we consider the restricted wreath product $G \wr \mathbf{Z}$, where $\mathbf{Z} = \langle b \rangle$ and choose the generating set $\{bg\}_{g \in G}$, then the corresponding Cayley graph will be also weakly isomorphic to $\mathrm{Cay}(\mathcal{L}_k, X_k^{\pm})$ and thus to the Diestel-Leader graph $\mathrm{DL}(k, k)$. For the following, we will focus on the lamplighter group $\mathcal{L}_k$.

We immediately have

$$\exp_X(b^{-1}\bar{c}_1)^k = \exp_X b(b^{-1}\bar{c}_1)^i \bar{c}_i^{-1} = \exp_X([b^{-1}\bar{c}_1, b^{n-1}\bar{c}_1 b^{-n}]) = 0.$$

Hence, the presentation (‡) of $\mathcal{L}_k$ satisfies the hypothesis of Proposition 4.2.7 and we have the following special case of Proposition 4.2.9.

Proposition 4.2.13. *For all $k \geq 2$ and $M \in \overline{\mathbf{N}}$, every connected component of the digraph $\vec{\mathrm{Cay}}(\mathcal{L}_k, X_k) \otimes \vec{C}_M$ is strongly isomorphic to $\vec{\mathrm{Cay}}(\mathcal{L}_k, X_k)$.*

4.3 Limit of de Bruijn and Spider-web graphs

A slightly different version of spider-web graphs, called spider-web networks, was first introduced by Ikeno in [67]. The 2-parameter family $\{\mathcal{S}_{k,N,M}\}$ that we will presently define is a natural extension of the well known 1-parameter family of the de Bruijn graphs $\{\mathcal{B}_{k,N}\}$, $k \geq 2$. In [6], Balram and Dhar observed, in the special case $k = 2$, some link between spider-web graphs and the Cayley graph of the lamplighter group $\mathcal{L}_2$.

The aim of this section is to discuss the definition of spider-web graphs $\mathcal{S}_{k,N,M}$ and to show that they converge to the Cayley graph of the lamplighter group $\mathcal{L}_k$. This is Theorem 4.3.17 for the oriented case and Corollary 4.3.18 for the non-oriented case. In order to do that, we first prove Theorem 4.3.14 which shows that de Bruijn graphs are weakly isomorphic to Schreier graphs of the lamplighter group.

From now on, we fix a $k \geq 2$ and omit to write it when it is not necessary. We will use the notations $\mathbf{N} = \{1, 2, \ldots\}$, $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$ and $\overline{\mathbf{N}} = \mathbf{N} \cup \{\infty\}$.

De Bruijn Graphs

Definition 4.3.1. For every $N \in \mathbf{N}_0$ and $\in \mathbf{N}_{\geq 2}$, the *de Bruijn digraph* $\vec{\mathcal{B}}_{k,N}$ on k symbols is the labeled digraph with vertex set $\{0, \ldots, k-1\}^N$ and, for every vertex $x_1 \ldots x_N$, k outgoing edges labeled by 0 to $k-1$. The edge labeled by y has $x_2 \ldots x_N y$ as final vertex.

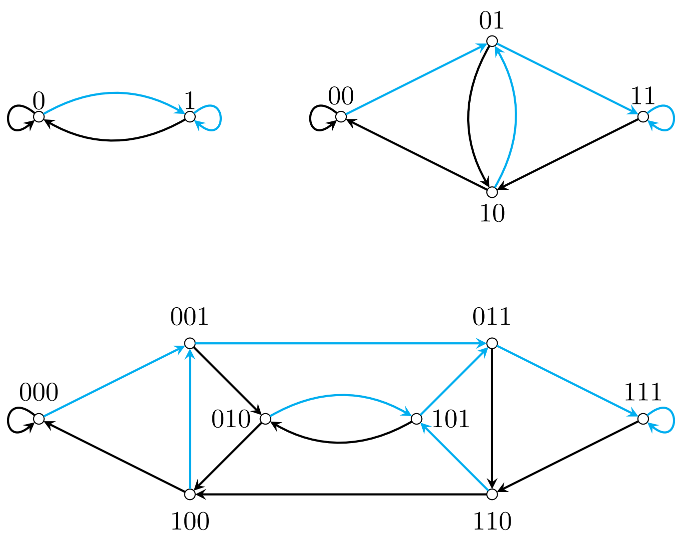


Figure 4.3: De Bruijn digraphs for $k = 2$ and $N \in \{1, 2, 3\}$. Black edges are labeled by 0 and cyan edges are labeled by 1.

Sometimes de Bruijn digraphs are defined as graphs. In the following, we will write $\vec{\mathcal{B}}_N = \vec{\mathcal{B}}_{k,N}$ for the oriented version and $\mathcal{B}_N = \mathcal{B}_{k,N}$ for the non-oriented one. Note that in our formalism, the graph $\mathcal{B}_N$ is $\underline{\vec{\mathcal{B}}_N}$.

De Bruijn digraphs are widely seen as representing overlaps between strings of symbols and are also combinatorial models of the Bernoulli map $x \mapsto kx \pmod 1$ and therefore are of interest in the theory of dynamical systems.

It is shown in [123] that each de Bruijn digraph $\vec{\mathcal{B}}_N$ is the line digraph (see Definition 2.3.3) of the previous one, $\vec{\mathcal{B}}_{N-1}$. For the sake of completeness we include here a proof of this fact, which is crucial for our purposes.

Lemma 4.3.2 ([123]). *For every $N \in \mathbf{N}_0$, the de Bruijn digraph $\vec{\mathcal{B}}_{N+1}$ is (weakly) isomorphic to the line digraph of $\vec{\mathcal{B}}_N$.*

Proof. It is clear from the definition that $\vec{\mathcal{B}}_1$ is weakly isomorphic to the complete digraph on k vertices, with loops. That is, $\vec{\mathcal{B}}_1$ has k vertices and for each pair (v, w) of vertices, there is exactly one edge from v to w. In particular, for every v there is a unique edge from v to itself. It is then obvious that $\vec{\mathcal{B}}_1$ is weakly isomorphic to the line digraph of $\vec{\mathcal{B}}_0$ (the rose).

Observe that, for any N, for each vertex v in $\vec{\mathcal{B}}_N$ and each label y, there is exactly one edge e with initial vertex v and label y. Therefore, there is a natural bijection between the vertex set of the line digraph of $\vec{\mathcal{B}}_N$ and the set of couples $\{(v, y) \mid v \text{ a vertex in } \vec{\mathcal{B}}_N, 0 \leq y < k\}$. Let $v = (x_1 \dots x_N)$ be a vertex in $\vec{\mathcal{B}}_N$. If $N \geq 1$, there is an edge in the line digraph from (v, y) to (w, z) if and only if $w = (x_2 \dots x_N y)$.

We construct now an explicit weak isomorphism φ from the line digraph of $\vec{\mathcal{B}}_N$ to $\vec{\mathcal{B}}_{N+1}$. We define φ on the vertices by $\varphi(v, y) := (x_1 \dots x_N y)$ if $v = (x_1 \dots x_N)$. This is obviously a bijection. If $N \geq 1$, there is a unique edge in the line digraph from $\big((x_1 \dots x_N), y\big)$ to $\big((x_2 \dots x_N y), z\big)$ (and all edges are of this form). Let the image of this edge by φ be the unique edge in $\vec{\mathcal{B}}_{N+1}$ with initial vertex $(x_1 \dots x_N y)$ and label z — see Figure 4.4. It is straightforward to see that φ is injective on the set of edges. Since the two graphs have the same finite number of edges ($k \cdot k^{N+1}$), φ is also bijective on the set of edges. Moreover, by definition, $\varphi\big(\iota(e)\big) = \iota\big(\varphi(e)\big)$ for any edge e in the line digraph. Hence, to show that φ is a weak isomorphism it only remains to check that $\varphi\big(\tau(e)\big) = \tau\big(\varphi(e)\big)$. If e is an edge from $\big((x_1 \dots x_N), y\big)$ to $\big((x_2 \dots x_N y), z\big)$, we have

$$\varphi\big(\tau(e)\big) = (x_2 x_3 \dots x_N y z).$$

On the other hand, $\varphi(e)$ has initial vertex $(x_1 \dots x_N y)$ and label z. Therefore,

$$\tau\big(\varphi(e)\big) = (x_2 x_3 \dots x_N y z) = \varphi\big(\tau(e)\big).$$

□

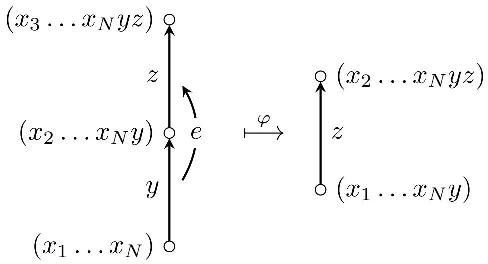


Figure 4.4: The edge e in the line digraph of $\vec{\mathcal{B}}_N$ and its image by φ.

Spider-web graphs

Definition 4.3.3. Let $k \geq 2$. For all $M \in \mathbf{N}$ and $N \in \mathbf{N}_0$, the *spider-web digraph* is the labeled digraph $\vec{\mathcal{S}}_{N,M} = \vec{\mathcal{S}}_{k,N,M}$ with vertex set $\{0, \dots, k-1\}^N \times \{1, 2, \dots, M\}$ and for every vertex $v = (x_1 \dots x_N, i)$ with k outgoing edges labeled by 0 to $k-1$. The edge labeled by y has $(x_2 \dots x_N y, i+1)$ as final vertex, where $i+1$ is taken modulo M. See figure 4.5 for an example.

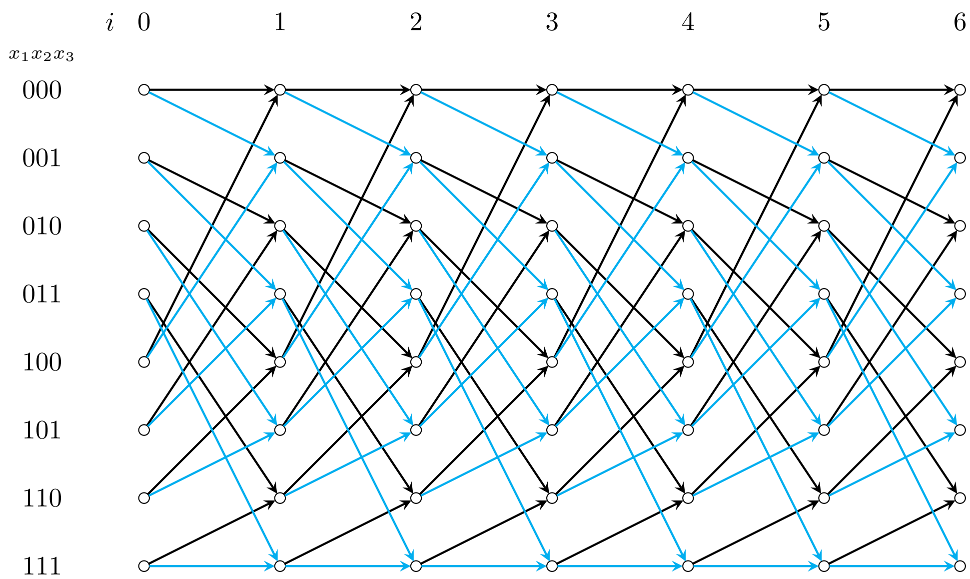


Figure 4.5: The digraph $\vec{\mathcal{S}}_{2,3,6}$, where vertices in the slice $i = 0$ are identified with vertices in the slice $i = M = 6$. Black edges correspond to edges with label 0 and cyan edges to edges with label 1.

As with de Bruijn digraphs, we write $\vec{\mathcal{S}}_{N,M}$ for the oriented version and $\mathcal{S}_{N,M}$ for the non-oriented one. The vertices of $\mathcal{S}_{N,M}$ are partitioned into slices $1, \dots, M$ and edges connect vertices in slice i to vertices in slice $i + 1 \pmod M$. Note that in our definition (unlike papers that talk about spider-web networks), the vertices of the slice M are connected to the vertices of the slice 1. Note also that it is possible to similarly define $\vec{\mathcal{S}}_{k,N,\infty}$ for all N (with vertex set $\{0, \dots, k-1\}^N \times \mathbf{Z}$).

Observe that the graph $\vec{\mathcal{S}}_{k,0,M}$ is a "thick" oriented circle (or line if $M = \infty$): the vertex set is M and for every vertex i there are k edges from i to $i + 1$. The graph $\vec{\mathcal{S}}_{1,0,M}$ is the usual oriented circle $\vec{C}_M$. Therefore, $\vec{\mathcal{S}}_{k,0,1}$ is the rose with one vertex and k oriented edges.

Lemma 4.3.4. *For all $M \in \overline{\mathbf{N}}$ and $N \in \mathbf{N}_0$ there is a strong isomorphism between $\vec{\mathcal{S}}_{N,M}$ and $\vec{\mathcal{S}}_{N,1} \otimes \vec{C}_M$.*

Remark 4.3.5. Observe that this lemma is not true if we consider the non-oriented spider-web graphs. This is the main reason why we are brought to work with digraphs in this chapter, even though the final result that we aim at and that we get are about graphs (Corollary 4.3.18).

Lemma 4.3.4 together with Theorem 2.3.6 ensures that in order to identify the limit of spider-web graphs when $M, N \to \infty$ it is enough to study the limit of spider-web digraphs with $M = 1$. It turns out that the spider-web digraphs with $M = 1$ are exactly de Bruijn digraphs.

Indeed the identification given by $(x_1 \dots x_N, 1) \mapsto (x_1 \dots x_N)$ induces a strong isomorphism between $\vec{\mathcal{S}}_{N,1}$ and $\vec{\mathcal{B}}_N$. The isomorphism between non-oriented versions follows. Hence we have the following.

Lemma 4.3.6. *For all $N \in \mathbf{N}_0$, the digraph $\vec{\mathcal{S}}_{N,1}$ is strongly isomorphic to $\vec{\mathcal{B}}_N$ and the graph $\mathcal{S}_{N,1}$ is strongly isomorphic to $\mathcal{B}_N$.*

Lemma 4.3.4 directly implies the following.

Corollary 4.3.7. *For all $M \in \overline{\mathbf{N}}$ and $N \in \mathbf{N}_0$, $\vec{\mathcal{S}}_{N,M}$ is strongly isomorphic to $\vec{\mathcal{B}}_N \otimes \vec{C}_M$.*

Lemma 4.3.8. *For all $M \in \overline{\mathbf{N}}$ and $N \in \mathbf{N}_0$, the graph $\mathcal{S}_{N,M}$ is connected.*

Proof. By the previous corollary, $\vec{\mathcal{S}}_{N,M} \simeq \vec{\mathcal{B}}_N \otimes \vec{C}_M$. On the other side, $\mathrm{der}(\vec{\mathcal{B}}_N) = 1$ since there is a loop (labeled by 0) at the vertex $(0 \dots 0)$. Therefore, by Proposition 4.2.4, the number of connected components of $\vec{\mathcal{S}}_{N,M}$ is 1. □

The group $\mathcal{L}_k$ and its action on the k-regular rooted tree

We will use the language of actions on rooted trees, see for instance [55] and [93].

Fix $k \geq 2$. The strings over the alphabet $\mathcal{A} = \{0, 1, \dots, k-1\}$ are in one-to-one correspondence with the vertices of the k-regular rooted tree T_k where the root vertex corresponds to the empty string. Under this correspondence, the n^{th} level of T_k is the set of strings over $\mathcal{A}$ of length n. The *boundary* ∂T_k of T_k is the set of right infinite strings over $\mathcal{A}$. We write $\overline{T}_k \coloneqq T_k \cup \partial T_k$.

We also have a one-to-one correspondence between T_k and the ring of polynomials $\mathbf{Z}/k\mathbf{Z}[t]$ given by

$$(x_0 \dots x_n) \mapsto x_0 + x_1 t \cdots + x_n t^n$$

and a one-to-one correspondence between ∂T_k and the ring of formal series $\mathbf{Z}/k\mathbf{Z}[[t]]$ given by

$$(\star) \qquad (x_0 x_1 x_2 \dots) \mapsto \sum_{i \geq 0} x_i t^i.$$

Let $G \leq \mathrm{Aut}(T_k)$ be a group acting on T_k by automorphisms. The action is said to be *spherically transitive* if it is transitive on each level.

Extending a result from Grigorchuk and Żuk [59] for $k = 2$, Silva and Steinberg showed in [114] that the lamplighter group $\mathcal{L}_k$ introduced in Subsection 4.2 above acts on $\overline{T}_k$. They showed that this action is faithful and spherically transitive and described some other interesting properties of this action. Here, we will look at the action given by

$$(x_1 x_2 x_3 \dots).\bar{c}_r = \big((x_1 + r)(x_2 + x_1)(x_3 + x_2) \dots\big)$$

where additions are taken modulo k. This action is slightly different from the one in [114], but remains faithful and spherically transitive.

Since the action of $\mathcal{L}$ on T is spherically transitive, for any generating set Y, for all N, the graph of the action on the N^{th} level are connected. Thus, they can be viewed as Schreier graphs $\vec{\mathrm{Sch}}(\mathcal{L}, H_N, Y)$, where $H_N = \mathrm{Stab}_{\mathcal{L}}(v_N)$, with v_N any vertex of the N^{th} level of T. On the other hand, it is obvious that the graph of the action on ∂T is not connected. Its connected components correspond to the orbits of the (countable) group $\mathcal{L}$ on the (uncountable) set ∂T. They can be viewed as Schreier graphs of the subgroups $\mathrm{Stab}_{\mathcal{L}}(\xi)$, $\xi \in \partial T$. If v, w are two vertices of T, with w lying on an infinite ray emanating from v, then $\mathrm{Stab}_{\mathcal{L}}(w)$ is a subgroup of $\mathrm{Stab}_{\mathcal{L}}(v)$. This implies that for every N, the graph $\vec{\mathrm{Sch}}(\mathcal{L}, H_{N+1}, Y)$ covers $\vec{\mathrm{Sch}}(\mathcal{L}, H_N, Y)$, and we deduce the following.

Proposition 4.3.9 ([51],[74])**.** *For any generating set Y and any $\xi = (x_1 x_2 \dots)$ ray in ∂T, the sequence of rooted graphs $\big(\vec{\mathrm{Sch}}(\mathcal{L}, H_N, Y), (x_1 \dots x_n)\big)$ converges (as labeled graphs) to*

$$\big(\vec{\mathrm{Sch}}(\mathcal{L}, \mathrm{Stab}_{\mathcal{L}}(\xi), Y), \xi\big).$$

We also have

Proposition 4.3.10 ([51],[74]). *For all but countably many rays $\xi \in \partial T$, the digraph $\vec{\mathrm{Sch}}(\mathcal{L}, \mathrm{Stab}_{\mathcal{L}}(\xi), Y)$ is strongly isomorphic to $\vec{\mathrm{Cay}}(\mathcal{L}, Y)$.*

Proof. the labeled digraph $\big(\vec{\mathrm{Sch}}(\mathcal{L}, \mathrm{Stab}_{\mathcal{L}}(\xi), Y), \xi\big)$ is strongly isomorphic to $\vec{\mathrm{Cay}}(\mathcal{L}, Y)$ if and only if $\mathrm{Stab}_{\mathcal{L}}(\xi) = \{1\}$. We will show that $\mathrm{Stab}_{\mathcal{L}}(\xi) \neq \{1\}$ for only countably many ξ. In order to prove that, we look at the equivalent action of $\mathcal{L}$ on $\mathbf{Z}/k\mathbf{Z}[[t]]$. Applying formula ($\star$) we set for any $F \in \mathbf{Z}/k\mathbf{Z}[[t]]$

$$F.c = F + 1$$
$$F.b = (1+t)F.$$

Hence for any i we have

$$F.b^{-i}cb^{i} = F + (1+t)^{i}.$$

Let $1 \neq g$ be an element in $\mathcal{L}$. Then g admits a unique decomposition as $g = b^{i}h$ for some $i \in \mathbf{Z}$ and $h \in A = \langle \{b^{j}cb^{-j}\}_{j\in Z} \rangle$. Therefore, there exists P a finite sum of $(1+t)^{j}$'s, $j \in \mathbf{Z}$, such that for any $F \in \mathbf{Z}/k\mathbf{Z}[[t]]$,

$$F.g = (1+t)^{i}F + P \qquad \text{and}$$
$$F.g^{-1} = (1+t)^{-i}F - (1+t)^{-i}P.$$

Since the action is faithful (and $g \neq 1$), it is not possible that $i = 0$ and $P = 0$ together. Now, suppose F has non-trivial stabilizer. Then there exists $1 \neq g \in \mathcal{L}$ such that $F = F.g = F.g^{-1}$ and therefore F is a solution to the equations

$$\big((1+t)^{i} - 1\big)x = -P$$
$$\big((1+t)^{-i} - 1\big)x = (1+t)^{-i}P.$$

We have $i \neq 0$, otherwise we would have $P = 0$, which is absurd. Hence, F is a solution of

$$(*) \qquad \big((1+t)^{i} - 1\big)x = Q,$$

with $i > 0$ and Q belonging to $L := \mathbf{Z}/k\mathbf{Z}[(1+t), (1+t)^{-1}]$, the subring of $\mathbf{Z}/k\mathbf{Z}[[t]]$ consisting of Laurent polynomials in $1+t$. Note that $(1+t)^{i} - 1$ is not invertible and thus we cannot write $x = Q/\big((1+t)^{i} - 1\big)$.

Suppose for a moment that k is prime. This is equivalent to $\mathbf{Z}/k\mathbf{Z}[[t]]$ being an integral domain. In this case, given $a \neq 0$ and b in $\mathbf{Z}/k\mathbf{Z}[[t]]$, the equation $ax = b$ has at most one solution. Now, if F has a non-trivial stabilizer, we had just proved that it satisfies equation ($*$). Since L is countable, there are only countably many equations of this form and, by unicity of solution, countably many solutions of such equation and hence countably many F with non-trivial stabilizer.

If k is not prime, we do not have the unicity of solution of equations in $\mathbf{Z}/k\mathbf{Z}[[t]]$. For example, for $k = 6$ the equation $2t \cdot x = 0$ admits uncountably many different solutions (all series x where all coefficients belongs to $\{0, 3\}$). But, in our special case, we claim that ($*$) has only finitely many solutions. Using that, we have again that the number of series F with non-trivial stabilizer is countable.

We now prove the claim. If equation ($*$) has no solution, then the claim is true. If there is as at least one solution, the solutions of ($*$) are in one-to-one correspondence with the solutions of

$$\big((1+t)^{i} - 1\big)x = 0.$$

But such an equation has only finitely many solutions by the following lemma. □

Lemma 4.3.11. *Let R be a finite ring and $P = \sum_{j=0}^{d} p_j t^j \in R[t]$ a polynomial. Then, in $R[[t]]$, the equation $Px = 0$ has*

1. *only one solution ($x = 0$) if the first non-zero coefficient of P does not divide 0;*
2. *at most $|R|^d$ solutions if p_d is invertible.*

Proof. The first statement is trivially true.

The proof of the second statement is by contradiction. Observe that if $x = \sum_{n\geq 0} x_n t^n$ is a solution, we have for all n

$$x_n = -p_d^{-1} \sum_{k=1}^{d} p_{d-k} x_{n+k}.$$

Now, suppose that the equation $Px = 0$ has more than $|R|^d$ solutions. Choose $|R|^d+1$ different solutions y_i. There exists an integer l such that all the $\bar{y}_i$ differ, where $\bar{x} = \sum_{n=0}^{l} x_n t^n$ is the series x up to degree l. We hence have $|R|^d + 1$ distinct polynomials of degree at most l, all satisfying the identities above. But this is not possible. Indeed, we have at most $|R|^d$ different choices for the coefficients of t^l to t^{l-d+1} and all other coefficients are uniquely determined by these ones. □

Until here, all the results of this subsection were true for any generating set Y of $\mathcal{L}_k$. In the following, we will work with our usual generating set $X_k = \{\bar{c}_i\}_{i=0}^{k-1}$.

Notation 4.3.12. Denote by $\vec{\Gamma}_k$ the labeled digraph of the action of $\mathcal{L}_k$ on $\overline{T}_k$, with respect to the generating set X_k. If we restrict this action to the N^{th} level of T_k, the corresponding labeled digraph of the action will be denoted by $\vec{\Gamma}_{k,N}$. The graph corresponding to the restriction of the action to ∂T_k will be denoted by $\vec{\Gamma}_{k,\infty}$.

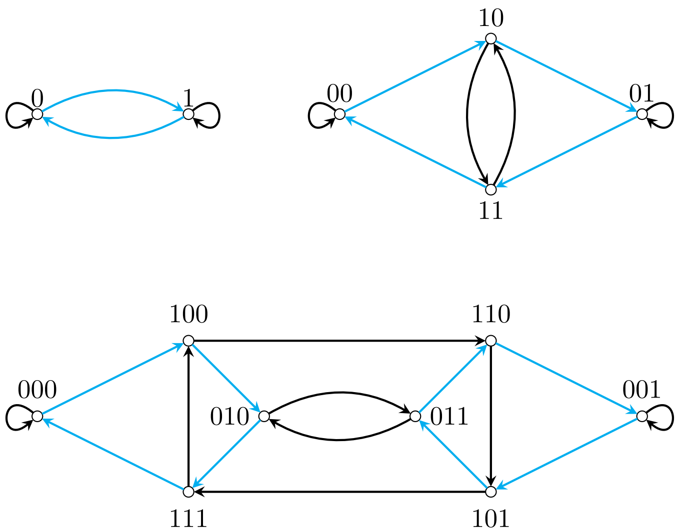


Figure 4.6: The digraphs $\vec{\Gamma}_{2,N}$ for $N \in \{1, 2, 3\}$. Black edges are labeled by $\bar{c}_0$ and cyan edges are labeled by $\bar{c}_1$.

As with spider-web digraphs and de Bruijn digraphs, from now on we omit k from our notation and write simply $\mathcal{L}$, T, X, $\vec{\Gamma}$, $\vec{\Gamma}_N$ and $\vec{\Gamma}_\infty$.

The following proposition will, together with Lemma 4.3.2, help to establish a connection between the lamplighter group and the de Bruijn digraphs.

Proposition 4.3.13. *For every $N \in \mathbf{N}_0$, the digraph $\vec{\Gamma}_{N+1}$ is (weakly) isomorphic to the line digraph of $\vec{\Gamma}_N$.*

Proof. First, we have that $\vec{\Gamma}_0$ is the rose with k loops and $\vec{\Gamma}_1$ is the complete digraph (with loops) on k vertices. Therefore $\vec{\Gamma}_1$ is weakly isomorphic to the line digraph of $\vec{\Gamma}_0$.

We have that the set of vertices in the line digraph of $\vec{\Gamma}_N$ is in bijection with the set of couples

$$\{(v, \bar{c}_i) \mid v \text{ a vertex in } \vec{\Gamma}_N, 0 \leq i < k\}.$$

Let $v = (x_1 \dots x_N)$ be a vertex in $\vec{\Gamma}_N$. If $N \geq 1$, there is an edge in the line digraph from $(v, \bar{c}_i)$ to $(w, \bar{c}_j)$ if and only if $w = v.\bar{c}_i = \big((x_1 + i)(x_2 + x_1) \dots (x_N + x_{N-1})\big)$.

We construct now an explicit weak isomorphism φ from the line digraph of $\vec{\Gamma}_N$ to $\vec{\Gamma}_{N+1}$, see Figure 4.7. We define φ on the vertices by $\varphi\big((x_1 \dots x_N, \bar{c}_i)\big) := (ix_1 \dots x_N)$. It is easy to see that φ is injective (and hence bijective) on vertices. If $N \geq 1$, there is a unique edge in the line digraph from $\big((x_1 \dots x_N), \bar{c}_i\big)$ to $\big((x_1 + i)(x_2 + x_1) \dots (x_N + x_{N-1}), \bar{c}_j\big)$ (and all edges are of this form). Let the image of this edge by φ be the unique edge in $\vec{\Gamma}_{N+1}$ with initial vertex $(ix_1 \dots x_N)$ and label $\bar{c}_{j-i}$ — see Figure 4.7. It is straightforward to see that φ is injective (and thus bijective) on the set of edges. Moreover, by definition, $\varphi\big(\iota(e)\big) = \iota\big(\varphi(e)\big)$ for any edge e in the line digraph. Hence, to show that φ is a weak isomorphism it only remains to check that $\varphi\big(\tau(e)\big) = \tau\big(\varphi(e)\big)$. If e is an edge from $\big((x_1 \dots x_N), \bar{c}_i\big)$ to $\big((x_1 + i)(x_2 + x_1) \dots (x_N + x_{N-1}), \bar{c}_j\big)$, we have

$$\varphi\big(\tau(e)\big) = \big(j(x_1 + i)(x_2 + x_1) \dots (x_N + x_{N-1})\big).$$

On the other hand, $\varphi(e)$ has initial vertex $(ix_1 \dots x_N)$ and label $\bar{c}_{j-i}$. Therefore,

$$\tau\big(\varphi(e)\big) = \big((i + j - i)(x_1 + i)(x_2 + x_1) \dots (x_N + x_{N-1})\big) = \varphi\big(\tau(e)\big).$$

□

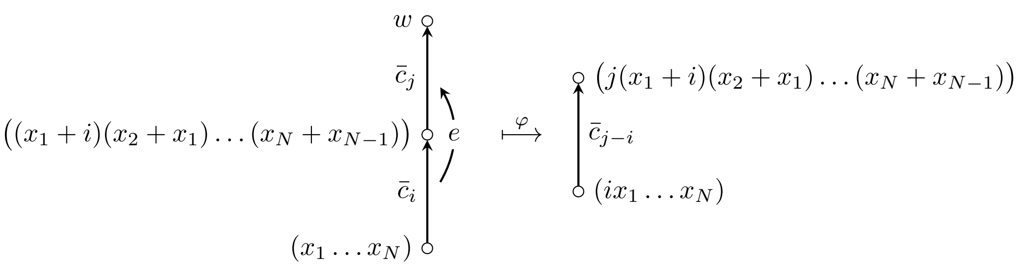


Figure 4.7: The edge e in the line digraph of $\vec{\Gamma}_N$ and its image by φ.

Convergence of the spider-web graphs to the Cayley graph of the lamplighter group

In this subsection, we use results of the last two subsections to finally establish a link between the spider-web digraphs $\vec{\mathcal{S}}_{k,N,M}$ and the digraph $\vec{\Gamma}_k$ of the action of the lamplighter group $\mathcal{L}_k$ on the k-regular tree $\overline{T}_k$ and to prove our main results.

Theorem 4.3.14. *Let $k \geq 2$. For all $N \in \mathbf{N}_0$, the de Bruijn digraph $\vec{\mathcal{B}}_{k,N}$ is weakly isomorphic to $\vec{\Gamma}_{k,N}$, the digraph of the action of $\mathcal{L}_k$ on the N^{th} level of T_k, with respect to the generating set $X_k = \{\bar{c}_r\}_{r=0}^{k-1}$ (see (‡) on page 54).*

Proof. The proof goes by induction on N. For $N = 0$, both graphs are weakly isomorphic to the rose with k loops. For $N \geq 1$ we have by Lemma 4.3.2 that $\vec{\mathcal{B}}_{k,N}$ is weakly isomorphic to the line digraph of $\vec{\mathcal{B}}_{k,N-1}$. Since $\vec{\mathcal{B}}_{k,N-1}$ is (by induction) weakly isomorphic to $\vec{\Gamma}_{k,N-1}$ and that the line digraph does not depend on the labeling we have a weak isomorphism between $\vec{\mathcal{B}}_{k,N}$ and the line digraph of $(\vec{\Gamma}_k)_{N-1}$. By Proposition 4.3.13, this graph is itself weakly isomorphic to $(\vec{\Gamma}_k)_N$. □

Remark 4.3.15. These graphs are not strongly isomorphic. Indeed, in $\vec{\mathcal{B}}_{k,N}$ for every vertex v, all edges ending at v have the same label, but in a Schreier graph two edges having the same end-vertex have distinct labels. (See Figures 4.3 and 4.6.)

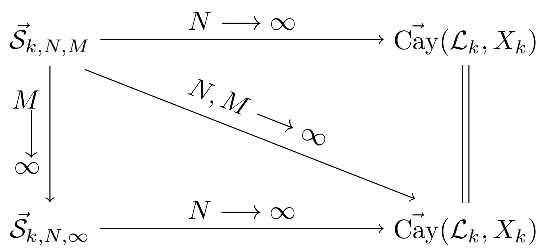
Theorem 4.3.14 and the fact that $\vec{\Gamma}_{k,N}$ covers $\vec{\Gamma}_{k,N-1}$ (see the discussion before Proposition 4.3.9), imply the following property of de Bruijn digraphs that, as far we know, was not observed before.

Corollary 4.3.16. *For all $N \in \mathbf{N}$ the graph $\vec{\mathcal{B}}_N$ (weakly) covers $\vec{\mathcal{B}}_{N-1}$.*

We are now able to prove the main theorem of this chapter.

Theorem 4.3.17. *Let $k \geq 2$. Recall that $\mathcal{L}_k$ denotes the lamplighter group $(\mathbf{Z}/k\mathbf{Z}) \wr \mathbf{Z}$ and $X_k = \{\bar{c}_r\}_{r=0}^{k-1}$ is the generating system given by (‡), see page 54.*

1. *The unlabeled de Bruijn digraphs $\vec{\mathcal{B}}_{k,N}$ converge to $\vec{\mathrm{Cay}}(\mathcal{L}_k, X_k)$ in the sense of Benjamini-Schramm convergence (see Definition 2.2.2).*
2. *The following diagram commutes, where the arrows stand for Benjamini-Schramm convergence of unlabeled digraphs.*



Proof. By Proposition 4.3.10, for all but countably many $\xi = (x_1 x_2 \dots)$ in ∂T_k, the digraph $(\vec{\Gamma}_{k,\infty}, \xi)^0$ is strongly isomorphic to the Cayley digraph $\vec{\mathrm{Cay}}(\mathcal{L}_k, X_k)$. On the other hand, by Proposition 4.3.9, the graphs $\big(\vec{\Gamma}_{k,N}, (x_1 \dots x_N)\big)$ strongly converge to $(\vec{\Gamma}_{k,\infty}, \xi)^0$. Theorem 4.3.14 gives us a weak isomorphism between $\vec{\mathcal{B}}_{k,N}$ and $\vec{\Gamma}_{k,N}$. Therefore, $\vec{\mathcal{B}}_{k,N}$ converge to $\vec{\mathrm{Cay}}(\mathcal{L}_k, X_k)$. This ends the proof of the first part of the theorem.

In order to prove the second part of the theorem, we should consider an auxiliary diagram, see Figure 4.8. First note that we already know that when $N \to \infty$ de Bruijn digraphs $(\vec{\mathcal{B}}_{k,N}, v_N)$ weakly converge to the Cayley digraph $(\vec{\mathrm{Cay}}(\mathcal{L}_k, X_k), 1_{\mathcal{L}_k})$ for nearly all choices of the v_N and that it is obvious that $(\vec{C}_M, 0)$ weakly converge to $(\vec{\mathbf{Z}}, 0)$ when $M \to \infty$. Hence, by Theorem 2.3.6, the diagram in Figure 4.8 is commutative. Finally, since this statement is true for nearly all choices of roots, we have the convergence in the sense of Benjamini-Schramm

$$\begin{array}{ccc}
\big(\vec{\mathcal{B}}_{k,N} \otimes \vec{C}_M, (v_{k,N},0)\big)^0 & \xrightarrow{n \longrightarrow \infty} & \big(\vec{\mathrm{Cay}}(\mathcal{L}_k, X_k) \otimes \vec{C}_M, (1_{\mathcal{L}_k},0)\big)^0 \\
{\scriptstyle M \to \infty}\big\downarrow & \searrow^{n,m \longrightarrow \infty} & \big\downarrow{\scriptstyle M \to \infty} \\
\big(\vec{\mathcal{B}}_{k,N} \otimes \vec{\mathbf{Z}}, (v_{k,N},0)\big)^0 & \xrightarrow{n \longrightarrow \infty} & \big(\vec{\mathrm{Cay}}(\mathcal{L}_k, X_k) \otimes \vec{\mathbf{Z}}, (1_{\mathcal{L}_k},0)\big)^0
\end{array}$$

Figure 4.8: Limit of $\vec{\mathcal{B}}_{k,N} \otimes \vec{C}_M$.

when we choose the roots uniformly. By Proposition 4.2.13, for every $M \in \overline{\mathbf{N}}$, any connected component of $\vec{\mathrm{Cay}}(\mathcal{L}_k, X_k) \otimes \vec{C}_M$ is strongly isomorphic to $\vec{\mathrm{Cay}}(\mathcal{L}_k, X_k)$; in particular, this does not depend on M. On the other hand, for all $M \in \overline{\mathbf{N}}$, $\vec{\mathcal{B}}_{k,N} \otimes \vec{C}_M$ is strongly isomorphic to $\vec{\mathcal{S}}_{k,N,M}$ by Corollary 4.3.7, and therefore is connected. □

Using Proposition 2.2.1, we obtain the same result for non-oriented versions of de Bruijn and spider-web graphs as an immediate corollary.

Corollary 4.3.18.

1. *The unlabeled de Bruijn graphs* $\mathcal{B}_{k,N}$ *converge to* $\mathrm{Cay}(\mathcal{L}_k, X_k^{\pm})$ *in the sense of Benjamini-Schramm convergence (see Definition 2.2.2).*

2. *The following diagram commutes, where the arrows stand for Benjamini-Schramm convergence of unlabeled graphs.*

$$\begin{array}{ccc}
\mathcal{S}_{k,N,M} & \xrightarrow{N \longrightarrow \infty} & \mathrm{Cay}(\mathcal{L}_k, X_k^{\pm}) \\
{\scriptstyle M \to \infty}\big\downarrow & \searrow^{N,M \longrightarrow \infty} & \big\Vert \\
\mathcal{S}_{k,N,\infty} & \xrightarrow{N \longrightarrow \infty} & \mathrm{Cay}(\mathcal{L}_k, X_k^{\pm})
\end{array}$$

Remark 4.3.19. Remark 4.2.12 implies that in Theorem 4.3.17 and its Corollary 4.3.18, $\mathcal{L}_k$ can be replaced by any wreath product $G \wr \mathbf{Z}$, where G is a finite group of cardinality k. However, Theorem 4.3.14 does not hold necessarily if G is not abelian.

4.4 Computation of spectra

In this section we compute the characteristic polynomial and the spectrum of the adjacency matrix of $\mathcal{S}_{N,M}$ for all $M, N \in \mathbf{N}$. The spectra of the graphs $\{\Gamma_N\}$ of the action of $\mathcal{L}_2$ on the levels of the binary rooted tree were first computed by Grigorchuk and Żuk in [59] using the fact that they form a tower of coverings, see the discussion just before Proposition 4.3.9. (Note that the multiplicity in their formula is not completely correct – compare with Theorem 4.4.3 below.) These computations were extended to any wreath product $G \wr \mathbf{Z}$, with $G \neq \{1\}$ finite, by Kambites, Silva and Steinberg in [74], using automata theory. Dicks and Schick [35] computed the spectral measures for random walks on $G \wr \mathbf{Z}$ using entirely different methods (see also [17]).

On the other hand, Delorme and Tillich computed the spectra of $\vec{\mathcal{B}}_N$ in [34] using simple matrix transformations. It is well known that for any pair of square matrices A and B with respective eigenvalues $(\lambda_i)_{i=1}^n$ and $(\mu_j)_{j=1}^m$, the spectrum of $A \otimes B$ is $\{\lambda_i \mu_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$. However, this formula cannot be applied in our case since we want to compute the spectrum of $\underline{\vec{\mathcal{B}}_N \otimes \vec{C}_M}$ and there is no formula relating eigenvalues of a matrix A (the adjacency matrix of a digraph) and of its symmetrized matrix $A + A^*$ (the adjacency matrix of the underlying graph). Instead we generalize Delorme and Tillich method directly to all $\mathcal{S}_{N,M}$.

For A a matrix, we denote its characteristic polynomial by $\chi(A)$. For a (di)graph Γ, the *characteristic polynomial* $\chi(\Gamma)$ of Γ is the characteristic polynomial of its adjacency matrix. The following lemma summarizes discussions 2.(1), 2.(2) and 2.(3) from [34].

Lemma 4.4.1. *Let $\vec{\Gamma}$ be a digraph, $\underline{\vec{\Gamma}}$ the underlying graph and $\vec{A}$ and A their respective adjacency matrices. Suppose that there exist complex matrices D and U with U unitary such that $\vec{A} = U^* D U$. Then $\chi(\vec{\Gamma}) = \chi(D)$. Moreover, $A = U^*(D + D^*)U$ and $\chi(\underline{\vec{\Gamma}}) = \chi(D + D^*)$.*

Proof. We have $A = \vec{A} + (\vec{A})^* = U^* D U + U^* D^* U = U^*(D + D^*)U$. On the other hand, A and D are equivalent matrices and therefore have the same characteristic polynomial. ☐

Observe that for a given complex matrix D we can construct a weighted digraph $\vec{\Theta}$ with adjacency matrix D, where a *weighted graph* is a (di)graph $\Gamma = (V, E)$ (labeled or not, etc.) with a map $w \colon E \to \mathbf{C}$ which assign to each edge a complex number (a weight) such that $w(\bar{e})$ is the complex conjugate of $w(e)$. In this case, $D + D^*$ is the adjacency matrix of $\underline{\vec{\Theta}}$.

In their article, Delorme and Tillich use this to compute the spectrum of de Bruijn graphs $\mathcal{B}_N$ and prove the following.

Proposition 4.4.2 (Dellorme-Tillich)**.** *For all $N \in \mathbf{N}_0$, let $\vec{\Theta}_{k,N}$ be the weighted digraph which is the disjoint union of*

1. *one oriented loop,*
2. *for all $0 \leq i \leq N-2$, $(k-1)^2 k^{N-i-2}$ disjoint oriented paths of length i,*
3. *$k-1$ disjoint oriented paths of length $N-1$;*

where all edges have weight k and an oriented path of length i is the digraph with vertex set $V := \{0, \ldots, i\}$ and for every vertex $j \in V$ a unique edge from j to $j+1$ (see Figure 4.9). Let $\vec{D}_N$ be the adjacency matrix of this graph and $\vec{B}_N$ be the adjacency matrix of $\vec{\mathcal{B}}_N$. Then there exists $U = U_N$ unitary with $\vec{B}_N = U^* \vec{D}_N U$.

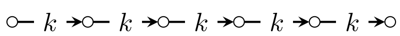


Figure 4.9: An oriented weighted path of length 5, where all edges have weight k.

See Figure 4.10 for the example of $\vec{\Theta}_{2,3}$.

The only thing remaining to do in order to compute the characteristic polynomial of $\vec{\mathcal{S}}_{N,M}$ is to express the adjacency matrix $\vec{S}_{N,M}$ of $\vec{\mathcal{S}}_{N,M}$ using $\vec{B}_N$. But it is well known that, for graphs, the adjacency matrix of a tensor product is the tensor (or Kronecker) product of adjacency matrices. This is also trivially true for digraphs. Therefore, we have

$$\vec{S}_{N,M} = \vec{B}_N \otimes \vec{A}_M,$$

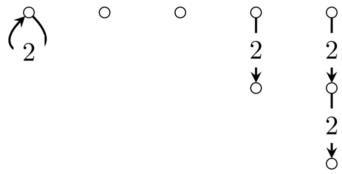


Figure 4.10: The weighted digraph $\vec{\Theta}_{2,3}$.

where $\vec{A}_M$, the adjacency matrix of the oriented cycle $\vec{C}_M$, has a 1 in position (i, j) if and only if $j \equiv i + 1 \pmod{M}$. Denoting $\vec{D}_{N,M} \coloneqq \vec{D}_N \otimes \vec{A}_M$, a simple computation gives us

$$\begin{aligned}\vec{S}_{N,M} &= \vec{B}_N \otimes \vec{A}_M \\ &= (U^* \vec{D}_N U) \otimes (\mathrm{Id}^* \vec{A}_M \,\mathrm{Id}) \\ &= (U^* \otimes \mathrm{Id}^*)(\vec{D}_N \otimes \vec{A}_M)(U \otimes \mathrm{Id}) \\ &= (U \otimes \mathrm{Id})^*(\vec{D}_{N,M})(U \otimes \mathrm{Id}),\end{aligned}$$

where Id is the identity matrix of size M. Since U and the identity matrix are both unitary, their tensor product $U \otimes \mathrm{Id}$ is also unitary. Thus, by Lemma 4.4.1, the characteristic polynomial of $\mathcal{S}_{N,M}$ is equal to $\chi(\vec{D}_{N,M} + \vec{D}^*_{N,M})$.

