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# Option pricing with discrete rebalancing

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## Abstract

We consider option pricing when dynamic portfolios are discretely rebalanced. The portfolio adjustments only occur after fixed relative changes in the stock price. The stock price follows a marked point process (MPP) and the market is incomplete. We first characterise the equivalent martingale measures. An explicit pricing formula based on the minimal martingale measure (MMM) is then provided together with the hedging strategy underlying portfolio adjustments. Two examples illustrate our pricing framework: a jump process driven by a latent geometric Brownian motion and a marked Poisson process. We establish the convergence to the Black–Scholes model when the triggering price increment shrinks to zero. For the empirical application, we use IBM, France Telecom (FT) and CAC 40 intraday transaction data, and compare option prices given by the marked Poisson model, the Black–Scholes model and observed option prices.

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*Keywords:* Incomplete market; Option pricing; Minimal martingale measure; Discrete rebalancing; Marked point process

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## 1. Introduction

Due to practical constraints, portfolios are not continuously rebalanced. Traders usually readjust their positions after significant moves in the underlying asset price. It means that hedging portfolios are discretely rebalanced according to relative price movements, i.e. after an increase (or decrease) by a given percentage  $a$ . Price movements leading to these

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portfolio modifications are best described by a process whose logarithmic variations have fixed sizes ( $a$  or  $-a$ ) and occur at random times. Such a process is a marked point process on the real line with one-dimensional mark space  $E=\{a, -a\}$ . The real line depicts time while the marks describe the random jumps taking place at random times of occurrence of events. In this paper, we analyze the pricing of an option when stock prices follow this kind of processes also called space–time point processes. The dynamics of such processes can be estimated on high frequency transaction data. As will become apparent in the empirical section of the paper, while being in phase with market practice, our pricing methodology also allows an adequate fit of the empirical features of IBM, France Telecom and CAC 40 price data.

The paper is organized as follows. In Section 2, we outline our framework and briefly portray the basic structure of a marked point process using the martingale approach. In Section 3, we characterise the set of equivalent martingale measures (the market is incomplete due to the presence of jumps and option prices are no more uniquely determined). In Section 4, we give a pricing formula when a particular choice is made within this set, namely the minimal martingale measure introduced by Föllmer and Schweizer (1991). An explicit form for the trading strategy is derived when this choice is adopted to build hedging portfolios. This is an obvious practical advantage of our pricing approach. Both sections are closely related to the work of Buhlman et al. (1996) and Colwell and Elliott (1993). Two examples are further analyzed. The first example consists of a jump process built from the random crossings of a latent (hidden or unobservable) geometric Brownian motion. This model corresponds to the intuitive case where the true stock price follows the same process as in Black–Scholes but traders only rebalance when the underlying has changed significantly. Although intuitive from a financial theory point of view, this specification is not appropriate for empirical purposes because of its complexity. We thus propose a simpler, readily implementable specification as our second example: a marked Poisson process with independent binomial increments. This second parametric model is easy to estimate and able to capture some empirical features of stock prices. In Section 5, we examine the convergence of the minimal pricing model to the standard Black and Scholes (1973) model when the jump size shrinks to zero. In Section 6, three empirical applications are provided. We first study IBM intraday transaction data from the New York Stock Exchange. Parameters of the marked Poisson case are estimated and used as input to compute European call option prices. These prices are compared with a Black–Scholes pricing based on an historical volatility estimate from daily closing prices. In our second and third examples, we compare the ability of the Black–Scholes and of the marked Poisson model to replicate observed prices of options on the France Telecom stock and the CAC 40 index listed of the Paris bourse. Section 7 concludes. All proofs and technical details are gathered in: Appendices A and B.

## 2. Framework

Before reviewing the concepts needed to proceed further, we would like to refer the reader to Brémaud (1981), Karr (1986) and Last and Brandt (LB) (1995) for more extensive treatments of (marked) point processes. Besides, the exposition of Jacod and

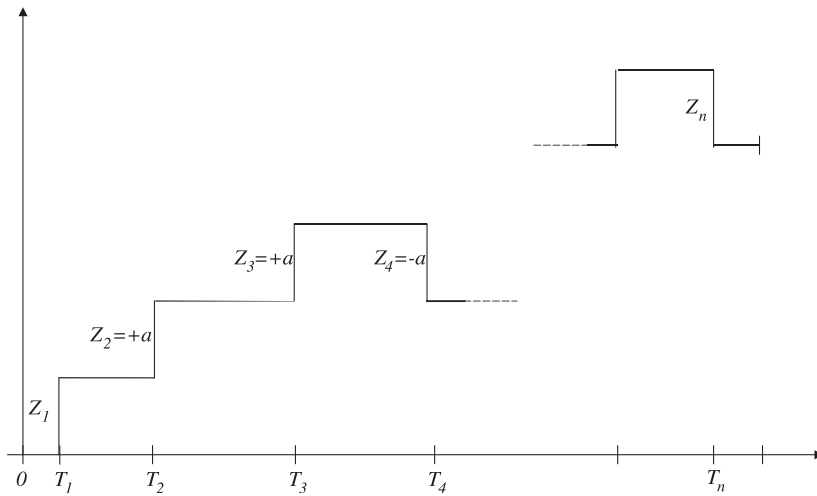


Fig. 1. The process  $X_t$  such that  $S_t = S_0 e^{X_t}$ .

Shiryaev (JS) (1987) is of paramount interest for the background of this paper in terms of notations and concepts.

Let us consider a sequence of positive random variables  $T_j$  satisfying  $T_j < T_{j+1}$ ,  $j = 1, \dots$ . Random elements  $Z_j$ , called the marks, take values in the mark space  $E$ , and are associated with the random times  $T_j$ . The random elements are defined on the same probability space:  $(\Omega, \mathcal{F}, P)$ . Each  $(T_j, Z_j)$  is said to be a marked point and the sequence  $((T_j, Z_j))_j$  of marked points is referred to as a marked point process (MPP).

The mark space  $E$  is here equal to  $\{a, -a\}$  and the MPP describes the dynamics of a stock price:

$$S_t = S_0 e^{X_t}, \quad (1)$$

through the random variable:

$$X_t = \sum_{j: T_j \leq t} Z_j, \quad (2)$$

built with the marked points  $(T_j, Z_j)$ . The process  $X$  is a purely discontinuous process with jumps  $Z_j = \Delta X_{T_j}$  at random times  $T_j$ . The jumps take the values  $a$  or  $-a$ , and so do the logarithmic variations of the stock price:  $\Delta \log S_{T_j} = \Delta X_{T_j}$ . A possible realisation of the process  $X_t$  is shown in Fig. 1.

The process  $X$  takes the form of an integral process defined as the sum of jumps:

$$X_t = \sum_{j: T_j \leq t} \Delta X_{T_j} = \int_0^t \int_E x d\mu.$$

The integer-valued random measure  $\mu(dt, dx)$  on  $\mathbb{R}_+ \times E$  is the counting measure associated to the marked point process. It is given by (Jacod and Shiryaev, 1987, p. 69):

$$\mu(dt, dx) = \sum_{s \geq 0} \mathbf{I}_{\{\Delta X_s \neq 0\}} \epsilon_{(s, \Delta X_s)}(dt, dx),$$

where  $\epsilon_v$  denotes the Dirac measure at point  $v$ .

If  $E[\sum_{j: T_j \leq t} |Z_j|]$  is finite for all  $t$  (Jacod and Shiryaev, 1987, p. 72), such a process can be decomposed as:

$$X = \int_0^t \int_E x dv + \int_0^t \int_E x d(\mu - \nu).$$

The measure  $\nu$  is a predictable measure, called the compensator, with the property that  $\mu - \nu$  is a local martingale measure. This measure can be disintegrated as (Jacod and Shiryaev, 1987, p. 67):

$$\nu(dt, dx) = dA_t K(t, dx),$$

where  $A$  is a predictable integrable increasing process and  $K$  is a transition kernel. In the sequel, we work with position-dependent marking (Last and Brandt, 1995, pp. 18 and 186). This corresponds to the following assumption for the specification of our model.

**Assumption 1 (Compensator specification).** The compensator  $\nu(dt, dx)$  on  $\mathbb{R}_+ \times E$  satisfies:

$$\nu(dt, dx) = dA_t K(t, dx),$$

where  $A$  is a predictable integrable increasing process absolutely continuous w.r.t. time,  $K$  is a probability kernel on  $E$  such that  $\int_E (x^2 \wedge 1) K(t, dx) < +\infty$ , and  $Z_1, Z_2, \dots$  are conditionally independent given  $(T_j)_j$  such that  $P(Z_j \in dx | (T_j)_j) = K(T_j, dx)$ .

The marked point process  $(T_j, Z_j)_j$  is called a position-dependent  $K$ -marking of the point process  $(T_j)_j$ . The marks have a probability kernel  $K$  and the arrivals of the increments are governed by the process  $A$ . The process  $A$  represents the intensity of the arrival times of jumps and is the compensator associated to the counting measure of the point process  $(T_j)_j$ . The absolute continuity hypothesis ensures that the set of fixed times of discontinuity of  $X$  is empty and that the Radon–Nikodym derivative  $\lambda_t = dA_t/dt$  is well defined. The process  $\lambda$  is called the directing intensity.

Now that the setting is given, we may turn to the next step: the characterisation of the equivalent martingale measures.