On the other hand, $\vec{D}_{N,M}$ is the adjacency matrix of the weighted graph $\vec{\Theta}_{k,N,M} \coloneqq \vec{\Theta}_{k,N} \otimes \vec{C}_M$. Computing this tensor product we have that the weighted graph $\vec{\Theta}_{k,N,M}$ is the disjoint union of

1. one oriented cycle of length M,
2. for all $0 \leq i \leq N-2$, $M(k-1)^2 k^{N-i-2}$ disjoint oriented paths of length i,
3. $M(k-1)$ disjoint oriented paths of length $N-1$;

where all edges have weight k — see Figure 4.11 for an example. Hence, $\vec{D}_{N,M} + \vec{D}^*_{N,M}$ is the adjacency matrix of $\underline{\vec{\Theta}_{k,N,M}}$. Therefore

$$\begin{aligned}\chi(\mathcal{S}_{N,M}) &= \chi(\vec{D}_{N,M} + \vec{D}^*_{N,M}) \\ &= Q(x) \cdot P_N(x)^{M(k-1)} \prod_{i=1}^{N-1} P_i(x)^{M(k-1)^2 k^{N-i-1}},\end{aligned}$$

where $Q(x) = Q_{k,M}(x)$ is the characteristic polynomial of the non-oriented cycle of length M with all edges of weight k and $P_i(x) = P_{i,k}(x)$ is the characteristic polynomial of the non-oriented path of length $i-1$ with all edges of weight k. We now want to have an explicit form for the P_i's and Q. In [34], Delorme and Tillich showed that $P_i(x) = k^i V_i(x/k)$ with V_i the Chebyshev polynomial of the second kind of degree i. The set of roots of $P_i(x)$ is exactly

$$\{2k \cos\Big(\frac{t\pi}{i+1}\Big) \mid 1 \leq t \leq i\}$$

and all roots are simple. On the other hand, $Q(x)$ is the characteristic polynomial of the adjacency matrix of a non-oriented cycle of length M with edges of weight k. If $M \neq 1$, all the non-zero entries of this matrix have value k and are in position (i, j) with $|i - j| = 1$.

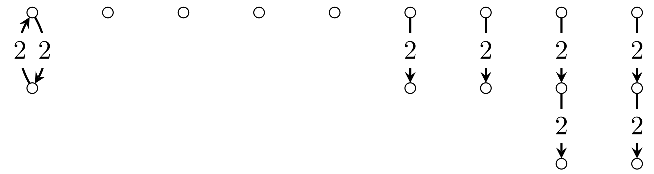


Figure 4.11: The weighted digraph $\vec{\Theta}_{2,3,2}$.

If $M = 1$, the cycle is a loop of length k and the adjacency matrix consists of a unique entry: $2k$. In both cases, the adjacency matrix is a circulant matrix of size M and it has characteristic polynomial

$$Q(x) = \prod_{l=1}^{M}(x - 2k\cos\Big(\frac{2\pi l}{M}\Big)).$$

That is, the root $2k$ has multiplicity 1, the root $-2k$ has multiplicity 1 if M is even and multiplicity 0 otherwise and for all $1 \leq l < M/2$ the root $2k\cos(\frac{2\pi l}{M})$ has multiplicity 2. Therefore we have just proved the following.

For every $k \geq 2$, $N \in \mathbf{N}_0$, and $M \in \mathbf{N}$, the spectrum of $\mathcal{S}_{N,M}$ consist of $2k$ with multiplicity 1, of

$$\{2k\cos\Big(\frac{p}{q}\pi\Big) \mid 1 \leq p < q \leq N+1; p \text{ and } q \text{ relatively prime}\},$$

with multiplicity not specified yet and, if M is even, also of $-2k$ with multiplicity 1.

The computation of the multiplicity of $2k\cos\Big(\frac{p}{q}\pi\Big)$ for a given p and q is done in four steps. Step one: compute its multiplicity in eigenvalues (interpreted as roots) of $Q(x)$; it is either 0 or 2. Step two: compute its multiplicity in $P_N(x)^{M(k-1)}$; it is either 0 or $M(k-1)$. Step three: for all $1 \leq i \leq N-1$, compute its multiplicity in $P_i(x)^{M(k-1)^2k^{N-i-1}}$; it is either 0 or $M(k-1)^2k^{N-i-1}$. Step four: add the results of the three previous steps.

In step one, the multiplicity is non-zero if and only if there exists $1 \leq l < \frac{M}{2}$ such that $\cos(2\pi l/M) = \cos(p\pi/q)$. But this is possible if and only if $l = Mp/2q \geq 1$. Since l is an integer and p and q are relatively prime, the multiplicity is 2 if and only if $2q$ divides Mp.

In step two, the multiplicity is non-zero if and only if $t = p(N+1)/q$, if and only if q divides $N+1$.

In step three, the multiplicity is non-zero if and only if $t = p(i+1)/q$, if and only if q divides $i+1$.

Summing up all these quantities we conclude that for $1 \leq p < q \leq N+1$, the multiplicity of $2k\cos\Big(\frac{p}{q}\pi\Big)$ in the spectrum of $\mathcal{S}_{N,M}$ is

$$M(k-1)^2 \cdot \left(\sum_{j=1}^{\lfloor \frac{N}{q} \rfloor} k^{N-jq}\right) + M(k-1)r_1 + 2r_2,$$

with

$$r_1 = r_1(q,N) = \begin{cases} 1 & \text{if } q \text{ divides } N+1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$r_2 = r_2(p,q,M) = \begin{cases} 1 & 2q \text{ divides } Mp \\ 0 & \text{otherwise.} \end{cases}$$

Observe that in the above sum, the first summand is equal to

$$\begin{aligned} M(k-1)^2 \cdot \left(\sum_{j=1}^{\lfloor \frac{N}{q} \rfloor} k^{N-jq} \right) &= M(k-1)^2 k^N \cdot \sum_{j=1}^{\lfloor \frac{N}{q} \rfloor} k^{-jq} \\ &= M(k-1)^2 k^N \left(\frac{1-k^{-q(\lfloor \frac{N}{q} \rfloor+1)}}{1-k^{-q}} - 1 \right). \end{aligned}$$

In the case where $q = N+1$, the multiplicity of $2k\cos(\frac{p}{N+1}\pi)$ is $M(k-1)+2r_2$. If $M=1$, q cannot divide M (q is at least 2), thus in this case r_2 is always equal to 0.

If Γ is a finite graph with m vertices and with eigenvalues of the adjacency matrix $\lambda_1 \geq \cdots \geq \lambda_m$, we write

$$\mu_\Gamma = \frac{1}{m} \sum_{i=1}^{m} \delta_{\lambda_i}$$

for the *spectral measure* on Γ, where δ_x denotes the Dirac mass on x. Then we have the following.

Theorem 4.4.3. *The spectral measure $\mu_{\mathcal{S}_{N,M}}$ of $\mathcal{S}_{N,M}$ is, if M is odd,*

$$\frac{1}{Mk^N}\delta_{2k} + \sum \delta_{2k\cos(\frac{p}{q}\pi)} \left((k-1)^2 \left(\frac{1-k^{-q(\lfloor \frac{N}{q} \rfloor+1)}}{1-k^{-q}} - 1 \right) + \frac{k-1}{k^N} r_1 + \frac{2}{Mk^N} r_2 \right)$$

where the sum is over all $1 \leq p < q \leq N+1$ with $(p,q)=1$.

If M is even, there is one more summand: $\frac{1}{Mk^N}\delta_{-2k}$.

Remark 4.4.4. It directly follows from the formula in the above theorem that for k and N fixed, $\mathcal{S}_{N,M}$ and $\mathcal{S}_{N,M'}$ have the same spectrum, except maybe for the value $-2k$, and that the total variation distance between $\mu_{\mathcal{S}_{N,M}}$ and $\mu_{\mathcal{S}_{N,M'}}$ is bounded by $\frac{2}{k^N}$, independently from M and M'.

Since spider-web graphs converge, in the sense of Benjamini-Schramm, to the Cayley graph $\mathrm{Cay}(\mathcal{L}_k, X_k^\pm)$ we retrieve the Kesten spectral measure of the graph $\mathrm{Cay}(\mathcal{L}_k, X_k^\pm)$. This measure was first computed by Grigorchuk and Żuk in [59] for $k=2$ and then by Dicks and Schick in [35] and by Kambites, Silva and Steinberg in [74] for the more general case $G \wr \mathbf{Z}$, with $G \neq \{1\}$ a finite group.

$$(\#) \qquad \mu_{\mathrm{Cay}(\mathcal{L}_k, X_k^\pm)} = (k-1)^2 \sum_{q\geq 2} \frac{1}{k^q - 1} \left(\sum_{\substack{1\leq p<q \\ (p,q)=1}} \delta_{2k\cos(\frac{p}{q}\pi)} \right).$$

4.5 General results on spider-web graphs

In 4.3.14 we proved that $\vec{\mathcal{S}}_{N,1}$ are weakly isomorphic to Schreier graphs of the lamplighter group $\mathcal{L}$. Other spider-web digraphs are so far described as $\vec{\mathcal{S}}_{N,M} \simeq \vec{\mathcal{S}}_{N,1} \otimes \vec{C}_M$. In this section we will show that $\vec{\mathcal{S}}_{N,M}$ is also a Schreier graph of $\mathcal{L}$ for each M, N. Then we characterize which of the $\vec{\mathcal{S}}_{N,M}$ are transitive and give bound on the number of orbits in $\vec{\mathcal{S}}_{N,M}$ and in $\mathcal{S}_{N,M}$. Finally, we generalize to spider-web digraphs some statements that are known for de Bruijn digraphs: existence of Eulerian and Hamiltonian paths, the property of being a line digraph and some facts about covering.

As before we fix a $k \geq 2$ and omit to write it when it is not necessary. We will write $\mathbf{N}_0$ for $\{0, 1, 2, \ldots\}$ and $\overline{\mathbf{N}}$ for $\{1, 2, \ldots, \infty\}$. We also take $X = \{\bar{c}_i\}_{i=0}^{k-1}$ (see (‡) on page 54) as generating set for the lamplighter group $\mathcal{L}$.

Spider-web graphs as Schreier graphs of lamplighter groups

In Theorem 4.3.14 we proved that the de Bruijn digraph $\vec{\mathcal{B}}_N$ is weakly isomorphic to a Schreier graph of $\mathcal{L}$ by using line digraphs. Let us denote $W_{N,1} := \mathrm{Stab}_{\mathcal{L}}(0^N)$. Then we have the following.

Theorem 4.5.1. *For all $M \in \overline{\mathbf{N}}$ and $N \in \mathbf{N}_0$, the spider-web digraph $\vec{\mathcal{S}}_{N,M}$ is weakly isomorphic to*

$$\vec{\mathrm{Sch}}(\mathcal{L}, W_{N,M}, X)$$

where $W_{N,M} = \{g \in W_{N,1} \mid \exp_X(g) \equiv 0 \pmod M\}$.

Proof. Since $\vec{\mathcal{B}}_N$ is weakly isomorphic to $\vec{\Gamma}_N$, the spider-web digraph $\vec{\mathcal{S}}_{N,M} \simeq \vec{\mathcal{B}}_N \otimes \vec{C}_M$ is weakly isomorphic to $\vec{\Gamma}_N \otimes \vec{C}_M$ which is a Schreier graph of $\mathcal{L}$ by Proposition 4.2.9 and the description of $W_{N,M}$ follows. □

Remark 4.5.2. Geometrically, we are here in the situation described in Remark 4.2.10, where the action of $\mathcal{L}$ on $\vec{C}_M$ is given by $j.(\prod_{t=1}^{j}(b^{i_t}cb^{-i_t})^{r_t} \cdot b^{r_b}) = j + r_b \pmod M$. Indeed, $b^{i_t}cb^{-i_t} = \bar{c}_0^{i_t-1}\bar{c}_1\bar{c}_0^{-i_t}$ and therefore we have $\exp_X(\prod_{t=1}^{j}(b^{i_t}cb^{-i_t})^{r_t} \cdot b^{r_b}) = r_b$.

Given a graph, a group and a generating set, there could be a priori many different ways to represent the graph as a Schreier graph of the group. It is easy to check that every $g \neq 1$ in $\mathcal{L}$, can be written in a unique way as $\prod_{t=1}^{j}(b^{i_t}cb^{-i_t})^{r_t} \cdot b^{r_b}$, where $j, r_b, r_t, i_t \in \mathbf{Z}$ with $j \geq 0$ and $0 < r_t < k$ for all $1 \leq t \leq j$. This allows us to define a subgroup $H_{N,M}$ of $\mathcal{L}$ as the following set:

$$H_{N,M} := \left\{ g = \prod_{t=1}^{j}(b^{i_t}cb^{-i_t})^{r_t} \cdot b^{r_b} \in \mathcal{L} \;\middle|\; \begin{array}{c} r_b \equiv 0 \pmod M \\ \forall i, 1 \leq i \leq N, \displaystyle\sum_{i_t \overset{N}{\equiv} i} r_t \equiv 0 \pmod k \end{array} \right\},$$

where the second sum is over all $i_t \equiv i \pmod N$.

Theorem 4.5.3. *For every $M \in \overline{\mathbf{N}}$, and every $N \in \mathbf{N}_0$, the spider-web digraph $\vec{\mathcal{S}}_{N,M}$ is weakly isomorphic to*

$$\vec{\mathrm{Sch}}(\mathcal{L}, H_{N,M}, X^{-1})$$

where $X^{-1} = \{x^{-1} \mid x \in X\}$.

Proof. Define the following permutations on the vertex set V of $\vec{\mathcal{S}}_{N,M}$:

$$(x_1 \ldots x_N, j).b := (x_N x_1 \ldots x_{N-1}, j-1)$$
$$(x_1 \ldots x_N, j).c := \big((x_1 - 1)x_2 \ldots x_N, j\big)$$

where $x_1 - 1$ is taken modulo k and $j - 1$ modulo M. The group G generated by b and c acts on V.

An straightforward check shows that b and c satisfies the relations in the presentation (†) (page 54) of $\mathcal{L}$. Therefore G is a quotient of $\mathcal{L}$, which implies that $\mathcal{L}$ acts too on V.

Moreover, for the generating set $X^{-1} = \{\bar{c}_i^{-1}\}$ there are exactly k edges in the digraph of the action with initial vertex $(x_1 \dots x_N, j)$: the one labeled by $(\bar{c}_r)^{-1} = c^{-r}b^{-1}$ having $(x_2 \dots x_N(x_1 + r), j+1)$ as final vertex. On other hand, in $\vec{\mathcal{S}}_{N,M}$ there are also exactly k edges with initial vertex $(x_1 \dots x_N, j)$: the one labeled by y having $(x_2 \dots x_N y, j+1)$ as final vertex. Since vertex sets and adjacency relations (if we forget about labeling) are the same, $\vec{\mathcal{S}}_{N,M}$ is weakly isomorphic to the digraph of the action of $\mathcal{L}$ on V with respect to the generating set X^{-1}. Moreover, this graph being connected, it is also weakly isomorphic to the Schreier graph $\vec{\mathrm{Sch}}(\mathcal{L}, H_{N,M}, X^{-1})$, with $H_{N,M} = \mathrm{Stab}_{\mathcal{L}}(0 \dots 0, 0)$. A straightforward calculation gives us $H_{N,M}$. □

This theorem directly implies its unoriented version: $\mathcal{S}_{N,M} \simeq \mathrm{Sch}(\mathcal{L}, H_{N,M}, X^{\pm})$.

In [51] Grigorchuk and Kravchenko classified subgroups of $\mathcal{L}$ and gave a criterion for normality.

We now recall these two results and identify $H_{N,M}$ and $W_{N,M}$ according to this classification. We then are able to see which of the subgroups are normal (in which case the corresponding Schreier graphs are in fact Cayley graphs).

Recall that $A = \oplus_{\mathbf{Z}} \mathbf{Z}/k\mathbf{Z}$ is the abelian part of $\mathcal{L}$ and that b acts on A by shift.

Lemma 4.5.4 ([51], Lemma 3.1)**.** *Let H be a subgroup of $\mathcal{L}$. Then it defines the triple (s, H^0, v) where $s \in \mathbf{N}$ is such that $s\mathbf{Z}$ is the image of the projection of H on $\mathbf{Z}$, $H^0 = H \cap A$, satisfying $H^0.b^s = H^0$, and $v \in A$ is such that $vb^s \in H$. The v is uniquely defined up to addition of elements from H^0. For $s = 0$ one can choose $v = 1_{\mathcal{L}}$.*

Conversely any triple (s, H^0, v) with such properties gives rise to a subgroup of $\mathcal{L}_k$. Two triples (s, H^0, v) and (s', G^0, v') define the same subgroup if and only if $s = s'$, $H^0 = G^0$ and $vH^0 = v'G^0$. Moreover, $H \subseteq G$ if and only if $s'|s$, $H^0 \subseteq G^0$ and $v \equiv \prod_{i=0}^{s/s'}(v'.b^{is'})$ $\pmod{G^0}$.

Lemma 4.5.5 ([51], Lemma 3.2)**.** *Let H be a subgroup of $\mathcal{L}$. Then H is normal if and only if the corresponding triple (s, H^0, v), satisfies the additional properties that $H^0.b = H^0$, $v(v.b)^{-1} \in H^0$ and $g(g.b^s)^{-1} \in H^0$ for all $g \in A$.*

Note that in [51] only the case of k prime is treated. However both lemmas remain true for all k.

Proposition 4.5.6.

1. *For all $M \in \mathbf{N}$ and $N \in \mathbf{N}_0$ the subgroup $H_{N,M}$ corresponds to the triple $(M, H^0, 1_{\mathcal{L}})$, where $H^0 = {H_{N,M}}^0$ is the following subgroup of A*

$$H^0 = \left\{ \prod_{t=1}^{j} (b^{i_t} c b^{-i_t})^{r_t} \in \mathcal{L} \;\middle|\; \forall i, 1 \le i \le N, \sum_{i_t \overset{N}{\equiv} i} r_t \equiv 0 \pmod{k} \right\}.$$

   For all $N \in \mathbf{N}_0$ the subgroup $H_{N,\infty}$, corresponds to the triple $(0, H^0, 1_{\mathcal{L}})$.

2. *For all $M \in \mathbf{N}$ and $N \in \mathbf{N}_0$ the subgroup $W_{N,M}$ corresponds to the triple $(M, W^0, 1_{\mathcal{L}})$, with $W^0 = {W_{N,M}}^0 = W_{N,1} \cap A$. In particular, W^0 does not depend on M. For all $N \in \mathbf{N}_0$ the subgroup $W_{N,\infty}$ corresponds to the triple $(0, W^0, 1_{\mathcal{L}})$.*

Proof. We first prove the proposition for $H_{N,M}$. It is obvious that $H \cap A = H^0$.

Take $g = \prod_{t=1}^{j}(b^{i_t}cb^{-i_t})^{r_t} \cdot b^{r_b}$ in $H_{N,M}$. The projection of g onto $\mathbf{Z}$ is r_b, which is equal to 0 (mod M). If M is finite, since b^M belongs to $H_{N,M}$, the projection of $H_{N,M}$ onto $\mathbf{Z}$ is $M\mathbf{Z}$. If $M = \infty$, then the projection of $H_{N,M}$ onto $\mathbf{Z}$ is 0. Finally, $1_{\mathcal{L}}b^M$ belongs to $H_{N,M}$ as asked.

Now, for $W_{N,M}$ we first look at the case $M = 1$. Since b stabilizes $(0\dots0)$, it belongs to $W_{N,M}$, the stabilizer of $(0\dots0)$. Therefore, $W_{N,1}$ corresponds to the triple $(1, W^0, 1_{\mathcal{L}})$. For an arbitrary M, $W_{N,M}$ is the subgroup of $W_{N,1}$ consisting of elements with total exponent equal to 0 modulo M. Since $b^icb^{-i} = b^{i-1}\bar{c}_1b^{-i}$, we have $\exp_{X_k}(b^icb^{-i}) = 0$ and for any $g = \prod_{t=1}^{j}(b^{i_t}cb^{-i_t})^{r_t} \cdot b^{r_b} \in W_{1,M}$ the total exponent of g is precisely r_b. Therefore, $W_{N,M}$ corresponds to the triple $(M, W^0, 1_{\mathcal{L}})$ if M is finite and to the triple $(0, W^0, 1_{\mathcal{L}})$ if $M = \infty$. □

Corollary 4.5.7.

1. *For all $M \in \overline{\mathbf{N}}$ and $N \in \mathbf{N}_0$, the subgroup $H_{N,M}$ is normal if and only if N divides M. In particular $H_{N,\infty}$ is normal for every N.*

2. *For all $M \in \overline{\mathbf{N}}$ and $N \in \mathbf{N}_0$, the subgroup $W_{N,M}$ is normal if and only if $N = 1$.*

Proof. First, we prove that $H^0_{N,M}.b$ is always equal to $H^0_{N,M}$. Indeed, for all $g \in A$, $g = \prod_{t=1}^{j}(b^{i_t}cb^{-i_t})^{r_t}$ and we have $g.b = \prod_{t=1}^{j}(b^{i_t-1}cb^{-i_t+1})^{r_t}$. Hence, g belongs to $H^0_{N,M}$ if and only if $g.b$ belongs to $H^0_{N,M}$. We also trivially have that $1(b1b^{-1})^{-1} = 1$ always belongs to $H^0_{N,M}$. Therefore, $H_{N,M}$ is normal if and only if $g(g.b^M)^{-1} \in H^0$ for all $g \in A$.

Suppose that N does not divide M. Take $g = c \in A$. Then $c(c.b^M)^{-1} = c(b^Mcb^{-M})^{-1}$. The sum $\sum_{i_t\equiv 0} r_t = 1 \not\equiv 0 \pmod{k}$ since $k \geq 2$. Therefore $c(c.b^M)^{-1} \notin H^0_{N,M}$, which implies that $H_{N,M}$ is not normal.

On the other hand, suppose now that N divides M. Then for all $g = \prod_{t=1}^{j}(b^{i_t}cb^{-i_t})^{r_t} \in A$, we have

$$g(g.b^M)^{-1} = \prod_{t=1}^{j}(b^{i_t}cb^{-i_t})^{r_t} \cdot \prod_{t=1}^{j}(b^{i_t-M}cb^{-i_t+M})^{-r_t}$$

belongs to $H^0_{N,M}$.

Now, in the case of $W_{N,M}$, we have $(0\dots0,0).cbc^{-1}b^{-1} = (01{-}11\cdots\pm1,0)$. Therefore, as soon as $N \geq 2$, $cbc^{-1}b^{-1}$ does not belongs to W^0 and $W_{N,M}$ is not normal. For $N = 1$, we have $(x,i).b = (x,i+1)$ and $(x,i).c = (x+1,i)$. Therefore,

$$W_{1,M} = \left\{ g = \prod_{t=1}^{j}(b^{i_t}cb^{-i_t})^{r_t}b^{r_b} \;\middle|\; \begin{array}{l} r_b \equiv 0 \pmod{M} \\ \sum_{t=1}^{j} r_t \equiv 0 \pmod{k} \end{array} \right\}$$

is a normal subgroup. □

In particular, this implies that for $N > 1$ dividing M, the subgroups $H_{N,M}$ and $W_{N,M}$ are not conjugate (equivalently the Schreier graphs are not strongly isomorphic). On the other hand, an easy check shows that $H_{N,1} = W_{N,1}$. A careful look at the order of the image of c in the corresponding action on the vertex set of $\vec{\mathcal{S}}_{N,M}$ shows that for $M < \infty$ and $N > 1$, $H_{N,M}$ and $W_{N,M}$ are nearly never conjugate (this can happen only if k divides all the $\binom{\operatorname{lcm}(M,N)}{j}$ for $1 \leq j \leq N-1$).

Proposition 4.5.6 (Part 1) and Corollary 4.5.7 imply the following.

Theorem 4.5.8. *For $M = Nl$, the spider-web digraph $\vec{\mathcal{S}}_{N,Nl}$ is weakly isomorphic to the Cayley digraph of*

$$\left(\oplus_{i=1}^{N}\mathbf{Z}/k\mathbf{Z}\right) \rtimes \mathbf{Z}/Nl\mathbf{Z}$$

where the action of $\mathbf{Z}/Nl\mathbf{Z}$ on $\oplus_{i=1}^{N}\mathbf{Z}/k\mathbf{Z}$ is by shift, and with respect to the generating set $\{bc^i\}_{i=0}^{k-1}$ where b is a generator of $\mathbf{Z}/Nl\mathbf{Z}$ and c a generator of $\mathbf{Z}/k\mathbf{Z}$.

In particular, if we write $\mathcal{L}_{k,N} = \mathbf{Z}/k\mathbf{Z} \wr \mathbf{Z}/N\mathbf{Z}$ for the finite lamplighter group, we have for N and l coprime,

$$\vec{\mathcal{S}}_{k,N,Nl} \simeq \vec{\mathrm{Cay}}(\mathcal{L}_{k,N} \times \left(\mathbf{Z}/l\mathbf{Z}\right), \{(\bar{c}_r^{-1}, -1)\}_{r=0}^{k-1})$$

which, for $l = 1$, gives

$$\vec{\mathcal{S}}_{k,M,M} \simeq \vec{\mathrm{Cay}}(\mathcal{L}_{k,M}, X^{-1})$$

and for $N = 1$

$$\vec{\mathcal{S}}_{k,1,M} \simeq \vec{\mathrm{Cay}}(\mathbf{Z}/k\mathbf{Z} \times \mathbf{Z}/M\mathbf{Z}, \{(-r, -1)\}_{r=0}^{k-1}).$$

We also have

$$\vec{\mathcal{S}}_{k,1,\infty} \simeq \vec{\mathrm{Cay}}(\mathbf{Z}/k\mathbf{Z} \times \mathbf{Z}, \{(-r, -1)\}_{r=0}^{k-1}).$$

Proof. The graph $\vec{\mathcal{S}}_{N,Nl}$ is weakly isomorphic to the graph $\vec{\mathrm{Sch}}(\mathcal{L}, H_{N,Nl}, X)$. Since $H_{N,Nl}$ is normal, this graph is strongly isomorphic to the Cayley digraph of $G = \mathcal{L}/H_{N,Nl}$. We know that $G = \langle b, c\rangle$ and that $c^k = 1$ in G. Moreover, b^{Nl} and $cb^N c^{-1} b^{-N}$ belong to $H_{N,Nl}$ and thus in G the relations $b^{Nl} = 1$ and $cb^N = b^N c$ are true. Therefore, G is a quotient of

$$L = \langle b, c \mid c^k, cb^N c^{-1} b^{-N}, b^{Nl}, [c, b^j cb^{-j}]; j \in \mathbf{N}\rangle \cong \left(\oplus_{i=1}^{N}\mathbf{Z}/k\mathbf{Z}\right) \rtimes \mathbf{Z}/Nl\mathbf{Z}$$

where the action of $\mathbf{Z}/Nl\mathbf{Z}$ is by shift. We have $|L| = k^N \cdot N \cdot l$ and $|G| = k^N \cdot Nl$ (the number of vertices of $\vec{\mathcal{S}}_{N,Nl}$). This implies $G = L$.

If l and N are coprime, $\mathbf{Z}/Nl\mathbf{Z} \cong \mathbf{Z}/N\mathbf{Z} \times \mathbf{Z}/l\mathbf{Z}$ acts on $\oplus_{i=1}^{N}\mathbf{Z}/k\mathbf{Z}$, with $\mathbf{Z}/N\mathbf{Z}$ acting by shift and $\mathbf{Z}/l\mathbf{Z}$ acting trivially. Hence, $L \cong \mathcal{L}_{k,N} \times \mathbf{Z}/l\mathbf{Z}$.

Generating sets are images of $\{\bar{c}_r\}$ in L.

Finally, $\vec{\mathcal{S}}_{1,\infty}$ and $\vec{\mathrm{Cay}}(\mathbf{Z}/k\mathbf{Z} \times \mathbf{Z}, \{(-r, -1)\}_{r=0}^{k-1})$ have the same vertex set: $\{1, \ldots, k\} \times \mathbf{Z}$. Moreover, in both graphs, there is an edge from (i, s) to (j, t) if and only if $t = s + 1$. □

Remark 4.5.9. It is interesting to observe that the family of spider-web digraphs $\vec{\mathcal{S}}_{k,N,M}$ interpolates between Cayley digraphs of direct products of finite cyclic groups and Cayley digraphs of wreath products of finite cyclic groups, with the corresponding generating sets.

Observe however that more graphs $\mathcal{S}_{k,N,M}$ can a priori be weakly isomorphic to Cayley graphs of some finite groups than those given in Theorem 4.5.8. For example, one can check by hand that this is the case of $\mathcal{S}_{2,2,3}$, thought $H_{2,2,3}$ is not normal.

Transitivity of spider-web graphs

We now investigate the vertex-transitivity of spider-web graphs. We already know from the last subsection that if N divides M the spider-web graphs are weakly isomorphic to Cayley graphs and therefore are weakly transitive. We will give a complete characterization of transitivity for spider-web graphs, but before that we need a technical lemma.

Theorem 4.5.10. *For every $N \in \mathbf{N}_0$ and every $M \in \overline{\mathbf{N}}$, the graphs $\vec{\mathcal{S}}_{N,M}$ and $\mathcal{S}_{N,M}$ are weakly transitive (i.e. is transitive by weak automorphisms) if and only if $M \geq N$.*

More precisely, the number of orbits of $\vec{\mathcal{S}}_{k,N,M}$ under its group of automorphisms is bounded from below by $\frac{N}{M}$ and from above by $\max(1, k^{N-M})$. On the other hand, if $M < N$, the number of orbits of $\mathcal{S}_{k,N,M}$ under its group of automorphisms is bounded from below by $\max(2, \frac{N}{2M})$ and from above by k^{N-M}.

In particular, for M fixed the number of orbits of $\mathcal{S}_{k,N,M}$ is unbounded.

The upper bound for $\vec{\mathcal{S}}_{N,M}$ is sharp. This is the case for example for $\vec{\mathcal{S}}_{2,2,1}$, $\vec{\mathcal{S}}_{2,3,1}$ (see Figure 4.3) and for $\vec{\mathcal{S}}_{3,2,1}$ (see figure 4.12). Observe that any automorphism of $\vec{\mathcal{S}}_{N,M}$ induces

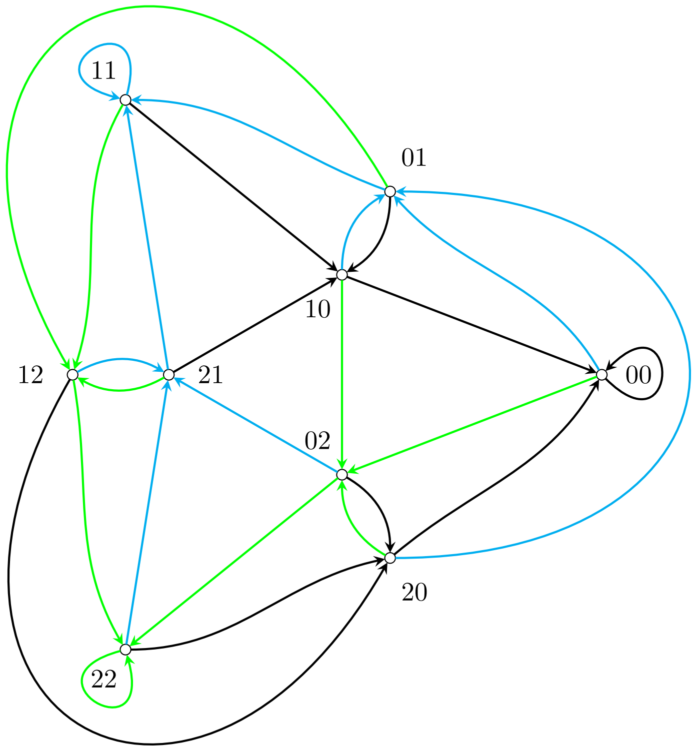


Figure 4.12: The digraph $\vec{\mathcal{S}}_{3,2,1}$. The vertex 01 cannot be send on the vertex 10 by an automorphism of the digraph.

an automorphism of $\mathcal{S}_{N,M}$, but that an automorphism of $\mathcal{S}_{N,M}$ does not necessarily come from an automorphism of $\vec{\mathcal{S}}_{N,M}$. Therefore, in order to prove the theorem we will successively show that $\vec{\mathcal{S}}_{N,M}$ is transitive if $M \geq N$, that if $M < N$ the number of orbits of $\vec{\mathcal{S}}_{N,M}$ is at most k^{N-M} and then we will prove the two lower bounds for the number of orbits.

Lemma 4.5.11. *If $M \geq N$, then $\vec{\mathcal{S}}_{N,M}$ is transitive.*

Proof. It is easy to check that for every M the function η on $\vec{C}_M$ defined by $\eta(i) := i + 1$ (where the addition is taken modulo M) is an automorphism. Therefore, $T := \mathrm{Id} \otimes \eta$ is an automorphism of $\vec{\mathcal{S}}_{N,1} \otimes \vec{C}_M \simeq \vec{\mathcal{S}}_{N,M}$. It is even a strong automorphism for every M — even for M smaller than N — since the labeling of $\vec{\mathcal{S}}_{N,M}$ comes from the labeling of $\vec{\mathcal{S}}_{N,1}$ and the fact that Id is a strong automorphism.

We now define an other function ψ on $\vec{\mathcal{S}}_{N,M}$ by the following formula on vertices:

$$\psi(x_1 \dots x_N, t) := \begin{cases} (x_1 \dots (x_{N-t} + 1) \dots x_N, t) & \text{if } 0 \leq t \leq N-1 \\ (x_1 \dots x_N, t) \text{ else.} \end{cases}$$

We define ψ on edges in the following way: the unique edge with initial vertex (x,t) and label i is sent on the unique edge with initial vertex $\psi(x,t)$ and label i if $t \not\equiv -1 \pmod M$, or on the edge with label $i+1$ if $t \equiv -1 \pmod M$. We claim that with this definition, ψ is a weak isomorphism if $M \geq N$. To prove that, it remains to check that for any edge e, $\tau\big(\psi(e)\big) = \psi\big(\tau(e)\big)$. Since the definition of ψ depends on t, we have four different cases. The first is when $0 \leq t \leq N-2$. The second is for $t = N-1$. The third when $N-1 < t < M-1$ and the last one when $t = M-1$. The first, second and fourth cases are easy computations left to the reader. In the third case, ψ acts as the identity and there is nothing to prove. Note that the only case where ψ does not preserve labeling is the fourth one and if $M = N$, then the third case does not exist and the second and the fourth cases are the same. We have

$$\begin{aligned} T^{-i}\psi^{x_i}T^{i}(0\dots0,0) &= T^{-i}\psi^{x_i}(0\dots0,i) \\ &= (0\dots x_i\dots 0,i) \\ &= (0\dots x_i\dots 0,0). \end{aligned}$$

Therefore, for any vertex $(x_1\dots x_N,t)$ we have

$$\Big(T^{i}\cdot\prod_{j=0}^{n}T^{-j}\psi^{x_j}T^{j}\Big)(0\dots0,0) = (x_1\dots x_N,t),$$

which proves the transitivity when $M \geq N$. □

Lemma 4.5.12. *The number of orbits of* $\vec{\mathcal{S}}_{N,M}$ *is at most* $\max(1,k^{N-M})$.

Proof. If $M \geq N$, this follows from transitivity. For $N > M$, we will show that every vertex of $\vec{\mathcal{S}}_{N,M}$ can be send by automorphisms on a vertex of the form $(0\dots0y_{M+1}\dots y_N,0)$. In order to do that we will use three different kind of automorphisms: *translations*, *additions* and *switches*. Translations are power of T (strong automorphisms), additions are power of A (weak automorphisms) and switches are power of U_t (weak automorphisms) with $T = \mathrm{Id}\otimes\eta$ as in the last lemma,

$$A(x_1\dots x_N,i) \coloneqq \big((x_1+1)\dots(x_N+1),i\big)$$

on vertices and A sends edges with label y to edges with label $y+1$. For all $0 \leq t \leq M-1$, we define

$$U_t(x_1\dots x_N,i) \coloneqq (y_1\dots y_N,i), \quad y_j = \begin{cases} x_j+1 & \text{if } N-j+t \equiv i \pmod M \\ x_j & \text{otherwise} \end{cases}$$

on vertices and U_t sends the only edge with label y and initial vertex $v = (x_1\dots x_N,i)$ to the edge with initial vertex $U_t(v)$ and label y if $t \not\equiv i+1 \pmod M$ and $y+1$ otherwise. It is trivial that T and A are automorphisms of $\vec{\mathcal{S}}_{N,M}$ and it is an easy verification for U_t.

Every vertex of $\vec{\mathcal{S}}_{N,M}$ can be send on a vertex of the form $(0x_2\dots x_N,0)$ by $\langle T,A\rangle$. Now, if $x_2 \neq 0$, then $U_t^{-x_2}(0x_2\dots x_N,0) = (00y_3\dots y_N,0)$ if and only if

$$\begin{aligned} N-1+t &\not\equiv 0 \pmod M \\ N-2+t &\equiv 0 \pmod M. \end{aligned}$$

There exists a t satisfying these equations if and only if $M \geq 2$. Repeating this argument we obtain that if $M < N$, every vertex of $\vec{\mathcal{S}}_{N,M}$ can be send by $\langle T,A,\{U_t\}_{t=0}^{M-1}\rangle$ on a vertex of the form $(0\dots0y_{M+1}\dots y_N,0)$. □

For the non-oriented case we may define one more automorphism: the flip $F(x_1 \dots x_N, i) \coloneqq (x_N \dots x_1, i)$. It is not an automorphism of $\vec{\mathcal{S}}_{N,M}$ if $N \geq 2$ as it changes the orientation of edges. Therefore, the number of orbits of $\mathcal{S}_{N,M}$ may be strictly smaller than the number of orbits of $\vec{\mathcal{S}}_{N,M}$ (it is the case for $\vec{\mathcal{S}}_{2,3,1}$) but it may also be the same (it is the case for $\vec{\mathcal{S}}_{2,2,1}$); see Figure 4.3 for examples.

Lemma 4.5.13. *The number of orbits of $\vec{\mathcal{S}}_{N,M}$ is at least $\frac{N}{M}$.*

Proof. If $N \geq jM$, then a path of length jM and derangement jM with initial vertex $(x_1 \dots x_N, 0)$ has final vertex $(x_{jM+1} \dots x_N y_1 \dots y_{jM}, 0)$, where y_t are some integers modulo k. Therefore, there exists a closed path of length jM with initial vertex $(x_1 \dots x_N, 0)$ consisting of edges of $\vec{\mathcal{S}}_{N,M}$ if and only if $x_1 \dots x_N$ is jM periodic. In particular, for the vertex $v = (0^{jM-1}10^{jM-1}1\dots, 0)$ there is a reduced closed path of length jM and derangement jM with initial vertex v, but no reduced closed path of length iM and derangement iM for $0 < i < j$. This implies that the vertices

$$(0^{M-1}10^{M-1}1\dots, 0)$$
$$(0^{2M-1}10^{2M-1}1\dots, 0)$$
$$\dots$$

are not in the same orbits. But there is only $\lfloor \frac{N}{M} \rfloor$ such vertices. If $M < N$ does not divide N, then we had the vertex $(0 \dots 01, 0)$ which is not jM periodic for every j. Therefore we have $\lceil \frac{N}{M} \rceil$ vertices in distinct orbits and this gives us the wanted lower bound. ☐

Lemma 4.5.14. *For any $d \in \mathbf{N}_0$ and any vertices v and w in $\mathcal{S}_{N,M}$, there is a bijection between closed reduced path of derangement 0 of length d with initial vertex v and the ones with initial vertex w.*

Proof. The bijection is easily seen on $\mathrm{Sch}(\mathcal{L}, H_{N,M}, X^{\pm})$. Recall that we have

$$(x_1 \dots x_N, i).\bar{c}_r = \big((x_N - r)x_1 \dots x_{N-1}, i - 1\big).$$

For any reduced path p with initial vertex $(x_1 \dots x_N, i)$, there is a unique path q with initial vertex $(y_1 \dots y_N, j)$ with the same label $l \in \mathcal{L}$. Since the derangement is 0, we have that

$$(z_1 \dots z_N, i).l = ((z_1 + a_1) \dots (z_N + a_N), i)$$

for some integers a_i. In particular, the final vertex of p is $((x_1 + a_1) \dots (x_N + a_N), i)$, while the final vertex of q is $((y_1 + a_1) \dots (y_N + a_N), j)$. Therefore, p is closed if and only if all the a_i's are equal to 0, if and only if q is closed. ☐

Lemma 4.5.15. *If $M < N$, the number of orbits of $\mathcal{S}_{N,M}$ is at least* $\max(2, \frac{N}{2M})$.

Proof. For $j > 0$ such that $N \geq jM + 1$, let us denote by v_j the vertex

$$(0^{jM-1}10^{jM-1}1\dots, 0)$$

and, if M does not divides N, by v_0 the vertex $(0 \dots 01, 0)$. Hence, the vertex v_j is jM periodic, but not iM periodic if $0 < i < j$ and v_0 is never iM periodic. We will show that if $0 < i < j \leq \frac{N}{2M}$ the number of closed reduced path of size iM with initial vertex v_i is strictly bigger than the number of such paths with initial vertex v_j. This will implies that v_i and v_j are not in the same orbit if $i \neq j$.

A closed path of length iM in $\mathcal{S}_{N,M}$ has derangement lM for some $-i \leq l \leq i$. Let us denote by $n_l(v)$ the number of closed paths with initial vertex v and derangement lM. Since there is an obvious bijection between paths with derangement l and paths with derangement $-l$ (taking the inverse path), we have $n_l(v) = n_{-l}(v)$. By the previous lemma, $n_0(v) = n_0(w)$ for all v and w. We will show that for $0 < l < i \leq \frac{N}{2M}$ we have $n_l(v_i) = 0$ and $n_i(v_i) \geq 1$. Hence, $n_l(v_i) = n_l(v_j) = 0$ if $\frac{N}{2M} \geq j > i$. This implies that the total number of closed paths of size iM with initial vertex v_i is $n_0(v_i) + n_i(v_i)$ which is strictly bigger than $n_0(v_j)$, the number of such paths with initial vertex v_j.

Let p be a path of length iM and derangement lM with initial vertex $(x_1 \dots x_N, 0)$. Therefore, in p there is $\frac{i+l}{2}M$ "positive edges" (edges in $\mathcal{S}_{N,M}$) and $\frac{i-l}{2}M$ "negative edges". Thus, p ends at a vertex of the form

$$(y_{lM+1} \dots y_{\frac{i+l}{2}M} x_{\frac{i+l}{2}M+1} \dots x_{N-\frac{i-l}{2}M} y_{N-\frac{i-l}{2}M+1} \dots y_{lM}, 0)$$

Hence, the digits in position between $\frac{i+l}{2}M + 1$ and $N - \frac{i-l}{2}M$ are translated, but their value is not altered. There is $N - \frac{i-l}{2}M - (\frac{i+l}{2}M + 1) + 1 = N - iM$ such digits. Therefore, if $i \leq \frac{N}{2M}$, there is at least one of this digit which has value 1 (the digit x_d such that $d \in [\frac{i+l}{2}M + 1, N - \frac{i-l}{2}M]$ is a multiple of iM). But that means that p ends on a vertex with a 1 in position $d - lM$, therefore this vertex is not v_i and p is not closed.