### 3. Equivalent martingale measures

From Harrison and Kreps (1979), Harrison and Pliska (1981), Delbaen and Schachermayer (1994), we know that in order to price derivative assets, we need to exhibit an equivalent measure under which discounted prices are martingales. We take here as

discount factor or numéraire a savings account whose growth rate  $r_{T_j}$  on the random time interval:  $]T_{j-1}, T_j]$  satisfies:

$$r_{T_j} = e^{\rho(T_j - T_{j-1})} - 1, \quad (3)$$

with  $\rho > 0$ .<sup>1</sup>

The discounted stock price is then equal to:

$$\tilde{S}_t = \frac{S_t}{\prod_{j: T_j \leq t} (1 + r_{T_j})}.$$

Let us introduce the discounted excess return process  $\delta(t, x)$  such that for  $t \in ]T_j, T_{j+1}]$ :

$$\delta(t, x) = \delta(T_j, Z_j) = \frac{e^{Z_j} - (1 + r_{T_j})}{1 + r_{T_j}}. \quad (4)$$

The equivalent martingale measures are then characterised by the following proposition (see also Buhlman et al., 1996).

**Proposition 1 (Equivalent martingale measures).** *Under Assumption 1, an equivalent martingale measure  $Q$  is characterised by its density process  $\eta$  relative to  $P$ :*

$$\eta_t = \varepsilon \left( \int_0^t \int_E H(s, x) d(\mu - \nu) \right), \quad (5)$$

where  $E(\cdot)$  denotes the Doléans–Dade exponential<sup>2</sup>,  $H$  is a predictable process satisfying  $H + 1 > 0$  a.s. and

$$0 = \int_0^t \int_E (H(s, x) + 1) \delta(s, x) d\nu. \quad (6)$$

The process  $\eta$  is a  $P$ -martingale such that  $\eta_t$  is the Radon–Nikodym derivative  $dQ_t/dP_t$ . The process  $H$  has the interpretation of a jump risk premium process. Eq. (6) introduces restrictions on this process but does not lead to a unique equivalent martingale measure (market incompleteness). In fact, for  $t \in ]T_j, T_{j+1}]$ , the process should satisfy  $H(t, x) = h_j(x)$  with  $h_j(x) + 1 > 0$  a.s. and

$$0 = \int_E (h_j(x) + 1) \delta(T_j, x) K(T_{j+1}, dx).$$

Then the discounted price  $\tilde{S}$  will be a martingale under the corresponding measure  $Q$ .

<sup>1</sup> For a marked point process where the directing intensity  $\lambda_t$  and the kernel  $K(t, dx)$  depend on time, we cannot use the usual deterministic savings account  $\beta_t = \exp(\rho t)$ . Arbitrage is not precluded in such a case.

<sup>2</sup> The Doléans–Dade exponential is the unique solution of the SDE  $d\varepsilon_t(U) = \varepsilon_{t-}(U) dU_t$ , namely  $\varepsilon_t(U) = \exp(U_t - 1/2[U]_t^c) \prod_{u \leq t} (1 + \Delta_u U) \exp(-\Delta_u U)$ , where  $[U]_t^c$  is the continuous part of  $[U]$  and  $\Delta_u U$  are the increments of its discontinuous part (see, e.g. Musiela and Rutkowski, 1997).

#### 4. Option pricing with the minimal martingale measure

Market incompleteness yields the non-uniqueness of prices and therefore requires delicate choices to be made in order to obtain a reasonable solution. An appealing approach is to consider super-replication. This consists of finding the trading strategy which enables the trader to be hedged irrespective of what the true martingale measure is and to derive bounds on the option price. Unfortunately, the choice of measures is often so wide that these bounds are typically close to the [Merton \(1973\)](#) bounds, which makes this approach unpractical (see [El Karoui and Quenez, 1995](#)).

A second approach, which we have retained in this paper, is to choose one measure and to motivate our choice. Both theoretical and practical considerations enter this choice as will be shown below.

A third approach has been recently introduced in the literature (see [Chernov and Ghysels, 2000](#), for example). The idea is to use time series of both the underlying asset price and of traded options and to use them as a guide to the appropriate martingale measure. Unfortunately, all explicit solutions are then lost and there is no possibility to recover what the hedging strategy should be under this measure. This questions the relevance of such methods in practical option trading.

In this paper, we have chosen to work under the minimal martingale measure (MMM), a very popular choice among equivalent martingale measures, which was introduced by [Föllmer and Schweizer \(1991\)](#). This measure is minimal among all martingale measures in the sense that, apart from turning discounted prices into martingales, it leaves unchanged the remaining structure of the model. In particular orthogonality relations are kept in a continuous framework (see [Schweizer, 1991, 1992a](#)) for a full discussion, and [Hofmann et al., 1992](#) for an illuminating use in stochastic volatility models). This measure is computationally convenient and induces nice convergence properties ([Runggaldier and Schweizer, 1995](#); [Prigent, 1999](#); [Mercurio and Vorst, 1996](#); [Lesne et al., 2000](#)). [Schweizer \(1993\)](#) shows that in some cases, the expectation of the final payoff under the minimal measure is equal to the value of the variance optimal hedging strategy (for the variance optimal measure, see, e.g. [Föllmer and Sondermann, 1986](#); [Bouleau and Lamberton, 1989](#); [Duffie and Richardson, 1991](#); [Schweizer, 1992b, 1994](#); [Gouriéroux et al., 1998](#); [Laurent and Scaillet, 1998](#)). This value or approximation price gives the initial amount required to implement a risk-minimizing strategy. Such a strategy is in general not self-financing and involves a non-vanishing hedging cost. However, as soon as the minimal measure is a probability measure, this value is an actual no-arbitrage price (it is the case below since the jump size is bounded).

Although the choice of a measure (which is similar to the choice of a utility function) will remain debatable, the MMM has a great practical advantage over other choices. It enables to write an explicit form for the change of measure and above all to find an explicit trading strategy (delta) which is crucial for trading implementation. This is a very rare property among equivalent martingale measures.

Concerning existence and uniqueness of the minimal measure, we refer to [Ansel and Stricker \(1992, 1993\)](#). We can now proceed to answer what form this measure takes in our framework (see also [Colwell and Elliott, 1993](#)).

**Proposition 2 (Minimal probability measure).** *Under Assumption 1, the minimal martingale measure  $\hat{P}$  is a probability measure characterised by its density process  $\hat{\eta}$  relative to  $P$ :*

$$\hat{\eta}_t = \varepsilon \left( \int_0^t \int_E -\alpha \tilde{S} \delta d(\mu - \nu) \right), \quad (7)$$

where for  $t \in ]T_j, T_{j+1}]$ :

$$\alpha_t = \frac{\int_E \delta(T_j, x) K(T_{j+1}, dx)}{\tilde{S}_{T_j} \int_E \delta^2(T_j, x) K(T_{j+1}, dx)}. \quad (8)$$

If we relate this proposition to Proposition 1, we have for  $t \in ]T_j, T_{j+1}]$ ,  $\hat{H}(t, x) = \hat{h}_j(x)$  with:

$$\hat{h}_j(x) = - \frac{\int_E \delta(T_j, \nu) K(T_{j+1}, d\nu)}{\int_E \delta^2(T_j, \nu) K(T_{j+1}, d\nu)} \delta(T_j, x).$$

Once the Radon–Nikodym derivative  $\hat{\eta}_t$  is computed, it is straightforward to derive the price  $C_t = C(t, S_t)$  of a contingent claim with final payoff  $C(T, S_T)$ . For a European call option with maturity  $T$  and strike price  $K$ , the final payoff is  $(S_T - K)_+ = \max(0, S_T - K)$ . Taking its expectation under  $\hat{P}$  after an adequate discounting gives the option price:

**Proposition 3 (Minimal option price).** *Under Assumption 1, the call price given by the minimal martingale measure is:*

$$C(t, S_t) = E^{\hat{P}} \left[ (S_T - K)_+ \prod_{j: T_j < T_j \leq T} (1 + r_{T_j})^{-1} | F_t \right]. \quad (9)$$

Hence, option pricing in our setup requires the following three steps:

1. specifying the marked point process followed by the stock price (i.e. the intensity and the kernel in the compensator): Assumption 1;
2. calculating the Radon–Nikodym derivative  $\hat{\eta}_t$ : Proposition 2;
3. pricing the options as expectations (under the minimal measure) of their discounted payoffs: Proposition 3.

Note that the dynamics of the stock price process under  $\hat{P}$  can be obtained by applying Girsanov theorems for jumps (see JS, p. 157 and details in Appendix B). Let us now specialize the above option pricing formula on two examples.



#### 4.1. Example 1: Latent geometric Brownian motion

As mentioned above, our motivation is to price options when traders rebalance their portfolios at random times triggered by significant changes in the stock price. It is natural from a financial theory point of view to start with the classic Black–Scholes setup of an underlying asset whose price follows a geometric Brownian motion. We assume that this process is unobservable but that the trader knows (he may be warned by his broker or by an automatic trading device) when the price has crossed some barriers at which times he rebalances his position. Let us constitute the set of barriers  $B = \{S_0 \exp ja, j \in \mathbb{Z}\}$ . The random crossings of such barriers by the continuous process  $\tilde{S}$  satisfying  $d\tilde{S}_t/\tilde{S}_t = mdt + sdW_t$  define a price process  $S$  of the forms Eqs. (1) and (2). It is enough to take:  $\exp X_t = \tilde{S}_t \cap B$ . The entire path of the geometric Brownian motion is assumed to be not observable and the continuous time process driving the jumps is thus latent (hidden). The distributional features of the (geometric) Brownian motion can nevertheless be used to identify the distribution of the jump process  $X$ . Fig. 2 describes the latent Brownian motion with drift (logarithm of the stock price) and its associated MPP.