On the other hand, we have

$$v_i.\bar{c}_{r_1} \dots \bar{c}_{r_{iM}} = (0^{iM-1}10^{iM-1}1 \dots r_1 r_2 \dots r_{iM}, 0) = v_i$$

for a suitable choice of $\bar{c}_{r_j}$'s. This implies that $n_i(v_i) \geq 1$.

Finally, v_i and v_0 are never in the same orbit since $n_0(v_i) = n_0(v_0)$ by the previous lemma and $n_i(v_i) \geq 1 > 0 = n_i(v_0)$. □

Line graphs, Eulerian and Hamiltonian cycles and coverings

The family of de Bruijn digraphs is well known to enjoy some nice graph-theoretic properties. The aim of this subsection is to verify that the family of spider-web graphs share many of them and can thus be indeed viewed as a natural extension of de Bruijn digraphs.

Proposition 4.5.16. *For all $M \in \overline{\mathbf{N}}$ and $N \in \mathbf{N}_0$, the spider-web digraph $\vec{\mathcal{S}}_{N+1,M}$ is (weakly) isomorphic to the line digraph of $\vec{\mathcal{S}}_{N,M}$.*

Proof. This follows from the same result for de Bruijn digraphs (Lemma 4.3.2), the fact that $\vec{\mathcal{S}}_{N,M}$ is the tensor product $\vec{\mathcal{B}}_N \otimes \vec{C}_M$, Lemma 2.3.4 and the fact that $\vec{C}_M$ is its own line digraph. □

Proposition 4.5.17. *For all $M \in \overline{\mathbf{N}}$ and $N \in \mathbf{N}_0$, the spider-web digraph $\vec{\mathcal{S}}_{N,M}$ is Eulerian (there exists a closed path p consisting of edges of $\vec{\mathcal{S}}_{N,M}$ that visits each edge exactly once) and Hamiltonian (there exists a closed path that visits each vertex exactly once)*

Proof. The directed graph $\vec{\mathcal{S}}_{N,M}$ is finite, connected and for every vertex v in $\vec{\mathcal{S}}_{N,M}$ the number of outgoing edges is equal to the number of ingoing edges. Therefore, $\vec{\mathcal{S}}_{N,M}$ is Eulerian.

For $N \geq 1$, the graph $\vec{\mathcal{S}}_{N,M}$ is isomorphic to the line digraph of $\vec{\mathcal{S}}_{N-1,M}$. This line digraph is Hamiltonian since $\vec{\mathcal{S}}_{N-1,M}$ is Eulerian. Finally, $\vec{\mathcal{S}}_{0,M}$ is a "thick" oriented circle: the vertex set is M and for every vertex i there is k edges from i to $i + 1$. This graph is obviously Hamiltonian. □

We proved that $\vec{\mathcal{S}}_{N,M}$ is Eulerian and Hamiltonian as a digraph. That is, the closed path in question consists only of edges of $\vec{\mathcal{S}}_{N,M}$. This trivially implies that $\mathcal{S}_{N,M}$ is Eulerian and Hamiltonian. Indeed, for a digraph $\vec{\Theta}$, being Eulerian (or Hamiltonian) as a digraph is a stronger property that $\underline{\Theta}$ being Eulerian (or Hamiltonian) as a graph. Finally, we generalize Corollary 4.3.16 and show that spider-web digraphs form towers of graphs coverings, both in N and, in a certain sense, in M.

Proposition 4.5.18. *For all $M \in \overline{\mathbf{N}}$ and $N \in \mathbf{N}$, the digraph $\vec{\mathcal{S}}_{N,M}$ (weakly) covers $\vec{\mathcal{S}}_{N-1,M}$.*

For every $i \in \mathbf{N}$, the digraph $\vec{\mathcal{S}}_{N,iM}$ (weakly) covers $\vec{\mathcal{S}}_{N,M}$.

Proof. By Corollary 4.3.16, we know that $\vec{\mathcal{B}}_{N+1}$ covers $\vec{\mathcal{B}}_N$ and it is easily seen that $\vec{C}_{iM}$ covers $\vec{C}_M$. A simple application of Lemma 2.3.2 gives the desired result. □

Note that any covering of digraphs $\varphi\colon \vec{\Delta}_1 \to \vec{\Delta}_2$ naturally induces a covering between the underlying graphs.

4.6 Complexity and spectral zeta function

In this section we compute the complexity (the number of covering trees) of both spider-web digraphs and spider-web graphs. For spider-web graphs we also show the convergence of the spectral zeta functions. After that, we turn our attention to spectral zeta function of spider-web graphs. We compute them and show that they converge to the spectral zeta function of $\mathrm{Cay}(\mathcal{L}_k, X_k^{\pm}) \simeq \mathrm{DL}(k,k)$. This allows us to compute the Fuglede-Kadison determinant of $\mathrm{DL}(k,k)$. Finally, we make a breve comment on the Green function of $\mathrm{DL}(k,k)$.

Complexity of spider-web digraphs

Definition 4.6.1. An oriented rooted spanning tree of $\vec{\Gamma}$ is a subdigraph containing all of the vertices of $\vec{\Gamma}$, having no directed cycles, in which one vertex, the root, has outdegree 0, and every other vertex has outdegree 1. The number of such trees is called the *complexity* of $\vec{\Gamma}$ and is written $\kappa(\vec{G})$. The number of such trees rooted at v a particular vertex of Γ is $\kappa(\vec{G}, v)$. The *asymptotical complexity* of a family of digraphs $\vec{\Gamma}_n$ is defined by $\kappa_{\mathrm{as}} := \lim \frac{\log\left(\kappa(\vec{\Gamma}_n)\right)}{|V(\vec{\Gamma}_n)|}$.

We know that the digraph $\vec{\mathcal{S}}_{M,N}$ is (weakly) isomorphic to the oriented line graph $L(\vec{\mathcal{S}}_{M,N-1})$ and therefore to $L^N(\vec{\mathcal{S}}_{M,0})$ (the iterated oriented line graph). On the other hand, $\vec{\mathcal{S}}_{1,0}$ is isomorphic to $\vec{R}_k$ the rose with k petals and, for $M \geq 2$, $\vec{\mathcal{S}}_{M,0}$ is isomorphic to a thick oriented circle: the vertices are $\{1, \ldots, M\}$ and for each i there is k edges from i to $i+1$ (mod M).

Since in $\vec{\mathcal{S}}_{M,0}$ every vertex has k-outgoing edges and k-ingoing edges, by Levine's formula for the complexity of line digraphs [87], we have $\kappa(\vec{\mathcal{S}}_{M,N}) = \kappa(\vec{\mathcal{S}}_{M,0}) k^{(k^N-1)M}$. We also have $\kappa(\vec{\mathcal{S}}_{1,0}) = 1$ and for $M \geq 2$, $\kappa(\vec{\mathcal{S}}_{M,0}) = M \cdot k^{M-1}$ (we choose the starting point and for each segment of the circle one of the k edges). Since $\vec{\mathcal{S}}_{M,N}$ is Eulerian, $\kappa(\vec{G}, v)$ does not depend on v [87]. Therefore, direct computations give us the following

Proposition 4.6.2. *For spider-web digraphs and for k fixed, we have*

$$\kappa(\vec{\mathcal{S}}_{M,N}) = M \cdot k^{k^N M - 1} \qquad \kappa(\vec{\mathcal{S}}_{M,N}, v) = k^{Mk^N - (N+1)}$$
$$\kappa_{\mathrm{as}}(\vec{\mathcal{S}}_{M,N}) = \log(k)$$

where the limit does not depend on the way we choose to take the limit $Mk^N \to \infty$.

For $\vec{\Gamma} = (V, \vec{E})$ a strongly connected finite digraph a v a vertex, define Δ_v by $\sum_{\iota(e)=v}(\tau(e) - v) \in \mathbf{Z}[V]$, and let $L_w \leq \mathbf{Z}[V]$ be the subgroup generated by w and $\{\Delta_v\}_{v\neq w}$. The *critical group* (or *sandpile group*) of $\vec{\Gamma}$ at v is $\mathbf{Z}[V]/L_V$. If $\vec{\Gamma}$ is Eulerian, then this group does not depends on v and is noted $\mathrm{Crit}(\vec{\Gamma})$. Observe that the cardinality of $\mathrm{Crit}(\vec{\Gamma})$ is exactly $\kappa(\vec{\Gamma})$.

The following proposition generalizes [87] for every prime k and every M.

Proposition 4.6.3. *For every k we have $\mathrm{Crit}(\vec{\mathcal{S}}_{M,0}) \cong (\mathbf{Z}/k\mathbf{Z})^{M-1}$. If moreover k is prime, then*

$$\mathrm{Crit}(\vec{\mathcal{S}}_{M,N}) \cong \bigoplus_{j=1}^{N-1} (\mathbf{Z}/k^j\mathbf{Z})^{M(k^{N-j+1}-2k^{N-j}+k^{N-j-1})} \oplus (\mathbf{Z}/k^N\mathbf{Z})^{M(k-2)} \oplus (\mathbf{Z}/k^{N+1}\mathbf{Z})^{M-1}$$

Proof. The formula for $M = 1$ and $N = 0$ directly follows from the definition. For $M \geq 2$ and $N = 0$, we have

$$\begin{aligned} L_v &= \langle v, k(v_1 - v), \dots, k(v_{M-1} - v_{M-2}), k(v - v_{M-1})\rangle \\ &= \langle v, kv_1, \dots, kv_{M-1})\rangle \end{aligned}$$

which gives us $\mathrm{Crit}(\vec{\mathcal{S}}_{M,0}) \cong (\mathbf{Z}/k\mathbf{Z})^{M-1}$.

Suppose now that k is prime. Then $\mathrm{Crit}(\vec{\mathcal{S}}_{M,N}) = \bigoplus_{j=1}^{l}(\mathbf{Z}/k^j\mathbf{Z})^{a_j}$ for some nonnegative integers l and $a_1, \dots, a_l$ satisfying $\sum_{j=1}^{l} ja_j = Mk^N - (N+1)$. Moreover, by [87], we have

$$k\bigoplus_{j=1}^{l}(\mathbf{Z}/k^j\mathbf{Z})^{a_j} = \bigoplus_{j=1}^{l-1}(\mathbf{Z}/k^j\mathbf{Z})^{a_{j+1}} \cong k\,\mathrm{Crit}(\vec{\mathcal{S}}_{M,N}) \cong \mathrm{Crit}(\vec{\mathcal{S}}_{M,N-1})$$

That is, for $j \geq 2$, we have $a_j(N) = a_{j-1}(N-1)$. For $N = 1$, this gives $a_2 = M-1$, and using the relation on the a_j we have $a_1 = Mk - 2 - 2(M-1) = M(k-2)$. Similarly, for $N = 2$, we have $a_3 = M - 1$, $a_2 = M(k-2)$ and $a_1 = M(k^2 - 2k + 1)$. Finally, by induction we obtain $a_{N+1} = M-1$, $a_N = M(k-2)$ and for $1 \leq j < N$, $a_j = M(k^{N-j+1} - 2k^{N-j} + k^{N-j-1})$. □

Complexity of spider-web graphs

If Γ is a graph (i.e. a non-oriented graph), then we define its *complexity* $\kappa(\Gamma)$ to be the number of its non-rooted spanning trees. The *asymptotical complexity* of a family of graphs is $\kappa_{\mathrm{as}} := \lim \frac{\log\left(\kappa(\Gamma_n)\right)}{|V(\Gamma_n)|}$.

Levine criterion on line graphs can a priori be used for the unoriented case $\mathcal{S}_{M,N}$. But the graph $\mathcal{S}_{M,N}$ is not isomorphic to the line graph of $\mathcal{S}_{M,N-1}$, we therefore need another approach.

Let $\mathrm{Li}_s(z) := \sum_{j\geq 1} \frac{z^j}{j^s}$ be the *polylogarithm*. Note that, for all s with $\mathrm{Re}(s) > 1$, we have $\mathrm{Li}_s(1) = \zeta(s)$ the Riemann zeta function.

Proposition 4.6.4. *For spider-web graphs and for k fixed, we have*

$$\begin{aligned} \kappa(\mathcal{S}_{M,N}) &= Mk^{M-1}(N+1)^{M(k-1)} \prod_{i=1}^{N-1} \left(k^i(i+1)\right)^{M(k-1)^2k^{N-1-i}} \\ \kappa_{\mathrm{as}}(\mathcal{S}_{M,N}) &= \log(k) + (k-1)^2 \sum_{i\geq 2} \frac{\log(i)}{k^i} \\ &= \log(k) - (k-1)^2 \frac{d}{ds} \mathrm{Li}_s\left(\frac{1}{k}\right)\Big|_{s=0} \end{aligned}$$

where the limit does not depend on the way we choose to take the limit $Mk^N \to \infty$.

Proof. We will us the Hutschenreuther statement who says that for any r-regular graph Γ with n vertices, we have

$$\kappa(\Gamma) = \frac{1}{n}\prod_{i=2}^{n}(r-\lambda_i) = \frac{1}{n}\chi'_\Gamma(r)$$

where the $r = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of the adjacency matrix and χ_Γ is the characteristic polynomial. The left equality is well-known in the following form :

$$\kappa(\Gamma) = \frac{1}{n}\prod_{i=2}^{n}\mu_i$$

where $0 = \mu_1 < \mu_2 \leq \cdots \leq \mu_n$ are the eigenvalues of the Laplacian Δ. Since Γ is r-regular, we have $\mu_i = r - \lambda_i$ for all i. Therefore, $\chi'_\Gamma(r) = \chi'_{\Delta_\Gamma}(0) = \prod_{i=2}^{n}(\mu_i)$ where χ_{Δ_Γ} is the characteristic polynomial of the Laplacian.

The case $M = 1$ is treated in [116] and gives

$$\kappa(\mathcal{B}_{k,N}) = (N+1)^{k-1}\prod_{i=1}^{N-1}\left(k^i(i+1)\right)^{(k-1)^2k^{N-1-i}}$$

By Section 4.4, we have

$$\chi_{\mathcal{S}_{M,N}}(x) = P(x)^M Q_M(x), \qquad Q_M(x) = \prod_{l=1}^{M}(x - 2k\cos\left(\frac{2\pi l}{M}\right))$$

where $P(x)$ does not depend on M and $Q_1(x) = x - 2k$. Therefore, $\kappa(\mathcal{S}_{M,N}) = \kappa(\mathcal{B}_N)^M\kappa(C_M)$ where C_M is the "thick cycle" of length M with k edges between two consecutive vertices. A direct computation gives the formulas for $\kappa(\mathcal{S}_{M,N})$ and $\kappa_{\mathrm{as}}(\mathcal{S}_{M,N})$. □

Spectral zeta function and Fuglede-Kadison determinant

For a graph (finite or infinite) Γ with spectral measure μ according to the (non-normalized) Laplace operator Δ,the *spectral zeta function* of G is define as the integral

$$\zeta_\Gamma(s) := \int_{\mathrm{Spec}(\Delta)\setminus\{0\}} \lambda^{-s}\mathrm{d}\mu(\lambda)$$

Spectral zeta functions are related to complexity. For a finite connected graph $\Gamma = (V, E)$, the Kirchhoff's formula (also known as matrix-tree theorem) says that

$$\kappa(\Gamma)\cdot|V| = \frac{1}{n}\prod_{i=2}^{n}\mu_i = \tilde{\det}(\Delta_\Gamma) = e^{-|V|\cdot\zeta'_\Gamma(0)}$$

where the μ_i are the eigenvalues of the Laplacian. For infinite graph Γ, $e^{-\zeta'_\Gamma(0)}$ can be understood in terms of the so-called Fuglede-Kadison determinant of the Laplacian, see [88].

If Γ is r regular, then $\mu(A) = \nu(r - A)$ where ν is the spectral measure according to the adjacency operator. If $(\Gamma_n)_n$ is a sequence of finite graphs converging to a vertex-transitive graph Γ, then the ν_n's weakly converge to ν and for all $x \in \mathbf{R}$, $\nu_n(\{x\})$ tend to $\nu(\{x\})$, see [1]. Thus, if the Γ_n are r-regular, the μ_n's weakly converge to μ and for all $x \in \mathbf{R}$, $\mu_n(\{x\})$ tend to $\mu(\{x\})$. Moreover, in this case, the support of μ_n and μ are in $[0, 2r]$. This implies that $\int_0^{2r} f(x)\mathrm{d}\mu_n(x)$ converge to $\int_0^{2r} f(x)\mathrm{d}\mu(x)$ for all function $f\colon [0, 2r] \to \mathbf{R}$ continuous and bounded. For all $\epsilon > 0$, the function λ^{-s} is continuous and bounded on $[\epsilon, 2r]$. If moreover, $s < 0$, then it is bounded and continuous on $[0, 2r]$. We therefore have the following lemma

Lemma 4.6.5. *Let* $(\Gamma_n)_{n\in\mathbf{N}}$ *be a family of regular finite graphs with limit a transitive graph* Γ*. Suppose that* $s \in \mathbf{R}$ *is such that we have* $\int_0^\varepsilon \lambda^{-s} \mathrm{d}\mu_\Gamma(\lambda) \xrightarrow{\varepsilon\to 0} 0$ *and*

$$\limsup_n \Big(\int_0^\varepsilon \lambda^{-s} \mathrm{d}\mu_{\Gamma_n}(\lambda) \Big) \xrightarrow{\varepsilon\to 0} 0.$$

Then

$$\zeta_{\Gamma_n}(s) \longrightarrow \zeta_\Gamma(s) \qquad \text{and} \qquad \kappa_{\mathrm{as}}(\Gamma_n) = -\zeta_\Gamma'(0)$$

The second equality was obtained in a more general context by Lyons in [88].

Corollary 4.6.6. *The spectral zeta functions of* $\mathcal{S}_{k,M,N}$ *converges to the spectral zeta function of the Cayley graph of* $\mathcal{L}_k$ *and*

$$\begin{aligned}\kappa_{\mathrm{as}}(\mathcal{S}_{k,M,N}) &= \log(2k) + (k-1)^2 \sum_{q\geq 2} \frac{1}{k^q - 1} \Big(\sum_{\substack{1\leq p<q \\ (p,q)=1}} \log(1 - 1\cos(\frac{p}{q}\pi)) \Big) \\ &= \log(k) - (k-1)^2 \frac{d}{ds} \operatorname{Li}_s\Big(\frac{1}{k}\Big)\Big|_{s=0}\end{aligned}$$

Corollary 4.6.7. *The Fuglede-Kadison determinant of the Laplacian of* $\overrightarrow{\mathrm{Cay}}(\mathcal{L}_k, X_k) \simeq \mathrm{DL}(k,k)$ *is equal to*

$$k \cdot e^{-(k-1)^2 \frac{d}{ds} \operatorname{Li}_s\left(\frac{1}{k}\right)\Big|_{s=0}}$$

Finally, we obtain following strange formula

$$-\sum_{j\geq 1} \frac{\log(j)}{k^j} = \sum_{q\geq 2} \frac{2}{k^q - 1} \Big(\sum_{\substack{1\leq p<q \\ (p,q)=1}} \sum_{j\geq 1} \frac{\cos(\frac{p}{q} j\pi)}{j} \Big)$$

where we used the following facts: $\log(1-\cos(x)) = \log(2\sin^2(x/2))$ and $\log(\sin(x)) = -\sum_{t\geq 1} \frac{\cos(2tx)}{t} - \log(2)$. Let Γ be a graph. We define $p_t(v;\Gamma)$ as the probability that the simple random walk on Γ starting at v ends at v after t steps and the *return probability generating function* as

$$\mathscr{G}(z;\Gamma) = \sum_{t\geq 0} p_t(v;G) z^t$$

It follows from definitions that

$$\int_0^1 \frac{\mathcal{G}(z) - 1}{z} dz = \sum_{t\geq 1} \frac{p_t(v)}{t}$$

Moreover, in [88], Lyons proved that for $(\Gamma_n)_n$ a sequence of graphs with limit Γ a vertex-transitive locally finite graph, the following holds

$$\lim_{n\to\infty} \frac{\log(\kappa(G_N))}{|V_N|} = \kappa_{as} = \log(\deg_G(v_0)) - \sum_{t\geq 1} \frac{p_t(v_0)}{t}$$

for all v_0 in G.

For spider-web graphs, this gives

$$\int_0^1 \frac{\mathcal{G}(z;\mathrm{Cay}(\mathcal{L}_k)) - 1}{z} dz = \log(2) + (k-1)^2 \frac{d}{ds} \operatorname{Li}_s\Big(\frac{1}{k}\Big)\Big|_{s=0}$$

4.7 Rauzy graphs

De Bruijn (di)graphs represent overlaps between strings of symbols and are therefore related to the full shift on $\mathbf{Z}$. A natural generalization of de Bruijn digraphs that is related to subshifts on $\mathbf{Z}$ is the Rauzy digraphs corresponding to subshifts of finite type.

The aim of this section is to compute the limit of sequences of Rauzy graphs. Theorem 4.7.15 deals with local limit while Theorem 4.7.18 gives a description of the Benjamini-Schramm limit when it exists. Theorem 4.7.21 gives sufficient conditions for the existence of a Benjamini-Schramm limit. We also study in Proposition 4.7.33 the spider-web version of Rauzy graphs.

The main tools are Theorem 4.7.9 which makes a link between horospheric product of trees and infinite version of Rauzy graphs, and the definition of a (non-regular) rooted tree (see Definition 4.7.10) which serves as an analogue of the k-regular rooted tree on which $\mathcal{L}_k$ acts.

Let $\mathcal{A}^*$ be the set of finite words over the alphabet $\mathcal{A} = \{0, \ldots, k-1\}$ (that is the free monoid on $\mathcal{A}$) and $\mathcal{A}^{\mathbf{N}}$, respectively $\mathcal{A}^{-\mathbf{N}}$, be the set of right, respectively left, infinite sequences on $\mathcal{A}$. Let F be a subset of $\mathcal{A}^*$, we think of F as *forbidden words*. Then $\mathcal{A}_F^N$ is the set of words of length N that do not contain elements of F as subwords. Similar definitions hold for $\mathcal{A}_F^*$, $\mathcal{A}_F^{\mathbf{N}}$ and $\mathcal{A}_F^{-\mathbf{N}}$.

Definition 4.7.1. For F in $\mathcal{A}^*$ and N an integer, the corresponding *labeled Rauzy digraph* is $\vec{R}_{F,k,N} \coloneqq (\mathcal{A}_F^N, \mathcal{A}_F^{N+1})$, where an edge $(x_1 \ldots x_{N+1})$ has initial vertex $(x_1 \ldots x_N)$, final vertex $(x_2 \ldots x_{N+1})$ and label x_{N+1}.

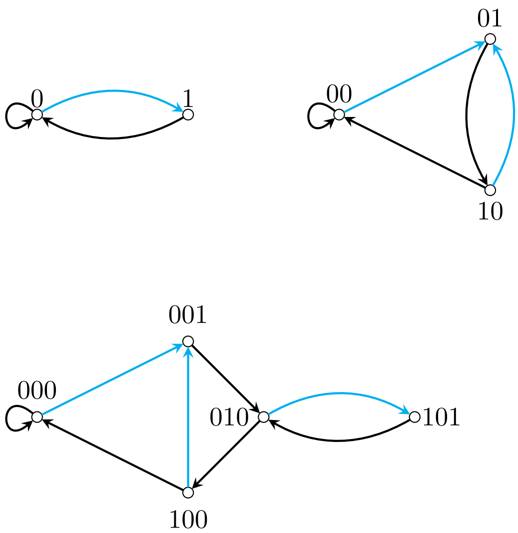


Figure 4.13: Rauzy digraphs for $k = 2$, $N \in \{1, 2, 3\}$ and $F = \{11\}$. Black edges are labeled by 0 and cyan edges are labeled by 1.

It directly follows from the definition that $\vec{R}_{F,k,N}$ is a subgraph of $\vec{\mathcal{B}}_{k,N}$, with equality if and only if $F = \emptyset$.

If F is finite, then the corresponding subshift is said to be of *finite type*. If F is a subshift of finite type over a finite alphabet $\mathcal{A}$ on k letter and n is the biggest length of a word in F, for all $N \geq n$, the digraph $\vec{R}_{F,k,N}$ is the line digraph of $\vec{R}_{F,k,N-1}$ — see Lemma 4.3.2

and Proposition 4.7.30. Therefore, if we take $\mathcal{A}'$ as the vertex set of $\vec{R}_{F,k,n-1}$ and F' as the missing vertices in $\vec{R}_{F,k,n-1}$, we have for all $N \geq n+1$ that the digraphs $\vec{R}_{F,k,N+n-2}$ and $\vec{R}_{F',k',N}$ are isomorphic. In the following, we will always assume that $F \subseteq \mathcal{A}^2$ is fixed, and will omit to write k when it is not necessary. In this case, $\vec{R}_{F,N}$ is a complete subdigraph of $\vec{\mathcal{B}}_N$; i.e. if v and w are two vertices of $\vec{R}_{F,N}$, then the number of edges from v to w is the same in $\vec{R}_{F,N}$ and in $\vec{\mathcal{B}}_N$.

Definition 4.7.2. Let $F \subseteq \mathcal{A}^2$. The associated matrix $A_F = (a_{ij})_{i,j}$ is defined by $a_{ij} = 1$ if $ij \notin F$ and $a_{ij} = 0$ otherwise. The *digraph associated* to F is the $\vec{\Gamma}_F$, the digraph with adjacency matrix $B_F := A_F \otimes A_F^{\mathrm{T}}$.

Observe that there is a bijection between $F \subseteq \mathcal{A}^2$ and k-by-k matrices with coefficients in $\{0, 1\}$.

By definition, the matrix B_F is non-negative. This matrix will turn out to be very useful in the study of the limit of Rauzy digraphs. Recall that such a matrix is said *irreducible* if the corresponding digraph is strongly connected and of *period* p if in the corresponding digraph, the gcd of length of closed paths is equal to p. If the period is equal to 1, then the digraph is said to be *aperiodic* We say that such a matrix is *weakly irreducible* if the corresponding digraph is connected and the decomposition into strongly connected component is as follow. One "big" connected component and small connected components consisting of a vertex v such that there is a path from v to the big connected component.

Weakly irreducible and aperiodic finite digraphs are example of digraphs that admits an equilibrium distribution. That is, an eigenvector vector v_0 of norm 1 such that every positive vectors converge to v_0. For irreducible and aperiodic finite digraphs, this is a well-known result that follows from Perron-Frobenius Theorem. For weakly irreducible, aperiodic digraph, this is a consequence of the fact that after a finite number of steps we are in the irreducible part (the "big" connected component). The corresponding eigenvalue is called the Perron-Frobenius eigenvalue.

We showed that the limit of de Bruijn digraphs is the Cayley digraph of the Lamplighter group. This graph is also isomorphic to the horospheric product of two regular oriented trees. We will show that in general, the limit of $\vec{R}_{F,N}$ is a probability measure concentrated on horospheric product of two (non-necessarily regular) oriented trees.

In order to do that, we define "infinite Rauzy graphs" and we will show that they are isomorphic to horospheric product of two oriented trees.

Definition 4.7.3. For $F \subseteq \mathcal{A}^2$, we define an infinite labeled (non connected) digraph $\vec{R}_{F,\infty}$ with vertex set $\mathcal{A}_F^{\mathbf{N}} \times \mathcal{A}_F^{-\mathbf{N}}$ and with an edge labeled by i from $(x_1x_2\ldots;\ldots x_{-2}x_{-1})$ to $(x_2x_3\ldots;\ldots x_{-1}i)$. We also define the subdigraph $\vec{R}'_{F,\infty} \subseteq \vec{R}_{F,\infty}$ as the complete subdigraph on the vertex set consisting of paires (ξ, η) such that at least one of the sequence is not eventually periodic.

We have that $\vec{R}'_{F,\infty}$ is a disjoint union of some connected components of $\vec{R}_{F,\infty}$. We will show that, under some mild assumptions, any connected component of $\vec{R}'_{F,\infty}$ is a limit of the sequence $(\vec{R}_{F,n})_{n\geq 1}$. Before doing that, we show that connected components of $\vec{R}_{F,\infty}$ are horospheric products of tree.

Definition 4.7.4. Let $F \subseteq \mathcal{A}^2$ be such that $\mathcal{A}_F^{\mathbf{N}}$ is non empty and let $\xi = (x_1x_2\ldots)$ be an element of $\mathcal{A}_F^{\mathbf{N}}$. Then $\vec{T}_{F,\xi}$ is the *cofinal tree of* ξ: i.e. the directed tree of elements of $\mathcal{A}_F^{\mathbf{N}}$ that are cofinals with ξ.

If $\eta = \ldots x_{-2}x_{-1}$ is an element of $\mathcal{A}_F^{-\mathbf{N}}$, the tree $\vec{T}_{F,\eta}$ is the *coinitial tree of* η: i.e. the directed tree of elements of $\mathcal{A}_F^{-\mathbf{N}}$ that are coinitial with η.

From the point of view of symbolic dynamics, the tree $\vec{T}_{F,\xi}$ consists of all trajectories of the shift $\mathcal{A}^{\mathbf{Z}}$ that share a "common direction".

The tree $\vec{T}_{F,\xi}$ can be construct explicitly as follow. Start with an infinite ray, directed upstair, where vertices are labeled by $x_1, x_2, \ldots$. Under each vertex v already constructed and labeled by x, for each $i \in \mathcal{A}$ such that $ix \notin F$, if there is not already a vertex with label i under v, attach a new vertex with label i. Repeat this construction infinitely many time to obtain a tree. Finally, we put an orientation on $T_{F,\xi}$ such that each edge is going up and we label each edge by its initial vertex. For $\vec{T}_{F,\eta}$, start with an infinite ray, directed upstair, where vertices are labeled by $x_{-1}, x_{-2}, \ldots$. Under each vertex v already constructed and labeled by x, for each $i \in \mathcal{A}$ such that $xi \notin F$, if there is not already a vertex with label i under v, attach a new vertex with label i. Repeat this construction infinitely many time to obtain a tree. Finally, we put an orientation on $T_{F,\eta}$ such that each edge is going down and we label each edge by its terminal vertex. See Figure 4.14 for an example.

An important but rather obvious property of $\vec{T}_{F,\xi}$ is that all paths consisting of positive edges are labeled by elements in $\mathcal{A}_F^*$. Similarly, infinite rays are labeled by elements in $\mathcal{A}_F^{\mathbf{N}}$ and biinfinite rays are labeled by elements in $\mathcal{A}_F^{\mathbf{Z}}$.

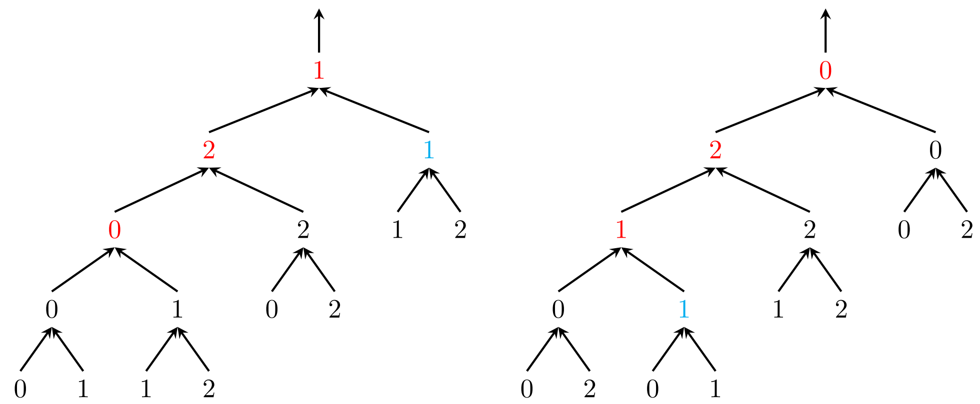


Figure 4.14: Part of the tree $\vec{T}_{\{01,12,20\},021\ldots}$ and $\vec{T}_{\{01,12,20\},\ldots 021}$ on the alphabet $\{0,1,2\}$. Labeling of the edges are not shown. In blue, the image of $w = (11\ldots;\ldots 0211)$.

For an infinite tree T, and a right infinite ray ξ, it is possible to define the *Busemann rank function* $\mathfrak{h}\colon V(T) \to \mathbf{Z}$ by $\mathfrak{h}(v) := d(p_\xi(v), \xi_1) - d(p_\xi(v), v)$, where $p_\xi(v)$ is the projection of v onto ξ. This naturally endows T with an orientation: the initial vertex of an edge between v and w is the vertex with the lower Busemann rank. Observe that if $T = \vec{T}_{F,\xi}$, then the orientation coming from the Busemann rank function with respect to ξ is the original orientation of $\vec{T}_{F,\xi}$. If η is a left infinite ray, we define $\mathfrak{h}$ by $\mathfrak{h}(v) := -d(p_\eta(v), \eta_1) + d(p_\eta(v), v)$.

Definition 4.7.5. Let (T_1, ξ) be a pair consisting of a infinite tree with a right infinite ray and (T_2, η) be a pair consisting of a infinite tree with a left infinite ray. The *horospheric product* (also called *horocyclic product*) $(T_1, \xi) \circledast (T_2, \eta)$ of (T_1, ξ) and (T_2, η) is digraph with vertex set $\{(v, w) \in V(T_1) \times V(T_2) \mid \mathfrak{h}(v) = \mathfrak{h}(w)\}$. There is an edge from (v, w) to (x, z) if and only if there is an edge from v to x and an edge from w to z, for the orientation coming from the Busemann rank function. This digraph is naturally rooted at (ξ_1, η_1).

If T_2 is a labeled graph, then the horospheric product is a labeled digraph where the label of (e, f) is the label of f.

A direct consequence of the definition is that $(T_1,\xi) \circledS (T_2,\eta)$ is a connected component of the digraph $(T_1,\xi) \otimes (T_2,\eta)$. Observe that if T_1 and T_2 are labeled, we may define a label on $(T_1,\xi) \circledS (T_2,\eta)$ either by the label of T_1, or by the label of T_2 or by the product of the labels. Labeling by T_1 is related to the initial vertex, while labeling by T_2 is related to final vertex, hence our choice by analogy with the labeling of Rauzy digraphs.

Horospheric product of regular trees of degree $n+1$ and $m+1$ are usually studied under the name of Diestel-Leader graphs and noted $\vec{\mathrm{DL}}(n,m)$. In [37], Diestel and Leader showed that for all n and m these graphs are vertex-transitive, that $\vec{\mathrm{DL}}(n,n)$ is isomorphic to the Cayley digraph of $\mathcal{L}_n$ with generating set $\{\bar{c}_r\}_{r=0}^{k-1}$ and conjectured that for $n \neq m$, $\mathrm{DL}(n,m)$ is not quasi-isometric to any Cayley graph. This was proven by Eskin, Fisher and Whyte in [39]. Since $\vec{\mathrm{DL}}(k,k)$ is isomorphic to the Cayley graph of $\mathcal{L}_k$, it is the limit of the sequence $(\vec{\mathcal{B}}_{k,N})_k$. Therefore, horospheric products may provide the right framework to study limit of Rauzy graphs.

Lemma 4.7.6. *Let $(\xi,\eta) = (x_1x_2\ldots;\ldots x_{-2}x_{-1})$ be a vertex in $\vec{R}'_{F,\infty}$. Let p and q be two paths with initial vertex v and with same final vertex. Then,* $\mathrm{der}(p) = \mathrm{der}(q)$.

Proof. Observe that positive edges shift both sequences $x_1x_2\ldots$ and $\ldots x_{-2}x_{-1}$ to the left (and change a finite number of digits), while negative edges shift the sequence to the right.

Therefore, if $\tau(p) = \tau(q)$ but $\mathrm{der}(p) \neq \mathrm{der}(q)$, both sequences ξ and η are eventually periodic, which is absurd. □

Definition 4.7.7. Let v be a vertex in $\vec{R}'_{F,\infty}$. We define the associated *rank function*: $\mathrm{rk}_v\colon (\vec{R}'_{F,\infty},v)^0 \to \mathbf{Z}$ by $\mathrm{rk}_v(w) = \mathrm{der}(p)$ where p is any path from v to w.

By the last lemma, this is well defined. The following formula directly follows from definitions and will be useful later.

Lemma 4.7.8. *Let w be a vertex in $(\vec{R}'_{F,\infty},v)^0$. If $\mathrm{rk}_v(w) \geq 0$, we have*

$$w = (y_{1+\mathrm{rk}_v(w)}\cdots y_n v_{n+1}\cdots;\ldots v_{-m}y_{-m+1}\cdots y_{-1}z_1\ldots z_{\mathrm{rk}_v(w)})$$

for some y_j's and z_j's in $\mathcal{A}$, while if $\mathrm{rk}_v(w) \leq 0$ we have

$$w = (z_1\ldots z_{-\mathrm{rk}_v(w)}y_1\cdots y_n v_{n+1}\cdots;\ldots v_{-m}y_{-m+1}\cdots y_{-1+\mathrm{rk}_v(w)}).$$

We are now able to show the link between Rauzy graphs and horospheric products.

Theorem 4.7.9. *For each vertex $v = (\xi,\eta)$ in $\vec{R}'_{F,\infty}$, there is a rooted strong isomorphism between $(\vec{R}'_{F,\infty},v)^0$ and the horospheric product $T_{F,\xi} \circledS T_{F,\eta}$.*

Proof. For a vertex x in $T_{F,\xi}$, the projection of x on ξ is noted by $p_\xi(x)$.

Let w be a vertex in $(\vec{R}'_{F,\infty},v)^0$. By the last lemma, if $\mathrm{rk}_v(w) \geq 0$, then w determines a vertex in $T_{F,\xi}$: the unique vertex x such that $p_\xi(x) = v_{n+1}$ and the label of vertices between x and v_{n+1} is given by $y_{1+\mathrm{rk}_v(w)}\cdots y_n$. This also determines a vertex in $T_{F,\eta}$: the unique vertex y such that $p_\eta(y) = v_{-m}$ and the label of vertices between x and v_{n+1} is given by $y_{-m+1}\cdots y_{-1}z_1\ldots z_{\mathrm{rk}_v(w)}$. Under this correspondence, we have

$$\begin{aligned}\mathfrak{h}(x) &= n - (n - \mathrm{rk}_v(w)) = \mathrm{rk}_v(w)\\ \mathfrak{h}(y) &= -(m-1) + (\mathrm{rk}_v(w) + m - 1) = \mathrm{rk}_v(w)\end{aligned}$$

A similar argument takes care of the case $\mathrm{rk}_v(w) \leq 0$.

Hence, we have a function φ from vertices of $(\vec{R}'_{F,\infty}, v)^0$ to vertices of $T_{F,\xi} \textcircled{h} T_{F,\eta}$. By definition, φ preserves root and is injective.

In $\vec{R}'_{F,\infty}$, there is an outgoing edge e from w with label i if and only if $w_{-1}i$ does not belongs to F, if and only if in $T_{F,\xi} \textcircled{h} T_{F,\eta}$ there is an outgoing edge f from $\varphi(w)$ with label i. Moreover, in this case, the terminal vertex of e is $(w_2 \dots ; \dots w_{-1}i)$ which is sent by φ to the terminal vertex of f. Therefore, the function φ naturally extends to a labeled graph homomorphism, which is thus locally bijective on outgoing edges. The injectivity of φ implies that it is locally injective and thus locally injective on ingoing edges.

It remains to show that φ is locally surjective on ingoing edges. Indeed, in this case φ is locally bijective and hence surjective. If w is a vertex in $\vec{R}'_{F,\infty}$, then the number of ingoing edges is $|\{i \mid iw_1 \notin F\}|$, which is exactly the number of ingoing edges for $\varphi(w)$. Since φ is locally injective on ingoing edges, it is also locally bijective on ingoing edges. $\square$

For an illustration of the proof, see Figure 4.14. It shows the two oriented tree corresponding to $v = (021 \dots , \dots 021)$ and in blue the vertices corresponding to $w = (1v_3 \dots ; \dots v_{-1}1)$. This w is connected to v by at least two paths in the underlying graph, given by

$$v \xrightarrow{1} (v_2 \dots ; \dots v_{-1}1) \xrightarrow{j} (v_3 \dots ; \dots v_{-1}1j) \xleftarrow{j} w$$

for $j \in 0, 1$.

Now that we have a description of connected components of $\vec{R}'_{F,\infty}$ as horospheric products, we are going to investigate the relations between the $\vec{R}_{F,n}$'s and $\vec{R}'_{F,\infty}$. In order to do that, we are going to construct a labeled rooted tree where (some of the) vertices of level n will correspond to vertices of $\vec{R}_{F,n}$.

Definition 4.7.10. Let $F \subseteq \mathcal{A}^2$ be a set of forbidden words. The associated infinite rooted tree T_F is constructed by "insertion in the middle". More precisely, it is defined inductively on level as follows. The root is the empty set. Every vertex $x_1 \dots x_{2n}$ of level $2n$ has as children all the $x_1 \dots x_n i x_{n+1} \dots x_{2n}$ such that $x_n i \notin F$. On the other hand, every vertex vertex $x_1 \dots x_{2n+1}$ of level $2n+1$ has as children all the $x_1 \dots x_{n+1} i x_{n+2} \dots x_{2n+1}$ such that $i x_{n+1} \notin F$. The vertices in $\mathcal{A}_F^*$ are called *good vertices* and the ones not in $\mathcal{A}_F^*$ the *bad vertices*.

Observe that in a bad vertex $x_1 \dots x_n$, the only forbidden word that appears is in the middle, i.e. is $x_{\lceil \frac{n}{2} \rceil} x_{\lceil \frac{n}{2} \rceil + 1}$. See Figures 4.15 and 4.16 for examples.

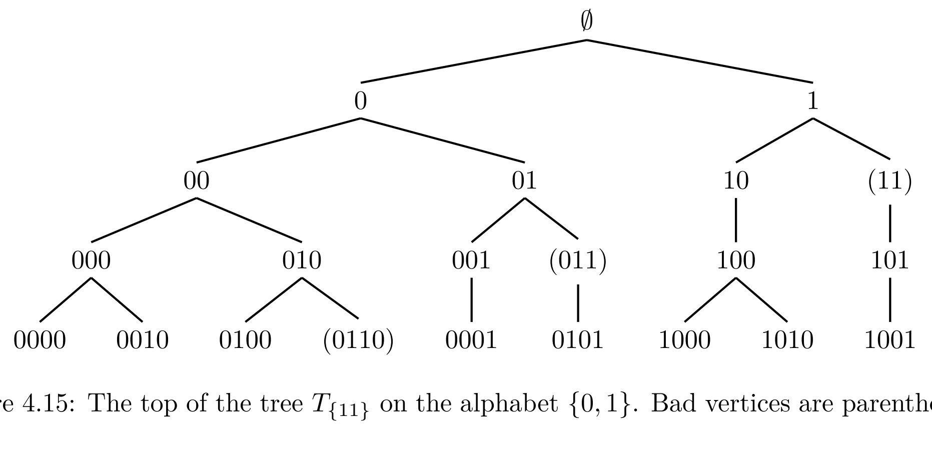


Figure 4.15: The top of the tree $T_{\{11\}}$ on the alphabet $\{0, 1\}$. Bad vertices are parenthesized.

By construction, good vertices of level n are in one-to-one correspondence with vertices of $\vec{R}_{F,n}$. On the other hand, infinite rays in T_F are parametrized by a pair of infinite sequences $(x_1x_2\dots;\dots x_{-2}x_{-1})$ and are in one-to-one correspondence with vertices (ξ,η) in $\vec{R}_{F,\infty}$.

Definition 4.7.11. Let v be a vertex in $\vec{R}'_{F,\infty}$ and $r\in\mathbf{N}$ an integer. For all $n\geq 2r+2$, define a function on vertices

$$\begin{aligned}\pi_n=\pi_{n,r}\colon\ &\mathrm{Ball}_{\vec{R}'_{F,\infty}}(v,r)\to\vec{R}_{\emptyset,n}\\ &w=(w_1\dots;\dots w_{-1})\mapsto(w_1\dots w_{\lceil\frac{n}{2}\rceil-\mathrm{rk}_v(w)}w_{-\lfloor\frac{n}{2}\rfloor-\mathrm{rk}_v(w)}\dots w_{-1})\end{aligned}$$

Lemma 4.7.12. *Let $v\in\vec{R}'_{F,\infty}$. For all $n\geq 2r+2$, the function π_n*

1. *Is well defined and has values in* $\mathrm{Ball}_{\vec{R}_{\emptyset,n}}(\pi_n(v),r)$*;*
2. *Naturally extends to a strong digraph homomorphism;*
3. *Is locally injective;*
4. *Is locally bijective if $v_{\lceil\frac{n}{2}\rceil}v_{-\lfloor\frac{n}{2}\rfloor}\notin F$;*
5. *We have* $\mathrm{Im}(\pi_n)=\mathrm{Ball}_{\vec{R}_{F,n}}(\pi_n(v),r)$ *if and only if $v_{\lceil\frac{n}{2}\rceil}v_{-\lfloor\frac{n}{2}\rfloor}\notin F$.*

Proof. We will prove the assertions for $2n$, in this case $n=\lceil\frac{2n}{2}\rceil=\lfloor\frac{2n}{2}\rfloor\geq r+1$. The proofs for $2n+1$ are similar.

Since w is at distance at most r from v, it has rank between $-r$ and r. Hence, π_{2n} is well defined.

If there is an edge labeled by i with initial vertex $w=(w_1\dots;\dots w_{-1})$, then it has final vertex $u=(w_2\dots;\dots w_{-1}i)$ and $\mathrm{der}(u)=\mathrm{der}(w)+1$. Therefore, these vertices are send respectively onto $(w_1\dots w_{n-\mathrm{rk}_v(w)}w_{-n-\mathrm{rk}_v(w)}\dots w_{-1})$ and $(w_2\dots w_{n-\mathrm{rk}_v(w)}w_{-n-\mathrm{rk}_v(w)}\dots w_{-1}i)$. Hence, there is a natural extension of π_{2n} to a strong digraph homomorphism. This directly implies that it has values in $\mathrm{Ball}_{\vec{R}_{\emptyset,2n}}(\pi_{2n}(v),r)$.

The local injectivity for outgoing edges follow from the fact that π_{2n} preserves the labeling and that for each vertex there is at most one outgoing edge with label i. On the other hand, for any vertex $w\in\vec{R}'_{F,\infty}$, the set of incoming edges (they all have label w_{-1}) is $\{(jw_1\dots;\dots w_{-2}\mid jw_1\notin F\}$. Since $2n\geq 2r+2$, these edges are sent onto $j\dots w_{-2}$ which are pairwise distinct.

If v_nv_{-n} is in F, then $\pi_{2n}(v)$ is not in $\mathcal{A}_F^*$. On the other hand, if v_nv_{-n} is not in F, we have $\mathrm{Im}(\pi_{2n})\subseteq\vec{R}_{F,2n}$. Indeed, in this case, a vertex w in $\mathrm{Ball}_{\vec{R}_{F,\infty}}(v,r)$ is of the form $(*\cdots*v_{r+1}\dots v_n\dots;\dots v_{-n}\dots v_{-r-1}*\cdots*)$ and is therefore send onto $*\cdots*$ $v_{r+1}\dots v_nv_{-n}\dots v_{-r-1}*\cdots*$. Since w belongs to $\vec{R}_{F,\infty}$, there is no forbidden words in $*\cdots*v_{r+1}\dots v_n$ and in $v_{-n}\dots v_{-r-1}*\cdots*$ and v_nv_{-n} is not forbidden by assumption.

It remains to show that if v_nv_{-n} is not in F, then π_{2n} is locally surjective. Indeed, in this case π_{2n} is locally bijective and hence surjective. If w is a vertex in $\vec{R}_{F,\infty}$, then the number of outgoing edges is $|\{i\mid w_{-1}i\notin F\}|$, which is exactly the number of outgoing edges for $\pi_{2n}(w)$. The same kind of relation holds for ingoing edges. Since π_{2n} is locally injective, it is therefore locally bijective. □

We want to prove that the sequence of $\big(\vec{R}_{F,n},\pi_n(v)\big)^0$'s (or at least a subsequence of it) locally converges to $\big(\vec{R}_{F,\infty},v\big)^0$ for n going to infinity. Therefore, we want to prove that for all r, there exists n big enough such that π_n is an isomorphism onto $\mathrm{Ball}_{\vec{R}_{F,n}}(\pi_n(v),r)$. That is, π_n injective and, by the last lemma, $v_{\lceil\frac{n}{2}\rceil}v_{-\lfloor\frac{n}{2}\rfloor}$ is not in F.

Lemma 4.7.13. *For all vertex v of $\vec{R}'_{F,\infty}$ and all r, there exists n_0 such that for all $n \geq n_0$, the function π_n is injective.*

Proof. Let w and u be two distinct vertices in the ball or radius r around v. Suppose that $\mathrm{rk}_v(w) = \mathrm{rk}_v(u) = i \geq 0$. In this case we have

$$w = (y_{1+i} \dots y_r v_{r+1} \dots ; \dots v_{-r-1} y_{-r} \dots y_{-1} z_1 \dots z_i)$$

and a similar formula for u. This implies that $w_j = u_j$ for all $j \geq r - i$ and all $j \leq -r - i$, and therefore that there exists $-r - i \leq j \leq r - i$ such that $w_j \neq u_j$. By definition of π_n, we have $\pi_n(w) \neq \pi_n(u)$ as soon as $\lfloor \frac{n}{2} \rfloor \geq r$, i.e. as soon as $n \geq 2r$. If $\mathrm{rk}_v(w) = \mathrm{rk}_v(u) = i \leq 0$, a similar proof gives the same result.

Suppose now that $\mathrm{rk}_v(u) - \mathrm{rk}_v(w) = i \neq 0$. By Lemma 4.7.8, if $\pi_n(u) = \pi_n(w)$, then for all $r < j \leq \lfloor \frac{n}{2} \rfloor - i$ we have $v_j = v_{j+i}$ and $v_{-j} = v_{-j-i}$. That is, between the $r + 1^{\text{th}}$ and the $\lfloor \frac{n}{2} \rfloor^{\text{th}}$ digit, both sequences $(v_1 v_2 \dots)$ and $(v_{-1} v_{-2} \dots)$ are periodic of period i. Since at least one of this sequence is not eventually periodic, there exists $n_i(v)$ such that for all $n \geq n_i(v)$, we have $\pi_n(u) \neq \pi_n(w)$.

Finally, for $n_0 = 2 \cdot \max_{-r \leq i \leq r}\{r + 1, n_i\}$ we have that π_{n_0} is injective and so is π_n for $n \geq n_0$. □

Observe that in the proof we can chose $n_0 = 2j$, where j is the smallest integer $j \geq r + 1$ such that at least one of the sequences $(v_{r+1} v_{r+2} \dots v_j)$ or $(v_{-r-1} v_{-r-2} \dots v_{-j})$ is not $r!$ periodic. In order to obtain a better, but less useful in practice, estimate, one can replace the $r!$ periodicity by $\mathrm{lcm}_{l \leq r}\{l\}$ periodicity.

Corollary 4.7.14. *Let $v = (v_1 \dots ; \dots v_{-1})$ be a vertex of $\vec{R}'_{F,\infty}$ such that infinitely many $v_i v_{-i}$ are not in F. Then, for all r, there exists n such that for every vertex $w = (v_1 \dots v_n w_{n+1} \dots ; \dots w_{-n-1} v_{-n} \dots v_{-1})$, $\mathrm{Ball}_{\vec{R}_{F,\infty}}(v, r)$ and $\mathrm{Ball}_{\vec{R}_{F,\infty}}(w, r)$ are strongly isomorphic.*

Proof. Let $n \geq n_0$ be such that $v_n v_{-n} \notin F$, where n_0 is given by Lemma 4.7.13. Then, since the proof of Lemmas 4.7.12 and 4.7.13 only look at $(v_1 \dots v_n v_{-n} \dots v_{-1})$, we have that both $\mathrm{Ball}_{\vec{R}_{F,\infty}}(v, r)$ and $\mathrm{Ball}_{\vec{R}_{F,\infty}}(w, r)$ are strongly isomorphic to $\mathrm{Ball}_{\vec{R}_{F,n}}(\pi_n(v), r)$. □

Theorem 4.7.15. *Let $v = (\xi, \eta)$ be a ray in T_F such that at least one of ξ or η is not eventually periodic. Let $w_1 = \emptyset$, w_2, ... denotes its successive vertices. Suppose that infinitely many w_i are good and let w_{m_j} be the corresponding subsequence. Then the sequence $\left(\vec{R}_{F,m_j}, w_{m_j}\right)^0$ locally converges to $(\vec{R}_{F,\infty}, v)^0 \simeq T_{F,\xi} \circledast T_{F,\eta}$ as j goes to infinity.*

Proof. The ray $v = (v_1 \dots ; \dots v_{-1})$ correspond to the vertex $(v_1 \dots ; \dots v_{-1}) \in \vec{R}_{F,\infty})$. If at least one of the sequence $(v_1 v_2 \dots)$ or $(v_{-1} v_{-2} \dots)$ is not eventually periodic, then (ξ, η) is in $\vec{R}'_{F,\infty}$. Under this correspondence, $w_m = \pi_m(v) = (v_1 \dots v_{\lceil \frac{n}{2} \rceil} v_{-\lfloor \frac{n}{2} \rfloor} \dots v_{-1})$. It is bad if and only if $v_{\lceil \frac{n}{2} \rceil} v_{-\lfloor \frac{n}{2} \rfloor}$ belongs to F. An application of Lemmas 4.7.12 and 4.7.13 show the convergence to $(\vec{R}_{F,\infty}, v)^0$. Theorem 4.7.9 implies the isomorphism with the horospheric product. □

We have just proved local convergence of some sequence of rooted Rauzy digraphs. By the discussion after Definition 4.7.10, points of ∂T_F are in bijection with vertices of $\vec{R}_{F,\infty}$. Therefore, we have a map $f \colon \partial T_F \to \vec{\mathcal{G}}_\bullet$, where $f(v)$ is the equivalence class of $(\vec{R}_{F,\infty}, v)^0$. In order to show Benjamini-Schramm convergence of Rauzy digraphs, we are going to associated to every digraph $\vec{R}_{F,N}$ a measure μ_N on ∂T_F, show that these measures weakly converge to a measure μ and show that the Benjamini-Schramm limit of $\vec{R}_{F,N}$ is the pushforward $f_*(\mu)$.

Definition 4.7.16. We note $\partial T'$ the subset of ∂T consisting of rays $v = (v_1 \dots ; \dots v_{-1})$ with at least one of the sequence non-periodic. We note $\partial \tilde{T}$ the subset of $\partial T'$ consisting of rays passing trough infinitely many good vertices.

Lemma 4.7.17. *When restricted to* $\partial \tilde{T} \cup \{\text{isolated points of } \partial T\}$*, the function* f *is continuous.*

Proof. Let $v = (v_1 \dots ; \dots v_{-1})$ be an element of $\partial \tilde{T}$. By Corollary 4.7.14 for any $\eta \in \partial \tilde{T}$ at distance at most 2^{-2n} of ξ, the graphs $f(\eta)$ and $f(\xi)$ have isomorphic balls of radius r and are therefore at distance less than 2^{-r}, which proves the continuity on $\partial \tilde{T}$. The continuity on the set of isolated points is trivial. □

Since the function f is continuous on $\partial \tilde{T}$, it is measurable: for any measure μ on ∂T_F such that $\mu(\partial \tilde{T}) = 1$, the pushforward $f_*(\mu)$ of μ gives a measure on $\mathcal{G}_\bullet$. Since good vertices of level N in T_F are identified with vertices of $\vec{R}_{F,N}$, the uniform measure μ_N on the vertices of $\vec{R}_{F,N}$ transpose the a measure (also noted μ_N) on the N^{th} level of T. This induces a measure on ∂T_F, also noted μ_n, by $\mu_n(C_v) \coloneqq \mu_n(v)$ for any vertex v of level N, where $C_v = \{w \mid w \leq v\}$ is a cylinder set, and by putting the uniform measure on vertices below level N. That is, if w is of level at most n, then $\mu_n(C_w)$ simply counts the percentage of good vertices of level n which are under w. On the other hand, for $w \leq v$ with v of level n, then μ_n is $\mu_n(C_v)$ times the uniform measure on $(T_F)_v$ (the subtree of T_F of vertices under v).

$$\mu_n(C_w) \coloneqq \begin{cases} \frac{|\{i | v_i \leq w\}|}{m} & \text{if } w \text{ is of level at most } n \\ \frac{1}{m} \frac{1}{|\{v \leq v_i | v \text{ of same level as } w\}|} & \text{if } w \leq v_i \text{ for some } i \\ 0 & \text{otherwise (i.e. } w \leq u_i \text{ for some } i) \end{cases}$$

Since ∂T_F is a metric space which is separable, compact and complete, by Prokhorov's Theorem [104] the space of Borel probability measures on it is compact in the weak topology. In particular, since the cylinder sets are clopen, for any converging sequence $\nu_n \to \nu$, the Portmanteau theorem implies that for any $v \in T_F$, the $\lim \nu_m(C_v)$ exists and is equal to $\nu(C_v)$. On the other hand, if a sequence of cylindrical measure converges on all (more generally on a dense subset of) cylinders, then this sequence is convergent. Recall that $\partial T'$ is the subset of ∂T consisting of rays (ξ, η) with at least on of the sequence not eventually periodic and that $\partial \tilde{T}$ is the subset of $\partial T'$ consisting of rays passing through infinitely many goof vertices.

Theorem 4.7.18. *Let* $\partial T = I \sqcup J$ *be the decomposition into isolated points* (I) *and non-isolated points* (J) *of* ∂T*. Suppose that* $(\mu_{N_j})_j$ *is a subsequence weakly converging to a measure* μ *on* ∂T_F*. Then,*

1. *the Benjamini-Schramm limit of the* $\vec{R}_{F,N_j}$ *is* $f_*(\mu)$*;*
2. $\mu(\partial \tilde{T}) = \mu(\partial T') = \mu(J)$*;*
3. $\operatorname{Supp}\big(f_*(\mu)\big) = \overline{f\big(\operatorname{Supp} \mu\big)}$*.*
4. *the support of* $f_*(\mu)$ *is contained in* $\overline{\{T_{F,\xi} \circledR T_{F,\eta} \mid (\xi, \eta) \in \partial T'\}} \cup \overline{f(I)}$*.*

Proof. Firstly, if $v = (v_1 \dots ; \dots v_{-1})$ is in I, then both sequences are eventually periodic, of the same period. Indeed, since v is in I, it means that there exists some n such that there is a unique way to continue the sequence $(v_1 \dots v_n ; v_{-n} \dots v_{-1})$. Since our alphabet is finite, there exist m and m' bigger than n such that $v_m = v_{m'}$ and $v_{-m} = v_{-m'}$. Both sequences

$(v_1 \dots v_m; v_{-m} \dots v_{-1})$ and $(v_1 \dots v_{m'}; v_{-m'} \dots v_{-1})$ can be extended in the same unique way. This implies that $v_{m+1} = v_{m'+1}$ and so on. Therefore, both sequences are eventually periodic of period $m' - m$. Therefore, $J \subseteq \partial T'$.

By definition of the μ_N, for all ray ω in J, we have $\mu_N(\omega) = 0$ and thus $\mu(\omega) = 0$. This implies that $\mu(\partial T') = \mu(J)$ since $J \setminus \partial T'$ is a countable union of rays. On the other hand, $\partial T' \setminus \partial \tilde{T}$ is contained in

$$\bigcup_{m \geq 0} \{\omega_1\omega_2 \dots \mid \omega_m \text{ is good}, \forall i > m, \omega_i \text{ is bad}\}$$

which is a countable union of countable sets of rays, and hence of measure 0. In particular, $\mu(\partial\tilde{T}) = \mu(\partial T')$. This implies that the set of discontinuity points of $f \colon \partial T \to \vec{\mathcal{G}}_\bullet$ has measure 0, with respect to μ and any μ_N.

For any finite rooted subgraph α and any positive radius r, let $B \coloneqq B_\alpha = \{\omega \in \partial T = \vec{R}_{F,\infty} \mid \mathrm{Ball}(\omega, r) \simeq \alpha\}$. By definition, we have $\mu(B \cap \partial T') + \mu(B \cap I) = \mu(B) = \mathbf{P}(f_*(\mu), \alpha, r)$ and $\mu_N(B \cap \partial T') + \mu_N(B \cap I) = \mu_N(B) = \mathbf{P}(f_*(\mu_n), \alpha, r)$. On the other hand, $f_*(\mu_{N_j})$ is exactly the measure on $\vec{\mathcal{G}}_\bullet$ given by choosing a vertex of $\vec{R}_{F,N_j}$ at random. It follows from Corollary 4.7.14 and Lemmas 4.7.12 and 4.7.13, that $B \cap \partial T'$ is clopen in $\partial T'$. On the other hand, $B \cap I$ is a countable union if isolated points that are clopen. It is therefore clopen in ∂T. Hence, by the Portmanteau theorem we have

$$\mathbf{P}(f_*(\mu_{N_j}), \alpha, r) \xrightarrow{j \to \infty} \mathbf{P}(f_*(\mu), \alpha, r)$$

which is the neighborhood sampling statistics convergence, which is equivalent to the Benjamini-Schramm convergence of the $f_*(\mu_{n_j})$.

Let v be in ∂T. If v is in J, for each n, there is uncountably many w in ∂T that agree with v up to level n. Since $\partial T \setminus \partial\tilde{T}$ is countable, there is at least one such w in $\partial\tilde{T}$. We have just proven that $\partial\tilde{T}$ is dense in J, hence $\partial\tilde{T} \cup I$ is dense in ∂T. Let us denote by $\tilde{\mu}$ the restriction of μ to $\partial\tilde{T} \cup I$. By continuity of f on $\partial\tilde{T} \cup I$, we have $\mathrm{Supp}(f_*(\mu)) \supseteq \overline{f\big(\mathrm{Supp}(\tilde{\mu})\big)} = \overline{f\big(\mathrm{Supp}(\mu)\big)}$. On the other hand, we have

$$\begin{aligned} f_*(\mu)(\vec{\mathcal{G}}_\bullet \setminus \overline{f(\mathrm{Supp}(\tilde{\mu})})) &= \mu\big(f^{-1}(\vec{\mathcal{G}}_\bullet \setminus \overline{f(\mathrm{Supp}(\tilde{\mu}))})\big) \\ &= \mu\big(\partial T \setminus f^{-1}(\overline{f(\mathrm{Supp}(\tilde{\mu}))})\big) \\ &\leq \mu\big(\partial T \setminus \mathrm{Supp}(\mu)\big) = 0. \end{aligned}$$

Since every set and every measure that we considered are living in $\vec{\mathcal{G}}_{\bullet,2k}$ which is second-countable, we can conclude that the support of $f_*(\mu)$ is contained in $\overline{f(\mathrm{Supp}(\tilde{\mu})}$, and thus equal to it.

Finally, it is trivial that $\mathrm{Supp}(f_*(\mu)) \subseteq \overline{f(\partial T)}$. Since $\partial T' \cup I$ is dense in ∂T, we have that $\overline{\partial T} = \overline{\{T_{F,\xi} \circledast T_{F,\eta} \mid (\xi, \eta) \in \partial T'\}} \cup f(I)$. ☐

Corollary 4.7.19. *Let $F \subseteq \mathcal{A}^2$. Suppose that in A_F, for each row the sum of coefficients is equal to d and that for each column the sum of coefficients is equal to d'. Then $d = d'$, each vertex in the Rauzy digraphs has d ingoing and d outgoing edges. Moreover, the Benjamini-Schramm limit of the Rauzy digraphs $\vec{R}_{F,n}$ exists and is a Dirac measure concentrated on $\vec{\mathrm{DL}}(d, d)$, where $d = |\mathcal{A}| - i_a$.*

Proof. Let $k = |\mathcal{A}|$. Then $|F| = k \cdot d = d' \cdot k$ and the first assertion is proven.

It is clear that for any ray (ξ, η) in $\vec{R}_{F,\infty}$, the trees $T_{F,\xi}$ and $T_{F,\eta}$ are both $d + 1$ regular. Therefore, $T_{F,\xi} \circledast T_{F,\eta} \simeq \vec{\mathrm{DL}}(d, d)$. The sequence $(\mu_N)_N$ can be decomposed into a disjoint union of countably many converging subsequences. Each of this subsequence converges to some μ^j. By regularity of the tree, $\mu^j(C_v) > 0$ for all cylinder. Therefore, $\mathrm{Supp}(\mu_j) = \partial T$ and $f_*(\mu_j) = \delta_{\vec{\mathrm{DL}}(d,d)}$, where δ is the Dirac measure. Finally, the sequence $\vec{R}_{F,n}$ is convergent, with limit $\delta_{\vec{\mathrm{DL}}(d,d)}$. □

As a special corollary, we have another proof of

Theorem 4.3.17. *1. The unlabeled de Bruijn digraphs $\vec{\mathcal{B}}_{k,N}$ converge to $\vec{\mathrm{DL}}(d,d)$ in the sense of Benjamini-Schramm convergence.*

2. The following diagram commutes, where the arrows stand for Benjamini-Schramm convergence of unlabeled digraphs.

$$\begin{array}{ccc} \vec{\mathcal{S}}_{k,N,M} & \xrightarrow{\quad N \longrightarrow \infty \quad} & \vec{\mathrm{DL}}(k,k) \\ {\scriptstyle M \to \infty}\big\downarrow & \searrow^{N,M \to \infty} & \big\| \\ \vec{\mathcal{S}}_{k,N,\infty} & \xrightarrow{\quad N \longrightarrow \infty \quad} & \vec{\mathrm{DL}}(k,k) \end{array}$$

Remark 4.7.20. Corollary 4.7.19 gives new approximations of $\vec{\mathrm{DL}}(d,d)$ that are not coming from spider-web digraphs. Indeed, there are examples of F satisfying the hypothesis such that for all N, the graph $R_{F,k,N}$ is not isomorphic to any spider-web graph $\mathcal{S}_{k',N',M}$. For example, for $\mathcal{A} = \{0,1,2\}$ and $F = \{01,12,20\}$. In this case, $i_a = 1 = o_a$, the hypothesis of Theorem 4.7.18 is satisfied and the limit is $\vec{\mathrm{DL}}(2,2)$. On the other hand, $R_{F,n}$ are 4 regular graphs with exactly 3 loops (on $0\dots0$, $1\dots1$ and $2\dots2$). But the only spider-web graphs $\mathcal{S}_{k',N',M}$ with 3 loops are the $\vec{\mathcal{B}}_{3,N'}$ which are 6 regular.

We now show that, under some technical assumptions, the sequence $(\mu_{2N})_N$ is indeed convergent. Recall that that the graph $\vec{\Gamma}_F$ is the graph with adjacency matrix $B_F = A_F \otimes A_F^{\mathrm{T}}$, where $(A_F)_{ij} = 1$ if and only if $ij \neq F$.

Theorem 4.7.21. *Let $F \subset \mathcal{A}^2$. Suppose that $\vec{\Gamma}_F$ is weakly irreducible and of period p. Then the for all $0 \leq l < p$, the sequences $(\vec{R}_{F,2(pN+l)})_N$ and $(\vec{R}_{F,2(pN+l)+1})_N$ are convergent.*

For the proof, see page 90.

Remark 4.7.22. In general, it is possible that $\lim \mu_{2N}$ and $\lim \mu_{2N+1}$ exists, are distinct, but that both sequences $(\vec{R}_{F,2N})_N$ and $(\vec{R}_{F,2N+1})_N$ have the same limit and therefore the sequence $(\vec{R}_{F,N})_N$ converges.

An example of this behavior is given by $F := \{00,11\} \subseteq \{0,1\}^2$. All the $\vec{R}_{F,N}$ (for $N \geq 2$) are isomorphic and therefore the limit exists, see Lemma 4.7.26. On the other hand, $\lim \mu_{2N}$ and $\lim \mu_{2N+1}$ have disjoint supports. Indeed, the tree T_F is depicted in Figure 4.16 and has only four rays: $\omega_1, \dots, \omega_4$. It is immediate that $\lim \mu_{2N} = \frac{1}{2}(\delta_{\omega_2} + \delta_{\omega_4})$, while $\lim \mu_{2N+1} = \frac{1}{2}(\delta_{\omega_1} + \delta_{\omega_3})$, where δ is the Kronecker measure.

Observe that $F := \{00,11\} \subseteq \{0,1\}^2$ is an example where $\vec{\Gamma}_F$ is of period two, but nevertheless the limit $\lim \mu_{2N}$ exists.

It is then natural to ask the following.

Question 4.7.23. *What is a necessary and sufficient condition to have the existence of the limit for $(\vec{R}_{F,2N})_N$? and for $(\vec{R}_{F,N})_N$?*

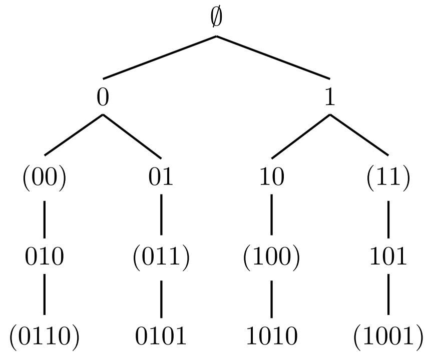


Figure 4.16: The top of the tree $T_{\{00,11\}}$ on the alphabet $\{0,1\}$. Bad vertices are parenthesized.

Definition 4.7.24. For any vertex $v = (v_1 \dots v_n)$ of $T = T_F$, define its *type* to be $v_{\lceil \frac{n}{2} \rceil}\overline{v}_{\lceil \frac{n}{2} \rceil+1}$ if v is of even level and $\overline{v}_{\lceil \frac{n}{2} \rceil}v_{\lceil \frac{n}{2} \rceil+1}$ if v is of odd level, with the convention that the type of the root is $\emptyset$ and the type of a vertex a of the first level is $\overline{a}$.

By construction of T, the types of children of v depend only on the type of v. Indeed, if v has type $\overline{a}b$, then it has one descendant of each type $a\overline{c}$ such that cb does not belongs to F. Therefore, a type t is said to be *good* if it corresponds to a good vertex. If v has type $a\bar{b}$, then it has one descendant of each type $\overline{c}b$ such that ac does not belongs to F. Therefore, The matrix A_F encodes the transition from level $2N \geq 2$ to level $2N+1$. Indeed, there is a vertex of type $\bar{j}k$ under $i\bar{k}$, if and only if $ij \notin F$, if and only if $(A_F)_{ij} = 1$. Similarly, the matrix A_F^{T} encodes the transition from level $2N+1$ to level $2N+2$. Therefore, the matrix $B_F := A_F \otimes A_F^{\mathrm{T}}$ encodes the transition from level $2N$ to level $2N+2$. In particular, the matrix $A_F \otimes A_F^{\mathrm{T}}$ is the adjacency matrix of the digraph where vertices are types of even non-zero levels, and there is an edge from t to t' if and only if in T, a vertex of type t has a grand-children of type t'.

Proof of Theorem 4.7.21. Our matrix B_F does not say anything about what happen to the root. We are going to change B_F a little bit to take them in account. Add a vertex $t_\emptyset$ to $\vec{\Gamma}_F$ and an edge from $t_\emptyset$ to t_v for each v of level 2. The digraph that we obtain with this construction is still weakly irreducible. Let $\tilde{B}_F$ be the adjacency matrix of this digraph, it has the same period as B.

Let $\vec{b} = (b_1, \dots, b_k)^{\mathrm{T}}$ be the column vector of good types, i.e. $b_i = 1$ if and only if t_i is good, otherwise $b_i = 0$. Let $\vec{v} = (0, \dots, 1, \dots 0)^{\mathrm{T}}$ be the column vector corresponding to v, i.e. there is a 1 in position i, where t_i is the type of v, and 0 otherwhere. By definition of the μ_N, if $M \leq N$ and v is of level $2M$, we have

$$\mu_{2N}(C_v) = \frac{\langle \vec{b}; (\tilde{B}_F^{(N-M)}\vec{v}) \rangle}{\langle \vec{b}; (\tilde{B}_F^{N}\vec{v}_\emptyset) \rangle}$$

Indeed, the coordinate i of $\tilde{B}_F^{(N-M)}\vec{v}$ is exactly the number of vertices of type t_i and of level $2N$ that are under v. The scalar product with $\vec{b}$ gives the number of all good vertices of level $2N$ that are under v.

By assumption, we have $\vec{v}_0$ such that $\|v_0\| = 1$ and $\tilde{B}_F^{p}\vec{v}_0 = h\vec{v}_0$, where p is the period of B_F and h is the unique Perron-Frobenius eigenvalue of $\tilde{B}_F^{p}$.

$\emptyset, \{0,1\}^2$
$\{11\}, \{01\}, \{00,01,10\}, \{00,01,11\}$
$\{00,11\}, \{00,10\}, \{00,01\}, \{01,10\}$

Table 4.1: Classification of $F \subseteq \{0,1\}^2$ up to strong isomorphisms of Rauzy digraphs.

Therefore, we have that for all $0 \leq l < p$, the following limit exists

$$\mu_{2(pN+l)}(C_v) \xrightarrow{N\to\infty} \frac{1}{h^M} \frac{\langle \vec{v}_0; (\tilde{B}_F^p)^{(N-M)} \vec{v} \rangle}{\langle \vec{v}_0; (\tilde{B}_F^p)^N \vec{v}_\emptyset \rangle}$$

This implies that the $\mu_{2(pN+l)}$ converge to a measure μ on ∂T.

The proof for the odd case is similar, except that this time we look at $A_F^{\mathrm{T}} \otimes A_F$; but the corresponding graph is isomorphic to the one that corresponds to $A_F \otimes A_F^{\mathrm{T}}$. □

Remark 4.7.25. The above proof gives in fact an algorithm which allows to explicitly compute the measure $\mu := \lim_N \mu_{2(pN+l)}$.

Let $\vec{\Delta}_F$ be the finite digraph corresponding to $\tilde{B}_F^p$ in the above proof, that is $\vec{\Delta}_F$ is the p-power of the graph obtained from $\vec{\Gamma}_F$ by adding a new vertex $t_\emptyset$ an an edge from $t_\emptyset$ to any vertex of $\vec{\Gamma}_F$. This graph is weakly irreducible. The value of $\mu(C_v)$ depends only on the level of v and its type. Therefore, for all v of level $2pN$ and w of level $2p(N+1)$, the value $\frac{\mu(C_w)}{\mu(C_v)}$ depends only on the type of v and w. This implies that all the information about μ is contained in the finite labeled digraph $\vec{\Delta}_F$, where the label is given by

$$l(t_v \to t_w) := \frac{1}{h} \frac{\langle \vec{v}_0; \tilde{B}_F^N \vec{w} \rangle}{\langle \vec{v}_0; \tilde{B}_F^N \vec{v} \rangle}$$

Indeed, we can pullback the labelling of $\vec{\Delta}_F$ to a "labelling" of T_F. Formally speaking, we label paths of length $2p$ between vertex of level $2pN$ and $2p(N+1)$. Then we have that for all v of level $2pN$ in T_F the measure $\mu(C_v)$ is exactly the product of the labels along the unique path from the root to v. See Proposition 4.7.29 and its proof for a concrete example.

Nevertheless, this algorithme is not necessarily sufficient to obtain informations on $f_*(\mu) = \lim \vec{R}_(F, 2(pN+l))$.

Examples for $k = 2$

In this subsection we will compute all the limit for $F \subseteq \mathcal{A}^2 = \{0,1\}^2$. Some of these limits will follows from the theorems, and some will be obtained by hand.

Let σ be a bijection on $\mathcal{A}$. Then we can extend σ to a bijection on $\mathcal{A}^2$ by $\sigma(ab) = \sigma(a)\sigma(b)$. Under this correspondence, it is clear that $\vec{R}_{F,N} \simeq \vec{R}_{\sigma(F),N}$.

Therefore, up to strong isomorphism of the Rauzy digraphs, there is 10 $F \subseteq \{0,1\}^2$. See Table 4.1 for a list.

Lemma 4.7.26. *Let $F \subseteq \{0,1\}^2$ be different from $\{11\}$ (or $\{00\}$). Then the limit of $(\vec{R}_{F,2,N})_N$ exists and is given in Table 4.2.*

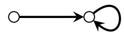
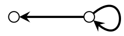
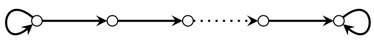
F	limit
$\emptyset$	$\vec{\mathrm{DL}}(2,2)$
$\{01\}$	$\vec{\mathbf{Z}} \simeq \vec{\mathrm{DL}}(1,1)$
$\{00,11\}$	
$\{00,10\}$	
$\{00,01\}$	
$\{01,10\}$	
$\{00,01,10\}$	
$\{00,01,11\}$	
$\{0,1\}^2$	$\emptyset$

Table 4.2: Limits of $(\vec{R}_{F,2,N})_N$ for $\{00\} \neq F \subseteq \{0,1\}^2$, where $\vec{\mathbf{Z}}$ is an infinite directed path.

Proof. We have $\vec{R}_{\emptyset,2,N} \simeq \vec{\mathcal{B}}_{2,N}$, and thus it has limit $\vec{\mathrm{DL}}(2,2)$. In the following, we will always suppose that $N \geq 2$. The digraph $\vec{R}_{\{01,10\},2,2}$ is a disjoint union of two loops. This digraph is isomorphic to its line digraph, and therefore, all $\vec{R}_{\{01,10\},2,N}$ are isomorphic. Since the two connected component are isomorphic, the limit consists of only one vertex with one loop. The digraph $\vec{R}_{\{01\},2,N}$ is

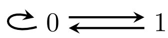


and the limit is obvious.

Finally, for all the remaining F's, the digraph $\vec{R}_{F,2,2}$ is connected and isomorphic to its own line digraph. Therefore, for a given F, all $\vec{R}_{F,2,N}$ are isomorphic and therefore the limit is $\vec{R}_{F,2,2}$. □

It remains to analyse the case $F = \{11\}$, see Figure 4.13. This is the only example of $F \subseteq \{0,1\}^2$ which is neither trivial, nor follows from Theorem 4.3.17. The subshift of $\{0,1\}^{\mathbf{Z}}$ associated to $F \subseteq \{0,1\}^2$ is known is symbolic dynamic as the *golden-mean shift*. We will see that the the golden ratio and the Fibonacci sequence also plays an important role in the study of $\vec{R}_{\{11\},2,N}$. For example, if $\mathscr{F}_N$ denotes the N^{th} Fibonacci number, then

$$A_{\{11\}}^N = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^N = \begin{pmatrix} \mathscr{F}_{N+1} & \mathscr{F}_N \\ \mathscr{F}_N & \mathscr{F}_{N-1} \end{pmatrix}$$

and the matrix A_F corresponds to the graph 

On the other hand, we have

$$B \coloneqq B_{\{11\}} = A_{\{11\}} \otimes A_{\{11\}}^{\mathrm{T}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

which corresponds to the graph

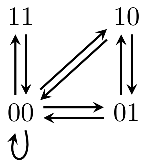


Figure 4.17: The graph of $B_F = A_F \otimes A_F^{\mathrm{T}}$ for $F = \{11\}$, the golden-mean shift.

Lemma 4.7.27. *Let $\mathscr{F}_N$ be the Fibonacci numbers. Then, the number of vertices of $\vec{R}_{\{11\},2,N}$ is $\mathscr{F}_{N+2}$, while its number of edges is $\mathscr{F}_{N+3}$.*

Proof. The proof is by induction. The formulas are true for $N = 0$ and $N = 1$. Recall that if $N \geq 1$, then $\vec{R}_{\{11\},2,N+1}$ is the line digraph of $\vec{R}_{\{11\},2,N}$.

Let a_N, respectively b_N be the number of edges, respectively of $\vec{R}_{\{11\},2,N}$ labelled by 0, respectively by 1. Then the number of edges of $\vec{R}_{\{11\},2,N}$ is $a_N + b_N$. When passing to the line digraph, the edges labeled by 0 give vertices where the final digit is 0. Every such vertex has two outgoing edges, one labeled by 0 and one labeled by 1. On the other hand, the edges labeled by 1 give in the line digraph vertices where the final digit is 1. Every such vertex has only one outgoing edge, labeled by 0. This implies that $a_{N+1} = a_N + b_N$ and $b_{N+1} = a_N$, that is $a_{n+1} = a_n + a_{n-1}$. Since $a_1 = 2$ and $a_3 = 3$, we have $a_n = \mathscr{F}_{N+2}$ and $a_n + b_n = \mathscr{F}_{N+3}$. This concludes the proof for the number of edges. The number of vertices follows by the line digraph structure. ☐

We say that an element $(v_1 \ldots ; \ldots v_{-1})$ in $\mathcal{A}^{\mathbf{N}} \times \mathcal{A}^{-\mathbf{N}}$ is a *palindrome* if for all i we have $v_i = v_{-i}$.

Lemma 4.7.28. *The function $f \colon T_{\{11\}} \to \vec{\mathcal{G}}_{\bullet}$ is injective, while the function $\hat{f} \colon T_{\{11\}} \to \mathcal{G}_{\bullet}$ is injective on palindromes and $2-1$ otherwise.*

Proof. For any k and F, we always have that $\big(R_{k,F,\infty}, (v_1 v_2 \ldots ; \ldots v_{-1})\big)^0$ is isomorphic to $\big(R_{k,F,\infty}, (v_{-1} \ldots ; \ldots v_2 v_1)\big)^0$. Therefore, the function $\hat{f}$ is at least $2-1$ on non palindromes.

It remains to prove that there is no other identifications. The proof is done by looking at small balls around $v = (v_1 v_2 \ldots ; \ldots v_{-2} v_{-1})$ in $R'_{\{11\},\infty}$. We explicitly draw all the possible balls and see what happens if the ball around v and w are isomorphic. Observe that in general the ball around $v = (v_1 v_2 \ldots ; \ldots v_{-2} v_{-1})$ is exactly the ball around $v = (v_{-1} v_{-2} \ldots ; \ldots v_2 v_1)$ where all the edges are inverted. But the corresponding graphs are naturally isomorphic via the switch $(v_1 \ldots ; \ldots v_{-1}) \mapsto (v_{-1} \ldots ; \ldots v_{11})$. Therefore, if we want to analyse all balls around $v = (v_1 v_2 \ldots ; \ldots v_{-2} v_{-1})$, it is sufficient to draw 6 of the 12 possible balls. See Figures 4.18 to 4.20. If we choose two sequences that differ in the first two or the lat two digits, then the corresponding balls are not isomorphic.

Now, for $(10 \ldots ; \ldots 00)$, $(00 \ldots ; \ldots 01)$ and $(01 \ldots ; \ldots 01)$ we can see on balls of radius 2 (see Figure 4.18) that if we fix the root, then we fix the ball of radius 1 around the root. We details here the case of the ball around $v = (01 \ldots ; \ldots 00)$, other cases are left to the reader. In this case, fixing the root fixes the ball of radius one. Indeed, there is only one vertex at distance one from the root with degree 2: the vertex $(1 \ldots ; \ldots 001)$. Therefore, this vertex is fixed. Similarly, there is only one vertex u of degree 4 at distance 1 of the root ans this vertex is fixed. Finally, there is two vertex of degree 3 at distance 1 of the root. But for one of them there is exactly one path of length 2 to u and the other one has two paths of length 2 to u. This implies that both these vertices are fixed.

For $(00\ldots;\ldots00)$, $(10\ldots;\ldots01)$ and $(01\ldots;\ldots10)$ if we fix the root, then the action on the ball of radius 1 around the root is determined by the image of any vertex at distance 1 from the root. Moreover, in this case, any vertex at distance 1 is either send onto itself, in which case the isomorphism is the identity when restricted on the ball of radius 1, or send to its switch version, in which case the isomorphism is the switch when restricted the the ball of radius 1. This is obvious for $(00\ldots;\ldots00)$ and $(10\ldots;\ldots01)$, but not for $(01\ldots;\ldots10)$. Indeed, in this case, each of the four vertices at distance 1 from the root as at most 2 image, but this a priori unclear if we can make the choices independently. By looking at the ball of radius 4, it is possible to see that these choices are not independent.

Suppose now that the graph $\left(R_{k,F,\infty},(v_1v_2\ldots;\ldots v_{-1})\right)^0$ is isomorphic to the graph $\left(R_{k,F,\infty},(v_{-1}\ldots;\ldots v_2v_1)\right)^0$. Then the isomorphism send the root the the root. It acts on the ball of radius 1 either as the identity or as the switch. If it acts as the identity (respectively as the switch), then by induction it acts as the identity (respectively as the switch) on all the graphs. This conclude the proof for $\hat{f}$. The conclusion for f follows from the fact that the flip is not an isomorphism of the digraphs.

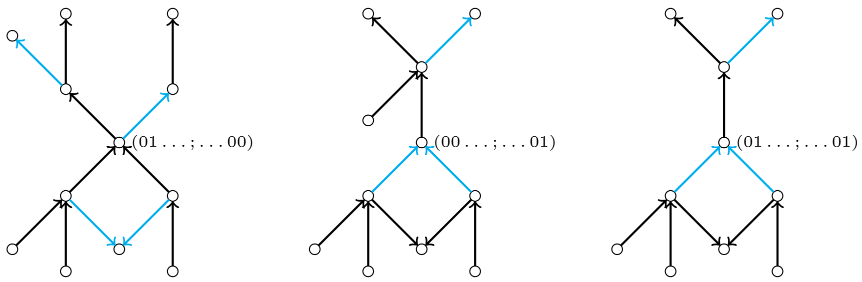


Figure 4.18: Ball of radius 2 around $(10\ldots;\ldots00)$, $(00\ldots;\ldots01)$ and $(01\ldots;\ldots01)$. Black edges are labeled by 0, while cyan edges are labeled by 1.

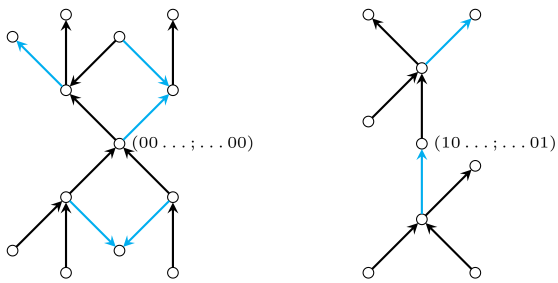


Figure 4.19: Ball of radius 2 around $(00\ldots;\ldots00)$ and $(10\ldots;\ldots01)$. Black edges are labeled by 0, while cyan edges are labeled by 1.

□

The following proposition finishes the study of the golden-mean shift.

Proposition 4.7.29. *The graphs $\vec{R}_{\{11\},2,N}$ converge to a continuous measure. The measure μ is continuous and showed in Figure 4.22.*

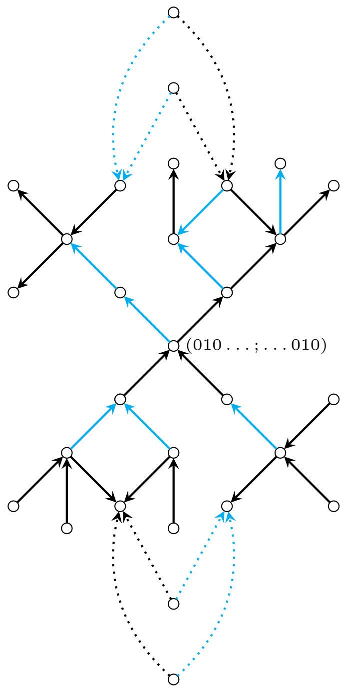


Figure 4.20: Ball of radius 3 around $(01\ldots;\ldots10)$ with some edges from the ball of radius 4. Black edges are labeled by 0, while cyan edges are labeled by 1.