The conditional distribution of arrival times is characterised by the probability that a Brownian motion with drift escapes the corridor  $\{a, -a\}$ . This probability can be deduced from the trivariate distribution of the running minimum, running maximum and end value of a Brownian motion (Revuz and Yor, 1994, p. 104) after a suitable change of measure to incorporate the presence of a drift (see also Kunitomo and Ikeda, 1992; Géman and Yor, 1994; He et al., 1998). The conditional distribution of marks is given by the probability that a Brownian motion with drift hits one barrier before the other (Karlin and Taylor, 1975, p. 361).

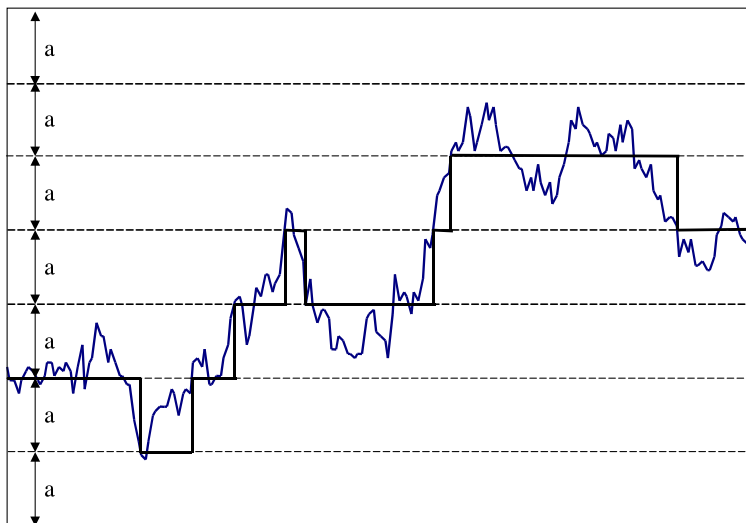


Fig. 2. Stock price logarithm and associated MPP.

Consequently, we replace in this example Assumption 1 by the more precise assumption:

**Assumption 2 (Latent geometric Brownian motion).** The compensator  $v(dt, dx)$  on  $\mathbb{R}_+ \times \{a, -a\}$  satisfies:

$$v(dt, dx) = \lambda_t dt K(dx),$$

where for  $t \in ]T_j, T_{j+1}]$ :

$$\lambda_t = \left[ -\frac{d}{dt} S(t - T_j) \right] / S(t - T_j),$$

and

$$K(dx) = \frac{e^{m'a/s^2}}{e^{-m'a/s^2} + e^{m'a/s^2}} \text{ if } dx = a,$$

$$= \frac{e^{-m'a/s^2}}{e^{-m'a/s^2} + e^{m'a/s^2}} \text{ if } dx = -a,$$

with  $m' = m - s^2/2$  and

$$S(u) = \sum_{k=-\infty}^{+\infty} e^{4km'a/s^2} \left\{ \left[ \Phi\left(\frac{a - m'u - 4ka}{s\sqrt{u}}\right) - \Phi\left(\frac{-a - m'u - 4ka}{s\sqrt{u}}\right) \right] \right. \\ \left. - e^{-2am'/s^2} \left[ \Phi\left(\frac{3a - m'u - 4ka}{s\sqrt{u}}\right) - \Phi\left(\frac{a - m'u - 4ka}{s\sqrt{u}}\right) \right] \right\}.$$

In this setting, the survival function  $S(u)$  is equal to the probability that a Brownian motion lives during a time period  $u$  between  $-a$  and  $a$ , or equivalently to the probability that the running minimum and maximum stay above  $-a$  and below  $a$ , respectively (see Duffie and Lando, 2001 for similar computations of hazard rate processes in default event modelling). We are now able to give the option pricing formula.

**Corollary 1 (Option price: latent geometric Brownian motion).** Under Assumption 2, the minimal call price is given by Eq. (9) with  $\hat{P}$  characterised by:

$$\frac{\hat{\eta}_T}{\hat{\eta}_t} = \prod_{j:t < T_j \leq T} \left( 1 - \frac{\delta(T_j, a)K(a) + \delta(T_j, -a)K(-a)}{\delta(T_j, a)^2 K(a) + \delta(T_j, -a)^2 K(-a)} \delta(T_j, Z_j) \right) \\ \times \exp \left( -\frac{(\delta(T_j, a)K(a) + \delta(T_j, -a)K(-a))^2}{\delta(T_j, a)^2 K(a) + \delta(T_j, -a)^2 K(-a)} \log S((T_{j+1} \wedge T) - T_j) \right).$$

We will establish the convergence of this model to the Black–Scholes model as  $a \rightarrow 0$  in Section 5. This example is intuitive but of little practical interest as the directing

intensity is too complex to be computed easily on real data. Estimating the parameters of the underlying geometric Brownian motion would require using estimation methods designed for latent variable models such as simulation based methods. These methods work well in large samples only. This would impose choosing small values for  $a$  so that a sufficiently large number of barrier crossing times could be observed.

To overcome these practical difficulties, we now abstract from the true stock price dynamics and model directly the marked point process corresponding to a choice of  $a$ . Our second example thus enables us to derive an easily implementable model.

#### 4.2. Example 2: Marked Poisson process

Our second example consists of a marked Poisson process with independent binomial marks. The corresponding compensator specification is:

**Assumption 3 (Marked Poisson process).** The compensator  $\nu(dt, dx)$  on  $\mathbb{R}_+ \times \{a, -a\}$  satisfies:

$$\nu(dt, dx) = \lambda dt K(dx),$$

with

$$K(dx) = p \quad \text{if } dx = a,$$

$$K(dx) = 1 - p \quad \text{if } dx = -a.$$

This example is particularly suited for empirical purposes since the constant parameters  $\lambda$  and  $p$  can easily be estimated from intraday price data (see the empirical illustration below). We get immediately from Proposition 3:

**Corollary 2 (Option price: marked Poisson process).** Under Assumption 3, the minimal call price is given by Eq. (9) with  $\hat{P}$  characterised by:

$$\begin{aligned} \frac{\hat{\eta}_T}{\hat{\eta}_t} = & \prod_{j:t < T_j \leq T} \left( 1 - \frac{\delta(T_j, a)p + \delta(T_j, -a)(1-p)}{\delta(T_j, a)^2 p + \delta(T_j, -a)^2 (1-p)} \delta(T_j, Z_j) \right) \\ & \times \exp \left( \lambda((T_{j+1} \wedge T) - T_j) \frac{(\delta(T_j, a)p + \delta(T_j, -a)(1-p))^2}{\delta(T_j, a)^2 p + \delta(T_j, -a)^2 (1-p)} \right). \end{aligned} \quad (10)$$

We will now investigate further the specific case when the interest rate is equal to zero before considering the more general case which will be used in the empirical applications. When the interest rate is set equal to zero ( $\rho=0$ ), we have an analytic formula for the call price. It is obtained by taking a weighted sum of recombining trees:

$$C(t, S_T) = \sum_{n=0}^{\infty} \frac{(\hat{\lambda}T)^n}{n!} e^{-\hat{\lambda}T} \left( \sum_{i=0}^n \frac{n!}{i!(n-i)!} \hat{p}^i (1-\hat{p})^{n-i} (S_0 e^{((2i-n)a)} - X)_+ \right), \quad (11)$$

with:

$$\hat{p} = \frac{1}{e^a + 1}, \quad (12)$$

$$\hat{\lambda} = (1 - \gamma)\lambda, \quad (13)$$

$$\gamma = \frac{(p(\exp(a) - 1) + (1 - p)(\exp(-a) - 1))^2}{p(\exp(a) - 1)^2 + (1 - p)(\exp(-a) - 1)^2}. \quad (14)$$

We recover the property of the standard binomial model (Cox et al., 1979) that the risk-neutral upmove ( $\hat{p}$ ) and downmove ( $1 - \hat{p}$ ) probabilities do not depend on historical probabilities. Formula (11) is a sum of standard binomial pricing formulae weighted by the probabilities of the exponential distribution. The pricing formula can thus be derived analytically by integrating w.r.t. to the exponential distribution and by weighting the different possible option payoffs by their binomial probabilities (see Fig. 3).