Proof. As we can see on Figure 4.17, F satisfies the hypothesis of Theorem 4.7.21. By the Theorem, both $(\mu_{2N})_N$ and $(\mu_{2N+1})_N$ are convergent. We know the graph $\vec{\Delta}_F$ of Theorem 4.7.21 and Remark Rauzy:Remark:Algo. We can therefore explicitly compute the Perron-Froebenuis eigenvalue and eigenvector. This gives us the labeling of $\vec{\Delta}_F$ depicted in Figure 4.21. This label can be pullback on $T_{\{11\}}$. Note that for each node in Figure 4.21 we have that the sum of weight of outgoing edges is indeed equal to 1.

A simple check shows that the labeling corresponding to $\lim \mu_{2N}$ and $\lim \mu_{2N+1}$ are the same and gives the weighted tree of Figure 4.22. This measure is continuous, as is the measure $f_*(\mu)$ by Lemma 4.7.28. □

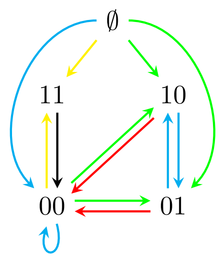


Figure 4.21: The graph $\vec{\Delta}_F$ for $F = \{11\} \subseteq \{0,1\}^2$, the golden-mean shift. The black edge has weight 1, red edges have weight $1/\varphi$, cyan edges weight $1/\varphi^2$, green edges weight $1/\varphi^3$ and yellow edges weight $1/\varphi^4$.

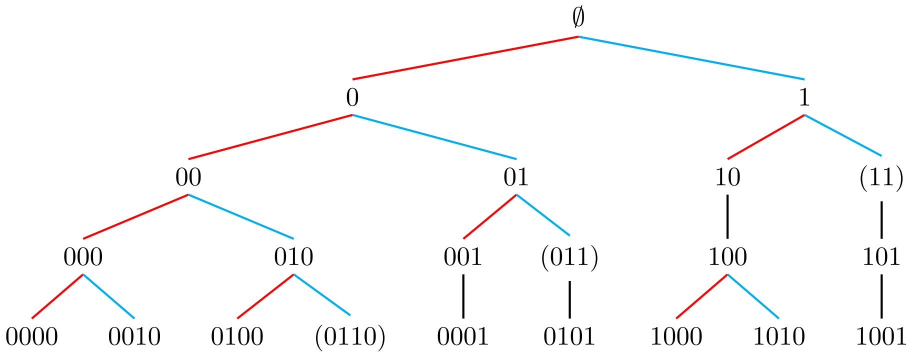


Figure 4.22: The limit measure μ on $T_{\{11\}}$. Black edges have weight 1, cyan edges have weight $1/\varphi^2$ and red edges have weight $1/\varphi$. For example, $\mu(C_{1010}) = 1/\varphi^5$.

Spider-web version of Rauzy digraphs

For any M, it is possible to construct the tensor product $\vec{R}_{F,N} \otimes \vec{C}_M$. This is an obvious generalization of spider-web digraphs and we denote it by $\vec{\mathcal{S}}_{F,N,M}$. These generalized spider-web digraphs retain some nice properties.

The following proposition is a generalization of Proposition 4.5.16. Since the proof is similar it is not repeated here.

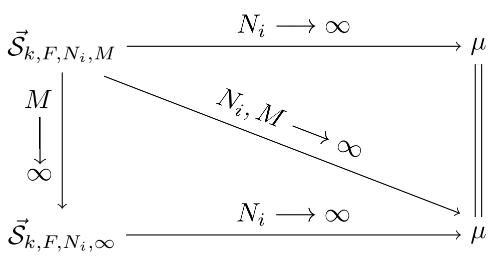
Proposition 4.7.30. *Let $F \subseteq \mathcal{A}^2$. For all $M \in \overline{\mathbf{N}}$, $N \in \mathbf{N}$, we have $\vec{\mathcal{S}}_{F,N+1,M} \simeq L(\vec{\mathcal{S}}_{F,N,M})$, the line graph of $\vec{\mathcal{S}}_{F,N,M}$.*

Remark 4.7.31. For all F, the digraph $\vec{\mathcal{S}}_{F,0,M}$ is a thick oriented circle. Therefore, the above proposition holds for $N = 0$ if and only if $F = \emptyset$.

We also have a generalization of Lemma 4.3.8.

Lemma 4.7.32. *For all $M \in \overline{\mathbf{N}}$ and $N \in \mathbf{N}_0$, all connected components of $\vec{\mathcal{S}}_{F,N,M}$ are isomorphic. Moreover, if $\vec{R}_{F,N}$ is connected and has derangement $\pm 1 \pmod{M}$, then $\vec{\mathcal{S}}_{F,N,M}$ is connected.*

Proposition 4.7.33. *Let $F \subseteq \mathcal{A}^2$ be such that in A_F there is at least one 1 on the diagonal. Suppose that $\vec{R}_{F,N}$ are connected and convergent with Benjamini-Schramm limit μ. Then, the following diagram is commutative.*



Proof. The condition on F ensures that $\mathrm{der}(\vec{R}_{F,N}) = 1$ for all N. Therefore, the digraphs $\vec{\mathcal{S}}_{F,N,M}$ are connected and $\big(\vec{\mathcal{S}}_{F,N,M}, (v,0)\big)$ and $\big(\vec{\mathcal{S}}_{F,N,M}, (v,i)\big)$ are isomorphic as rooted digraphs. This implies that they have the same limit as $\vec{R}_{F,N}$. □

4.8 Open questions and further research directions

In this section, we list the open questions that appeared during this chapter and propose further direction of research.

For de Bruijn graphs, the main open question is raised by Corollary 4.6.6 and Lemma 4.6.5 which together imply:

$$\zeta'_{\mathrm{DL}(k,k)}(0) = (k-1)^2 \frac{d}{ds} \operatorname{Li}_s\left(\frac{1}{k}\right)\Big|_{s=0} - \log(k)$$

Since for all s with $\operatorname{Re}(s) > 1$, we have $\operatorname{Li}_s(1) = \zeta(s)$ the Riemann zeta function it is natural to ask

Question 4.8.1. *What is the exact relation between* $\zeta_{\mathrm{DL}(k,k)}$ *and* $\operatorname{Li}_s\left(\frac{1}{k}\right)$*?*

For Rauzy graphs, the first thing to do is, if possible, to remove all the technical assumptions on F needed in the theorems. This includes for example Proposition 4.7.33. This also concerns Theorem 4.7.21. It would be nice to show that the limit of Rauzy digraphs always exists, or the fact that the μ_{2N} and μ_{2N+1} always converge.

Another interesting result would be the computation of the spectral measure or Rauzy graphs. For Rauzy digraphs, this is easy and follows from the fact that they are all line digraphs, but for the unoriented Rauzy graphs this remains open.

We proved that if the Rauzy graphs are k in-regular and k out-regular, then they converge to $\mathrm{DL}(k,k)$. Recall that $\mathrm{DL}(k,l)$ is unimodular if and only if $k = l$ and that Benjamini-Schramm limit are always unimodular. In particular, for $k \neq l$, the graph $\mathrm{DL}(k,l)$ cannot be obtained as a limit of Rauzy graphs. Moreover there are only countably many subshifts, while they are a continuum of unimodular measures supported on horospheric product of trees, see [72]. Nevertheless, we can ask

Question 4.8.2. *Describe which unimodular measure supported on horospheric product of trees arise as limit of Rauzy graphs.*

We may also want to analyse all F for $|\mathcal{A}| = 3$ and see if there is any new behavior in comparison with the case $|\mathcal{A}| = 2$. Another interesting computation would be to compute the spectrum of $\vec{R}_{\{00,11,22\},n}$ and compare with $\vec{\mathcal{B}}_{2,n}$ since these families have the same limit $\mathrm{DL}(2,2)$.

Another interesting direction of research is the following conjecture which was suggested by Kaimanovich

Conjecture 4.8.3. *Let* $F \subseteq \mathcal{A}^2$*. Then the Benjamini-Schramm limit of the Rauzy graphs is the unimodular measure (supported on* $\vec{R}'_{F,\infty}$*) which is the image of the shift-invariant measure on* $\mathcal{A}_F$ *of maximal entropy.*

Here the entropy of a shift-invariant measure is its topological entropy. If the subshift is irreducible, then there exists a unique shift-invariant measure of maximal entropy, [78] for a detailed exposition. Such a measure μ gives raise to a measure $\Phi_*(\mu)$ on $\mathcal{G}_\bullet$ supported on horospheric product of trees, where $\Phi(\xi\eta) = T_\xi \,\textcircled{h}\, T_\eta$.

Finally, we may want to generalize our results to subshifts of finite type defined over other groups than $\mathbf{Z}$, for example on free groups.

Chapter **5**

Weakly maximal subgroups of branch groups

In 1980, Grigorchuk constructed in [60] a group $\mathcal{G}$, which is known as the *(first) Grigorchuk group*, and proved in [47] that $\mathcal{G}$ has intermediate growth between polynomial and exponential, thus providing an answer to an open problem posed by John Milnor in 1968.

Since then, this group continues to be extensively studied and gave rise to the definition of branch groups (which naturally appear in the classification of just-infinite groups). All these groups appear as automorphism groups of infinite rooted trees. Branch groups admit a definition in terms of stabilizers of points and of stabilizers of rays (also called *parabolic subgroups*) for this action and a major open problem about them is to describe their lattice of subgroups. In the case of the Grigorchuk group, a deep result in this direction, see [97], is the rigidity of maximal subgroups — a behavior which is completely different from the linear case of, say $SL(n, \mathbf{Z})$.

If G acts on T in a weakly branched way, then all parabolic subgroups are infinite and pairwise distinct [49]. Moreover, if the action of G is branched, then all parabolic subgroups are weakly maximal [12]. In particular, a finitely generated branch group has uncountably many weakly maximal subgroups, and thus uncountably many automorphism equivalence classes of weakly maximal subgroups. In the Grigorchuk group $\mathcal{G}$, the class of weakly maximal subgroups is not reduced to the class of parabolic subgroups: some sporadic examples were constructed [48, 49]. There are however no classification results for weakly maximal subgroups of $\mathcal{G}$ or other (weakly) branch groups (Problem 6.3 in [48]).

5.1 Overview of the results

One of the main results (Corollary 5.1.2) of this chapter is to show that a finitely generated regular branch group (subject to a minor technical condition) contains uncountably many non parabolic weakly maximal subgroups up to automorphism equivalence. More precisely, we show that any finite subgroup $Q \leq G$ is contained in uncountably many weakly maximal subgroups.

Theorem 5.1.1. *Let T be a regular rooted tree and $G \leq \mathrm{Aut}(T)$ be a finitely generated regular branch group. Then, for any finite subgroup $Q \leq G$ there exist uncountably many automorphism equivalence classes of weakly maximal subgroups of G containing Q.*

As an immediate corollary of the theorem, we have the following result. It indicates that a full classification of weakly maximal subgroups must involve, in a significant way, subgroups that are not parabolic.

Corollary 5.1.2. *Let T be a regular rooted tree and $G \leq \mathrm{Aut}(T)$ be a finitely generated regular branch group. Suppose that G contains a finite subgroup Q that does not fix any point in ∂T. Then there exist uncountably many automorphism equivalence classes of weakly maximal subgroups of G, all distinct from classes of parabolic subgroups associated with the action of G on T.*

These results hold in a large class of groups. Indeed, all known examples of fractal branch groups are regular branch. In fact, these results hold for the first Grigorchuk group $\mathcal{G}$ as well as for groups in $\mathscr{C} \setminus \mathscr{G}$ ("non-trivial" generalized multi-edge spinal groups, see Definition 5.3.2). Such groups include Gupta-Sidki groups [12] and many of the GGS groups.

Moreover, all these examples essentially admit a unique branch action on a spherically homogeneous tree [57, 79]. (An infinite rooted tree is *spherically homogeneous* if the degree of a vertex depends only on its distance from the root.) The unicity of branch action, together with our Theorem 5.1.1 allow us to deduce the following.

Theorem 5.1.3. *Let G be either the first Grigorchuk group, or a group in $\mathscr{C} \setminus \mathscr{G}$. Then G has uncountably many automorphism equivalence classes of weakly maximal subgroups, all distinct from classes of parabolic subgroups of any branch action of G on a spherically regular tree.*

We also demonstrate that, loosely speaking, weakly maximal subgroups of $\mathcal{G}$ and groups in $\mathscr{C} \setminus \mathscr{G}$ live as deep in the tree as one desires.

Theorem 5.1.4. *Let T be a regular rooted tree and $G \leq \mathrm{Aut}(T)$ be a finitely generated regular branch group. Suppose that G contains an infinite index subgroup Q that does not fix any point in ∂T. Then there exists an integer l such that for any $k \in \mathbf{N}$ and any vertex v of level k, there exists a weakly maximal subgroup of G which stabilizes v and does not stabilizes any vertex of the $(k + l)$th level.*

More precisely, we can choose l to be the minimal integer such that there exists an infinite index subgroup Q that does not stabilize any vertex of the l^{th} level.

If G is the first Grigorchuk group or a group in $\mathscr{C} \setminus \mathscr{G}$, then it is possible to take $l = 1$.

The rich structure of weakly maximal subgroups can be used to study the subgroup structure of branch groups from the statistical viewpoint. Namely, a very active area of research has lately developed around the notion of IRS – invariant random subgroup in a locally compact group. An IRS is a probability distribution on the space of all subgroups of the group which is required to be invariant under the natural action of the group on its subgroups by conjugation. This is a natural example of an invariant measure, but moreover it also has a nice algebraic meaning: it generalizes the notion of a normal subgroup that corresponds to the Dirac delta measures. It is very interesting and important to understand how the algebraic subgroup structure of the group is related to the variety of IRSs on it. Recent new examples of Juschenko and Monod [70] provide first examples of simple groups (i.e. groups that have no non-trivial normal subgroups) with non-trivial IRS. Benli, Grigorchuk and Nagnibeda [21] constructed first examples of groups of intermediate growth with uncountably many distinct IRS. It is still an open question whether a just-infinite group, and in particular the Grigorchuk group $\mathcal{G}$ can have uncountably many IRS (observe that being just-infinite it can only have countably many normal subgroups).

This problem is connected with the above results about variety of weakly maximal subgroups. There is indeed a strategy that potentially allows to construct an IRS out of a weakly maximal subgroup, as follows. Given a finitely generated group G and a weakly maximal subgroup H which is closed for the profinite topology (such an H is said to be *wmc*), it is possible to construct a binary rooted tree T_H on which G acts and such that H

is the stabilizer of the leftmost ray. If H is not parabolic for the original action, then the action of G on T_H is not necessarily branched, but we can ask the following.

Question 5.1.5. *Let G be a finitely generated group, H a wmc subgroup. Is it true that for the action of G on T_H*

1. *All parabolic subgroups are pairwise distinct?*
2. *All parabolic subgroups are wmc?*

Positive answer to these questions will imply the construction of new non-trivial invariant random subgroups of G.

These questions are deeply related with the profinite completion $\hat{G}$ of G. One important remark is that for the Grigorchuk group as for groups in $\mathscr{C} \setminus \mathscr{G}$ (and other groups where all maximal subgroups are of finite index), wmc subgroups coincide with weakly maximal subgroups. If we denote by $\bar{H}$ the closure of $H \leq G$ in $\hat{G}$, we have the following results.

Proposition 5.1.6. *Let T be a spherically homogeneous rooted tree and $G \leq Aut(T)$ be a finitely generated subgroup which is torsion. Suppose that the action of G is spherically transitive and there exists a ray ξ such that $H = \mathrm{Stab}_G(\xi)$ is wmc and $\bar{H} \leq \hat{G}$ is wmc. Then*

1. *All $\hat{G}$-parabolic subgroups are pairwise distinct and wmc;*
2. *All G-parabolic subgroups are distinct from H;*
3. *Suppose moreover that for all $g \in \hat{G}$ we have $\overline{\bar{H} \cap G^g} = \bar{H}$. Then all G-parabolic subgroups are pairwise distinct.*

The following theorem does not make any assumption on the weak maximality of $\bar{H}$.

Theorem 5.1.7. *Le G be a residually finite just-infinite p-group and H be a wmc subgroup. Then there exists a coset tree T_H such that the action of G on T_H is spherically transitive, $\mathrm{Stab}_G(\bar{0}) = H$ and*

1. *All $\hat{G}$-parabolic subgroups are wmc and pairwise distinct.*
2. *Any two rays in the $N_{\hat{G}}(G)$-orbit of $\bar{0}$ have distinct stabilizers.*

These results are mainly results on $\hat{G}$, while we would like results on G. A way to go from $\hat{G}$ to G is by the two applications $H \mapsto \bar{H}$, respectively $M \mapsto M \cap G$, from closed subgroups of G to closed subgroups of $\hat{G}$, respectively, from closed subgroups of $\hat{G}$ to closed subgroups of G. These functions are lattice isomorphisms whose preserve indices when restricted on finite index subgroups (Corollary 5.7.8). It is thus natural to

Conjecture 5.1.8. *The two functions $H \mapsto \bar{H}$ and $M \mapsto M \cap G$ send wmc subgroups to wmc subgroups.*

Lemma 5.7.11 is the first step in proving this conjecture. This conjecture is of great interest. Indeed, if it is true, then Proposition 5.1.6 directly implies

Proposition 5.1.9. *Le G be a residually finite just-infinite p-group and H be a wmc subgroup. Suppose that Conjecture 5.1.8 is true. Then there exists a coset tree T_H such that the action of G on T_H is spherically transitive, $\mathrm{Stab}_G(\bar{0}) = H$ and all G-parabolic subgroups are wmc and pairwise distinct.*

Finally, the following proposition, gives some sufficient criterion to have uncountably many distinct parabolic subgroups. For every vertex v, $C_v \subseteq \partial T$ is the cylinder under w and $\mathrm{Stab}_G(C_v)$ denotes the pointwise stabilizer of C_v. That is, $\mathrm{Stab}_G(C_v) = \bigcap_{\xi : v \in \xi} \mathrm{Stab}_G(\xi)$.

Proposition 5.1.10. *Let G, H and T be as in Theorem 5.1.7. For $A \leq G$, let $S_A \coloneqq \{\xi \in \partial T \mid \mathrm{Stab}_G(\xi) = A\}$. The implications between the following assumptions are resumed in the diagram below.*

1. *All parabolic subgroups are wmc;*
2. *For all ξ and η in ∂T, if $\mathrm{Stab}_G(\xi) \leq \mathrm{Stab}_G(\eta)$, then they are equal;*
3. *For all $A \leq G$, the subset S_A is closed;*
4. *The group G is hereditarily just-infinite and there is no ray with trivial stabilizer;*
5. *For all $\xi \in \partial T$ and all $v \in \xi$, we have $\mathrm{Stab}_G(C_v) \lneqq \mathrm{Stab}_G(\xi)$;*
6. *For all A and all v in T, there exists $w \leq v$, such that $C_w \subset \partial T \setminus S_A$;*
7. *There is an uncountable number of pairwise distinct parabolic subgroups.*

$$\begin{array}{ccccccccc} 1 & \Longrightarrow & 2 & \Longrightarrow & 3 & \Longrightarrow & 6 & \Longrightarrow & 7 \\ & & & & & & \Uparrow & & \\ & & & & 4 & \Longrightarrow & 5 & & \end{array}$$

An important remark is that this proposition does not guaranty the existence of a subset X of ∂T of positive measure such that any two rays in X have distinct stabilizers. On the other hand, we do not know yet any examples of $H \leq G$ that is not a parabolic subgroup of a branch action but that still satisfy any assumption of the proposition.

Throughout this chapter and unless specified otherwise, G will denote a finitely generated group, T a locally finite rooted tree and $\mathcal{G}$ the first Grigorchuk group.

Section 5.2 contains miscellaneous results that will be use later. Section 5.3 introduces the definition of branch groups as well as examples. In Section 5.4 we prove various results on weakly maximal subgroups. Sections 5.5 and 5.6 are devoted to the proofs of Theorems 5.1.1, 5.1.3 and 5.1.4 and of Corollary 5.1.2. This is a common work with Bou-Rabee and Nagnibeda and was published in [24]. Section 5.7 is consecrated to the study of coset trees and of the proof of Theorem 5.1.7 and of Propositions 5.1.6 and 5.1.10. Finally, Section 5.8 contains all the open questions appearing in this chapter as well as ideas for further research.

5.2 Preliminaries results

This section contains miscellaneous definitions and results that will be useful later.

A group G is residually finite, if the intersection of all its finite index subgroups (equivalently all its normal finite index subgroups) is trivial. This is equivalent to be a subgroup of the automorphism group of a rooted tree.

A group G is *just-infinite* if all its proper normal subgroups are of finite index (equivalently if all its non-trivial quotient are finite).

Lemma 5.2.1. *Let G be a just-infinite group, A be an infinite index subgroup of G and $\gamma \in G$ be any nontrivial element. Then there is a conjugate of γ that is not in A.*

Lemma 5.2.2. *Let G be a p-group. And $S \geq T$ two subgroups of index p^i and p^j. Then for all $i \leq s \leq j$, there exists a subgroup L of index p^s and with $S \geq L \geq T$.*

Proof. Observe that since G is a p-group, every finite index subgroup has index a power of p. Let $N = \mathrm{Core}_S(T)$ the normal core of T in S. The normal subgroup N has finite index in S, and therefore the group S/N is a finite p-group, and hence nilpotent. This implies that $\bar{T}$, the image of T in S/N is subnormal. Therefore, there exists a composition series $S_0 = S \geq S_1 \geq \cdots \geq S_{j-i} = \bar{T} \geq \cdots \geq \{1\}$. The image of this series in S gives the wanted subgroups. □

The following lemma is a well-known result on topological group.

Lemma 5.2.3. *If G is a topological group group, then for all closed subgroups H, the normalizer $N_G(H)$ is a closed subgroup.*

Proof. For any $h \in G$, let φ_h be the continuous function $\varphi_h(g) = ghg^{-1}$. Then

$$\begin{aligned} N_G(H) &= \{g \mid gHg^{-1} \leq H \text{ and } g^{-1}Hg \leq H\} \\ &= \bigcap_{g\in H} \varphi_g^{-1}(H) \cap (\bigcap_{g\in H} \varphi_g^{-1}(H))^{-1} \end{aligned}$$

is closed. □

The *profinite topology* on a group G is the topology generated by cosets of finite index subgroups. In such a topology, a subgroup is open if and only if it is of finite index. Any group is a topological group for the profinite topology.

Lemma 5.2.4. *In a group G with the profinite topology, finite index subgroups are exactly clopen subgroups.*

Proof. First, suppose that H is of finite index, then it is open.

Since H is of finite index, we have a finite system of transversal: $g_1 = 1, \ldots, g_n$. Therefore $H = G - \bigcup_{i=2}^{n} g_i H$ is also closed.

On the other hand, if H is clopen, it is open and thus of finite index. □

5.3 Groups acting on rooted trees

Let T be an infinite rooted tree — we will always assume that T is locally finite. We can divide T in level: $\mathscr{L}_n$ consists of all vertices at distance n from the root. We also have a natural order on the vertices: $w \leq v$ if and only if there is "descending path" from v to w — i.e. a path $v = v_0, v_1, \ldots, v_n = w$ with the level of v_i being 1 plus the level of v_{i-1}. For any vertex v, let T_v denotes the subtree (rooted at v) consisting of all vertices below v. If the degree of a vertex depends only on its level (that is if $\mathrm{Aut}(T)$ acts transitively on level), we say that T is *spherically homogeneous*.

Let G be a group acting on a infinite rooted tree T. If the action of G is transitive on each level, we say that the action is *spherically transitive*. Recall that for any vertex v of T, the *stabilizer* $\mathrm{Stab}_G(v)$ is the subgroup of elements fixing v. The stabilizer of a level, $\mathrm{Stab}_G(n)$ is equal to the intersection of stabilizer of vertices on this level, this is a normal subgroup.

$$\mathrm{Stab}_G(n) := \bigcap_{v\in\mathscr{L}_n} \mathrm{Stab}_G(v)$$

The *rigid stabilizer* $\mathrm{Rist}_G(v)$ of a vertex v consist of elements g of G fixing all vertices outside T_v. This is a normal subgroup of $\mathrm{Stab}_G(v)$. The *rigid stabilizer of level n* is

$$\mathrm{Rist}_G(n) = \langle \mathrm{Rist}_G(v) \mid v \in \mathscr{L}_n \rangle = \prod_{v \in \mathscr{L}_n} \mathrm{Rist}_G(v)$$

this is a normal subgroup of G.

Branch groups

An action of G on T is *weakly branch*, if G acts level transitively and $\mathrm{Rist}_G(v)$ is infinite for all v (equivalently, $\mathrm{Rist}_G(v)$ is never trivial). The action is said to be *branch* if G acts level transitively and $\mathrm{Rist}_G(n)$ is a finite index subgroup of G for all n. A group is *(weakly) branch* if it admits a faithful (weakly) branch action

For a group G acting on some infinite rooted tree, stabilizers of rays are called parabolic subgroups. Hence, parabolic subgroup depends on the action on not only on the group.

If T is a d-regular rooted tree and G a subgroup of $\mathrm{Aut}(T)$, we have injective maps

$$\begin{aligned} \psi_v \colon \mathrm{Stab}_G(v) &\to \mathrm{Aut}(T) & \qquad \psi_1 \colon \mathrm{Stab}_G(1) &\to \mathrm{Aut}(T)^d \\ \varphi &\to \varphi|_{T_v} & \varphi &\mapsto (\varphi|_{T_{v_1}}, \dots, \varphi|_{T_{v_d}}) \end{aligned}$$

where v is any vertex of T, $v_1, \dots, v_d$ are the vertices of the first level. For a d-regular rooted tree T, a group $G \leq \mathrm{Aut}(T)$ is said to be *fractal* if for any v, $\psi_v(\mathrm{Stab}_G(v))$ is equal to G, after an identification of T with T_v.

Definition 5.3.1. For a d-regular rooted tree T, a group $G \leq \mathrm{Aut}(T)$ is said to be *regular branch (over K)* if it satisfies the following conditions:

- G is spherically transitive;
- G is fractal;
- there exists a finite index subgroup K of G such that K^d is contained in $\psi_1(K \cap \mathrm{Stab}_G(1))$ as a subgroup of finite index.

Since G is fractal, the image of ψ_1 is included in G^d and we can therefore iterate ψ_1 to have a sequence

$$\psi_n \colon \mathrm{Stab}_G(n) \hookrightarrow G^{d^n}$$

It follows from the definition that K^{d^n} is contained in $\psi_n(\mathrm{Stab}_G(n))$ as a subgroup of finite index and therefore that for all level n, $\mathrm{Rist}_G(n)$ is of finite index in G [49]. This proves that a regular branch group is a branch group, and it follows directly from definitions that a branch group is weakly branch. Since $K^{d^n} \leq \psi_n(\mathrm{Stab}_G(n)) \leq G^{d^n}$, the image of ψ_n is of finite index in G^{d^n}.

Grigorchuk group and generalized multi-edge spinal groups

An important example of a finitely generated regular branch group is the first Grigorchuk group $\mathcal{G}$. It is a 2-group and it has word growth strictly between polynomial and exponential [60]. A detailed introduction to the group $\mathcal{G}$ and for proofs of the properties of $\mathcal{G}$ mentioned in this section can be found in [65, Chapter VIII].

The group $\mathcal{G}$ can be defined as a subgroup of the group of automorphisms of the infinite binary tree, $\mathcal{G} := \langle a, b, c, d\rangle$, generated by four automorphisms a, b, c, d defined recursively, as follows

$$\begin{aligned} a(xw) &= (\overline{x}w) & & \\ b(0w) &= 0a(w) & b(1w) &= 1c(w) \\ c(0w) &= 0a(w) & c(1w) &= 1d(w) \\ d(0w) &= 0w & (1w) &= 1b(w) \end{aligned}$$

where $x \in \{0, 1\}$; w denotes an arbitrary binary word; $\overline{1} = 0$ and $\overline{0} = 1$.

Let T be a d-regular rooted tree, and let $\mathcal{A} := \{0, \ldots, d-1\}$. For a vertex u, represented by a word over $\mathcal{A}$, and a letter $x \in \mathcal{A}$, we have $f(ux) = f(u)x'$ where $x' \in \mathcal{A}$ is uniquely determined by u and f. This induces a permutation f_u of X so that $f(ux) = f(u)f_u(x)$. The automorphism f is *rooted* if $f_u = 1$ for $u \neq \emptyset$. It is *directed*, with directing path $\xi \in \partial T$, if the support $\{u \mid f_u \neq 1\}$ of its labelling is infinite and contains only vertices at distance 1 from ξ.

For the Grigorchuk group, a is rooted and $f_\emptyset = (01)$ the permutation of the first level. On the other hands, b, c and d are directed, with directing path 1^∞. See Figures 5.1 and 5.2.

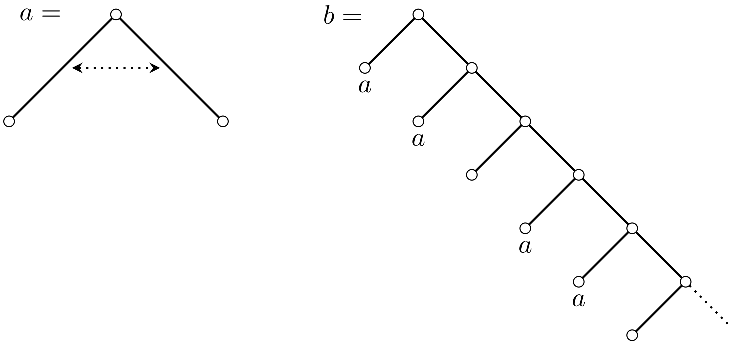


Figure 5.1: The rooted automorphism a and the directed automorphism b.

In the case when $g \in \mathrm{Aut}(T)$ fixes the first k levels of the tree, we will say that g has *level* k and we will write $g = (\gamma_1, \ldots, \gamma_{2^k})_k$ in order to record the action beyond level k only. For example, $b = (a, c)_1$, $c = (a, d)_1$, and $d = (1, b)_1$ all have level 1.

The action of $\mathcal{G}$ on the infinite binary tree is branched, and it can be shown moreover that $\mathcal{G}$ is regular branch over the subgroup K, the normal closure of the element $(ab)^2$, see [12]. This can be used to prove that $\mathcal{G}$ is just-infinite.

Grigorchuk and Wilson proved in [57] that $\mathcal{G}$ admits a unique branch action on a tree.

Other interesting examples of groups acting on a regular tree, like the second Grigorchuk group [60], (extended) Gupta-Sidki groups or GGS groups (Grigorchuk-Gupta-Sidki groups, the terminology comes from [18]) arise as generalized multi-edge spinal groups. For a general treatment and more examples, see [79].

Definition 5.3.2 ([79]). A *generalised multi-edge spinal group*

$$G = \langle \{a\} \cup \{b_i^{(j)} \mid 1 \leq j \leq p, 1 \leq i \leq r_j\}\rangle$$

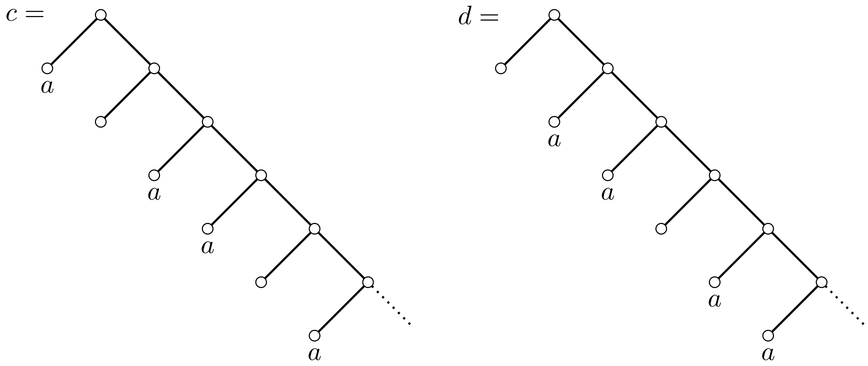


Figure 5.2: The directed automorphisms c and d.

is an infinite subgroup of (a Sylow-pro-p subgroup of) the profinite group Aut(T) that is generated by

- a rooted automorphism a of order p permuting cyclically the vertices of the first level of T, and
- families $\boldsymbol{b}^{(j)} = \{b_1^{(j)}, ..., b_{r_j}^{(j)}\}$, $j \in \{1, ..., p\}$, of directed automorphisms sharing a common directed path ξ_j in T.

The paths $\xi_1, \dots, \xi_p$ are required to be mutually disjoint.

The class $\mathscr{C}$ is the class of subgroups of $\mathrm{Aut}(T)$ conjugated (in $\mathrm{Aut}(T)$) to a generalized multi-edge spinal group. By construction such a group is a finitely generated, residually-(finite p) infinite group. Regarded as a subgroup of $\mathrm{Aut}(T)$ it is fractal.

The subclass $\mathscr{G}$ of $\mathscr{C}$ consists of all groups conjugated to a group $\langle a, \boldsymbol{b}^{(1)}, \dots, \boldsymbol{b}^{(p)} \rangle$ in $\mathscr{C}$, where for at least one superscript $j \in \{1, \dots, p\}$ the generator family $\boldsymbol{b}^{(j)} = (b_1^{(j)})$ consists of a single directed automorphism with 'constant' defining vector $e_1^{(j)} = (1, \dots, 1)$. For more details and a rigorous definition, see [79].

Klopsch and Thillaisundaram showed in [79] that if G is a torsion groups in $\mathscr{C}$, then it is just-infinite, branch, has all maximal subgroups of finite index (and the same holds for groups commensurable with G). If moreover G is in $\mathscr{C} \setminus \mathscr{G}$, then G is regular branch. They also proved that if G is in $\mathscr{C}$ and is branch (for example, G in $\mathscr{C} \setminus \mathscr{G}$), then it admits a unique branch action.

5.4 Maximal and weakly maximal subgroups

Maximal subgroups (that is maximal proper subgroups) are of great interest in group theory and have been widely studied.

In [92] Margulis and Soĭfer proved that a linear group has a maximal subgroup of infinite index if and only if it is not virtually solvable. The also proved that linear group that are not virtually solvable admit uncountably many, non-conjugated, maximal subgroups. For free groups, a simple construction using Schreier graphs even shows the existence of a continuum of such subgroups.

Lemma 5.4.1. *Let $\mathcal{F} \coloneqq \mathcal{F}_{Z \sqcup Y} = (*_{|Z|}\, \mathbf{Z}) * (*_{|Y|}\, \mathbf{Z}/2\mathbf{Z}) = \langle y_i \in Y,\ z_j \in Z \mid y_i^2 \rangle$, with Y and Z are any sets (not necessary finite). Suppose that $|Z| \geq 1$ and $|Z| + |Y| \geq 2$. The $\mathcal{F}$ has a continuum of non-conjugated maximal subgroups of infinite index and of infinite rank.*

Proof. The proof is done by looking at Schreier graphs.

Let H be a subgroup and Γ its Schreier graphs. Then H is of infinite index if and only if Γ has infinitely many vertices. By Subsection 2.4 and Section 3.4, a Schreier graph Γ of $\mathcal{F}$ correspond to a maximal subgroup if and only if the only graph strongly covered by Γ is a $(|Z|,|Y|)$-rose. Finally, H and H' are conjugated if and only if Γ and Γ' are strongly isomorphic.

In view of this, we want to construct infinite graphs that do not strongly covers other graphs (except for the rose) and that are not strongly isomorphic. Since $|Z|\geq 1$, we have at least one generator of infinite order, call it z. Since $|Z|+|Y|\geq 2$, we have a second generator, call it x. We now construct Γ a Schreier graph of $\mathcal{F}_{(Z\sqcup Y)\setminus\{x\}}$. Start with the Cayley graph of $\mathbf{Z}=\langle z\rangle$ and for each b in $(Z\sqcup Y)\setminus\{z,x\}$ add to each vertex a loop labeled by b. This loop is degenerate if and only if $b\in Y$. Since we started with the Cayley graph of $\mathbf{Z}$, the only graph covered by Γ are itself, the rose, and cycle (Cayley graph of $\mathbf{Z}/n\mathbf{Z}=\langle z\rangle$) with loops.

We now want to add edges labeled by x to Γ in order to have a Schreier graph of $\mathcal{F}$ corresponding to a maximal subgroup of infinite index. Let $(r_i)_{\{i\geq 0\}}$ be an infinite sequence of 0 and 1. Since Γ is the usual Cayley graph of $\mathbf{Z}$ with some loops, its vertices are labeled by $\mathbf{Z}$. For every vertex smaller than 0, add a loop labeled by x. Now, if $r_i=0$ add a loop labeled by x to the vertex $2i$ and the vertex $2i+1$. If $r_i=1$ and x is of order 2 add an edge from $2i$ to $2i+1$. If $r_i=1$ and x is of infinite order, add a pair of edges from $2i$ to $2i+1$. The new graph Δ obtained like this is by construction a Schreier graph of $\mathcal{F}$ corresponding to a subgroup of infinite index and of infinite rank, see Figure 5.3 for an example. Moreover, if $(r_i)_{\{i\geq 0\}}$ is not the zero sequence, then the only graphs covered by Δ are Δ and the rose. This directly follows from the fact that there loops labeled by x on negative vertices and at least one edge labeled by x that is not a loop. Therefore, the corresponding subgroup is maximal. Since there is a continuum of sequences $(r_i)_{\{i\geq 0\}}$ we have a continuum of such graphs.

Finally, the only possibility for Δ and Δ' to be strongly isomorphic is that the sequence $(r_i)_{\{i\geq 0\}}$ and $(r'_i)_{\{}i\geq 0\}$ have finitely many non-zero terms and that there exists j such that $r_i=r'_{j-i}$. We therefore have a continuum of non-strongly isomorphic such graphs and a continuum of non-conjugated maximal subgroups of infinite index and of infinite rank. □

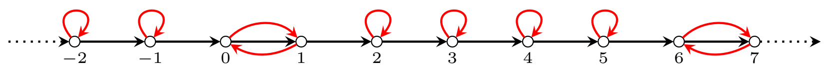


Figure 5.3: Fragment of the Schreier graph of the maximal subgroup of $F_2=\langle z,x\rangle$ associated to the sequence $1001\ldots$. Edges in black are labeled by z and edges in red labeled by x. For the construction, see the proof of 5.4.1.

On the other hand, Pervova showed [97] that all maximal subgroups of $\mathcal{G}$ are of finite index (there are seven of them, all of index 2). Pervova's result extends to groups abstractly commensurable with $\mathcal{G}$ [56], as well as to torsion generalized multi-edge spinal groups and group commensurable with them [79]; but not to all branch groups (see [22]).

All these results are related to the profinite topology. Indeed, a group G has an infinite index maximal subgroup if and only if it has a proper subgroup which is dense in the profinite topology.

The next step in understanding the subgroup structure of a group G is the study of weakly maximal subgroups.

Definition 5.4.2. Let G be a group. A subgroup A is *weakly maximal* if it is of infinite index and maximal among such subgroups.

If G is a topological group, then a subgroup A is *weakly maximal closed*, or *wmc* for short, if it is closed, of infinite index and maximal for these properties.

If G is endowed with the discrete topology, then weakly maximal and wmc subgroups coincide.

In the following, when not specified otherwise, all our topological groups will have the profinite topology. In particular, wmc will stand for wmc in the profinite topology.

If G is finitely generated, then by Zorn Lemma, every infinite index subgroup is contained in a weakly maximal subgroup. On the other hand, it is false that in a (topologically) finitely generated profinite group every closed subgroup of infinite index is contained in a wmc subgroup. Nevertheless, this becomes true if G is virtually pro-p, [106].

The rigidity of maximal subgroups in torsion generalized multi-edge spinal groups implies that in such a group G, the class of weakly maximal subgroups and the class of wmc subgroups coincide, as shown in the next lemma.

Lemma 5.4.3. *Let G be a group. Assume that for each finite index subgroup $H \leq G$, maximal subgroups of H are of finite index. Then weakly maximal subgroups of G coincide with wmc subgroups for the profinite topology on G.*

Proof. Since every wmc subgroup is contained in a weakly maximal subgroup, it is sufficient to prove that every weakly maximal subgroup is closed in the profinite topology.

If W is a weakly maximal subgroup, it is contained in a maximal subgroup M_1 of G. By assumption this subgroup is of finite index in G, therefore $W < M_1$. This implies that W is contains in a maximal subgroup M_2 of M_1, which is by assumption of finite index. Thus we have that $W \leq \bigcap M_i$ with $G > M_1 > M_2 > \dots$ with all indices finite. But then $\bigcap M_i$ is an infinite index subgroup of G containing W. By maximality we have $W = \bigcap M_i$ is closed in the profinite topology. □

Proposition 5.4.4 ([49, 11, 10])**.** *If G acts on T in a weakly branched way, then all parabolic subgroups are infinite and pairwise distinct.*

Moreover, if the action of G is branched, then all parabolic subgroups are weakly maximal.

A measure preserving action of a group G on a Lebesgue space (X, Σ, μ) is called *extremely nonfree* if there exists $A \subseteq X$, $\mu(A) = 1$ such that for each $x, y \in A$, $x \neq y$ one has $\mathrm{Stab}_G(x) \neq \mathrm{Stab}_G(y)$. Weakly branch actions are extremely nonfree.

Question 5.4.5. *Suppose that $G \leq \mathrm{Aut}(T)$ is such that all parabolic subgroups are infinite, pairwise distinct and wmc. Does it imply that the action of G is (weakly) branched?*

This question is of a particular interest in the case of branch generalized multi-edge spinal groups (and other groups considered in [57]) since there exists only one weakly branched action, [57, 79].

Since parabolic subgroups are weakly maximal and pairwise disjoint, finitely generated branch groups have uncountably many automorphism equivalence classes of weakly maximal subgroups. In the Grigorchuk group $\mathcal{G}$, the class of weakly maximal subgroups is not reduced to the class of parabolic subgroups: some sporadic examples were constructed [48, 49], see Section 6.4. There are however no classification results for weakly maximal subgroups of $\mathcal{G}$ or other (weakly) branch groups (Problem 6.3 in [48]).

The following lemmas and corollaries generalize results from [24]. They are particularly useful for torsion groups.

Lemma 5.4.6. *Let G be a topological group, and A a wmc subgroup, $g \notin A$ with $g^k \in A$ for some k (for example, g with finite order). Then $gAg^{-1} \neq A$.*

Proof. If it was the case, then $A < \langle A, g\rangle = A \sqcup gA \sqcup \cdots \sqcup g^{k-1}A$ would be closed and thus of finite index in G. We have $A < \langle A, g\rangle$ of index $k > 1$, therefore A is of finite index in G. This is the desired contradiction. □

Corollary 5.4.7. *Let G be a topological group and A a wmc subgroup such that for all $g \in G$, there exists k with $g^k \in A$. Then A is self-normalizing.*

Observe that this deeply contrast with the situation of proper subgroups in finite p-groups. Indeed, finite p-groups do not have any self-normalizing proper subgroups.

Corollary 5.4.8. *Let G be a topological group acting on a infinite rooted tree and $\xi \in \partial T$. If $A = \mathrm{Stab}(\xi)$ is wmc and $g \in G \setminus A$ is such that $g^k \in A$ for some k, then $\mathrm{Stab}(g\xi)$ is distinct from $\mathrm{Stab}(\xi)$. In particular, if G is torsion, in each orbit of ∂T with a wmc stabilizer, all stabilizers are pairwise distinct.*

Proof. The proof is by contradiction. If there were not distinct, then we have

$$\mathrm{Stab}(\xi) = \mathrm{Stab}(g\xi) = g\,\mathrm{Stab}(\xi)g^{-1}$$

and therefore $g \in N_G(A) \setminus A$. But this is not possible by last corollary. □

Proof. Suppose that A contains all the conjugates of γ. Then A contains $\langle\gamma\rangle^G$, the normal closure of γ in $\mathcal{G}$. This subgroup is normal and therefore of finite index since G is just-infinite. But this contradicts the fact that A is of infinite index. □

Proposition 5.4.9. *Let G be a just-infinite group and let $s = \sup\{p^n \mid p \text{ prime}, n \in \mathbf{N}, \exists\gamma \in G \text{ of order } p^n\}$. Then every weakly maximal subgroup of G has at least s conjugates. The same holds for wmc subgroups.*

Proof. If $s = 0$, the conclusion is trivially true. If $s > 0$, take γ a non-trivial element of order p^n. If $n = 1$, by Lemma 5.2.1, given a weakly maximal subgroup A, there exists $f\gamma f^{-1}$ a conjugate of γ which is outside H. Since p is prime, for all $0 < i < p$, the element $f\gamma^i f^{-1}$ is not in A. If $n \geq 2$, we apply Lemma 5.2.1 to $\gamma^{p^{n-1}}$ to find $f \in G$ such that $h = f\gamma^{p^{n-1}}f^{-1}$ is outside A. Since for all $1 < i < p^n$, h belongs to the subgroup generated by $f\gamma^i f^{-1}$, none of the $f\gamma^i f^{-1}$ are in A.

By Lemma 5.4.6, for each $f\gamma^i f^{-1} \neq f\gamma^j f^{-1}$, we have

$$f\gamma^i f^{-1}Af\gamma^{-i}f^{-1} \neq f\gamma^j f^{-1}Af\gamma^{-j}f^{-1}.$$

Thus the conjugacy class of A contains at least $\mathrm{ord}(\gamma)$ elements. □

If we have a faithful action of a group G, we may try to classify subgroups of G by their fix points. If G is a branch group, this will fail. Indeed, we have the following lemma.

Lemma 5.4.10. *Let $G \leq \mathrm{Aut}(T)$ be a spherically transitive group and A a weakly maximal subgroup. Then either A is parabolic, or it fixes only a finite number of vertices of T.*

Proof. If A fixes infinitely many vertices of T, then by König's lemma it fixes a ray. But then $A \leq \mathrm{Stab}_G(\xi) <_\infty G$ by spherical transitivity of G. The weak maximality of A implies then that $A = \mathrm{Stab}_G(\xi)$. □

We will show in Section 5.6 that for $G \in \mathscr{C} \setminus \mathscr{G}$, there is uncountably many non-parabolic weakly maximal subgroups. In particular, there are uncountably many weakly maximal subgroups that fixes the same subset of T.

Coset tree

Proposition 5.4.4 shows that if G acts on a rooted tree in a branch way, then parabolic subgroups are weakly maximal. In this subsection we want to go the other way around. That is, start with a weakly maximal subgroup H and construct an action of G on some rooted tree T_H such that H is a parabolic subgroup for this action. In general, this action will not be branch, but we can still study it and test if it satisfies good properties.

Definition 5.4.11. Let G be a group and H a subgroup. A *descending chain* for H, is a chain $A_0 = G \geq A_1 \geq \dots$ of subgroups of finite index such that $\bigcap A_i = H$. The associated *coset tree* is the rooted tree where vertices of the n^{th} level correspond to the right cosets of A_n, and where there is an edge from $A_i g$ to $A_j h$ if and only if $j = i+1$ and $A_j h \subseteq A_i g$.

Observe that in general we can define a coset tree for any chain $A_0 = G \geq A_1 \geq \dots$, but the tree will be locally finite if and only if the subgroups are of finite index.

If $(g_i^{(j)})_{j=1}^{r_j}$ is a transversal system for $A_i \leq A_{i-1}$, then the set of vertices of the i^{th} level is $\{A_i g_{l_n} \dots g_{l_2} g_{l_1} \mid n \leq r_1 r_2 \dots r_i \text{ and } l_1 < l_2 < \dots < l_n\}$. See Figure 5.4 for an example. We will often make the abuse of notation of writing T_H for the corresponding coset tree,

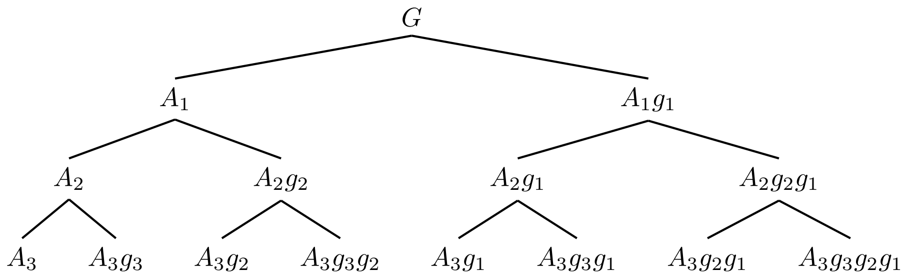


Figure 5.4: The coset tree of $G >_2 A_1 >_2 A_2 >_2 A_3$.

and we will always make the identification of the ray 0^∞, and the ray corresponding to the sequence of cosets $(A_i)_i$. This tree is locally finite and spherically homogeneous: the degree of a vertex depends only on its level. If H is of infinite index, then the tree is infinite. There is a natural action by multiplication on right of G on T_H. This action is spherically transitive and $\mathrm{Stab}_G(0^\infty) = H$. By spherical transitivity, $\mathrm{Stab}_G(n) > \mathrm{Stab}_G(n+1)$ for all n. Therefore, if G is just-infinite, then the action is faithful.

Lemma 5.4.12. *Let G be a finitely generated group and H be a closed subgroup for the profinite topology. Then there exists a descending chain for H.*

Moreover, if G is a p-group, then every such chain can be completed in a chain such that for all i, A_{i+1} is a normal subgroup of A_i of index p.

Proof. Since H is closed, then H is the intersection of all subgroups of finite index that contain it. Let A_i be the intersection of all such subgroups of index at most i. It is trivial that $A_i \geq A_{i+1}$ and that the intersection is H. Since G is finitely generated, N_i is a finite intersection of finite index subgroups and thus of finite index.

Now, if G is a p-group and A is a subgroup of finite index, then its normalizer N is a normal subgroup of finite index. In particular, G/N is a finite p-group. But in a finite p-group, every proper subgroup is contained in a normal subgroup of index p. By applying this to A/N and using the correspondence theorem, we have $G > A' \geq A$, where A' is normal subgroup of index p. An induction finishes the proof. □

In particular, if H is a wmc subgroup in a p-group, then it is always possible to construct a coset tree which is p-regular.

5.5 Weakly maximal subgroups living deep in the tree

This section is consecrated to the proof of Theorem 5.1.4. Let G be a finitely generated regular branch group acting on T and Q an infinite index subgroup that does not fixes any point in ∂T. Let l be the minimal integer such that Q does not fixes any vertex of level l. For all vertex v of level k, we want to construct a weakly maximal subgroup that stabilizes v but does not stabilize any vertex of level $k+l$.

We first demonstrate that if $Q \leq \mathrm{Aut}(T)$ stabilizes no point in ∂T, then there exists an integer l such that Q stabilizes no vertex of level l. The proof is done by contradiction. Let T_Q be the full subgraph of T consisting of all vertices stabilized by Q, with edges coming from edges in T. Assume that for each integer l, the group Q stabilizes a vertex v of level l. Then Q stabilizes the path between the root and v. Therefore, T_Q is a locally finite tree with paths of arbitrary large length. By König's lemma, T_Q contains an infinite path ξ which is stabilized by Q — a contradiction.

Proof of Theorem 5.1.4. Let d be the degree of T. Using Q we will, for any integer $k \geq 1$, produce the desired subgroup W of $\mathrm{Stab}_G(k)$. For any k, let ψ_k be the map

$$\mathrm{Stab}_G(k) \hookrightarrow G^{d^k}$$

and for any $i = 1, \ldots, d^k$, let π_i be the projection of G^{d^k} onto the ith factor. Let Δ be a weakly maximal subgroup of G containing Q and look at $(\pi_1 \circ \psi_k)^{-1}(\Delta)$. Since G is regular branch, the image of ψ_k has finite index in G^{d^k}. This and the surjectivity of π_1 implies that $(\pi_1 \circ \psi_k)^{-1}(\Delta)$ is an infinite index subgroup of $\mathrm{Stab}_G(k)$. Let H be any weakly maximal subgroup containing it. We claim that $H \leq \mathrm{Stab}_G(0^k)$, where 0^k is the leftmost vertex of level k. Suppose, for the sake of contradiction, that there exists $\gamma = (g_1, \ldots, g_{d^k})_k \tau \in H \setminus \mathrm{Stab}_G(0^k)$ for some τ permuting the d^k coordinates. Then for any $(a_1, \ldots, a_{d^k})_k \in H$, we have $\gamma (a_1, \ldots, a_{d^k})_k \gamma^{-1} \in H$. Since γ is not in $\mathrm{Stab}_G(v)$, we have that

$$\gamma (a_1, \ldots, a_{d^k})_k \gamma^{-1} = (g_j a_j g_j^{-1}, b_2, \ldots, g_1 a_1 g_1^{-1}, \ldots, b_{d^k})_k$$

for some $j \in 2, 3, \ldots, d^k$; with b_l's being a permutation of conjugates of the remaining a_i's. However, since G is regular branch, the image of ψ_k is of finite index in G^{d^k}. Thus, the image of ψ_k contains $Q_1 \times Q_2 \cdots \times Q_{d^k}$, where each Q_i is a finite index subgroup of G for $i = 1, \ldots, d^k$. It follows, then that $\psi_k\big(H \cap \mathrm{Stab}_G(k)\big)$ contains $1 \times Q_2 \cdots \times Q_{d^k}$, as $1 \times Q_2 \times \cdots \times Q_{d^k}$ is in the kernel of π_1. But then $\gamma(H \cap \mathrm{Stab}_G(k))\gamma^{-1} = H \cap \mathrm{Stab}_G(k)$ and $\psi_k(\gamma(H \cap \mathrm{Stab}_G(k))\gamma^{-1})$ contains a conjugate of Q_j in the first factor. Therefore, $\psi_k(H \cap \mathrm{Stab}_G(k))$ is of finite index in G^{d^k}. It follows, because ψ_k is injective, that $H \cap \mathrm{Stab}_G(k)$ is of finite index in $\mathrm{Stab}_G(k)$, which is impossible as H is weakly maximal and $\mathrm{Stab}_G(k)$ is a subgroup of G of finite index. So we have found our desired contradiction and proved that $H \leq \mathrm{Stab}_G(0^k)$. By replacing π_1 with a suitable π_i, we show that for every k and every vertex v of level k, there exists a weakly maximal subgroup $H \leq \mathrm{Stab}_G(v)$ that stabilizes no vertices of level $k+l$.

Finally, the map $\pi_i \circ \psi_k$ maps H onto all of G for $i \neq 1$. And $\pi_1 \circ \psi_k(H)$ contains Q. Thus, it is impossible for H to stabilize any vertex of the $(k+l)^{\mathrm{th}}$ level, as desired.

When G is the first Grigorchuk group or a group in $\mathscr{C} \setminus \mathscr{G}$, it is possible to take $Q = \langle a \rangle$. □

We remark here that it is now possible to produce countably many groups each of which is not conjugate to any parabolic subgroup of $\mathcal{G}$. For this, set $\mathrm{Fix}(H)$ to be the points in T fixed by every element in H. For any $g \in \mathcal{G}$ and $H \leq \mathcal{G}$, recall that $\mathrm{Fix}(gHg^{-1}) = g\,\mathrm{Fix}(H)$. By this relation, the conjugate of any parabolic group is parabolic. Thus, if H_k is conjugate to a parabolic group, it must be parabolic itself. However, H_k does not fix any vertex of level $k+1$ and conjugation preserves levels of the tree, so this is impossible.

Next, we show that for $i \neq j$, the groups H_i and H_j are never conjugate. Suppose, for the sake of contradiction, that H_i and H_j are conjugate and $i < j$. Then $\mathrm{Fix}(H_i) = g\,\mathrm{Fix}(H_j)$ for some $g \in \mathcal{G}$. However, by construction, H_j fixes a vertex, v, of level j. It follows then that H_i fixed gv, but H_i does not fix any vertex at level $i+1$, and hence cannot fix a vertex at level $j \geq i+1$. It follows then that H_i and H_j are not conjugate.

5.6 Weakly maximal subgroups containing a given finite subgroup

This section is devoted to the proof of Theorems 5.1.1, 5.1.3 and of Corollary 5.1.2.

Let T be a regular rooted tree, $G \leq \mathrm{Aut}(T)$ be a finitely generated regular branch group and $Q \leq G$ a finite subgroup of G. We will prove that there are uncountably many weakly maximal subgroups of G containing Q. This is sufficient to prove Theorem 5.1.1 since G is finitely generated and therefore $\mathrm{Aut}(G)$ is countable.

Our proof here is a diagonal argument, similar, in a sense, to Cantor's proof of uncountability of the real numbers. In many ways, this proof is also to Margulis and Soĭfer's proof that in a non virtually solvable linear group there is uncountably many maximal subgroups of infinite index. In both cases, the main idea is to find a subgroup of infinite index H_i with "good properties" such that for any weakly maximal (respectively maximal of infinite index) subgroup M, there exists $g \notin M$ such that $H_{i+1} \coloneqq \langle H_i, g\rangle$ is still a subgroup of infinite index with the same good properties. This general idea is a priori applicable for any class of groups, we only need to properly define what would be the "good properties" of the H_i and show that it is possible to do an induction. Since regular branch groups and linear groups stand far away one from the other, the "good properties" involved and the technical tools used are totally distinct.

Suppose, for the sake of contradiction, that there exists countably many non-parabolic weakly maximal subgroups that contain Q. Enumerate them $\{W_i\}_{i\geq 1}$. Since Q is finite, there exists k_1 such that $\mathrm{Stab}_Q(k_1) = \{1\}$ and Q does not act transitively on $\mathscr{L}_{k_1}$. If W_1 contains $\mathrm{Rist}_G(v)$ for all $v \in \mathscr{L}_{k_1}$, then W_1 contains $\mathrm{Rist}_G(k_1)$. This is impossible as $\mathrm{Rist}_G(k_1)$ is of finite index in G because G is branch. Hence, there exists some $v_0 \in \mathscr{L}_{k_1}$ such that $\mathrm{Rist}_G(v_0) \setminus W_1$ is non-empty. Pick an element in this non-empty set and call it w_1.

Claim 5.6.1. Set $H_1 \coloneqq \langle Q, w_1\rangle$. Then there exists a vertex u of level k_1 such that the subtrees $\{T_{q(u)} : q \in Q\}$ are all fixed by $\mathrm{Stab}_{H_1}(k_1)$. Moreover, $\mathrm{Stab}_{H_1}(k_1)$ is the normal closure of w_1 in H_1.

Proof. Since w_1 is in $\mathrm{Rist}_G(v_0)$, for any $q \in Q$, qw_1q^{-1} belongs to $\mathrm{Rist}_G\big(q(v_0)\big)$. Hence, for any $g \in H_1$, gw_1g^{-1} fixes T_v for $v \in \mathscr{L}_{k_1} \setminus Q(v_0)$. By the choice of k_1, $Q(v_0)$ cannot contain all vertices of level k_1, thus there exists a vertex $u \in \mathscr{L}_{k_1} \setminus Q(v_0)$ such that $\{T_{q(u)} : q \in Q\}$ are all fixed by the normal closure of w_1 in H_1.

We conclude by showing that $\mathrm{Stab}_{H_1}(k_1)$ is exactly the normal closure of w_1 in H_1. By definition, the element w_1, and hence any of its conjugates, are in $\mathrm{Stab}_{H_1}(k_1)$. Since $\mathrm{Stab}_Q(k_1) = \{1\}$, we have that $H_1 = Q \ltimes \langle w\rangle_1^{H_1} = Q \ltimes \mathrm{Stab}_{H_1}(k_1)$ with $\langle w\rangle_1^{H_1} \leq \mathrm{Stab}_{H_1}(k_1)$. This implies that $\langle w\rangle_1^{H_1} = \mathrm{Stab}_{H_1}(k_1)$ as desired. □

For a vertex w and a set S of vertices in T, we write $w \leq S$ if $w \leq v$ for some $v \in S$. Suppose $w_1, \ldots, w_i \in G$ of level $k_1, \ldots, k_i$ have been constructed so that $H_i \coloneqq \langle Q, w_1, \ldots, w_i \rangle$ with

1. w_j does not belongs to W_j for all $1 \leq j \leq i$,
2. the normal closure of $\{w_1, \ldots, w_i\}$ in H_i is $\mathrm{Stab}_{H_i}(k_1)$, and
3. for $2 \leq j \leq i$, there exist vertices u_j of level k_j with $u_j \leq Q(u_{j-1})$, such that $\mathrm{Stab}_{H_j}(k_1)$ fixes every point in $\{T_{q(u_j)} : q \in Q\}$.

Claim 5.6.2. For all k_{i+1} big enough there exists $w_{i+1} \in \mathrm{Stab}_G(k_{i+1})$ such that $H_{i+1} \coloneqq \langle Q, w_1, \ldots, w_{i+1} \rangle$ is not included in any $W_1, \ldots, W_i, W_{i+1}$, and, further, there exists a vertex $u_{i+1} \leq Q(u_i)$ of level k_{i+1} such that $\mathrm{Stab}_{H_{i+1}}(k_1)$ fixes every element in $\{T_{q(u_{i+1})} : q \in Q\}$. Moreover, $\mathrm{Stab}_{H_{i+1}}(k_1)$ is the normal closure of $w_1, w_2, \ldots, w_i, w_{i+1}$ in H_{i+1}.

Proof. Since Q is finite, it is possible to find $k_{i+1} > k_i$ such that Q does not acts transitively on $\mathscr{L}_{k_{i+1}} \cap \{T_{q(u_i)} : q \in Q\}$. Since W_{i+1} is of infinite index in G and $\mathrm{Rist}_G(k_{i+1})$ is of finite index, there exist an element $w_{i+1} \in \mathrm{Rist}_G(v_0) \setminus W_{i+1}$ for some $v_0 \in \mathscr{L}_{k_{i+1}}$.

From now on, when we write v, we will always assume it is a vertex of level k_{i+1}. By assumption, $\mathrm{Stab}_{H_i}(k_1)$ acts trivially on $\{T_{q(u_i)} : q \in Q\}$. Therefore, for any $p \in \mathrm{Stab}_{H_i}(k_1)$, $pw_{i+1}p^{-1}$ fixes T_v for all $v \leq Q(u_i)$ with $v \neq v_0$. By definition of k_{i+1}, there exists $u_{i+1} \leq Q(u_i)$ of level k_{i+1} with $u_{i+1} \notin Q(v_0)$. For such a u_{i+1}, every element in $\{T_{q(u_{i+1})} : q \in Q\}$ is fixed by the normal closure of $\{w_1, \ldots, w_{i+1}\}$ in H_{i+1}. In fact, since each of the trees in $\{T_{q(u_{i+1})} : q \in Q\}$ are contained in $\{T_{q(u_i)} : q \in Q\}$, it follows that for all $q \in Q$ and $j = 1, \ldots, i+1$, qw_jq^{-1} fixes $\{T_{q(u_{i+1})} : q \in Q\}$. Set N to be the normal closure of $\{w_1, \ldots, w_{i+1}\}$ in H_{i+1}. Since N is generated by $\{qw_jq^{-1} : q \in Q \text{ and } j = 1, \ldots, i+1\}$, it follows that N fixes $\{T_{q(u_{i+1})} : q \in Q\}$. Since $\mathrm{Stab}_Q(k_1)$ is trivial and $N \leq \mathrm{Stab}_G(k_1)$, it follows that $N = \mathrm{Stab}_{H_{i+1}}(k_1)$, as desired. ☐

By applying Claims 5.6.1 and 5.6.2, we construct an infinite sequence $k_1 < k_2 < \ldots$ and an infinite sequence $\{w_i\}_{i \geq 1}$ in G such that:

- $H_i \coloneqq \langle Q, w_1, \ldots, w_i \rangle$ is not included in any $W_1, \ldots, W_i$,
- there exists a vertex $u_i \in T$ of level k_i, such that $H_i \cap \mathrm{Stab}_G(k_1)$ fixes every point in T_{u_i}.

We further claim that for any $i \in \mathbf{N}$, the associated group H_i has infinite index in G. Suppose, for the sake of contradiction, that H_i is of finite index in G for some i.

Note that $H_i \cap \mathrm{Stab}_G(k_1)$ acts trivially on some T_v where $v \in T$ has level $k_i > k_1$. Since G is regular branch over K, $K^{d^{k_i}}$ is a finite index subgroup of $\psi_{k_i}(\mathrm{Stab}_G(k_i) \cap K)$. This implies that $\psi_{k_i}\big(\mathrm{Stab}_G(k_i)\big)$ is a finite index subgroup of $G^{d^{k_i}}$. The subgroups H_i and $\mathrm{Stab}_G(k_i)$ are of finite index in G, thus $\mathrm{Stab}_G(k_i) \cap H_i$ has finite index in $\mathrm{Stab}_G(k_i)$. Since $\psi_{k_i}\big(\mathrm{Stab}_G(k_i)\big)$ is a finite index subgroup of $G^{d^{k_i}}$, $\psi_{k_i}(\mathrm{Stab}_G(k_i) \cap H_i)$ has finite index in $G^{d^{k_i}}$. This conclusion is impossible as every element in $H_i \cap \mathrm{Stab}_G(k_i) \leq H_i \cap \mathrm{Stab}_G(k_1)$ fixes every point in T_{u_i}. The claim is now shown.

Now, given that H_i is never of finite index, we claim that $H = \cup_{i \geq 1} H_i$ is an infinite-index subgroup of G. Suppose not, then H, being of finite index in a finitely generated group G, is finitely generated, say, by elements $a_1, \ldots, a_m$. Thus, as the sequence of groups $\{H_i\}_{i \geq 1}$ is increasing, there is a single H_i that must contain each $a_1, \ldots, a_m$. It follows that H_i is of finite index – a contradiction.

Since G is finitely generated, there exists a weakly maximal subgroup A containing H in G. Therefore, A contains Q and hence equal to one of the W_i. On the other hand, for any W_i, H contains an element that is not in W_i, which give us the desired contradiction. Thus, there cannot be countably many weakly maximal subgroups containing Q. This completes the proof of Theorem 5.1.1. □

We now prove Corollary 5.1.2 and Theorem 5.1.3 as direct corollaries of Theorem 5.1.1.

Corollary 5.1.2. *Let T be a regular rooted tree and $G \leq \mathrm{Aut}(T)$ be a finitely generated regular branch group. Suppose that G contains a finite subgroup Q that does not fix any point in ∂T. Then there exist uncountably many automorphism equivalence classes of weakly maximal subgroups of G, all distinct from classes of parabolic subgroups associated with the action of G on T.*

Proof. Let φ be an automorphism of G. Since G is weakly branch, by Theorem 7.3 of [81] there exists ψ a homeomorphism of ∂T such that for all $g \in G$ and $\xi \in \partial T$ we have $\varphi(g)(\xi) = \psi^{-1} \cdot g \cdot \psi(\xi)$. Therefore, automorphisms of G send parabolic subgroups to parabolic subgroups. On the other hand, if W is one of the uncountably many weakly maximal subgroups containing Q, it fixes no ray and therefore is not parabolic. □

Theorem 5.1.3. *Let G be either the first Grigorchuk group, or a GGS group with $E = (1, -1)$ or $E = (\epsilon_1, \dots, \epsilon_{p-1})$, p prime, such that the ϵ_i are not all zero and not all non-zero. Then G has uncountably many automorphism equivalence classes of weakly maximal subgroups, all distinct from classes of parabolic subgroups of any branch action of G on a spherically regular tree.*

Proof. Take $Q = \langle a \rangle$. Then, there exist uncountably many automorphism equivalence classes of weakly maximal subgroups of G all distinct from classes of parabolic subgroups of the action of G on T. The rigidity described in [57] implies that for any branch action of G on a spherically regular tree, the parabolic subgroups are the same as the ones from the original action. □

5.7 From wmc subgroups to coset trees

In this section we will work with groups acting faithfully on locally finite rooted trees. Such a group is residually finite and its profinite completion also acts on the tree. We will show that there is a bijection between closed subgroups of finite index in G and closed subgroups of finite index in $\hat{G}$ its profinite completion.

We will then study the relation between wmc subgroups of G and wmc subgroups of $\hat{G}$. Finally, given G a residually finite just-infinite p-group and H a wmc subgroup, we construct in Theorem 5.7.20 an action of G on an infinite rooted tree such that there is a ray with $\mathrm{Stab}_G(\xi) = H$ and all $\hat{G}$ parabolic subgroup are distinct and wmc.

First we begin with some results on topological groups.

Topological groups

Throughout this subsection, G will be a topological group not necessarily (topologically) finitely generated. For us a subgroup of G will always be an abstract subgroup not necessarily closed.

Lemma 5.7.1. *Let X be a topological space, Y a subspace with the subspace topology, $\mathcal{F}(Y)$ the collection of closed subsets of Y and $\bar{\cdot}$ the closure in X. Then we have two applications:*

$$\begin{aligned} \Theta\colon \mathcal{F}(Y) &\to \mathcal{F}(X) \\ A &\mapsto \bar{A} \end{aligned} \qquad\qquad \begin{aligned} \Psi\colon \mathcal{F}(X) &\to \mathcal{F}(Y) \\ B &\mapsto B \cap Y \end{aligned}$$

which satisfy

1. $\Psi \cdot \Theta = \mathrm{Id}_{\mathcal{F}(Y)}$*;*
2. Ψ *is increasing, while* Θ *is strictly increasing;*
3. $\Theta \cdot \Psi(B) \subseteq B$*;*
4. *If we restrict ourselves to* $\mathrm{Im}(\Theta)$*, then* Θ *and* Ψ *are isomorphism of lattices and* $\Theta = \Psi^{-1}$*;*
5. *If* Y *is dense and* B *open,* $\Theta \cdot \Psi(B) = B$*;*
6. *If* Y *is dense,* B *open and* φ *an homeomorphism of* X*, we have* $\varphi(B) = \overline{\varphi(B) \cap Y}$*.*

Proof. For $A \in \mathcal{F}(Y)$, we have $A \subseteq \bar{A}$ and $A \subseteq Y$, thus $A \subseteq \bar{A} \cap Y$. For the other direction, remember that $\overline{U \cap V} \subseteq \bar{U} \cap \bar{V}$. Now, by definition of the induced topology, $A = B \cap Y$ with B a closed subset of X. Therefore, $\bar{A} \cap Y = \overline{B \cap Y} \cap Y \subseteq \bar{B} \cap \bar{Y} \cap Y = B \cap Y = A$; and this prooves the first statement. It is evident that both Ψ and Θ are increasing. The equality $\Psi \cdot \Theta = \mathrm{Id}_{\mathcal{F}(Y)}$ implies that Θ is injective and thus strictly increasing.

The third and fourth statements are trivial.

For the fifth statement, we have to prove that $B \cap Y$ is dense in B. Take any open set U of B (with the induced topology). Since B is open, U is also an open set of X. Therefore, $U \cap (B \cap Y) = (U \cap B) \cap Y = U \cap Y \neq \emptyset$ by density of Y.

For the last statement, since φ is an homeomorphism, $\varphi^{-1}(Y)$ is dense in X and $\varphi(B)$ is clopen. Therefore, $\overline{\varphi(B) \cap Y} = \varphi(\overline{B \cap \varphi^{-1}(Y)}) = \varphi(B)$. ◻

This lemma means that (when restricted to closed subset of X and Y), $\bar{\cdot}$ is section $\cdot \cap Y$. Therefore, $\bar{\cdot}$ is injective and $\cdot \cap Y$ is surjective. Observe that even in the particular case of X a topological group, Y a dense subgroup and B a closed subgroup, it is not necessarily true that $\Theta \cdot \Psi(B) = B$. For example, take $X = \mathbf{R}$, $Y = \mathbf{Q}$ and $B = \sqrt{2}\mathbf{Z}$.

Proposition 5.7.2. *Let K be a topological group and $G \leq K$ an abstract subgroup. For every pair of subgroups $H \leq L$ of G, if $[L : H]$ is finite then $[\bar{L} : \bar{H}] \leq [L : H]$. If H is a closed subgroup of G, then $[L : H]$ is finite if and only if $[\bar{L} : \bar{H}]$ is finite, in which case they are equal. Moreover, if H is a closed subgroup of finite index of G and $\{g_i\}_{i=1}^n$ is a transversal for $H \leq L$, it is also a transversal for $\bar{H} \leq \bar{L}$*

Proof. First suppose that $[L : H]$ is finite. We have $L = \sqcup_{i=1}^n g_i H$. Thus $\bar{L} = \overline{\sqcup g_i H} = \cup_{i=1}^n \overline{g_i H} = \cup g_i \bar{H}$, where the second equality holds because a finite union of closed subset is closed and the last equality holds since left multiplication by g_i is an homeomorphism.

Moreover, if H is closed in G and $g_i \bar{H} \cap g_j \bar{H}$ is non-empty, then $g_i g_j^{-1} \in \bar{H} \cap G = H$ which implies $i = j$. This prove that in this case $\bar{L} = \sqcup g_i \bar{H}$.

Now suppose that H is closed in G and $\bar{H}$ is of finite index in $\bar{L}$. Then $\bar{H} \cap L$ is of finite index in $\bar{L} \cap L = L$. We have $L \leq \bar{L}$ and H closed in G implies that H is closed in L for the subspace topology coming from L. Thus, we can apply Lemma 5.7.1 to $L \leq \bar{L}$ in order to have $\bar{H} \cap L = \bar{H}^L \cap L = H$ of finite index in L. ◻

Let K be a topological group. We denote by $\mathrm{Sub}_{\mathrm{cl}}(K)$ the collection of closed subgroups of K. This is a lattice with $A \wedge B = A \cap B$ and $A \vee B = \overline{\langle A \cup B\rangle}$.

Corollary 5.7.3. *If K is topological group and G a dense subgroup we have two applications:*

$$\begin{aligned}\Theta\colon \mathrm{Sub}_{\mathrm{cl}}(G) &\to \mathrm{Sub}_{\mathrm{cl}}(K) \\ H &\mapsto \bar{H}\end{aligned} \qquad \begin{aligned}\Psi\colon \mathrm{Sub}_{\mathrm{cl}}(K) &\to \mathrm{Sub}_{\mathrm{cl}}(G) \\ M &\mapsto M \cap G\end{aligned}$$

which satisfy

1. $\Psi \cdot \Theta = \mathrm{Id}_{\mathrm{Sub}_{\mathrm{cl}}(G)}$*;*
2. Ψ *is increasing, while* Θ *is strictly increasing;*
3. $\Theta \cdot \Psi(M) \subseteq M$*;*
4. *If we restrict ourselves to* $\mathrm{Im}(\Theta)$*, then* Θ *and* Ψ *are isomorphism of lattices and* $\Theta = \Psi^{-1}$*;*
5. *If* M *is clopen,* $\Theta \cdot \Psi(M) = M$*;*
6. *For all* H*,* L *in* $\mathrm{Sub}_{\mathrm{cl}}(G)$*, the index* $[L : H]$ *is finite if and only if* $[\bar{L} : \bar{H}]$ *is finite, in which case they are equal;*
7. *For all* M *in* $\mathrm{Sub}_{\mathrm{cl}}(K)$*, the index* $[K : M]$ *is finite if and only if* $[G : M \cap G]$ *is finite. In which case they are equal;*
8. *For all* H*,* L *in* $\mathrm{Sub}_{\mathrm{cl}}(G)$ *and* M*,* N *in* $\mathrm{Sub}_{\mathrm{cl}}(K)$ *we have*

$$\begin{aligned}\Theta(H \vee L) &= \Theta(H) \vee \Theta(L) \\ \Theta(H \wedge L) &\leq \Theta(H) \wedge \Theta(L)\end{aligned} \qquad \begin{aligned}\Psi(M \vee N) &\geq \Psi(N) \vee \Psi(N) \\ \Psi(M \wedge N) &= \Psi(N) \wedge \Psi(N)\end{aligned}$$

Proof. Assertions 1 to 6 are obvious.

For 7, we already know that $[K : M] \geq [G : M \cap G]$. We also have that $\overline{M \cap G} \leq M$. Therefore, we have $[G : M \cap G] \leq [\bar{G} : M] \leq [\bar{G} : \overline{M \cap G}]$. If $[K : M]$ is finite, so is $[G : M \cap G]$ and we can use assertion 6. If $[K : M]$ is infinite, $[G : M \cap G]$ can not be finite, otherwise we would have $[K : M] \leq [\bar{G} : \overline{M \cap G}] = [G : M \cap G]$.

For Assertion 8, the only non-obvious inequality is $\Theta(A) \vee \Theta(B) \leq \Theta(A \vee B)$. If g is in $\langle \bar{A} \cup \bar{B}\rangle$, then $g = a_1 b_1 \dots a_n b_n$ with the a_i in $\bar{A}$ and the b_i in B. Therefore, $g = \lim_{j\to\infty} a_{1,j} b_{1,j} \dots a_{n,j} b_{n,j}$, where for all i, $a_i = \lim_{j\to\infty} a_{i,j}$ with the $a_{i,j}$ in A. This shows that $\langle \bar{A} \cup \bar{B}\rangle \leq \overline{\langle A \cup B\rangle} = \overline{\overline{\langle A \cup B\rangle}^{G}} = \Theta(A \vee B)$. Since this last subset is closed, we conclude that $\Theta(A) \vee \Theta(B) = \overline{\langle \bar{A} \cup \bar{B}\rangle} \leq \Theta(A \vee B)$.

□

Profinite groups

Definition 5.7.4. A *profinite group* is a topological group that is isomorphic to the inverse limit of an inverse system of discrete finite groups.

Let G be an abstract group. Its *profinite completion* $\hat{G}$ is the inverse limit of the system $(G/N)_{N\in\mathcal{N}}$ where $\mathcal{N}$ is the collection of finite index normal subgroup of G.

Remark 5.7.5. Profinite groups are topological groups, but in general this topology is coarser than the profinite topology: it is possible that some subgroup of finite index are not open. However, using classification of finite simple groups, Nokilov and Segal showed in [94, 95] that if G is topologically finitely generated then both topology coincide.

Theorem 5.7.6 (Nikolov and Segal)**.** *Let G be a (topologically) finitely generated profinite group. Then every finite index subgroup is open.*

An important corollary of this result is that if two topologically finitely generated profinite groups are abstractly isomorphic, then they are homeomorphic. Hence, the topology of topologically finitely generated profinite groups is uniquely determined by their algebraic structure. Therefore, in a topologically finitely generated profinite group G, finite index subgroups are exactly clopen subgroups.

There is always a natural map $\eta\colon G \to \hat{G}$, which is injective if and only if G is residually finite. If G is residually finite, the profinite topology on G is the same as the subspace topology induced by $\hat{G}$ (this follows from a universal property of $\hat{G}$) and $\bar{G} = \hat{G}$, where $\bar{\cdot}$ denotes the closure in $\hat{G}$.

Profinite groups have a rather interesting topology, see [107] and [120] for a detailed expositions of the following facts. It turns out that a group is profinite if and only if it is Hausdorff, compact and totally disconnected. Since it is compact, its open subgroups are exactly finite index closed subgroups. We also have that in a profinite group, the intersection of all open normal subgroups is trivial and that the closure of a subgroup H is the intersection of all open subgroups containing H. Finally, if G is a topologically finitely generated profinite group, then it is metrizable.

Proposition 5.7.7. *If G is a residually finite group and $\hat{G}$ its profinite completion, the two applications*

$$\begin{aligned}\Theta\colon \operatorname{Sub}_{\mathrm{cl}}(G) &\to \operatorname{Sub}_{\mathrm{cl}}(\hat{G}) \\ H &\mapsto \bar{H}\end{aligned} \qquad\qquad \begin{aligned}\Psi\colon \operatorname{Sub}_{\mathrm{cl}}(\hat{G}) &\to \operatorname{Sub}_{\mathrm{cl}}(G) \\ M &\mapsto M\cap G\end{aligned}$$

of Corollary 5.7.3 satisfies that if M is of finite index, $\Theta\cdot\Psi(M) = M$.

Proof. This is Corollary 5.7.3 and the fact that finite index subgroups of $\hat{G}$ are clopen subgroups. □

Corollary 5.7.8. *If G is a residually finite group and $\hat{G}$ its completion, their lattice of clopen subgroups are isomorphic, by $\bar{\cdot}$ and $\cdot\cap G$.*

If moreover G is finitely generated, their lattice of finite index subgroups are isomorphic, by $\bar{\cdot}$ and $\cdot\cap G$.

Lemma 5.7.9. *Let G be a residually finite group. Then*

1. *for any subgroup $M \leq \hat{G}$, we have $N_G(M\cap G) \geq N_{\hat{G}}(M)\cap G$;*
2. *for any subgroup $H \leq G$, we have $N_G(H) = N_{\hat{G}}(\bar{H})\cap G$.*

Proof. Take $g \in N_{\hat{G}}(M) \cap G$ then, for all $h \in M \cap G \leq M$, ghg^{-1} is in M. But since g is in G and h too, we have $ghg^{-1} \in M \cap G$. This prove the first statement an half of the second statement.

On the other hand, if g is in $N_G(H)$ and $h \in \bar{H}$, we have $h = \lim_i h_i$ with all the h_i in H. Therefore, $ghg^{-1} = g(\lim h_i)g^{-1} = \lim gh_ig^{-1}$ belongs to $\bar{H}$ since all the gh_ig^{-1} are in H. ☐

Corollary 5.7.10. *Let G be a finitely generated residually finite group. Then if $H \leq G$ is of finite index, we have $\overline{N_G(H)} = N_{\hat{G}}(\bar{H})$.*

Proof. Since H is of finite index, so is $\bar{H}$. Hence, both $N_{\hat{G}}(\bar{H})$ and $\overline{N_G(H)}$ are of finite index. Therefore, $N_{\hat{G}}(\bar{H}) = \overline{N_G(H)}$ if and only if $N_{\hat{G}}(\bar{H}) \cap G = N_G(H)$ and the conclusion follows. ☐

The following lemma gives us some link between wmc subgroups in G and wmc subgroups in $\hat{G}$.

Lemma 5.7.11. *Let G be a residually finite group, $H \leq G$ a closed subgroup and $M \leq \hat{G}$ a closed subgroup of the profinite completion.*

1. *If $\bar{H}$ is a wmc subgroup of $\hat{G}$, then H is a wmc subgroup of G.*
2. *If $M \cap G$ is wmc in G, then $\overline{M \cap G}$ is weakly maximal among closed subgroups of the form $\overline{A \cap G}$.*

Proof. If $\bar{H}$ is a wmc subgroup, then it has infinite index and H has infinite index too. If H was not a wmc subgroup, then we have $H < A <_\infty G$, which gives us $\bar{H} < \bar{A} <_\infty G$.

On the other hand, suppose that $M \cap G$ is a wmc subgroup. This implies that, $\overline{M \cap G}$ is of infinite index. If there is $\overline{M \cap G} \leq \overline{A \cap G} <_\infty \hat{G}$, then $M \cap G \leq A \cap G <_\infty G$. By maximality, $M \cap G = A \cap G$ and thus $\overline{M \cap G} = \overline{A \cap G}$. ☐

This lemma and the fact that applications $\bar{\cdot}$ and $\cdot \cap G$ are bijections that preserves indices when restricted to finite index subgroups raise the following questions.

Question 5.7.12. *Do the applications $\bar{\cdot}$ and $\cdot \cap G$ send wmc subgroups onto wmc subgroups? If it is the case, are they bijections when restricted to wmc subgroups.*

If the answer to the first question is yes, then using proposition 5.1.6 we have that if H is a wmc subgroup of G, then all parabolic subgroups for the action $G \curvearrowright T_H$ are wmc and two-by-two distinct.

Groups acting on infinite rooted tree

Recall that if G is a subgroup of $\mathrm{Aut}(T)$, then G is residually finite. In this case, we also have that $\bar{G}^T := \bar{G}^{\mathrm{Aut}(T)}$, the closure of G in $\mathrm{Aut}(T)$, is a quotient of $\hat{G}$. This implies that $\hat{G}$ acts too on T, but this action is not necessarily faithful. If the action of G is transitive on each level, then $\bar{G}^T$ acts transitively on the boundary ∂T. The following lemma follows immediately from the definition of just-infinite.

Lemma 5.7.13. *Let $G \leq \mathrm{Aut}(T)$ be an infinite residually finite group such that $\hat{G}$ is just-infinite. Then $\hat{G} = \bar{G}^T$.*

Remark 5.7.14. Let H be any subgroup of G. Then we can consider different closure for H. Firstly the closure in $\hat{G}$, which we will denote by $\bar{H}$. There is also the closure in $\mathrm{Aut}(T)$ which we denote by $\bar{H}^T$. Finally, there is the closure of H in G, written $\bar{H}^G$, where the topology on G is the profinite one.

Lemma 5.7.15. *Let T be a rooted tree and let $G \leq \mathrm{Aut}(T)$ be a finitely generated group. Then for all vertex v and all ray ξ we have*

1. $\mathrm{Stab}_G(v) = \mathrm{Stab}_{\hat{G}}(v) \cap G$ *and* $\mathrm{Stab}_G(\xi) = \mathrm{Stab}_{\hat{G}}(\xi) \cap G$*;*
2. $\overline{\mathrm{Stab}_G(v)} = \mathrm{Stab}_{\hat{G}}(v)$*.*

Proof. The first statement is obvious.

The group G is dense in $\hat{G}$ which is a finitely generated profinite group. Therefore, the statement on $\mathrm{Stab}(v)$ follows from the fact that it is a finite index subgroup. □

Lemma 5.7.16. *Let T be a spherically homogeneous rooted tree and let $G \leq Aut(T)$ be a finitely generated subgroup. Suppose that this action if spherically transitive. Let ξ be any ray in T and $M = \mathrm{Stab}_{\hat{G}}(\xi)$ be the corresponding parabolic subgroup in $\hat{G}$. Then there exists a partition of ∂T in $[N_{\hat{G}}(M) \colon M]$ subsets (A_λ) of size $[\hat{G} : N_{\hat{G}}(M)]$ such that two rays in the same A_λ have distinct $\hat{G}$-stabilizers and $\{\mathrm{Stab}_{\hat{G}}(\xi) \mid \xi \in A_\lambda\}$ is the same for all β.*

Proof. Let $N = N_{\hat{G}}(M)$. Since G acts spherically transitive on T, $\hat{G}$ acts transitively on ∂T — this action is not necessarily faithful.

It is evident that for this action, $\mathrm{Stab}_{\hat{G}}(\xi.g) = \mathrm{Stab}_{\hat{G}}(\xi)^g$ and the conclusion follows. □

Remark 5.7.17. With the hypothesis of last lemma, the action of $\hat{G}$ on ∂T is extremely nonfree if and only if $N_{\hat{G}}(M) = M$, that is if and only if all stabilizers are pairwise distinct.