When  $\rho \neq 0$ , one cannot build recombining trees. However, the price of our option can be computed via Monte-Carlo integration by simulating the stock price process directly

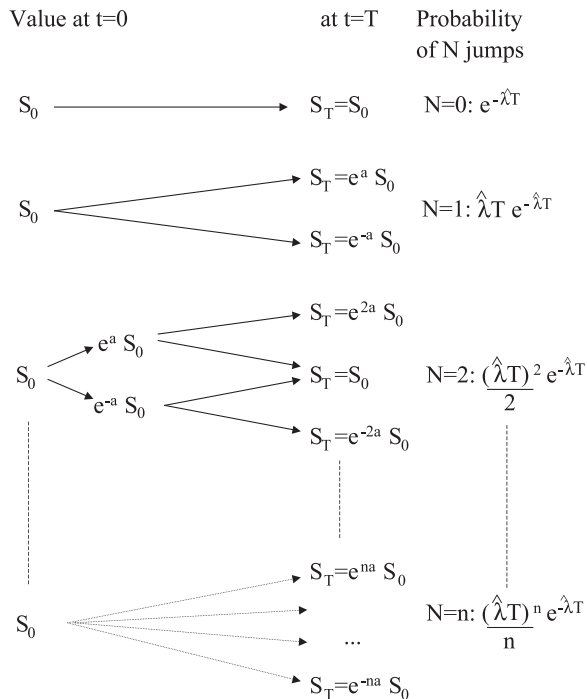


Fig. 3. Calculating the marked Poisson option price.

under  $\hat{P}$  (cf. Eq. (9)). The intensity and transition kernel under  $\hat{P}$  are characterised by (see proof in Appendix B):

$$\begin{aligned} d\hat{\lambda}_t &= \hat{\lambda}dt = (1 - \gamma)\lambda dt, \\ \hat{K}(t, a) &= \hat{p} = \frac{p(\hat{h}(T_j, a) + 1)}{I(T_j, a)}, \end{aligned}$$

with:

$$\begin{aligned} \hat{h}(T_j, x) &= -\delta(T_{j+1}, x) \frac{p\delta(T_{j+1}, a) + (1 - p)\delta(T_{j+1}, -a)}{p\delta^2(T_{j+1}, a) + (1 - p)\delta^2(T_{j+1}, -a)}, \\ I(T_j, a) &= 1 - \frac{(p\delta(T_{j+1}, a) + (1 - p)\delta(T_{j+1}, -a))^2}{p\delta^2(T_{j+1}, a) + (1 - p)\delta^2(T_{j+1}, -a)}. \end{aligned}$$

To close this section, it remains to exhibit the composition  $(\phi_t, \psi_t)$  of the hedging strategy which generates  $C(T, S_T)$ . The processes  $\phi$  and  $\psi$  represent the quantities held in the risky and riskless asset, respectively. The discounted price of the contingent claim is denoted:  $\tilde{C}(t, S_t) = C(t, S_t) \prod_{j: T_j \leq t} (1 + r_{T_j})^{-1}$ .

**Proposition 4 (Minimal trading strategy).** *Under Assumption 2, the minimal trading strategy  $(\phi_t, \psi_t)$  is given by:*

$$\phi_t = \frac{\int_E \tilde{\phi}_{T_j}(x) \delta(T_j, x) K(T_{j+1}, dx)}{\tilde{S}_{T_j} \int_E \delta^2(T_j, x) K(T_{j+1}, dx)}, \quad (15)$$

and:

$$\psi_t = \tilde{C}(t, S_t) - \phi_t \tilde{S}_t, \quad (16)$$

for  $t \in ]]T_j, T_{j+1}]$  with:  $\tilde{\phi}_{T_j}(x) = \tilde{C}(T_j, S_{T_j} e^x) - \tilde{C}(T_j, S_{T_j})$ .

Using the minimal martingale measure thus enables us to obtain an explicit form for the quantities to be held in the stock and in the riskless asset. This is one of the main advantages of the use of this measure over many other possible choices of EMM, such as the variance optimal measure or that extracted from time series of the underlying asset price and traded options (see Chernov and Ghysels, 2000, for example). Without these quantities, it is impossible to implement a hedging strategy.

As mentioned above, the cumbersome formula for the intensity of the MPP driven by the geometric Brownian motion makes the empirical estimation of the GBM-driven process difficult, while the marked Poisson process is particularly simple to implement. The two models are nonnested and we therefore cannot compare their performance without relying on complicated tests for nonnested hypotheses which are beyond the scope of this paper.

## 5. Convergence to the Black–Scholes model

We now study the convergence to the Black–Scholes model when the increment  $a$  shrinks to zero. In the Black–Scholes model, the stock price evolves according to a geometric Brownian motion  $\tilde{S}_t = S_0 \exp((m - (s^2/2)t + sW_t)$ , and the savings account value according to:  $\exp(\rho t)$ . We use  $\xrightarrow{\mathcal{P}}$  for convergence in probability, and  $\xrightarrow{\mathcal{L}(\mathbb{D})}$  for weak convergence on  $\mathbb{D}$  the space of cadlag functions endowed with the customary Skorokhod topology. We index all relevant quantities by  $a$ , and convergence should be understood when letting  $a$  go to zero.

**Proposition 5 (Convergence).** *Under Assumption 1, if:*

$$\int_0^t \int_E \delta v^a(dt, dx) \xrightarrow{\mathcal{P}} (m - \rho)t, \quad \int_0^t \int_E \delta^2 v^a(dt, dx) \xrightarrow{\mathcal{P}} s^2 t, \quad (17)$$

then:

$$(\tilde{S}_t^a, \hat{\eta}_t^a) \xrightarrow{L(\mathbb{D}^2)} \left( \tilde{S}_t e^{-\rho t}, \varepsilon \left( \frac{m - \rho}{s} W_t \right) \right). \quad (18)$$

Proposition 5 embodies the convergence of the incomplete model based on the marked point process to the Black–Scholes model. Indeed the proposition states the joint convergence of the sequence of discounted stock prices and Radon–Nikodym derivatives of the minimal measure. We recognize in Eq. (18) the well-known change of measure from the historical probability to the risk-neutral measure of the limiting complete model. As usual, convergence of contingent claim prices will be ensured under appropriate continuity and equiintegrability conditions on the payoffs.

Now we analyze in some details the simple example of the marked Poisson model. For computational convenience, we set the interest rate equal to zero.

**Proposition 6 (Convergence: marked Poisson process).** *Under Assumption 3 and  $\rho = 0$ , if  $p^a - 1/2 \sim ma/(2s^2)$  and  $\lambda^a \sim s^2/a^2$ , then:*

$$(S_t^a, \hat{\eta}_t^a) \xrightarrow{L(\mathbb{D}^2)} \left( S_0 \varepsilon(mt + sW_t), \varepsilon \left( \frac{m}{s} W_t \right) \right).$$

Proposition 6 gives the condition on the probability  $p^a$  and the directing intensity  $\lambda^a$  so that the marked Poisson model coincides with the Black–Scholes model in the limit.

Fig. 4 shows the smooth form of the convergence obtained for European call option prices when using the closed form expression (Eq. (11)). The numerical example is designed with:  $S_0 = 100$ ,  $m = 5\%$ ,  $s = 25\%$ ,  $K = 100\%$ , and  $T = 0.25$ . The straight line is the Black–Scholes price equal to 4.9835, while the dashed line gives the price computed by the marked Poisson model with  $p^a = (ma^2/s^2 + (1 - e^{-a}))/e^a$  and  $\lambda^a = s^2/a^2$ .

Finally, we are able to deliver the analogous result to Proposition 6 for the more complex latent geometric Brownian motion case. Some numerical results (not reported

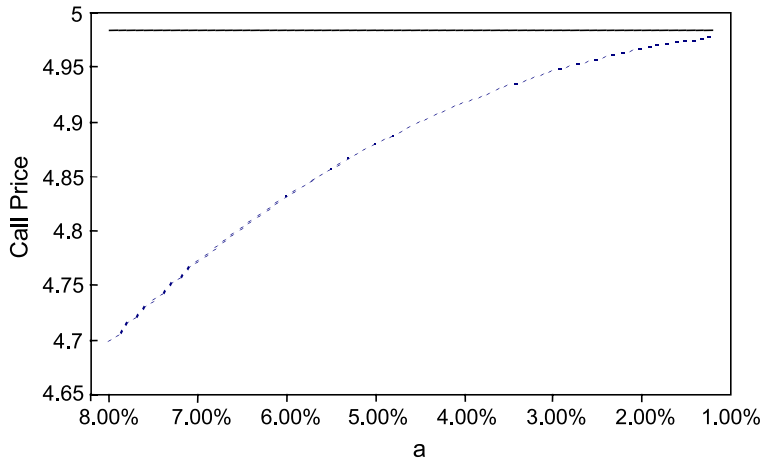


Fig. 4. Convergence of option prices: marked Poisson process.

here) show that the convergence is very similar but less smooth than the previous case because of the use of simulations.

**Proposition 7 (Convergence: latent geometric Brownian motion).** *Under Assumption 3 and  $\rho = 0$ :*

$$(\tilde{S}_t^a, \hat{\eta}_t^a) \xrightarrow{L(\mathbb{D}^2)} \left( S_0 \varepsilon(mt + sW_t), \varepsilon\left(\frac{m}{s} W_t\right) \right).$$

## 6. Empirical illustrations

This section illustrates the empirical application of the marked Poisson model (Example 2) to the pricing of European call options.<sup>3</sup> The model parameters  $\lambda$  and  $p$  are estimated from intraday transaction data first on the IBM stock price listed on the New York Stock Exchange (NYSE) and then on France Telecom stock price listed on the Paris Bourse. We also consider the CAC 40 index as a final example.

### 6.1. IBM transaction data

We start by focussing on IBM transaction data. The data were extracted from the Trades and Quotes Database (TAQ Database) released by the NYSE. The observations are the trades of IBM stock recorded every second from market opening (9:30:00) to market closure (16:00:00). They consist of 486,513 transactions beginning on Thursday January 2nd, 1997 and ending on Wednesday September 30th, 1997 (9 months).