Proposition 5.7.18. *Let T be a spherically homogeneous rooted tree and $G \leq Aut(T)$ be a finitely generated subgroup which is torsion. Suppose that the action of G is spherically transitive and there exists a ray ξ such that $H = \mathrm{Stab}_G(\xi)$ is wmc and $\bar{H} \leq \hat{G}$ is wmc. Then*

1. *All $\hat{G}$-parabolic subgroups are pairwise distinct and wmc;*
2. *All G-parabolic subgroups are distinct from H;*
3. *Suppose moreover that for all $g \in \hat{G}$ we have $\overline{\bar{H} \cap G^g} = \bar{H}$. Then all G parabolic subgroups are pairwise distinct.*

Proof. Let $M = \mathrm{Stab}_{\hat{G}}(\xi)$. This is an infinite index closed subgroup of $\hat{G}$ and we have $M \cap G = H$. By maximality of $\bar{H}$, we have $M = \bar{H}$. Since G is a torsion group, we have $H = N_G(H) = N_{\hat{G}}(M) \cap G$. Since $M \leq N_{\hat{G}}(M)$ is wmc, either $M = N_{\hat{G}}(M)$ or $N_{\hat{G}}(M)$ is of finite index in $\hat{G}$. But this last possibility would implies that $H = N_{\hat{G}}(M) \cap G$ is of finite index in G which is impossible. Thus $M = N_{\hat{G}}(M)$ and all $\hat{G}$-parabolic subgroups are wmc and pairwise distinct.

Now, let η be a ray. There exists $g \in \hat{G}$ such that $M_\eta \coloneqq \mathrm{Stab}_{\hat{G}}(\eta) = g\bar{M}g^{-1}$. Therefore, $H_\eta \coloneqq \mathrm{Stab}_G(\eta) = M_\eta \cap G = gMg^{-1} \cap G$.

Suppose that $H_\eta = H$. This implies that $\overline{gMg^{-1} \cap G} = \bar{H} = M$ and thus, $M \leq M^g$. By maximality, we have $M = M^g$ which implies that g fixes η.

Finally, observe that $\overline{\bar{H} \cap G^g} = \bar{H}$ for all g if and only if $\overline{\bar{H}^g \cap G} = \bar{H}^g$ for all g. Hence, we can apply the same argument as above to prove that G stabilizers are pairwise distinct. □

p-Groups acting on rooted trees

In the following, we will consider finitely generated residually finite p-group[1]. In this case, the pro-finite completion and the pro-p completion of G coincide.

Let G be a residually finite and $\hat{G}$ be its profinite completion. Suppose that $\hat{G}$ is virtually pro-p (for example G a p-group). Let H be a wmc subgroup of G. Then $\bar{H}$ is an infinite index subgroup of $\hat{G}$ and therefore contained in a wmc subgroup M of $\hat{G}$. Moreover, $M \cap G = H$. If $(M_i)_i$ is a descending chain for M, then $(H_i = M_i \cap G)_i$ is a descending chain for H. Since all the M_i are of finite index, we can choose the same transversal system for H_i and M_i. Therefore, the coset tree T_H and T_M are isomorphic and the action and the actions of $\hat{G}$ on T_H and on T_M are the same.

On the other hand, if $(H_i)_i$ is a descending chain for some wmc subgroup $H \leq G$, then $(\bar{H}_i = M_i)_i$ is a descending chain for $M \coloneqq \bigcap M_i$. This subgroup is of infinite index, but a priori not necessarily wmc.

Proposition 5.7.19. *Let G be a finitely generated residually finite p-group and $M < \hat{G}$ be a wmc subgroup such that $M \cap G$ is wmc. For the action of $\hat{G}$ on T_M, all parabolic subgroups are wmc and they are pairwise distinct.*

Proof. Let $H = M \cap G$. Since H is wmc and G is torsion, we have $H = N_G(H)$. On the other hand, $M \leq N_{\hat{G}}(M)$. By Lemma 5.7.9, $N_{\hat{G}}(M) \cap G \leq N_G(H) = H$ is of infinite index. Therefore, $M = N_{\hat{G}}(M)$ and we conclude by Lemma 5.7.16 and Remark 5.7.17. □

Theorem 5.7.20. *Le G be a residually finite just-infinite p-group and H be a wmc subgroup. Then there exists a coset tree T_H such that $\mathrm{Stab}_G(\bar{0}) = H$ and*

1. *All $\hat{G}$ parabolic subgroups are wmc and pairwise distinct.*
2. *Any two rays in the $N_{\hat{G}}(G)$-orbit of $\bar{0}$ have distinct stabilizers.*

Proof. We have $H = N_G(H)$ by Corollary 5.4.7. Since $N_{\hat{G}}(\bar{H}) \cap G = N_G(H)$, the subgroup $N_{\hat{G}}(\bar{H})$ is of infinite index in $\hat{G}$ and contained in some wmc M. We have $M \cap G = H$ (it is an infinite index closed subgroup containing H).

Now, for the tree $T_H = T_M$, $\mathrm{Stab}_G(\bar{0}) = H$ while $\mathrm{Stab}_{\hat{G}}(\bar{0}) = M$. By Proposition 5.7.19, all $\hat{G}$-parabolic subgroups are pairwise distinct and wmc. For all $g \in \hat{G}$, we have $\mathrm{Stab}_G(g\bar{0}) = \mathrm{Stab}_{\hat{G}}(g\bar{0}) \cap G = M^g \cap G$. If ξ and ζ are in both in the $N_{\hat{G}}(G)$-orbit of $\bar{0}$, then $\mathrm{Stab}_G(\xi) = H^g$ and $\mathrm{Stab}_G(\eta) = H^f$ for some g, f in $N_{\hat{G}}(G)$. If these subgroups are equal, we have $\bar{H}^f = \bar{H}^g$. Therefore gf^{-1} is in $N_{\hat{G}}(\bar{H}) \leq M$, but this implies that $\xi = \eta$. □

Remark 5.7.21. For $\mathcal{G}$ the grigorchuk group, $N_{\hat{\mathcal{G}}}(\mathcal{G})$ is countable. Indeed, Grigorchuk and Sidki showed that for the original action on T we have $N_{\mathrm{Aut}(T)}(\mathcal{G}) = \mathrm{Aut}(\mathcal{G})$ is countable and we obviously have $N_{\hat{\mathcal{G}}}(\mathcal{G}) \leq N_{\mathrm{Aut}(T)}(\mathcal{G})$.

Theorem 5.7.20 is mainly a result on actions of $\hat{G}$ while we would like a result on actions of G. This raises the following question.

Question 5.7.22. *Let $G \leq Aut(T)$ be a finitely generated spherically transitive group and such that all $\hat{G}$-parabolic subgroups are wmc and at least one G-parabolic subgroup is wmc. What can be said about G-parabolic subgroups? Are they pairwise distinct, wmc? If not, is this true for a subset of positive measure?*

[1] In fact, it is sufficient to take G finitely generated residually finite torsion group such such that $\hat{G}$ is virtually pro-p.

One possible direction to answer this question is to look at the restriction of the function $\Psi(M) = M \cap G$ to the set of conjugates of M. Is this function injective? And if not, how many preimage can have a subgroup $M^g \cap G$?

Another possibility is to look directly at $\mathrm{Stab}_G(\xi)$ and at the partion of ∂T by the subset $S_A := \{\xi \mid \mathrm{Stab}_G(\xi) = A\}$. We have the following proposition.

Proposition 5.7.23. *Let G be a residually finite just-infinite p-group, H a wmc subgroup and T_H the coset tree constructed in Theorem 5.7.20. Suppose that one of the following conditions is true.*

1. *All parabolic subgroups are wmc;*
2. *For all ξ and η in ∂T, if $\mathrm{Stab}_G(\xi) \leq \mathrm{Stab}_G(\eta)$, then they are equal;*
3. *For all $A \leq G$, the subset S_A is closed;*
4. *The group G is hereditarily just-infinite and there is no ray with trivial stabilizer;*
5. *For all $\xi \in \partial T$ and all $v \in \xi$, we have $\mathrm{Stab}_G(C_v) \subsetneq \mathrm{Stab}_G(\xi)$;*
6. *For all A and all v in T, there exists $w \leq v$, such that $C_w \subset \partial T \setminus S_A$.*

Then, there is an uncountable number of pairwise distinct parabolic subgroups.

Proof. First of all, for all A, S_A contains at most one ray in the G-orbit of $\bar{0}$. In particular, for all A and all $v \in T$, the cylinder subset C_v of ∂T is not contained in S_A. Indeed, by Theorem 5.7.20, all rays in the G-orbit of $\bar{0}$ have pairwise distinct stabilizers.

It is clear that Hypothesis 1 implies Hypothesis 2. If ξ_i are all in S_A and ξ_i converge to ξ, we have $A \leq \mathrm{Stab}_G(\xi)$. Hence, Hypothesis 2 implies Hypothesis 3.

Now, take A such that S_A is closed. Since S_A is closed, $S_A = \partial T \setminus \bigcup_{\lambda \in \Lambda} C_{w_\lambda}$. We already know that for any v we have $C_v \not\subset S_A$ and thus, there exists ξ in $C_v \setminus S_A$. Therefore, there exists $\lambda \in \Lambda$ such that ξ belongs to C_{w_λ}. Since ξ belongs to C_v and to C_{w_λ}, we have either $v \leq w_\lambda$ or $w_\lambda \leq v$. Let z be the smallest of the two vertices. Then $z \leq v$ and $C_z \subseteq \partial T \setminus S_A$. This shows that Hypothesis 3 implies Hypothesis 6.

Observe that $\mathrm{Stab}_G(C_v)$ is a normal subgroup of $\mathrm{Stab}_G(v)$, a finite index subgroup of G. Therefore, if G is herediteraly just-infinite, we have $\mathrm{Stab}_G(C_v) = \{1\}$ for all v. Hence, Hypothesis 4 implies Hypothesis 5.

For all ξ and all $v \in \xi$ we have $\mathrm{Stab}_G(C_v) \leq \mathrm{Stab}_G(\xi)$. Suppose that Hypothesis 6 does not hold. That implies that there exists A and v such that for all $w \leq v$ there exists ξ_w with $w \in \xi_w$ and $\xi_w \in S_A$. This implies that $\mathrm{Stab}_G(\xi_w) = A$ and that $\mathrm{Stab}_G(\xi_v) = A \leq \mathrm{Stab}_G(C_v)$. We just proved that Hypothesis 5 implies Hypothesis 6.

Finally, suppose that Hypothesis 6 holds and that there is only a countable number of pairwise distinct parabolic subgroups $(A_i)_{i \geq 0}$. We thus have that $\partial T = \bigcup_{i \geq 0} S_{A_i}$. Let v_0 be the root of the tree. By hypothesis, there exists $v_1 \leq v_0$ with $C_{v_1} \subset \partial T \setminus S_{A_1}$. There also exists $v_2 \leq v_1$ with $C_{v_2} \subset \partial T \setminus S_{A_2}$, and so on. Let $\xi = (v_1 v_2 v_2 \dots)$. By definition, ξ belongs to ∂T but does not belong to any S_{A_i}, which is absurd. ☐

In order to prove that all G parabolic subgroups are distinct from H, we want $\overline{\mathrm{Stab}_G(g\bar{0})} = \bar{H}^g$ (Theorem 5.7.20). But in general, $\overline{\mathrm{Stab}_G(g\bar{0})} = \overline{M^g \cap G} \geq \overline{\bar{H}^g \cap G}$ and $\overline{\bar{H}^g \cap G} \leq \bar{H}^g$.

Remark 5.7.24. Suppose that the following holds: if $M \not\subset M^g$ then $M \cap G \not\subset M^g \cap G$. Then all G parabolic subgroups are pairwise distinct.

5.8 Open questions and further research directions

In Section 5.6 we proved that in a regular branch group containing a finite subgroup that does not fix any rays, there is uncountably many distinct weakly maximal subgroups that are not parabolic subgroups of the original action. Each such subgroup is contained in a maximal subgroup. In the Grigorchuk group, all maximal subgroups are of infinite index, but this is not the case for example in some Šunić groups. Francoeur and Garrido [42] proved that every Šunić group with $p = 2$ such that it has an element of infinite order contains at least countably many maximal subgroups of infinite index, which are all finitely generated. For example this applies to the so-called Grigorchuk-Erschler group — the only self-similar group in the uncountable family of groups of intermediate growth constructed by Grigorchuk, beside the so-called "first Grigorhuk group".

Question 5.8.1. *Does all these groups have uncountably many maximal subgroups of infinite index?*

A possible approach to this question is to use techniques of Section 5.6. Indeed, given countably many subgroups B_i of infinite index, we are able to produce a new subgroups of infinite index A such that for each i there exists $g_i \in A \setminus B_i$. The subgroup A is the increasing union of subgroups A_i. To pass from A_i to A_{i+1} we added g_i and showed that A_{i+1} retains some "good properties". Such an A is contained in a weakly maximal subgroup which is itself contained in a maximal subgroup M, but a priori M could be of finite index. In order to force M to be of infinite index, it is sufficient to have A dense in the profinite topology. This can be done if for all i, for all finite index subgroup and all left coset gC there is an element $f \in gC$ such that $\langle A_i, f\rangle$ still have the desired "good properties".

In Section 5.7 we proved that given a wmc subgroup of G, the action of $\hat{G}$ on the coset tree has all parabolic subgroups wmc and distinct. Nevertheless, the original question remains.

Question 5.8.2. *Le G be a residually finite just-infinite p-group and H be a wmc subgroup. Then the action of G on the coset tree T_H is spherically transitive and* $\mathrm{Stab}_G(\bar{0}) = H$*.*

Is it true that all G-parabolic subgroups for this action are

1. *pairwise distinct?*

2. *wmc?*

Positive answer to the first question would imply the construction of new IRSs.

Partial results are given in Proposition 5.1.6 and Theorem 5.1.7. As discussed in Section 5.1, the following conjecture implies positive answers to Question 5.8.2.

Conjecture 5.8.3. *The two functions $H \mapsto \bar{H}$ and $M \mapsto M \cap G$ send wmc subgroups to wmc subgroups.*

It is thus of interest to study the behavior of the functions $H \mapsto \bar{H}$ and $M \mapsto M \cap G$, even in the specific case of the Grigorchuk group.

Chapter **6**

Weakly maximal subgroups of the Grigorchuk group

In Chapter 5 we have shown that there exist, in any finitely generated regular branch group, uncountably many weakly max subgroups different from parabolic ones. The question of their classification remains largely open. In this chapter we present some partial results towards such a classification on the example of Grigorchuk's group.

We begin with Section 6.1 which contain preliminary results on Schreier graphs of the Grigorchuk group. We also describe the lattice of all subgroups containing $B = \langle b \rangle^{\mathcal{G}}$, see Figure 6.10. These subgroups are of special interest since they naturally appear in the classification of weakly maximal subgroups by projections in Section 6.5.

In Section 6.2 we turn our attention to non-linear Schreier graphs of $\mathcal{G}$. In particular, we prove that there exist uncountably many weakly maximal subgroups with non-linear Schreier graphs. This contrasts the fact that all Schreier graphs of parabolic subgroup for the original action (which are weakly maximal) are linear, [118].

Section 6.3 studies a broad equivalence relation on weakly maximal subgroups. Namely, two subgroups are said to be tree equivalent if they arises as parabolic subgroups for the same action of $\mathcal{G}$ on a tree. In particular, parabolic subgroups off the original action are one tree equivalence class. We show that there is at least 4 distinct tree equivalence classes.

In Section 6.4 we discuss the case of finitely generated weakly maximal subgroup in $\mathcal{G}$. We start with a particular example due to Pervova that was presented in [49]. We then use it to construct infinitely many non-conjugated weakly maximal subgroups that we conjecture to be finitely generated.

Finally, in Section 6.5 we give a quick look at right and left projections of weakly maximal subgroups, following an idea of Grigorchuk. Parabolic subgroups of the original action are characterized by their projections. Therefore, a better understanding of the general case may lead to the beginning of a classification of weakly maximal subgroups of Γ.

6.1 Schreier graphs of $\mathcal{G}$

The aim of this section is to describe subgroups of $\mathcal{G}$ via their Schreier graph. This allows to describe the top of the lattice of subgroups of the Grigorchuk group and to exhibit weakly maximal subgroups with "non-linear" Schreier graph.

label of the edge	color of the edge
a	red
b	black
c	cyan
d	green

Table 6.1: Label of edges in Schreier graph of the Grigorchuk group.

Introduction

The description of the top of the lattice of normal subgroups was done by Ceccherini-Silberstein, Scarabotti and Tolli in [28] and by Bartholdi in [8]. In particular, there are seven subgroups of index 2, all of them normal and seven normal subgroups of index 4. Bartholdi also wrote a program in GAP that listes all subgroups of finite index by looking at finite quotient of $\mathcal{G}$.

In this section, we will describe all Schreier graphs and give the lattice of subgroups of index at most 4. We will also describe all subgroups containing $B = \langle b\rangle^{\mathcal{G}}$ since they naturally appears in the classification of weakly maximal subgroups. Finally, we will look at "non-linear" Schreier graphs of $\mathcal{G}$.

In the following, we will use the notation of [28] for normal subgroups. We will make extensive use of the following presentation of the Grigorchuk group due to Lysenok [89]. Let A be the set of words on $\{a, b, c, d\}$ such that one letter out of two is an a and the other belongs to $\{b, c, d\}$. Define $\sigma\colon A \to A$ by $\sigma(ww') = \sigma(w)\sigma(w')$ and

$$\sigma\colon a \mapsto aca \qquad b \mapsto d$$
$$c \mapsto b \qquad d \mapsto c.$$

Let $w_0 = ad$ and for all n, define $w_{n+1} = \sigma(w_n) = \sigma^n(ad)$. The the first Grigorchuk group admits the following presentation

$$\mathcal{G} = \left\langle a, b, c, d \;\middle|\; \begin{array}{c} a^2 = b^2 = c^2 = d^2 = bcd = 1 \\ \forall n \geq 0, (w_n)^4 = (w_n w_{n+1})^4 = 1 \end{array} \right\rangle$$

Notice that the relators in this presentation come naturally in two distinct groups:

$$a^2 = b^2 = c^2 = d^2 = bcd = 1 \tag{6.1}$$

$$\forall n \geq 0, (w_n)^4 = (w_n w_{n+1})^4 = 1. \tag{6.2}$$

Since $X = \{a, b, c, d\}$ contains only element of order 2, any Schreier graph of $\mathcal{G}$ is a 4-regular graph such that at every vertex there is exactly one edge with label x for all $x \in X$.

In the following, label edges will be drawn in colors, according to the Table 6.1. Recall that for any group $G = \langle X \mid R\rangle$ and subgroups H_1 and H_2, we have $H_1 = H_2$ if and only if the corresponding Schreier graphs are isomorphic as labeled rooted graphs. Moreover, H_1 and H_2 are conjugated if and only if their Scheier graphs are isomorphic as labeled graphs. We also have that $H_1 \leq H_2$ of and only if there is a X-covering sending root to root from the Schreier graph of H_1 to the one of H_2. Therefore, there is a X-covering from the Schreier graph of H_1 to the one of H_2 if and only if H_1 is a subgroup of a conjugate of H_2. A X-*graph* is a graph labeled by X such that for any vertex v and any $x \in X$, there is exactly one outgoing edge with label x. Recall also that a (rooted) X-graph is a Schreier graph of $\langle X \mid R\rangle$ if and only if, for all vertices v and all $r \in R$, the unique path with initial vertex v and label r has final vertex v.

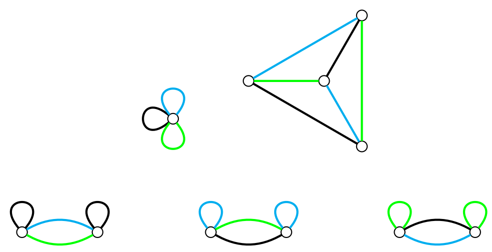


Figure 6.1: $\{b, c, d\}$-components arising in Schreier graphs of the Grigorchuk group: the rose, the tetrahedron and 3 "mickeys" Black edges are labeled by b, cyan by c and green by d.

For a X-graph (V, E) and $T \subseteq X$, the T-*components* of (V, E) are the connected components of the subgraph of (V, E) consisting of all edges labeled by elements of T.

Now, let us have a look at the relators of (6.1). They exactly state that $\langle b, c, d\rangle$ is the Vierergruppe. If (V, E) a Schreier graph of $\mathcal{G}$, then this implies that each $\{b, c, d\}$-component is isomorphic to a quotient of the Cayley graph of the Vierergruppe with generating set $\{b, c, d\}$, see Figure 6.1. This gives us a total of $1+3+1 = 5$ possibilities. On the other hand, $(ad)^4 = 1$, and therefore $\{a, d\}$-components are quotient to the octogone, see Figure 6.2.

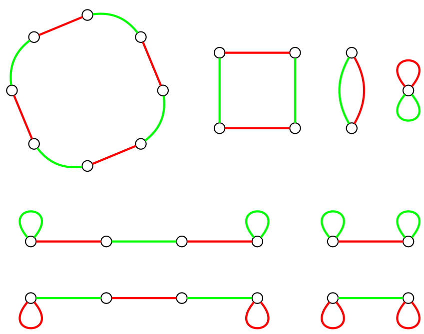


Figure 6.2: $\{a, d\}$-components arising in Schreier graphs of the Grigorchuk group.

The classification of Schreier graphs of finite index subgroups of $\mathcal{G}$ can be done by looking at 2-covering. Indeed, by Lemma 5.2.2, if Γ is a Schreier graph of a finite index proper subgroup of $\mathcal{G}$, then it is a double strong cover of another Schreier graph of $\mathcal{G}$. By Relations 6.1, Schreier graphs corresponding to subgroup of index 2^i can be parametrized by triple of involutions (p_a, p_b, p_c) of the symmetric group Sym_{2^i}, where p_a is the involution given by $p_a(v) = v.a$, where $v \in \{1, \dots, 2^i\}$ is view as a vertex of the Schreier graph. Given such a triple, it is easy to find all double strong covers of the corresponding graph. It then remains to keep only the one that are connected and to test if they indeed correspond to subgroups of $\mathcal{G}$. Since everything happen in a finite quotient of $\mathcal{G}$, this can be done by a finite algorithme.

Before looking at subgroups of index 4, let us draw the graphs of subgroups of index at most 2, see Figure 6.3.

In the notation of [28], the first line of figure 6.3 corresponds (from the left to the right) to $\mathcal{G}$ and subgroups $J_{0,1}$ to $J_{0,4}$, the second line to $J_{1,2} = H$ (the stabilizer of the first level of the 2 regular rooted tree) and subgroups $J_{0,5}$ to $J_{0,7}$.

Recall that there is a $1-1$ correspondence between Schreier graphs of $G = \langle X\rangle$ and conjugacy classes of subgroups. Under this correspondence, the number of conjugates of

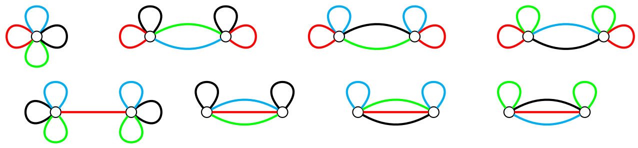


Figure 6.3: Schreier graphs of subgroups of index at most 2. Edges in red are labeled by a, in black by b, in cyan by c and in green by d.

A is the number of orbits in $\mathrm{Sch}(G, A, X^{\pm})$ under its group of strong automorphisms (i.e. label preserving automorphisms) and changing the root in the Schreier graph corresponds to conjugating A. We also have that $A \leq B$ if and only if $\mathrm{Sch}(G, A, X^{\pm})$ covers $\mathrm{Sch}(G, B, X^{\pm})$, where the covering preserves the labeling and the root. Finally, generators of A can be read on the Schreier graph. Therefore, Schreier graphs encode all the information about the lattice of subgroups and their generators.

Subgroups of index 4

For index 4 subgroups, the corresponding Schreier graph has 4 vertices and we can do the classification by hands. The following technical proposition will be useful in order to classify $\{a, b, c, d\}$-graphs over 4 vertices that are Schreier graphs of $\mathcal{G}$.

Proposition 6.1.1. *Let $X = \{a_1, \dots a_n\}$ be a finite alphabet (we allow $a_i^2 = 1$ for some i's). Let (V, E) be a X-graph on 4 vertices such that under its group of X-automorphisms, there is either only one orbit, or two orbits consisting each of them of two vertices. Then for all $w \in X^*$, p_{w^4}, the permutation induced on V by w^4 is trivial.*

Proof. First, observe that for any X-automorphism α of the graph, if $p_w : i \mapsto j$, then $p_w : \alpha(i) \mapsto \alpha(j)$.

If the graph is not connected, then it consist of two X-transitive graphs on 2 vertices each. Since Sym_2 has only elements of order 1 or 2, the proposition is trivially true. Therefore, we may assume (V, E) to be connected.

Now, if there is 2 orbits of 2 vertices each, there is an X-automorphism of order 2. If there is only 1 orbit, the groups of X-automorphism is a transitive subgroup of Sym_4 and thus also have an element of order 2. Let $\bar{}$ be this X-automorphism of order 2. This allows us to label the vertices of (V, E) by $0, \bar{0}, 1, \bar{1}$. Indeed, if the vertices are $\{0, 1, 2, 3\}$ we may assume, up to relabeling that $\bar{0} = 3$. This implies that either $\bar{1} = 2$ or $\bar{1} = 1$. Assume fo the sake of contradiction that $\bar{1} = 1$. This implies that $\bar{2} = 2$. Now, if there is an edge from 1 to 0, say labeled by l, then there is an edge labeled by l from $\bar{1} = 1$ to $\bar{0} \neq 0$. This is not possible. Repeating the same argument shows that 1 and 2 do not belong to the connected component of $\{0, \bar{0}\}$, which is absurd.

Finally, take $w \in X^*$ and $x \in \{0, \bar{0}, 1, \bar{1}\}$ If $p_w(x) = x$ or $p_w(x) = \bar{x}$, then ${p_w}^4(x) = {p_w}^2(x) = x$. Otherwise, $p_w(x) = y$ with $y \neq x, \bar{x}$. This implies that $p_w(\bar{x}) = \bar{y}$ and therefore that $p_w(y) \in \{x, \bar{x}\}$. In both case, we have $p_{w^4}(x) = x$. □

We now look at all 4 vertices X-graphs such that every path labeled by a relation from (6.1) is closed. Such graphs are exactly the one such that every $\{b, c, d\}$ connected component is one of the 5 graphs in figure 6.1. A quick investigation gives us 16 possible graphs. Among these 16 graphs, 7 are X-transitive. They correspond to normal subgroup of index 4, see Figure 6.3. The 9 remaining graphs split into 3 family consisting of 3 graphs each, see Figure

6.4. Such a family consist of graphs that can be obtained one from another by changing the labeling according to any permutation of $\{b, c, d\}$. The graphs of the first two families have 2 X-orbits, while the graph in the third family have 4 X-orbits.

Proposition 6.1.2. *All the* 12 *graphs defined by Figure 6.4 are Schreier graphs of the Grigorchuk group.*

Proof. For all these graphs, the paths labeled by relators in (6.1) are closed. The 9 graphs defined by the first three diagrams satisfy the hypothesis of the Proposition 6.1.1. Therefore, for these graphs, the paths labeled by relators in (6.2) are also closed.

Finally, we look at the last 3 graphs, corresponding to the last diagram in Figure 6.4. These graphs have 4 X-orbits. But we know that any letter in $\{b, c, d\}$ is a product of the two others. Therefore, for any $x \in \{b, c, d\}$, a $\{a, b, c, d\}$-graph is a Schreier graph of $\mathcal{G}$ if and only if the graph obtained by deleting all edges with label x is a Schreier graph of $\mathcal{G}$. In our case, if we delete gray edges of (V, E), we obtain a graph (V, E') satisfying the hypothesis of Proposition 6.1.1. This implies that (V, E') is a Schreier graph of $\mathcal{G}$ (paths labeled by relators in (6.1) and (6.2) are closed) and hence that (V, E) is itself a Schreier graph of $\mathcal{G}$. ☐

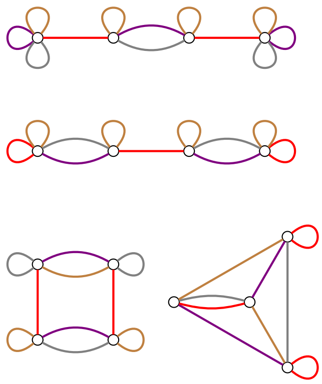


Figure 6.4: Schreir graphs of non-normal subgroups of index 4. Edges in red are labeled by a. Violet, gray and brown edges correspond to any choice of 3 distinct letters in $\{b, c, d\}$.

As a corollary, using relation true in $\mathcal{G}$ to reduce the number of generators, we obtain

Proposition 6.1.3. *There is* 37 *subgroups of index* 4 *of the Grigorchuk group, which belong to* 19 *conjugacy classes. More precisely, there is* 7 *normal subgroups,* 9 *conjugacy classes of size* 2 *and* 3 *conjugacy classes of size* 4*. They are all listed in Table 6.2.*

The corresponding Schreier graphs are depicted in Figures 6.5 to 6.9. Where we adopted the following notation. A subgroup of finite index of $\mathcal{G}$ is be denoted by $S_{i,j,k,l}$ with the following conventions. The index i will denote the “level” of this subgroup, that is $S_{i,j,k,l}$ is of index 2^i. The index j will classify class of $\{a, b, c, d\}$-length-isomorphic subgroups (i.e. subgroups with isomorphic unlabeled Schreier graphs on generating set $\{a, b, c, d\}$), the index k class of conjugated subgroups (i.e. subgroups with isomorphic labeled Schreier graphs) and the index l distinct subgroups in the same conjugation class. We will omit the last index for normal subgroups and the third index if there is only one subgroup in the length-isomorphic class. By convention, for each level we will firstly list the normal subgroups and then the non-normal ones.

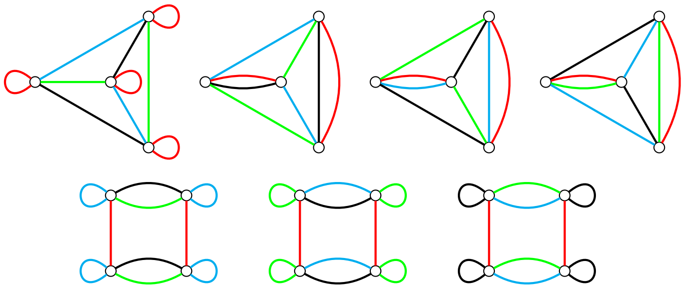


Figure 6.5: Schreir graphs of normal subgroups of index 4. From left to right we have (in the notation from [28]), $J_{0,8}$ to $J_{0,11}$ on the first line and $J_{1,3}$, $J_{1,9}$ and $J_{1,5}$ on the second line.

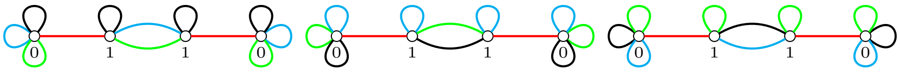


Figure 6.6: From left to right, the subgroups $S_{2,3,0,l}$ to $S_{2,3,2,l}$ with $0 \leq l \leq 1$ depending on the choice of the root.

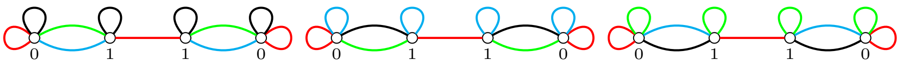


Figure 6.7: From left to right, the subgroups $S_{2,4,0,l}$ to $S_{2,4,2,l}$ with $0 \leq l \leq 1$ depending on the choice of the root.

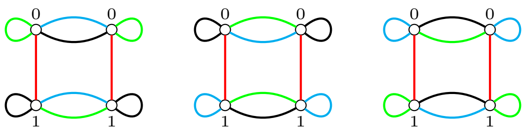


Figure 6.8: From left to right, the subgroups $S_{2,5,0,l}$ to $S_{2,5,2,l}$ with $0 \leq l \leq 1$ depending on the choice of the root.

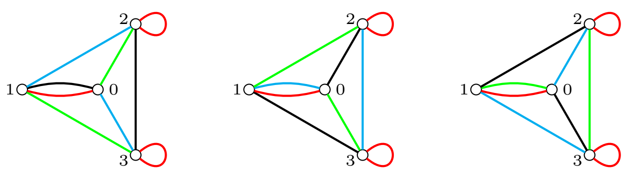


Figure 6.9: From left to right, the subgroups $S_{2,6,0,l}$ to $S_{2,6,2,l}$ with $0 \leq l \leq 3$ depending on the choice of the root.

Subgroup	Generators	Transversal System
$S_{2,0}$	a, a^b, a^c, a^d	$\{1\}$
$S_{2,1,1}$	ba, dac	$\{1\}$
$S_{2,1,2}$	ca, dab	$\{1\}$
$S_{2,1,3}$	da, bac	$\{1\}$
$S_{2,2,1}$	c, c^a, bb^a	$\{1\}$
$S_{2,2,2}$	d, d^a, cc^a	$\{1\}$
$S_{2,2,3}$	b, b^a, dd^a	$\{1\}$
$S_{2,3,0,0}$	$b, c, b^a, b^{aca}, c^{aca}$	$\{1, a\}$
$S_{2,3,1,0}$	$c, d, c^a, c^{ada}, d^{ada}$	$\{1, a\}$
$S_{2,3,2,0}$	$d, b, d^a, d^{aba}, b^{aba}$	$\{1, a\}$
$S_{2,4,0,0}$	$a, b, b^d, b^{da}, a^{dad}$	$\{1, d\}$
$S_{2,4,1,0}$	$a, c, c^b, c^{ba}, a^{bab}$	$\{1, b\}$
$S_{2,4,2,0}$	$a, d, d^c, d^{ca}, a^{cac}$	$\{1, c\}$
$S_{2,5,0,0}$	$d, (ca)^2, b^{ca}$	$\{1, a\}$
$S_{2,5,1,0}$	$b, (da)^2, c^{da}$	$\{1, a\}$
$S_{2,5,2,0}$	$c, (ba)^2, d^{ba}$	$\{1, a\}$
$S_{2,6,0,0}$	ab, a^d, a^c	$\{1, a, b, c\}$
$S_{2,6,1,0}$	ac, a^b, a^d	$\{1, a, c, d\}$
$S_{2,6,2,0}$	ad, a^c, a^b	$\{1, a, b, d\}$

Table 6.2: The 37 subgroups of index 4 of $\mathcal{G}$ and their generators.

We want to explore the lattice of subgroups of index at most 4 and of subgroups containing $B = \langle b \rangle^{\mathcal{G}}$. For the lattice of all normal subgroups up to index 2^8, see for example [28]. Therefore, in order to describe the lattice of subgroups of index at most 4 it only remains to describe in which subgroups of index 2 live the non-normal subgroups of index 4.

For each subgroup $S_{2,3,k,l}$, its Schreier graph is a strong cover of degree 2 of only one graph. The resulting graph is obtained by identifying vertices of "type" 0 together and vertices of "type" 1 together. This is exactly the Schreier graph of H.

For each subgroup $S_{2,4,k,l}$, its Schreier graph is a strong cover of degree 2 of only one graph. The resulting graph is obtained by identifying vertices of "type" 0 together and vertices of "type" 1 together. This is exactly the Schreier graph of $S_{1,0,k}$.

For each subgroup $S_{2,5,k,l}$, its Schreier graph is a strong cover of degree 2 of only one graph. As before, the resulting graph is obtained by identifying vertices of "type" 0 together and vertices of "type" 1 together and correspond to the Schreier graph of H.

For each subgroup $S_{2,6,k,l}$, its Schreier graph is a strong cover of degree 2 of only one graph. Indeed, the vertex 2 can only be identified with the vertex 3 (the only other vertex with a loop labeled by a). The resulting graph is a Schreier graph of $S_{1,0,k}$.

We hence have that H contains 12 non-normal subgroups of index 4 and for each $0 \leq k \leq 2$, the subgroup $S_{1,0,k}$ contains 6 non-normal subgroups. The 3 remaining subgroups of index 2 do not contain non-normal subgroups of index 4. Finally, we turn our attention to one particular subgroup of index 8, namely $B = \langle b \rangle^{\mathcal{G}}$. Since B is a normal subgroup containing b, there is a loop labeled by b at each vertex of its Schreier graph. Since the $\{a, b, d\}$-component of the Schreier graph of B is connected, its $\{a, d\}$ is also connected and is therefore an octogone. This gives us the graph at the bottom of Figure 6.10.

Since there is a loop labeled by b at each vertex of its Schreier graph, any subgroup A containing B has also a loop labeled by b at each vertex of its Schreier graph. This implies that A is either $S_{2,2,3} = J_{1,5}$, $S_{2,3,0,l}$ or $S_{2,4,0,l}$, $l \in \{0,1\}$. An easy verification shows that the Schreier graph of B indeed strongly covers the Schreier graphs of $S_{2,3,0}$ and $S_{2,4,0}$, and this was already known for the others subgroups, see [28]. Therefore, there is exactly 10 subgroups containing B, of which 5 are of index 4. The lattice of Schreier graphs of $\mathcal{G}$ covered by the Schreier graph corresponding to B is shown in Figure 6.10.

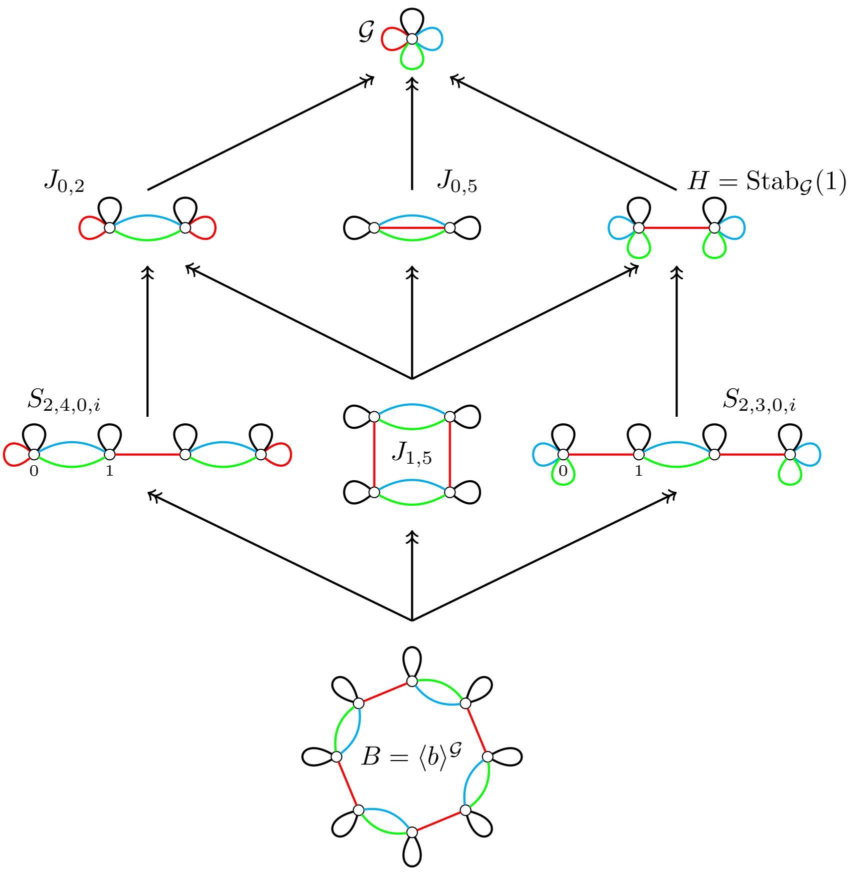


Figure 6.10: The lattice of Schreier graphs of $\mathcal{G}$ that are covered by the Schreier graph of B. Two of these graphs have two orbits under the group of X-automorphisms. Therefore, each of them corresponds to two (conjugated) subgroups of $\mathcal{G}$.

6.2 Non-linear subgroups

In this section, we focus on subgroups with *non-linear* Schreier graphs, where a graph is *linear* if after erasing loops and identifying multiple edges it is isomorphic to a (possibly infinite or bi-infinite) path. Parabolic subgroups for the original branch action of $\mathcal{G}$ are weakly maximal and their Schreier graphs are know to be linear, [118]. There is uncountably many non-strongly isomorphic such graphs, but if we forget the labelling, there is only two isomorphism classes of such graphs. We will prove that there exist uncountably many (non-strongly isomorphic) non-linear Schreier graphs corresponding to weakly maximal subgroups.

For the relevant definitions, see Section 6.1.

Our conclusion will follows from the fact that in $\mathcal{G}$ the following relations holds

$$a^2 = b^2 = c^2 = d^2 = 1 \tag{6.3}$$
$$bcd = 1 \tag{6.4}$$
$$(ad)^4 = (ac)^8 = 1 \tag{6.5}$$

In the following, G will be any group that is a quotient of

$$\langle a, b, c, d \mid a^2, b^2, c^2, d^2, bcd, (ad)^4, (ac)^8 \rangle \tag{6.6}$$

with generating set $X = \{a, b, c, d\}$. We will list all possible T-components for $T = \{b, c, d\}$, $\{a, d\}$ or $\{a, c\}$. Observe that since G is a quotient of the group given by Presentation (6.6), some listed T-components may actually never appear in Schreier graphs of G.

Let us have a look at the relators of (6.3) and (6.4). They exactly state that $\langle b, c, d\rangle$ is a quotient of the Vierergruppe $V_4 = (\mathbf{Z}/2\mathbf{Z})^2$. If (V, E) a a Schreier graph of G, then the Relations (6.3) and (6.4) implies that each $\{b, c, d\}$-component is isomorphic to a quotient of the Cayley graph of $\{b, c, d\}$, see Figure 6.1. This gives us a total of $1+3+1 = 5$ possibilities: a rose, a tetrahedron and three "bigons".

Now, for the $\{a, d\}$-components of Schreier graphs of G, we have that $(ad)^4 = 1$. Therefore, they are all quotients of an octagon, leaving us with eight possibilities, see Figure 6.2.

On the other hand, we have $(ac)^8 = 1$ which implies that in Schreier graphs of G, $\{a, c\}$-components are quotients of a 16-gone. See Figure 6.11 for the list of possible $\{a, c\}$-components with a loop labeled by a.

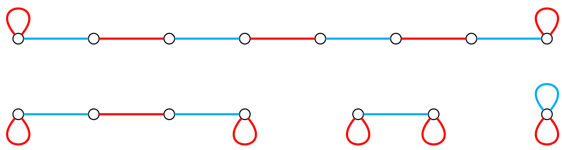


Figure 6.11: $\{a, c\}$-components with a a-loop that may arise in Schreier graphs of G.

Proposition 6.2.1. *Let $A \le G$ be a subgroup that contains a conjugate of a. Then either the corresponding Schreier graph has at least one $\{b, c, d\}$-component which is a tetrahedron, or A has index 2^i for some $0 \le i \le 3$.*

More precisely, if the index of A does not divide 8, then each a-loop is at distance 0, 2, 4 or 6 from a tetrahedron. On the other hand, if the index of A divides 8, then the corresponding Schreier graph is linear, except for A one of the four conjugates of $\langle a, d, d^c, d^{(ca)^3}, d^{(ca)^3c}, a^{(ca)^3c}, (bac)^{ca}, (da)^{cac}\rangle$.

Proof. The subgroup A contains a conjugate of a if and only if the corresponding Schreier graphs as a loop labeled by a. There is only 3 choices for the corresponding $\{a, d\}$-component and 4 choices for the $\{a, c\}$-component. We now assume that the graph has no tetrahedron. Therefore, there is only 4 possibilities for the $\{b, c, d\}$-components.

If both the $\{a, d\}$-component and the $\{a, c\}$-component are rose with 2 petals, then $A \ge \langle a, d, c\rangle = G$.

If the $\{a, c\}$-component is not a rose with 2 petals, it is a path of length 1, 3 or 5. In each case, gluing $\{b, c, d\}$-components on it does not add more vertex and we obtain the full Schreier graph. In this case, A is of index 2, 4 or 8.

We proved that each a-loop is in a connected component of at most 8 vertices, or at distance 0, 2, 4 or 6 from a tetrahedron.