The estimators are obtained by maximum likelihood. The estimator of  $\lambda$  is the inverse of the empirical average of durations between two successive relative price changes (recall

<sup>3</sup> Gauss programs developed for this section are available on request.

that if events follow a Poisson distribution, time intervals are exponentially distributed). The estimator of  $p$  is the observed proportion of positive variations. To test the null hypothesis that the durations are exponentially distributed, we use a test statistic based on the equality between the mean and the standard deviation (see, e.g. Chesher and Spady, 1991; Chesher et al., 1999), and a Bootstrap procedure to account for finite sample level distortions. We consider models with absolute variation sizes  $a$  ranging from 0.25% to 5%. Common practice on the market is to rebalance after relative variations by 3% or 4%. The observed path of  $\log S_t$  for the IBM stock is plotted in Fig. 5. It corresponds to the period from January to September 97 with the jump size  $a = 3\%$ .

Table 1 reports the estimators of  $\lambda$  and  $p$ , with their respective standard deviations within brackets, and the number (Nb.) of observed relative price changes for each absolute variation size. The intensity estimate is in 1/s and should be multiplied by 60 (s)  $\times$  60 (min)  $\times$  6.5 (h)  $\times$  250 (days) = 5,850,000 in order to get an annualized intensity. The hypothesized exponential behavior of the durations is rejected at the 5% significance level for models with too narrow rebalancing bounds ( $a \leq 2\%$ ). Values of the test statistic are also reported in Table 1. For the significant models, a jump is expected every 2, 4 and 6 days for  $a = 3\%$ , 4%, and 5%, respectively. The upmove probabilities are not far from one half.

With these estimates, we may compare option prices given by the marked Poisson model and the Black–Scholes model. We use a historical approach to estimate the volatility in the Black–Scholes model (unfortunately, we do not have option data to compute implied volatilities in this first empirical illustration). The volatility estimator is the standard deviation of daily closing prices from Thursday January 2nd, 1997 to Thursday March 27th, 1997 (3 months). This choice corresponds to market standards. We get a volatility estimate of 2.19%, which gives an annualized volatility ( $\times \sqrt{250}$ ) of 34.67%. This is also close to the annualized volatility computed on the whole dataset: 31.98%. The interest rate is taken equal to 5%. The maturity of the European call option is 3 months. The initial stock price  $S_0$  is normalized to 100 and strike prices  $K$  expressed in

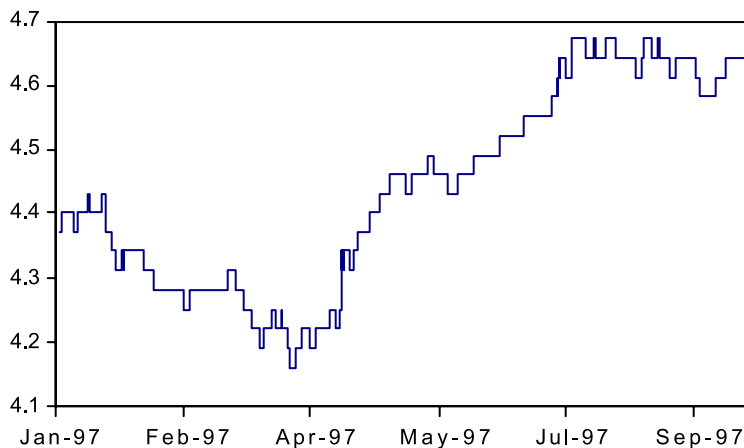


Fig. 5. MPP based on IBM stock price with  $a = 3\%$ .



Table 1

Parameter estimates for IBM stock (in percent)

| $a$  | $\lambda$       | $p$          | Test statistic | Nb.   |
|------|-----------------|--------------|----------------|-------|
| 0.25 | 0.2849 (0.0054) | 50.52 (0.45) | 39891.99*      | 12501 |
| 0.5  | 0.0846 (0.0027) | 50.87 (0.82) | 8455.86*       | 3715  |
| 0.75 | 0.0407 (0.0017) | 51.20 (1.18) | 2083.03*       | 1789  |
| 1    | 0.0242 (0.0013) | 51.55 (1.53) | 1241.35*       | 1065  |
| 2    | 0.0050 (0.0004) | 53.60 (3.35) | 14.04*         | 222   |
| 3    | 0.0021 (0.0002) | 55.91 (5.15) | 0.03           | 93    |
| 4    | 0.0010 (0.0001) | 59.09 (7.41) | 0.95           | 44    |
| 5    | 0.0007 (0.0001) | 60.00 (8.94) | 0.59           | 30    |

\* Exponential assumption rejected at 5%.

percentage of spot price (spot moneyness degree) are between 90% and 110%. Table 2 only gives the results for the statistically significant models of Table 1. Monte-Carlo integration with 200,000 replications is used to calculate expectation (under MMM) in a few seconds. To reduce the variance of the simulations, we use  $\tilde{S}_T$  as control variate device. Note that the empirical martingality of  $\tilde{S}_T$  is satisfied up to a precision of at least  $10^{-3}$ . All call prices are below their Black–Scholes equivalents. From put-call parity, it also implies that all put prices will be below their Black–Scholes counterparts. The prices for  $a = 5\%$  are higher than for  $a = 4\%$ . This is due to the compensation of the lower intensity by the higher upmove probability.

Tick-by-tick transaction data have been reported to exhibit daily seasonality (see, e.g. Engle and Russell, 1997). On the US stock market, the activity is higher at the beginning and at the end of the trading sessions but lower around lunch time. Our pricing approach is robust to this feature (although seasonality has no impact when we consider  $a \geq 2\%$  as the average duration between two jumps is larger than 1 day). In order to take seasonality in our setup, we would just have to include a deterministic time component in the intensity.

Table 2

Option prices (in percent)

| $a$ | $K/S_0$ | Poisson | BS      |
|-----|---------|---------|---------|
| 3   | 90      | 0.13314 | 0.13529 |
|     | 95      | 0.09976 | 0.10231 |
|     | 100     | 0.07215 | 0.07503 |
|     | 105     | 0.05097 | 0.05340 |
|     | 110     | 0.03482 | 0.03693 |
| 4   | 90      | 0.12858 | 0.13529 |
|     | 95      | 0.09415 | 0.10231 |
|     | 100     | 0.06583 | 0.07503 |
|     | 105     | 0.04491 | 0.05340 |
|     | 110     | 0.02973 | 0.03693 |
| 5   | 90      | 0.13052 | 0.13529 |
|     | 95      | 0.09660 | 0.10231 |
|     | 100     | 0.06907 | 0.07503 |
|     | 105     | 0.04805 | 0.05340 |
|     | 110     | 0.03284 | 0.03693 |

## 6.2. France Telecom stock options

We now want to confront our model to observed option prices. We have collected intraday France Telecom (FT) stock data for the period from February 22nd, 1999 to end July 1999. After removing all transactions outside trading hours, we obtain a set of 487,981 observations.

The FT stock is used as underlying asset for various options listed on the MONEP in Paris. Those we consider in this paper are long-term European type calls and puts. We have collected the prices of 24 transactions which have occurred over the period August 2nd, 1999 to September 23rd, 1999. These trades are based on 12 calls and 12 puts with various strike prices ranging from 55 to 100 and times-to-maturity from 6 to 20 months.

Parameters of the marked Poisson process are estimated over the entire sample with  $\alpha = 3\%$ . The historical volatility to be used in the Black–Scholes model is calculated over the 3 months preceeding our option sample. The 3-month PIBOR was used as risk-free interest rate. The estimates for  $\lambda$  and  $p$  are respectively found to be 112.5% and 49.3%, while the historical volatility is 33.52%.

We calculate option prices using 50,000 Monte-Carlo simulations directly under the MMM with control variate.

Table 3 summarises our main findings. Both absolute (based on the absolute difference in prices between the model and the market) and relative (based on percentage difference) mean squared errors (MSE) are reported for the marked Poisson model, the Black–Scholes model with historical volatility and the Black–Scholes model with “best” volatility. By “best”, we mean the volatility which minimises the MSE in the Black–Scholes model.

We see that in our sample, the marked Poisson model outperforms not only the Black–Scholes model with historical volatility but also the Black–Scholes model with any choice of volatility parameter. This is probably due to the fact that our pure jump model generates smiles (as most jump models do) and is therefore able to price more accurately deep in- and out-of-the-money options. Fig. 6 is a plot of the smile generated by our model for a 1-year horizon at the end of August 1999 ( $r = 2.68\%$ ), with  $\alpha = 3\%$  and  $S_0 = 100$ . It has the “smirk” shape frequently reported in equity derivatives markets (see, e.g. Dumas et al., 1998).

Finally, we can turn to Fig. 7 which compares frequency plots for terminal stock returns ( $\ln(S_T/S_0)$ ) for the marked Poisson model under  $\hat{P}$  and for the Black–Scholes model taking a 1-year horizon. The risk-neutral marked Poisson distribution is fat tailed, as expected from stock prices dynamics exhibiting jumps.