Now, if the index of A divides 8, a quick check of possible components give us the graphs of Figures 6.12 and 6.13 as only possibilities. Except one, all these graphs are linear. □

If G is a 2-group (for example, the Grigorchuk group), then "the index of A divides 8" is equivalent to "A is of index at most 8".

Corollary 6.2.2. *There exists uncountably many weakly maximal subgroups $H \leq \mathcal{G}$ with non strongly-isomorphic Schreier graphs such that the graphs are non-linear.*

Proof. By Theorem 5.1.1, there exist uncountably many weakly maximal subgroups of $\mathcal{G}$ containing a. For all this subgroups, the corresponding Schreier graph contains a tetrahedron and is therefore non-linear. Since $\mathcal{G}$ is countable, there are uncountably many non strongly isomorphic such graphs. □

Remark 6.2.3. It is possible to list all subgroups A that contains a conjugate of a such that the Schreier graph does not have a tetrahedron. Indeed, the proof of Proposition 6.2.1 give us the list of all candidates, see Figure 6.12 It then remains to show that they indeed correspond to subgroups of G. Table 6.3 give the list of all these subgroups that contains a for the group $\langle a,b,c,d \mid a^2,b^2,c^2,d^2,abc,(ad)^4,(ac)^8\rangle$. Observe that all subgroups of index at most 4 are subgroups of $\mathcal{G}$, the first Grigorchuk group.

Subgroup	index	# of conjugates
G	1	1
$\langle a,b,a^c\rangle$	2	1
$\langle a,c,a^d\rangle$	2	1
$\langle a,d,a^b\rangle$	2	1
$\langle a,b,b^d,b^{da},a^{dad}\rangle$	4	2
$\langle a,c,c^b,c^{ba},a^{bab}\rangle$	4	2
$\langle a,d,d^c,d^{ca},a^{cac}\rangle$	4	2
$\langle a,d,d^c,(dc)^{ca},d^{(ca)^2},d^{(ca)^2c},d^{(ca)^3},d^{(ca)^3c},a^{(ca)^3c}\rangle$	8	8
$\langle a,d,d^c,d^{ca},d^{cac},d^{(ca)^2},d^{(ca)^2c},d^{(ca)^3},d^{(ca)^3c},a^{(ca)^3c}\rangle$	8	4

Table 6.3: Subgroups of $G = \langle a,b,c,d \mid a^2,b^2,c^2,d^2,abc,(ad)^4,(ac)^8\rangle$ that contain a a and do not have tetrahedron in the Schreier graph.

Corollary 6.2.2 shows that Schreier graphs of weakly maximal subgroups of $\mathcal{G}$ may be non-linear, contrasting with the situation for parabolic subgroups. This is done by showing that these graphs have one tetrahedron. Nevertheless, this results does not say anything about the large-scale geometry of such graphs. In particular, it is unclear if such graphs are always quasi-isometric to either $\mathbf{N}$ or $\mathbf{Z}$. A better study of these graphs would lead to a better understanding of how far away from parabolic subgroups are general weakly maximal subgroups. It may also be possible to classify weakly maximal subgroups in term of the geometry of their Schreier graphs.

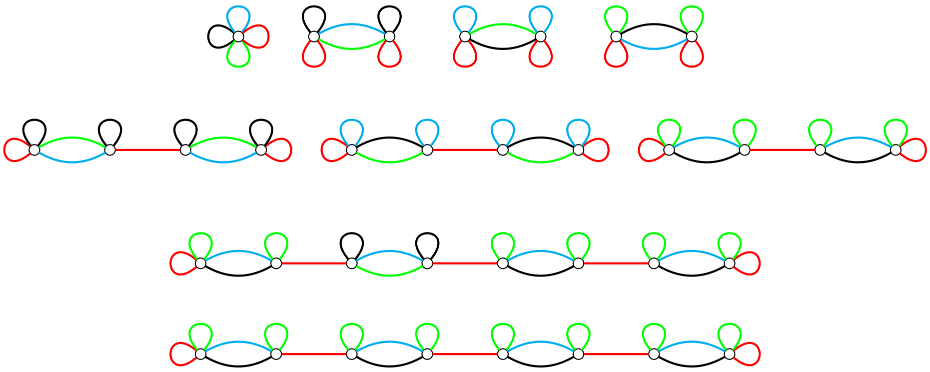


Figure 6.12: Schreier graphs of $G = \langle a, b, c, d \mid a^2, b^2, c^2, d^2, abc, (ad)^4, (ac)^8\rangle$ that contain an a-loop and are linear.

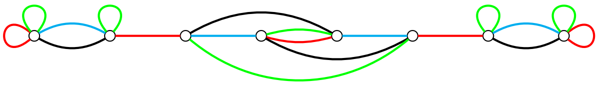


Figure 6.13: Schreier graph of $\langle a, d, d^c, d^{(ca)^3}, d^{(ca)^3c}, a^{(ca)^3c}, (bac)^{ca}, (da)^{cac}\rangle$.

6.3 Counting weakly maximal subgroups in $\mathcal{G}$

We proved in Chapter 5 that in $\mathcal{G}$ there are uncountably many weakly maximal subgroups containing a given finite subgroup Q. Since $\mathcal{G}$ is finitely generated, there are uncountably many such subgroups, up to conjugacy, and even up to $\mathrm{Aut}(\mathcal{G})$ equivalence. Nevertheless, all parabolic subgroups for the original action come together even if they are not $\mathrm{Aut}(\mathcal{G})$-equivalent. This motivates the definition of a broader equivalence relation on the set of subgroups.

Definition 6.3.1. Let G be a group endowed with the profinite topology. Two closed subgroups A and B are tree equivalent if there exists a coset tree T_A with B a parabolic subgroup for the G action on T_A.

Observe, that this is an equivalence relation. Indeed, if $B = \mathrm{Stab}_G(x_1x_2\dots)$ is a parabolic subgroup, then $B = \bigcap_{i\geq 1} \mathrm{Stab}_G(x_1x_2\dots x_i)$ and thus T_A is a coset tree for B. This implies that $\sim$ is symmetric. On the other hand, if $A \sim B$ and $B \sim C$, then A and C are both parabolic subgroups of $T_B = T_C$.

If $G \leq \mathrm{Aut}(T)$ is branch, then all parabolic subgroups (of this branch action) are weakly maximal and tree equivalent. For finitely generated residually finite p-group, tree equivalence roughly corresponds to conjugation in $\hat{G}$.

Lemma 6.3.2. *Let G be a finitely generated residually finite p-group and A and B two closed subgroups for the profinite topology. Then A and B are tree equivalent if and only if there exists M a wmc subgroup of $\hat{G}$ containing A and $g \in \hat{G}$ such that $B = M^g \cap G$.*

Proof. Suppose that A and B are tree equivalent. The tree T_A corresponds to $\bigcap A_i$. Therefore, it is isomorphic to the tree T_M corresponding to $M = \bigcap \bar{A}_i$, see the discussion before Proposition 5.7.19. In particular, we have $A = \mathrm{Stab}_G(\xi)$ and $M = \mathrm{Stab}_{\hat{G}}(\xi)$. On the other hand, $B = \mathrm{Stab}_G(\eta) = \mathrm{Stab}_{\hat{G}}(\eta) \cap G$. Since $\hat{G}$ acts transitively on ∂T_M, there exists $g \in \hat{G}$ such that $\mathrm{Stab}_{\hat{G}}(\eta) = M^g$.

On the other hand, if $B = M^g \cap G$, then we can construct $T_M \simeq T_A$ which gives us the tree equivalence. □

It may be interesting to classify weakly maximal subgroups up to tree equivalence. This raises the following question.

Question 6.3.3. *How many tree equivalence classes of weakly maximal subgroups has $\mathcal{G}$?*

It is possible that such a class is uncountable (for example, for the original action, the class as the cardinality of the continuum), and therefore the number of such classes is a priori smaller than the number of classes under automorphism equivalence.

Lemma 6.3.4. *If $A \sim B$ are equivalent by a coset tree T_A corresponding to $A_0 > A_1 > \dots$, then they are equivalent by T'_A corresponding to $G > A'_1 > A'_2 > \dots$ where each A'_{i+1} is maximal in A_i.*

Proof. Refining the sequence $G = A_0 > A_1 > \dots$ does not change the set of rays and thus does not change the set of parabolic subgroups. □

Proposition 6.3.5. *The Grigorchuk group $\mathcal{G}$ has at least 4 tree equivalence classes of weakly maximal subgroups.*

Proof. In this special case, all maximal subgroups are of index two and hence normal. Now, take $A_1 = \langle b, c\rangle$, $A_2 = \langle a, b\rangle$, $A_3 = \langle a, c\rangle$ and $A_4 = \langle a, d\rangle$. These subgroups are finite. In fact, they are respectively isomorphic to the dihedral group of order 4, 16, 8 and 32 [65]. Therefore, each A_i is contained in a weakly maximal subgroups A'_i. Since $\mathcal{G} = \langle a, x, y\rangle$ for any choice of $x \neq y \in \{b, c, d\}$, all the A'_i are pairwise distinct.

It is easy to see that for each i, there exists only one index 2 subgroup $\tilde{A}_i$ of $\mathcal{G}$ containing A_i and that all the $\tilde{A}_i$ are pairwise distinct. It can for example be done by looking at the Schreier graphs of maximal subgroups, see Figure 6.3. Therefore, if $i \neq j$, the subgroups A'_i and A'_j are not tree equivalent. Indeed, if it was the case, we would have A'_j a parabolic subgroup of a coset tree $T_{A'_i}$. By Lemma 6.3.4, we could suppose that the stabilizer of any first level vertex in $T_{A'_i}$ is $\tilde{A}_i$. This implies that $A'_j \leq \tilde{A}_i$ which is absurd. □

Observe that with the same kind of proof we may hope to find up to 7 tree equivalence classes of weakly maximal subgroups, one for each maximal subgroup. Indeed, in order to have n different classes, it is sufficient to find n weakly maximal subgroups such that each of them is contained in a unique maximal subgroup (and that all these maximal subgroups are pairwise distinct).

It is natural to ask the following

Question 6.3.6. *Does there exist uncountably many tree equivalence classes of subgroups?*

6.4 Finitely generated weakly maximal subgroups of $\mathcal{G}$

In this section we investigate a particular example, due to Pervova, of a weakly maximal subgroup of the Grigorchuk group. The first published proof of the weak maximality of this example can be found in [49]. After that, we turn our attention to finitely generated weakly maximal subgroups.

Recall that $\mathcal{G} = \langle a,b,c,d\rangle$ is the first Grigorchuk group. This group is branch over $K := \langle (ab)^2\rangle^{\mathcal{G}}$ and we have[1] $K <_2 B := \langle b\rangle^G <_2 \tilde{B} := \langle B, (ad)^2\rangle <_4 \mathcal{G}$, where all this subgroups are normal. Another important subgroup of $\mathcal{G}$ is $H := \mathrm{Stab}_{\mathcal{G}}(1) <_2 \mathcal{G}$, the stabilizer of the first level.

The left and right projections are denoted by $\pi_0, \pi_1 \colon G \to G$.

Pervova's example

Lemma 6.4.1. *Let $1 \neq x \in \mathrm{Stab}_{\mathcal{G}}(1)$. Then $\langle x\rangle^{\tilde{B}}$ has infinite index in $\mathcal{G}$ if and only if $\pi_i(x) = 1$ for some $i \in \{0,1\}$.*

Proof. If $\pi_i(x) = 1$, then $\langle x\rangle^{\tilde{B}} \leq \{1\}\times\mathcal{G}$ (or $\langle x\rangle^{\tilde{B}} \leq \mathcal{G}\times\{1\}$) and thus cannot contain $\mathrm{Stab}_{\mathcal{G}}(n)$ for any n. Therefore, if $\pi_i(x) = 1$, then $\langle x\rangle^{\tilde{B}}$ has infinite index in $\mathcal{G}$ by the congruence subgroup property.

On the other hand, suppose that $x = (x_0, x_1)$ with $x_i \neq 1$. In this case, by [109], the centralizer $C_{\mathcal{G}}(x_i)$ has infinite index in $\mathcal{G}$. This implies that there exists $y_i \in K$, $i \in \{0,1\}$ such that y_i does not belongs to the centralizer of x_i. We then have $[x_i, y_i] \neq 1$ and thus $[x,(1,y_1)] = (1,[x_1,y_1])$ and $[x,(y_0,1)] = ([x_0,y_0],1)$ with both $[x_i,y_i]$ belonging to K. Since $\pi_i(\tilde{B}) = \mathcal{G}$, we have $\langle x\rangle^{\tilde{B}} \geq \langle [x_0,y_0]\rangle^{\mathcal{G}} \times \langle [x_1,y_1]\rangle^{\mathcal{G}}$. Both $\langle [x_i,y_i]\rangle^{\mathcal{G}}$ are non-trivial normal subgroups of $\mathcal{G}$ and therefore of finite index since $\mathcal{G}$ is just-infinite. This shows that $\langle x\rangle^{\tilde{B}}$ itself is of finite index in $\mathcal{G}$. □

Conjecture 6.4.2. *Let $1 \neq x$ be any element of $\hat{\mathcal{G}}$. Then $[\mathcal{G} : C_{\mathcal{G}}(x)] = \infty$.*

Observe that the above conjecture is true if x belongs to $\mathcal{G}$. In general, this implies that for every $1 \neq x \in \hat{\mathcal{G}}$, there exists $y \in \bar{K}$ such that $[x,y] \neq 1$. Indeed, $[\hat{\mathcal{G}} : C_{\hat{\mathcal{G}}}(x)] = \infty$ if and only if $[\mathcal{G} : C_{\hat{\mathcal{G}}}(x) \cap \mathcal{G}] = [\mathcal{G} : C_{\mathcal{G}}(x)] = \infty$. In this case, the finite index subgroup $\bar{K}$ cannot be contained in $C_{\hat{\mathcal{G}}}(x)$.

If this conjecture is true, then as a direct corollary we have.

Conjecture 6.4.3. *Let $1 \neq x \in \mathrm{Stab}_{\hat{\mathcal{G}}}(1)$. Then $\overline{\langle x\rangle^{\tilde{B}}}$ has infinite index in $\hat{\mathcal{G}}$ if and only if $\pi_i(x) = 1$ for some $i \in \{0,1\}$.*

We now turn our attention on the following example of weakly maximal subgroup due to Pervova:

$$W := \langle a, \mathrm{diag}(\tilde{B}\times\tilde{B}), \{1\}\times K\times\{1\}\times K\rangle$$

where $\mathrm{diag}(\tilde{B}\times\tilde{B}) = \{(x,x) \mid x \in \tilde{B}\} \leq \mathcal{G}\times\mathcal{G}$ is the diagonal subgroup.

Lemma 6.4.4. *The group W is a subgroup of $\mathcal{G}$.*

Proof. By definition a belongs to $\mathcal{G}$ and it is well known that $\{1\}\times K\times\{1\}\times K$ is a subgroup of $\mathcal{G}$. Therefore, it remains to check that $\mathrm{diag}\,\tilde{B}\times\tilde{B}$ is also a subgroup of $\mathcal{G}$. We have $B = \langle K, b, (ad)^2\rangle$ and we know that for every $k \in K$ the element (k,k) belongs to $\mathcal{G}$. On the other hand, $(b,b) = d\cdot d^a$ and $((ad)^2,(ad)^2) = ((ad)^2,(da)^2) = (c\cdot c^a)^2$ also belong to $\mathcal{G}$. □

[1] The fact that $K <_2 B <_8 \mathcal{G}$ is well known. The other indices are easily computed on Schreier graphs.

Proposition 6.4.5. *The subgroup W is a finitely generated weakly maximal subgroup of $\mathcal{G}$.*

Proof. The subgroup K and $\tilde{B}$ being of finite index in $\mathcal{G}$, a finitely generated group, are finitely generated and so is W. Now, if g is an element of $W \cap (K \times \{1\} \times \{1\} \times \{1\})$, we have $g = (k, 1, 1, 1)$ and also $g = (g_0, g_1, g_0, g_3)$ which implies $g = 1$.We have shown that $W \cap (K \times \{1\} \times \{1\} \times \{1\}) = \{1\}$ and therefore that W is of infinite index.

We now want to prove that W is weakly maximal. That is, for all $x \in \mathcal{G} \setminus W$, the subgroup $\widetilde{W} := \langle W, x\rangle$ is of finite index in $\mathcal{G}$. Since a belongs to W, we can assume that x belongs to H and $x = (x_0, x_1)$. We have $\mathcal{G}/B = \{1, a, d, ad, ada, \dots, (ad)^3a\} \cong D_{2\cdot 4}$, the dihedral group of order 8, and hence $\mathcal{G}/\tilde{B} = \{1, a, d, ad\}$. By factorizing the first coordinate by $\pi_0(W) \geq \tilde{B}$ we can assume that x_0 belongs to $\{1, a, d, ad\}$ which leave us with four cases to check. If x_0 is not in H, then $\widetilde{W}$ contains $(\{1\} \times K \times \{1\} \times \{1\})^{x_0} = K \times \{1\} \times \{1\} \times \{1\}$ and a. In this case, $\widetilde{W}$ contains $K \times K \times K \times K$ and is therefore of finite index. We can hence suppose that x_0 is in H, and by symmetry, that x_1 is also in H. It thus remains to check to cases: $(1, x_1)$ and (d, x_1) with x_1 in H.

(i) If $x_0 = 1$, then $x_1 \neq 1$. In this case, $\widetilde{W}$ contains $\{1\} \times \langle x_1\rangle^{\tilde{B}}$ and $\mathrm{diag}(\langle x_1\rangle^{\tilde{B}} \times \langle x_1\rangle^{\tilde{B}}) \leq \mathrm{diag}(\tilde{B} \times \tilde{B})$. This implies $\widetilde{W} \geq \langle x_1\rangle^{\tilde{B}} \times \langle x_1\rangle^{\tilde{B}}$. If $\langle x_1\rangle^{\tilde{B}}$ has finite index in $\mathcal{G}$, then $\widetilde{W}$ has also finite index in $\mathcal{G}$. We can therefore assume that $\langle x_1\rangle^{\tilde{B}}$ has infinite index in $\mathcal{G}$, which implies by Lemma 6.4.1 that $x_1 = (1, z)$ or $x_1 = (z, 1)$, $z \neq 1$. In both cases, we have $x = (1, x_1)$ is an element of $\mathrm{Rist}_{\mathcal{G}}(2) = K \times K \times K \times K$ which implies that z is in K. This rules out the case $x_1 = (1, z)$, since in this case we would have $x = (1, 1, 1, z) \in \{1\} \times K \times \{1\} \times K \leq W$. On the other hand, $\langle z\rangle^{\mathcal{G}}$ is a non-trivial normal subgroup of $\mathcal{G}$ and thus of finite index. Therefore, $\widetilde{W}$ contains $A := \langle z\rangle^{\mathcal{G}} \times K$, a finite index subgroup of $\tilde{B}$, and hence it also contains $\mathrm{diag}(A \times A)$. Altogether, we have $\widetilde{W} \geq A \times A$, where $A = \langle z\rangle^{\mathcal{G}} \times K$ is a finite index subgroup of $\mathcal{G}$. This implies that $\widetilde{W}$ is of finite index.

(ii) We will now show that the case $x_0 = d$ cannot happen if x_i is in H. Indeed, (d, x_1) belongs to $\mathcal{G}$ if and only if $(1, ax_1) = c^a \cdot (d, x_1)$ is in $\mathcal{G}$. But in this case, $(1, ax_1)$ belongs to $\mathrm{Rist}_{\mathcal{G}}(1) = B \times B$ and so ax_1 is in $B \leq H$, which is impossible if $x_1 \in H$.

□

Lemma 6.4.6. *The subgroup $W \cap H$ is a weakly maximal subgroup of H with both left and right projections equal to $\tilde{B} <_4 \mathcal{G}$.*

Proof. We have $W \cap H = \langle \mathrm{diag}(\tilde{B} \times \tilde{B}), \{1\} \times K \times \{1\} \times K\rangle$ and the result on projections follows directly.

For the weak maximality, let x be in $H \setminus W$ and look at $\widetilde{W} := \langle x, W\rangle$. Then $x = (x_0, x_1)$ and factorizing the first factor by $\tilde{B}$, we can assume that x_0 is either 1 or d. The rest of the proof is the same as the proof of the weak maximality of W in $\mathcal{G}$. □

Conjecture 6.4.7.

1. *$\overline{W}$ is a (topologically) finitely generated wmc subgroup of $\hat{\mathcal{G}}$;*
2. *There is a continuum of two-by-two distinct conjugates of $\overline{W}$ in $\bar{\mathcal{G}}$ (and all of them are weakly maximal).*

Proof. We will prove the conjecture under the assumption that Conjecture 6.4.3 is true.

The application $\overline{}\colon \mathrm{Sub}_{\mathrm{cl}}(G) \to \mathrm{Sub}_{\mathrm{cl}}(\hat{\mathcal{G}})$ that send a close (in the profinite topology) subgroup of $\mathcal{G}$ to its closure in $\hat{\mathcal{G}}$ send infinite index subgroups to infinite index subgroups. Since W is weakly maximal in $\mathcal{G}$, it is close and we have that $\overline{W}$ is an infinite index subgroup of $\hat{\mathcal{G}}$. The element a normalizes both $\mathrm{diag}(\tilde{B}\times\tilde{B})$ and $\{1\}\times K\times\{1\}\times K$ and since K is normal and $\tilde{B} \leq H$, the subgroup $\{1\}\times K\times\{1\}\times K$ is normalized by $\mathrm{diag}(\tilde{B}\times\tilde{B})$ Therefore, $W = \{1,a\}\cdot\mathrm{diag}(\tilde{B}\times\tilde{B})\cdot(\{1\}\times K\times\{1\}\times K)$ and $\overline{W} = \overline{\{1,a\}}\cdot\overline{\mathrm{diag}(\tilde{B}\times\tilde{B})}\cdot\overline{\{1\}\times K\times\{1\}\times K}$. The subgroup $\{1,a\}$ is closed and we have $\overline{\{1\}\times K\times\{1\}\times K} = \{1\}\times\bar{K}\times\{1\}\times\bar{K}$ and $\overline{\mathrm{diag}(\tilde{B}\times\tilde{B})} = \mathrm{diag}(\bar{\tilde{B}}\times\bar{\tilde{B}})$. Altogether, we have

$$\overline{W} = \langle a, \mathrm{diag}(\bar{\tilde{B}}\times\bar{\tilde{B}}), \{1\}\times\bar{K}\times\{1\}\times\bar{K}\rangle$$

is a (topologically) finitely generated subgroup of $\hat{\mathcal{G}}$.

For all n we have $\mathrm{Stab}_{\mathcal{G}}(n) = \mathrm{Stab}_{\hat{\mathcal{G}}}(n)\cap\mathcal{G}$ and $\mathrm{Rist}_{\mathcal{G}}(n) = \mathrm{Rist}_{\hat{\mathcal{G}}}(n)\cap\mathcal{G}$. Since these subgroups are of finite index, for all n we have

$$\overline{\mathrm{Stab}_{\mathcal{G}}(n)} = \mathrm{Stab}_{\hat{\mathcal{G}}}(n) \qquad \overline{\mathrm{Rist}_{\mathcal{G}}(n)} = \mathrm{Rist}_{\hat{\mathcal{G}}}(n)$$

In particular, we have

$$\mathrm{Rist}_{\hat{\mathcal{G}}}(1) = \bar{B}\times\bar{B} \qquad \mathrm{Rist}_{\hat{\mathcal{G}}}(2) = \bar{K}\times\bar{K}\times\bar{K}\times\bar{K}$$

On the other hand, the closure preserves transversals for finite index subgroups. In particular, $\hat{\mathcal{G}}/\bar{H} = \{1,a\}$ and $\hat{\mathcal{G}}/\bar{\tilde{B}} = \{1,a,ad,d\}$. Let x be an element from $\mathcal{G}\setminus\overline{W}$ and look at $\widetilde{\overline{W}} := \overline{\langle\overline{W},x\rangle}$. The proof that $\widetilde{\overline{W}}$ is of finite index is the same as the one for $\widetilde{W}$, where Lemma 6.4.1 is replaced by Conjecture 6.4.3.

Since W is weakly maximal we can construct a 2-regular coset tree T_W. Since $\overline{W}$ is wmc, then for the action of $\hat{\mathcal{G}}$ on T_W, parabolic subgroups are 2-by-2 distinct; and there is $2^{\aleph_0}$ many such subgroups. ☐

Remark 6.4.8. We have te following alternative. Either some parabolic subgroups for the action $\mathcal{G}$ on T_W are finitely generated and some are infinitely generated, or there is only a countable number of distinct parabolic subgroups. Indeed, since $\mathcal{G}$ is finitely generated, there are only countably many finitely generated subgroups.

In both cases we have a comportment that is far away from the classical action, where all parabolic subgroups are of infinite rank and two-by-two distinct.

Block subgroups

Since $\mathcal{G}$ is finitely generated, it has at most countably many finitely generated subgroups. In particular, it has at most countably many finitely generated weakly maximal subgroups.

Conjecture 6.4.9. *The group $\mathcal{G}$ has infinitely many distinct, up to* $\mathrm{Aut}(\mathcal{G})$*, finitely generated weakly maximal subgroups.*

An important tool for the study of finitely generated subgroups of $\mathcal{G}$ is the notion of block subgroups due to Grigorchuk and Nagnibeda [54]. Write K_u for the subgroup of $\mathcal{G}$ acting as K on T_u and trivially outside. Observe that if u is the root, then $K_u = K$ and if u is of level at least 2, then $K_u = \mathrm{Rist}_{\mathcal{G}}(u)$.

Definition 6.4.10. Let S be a section of T and $S = E \sqcup \{E_i\}_{i\in I} \sqcup S'$ a covering of S with all E_i of cardinality at least 2. Observe that S is necessarily finite. If $E_i = u_1, \dots, u_j$ we can define a *diagonal subgroup* $D_i \coloneqq \operatorname{diag}(K_{u_1} \times \varphi_2(K_{u_2}) \times \cdots \times \varphi_j(K_{u_j}))$ where the φ_t are automorphism of $\mathcal{G}$. These datas (the covering of S and the choices of φ_t) determine a *block subgroup* U of $\mathcal{G}$ by

$$U \coloneqq \prod_{u\in E} K_u \times \prod_{i\in I} D_i \times \prod_{u\in S'} \{1\}$$

It is obvious that block subgroups are finitely generated. On the other hand, we have the following result.

Proposition 6.4.11 (Grigorchuk-Nagnibeda [54])**.** *A subgroup of $\mathcal{G}$ is finitely generated if and only if it contains a block subgroup as a finite index subgroup.*

In order to find weakly maximal subgroups that are finitely generated, it may be interesting to start with infinite index block subgroups that are maximal among infinite index block subgroups. The following lemma directly follows from the congruence subgroup property.

Lemma 6.4.12. *A block subgroup*

$$U = \prod_{u\in E} K_u \times \prod_{i\in I} D_i \times \prod_{u\in S'} \{1\}$$

is of finite index if and only if $E = S$, that is if $S' = I = \emptyset$

Lemma 6.4.13. *Let*

$$U = \prod_{u\in E} K_u \times \prod_{i\in I} D_i \times \prod_{u\in S'} \{1\}$$

be a block subgroup that is maximal among block subgroup of infinite index. Then, $S = E \sqcup E_1$ with $E_1 = \{v, w\}$ and S is the "uppermost" section that contains v and w.

Proof. By assumption, U is of infinite index and either S' or I is non-empty. If S' is non-empty, then U is not maximal. Indeed, let $u \in S'$ and v and w be the two children of u. Then U is properly contained in U', the block subgroup obtained from U by replacing the factor $\{1\}_u$ by $\{1\}_v \times \{1\}_w$. By the lemma, U' is still of infinite index. Therefore, S' is empty and I is non-empty.

If I has more than one element, than U is contained in U', the block subgroup obtained by replacing D_2 by $K_{u_1} \times K_{u_j}$, which is still of infinite index. This implies that I has exactly one element.

Now, if E_1 has more than two elements, we can construct $U < U'$ by replacing D_1 by $\operatorname{diag}(K_{u_1} \times \varphi_2(K_{u_2})) \times K_{u_3} \times \cdots \times K_{u_t}$.

Finally, we want to prove that S is the "uppermost" section containing v and w. To be precise, we can put a partial order on section by saying that $S \leq \tilde{S}$ if every element of S is below some elements of $\tilde{S}$. Among sections containing v and w, there is a greatest element $\tilde{S}$ which consist of v, w and all roots of maximal trees in $T \setminus$ path from v to the root, path from w to the root. It is evident that if our block subgroup correspond to a section $S < \tilde{S}$, then it is a subgroup of the block subgroup corresponding to $\tilde{S}$. ☐

We know turn our attention to a special case of block subgroups. Define U_n by $E = \{0, 10, \dots, 1^{n-1}0\}$, $E_1 = \{1^n0, 1^{n+1}\}$ and $D_1 = \operatorname{diag}(K_{1^n0} \times K_{1^{n+1}})$. The subgroups U_n are maximal among block subgroups of infinite index and they are in the stabilizer of 1^n. Moreover, U_0 is the block subgroup corresponding to W, the weakly maximal subgroup of the last subsection. In order to construct more weakly maximal subgroups, we are going to extend U_n to a bigger subgroup W_n which will be weakly maximal. Since U_n is of infinite index, such a W_n can always be founded, but we want to ensure that it will be finitely generated.

In order to do that, it is useful to have a description of $\mathrm{Stab}_{\mathcal{G}}(1^n)$. This was done by Kravchenko [80] (see also [10, Theorem 4.4]) who describes $\mathrm{Stab}_{\mathcal{G}}(1^n)$ has an iterated semi-direct product, as depicted in Figure 6.14.

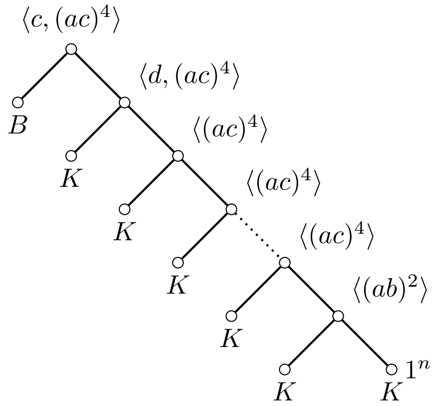


Figure 6.14: The stabilizer in $\mathcal{G}$ of the vertex 1^n.

Observe that the projection of $\mathrm{Stab}_{\mathcal{G}}(1^n)$ on 1^n is equal to $\mathcal{G}$. Therefore, for all weakly maximal subgroup M of $\mathcal{G}$, we can define M_n as the subgroup of $\mathrm{Stab}_{\mathcal{G}}(1^n)$ consisting of elements g such that the projection of g on 1^n is in M.

Lemma 6.4.14. *For all weakly maximal subgroup M and all n, the subgroup M_n is weakly maximal.*

Proof. Let g be in $\mathcal{G} \setminus M_n$. If g does not stabilizes 1^n, then M_n contains $K_{g(1^n)}$. Therefore, $\langle g, M_n\rangle$ contains K_{1^n} and also K_v for every $v \neq 1^n$ of level n. This implies that $\langle g, M_n\rangle$ has finite index in $\mathcal{G}$. On the other hand, if g is in $\mathrm{Stab}_{\mathcal{G}}(1^n)$, then by definition, g_{1^n} does not belong to M. This implies that the projection of M_n on q^n, which is equal to $\langle g_{1^n}, M\rangle$, has finite index in $\mathcal{G}$. Therefore, $\langle g, M\rangle$ is a finite index subgroup of $\mathcal{G}$. This concludes the proof of the weak maximality of M_n. □

Proposition 6.4.15. *Let $(W_n)_{n\in\mathbf{N}}$ be the subgroups constructed from W, the Pervova's example. Then they are all weakly maximal and if $n \neq m$, the subgroups W_n and W_m are distinct, non-conjugate and even not in the same class of equivalence under* $\mathrm{Aut}(\mathcal{G})$.

Proof. Grigorchuk and Sidki showed that $N_{\mathrm{Aut}(T)}(\mathcal{G}) = \mathrm{Aut}(\mathcal{G})$. We are going to prove that if $m > n$, then for every $1 \neq \varphi \in \mathrm{Aut}(\mathcal{G})$, the subgroup $\langle W_n, \varphi(W_m)\rangle$ has finite index in $\mathcal{G}$. If φ does not stabilizes 1^n, then $\langle W_n, \varphi(W_m)\rangle$ contains K_v for every vertex of level n and is thus of finite index.

If φ does stabilizes 1^n, then $\varphi(W_m)$ contains either $K_{1^{n+1}}$ or K_{1^n0}. On the other hand, W_n contains $\mathrm{diag}(K_{1^{n+1}} \times K_{1^n0})$. Hence, $\langle W_n, \varphi(W_m)\rangle$ contains K_v for every vertex of level $n+1$ and is thus of finite index. □

Conjecture 6.4.16. *For each n, the subgroup W_n is finitely generated.*

Remark 6.4.17. For $n = 0$ we have $W_0 = W$ which is obviously finitely generated. In general, $\mathrm{Stab}_{\mathcal{G}}(1^n)$ is generated by the normal subgroup $N := B_0 \times K_{10} \times \cdots \times K_{1^{n-1}0}$ and by $L := \langle K_{1^n}, (c)_1, (b)_2, ((ab)^2)_{n-1}((ac)^4)_i, 0 \leq i \leq n-2\rangle$, where $(g)_n$ denotes the element $(1, \dots, 1, g)$ in $\mathrm{Stab}_{\mathcal{G}}(n)$ — for example $(b)_2 = (1, b) = d$. Therefore, $W_n = \langle N, L'\rangle$ where $L' := \{g \in L \mid g_{1^n} \in W\}$ and W_n is finitely generated if L' is finitely generated.

One interesting open question of this section is the existence of infinitely many non-conjugated finitely generated weakly maximal subgroups. This is implied by Conjecture 6.4.16. Another interesting question is the weak maximality of $\bar{W}$, see 6.4.7. Indeed, this is the first step in solving Conjecture 5.8.3 and answering Question 5.8.2.

6.5 Projections of weakly maximal subgroups of $\mathcal{G}$

In this section, following an idea of Grigorchuk we turn our attention to left and right projections of weakly maximal subgroups. Progress in this direction may lead to a classification of weakly maximal subgroups of $\mathcal{G}$.

Here $\mathcal{G} = \langle a, b, c, d\rangle$ is the first Grigorchuk group, $B := \langle b\rangle^{\mathcal{G}}$ the normal closure of b and $H := \mathrm{Stab}_{\mathcal{G}}(1) = \langle b, c, d, aba, aca, ada\rangle$ the stabilizer of the first level. Let $A < \mathcal{G}$ be a weakly maximal subgroup. There is two possibilities. Either A is a subgroup of H or not. If A is a subgroup of H, we can look at A_0 and A_1, the action of A on the left and right subtrees. Since $\mathcal{G}$ is self-replicating, A_0 and A_1 are subgroups of $\mathcal{G}$. Thus $A \leq_S A_0 \times A_1$ is a subgroup of the product.

Lemma 6.5.1. *Suppose that $A \leq H$ and $[\mathcal{G} : A_0]$ is of infinite index. Then A_0 is a weakly maximal subgroup of $\mathcal{G}$ and A_1 contains B.*

Proof. Suppose that $A_0 < A_0'$ is of infinite index. Then for $\alpha \in A_0' \setminus A_0$ there exists $\beta \in \mathcal{G}$ such that $(\alpha, \beta) \in \mathcal{G}$ and we have

$$A < \langle A, (\alpha, \beta)\rangle \leq_S A_0' \times \langle A_1, \beta\rangle \qquad [\mathcal{G} : \langle A, (\alpha, \beta)\rangle] = \infty$$

since A_0' is of infinite index. This proves the weak maximality of A_0.

On the other hand, if there exists $\beta \in B \setminus A_1$, then $(1, \beta) \in \mathcal{G}$ and

$$A < \langle A, (1, \beta)\rangle \leq_S A_0 \times \langle A_1, \beta\rangle \qquad [\mathcal{G} : \langle A, (1, \beta)\rangle] = \infty$$

since A_0' is of infinite index.

We even have:

$$B \leq D := \{\beta \in \mathcal{G} \mid \exists \alpha \in A_0 : (\alpha, \beta) \in \mathcal{G}\} \leq A_1.$$

□

Since in this case A_1 contains B, it is interesting to know all subgroups of $\mathcal{G}$ containing B. The classification of such subgroups is done in [28] for normal subgroups and in Section 6.1 for the general case. There is exactly 10 such subgroups: $\mathcal{G}$, 3 subgroups of index 2 — $J_{0,2}$, H, $J_{0,5}$ — 5 of index 4 — $J_{1,5}$ (normal), $S_{2,3,0,0}$, $S_{2,3,0,1}$, $S_{2,4,0,0}$ and $S_{2,4,0,1}$ and B itself, see Figure 6.10. The following questions remain open.

Question 6.5.2. *Which of the* 10 *subgroups containing B could appear as A_1?*

Question 6.5.3. *Is it possible that $A \leq H$ if A is not parabolic?*

Question 6.5.4. *What about the case where both A_i are of finite index? Is it possible? Does it imply that A is finitely generated?*

Now, if A is not a subgroup of H, we can look at $A' \coloneqq A \cap H$, A_0 and A_1, the action of A' on the left and right subtrees. Since $\mathcal{G}$ is self-replicating, A_0 and A_1 are subgroups of $\mathcal{G}$. Thus $A' \leq_S A_0 \times A_1$ is a subgroup of the product.

Lemma 6.5.5. *Suppose that A' is weakly maximal in H and that $[\mathcal{G} : A_0]$ is of infinite index. Then A_0 is a weakly maximal subgroup of $\mathcal{G}$ and A_1 contains B.*

The proof is the same as the for Lemma 6.5.1.

Question 6.5.6. *What about the general case: i.e. A' not weakly maximal in H? Can it happen?*

Lemma 6.4.6, shows that in the case of W, the Pervova's example, $W \cap H$ is weakly maximal and has both projection of index 4 in $\mathcal{G}$.

Parabolic subgroups

Let A be a weakly maximal subgroup of $\mathcal{G}$. We can look as before at A_0 and A_1 the left and right projections (of $A \cap \mathrm{Stab}_{\mathcal{G}}(1)$). But, for any vertex $v \in T$ of level n we can also look at A_v the projection on v (of $A \cap \mathrm{Stab}_{\mathcal{G}}(n)$). It is obvious that if $v \leq w$, then $A_v = (A_w)_v$.

The following lemma shows that the closure (in $\hat{\mathcal{G}}$) of the projection and the projection of the closure agree

Lemma 6.5.7. *For all subgroup $A \leq \mathcal{G}$ and all $v \in T$, we have $(\bar{A})_v = \overline{A_v}$.*

Proof. Observe that since H is maximal of index 2 in $\mathcal{G}$ we have either $A \leq H$ — in which case $\bar{A} \leq \bar{H}$ — or $A \cap H <_2 A$ — in which case $\bar{A} \cap \bar{H} <_2 \bar{A}$. Since the closure preserves finite index, if $A \not\leq H$, we have $\overline{A \cap H} <_2 A$ and $\overline{A \cap H} \leq \bar{A} \cap \bar{H} <_2 A$. Therefore, we always have $\overline{A \cap H} = \bar{A} \cap \bar{H}$.

Suppose now, that v is a vertex of the first level; say $v = 0$. If g belongs to $(\bar{A})_0$, there exists $h \in \bar{\mathcal{G}}$ such that (g, h) is an element of $\bar{A} \cap \bar{H} = \overline{A \cap H}$. This implies that $(g, h) = \lim_i (g_i, h_i)$ with $(g_i, h_i) \in A \cap H$. In particular, $g = \lim g_i$ with $g_i \in A_0$ and therefore $(\bar{A})_0 \leq \overline{A_0}$.

On the other hand, if g is in $\overline{A_v}$, then $g = \lim_i g_i$ for some g_i in A_v. For all i, there exists h_i such that (g_i, h_i) is in A. By compacity, there exists a converging subsequence of the h_i such that $\lim_i (g_i, h_i)$ belongs to $\bar{A}$. This proves that $(\bar{A})_v \geq \overline{A_v}$.

For the general case where v is of level n, we do an induction, using the fact that for $v \leq w$ we have $A_v = (A_w)_v$. □

Lemma 6.5.8. *Let $A \leq \mathcal{G}$ be a weakly maximal subgroup. Then the following are equivalent.*

1. *A is a parabolic subgroup of the original action;*
2. *There exists a infinite ray $\xi = x_1 x_2 \ldots$ such that for all $j \geq 0$, the subgroup $A_{x_1 \ldots x_j}$ is contained in H (where $A_\emptyset = A$).*

Proof. The condition on $A_{x_1 \ldots x_j} \leq H$ implies that A stabilizes the ray ξ for the original action. Therefore, $A \leq \mathrm{Stab}_{\mathcal{G}}(\xi)$ and we conclude by weak maximality. The other direction is obvious. □

Finally, we have a full description of the projections of parabolic subgroups.

Proposition 6.5.9. *If $A = \mathrm{Stab}_{\mathcal{G}}(\xi)$, with $\xi = x_1x_2\dots$, then for all j we have*

$$A_{x_1\dots x_j} = \mathrm{Stab}_{\mathcal{G}}(\sigma^j(\xi)) \qquad A_{x_1\dots x_{j-1}\bar{x}_j} = J_{0,2} <_2 \mathcal{G}$$

Proof. If $A = \mathrm{Stab}_{\mathcal{G}}(\xi)$ with $\xi = x_1x_2\dots$, then A_{x_1} stabilizes $\sigma(\xi)$. Therefore, A_{x_1} is an infinite index subgroup of H. This implies that $A_{\bar{x}_1}$ contains B and that A_{x_1} is weakly maximal and hence equal to $\mathrm{Stab}_{\mathcal{G}}(\sigma(\xi))$. An easy induction shows that $A_{x_1\dots x_j} = \mathrm{Stab}_{\mathcal{G}}(\sigma^j(\xi))$ and $B \leq A_{x_1\dots x_{j-1}\bar{x}_j} \leq \mathcal{G}$.

Let $P \coloneqq \mathrm{Stab}_{\mathcal{G}}(1^\infty)$. We have the following description of P from [10, Theorem 4.4]

$$P = \Big(B \times \big((K \times ((K \times \dots) \rtimes \langle (ac)^4 \rangle)) \rtimes \langle b, (ac)^4 \rangle\big)\Big) \rtimes \langle c, (ac)^4 \rangle$$

This immediately implies that for all j we have

$$P_{1^j} = P \qquad P_{1^{j-1}0} = \langle B, (ad)^2, a \rangle = J_{0,2} <_2 \mathcal{G}$$

For the general case, let ξ be any ray in T and $A = \mathrm{Stab}_{\mathcal{G}}(\xi)$. There exists $g \in \hat{\mathcal{G}}$ such that $\bar{A} = \mathrm{Stab}_{\hat{\mathcal{G}}}(\xi) = \mathrm{Stab}_{\hat{\mathcal{G}}}(1^\infty) = \bar{P}$. Therefore, for all j, we have $\bar{A}_{x_1\dots x_{j-1}\bar{x}_j} = \overline{J_{0,2}}^g = \overline{J_{0,2}}$. $\square$

There is still a long way before understanding the structure of weakly maximal subgroups in $\mathcal{G}$. The first important step would be to understand the structure of subgroups contained in H and in particular to answer the following

Question 6.5.10. *Let A be a weakly maximal subgroup of $\mathcal{G}$ contained in H and A_1 and A_2 its left and right projections.*

1. *If A_2 is of infinite index, which of the 10 subgroups of $\mathcal{G}$ containing B could appear as A_1?*
2. *Is it possible that $A \leq H$ if A is not parabolic?*
3. *What about the case where both A_i are of finite index? Is it possible? Does it imply that A is finitely generated?*

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