Table 3  
Mean squared errors: Black–Scholes and marked Poisson

| Model                      | Absolute | Relative |
|----------------------------|----------|----------|
| Marked Poisson             | 0.090    | 0.0024   |
| BS (historical volatility) | 0.303    | 0.0077   |
| BS (best volatility)       | 0.100    | 0.0028   |

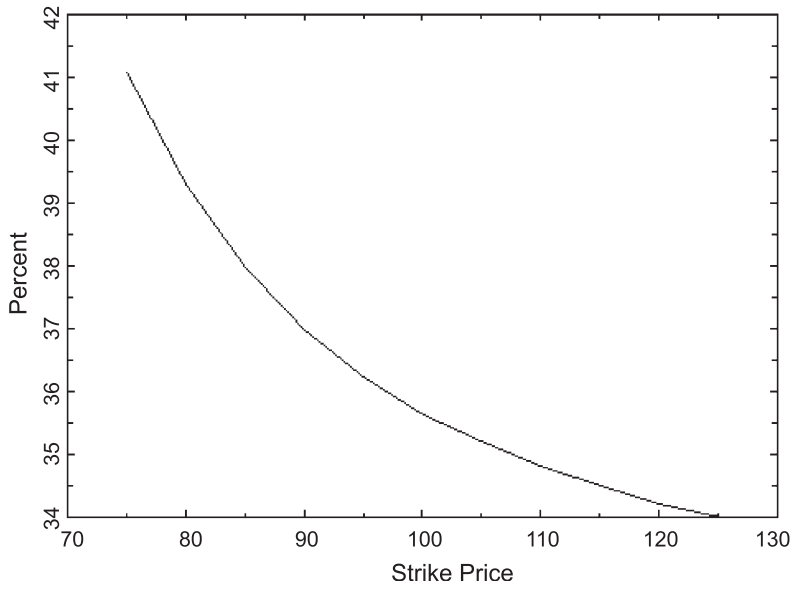


Fig. 6. Implied volatility smile from marked Poisson model on France Telecom calls.

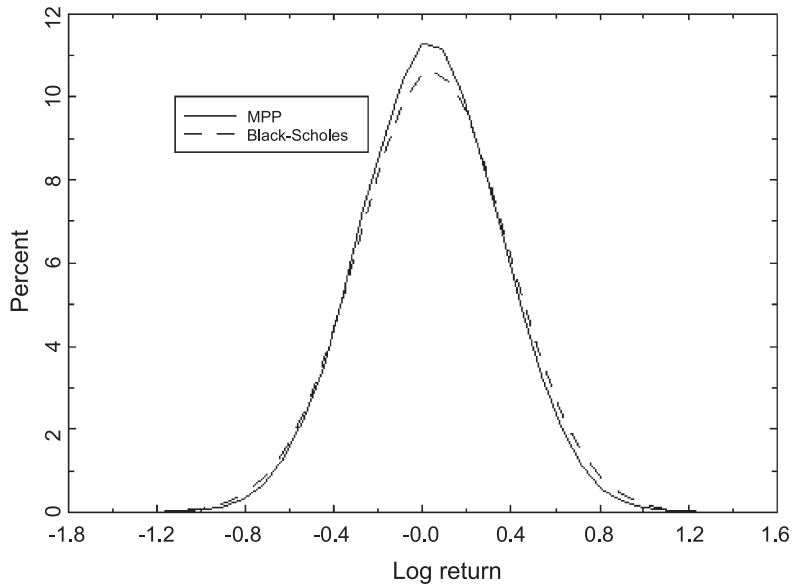


Fig. 7. Marked Poisson versus Black-Scholes risk-neutral frequency plots.

### 6.3. CAC 40 index options

The above two examples have demonstrated the ability of MPP-based models to price options on individual stocks. As a last example, we now consider index options. This will enable us to compare the pricing performance of the Black–Scholes model and the MPP model when many strike prices are available. Options on individual stocks are typically only traded with a limited number of exercise prices while index options offer a wider choice.

The options data consist of prices of European calls and puts maturing at the end of November and December 1999. These options are again traded on the MONEP and the transactions were recorded during the month of October 1999. The options have therefore between 1 and 3 months before expiration. Overall, we have 4000 observations: 3456 with maturity November and 544 for December. Fig. 8 is a plot of the implied volatilities of the December options as a function of (call) moneyness. We can see each “layer” as one volatility smile observed on a given day.

In the following, we define an option to be at-the-money (ATM) if the current stock price is within  $\pm 2\%$  of its strike price. Call (resp. put) options will be considered in-the-money (ITM) if the stock price is above 102% (resp. below 98%) of the strike price and out-of-the-money otherwise.

We replicate the calculations described in the previous section. Again, we choose  $\alpha = 3\%$ , and simulate 50,000 paths directly under the MMM. The parameters of the MPP are  $\lambda = 45.97$  and  $\hat{p} = 61.11\%$ . The results are summarised in Table 4.

The “overall” column corresponds to all strike prices, while ITM, ATM and OTM columns are the subsamples defined above. The superiority of the MPP model over the *historical* Black–Scholes model is again obvious as far as mean squared errors are concerned. This is not surprising as historical volatility tended to be much lower than

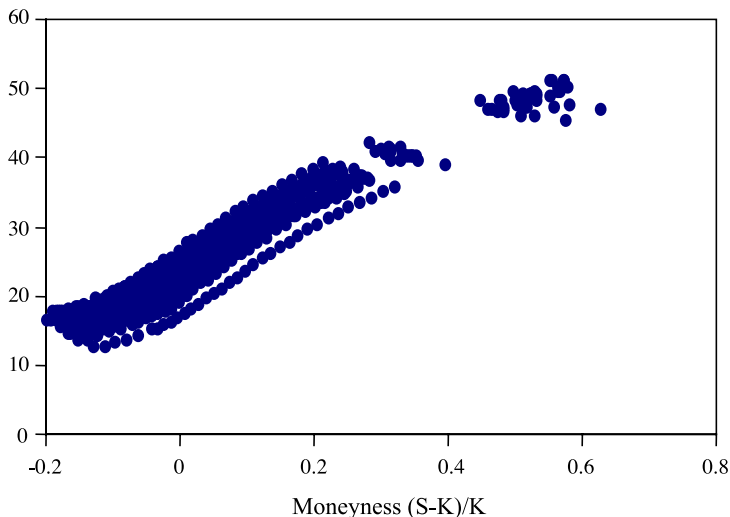


Fig. 8. Volatility smile on CAC 40 options (October 1999).

Table 4

Mean squared errors: Black–Scholes and MPP

| Model                      | Overall | ITM   | ATM   | OTM   |
|----------------------------|---------|-------|-------|-------|
| <i>November maturity</i>   |         |       |       |       |
| Marked Poisson             | 0.266   | 0.026 | 0.101 | 0.332 |
| BS (historical volatility) | 0.347   | 0.039 | 0.230 | 0.402 |
| BS (best volatility)       | 0.338   | 0.029 | 0.105 | 0.401 |
| <i>December maturity</i>   |         |       |       |       |
| Marked Poisson             | 0.303   | 0.010 | 0.027 | 0.405 |
| BS (historical volatility) | 0.486   | 0.034 | 0.130 | 0.627 |
| BS (best volatility)       | 0.332   | 0.010 | 0.035 | 0.441 |

implied volatility over this period. This led to an underpricing of options using the historical Black–Scholes model.

More interestingly, we again find that no choice of volatility (constant across options in a given subsample) leads to the Black–Scholes model outperforming the MPP model on our sample. There does not seem to be any significant difference between shorter and longer pricing performance.

## 7. Concluding remarks

This paper proposes explicit formulae for pricing contingent claims when the underlying price follows a marked point process. The formulae rely on a minimal martingale approach. It is particularly suited for discrete hedging in which portfolios are only rebalanced after fixed relative price variations. The practical use of the pricing model is illustrated with a marked Poisson model estimated on high frequency transaction data. The marked Poisson model is shown to be able to provide a good fit for large variation sizes but not for small sizes. However, our theoretical framework encompasses very general dynamics for the stock price. A wide range of empirical applications based on more sophisticated econometric models, e.g. developed for microstructure theory, can thus be proposed. For example, [Prigent et al. \(1999, 2001\)](#) use a specification where durations between jumps follow an Autoregressive Conditional Duration (ACD) model as introduced by [Engle and Russell \(1997, 1998\)](#) and compare the prices of vanilla and exotic options to their Black–Scholes counterparts. These refinements are useful to incorporate clustering in durations as well as time varying upmove probabilities which are usually observed for small values of  $a$ . These extensions however come at the cost of greater complexity for practical implementation.

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## Appendix A. Proofs of propositions

In order to obtain concise expressions in the proofs, we will often use (as in JS) the notation  $x^* \mu_t$  for  $\int_0^t \int_E x d\mu$ . Recall that  $\varepsilon(\cdot)$  denotes the Doléans–Dade exponential.

**Proof of Proposition 1.** The proof consists in finding the adequate restriction on the change of measure  $dQ/dP$  so that the discounted stock price  $\tilde{S}$  is a local martingale under the new measure  $Q$ . It parallels the developments in the work of Buhlman et al. (1996).

We denote by  $M_{\text{loc}}(Q)$  the set of all local martingales under  $Q$ . The density process  $\eta$  of  $Q$  relative to  $P$  can be rewritten thanks to the theorem of representation of martingales:

$$\eta_t = \varepsilon \left( \int_0^t \int_E H(s, x) d(\mu - \nu) \right),$$

where the predictable process  $H$  satisfies some integrability conditions (JS, p. 172) and  $H+1 > 0$  a.s. since  $\eta$  is strictly positive.

Furthermore,  $\tilde{S}_t \in M_{\text{loc}}(Q)$  if and only if  $\tilde{S}_t \eta_t \in M_{\text{loc}}(P)$ . From Ito's and Yor's formulas (Protter, 1990 p. 7; Revuz and Yor, 1994 p. 354), we have:

$$\tilde{S}_t \eta_t = \varepsilon \left( \int_0^t \int_E \delta(s, x) d\mu + H(s, x) d(\mu - \nu) + H(s, x) \delta(s, x) d\nu \right).$$

This implies that  $\int_0^t \int_E (H(s, x) + 1) \delta(s, x) d\nu$  should be equal to zero in order to get  $\tilde{S}_t \eta_t \in M_{\text{loc}}(P)$ , which ends the proof.  $\square$

**Proof of Proposition 2.** We know from Ansel and Stricker (1992, 1993) that if  $1 - \alpha \Delta \tilde{M} > 0$  a.s. and  $E^P[\sup_{0 \leq s \leq T} \tilde{S}_s^2] < +\infty$ , the minimal martingale measure is a probability measure characterised by its density process  $\eta$  relative to  $P$ :

$$\hat{\eta} = \varepsilon(-\alpha \cdot \tilde{M}),$$

with  $\tilde{M}$  a locally bounded integrable martingale and  $\alpha$  a predictable process satisfying:

$$\tilde{S} = \tilde{M} + \alpha \cdot \langle \tilde{M}, \tilde{M} \rangle,$$

and where the dot denotes the stochastic integration of a predictable process w.r.t. a semimartingale. The predictable process  $\alpha$  is given by the relation:

$$\alpha_t d\langle \tilde{M}, \tilde{M} \rangle_t = d\tilde{A}_t, \quad (19)$$

where  $\tilde{A}$  is the predictable bounded part of  $\tilde{S}$ .

In our case, the martingale part of  $\tilde{S}$  is equal to:  $\tilde{M} = (\tilde{S}\delta)^*(\mu - \nu)$  and its predictable quadratic variation (angle bracket) equal to:  $\langle \tilde{M}, \tilde{M} \rangle = (\tilde{S}\delta)^2 * \nu$  (JS, 1.33, p. 73). Furthermore,  $\tilde{A} = (\tilde{S}\delta)^* \nu$  which gives Eq. (8) using relation (??). Besides, since the jump size is bounded,  $\hat{\eta}$  is a strictly positive local martingale and the minimal martingale measure is a probability measure.  $\square$

**Proof of Proposition 4.** We follow Colwell and Elliott (1993). On the one hand, since  $\tilde{C}$  is a  $\tilde{P}$ -martingale, it comes from the martingale representation theorem and Ito's lemma that:

$$\tilde{C}(t, S_t) = \tilde{C}(0, S_0) + \tilde{\phi}^*(\mu - \hat{\nu})_t, \quad (20)$$

with  $\tilde{\phi}_t(x) = \tilde{C}(T_j, S_{T_j} e^x) - \tilde{C}(T_j, S_{T_j})$  for  $t \in ]T_j, T_{j+1}]$ . On the other hand, we have (see, Schweizer, 1991):

$$\tilde{C}(t, S_t) = \tilde{C}(0, S_0) + (\phi \tilde{S} \delta)^* \mu_t + \Gamma_t, \quad (21)$$

where  $\Gamma_t$  is a martingale under  $P$ .

This process represents the cost associated with the trading strategy, and can be rewritten from Eqs. (20) and (21):

$$\Gamma = \tilde{\phi}^*(\mu - \hat{\nu}) - (\phi \tilde{S} \delta)^* \mu.$$

Since it is a martingale under  $P$ , we deduce from:

$$\tilde{\phi}^*(\mu - \hat{\nu}) = \tilde{\phi}^*(\mu - \nu) + \tilde{\phi}^*(\nu - \hat{\nu}),$$

and:

$$(\phi \tilde{S} \delta)^* \mu = (\phi \tilde{S} \delta)^*(\mu - \nu) + (\phi \tilde{S} \delta)^* \nu,$$

that the following equality should hold:

$$\tilde{\phi}^*(\nu - \hat{\nu}) = (\phi \tilde{S} \delta)^* \nu.$$

Using  $\hat{\nu} = (\hat{H} + 1)\nu$  and the expression of  $\hat{H}$ , the result follows.  $\square$

**Proof of Proposition 5.** Condition (17) and the jump boundedness ensure the convergence of the first Doléans–Dade exponential  $\varepsilon(\delta^* \mu_t^a)$  to  $\varepsilon((m - \rho)t + sW_t)$  (JS, 3.11, p. 432). Since the martingale part of the discounted price is uniformly tight (the jumps are bounded and the predictable part is increasing), we conclude the convergence of the second Doléans–Dade exponential  $\hat{\eta}_t^a = \varepsilon((- \alpha \tilde{S} \delta)^*(\mu^a - \nu^a)_t)$  as in Prigent (1999, Proposition 3.3) or Lesne et al. (2000, Proposition 1).  $\square$

**Proof of Proposition 6.** Condition (17) of Proposition 5 can be rewritten as:

$$\lambda^a[(e^a - 1)p^a + (e^{-a} - 1)(1 - p^a)] \rightarrow m,$$

$$\lambda^a[(e^a - 1)^2 p^a + (e^{-a} - 1)^2 (1 - p^a)] \rightarrow s^2.$$

From Taylor expansions, it can be verified that both conditions are satisfied if:  $p^a - 1/2 \sim ma/(2s^2)$  and  $\lambda^a \sim s^2/a^2$ . The stated result is then a direct consequence of Proposition 5.  $\square$

**Proof of Proposition 7.**

- (i) The weak convergence of the stock price is immediately deduced from the construction of  $X^a$  since:

$$\sup_{t \in [0, T]} \left| X_t^a - \left( \left( m - \frac{s^2}{2} \right) t + s W_t \right) \right| \leq a$$

(see JS, 3.30 and 3.31, p. 316 with  $t^a = X_t^a - (m - (s^2/2)t + sW_t)$  and  $Y_t^a = (m - (s^2/2)t + sW_t)$ ).

- (ii) To deduce the weak convergence of the density process, we need first to establish that  $X^a$  has the property UT (uniform tightness, see [Mémén and Slominsky, 1991](#) for a definition). In order to do so, we will need the following key lemma which gives the behavior of the Laplace transform of the expectation of the number of jumps for small  $a$ . Its proof is given below.

**Lemma.** If  $N_t^a$  denotes the number of jumps on the time interval  $[0, t]$ , we have for small  $a$ :

$$\int_0^\infty e^{-\xi t} \mathbb{E}[N_t^a] dt \sim \frac{1}{\xi^2} \frac{s^2}{a^2}.$$

Now, recall that  $X_t^a = (x * v^a)_t + x * (\mu^a - v^a)_t$ . From [Condition \(ii\) 2\) of Theorems 1–4 of Mémén and Slominsky \(1991\)](#), we have to check that  $\text{Var}(B^a)$  is  $\mathbb{P}^a$ -stochastically bounded where  $B^a$  is the bounded variation part of  $X^a$  (i.e.  $x * v^a$ ). Here, since  $x * v^a$  is an increasing process, we have:

$$\sup_{[0, T]} \text{Var}(x * v^a) = \left( \int_E x K^a(dx) \right) \times \int_0^T \lambda_t^a dt.$$

Moreover, for all  $L > 0$ , we get:

$$\mathbb{P} \left[ \left( \int_E x K^a(dx) \right) \times \int_0^T \lambda_t^a dt > L \right] \leq \frac{1}{L} \left( \int_E x K^a(dx) \right) \times \mathbb{E} \left[ \int_0^T \lambda_t^a dt \right].$$

Recall that  $\mathbb{E}[\int_0^t \lambda_s^a ds] = \mathbb{E}[N_t^a]$ . Hence, from the above lemma, we can deduce by continuity of the inverse Laplace transform that, for small  $a$  and fixed  $t$ ,  $\mathbb{E}[\int_0^t \lambda_s^a ds] \sim (s^2/$



$a^2)t$ . Note also that  $\int_E xK^a(dx) \sim (m'/s^2)a^2$  since  $K(a) \sim (1/2)+(1/2) (m'a/s^2)$ . Therefore, we get:

$$\forall \epsilon > 0, \exists L, \mathbb{P}[\sup_{[0,T]} \text{Var}(x^*v^a) > L] < \epsilon.$$

So  $\text{Var}(B^a)$  is  $\mathbb{P}^a$ -stochastically bounded, and we conclude to the uniform tightness of  $X^a$ .

Second, since  $(X_t)_{t \leq m't+sW_t}$  is continuous, and since  $X^a$  converges to  $X$  and satisfies the property UT, we deduce from [Proposition 2.2 of Mémin and Slominsky \(1991\)](#) that the martingale part  $x^*(\mu^a - v^a)$  of  $X^a$  converges to the martingale  $(sW_t)_t$  and that the bounded variation part  $x^*v^a$  converges to the bounded variation part  $m't$ . Therefore, we get from direct Taylor expansions:

$$\delta^*v_t^a \xrightarrow{P} m't,$$

$$\delta^2*v_t^a \xrightarrow{P} s^2t.$$

The last condition guarantees the convergence of the predictable compensator of the martingales  $\delta^*(\mu^a - v^a)$ . Moreover, since the jumps of these martingales are uniformly bounded, the sequence  $(\delta^*(\mu^a - v^a))_a$  satisfies the UT (uniform tightness) condition. From Eq. (8), a standard Taylor expansion of  $\alpha_t^a$  proves that  $\alpha^a$  weakly converges to  $\alpha$  with  $\alpha_t^a = (1/\tilde{S}_t)(m/s^2)$ .

From all these results, the convergence of the second Doléans–Dade exponential  $\hat{\eta}_t^a = \varepsilon((-\alpha\tilde{S}\delta^*)(\mu^a - v^a))_t$  follows immediately as in [Prigent \(1999, Proposition 3.3\)](#) or [Lesne et al. \(2000, Proposition 1\)](#).

**Proof of Lemma.** Let us first compute the Laplace transform of  $\mathbb{E}[N_t^a]$ . We denote by  $T_{-a,+a}$  the first time the Brownian motion with drift escapes from the interval  $[-a, +a]$ :

$$T_{-a,+a} = \inf\{u > 0, W_u + m'u \notin [-a, +a]\}.$$

Consider the sequence  $(\epsilon_k)_k$  of i.i.d. random variables where all  $\epsilon_k$  have the same distribution as  $T_{-a,+a}$ . Then for all  $k \in \mathbb{N}^*$ , the following equality is verified:

$$\mathbb{P}[N_t^a \geq k] = \mathbb{P}[\epsilon_1 + \dots + \epsilon_k \leq t].$$

Moreover, recall that if  $\mathbb{P}$  is a probability on  $\mathbb{R}_*^+$  with a density  $p$  then:

$$\begin{aligned} \int_0^\infty e^{-\xi t} P(t) dt &= \int_0^\infty e^{-\xi t} dP(t) = [e^{-\xi t} P(t)]_0^\infty + \xi \int_0^\infty e^{-\xi t} P(t) dt \\ &= \xi \int_0^\infty e^{-\xi t} P(t) dt. \end{aligned}$$

Let us denote:

$$L_k(\xi) = \xi \int_0^\infty e^{-\xi t} \mathbb{P}[\epsilon_1 + \dots + \epsilon_k \leq t] dt,$$

which is equal to  $\xi \int_0^\infty e^{-\xi t} \mathbb{P}[N_t^a \geq k] dt$ , and also equal to  $(L_1(\xi))^k$  by independence of the  $\epsilon_k$ . Since:

$$\mathbb{P}[N_t^a = k] = \mathbb{P}[N_t^a \geq k] - \mathbb{P}[N_t^a \geq k + 1],$$

we get:

$$\begin{aligned} \xi \int_0^\infty e^{-\xi t} \mathbb{E}[N_t^a] dt &= \xi \int_0^\infty e^{-\xi t} \left( \sum_{k=1}^\infty k \mathbb{P}[N_t^a = k] \right) dt \\ &= \sum_{k=1}^\infty k \left( \xi \int_0^\infty e^{-\xi t} \mathbb{P}[N_t^a \geq k] dt - \xi \int_0^\infty e^{-\xi t} \mathbb{P}[N_t^a \geq k + 1] dt \right) \\ &= \sum_{k=1}^\infty k (L_k(\xi) - L_{k+1}(\xi)) = \sum_{k=1}^\infty k L_k(\xi) - \sum_{k=1}^\infty (k + 1) L_{k+1}(\xi) + \sum_{k=1}^\infty L_{k+1}(\xi), \end{aligned}$$

which finally gives:

$$\xi \int_0^\infty e^{-\xi t} \mathbb{E}[N_t^a] dt = \sum_{k=1}^\infty L_1(\xi)^k = \frac{L_1(\xi)}{1 - L_1(\xi)}.$$

For  $m' = 0$ , we have (exponential martingale):

$$\mathbb{E} \left[ \exp \left( \xi W_{T_{-a,+a}} - \frac{\xi^2}{2} T_{-a,+a} \right) \right] = 1,$$

from which we deduce:

$$\mathbb{E} \left[ \mathbb{E}[\exp(\xi W_{T_{-a,+a}}) | \mathcal{F}_{T_{-a,+a}}] \exp \left( -\frac{\xi^2}{2} T_{-a,+a} \right) \right] = 1,$$

and:

$$Ch(\xi a) \mathbb{E} \left[ \exp \left( -\frac{\xi^2}{2} T_{-a,+a} \right) \right] = 1.$$

Therefore:

$$L_1(\xi) = \frac{1}{Ch(a\sqrt{2\xi})}.$$

For  $m' \neq 0$  and for all  $t > 0$ ,

$$\mathbb{E}[\exp(-\xi T_{-a,+a}) \mathbf{1}_{T_{-a,+a} < t}] = \mathbb{E} \left[ \frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} \Big|_{\mathcal{F}_t} \exp(-\xi T_{-a,+a}) \mathbf{1}_{T_{-a,+a} < t} \right]$$

where  $\tilde{W}_u = m'u + W_u$  is a Brownian motion under  $\tilde{\mathbb{P}}$  by applying Girsanov theorem. Now, since  $d\mathbb{P}/d\tilde{\mathbb{P}}|_{\mathcal{F}_t} = \exp(m' \tilde{W}_t - (m'^2/2)t)$ , we get:

$$\begin{aligned}
 & \mathbb{E} \left[ \frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} \middle|_{\mathcal{F}_t} \exp(-\xi T_{-a,+a}) \mathbf{I}_{T_{-a,+a} < t} \right] \\
 &= \mathbb{E} \left[ \mathbb{E} \left[ \exp \left( m' \tilde{W}_t - \frac{m'^2}{2} t \right) \middle| \mathcal{F}_{T_{-a,+a}} \right] \exp(-\xi T_{-a,+a}) \mathbf{I}_{T_{-a,+a} < t} \right] \\
 &= \mathbb{E} \left[ \exp \left( m' \tilde{W}_{T_{-a,+a}} - \left( \xi + \frac{m'^2}{2} \right) T_{-a,+a} \right) \mathbf{I}_{T_{-a,+a} < t} \right] \\
 &= \mathbb{E} \left[ \mathbb{E} [\exp(m' \tilde{W}_{T_{-a,+a}}) | \mathcal{F}_{T_{-a,+a}}] \exp \left[ - \left( \xi + \frac{m'^2}{2} \right) T_{-a,+a} \right] \mathbf{I}_{T_{-a,+a} < t} \right] \\
 &= Ch(m'a) \mathbb{E} \left[ \exp \left[ - \left( \xi + \frac{m'^2}{2} \right) T_{-a,+a} \right] \mathbf{I}_{T_{-a,+a} < t} \right].
 \end{aligned}$$

Then, by convergence with respect to  $t$ , we get:

$$\begin{aligned}
 \mathbb{E} [\exp(-\xi T_{-a,+a})] &= Ch(m'a) \mathbb{E} \left[ \exp \left( \left( -\xi - \frac{m'^2}{2} \right) T_{-a,+a} \right) \right] \\
 &= \frac{Ch(m'a)}{Ch(a\sqrt{2\xi + m'^2})}.
 \end{aligned}$$

Finally:

$$\int_0^\infty e^{-\xi t} \mathbb{E}[N_t^a] dt = \frac{1}{\xi} \frac{Ch(m'a)}{Ch(a\sqrt{2\xi + m'^2}) - Ch(m'a)},$$

and, for the process  $(m' t + s W_t)_t$  with the volatility  $s$ :

$$\int_0^\infty e^{-\xi t} \mathbb{E}[N_t^a] dt = \frac{1}{\xi} \frac{Ch(m'a/s^2)}{Ch(a/s\sqrt{2\xi + (m'^2/s^2)}) - Ch(m'a/s^2)}.$$

Then a Taylor expansion leads to the stated result:

$$\int_0^\infty e^{-\xi t} \mathbb{E}[N_t^a] dt \sim \frac{1}{\xi^2} \frac{s^2}{a^2}.$$

□

## Appendix B. $\hat{P}$ -dynamics of the price process

In this appendix, we derive the dynamics of the price process under  $\hat{P}$ . This allows us to use expression (9) when computing option prices. We know that the compensator under the MMM is linked to the compensator under the historical measure (see Girsanov theorem for jumps in JS, p. 157) by the following relation:

$$\hat{v}(dt, dx) = v(dt, dx)(\hat{h}(t, x) + 1),$$

where  $v(dt, dx) = dA_t K(t, dx)$ . Besides, the compensator  $\hat{v}(dt, dx)$  under  $\hat{P}$  can also be disintegrated into a kernel part  $\hat{K}(t, dx)$  and an intensity component  $d\hat{\lambda}_t$ . Hence, after identification, we deduce:

$$d\hat{\lambda}_t = \left( \int_E (\hat{h}(t, x) + 1) K(t, dx) \right) dA_t, \quad (22)$$

$$\hat{K}(t, dx) = \frac{K(t, dx)(\hat{h}(t, x) + 1)}{\int_E (\hat{h}(t, x) + 1) K(t, dx)}, \quad (23)$$

which characterises the dynamics of the stock price under  $\hat{P}$ .

The normalisation factor  $\int_E (\hat{h}(t, x) + 1) K(t, dx)$  in Eq. (23) comes from the condition on the transition kernel to integrate to 1. In the marked Poisson case, substituting for  $\hat{h}$ ,  $K(t, dx)$ ,  $I(t, dx)$ ,  $\gamma$  and  $A_t$ , yields:

$$d\hat{\lambda}_t = \hat{\lambda} dt = (1 - \gamma) \lambda dt,$$

$$\hat{K}(t, a) = \hat{p} = \frac{p(\hat{h}(T_j, a) + 1)}{I(T_j, a)}.$$

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